On Some Nonlocal in Time and Space Parabolic Problem

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Abstract

The goal of this note is to study nonlinear parabolic problems nonlocal in time and space. We first establish the existence of a solution and its uniqueness in certain cases. Finally we consider its asymptotic behaviour.

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1 Introduction and notation

We will denote by Ω a smooth bounded open set of $\mathbb{R}^n$, $n \geq 2$ with boundary $\partial\Omega$. We would like to consider the following problem. Find $u = u(x,t)$ such that

$$\begin{cases} u_t - \alpha\big(\int_\Omega g(x)u(x,t)dx\big)\Delta u + \beta\big(\int_0^t h(s)u(x,s)ds\big)u = f \quad \text{in } \Omega \times (0,T), \\ \\ u(\cdot,t) = 0 \text{ on } \partial\Omega,\ t \in (0,T),\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

T is a positive number, u_0, f, g, h are given data. The equation could be regarded as a model of population dynamics where $u(x,t)$ is the density of a population at the location x, at the time t. The nonlinear terms are an account for a death or diffusion rate which at time t depends on the total population having been at the location x in the past or in Ω at time t. To study this issue we were inspired by the papers [7]-[10] where a similar problem was introduced at the difference that in (1.1) the integral goes up to T which we think is an interesting point of view but perhaps a bit surprising from a realistic one in our framework.

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The paper is divided as follows. In the next section we prove existence of a weak solution to (1.1). In the subsequent section we establish a result of uniqueness. Note that in comparison to [7]-[10] our result is global. Finally we study in a simple case the asymptotic behaviour of the solution to (1.1).

2 A result of existence

We denote by $L^p(\Omega), 1 \leq p \leq +\infty$ the usual L^p-space on Ω. It is equipped with its usual norm and for instance, in the case where $p = 2$, we denote it by $| \ |_2$ i.e.

$$|v|_2^2 = \int_\Omega v(x)^2 dx \quad \forall v \in L^2(\Omega).$$

We refer the reader to [2]-[6] for the notation used in the sequel, for instance for $H_0^1(\Omega)$ or its dual $H^{-1}(\Omega)$ or the spaces $L^2(0,T;V)$, $L^2(0,T;V')$ when V is a Banach space.

The main result of this section is the following :

Theorem 2.1. *Set $V = H_0^1(\Omega), V' = H^{-1}(\Omega)$. Suppose*

$$u_0 \in L^2(\Omega),\ f \in L^2(0,T;V'),\ \alpha, \beta \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ g \in L^\infty(\Omega), h \in L^\infty(0,T),\ \forall T, \tag{2.1}$$

and that for some positive constant a one has

$$0 < a \leq \alpha. \tag{2.2}$$

Then there exists a weak solution to (1.1). *$C(\mathbb{R})$ denotes the space of continuous functions.*

Proof. 1. One can assume that $\beta \geq 1$.
Indeed, suppose that we can solve (1.1) in this case. u is solution to (1.1) iff

$$\tilde{u} = e^{-\lambda t} u \tag{2.3}$$

satisfies

$$\begin{aligned}
&(e^{\lambda t}\tilde{u})_t - \alpha\Big(\int_\Omega g(x) e^{\lambda t}\tilde{u}(x,t)dx\Big) e^{\lambda t}\Delta\tilde{u} + \beta\Big(\int_0^t h(s)e^{\lambda s}\tilde{u}(x,s)ds\Big)e^{\lambda t}\tilde{u} = f, \\
\Leftrightarrow \quad & e^{\lambda t}\tilde{u}_t + \lambda e^{\lambda t}\tilde{u} - \alpha\Big(\int_\Omega g(x) e^{\lambda t}\tilde{u}(x,t)dx\Big) e^{\lambda t}\Delta\tilde{u} + \beta\Big(\int_0^t h(s)e^{\lambda s}\tilde{u}(x,s)ds\Big)e^{\lambda t}\tilde{u} = f, \\
\Leftrightarrow \quad & \tilde{u}_t - \alpha\Big(\int_\Omega g(x) e^{\lambda t}\tilde{u}(x,t)dx\Big)\Delta\tilde{u} + \Big\{\lambda + \beta\Big(\int_0^t h(s)e^{\lambda s}\tilde{u}(x,s)ds\Big)\Big\}\tilde{u} = e^{-\lambda t} f,
\end{aligned}$$

i.e. iff $\tilde{u}$ satisfies (1.1) with f, h, α replaced respectively by $e^{-\lambda t} f, e^{\lambda t} h, \alpha(e^{\lambda t}.)$ and β by $\lambda + \beta$ which is greater than 1 for λ large enough.

