# On some singular nonlinear problems for monotone elliptic operators

Michel Chipot *

**Abstract**

The goal of this note is to study some class of problems associated to nonlinear operators of the $p$-Laplacian type with source term having a singularity at the origin.

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## 1 Introduction and notation

We will denote by $\Omega$ a smooth open set of $\mathbb{R}^N$, $N \geq 1$. For $1 < p < \infty$, $W_0^{1,p}(\Omega)$ will be the usual Sobolev space of functions in $L^p(\Omega)$ with derivatives in $L^p(\Omega)$ and vanishing on the boundary of $\Omega$. We refer the reader to [6], [10], [11] for more details. $W_0^{1,p}(\Omega)$ is a Banach space that we will suppose equipped with the norm

$$\big|\big|\nabla u\big|\big|_p = \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}}. \tag{1.1}$$

(In what follows $|\ \ |$ denotes the usual Euclidean norm in $\mathbb{R}^N$, the corresponding scalar product will be denoted by a dot i.e. "$\cdot$"). If $p'$ denotes the conjugate of $p$ i.e. if $\frac{1}{p} + \frac{1}{p'} = 1$ we will denote by $W^{-1,p'}(\Omega)$ the dual of $W_0^{1,p}(\Omega)$ that we will suppose equipped with its strong dual norm associated to $\big|\big|\nabla u\big|\big|_p$ and denoted by $|\ \ |_{-1,p'}$

Let $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying for $\nu \in L^{p'}(\Omega), \nu \geq 0$, $\alpha, \beta > 0$,

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p \quad \text{a.e. } x \text{ in } \Omega, \ \ \forall\, \xi \in \mathbb{R}^N, \tag{1.2}$$

$$(a(x,\xi) - a(x,\xi'))\cdot(\xi - \xi') \geq 0 \quad \text{a.e. } x \text{ in } \Omega, \ \ \forall\, \xi, \xi' \in \mathbb{R}^N, \tag{1.3}$$

$$|a(x,\xi)| \leq \nu(x) + \beta|\xi|^{p-1} \quad \text{a.e. } x \text{ in } \Omega, \ \ \forall\, \xi \in \mathbb{R}^N. \tag{1.4}$$

*Institute of Mathematics, University of Zürich, Winterthurerstr.190, CH-8057 Zürich, email : m.m.chipot@math.uzh.ch

For $u, v \in W_0^{1,p}(\Omega)$ setting

$$\langle Au, v\rangle = \int_\Omega a(x, \nabla u) \cdot \nabla v dx \tag{1.5}$$

defines an operator from $W_0^{1,p}(\Omega)$ into its dual which is monotone, hemicontinuous, coercive.

The goal of this paper is to study problems of the type

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \langle Au, v\rangle = \langle \mu, H(u)v\rangle \quad \forall v \in V, \end{cases} \tag{1.6}$$

where $\mu \in W^{-1,p'}(\Omega)$ and $V$ is some space allowing the equality above to hold in the distributional sense. These kinds of problems were investigated in the linear case - i.e. when the operator $A$ is linear - in [3] and in the case where $\mu$ is a measure in [1], [4], [5], [7], [8], [9], [13], [15], [16], [17]. Our note borrows some ideas contained in [3] however the nonlinearity of the operators induces unexpected difficulties. All the results below include in particular the case of operators of the $p$-Laplacian type more precisely of the form

$$-\nabla \cdot (b(x)|\nabla u|^{p-2}\nabla u) \tag{1.7}$$

where $b$ denotes some uniformly positive function.

## 2 A singular perturbation problem

We suppose in this section that $\Omega$ is bounded and that

$$N \leq 2, \text{ or } p > \frac{2N}{N+2}. \tag{2.1}$$

This implies that

$$W_0^{1,p}(\Omega) \subset W^{-1,p'}(\Omega). \tag{2.2}$$

Indeed if $N = 1$ or $p \geq N$ this is due to the fact that $W_0^{1,p}(\Omega) \subset L^{p'}(\Omega) \subset W^{-1,p'}(\Omega)$.

Else if $p < N$, one has $f \in W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. Since

$$\frac{1}{p^*} < \frac{1}{2} \Leftrightarrow \frac{1}{p} - \frac{1}{N} < \frac{1}{2} \Leftrightarrow \frac{1}{p} < \frac{2+N}{2N}$$

one has $p^* > 2$. Denote then by $u$ the solution to

$$-\Delta u = -\partial_{x_i}(\partial_{x_i} u) = f \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

(we used the summation convention in $i$). Clearly, (see [11]), $u \in W^{2,p^*}(\Omega)$ and thus for $i = 1, \ldots, N$, $\partial_{x_i} u \in W^{1,p^*}(\Omega) \subset L^{p^{**}}(\Omega)$ where $p^{**}$ is arbitrary for $p^* \geq N$ i.e. if $\frac{1}{p} - \frac{1}{N} \leq \frac{1}{N}$ which is equivalent to $\frac{1}{p} \leq \frac{2}{N}$. Else since

$$p^{**} > p' \Leftrightarrow \frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{N} < 1 - \frac{1}{p} \Leftrightarrow p > \frac{2N}{N+2}$$

one has clearly, $f \in W^{-1,p'}(\Omega)$.

Let $\hat{\mu} \in W_0^{1,p}(\Omega)$, $\epsilon > 0$. Thus under the condition (2.1), $\hat{\mu} \in W^{-1,p'}(\Omega)$ and the operator $u \to \epsilon Au + u$ is strictly monotone, hemicontinuous and coercive from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ thus there exists a unique $u_\epsilon$ solution to

$$\begin{cases} u_\epsilon \in W_0^{1,p}(\Omega), \\ \epsilon Au_\epsilon + u_\epsilon = \hat{\mu} \quad \text{in } \Omega. \end{cases} \tag{2.3}$$

The awkward notation $\hat{\mu}$ for the right hand side of the equation will be clear later on. Then we can show

**Proposition 2.1.** *Under the assumptions above one has*

$$u_\epsilon \rightharpoonup \hat{\mu} \text{ in } W_0^{1,p}(\Omega), Au_\epsilon \rightharpoonup A\hat{\mu} \text{ in } W^{-1,p'}(\Omega) \text{ when } \epsilon \to 0. \tag{2.4}$$

*Moreover if $A\hat{\mu} \geq 0$ then $Au_\epsilon \geq 0$ and $u_\epsilon \leq \hat{\mu}$.*

*Proof.* Note that by the second equation of (2.3), $Au_\epsilon \in W_0^{1,p}(\Omega)$. Taking as test function in the weak formulation $u_\epsilon - \hat{\mu}$ we get

$$\langle Au_\epsilon, u_\epsilon - \hat{\mu} \rangle = -\frac{1}{\epsilon} \int_\Omega (\hat{\mu} - u_\epsilon)^2 dx \leq 0. \tag{2.5}$$

Thus

$$\int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla u_\epsilon dx \leq \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla \hat{\mu} dx.$$

