ON SOME ANISOTROPIC SINGULAR PERTURBATION PROBLEMS

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Abstract. We investigate the asymptotic behavior of some anisotropic diffusion problems and give some estimates on the rate of convergence of the solution toward its limit. We also relate this type of elliptic problems to problems set in cylinder becoming unbounded in some directions and show how some information on one type leads to information for the other type and conversely.

1. A model problem

The goal of this note is to study diffusion problems for which the diffusion in some directions is very small. More precisely we would like to find the limit behavior of the solution of such a problem when the small diffusion parameters approach zero. First we would like to explain the issues on a very simple example in two dimensions.

Let us denote by \( \omega \) the interval \((-1, 1)\) and for \( a > 0 \) by \( \Omega \) the rectangle \((-a, a) \times \omega\). The points in \( \Omega \) will be labeled by \( x = (x_1, x_2) \). We denote by \( H^1_0(\omega) \) the usual Sobolev space of functions vanishing on \( \partial \omega \) and by \( H^{-1}(\omega) \) its dual (see [6]). The duality bracket between \( H^{-1}(\omega) \) and \( H^1_0(\omega) \) will be denoted by \( \langle , \rangle \) or simply \( \langle , \rangle \).

Let us consider \( f \) such that

\[
W^{1,2}(\omega; H^{-1}(\omega)) \subset C^0(\omega; H^{-1}(\omega)),
\]

with an obvious notation for this space (see [1], [6]). Note that clearly

\[
W^{1,2}(\omega; H^{-1}(\omega)) \subset C^0(\omega; H^{-1}(\omega)),
\]

where \( C^0(\omega; H^{-1}(\omega)) \) denotes the space of continuous functions from \( \omega \) into \( H^{-1}(\omega) \).

We define a continuous linear form on \( H^1_0(\Omega) \) – i.e. an element of \( H^{-1}(\Omega) \) by setting

\[
\langle f, v \rangle_{\Omega} = \int_\omega \langle f, v(x_1, \cdot) \rangle dx_1, \quad \forall v \in H^1_0(\Omega).
\]

Since there is no ambiguity this linear form will be also denoted by \( f \). Then we would like to consider for \( \varepsilon > 0 \), \( u_\varepsilon \) the weak solution to

\[
\begin{cases}
-\varepsilon^2 \partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon = f & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}
\]

1.
that is to say the solution to
\[
\begin{cases}
\int_{\Omega} \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} v + \partial_{x_2} u_\varepsilon \partial_{x_2} v \, dx = (f, v)_{\Omega_1} \quad \forall v \in H^1_0(\Omega), \\
u_\varepsilon \in H^1_0(\Omega).
\end{cases}
\]
(1.5)

It is clear that the existence and uniqueness of a solution to (1.4), (1.5) follows from the Lax–Milgram theorem. We are interested in the limit of $u_\varepsilon$ when $\varepsilon \to 0$. The natural candidate for the limit is of course $u_0 = u_0(x_1, \cdot)$ the weak solution of the problem
\[
\begin{cases}
-\partial_{x_2}^2 u_0 = f \quad \text{in } \omega, \\
u_0 = 0 \quad \text{on } \partial \omega,
\end{cases}
\]
(1.6)
i.e.
\[
\begin{cases}
\int_{\omega} \partial_{x_2} u_0 \partial_{x_2} v \, dx_2 = (f, v)_\omega \quad \forall v \in H^1_0(\omega), \\
u_0 \in H^1_0(\omega).
\end{cases}
\]
(1.7)

Note that the solution of (1.6), (1.7) depends on a parameter $x_1$. In principle it is defined for almost all $x_1$, but by (1.2) it can be defined for every $x_1 \in \omega$.

Then we can show the following.

**Theorem 1.1.** Under the above assumptions, for any $0 < a < 1$ there exists a constant $C = C(a, f)$ independent of $\varepsilon$ such that it holds that
\[
|\partial_{x_2} (u_\varepsilon - u_0)|^2_{2, \Omega_1} \leq C \varepsilon^2
\]
(1.8)
and
\[
|u_\varepsilon - u_0|^2_{2, \Omega_1} \leq 2 C \varepsilon^2.
\]
(1.9)

($|\cdot|_{2, A}$ denotes the usual norm on $L^2(A)$ the space of class of functions which square is integrable on the measurable subset $A$).

**Proof.** First we claim that
\[
u_0 \in H^1(\Omega_1).
\]
(1.10)

Indeed, taking $v = u_0(x_1, \cdot)$ in (1.7) we obtain
\[
\int_{\omega} \partial_{x_2} u_0(x_1, x_2)^2 \, dx_2 = (f, u_0(x_1, \cdot))_\omega
\]
(1.11)
\[
\leq |f(x_1, \cdot)|_{H^{-1}(\omega)}|\partial_{x_2} u_0(x_1, \cdot)|_{2, \omega}.
\]
(1.12)

($|\cdot|_{H^{-1}(\omega)}$ denotes the strong dual norm in $H^{-1}(\omega)$ when $H^1_0(\omega)$ is equipped with the norm $|\partial_{x_2} v|_{2, \omega}$). It follows from (1.11) that it holds that
\[
|\partial_{x_2} u_0(x_1, \cdot)|_{2, \omega} \leq |f(x_1, \cdot)|_{H^{-1}(\omega)}.
\]
(1.13)

Applying the Poincaré inequality in $\omega$ we have (see [2])
\[
|u_0(x_1, \cdot)|_{2, \omega} \leq \sqrt{2}|\partial_{x_2} u_0(x_1, \cdot)|_{2, \omega} \leq \sqrt{2}|f(x_1, \cdot)|_{H^{-1}(\omega)}
\]
and from (1.12), (1.13) since $f \in L^2(\omega; H^{-1}(\omega))$ we derive easily that it holds
\[
u_0, \partial_{x_2} u_0 \in L^2(\Omega_1),
\]
(1.14)

with for instance
\[
|\partial_{x_2} u_0|_{2, \Omega_1} \leq |f|_{H^{-1}(\omega)}|_{2, \omega} = \left(\int_{\omega} |f(x_1, \cdot)|_{H^{-1}(\omega)}^2 \, dx_1\right)^{1/2}.
\]
(1.15)
Next, setting \( \tau_h u_0(x_1, x_2) = u_0(x_1 + h, x_2) \), from (1.6), (1.7) we derive

\[
- \partial^2_{x_2}(\tau_h u_0 - u_0) = \tau_h f - f,
\]

with an obvious notation for \( \tau_h f \). Note that \( \tau_h u_0 - u_0 \in H^1_0(\Omega) \) and that this equation holds in \( \omega \) for \( x_1 \in \omega' \subset \subset \omega \) and \( h < \text{dist}(\omega', \partial \omega) \). Arguing as in (1.11) – i.e. taking \( v = \tau_h u_0 - u_0 \) in the weak form of the above equation – it comes

\[
|\tau_h u_0 - u_0|_{2, \omega} \leq \sqrt{2} |\partial_{x_2}(\tau_h u_0 - u_0)|_{2, \omega} \leq \sqrt{2} |\tau_h f - f|_{H^{-1}(\omega)}.
\]

