On a Class of Intermediate Local-Nonlocal Elliptic Problems

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Abstract

This paper is concerned with the existence of solutions for a class of local-nonlocal boundary value problems of the following type

$$(IP) \qquad -div\left[a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right] = f(x,u,\nabla u) \text{ in } \Omega, \quad u \in H_0^1(\Omega)$$

where Ω is a bounded domain of $\mathbb{R}^N$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N$ is a given function, $r > 0$ is a fixed number, $\Omega(x,r) = \Omega \cap B(x,r)$, where $B(x,r) = \left\{y \in \mathbb{R}^N; |y-x| < r\right\}$. Here $|\cdot|$ is the Euclidian norm, $\fint_{\Omega(x,r)} u(y)dy = \frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} u(y)dy$ and $|X|$ denotes the Lebesgue measure of a measurable set $X \subset \mathbb{R}^N$.

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1 Introduction

In this work we will be concerned with the intermediate class of local-nonlocal elliptic problem

$$(IP) \qquad -div\left(a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right) = f(x,u,\nabla u) \ \text{ in } \ \Omega, \ u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a bounded domain, $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, $r > 0$ is a fixed real number,

$$\Omega(x,r) := \Omega \cap B(x,r),$$

with

$$B(x,r) := \left\{y \in \mathbb{R}^N; |y-x| < r\right\}.$$

Here $|\cdot|$ is the usual Euclidian norm of $\mathbb{R}^N$ and

$$\fint_{\Omega(x,r)} u(y)dy = \frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} u(y)dy,$$

where $|\Omega(x,r)|$ is the Lebesgue measure of the set $\Omega(x,r)$.

Note that (IP) is a class of interpolating problem between the purely local problem

$$(L) \qquad -div\,(a(u(x))\nabla u) = f(x,u,\nabla u) \ \text{ in } \ \Omega, \ u \in H_0^1(\Omega)$$

and the nonlocal problem

$$(NL) \qquad -div\left(a\left(\fint_{\Omega} u(x)dx\right)\nabla u\right) = f(x,u,\nabla u) \ \text{ in } \ \Omega, \ u \in H_0^1(\Omega).$$

Note that in our case, we are considering a nonlocal quantity $\fint_{\Omega(x,r)} u(y)dy$ which is calculated locally in neighborhoods of the form $\Omega(x,r)$.

Remark 1.1. *Although we are working in the space $H_0^1(\Omega)$, we may treat problem (IP) in the space $H_0^1(\Omega;\Gamma_0)$, where $\Gamma_0 \subset \partial\Omega$ is a part of $\partial\Omega$ of positive measure. See, for example [4].*

The purely nonlocal counterpart of problem problem (IP) is given by problem (NL) and has been studied by several authors like [8], [6] and [7] among others. Equations like (NL) appears in several phenomena. For instance, $u = u(x)$ may represent a density of population (for instance of bacteria) subject to spreading and because we are considering homogeneous Dirichlet boundary condition ($u \in H_0^1(\Omega)$) it means that the domain Ω is surrounded by inhospitable environment. Contrary to the local model in which the crowding effect of the population u at x only depends on the value of the population in the same point, the model (NL) considers the case in which the crowding effect depends on the total population in Ω. In the present model (IP) the crowding effect depends also on the value of the population in neighborhoods of x. According to [5], see also [1], such a model seems to be more realistic.

In the present paper, we use mainly Galerkin's method in order to attack problem (IP). For this, our approach relies on a variant of the Brouwer Fixed Point Theorem which will be quoted below. Its proof may be found in Lions [9], p. 53.

Proposition 1.2. *Suppose that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function such that $(F(\xi), \xi) \geq 0$ on $|\xi| = r$, where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^m$ and $|\cdot|$ its corresponding norm. Then there exists $\xi_0 \in \overline{B_r(0)}$ such that $F(\xi_0) = 0$.*

This paper is organized as follows. In Section 2, we consider the existence of solution for a class of pseudo-linear problem, while in Section 3 we prove the existence of solution for a large class of nonlinearity involving a convective term.

2 A Pseudo-Linear Problem

We first study the pseudo-linear version of the problem (IP). More precisely, for each $f \in H^{-1}(\Omega)$, we search weak solutions of the problem

$$(PL) \qquad -div\left(a\left(⨍_{\Omega(x,r)} u(y)dy\right)\nabla u\right) = f(x) \text{ in } \Omega, \quad u \in H_0^1(\Omega).$$

Here $H_0^1(\Omega)$ is understood as the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ and is supposed to be equipped with the Dirichlet norm $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$. $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$ and $< \ , \ >$ will denote the duality bracket between these spaces. We will suppose

(H_1) $\quad a$ is continuous and there exists $\lambda > 0$ such that $a(s) \geq \lambda > 0 \; \forall s \in \mathbb{R}$.