2. We suppose that $\beta \geq 1$.

Let $w \in L^2(0,T;L^2(\Omega)) \subset L^1(0,T;L^1(\Omega))$. Then, see [2], there exists a unique $u = S(w)$ solution to

$$\begin{cases} u \in L^2(0,T;V),\ u_t \in L^2(0,T;V'), \\ \frac{d}{dt}(u,v) + \alpha\big(\int_\Omega g(x)w(x,t)dx\big)\int_\Omega \nabla u \cdot \nabla v dx + \big(\beta\big(\int_0^t h(s)w(x,s)ds\big)u, v\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega),\ \text{in } \mathcal{D}'(0,T). \end{cases} \tag{2.4}$$

In the equation above we denote by (,) the canonical scalar product in $L^2(\Omega)$ and by $\langle\ ,\ \rangle$ the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $\mathcal{D}(0,T)$ and $\mathcal{D}'(0,T)$ denote respectively the space of C^∞ functions with compact support in $(0,T)$ and its dual, the usual space of distributions on $(0,T)$. (Cf. for instance [2]). We will be done if we can show that S has a fixed point. Taking in the equation above $v = u$ we get easily if $a \wedge 1$ denotes the minimum of a and 1

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u|_2^2 + a \wedge 1 \int_\Omega (|\nabla u|^2 + u^2)dx &\le \langle f, u\rangle \le |f|_{V'}\big||\nabla u|\big|_2 \\ &\le \frac{1}{2(a\wedge 1)}|f|_{V'}^2 + \frac{a\wedge 1}{2}\big||\nabla u|\big|_2^2. \end{aligned}$$

$|f|_{V'}$ denotes the strong dual norm of f in $H^{-1}(\Omega)$ associated to the norm $\big||\nabla u|\big|_2$ in $H_0^1(\Omega)$. From this we derive

$$\frac{d}{dt}|u|_2^2 + (a\wedge 1)\int_\Omega (|\nabla u|^2 + u^2)dx \le \frac{1}{a\wedge 1}|f|_{V'}^2$$

and after an integration in t

$$|u|_2^2 + (a\wedge 1)\int_0^t\int_\Omega (|\nabla u|^2 + u^2)dxds \le |u_0|_2^2 + \frac{1}{a\wedge 1}\int_0^t |f(\cdot,s)|_{V'}^2 ds.$$

It follows that

$$|u|_{L^2(0,T;V)}^2,\ |u|_{L^2(0,T;L^2(\Omega))}^2 \le C^2 = \frac{1}{(a\wedge 1)}\Big(|u_0|_2^2 + \frac{1}{a\wedge 1}\int_0^T |f(\cdot,s)|_{V'}^2 ds\Big).$$

Set

$$B = \{v \in L^2(0,T;L^2(\Omega)) \mid |v|_{L^2(0,T;L^2(\Omega))} \le C\}.$$

Clearly, S maps B into itself. Moreover since

$$u_t = \alpha\Big(\int_\Omega g(x)w(x,t)dx\Big)\Delta u - \beta\Big(\int_0^t h(s)w(x,s)ds\Big)u + f \quad \text{in } V'$$

u_t is uniformly bounded in $L^2(0,T;V')$ and $S(B)$ is relatively compact in B. The existence of a weak solution to (1.1) will follow by the Schauder fixed point theorem if S is continuous. To show that, let $w_n \in L^2(0,T;L^2(\Omega))$ such that

$$w_n \to w \text{ in } L^2(0,T;L^2(\Omega)).$$

Denote $u_n = S(w_n)$. The estimates above hold and one can extract a subsequence such that, if we still label it by n

