

# ON THE ASYMPTOTIC BEHAVIOR OF VARIATIONAL INEQUALITIES SET IN CYLINDERS

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ABSTRACT. We study the asymptotic behavior of solutions to variational inequalities with pointwise constraint on the value and gradient of the functions as the domain becomes unbounded. First, as a model problem, we consider the case when the constraint is only on the value of the functions. Then we consider the more general case of constraint also on the gradient. At the end we consider the case when there is no force term which corresponds to Saint-Venant principle for linear problems.

## 1. INTRODUCTION

For  $x \in \mathbb{R}^n$  we denote by  $X_1$  the  $p$  first coordinates of  $x$  and by  $X_2$  the  $n - p$  last ones ( $n \geq 2, p \geq 1$ ) i.e. we set

$$x = (X_1, X_2) \text{ with } X_1 = (x_1, \dots, x_p), X_2 = (x_{p+1}, \dots, x_n).$$

Let us then consider  $\omega_1$  and  $\omega_2$  two bounded open subsets of  $\mathbb{R}^p$  and  $\mathbb{R}^{n-p}$  respectively such that

$$(1.1) \quad \omega_1 \text{ is convex containing } 0.$$

For  $\ell > 0$  let us set

$$\Omega_\ell = \ell\omega_1 \times \omega_2$$

so  $\Omega_\ell$  is a generalized cylinder. For instance when  $\omega_1 = \omega_2 = (-1, 1)$  then  $\Omega_\ell$  is the rectangle  $(-\ell, \ell) \times (-1, 1)$  and  $\omega_2$  is the section of the cylinder.

For each  $X_2 \in \omega_2$  let  $K(X_2)$  be a closed convex subset of  $\mathbb{R}^{1+n}$  and let  $g \in H_0^1(\omega_2)$ , then we define

$$\mathcal{K}_\ell = \left\{ v \in H^1(\Omega_\ell) \mid v = g \text{ on } \partial\Omega_\ell, (v, \nabla v)(X_1, X_2) \in K(X_2) \text{ a.e. in } \Omega_\ell \right\}.$$

This is a closed convex subset of  $H^1(\Omega_\ell)$ .

Now for  $f \in L^2(\omega_2)$  let us consider  $u_\ell$  the solution of the following variational inequality

$$(1.2) \quad \begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(v - u_\ell) dx \geq \int_{\Omega_\ell} f(X_2)(v - u_\ell) dx, \forall v \in \mathcal{K}_\ell. \end{cases}$$

(For existence and uniqueness of the solution to this problem we refer for instance to [KS], [LS]). The goal of this note is to analyse the asymptotic behavior of  $u_\ell$  when  $\ell$  goes to plus infinity under different assumptions on  $K$ ,  $g$  and  $f$ . The asymptotic behavior of problems set in domains becoming unbounded has been addressed by

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*Date:* September 10, 2011.

*2010 Mathematics Subject Classification.* Primary 35B40; Secondary 35J86, 35B09.

*Key words and phrases.* Variational inequality, Asymptotic behavior.

many authors. One could quote [BG], [C], [CM], [CR1], [CR1], [CR2], [CX1], [CX2], [CY], [HK], [HP], [HW], [K], [OY], [T], [KY].

In the following we introduce the expected limit  $u_\infty$  of our solution. Let us define

$$\mathcal{K}_\infty = \left\{ v \in H_0^1(\omega_2) \mid (v, 0, \nabla_{X_2} v)(X_2) \in K(X_2) \text{ a.e. in } \omega_2 \right\}.$$

In the above, one has denoted by  $\nabla_{X_2}$  the gradient in  $X_2$  i.e.