Moreover, we will say that Ω is regular, if there is $\tau > 0$ such that

$$|\Omega(x, r)| \geq \tau = \tau(r) > 0, \quad \forall x \in \overline{\Omega}. \tag{2.1}$$

Note that this is the case for a smooth domain.

Our main result in this section is the following:

Theorem 2.1. *If a satisfies (H_1) and if*

(i) a is bounded

or

(ii) Ω is regular,

then for each $f \in H^{-1}(\Omega)$, the problem (PL) possesses a weak solution $u \in H_0^1(\Omega)$.

Proof: Since the operator

$$Lu = -div\left(a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right)$$

has no variational structure, we will attack the problem (PL) by using a Galerkin method. For that, let

$$\mathbb{B} = \{e_1, e_2, \ldots\}$$

be an Hilbertian basis of $H_0^1(\Omega)$ satisfying

$$((e_i, e_j)) = \delta_{ij},$$

where $((\cdot,\cdot))$ is the usual inner product in $H_0^1(\Omega)$ and δ_{ij} is the Kroenecker symbol. Setting

$$\mathbb{V}_m := [e_1, \ldots, e_m],$$

the span of the set $\{e_1, \ldots, e_m\}$, for each $u \in \mathbb{V}_m$ there is $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ such that

$$u = \sum_{j=1}^{m} \xi_j e_j.$$

Thus $\|u\| = |\xi|$, where

$$\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} \quad \text{and} \quad |\xi| = \left(\sum_{j=1}^{m} \xi_j^2\right)^{\frac{1}{2}}.$$

Consequently, $\mathbb{V}_m$ and $\mathbb{R}^m$ are isometrically isomorphic finite dimensional vector spaces. Unless we say something on the contrary, we identify $u \longleftrightarrow \xi, u \in \mathbb{V}_m, \xi \in \mathbb{R}^m$.

Let $F : \mathbb{R}^m \to \mathbb{R}^m, F = (F_1, \ldots, F_m)$ given by

$$F_i(\xi) = \int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u \cdot \nabla e_i - < f, e_i >,\ i = 1, 2, \ldots, m \tag{2.2}$$

so that

$$F_i(\xi)\xi_i = \int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u \cdot \nabla(\xi_i e_i) - < f, (\xi_i e_i) >,\ i = 1, 2, \ldots, m. \tag{2.3}$$

Consequently,

$$((F(\xi),\xi)) = \int_\Omega a\left(⨍_{\Omega(x,r)} u(y)dy\right)|\nabla u|^2 - < f, u >, \ \forall u \in \mathbb{V}_m. \tag{2.4}$$

In view of the assumption (H_1)

$$((F(\xi),\xi)) \geq \lambda\|u\|^2 - ||f||^*\|u\|, \ \forall u \in \mathbb{V}_m. \tag{2.5}$$

where $||f||^*$ denotes the strong dual norm of f. Then

$$((F(\xi),\xi)) > 0, \quad \text{if } \ \|u\| > \frac{||f||^*}{\lambda}.$$

Therefore, there is $u_m \in \mathbb{V}_m$ with $\|u_m\| \leq \frac{||f||^*}{\lambda}$, such that $F(u_m) = 0$, i.e.,

$$0 = F_i(u_m) = \int_\Omega a\left(⨍_{\Omega(x,r)} u_m(y)dy\right)\nabla u_m \nabla e_i - \int_\Omega f e_i, \ \forall i = 1,\dots,m. \tag{2.6}$$

Hence,

$$\int_\Omega a\left(⨍_{\Omega(x,r)} u_m(y)dy\right)\nabla u_m \nabla \varphi = \int_\Omega f\varphi, \quad \forall \varphi \in \mathbb{V}_k, \quad k \leq m. \tag{2.7}$$

In what follows we fix k. From the boundedness of $\|u_m\|$, it follows that there is a subsequence of (u_m), still labelled by m, such that

$$u_m \rightharpoonup u \ \text{ in } \ H_0^1(\Omega),$$

and

$$u_m \to u \ \text{ in } \ L^2(\Omega).$$

As $u_m \to u$ also in $L^1(\Omega)$ and Ω is bounded, we have that

$$\begin{aligned}\left|\int_{\Omega(x,r)} u_m dy - \int_{\Omega(x,r)} u dy\right| &\leq \int_{\Omega(x,r)} |u_m - u| dy \\ &\leq \int_\Omega |u_m - u|dy \to 0, \ \text{uniformly for} \quad x \in \Omega.\end{aligned}$$

