

when it comes to finding a fixed point for S some compactness for this mapping is missing. One way to overcome this is to introduce for instance the following anisotropic singular perturbations problem

$$\begin{cases} \partial_t u_\varepsilon - a \left(\int_{\Omega} u_\varepsilon(\cdot, x) dx \right) (\varepsilon^2 \partial_{x_1}^2 u_\varepsilon + \partial_{x_2}^2 u_\varepsilon) = f \text{ in } (0, T) \times \Omega, \\ u_\varepsilon = 0 \text{ on } (0, T) \times \partial\Omega, \\ u_\varepsilon(0) = u^0 \text{ in } \Omega, \end{cases} \quad (2)$$

and to see what can be said on the behaviour of u_ε when $\varepsilon > 0$ goes to 0. This is what we would like to do in this note. For more details about the anisotropic singular perturbations problems from different point of view, the reader is referred to [1, 3–6, 8, 9] (see [10] for the basic theory of singular perturbations).

2. A more general setting

Now we suppose that Ω is a bounded open subset of \mathbb{R}^n and introduce a problem generalizing (2). We split the components of a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ into two parts

$$X_1 = (x_1, \dots, x_p) \quad \text{and} \quad X_2 = (x_{p+1}, \dots, x_n),$$

so that $x = (X_1, X_2)$ (p and n are positive integers such that $p < n$). Similarly setting

$$\nabla_{X_1} u = (\partial_{x_1} u, \dots, \partial_{x_p} u)^T, \quad \nabla_{X_2} u = (\partial_{x_{p+1}} u, \dots, \partial_{x_n} u)^T,$$

where T denotes the transpose operation and ∂_{x_i} the partial derivative in the direction x_i , we have if ∇ denotes the usual gradient operator

$$\nabla u = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}. \quad (3)$$

It will be convenient to simplify the notation to set

$$z = (t, x) \in \mathbb{R}^{n+1}, \quad Q = (0, T) \times \Omega. \quad (4)$$

Let

$$A = A(z, s) = (a_{ij}(z, s)) \quad (5)$$

be a $n \times n$ matrix defined for $(z, s) \in Q \times \mathbb{R}$ and such that A is of Carathéodory type that is to say

$$z \mapsto a_{ij}(z, s) \text{ is measurable } \forall s \in \mathbb{R}, \quad \forall i, j = 1, \dots, n, \quad (6)$$

$$s \mapsto a_{ij}(z, s) \text{ is continuous a.e. } z \in Q, \quad \forall i, j = 1, \dots, n. \quad (7)$$

Moreover we assume that there exist two positive constants λ, Λ such that

$$\lambda |\xi|^2 \leq A(z, s) \xi \cdot \xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } z \in Q, \quad \forall s \in \mathbb{R}. \quad (8)$$

In the inequality above we have denoted the scalar product by a dot and the usual Euclidean norm by $|\cdot|$. Note that (8) implies that the coefficients a_{ij} are uniformly bounded.

We split A into four blocks in such way that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (9)$$

where A_{11} , A_{22} are respectively $p \times p$ and $(n - p) \times (n - p)$ matrices. We then set for $\varepsilon > 0$

$$A_\varepsilon = A_\varepsilon(z, s) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \quad (10)$$

If $\xi = \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix} \in \mathbb{R}^n$ where $\bar{\xi}_1 = (\xi_1, \dots, \xi_p)^T$, $\bar{\xi}_2 = (\xi_{p+1}, \dots, \xi_n)^T$ by (8) one has for $\xi_\varepsilon = \begin{pmatrix} \varepsilon \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix}$

$$\lambda |\xi_\varepsilon|^2 = \lambda \{ \varepsilon^2 |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2 \} \leq A \xi_\varepsilon \cdot \xi_\varepsilon = A_\varepsilon \xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } z \in Q, \forall s \in \mathbb{R}, \quad (11)$$

$$\lambda |\bar{\xi}_2|^2 \leq A_{22} \bar{\xi}_2 \cdot \bar{\xi}_2 \quad \forall \bar{\xi}_2 \in \mathbb{R}^{n-p}, \text{ a.e. } z \in Q, \forall s \in \mathbb{R}. \quad (12)$$

