On some singular nonlinear problems
for monotone elliptic operators

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Abstract

The goal of this note is to study some class of problems associated to nonlinear operators
of the p-Laplacian type with source term having a singularity at the origin.

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1 Introduction and notation

We will denote by $\Omega$ a smooth open set of $\mathbb{R}^N$, $N \geq 1$. For $1 < p < \infty$, $W^{1,p}_0(\Omega)$ will be the usual Sobolev space of functions in $L^p(\Omega)$ with derivatives in $L^p(\Omega)$ and vanishing on the boundary of $\Omega$. We refer the reader to [6], [10], [11] for more details. $W^{1,p}_0(\Omega)$ is a Banach space that we will suppose equipped with the norm

$$||\nabla u||_p = \left( \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}.$$  \hspace{1cm} (1.1)

(In what follows $| |$ denotes the usual Euclidean norm in $\mathbb{R}^N$, the corresponding scalar product will be denoted by a dot i.e. "$\cdot$".) If $p'$ denotes the conjugate of $p$ i.e. if $\frac{1}{p} + \frac{1}{p'} = 1$ we will denote by $W^{-1,p'}(\Omega)$ the dual of $W^{1,p}_0(\Omega)$ that we will suppose equipped with its strong dual norm associated to $||\nabla u||_{-1,p'}$, and denoted by $| |_{-1,p'}$.

Let $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying for $\nu \in L^{p'}(\Omega), \nu \geq 0$, $\alpha, \beta > 0$,

$$a(x,\xi) \cdot \xi \geq \alpha |\xi|^p \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N,$$ \hspace{1cm} (1.2)

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall \xi, \xi' \in \mathbb{R}^N,$$ \hspace{1cm} (1.3)

$$|a(x,\xi)| \leq \nu(x) + \beta |\xi|^{p-1} \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N.$$ \hspace{1cm} (1.4)

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For \( u, v \in W^{1,p}_0(\Omega) \) setting
\[
\langle Au, v \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v dx
\] (1.5)
defines an operator from \( W^{1,p}_0(\Omega) \) into its dual which is monotone, hemicontinuous, coercive.

The goal of this paper is to study problems of the type
\[
\begin{aligned}
  u &\in W^{1,p}_0(\Omega), \\
  \langle Au, v \rangle &= \langle \mu, H(u)v \rangle \quad \forall v \in V,
\end{aligned}
\] (1.6)
where \( \mu \in W^{-1,p'}(\Omega) \) and \( V \) is some space allowing the equality above to hold in the distributional sense. These kinds of problems were investigated in the linear case - i.e. when the operator \( A \) is linear - in [3] and in the case where \( \mu \) is a measure in [1], [4], [5], [7], [8], [9], [13], [15], [16], [17]. Our note borrows some ideas contained in [3] however the nonlinearity of the operators induces unexpected difficulties. All the results below include in particular the case of operators of the \( p \)-Laplacian type more precisely of the form
\[
-\nabla \cdot (b(x)|\nabla u|^{p-2}\nabla u)
\] (1.7)
where \( b \) denotes some uniformly positive function.

2 A singular perturbation problem

We suppose in this section that \( \Omega \) is bounded and that
\[
N \leq 2, \text{ or } p > \frac{2N}{N+2}.
\] (2.1)
This implies that
\[
W^{1,p}_0(\Omega) \subset W^{-1,p'}(\Omega).
\] (2.2)
Indeed if \( N = 1 \) or \( p \geq N \) this is due to the fact that \( W^{1,p}_0(\Omega) \subset L^p(\Omega) \subset W^{-1,p'}(\Omega) \).

Else if \( p < N \), one has \( f \in W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega) \) with \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \). Since
\[
\frac{1}{p^*} < \frac{1}{2} \iff \frac{1}{p} - \frac{1}{N} < \frac{1}{2} \iff \frac{1}{p} < \frac{2 + N}{2N}
\]
one has \( p^* > 2 \). Denote then by \( u \) the solution to
\[
-\Delta u = -\partial_{x_i}(\partial_{x_i} u) = f \text{ in } \Omega, \text{ } u = 0 \text{ on } \partial \Omega,
\]
(we used the summation convention in \( i \)). Clearly, (see [11]), \( u \in W^{2,p^*}(\Omega) \) and thus for \( i = 1, \ldots, N, \partial_{x_i} u \in W^{1,p^*}(\Omega) \subset L^{p^{**}}(\Omega) \) where \( p^{**} \) is arbitrary for \( p^* \geq N \) i.e. if \( \frac{1}{p} - \frac{1}{N} \leq \frac{1}{N} \) which is equivalent to \( \frac{1}{p} \leq \frac{2}{N} \). Else since
\[
p^{**} > p' \iff \frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{N} < 1 - \frac{1}{p} \iff p > \frac{2N}{N+2}
\]
one has clearly, \( f \in W^{-1,p'}(\Omega) \).

Let \( \tilde{\mu} \in W_0^{1,p}(\Omega) \), \( \epsilon > 0 \). Thus under the condition (2.1), \( \tilde{\mu} \in W^{-1,p'}(\Omega) \) and the operator \( u \to \epsilon Au + u \) is strictly monotone, hemicontinuous and coercive from \( W_0^{1,p}(\Omega) \) into \( W^{-1,p'}(\Omega) \) thus there exists a unique \( u_\epsilon \) solution to

\[
\begin{aligned}
u 
\begin{cases}
u
u_\epsilon \in W_0^{1,p}(\Omega), 
\epsilon Au_\epsilon + u_\epsilon = \tilde{\mu} 
\text{ in } \Omega.
\end{cases}
\end{aligned}
\]

The awkward notation \( \tilde{\mu} \) for the right hand side of the equation will be clear later on. Then we can show

**Proposition 2.1.** Under the assumptions above one has

\[
u
\begin{aligned}
u
u_\epsilon \rightharpoonup \tilde{\mu} \text{ in } W_0^{1,p}(\Omega), 
\epsilon Au_\epsilon \rightharpoonup A\tilde{\mu} \text{ in } W^{-1,p'}(\Omega) \quad \text{when } \epsilon \to 0.
\end{aligned}
\]

Moreover if \( A\tilde{\mu} \geq 0 \) then \( Au_\epsilon \geq 0 \) and \( u_\epsilon \leq \tilde{\mu} \).