In view of the continuity of a it follows that

$$a\left(⨍_{\Omega(x,r)} u_m dy\right) \to a\left(⨍_{\Omega(x,r)} u dy\right), \ \text{for each} \quad x \in \Omega. \tag{2.8}$$

It is easy to see that in both cases (i) or (ii), $a\left(\fint_{\Omega(x,r)} u_m dy\right)$ is bounded independently of m. Thus by the Lebesgue theorem

$$a\left(\fint_{\Omega(x,r)} u_m(y)dy\right)\nabla\varphi \to a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla\varphi \ \text{ in } \ L^2(\Omega). \tag{2.9}$$

As

$$\nabla u_m \rightharpoonup \nabla u \ \text{ in } \ L^2(\Omega), \tag{2.10}$$

taking the limit of $m \to +\infty$ in (2.7), we get

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\nabla\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in \mathbb{V}_k. \tag{2.11}$$

Since k is arbitrary, we obtain

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\nabla\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in H_0^1(\Omega), \tag{2.12}$$

showing that u is a weak solution of the problem (PL). ■

Here, we would like to point out that one could use also the Schauder fixed point theorem in the spirit of [4] in order to get the existence result above. However the technique we developed here will be useful in the second part of the paper.

3 A Sublinear Singular Problem with a Convective Term

In this section, our main goal is to study a problem involving sublinear, singular and convective terms. More precisely, we will be concerned with the existence of positive solutions to the problem

$$\begin{cases} -div\left(a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right) = H(x)u^\alpha + \frac{K(x)}{u^\gamma} + L(x)|\nabla u|^\theta \text{ in } \Omega, \\ \\ u \in H_0^1(\Omega), \end{cases} \tag{3.1}$$

where $H(x), K(x), L(x) \geq 0$, for all $x \in \Omega$, are given functions whose properties will be timely introduced and α, γ and θ are positive numbers suitably chosen.

Remark 3.1. *We should remark that it would be more natural, before studying the problem (3.1), to attack problems like*

$$\begin{cases} -div\left(a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right) = a(x)u^\alpha + b(x)u^\beta, \ \mathit{in}\ \Omega, \\ u > 0, \ \mathit{in}\ \Omega, \\ u = 0, \ \mathit{on}\ \partial\Omega, \end{cases} \tag{3.2}$$

where a and b are given functions and $\alpha, \beta > 0$ are real numbers.

Note that if $0 < \alpha < 1$ and $b \equiv 0$ we have a typical sublinear problem. If $a \equiv 0$ and $1 < \beta \leq 2^*$ we are in the presence of a superlinear problem. If both a, b are not simultaneously vanishing and $0 < \alpha < 1 < \beta \leq 2^*$ we have a concave-convex problem which is studied, for example, by Ambrosetti-Brezis-Cerami. Due to some technical difficulties we were not able yet to deal with it.

In order to attack problem (3.1), let us begin by considering the auxiliary problem

$$\begin{cases} -div\left(a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u\right) & = H(x)(u^+)^\alpha \\ & \quad + \frac{K(x)}{(|u|+\epsilon)^\gamma} + L(x)|\nabla u|^\theta \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{3.3}$$

where $0 < \epsilon < 1$ is a fixed number.

We will consider the following assumptions

$$(H_2) \qquad 0 < \alpha, \gamma < 1,$$

$$(H_3) \qquad H, K, L \in L^\infty(\Omega), \text{ and for } h_0 > 0, \quad H(x), K(x), L(x) \geq h_0, \text{ a.e. } x \in \Omega,$$

$$(H_4) \qquad 0 < \theta < 1.$$

Theorem 3.2. *Under the assumptions of Theorem 2.1 and $(H_1) - (H_4)$ the problem (3.3) possesses a positive solution.*

Proof: As in the previous section, we introduce functions $F_i(\xi)$, given now by

$$\begin{aligned} F_i(\xi) = \int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)\nabla u \nabla e_i - \int_\Omega H(u^+)^\alpha e_i \\ - \int_\Omega \frac{K}{(|u|+\epsilon)^\gamma} e_i - \int_\Omega L|\nabla u|^\theta e_i, \end{aligned} \tag{3.4}$$

for all $i = 1, \ldots, m$. Hence

$$\begin{aligned} ((F(\xi), \xi)) = \int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)|\nabla u|^2 - \int_\Omega H(u^+)^\alpha u \\ - \int_\Omega K \frac{u}{(|u|+\epsilon)^\gamma} - \int_\Omega L|\nabla u|^\theta u. \end{aligned} \tag{3.5}$$

We recall that, as before, we are identifying $u \in \mathbb{V}_m$ with $\xi \in \mathbb{R}^m$. As $a(s) \geq \lambda > 0$ for all $s \in \mathbb{R}$, we have