From this it follows that it holds

\[
\left| \frac{\tau_h u_0 - u_0}{h} \right|_{2, \omega} \leq \sqrt{2} \left| \frac{\tau_h f - f}{h} \right|_{H^{-1}(\omega)}.
\]

Since \( f \in W^{1,2}(\omega; H^{-1}(\omega)) \) we obtain – see [7] – for \( x_1 \in \omega' \subset \subset \omega \) and \( h < \text{dist}(\omega', \partial \omega) \)

\[
\left| \frac{\tau_h u_0 - u_0}{h} \right|_{2, \omega} \leq C(f)
\]

where \( C(f) \) is some constant independent of \( h \). Integrating in \( x_1 \) we derive easily for any set \( K \subset \subset \Omega_1 \) that it holds

\[
\left| \frac{\tau_h u_0 - u_0}{h} \right|_{2, K} \leq C
\]

where \( C \) is independent of \( K \) and \( h \) such that \( h < \text{dist}(K, \partial \Omega_1) \). It follows – see [7] – that

\[
\partial_{x_1} u_0 \in L^2(\Omega_1)
\]

and this completes the proof of (1.10) by (1.14).

Next, taking \( v = u_\varepsilon \) in (1.5) we derive (see (1.3))

\[
\varepsilon^2 |\partial_{x_1} u_\varepsilon|^2_{2, \Omega_1} + |\partial_{x_2} u_\varepsilon|^2_{2, \Omega_1} \leq \int_{\omega} \langle f, u_\varepsilon(x_1, \cdot) \rangle_{\omega} \, dx_1
\]

\[
\leq \int_{\omega} |f|_{H^{-1}(\omega)} |\partial_{x_2} u_\varepsilon(x_1, \cdot)|_{2, \omega}
\]

\[
\leq |\partial_{x_2} u_\varepsilon|_{2, \Omega_1} \|f\|_{H^{-1}(\omega)}_{2, \omega}
\]

by the Cauchy–Schwarz inequality. It follows then that

\[
|\partial_{x_2} u_\varepsilon|_{2, \Omega_1} \leq \|f\|_{H^{-1}(\omega)}_{2, \omega}.
\]

(Note that \( |f|_{H^{-1}(\omega)} = |f(x_1, \cdot)|_{H^{-1}(\omega)} \).

Next we consider a smooth function \( \rho = \rho(x_1) \) such that

\[
0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } (-a, a), \quad \rho = 0 \text{ outside } \left(-a - \frac{1}{q}, a + \frac{1}{q}\right),
\]

\[
|\rho'| \leq c_q, \quad a + \frac{1}{q} < 1,
\]

\( (q \) will be adjusted later on \( ) \). By (1.10) it is clear that

\[
(u_\varepsilon - u_0)\rho^2(x_1) \in H^1_0(\Omega_1).
\]
Using (1.6), (1.7) we derive easily that it holds that

\[
\int_{\Omega_1} \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} \{ (u_\varepsilon - u_0) \rho^2 \} + \partial_{x_2} u_\varepsilon \partial_{x_2} \{ (u_\varepsilon - u_0) \rho^2 \} \, dx
\]

\[
= \int_{\Omega_1} \partial_{x_2} u_0 \partial_{x_2} \{ (u_\varepsilon - u_0) \rho^2 \} \, dx.
\]

Setting \( w_\varepsilon = u_\varepsilon - u_0 \), this can be rewritten as

\[
\int_{\Omega_1} \{ \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} \{ w_\varepsilon + (\partial_{x_2} w_\varepsilon)^2 \} \rho^2 \, dx = - \int_{\Omega_1} \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} \rho^2 w_\varepsilon \, dx
\]

and finally

\[
\int_{\Omega_1} \{ \varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 + (\partial_{x_2} w_\varepsilon)^2 \} \rho^2 \, dx
\]

\[
= - \varepsilon^2 \int_{\Omega_1} \partial_{x_1} u_\varepsilon 2 \partial_{x_1} \rho w_\varepsilon \, dx - \varepsilon^2 \int_{\Omega_1} \partial_{x_1} u_0 \partial_{x_1} \rho^2 w_\varepsilon \, dx.
\]

Using the Young inequality it comes

\[
\int_{\Omega_1} \{ \varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 + (\partial_{x_2} w_\varepsilon)^2 \} \rho^2 \, dx
\]

\[
\leq \varepsilon^2 \int_{\Omega_1} c \varepsilon |\partial_{x_1} u_\varepsilon| \frac{\rho |w_\varepsilon|}{\varepsilon} \, dx + \varepsilon^2 \int_{\Omega_1} |\partial_{x_1} u_0| \rho |\partial_{x_1} w_\varepsilon| \rho \, dx
\]

\[
\leq \varepsilon^2 \int_{\Omega_1} \frac{\rho^2 w_\varepsilon^2}{4 \varepsilon^2} + c^2 \varepsilon^2 (\partial_{x_1} u_\varepsilon)^2 \, dx + \frac{(\partial_{x_1} w_\varepsilon)^2}{2} \rho^2 + \frac{(\partial_{x_1} u_0)^2}{2} \rho^2 \, dx,
\]

with \( c \) a constant depending on \( \partial_{x_1} \rho = \rho' \) i.e. depending on \( a \). Noting that by the Poincaré inequality it holds

\[
\int_{\Omega_1} w_\varepsilon^2 \rho^2 \, dx \leq 2 \int_{\Omega_1} (\partial_{x_2} w_\varepsilon)^2 \rho^2 \, dx,
\]

we obtain easily

\[
\int_{\Omega_1} \{ \varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 + (\partial_{x_2} w_\varepsilon)^2 \} \rho^2 \, dx \leq 2 \varepsilon^2 \varepsilon^4 |\partial_{x_1} u_\varepsilon|^2_{2,\Omega_1} + \varepsilon^2 |\partial_{x_1} u_0|^2_{2,\Omega_1}.
\]

Using the estimates (1.15)-(1.17) we derive for \( c' = c'(a) \)

\[
\int_{\Omega_1} (\partial_{x_2} w_\varepsilon)^2 \rho^2 \, dx \leq c' \varepsilon^2 |f|_{H^{-1}(\omega)}^2_{2,\omega}.
\]

By the Poincaré inequality it follows that it holds

\[
|w_\varepsilon|_{2,\Omega_\varepsilon}^2 \leq 2 |\partial_{x_2} w_\varepsilon|^2_{2,\Omega_\varepsilon} \leq 2 \int_{\Omega_1} (\partial_{x_2} w_\varepsilon)^2 \rho^2 \, dx \leq 2 c' \varepsilon^2 |f|_{H^{-1}(\omega)}^2_{2,\omega}.
\]

(see (1.18)). Then (1.8), (1.9) follow with \( C = C(a, f) = c' |f|_{H^{-1}(\omega)}^2_{2,\omega} \). This completes the proof of the theorem.