Using (1.2), (1.4) this leads to

$$\alpha \int_\Omega |\nabla u_\epsilon|^p \, dx \leq \int_\Omega \big(\nu(x) + \beta |\nabla u_\epsilon|^{p-1}\big) |\nabla \hat{\mu}| \, dx.$$

By Hölder's inequality it comes

$$\alpha \int_\Omega |\nabla u_\epsilon|^p dx \leq \int_\Omega \nu(x) |\nabla \hat{\mu}| dx + \beta \Big( \int_\Omega |\nabla u_\epsilon|^p dx \Big)^{\frac{p-1}{p}} \Big( \int_\Omega |\nabla \hat{\mu}|^p \, dx \Big)^{\frac{1}{p}}$$

i.e.

$$\alpha \big|\big|\nabla u_\epsilon\big|\big|_p^p \leq C_1 + C_2 \big|\big|\nabla u_\epsilon\big|\big|_p^{p-1}$$

with $C_1 = \int_\Omega \nu(x) |\nabla \hat{\mu}| dx$, $C_2 = \beta \big|\big|\nabla \hat{\mu}\big|\big|_p$ independent of $\epsilon$. Thus, clearly, $u_\epsilon$ is bounded in $W_0^{1,p}(\Omega)$ independently of $\epsilon$ and up to a "subsequence" one has for some $u$ in $W_0^{1,p}(\Omega)$

$$u_\epsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega).$$

Moreover it is also clear, since $a(x, \nabla u_\epsilon)$ is bounded in $L^{p'}(\Omega)$, that $Au_\epsilon$ is also bounded in $W^{-1,p'}(\Omega)$ and from (2.3) one deduces that

$$u_\epsilon - \hat{\mu} = -\epsilon Au_\epsilon \to 0 \text{ in } W^{-1,p'}(\Omega).$$

Due to the uniqueness of the limit in the distributional sense one has $u = \hat{\mu}$ and since the limit of $u_\epsilon$ is uniquely determined the whole "sequence" $u_\epsilon$ converges weakly towards $\hat{\mu}$.

We claim now that $Au_\epsilon \rightharpoonup A\hat{\mu}$ in $W^{-1,p'}(\Omega)$ when $\epsilon \to 0$. Indeed since $Au_\epsilon$ is bounded in $W^{-1,p'}(\Omega)$ one has for some "subsequence" and some $\chi \in W^{-1,p'}(\Omega)$

$$Au_\epsilon \rightharpoonup \chi \text{ in } W^{-1,p'}(\Omega).$$

Using (2.5) i.e.

$$\langle Au_\epsilon, u_\epsilon - \hat{\mu}\rangle \leq 0$$

one deduces that for this subsequence

$$\limsup_{\epsilon\to 0}\langle Au_\epsilon, u_\epsilon\rangle \leq \langle \chi, \hat{\mu}\rangle. \tag{2.6}$$

From the monotonicity assumption one has

$$\langle Au_\epsilon - Av, u_\epsilon - v\rangle \geq 0 \quad \forall v \in W_0^{1,p}(\Omega).$$

This implies

$$\langle Au_\epsilon, u_\epsilon\rangle \geq \langle Au_\epsilon, v\rangle + \langle Av, u_\epsilon - v\rangle \quad \forall v \in W_0^{1,p}(\Omega).$$

Passing to the lim sup we get by (2.6)

$$\langle \chi, \hat{\mu}\rangle \geq \langle \chi, v\rangle + \langle Av, \hat{\mu} - v\rangle \quad \forall v \in W_0^{1,p}(\Omega)$$

i.e

$$\langle \chi - Av, \hat{\mu} - v\rangle \geq 0 \quad \forall v \in W_0^{1,p}(\Omega).$$

Taking $v = \hat{\mu} - tw$, $w \in W_0^{1,p}(\Omega)$, $t > 0$ we deduce

$$\langle \chi - A(\hat{\mu} - tw), w\rangle \geq 0 \quad \forall w \in W_0^{1,p}(\Omega).$$

Letting $t \to 0$ it follows

$$\langle \chi - A\hat{\mu}, w\rangle \geq 0 \quad \forall w \in W_0^{1,p}(\Omega)$$

and thus $\chi = A\hat{\mu}$. From the uniqueness of the limit it follows that the whole sequence converges toward $A\hat{\mu}$ and thus we have

$$Au_\epsilon \rightharpoonup A\hat{\mu} \text{ in } W^{-1,p'}(\Omega).$$

To prove the last point of the proposition note that considering the test function $(u_\epsilon - \hat{\mu})^+$ one has

$$\epsilon\langle Au_\epsilon, (u_\epsilon - \hat{\mu})^+\rangle + \int_\Omega \{(u_\epsilon - \hat{\mu})^+\}^2 dx \leq 0.$$

This reads also

$$\int_\Omega \{(u_\epsilon - \hat{\mu})^+\}^2 dx \leq -\epsilon \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla (u_\epsilon - \hat{\mu})^+ dx.$$

From the monotinicity assumption one has

$$\int_\Omega \big(a(x, \nabla u_\epsilon) - a(x, \nabla \hat{\mu})\big) \cdot \nabla (u_\epsilon - \hat{\mu})^+ dx \geq 0$$

so that it comes

$$\int_\Omega \{(u_\epsilon - \hat{\mu})^+\}^2 dx \le -\epsilon \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(u_\epsilon - \hat{\mu})^+ dx = -\epsilon\langle A\hat{\mu}, (u_\epsilon - \hat{\mu})^+\rangle.$$

Hence, if $A\hat{\mu} \ge 0$ it follows that $u_\epsilon \le \hat{\mu}$ and by (2.3) that $Au_\epsilon \ge 0$. This completes the proof of the proposition. □

Under some stronger assumptions one can improve the above convergence. Indeed one has

**Proposition 2.2.** *Suppose that for some constant $\lambda = \lambda_p$ one has*

$$\text{if } p \ge 2, \quad (a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') \ge \lambda_p |\xi - \xi'|^p \quad a.e.\ x \text{ in } \Omega, \quad \forall\, \xi, \xi' \in \mathbb{R}^N, \tag{2.7}$$

$$\text{if } p < 2 \quad (a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') \ge \lambda_p |\xi - \xi'|^2 \{|\xi| + |\xi'|\}^{p-2} \quad a.e.\ x \text{ in } \Omega, \quad \forall\, \xi, \xi' \in \mathbb{R}^N. \tag{2.8}$$

*Then the convergences in* (2.4) *are strong.*

*Proof.* The inequality (2.5) can be written as

$$\langle Au_\epsilon - A\hat{\mu}, u_\epsilon - \hat{\mu}\rangle \le -\langle A\hat{\mu}, u_\epsilon - \hat{\mu}\rangle. \tag{2.9}$$

Suppose first that $p \ge 2$. Then from (2.7) one deduces

$$\lambda_p \int_\Omega |\nabla(u_\epsilon - \hat{\mu})|^p dx \le \langle -A\hat{\mu}, u_\epsilon - \hat{\mu}\rangle. \tag{2.10}$$

