ANISOTROPIC EQUATIONS: UNIQUENESS AND EXISTENCE RESULTS

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Abstract. We study uniqueness of weak solutions for elliptic equations of the following type

\[-\partial_x a_i(x, u) |\partial_x u|^{p_i-2} \partial_x u + b(x, u) = f(x)\]

in a bounded domain \(\Omega \subset \mathbb{R}^n\) with Lipschitz continuous boundary \(\Gamma = \partial \Omega\). We consider in particular mixed boundary conditions -i.e., Dirichlet condition on one part of the boundary and Neumann condition on the other part. We study also uniqueness of weak solutions for the parabolic equations

\[
\begin{cases}
\partial_t u = \partial_x \left( a_i(x, t, u) |\partial_x u|^{p_i-2} \partial_x u \right) + f & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \Gamma \times (0, T) = \partial \Omega \times (0, T), \\
u(x, 0) = u_0 & x \in \Omega.
\end{cases}
\]

It is assumed that the constant exponents \(p_i\) satisfy \(1 < p_i < \infty\) and the coefficients \(a_i\) are such that \(0 < \lambda \leq \lambda_i \leq a_i(x, u) < \infty, \forall i, a.e. x \in \Omega, (a.e. t \in (0, T)), \forall u \in \mathbb{R}\).

We indicate also conditions which guarantee existence of solutions.

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1. Introduction

In recent years an increasing interest has turned towards anisotropic elliptic and parabolic equations. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. We refer to the recent works [2, 3], [6]-[13], [17]-[19], [24, 25] where it is possible to find some references. In these papers various types of nonlinear anisotropic elliptic and parabolic equations are studied from the point of view of existence and nonexistence, regularity and qualitative properties with respect to the data.

The main aim of this note is to prove uniqueness of weak solutions to the above mentioned equations under weak feeble restrictions on the coefficients $a_i$. To prove it we will follow some of the ideas of [1, 15, 16].

We do not discuss the existence and the regularity of such solutions under generic conditions. We just underline some scheme to prove it. In fact, existence results are technical improvements of classical results stated in [21]-[23]. For more exotic equations existence results can be found in [2]-[8], [10]-[13], [17]-[20], [24, 25].

The paper is organized as follows.

In Section 2 we consider elliptic equations and prove uniqueness results when at least one $p_i \leq 2$, and when a lower order monotone term $b(x, u)$ exists and $p_i > 2, \forall i$. We give also an existence theorem and consider more general equations.

The similar scheme is developed in Section 3 for parabolic equations.

2. Elliptic equations

2.1. Uniqueness result when at least one $p_i \leq 2$. We consider the Dirichlet problem

\[
\begin{aligned}
-\partial_{x_i} \left( a_i(x, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right) &= f \quad \text{in } \Omega, \\
\partial_{x_i} u &= 0 \quad \text{on } \Gamma = \partial \Omega,
\end{aligned}
\]

in a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary $\Gamma = \partial \Omega$.

We assume that the components of the constant vector $\vec{p} = (p_1, p_2, ..., p_n)$ satisfy

\[
1 < p_1 \leq p_2 \leq ... \leq p_n < \infty.
\]

We define

\[
W^{1, \vec{p}}_0(\Omega) = \left\{ u \in W^{1, p_1}_0(\Omega) : \partial_{x_i} u \in L^{p_i}(\Omega), \ i = 1, ..., n \right\}
\]

and we equip this space with the norm

\[
\| u \|_{W^{1, \vec{p}}_0(\Omega)} = \sum_{i=1}^{n} \| \partial_{x_i} u \|_{L^{p_i}(\Omega)}.
\]
By $W_0^{-1, \vec{p}' \cdot} (\Omega)$ we denote the dual space of $W_0^{1, \vec{p}} (\Omega)$. A weak solution $u \in W_0^{1, \vec{p}} (\Omega)$ of problem (2.1) is a function $u$ such that

$$\int_{\Omega} a_i(x, u) |\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \partial_{x_i} v = < f, v >, \forall v \in W_0^{1, \vec{p}} (\Omega).$$

(2.3)

We are going to prove uniqueness and existence of a solution $u \in W_0^{1, \vec{p}} (\Omega)$ to (2.3).

We will suppose

$$0 < \lambda_i \leq a_i(x, u) \leq \Lambda < \infty, \forall i, a.e. x \in \Omega, \forall u \in \mathbb{R},$$

(2.4)

$$|a_i(x, u) - a_i(x, v)| \leq \omega(|u - v|), \forall i, a.e. x \in \Omega, \forall u, v \in \mathbb{R},$$

(2.5)

where

$$\int_{0^+} \frac{ds}{\omega(s)} = +\infty.$$  

(2.6)

We will use the Young inequality ($q \in (1, \infty)$, $0 \leq a, b, \delta < \infty$)

$$ab \leq \delta a^q + \frac{(q\delta)^{-q'}}{q'} b^{q'}, \frac{1}{q} + \frac{1}{q'} = 1.$$  

(2.7)

and the coerciveness inequality

$$\gamma |\xi_i - \eta_i|^{2+\sigma} \left\{|\xi_i| + |\eta_i|\right\}^{p_i - 2 - \sigma} \leq \left\{|\xi_i|^{p_i - 2} \xi_i - |\eta_i|^{p_i - 2} \eta_i\right\} \cdot (\xi_i - \eta_i).$$

(2.8)

which holds for some positive constant $\gamma$ and any nonnegative constant $\sigma$, for every $\xi_i, \eta_i, i = 1, ..., n$, (see Lemma 2.2, [5]).

**Theorem 2.1.** Under the conditions (2.4), (2.5), any weak solution $u \in W_0^{1, \vec{p}} (\Omega)$ to (2.1)(or (2.3)) is unique if at least one $p_i \leq 2$.