**Remark 1.1.** The problem (1.4) describes a diffusion process (of heat, population ...) where the diffusion in the direction \( x_1 \) is very small. Thus, at the limit, it is normal to expect that the diffusion is taking place in the \( x_2 \) direction only. This is what states Theorem 1.1. Note that this kind of problem was not addressed in the monograph [9].

If one is not concerned by the rate of convergence we have:
Theorem 1.2. Under the assumptions of Theorem 1.1 it holds that
\[ u_\varepsilon \rightarrow u_0, \quad \partial_{x_2} u_\varepsilon \rightarrow \partial_{x_2} u_0 \quad \text{in} \quad L^2(\Omega_1). \] (1.21)

Proof. From the estimate (1.17) and Theorem 1.1 we have
\[ \partial_{x_2} u_\varepsilon \rightarrow \partial_{x_2} u_0 \quad \text{in} \quad L^2(\Omega_1). \] (1.22)
Moreover, taking \( v = u_\varepsilon \) in (1.5) we obtain taking into account (1.7)
\[ \varepsilon^2 \int_{\Omega_1} (\partial_{x_1} u_\varepsilon)^2 \, dx + \int_{\Omega_1} (\partial_{x_2} u_\varepsilon)^2 \, dx = \int_{\Omega_1} \partial_{x_2} u_0 \partial_{x_2} u_\varepsilon \, dx. \]
It follows that it holds
\[ \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_1} (\partial_{x_2} u_\varepsilon)^2 \, dx \leq \int_{\Omega_1} (\partial_{x_2} u_0)^2 \, dx. \]
Now (1.22) implies that
\[ \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_1} (\partial_{x_2} u_\varepsilon)^2 \, dx \geq \int_{\Omega_1} (\partial_{x_2} u_0)^2 \, dx. \]
Thus, since \( |\partial_{x_2} u_\varepsilon|^2 \leq |\partial_{x_2} u_0|^2 \) we have
\[ \partial_{x_2} u_\varepsilon \rightarrow \partial_{x_2} u_0 \quad \text{in} \quad L^2(\Omega_1), \]
when \( \varepsilon \rightarrow 0 \). The convergence of \( u_\varepsilon \) in \( L^2(\Omega_1) \) follows from the Poincaré inequality. This completes the proof of the theorem. \( \square \)

In the case where
\[ f = f(x_2) \in H^{-1}(\omega) \] (1.23)
i.e. in the case where \( f \) is independent of \( x_1 \) the convergence obtained in Theorem 1.1 can be improved. Indeed, first in this case one should notice that \( u_0 \) is independent of \( x_1 \). Then we have:

Theorem 1.3. Suppose that (1.23) holds. Let \( u_\varepsilon, \, u_0 \) be the solutions to (1.4), (1.6) respectively. Then for every \( 0 < a < 1 \) and every \( k \) there exists a constant \( C_k(a) \) such that
\[ |\partial_{x_2}(u_\varepsilon - u_0)|^2 \leq C_k(a)\varepsilon^2|f|^2_{H^{-1}(\omega)} \] (1.24)
\[ |u_\varepsilon - u_0|^2 \leq 2C_k(a)\varepsilon^2|f|^2_{H^{-1}(\omega)} \] (1.25)
i.e. on \( \Omega_a \) the speed of convergence of \( u_\varepsilon \) toward \( u_0 \) is as fast as one wishes in terms of power of \( \varepsilon \).

Proof. We recall (1.18) – at this stage we did not use \( q \). Going back to (1.20) we obtain since \( u_0 \) is independent of \( x_1 \)
\[ \int_{\Omega_1} \{\varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 + (\partial_{x_2} w_\varepsilon)^2\} \rho^2 \, dx \leq 2\varepsilon^2 c_0 \int_{\Omega_1} |\partial_{x_1} w_\varepsilon| \rho \, w_\varepsilon \, dx. \]
Since in fact we integrate on \( \Omega_{a + \frac{1}{q}} \), by the Young inequality it follows
\[ \int_{\Omega_1} \{\varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 + (\partial_{x_2} w_\varepsilon)^2\} \rho^2 \, dx \leq \frac{1}{2} \int_{\Omega_1} \varepsilon^2 (\partial_{x_1} w_\varepsilon)^2 \rho^2 \, dx + 2c_q^2 \varepsilon^2 \int_{\Omega_{a + \frac{1}{q}}} w_\varepsilon^2 \, dx. \] (1.26)
This implies that we have
\[
\int_{\Omega_a} (\partial_{x_2} w_\varepsilon)^2 \, dx \leq 2c_q^2 \varepsilon^2 \int_{\Omega_a+\frac{1}{q}} w_\varepsilon^2 \, dx \leq 4c_q^2 \varepsilon^2 \int_{\Omega_a+\frac{1}{q}} (\partial_{x_2} w_\varepsilon)^2 \, dx
\]
by the Poincaré inequality. Iterating this formula \(k\)-times for \(k\) such that
\[
a + \frac{k}{q} < 1
\]
which is possible for \(q\) large enough we get
\[
\int_{\Omega_a} (\partial_{x_2} w_\varepsilon)^2 \, dx \leq C_k(a) \varepsilon^{2k} \int_{\Omega_1} (\partial_{x_2} w_\varepsilon)^2 \, dx.
\]
Going back to the estimates (1.15), (1.17) we obtain
\[
|\partial_{x_2}(u_\varepsilon - u_0)|^2_{L^2(\Omega_a)} \leq C_k(a) \varepsilon^{2k} |f|_{H^{-1}(\omega)}^2
\]
which is (1.24) (we used here the fact that \(f\) is independent of \(x_1\)). The inequality (1.25) is a consequence of the Poincaré inequality. This completes the proof of the theorem. \(\Box\)

**Remark 1.2.** Note that by (1.26) we also have for some constant \(C_k(a)\)
\[
\int_{\Omega_a} (\partial_{x_2} w_\varepsilon)^2 \, dx = \int_{\Omega_a} \partial_{x_1}(u_\varepsilon - u_0)^2 \, dx \leq C_k(a) \varepsilon^{2k} |f|_{H^{-1}(\omega)}^2
\]
and thus the convergence of \(u_\varepsilon\) toward \(u_0\) in \(H^1(\Omega_a)\) with a speed in \(\varepsilon^k\) for arbitrary \(k\). This speed can in fact be improved to an exponential rate of convergence for (1.24), (1.25) as well. We refer the reader to section 4 and [5].