$$\nabla_{X_2} = (\partial_{x_{p+1}}, \dots, \partial_{x_n}).$$

The set  $\mathcal{K}_\infty$  is a closed convex subset of  $H_0^1(\omega_2)$ . Then  $u_\infty$  is the solution of the variational inequality

$$(1.3) \quad \begin{cases} u_\infty \in \mathcal{K}_\infty, \\ \int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} (v - u_\infty) dX_2 \geq \int_{\omega_2} f(X_2)(v - u_\infty) dX_2, \forall v \in \mathcal{K}_\infty. \end{cases}$$

In section 2 as a model problem we consider the case when  $g = 0$  and the constraint is only on the value of the functions, i.e.  $K(X_2) = K_0(X_2) \times \mathbb{R}^n$  where  $K_0(X_2)$  is a closed real interval. In this case we show that the solution in the middle of the cylinder converges exponentially fast towards  $u_\infty$ .

In section 3 we consider the case when  $g = 0$ ,  $f \geq 0$  and first the case when  $K(X_2) = \mathbb{R} \times \overline{B_n(0, 1)}$  where  $B_n(0, 1)$  is the unit euclidean open ball in  $\mathbb{R}^n$ , in this case  $\mathcal{K}_\ell$  consists of the functions in  $H_0^1(\Omega_\ell)$  such that the euclidean length of their gradient is less than or equal to 1. We show that  $u_\ell$  converges again to  $u_\infty$ , but the rate of convergence is not known. Then we generalize our result to general  $K(X_2)$  with mild assumptions.

In section 4 we consider the case when  $f = 0$  and  $K(X_2) = \mathbb{R} \times K'$  and show that the solution converges exponentially fast towards 0 in the middle of the cylinder.

## 2. A MODEL PROBLEM

As mentioned above for  $X_2 \in \omega_2$  let us denote by  $K_0(X_2)$  a closed real interval such that  $0 \in K_0(X_2)$ . Then rewriting  $\mathcal{K}_\ell$  and  $\mathcal{K}_\infty$  when  $g = 0$ ,  $K(X_2) = K_0(X_2) \times \mathbb{R}^n$  we have

$$\mathcal{K}_\ell = \left\{ v \in H_0^1(\Omega_\ell) \mid v(x) \in K_0(X_2) \text{ a.e. in } \Omega_\ell \right\}$$

and

$$\mathcal{K}_\infty = \left\{ v \in H_0^1(\omega_2) \mid v(X_2) \in K_0(X_2) \text{ a.e. in } \omega_2 \right\}.$$

Then, if  $u_\ell$  and  $u_\infty$  denote respectively the solutions of the problems (1.2) and (1.3), we have

**Theorem 1.** *There exist positive constants  $C$  and  $\alpha$  independent of  $\ell$  such that*

$$(2.1) \quad \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq C e^{-\alpha \ell} |f|_{2, \omega_2}^2$$

where  $|f|_{2, \omega_2}^2 = \int_{\omega_2} f^2(X_2) dX_2$ .

*Proof.* We use the technique developed in [CY].

Let  $0 < \ell_1 \leq \ell - 1$  and  $\rho$  be a smooth function in  $X_1$  such that

$$(2.2) \quad 0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } \ell_1 \omega_1, \quad \rho = 0 \text{ on } \mathbb{R}^p \setminus (\ell_1 + 1) \omega_1, \quad |\nabla_{X_1} \rho| \leq c$$

where  $c$  is some constant independent of  $\ell_1$ ,  $\nabla_{X_1} = (\partial_{x_1}, \dots, \partial_{x_p})$  is the gradient in  $X_1$ ,  $|\cdot|$  the euclidean norm in  $\mathbb{R}^p$ . Then one has clearly

$$(2.3) \quad u_\ell - \rho^2(u_\ell - u_\infty) \in \mathcal{K}_\ell$$

from which we derive by (1.2)

$$(2.4) \quad \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(-\rho^2(u_\ell - u_\infty)) dx \geq - \int_{\Omega_\ell} f \rho^2(u_\ell - u_\infty) dx.$$