**Proof.** We define

$$F_\varepsilon(x) = \begin{cases} \int_{\varepsilon}^{x} \frac{ds}{\omega(s)} & x \geq \varepsilon, \\ 0 & x \leq \varepsilon. \end{cases}$$

(2.9)

Let $u, v$ be two solutions to (2.3). Set $w = u - v$. We take as test function in (2.3) $F_\varepsilon(w)$ (for $u, v$). We get by subtraction

$$\int_{\Omega_\varepsilon} a_i(x, u) \left\{|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u - |\partial_{x_i} v|^{p_i - 2} \partial_{x_i} v\right\} \frac{\partial_{x_i} w}{\omega^2} =$$

(2.10)

$$\int_{\Omega_\varepsilon} \left\{a_i(x, v) - a_i(x, u)\right\} |\partial_{x_i} v|^{p_i - 2} \partial_{x_i} v \frac{\partial_{x_i} w}{\omega^2},$$
where
\[ \Omega_\varepsilon = \{ x \in \Omega \mid w(x) > \varepsilon \}. \tag{2.11} \]
(In the above we have a summation in \(i\)). Using the ellipticity conditions (2.4), (2.5) and (2.8), we get with summation in \(i\) and for \(\mu = \gamma \lambda\).

\[ \mu \int_{\Omega_\varepsilon} \frac{|\partial_x w|^2}{\omega} \left( |\partial_x u| + |\partial_x v| \right)^{p_i - 2} \leq \int_{\Omega_\varepsilon} |\partial_x v|^{p_i - 1} \left| \frac{\partial_x w}{\omega} \right|. \tag{2.12} \]

· Suppose that \(p_i \geq 2\). Then

\[ I_i := \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right| |\partial_x v|^{p_i - 1} \leq \delta \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^2 |\partial_x v|^{p_i - 2} + C(\delta, p_i) \int_{\Omega_\varepsilon} |\partial_x v|^{p_i}, \tag{2.13} \]

for every \(\delta > 0\) by the Young inequality (2.7) with \(q = 2\).

· Suppose that \(p_i \leq 2\). Then

\[ I_i = \int_{\Omega_\varepsilon} |\partial_x v|^{p_i - 1} \left| \frac{\partial_x w}{\omega} \right| \leq \delta \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^{p_i} + C(\delta, p_i) \int_{\Omega_\varepsilon} |\partial_x v|^{p_i}, \tag{2.14} \]

for any \(\delta > 0\) we used (2.7) with \(q = p_i\). Moreover we have

\[ \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^{p_i} = \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^{p_i} \left( |\partial_x u| + |\partial_x v| \right)^{p_i - 2} \left( |\partial_x u| + |\partial_x v| \right)^{-\frac{p_i - 2}{2}} \left( |\partial_x u| + |\partial_x v| \right)^{-\frac{p_i - 2}{2}} \tag{2.15} \]

\[ \leq \delta \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^2 \left( |\partial_x u| + |\partial_x v| \right)^{p_i - 2} + C(\delta, p_i) \int_{\Omega_\varepsilon} \left( |\partial_x u| + |\partial_x v| \right)^{p_i} \]

for any \(\delta > 0\) (we used (2.7) with \(q = 2/p_i\) with a slight modification if \(p_i = 2\)).

Combining (2.14), (2.15), we come to the same estimate which was obtained for \(p_i \geq 4\).
2 (compare with (2.13))

\[ I_i \leq \delta \int_{\Omega_\varepsilon} \left| \frac{\partial x_i w}{\omega} \right|^2 (|\partial x_i u| + |\partial x_i v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial x_i u| + |\partial x_i v|)^{p_i}. \]  

(2.16)

Finally we get for every \( i \)

\[ I_i \leq \delta \int_{\Omega_\varepsilon} \left| \frac{\partial x_i w}{\omega} \right|^2 (|\partial x_i u| + |\partial x_i v|)^{p_i-2} + C(\delta, p_i) \int_{\Omega_\varepsilon} (|\partial x_i u| + |\partial x_i v|)^{p_i}, \]  

(2.17)

for every \( \delta > 0 \). We collect now (2.12)-(2.16) with a suitable choice of \( \delta > 0 \). Then we obtain

\[ \int_{\Omega_\varepsilon} \left| \frac{\partial x_i w}{\omega} \right|^2 (|\partial x_i u| + |\partial x_i v|)^{p_i-2} \leq C \int_{\Omega_\varepsilon} (|\partial x_i u| + |\partial x_i v|)^{p_i} \leq C', \]  

(2.18)

with some constant \( C' = C'(\delta, p_i) \) independent of \( \varepsilon \). If there is one \( p_i \) say \( p_k < 2 \), applying the Hölder inequality with \( q = 2/p_k, q' = 2/(2 - p_k) \) we obtain

\[ \int_{\Omega_\varepsilon} \left| \frac{\partial x_k w}{\omega} \right|^{p_k} = \int_{\Omega_\varepsilon} \left| \frac{\partial x_k w}{\omega} \right|^{p_k} (|\partial x_k u| + |\partial x_k v|) \frac{p_k - 2}{2} (|\partial x_k u| + |\partial x_k v|)^{2 - p_k} \]  

(2.19)

\[ \leq \left( \int_{\Omega_\varepsilon} \left| \frac{\partial x_k w}{\omega} \right|^2 (|\partial x_k u| + |\partial x_k v|)^{p_k - 2} \right)^{p_k \over 2} \left( \int_{\Omega_\varepsilon} (|\partial x_k u| + |\partial x_k v|)^{p_k} \right)^{2 - p_k \over 2}. \]

By (2.18), (2.19) we deduce

\[ \int_{\Omega_\varepsilon} \left| \frac{\partial x_k w}{\omega} \right|^{p_k} \leq C'. \]  

(2.20)

We introduce the function \( G_\epsilon \) defined by

\[ G_\epsilon(x) = \begin{cases} \int_\epsilon^x {ds \over \omega(s)} & x > \epsilon, \\ 0 & x \leq \epsilon, \end{cases} \]

which satisfies from above

\[ \int_{\Omega_\varepsilon} |\partial x_k G_\epsilon(w)|^{p_k} \leq C'. \]  

(2.21)
By the Poincaré inequality it follows that
\[
\int_{\Omega} |G_\epsilon(w)|^{pk} \leq C''.
\] (2.22)

If \( \{w = u - v > \epsilon\} \) has a positive measure, letting \( \epsilon \to 0 \), we obtain in accord to (2.6) that
\[
\int_{\Omega} |G_\epsilon(w)|^{pk} \to \infty,
\]
which contradicts the above inequality. Hence we have \( u \geq v \).