2. **The general elliptic case**

Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\). We denote by \(x = (x_1, \ldots, x_n)\) the points in \(\mathbb{R}^n\) and use the decomposition
\[
x = (X_1, X_2), \quad X_1 = x_1, \ldots, x_p, \quad X_2 = x_{p+1}, \ldots, x_n.
\]
With this notation we set
\[
\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}
\]
with
\[
\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T, \quad \nabla_{X_2} u = (\partial_{x_{p+1}} u, \ldots, \partial_{x_n} u)^T.
\]
We denote by \(A = (a_{ij}(x))\) a \(n \times n\) matrix such that
\[
a_{ij} \in L^\infty(\Omega) \quad \forall \ i, j = 1, \ldots, n,
\]
and for some \(\lambda > 0\) it holds that
\[
\lambda |\xi|^2 \leq (A\xi \cdot \xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega.
\]
In the above formula \((\cdot, \cdot)\) denotes the usual scalar product in \(\mathbb{R}^n\). We decompose \(A\) into four blocks by writing
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}
\] (2.6)
where \(A_{11}, A_{22}\) are respectively \(p \times p\) and \(n - p \times n - p\) matrices. We then set for \(\varepsilon > 0\)
\[
A_\varepsilon = A_\varepsilon(x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}.
\] (2.7)
For \(\xi \in \mathbb{R}^n\) if we set \(\xi = (\xi_1, \ldots, \xi_p)^T\), \(\xi_2 = (\xi_{p+1}, \ldots, \xi_n)^T\) we have for a.e. \(x \in \Omega\) and every \(\xi \in \mathbb{R}^n\)
\[
(A_\varepsilon \xi \cdot \xi) = (A\xi_1 \cdot \xi_1) \geq \lambda |\xi_1|^2 = \lambda \{\varepsilon^2 |\xi_1|^2 + |\xi_2|^2\} \tag{2.8}
\]
where we have set \(\xi_\varepsilon = (\varepsilon \xi_1, \xi_2)\). Thus, it holds
\[
(A_\varepsilon \xi \cdot \xi) \geq \lambda (\varepsilon^2 \wedge 1)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega. \tag{2.9}
\]
(\(\wedge\) denotes the minimum of two numbers). It follows that \(A_\varepsilon\) is positive definite and for \(f \in H^{-1}(\Omega)\) there exists a unique \(u_\varepsilon\) solution to
\[
\begin{cases}
\int_{\Omega} A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle & \forall v \in H^1_0(\Omega), \\
u_\varepsilon \in H^1_0(\Omega) & \tag{2.10}
\end{cases}
\]
Clearly \(u_\varepsilon\) is the solution of a problem generalizing (1.5). We would like then to study the behavior of \(u_\varepsilon\) when \(\varepsilon \to 0\).

Let us denote by \(\Pi_{X_1}\) the orthogonal projection from \(\mathbb{R}^n\) onto the space \(X_2 = 0\). For any \(X_1 \in \Pi_{X_1}(\Omega) := \Pi_{X_1}\) we denote by \(\Omega_{X_1}\) the section of \(\Omega\) above \(X_1\) i.e.
\[
\Omega_{X_1} = \{ X_2 \mid (X_1, X_2) \in \Omega \}.
\]
To avoid unnecessary complications we will suppose that
\[
f \in L^2(\Omega). \tag{2.11}
\]
Then for a.e. \(X_1 \in \Pi_{\Omega}\) we have \(f(X_1, \cdot) \in L^2(\Omega_{X_1})\) and there exists a unique \(u_0 = u_0(X_1, \cdot)\) solution to
\[
\begin{cases}
\int_{\Omega_{X_1}} A_{22} \nabla X_2 u_0(X_1, X_2) \cdot \nabla X_2 v(X_2) \, dX_2 \\
u_0(X_1, \cdot) \in H^1_0(\Omega_{X_1}). \tag{2.12}
\end{cases}
\]
Clearly \(u_0\) is the natural candidate for the limit of \(u_\varepsilon\). Indeed we have

**Theorem 2.1.** Under the assumption (2.11) and if
\[
\begin{cases}
\sum_{i=p+1}^{n} \partial_{x_i} a_{ij} \in L^2(\Omega) \quad \forall j = 1, \ldots, p, \\
\sum_{j=1}^{p} \partial_{x_i} a_{ij} \in L^2(\Omega) \quad \forall i = p + 1, \ldots, n, \tag{2.13}
\end{cases}
\]
then we have
\[
u_\varepsilon \rightharpoonup u_0, \quad \nabla X_2 u_\varepsilon \rightharpoonup \nabla X_2 u_0 \quad \text{in} \quad L^2(\Omega). \tag{2.14}
\]
Proof. Take \( v = u_\varepsilon \) in (2.10) – recalling (2.8), (2.1)-(2.3), it comes
\[
\lambda \int_\Omega \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + |\nabla X_2 u_\varepsilon|^2 \, dx = \int_\Omega f u_\varepsilon \, dx \leq |f|_{2,\Omega} |u_\varepsilon|_{2,\Omega}. \tag{2.15}
\]
Since \( \Omega \) is bounded in the \( X_2 \)-direction, we have for some constant \( C = C_\Omega \) the Poincaré inequality
\[
|v|_{2,\Omega} \leq C|\nabla X_2 v|_{2,\Omega} \quad \forall \, v \in H^1_0(\Omega), \tag{2.16}
\]
(see [2]). Thus, from (2.15) we derive
\[
\lambda |\nabla X_2 u_\varepsilon|_{2,\Omega}^2 \leq C|f|_{2,\Omega} |\nabla X_2 u_\varepsilon|_{2,\Omega}.
\]
From this it follows that
\[
|\nabla X_2 u_\varepsilon|_{2,\Omega} \leq C \frac{|f|_{2,\Omega}}{\lambda}
\]
and by (2.15), (2.16)
\[
\varepsilon^2 |\nabla X_1 u_\varepsilon|_{2,\Omega}^2 + |\nabla X_2 u_\varepsilon|_{2,\Omega}^2 \leq \frac{C^2}{\lambda^2} |f|_{2,\Omega}^2 \tag{2.17}
\]
\[
|u_\varepsilon|_{2,\Omega} \leq C \frac{|f|_{2,\Omega}}{\lambda}. \tag{2.18}
\]
Thus – up to a subsequence – there is a \( v_0 \in L^2(\Omega) \) such that
\[
u_\varepsilon \rightharpoonup v_0 \quad \nabla X_2 u_\varepsilon \rightharpoonup \nabla X_2 v_0 \quad \text{in} \quad L^2(\Omega).
\]
Since \( u_\varepsilon \in H^1_0(\Omega) \), for a.e. \( X_1 \in \Pi_\Omega \) it holds that
\[
u_\varepsilon(X_1, \cdot) \in H^1_0(\Omega X_1)
\]
(see [1]). Let us denote by \( B_\Omega \) a ball in \( \mathbb{R}^{n-p} \) such that
\[
\Omega X_1 \subset B_\Omega \quad \forall X_1 \in \Pi_\Omega
\]
and suppose \( u_\varepsilon \) extended by 0 outside \( \Omega \). Then we have
\[
\int_{\Pi_\Omega} \int_{B_\Omega} |\nabla X_2 u_\varepsilon(X_1, X_2)|^2 \, dX_1 \, dX_2 \leq \frac{C^2}{\lambda^2} |f|_{2,\Omega}^2
\]
and there exists a subsequence of \( \varepsilon \) such that
\[
u_\varepsilon \rightharpoonup \tilde{v}_0 \quad \text{in} \quad L^2(\Pi_\Omega; H^1_0(B_\Omega)).
\]
Since the above convergence implies the convergence in \( \mathcal{D}'(\Pi_\Omega \times B_\Omega) \) if we extend \( v_0 \) by 0 outside \( \tilde{\Omega} \) we have \( \tilde{v}_0 = v_0 \) and in particular for a.e. \( X_1 \in \Pi_\Omega \) we get
\[
\nu_\varepsilon(X_1, \cdot) \in H^1_0(\Omega X_1).
\]
We can now pass to the limit. For this let \( \varphi \in \mathcal{D}(\Omega) \). We have by (2.10)
\[
\varepsilon^2 \int_\Omega A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 \varphi \, dx + \varepsilon \int_\Omega A_{12} \nabla X_2 u_\varepsilon \cdot \nabla X_1 \varphi \, dx + \varepsilon \int_\Omega A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 \varphi \, dx + \int_\Omega A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 \varphi \, dx = \int_\Omega f \varphi \, dx.
\]
Since by (2.17), (2.18)
\[
\varepsilon |\nabla X_1 u_\varepsilon|_{2,\Omega}, \quad |\nabla X_2 u_\varepsilon|_{2,\Omega} \leq C
\]
(2.22)
where $C$ is a constant independent of $\varepsilon$, when $\varepsilon \to 0$ the two first terms of (2.21) are converging toward 0. Next we consider the term