The result follows then from the weak convergence of $u_\epsilon$ toward $\hat{\mu}$ since $A\hat{\mu} \in W^{-1,p'}(\Omega)$. If now $p < 2$ the inequality (2.8) leads to

$$\lambda_p \int_\Omega |\nabla(u_\epsilon - \hat{\mu})|^2 \{|\nabla u_\epsilon| + |\nabla\hat{\mu}|\}^{p-2} dx \le \langle -A\hat{\mu}, u_\epsilon - \hat{\mu}\rangle. \tag{2.11}$$

Thus the left hand side integral of this inequality goes to 0 when $\epsilon \to 0$. Using Hölder's inequality one has

$$\begin{aligned}\int_\Omega |\nabla(u_\epsilon - \hat{\mu})|^p &= \int_\Omega |\nabla(u_\epsilon - \hat{\mu})|^p \{|\nabla u_\epsilon| + |\nabla\hat{\mu}|\}^{\frac{(p-2)p}{2}} \{|\nabla u_\epsilon| + |\nabla\hat{\mu}|\}^{\frac{(2-p)p}{2}} dx \\ &\le \Big(\int_\Omega |\nabla(u_\epsilon - \hat{\mu})|^2 \{|\nabla u_\epsilon| + |\nabla\hat{\mu}|\}^{p-2} dx\Big)^{\frac{p}{2}} \Big(\int_\Omega \{|\nabla u_\epsilon| + |\nabla\hat{\mu}|\}^p dx\Big)^{\frac{2}{2-p}}.\end{aligned}$$

Since the last integral above is bounded independently of $\epsilon$ it follows by (2.11) that the left hand side integral goes to 0 when $\epsilon \to 0$. This completes the proof of the proposition. □

**Remark 1.** *In the case when* (2.7), (2.8) *hold one can drop the assumption* (1.2), (1.3) *in the proposition 2.1.*

Let us suppose that $a$ is strictly monotone in the sense that

$$(a(x,\xi)-a(x,\xi'))\cdot(\xi-\xi')>0 \quad \text{a.e. } x \text{ in } \Omega, \quad \forall\, \xi\neq\xi'\in\mathbb{R}^N. \tag{2.12}$$

Then the operator $A$ defined above is strictly monotone itself and for $f\in W^{-1,p'}(\Omega)$ there exists a unique $u$ such that

$$\begin{cases} u\in W_0^{1,p}(\Omega), \\ \langle Au,v\rangle=\langle f,v\rangle \quad \forall v\in W_0^{1,p}(\Omega). \end{cases} \tag{2.13}$$

It is convenient to mention also here the following lemma.

**Lemma 2.1.** *Suppose that* (1.2), (1.4) *and* (2.12) *hold. For* $f\in W^{-1,p'}(\Omega)$ *let* $u=T(f)$ *be the solution to* (2.13). *Then* $T$ *is continuous from* $W^{-1,p'}(\Omega)$ *equipped with the strong topology into* $W_0^{1,p}(\Omega)$ *equipped with the weak topology.*

*Proof.* Suppose that $f_n\to f$ in $W^{-1,p'}(\Omega)$. Set $u_n=T(f_n)$. One has

$$\langle Au_n,u_n\rangle=\langle f_n,u_n\rangle.$$

Hence from (1.2) it follows that

$$\alpha\big|\big|\nabla u_n\big|\big|_p^p\leq |f_n|_{-1,p'}\big|\big|\nabla u_n\big|\big|_p.$$

Thus $u_n$ is bounded in $W_0^{1,p}(\Omega)$ independently of $n$ and there exists $u\in W_0^{1,p}(\Omega)$ such that $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Since $Au_n$ is also bounded in $W^{-1,p'}(\Omega)$ one has - up to a subsequence and for some $\chi\in W^{-1,p'}(\Omega)$

$$u_n\rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \quad Au_n\rightharpoonup\chi \text{ in } W^{-1,p'}(\Omega).$$

We claim that $\chi=Au$ where $u$ is the solution to (2.13). We use the same arguments as before. Indeed from the monotonicity assumption (2.3) one has

$$\langle Au_n,u_n\rangle\geq\langle Au_n,v\rangle+\langle Av,u_n-v\rangle \quad \forall v\in W_0^{1,p}(\Omega). \tag{2.14}$$

One has also

$$\langle Au_n,u_n\rangle=\langle f_n,u_n\rangle\to\langle f,u\rangle$$

and since $\langle Au_n,u\rangle=\langle f_n,u\rangle$ by passing to the limit $\langle f,u\rangle=\langle\chi,u\rangle$. Thus passing to the limit in (2.14) one deduces that

$$\langle\chi-Av,u-v\rangle\geq 0 \quad \forall v\in W_0^{1,p}(\Omega).$$

Choosing $v=u-tw,\ t>0$ one gets easily

$$\langle\chi-A(u-tw),w\rangle\geq 0 \quad \forall w\in W_0^{1,p}(\Omega).$$

Letting $t\to 0$ we get

$$\langle\chi-Au,w\rangle\geq 0 \quad \forall w\in W_0^{1,p}(\Omega)$$

which implies that $\chi=Au$ and the result follows since the limit of $Au_n$ is uniquely determined. □

**Remark 2.** *In the case of the assumptions* (2.7), (2.8) *the convergence is strong. This follows from*

$$\langle Au_n-Au,u_n-u\rangle=\langle f_n-f,u_n-u\rangle\to 0$$

*and the coerciveness assumptions involved (Cf.* (2.10), (2.11)*).*

## 3 $\Omega$ and $H$ bounded

Let us suppose that $a$ is strictly monotone in the sense of (2.12). Then, as seen in the previous section, for $\mu \in W^{-1,p'}(\Omega)$ there exists a unique $\hat{\mu}$ such that

$$\begin{cases} \hat{\mu} \in W_0^{1,p}(\Omega), \\ \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla v dx = \langle \mu, v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.1}$$

We can show

**Theorem 3.1.** *Suppose that* (1.2), (1.4), (2.7), (2.8) *hold. Let $H$ be a function from $\mathbb{R}$ into itself, bounded, Lipschitz continuous and such that*

$$(sH(s))' \in L^\infty(\Omega). \tag{3.2}$$

*Let $\mu \in W^{-1,p'}(\Omega)$, with $p > \frac{2N}{N+2}$ and $\hat{\mu}$ defined by* (3.1)*. Then there exists $u$ solution to*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \langle Au, v \rangle = \langle A\hat{\mu}, H(u)v \rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.3}$$

*Proof.* We denote by $\hat{\mu}_n$ the solution $u_{\frac{1}{n}}$ to (2.3) with $\epsilon = \frac{1}{n}$. We claim that there exists a solution $u_n$ to

$$\begin{cases} u_n \in W_0^{1,p}(\Omega), \\ \langle Au_n, v \rangle = \langle A\hat{\mu}_n, H(u_n)v \rangle \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.4}$$