**Proof.** Note that by the second equation of (2.3), \( Au_\epsilon \in W_0^{1,p}(\Omega) \). Taking as test function in the weak formulation \( u_\epsilon - \tilde{\mu} \) we get

\[
\langle Au_\epsilon, u_\epsilon - \tilde{\mu} \rangle = -\frac{1}{\epsilon} \int_\Omega (\tilde{\mu} - u_\epsilon)^2 dx \leq 0.
\]

Thus

\[
\int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla u_\epsilon dx \leq \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla \tilde{\mu} dx.
\]

Using (1.2), (1.4) this leads to

\[
\alpha \int_\Omega |\nabla u_\epsilon|^p dx \leq \int_\Omega (\nu(x) + \beta |\nabla u_\epsilon|^{p-1}) |\nabla \tilde{\mu}| dx.
\]

By Hölder’s inequality it comes

\[
\alpha \int_\Omega |\nabla u_\epsilon|^p dx \leq \int_\Omega \nu(x) |\nabla \tilde{\mu}| dx + \beta \left( \int_\Omega |\nabla u_\epsilon|^p dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |\nabla \tilde{\mu}|^p dx \right)^{\frac{1}{p}}
\]

i.e.

\[
\alpha ||\nabla u_\epsilon||_p^p \leq C_1 + C_2 ||\nabla \tilde{\mu}||_p^{p-1}
\]

with \( C_1 = \int_\Omega \nu(x) |\nabla \tilde{\mu}| dx \), \( C_2 = \beta ||\nabla \tilde{\mu}||_p^p \) independent of \( \epsilon \). Thus, clearly, \( u_\epsilon \) is bounded in \( W_0^{1,p}(\Omega) \) independently of \( \epsilon \) and up to a "subsequence" one has for some \( u \) in \( W_0^{1,p}(\Omega) \)

\[
u
\begin{aligned}
u
u_\epsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega).
\end{aligned}
\]

Moreover it is also clear, since \( a(x, \nabla u_\epsilon) \) is bounded in \( L^p(\Omega) \), that \( Au_\epsilon \) is also bounded in \( W^{-1,p'}(\Omega) \) and from (2.3) one deduces that

\[
\begin{aligned}
u
\begin{cases}
u
u_\epsilon - \tilde{\mu} = -\epsilon Au_\epsilon \rightharpoonup 0 \text{ in } W^{-1,p'}(\Omega).
\end{cases}
\end{aligned}
\]
Due to the uniqueness of the limit in the distributional sense one has $u = \hat{\mu}$ and since the limit of $u_\epsilon$ is uniquely determined the whole "sequence" $u_\epsilon$ converges weakly towards $\hat{\mu}$.

We claim now that $Au_\epsilon \rightharpoonup A\hat{\mu}$ in $W^{-1,p'}(\Omega)$ when $\epsilon \to 0$. Indeed since $Au_\epsilon$ is bounded in $W^{-1,p'}(\Omega)$ one has for some "subsequence" and some $\chi \in W^{-1,p'}(\Omega)$

$$Au_\epsilon \rightharpoonup \chi \text{ in } W^{-1,p'}(\Omega).$$

Using (2.5) i.e.

$$\langle Au_\epsilon, u_\epsilon - \hat{\mu} \rangle \leq 0$$

one deduces that for this subsequence

$$\limsup_{\epsilon \to 0} \langle Au_\epsilon, u_\epsilon \rangle \leq \langle \chi, \hat{\mu} \rangle.$$ (2.6)

From the monotonicity assumption one has

$$\langle Au_\epsilon - Av, u_\epsilon - v \rangle \geq 0 \quad \forall v \in W^{1,p}_0(\Omega).$$

This implies

$$\langle Au_\epsilon, u_\epsilon \rangle \geq \langle Au_\epsilon, v \rangle + \langle Av, u_\epsilon - v \rangle \quad \forall v \in W^{1,p}_0(\Omega).$$

Passing to the lim sup we get by (2.6)

$$\langle \chi, \hat{\mu} \rangle \geq \langle \chi, v \rangle + \langle Av, \hat{\mu} - v \rangle \quad \forall v \in W^{1,p}_0(\Omega)$$

i.e

$$\langle \chi - Av, \hat{\mu} - v \rangle \geq 0 \quad \forall v \in W^{1,p}_0(\Omega).$$

Taking $v = \hat{\mu} - tw, w \in W^{1,p}_0(\Omega), t > 0$ we deduce

$$\langle \chi - A(\hat{\mu} - tw), w \rangle \geq 0 \quad \forall w \in W^{1,p}_0(\Omega).$$

Letting $t \to 0$ it follows

$$\langle \chi - A\hat{\mu}, w \rangle \geq 0 \quad \forall w \in W^{1,p}_0(\Omega)$$

and thus $\chi = A\hat{\mu}$. From the uniqueness of the limit it follows that the whole sequence converges toward $A\hat{\mu}$ and thus we have

$$Au_\epsilon \rightharpoonup A\hat{\mu} \text{ in } W^{-1,p'}(\Omega).$$

To prove the last point of the proposition note that considering the test function $(u_\epsilon - \hat{\mu})^+$ one has

$$\epsilon \langle Au_\epsilon, (u_\epsilon - \hat{\mu})^+ \rangle + \int_\Omega \{(u_\epsilon - \hat{\mu})^+\}^2 dx \leq 0.$$ 

This reads also

$$\int_\Omega \{(u_\epsilon - \hat{\mu})^+\}^2 dx \leq -\epsilon \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla (u_\epsilon - \hat{\mu})^+ dx.$$ 

From the monotonicity assumption one has

$$\int_\Omega (a(x, \nabla u_\epsilon) - a(x, \nabla \hat{\mu})) \cdot \nabla (u_\epsilon - \hat{\mu})^+ dx \geq 0$$
so that it comes
\[
\int_{\Omega} \{(u_{\epsilon} - \hat{\mu})^+\}^2 dx \leq -\epsilon \int_{\Omega} a(x, \nabla \hat{\mu}) \cdot \nabla (u_{\epsilon} - \hat{\mu})^+ dx = -\epsilon \langle A\hat{\mu}, (u_{\epsilon} - \hat{\mu})^+ \rangle.
\]

Hence, if $A\hat{\mu} \geq 0$ it follows that $u_{\epsilon} \leq \hat{\mu}$ and by (2.3) that $Au_{\epsilon} \geq 0$. This completes the proof of the proposition.

Under some stronger assumptions one can improve the above convergence. Indeed one has

**Proposition 2.2.** Suppose that for some constant $\lambda = \lambda_p$ one has

if $p \geq 2$,
\[
(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \lambda_p |\xi - \xi'|^2 a.e. x \in \Omega, \forall \xi, \xi' \in \mathbb{R}^N, (2.7)
\]

if $p < 2$
\[
(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \lambda_p |\xi - \xi'|^2 \{\|\xi\| + |\xi'|\}^{p-2} a.e. x \in \Omega, \forall \xi, \xi' \in \mathbb{R}^N. (2.8)
\]

Then the convergences in (2.4) are strong.