$$\int_{\Omega} A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 \varphi \, dx.$$  

We claim that thanks to our assumption (2.13) it is bounded independently of $\varepsilon$. Indeed we have

$$\int_{\Omega} A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 \varphi \, dx = \sum_{j=1}^{p} \sum_{i=p+1}^{n} \int_{\Omega} a_{ij} \partial_{x_j} u_\varepsilon \partial_{x_i} \varphi \, dx$$

$$= \sum_{j} \sum_{i} \int_{\Omega} \partial_{x_j}(a_{ij} u_\varepsilon) \partial_{x_j} \varphi - (\partial_{x_j} a_{ij}) u_\varepsilon \partial_{x_j} \varphi \, dx$$

$$= \sum_{j} \sum_{i} \int_{\Omega} \partial_{x_i}(a_{ij} u_\varepsilon) \partial_{x_i} \varphi - (\partial_{x_i} a_{ij}) u_\varepsilon \partial_{x_i} \varphi \, dx$$

$$= \sum_{j} \sum_{i} \int_{\Omega} a_{ij} \partial_{x_j} u_\varepsilon \partial_{x_j} \varphi + (\partial_{x_j} a_{ij}) u_\varepsilon \partial_{x_j} \varphi - (\partial_{x_j} a_{ij}) u_\varepsilon \partial_{x_j} \varphi \, dx$$

and this is bounded thanks to (2.21). Note that the above integrations can be performed first when the data are smooth and then in the general case by approximation. Thus, passing to the limit in (2.20) we get

$$\int_{\Omega} A_{22} \nabla X_2 v_0 \cdot \nabla X_2 \varphi \, dx = \int_{\Omega} f \varphi \, dx.$$  

Let $X_1 \in \Pi_\Omega$ and $\psi \in D(\Omega X_1')$. It is clear that $\psi \in D(\Omega X_1')$ for $X_1'$ in a neighbourhood $U_{X_1}$ of $X_1$. Then if we choose $\eta \in D(U_{X_1})$ we have $\psi \eta \in D(\Omega)$ and

$$\int_{U_{X_1}} \eta \int_{\Omega X_1} A_{22} \nabla X_2 v_0 \cdot \nabla X_2 \psi \, dx = \int_{U_{X_1}} \eta \int_{\Omega X_1} f \psi \, dx \quad \forall \eta \in D(U_{X_1}).$$

This clearly implies that $v_0 = u_0(X_1, \cdot)$ and thus the whole sequence satisfies (2.19). This completes the proof of the theorem.

**Remark 2.1.** In the case where

$$A_{12} = A_{21} = 0 \quad (2.23)$$

we have $u_\varepsilon, \nabla X_2 u_\varepsilon \to u_0, \nabla X_2 u_0$ strongly in $L^2(\Omega)$. Indeed this follows the lines of Theorem 1.2. Taking $v = u_\varepsilon$ in (2.10) we get

$$\varepsilon^2 \int_{\Omega} A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 u_\varepsilon \, dx + \int_{\Omega} A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx = \int_{\Omega} f u_\varepsilon \, dx.$$  

From this it follows that

$$\limsup_{\varepsilon \to 0} \int_{\Omega} A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx \leq \int_{\Omega} f u_0 \, dx = \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx.$$  

From the coerciveness of $A_{22}$ one has also

$$\liminf_{\varepsilon \to 0} \int_{\Omega} A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx \geq \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx,$$
and thus
\[
\lim_{\varepsilon \to 0} \int_{\Omega} A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx = \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx.
\]
This leads to
\[
\nabla X_2 u_\varepsilon \to \nabla X_2 u_0 \quad \text{in} \quad L^2(\Omega)
\]
and by (2.16) to \( u_\varepsilon \to u_0 \) in \( L^2(\Omega) \).

**Remark 2.2.** One can assume that \( A_{11} = A_{11}(\varepsilon, x) \), \( A_{12} = A_{12}(\varepsilon, x) \), \( A_{21} = A_{21}(\varepsilon, x) \) and get the same results provided (2.4), (2.5) hold and the functions in (2.13) are bounded independently of \( \varepsilon \).

**Remark 2.3.** One can also consider the problem with a lower order term, i.e.
\[
\begin{aligned}
- \text{div}(A_{\varepsilon} \nabla u_\varepsilon) + au_\varepsilon &= f \quad \text{in} \ \Omega, \\
u_\varepsilon &\in H^1_0(\Omega),
\end{aligned}
\]
where \( a \in L^\infty(\Omega) \), \( a \geq 0 \). The convergence analysis can be carried out the same way.

### 3. A particular case

In this section we assume that
\[
\Omega = \omega_1 \times \omega
\]
where \( \omega_1 \) and \( \omega \) are bounded open sets of \( \mathbb{R}^p, \mathbb{R}^{n-p} \) respectively. We will assume these open sets of class \( C^1 \) and we will use the notation introduced in (2.1) to denote the points in \( \mathbb{R}^n \).