Note that by (2.3) one has

$$A\hat{\mu}_n \in W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega) \tag{3.5}$$

$p^*$ being any real number in the case $N = 1$ or $p \geq N$. If $N \geq 2, p < N$ we claim that

$$p^* > (p^*)'. \tag{3.6}$$

Indeed this inequality reads

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} < 1 - (\frac{1}{p} - \frac{1}{N}) \Leftrightarrow p > \frac{2N}{N+2}.$$

Let us then choose $r$ such that

$$r = p \text{ if } N = 1 \text{ or } p \geq N, \quad p^* > r > (p^*)' \text{ else.} \tag{3.7}$$

For $w \in L^r(\Omega)$ we claim that there exists a unique solution $u = S_n(w)$ to

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \langle Au, v \rangle = \int_\Omega A\hat{\mu}_n H(w)v \; dx \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.8}$$

Indeed, in the case $p < N$, since by (3.5) $A\hat{\mu}_n H(w) \in L^{p^*}(\Omega)$ one has for $v \in W_0^{1,p}(\Omega)$

$$|\int_\Omega A\hat{\mu}_n H(w)v \; dx| \leq |A\hat{\mu}_n H(w)|_{p^*} |v|_{(p^*)'} \leq C|A\hat{\mu}_n|_{p^*} |H|_\infty |v|_{p^*} \leq C' \big||\nabla v|\big|_p$$

where $C, C'$ are constants independent of $w$ (recall that $n$ is here fixed, $|H|_\infty$ denotes the $L^\infty(\Omega)$-norm of $H$). In the case when $r = p$, since $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall q$, the inequality above reads

$$|\int_\Omega A\hat{\mu}_n H(w) v \; dx| \leq |A\hat{\mu}_n H(w)|_p |v|_{p'} \leq C|A\hat{\mu}_n|_p |H|_\infty |v|_{p'} \leq C' \big||\nabla v|\big|_p.$$

Thus in both cases the right hand side of the equation in (3.8) defines an element of $W^{-1,p'}(\Omega)$ and the existence and uniqueness of $u$ is clear. In addition, taking $v = u$ in (3.8) one derives easily for some other constants $C, C'$, when $p < N$

$$\alpha \big||\nabla u|\big|_p^p \leq |A\hat{\mu}_n H(w)|_{p^*} |u|_{(p^*)'} \leq C|A\hat{\mu}_n|_{p^*} |H|_\infty |u|_r \leq C'|u|_r$$

and when $r = p$

$$\alpha \big||\nabla u|\big|_p^p \leq |A\hat{\mu}_n H(w)|_{p'} |u|_p \leq C|A\hat{\mu}_n|_{p'} |H|_\infty |u|_r \leq C'|u|_r$$

where $C, C'$ are independent of $w$. Using again the Sobolev embedding theorem it results that

$$\big||\nabla u|\big|_p, |u|_r \leq C$$

where $C$ is some other constant independent of $w$. Thus $S_n$ maps the ball

$$B = \{v \in L^r(\Omega) \mid |v|_r \leq C\}$$

into itself and moreover $S_n(B)$ is relatively compact in $B$. The existence of $u_n$, fixed point of $S_n$, will follow by the Schauder fixed point theorem if we can show the continuity of $S_n$ in $L^r(\Omega)$.

If $w_p \to w$ in $L^r(\Omega)$ one has - up to a subsequence - $w_p \to w$ a.e. in $\Omega$ and by the Lebesgue theorem (recall that $n$ is fixed and $H$ is bounded and continuous)

$$A\hat{\mu}_n H(w_p) \to A\hat{\mu}_n H(w) \text{ in } L^{p^*}(\Omega) \text{ if } p < N, \quad (\text{in } L^{p'}(\Omega) \text{ if } r = p).$$

But $L^{p^*}(\Omega) \subset L^{(p^*)'}$, ($L^{p'}(\Omega)$ if $r = p$), is continuously imbedded in $W^{-1,p'}(\Omega)$ (Cf. (2.2)). Indeed to see that in the case $p < N$ note that for $f \in L^{p^*}(\Omega)$ (Cf. (3.6)) one has

$$|\int_\Omega f v \; dx| \leq |f|_{(p^*)'} |v|_{p^*} \leq C|f|_{p^*} \big||\nabla v|\big|_p$$

for some constant $C$. It follows then from the lemma 2.1 that

$$S_n(w_p) \rightharpoonup S_n(w)$$

in $W_0^{1,p}(\Omega)$ and also strongly in $L^r(\Omega)$ due to the compactness of the Sobolev embedding and the unique possible limit. This completes the existence of a solution to (3.4).

Taking now $v = u_n$ in (3.4) and using (1.2), (1.4) we deduce

$$\alpha \int_\Omega |\nabla u_n|^p \; dx \leq \int_\Omega a(x, \nabla\hat{\mu}_n) \cdot \nabla(H(u_n)u_n) \; dx$$
$$\int_\Omega \Big(\nu(x) + \beta|\nabla\hat{\mu}_n|^{p-1}\Big) C_\infty |\nabla u_n| \; dx.$$

($C_\infty$ is the Lipschitz constant of $sH(s)$). Since $\hat{\mu}_n$ is bounded in $W_0^{1,p}(\Omega)$ one gets easily that $u_n$ is also bounded in $W_0^{1,p}(\Omega)$ and - up to a subsequence - one has for some $u$ in $W_0^{1,p}(\Omega)$ when $n \to \infty$

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), u_n \to u \text{ in } L^p(\Omega), u_n \to u \text{ a.e. in } \Omega. \tag{3.9}$$

One has then to pass to the limit in

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v \; dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n) v) \; dx \quad \forall v \in W_0^{1,p}(\Omega).$$

Denote by $(\;\;)_k$ the truncation at the level $k > 0$, i.e. $(w)_k = (-k) \vee w \wedge k$ where $\vee$ denotes the maximum of two numbers and $\wedge$ the minimum of two numbers.

Taking $v = (v - u_n)_k$ one has

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla (v - u_n)_k \; dx = \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n)(v - u_n)_k) \; dx \quad \forall v \in W_0^{1,p}(\Omega).$$

By the monotonicity of $a$ this implies

$$\int_\Omega a(x, \nabla v) \cdot \nabla (v - u_n)_k \; dx \geq \int_\Omega a(x, \nabla \hat{\mu}_n) \cdot \nabla (H(u_n)(v - u_n)_k) \; dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.10}$$

Clearly $(v - u_n)_k$ is bounded in $W_0^{1,p}(\Omega)$ independently of $n$ and converges -up to a subsequence- toward some $\kappa \in W_0^{1,p}(\Omega)$ weakly in $W_0^{1,p}(\Omega)$. Due to (3.9) this $\kappa$ is necessarily equal to $(v - u)_k$ and thus

$$(v - u_n)_k \rightharpoonup (v - u)_k \text{ in } W_0^{1,p}(\Omega). \tag{3.11}$$