**Proof.** The inequality (2.5) can be written as
\[
\langle Au_{\epsilon} - A\hat{\mu}, u_{\epsilon} - \hat{\mu} \rangle \leq -\langle A\hat{\mu}, u_{\epsilon} - \hat{\mu} \rangle. (2.9)
\]

Suppose first that $p \geq 2$. Then from (2.7) one deduces
\[
\lambda_p \int_{\Omega} |\nabla (u_{\epsilon} - \hat{\mu})|^p dx \leq \langle -A\hat{\mu}, u_{\epsilon} - \hat{\mu} \rangle. (2.10)
\]

The result follows then from the weak convergence of $u_{\epsilon}$ toward $\hat{\mu}$ since $A\hat{\mu} \in W^{-1, p'}(\Omega)$. If now $p < 2$ the inequality (2.8) leads to
\[
\lambda_p \int_{\Omega} |\nabla (u_{\epsilon} - \hat{\mu})|^2 \{\|\nabla u_{\epsilon}\| + |\nabla \hat{\mu}|\}^{p-2} dx \leq \langle -A\hat{\mu}, u_{\epsilon} - \hat{\mu} \rangle. (2.11)
\]

Thus the left hand side integral of this inequality goes to 0 when $\epsilon \to 0$. Using Hölder’s inequality one has
\[
\int_{\Omega} |\nabla (u_{\epsilon} - \hat{\mu})|^p dx = \int_{\Omega} |\nabla (u_{\epsilon} - \hat{\mu})|^p \left\{\|\nabla u_{\epsilon}\| + |\nabla \hat{\mu}|\right\}^{\frac{(p-2)p}{2}} \left\{\|\nabla u_{\epsilon}\| + |\nabla \hat{\mu}|\right\}^{\frac{(2-p)p}{2}} dx \\
\leq \left(\int_{\Omega} |\nabla (u_{\epsilon} - \hat{\mu})|^2 \{\|\nabla u_{\epsilon}\| + |\nabla \hat{\mu}|\}^{p-2} dx\right)^{\frac{p}{2}} \left(\int_{\Omega} \{\|\nabla u_{\epsilon}\| + |\nabla \hat{\mu}|\}^p dx\right)^{\frac{2}{2-p}}.
\]

Since the last integral above is bounded independently of $\epsilon$ it follows by (2.11) that the left hand side integral goes to 0 when $\epsilon \to 0$. This completes the proof of the proposition.

**Remark 1.** In the case when (2.7), (2.8) hold one can drop the assumption (1.2), (1.3) in the proposition 2.1.
Let us suppose that \( a \) is strictly monotone in the sense that
\[
(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') > 0 \quad \text{a.e. in } \Omega, \quad \forall \xi \neq \xi' \in \mathbb{R}^N. \tag{2.12}
\]
Then the operator \( A \) defined above is strictly monotone itself and for \( f \in W^{-1, p'}(\Omega) \) there exists a unique \( u \) such that
\[
\begin{aligned}
\{ u & \in W^{1, p}_0(\Omega), \\
\langle Au, v \rangle &= \langle f, v \rangle \quad \forall v \in W^{1, p}_0(\Omega) .
\}
\tag{2.13}
\end{aligned}
\]

It is convenient to mention also here the following lemma.

**Lemma 2.1.** Suppose that (1.2), (1.4) and (2.12) hold. For \( f \in W^{-1, p'}(\Omega) \) let \( u = T(f) \) be the solution to (2.13). Then \( T \) is continuous from \( W^{-1, p'}(\Omega) \) equipped with the strong topology into \( W^{1, p}_0(\Omega) \) equipped with the weak topology.

**Proof.** Suppose that \( f_n \to f \) in \( W^{-1, p'}(\Omega) \). Set \( u_n = T(f_n) \). One has
\[
\langle Au_n, u_n \rangle = \langle f_n, u_n \rangle .
\]

Hence from (1.2) it follows that
\[
\alpha \left| |\nabla u_n|_p \right|_p \leq |f_n|_{-1, p'} \left| |\nabla u_n|_p \right|_p .
\]
Thus \( u_n \) is bounded in \( W^{1, p}_0(\Omega) \) independently of \( n \) and there exists \( u \in W^{1, p}_0(\Omega) \) such that \( u_n \rightharpoonup u \) in \( W^{1, p}_0(\Omega) \). Since \( Au_n \) is also bounded in \( W^{-1, p'}(\Omega) \) one has - up to a subsequence and for some \( \chi \in W^{-1, p'}(\Omega) \)
\[
u_n \rightharpoonup u \text{ in } W^{1, p}_0(\Omega), \quad Au_n \rightharpoonup \chi \text{ in } W^{-1, p'}(\Omega).
\]
We claim that \( \chi = Au \) where \( u \) is the solution to (2.13). We use the same arguments as before. Indeed from the monotonicity assumption (2.3) one has
\[
\langle Au_n, u_n \rangle \geq \langle Au_n, v \rangle + \langle Av, u_n - v \rangle \quad \forall v \in W^{1, p}_0(\Omega) . \tag{2.14}
\]
One has also
\[
\langle Au_n, u_n \rangle = \langle f_n, u_n \rangle \to \langle f, u \rangle
\]
and since \( \langle Au_n, u \rangle = \langle f_n, u \rangle \) by passing to the limit \( \langle f, u \rangle = \langle \chi, u \rangle \). Thus passing to the limit in (2.14) one deduces that
\[
\langle \chi - Av, u - v \rangle \geq 0 \quad \forall v \in W^{1, p}_0(\Omega) .
\]
Choosing \( v = u - tw, \ t > 0 \) one gets easily
\[
\langle \chi - A(u - tw), w \rangle \geq 0 \quad \forall w \in W^{1, p}_0(\Omega) .
\]
Letting \( t \to 0 \) we get
\[
\langle \chi - Au, w \rangle \geq 0 \quad \forall w \in W^{1, p}_0(\Omega) ,
\]
which implies that \( \chi = Au \) and the result follows since the limit of \( Au_n \) is uniquely determined.

**Remark 2.** In the case of the assumptions (2.7), (2.8) the convergence is strong. This follows from
\[
\langle Au_n - Au, u_n - u \rangle = \langle f_n - f, u_n - u \rangle \to 0
\]
and the coerciveness assumptions involved (Cf. (2.10), (2.11)).
3 Ω and H bounded

Let us suppose that \( a \) is strictly monotone in the sense of (2.12). Then, as seen in the previous section, for \( \mu \in W^{-1,p'}(\Omega) \) there exists a unique \( \hat{\mu} \) such that

\[
\begin{cases}
\hat{\mu} \in W^{1,p}_0(\Omega), \\
\int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla v dx = \langle \mu, v \rangle \quad \forall v \in W^{1,p}_0(\Omega).
\end{cases}
\] (3.1)

We can show

**Theorem 3.1.** Suppose that (1.2), (1.4), (2.7), (2.8) hold. Let \( H \) be a function from \( \mathbb{R} \) into itself, bounded, Lipschitz continuous and such that

\[
(sH(s))' \in L^\infty(\Omega).
\] (3.2)