We would like to consider the family of parabolic, nonlinear, nonlocal problems

$$\begin{cases} \partial_t u_\varepsilon - \nabla \cdot (A_\varepsilon(\cdot, l(u_\varepsilon)) \nabla u_\varepsilon) = f & \text{in } Q, \\ u_\varepsilon = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_\varepsilon(0, \cdot) = u^0 & \text{in } \Omega, \end{cases} \quad (13)$$

where

$$l(u) = \int_\Omega h(x) u(t, x) dx \quad (14)$$

for some $h \in L^2(\Omega)$ i.e. l is a continuous linear form on $L^2(\Omega)$. We are in particular interested to show that u_ε - perhaps up to subsequence - possesses a limit when $\varepsilon \rightarrow 0$. Note that for $A(z, s) = a(s) Id$ where Id is the identity matrix in \mathbb{R}^2 , $h \equiv 1$ one recognizes the problem (2) of the introduction and the limit of u_ε will provide us with a solution to problem (1).

We suppose here that

$$f \in L^2(Q), \quad u^0 \in L^2(\Omega). \quad (15)$$

Then we have

Theorem 2.1: *Under the assumptions above for any $\varepsilon > 0$ there exists a weak*

deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_\varepsilon|_{2,\Omega}^2 + \lambda \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 dx &\leq C |f|_{2,\Omega} |\nabla_{X_2} u_\varepsilon|_{2,\Omega} \\ &\leq \frac{C^2}{2\lambda} |f|_{2,\Omega}^2 + \frac{\lambda}{2} |\nabla_{X_2} u_\varepsilon|_{2,\Omega}^2 \end{aligned}$$

and in particular

$$\frac{1}{2} \frac{d}{dt} |u_\varepsilon|_{2,\Omega}^2 + \frac{\lambda}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 dx \leq \frac{C^2}{2\lambda} |f|_{2,\Omega}^2. \quad (24)$$

Integrating in t between 0 and T we get

$$\frac{1}{2} |u_\varepsilon|_{2,\Omega}^2(T) + \frac{\lambda}{2} \int_0^T \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 dt dx \leq \frac{1}{2} |u^0|_{2,\Omega}^2 + \frac{C^2}{2\lambda} |f|_{2,Q}^2, \quad (25)$$

and thus

$$u_\varepsilon, \quad |\varepsilon \nabla_{X_1} u_\varepsilon|, \quad |\nabla_{X_2} u_\varepsilon| \quad \text{are bounded in } L^2(Q), \quad (26)$$

(this of course independently of ε). From the equation in (16) we derive for $v \in H_0^1(\Omega)$,

$$\begin{aligned} |\langle \partial_t u_\varepsilon, v \rangle| &\leq |(f, v)| + \left| \int_{\Omega} A_\varepsilon(t, \cdot, l(u_\varepsilon)) \nabla u_\varepsilon \cdot \nabla v dx \right| \\ &\leq |f(t, \cdot)|_{2,\Omega} |v|_{2,\Omega} + C (|\varepsilon \nabla_{X_1} u_\varepsilon|_{2,\Omega} + |\nabla_{X_2} u_\varepsilon|_{2,\Omega}) |\nabla_{X_2} v|_{2,\Omega} \\ &\leq C' (|f(t, \cdot)|_{2,\Omega} + |\varepsilon \nabla_{X_1} u_\varepsilon|_{2,\Omega} + |\nabla_{X_2} u_\varepsilon|_{2,\Omega}) |v|_{H^1(\Omega)}, \end{aligned}$$

where C, C' are constants independent of $\varepsilon \rightarrow 0$, $|v|_{H^1(\Omega)}$ the usual $H^1(\Omega)$ -norm. This implies

$$|\partial_t u_\varepsilon|_{L^2(0,T;H^{-1}(\Omega))} \leq C' (|f|_{2,Q} + |\varepsilon \nabla_{X_1} u_\varepsilon|_{2,Q} + |\nabla_{X_2} u_\varepsilon|_{2,Q}).$$

It follows from (26) that

$$\partial_t u_\varepsilon \quad \text{is bounded in } L^2(0, T; H^{-1}(\Omega)). \quad (27)$$

Up to a subsequence we deduce from (26), (27) that there exist $u_0, u_1, u_2, \in L^2(Q)$ and $u_3 \in L^2(0, T; H^{-1}(\Omega))$ - i.e. with components in $L^2(Q)$ for the vectors - such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0, & \varepsilon \nabla_{X_1} u_\varepsilon &\rightharpoonup u_1, & \nabla_{X_2} u_\varepsilon &\rightharpoonup u_2, & \text{in } L^2(Q), \\ \partial_t u_\varepsilon &\rightharpoonup u_3 & \text{in } L^2(0, T; H^{-1}(\Omega)). \end{aligned}$$