One has

$$\nabla (H(u_n)(v - u_n)_k) = H'(u_n)(v - u_n)_k \nabla u_n + H(u_n) \nabla (v - u_n)_k.$$

Hence, if $H'_\infty$ denotes the Lipschitz constant of $H$,

$$|\nabla (H(u_n)(v - u_n)_k)| \leq H'_\infty k |\nabla u_n| + |H|_\infty |\nabla (v - u_n)_k|.$$

Thus, $H(u_n)(v - u_n)_k$ is also bounded in $W_0^{1,p}(\Omega)$ independently of $n$. There exists $h$ in $W_0^{1,p}(\Omega)$ such that

$$H(u_n)(v - u_n)_k \rightharpoonup h \text{ in } W_0^{1,p}(\Omega), H(u_n)(v - u_n)_k \to H(u)(v - u)_k \text{ in } L^p(\Omega)$$

the last convergence being due to the Lebesgue theorem. Both convergence taking also part in the distributional sense one has $h = H(u_n)(v - u)_k$ and

$$H(u_n)(v - u_n)_k \rightharpoonup H(u)(v - u)_k \text{ in } W_0^{1,p}(\Omega).$$

Since $\hat{\mu}_n \to \hat{\mu}$ in $W_0^{1,p}(\Omega)$, Cf. Proposition 2.2, one can pass to the limit in (3.10) to get that

$$\int_\Omega a(x, \nabla v) \cdot \nabla (v - u)_k \; dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla (H(u)(v - u)_k) \; dx \quad \forall v \in W_0^{1,p}(\Omega).$$

If we take in this inequality $v = u + tw$ where $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ we get

$$\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla(tw)_k \, dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(H(u)(tw)_k) \, dx \quad \forall w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

But for $t$ small we have $(tw)_k = tw$ and thus we get

$$\int_\Omega a(x, \nabla(u + tw)) \cdot \nabla w \, dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(H(u)w) \, dx \quad \forall w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Letting $t \to 0$ leads to

$$\int_\Omega a(x, \nabla u) \cdot \nabla w \, dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla(H(u)w) \, dx \quad \forall w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

The existence of a solution $u$ to (3.3) follows by changing $w$ into $-w$. □

If $\mu$ is a nonnegative measure we have the following complementary results.

**Theorem 3.2.** *Suppose that $\mu \in W^{-1,p'}(\Omega)$ is a nonnegative measure. Then under the assumptions of theorem 3.1 if*

$$H(s) \geq 0 \quad \forall s \leq 0, \tag{3.12}$$

*a solution to* (3.3) *satisfies*

$$u \geq 0. \tag{3.13}$$

*Proof.* For $k > 0$ one considers $v = u^- \wedge k$ where $\wedge$ denotes the minimum of two numbers. Since

$$\begin{aligned} H(u)(u^- \wedge k) &= 0 \text{ when } u \geq 0, \\ &= H(-u^-)(u^- \wedge k) \text{ when } u \leq 0, \end{aligned}$$

this function is nonnegative and thus from (3.3) one deduces

$$\langle Au, u^- \wedge k \rangle \geq 0.$$

Denoting by $\{-k \leq u \leq 0\}$ the set $\{x \mid -k \leq u(x) \leq 0 \text{ a.e. in } \Omega\}$ this implies

$$\int_{\{-k \leq u \leq 0\}} a(x, -\nabla u^-) \cdot \nabla u^- dx = \int_{\{-k \leq u \leq 0\}} a(x, \nabla u) \cdot \nabla u^- dx \geq 0.$$

Thus

$$0 \geq \int_{\{-k \leq u \leq 0\}} a(x, -\nabla u^-) \cdot -\nabla u^- dx \geq \alpha \int_{\{-k \leq u \leq 0\}} |\nabla u^-|^p dx = \int_\Omega |\nabla(u^- \wedge k)|^p dx.$$

It follows that for any $k > 0$, $u^- \wedge k = 0$. This implies that $u^- = 0$ and this completes the proof. □

**Theorem 3.3.** *Suppose that $\mu \in W^{-1,p'}(\Omega)$ is a nonnegative measure. Suppose that we are under the assumptions of Theorem 3.1 and that $H_1$ and $H_2$ are two functions satisfying the assumptions on $H$. Let $u_1$, $u_2$ be two solutions to* (3.3) *corresponding to $H_1$ and $H_2$ respectively, then if*

$$H_1 \geq H_2, \quad H_1 \text{ or } H_2 \text{ is nonincreasing,} \tag{3.14}$$

*then one has*

$$u_1 \geq u_2. \tag{3.15}$$

*Proof.* From (3.3) written for $H_1$ and $H_2$ one deduces by subtraction

$$\langle Au_1 - Au_2, v\rangle = \langle A\hat{\mu}, (H_1(u_1) - H_2(u_2))v\rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Taking $v = (u_1 - u_2)^- \wedge k$, $k > 0$ one has

$$(H_1(u_1) - H_2(u_2))\{(u_1 - u_2)^- \wedge k\} = 0 \text{ if } u_1 - u_2 \geq 0.$$

– If $H_1$ is nonincreasing

$$(H_1(u_1) - H_2(u_2))v = (H_1(u_1) - H_1(u_2) + H_1(u_2) - H_2(u_2))\{(u_1 - u_2)^- \wedge k\} \geq 0 \text{ if } u_1 - u_2 \leq 0.$$

– If $H_2$ is nonincreasing

$$(H_1(u_1) - H_2(u_2))v = (H_1(u_1) - H_2(u_1) + H_2(u_1) - H_2(u_2))\{(u_1 - u_2)^- \wedge k\} \geq 0 \text{ if } u_1 - u_2 \leq 0.$$

Thus since $(H_1(u_1) - H_2(u_2))v \geq 0$ in all cases, one has

$$\langle Au_1 - Au_2, \{(u_1 - u_2)^- \wedge k\}\rangle \geq 0.$$

With obvious notation this leads to

$$\int_{\{-k \leq u_1 - u_2 \leq 0\}} a(x, \nabla u_1) - a(x, \nabla u_2) \cdot \nabla(u_1 - u_2) dx \leq 0.$$

Thus $\nabla u_1 = \nabla u_2$ a.e. on the set $\{-k \leq u_1 - u_2 \leq 0\}$. This implies that $\nabla\{(u_1 - u_2)^- \wedge k\} = 0$ a.e. in $\Omega$. Thus $(u_1 - u_2)^- \wedge k = 0$ for every $k > 0$ and so $(u_1 - u_2)^- = 0$. This completes the proof of the theorem.

□

**Remark 3.** *Under the assumptions of Theorem 3.1 if $\mu$ is a nonnegative measure in $W^{-1,p'}(\Omega)$ and $H$ a nonincreasing function then the solution to* (3.2) *is unique. This follows immediately from Theorem 3.3.*

## 4 $\Omega$ bounded and $H$ unbounded

Now we would like to allow $H$ to be unbounded near zero.

