SOME RESULTS ON THE $p(u)$-LAPLACIAN PROBLEM

M. CHIPOT[(1)]   H. B. DE OLIVEIRA[(2,3)]

[(1)] *IMath, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland*

[(2)] *FCT - Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal*

[(3)] *CMAFCIO - Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal*

Abstract. The $p-$Laplacian problem with the exponent of nonlinearity p depending on the solution u itself is considered in this work. Both situations when $p(u)$ is a local quantity or when $p(u)$ is nonlocal are studied here. For the associated boundary-value local problem, we prove the existence of weak solutions by using a singular perturbation technique. We also prove the existence of weak solutions to the nonlocal version of the associated boundary-value problem. The issue of uniqueness for these problems is addressed in this work as well, in particular by working out the uniqueness for a one dimensional local problem and by showing that the uniqueness is easily lost in the nonlocal problem.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with its boundary denoted by $\partial\Omega$. We consider the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a given data and

$$p : \mathbb{R} \longrightarrow [1, +\infty) \tag{1.2}$$

is the exponent function of nonlinearity.

Problems of the type (1.1) appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [3, 4, 14]. Variational models exhibit the solution of these problems as minimizers of appropriately chosen functionals and the minimization technique of such models involves the solution of nonlinear partial differential equations derived as necessary optimality conditions [13]. Several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u are presented in [14, Section 8]. A related, although far more complicated, minimization problem with the exponent of the regularization term depending on the gradient of the reconstructed image u, was considered in the works [3, 4]. In particular, the pioneer work [3, Section 3.3] presents a numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$.

E-mail address: `m.m.chipot@math.uzh.ch, holivei@ualg.pt`.

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Along with problem (1.1), we consider also in this work its nonlocal version,

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where f is again a given data and

$$p : \mathbb{R} \longrightarrow [1, +\infty), \tag{1.4}$$

$$b : W_0^{1,\alpha}(\Omega) \longrightarrow \mathbb{R} \tag{1.5}$$

are the functions involved in the exponent of nonlinearity, for some constant exponent α such that $1 < \alpha < \infty$. In this case, suitable examples for the mapping b in (1.5) are for instance

$$b(u) = \|\nabla u\|_\alpha,$$

or else

$$b(u) = \|u\|_q, \quad \text{for } q \leq \alpha^*, \quad \text{where } \frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{1}{d}.$$

where $\|\cdot\|_r$ denotes the usual $L^r(\Omega)$-norm. Problem (1.1) is the natural extension of the $p(x)-$Laplacian problem introduced by Zhikov [15] and for which the revival of interest in the almost last two decades came from modelling applications such as thermo or electro-rheological fluids [2, 11] and image restoration [10]. For the $p(x)-$Laplacian problem several issues of existence, uniqueness and regularity were already addressed in many works and by different authors (see again [2, 10, 11] and the references cited therein). However for the $p(u)-$Laplacian problem (1.1), and to our best knowledge, the only work is due to Andreianov *et al.* [1], where the considered prototype problem is

$$\begin{cases} u - \operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

As pointed out in [1], the main difficulty in the analysis of these $p(u)$-problems relies in the fact that neither the weak formulation of (1.1) nor (1.6) can be written as equalities in terms of duality in fixed Banach spaces. In particular, sequences of solutions u_n to these problems correspond to different exponents p_n and therefore belong to possible distinct Sobolev spaces. In the proof of [1, Theorem 2.8], the authors could not have used the abstract Minty argument on such a sequence of solutions, preferring instead a tool that pulls everything down to the L^1 space. Then they have used the description of weakly L^1 convergent sequences in terms of Young measures and their reduction using the monotonicity of the nonlinearity $\xi \mapsto |\xi|^{p-2}\xi$. We prove, in Theorem 5.1, the existence of weak solutions to the problem (1.1) by using the Minty trick together with the technique of Zhikov [16] for passing to the limit in our sequence of $p(u_n)-$Laplacian problems. For the sake of completeness we give an elementary proof, in Lemma 3.1, of a version of [16, Lemma 3.3] that is suitable to be applied in our problem. The technique we use here to prove the existence result for the problem (1.1) is rather simpler and more general than the one used in [1]. By assuming that $\partial\Omega$ belongs to some Hölder class $C^{0,\alpha}$ and the source term f belongs to $L^\infty(\Omega)$, it is established in [1, Theorem 2.9] the uniqueness of weak solutions that are Lipschitz-continuous. Establishing an uniqueness result without these assumptions seems to be a rather difficult task since there is *a priori* no guarantee that distinct solutions u_1 and u_2 are in the same test function space. With no further assumptions, we prove, in Theorem 6.1, a uniqueness result for a one dimensional version of our problem (1.1).

Regarding the nonlocal problem (1.3), it should be stressed that many diffusion, or reaction-diffusion, equations with distinct nonlocal terms have been studied in many works and by different authors since the pioneer works by Chipot *et al.* [8, 9]. However, we could not find in the literature any $p-$Laplacian problem with a nonlocal exponent of nonlinearity p as we consider here. Usually, the motivation to study nonlocal problems relies in the physical fact that in reality the measurements of some quantities are not made pointwise but through some local averages. In this work, we prove, in Theorem 5.1, the existence of weak solutions to the nonlocal problem (1.3), and we show in the final section how the uniqueness for this problem is easily lost.

This paper is organised as follows. In the next Section 2 we introduce the basic properties of generalised Sobolev spaces that we will used later. The Section 3 is devoted to two auxiliary lemmas. In the Section 4 we give a proof of existence of a solution to the local problem (1.1) using a singular perturbation technique.

The Section 5 is devoted to the existence of a solution to some nonlocal version of the problems we are interested in, i.e. to (1.3). Finally in the last Section 6 we evoke the issue of uniqueness for these problems working out in particular a one dimensional example. The notation used throughout the paper is nowadays rather standard in the analysis of Partial Differential Equations and therefore we address the reader to the monographs [2, 5, 6, 10, 11] for any question related to that matter.

2. generalised Sobolev Spaces

From the statement of the local problem (1.1), we can see that the exponent function p depends on the solution u and therefore it depends ultimately on the space variable x. As a consequence, the power p can be written as a variable exponent $q(x)$ in the following sense,

$$q(x) = p(u(x)).$$

This motivates us to look for the weak solutions to the problem (1.1) in a Sobolev space with variable exponents. The mathematical theory of these function spaces has been so developed during the last 20 years that we can now analyze the problem (1.1) in the light of this theory. In this section, we recall the properties of Lebesgue and Sobolev spaces with variable exponents which shall be used in the sequel. For this review, we have followed the monograph [2] (see also [10, 11, 17]).