By the continuity of the derivation in $\mathcal{D}'(Q)$ we derive that

$$u_\varepsilon \rightharpoonup u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \quad \text{in } L^2(Q), \quad (28)$$

$$\partial_t u_\varepsilon \rightharpoonup \partial_t u_0 \quad \text{in } L^2(0, T; H^{-1}(\Omega)). \quad (29)$$

A delicate issue is now to show that

$$u_0(t, X_1, \cdot) \in H_0^1(\Omega_{X_1}) \quad \text{a.e.} \quad (t, X_1) \in (0, T) \times \Pi_1(\Omega).$$

For that we denote by \mathcal{B} an open ball of \mathbb{R}^{n-p} such that

$$\Omega_{X_1} \subset \mathcal{B} \quad \forall X_1 \in \Pi_1(\Omega). \quad (30)$$

We suppose that $u_\varepsilon(t, \cdot)$ is extended by 0 outside Ω . Then from (26) we derive that

$$\int_0^T \int_{\Pi_1(\Omega) \times \mathcal{B}} |\nabla_{X_2} u_\varepsilon|^2 dt dx \leq C, \quad (31)$$

where C is a constant independent of ε . This can also be written as

$$|u_\varepsilon|_{L^2((0, T) \times \Pi_1(\Omega); H_0^1(\mathcal{B}))} \leq C.$$

Thus there exists a function $\bar{u}_0 \in L^2((0, T) \times \Pi_1(\Omega); H_0^1(\mathcal{B}))$, a subsequence of the subsequence above and still labelled ε such that

$$u_\varepsilon \rightharpoonup \bar{u}_0 \quad \text{in } L^2((0, T) \times \Pi_1(\Omega); H_0^1(\mathcal{B}))$$

and in particular

$$u_\varepsilon \rightarrow \bar{u}_0 \quad \text{in } \mathcal{D}'((0, T) \times \Pi_1(\Omega) \times \mathcal{B}).$$

If u_0 is also extended by 0 outside Ω one has of course also

$$u_\varepsilon \rightarrow u_0 \quad \text{in } \mathcal{D}'((0, T) \times \Pi_1(\Omega) \times \mathcal{B})$$

and thus

$$u_0 = \bar{u}_0 \in L^2((0, T) \times \Pi_1(\Omega); H_0^1(\mathcal{B})).$$

It follows that

$$u_0(t, X_1, \cdot) \in H_0^1(\mathcal{B}) \quad \text{a.e.} \quad (t, X_1) \in (0, T) \times \Pi_1(\Omega),$$

i.e.

$$u_0(t, X_1, \cdot) \in H_0^1(\Omega_{X_1}) \quad \text{a.e.} \quad (t, X_1) \in (0, T) \times \Pi_1(\Omega). \quad (32)$$

We next show that

$$l(u_\varepsilon) \rightarrow l(u_0).$$

This will follow from the lemma:

Lemma 3.3: *For any $v \in H_0^1(\Omega)$, the functions*

$$t \mapsto \int_{\Omega} u_\varepsilon v dx, \quad t \mapsto \int_{\Omega} u_0 v dx \quad (33)$$

belong to $H^1(0, T)$ and for the subsequence above we have

$$\begin{aligned} (i) \quad & \int_{\Omega} u_{\varepsilon} v dx \rightarrow \int_{\Omega} u_0 v dx \quad \text{in } L^2(0, T), \mathcal{C}(0, T), \\ (ii) \quad & \int_{\Omega} u_{\varepsilon} v dx \rightharpoonup \int_{\Omega} u_0 v dx \quad \text{in } H^1(0, T), \\ (iii) \quad & l(u_{\varepsilon}) \rightarrow l(u_0) \quad \text{in } L^2(0, T). \end{aligned}$$

($\mathcal{C}(0, T)$ denotes the space of continuous functions on $[0, T]$ for the uniform norm).