**Theorem 4.1.** *Let $\mu \in W^{-1,p'}(\Omega)$ be a nonnegative bounded measure, $A$ be an operator satisfying* (1.2), (2.7), (2.8) *and* (1.4) *with $\nu = 0$ and let $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative, nonincreasing function such that*

$$\begin{cases} \lim_{s\to 0^+} H(s) = +\infty, \\ \forall \epsilon > 0,\ H \text{ is Lipschitz continuous on } (\epsilon, +\infty), \end{cases} \tag{4.1}$$

$$\exists \mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+ \text{ s.t. } H \le \mathcal{K} \text{ and s.t. } (s\mathcal{K}(s)) \in W^{1,\infty}(\mathbb{R}^+). \tag{4.2}$$

*Then there exists $u$ solution to*

$$\begin{cases} u \in W_0^{1,p}(\Omega), u \ge 0, \\ \langle Au, v\rangle = \langle A\hat{\mu}, H(u)v\rangle \quad \forall v \in C_c^1(\Omega). \end{cases} \tag{4.3}$$

*Proof.* Some of the arguments developed here are similar to the ones in [3]. However the nonlinear operator forces tricky modifications. We first choose $n$ large enough, more precisely $n \ge n_0$, where $n_0$ is such that $H(n_0) < n_0$. Then we define a function $H_n$ by setting

$$H_n(s) = \begin{cases} n \ \text{ for } s \le 0, \\ n \wedge H(s) \ \text{ for } 0 < s \le n, \\ \{-H(n)(s-(n+1))\} \wedge H(s) \ \text{ for } n \le s \le n+1, \\ 0 \ \text{ for } n+1 \le s. \end{cases} \tag{4.4}$$

Clearly for every $n$ one has

$$H_n(s) \le H_{n+1}(s) \le H(s) \quad \forall s > 0. \tag{4.5}$$

Moreover $H_n$ satisfies the assumptions of theorem 3.1 (Cf. also theorem 3.2) and thus there exists $u_n$ solution to

$$\begin{cases} u_n \in W_0^{1,p}(\Omega), u_n \ge 0, \\ \langle Au_n, v\rangle = \langle A\hat{\mu}, H_n(u_n)v\rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{4.6}$$

Taking in (4.6) $v = u_n \wedge k$ with $k > 0$ leads to

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n \wedge k)dx = \langle A\hat{\mu}, H_n(u_n)(u_n \wedge k)\rangle. \tag{4.7}$$

One has of course

$$\alpha \int_\Omega |\nabla(u_n \wedge k)|^p \le \int_\Omega a(x, \nabla(u_n \wedge k)) \cdot \nabla(u_n \wedge k)dx = \int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n \wedge k)dx. \tag{4.8}$$

Moreover

$$\begin{aligned} \langle A\hat{\mu}, H_n(u_n)(u_n \wedge k)\rangle &= \langle A\hat{\mu}, \{H_n(u_n) - H_n(u_n \wedge k) + H_n(u_n \wedge k)\}(u_n \wedge k)\rangle \\ &\le \langle A\hat{\mu}, H_n(u_n \wedge k)(u_n \wedge k)\rangle, \end{aligned} \tag{4.9}$$

since $\{H_n(u_n) - H_n(u_n \wedge k)\}(u_n \wedge k) \leq 0$ and $\mu$ is a nonnegative measure. Combining (4.7)-(4.9) we get easily by (4.2)

$$\alpha \int_\Omega |\nabla(u_n \wedge k)|^p \leq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{\mathcal{K}(u_n \wedge k)(u_n \wedge k)\} dx. \tag{4.10}$$

Since the function $s\mathcal{K}(s)$ is supposed to be Lipschitz continuous one has for some $f \in L^\infty(\mathbb{R})$ and some constant $C_\mathcal{K}$

$$s\mathcal{K}(s) = \int_0^s f(\xi) d\xi + C_\mathcal{K}.$$

It follows that $\mathcal{K}(u_n \wedge k)(u_n \wedge k) - C_\mathcal{K} \in W_0^{1,p}(\Omega)$ and from (4.10) we derive

$$\begin{aligned}
\alpha \int_\Omega |\nabla(u_n \wedge k)|^p &\leq \langle \mu, \mathcal{K}(u_n \wedge k)(u_n \wedge k) - C_\mathcal{K} + C_\mathcal{K} \rangle \\
&= \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{\mathcal{K}(u_n \wedge k)(u_n \wedge k) - C_\mathcal{K}\} + \langle \mu, C_\mathcal{K} \rangle \\
&\leq \beta |f|_\infty \int_\Omega |\nabla\hat{\mu}|^{p-1} |\nabla(u_n \wedge k)| dx + \langle \mu, C_\mathcal{K} \rangle \\
&\leq \beta |f|_\infty \big|\big|\nabla\hat{\mu}\big|\big|_p^{p-1} \Big( \int_\Omega |\nabla(u_n \wedge k)|^p dx \Big)^{\frac{1}{p}} + \langle \mu, C_\mathcal{K} \rangle.
\end{aligned}$$

It follows easily that for some constant $C$ independent of $n$ and $k$ one has

$$\int_\Omega |\nabla(u_n \wedge k)|^p dx \leq C.$$

Letting $k \to \infty$ we deduce

$$\int_\Omega |\nabla u_n|^p dx \leq C.$$

Thus $u_n$ is bounded in $W_0^{1,p}(\Omega)$ and - up to a subsequence - one has for some $u \in W_0^{1,p}(\Omega)$

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \ u_n \to u \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega.$$

We would like to show that $u$ is solution to (4.3).

One has

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v dx = \langle A\hat{\mu}, H_n(u_n) v \rangle \ \ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Taking $v = (v - u_n)_k$ one deduces that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla (v - u_n)_k dx = \langle A\hat{\mu}, H_n(u_n)(v - u_n)_k \rangle \ \ \forall v \in W_0^{1,p}(\Omega).$$

It is clear from (4.5) and the monotonicity of $H_n$ that $u_n$ is nondecreasing with $n$, i.e. one has

$$u_{n+1} \geq u_n \geq u_{n_0}, \ \ \forall n \geq n_0. \tag{4.11}$$

Thus one has also $u \geq u_n$ for every $n \geq n_0$ and thus also $(v-u_n)_k \geq (v-u)_k$ for any $v$. Using the nonnegativity of $A\hat{\mu}$ and $H_n(u_n)$ one derives that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla (v-u_n)_k dx \geq \langle A\hat{\mu}, H_n(u_n)(v-u)_k \rangle \quad \forall v \in W_0^{1,p}(\Omega)$$

and by the monotonicity assumption of $a$ it comes

$$\int_\Omega a(x, \nabla v) \cdot \nabla (v-u_n)_k dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{H_n(u_n)(v-u)_k\} dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.12}$$

By definition of $u_{n_0}$ one has

$$-\mathrm{div}\{a(x, \nabla u_{n_0})\} \geq 0, \quad -\mathrm{div}\{a(x, \nabla u_{n_0})\} \not\equiv 0.$$