To start with, we denote by $\mathcal{Q}(\Omega)$ the set of all Lebesgue-measurable functions $q : \Omega \to [1, \infty)$ and define

$$q_- := \operatorname*{ess\,inf}_{x\in\Omega} q(x), \quad q_+ := \operatorname*{ess\,sup}_{x\in\Omega} q(x). \tag{2.1}$$

Given $q \in \mathcal{Q}(\Omega)$, we denote by $L^{q(\cdot)}(\Omega)$ the space of all Lebesgue-measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that the modular is finite, i.e.

$$\rho_{q(\cdot)}(u) := \int_\Omega |u(x)|^{q(x)}\, dx < \infty. \tag{2.2}$$

Equipped with the Luxembourg norm

$$\|u\|_{q(\cdot)} := \inf\left\{\lambda > 0 : \rho_{q(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}, \tag{2.3}$$

$L^{q(\cdot)}(\Omega)$ becomes a Banach space. Note that the infimum in (2.3) is attained if (2.2) is positive. The space $L^{q(\cdot)}(\Omega)$ is a sort of Musielak-Orlicz space that we shall denote here by generalised Lebesgue space, because many of its properties are inherited from the classical Lebesgue spaces. If

$$1 \le q_- \le q_+ < \infty, \tag{2.4}$$

$L^{q(\cdot)}(\Omega)$ is separable and, in particular, $C_0^\infty(\Omega)$ is dense in $L^{q(\cdot)}(\Omega)$. Moreover, under assumption (2.4), $L^\infty(\Omega) \cap L^{q(\cdot)}(\Omega)$ is also dense in $L^{q(\cdot)}(\Omega)$. If we restrict (2.4) to

$$1 < q_- \le q_+ < \infty, \tag{2.5}$$

then $L^{q(\cdot)}(\Omega)$ is reflexive. In this case, the dual space of $L^{q(\cdot)}(\Omega)$ is identified with $L^{q'(\cdot)}(\Omega)$, where $q'(x)$ is the generalised Hölder conjugate of $q(x)$,

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1.$$

Note that from (2.1) and (2.5), we have

$$1 < (q_+)' \le \operatorname*{ess\,inf}_{x\in\Omega} q'(x) \le \operatorname*{ess\,sup}_{x\in\Omega} q'(x) \le (q_-)' < \infty.$$

One problem in generalised Lebesgue spaces, is the relation between the modular (2.2) and the norm (2.3) that is not so direct as in the classical Lebesgue spaces. However, if (2.5) holds, it can be proved, from its definitions in (2.2) and (2.3), that

$$\begin{aligned} &\min\left\{\|u\|_{q(\cdot)}^{q_-}, \|u\|_{q(\cdot)}^{q_+}\right\} \le \rho_{q(\cdot)}(u) \le \max\left\{\|u\|_{q(\cdot)}^{q_-}, \|u\|_{q(\cdot)}^{q_+}\right\}, \\ &\min\left\{\rho_{q(\cdot)}(u)^{\frac{1}{q_-}}, \rho_{q(\cdot)}(u)^{\frac{1}{q_+}}\right\} \le \|u\|_{q(\cdot)} \le \max\left\{\rho_{q(\cdot)}(u)^{\frac{1}{q_-}}, \rho_{q(\cdot)}(u)^{\frac{1}{q_+}}\right\}. \end{aligned} \tag{2.6}$$

When proving some estimates the following consequence of (2.6) is very useful,

$$\|u\|_{q(\cdot)}^{q_-}-1\leq\rho_{q(\cdot)}(u)\leq\|u\|_{q(\cdot)}^{q_+}+1. \tag{2.7}$$

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|u\,v|\leq\delta\frac{|u|^{q(x)}}{q(x)}+C(\delta)\frac{|v|^{q'(x)}}{q'(x)},$$

valid for some positive constant $C(\delta)$ and any $\delta>0$, and a version of Hölder's inequality,

$$\int_\Omega uv\,dx\leq\left(\frac{1}{q_-}+\frac{1}{q'_-}\right)\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)}\leq 2\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)}, \tag{2.8}$$

valid for $u\in L^{q(\cdot)}(\Omega)$ and $v\in L^{q'(\cdot)}(\Omega)$. As a consequence of (2.8), we have, for a bounded domain Ω and q satisfying to (2.4), the following continuous imbedding,

$$L^{q(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)\quad\text{whenever}\quad q(x)\geq r(x)\ \text{ for a.e. } x\in\Omega. \tag{2.9}$$

Assuming the weak derivatives $\dfrac{\partial u}{\partial x_i}$ exist for any $i\in\{1,\dots,d\}$, we define

$$W^{1,q(\cdot)}(\Omega):=\left\{u\in L^{q(\cdot)}(\Omega):\boldsymbol{\nabla}u\in L^{q(\cdot)}(\Omega)\right\},$$

which is a Banach space for the norm

$$\|u\|_{1,q(\cdot)}:=\|u\|_{q(\cdot)}+\|\boldsymbol{\nabla}u\|_{q(\cdot)}. \tag{2.10}$$

This space belongs to a special class of Sobolev-Orlicz spaces so called her generalised Sobolev spaces. The generalised Sobolev spaces $W^{1,q(\cdot)}(\Omega)$ inherit many of the properties of the generalised Lebesgue spaces $L^{q(\cdot)}(\Omega)$. In particular, $W^{1,q(\cdot)}(\Omega)$ is separable if (2.4) holds, and is reflexive when (2.5) is fulfilled. We have as in (2.9)

$$W^{1,q(\cdot)}(\Omega)\hookrightarrow W^{1,r(\cdot)}(\Omega)\quad\text{whenever}\quad q(x)\geq r(x)\ \text{ for a.e. } x\in\Omega. \tag{2.11}$$

We now introduce the following function space

$$W_0^{1,q(\cdot)}(\Omega):=\left\{u\in \mathrm{W}_0^{1,1}(\Omega):\boldsymbol{\nabla}u\in L^{q(\cdot)}(\Omega)\right\},$$

which we endow with the norm

$$\|u\|_{W_0^{1,q(\cdot)}(\Omega)}:=\|u\|_1+\|\boldsymbol{\nabla}u\|_{q(\cdot)}. \tag{2.12}$$

If $q\in C(\overline{\Omega})$, then an equivalent norm in $W_0^{1,q(\cdot)}(\Omega)$ is $\|\boldsymbol{\nabla}u\|_{q(\cdot)}$.

Unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W_0^{1,q(\cdot)}(\Omega)$. So, defining

$$H_0^{1,q(\cdot)}(\Omega):=\ \text{the closure of } C_0^\infty(\Omega) \text{ in the norm (2.10)},$$

where $C_0^\infty(\Omega)$ denotes the space of C^∞-functions with compact support in Ω we have generally

$$H_0^{1,q(\cdot)}(\Omega)\subsetneq W_0^{1,q(\cdot)}(\Omega).$$

However, if Ω is a bounded domain with $\partial\Omega$ Lipschitz-continuous and q is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,q(\cdot)}(\Omega)$. Recall that a function q is log-Hölder continuous, if

$$\exists\ C>0\ :\ |q(x)-q(y)|\leq\frac{C}{\ln\left(\frac{1}{|x-y|}\right)}\quad\forall\ x,\ y\in\Omega,\ |x-y|<\frac{1}{2}. \tag{2.13}$$

This means that

$$|q(x)-q(y)|\leq\omega(|x-y|)\quad\forall\ x,\ y\in\Omega,$$

for the modulus of continuity $\omega:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ defined by

$$\omega(t):=\frac{C}{\ln\left(\frac{1}{t}\right)},$$

which is an increasing and continuous function for $t < \frac{1}{2}$, and such that $\lim_{t\to 0^+} \omega(t) = 0$. If (2.13) holds, then we have

$$H_0^{1,q(\cdot)}(\Omega) = W_0^{1,q(\cdot)}(\Omega).$$

Note that in particular,

$$q \in C^{0,\lambda}(\Omega) \text{ for some } \lambda \in (0,1) \quad \Rightarrow \quad q \text{ is log-Hölder continuous.} \tag{2.14}$$

The Log-Hölder continuity property (2.13) is also very important to establish Sobolev inequalities in the framework of Sobolev spaces with variable exponents. Let us define the pointwise Sobolev conjugate of $q(x)$ by

$$q^*(x) := \begin{cases} \frac{dq(x)}{d-q(x)} & \text{if } q(x) < d \\ \infty & \text{if } q(x) \geq d. \end{cases}$$

If q is a measurable function in Ω satisfying to $1 \leq q_- \leq q_+ < d$ and (2.13), then

$$\|u\|_{q^*(\cdot)} \leq C\|\boldsymbol{\nabla} u\|_{q(\cdot)} \qquad \forall\, u \in W_0^{1,q(\cdot)}(\Omega),$$

for some positive constant C depending on q_+, d and on the constant of (2.13). On the other hand, if q satisfies (2.13) and $q_- > d$, then

$$\|u\|_\infty \leq C\|\boldsymbol{\nabla} u\|_{q(\cdot)} \qquad \forall\, u \in W_0^{1,q(\cdot)}(\Omega),$$

and where C is another positive constant depending on q_-, d and on the constant of (2.13).