Proof: [Proof of the lemma 3.3]

From (16) we derive for $v \in H_0^1(\Omega)$

$$\frac{d}{dt}(u_{\varepsilon}(t, \cdot), v) = (f(t, \cdot), v) - \int_{\Omega} A_{\varepsilon}(t, x, l(u_{\varepsilon})) \nabla u_{\varepsilon}(t, x) \cdot \nabla v(x) dx.$$

Thus we have (see (8)) for some constant C independent of ε

$$\left| \frac{d}{dt}(u_{\varepsilon}, v) \right| \leq |f|_{2, \Omega} |v|_{2, \Omega} + C(\varepsilon \|\nabla_{X_1} u_{\varepsilon}\|_{2, \Omega} + \|\nabla_{X_2} u_{\varepsilon}\|_{2, \Omega}) \|\nabla v\|_{2, \Omega}, \quad (34)$$

whence

$$\frac{d}{dt}(u_{\varepsilon}, v) \in L^2(0, T).$$

Similarly one has

$$|(u_{\varepsilon}, v)| \leq |u_{\varepsilon}|_{2, \Omega} |v|_{2, \Omega}$$

and by (26), (34) we conclude that

$$(u_{\varepsilon}, v) \in H^1(0, T)$$

and for some constant D independent of ε it holds

$$|(u_{\varepsilon}, v)|_{H^1(0, T)} \leq D.$$

Thus - up to a subsequence of the above subsequence - we have

$$(u_{\varepsilon}, v) \rightarrow L_v \quad \text{in } L^2(0, T), \mathcal{C}(0, T), \quad (35)$$

$$(u_{\varepsilon}, v) \rightharpoonup L_v \quad \text{in } H^1(0, T). \quad (36)$$

Let us choose $\varphi \in \mathcal{D}(0, T)$ then one has

$$\int_0^T (u_{\varepsilon}, v) \varphi dt = \int_0^T \int_{\Omega} u_{\varepsilon} v \varphi dt dx \rightarrow \int_0^T \int_{\Omega} u_0 v \varphi dt dx = \int_0^T (u_0, v) \varphi dt,$$

by (28). On the other hand by (35) we have

$$\int_0^T (u_{\varepsilon}, v) \varphi dt \rightarrow \int_0^T L_v \varphi dt.$$

It follows that

$$L_v = \int_{\Omega} u_0 v dx$$

and by uniqueness of the limit, the convergences in (35), (36) are not up to a subsequence. This completes the proof of (i), (ii) and (iii) for $h \in H_0^1(\Omega)$ (recall (14)). To obtain (iii) for $h \in L^2(\Omega)$ it is enough to rely in the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ taking into account (26). \square

End of the proof of Lemma 3.2 and Theorem 3.1

For $\varphi \in \mathcal{D}(0, T)$, $v \in H_0^1(\Omega)$ we have by (16)

$$\begin{aligned} - \int_0^T (u_\varepsilon, v) \varphi' dt + \int_0^T \int_{\Omega} \varepsilon A_{11}(z, l(u_\varepsilon)) \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v \varphi dt dx \\ + \int_0^T \int_{\Omega} A_{12}(z, l(u_\varepsilon)) \varepsilon \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \varphi dt dx \\ + \int_0^T \int_{\Omega} A_{21}(z, l(u_\varepsilon)) \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v \varphi dt dx \\ + \int_0^T \int_{\Omega} A_{22}(z, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \varphi dt dx = \int_0^T \int_{\Omega} f v \varphi dt dx. \end{aligned} \quad (37)$$

Noting (iii) of Lemma 3.3 we have -up to a subsequence-

$$A(z, l(u_\varepsilon)) \rightarrow A(z, l(u_0)) \quad \text{a.e. } z \in Q, \quad (38)$$

i.e. we have this convergence for all coefficients of A . It follows that for $v \in H_0^1(\Omega)$

$$a_{ij}(\cdot, l(u_\varepsilon)) \partial_{x_k} v \rightarrow a_{ij}(\cdot, l(u_0)) \partial_{x_k} v \quad \text{in } L^2(Q), \quad \forall i, j, k = 1, \dots, n.$$

Then by (28), (29) one can easily pass to the limit in (37) to get

$$- \int_0^T (u_0, v) \varphi' dt + \int_0^T \int_{\Omega} A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \varphi dt dx = \int_0^T \int_{\Omega} f v \varphi dt dx. \quad (39)$$