Let \( \mu \in W^{-1,p'}(\Omega) \), with \( p > \frac{2N}{N+2} \) and \( \hat{\mu} \) defined by (3.1). Then there exists \( u \) solution to

\[
\begin{cases}
u \in W^{1,p}_0(\Omega), \\
\langle Au, v \rangle = \langle A\hat{\mu}, H(u)v \rangle \quad \forall v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\end{cases}
\] (3.3)

**Proof.** We denote by \( \hat{\mu}_n \) the solution \( u_{1/n} \) to (2.3) with \( \epsilon = \frac{1}{n} \). We claim that there exists a solution \( u_n \) to

\[
\begin{cases}
u \in W^{1,p}_0(\Omega), \\
\langle Au_n, v \rangle = \langle A\hat{\mu}_n, H(u_n)v \rangle \quad \forall v \in W^{1,p}_0(\Omega).
\end{cases}
\] (3.4)

Note that by (2.3) one has

\[A\hat{\mu}_n \in W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega)\] (3.5)

\( p^* \) being any real number in the case \( N = 1 \) or \( p \geq N \). If \( N \geq 2 \), \( p < N \) we claim that

\[
p^* > (p^*)'.
\] (3.6)

Indeed this inequality reads

\[
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} < 1 - \left( \frac{1}{p} - \frac{1}{N} \right) \Leftrightarrow p > \frac{2N}{N+2}.
\]

Let us then choose \( r \) such that

\[
r = p \text{ if } N = 1 \text{ or } p \geq N, \quad p^* > r > (p^*)' \text{ else.}
\] (3.7)

For \( w \in L^r(\Omega) \) we claim that there exists a unique solution \( u = S_n(w) \) to

\[
\begin{cases}
u \in W^{1,p}_0(\Omega), \\
\langle Au, v \rangle = \int_\Omega A\hat{\mu}_n H(w)v \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\end{cases}
\] (3.8)

Indeed, in the case \( p < N \), since by (3.5) \( A\hat{\mu}_n H(w) \in L^{p^*}(\Omega) \) one has for \( v \in W^{1,p}_0(\Omega) \)

\[
|\int_\Omega A\hat{\mu}_n H(w)v \, dx| \leq |A\hat{\mu}_n H(w)|_{p^*} |v|_{(p^*)'} \leq C |A\hat{\mu}_n|_{p^*} |H|_\infty |v|_{p^*} \leq C |\nabla v|_{p^*}
\]
where \( C, C' \) are constants independent of \( w \) (recall that \( n \) is here fixed, \( |H|_\infty \) denotes the \( L^\infty(\Omega) \)-norm of \( H \)). In the case when \( r = p \), since \( W_0^{1,p}(\Omega) \subset L^q(\Omega) \ \forall q \), the inequality above reads
\[
|\int_\Omega A\mu_n H(w) v \, dx| \leq |A\mu_n H(w)|_{[p]} v_{[p']} \leq C|A\mu_n|_{[p]} |H|_{\infty} v_{[p']} \leq C'\|\nabla v\|_p.
\]
Thus in both cases the right hand side of the equation in (3.8) defines an element of \( W^{-1,p'}(\Omega) \) and the existence and uniqueness of \( u \) is clear. In addition, taking \( v = u \) in (3.8) one derives easily for some other constants \( C, C' \), when \( p < N \)
\[
\alpha \|\nabla u\|_{p'}^p \leq |A\mu_n H(w)|_{p'} |u|_{[p']} \leq C|A\mu_n|_{[p']} |H|_{\infty} |u|_r \leq C' |u|_r
\]
and when \( r = p \)
\[
\alpha \|\nabla u\|_{p'}^p \leq |A\mu_n H(w)|_{p'} |u|_{[p']} \leq C|A\mu_n|_{[p']} |H|_{\infty} |u|_r \leq C' |u|_r
\]
where \( C, C' \) are independent of \( w \). Using again the Sobolev embedding theorem it results that
\[
\|\nabla u\|_{p'} |u|_r \leq C
\]
where \( C \) is some other constant independent of \( w \). Thus \( S_n \) maps the ball
\[
B = \{ v \in L^r(\Omega) \mid |v|_r \leq C \}
\]
into itself and moreover \( S_n(B) \) is relatively compact in \( B \). The existence of \( u_n \), fixed point of \( S_n \), will follow by the Schauder fixed point theorem if we can show the continuity of \( S_n \) in \( L^r(\Omega) \).

If \( w_p \to w \) in \( L^r(\Omega) \) one has - up to a subsequence - \( w_p \to w \) a.e. in \( \Omega \) and by the Lebesgue theorem (recall that \( n \) is fixed and \( H \) is bounded and continuous)
\[
A\mu_n H(w_p) \to A\mu_n H(w) \text{ in } L^{p^*}(\Omega) \text{ if } p < N, \quad \text{(in } L^{p'}(\Omega) \text{ if } r = p).
\]
But \( L^{p^*}(\Omega) \subset L^{(p')^*} \), \( (L^{p'}(\Omega) \text{ if } r = p) \), is continuously imbedded in \( W^{-1,p'}(\Omega) \) (Cf. (2.2)). Indeed to see that in the case \( p < N \) note that for \( f \in L^{p^*}(\Omega) \) (Cf. (3.6)) one has
\[
|\int_\Omega f v \, dx| \leq |f|_{(p')^*} |v|_{p'} \leq C|f|_{p^*} \|\nabla v\|_p
\]
for some constant \( C \). It follows then from the lemma 2.1 that
\[
S_n(w_p) \to S_n(w)
\]
in \( W_0^{1,p}(\Omega) \) and also strongly in \( L^r(\Omega) \) due to the compactness of the Sobolev embedding and the unique possible limit. This completes the existence of a solution to (3.4).

Taking now \( v = u_n \) in (3.4) and using (1.2), (1.4) we deduce
\[
\alpha \int_\Omega |\nabla u_n|^p \, dx \leq \int_\Omega a(x, \nabla \mu_n) \cdot \nabla (H(u_n)u_n) \, dx
\]
\[
\int_\Omega \left( \nu(x) + \beta |\nabla \mu_n|^{p-1} \right) C_\infty |\nabla u_n| \, dx.
\]
\(C_\infty\) is the Lipschitz constant of \(sH(s)\). Since \(\hat{\mu}_n\) is bounded in \(W^{1,p}_0(\Omega)\) one gets easily that \(u_n\) is also bounded in \(W^{1,p}_0(\Omega)\) and - up to a subsequence - one has for some \(u\) in \(W^{1,p}_0(\Omega)\) when \(n \to \infty\)
\[
u_n \rightharpoonup u \quad \text{in} \quad W^{1,p}_0(\Omega), \quad u_n \to u \quad \text{in} \quad L^p(\Omega), \quad u_n \to u \quad \text{a.e. in} \quad \Omega. \tag{3.9}
\]

One has then to pass to the limit in
\[
\int_\Omega a(x, \nabla u_n) \cdot \nabla v \, dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n)v) \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\]

Denote by \((\ )_k\) the truncation at the level \(k > 0\), i.e. \((w)_k = (-k) \vee w \wedge k\) where \(\vee\) denotes the maximum of two numbers and \(\wedge\) the minimum of two numbers.