We consider a matrix \( A = A(\varepsilon) = (a_{ij}(\varepsilon, x)) \)
\[
A = \begin{pmatrix}
A_{11}(\varepsilon, x) & A_{12}(\varepsilon, x) \\
A_{21}(\varepsilon, x) & A_{22}(X_2)
\end{pmatrix}
\]
and we suppose that
\[
|a_{ij}(\varepsilon, \cdot)|_\infty \leq \Lambda \quad \forall \ i, j = 1, \ldots, n, \quad (3.3)
\]
\[
(A\xi \cdot \xi) \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \quad (3.4)
\]
where \( \lambda, \Lambda \) are independent of \( \varepsilon \), \( | \cdot \infty \) denotes the usual \( L^\infty(\Omega) \)-norm.

Moreover we assume that it holds that
\[
\sum_{i=1}^{p} \partial_{x_i} \{a_{ij}(\varepsilon, x)\} = 0 \quad \forall \ j = p + 1, \ldots, n. \quad (3.5)
\]
Note that the above assumption holds when for instance
\[
A_{12}(\varepsilon, x) = A_{12}(\varepsilon, X_2). \quad (3.6)
\]
As in the previous section we denote by \( A_\varepsilon \) the matrix
\[
A_\varepsilon = \begin{pmatrix}
\varepsilon^2 A_{11} & \varepsilon A_{12} \\
\varepsilon A_{21} & A_{22}
\end{pmatrix}. \quad (3.7)
\]
It is easy to see that (2.8) is still valid. For \( f \in H^{-1}(\omega) \) there exists a unique \( u_0 \) solution to
\[
\begin{cases}
  u_0 \in H^1_0(\omega), \\
  \int_\omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 v \, dX = \langle f, v \rangle \quad \forall v \in H^1_0(\omega).
\end{cases}
\] (3.8)

Moreover, for \( v \in H^1_0(\Omega) \) we have for almost every \( X_1 \) of \( \omega_1 \)
\[
v(X_1, \cdot) \in H^1_0(\omega).
\] (3.9)

Thus,
\[
\langle f, v \rangle = \int_{\omega_1} \langle f, v(X_1, \cdot) \rangle \, dX_1
\] (3.10)
is clearly a continuous linear form on \( H^1_0(\Omega) \) and by the Lax–Milgram theorem there exists a unique \( u_\varepsilon \) solution to
\[
\begin{cases}
  u_\varepsilon \in H^1_0(\Omega), \\
  \int_\Omega A_{\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).
\end{cases}
\] (3.11)

Under the above assumptions we have the following Caccioppoli-type inequality

**Lemma 3.1.** There exists a constant \( C = C(\lambda, \Lambda, |\nabla X_1 \rho|) \) independent of \( \varepsilon \) such that it holds
\[
\int_\Omega \{ \varepsilon^2 |\nabla X_1 (u_\varepsilon - u_0)|^2 + |\nabla X_2 (u_\varepsilon - u_0)|^2 \} \rho^2 \, dx \\
\leq C \varepsilon^2 \int_{\text{Supp}(\rho)} (u_\varepsilon - u_0)^2 \, dx
\] (3.12)
for any \( \rho \in H^1_0(\omega_1) \cap W^{1,\infty}(\omega_1), \text{Supp}(\rho) = \{ x \mid \rho(x) > 0 \} \).

**Proof.** From (3.8)–(3.11) we derive that
\[
\int_\Omega A_{\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 v \, dx \quad \forall v \in H^1_0(\Omega).
\] (3.13)

Noticing that \( u_0 \) is independent of \( X_1 \) this implies
\[
\int_\Omega A_{\varepsilon} \nabla (u_\varepsilon - u_0) \cdot \nabla v \, dx = -\varepsilon \int_\Omega A_{12} \nabla X_2 u_0 \cdot \nabla X_1 v \, dx \quad \forall v \in H^1_0(\Omega).
\] (3.14)
The last integral can be written as
\[
\int_\Omega A_{12} \nabla X_2 u_0 \cdot \nabla X_1 v \, dx = \int_\Omega \sum_{i=1}^{p} \sum_{j=p+1}^{n} a_{ij} \partial_{x_j} u_0 \partial_{x_i} v \, dx = 0
\] by (3.5). Thus, from (3.14) we obtain
\[
\int_\Omega A_{\varepsilon} \nabla (u_\varepsilon - u_0) \cdot \nabla v \, dx = 0 \quad \forall v \in H^1_0(\Omega).
\] (3.15)
If \( \rho \in W^{1,\infty}(\omega_1) \cap H^1_0(\omega_1) \) we have
\[
(u_\varepsilon - u_0) \rho^2 \in H^1_0(\Omega)
\] (3.16)
and from (3.15) we get – recall that $\rho$ is depending on $X_1$ only –
\[
\int_{\Omega} \{ A_\varepsilon \nabla (u_\varepsilon - u_0) \cdot \nabla (u_\varepsilon - u_0) \} \rho^2 \, dx \\
= -2 \int_{\Omega} \varepsilon^2 A_{11} \nabla X_1 (u_\varepsilon - u_0) \cdot \nabla X_1 \rho (u_\varepsilon - u_0) \rho \, dx \\
- 2 \int_{\Omega} \varepsilon A_{12} \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_1 \rho (u_\varepsilon - u_0) \rho \, dx.
\]

Using (2.8), (3.3) we get for some constant $C = C(\lambda, \Lambda, |\nabla X_1 \rho|_\infty)$
\[
\int_{\Omega} \{ \varepsilon^2 |\nabla X_1 (u_\varepsilon - u_0)|^2 + |\nabla X_2 (u_\varepsilon - u_0)|^2 \} \rho^2 \, dx \\
\leq \int_{\Omega} \varepsilon |\nabla X_1 (u_\varepsilon - u_0)| \rho C \varepsilon |u_\varepsilon - u_0| \, dx \\
+ \int_{\Omega} |\nabla X_2 (u_\varepsilon - u_0)| \rho C \varepsilon |u_\varepsilon - u_0| \, dx.
\]

Using the Cauchy–Young inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ we obtain
\[
\int_{\Omega} \{ \varepsilon^2 |\nabla X_1 (u_\varepsilon - u_0)|^2 + |\nabla X_2 (u_\varepsilon - u_0)|^2 \} \rho^2 \, dx \\
\leq \frac{1}{2} \int_{\Omega} \{ \varepsilon^2 |\nabla X_1 (u_\varepsilon - u_0)|^2 + |\nabla X_2 (u_\varepsilon - u_0)|^2 \} \rho^2 \, dx \\
+ \varepsilon^2 C^2 \int_{\text{Supp}(\rho)} (u_\varepsilon - u_0)^2 \, dx.
\]

This completes the proof of the lemma taking $2C^2$ as the constant $C$ in (3.12).