By changing the rôle of \( u \) and \( v \) this completes the proof. \( \square \)

**Remark 2.1.** With the same proof as above one could allow in (2.1) a monotone lower order term \( b(x, u) \) where \( u \mapsto b(x, u) \) is nonincreasing (see also below).

### 2.2. Uniqueness result when a lower order term exists and \( p_i > 2, \forall i \).

Let us consider the problem
\[
\begin{cases}
-\partial_{x_i} \left( a_i(x, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right) + b(x, u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma = \partial \Omega.
\end{cases}
\] (2.23)

That is to say \( u \in W^{1, \tilde{p}}(\Omega) \) is a weak solution in the sense that
\[
\int_{\Omega} a_i(x, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \partial_{x_i} v + b(x, u)v = < f, v > \quad \forall v \in W^{1, \tilde{p}}(\Omega).
\] (2.24)

Let us denote by \( m(t) \) the uniform modulus of continuity of the \( a_i \)'s i.e.
\[
m(t) = \max_i \sup_{x \in \Omega, |u-v| \leq t} |a_i(x, u) - a_i(x, v)|.
\] (2.25)

If the \( a_i(x, u) \) are Carathéodory functions uniformly continuous with respect to \( x \) we have
\[
m(t) \to 0 \quad \text{as } t \to 0.
\] (2.26)

Moreover \( m(t) \) is nondecreasing with \( t \). We denote by \( \omega \) the function
\[
\omega(t) = \min(m(t), 1).
\]

We assume that (2.4) holds and since the \( a_i \)'s are bounded for some constant \( C \) we have
\[
|a_i(x, u) - a_i(x, v)| \leq C \omega(|u - v|) \quad \forall i, \text{ a.e. } x \in \Omega, \forall u, v \in \mathbb{R}.
\] (2.27)

We suppose that there exists \( \alpha \) such that
\[
\int_{0^+} \frac{ds}{\omega^\alpha(s)} = +\infty, \quad 1 < \alpha \leq \frac{p_n}{p_n - 1} \leq \frac{p_i}{p_i - 1}, \quad \forall i.
\] (2.28)
Then, for every \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) \) such that
\[
\int_{\delta(\varepsilon)}^{\varepsilon} \frac{ds}{\omega_\alpha(s)} = 1. 
\]  
(2.29)

We then define \( F_\varepsilon(x) \) as
\[
F_\varepsilon(x) = \begin{cases} 
0 & x \leq \delta(\varepsilon), \\
\int_{\delta(\varepsilon)}^{x} \frac{ds}{\omega_\alpha(s)} \delta(\varepsilon) \leq x \leq \varepsilon, \\
1 \varepsilon \leq x. 
\end{cases} 
\]  
(2.30)

Clearly this function is Lipschitz continuous. We assume also
\[
u \rightarrow b(x,u) \text{ is increasing.} \]  
(2.31)

**Theorem 2.2.** Under the conditions (2.4), (2.27), (2.28), (2.31) the weak solution \( u \in W^{1, \tilde{p}}(\Omega) \) to (2.24) is unique.

**Proof.** Let \( u, v \) be two solutions to (2.23). Setting \( w = u - v \), we take as test function in (2.24) \( F_\varepsilon(w) \). Thus we have, dropping \( x \) in \( b(x,u) \)
\[
\int_{\Omega_\varepsilon} a_i(x,u) \left\{ |\partial_x u|^{p_i-2} \partial_x u - |\partial_x v|^{p_i-2} \partial_x v \right\} \partial_x w F_\varepsilon'(w) + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w) 
\]  
(2.32)
\[
= \int_{\Omega_\varepsilon} \left\{ a_i(x,v) - a_i(x,u) \right\} |\partial_x v|^{p_i-2} \partial_x v \partial_x w F_\varepsilon'(w). 
\]

The integration in the integral containing \( F_\varepsilon' \) is taking place only on the set
\[
\Omega_\varepsilon = \{ x \mid \delta(\varepsilon) < w(x) < \varepsilon \} 
\]  
(2.33)
when some derivative occurs. Using (2.4), (2.27), we derive for some constant \( \mu > 0 \)
\[
\mu \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_x w|^{p_i}}{\omega_\alpha} dx + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w) 
\]  
(2.34)
\[
\leq \sum_i \int_{\Omega_\varepsilon} |a_i(x,v) - a_i(x,u)| |\partial_x v|^{p_i-1} \frac{|\partial_x w|}{\omega_\alpha} 
\]  
\[
\leq C \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_x w|}{\omega^{(\alpha-1)}} |\partial_x v|^{p_i-1}. 
\]
Next we use the Young inequality (2.7) with \( q = p_i \) to evaluate the left hand side. From (2.34) we get

\[
\mu \sum_i \int_{\Omega_{\varepsilon}} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} \, dx + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w)
\]

\[
\leq \delta \sum_i \int_{\Omega_{\varepsilon}} \frac{|\partial_{x_i} w|^{p_i}}{\omega^{(\alpha-1)p_i}} + C \sum_i \int_{\Omega_{\varepsilon}} |\partial_{x_i} v|^{p_i}
\]

for any \( \delta > 0 \) and for some constant \( C = C(\delta, p_i) \). We know that \( \omega \leq 1 \) and thus

\[
\omega^{(\alpha-1)p_i} \geq \omega^\alpha
\]

since

\[
\alpha \geq (\alpha - 1)p_i \iff \alpha \leq \frac{p_i}{p_i - 1}.
\]