Taking \(v = (v - u_n)_k\) one has
\[
\int_\Omega a(x, \nabla u_n) \cdot \nabla (v - u_n)_k \, dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n)(v - u_n)_k) \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\]

By the monotonicity of \(a\) this implies
\[
\int_\Omega a(x, \nabla v) \cdot \nabla (v - u_n)_k \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n)(v - u_n)_k) \, dx \quad \forall v \in W^{1,p}_0(\Omega). \tag{3.10}
\]

Clearly \((v - u_n)_k\) is bounded in \(W^{1,p}_0(\Omega)\) independently of \(n\) and converges - up to a subsequence - toward some \(\kappa \in W^{1,p}_0(\Omega)\) weakly in \(W^{1,p}_0(\Omega)\). Due to (3.9) this \(\kappa\) is necessarily equal to \((v - u)_k\) and thus
\[
(v - u_n)_k \to (v - u)_k \quad \text{in} \quad W^{1,p}_0(\Omega). \tag{3.11}
\]

One has
\[
\nabla (H(u_n)(v - u_n)_k) = H'(u_n)(v - u_n)_k \nabla u_n + H(u_n) \nabla (v - u_n)_k.
\]

Hence, if \(H'\) denotes the Lipschitz constant of \(H\),
\[
|\nabla (H(u_n)(v - u_n)_k)| \leq H'_\infty k |\nabla u_n| + |H|_\infty |\nabla (v - u_n)_k|.
\]

Thus, \(H(u_n)(v - u_n)_k\) is also bounded in \(W^{1,p}_0(\Omega)\) independently of \(n\). There exists \(h\) in \(W^{1,p}_0(\Omega)\) such that
\[
H(u_n)(v - u_n)_k \rightharpoonup h \quad \text{in} \quad W^{1,p}_0(\Omega), \quad H(u_n)(v - u_n)_k \to H(u)(v - u)_k \quad \text{in} \quad L^p(\Omega)
\]

the last convergence being due to the Lebesgue theorem. Both convergence taking also part in the distributional sense one has \(h = H(u_n)(v - u)_k\) and
\[
H(u_n)(v - u_n)_k \rightharpoondown H(u)(v - u)_k \quad \text{in} \quad W^{1,p}_0(\Omega).
\]

Since \(\hat{\mu}_n \to \hat{\mu}\) in \(W^{1,p}_0(\Omega)\), Cf. Proposition 2.2, one can pass to the limit in (3.10) to get that
\[
\int_\Omega a(x, \nabla v) \cdot \nabla (v - u)_k \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla (H(u)(v - u)_k) \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\]
If we take in this inequality $v = u + tw$ where $w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ we get
\[
\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla(tw) \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla (H(u)(tw)_k) \, dx \quad \forall w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\]
But for $t$ small we have $(tw)_k = tw$ and thus we get
\[
\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla w \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla (H(u)w) \, dx \quad \forall w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\]
Letting $t \to 0$ leads to
\[
\int_\Omega a(x, \nabla u) \cdot \nabla w \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla (H(u)w) \, dx \quad \forall w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\]
The existence of a solution $u$ to (3.3) follows by changing $w$ into $-w$. 

If $\mu$ is a nonnegative measure we have the following complementary results.

**Theorem 3.2.** Suppose that $\mu \in W^{-1,p'}(\Omega)$ is a nonnegative measure. Then under the assumptions of theorem 3.1 if
\[
H(s) \geq 0 \quad \forall s \leq 0,
\]
a solution to (3.3) satisfies
\[
u \geq 0.
\]

**Proof.** For $k > 0$ one considers $v = u^- \land k$ where $\land$ denotes the minimum of two numbers. Since
\[
H(u)(u^- \land k) = 0 \text{ when } u \geq 0, \\
= H(-u^-)(u^- \land k) \text{ when } u \leq 0,
\]
this function is nonnegative and thus from (3.3) one deduces
\[
\langle Au, u^- \land k \rangle \geq 0.
\]
Denoting by $\{ -k \leq u \leq 0 \}$ the set \{ $x \mid -k \leq u(x) \leq 0$ a.e. in $\Omega$ \} this implies
\[
\int_{\{ -k \leq u \leq 0 \}} a(x, -\nabla u^-) \cdot \nabla u^- \, dx = \int_{\{ -k \leq u \leq 0 \}} a(x, \nabla u) \cdot \nabla u^- \, dx \geq 0.
\]
Thus
\[
0 \geq \int_{\{ -k \leq u \leq 0 \}} a(x, -\nabla u^-) \cdot -\nabla u^- \, dx \geq \alpha \int_{\{ -k \leq u \leq 0 \}} |\nabla u^-|^p \, dx = \int \Omega |\nabla (u^- \land k)|^p \, dx.
\]
It follows that for any $k > 0$, $u^- \land k = 0$. This implies that $u^- = 0$ and this completes the proof. 

\[10\]
Theorem 3.3. Suppose that $\mu \in W^{-1,p'}(\Omega)$ is a nonnegative measure. Suppose that we are under the assumptions of Theorem 3.1 and that $H_1$ and $H_2$ are two functions satisfying the assumptions on $H$. Let $u_1, u_2$ be two solutions to (3.3) corresponding to $H_1$ and $H_2$ respectively, then if

\[ H_1 \geq H_2, \quad H_1 \text{ or } H_2 \text{ is nonincreasing,} \tag{3.14} \]

then one has

\[ u_1 \geq u_2. \tag{3.15} \]

Proof. From (3.3) written for $H_1$ and $H_2$ one deduces by subtraction

\[ \langle Au_1 - Au_2, v \rangle = \langle A\hat{\mu}, (H_1(u_1) - H_2(u_2))v \rangle \quad \forall v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega). \]

Taking $v = (u_1 - u_2)^- \land k, k > 0$ one has

\[ (H_1(u_1) - H_2(u_2))\{(u_1 - u_2)^- \land k\} = 0 \text{ if } u_1 - u_2 \geq 0. \]

- If $H_1$ is nonincreasing

\[ (H_1(u_1) - H_2(u_2))v = (H_1(u_1) - H_1(u_2) + H_1(u_2) - H_2(u_2))\{(u_1 - u_2)^- \land k\} \geq 0 \text{ if } u_1 - u_2 \leq 0. \]

- If $H_2$ is nonincreasing

\[ (H_1(u_1) - H_2(u_2))v = (H_1(u_1) - H_2(u_1) + H_2(u_1) - H_2(u_2))\{(u_1 - u_2)^- \land k\} \geq 0 \text{ if } u_1 - u_2 \leq 0. \]

Thus since $(H_1(u_1) - H_2(u_2))v \geq 0$ in all cases, one has

\[ \langle Au_1 - Au_2, \{(u_1 - u_2)^- \land k\} \rangle \geq 0. \]

With obvious notation this leads to

\[ \int_{\{-k \leq u_1 - u_2 \leq 0\}} a(x, \nabla u_1) - a(x, \nabla u_2) \cdot \nabla (u_1 - u_2) dx \leq 0. \]

Thus $\nabla u_1 = \nabla u_2$ a.e. on the set $\{-k \leq u_1 - u_2 \leq 0\}$. This implies that $\nabla \{(u_1 - u_2)^- \land k\} = 0$ a.e. in $\Omega$. Thus $(u_1 - u_2)^- \land k = 0$ for every $k > 0$ and so $(u_1 - u_2)^- = 0$. This completes the proof of the theorem.