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u(y)dy\right)|\nabla u|^2 \geq \lambda \int_\Omega |\nabla u|^2.$$

On the other hand, by Sobolev's continuous embedding and Poincaré's inequality

$$\int_\Omega H(u^+)^\alpha u \le C\|H\|_\infty \left(|\nabla u|^2\right)^{\frac{\alpha+1}{2}} = C\|H\|_\infty \|u\|^{\alpha+1}$$

and

$$\int_\Omega K\frac{u}{(|u|+\epsilon)^\gamma} \le \int_\Omega K|u|^{1-\gamma} \le C\|K\|_\infty\|u\|^{1-\gamma},$$

for some positive constant C, which is independent of ϵ. Here, we point out that, at this stage, $0<\epsilon<1$ is fixed. In view of (H_4), one has $0<\theta<1<\frac{N+2}{N}\le 2$ if $N\ge 2$, that is, in particular $\theta<2$. Thus,

$$\left|\int_\Omega L|\nabla u|^\theta u\right| \le \|L\|_\infty \left[\int_\Omega (|\nabla u|^\theta)^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}} \left(\int_\Omega |u|^{\frac{2}{2-\theta}}\right)^{\frac{2-\theta}{2}}.$$

Since $0<\theta<\frac{N+2}{N}, N\ge 2$, we also have $\frac{2}{2-\theta}<2^*=\frac{2N}{N-2}$ and so $H_0^1(\Omega)\hookrightarrow L^{\frac{2}{2-\theta}}(\Omega)$. So,

$$\left|\int_\Omega L|\nabla u|^\theta u\right| \le \|L\|_\infty\|u\|^\theta |u|_{\frac{2}{2-\theta}} \le C\|u\|^{\theta+1}.$$

These last inequalities imply that

$$((F(\xi),\xi)) \ge \lambda\|u\|^2 - C\|H\|_\infty\|u\|^{\alpha+1} - C\|K\|_\infty\|u\|^{1-\gamma} - C\|u\|^{\theta+1}. \tag{3.6}$$

In view of assumptions $(H_2)-(H_4)$, we may find a real constant $R>0$ such that

$$((F(\xi),\xi)) > 0 \text{ if } \|u\| = |\xi| = R. \tag{3.7}$$

Here is important to observe that R does not depend on m or ϵ. By the Brouwer Fixed Point Theorem, there is $u_m\in\mathbb{V}_m$ such that

$$F(u_m)=0,\ \ \|u_m\|\le R,\ m=1,2,\dots \tag{3.8}$$

that is,

$$\begin{aligned}\int_\Omega a\left(⨍_\Omega u_m(y)dy\right)\nabla u_m\nabla\varphi &= \int_\Omega H(u_m^+)^\alpha\varphi \\ &+ \int_\Omega \frac{K}{(|u_m|+\epsilon)^\gamma}\varphi + \int_\Omega L|\nabla u_m|^\theta\varphi,\ \forall\varphi\in\mathbb{V}_m.\end{aligned} \tag{3.9}$$

Since $\|u_m\|\le R$ for all $m\in\mathbb{N}$, there is $u_\epsilon\in H_0^1(\Omega)$ such that, perhaps for some subsequence,

$$u_m \rightharpoonup u_\epsilon \text{ in } H_0^1(\Omega),$$

$$u_m \to u_\epsilon \text{ in } L^q(\Omega), 1\le q<2^*,$$

$$u_m(x)\to u_\epsilon(x) \text{ a.e. in } \Omega.$$

(We have a conflict of notation between u_m and u_ϵ but it should be no trouble). We now fix $1 \leq k < m$ and $\varphi \in \mathbb{V}_k$. As in the previous section,

$$\int_\Omega a\left(\fint_{\Omega((x,r)} u_m(y)dy\right)\nabla u_m \nabla\varphi \to \int_\Omega a\left(\fint_{\Omega((x,r)} u_\epsilon(y)dy\right)\nabla u_\epsilon \nabla\varphi,\ \forall\varphi\in\mathbb{V}_k.$$

At the expense of extracting a subsequence we can assume that

$$u_m \to u_\epsilon \text{ in } L^q(\Omega), \quad \text{and} \quad |u_m| \leq h \quad \text{a.e.}$$

for some $h \in L^q(\Omega)$. Since for $q > 2$, $h^\alpha\varphi \in L^1(\Omega)$, by the Lebesgue dominated convergence theorem, for each $\varphi \in \mathbb{V}_k$ we have

$$\int_\Omega H(u_m^+)^\alpha\varphi \to \int_\Omega H(u_\epsilon^+)^\alpha\varphi$$

and

$$\int_\Omega \frac{K}{(|u_m|+\epsilon)^\gamma}\varphi \to \int_\Omega \frac{K}{(|u_\epsilon|+\epsilon)^\gamma}\varphi.$$