3. Auxiliary results

To prove later that $|\boldsymbol{\nabla} u|^{p(u)} \in L^1(\Omega)$, we shall make use of the following result which is a particular case of a more general one established by Zhikov [16]. We give here an elementary proof of this result which does not require all the assumptions considered in [16, Lemma 3.3].

Lemma 3.1. Assume that

$$1 < \alpha \leq q_n(x) \leq \beta < \infty \qquad \forall\, n \in \mathbb{N}, \quad \text{for a.e. } x \in \Omega, \qquad \text{for some constants } \alpha \text{ and } \beta, \tag{3.1}$$

$$q_n \to q \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty, \tag{3.2}$$

$$\boldsymbol{\nabla} u_n \rightharpoonup \boldsymbol{\nabla} u \quad \text{in } L^1(\Omega)^d, \quad \text{as } n \to \infty, \tag{3.3}$$

$$\|\, |\boldsymbol{\nabla} u_n|^{q_n(x)}\|_1 \leq C, \quad \text{for some positive constant } C \text{ not depending on } n. \tag{3.4}$$

Then $\boldsymbol{\nabla} u \in L^{q(\cdot)}(\Omega)^d$ and

$$\liminf_{n\to\infty} \int_\Omega |\boldsymbol{\nabla} u_n|^{q_n(x)} dx \geq \int_\Omega |\boldsymbol{\nabla} u|^{q(x)} dx. \tag{3.5}$$

Proof. By Young's inequality one has for $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^d$ and $1 < q < \infty$,

$$\mathbf{a}\cdot\mathbf{b} \leq |\mathbf{a}|\,|\mathbf{b}| \leq |\mathbf{a}|^q + \frac{|\mathbf{b}|^{q'}}{q' q^{\frac{q'}{q}}}, \qquad \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.6}$$

If now $\mathbf{b}$ is a function in $L^\infty(\Omega)^d$ and we make $q = q_n$ in (3.6) and use assumption (3.1), one derives

$$\int_\Omega \left(\boldsymbol{\nabla} v_n \cdot \mathbf{b} - \frac{|\mathbf{b}|^{q_n'(x)}}{q_n'(x) q_n(x)^{\frac{q_n'(x)}{q_n(x)}}} \right) dx \leq \int_\Omega |\boldsymbol{\nabla} v_n|^{q_n(x)}\, dx. \tag{3.7}$$

Using assumptions (3.2) and (3.3), we can pass to the limit in (3.7) as $n \to \infty$, so that

$$\int_\Omega \left(\boldsymbol{\nabla} v \cdot \mathbf{b} - \frac{|\mathbf{b}|^{q'(x)}}{q'(x) q(x)^{\frac{q'(x)}{q(x)}}} \right) dx \leq \liminf_{n\to\infty} \int_\Omega |\boldsymbol{\nabla} v_n|^{q_n(x)}\, dx := L. \tag{3.8}$$

Then we consider the following function $\mathbf{b}$,

$$\mathbf{b} := \frac{\boldsymbol{\nabla} v}{|\boldsymbol{\nabla} v|} q(x) |\boldsymbol{\nabla} v|_k^{\frac{1}{q'(x)-1}}, \qquad \text{with } |\boldsymbol{\nabla} v|_k := |\boldsymbol{\nabla} v| \wedge k, \quad k > 0,$$

and where $u \wedge v := \min\{u, v\}$. Inserting this function $\mathbf{b}$ into (3.8), one obtains

$$\int_\Omega \left(|\nabla v|_k q(x) |\nabla v|_k^{\frac{1}{q'(x)-1}} - |\nabla v|_k^{\frac{q'(x)}{q'(x)-1}} \frac{q(x)}{q'(x)} \right) dx \leq L,$$

which implies

$$\int_\Omega |\nabla v|_k^{\frac{1}{q'(x)-1}+1} \, dx \leq L.$$

Observing that $\frac{1}{q'(x)-1} + 1 = q(x)$, we arrive at

$$\int_\Omega |\nabla v|_k^{q(x)} \, dx \leq L. \tag{3.9}$$

Finally (3.5) follows by letting $k \to \infty$ in (3.9), and $\nabla u \in L^{q(\cdot)}(\Omega)^d$ due to assumption (3.4). □

We recall also the following inequalities which are classical in the theory of p-Laplace equations.

Lemma 3.2. For all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$, the following assertions hold true:

$$2 \leq p < \infty \quad \Rightarrow \quad \frac{1}{2^{p-1}} |\boldsymbol{\xi} - \boldsymbol{\eta}|^p \leq \left(|\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi} - |\boldsymbol{\eta}|^{p-2}\boldsymbol{\eta} \right) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}); \tag{3.10}$$

$$1 < p < 2 \quad \Rightarrow \quad (p-1)|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \leq \left(|\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi} - |\boldsymbol{\eta}|^{p-2}\boldsymbol{\eta} \right) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \left(|\boldsymbol{\xi}|^p + |\boldsymbol{\eta}|^p \right)^{\frac{2-p}{p}}. \tag{3.11}$$

Proof. See, for instance, [6, 12]. □

4. Existence for the local problem

We define the set where we are going to look for the solutions to the problem (1.1) as

$$W_0^{1,p(u)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} dx < \infty \right\}.$$

If $1 < p(u) < \infty$ for all $u \in \mathbb{R}$, this set is a Banach space for the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined at (2.12) which is equivalent to $\|\nabla u\|_{p(u)}$ in the case of $p(u) \in C(\overline{\Omega})$. If for some constant α, $p \geq \alpha > 1$, p continuous, then, in view of (2.11), $W_0^{1,p(u)}(\Omega)$ is a closed subspace of $W_0^{1,\alpha}(\Omega)$ and therefore it is separable and reflexive. In what follows, $W^{-1,\gamma'}(\Omega)$, with $1 < \gamma < \infty$, denotes, as usual, the dual space of $W_0^{1,\gamma}(\Omega)$.

Definition 4.1. Let the function p given in (1.2) be continuous and assume that

$$1 < \alpha \leq p(u) \leq \beta < \infty \quad \forall u \in \mathbb{R}, \tag{4.1}$$

for some constants α and β. Assume also that

$$f \in W^{-1,\alpha'}(\Omega). \tag{4.2}$$

We say a function u is a weak solution to the problem (1.1) if

$$\begin{cases} u \in W_0^{1,p(u)}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(u)-2} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \qquad \forall\, v \in W_0^{1,p(u)}(\Omega), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_0^{1,p(u)}(\Omega))'$ and $W_0^{1,p(u)}(\Omega)$.

Note that, in view of this definition, $q = p(u) \in \mathcal{Q}(\Omega)$ and q_-, q_+, defined in (2.1), satisfy (2.4) for all $u \in W_0^{1,p(u)}(\Omega)$.