$$\begin{aligned}
&gw_n \to gw \quad \text{in} \quad L^2(0,T;L^2(\Omega)),\\
&hw_n \to hw \quad \text{in} \quad L^2(0,T;L^2(\Omega)),\\
&u_n \to u_\infty \quad \text{in} \quad L^2(0,T;L^2(\Omega)),\\
&\nabla u_n \rightharpoonup \nabla u_\infty \quad \text{in} \quad L^2(0,T;L^2(\Omega)),\\
&(u_n)_t \rightharpoonup (u_\infty)_t \quad \text{in} \quad L^2(0,T;V').
\end{aligned} \tag{2.5}$$

By definition of u_n we have for every $v \in H_0^1(\Omega)$ and every $\varphi \in \mathcal{D}(0,T)$

$$\int_0^T -(u_n,v)\varphi'(t)dt + \int_0^T \varphi(t)\int_\Omega \alpha\Big(\int_\Omega g(x)w(x,t)dx\Big)\nabla u_n\cdot\nabla v\ dxdt \\ + \int_0^T \varphi(t)\int_\Omega \beta\Big(\int_0^t h(s)w_n(x,s)ds\Big)u_n v\ dxdt = \int_0^T \langle f,v\rangle\varphi(t)dt.$$

By the Lebesgue theorem

$$\varphi(t)\beta\Big(\int_0^t h(s)w_n(x,s)ds\Big)v \to \varphi(t)\beta\Big(\int_0^t h(s)w(x,s)ds\Big)v \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{2.6}$$

Indeed, note that

$$\begin{aligned}
\Big|\int_0^t h(s)w_n(x,s)ds - \int_0^t h(s)w(x,s)ds\Big| &\le \int_0^T |h|_\infty |w_n - w|(x,s)ds \\
&\le |h|_\infty \sqrt{T}\Big\{\int_0^T (w_n-w)^2(x,s)ds\Big\}^{\frac12} \to 0 \ \text{ a.e.}
\end{aligned}$$

up to a subsequence. $|h|_\infty$ is the $L^\infty(0,T)$-norm of h. Using (2.6) and the analogue written for α and g, one can pass to the limit in the equation satisfied by u_n. It follows that $u_\infty = S(w)$. Since the limit of u_n is unique the whole sequence u_n converges toward $u_\infty = S(w)$ and thus S is continuous. This completes the proof of the theorem. □

Remark 1. *The same existence result holds if in* (1.1) *one replaces the integral on* $(0,t)$ *by*

$$\int_0^{t'} h(s)u(x,s)ds$$

where t' *is any real number in* $(0,T]$.

3 Uniqueness issue

One has the following estimate for the solution to (1.1):

Proposition 3.1. *Suppose that $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(\Omega \times (0,T))$, $\beta \geq 1$. Then it holds*

$$|u| \leq K = |f|_\infty \vee |u_0|_\infty. \tag{3.1}$$

($\vee$ stands for the maximum of two numbers).

Proof. One has

$$\frac{d}{dt}(u-K) - \nabla \cdot \Big(\alpha\big(\int_\Omega gu\ dx\big)\nabla(u-K)\Big) + \beta\big(\int_0^t hu\ ds\big)u - K = f - K \leq 0.$$

It follows, using as test function $(u-K)^+$ where $(\ )^+$ denotes the positive part of a function

$$\frac{1}{2}\frac{d}{dt}|(u-K)^+|_2^2 + a \wedge 1 \int_\Omega |\nabla(u-K)^+|^2 + ((u-K)^+)^2 \leq 0.$$

This implies that

$$\frac{d}{dt}\big(|(u-K)^+|_2^2 e^{2(a\wedge 1)t}\big) \leq 0$$

and since this quantity vanishes at 0 it vanishes for all time. This shows that $u \leq K$. Since $-u$ satisfies a similar equation one has also $-u \leq K$. This completes the proof of the proposition. $\square$

One can then prove the following uniqueness result :