It follows from the maximum principle (Cf. [18] Theorems 7.1.3, 7.2.2) that for each subdomain $\omega \subset\subset \Omega$ one has for some positive constant $c_\omega$

$$u \geq u_n \geq u_{n_0} \geq c_\omega \quad \text{a.e. in } \omega. \tag{4.13}$$

Consider then $w \in C_c^1(\Omega)$ with support included in $\omega$. For $t > 0$ choose in (4.12) $v = u + tw$. It comes

$$\int_\Omega a(x, \nabla(u+tw)) \cdot \nabla (u+tw-u_n)_k dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{H_n(u_n)(tw)_k\} dx. \tag{4.14}$$

One has of course when $n \to \infty$, $\nabla(u+tw-u_n)_k \rightharpoonup \nabla(tw)_k$ in $L^p(\Omega)$. Moreover one has by the definition of $H_n$

$$0 \leq -H_n' \leq C \text{ on } [c_\omega, \infty)$$

where $C$ is independent of $n$. It follows that

$$\nabla\{H_n(u_n)(tw)_k\} = H_n(u_n)\nabla(tw)_k + H_n'(u_n)(tw)_k \nabla u_n$$

is bounded in $L^p(\Omega)$ independently of $n$. Thus it converges - up to a subsequence - weakly in $L^p(\Omega)$. But clearly it follows from the convergence of $u_n$ that the limit can only be $\nabla\{H(u)(tw)_k\}$ - recall that $H_n \to H$ when $n \to \infty$. Passing to the limit in (4.14) we obtain

$$\int_\Omega a(x, \nabla(u+tw)) \cdot \nabla (tw)_k dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{H(u)(tw)_k\} dx.$$

Since for $t$ small enough $(tw)_k = tw$ letting $t \to 0$ it comes

$$\int_\Omega a(x, \nabla u) \cdot \nabla w dx \geq \int_\Omega a(x, \nabla \hat{\mu}) \cdot \nabla \{H(u)w\} dx.$$

This is true for any $w$ and changing $w$ into $-w$ it results that $u$ is solution to (4.3). This completes the proof of the theorem. □

**Remark 4.** *A function $H$ satisfying our assumptions is for instance given by*

$$H(s) = s^{-\gamma} \tag{4.15}$$

*with $\gamma \leq 1$. This function gives rise to a solution of finite energy - i.e. belonging to $W_0^{1,p}(\Omega)$. In the case where the singularity is stronger for instance when $\gamma > 1$ one can have a weaker solution (Cf. [4], [13]). Suppose that we are restricting ourselves to the simple case of the Laplace operator. If $H$ is given by* (4.15) *let us denote by $H_n$ the function defined in* (4.4)*. Then there exists $u_n$ solution to*

$$\begin{cases} u_n \in W_0^{1,2}(\Omega), \\ \langle -\Delta u_n, v\rangle = \langle A\hat{\mu}, H_n(u_n)v\rangle \quad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{4.16}$$

*Now by Theorems 3.2, 3.3, $u_n$ is a nondecreasing sequence which converges point-wise and satisfies for any domain $\omega \subset\subset \Omega$ and for some constant $c_\omega$ independent of $n$*

$$u_n \geq c_\omega.$$

*Using Harnack inequality when $p < n$ (Cf. [18] Theorem. 7.1.2), $u_n$ is locally bounded in some $L^q$. Then for $v \in C_c^\infty(\Omega)$ with support in $\omega$ one can pass to the limit in* (4.16) *to get $u$ to satisfy*

$$\langle u, -\Delta v\rangle = \langle A\hat{\mu}, H(u)v\rangle \quad \forall v \in C_c^\infty(\Omega).$$

# 5 $\Omega$ unbounded

In this section we extend some of our previous results in the case when $\Omega$ is unbounded.

**Theorem 5.1.** *Let $\Omega \subset \mathbb{R}^N$ be an unbounded open set and for $p > \frac{2N}{N+2}$, $\mu \in W_0^{1,p}(\Omega_\ell)$ be a nonnegative measure. Let $a$ be a Carathéodory function satisfying* (1.2), (1.4), (2.7), (2.8) *and $A$ the operator defined by* (1.5)*. Let $H : \mathbb{R} \to \mathbb{R}$ be a function such that*

$$\begin{cases} H \text{ is nonincreasing, bounded and Lipschitz continuous}, \\ (sH(s))' \in L^\infty(\Omega), \\ H(s) \geq 0 \quad \forall s \leq 0. \end{cases} \tag{5.1}$$

*If $\hat{\mu}$ is defined as in* (3.1)*, then there exists a unique $u$ solution to*

$$\begin{cases} u \in W_0^{1,p}(\Omega), u \geq 0, \\ \langle Au, v\rangle = \langle A\hat{\mu}, H(u)v\rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.2}$$

*Proof.* For $\ell > 0$ if $B_\ell$ denotes the open ball centred at 0 and radius $\ell$ set

$$\Omega_\ell = \Omega \cap B_\ell, \quad V_\ell = W_0^{1,p}(\Omega_\ell). \tag{5.3}$$

If one assumes the functions of $W_0^{1,p}(\Omega_\ell)$ extended by 0 outside of $\Omega_\ell$ it is clear that

$$W_0^{1,p}(\Omega_\ell) \subset W_0^{1,p}(\Omega)$$

and thus $\mu \in W^{-1,p'}(\Omega_\ell)$. By Theorems 3.1, 3.2 and Remark 3 there exists a unique $u_\ell$ solution to

$$\begin{cases} u_\ell \in W_0^{1,p}(\Omega_\ell), u_\ell \geq 0, \\ \langle Au_\ell, v\rangle = \langle A\hat{\mu}, H(u_\ell)v\rangle \quad \forall v \in W_0^{1,p}(\Omega_\ell) \cap L^\infty(\Omega_\ell). \end{cases} \tag{5.4}$$

Taking $v = u_\ell \wedge k$ for $k > 0$ one obtains easily

$$\begin{aligned} \alpha \int_{\Omega_\ell} |\nabla(u_\ell \wedge k)|^p dx &\leq \langle Au_\ell, (u_\ell \wedge k)\rangle = \langle A\hat{\mu}, H(u_\ell)(u_\ell \wedge k)\rangle = \langle \mu, H(u_\ell)(u_\ell \wedge k)\rangle \\ &= \langle \mu, \{H(u_\ell) - H(u_\ell \wedge k) + H(u_\ell \wedge k)\}(u_\ell \wedge k)\rangle \\ &\leq \langle \mu, H(u_\ell \wedge k)(u_\ell \wedge k)\rangle \end{aligned} \tag{5.5}$$

since $\mu$ is nonnegative and $\{H(u_\ell) - H(u_\ell \wedge k)\}(u_\ell \wedge k) \leq 0$. Thus it comes

$$\begin{aligned} \alpha \int_{\Omega_\ell} |\nabla(u_\ell \wedge k)|^p dx &\leq |\mu|_{-1,p'} \{\int_{\Omega_\ell} |\nabla H(u_\ell \wedge k)(u_\ell \wedge k)|^p dx\}^{\frac{1}{p}} \\ &\leq C_\infty |\mu|_{-1,p'} \{\int_{\Omega_\ell} |\nabla(u_\ell \wedge k)|^p dx\}^{\frac{1}{p}} \end{aligned} \tag{5.6}$$

where $C_\infty$ denotes the Lipschitz constant of the function $sH(s)$ and $|\mu|_{-1,p'}$ the norm of $\mu$ in $W^{-1,p'}(\Omega)$. It follows easily that

$$\int_{\Omega_\ell} |\nabla(u_\ell \wedge k)|^p dx \leq \Big(\frac{C_\infty |\mu|_{-1,p'}}{\alpha}\Big)^{p'}.$$