Theorem 4.1. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with $\partial\Omega$ Lipschitz-continuous. Assume that

$$p : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{is a Lipschitz-continuous function} \tag{4.3}$$

and that condition (4.2) holds. If

$$d < \alpha \leq p(u) \leq \beta < \infty \quad \forall u \in \mathbb{R}, \tag{4.4}$$

then there exists, at least, one weak solution to the problem (1.1) in the sense of Definition 4.1.

Proof. The proof of Theorem 4.1 will be split into two main steps.

1. Approximation: For each $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) - \varepsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u\right) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where β is the upper bound constant from assumption (4.4).

For an exponent function p satisfying (4.3) and (4.4), we say that a function u is a weak solution to the regularized problem (4.5), if

$$\begin{cases} u \in W_0^{1,\beta}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla v \, dx + \varepsilon \int_\Omega |\nabla u|^{\beta-2}\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,\beta}(\Omega), \end{cases}$$

where here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,\alpha'}(\Omega)$ and $W_0^{1,\alpha}(\Omega)$.

Claim 4.1. For each $\varepsilon > 0$ there exists a weak solution u_ε to the problem (4.5).

Proof. (Claim 4.1) Let $w \in L^2(\Omega)$ be given. From (4.4), we have

$$d < \alpha \leq p(w) \leq \beta < \infty \qquad \text{for a.e. } \ x \in \Omega. \tag{4.6}$$

Observing that, in view of the assumption (4.2) and of (4.6), we have

$$f \in W^{-1,\alpha'}(\Omega) \subset W^{-1,\beta'}(\Omega),$$

and, by the usual theory of monotone operators, there exists a unique $u = u_w$ solution to the problem

$$\begin{cases} u \in W_0^{1,\beta}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(w)-2}\nabla u \cdot \nabla v \, dx + \varepsilon \int_\Omega |\nabla u|^{\beta-2}\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,\beta}(\Omega). \end{cases} \tag{4.7}$$

Taking $v = u$ in the second line of (4.7) we derive using the Hölder inequality

$$\int_\Omega |\nabla u|^{p(w)} dx + \varepsilon \int_\Omega |\nabla u|^\beta dx \leq \|f\|_{-1,\alpha'} \|\nabla u\|_\alpha \leq C\|\nabla u\|_\beta,$$

for some positive constant $C = C(\alpha, \beta, \Omega, f)$, and where $\|\cdot\|_{-1,\alpha'}$ is the operator norm associated to the norm $\|\nabla \cdot\|_\alpha$. Thus

$$\varepsilon \|\nabla u\|_\beta^\beta \leq C\|\nabla u\|_\beta,$$

and

$$\|\nabla u\|_\beta \leq C, \tag{4.8}$$

for some positive constant $C = C(\alpha, \beta, \Omega, \varepsilon, f)$ independent of w. Since $\beta > d \geq 2$ one has $W_0^{1,\beta}(\Omega) \hookrightarrow L^2(\Omega)$, compactly and

$$\|u\|_2 = \|u_w\|_2 \leq C,$$

for some positive constant $C = C(\alpha, \beta, \Omega, \varepsilon, f, d)$ independent of w. Let us now consider the mapping

$$B \ni w \mapsto u_w \in B, \tag{4.9}$$

where $B := \{v \in L^2(\Omega) : \|v\|_2 \leq C\}$. From Schauder's fixed point theorem, it is clear that this mapping will have a fixed point provided it is continuous. To prove this, let us assume that w_n is a sequence in $L^2(\Omega)$ such that

$$w_n \to w \qquad \text{in } L^2(\Omega), \quad \text{as } n \to \infty. \tag{4.10}$$

For every $n \in \mathbb{N}$, let u_n be the solution to the problem (4.7) associated to $w = w_n$. By (4.8), one has

$$\|\boldsymbol{\nabla} u_n\|_\beta \leq C,$$

for some positive constant C which does not depend on n. Hence, for some subsequence still labelled by n and some u we have

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\beta}(\Omega), \quad \text{as } n \to \infty, \tag{4.11}$$

$$u_n \to u \quad \text{in } L^2(\Omega), \quad \text{as } n \to \infty. \tag{4.12}$$

By (4.7) written with u_n and w_n in the places of u and w, one has

$$\int_\Omega \left(|\boldsymbol{\nabla} u_n|^{p(w_n)-2}\boldsymbol{\nabla} u_n + \varepsilon|\boldsymbol{\nabla} u_n|^{\beta-2}\boldsymbol{\nabla} u_n\right)\cdot\boldsymbol{\nabla} v\,dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,\beta}(\Omega). \tag{4.13}$$

Then, by monotonicity, one has also

$$\begin{aligned}&\int_\Omega \left(|\boldsymbol{\nabla} u_n|^{p(w_n)-2}\boldsymbol{\nabla} u_n + \varepsilon|\boldsymbol{\nabla} u_n|^{\beta-2}\boldsymbol{\nabla} u_n\right)\cdot\boldsymbol{\nabla}(u_n - v)\,dx \\ &-\int_\Omega \left(|\boldsymbol{\nabla} v|^{p(w_n)-2}\boldsymbol{\nabla} v + \varepsilon|\boldsymbol{\nabla} v|^{\beta-2}\boldsymbol{\nabla} v\right)\cdot\boldsymbol{\nabla}(u_n - v)\,dx \geq 0 \qquad \forall\ v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{4.14}$$

Taking $v = u_n - v$ in (4.13) and using the resulting equation in (4.14), we derive

$$\langle f, u_n - v\rangle - \int_\Omega \left(|\boldsymbol{\nabla} v|^{p(w_n)-2}\boldsymbol{\nabla} v + \varepsilon|\boldsymbol{\nabla} v|^{\beta-2}\boldsymbol{\nabla} v\right)\cdot\boldsymbol{\nabla}(u_n - v)\,dx \geq 0 \qquad \forall\ v \in W_0^{1,\beta}(\Omega). \tag{4.15}$$

In view of (4.10) one can assume that for some subsequence

$$w_n \to w \qquad \text{a.e. in } \Omega, \quad \text{as } n \to \infty.$$

By virtue of this and by assumption (4.3), we can apply Lebesgue's theorem so that

$$|\boldsymbol{\nabla} v|^{p(w_n)-2}\boldsymbol{\nabla} v \to |\boldsymbol{\nabla} v|^{p(w)-2}\boldsymbol{\nabla} v \quad \text{strongly in } L^{\beta'}(\Omega)^d, \quad \text{as } n \to \infty, \tag{4.16}$$

for all $v \in W_0^{1,\beta}(\Omega)$. Using (4.11) and (4.16) we can pass to the limit in (4.15) to get

$$\langle f, u - v\rangle - \int_\Omega \left(|\boldsymbol{\nabla} v|^{p(w)-2}\boldsymbol{\nabla} v + \varepsilon|\boldsymbol{\nabla} v|^{\beta-2}\boldsymbol{\nabla} v\right)\cdot\boldsymbol{\nabla}(u - v)\,dx \geq 0 \qquad \forall\ v \in W_0^{1,\beta}(\Omega). \tag{4.17}$$

Taking $v = u \mp \delta z$, with $z \in W_0^{1,\beta}(\Omega)$ and $\delta > 0$, we obtain from (4.17)

$$\pm\left[\langle f, z\rangle - \int_\Omega \left(|\boldsymbol{\nabla}(u \mp \delta z)|^{p(w)-2}\boldsymbol{\nabla}(u \mp \delta z) + \varepsilon|\boldsymbol{\nabla}(u \mp \delta z)|^{\beta-2}\boldsymbol{\nabla}(u \mp \delta z)\right)\cdot\boldsymbol{\nabla} z\,dx\right] \geq 0. \tag{4.18}$$