Using an argument as in [3] or [5] one can easily conclude that

$$\begin{aligned} \int_0^T (u_0, v)_{\Omega_{X_1}} \varphi' dt + \int_0^T \int_{\Omega_{X_1}} A_{22}(t, X_1, X_2, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \varphi dt dx \\ = \int_0^T \int_{\Omega_{X_1}} f(t, X_1, X_2) v \varphi dt dx \\ \forall \varphi \in \mathcal{D}(0, T), \quad \forall v \in H_0^1(\Omega_{X_1}), \quad \text{a.e. } X_1 \in \Pi_1(\Omega), \end{aligned} \quad (40)$$

which is the last equation in (22). (We denoted by $(\cdot, \cdot)_{\Omega_{X_1}}$ the scalar product in $L^2(\Omega_{X_1})$). Clearly (40) implies that for a.e. $X_1 \in \Pi_1(\Omega)$

$$\partial_t u_0 = \nabla_{X_2} (A_{22}(\cdot, l(u_0)) \nabla_{X_2} u_0) + f \quad \text{in } \mathcal{D}'((0, T) \times \Omega_{X_1}) \quad (41)$$

and since $u_0 \in L^2((0, T); H_0^1(\Omega_{X_1}))$ we derive

$$\partial_t u_0 \in L^2(0, T; H^{-1}(\Omega_{X_1})) \text{ for a.e. } X_1 \in \Pi_1(\Omega).$$

Since for a.e. $X_1 \in \Pi_1(\Omega)$

$$u_0 \in L^2(0, T; L^2(\Omega_{X_1})) \quad (42)$$

by the usual embedding theorem the two first relations of (22) follow.

For $v \in H_0^1(\Omega)$ we have when $\varepsilon \rightarrow 0$

$$(u^0, v) = (u_\varepsilon(0, \cdot), v) = \int_{\Omega} u_\varepsilon(0, \cdot) v dx \rightarrow \int_{\Omega} u_0(0, \cdot) v dx$$

by (i) of Lemma 3.3. Thus

$$(u^0, v) = (u_0(0, \cdot), v) \quad \forall v \in H_0^1(\Omega). \quad (43)$$

By density of $H_0^1(\Omega)$ in $L^2(\Omega)$, (43) holds for every $v \in L^2(\Omega)$ and we get

$$u_0(0, \cdot) = u^0.$$

This completes the proof of Lemma 3.2. \square

Of course the proof of Theorem 3.1 is also completed. \square

4. Additional results

We now give additional properties of the solutions to (22). We have indeed

Theorem 4.1: *Let u_0 be solution to (22), then*

$$u_0 \in V = \{v \in L^2(Q) \mid |\nabla_{X_2} v| \in L^2(Q), \partial_t v \in L^2(0, T; H^{-1}(\Omega))\}, \quad (44)$$

$$u_0 \in L^\infty(0, T; L^2(\Omega)). \quad (45)$$

Proof: From (22) we have for a.e. $X_1 \in \Pi_1(\Omega)$, $v \in H_0^1(\Omega_{\Omega_{X_1}})$

$$\langle \partial_t u_0, v \rangle_{\Omega_{X_1}} + \int_{\Omega_{X_1}} A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dX_2 = \int_{\Omega_{X_1}} f v dX_2.$$

($\langle \cdot, \cdot \rangle_{\Omega_{X_1}}$ denotes the duality bracket between $H^{-1}(\Omega_{X_1})$ and $H_0^1(\Omega_{X_1})$). Choosing $v = u_0$ we deduce

$$\frac{1}{2} \frac{d}{dt} |u_0|_{2, \Omega_{X_1}}^2 + \int_{\Omega_{X_1}} A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dX_2 = \int_{\Omega_{X_1}} f u_0 dX_2.$$

Integrating on $(0, t_0) \times \Pi_1(\Omega)$ we get

$$\begin{aligned} \frac{1}{2} |u_0|_{2,\Omega}^2(t_0) + \int_0^{t_0} \int_{\Omega} A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx \\ = \frac{1}{2} |u^0|_{2,\Omega}^2 + \int_0^{t_0} \int_{\Omega} f u_0 dt dx. \end{aligned} \quad (46)$$

Choosing $t_0 = T$ and using (12) we obtain

$$\lambda |\nabla_{X_2} u_0|_{2,Q}^2 \leq \frac{1}{2} |u^0|_{2,\Omega}^2 + |f|_{2,Q} |u_0|_{2,Q}.$$

Applying the Poincaré inequality on each section Ω_{X_1} we obtain for a constant C

$$|u_0|_{2,Q}^2 \leq C |\nabla_{X_2} u_0|_{2,Q}^2$$

and thus by Young's inequality

$$\lambda |\nabla_{X_2} u_0|_{2,Q}^2 \leq \frac{1}{2} |u^0|_{2,\Omega}^2 + \frac{C^2}{2\lambda} |f|_{2,Q}^2 + \frac{\lambda}{2} |\nabla_{X_2} u_0|_{2,Q}^2.$$