Then we can show:

**Theorem 3.1.** Let $\omega'_1 \subset \subset \omega_1$ be an open subset of $\omega_1$. For any $r$ there exists a constant $C$ independent of $\varepsilon$ such that
\[
\int_{\omega'_1 \times \omega} |\nabla (u_\varepsilon - u_0)|^2 \, dx \leq C \varepsilon^{2r} |f|_{-1,\omega}^2
\]
where $|f|_{-1,\omega}$ denotes the strong dual norm of $f$ in $H^{-1}(\omega)$. Note that the above inequality implies the convergence of $u_\varepsilon$ toward $u_0$ in $H^1(\omega'_1 \times \omega)$ with an arbitrary speed of convergence in terms of power of $\varepsilon$.

**Proof.** We can suppose without loss of generality that $\omega'_1$ is smooth. Then we set
\[
\delta = \text{dist}(\omega'_1, \partial \omega_1) > 0,
\]
\[
D_0 = \omega'_1, \quad D_k = \{ x \mid \text{dist}(x, \omega'_1) < k \frac{\delta}{q} \} \quad k = 1, \ldots, q.
\]

We will choose $q$ later. It is clear that for every $k = 0, \ldots, q - 1$ one can find a function $\rho_k$ such that
\[
0 \leq \rho_k \leq 1, \quad \rho_k = 1 \text{ on } D_k, \quad \rho_k = 0 \text{ outside } D_{k+1}, \quad |\nabla \rho_k| \leq C_q.
\]
Using (3.12) we derive for $k = 0, \ldots, q - 1$ that it holds for various constants $C$

\[
\begin{align*}
\int_{D_k \times \omega} \varepsilon^2 |\nabla X_1(u_\varepsilon - u_0)|^2 + |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx \\
\leq C\varepsilon^2 \int_{D_{k+1} \times \omega} (u_\varepsilon - u_0)^2 \, dx \\
\leq C\varepsilon^2 \int_{D_{k+1} \times \omega} |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx \\
\leq C\varepsilon^2 \int_{D_{k+1} \times \omega} \varepsilon^2 |\nabla X_1(u_\varepsilon - u_0)|^2 + |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx.
\end{align*}
\] (3.21)

(We used here the Poincaré inequality in the direction $X_2$). Recalling that $D_0 = \omega_1'$ and iterating $q$-times this formula leads to

\[
\begin{align*}
\int_{\omega'_1 \times \omega} \varepsilon^2 |\nabla X_1(u_\varepsilon - u_0)|^2 + |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx \\
\leq C\varepsilon^{2q} \int_{\Omega} \varepsilon^2 |\nabla X_1(u_\varepsilon - u_0)|^2 + |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx.
\end{align*}
\] (3.22)

To evaluate this last integral one remarks taking $v = u_0$ in (3.8) that it holds

\[
\lambda \int_{\omega} |\nabla X_2 u_0|^2 \, dx \leq |\langle f, u_0 \rangle| \leq |f|_{-1, \omega}
\] \[
\left( \int_{\omega} |\nabla X_2 u_0|^2 \, dx \right)^{1/2}.
\] (We took the last integral as the norm in $H^1_0(\omega)$). It follows that

\[
\int_{\Omega} |\nabla X_2 u_0|^2 \, dx \leq |\omega_1| \frac{|f|_{-1, \omega}^2}{\lambda^2}
\]

where $|\omega_1|$ denotes the measure of $\omega_1$. Similarly, taking $v = u_\varepsilon$ in (3.11) and using the ellipticity condition we get

\[
\begin{align*}
\lambda \int_{\Omega} \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + |\nabla X_2 u_\varepsilon|^2 \, dx \\
\leq \int_{\omega_1} |f|_{-1, \omega}
\left( \int_{\omega} |\nabla X_2 u_\varepsilon(X_1, X_2)|^2 \, dX_2 \right)^{1/2} \, dX_1 \\
\leq |f|_{-1, \omega} |\omega_1|^{1/2} \left( \int_{\Omega} |\nabla X_2 u_\varepsilon|^2 \, dx \right)^{1/2}.
\end{align*}
\]

From this it follows easily that

\[
\int_{\Omega} \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + |\nabla X_2 u_\varepsilon|^2 \, dx \leq |\omega_1| \frac{|f|_{-1, \omega}^2}{\lambda^2}
\]

and from (3.22) we obtain

\[
\int_{\omega'_1 \times \omega} \varepsilon^2 |\nabla X_1(u_\varepsilon - u_0)|^2 + |\nabla X_2(u_\varepsilon - u_0)|^2 \, dx \leq C\varepsilon^{2q} |f|_{-1, \omega}^2. \quad (3.23)
\]

Choosing $q$ such that $q > r + 1$ the result follows.

**Remark 3.1.** As previously, the above rate of convergence can be improved to an exponential one (see [5]).
4. Some applications

For $\omega_1$ a star-shaped $C^1$ domain with respect to the origin set
$$\Omega_\ell = \ell \omega_1 \times \omega \quad (4.1)$$
where $\ell$ is a parameter which will go to $+\infty$. Let $A = A(x)$ be a matrix of the type
$$A = \begin{pmatrix} A_{11}(x) & A_{12}(X_2) \\ A_{21}(x) & A_{22}(X_2) \end{pmatrix}, \quad x \in \mathbb{R}^n \times \omega, \quad (4.2)$$
with
$$|a_{ij}(x)| \leq \Lambda \quad \text{a.e. } x \in \mathbb{R}^n \times \omega, \quad (4.3)$$
$$(A\xi \cdot \xi) \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^n \times \omega, \quad (4.4)$$
$$\sum_{i=1}^p \partial_{x_i} a_{ij}(x) = 0 \quad \forall j = p+1, \ldots, n. \quad (4.5)$$

Let $f \in H^{-1}(\omega)$. We suppose that $f$ has been extended to $\Omega_\ell$ by the formula (3.10) where $\omega_1$ is replaced by $\ell \omega_1$. Then as the consequence of the Lax–Milgram theorem there exists $u_\infty$ and $u_\ell$ solution to
$$\begin{cases}
\int_\omega A_{22} \nabla X_2 u_\infty \nabla X_2 v \, dX_2 = \langle f, v \rangle & \forall v \in H^1_0(\omega), \\
u_\infty \in H^1_0(\omega),
\end{cases} \quad (4.6)$$
$$\begin{cases}
\int_{\Omega_\ell} A \nabla u_\ell \nabla v \, dx = \langle f, v \rangle & \forall v \in H^1_0(\Omega_\ell), \\
u_\ell \in H^1_0(\Omega_\ell).
\end{cases} \quad (4.7)$$

Moreover we have (compare to [2], [3]):

**Theorem 4.1.** Under the above assumptions, for any $r > 0$ there exists a constant $C$ independent of $\ell$ such that
$$\int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq C |f|^2 \omega_1 / \ell^{2r}. \quad (4.8)$$

In particular $u_\ell$ converges toward $u_\infty$ with a speed arbitrary large in term of $\ell$ on any subdomain $\Omega_{\ell_0}$, $\ell_0$ fixed.