Letting $\delta \to 0$ in (4.18), it comes

$$\int_\Omega \left(|\boldsymbol{\nabla} u|^{p(w)-2}\boldsymbol{\nabla} u + \varepsilon|\boldsymbol{\nabla} u|^{\beta-2}\boldsymbol{\nabla} u\right)\cdot\boldsymbol{\nabla} z\,dx = \langle f, z\rangle \qquad \forall\ z \in W_0^{1,\beta}(\Omega).$$

Thus $u = u_w$. Since the limit is uniquely determined we have, in view of (4.12)

$$u_{w_n} \to u_w \quad \text{strongly in } L^2(\Omega), \quad \text{as } n \to \infty,$$

which proves the continuity of the mapping (4.9) and thus concludes the proof of the claim. □

So far, we have proven that for each $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ such that

$$\int_\Omega |\boldsymbol{\nabla} u_\varepsilon|^{p(u_\varepsilon)-2}\boldsymbol{\nabla} u_\varepsilon\cdot\boldsymbol{\nabla} v\,dx + \varepsilon\int_\Omega |\boldsymbol{\nabla} u_\varepsilon|^{\beta-2}\boldsymbol{\nabla} u_\varepsilon\cdot\boldsymbol{\nabla} v\,dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,\beta}(\Omega). \tag{4.19}$$

Moreover recall that

$$d < \alpha \leq p(u_\varepsilon) \leq \beta < \infty \quad \forall\ \varepsilon > 0, \quad \text{for a.e. } x \in \Omega.$$

2. Passage to the limit as $\varepsilon \to 0$: Taking $v = u_\varepsilon$ in (4.19), we get

$$\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + \varepsilon \|\nabla u_\varepsilon\|_\beta^\beta = \langle f, u_\varepsilon\rangle. \tag{4.20}$$

Note that the first inequality in (2.7) can be written as

$$\|u\|_{q(\cdot)} \leq (\rho_{q(\cdot)}(u)+1)^{\frac{1}{q_-}} = \left(\int_\Omega |\nabla u|^{q(x)}dx + 1\right)^{\frac{1}{q_-}}.$$

Thus by the Hölder inequality (2.8) one has

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^\alpha dx &\leq C \, \||\nabla u_\varepsilon|^\alpha\|_{\frac{p(u_\varepsilon)}{\alpha}} \leq C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right)^{\frac{1}{\left(\frac{p(u_\varepsilon)}{\alpha}\right)_-}} \\ &\leq C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right),\end{aligned} \tag{4.21}$$

for some positive constant $C = C(\alpha, \beta, \Omega)$. Thus we have

$$\langle f, u_\varepsilon\rangle \leq \|f\|_{-1,\alpha'}\|\nabla u_\varepsilon\|_\alpha \leq C\|f\|_{-1,\alpha'}\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right)^{\frac{1}{\alpha}}. \tag{4.22}$$

One deduces from (4.20), (4.22) and by using Young's inequality that

$$\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + \varepsilon\|\nabla u_\varepsilon\|_\beta^\beta \leq C, \tag{4.23}$$

for some positive constant C which does not depend on ε. From (4.21) and (4.22) one then also has

$$\|\nabla u_\varepsilon\|_\alpha \leq C, \tag{4.24}$$

for some positive constant C independent of ε. Thus by the compact imbedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ we have for some subsequence still labelled with n and some u

$$u_{\varepsilon_n} \rightharpoonup u \qquad \text{in } W_0^{1,\alpha}(\Omega), \quad \text{as } n \to \infty, \tag{4.25}$$

$$\nabla u_{\varepsilon_n} \rightharpoonup \nabla u \qquad \text{in } L^\alpha(\Omega)^d, \quad \text{as } n \to \infty, \tag{4.26}$$

$$u_{\varepsilon_n} \to u \qquad \text{in } L^2(\Omega), \quad \text{as } n \to \infty,$$

$$u_{\varepsilon_n} \to u \qquad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{4.27}$$

It should be noticed that, due to (4.4), u is Hölder-continuous and, in view of this and (4.3), so does $p(u)$. Due to (4.27), one has also

$$p(u_{\varepsilon_n}) \longrightarrow p(u) \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{4.28}$$

Moreover, recall that

$$d < \alpha \leq p(u_{\varepsilon_n}) \leq \beta < \infty \qquad \forall\, n \in \mathbb{N}, \quad \text{for a.e. } x \in \Omega. \tag{4.29}$$

We can then use (4.23), with u_{ε_n} in the place of u_ε, together with (4.23), (4.26), (4.28) and (4.29) so that, by the application of Lemma 3.1, we have

$$u \in W_0^{1,p(u)}(\Omega). \tag{4.30}$$

Using the monotonicity, one has

$$\begin{aligned}&\int_\Omega \left(|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}\nabla u_{\varepsilon_n} + \varepsilon_n|\nabla u_{\varepsilon_n}|^{\beta-2}\nabla u_{\varepsilon_n}\right)\cdot \nabla(u_{\varepsilon_n} - v)\,dx \\ &- \int_\Omega \left(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon_n|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla(u_{\varepsilon_n}-v)\,dx \geq 0 \qquad \forall\, v \in W_0^{1,\beta}(\Omega).\end{aligned} \tag{4.31}$$

Using the identity (4.19), with u_{ε_n} in the place of u_ε and $u_{\varepsilon_n} - v$ in the place of v, we can write the inequality (4.31) as

$$\langle f, u_{\varepsilon_n} - v\rangle - \int_\Omega |\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v\cdot\nabla(u_{\varepsilon_n}-v)\,dx - \varepsilon_n\int_\Omega |\nabla v|^{\beta-2}\nabla v\cdot\nabla(u_{\varepsilon_n}-v)\,dx \geq 0, \tag{4.32}$$

say for all $v \in C_0^\infty(\Omega)$. Note that, as in (4.16) but now using (4.28), by the Lebesgue theorem, we have for such a v

$$|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v \to |\nabla v|^{p(u)-2}\nabla v \quad \text{in } L^{\alpha'}(\Omega)^d, \quad \text{as } n \to \infty. \tag{4.33}$$

Using (4.24), (4.25) and (4.33), we can pass to the limit in (4.32) as $n \to \infty$ so that

$$\langle f, u-v\rangle - \int_\Omega |\nabla v|^{p(u)-2}\nabla v \cdot \nabla(u-v)\,dx \geq 0 \qquad \forall\, v \in C_0^\infty(\Omega). \tag{4.34}$$

As observed above, due to assumptions (4.3) and (4.4), $p(u)$ is Hölder-continuous and therefore $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(u)}(\Omega)$ due to (2.13)-(2.14). Thus, (4.34) holds true also for all $v \in W_0^{1,p(u)}(\Omega)$. Hence we can take $v = u \mp \delta z$, with $z \in W_0^{1,p(u)}(\Omega)$ and $\delta > 0$, in (4.34) so that

$$\pm\left(\langle f, z\rangle - \int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla z\,dx\right) \geq 0.$$

As a consequence,

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla z\,dx = \langle f, z\rangle \qquad \forall\, z \in W_0^{1,p(u)}(\Omega),$$

which, together with (4.30), completes the proof of Theorem 4.1. □

5. Nonlocal problems

In this section we consider a real function p such that

$$p \text{ is continuous, } 1 < \alpha \leq p \leq \beta, \tag{5.1}$$

for some constants α, β. We denote by b a mapping from $W_0^{1,\alpha}(\Omega)$ into $\mathbb{R}$ such that

$$b \text{ is continuous, } b \text{ is bounded,} \tag{5.2}$$

i.e. b sends bounded sets of $W_0^{1,\alpha}(\Omega)$ into bounded sets of $\mathbb{R}$.