From this it follows that

$$|\nabla_{X_2} u_0|_{2,Q}^2 \leq \frac{1}{\lambda} |u^0|_{2,\Omega}^2 + \frac{C^2}{\lambda^2} |f|_{2,Q}^2.$$

On the other hand integrating the equation in (22) on $\Pi_1(\Omega)$ we have for $v \in H_0^1(\Omega)$

$$\begin{aligned} \langle \partial_t u_0, v \rangle &= \frac{d}{dt} (u_0, v) = \int_{\Pi_1(\Omega)} \partial_t (u_0, v)_{\mathcal{B}} dX_1 = \int_{\Pi_1(\Omega)} \partial_t (u_0, v)_{\Omega_{X_1}} dX_1 \\ &= \int_{\Omega} f v dx - \int_{\Omega} A_{22}(\cdot, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx. \end{aligned}$$

where \mathcal{B} is defined by (30) and u_0, v are supposed to be extended by 0 outside Ω . Then it follows that

$$\begin{aligned} |\langle \partial_t u_0, v \rangle| &\leq |(f, v)| + \left| \int_{\Omega} A_{22}(\cdot, l(u_0)) \nabla u_0 \cdot \nabla v dx \right| \\ &\leq C \left(|f(t, X_1, \cdot)|_{2,\Omega} + |\nabla_{X_2} u_0|_{2,\Omega} \right) |v|_{H^1(\Omega)}, \end{aligned}$$

where C is a positive constant. Then after integrating on $(0, T)$ we get

$$|\partial_t u_0|_{L^2(0,T;H^{-1}(\Omega))} \leq C \left(|f|_{2,Q} + |\nabla_{X_2} u_0|_{2,Q} \right).$$

Thus (44) follows and (45) is then a consequence of (46). \square

In the case where the solution of (22) is unique we can yet prove

Theorem 4.2: *Suppose that (22) has a unique solution then we have*

$$u_{\varepsilon} \longrightarrow u_0 \quad \text{in } V \quad (47)$$

i.e.

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{in } L^2(Q), \quad \partial_t u_\varepsilon \longrightarrow \partial_t u_0 \quad \text{in } L^2(0, T; H^{-1}(\Omega))$$

and

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(Q). \quad (48)$$

(The convergences hold for the whole sequence).

Proof: We introduce

$$I_\varepsilon = \int_Q A_\varepsilon(z, l(u_\varepsilon)) \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} (u_\varepsilon - u_0) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2} (u_\varepsilon - u_0) \end{pmatrix} dt dx.$$

In what follows we just denote by A_ε the matrix $A_\varepsilon(z, l(u_\varepsilon))$ and drop the measures of integration to get

$$\begin{aligned} I_\varepsilon &= \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} - \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon \\ &+ \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \\ &= \int_Q f u_\varepsilon - \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle - \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} - \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon \\ &+ \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \\ &:= \int_Q f u_\varepsilon - \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle dt + J_\varepsilon. \end{aligned} \quad (49)$$

(We used (16) with $v = u_\varepsilon$). Remark now that

$$\begin{aligned} \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} &= \int_Q A_{21}(z, l(u_\varepsilon)) \varepsilon \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} u_0 \\ &+ \int_Q A_{22}(z, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0. \end{aligned}$$

Passing to the limit as we did in (37) we obtain that

$$\lim_{\varepsilon \rightarrow 0} \int_Q A_\varepsilon \nabla u_\varepsilon \cdot \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} = \int_Q A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0.$$

Note that since the possible limit is unique this is not up to a subsequence. We obtain similarly

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla u_\varepsilon &= \int_Q A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0, \\ \lim_{\varepsilon \rightarrow 0} \int_Q A_\varepsilon \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} \cdot \nabla \begin{pmatrix} 0 \\ \nabla_{X_2} u_0 \end{pmatrix} &= \int_Q A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \end{aligned}$$

and thus combining these three passages to the limit

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon = - \int_Q A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0. \quad (50)$$