Our next step is to pass to the limit in the gradient term. Since (u_n) is bounded in $H_0^1(\Omega)$, it is easy to prove that $(|\nabla u_m|^\theta)$ is bounded in $L^{\frac{2}{\theta}}(\Omega)$. Then, there is $g \in L^{\frac{2}{\theta}}(\Omega)$ such that

$$L|\nabla u_m|^\theta \rightharpoonup g \text{ in } L^{\frac{2}{\theta}}(\Omega), \tag{3.10}$$

or equivalently

$$\int_\Omega L|\nabla u_m|^\theta\varphi \to \int_\Omega g\varphi,\ \ \forall\ \ \varphi\in L^{(\frac{2}{\theta})'}(\Omega),$$

where $(\frac{2}{\theta})' = \frac{2}{2-\theta}$ is the conjugate exponent of $\frac{2}{\theta}$. Furthermore,

$$\int_\Omega L|\nabla u_m|^\theta u_m = \int_\Omega L|\nabla u_m|^\theta u_\epsilon + \int_\Omega L|\nabla u_m|^\theta(u_m - u_\epsilon).$$

In view of $u_m \to u_\epsilon$ in $L^{\frac{2}{2-\theta}}(\Omega)$ (note that $\frac{2}{2-\theta} < 2 < 2^*$), we obtain

$$\begin{aligned}\left|\int_\Omega L|\nabla u_m|^\theta(u_m-u_\epsilon)\right| &\leq \|L\|_\infty\left(\int_\Omega(|\nabla u_m|^\theta)^{\frac{2}{\theta}}\right)^{\frac{\theta}{2}}\left(\int_\Omega|u_m-u_\epsilon|^{\frac{2}{2-\theta}}\right)^{\frac{2-\theta}{2}}\\ &\leq C\|u_m-u_\epsilon\|_{L^{\frac{2}{2-\theta}}} \to 0.\end{aligned}$$

Consequently,

$$\int_\Omega L|\nabla u_m|^\theta u_m \to \int_\Omega g u_\epsilon.$$

Fixing e_j, we obtain, for $1 \leq j \leq k$,

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_m(y)dy\right)\nabla u_m\nabla e_j = \int_\Omega H(u_m^+)^\alpha e_j + \int_\Omega \frac{K}{(|u_m|+\epsilon)^\gamma}e_j + \int_\Omega L|\nabla u_m|^\theta e_j.$$

Taking limits as $m \to +\infty$, we get

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_\epsilon(y)dy\right)\nabla u_\epsilon \nabla e_j = \int_\Omega H(u_\epsilon^+)^\alpha e_j + \int_\Omega \frac{K}{(|u_\epsilon|+\epsilon)^\gamma} e_j + \int_\Omega Lge_j.$$

Since k is arbitrary, this last equality becomes

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_\epsilon(y)dy\right)\nabla u_\epsilon \nabla \varphi = \int_\Omega H(u_\epsilon^+)^\alpha \varphi + \int_\Omega \frac{K}{(|u_\epsilon|+\epsilon)^\gamma} \varphi + \int_\Omega Lg\varphi$$

for all $\varphi \in H_0^1(\Omega)$. Hence, u_ϵ is a weak solution of the problem

$$\begin{aligned} -div\left(a\left(\fint_{\Omega(x,r)} u_\epsilon(y)dy\right)\nabla u_\epsilon\right) &= \\ H(u_\epsilon^+)^\alpha + \frac{K}{(|u_\epsilon|+\epsilon)^\gamma} &+ Lg \text{ in } \Omega, \quad u_\epsilon \in H_0^1(\Omega). \end{aligned} \tag{3.11}$$

Since a, H, K and g are nonnegative functions, the maximum principle ensures that $u_\epsilon \geq 0$, and so, u_ϵ is solution to

$$\begin{aligned} -div\left(a\left(\fint_{\Omega(x,r)} u_\epsilon(y)dy\right)\nabla u_\epsilon\right) &= \\ H(x)u_\epsilon^\alpha + \frac{K(x)}{(u_\epsilon+\epsilon)^\gamma} &+ L(x)g \text{ in } \Omega, \quad u_\epsilon \in H_0^1(\Omega). \end{aligned} \tag{3.12}$$

Therefore,

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_\epsilon(y)dy\right)|\nabla u_\epsilon|^2 = \int_\Omega Hu_\epsilon^{\alpha+1} + \int_\Omega \frac{K}{(u_\epsilon+\epsilon)^\gamma} u_\epsilon + \int_\Omega Lgu_\epsilon. \tag{3.13}$$