**Proof.** We use a rescaling argument. We set
$$v_\ell(X_1, X_2) = u_\ell(\ell X_1, X_2). \quad (4.9)$$
Clearly $v_\ell$ is defined on $\omega_1 \times \omega$. Moreover it holds that
$$\nabla u_\ell(\ell X_1, X_2) = \left( \frac{1}{\ell} \nabla X_1 v_\ell(X_1, X_2), \nabla X_2 v_\ell(X_1, X_2) \right).$$

From (4.7) making the change of variable $(X_1, X_2) \to (\ell X_1, X_2)$ we derive
$$\int_{\Omega} A(\ell X_1, X_2) \nabla u_\ell(\ell X_1, X_2) \cdot \nabla v(\ell X_1, X_2) \, dx = \int_{\omega_1} \langle f, v(\ell X_1, \cdot) \rangle \, dX_1$$
for any $v \in H^1_0(\Omega_\ell)$ – recall that $\Omega$ is the set $\omega_1 \times \omega$ (see (3.1)). For $w \in H^1_0(\Omega)$ the function
$$v(X_1, X_2) = w \left( \frac{X_1}{\ell}, X_2 \right)$$
can be used as a test function above. Since
\[ \nabla v(X_1, X_2) = \left( \frac{1}{\ell} \nabla w(X_1, X_2) \nabla X_1, \nabla w(X_1, X_2) \nabla X_2 \right) \]
we obtain that \( v_\ell \) satisfies
\[
\begin{cases}
\int_\Omega A_{1/\ell} \nabla v_\ell \cdot \nabla w \, dx = \int_{\omega_1} \langle f, w(X_1, \cdot) \rangle \, dX_1 & \forall v \in H^1_0(\Omega), \\
v_\ell \in H^1_0(\Omega),
\end{cases}
\]
where \( A_{1/\ell} \) is the matrix
\[
A_{1/\ell} = \begin{pmatrix}
\frac{1}{\ell^2} A_{11}(\ell X_1, X_2) & \frac{1}{\ell} A_{12}(X_2) \\
\frac{1}{\ell} A_{21}(\ell X_1, X_2) & A_{22}(X_2)
\end{pmatrix}.
\]
Setting \( \varepsilon = 1/\ell \) we derive from Theorem 3.1 that it holds
\[
\int_{1/2 \omega_1 \times \omega} \ell^{-2} |\nabla X_1(v_\ell - u_\infty)|^2 + |\nabla X_2(v_\ell - u_\infty)|^2 \, dx \leq C \ell^{-2r-2-p} |f|_{-1,\omega}^2
\]
which implies
\[
\int_{1/2 \omega_1 \times \omega} |\nabla (u_\ell - u_\infty)(\ell X_1, X_2)|^2 \, dx \leq C \ell^{-2r-p} |f|_{-1,\omega}^2.
\]
Making the change of variable \((\ell X_1, X_2) \rightarrow (X_1, X_2)\) we get
\[
\int_{\Omega/2} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq C \ell^{-2r} |f|_{-1,\omega}^2.
\]
In particular if we fix \( \ell_0 < 1/2 \) then it holds, since \( \Omega_{\ell_0} \subset \Omega_{\ell/2} \)
\[
\int_{\Omega_{\ell_0}} |\nabla (u_\ell - u_\infty)|^2 \, dx \leq C \ell^{-2r} |f|_{-1,\omega}^2 \quad (4.10)
\]
which concludes the proof. \( \square \)

This relationship between \( u_\ell \) and \( u_\varepsilon \) allows to improve the rate of convergence of \( u_\varepsilon \) toward \( u_0 \). Indeed, consider the case where
\[
\Omega = \omega_1 \times \omega
\]
where \( \omega_1 \) is a bounded open subset of \( \mathbb{R}^p \) star-shaped with respect to the origin and \( \omega \) is a bounded open set of \( \mathbb{R}^{n-p} \). Consider for \( \varepsilon > 0, f \in H^{-1}(\omega) \) \( u_\varepsilon \) the weak solution to
\[
\begin{cases}
-\varepsilon^2 \Delta_1 u_\varepsilon - \Delta_2 u_\varepsilon = f & \text{in } \Omega, \\
u_\varepsilon \in H^1_0(\Omega).
\end{cases}
\]
(4.11)

In the above system we have denoted by \( \Delta_1 \) (resp. \( \Delta_2 \)) the Laplace operator in \( X_1 \) (resp. in \( X_2 \)) i.e.
\[
\Delta_1 = \sum_{i=1}^p \partial^2_{x_i}, \quad \Delta_2 = \sum_{i=p+1}^n \partial^2_{x_i}
\]
and \( f \) is supposed to be extended in \( X_1 \) by (3.10).

Then we have
Theorem 4.2. Let $\omega'_1 \subset \subset \omega$ be a bounded open set. There exist positive constants $c, C$ independent of $\varepsilon$ such that it holds
\begin{equation}
\int_{\omega'_1 \times \omega} |\nabla (u_\varepsilon - u_0)|^2 \, dx \leq Ce^{-c/\varepsilon},
\end{equation}
where $u_0$ is the weak solution to
\begin{equation}
\begin{cases}
-\Delta_2 u_0 = f & \text{in } \omega, \\
u_0 \in H^1_0(\omega).
\end{cases}
\end{equation}

Proof. One introduces $u_\ell$ the solution to
\begin{equation}
\begin{cases}
-\Delta u_\ell = f & \text{in } \Omega_\ell = \ell \omega_1 \times \omega, \\
u_\ell \in H^1_0(\Omega_\ell).
\end{cases}
\end{equation}
From [2] or [3] one has
\begin{equation}
\int_{\ell \omega'_1 \times \omega} |\nabla (u_\ell - u_0)|^2 \, dx \leq Ce^{-c\ell}
\end{equation}
for some constants $c, C$. Setting then
\begin{equation*}
\varepsilon = \frac{1}{\ell}, \quad u_\varepsilon = u_\ell \left( \frac{1}{\varepsilon} X_1, X_2 \right)
\end{equation*}
one sees easily as above that $u_\varepsilon$ coincides with $u_\ell$ introduced in (4.11). Then the result follows after an easy change of variable. \hfill \Box

Remark 4.1. With the same type of argument one can also consider the case of nonlinear singular perturbation problems (see [2], [3], [4], [8], [10]). Some results of this section were obtained independently in [8]. The exponential rate of convergence can be obtained in more general situations. We refer the reader to [5]

Added in proofs: In Theorem 2.1 the assumption (2.13) can be removed and strong convergence obtained.

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