Remark 3. Under the assumptions of Theorem 3.1 if $\mu$ is a nonnegative measure in $W^{-1,p'}(\Omega)$ and $H$ a nonincreasing function then the solution to (3.2) is unique. This follows immediately from Theorem 3.3.
4 \( \Omega \) bounded and \( H \) unbounded

Now we would like to allow \( H \) to be unbounded near zero.

**Theorem 4.1.** Let \( \mu \in W^{-1,p'}(\Omega) \) be a nonnegative bounded measure, \( A \) be an operator satisfying (1.2), (2.7), (2.8) and (1.4) with \( \nu = 0 \) and let \( H : \mathbb{R}^+ \to \mathbb{R}^+ \) be a nonnegative, nonincreasing function such that

\[
\begin{align*}
\lim_{s \to 0^+} H(s) &= +\infty, \\
\forall \epsilon > 0, \ H \text{ is Lipschitz continuous on } (\epsilon, +\infty), \\
\exists \mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+, \text{ s.t. } H \leq \mathcal{K} \text{ and s.t. } (s\mathcal{K}(s)) \in W^{1,\infty}(\mathbb{R}^+).
\end{align*}
\]

Then there exists \( u \) solution to

\[
\begin{cases}
    u \in W^{1,p}(\Omega), u \geq 0, \\
    \langle Au, v \rangle = \langle A\mu, H(u)v \rangle \ \forall v \in C^1_c(\Omega).
\end{cases}
\]

**Proof.** Some of the arguments developed here are similar to the ones in [3]. However the nonlinear operator forces tricky modifications. We first choose \( n \) large enough, more precisely \( n \geq n_0 \), where \( n_0 \) is such that \( H(n_0) < n_0 \). Then we define a function \( H_n \) by setting

\[
H_n(s) = \begin{cases}
  n & \text{for } s \leq 0, \\
  n \land H(s) & \text{for } 0 < s \leq n, \\
  \{-H(n)(s - (n + 1))\} \land H(s) & \text{for } n \leq s \leq n + 1, \\
  0 & \text{for } n + 1 \leq s.
\end{cases}
\]

Clearly for every \( n \) one has

\[
H_n(s) \leq H_{n+1}(s) \leq H(s) \ \forall s > 0.
\]

Moreover \( H_n \) satisfies the assumptions of theorem 3.1 (Cf. also theorem 3.2) and thus there exists \( u_n \) solution to

\[
\begin{cases}
    u_n \in W^{1,p}_0(\Omega), u_n \geq 0, \\
    \langle Au_n, v \rangle = \langle A\hat{\mu}, H_n(u_n)v \rangle \ \forall v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\end{cases}
\]

Taking in (4.6) \( v = u_n \land k \) with \( k > 0 \) leads to

\[
\int_\Omega a(x, \nabla u_n) \cdot \nabla (u_n \land k)dx = \langle A\hat{\mu}, H_n(u_n)(u_n \land k) \rangle.
\]

One has of course

\[
\alpha \int_\Omega |\nabla (u_n \land k)|^p \leq \int_\Omega a(x, \nabla (u_n \land k)) \cdot \nabla (u_n \land k)dx = \int_\Omega a(x, \nabla u_n) \cdot \nabla (u_n \land k)dx.
\]

Moreover

\[
\langle A\hat{\mu}, H_n(u_n)(u_n \land k) \rangle = \langle A\hat{\mu}, \{H_n(u_n) - H_n(u_n \land k) + H_n(u_n \land k)\}(u_n \land k) \rangle \\
\leq \langle A\hat{\mu}, H_n(u_n \land k)(u_n \land k) \rangle.
\]

\[12\]
since \( \{ H_n(u_n) - H_n(u_n \wedge k) \}(u_n \wedge k) \leq 0 \) and \( \mu \) is a nonnegative measure. Combining (4.7)-(4.9) we get easily by (4.2)

\[
\alpha \int_\Omega |\nabla (u_n \wedge k)|^p \leq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{ K(u_n \wedge k)(u_n \wedge k) \} dx.
\]

(4.10)

Since the function \( sK(s) \) is supposed to be Lipschitz continuous one has for some \( f \in L^\infty(\mathbb{R}) \) and some constant \( C_K \)

\[
sK(s) = \int_0^s f(\xi) d\xi + C_K.
\]

It follows that \( K(u_n \wedge k)(u_n \wedge k) - C_K \in W^{1,p}_0(\Omega) \) and from (4.10) we derive

\[
\alpha \int_\Omega |\nabla (u_n \wedge k)|^p \leq \langle \mu, K(u_n \wedge k)(u_n \wedge k) - C_K + C_K \rangle
\]

\[
= \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{ K(u_n \wedge k)(u_n \wedge k) - C_K \} + \langle \mu, C_K \rangle
\]

\[
\leq \beta |f|_{\infty} \int_\Omega |\nabla \hat{\mu}|^{p-1}|\nabla (u_n \wedge k)| dx + \langle \mu, C_K \rangle
\]

\[
\leq \beta |f|_{\infty} ||\nabla \hat{\mu}|^{p-1} \left( \int_\Omega |\nabla (u_n \wedge k)|^p dx \right)^{\frac{1}{p}} + \langle \mu, C_K \rangle.
\]

It follows easily that for some constant \( C \) independent of \( n \) and \( k \) one has

\[
\int_\Omega |\nabla (u_n \wedge k)|^p dx \leq C.
\]

Letting \( k \to \infty \) we deduce

\[
\int_\Omega |\nabla u_n|^p dx \leq C.
\]

Thus \( u_n \) is bounded in \( W^{1,p}_0(\Omega) \) and - up to a subsequence - one has for some \( u \in W^{1,p}_0(\Omega) \)

\[
u_n \rightharpoonup u \text{ in } W^{1,p}_0(\Omega), \quad u_n \to u \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega.
\]

We would like to show that \( u \) is solution to (4.3).