Theorem 3.1. *Suppose that $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(\Omega \times (0,T))$, $g \in L^\infty(\Omega)$, $h \in L^\infty(0.T)$. Suppose that $\beta \geq 1$, α are Lipschitz continuous in the sense that for some positive constant C_α, C_β*

$$|\alpha(\xi) - \alpha(\eta)| \leq C_\alpha|\xi - \eta|, \quad |\beta(\xi) - \beta(\eta)| \leq C_\beta|\xi - \eta| \ \ \forall \xi, \eta \in \mathbb{R}, \tag{3.2}$$

then the weak solution to (1.1) *is unique.*

Proof. Let u_1, u_2 be two solutions to (1.1). By subtraction one gets

$$\begin{aligned}
\frac{d}{dt}(u_1 - u_2) &- \alpha\big(\int_\Omega g(x)u_1(x,t)dx\big)\Delta(u_1-u_2) + \beta\big(\int_0^t h(s)u_1(x,s)ds\big)(u_1-u_2) \\
&= \Big(\alpha\big(\int_\Omega g(x)u_1(x,t)dx\big) - \alpha\big(\int_\Omega g(x)u_2(x,t)dx\big)\Big)\Delta u_2 \\
&\qquad - \{\beta\big(\int_0^t h(s)u_1(x,s)ds\big) - \beta\big(\int_0^t h(s)u_2(x,s)ds\big)\}u_2.
\end{aligned}$$

Multiplying by $(u_1 - u_2)$ and integrating on Ω we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 &+ \alpha\big(\int_\Omega g(x)u_1(x,t)dx\big)\int_\Omega |\nabla(u_1-u_2)|^2dx \\
&\qquad + \int_\Omega \beta\big(\int_0^t h(s)u_1(x,s)ds\big)(u_1-u_2)^2dx \\
&= -\int_\Omega \Big(\alpha\big(\int_\Omega g(x)u_1(x,t)dx\big) - \alpha\big(\int_\Omega g(x)u_2(x,t)dx\big)\Big)\nabla u_2 \cdot \nabla(u_1-u_2)dx \\
&\qquad - \int_\Omega \{\beta\big(\int_0^t h(s)u_1(x,s)ds\big) - \beta\big(\int_0^t h(s)u_2(x,s)ds\big)\}u_2(u_1-u_2)dx.
\end{aligned}$$

By (2.2) and since $\beta \geq 1$ we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + a\int_\Omega |\nabla(u_1-u_2)|^2dx + \int_\Omega (u_1-u_2)^2dx \\
&\qquad \leq \int_\Omega |\alpha\big(\int_\Omega g(x)u_1(x,t)dx\big) - \alpha\big(\int_\Omega g(x)u_2(x,t)dx\big)|\ |\nabla u_2|\ |\nabla(u_1-u_2)|dx \\
&\qquad\qquad + \int_\Omega |\{\beta\big(\int_0^t h(s)u_1(x,s)ds\big) - \beta\big(\int_0^t h(s)u_2(x,s)ds\big)\}|\ |u_2|\ |(u_1-u_2)|dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + a\int_\Omega |\nabla(u_1-u_2)|^2dx + \int_\Omega (u_1-u_2)^2dx \\
&\qquad \leq \int_\Omega C_\alpha|\int_\Omega g(x)(u_1(x,t)-u_2(x,t))dx|\ |\nabla u_2|\ |\nabla(u_1-u_2)|dx \\
&\qquad\qquad + \int_\Omega C_\beta|\int_0^t h(s)(u_1(x,s)-u_2(x,s))ds|\ |u_2|\ |(u_1-u_2)|dx \\
&\qquad \leq \int_\Omega C_\alpha g_\infty\Big(\int_\Omega |u_1(x,t)-u_2(x,t)|dx\Big)|\nabla u_2|\ |\nabla(u_1-u_2)|dx \\
&\qquad\qquad + \int_\Omega C_\beta h_\infty \int_0^t |u_1(x,s)-u_2(x,s)|ds\ |u_2(x,t)||(u_1-u_2)(x,t)|dx
\end{aligned}$$

where g_∞, h_∞ denote the $L^\infty(\Omega)$ and $L^\infty(0,T)$ norms of g and h. Now we use (3.1) and the Young inequality