Definition 5.1. A function u is a weak solution to the problem (1.3) if

$$\begin{cases} u \in W_0^{1,p(b(u))}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(b(u))-2}\nabla u \cdot \nabla v\,dx = \langle f, v\rangle \qquad \forall\, v \in W_0^{1,p(b(u))}(\Omega), \end{cases} \tag{5.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(W_0^{1,p(b(u))}(\Omega))'$ and $W_0^{1,p(b(u))}(\Omega)$.

One should notice that $p(b(u))$ is here a real number and not a function so that the Sobolev spaces involved are the classical ones. We refer to [5, 7, 8, 9] for more on nonlocal problems.

Then one has:

Theorem 5.1. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain and assume that (5.1) and (5.2) hold together with

$$f \in W^{-1,\gamma'}(\Omega) \qquad \text{for } \ \gamma < \alpha.$$

Then there exists at least one weak solution to the problem (1.3) in the sense of Definition 5.1.

The proof of Theorem 5.1 is based on the following result.

Lemma 5.1. For $n \in \mathbb{N}$, let u_n be the solution to the problem

$$\begin{cases} u_n \in W_0^{1,p_n}(\Omega), \\ \displaystyle\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \cdot \nabla v\,dx = \langle f, v\rangle \qquad \forall\, v \in W_0^{1,p_n}(\Omega), \end{cases} \tag{5.4}$$

where $\langle\cdot,\cdot\rangle$ denotes here the duality pairing between $(W_0^{1,p_n}(\Omega))'$ and $W_0^{1,p_n}(\Omega)$. Suppose that

$$p_n \to p, \quad \text{as } n \to \infty, \qquad \text{where } \ p \in (1,\infty), \tag{5.5}$$

$$f \in W^{-1,q'}(\Omega) \quad \text{for some } \ q < p. \tag{5.6}$$

Then

$$u_n \to u \qquad \text{in} \;\; W_0^{1,q}(\Omega), \quad \text{as } n \to \infty, \tag{5.7}$$

where u is the solution to the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,p}(\Omega). \end{cases} \tag{5.8}$$

Proof. (Lemma 5.1) We shall split this proof into two steps.

1. Weak convergence: We first observe that, in view of $p_n \to p$, as $n \to \infty$, and $q < p$, we may assume that

$$p + 1 > p_n > q \qquad \forall\ n \in \mathbb{N}. \tag{5.9}$$

Taking $v = u_n$ in the equation of (5.4) we get

$$\int_\Omega |\nabla u_n|^{p_n} dx \le \|f\|_{-1,q'} \|\nabla u_n\|_q. \tag{5.10}$$

Recall that $\|f\|_{-1,q'}$ denotes the strong dual norm of f associated to the norm $\|\nabla \cdot\|_q$. On the other hand, by using Hölder's inequality and (5.9), we have

$$\|\nabla u_n\|_q \le C \|\nabla u_n\|_{p_n}, \tag{5.11}$$

for some positive constant $C = C(p, q, \Omega)$. Plugging (5.11) into (5.10) it comes

$$\|\nabla u_n\|_{p_n} \le C, \tag{5.12}$$

for some other positive constant $C = C(p, q, \Omega, f)$. Combining (5.11) with (5.12), it follows that

$$\|\nabla u_n\|_q \le C, \tag{5.13}$$

for some positive constant C independent of n. From (5.13) we deduce then that for some subsequence still labelled by n and for some $u \in W_0^{1,q}(\Omega)$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{ in } L^q(\Omega), \quad \text{as } n \to \infty. \tag{5.14}$$

Due to (5.5), (5.9), (5.12) and (5.14), we can also apply Lemma 3.1 so that

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p_n} dx \ge \int_\Omega |\nabla u|^p dx.$$

As a consequence we have

$$u \in W_0^{1,p}(\Omega). \tag{5.15}$$

Clearly the equation in (5.4) is equivalent to

$$\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \cdot \nabla(v - u_n)\, dx \ge \langle f, v - u_n\rangle \qquad \forall\ v \in W_0^{1,p_n}(\Omega).$$

and by the Minty lemma to

$$\int_\Omega |\nabla v|^{p_n-2}\nabla v \cdot \nabla(v - u_n)\, dx \ge \langle f, v - u_n\rangle \qquad \forall\ v \in W_0^{1,p_n}(\Omega). \tag{5.16}$$

Taking $v \in C_0^\infty(\Omega)$, one can use (5.5) and (5.14) to pass to the limit in (5.16), as $n \to \infty$, so that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u)\, dx \ge \langle f, v - u\rangle \qquad \forall\ v \in C_0^\infty(\Omega). \tag{5.17}$$

Using the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, we see that (5.17) also holds for all $v \in W_0^{1,p}(\Omega)$. In this case, taking $v = u \pm \delta z$, with $z \in W_0^{1,p}(\Omega)$ and $\delta > 0$, and letting $\delta \to 0$ after simplifying the resulting inequality, one obtains

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla z\, dx = \langle f, z\rangle \qquad \forall\ z \in W_0^{1,p}(\Omega).$$

Thus u is the solution to the problem (5.8).

2. Strong convergence: We want to show that the convergence (5.14) is in fact strong. To prove this, we first note that, taking $v = u_n$ in the equation of (5.4) and using (5.14) to pass to the limit, we obtain

$$\int_\Omega |\boldsymbol{\nabla} u_n|^{p_n} dx = \langle f, u_n\rangle \to \langle f, u\rangle = \int_\Omega |\boldsymbol{\nabla} u|^p dx, \text{ as } n \to \infty. \tag{5.18}$$

Consider the case of the p_n's such that

$$p_n \geq p \qquad \forall\, n \in \mathbb{N}.$$

One has by Hölder's inequality

$$\int_\Omega |\boldsymbol{\nabla} u_n|^p dx \leq \Big(\int_\Omega |\boldsymbol{\nabla} u_n|^{p_n} dx\Big)^{\frac{p}{p_n}} |\Omega|^{1-\frac{p}{p_n}},$$

where $|\Omega|$ denotes the $d-$Lebesgue measure of Ω. Thus by (5.18) for such a sequence

$$\limsup_{n\to\infty} \int_\Omega |\boldsymbol{\nabla} u_n|^p dx \leq \int_\Omega |\boldsymbol{\nabla} u|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\boldsymbol{\nabla} u_n|^p dx,$$

which shows (since $\|\boldsymbol{\nabla} u_n\|_p \to \|\boldsymbol{\nabla} u\|_p$, as $n \to \infty$)

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega), \quad \text{as } n \to \infty. \tag{5.19}$$

Since $W_0^{1,p}(\Omega) \subset W_0^{1,q}(\Omega)$, (5.19) implies (5.7).