We now consider the other terms in I_ε -see (49)- in order to pass to the limit. Let $\theta_k \in \mathcal{D}(\Omega)$ be a sequence of smooth functions independent of t that we will choose later on. We have

$$\begin{aligned} & \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle dt \\ &= \int_0^T \langle \partial_t (u_\varepsilon - \theta_k), u_\varepsilon - \theta_k \rangle dt + \int_0^T \langle \partial_t \theta_k, u_\varepsilon - \theta_k \rangle dt + \int_0^T \langle \partial_t u_\varepsilon, \theta_k \rangle dt \\ &= \frac{1}{2} |u_\varepsilon - \theta_k|_{2,\Omega}^2(T) - \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \int_0^T \langle \partial_t u_\varepsilon, \theta_k \rangle dt. \end{aligned}$$

Going back to (49) we have

$$R_\varepsilon := I_\varepsilon + \frac{1}{2} |u_\varepsilon - \theta_k|_{2,\Omega}^2(T) = \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \int_Q f u_\varepsilon - \int_0^T \langle \partial_t u_\varepsilon, \theta_k \rangle dt + J_\varepsilon. \quad (51)$$

Using (22), (28), (29), (50) we get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} R_\varepsilon \\ &= \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \int_Q f u_0 - \int_Q A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 - \int_0^T \langle \partial_t u_0, \theta_k \rangle dt \\ &= \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \int_0^T \int_{\Pi_1(\Omega)} \langle \partial_t u_0, u_0 \rangle_{\Omega_{X_1}}(t, X_1) dt - \int_0^T \langle \partial_t u_0, \theta_k \rangle dt. \end{aligned}$$

Note that since we have

$$\int_0^T \langle \partial_t u_0, \theta_k \rangle dt = \int_0^T \int_{\Pi_1(\Omega)} \langle \partial_t u_0, \theta_k \rangle_{\Omega_{X_1}}(t, X_1) dt dX_1$$

the limit above becomes

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} R_\varepsilon &= \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \int_{\Pi_1(\Omega)} \int_0^T \langle \partial_t (u_0 - \theta_k), (u_0 - \theta_k) \rangle_{\Omega_{X_1}}(t, X_1) dt dX_1 \\ &= \frac{1}{2} |u^0 - \theta_k|_{2,\Omega}^2 + \frac{1}{2} \int_{\Pi_1(\Omega)} |u_0 - \theta_k|_{2,\Omega_{X_1}}^2(T, X_1) - |u^0 - \theta_k|_{2,\Omega_{X_1}}^2(X_1) dt dX_1 \\ &= \frac{1}{2} |u_0 - \theta_k|_{2,\Omega}^2(T). \end{aligned} \quad (52)$$

Thus from (51) we deduce

$$\limsup_{\varepsilon \rightarrow 0} I_\varepsilon \leq \lim_{\varepsilon \rightarrow 0} R_\varepsilon = \frac{1}{2} |u_0 - \theta_k|_{2,\Omega}^2(T).$$

Choosing θ_k such that

$$\theta_k \rightarrow u_0(T) \quad \text{in } L^2(\Omega), \quad (53)$$

we deduce that

$$0 \leq \limsup_{\varepsilon \rightarrow 0} I_\varepsilon \leq 0$$

and thus

$$I_\varepsilon \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0.$$

Since, by the coerciveness assumption, we have

$$\lambda \int_0^T \int_\Omega |\varepsilon \nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \leq I_\varepsilon,$$

it follows that

$$\varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0, \quad \nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0 \quad \text{in } L^2(Q),$$

and thus (by Poincaré's inequality)

$$u_\varepsilon \rightarrow u_0 \quad \text{in } L^2(Q).$$

It remains to show that

$$\partial_t u_\varepsilon \rightarrow \partial_t u_0 \quad \text{in } L^2(0, T; H^{-1}(\Omega)).$$

Testing the equation in (22) with $v \in H_0^1(\Omega)$ and integrating on $\Pi_1(\Omega)$, we get

$$\int_{\Pi_1(\Omega)} \partial_t (u_0, v)_{\Omega_{X_1}} dX_1 + \int_\Omega A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = \int_\Omega f v dx,$$

whence

$$\frac{d}{dt} (u_0, v)_\Omega + \int_\Omega A_{22}(z, l(u_0)) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = \int_\Omega f v dx.$$

Then subtracting the above equality from (16) leads to

$$\begin{aligned} & \frac{d}{dt} (u_\varepsilon - u_0, v) + \int_\Omega \varepsilon^2 A_{11}(z, l(u_\varepsilon)) \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v dx \\ & + \int_\Omega \varepsilon A_{12}(z, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v dx \\ & + \int_\Omega \varepsilon A_{21}(z, l(u_\varepsilon)) \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v dx \\ & + \int_\Omega (A_{22}(z, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon - A_{22}(z, l(u_0)) \nabla_{X_2} u_0) \cdot \nabla_{X_2} v dx \\ & = 0. \end{aligned}$$