Then from (2.35) we obtain

\[
\mu \sum_i \int_{\Omega_{\varepsilon}} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} \, dx + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w)
\]

\[
\leq \delta \sum_i \int_{\Omega_{\varepsilon}} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} + C \sum_i \int_{\Omega_{\varepsilon}} |\partial_{x_i} v|^{p_i}.
\]

Let us choose \( \delta \) small enough such that \( \delta < \mu \), (\( \delta \) is now fixed). We obtain

\[
\int_{\Omega} (b(x, u) - b(x, v)) F_\varepsilon(w) \leq C' \sum_i \int_{\Omega_{\varepsilon}} |\partial_{x_i} v|^{p_i}
\]

for some constant \( C' \). Note that

\[
\chi_{\Omega_{\varepsilon}} \rightarrow 0 \text{ a.e., } F_\varepsilon(w) \rightarrow 1 \text{ on } w > 0.
\]

Passing to the limit we get

\[
\int_{u-v>0} (b(x, u) - b(x, v)) \leq 0,
\]
and thus $u - v \leq 0$ if $b$ is monotone increasing in $u$. Exchanging the rôle of $u, v$, we get $u = v$ provided $\alpha$ is chosen such that

$$\alpha \leq \frac{p_i}{p_i - 1} \forall i.$$  

□

Remark 2.2. If

$$\beta = \inf_i \frac{p_i}{p_i - 1} = \frac{p_n}{p_n - 1} > 1$$

one can take $\alpha = \beta$ and (2.28) holds for

$$m(t) \leq Ct^{\frac{1}{\beta}}$$

i.e. for $a_i$’s Hölder continuous in $u$ of exponent $\frac{1}{\beta}$.

2.3. Existence of solutions. In this section we discuss the existence of weak solutions for the problem (2.24). We do not consider this question under the weakest possible assumptions, our aim in this note being uniqueness.

To prove the desired existence results we follow [1, 2, 3], in which the Dirichlet problem was studied.

Considering the equation

$$\int_{\Omega} a_i(x, u)|\partial x_i u|^{p_i - 2} \partial x_i u \partial x_i v + b(x, u)v =< f, v > \forall v \in W_0^{1, \vec{p}} (\Omega),$$  

we assume that:

$$a_i, b : \Omega \times \mathbb{R} \to \mathbb{R} \text{ are Carathéodory functions},$$  

and that

$$0 < \lambda_i \leq a_i(x, u) \leq \Lambda < \infty, \forall i, \text{a.e. } x \in \Omega, \forall u \in \mathbb{R},$$  

(2.39)

$$|b(x, u)| \leq C_0 |u|^{\beta - 1} + h(x), \quad 0 \leq h(x) \in L^{\frac{\beta}{\beta - 1}}(\Omega), 1 \leq \beta \leq p_n,$$  

(2.40)

(recall that $p_n$ is the largest $p_i$)

$$f(x) \in W_0^{-1, \vec{p}' / \beta'}(\Omega).$$  

(2.41)
Theorem 2.3. Under the conditions (2.2), (2.38)-(2.41) the problem (2.37) has at least one weak solution \( u \in W^1_0, \vec{p}^\prime (\Omega) \).

Proof. Let \( v(x) \) be any given function such that \( v(x) \in L^{p_n} (\Omega) \). We define functions \( A_i \) by

\[
A_i (x) = a_i (x, v(x)) , \quad \forall i .
\]

We then introduce the operator \( \Lambda : W^1_0, \vec{p}^\prime (\Omega) \rightarrow W^{-1}_0, \vec{p}^\prime (\Omega) , \)

\[
(\Lambda u, \varphi) = \int_\Omega A_i (x) |\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \partial_{x_i} \varphi + b(x, u) \varphi .
\]

It is obvious that the mapping \( \Lambda : W^1_0, \vec{p}^\prime (\Omega) \rightarrow W^{-1}_0, \vec{p}^\prime (\Omega) \) is continuous and monotone. Let us verify that it is coercive. Using (2.4), (2.40) we get

\[
\lambda \sum_i \int_\Omega |\partial_{x_i} u|^{p_i} - \int_\Omega \left( C_0 |u|^\beta + h |u| \right) \leq (\Lambda u, u) .
\]

(2.42)

Next applying the Young, Hölder and Poincaré inequalities, we obtain

\[
\int_\Omega \left( C_0 |u|^\beta + h |u| \right) \leq C' \int_\Omega \left( |u|^\beta + h^{\beta / (\beta - 1)} \right) \)

\[
\leq C'' \left( \left( \int_\Omega |\partial_{x_n} u|^{p_n} \right)^{\beta / p_n} + \int_\Omega h^{\beta / (\beta - 1)} \right) \leq \delta \int_\Omega |\partial_{x_n} u|^{p_n} + C''' \left( 1 + \int_\Omega h^{\beta / (\beta - 1)} \right) .
\]

(2.43)

Combining (2.42), (2.43) with \( 2\delta = \lambda \), we come to

\[
\frac{\lambda}{2} \sum_i \int_\Omega |\partial_{x_i} u|^{p_i} - C \left( 1 + \int_\Omega h^{\beta / (\beta - 1)} \right) \leq (\Lambda u, u) ,
\]

(2.44)

which guarantee that the operator \( \Lambda \) is coercive. The space \( W^1_0, \vec{p}^\prime (\Omega) \) is reflexive. By the Browder-Minty theorem ([14], Theorem 7.3.2) the equation

\[
\Lambda u = f
\]

(2.45)
has at least one weak solution \( u \in W_0^{1, \bar{p}}(\Omega) \) for every \( f \in W_0^{-1, \bar{p}'}(\Omega) \). Thus the mapping

\[
\Phi : v \mapsto u,
\]

defines an operator from \( L^{p_n}(\Omega) \) into itself. Moreover if \( u \) is solution to (2.45) by (2.44) we have

\[
\frac{\lambda}{2} \sum_i \|\partial_x u\|_{L^{p_i}(\Omega)}^{p_i} \leq C \left( 1 + \int_{\Omega} h^\beta \right) + \|f\|_{W_0^{-1, \bar{p}'}(\Omega)} \sum_i \|\partial_x u\|_{L^{p_i}(\Omega)}.
\]