On the other hand, we know that

$$\begin{aligned} \int_\Omega a\left(\fint_{\Omega(x,r)} u_m(y)dy\right)|\nabla u_m|^2 &= \\ \int_\Omega H(u_m^+)^{\alpha+1} + \int_\Omega \frac{K}{(|u_m|+\epsilon)^\gamma} u_m &+ \int_\Omega L|\nabla u_m|^\theta u_m. \end{aligned} \tag{3.14}$$

Hence

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_m(y)dy\right)|\nabla u_m|^2 \to \int_\Omega Hu_\epsilon^{\alpha+1} + \int_\Omega \frac{K}{(u_\epsilon+\epsilon)^\gamma} + \int_\Omega Lgu_\epsilon. \tag{3.15}$$

From (3.13) and (3.15),

$$\int_\Omega a\left(\fint_{\Omega(x,r)} u_m(y)dy\right)|\nabla u_m|^2 \to \int_\Omega a\left(\fint_{\Omega(x,r)} u_\epsilon\right)|\nabla u_\epsilon|^2. \tag{3.16}$$

Arguing as in Section 1 one has $a\left(⨍_{\Omega(x,r)} u_m dy\right)$ is bounded independently of m and

$$a\left(⨍_{\Omega(x,r)} u_m(y)dy\right) \to a\left(⨍_{\Omega(x,r)} u_\epsilon(y)dy\right) \quad \text{for all} \quad x \in \Omega. \tag{3.17}$$

Hence

$$\begin{aligned}\int_\Omega |\nabla(u_m - u_\epsilon)|^2 &\leq \frac{1}{\lambda}\int_\Omega a\left(⨍_{\Omega(x,r)} u_m(y)dy\right)|\nabla u_m - u_\epsilon|^2 \\ &= \frac{1}{\lambda}\int_\Omega a\left(⨍_{\Omega(x,r)} u_m(y)dy\right)\{|\nabla u_m|^2 - 2\nabla u_m \cdot \nabla u_\epsilon + |\nabla u_\epsilon|^2\} \\ &\to 0.\end{aligned} \tag{3.18}$$

i.e.

$$u_m \to u_\epsilon \quad \text{in} \quad H_0^1(\Omega). \tag{3.19}$$

The above limit implies that up to a subsequence

$$L|\nabla u_m|^\theta \rightharpoonup L|\nabla u_\epsilon|^\theta \quad \text{in} \quad L^{\frac{2}{\theta}}(\Omega). \tag{3.20}$$

To see that, note first that from

$$\int_\Omega (|\nabla u_m| - |\nabla u_\epsilon|)^2 \leq \int_\Omega |\nabla u_m - \nabla u_\epsilon|^2$$

one derives that

$$|\nabla u_m| \to |\nabla u_\epsilon| \quad \text{in} \quad L^2(\Omega).$$

Thus, up to a subsequence one has

$$|\nabla u_m| \to |\nabla u_\epsilon| \ \text{ a.e. in } \Omega \quad , \quad |\nabla u_m| \leq h$$

for some $h \in L^2(\Omega)$. This implies that for any $\varphi \in L^{(\frac{2}{\theta})'}(\Omega)$

$$L|\nabla u_m|^\theta \varphi \leq L h^\theta \varphi$$

with $Lh^\theta\varphi \in L^1(\Omega)$. Then (3.20) follows from the Lebesgue dominated convergence theorem. Now, we recall that $\forall j = 1, 2, \ldots$

$$\int_\Omega a\left(⨍_\Omega u_m(y)dy\right)\nabla u_m \nabla e_j = \int_\Omega H(u_m^+)^\alpha e_j + \int_\Omega \frac{K}{(|u_m| + \epsilon)^\gamma} e_j + \int_\Omega L|\nabla u_m|^\theta e_j.$$

Gathering (3.17), (3.19), (3.20) and taking limits as $m \to +\infty$ on both sides of the last equality, we obtain

$$\int_\Omega a\left(⨍_\Omega u_\epsilon(y)dy\right)\nabla u_\epsilon \nabla e_j = \int_\Omega H(u_\epsilon^+)^\alpha e_j + \int_\Omega \frac{K}{(u_\epsilon + \epsilon)^\gamma} e_j + \int_\Omega L|\nabla u_\epsilon|^\theta e_j.$$

So, $u_\epsilon \in H_0^1(\Omega)$ is a positive weak solution of the auxiliary problem (3.3). ■

Now, we are ready to prove the main result of this section

Theorem 3.3. *Under the same assumptions as in Theorem 3.2, problem (3.1) possesses a positive solution*