Using the Cauchy-Schwarz inequality and by (44), it follows that

$$\begin{aligned} \langle \partial_t (u_\varepsilon - u_0), v \rangle &\leq \varepsilon^2 C \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega} \|\nabla_{X_1} v\|_{2,\Omega} \\ &\quad + \varepsilon C \|\nabla_{X_2} u_\varepsilon\|_{2,\Omega} \|\nabla_{X_1} v\|_{2,\Omega} + \varepsilon C \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega} \|\nabla_{X_2} v\|_{2,\Omega} \\ &\quad + \|A_{22}(\cdot, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0)) \nabla_{X_2} u_0\|_{2,\Omega} \|\nabla_{X_1} v\|_{2,\Omega} \\ &\leq (C (\varepsilon^2 \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega} + \varepsilon \|\nabla_{X_2} u_\varepsilon\|_{2,\Omega} + \varepsilon \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega}) \\ &\quad + \|A_{22}(\cdot, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0)) \nabla_{X_2} u_0\|_{2,\Omega}) \|v\|_{H^1(\Omega)}, \end{aligned}$$

for any $v \in H_0^1(\Omega)$. This implies

$$\begin{aligned} |\partial_t (u_\varepsilon - u_0)|_{H^{-1}(\Omega)} &\leq C (\varepsilon^2 \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega} + \varepsilon \|\nabla_{X_2} u_\varepsilon\|_{2,\Omega} + \varepsilon \|\nabla_{X_1} u_\varepsilon\|_{2,\Omega}) \\ &\quad + \|A_{22}(\cdot, l(u_\varepsilon)) \nabla_{X_2} u_\varepsilon - A_{22}(\cdot, l(u_0)) \nabla_{X_2} u_0\|_{2,\Omega}. \end{aligned}$$

Integrating over $(0, T)$, letting $\varepsilon \rightarrow 0$ and taking into account (iii) of Lemma 3.3, we deduce

$$|\partial_t (u_\varepsilon - u_0)|_{L^2(0,T;H^{-1}(\Omega))} \rightarrow 0.$$

This completes the proof of the theorem. \square

Remark 1: We also have

$$u_\varepsilon(t, \cdot) \longrightarrow u_0(t, \cdot) \quad \text{in } L^2(\Omega) \quad \forall t \in (0, T). \quad (54)$$

Indeed, if we replace T by $t_0 \in (0, T)$ in the proof of Theorem 4.2 we get from (51), (52),

$$\limsup_{\varepsilon \rightarrow 0} |\theta_k - u_\varepsilon(t_0, \cdot)|_{2,\Omega} \leq |\theta_k - u_0(t_0, \cdot)|_{2,\Omega},$$

thus

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} &\leq |\theta_k - u_0(t_0, \cdot)|_{2,\Omega} + \limsup_{\varepsilon \rightarrow 0} |\theta_k - u_\varepsilon(t_0, \cdot)|_{2,\Omega} \\ &\leq 2 |\theta_k - u_0(t_0, \cdot)|_{2,\Omega}. \end{aligned} \quad (55)$$

Then passing to the limit in the inequality above when $k \rightarrow \infty$ and taking into account (53) for $T = t_0$, we obtain

$$\limsup_{\varepsilon \rightarrow 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} \leq 0,$$

whence

$$\lim_{\varepsilon \rightarrow 0} |u_\varepsilon(t_0, \cdot) - u_0(t_0, \cdot)|_{2,\Omega} = 0.$$

It follows that

$$u_\varepsilon(t, \cdot) \longrightarrow u_0(t, \cdot) \quad \text{in } L^2(\Omega) \quad \forall t \in (0, T).$$

Remark 2: In the case when u_0 is not unique one can show that the set of solutions to (22) lies in a neighborhood of

$$\{u_\varepsilon \mid \varepsilon < \varepsilon_0\}$$

for the topology of V , that is to say from any subsequence of u_ε there exists another subsequence such that (47), (48) hold. In addition if there exists a subsequence of u_ε , still labeled by u_ε , and a function u_0 such that

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } \mathcal{D}'(Q),$$

then u_0 is a solution of (22) and (47), (48) hold.

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