(2.47)

where \( \|f\|_{W_0^{-1, \bar{p}'}(\Omega)} \) denotes the strong dual norm of \( f \). It follows by Young’s inequality that

\[
\frac{\lambda}{2} \sum_i \|\partial_x u\|_{L^{p_i}(\Omega)}^{p_i} \leq \delta \sum_i \|\partial_x u\|_{L^{p_i}(\Omega)}^{p_i} + C(h, f).
\]

(2.48)

Choosing \( \delta = \frac{\lambda}{4} \) we derive that if \( u \) is solution to (2.45)

\[
\frac{\lambda}{2} \sum_i \|\partial_x u\|_{L^{p_i}(\Omega)}^{p_i} \leq C(\lambda, h, f).
\]

(2.49)

Due to the Poincaré inequality

\[
\|u\|_{L^{p_n}(\Omega)} \leq C \|\partial_x u\|_{L^{p_n}(\Omega)}
\]

(2.50)

we obtain for some new constant

\[
\|u\|_{L^{p_n}(\Omega)} \leq C(\lambda, h, f) = R.
\]

(2.51)

It is easy to verify that the operator \( \Phi \) is compact and continuous and transforms the ball \( \{ u \in L^{p_n}(\Omega) \mid \|u\|_{L^{p_n}(\Omega)} \leq R \} \) into itself. According to the Schauder fixed point theorem the operator \( \Phi \) has at least one fixed point, which defines a weak solution of problem (2.37). This completes the proof of the theorem. \( \Box \)
2.4. Generalizations. Let us consider more generally the problem
\[
\begin{cases}
-\partial_{x_i} (a_i(x, u, \nabla u)) + b(x, u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma = \partial \Omega.
\end{cases}
\] (2.52)

A solution \( u \) is an element \( u \in W^{1, \bar{p}}_0(\Omega) \) such that
\[
\int_{\Omega} a_i(x, u, \nabla u) \partial_{x_i} v + \int_{\Omega} b(x, u)v = f, \forall v \in W^{1, \bar{p}}_0(\Omega).
\] (2.53)

Assume that
\[
a_i(x, u, \nabla u) \in L^{p_i}(\Omega), \ b(x, u) \in W^{-1, \bar{p}}(\Omega), \ \forall u \in W^{1, \bar{p}}_0(\Omega); \ f \in W^{-1, \bar{p}}(\Omega).
\] (2.54)

First we consider the case
\[
2 < p_i < \infty, \ i = 1, ..., n.
\] (2.55)

We assume that for some positive constant \( \mu > 0, \ a.e. \ x \in \Omega, \forall u, v \in \mathbb{R} \ \forall \eta, \xi \in \mathbb{R}^n \) we have
\[
\mu |\eta - \xi|^p - \omega(|u - v|) (|\eta| + |\xi|)^{p-1} |\eta - \xi|
\] (2.56)

where the function \( \omega(s) \), \( \alpha \) satisfy (2.28), (2.36). Let \( u, v \) be two solutions of (2.52) and \( w = u - v \). Then taking as test function in (2.53) \( F_\varepsilon(w) \) defined by (2.30), we come to an analog to (2.32). Indeed we have
\[
\int_{\Omega_\varepsilon} \{a_i(x, u, \nabla u) - a_i(x, v, \nabla v)\} \partial_{x_i} w F'_\varepsilon(w) + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w) = 0.
\] (2.57)

Using (2.56), we get
\[
\mu \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|^{p_i}}{\omega^\alpha} dx + \int_{\Omega} (b(u) - b(v)) F_\varepsilon(w) \leq \sum_i \int_{\Omega_\varepsilon} \frac{|\partial_{x_i} w|}{\omega^{\alpha-1}} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-1},
\] (2.58)

(compare with (2.34)). Next repeating the arguments of the proof of the Theorem 2.2 we have

**Theorem 2.4.** Under the conditions \((2.54)-(2.56)\), there exists at most one solution
\( u \in W^{1, \bar{p}}_0(\Omega) \) to (2.53).

Now we consider the case when at least one \( p_i \) is less or equal to 2. We assume that for some positive constant \( \mu > 0, \ a.e. \ x \in \Omega, \forall u, v \in \mathbb{R} \ \forall \eta, \xi \in \mathbb{R}^n \)
\[
\mu |\eta - \xi|^2 (|\eta| + |\xi|)^{p_i-2} - \omega(|u - v|) (|\eta| + |\xi|)^{p_i-1} |\eta - \xi|
\] (2.59)

where the function \( \omega(s) \) defined by (2.9). Repeating the arguments of the Theorem 2.1, we obtain
Theorem 2.5. Under the conditions (2.54), (2.59) there exists at most one solution $u \in W_0^{1, \vec{p}}(\Omega)$ to (2.53) if at least one $p_i$ is less or equal to 2.

Remark 2.3. The theorems 2.1-2.5 remain valid for mixed boundary conditions

$u = 0$ on $\Gamma_D$, $a_i(x, u, \nabla u)\nu_i = 0$ on $\Gamma_N$, \hspace{1cm} (2.60)

where $\vec{\nu} = (\nu_1, ..., \nu_1)$ is the unit normal vector to $\Gamma_N$, $\Gamma = \partial \Omega = \Gamma_D \cup \Gamma_N$ and $\text{mes} \, \Gamma_D > 0$. In this case we consider weak solutions $u \in W_0^{1, \vec{p}}(\Omega, \Gamma_D)$ where

$W_0^{1, \vec{p}}(\Omega, \Gamma_D) = \{ u \in W_0^{1, p_1}(\Omega, \Gamma_D) : \partial_{x_i} u \in L^{p_i}(\Omega), \ i = 1, ..., n \} \, .