Next, consider the p_n's such that

$$q < p_n < p \qquad \forall\, n \in \mathbb{N} \tag{5.20}$$

and set

$$A_n := \int_\Omega \big(|\boldsymbol{\nabla} u_n|^{p_n-2}\boldsymbol{\nabla} u_n - |\boldsymbol{\nabla} u|^{p_n-2}\boldsymbol{\nabla} u\big)\cdot(\boldsymbol{\nabla} u_n - \boldsymbol{\nabla} u)\; dx. \tag{5.21}$$

Due to the monotonicity, $A_n \geq 0$ and, because of (5.4), one has

$$A_n = \langle f, u_n - u\rangle - \int_\Omega |\boldsymbol{\nabla} u|^{p_n-2}\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla}(u_n - u)\, dx.$$

From (5.6) and (5.14), we have

$$\langle f, u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{5.22}$$

Moreover, from (5.15) one easily gets

$$\big||\boldsymbol{\nabla} u|^{p_n-2}\boldsymbol{\nabla} u\big| \leq \max\{1, |\boldsymbol{\nabla} u|\}^{p-1} \in L^{p'}(\Omega). \tag{5.23}$$

Hence, (5.20), (5.22) and (5.23) ensure that

$$A_n \to 0, \quad \text{as } n \to \infty. \tag{5.24}$$

Assume first that

$$p_n \geq 2.$$

This allows us to use property (3.10) of Lemma 3.2 in (5.21) so that

$$A_n \geq \frac{1}{2^{p_n-1}} \int_\Omega |\boldsymbol{\nabla}(u_n - u)|^{p_n}\, dx. \tag{5.25}$$

Since, by (5.20), $p_n > q$, we have by Hölder's inequality, (5.20), (5.24) and (5.25)

$$\int_\Omega |\boldsymbol{\nabla}(u_n - u)|^q\, dx \leq \Big(\int_\Omega |\boldsymbol{\nabla}(u_n - u)|^{p_n}\, dx\Big)^{\frac{q}{p_n}} |\Omega|^{1-\frac{q}{p_n}} \to 0,$$

when $n \to \infty$. This proves (5.7) in this case.

Consider now the case when

$$p_n < 2.$$

Here, we use Hölder's inequality as follows

$$\begin{aligned}&\int_\Omega |\boldsymbol{\nabla}(u_n-u)|^{p_n}\,dx =\\ &\int_\Omega |\boldsymbol{\nabla}(u_n-u)|^{p_n}\,(|\boldsymbol{\nabla}u_n|+|\boldsymbol{\nabla}u|)^{\frac{(p_n-2)p_n}{2}}\,(|\boldsymbol{\nabla}u_n|+|\boldsymbol{\nabla}u|)^{\frac{(2-p_n)p_n}{2}}\,dx \leq\\ &\left[\int_\Omega |\boldsymbol{\nabla}(u_n-u)|^2\,(|\boldsymbol{\nabla}u_n|+|\boldsymbol{\nabla}u|)^{p_n-2}\,dx\right]^{\frac{p_n}{2}}\left[\int_\Omega (|\boldsymbol{\nabla}u_n|+|\boldsymbol{\nabla}u|)^{p_n}\,dx\right]^{1-\frac{p_n}{2}}.\end{aligned} \tag{5.26}$$

Using property (3.11) of Lemma 3.2 we have

$$A_n \geq C\int_\Omega |\boldsymbol{\nabla}(u_n-u)|^2\,(|\boldsymbol{\nabla}u_n|+|\boldsymbol{\nabla}u|)^{p_n-2}\,dx, \tag{5.27}$$

for some positive constant $C = C(p_n)$. Now, by using (5.26), (5.27) together with (5.12) we deduce that

$$\int_\Omega |\boldsymbol{\nabla}(u_n-u)|^{p_n}\,dx \to 0, \quad \text{as } n\to\infty.$$

Thus, as above, (5.7) holds true also in this case. □

Let us now show how Lemma 5.1 can be applied to prove the existence of weak solutions to the nonlocal problem (1.3).

Proof. (Theorem 5.1) Note that $f \in (W_0^{1,\gamma}(\Omega))' \subset (W_0^{1,\delta}(\Omega))'$ for any $\delta > \gamma$. Thus for each $\lambda \in \mathbb{R}$, there exists a unique solution $u = u_\lambda$ to the $p(\lambda)$-Laplacian problem

$$\begin{cases} u \in W_0^{1,p(\lambda)}(\Omega),\\[2mm] \displaystyle\int_\Omega |\boldsymbol{\nabla}u|^{p(\lambda)-2}\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v\,dx = \langle f, v\rangle \qquad \forall\ v \in W_0^{1,p(\lambda)}(\Omega).\end{cases} \tag{5.28}$$

Taking $v = u = u_\lambda$ in (5.28) one derives

$$\int_\Omega |\boldsymbol{\nabla}u_\lambda|^{p(\lambda)}\,dx \leq \|f\|_{-1,\alpha'}\|\boldsymbol{\nabla}u_\lambda\|_\alpha. \tag{5.29}$$

By Hölder's inequality one has

$$\|\boldsymbol{\nabla}u_\lambda\|_\alpha \leq \|\boldsymbol{\nabla}u_\lambda\|_{p(\lambda)}|\Omega|^{\frac{1}{\alpha}-\frac{1}{p(\lambda)}}. \tag{5.30}$$

Thus by (5.29) it comes

$$\|\boldsymbol{\nabla}u_\lambda\|_{p(\lambda)}^{p(\lambda)-1} \leq \|f\|_{-1,\alpha'}|\Omega|^{\frac{1}{\alpha}-\frac{1}{p(\lambda)}}. \tag{5.31}$$

Gathering (5.30) and (5.31), and using (5.1) we obtain

$$\|\boldsymbol{\nabla}u_\lambda\|_\alpha \leq \|f\|_{-1,\alpha'}^{\frac{1}{p(\lambda)-1}}|\Omega|^{\left(\frac{1}{\alpha}-\frac{1}{p(\lambda)}\right)\frac{p(\lambda)}{p(\lambda)-1}} \leq \max_{p\in[\alpha,\beta]}\|f\|_{-1,\alpha'}^{\frac{1}{p-1}}|\Omega|^{\left(\frac{1}{\alpha}-\frac{1}{p}\right)\frac{p}{p-1}} = C, \tag{5.32}$$

for some positive constant $C = C(\alpha,\beta,\Omega,f)$. Due to the boundedness of b, see (5.2), and to (5.32), there exists $L \in \mathbb{R}$ such that

$$b(u_\lambda) \in [-L, L] \qquad \forall\ \lambda \in \mathbb{R}.$$

Let us now consider the map

$$\lambda \mapsto b(u_\lambda), \tag{5.33}$$

from $[-L, L]$ into itself. This map is continuous. Indeed, if $\lambda_n \to \lambda$ as $n \to \infty$, due to (5.1), we have $p(\lambda_n) \to p(\lambda)$. Applying now Lemma 5.1 with $p_n = p(\lambda_n)$, it follows that

$$u_{\lambda_n} \to u_\lambda \qquad \text{in} \quad W_0^{1,\alpha}(\Omega), \quad \text{as } n \to \infty.$$

Now, b being continuous (see (5.2)), it follows that $b(u_{\lambda_n}) \longrightarrow b(u_\lambda)$, as $n \to \infty$, and thus the map (5.33) is also continuous. It has then a fixed point λ_0 and u_{λ_0} is then solution to (5.3). □

6. Uniqueness

The proof of uniqueness of the solution to (1.1) in all generality does not seem to be straightforward. In fact, if we have two weak solutions u_1 and u_2 to the problem (1.1), in the sense of the Definition 4.1, there is a priori no guarantee that both functions u_1 and u_2 are in the same test function space, $W_0^{1,p(u_1)}(\Omega)$ or $W_0^{1,p(u_2)}(\Omega)$. Hence, we cannot use $u_1 - u_2$ as test function as usual. However, if we restrict ourselves to the 1-dimensional problem (1.1), it is possible to prove some uniqueness result. This is what we would like to do now.

Let us set, for instance,

$$\Omega = (-1, 1),$$

and consider the problem

$$\begin{cases} -\left(|u'|^{p(u)-2}u'\right)' = f & \text{in } \Omega, \\ u(-1) = u(1) = 0, \end{cases} \tag{6.1}$$

where u' denotes the derivative of u.