Proof: First of all we note that that we will use the notation introduced in the previous sections. Thus, we recall that $\|u_m\| \leq R$ for all $m = 1, 2, \ldots$ and R does not depend on ϵ. Hence

$$\|u_\epsilon\| \leq \liminf \|u_m\| \leq R.$$

Consequently, fixing $\epsilon_n = \frac{1}{n}$ and $v_n := u_{\epsilon_n}$, for some subequence still denoted by n, there exists $v \in H_0^1(\Omega)$ satisfying

$$v_n \rightharpoonup v \ \text{ in } \ H_0^1(\Omega),$$

$$v_n \to v \ \text{ in } \ L^q(\Omega), \ 1 \leq q < 2^*,$$

and

$$v_n(x) \to v(x) \ \text{ a.e. in } \ \Omega.$$

Let us consider the function

$$M(t) = h_0 t^\alpha + \frac{h_0}{(t+1)^\gamma} \text{ for } \quad t \geq 0, \tag{3.21}$$

where h_0 is defined in the assumption (H_3). Thus, there is $m_0 > 0$ such that

$$M(t) \geq m_0 > 0 \ \ \forall t \geq 0. \tag{3.22}$$

Noticing that

$$H(x)v_n^\alpha + \frac{K(x)}{(v_n+\epsilon_n)^\gamma} + L|\nabla v_n|^\theta \geq h_0 v_n^\alpha + \frac{h_0}{(v_n+\epsilon_n)^\gamma} \geq m_0 \quad \forall n \in \mathbb{N},$$

we obtain

$$-div\left(a\left(⨍_{\Omega(x,r)} v_n(y)dy\right)\nabla v_n\right) \geq m_0 \quad \text{in } \ \Omega \quad \forall n \in \mathbb{N}. \tag{3.23}$$

Let $\omega_n > 0$ be the unique solution of the problem

$$-div\left(a\left(⨍_{\Omega(x,r)} v_n(y)dy\right)\nabla w_n\right) = m_0 \ \text{ in } \ \Omega, w_n \in H_0^1(\Omega). \tag{3.24}$$

Note that, for each $n \in \mathbb{N}$, $a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)$ is a positive function, which belongs to $C(\overline{\Omega})$, this implies the positivity of w_n. Consequently,

$$-div\left(a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)\nabla v_n\right) \geq -div\left(a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)\nabla w_n\right) \quad (3.25)$$

i.e.

$$\int_\Omega a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)\nabla v_n \nabla\varphi \geq \int_\Omega a\left(\fint_{\Omega(x,r)} v_n\right)\nabla w_n \nabla\varphi \quad (3.26)$$

for all $\varphi \in H_0^1(\Omega), \varphi \geq 0$. This implies by the maximum principle

$$v_n \geq w_n \text{ in } \Omega. \quad (3.27)$$

Since

$$\int_\Omega a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)\nabla w_n \nabla\varphi = \int_\Omega m_0\varphi \ \ \forall\varphi \in H_0^1(\Omega) \quad (3.28)$$

we have

$$\lambda\|w_n\|^2 \leq C\|w_n\| \quad (3.29)$$

and so, $\|w_n\| \leq C$ for all $n \in \mathbb{N}$. As before, there is $w \in H_0^1(\Omega)$ such that $w_n \rightharpoonup w$ in $H_0^1(\Omega)$and

$$-div\left(a\left(\fint_{\Omega(x,r)} v(y)dy\right)\nabla w\right) = m_0 \text{ in } \Omega, w \in H_0^1(\Omega). \quad (3.30)$$

Consequently, $w > 0$ in Ω and $w \in C(\overline{\Omega})$. In view of (3.27), if $n \to \infty$, we obtain

$$v(x) \geq w(x) > 0 \text{ a.e in } \Omega. \quad (3.31)$$

We now claim that up to a subsequence

$$\nabla v_n(x) \to \nabla v(x) \quad \text{a.e in} \quad \Omega.$$

Indeed, given $\Omega' \subset\subset \Omega$, there is $\phi \in C_0^\infty(\Omega)$ such that

$$\phi(x) = 1 \quad \forall x \in \Omega'.$$

Repeating the arguments of the proof of the previous theorem and using (3.31) to control the singular term, we deduce also that for some $g \in L^{\frac{2}{\theta}}(\Omega)$ (see (3.10))

$$\int_\Omega a\left(\fint_\Omega v(y)dy\right)\nabla v\nabla\psi = \int_\Omega Hv^\alpha\psi + \int_\Omega \frac{K}{v^\gamma}\psi + \int_\Omega g\psi,$$

for all $\psi \in H_0^1(\Omega)$ with compact support. Taking $\psi = v\phi$ leads to

$$\int_\Omega a\left(\fint_\Omega v(y)dy\right)|\nabla v|^2\phi + a\left(\fint_\Omega v(y)dy\right)\nabla v\nabla\phi\, v$$
$$= \int_\Omega Hv^\alpha v\phi + \int_\Omega \frac{K}{v^\gamma}v\phi + \int_\Omega gv\phi,$$