3. PARABOLIC EQUATIONS

3.1. Uniqueness of solution. We consider the problem

$$
\begin{cases}
\partial_t u = \partial_{x_i} \left( a_i(x, t, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right) + f & \text{in } Q_T = \Omega \times (0, T), \\
u = 0 & \text{on } \Gamma_T = \partial \Omega \times (0, T), \\
\partial_x u(x, 0) = u_0 & \text{on } x \in \Omega.
\end{cases}
$$

We define

$V_0^{\vec{p}}(Q_T) = L^\infty(0, T; L^2(\Omega)) \cap L^{\vec{p}}(0, T; W_0^{1, \vec{p}}(\Omega))$

and we equip it with the norm

$$
\|u\|_{V_0^{\vec{p}}(Q_T)} = \sup_{0 \leq t \leq T} \|u\|_{L^2(\Omega)} + \sum_{i=1}^n \left( \int_0^T \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \right)^{\frac{1}{p_i}}.
$$

(Recall that $W_0^{1, \vec{p}}(\Omega)$ was defined in the section (2.1).) By $V_0^{-1, \vec{p}}(Q_T)$ we denote the dual space of $V_0^{1, \vec{p}}(Q_T)$. We define

$V(Q_T) = \{ u \in V_0^{1, \vec{p}}(Q_T), \ u_t \in V_0^{-1, \vec{p}}(Q_T) \} \, .

By a solution to (3.1) we mean a $u \in V(Q_T)$ such that

$$(u_t, \varphi)_{Q_T} + \sum_{i=1}^n \left( a_i(x, t, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u, \partial_{x_i} \varphi \right)_{Q_T} = (f, \varphi)_{Q_T}, \ u(x, 0) = u_0, \hspace{1cm} (3.2)$$

$$\forall \varphi \in V_0^{\vec{p}}(Q_T), \ a.e. \ t \in (0, T), \ (u, v)_{Q_t} = \int_{Q_t} uv.$$
Note that, according to (3.2), \( u \) satisfies \( u_t \in V_{0}^{-1, \varphi}(\Omega) \) and
\[
\left| (u_t, \varphi)_{Q_T} \right| \leq C \left\| \varphi \right\|_{V_{0}^{\varphi}(Q_T)} \left( \left\| u \right\|_{V_{0}^{\varphi}(Q_T)} + \left\| f \right\|_{V_{0}^{-1, \varphi}(Q_T)} \right).
\] (3.3)

We suppose that
\[
0 < \lambda \leq \lambda_i \leq a_i(x, t, u) \leq \Lambda \ \forall i, \forall u \in \mathbb{R}, \text{ a.e. } (x, t) \in \Omega \times (0, T). \tag{3.4}
\]

Denote by \( m(t) \) the uniform modulus of continuity of the \( a_i \) in \( u \) i.e. set
\[
m(s) = \sup_{i, x, t, \left| u - v \right| \leq s} \left| a_i(x, t, u) - a_i(x, t, v) \right|.
\] (3.5)

Modifying eventually \( m(t) \) for \( t \geq 1 \) one can find a function \( \omega \) which coincides with \( m(t) \) for \( t \leq 1 \) and which satisfies
\[
\int_1^\infty \frac{ds}{\omega^2(s)} < +\infty, \tag{3.6}
\]
\[
\forall i, \forall u, v \in \mathbb{R}, \text{ a.e. } (x, t) \in \Omega \times (0, T).
\]

Suppose that
\[
\int_{0^+} \frac{ds}{\omega^2(s)} = +\infty. \tag{3.7}
\]

Set
\[
F_\varepsilon(x) = \begin{cases} \int_{\varepsilon}^x \frac{ds}{\omega^2(s)} & x \geq \varepsilon, \\
0 & x \leq \varepsilon, \end{cases}
\]
\[
I_\varepsilon = \int_{\varepsilon}^\infty \frac{ds}{\omega^2(s)}, \quad H_\varepsilon(x) = \frac{F_\varepsilon(x)}{I_\varepsilon}. \tag{3.8}
\]

Since for \( x \geq \varepsilon \) one has
\[
\frac{H_\varepsilon(x)}{I_\varepsilon} = \frac{\int_{\varepsilon}^x \frac{ds}{\omega^2(s)}}{I_\varepsilon} = \frac{I_\varepsilon - \int_{\varepsilon}^\infty \frac{ds}{\omega^2(s)}}{I_\varepsilon}. \tag{3.9}
\]

It is clear that \( I_\varepsilon \to +\infty \) as \( \varepsilon \to 0 \) and thus
\[
H_\varepsilon(x) \to H(x) \text{ the Heaviside function.} \tag{3.10}
\]

**Theorem 3.1.** Let \( u, v \) be two solutions of (3.1) corresponding to initial values \( u_0, v_0 \). Under the conditions (2.2), (3.4), (3.6), (3.7), we have
\[
\int_{\Omega} (u - v)^+ dx \leq \int_{\Omega} (u_0 - v_0)^+ dx, \quad (u)^+ = \max(u, 0), \tag{3.11}
\]
which implies in particular uniqueness of the solution to (3.1).
Proof. Let \( u, v \) be two solutions of (3.1) corresponding to initial values \( u_0, v_0 \). Set \( w = u - v \). By difference of the equations (3.2) for \( u \) and \( v \) we have

\[
(w_t, \varphi)_Q + \sum_{i=1}^n \left( a_i(x, t, u) \left\{ |\partial_x u|^{p_i-2} \partial_x u - |\partial_x v|^{p_i-2} \partial_x v \right\} , \partial_x \varphi \right)_Q = \sum_{i=1}^n \left( \{ a_i(x, t, v) - a_i(x, t, u) \} |\partial_x v|^{p_i-2} \partial_x v , \partial_x \varphi \right)_Q . \tag{3.13}
\]

We take as test function in (3.13)

\[
H_\varepsilon(w) = \frac{F_\varepsilon(w)}{I_\varepsilon}. \tag{3.14}
\]