Theorem 6.1. Assume (4.1) and (4.3) and suppose that $f > 0$ is a continuous function on $[-1, 1]$. Then there exists a unique solution to (6.1) in the distributional sense.

Remark 6.1. One could weaken the assumptions on f. Note that from (6.1) it results that $|u'|^{p(u)-2}u'$ and also u' are bounded. u being a Lipschitz continuous function the boundary conditions are well defined.

Proof. (Theorem 6.1) In view of (6.1), one has

$$\left(|u'|^{p(u)-2}u'\right)' = -f < 0 \qquad \text{in } \Omega,$$

and therefore the function $|u'|^{p(u)-2}u'$ is decreasing in Ω. This function cannot have a constant sign, otherwise u' would have a constant sign too, i.e. u' would be always positive or negative, which in turn would render $u(-1) = u(1)$ impossible. Thus, $|u'|^{p(u)-2}u'$ and consequently u' vanish at some point $\xi \in (-1, 1)$. Since $|u'|^{p(u)-2}u'$ is decreasing in Ω, one has

$$u' > 0 \quad \text{in} \quad (-1, \xi) \qquad \text{and} \qquad u' < 0 \quad \text{in} \quad (\xi, 1). \tag{6.2}$$

Let us set

$$F(x) := \int_0^x f(s)\, ds. \tag{6.3}$$

Note that, due to the positivity of f, F is an increasing function. On the other hand, in view of (6.2), one derives from (6.1) that

$$|u'|^{p(u)-2}u' = F(\xi) - F(x)$$

for either $-1 < x < \xi$ or $\xi < x < 1$. Thus, (6.1) can be decoupled into two problems namely,

$$\begin{cases} u' = \left(F(\xi) - F(x)\right)^{\frac{1}{p(u)-1}} & \text{in } (-1, \xi), \\ u(-1) = 0, \end{cases} \tag{6.4}$$

and

$$\begin{cases} u' = -\left(F(x) - F(\xi)\right)^{\frac{1}{p(u)-1}} & \text{in } (\xi, 1), \\ u(1) = 0. \end{cases} \tag{6.5}$$

Claim 6.1. For a fixed ξ, the functions

$$G(x, u) = \left(F(\xi) - F(x)\right)^{\frac{1}{p(u)-1}} \qquad \text{and} \qquad H(x, u) = -\left(F(x) - F(\xi)\right)^{\frac{1}{p(u)-1}}$$

are uniformly Lipschitz-continuous with respect to u in the intervals $(-1, \xi)$ and $(\xi, 1)$, respectively.

Proof. (Claim 6.1) We shall only prove the claim for the function $G(x,u)$, the proof for the function $H(x,u)$ being analogous. Let us first write

$$G(x,u) - G(x,v) = e^{\frac{1}{p(u)-1}\ln(F(\xi)-F(x))} - e^{\frac{1}{p(v)-1}\ln(F(\xi)-F(x))}.$$

By the mean value theorem, one gets for some $\theta \in (0,1)$

$$G(x,u) - G(x,v) = e^{\left(\frac{\theta}{p(u)-1}+\frac{1-\theta}{p(v)-1}\right)\ln(F(\xi)-F(x))}\ln(F(\xi)-F(x))\left(\frac{1}{p(u)-1}+\frac{1}{p(v)-1}\right). \tag{6.6}$$

On the other hand, by the monotony of the ln function and due to assumption (4.1), one has

$$\ln(F(\xi)-F(x)) \in (-\infty, \ln(F(\xi)-F(-1))), \qquad \frac{\theta}{p(u)-1}+\frac{1-\theta}{p(v)-1} \in \left[\frac{1}{\beta-1}, \frac{1}{\alpha-1}\right]$$

and thus the product of the first two terms in (6.6) is uniformly bounded by a constant C, that does not depend on x. Observing this and using the assumption (4.3), one has

$$|G(x,u)-G(x,v)| \le C\left|\frac{1}{p(u)-1}-\frac{1}{p(v)-1}\right| = C\frac{|p(v)-p(u)|}{(p(v)-1)(p(u)-1)} \le C'|u-v|,$$

for another positive constant C', which proves the claim. □

As a consequence of Claim 6.1, the solutions to the problems (6.4) and (6.5) are unique.

Claim 6.2. The mapping $\xi \mapsto u(\xi)$ is an increasing function of ξ when ξ runs in the interval $(-1,1)$.

Proof. (Claim 6.2) Let us denote by $u(\xi,x)$ the unique solution to the problem (6.4). We would like to show that, whenever $\xi,\ \xi' \in (-1,1)$, we have

$$\xi' > \xi \ \Rightarrow\ u(\xi',\xi') > u(\xi,\xi). \tag{6.7}$$

Note again that, in view of the positivity of f the function F defined at (6.3) is increasing and therefore $F(\xi') > F(\xi)$. From (6.4) we derive then

$$u'(\xi,-1) = (F(\xi)-F(-1))^{\frac{1}{p(u)-1}} < (F(\xi')-F(-1))^{\frac{1}{p(u)-1}} = u'(\xi',-1), \tag{6.8}$$

which shows that $u(\xi,x) < u(\xi',x)$ near $x=-1$. If there is a first point $x_0 \in (-1,\xi)$ where $u(\xi',x_0) = u(\xi,x_0)$, we argue as we did for (6.8) so that one will have

$$u'(\xi,x_0) = (F(\xi)-F(x_0))^{\frac{1}{p(u)-1}} < (F(\xi')-F(x_0))^{\frac{1}{p(u)-1}} = u'(\xi',x_0),$$

which is impossible. Thus, $u(\xi,x) < u(\xi',x)$ on $(-1,\xi)$ and one has (6.7), since $u(\xi,\cdot)$ continues to grow after ξ.

With the same arguments, if $v(x,\xi)$ denotes the unique solution to the problem (6.5), one would show that, whenever $\xi,\ \xi' \in (-1,1)$,

$$\xi' > \xi \ \Rightarrow\ v(\xi',\xi') < v(\xi,\xi),$$

which completes the proof of the claim. □

Let us now show that the conclusion of Theorem 6.1 follows from Claim 6.2. In fact, since we know the solution to the problem (6.1) exists, there exists a $\xi \in (-1,1)$ such that

$$u(\xi,\xi) = v(\xi,\xi),$$

i.e. the solutions to the problems (6.4) and (6.5) merge at ξ. Then, for $\xi' > \xi$, Claim 6.2 ensures that

$$u(\xi',\xi') > u(\xi,\xi) = v(\xi,\xi) > v(\xi',\xi'),$$

and thus the two solutions could merge only at a point. The same argument applies for $\xi' < \xi$ and therefore the uniqueness for the problem (6.1) holds, which completes the proof of Theorem 6.1. □

The nonlocal case is completely different and one can see that uniqueness to the problem (1.3) is easily lost. To see this, let $p_1,\ p_2 \in (1,\infty)$, with $p_1 \neq p_2$, and consider u_1 and u_2 respectively the solutions to the problems

$$\begin{cases} -\operatorname{div}\left(|\boldsymbol{\nabla} u|^{p_i-2}\boldsymbol{\nabla} u\right) = f \quad \text{in} \quad \Omega, \\ \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

for $i = 1$ and $i = 2$. Choose a function b such that

$$b(u_1) \neq b(u_2).$$

Let us now consider a function p as in (1.4) and such that

$$p(b(u_1)) = p_1 \quad \text{and} \quad p(b(u_2)) = p_2.$$

Then u_1 and u_2 are both solutions to the problem (1.3). One can this way construct problems with infinitely many solutions.

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