One has

\[
\int_\Omega a(x, \nabla u_n) \cdot \nabla v dx = \langle A\hat{\mu}, H_n(u_n) v \rangle \quad \forall v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega).
\]

Taking \( v = (v - u_n)_k \) one deduces that

\[
\int_\Omega a(x, \nabla u_n) \cdot \nabla (v - u_n)_k dx = \langle A\hat{\mu}, H_n(u_n)(v - u_n)_k \rangle \quad \forall v \in W^{1,p}_0(\Omega).
\]

It is clear from (4.5) and the monotonicity of \( H_n \) that \( u_n \) is nondecreasing with \( n \), i.e. one has

\[
u_{n+1} \geq u_n \geq u_{n_0}, \quad \forall n \geq n_0.
\]

(4.11)
Thus one has also $u \geq u_n$ for every $n \geq n_0$ and thus also $(v - u_n)_k \geq (v - u)_k$ for any $v$. Using the nonnegativity of $A\hat{\mu}$ and $H_n(u_n)$ one derives that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla(v - u_n)_k dx \geq \langle A\hat{\mu}, H_n(u_n)(v - u)_k \rangle \quad \forall v \in W^{1,p}_0(\Omega)$$

and by the monotonicity assumption of $a$ it comes

$$\int_\Omega a(x, \nabla) \cdot \nabla(v - u_n)_k dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H_n(u_n)(v - u)_k\} dx \quad \forall v \in W^{1,p}_0(\Omega). \quad (4.12)$$

By definition of $u_{n_0}$ one has

$$-\text{div}\{a(x, \nabla u_{n_0})\} \geq 0, \quad -\text{div}\{a(x, \nabla u_{n_0})\} \neq 0.$$

It follows from the maximum principle (Cf. [18] Theorems 7.1.3, 7.2.2) that for each subdomain $\omega \subset \subset \Omega$ one has for some positive constant $c_\omega$

$$u \geq u_n \geq u_{n_0} \geq c_\omega \quad \text{a.e. in } \omega. \quad (4.13)$$

Consider then $w \in C^1_c(\Omega)$ with support included in $\omega$. For $t > 0$ choose in (4.12) $v = u + tw$. It comes

$$\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla(u + tw - u_n)_k dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H_n(u_n)(tw)_k\} dx. \quad (4.14)$$

One has of course when $n \to \infty$, $\nabla(u + tw - u_n)_k \to \nabla(tw)_k$ in $L^p(\Omega)$. Moreover one has by the definition of $H_n$

$$0 \leq -H'_n \leq C \quad \text{on } [c_\omega, \infty)$$

where $C$ is independent of $n$. It follows that

$$\nabla\{H_n(u_n)(tw)_k\} = H_n(u_n)\nabla(tw)_k + H'_n(u_n)(tw)_k \nabla u_n$$

is bounded in $L^p(\Omega)$ independently of $n$. Thus it converges - up to a subsequence - weakly in $L^p(\Omega)$. But clearly it follows from the convergence of $u_n$ that the limit can only be $\nabla\{H(u)(tw)_k\}$ - recall that $H_n \to H$ when $n \to \infty$. Passing to the limit in (4.14) we obtain

$$\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla(tw)_k dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H(u)(tw)_k\} dx.$$

Since for $t$ small enough $(tw)_k = tw$ letting $t \to 0$ it comes

$$\int_\Omega a(x, \nabla u) \cdot \nabla w dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H(u)w\} dx.$$

This is true for any $w$ and changing $w$ into $-w$ it results that $u$ is solution to (4.3). This completes the proof of the theorem. \qed
Remark 4. A function $H$ satisfying our assumptions is for instance given by

$$H(s) = s^{-\gamma}$$  \hspace{1cm} (4.15)

with $\gamma \leq 1$. This function gives rise to a solution of finite energy - i.e. belonging to $W^{1,p}_0(\Omega)$. In the case where the singularity is stronger for instance when $\gamma > 1$ one can have a weaker solution (Cf. [4], [13]). Suppose that we are restricting ourselves to the simple case of the Laplace operator. If $H$ is given by (4.15) let us denote by $H_n$ the function defined in (4.4). Then there exists $u_n$ solution to

$$\begin{cases} u_n \in W^{1,2}_0(\Omega), \\ \langle -\Delta u_n, v \rangle = \langle A\hat{\mu}, H_n(u_n)v \rangle \forall v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega). \end{cases}$$  \hspace{1cm} (4.16)

Now by Theorems 3.2, 3.3, $u_n$ is a nondecreasing sequence which converges point-wise and satisfies for any domain $\omega \subset \subset \Omega$ and for some constant $c_\omega$ independent of $n$

$$u_n \geq c_\omega.$$  

Using Harnack inequality when $p < n$ (Cf. [18] Theorem. 7.1.2), $u_n$ is locally bounded in some $L^q$. Then for $v \in C_c(\Omega)$ with support in $\omega$ one can pass to the limit in (4.16) to get $u$ to satisfy

$$\langle u, -\Delta v \rangle = \langle A\hat{\mu}, H(u)v \rangle \forall v \in C_c(\Omega).$$

5 \hspace{1cm} \Omega \hspace{1cm} unbounded

In this section we extend some of our previous results in the case when $\Omega$ is unbounded.

**Theorem 5.1.** Let $\Omega \subset \mathbb{R}^N$ be an unbounded open set and for $p > \frac{2N}{N+2}$, $\mu \in W^{1,p}_0(\Omega_\ell)$ be a nonnegative measure. Let $a$ be a Carathéodory function satisfying (1.2), (1.4), (2.7), (2.8) and $A$ the operator defined by (1.5). Let $H : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{cases} H \text{ is nonincreasing, bounded and Lipschitz continuous,} \\ (sH(s))' \in L^\infty(\Omega), \\ H(s) \geq 0 \ \forall s \leq 0. \end{cases}$$  \hspace{1cm} (5.1)

If $\hat{\mu}$ is defined as in (3.1), then there exists a unique $u$ solution to

$$\begin{cases} u \in W^{1,p}_0(\Omega), u \geq 0, \\ \langle Au, v \rangle = \langle A\hat{\mu}, H(u)v \rangle \forall v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega). \end{cases}$$  \hspace{1cm} (5.2)

**Proof.** For $\ell > 0$ if $B_\ell$ denotes the open ball centred at 0 and radius $\ell$ set

$$\Omega_\ell = \Omega \cap B_\ell, \quad V_\ell = W^{1,p}_0(\Omega_\ell).$$  \hspace{1cm} (5.3)

If one assumes the functions of $W^{1,p}_0(\Omega_\ell)$ extended by 0 outside of $\Omega_\ell$ it is clear that

$$W^{1,p}_0(\Omega_\ell) \subset W^{1,p}_0(\Omega)$$
and thus $\mu \in W^{-1,p'}(\Omega)$.