Then we obtain

\[
\int_0^t \int_{\Omega_\varepsilon} \partial_t w \ H_\varepsilon(w) = - \frac{1}{I_\varepsilon} \int_0^t \int_{\Omega_\varepsilon} a_i(x, s, u) \left\{ |\partial_x u|^{p_i-2} \partial_x u - |\partial_x v|^{p_i-2} \partial_x v \right\} \partial_x F_\varepsilon(w) \tag{3.15}
\]

where \( \Omega_\varepsilon \) is the set defined by

\[
\Omega_\varepsilon = \left\{ x \in \Omega \mid (u - v)(x, s) > \varepsilon \right\} . \tag{3.16}
\]

(outside of \( \Omega_\varepsilon \), \( F_\varepsilon(w) \) vanishes, see (3.8)). Using (3.4) and the definition of \( F_\varepsilon \) we have

for some \( \mu > 0 \)

\[
\mu \int_{\Omega_\varepsilon} \left| \frac{\partial_x w}{\omega} \right|^2 (|\partial_x u| + |\partial_x v|)^{p_i-2} \tag{3.17}
\]

and

\[
\leq \int_{\Omega_\varepsilon} a_i(x, s, u) \left\{ |\partial_x u|^{p_i-2} \partial_x u - |\partial_x v|^{p_i-2} \partial_x v \right\} \frac{\partial_x w}{\omega^2} \tag{3.18}
\]

By (3.6) we have also

\[
\left| \int_{\Omega_\varepsilon} \{ a_i(x, s, v) - a_i(x, s, u) \} |\partial_x v|^{p_i-2} \partial_x v \partial_x F_\varepsilon(w) \right| \tag{3.19}
\]
\[ \leq \int_{\Omega_{\varepsilon}} \left| a_i(x, s, v) - a_i(x, s, u) \right| \left| \partial_{x_i} v \right|^{p_i-1} \frac{\left| \partial_{x_i} w \right|}{\omega^2} \]

\[ \leq \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} v \right|^{p_i-1} \frac{\left| \partial_{x_i} w \right|}{\omega}. \]

From (3.14) we then derive
\[ \int_0^t \int_{\Omega_{\varepsilon}} \partial_t w \ H_{\varepsilon}(w) \leq -\frac{\mu}{\varepsilon} \int_0^t \sum_i \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right| \left( \left| \partial_{x_i} u \right| + \left| \partial_{x_i} v \right| \right)^{p_i-2} \tag{3.18} \]

\[ + \frac{1}{\varepsilon} \int_0^t \sum_i \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} v \right|^{p_i-1} \frac{\left| \partial_{x_i} w \right|}{\omega}. \]

(In the formulae before we did not write the summation in \( i \)). Then we proceed with the same estimates than in the elliptic case.

· Suppose that \( p_i < 2 \). Combining the estimates (2.12), (2.15) of the elliptic case we have

\[ \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right| \left| \partial_{x_i} v \right|^{p_i-1} \leq \delta \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right| + C(\delta, p_i) \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} v \right|^{p_i} \]

\[ \leq \delta^2 \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right|^2 \left( \left| \partial_{x_i} u \right| + \left| \partial_{x_i} v \right| \right)^{p_i-2} + C(\delta, p_i) \int_{\Omega_{\varepsilon}} \left( \left| \partial_{x_i} u \right| + \left| \partial_{x_i} v \right| \right)^{p_i}. \tag{3.19} \]

· Suppose that \( p_i \geq 2 \). From the estimate (2.13) of the elliptic case we have

\[ \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right| \left| \partial_{x_i} v \right|^{p_i-1} \leq \delta \int_{\Omega_{\varepsilon}} \left| \partial_{x_i} w \right|^2 \left( \left| \partial_{x_i} u \right| + \left| \partial_{x_i} v \right| \right)^{p_i-2} + C(\delta, p_i) \int_{\Omega_{\varepsilon}} \left( \left| \partial_{x_i} u \right| + \left| \partial_{x_i} v \right| \right)^{p_i}. \tag{3.20} \]

Choosing and fixing \( \delta \) such that

\[ \delta \text{ and } \delta^2 < \mu \]

in (3.19), (3.20) it is clear that from (3.18)-(3.20) we get for some constant \( C = C(\delta, p_i) \).
\[ \int_0^t \int_{\Omega_{\varepsilon}} \partial_t w H_{\varepsilon}(w) \leq \frac{C}{I_{\varepsilon}} \int_0^t \sum_i \int_{\Omega_{\varepsilon}} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}. \] (3.21)

Let us set
\[ G_{\varepsilon}(x) = \int_{-\infty}^{x} H_{\varepsilon}(t)dt. \]

We have
\[ \partial_t w H_{\varepsilon}(w) = \partial_t G_{\varepsilon}(w) \]
and the above inequality gives us
\[ \int_{\Omega_{\varepsilon}} G_{\varepsilon}(w)(t) - \int_{\Omega_{\varepsilon}} G_{\varepsilon}(w)(0) \leq \frac{C}{I_{\varepsilon}} \int_0^t \sum_i \int_{\Omega_{\varepsilon}} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}. \]

Letting \( \varepsilon \to 0 \), we get (3.12) since
\[ G_{\varepsilon}(w) \to (w)^+, \ I_{\varepsilon} \to +\infty. \]

\[ \square \]

**Remark 3.1.** The condition (3.7) allows the \( a_i(x, t, u) \) to be Hölder continuous with Hölder exponent greater or equal to 1/2.

### 3.2. Existence of solutions.

In this section we discuss the existence of weak solutions \( u \in V(Q_T) \) for the problem (3.1). We show here only a sketch of the proof for a simple case.