$$ab \leq \epsilon a^2 + C_\epsilon b^2$$

to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + a\int_\Omega |\nabla(u_1-u_2)|^2dx + \int_\Omega (u_1-u_2)^2dx \\
&\qquad \leq \int_\Omega C_\epsilon\Big\{C_\alpha g_\infty\Big(\int_\Omega |u_1(x,t)-u_2(x,t)|dx\Big)|\nabla u_2|\Big\}^2 + \epsilon|\nabla(u_1-u_2)|^2dx \\
&\qquad\qquad + \int_\Omega C_\beta h_\infty K\frac{1}{2}\frac{d}{dt}\Big(\int_0^t |u_1(x,s)-u_2(x,s)|ds\Big)^2dx.
\end{aligned}$$

Choosing $\epsilon = \frac{a}{2}$ it comes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2 + \frac{a}{2}\int_\Omega |\nabla(u_1-u_2)|^2dx + \int_\Omega (u_1-u_2)^2dx \\
&\qquad \leq \int_\Omega C_\epsilon C_\alpha^2 g_\infty^2|\nabla u_2|^2\Big(\int_\Omega |u_1(x,t)-u_2(x,t)|dx\Big)^2dx \\
&\qquad\qquad + \int_\Omega C_\beta h_\infty K\frac{1}{2}\frac{d}{dt}\Big(\int_0^t |u_1(x,s)-u_2(x,s)|ds\Big)^2dx \\
&\qquad \leq \int_\Omega C_\epsilon C_\alpha^2 g_\infty^2|\nabla u_2|^2|\Omega||u_1-u_2|_2^2dx \\
&\qquad\qquad + \int_\Omega C_\beta h_\infty K\frac{1}{2}\frac{d}{dt}\Big(\int_0^t |u_1(x,s)-u_2(x,s)|ds\Big)^2dx.
\end{aligned}$$

We used Hölder's inequality, $|\Omega|$ denotes the measure of Ω. Thus we obtain

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|_2^2+\frac{a}{2}\int_\Omega|\nabla(u_1-u_2)|^2dx+\int_\Omega(u_1-u_2)^2dx$$
$$\le C_\epsilon C_\alpha^2 g_\infty^2|\nabla u_2|_2^2|\Omega||u_1-u_2|_2^2$$
$$+\int_\Omega C_\beta h_\infty K\frac{1}{2}\frac{d}{dt}\Big(\int_0^t|u_1(x,s)-u_2(x,s)|ds\Big)^2dx.$$

Integrating between 0 and t we derive

$$|u_1-u_2|_2^2\le 2\int_0^t C_\epsilon C_\alpha^2 g_\infty^2|\nabla u_2|_2^2|\Omega||u_1-u_2|_2^2ds$$
$$+\int_\Omega C_\beta h_\infty K\Big(\int_0^t|u_1(x,s)-u_2(x,s)|ds\Big)^2dx$$
$$\le 2\int_0^t C_\epsilon C_\alpha^2 g_\infty^2|\nabla u_2|_2^2|\Omega||u_1(x,t)-u_2(x,t)|_2^2dt$$
$$+\int_\Omega C_\beta h_\infty Kt\int_0^t|u_1(x,s)-u_2(x,s)|^2dsdx$$
$$=\int_0^t\Big(2C_\epsilon C_\alpha^2 g_\infty^2|\Omega||\nabla u_2|_2^2+C_\beta h_\infty Kt\Big)|u_1-u_2|_2^2dt.$$

Since $\Big(2C_\epsilon C_\alpha^2 g_\infty^2|\Omega||\nabla u_2|_2^2+C_\beta h_\infty Kt\Big)\in L^1(0,T)$ the result follows from the Gronwall inequality. $\square$

4 Stationary problem

In this section we consider u solution to (1.1) and we assume

$$f,u_0,g,h\ge 0. \tag{4.1}$$

Moreover we assume that

$$\beta(z)\ \text{ admits a limit when } z\to+\infty. \tag{4.2}$$

First notice that (4.1) implies that $u\ge 0$. Indeed multiplying (1.1) by $-u^-$ we get

$$\frac{1}{2}\frac{d}{dt}|u^-|_2^2+\alpha(\int_\Omega g\ udx)\int_\Omega|\nabla u^-|^2dx+\int_\Omega\beta(\int_0^t h\ uds)(u^-)^2dx=-(f,u^-)\le 0.$$