By Theorems 3.1, 3.2 and Remark 3 there exists a unique $u_\ell$ solution to

$$
\begin{aligned}
&\left\{
\begin{array}{l}
u_\ell \in W^1_0(\Omega), u_\ell \geq 0,
\end{array}
\right.
\end{aligned}
$$

(5.4)

Taking $v = u_\ell \wedge k$ for $k > 0$ one obtains easily

$$
\alpha \int_{\Omega_\ell} |\nabla (u_\ell \wedge k)|^p dx \leq |\mu|^{-1,p'} \int_{\Omega_\ell} |\nabla H(u_\ell \wedge k)(u_\ell \wedge k)|^p dx,\nonumber
$$

(5.6)

and

$$
\leq C_\infty |\mu|^{-1,p'} \int_{\Omega_\ell} |\nabla (u_\ell \wedge k)|^p dx,\nonumber
$$

(5.6)

where $C_\infty$ denotes the Lipschitz constant of the function $sH(s)$ and $|\mu|^{-1,p'}$ the norm of $\mu$ in $W^{-1,p'}(\Omega)$. It follows easily that

$$
\int_{\Omega_\ell} |\nabla (u_\ell \wedge k)|^p dx \leq \left(\frac{C_\infty |\mu|^{-1,p'}}{\alpha}\right)^{p'}
$$

(5.6)

Letting $k \to +\infty$ we deduce

$$
\left|\nabla u_\ell\right|_p \leq \left(\frac{C_\infty |\mu|^{-1,p'}}{\alpha}\right)^{\frac{1}{p-2}}
$$

(5.7)

i.e. $u_\ell$ is bounded in $W^1_0(\Omega)$ independently of $\ell$.

Let us fix some $\ell' < \ell$ and consider $v \in W^1_0(\Omega_{\ell'})$. One has, for $k > 0$, $(v - u_\ell)_k \in W^1_0(\Omega_{\ell'}) \cap L^\infty(\Omega_{\ell'})$. From the equation of (5.4) it follows that

$$
\int_{\Omega} a(x, \nabla u_\ell) \cdot \nabla (v - u_\ell)_k dx = \int_{\Omega} a(x, \nabla \mu) \cdot \nabla \{H(u_\ell)(v - u_\ell)_k\} dx
$$

(5.8)

(We suppose $(v - u_\ell)_k$ extended by 0 outside $\Omega_{\ell'}$). Using the monotonicity of $a$ this implies

$$
\int_{\Omega} a(x, \nabla v) \cdot \nabla (v - u_\ell)_k dx \geq \int_{\Omega} a(x, \nabla \mu) \cdot \nabla \{H(u_\ell)(v - u_\ell)_k\} dx.
$$

(5.9)

From (5.7) we deduce that there exists $u \in W^1_0(\Omega)$ such that -up to a subsequence

$$
u_\ell \to u \text{ in } W^1_0(\Omega), \quad u_\ell \to u \text{ in } L^p_{\text{loc}}(\Omega).
$$

(5.10)

(Using for instance a diagonal process for a sequence of subdomains $\Omega_k$ absorbing $\Omega$).

Then since $(v - u_\ell)_k$ is also bounded in $W^1_0(\Omega)$, for a subsequence, one has

$$
\nabla (v - u_\ell)_k \to \nabla (v - u)_k \text{ in } (L^p(\Omega))^N.
$$

(5.11)
One has also
\[ \nabla \{ H(u_\ell)(v - u_\ell) \} = H(u_\ell) \nabla (v - u_\ell) + (v - u_\ell) H'(u_\ell) \nabla u_\ell. \]

Thus this gradient is uniformly bounded in \((L^p(\Omega))^N\) and thus there exists \(G \in (L^p(\Omega))^N\) such that
\[ \nabla \{ H(u_\ell)(v - u_\ell) \} \rightarrow G \text{ in } (L^p(\Omega))^N. \]

Since by the Lebesgue theorem one has up to a subsequence
\[ H(u_\ell)(v - u_\ell) \rightarrow H(u)(v - u) \text{ in } L^p_{\text{loc}}(\Omega) \]

\[ \text{it follows that} \]
\[ \nabla \{ H(u_\ell)(v - u_\ell) \} \rightarrow G = \nabla \{ H(u)(v - u) \} \text{ in } (L^p(\Omega))^N. \] (5.12)

Combining (5.11) and (5.12) one can pass to the limit in (5.9) to get
\[ \int_\Omega a(x, \nabla v) \cdot \nabla (v - u) \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{ H(u)(v - u) \} \, dx \quad \forall v \in W^{1,p}_0(\Omega'), \] (5.13)

\(\ell'\) being arbitrary.

\[ \text{Now if } v \in W^{1,p}_0(\Omega), \theta_{\ell'}(x) = 1 \land \text{dist}(x, \mathbb{R}^N \setminus B_{\ell'}) \text{ one has} \]
\[ \theta_{\ell'}(x)v \in W^{1,p}_0(\Omega'), \quad \theta_{\ell'}(x)v \rightarrow v \text{ in } W^{1,p}_0(\Omega) \]

when \(\ell' \rightarrow \infty\). Thus exchanging \(v\) in \(\theta_{\ell'}(x)v\) in (5.13) and passing to the limit in \(\ell'\) one obtains
\[ \int_\Omega a(x, \nabla v) \cdot \nabla (v - u) \, dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla H(u)(v - u) \, dx \quad \forall v \in W^{1,p}_0(\Omega). \] (5.14)

Then taking \(v = u + tw\), \(w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\) one concludes as in the proof of theorem 3.1.

As in theorem 3.3 one can show monotonicity of the solution to (5.2) with respect to \(H\) and uniqueness follows as in Remark 3. This completes the proof of the theorem. \(\square\)

The theorem 4.1 can be extended easily in the case when \(H\) is unbounded near 0. One has

**Theorem 5.2.** Let \(\Omega \subset \mathbb{R}^N\) be an unbounded open set and for \(p > \frac{2N}{N+2}\), \(\mu \in W^{1,p}_0(\Omega_\ell)\) be a nonnegative bounded measure. Let \(a \) be a Carathéodory function satisfying (1.2), (2.7), (2.8), (1.4) with \(\nu = 0\) and \(A\) the operator defined by (1.5). Let \(H : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a nonnegative, nonincreasing function such that
\[
\begin{align*}
\lim_{s \rightarrow 0^+} H(s) &= +\infty, \\
\forall \epsilon > 0, \ H \text{ is Lipschitz continuous on } (\epsilon, +\infty),
\end{align*}
\] (5.15)
\[ \exists K : \mathbb{R}^+ \to \mathbb{R}^+ \text{ s.t. } H \leq K \text{ and s.t. } (sK(s)) \in W^{1,\infty}(\mathbb{R}^+). \quad (5.16) \]

Then there exists a solution to

\[ \begin{cases} u \in W^{1,p}_0(\Omega), u \geq 0, \\ \langle Au, v \rangle = \langle A\hat{\mu}, H(u)v \rangle \quad \forall v \in C^1_c(\Omega). \end{cases} \quad (5.17) \]

Proof. One introduces the function \( H_n \) as in (4.4) and one follows basically all the steps of the proof of theorem 4.1.

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