Considering the problem
\[ (u_t, \varphi)_{Q_t} + \sum_{i=1}^{n} (a_i(x, t, u) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u, \partial_{x_i} \varphi)_{Q_t} = (f, \varphi)_{Q_t}, \ u(x, 0) = u_0, \] (3.22)
we assume that:
\[ a_i, b : Q_T \times \mathbb{R} \to \mathbb{R} \text{ are Carathéodory functions} \] (3.23)
such that a.e. \( (x, t) \in Q_T, \forall u \in \mathbb{R}, \)
\[ 0 < \lambda \leq \lambda_i \leq a_i(x, t, u) \leq \Lambda < \infty, \ \forall i. \] (3.24)
We suppose
\[ f(x, t) \in V_0^{-1}, \tilde{p}'(Q_T). \]  
Then we have

**Theorem 3.2.** Under the conditions (3.23)-(3.25) the problem (3.22) has at least one weak solution \( u \in V_0^\tilde{p}(Q_T) \).

**Proof.** Let \( v(x, t) \) be a function such that \( v(x, t) \in L^{p_n}(Q_T) \). We set
\[ A_i(x, t) = a_i(x, t, v(x, t)), \forall i. \]
Let us consider the problem
\[ u_t - \Lambda(u) = f, \quad u(x, 0) = u_0, \]  
\[ \Lambda(u) = \sum_{i=1}^{n} \partial_{x_i} \left( A_i(x, t) |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right). \]
It is easy to verify that the operator \( \Lambda : V_0^{1, \tilde{p}}(Q_T) \mapsto V_0^{-1, \tilde{p}'}(Q_T) \) is continuous, monotone and coercive. Thus according to ([23], Ch2.), the problem (3.26) has a unique solution which defines an operator
\[ \Phi : v \mapsto u. \]
According to (3.24) for all solution \( u \) the following estimate
\[ \|u\|_{V_0^{1, \tilde{p}'}(Q_T)} = \sup_{0 \leq t \leq T} \|u\|_{L^2(\Omega)} + \sum_{i=1}^{n} \left( \int_{0}^{T} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}^{p_i} \right)^{\frac{1}{p_i}} \]
\[ \leq C \left( \|u_0\|_{L^2(\Omega)} + \|f\|_{V_0^{-1, \tilde{p}'}(Q_T)} \right) = C' \]
holds. It is easy to verify that the operator \( \Phi \) is compact and continuous from \( L^{p_n}(Q_T) \) into itself and transforms, for some \( R \) large enough, the ball of center 0 and radius \( R \) in \( L^{p_n}(Q_T) \) into itself. According to the Schauder fixed point theorem the operator \( \Phi \) has
at least one fixed point, which is a weak solution of problem (3.22). The theorem is proved.

3.3. Generalizations. Let us consider more generally \( u \in V(Q_T) \) solution to

\[
(u_t, \varphi)_{Q_T} + \sum_{i=1}^{n} (a_i(x, t, u, \nabla u), \partial_{x_i} \varphi)_{Q_T} = (f, \varphi)_{Q_T}, \ u(x, 0) = u_0. \quad (3.28)
\]

Assume that

\[
a_i(x, u, t, \nabla u) \in L^{p_i'}(Q_T), \ \forall u \in V_0^{1, p}(Q_T); \ f \in V_0^{-1, p}(Q_T). \quad (3.29)
\]

We address here the case

\[
1 < p_i < \infty, \ i = 1, \ldots, n. \quad (3.30)
\]

We suppose that for some positive constant \( \mu > 0 \), a.e. \( x \in \Omega, \forall t, u, v \in \mathbb{R} \), \( \xi \in \mathbb{R}^n \)

\[
\mu |\eta_i - \xi_i|^2 (|\eta_i| + |\xi_i|)^{p_i-2} - \omega(|u - v|) (|\eta_i| + |\xi_i|)^{p_i-1} |\eta_i - \xi_i| \leq (a_i(x, t, u, \xi) - a_i(x, t, v, \xi), \eta_i - \xi_i),
\]

where the function \( \omega(s) \) satisfies (3.7). Let \( u, v \) be two solutions of (3.28) and \( w = u - v \). Then taking as test function in (3.28) \( H_\varepsilon(w) \) for \( u \) and \( v \), we come to an analog of (3.14)

\[
\int_0^t \int_{\Omega_\varepsilon} \partial_t w \ H_\varepsilon(w) = -\frac{1}{I_\varepsilon} \int_0^t \int_{\Omega_\varepsilon} \sum_{i=1}^{n} \{a_i(x, t, u, \nabla u) - a_i(x, t, v, \nabla v)\} \partial_{x_i} w \ F'_\varepsilon(w) \quad (3.32)
\]

Using (3.31), we get

\[
\sum_{i=1}^{n} \left( \mu \frac{\partial_{x_i} w}{\omega^2} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} - \frac{|\partial_{x_i} w|}{\omega} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-1} \right) \leq \sum_{i=1}^{n} \{a_i(x, t, u, \nabla u) - a_i(x, t, v, \nabla v)\} \partial_{x_i} w \ F'_\varepsilon(w). \quad (3.33)
\]

Thus we arrive to an inequality like (3.18). Next repeating the arguments of the proof of the Theorem 3.1 we obtain

**Theorem 3.3.** Let \( u, v \) be two solutions of (3.28) corresponding to the initial values \( u_0, v_0 \). Under conditions (3.29)-(3.31) we have

\[
\int_{\Omega} (u - v)^+ dx \leq \int_{\Omega} (u_0 - v_0)^+ dx, \ (u)^+ = \max(u, 0), \quad (3.34)
\]

which implies in particular uniqueness of the solution (3.28).
**Remark 3.2.** Assertions of Theorems 3.2, 3.3 remain valid for mixed boundary conditions

\[ u = 0 \text{ on } \Gamma_D, \quad a_i(x, t, u, \nabla u)\nu_i = 0 \text{ on } \Gamma_N, \]  

(3.35)

where \( \vec{\nu} = (\nu_1, \ldots, \nu_N) \) is the unit normal vector to \( \Gamma_N \), \( \Gamma = \partial \Omega = \Gamma_D \cup \Gamma_N \) and \( \text{mes } \Gamma_D > 0 \).

**References**


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