Since α,β are positive we get

$$\frac{1}{2}\frac{d}{dt}|u^-|_2^2\le 0$$

i.e. $u^-=0$ since $u^-(x,0)=0$. Since $u\ge 0$, then

$$t\to\int_0^t h(s)\ u(x,s)\ ds$$

is nondecreasing in time and has a limit when $t \to +\infty$ for almost every $x \in \Omega$ and so does

$$\beta\Big(\int_0^t h(s)\ u(x,s)\ ds \Big).$$

We denote by $\beta_\infty(x) \in L^\infty(\Omega)$ this limit. Then the stationary problem associated to (1.1) is : find u_∞ weak solution to

$$\begin{cases} -\alpha\big(\int_\Omega g(x) u_\infty(x) dx \big) \Delta u_\infty + \beta_\infty u_\infty = f(x) \text{ in } \Omega, \\ u_\infty = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.3}$$

For convenience we set

$$\ell(u) = \int_\Omega g(x) u(x) dx \tag{4.4}$$

and for any $\mu > 0$ we denote by u_μ the weak solution to

$$\begin{cases} -\mu \Delta u_\mu + \beta_\infty u_\mu = f(x) \text{ in } \Omega, \\ u_\mu = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

As usual, solving a problem like (4.3) reduces to solve an equation in $\mathbb{R}$ (see [4], [1]). Here arguing on $\ell(u)$ or $\alpha(\ell(u))$ offers two different equations. Indeed we have first

Theorem 4.1. *The mapping $u \to \ell(u)$ is a one-to-one mapping from the set of solutions to* (4.3) *into the set of solutions of the equation in* $\mathbb{R}$

$$\mu = \ell(u_{\alpha(\mu)}). \tag{4.6}$$

Proof. Suppose that u_∞ is solution to (4.3). Then, with our notation for u_μ

$$u_\infty = u_{\alpha(\ell(u_\infty))}$$

this implies

$$\ell(u_\infty) = \ell(u_{\alpha(\ell(u_\infty))})$$

i.e. $\ell(u_\infty)$ is solution to (4.6). Conversely, suppose that μ is solution to (4.6). Then, $u_{\alpha(\mu)}$ satisfies

$$\begin{cases} -\alpha(\mu) \Delta u_{\alpha(\mu)} + \beta_\infty u_{\alpha(\mu)} = f(x) \text{ in } \Omega, \\ u_{\alpha(\mu)} = 0 \text{ on } \partial\Omega. \end{cases}$$

Since, by (4.6), $\alpha(\mu) = \alpha(\ell(u_{\alpha(\mu)}))$, $u_{\alpha(\mu)}$ is solution to (4.3). The injectivity of the map $u \to \ell(u)$ is due to the fact that if $\ell(u_1) = \ell(u_2)$ when u_1 and u_2 are solutions to (4.3) then clearly $u_1 = u_2$. This completes the proof of the theorem. □

It is now interesting to remark that the set of solutions can also be characterised by another set of fixed points namely :

Theorem 4.2. *The mapping $u \to \alpha(\ell(u))$ is a one-to-one mapping from the set of solutions to* (4.3) *into the set of solutions of the equation in* $\mathbb{R}$

$$\mu = \alpha(\ell(u_\mu)). \tag{4.7}$$

Proof. Suppose that u_∞ is solution to (4.3). Then, with our definition for u_μ

$$u_\infty = u_{\alpha(\ell(u_\infty))}$$

this implies that

$$\alpha(\ell(u_\infty)) = \alpha(\ell(u_{\alpha(\ell(u_\infty))})$$

i.e. $\alpha(\ell(u_\infty))$ is solution to (4.7). Conversely, suppose that μ is solution to (4.7). Then u_μ is solution to

$$\begin{cases} -\alpha(\ell(u_\mu))\Delta u_\mu + \beta_\infty u_\mu = f(x) \text{ in } \Omega, \\ u_\mu = 0 \text{ on } \partial\Omega, \end{cases}$$

i.e. u_μ is solution to (4.3). To prove the injectivity of the map $u \to \alpha(\ell(u))$ one has just to notice that if $\alpha(\ell(u_1)) = \alpha(\ell(u_2))$ when u_1, u_2 are solutions to (4.3) then clearly $u_1 = u_2 = u_{\alpha(\ell(u_i))}$. This completes the proof of the theorem. □