Now taking $v_n\phi$ as test function in the equation satisfied by v_n one gets

$$\int_\Omega a\left(\fint_\Omega v_n(y)dy\right)|\nabla v_n|^2\phi + a\left(\fint_\Omega v_n(y)dy\right)\nabla v_n\nabla\phi\, v$$
$$= \int_\Omega Hv_n^\alpha v_n\phi + \int_\Omega \frac{K}{v_n^\gamma}v_n\phi + \int_\Omega |\nabla v_n|^\theta v_n\phi,$$

Taking the limit in n we deduce easily arguing as in the proof of Theorem 3.2

$$\int_\Omega a\left(\fint_\Omega v_n(y)dy\right)|\nabla v_n|^2\phi \to \int_\Omega a\left(\fint_\Omega v(y)dy\right)|\nabla v|^2\phi \tag{3.32}$$

We have also,

$$\int_{\Omega'} |\nabla(v_n - v)|^2 \leq \frac{1}{\lambda}\int_\Omega a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)|\nabla v_n - v|^2\phi \tag{3.33}$$

and

$$\int_\Omega a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)|\nabla v_n - v|^2\phi =$$
$$\int_\Omega a\left(\fint_{\Omega(x,r)} v_n(y)dy\right)\left(|\nabla v_n|^2 - 2\nabla v_n.\nabla v + |\nabla v|^2\right)\phi.$$

From (3.32) and (3.33) taking the limit in n we deduce

$$|\nabla v_n - \nabla v| \to 0 \quad \text{in} \quad L^2(\Omega').$$

Hence, for some subsequence,

$$\nabla v_n(x) \to \nabla v(x) \quad \text{a.e in} \quad \Omega'.$$

As Ω' is arbitrary, it follows that

$$\nabla v_n(x) \to \nabla v(x) \quad \text{a.e in} \quad \Omega.$$

Now, gathering this with the boundedness of $(|\nabla v_n|^\theta)$ in $L^{\frac{2}{\theta}}(\Omega)$ we can conclude as below (3.20) that the weak limit of $(|\nabla v_n|^\theta)$ in $L^{\frac{2}{\theta}}(\Omega)$ is $|\nabla v|^\theta$, that is,

$$\int_\Omega |\nabla v_n|^\theta\psi \to \int_\Omega |\nabla v|^\theta\psi, \quad \forall\psi \in L^{\frac{2}{\theta}}(\Omega).$$

Using this, we derive easily that v verifies

$$\begin{aligned}\int_\Omega a\left(⨍_\Omega v(y)dy\right)\nabla v\nabla\psi \\ = \int_\Omega Hv^\alpha\psi + \int_\Omega \frac{K}{v^\gamma}\psi + \int_\Omega L|\nabla v|^\theta\psi, \quad \forall\psi\in C_0^\infty(\Omega).\end{aligned} \tag{3.34}$$

From the above equality, there is $C>0$ such that

$$\left|\int_\Omega \frac{K\psi}{v^\gamma}\right| \le C\|\psi\|, \quad \forall\psi\in C_0^\infty(\Omega).$$

Combining the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ with the last inequality, we derive that

$$\left|\int_\Omega \frac{Kw}{v^\gamma}\right| \le C\|w\|, \quad \forall w\in H_0^1(\Omega).$$

Then, if $w\in H_0^1(\Omega)$ and $(\psi_n)\subset C_0^\infty(\Omega)$ verify

$$\psi_n\to w \quad \text{in} \quad H_0^1(\Omega),$$

we can infer that

$$\lim_{n\to\infty}\int_\Omega \frac{K\psi_n}{v^\gamma} = \int_\Omega \frac{Kw}{v^\gamma}.$$

The last limit combined with equality (3.34) and Sobolev embeddings gives

$$\begin{aligned}\int_\Omega a\left(⨍_\Omega v(y)dy\right)\nabla v\nabla\psi \\ = \int_\Omega Hv^\alpha\psi + \int_\Omega \frac{K}{v^\gamma}\psi + \int_\Omega L|\nabla v|^\theta\psi, \quad \forall\psi\in H_0^1(\Omega),\end{aligned} \tag{3.35}$$

showing that v is a solution of problem (3.1). ■

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