Letting $k \to +\infty$ we deduce

$$\big||\nabla u_\ell|\big|_p \leq \Big(\frac{C_\infty |\mu|_{-1,p'}}{\alpha}\Big)^{\frac{1}{p-1}} \tag{5.7}$$

i.e. $u_\ell$ is bounded in $W_0^{1,p}(\Omega)$ independently of $\ell$.

Let us fix some $\ell' < \ell$ and consider $v \in W_0^{1,p}(\Omega_{\ell'})$. One has, for $k > 0$, $(v - u_\ell)_k \in W_0^{1,p}(\Omega_\ell) \cap L^\infty(\Omega_\ell)$. From the equation of (5.4) it follows that

$$\int_\Omega a(x, \nabla u_\ell) \cdot \nabla(v - u_\ell)_k dx = \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H(u_\ell)(v - u_\ell)_k\} dx \tag{5.8}$$

(We suppose $(v - u_\ell)_k$ extended by 0 outside $\Omega_\ell$). Using the monotonicity of $a$ this implies

$$\int_\Omega a(x, \nabla v) \cdot \nabla(v - u_\ell)_k dx \geq \int_\Omega a(x, \nabla\hat{\mu}) \cdot \nabla\{H(u_\ell)(v - u_\ell)_k\} dx. \tag{5.9}$$

From (5.7) we deduce that there exists $u \in W_0^{1,p}(\Omega)$ such that -up to a subsequence

$$u_\ell \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \quad u_\ell \to u \text{ in } L^p_{loc}(\Omega). \tag{5.10}$$

(Using for instance a diagonal process for a sequence of subdomains $\Omega_k$ absorbing $\Omega$).

Then since $(v - u_\ell)_k$ is also bounded in $W_0^{1,p}(\Omega)$, for a subsequence, one has

$$\nabla(v - u_\ell)_k \rightharpoonup \nabla(v - u)_k \text{ in } (L^p(\Omega))^N. \tag{5.11}$$

One has also

$$\nabla\{H(u_\ell)(v-u_\ell)_k\} = H(u_\ell)\nabla(v-u_\ell)_k + (v-u_\ell)_k H'(u_\ell)\nabla u_\ell.$$

Thus this gradient is uniformly bounded in $(L^p(\Omega))^N$ and thus there exists $G \in (L^p(\Omega))^N$ such that

$$\nabla\{H(u_\ell)(v-u_\ell)_k\} \rightharpoonup G \text{ in } (L^p(\Omega))^N.$$

Since by the Lebesgue theorem one has up to a subsequence

$$H(u_\ell)(v-u_\ell)_k \to H(u)(v-u)_k \text{ in } L^p_{loc}(\Omega)$$

it follows that

$$\nabla\{H(u_\ell)(v-u_\ell)_k\} \rightharpoonup G = \nabla\{H(u)(v-u)_k\} \text{ in } (L^p(\Omega))^N. \tag{5.12}$$

Combining (5.11) and (5.12) one can pass to the limit in (5.9) to get

$$\int_\Omega a(x,\nabla v)\cdot\nabla(v-u)_k dx \geq \int_\Omega a(x,\nabla\hat{\mu})\cdot\nabla\{H(u)(v-u)_k\}dx.$$

Letting $k\to\infty$ leads to

$$\int_\Omega a(x,\nabla v)\cdot\nabla(v-u)dx \geq \int_\Omega a(x,\nabla\hat{\mu})\cdot\nabla\{H(u)(v-u)\}dx \quad \forall v\in W_0^{1,p}(\Omega_{\ell'}), \tag{5.13}$$

$\ell'$ being arbitrary.

Now if $v\in W_0^{1,p}(\Omega)$, $\theta_{\ell'}(x) = 1\wedge dist(x,\mathbb{R}^N\backslash B_{\ell'})$ one has

$$\theta_{\ell'}(x)v \in W_0^{1,p}(\Omega_{\ell'}), \quad \theta_{\ell'}(x)v \to v \text{ in } W_0^{1,p}(\Omega)$$

when $\ell'\to\infty$. Thus exchanging $v$ in $\theta_{\ell'}(x)v$ in (5.13) and passing to the limit in $\ell'$ one obtains

$$\int_\Omega a(x,\nabla v)\cdot\nabla(v-u)dx \geq \int_\Omega a(x,\nabla\hat{\mu})\cdot\nabla H(u)(v-u)dx \quad \forall v\in W_0^{1,p}(\Omega). \tag{5.14}$$

Then taking $v = u+tw$, $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ one concludes as in the proof of theorem 3.1.

As in theorem 3.3 one can show monotonicity of the solution to (5.2) with respect to $H$ and uniqueness follows as in Remark 3. This completes the proof of the theorem. □

The theorem 4.1 can be extended easily in the case when $H$ is unbounded near 0. One has

**Theorem 5.2.** *Let $\Omega\subset\mathbb{R}^N$ be an unbounded open set and for $p>\frac{2N}{N+2}$, $\mu\in W_0^{1,p}(\Omega_\ell)$ be a nonnegative bounded measure. Let $a$ be a Carathéodory function satisfying* (1.2), (2.7), (2.8), (1.4) *with $\nu=0$ and $A$ the operator defined by* (1.5). *Let $H:\mathbb{R}^+\to\mathbb{R}^+$ be a nonnegative, nonincreasing function such that*

$$\begin{cases}\lim_{s\to 0^+} H(s) = +\infty,\\ \forall\epsilon>0,\ H \text{ is Lipschitz continuous on } (\epsilon,+\infty),\end{cases} \tag{5.15}$$

$$\exists \mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+ \ s.t. \ H \leq \mathcal{K} \ and \ s.t. \ (s\mathcal{K}(s)) \in W^{1,\infty}(\mathbb{R}^+). \tag{5.16}$$

*Then there exists $u$ solution to*

$$\begin{cases} u \in W_0^{1,p}(\Omega), u \geq 0, \\ \langle Au, v \rangle = \langle A\hat{\mu}, H(u)v \rangle \quad \forall v \in C_c^1(\Omega). \end{cases} \tag{5.17}$$

*Proof.* One introduces the function $H_n$ as in (4.4) and one follows basically all the steps of the proof of theorem 4.1. □

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