Then we can now show

Theorem 4.3. *Suppose that α is continuous and for some constants α_0, α_1 one has*

$$0 < \alpha_0 \le \alpha \le \alpha_1, \tag{4.8}$$

then the problem (4.3) *admits at least one solution.*

Proof. Due to (4.8) the strait line $y = \mu$ is cutting the curve $y = \alpha(\ell(\mu))$ and the result follows from the theorem 4.2. □

Remark 2. *Of course* (4.7) *can have several solutions and even an infinity. In the case of a single solution it would be interesting and non trivial to show the convergence of $u(t)$ toward u_∞. In the next paragraph we address a simple case to show what is on stake. We made it voluntary simple in a didactic spirit.*

Let us suppose that g is an eigenvalue of the Dirichlet problem i.e. that for some $\lambda > 0$, g satisfies in a weak sense

$$-\Delta g = \lambda g \text{ in } \Omega, \quad g = 0 \text{ on } \partial\Omega. \tag{4.9}$$

Then we have

Theorem 4.4. *Let g be solution to* (4.9)*. Suppose that β is a positive constant, $(f, g) > 0$ and that the equation*

$$(\lambda\alpha(\mu) + \beta)\mu = (f, g) \tag{4.10}$$

admits a unique solution. Then if $u(x,t)$ is solution to (1.1) *and u_∞ solution to* (4.3) *one has*

$$|u(x,t) - u_\infty|_2 \to 0 \text{ when } t \to +\infty. \tag{4.11}$$

Proof. It is enough to show (see [2]) that $\ell(u(x,t)) \to \ell(u_\infty)$ when $t \to \infty$. Multiplying the equation (1.1) by g and integrating on Ω one gets

$$\frac{d}{dt}(u,g) + \alpha(\ell(u)) \int_\Omega \nabla u \nabla g dx + \beta(u,g) = (f,g)$$

i.e. using the definition of g and ℓ it comes

$$\frac{d}{dt}\ell(u) + \lambda\alpha(\ell(u))\ell(u) + \beta\ell(u) = (f,g).$$

Denote by μ_∞ the unique solution to (4.10). Since we assume $(f,g) > 0$ one has $\mu_\infty > 0$ and $(\lambda\alpha(\mu) + \beta)\mu < (f,g)$ for $\mu < \mu_\infty$. Indeed, for $\mu = 0$ $(\lambda\alpha(\mu) + \beta)\mu < (f,g)$ and the inequality follows for any $\mu < \mu_\infty$ since the solution to (4.10) is supposed to be unique. Suppose that

$$\ell(u_0) < \mu_\infty.$$

Since $\ell(u)$ is solution to the differential equation

$$\frac{d}{dt}\ell(u) = (f,g) - (\lambda\alpha(\ell(u)) + \beta)\ell(u)$$

$\ell(u)$ is increasing and of course converging toward μ_∞. Similarly $\ell(u_0) > \mu_\infty$ implies that $\ell(u)$ is decreasing toward μ_∞ (indeed for $\mu > \mu_\infty$, $(f,g) - (\lambda\alpha(\mu) + \beta)\mu$ cannot take another time the value 0, but also cannot be positive since in this case since $(f,g) - (\lambda\alpha(\mu) + \beta)\mu \le (f,g) - (\lambda\alpha_0 + \beta)\mu \to -\infty$ when $\mu \to +\infty$, $(f,g) - (\lambda\alpha(\mu) + \beta)\mu$ would have another 0). This completes the proof of the theorem. $\square$

Remark 3. *In the case that we just considered one could describe the asymptotic behaviour of* u *using the same argument when the equation* (4.10) *admits different isolated solutions. We leave the proof to the reader.*

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