□

Similarly for almost every  $X_1 \in \ell\omega_1$  one has

$$u_\infty + \rho^2(X_1)(u_\ell(X_1, \cdot) - u_\infty) \in \mathcal{K}_\infty$$

and by (1.3) for a.e.  $X_1$

$$\int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2}(\rho^2(u_\ell - u_\infty)) dX_2 \geq \int_{\omega_2} f \rho^2(u_\ell - u_\infty) dX_2.$$

Integrating this inequality with respect to  $X_1$  and using the fact that  $u_\infty$  is independent of  $X_1$  we get

$$(2.5) \quad \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla(\rho^2(u_\ell - u_\infty)) dx \geq \int_{\Omega_\ell} f \rho^2(u_\ell - u_\infty) dx.$$

Adding (2.4) and (2.5) it comes

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla(\rho^2(u_\ell - u_\infty)) dx \leq 0$$

which can also be written as

$$\begin{aligned} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \rho^2 dx &\leq -2 \int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla \rho \rho(u_\ell - u_\infty) dx \\ &\leq 2c \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} \rho |\nabla(u_\ell - u_\infty)| |u_\ell - u_\infty| dx \\ &\leq 2c \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} \left\{ \frac{1}{2c} \rho^2 |\nabla(u_\ell - u_\infty)|^2 + \frac{c}{2} (u_\ell - u_\infty)^2 \right\} dx \\ &= \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho^2 dx + c^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (u_\ell - u_\infty)^2 dx \end{aligned}$$

since  $\nabla \rho$  vanishes out of  $\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}$ .

And because by (2.2),  $\rho = 0$  on  $\Omega_\ell \setminus \Omega_{\ell_1+1}$  and  $\rho = 1$  on  $\Omega_{\ell_1}$  we get

$$(2.6) \quad \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \leq c^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (u_\ell - u_\infty)^2 dx.$$

Due to the Poincaré inequality in  $\omega_2$  one has for some constant  $c_p$

$$(2.7) \quad \int_{\omega_2} (u_\ell - u_\infty)^2 dX_2 \leq c_p^2 \int_{\omega_2} |\nabla_{X_2}(u_\ell - u_\infty)|^2 dX_2$$

for a.e.  $X_1$ . Integrating in  $X_1$  one derives easily that

$$\int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} (u_\ell - u_\infty)^2 dx \leq c_p^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

and thus by (2.6)

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \leq c^2 c_p^2 \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx.$$

This is equivalent to

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where we have set  $C = c^2 c_p^2$ . Iterating this formula starting from  $\ell_1 = \frac{\ell}{2}$  we get

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \left(\frac{C}{1+C}\right)^{[\frac{\ell}{2}]} \int_{\Omega_{\frac{\ell}{2} + [\frac{\ell}{2}]}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where  $[\frac{\ell}{2}]$  denotes the integer part of  $\frac{\ell}{2}$ .

Noting that

$$\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$$

it comes

$$\begin{aligned} \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx &\leq e^{(-\frac{\ell}{2}+1) \ln(\frac{1+C}{C})} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \\ &= \frac{1+C}{C} e^{-\left(\frac{1}{2} \ln(\frac{1+C}{C})\right) \ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx. \end{aligned}$$

Next we estimate the right hand side of the above inequality. If we take  $v = 0$  in (1.2) and (1.3) we get

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u_\ell|^2 dx &\leq \int_{\Omega_\ell} f u_\ell dx \leq \left(\int_{\Omega_\ell} f^2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega_\ell} u_\ell^2 dx\right)^{\frac{1}{2}}, \\ \int_{\omega_2} |\nabla u_\infty|^2 dX_2 &\leq \int_{\omega_2} f u_\infty dX_2 \leq \left(\int_{\omega_2} f^2 dX_2\right)^{\frac{1}{2}} \left(\int_{\omega_2} u_\infty^2 dX_2\right)^{\frac{1}{2}}. \end{aligned}$$

Using the Poincaré inequality in  $\omega_2$  we derive easily that

$$\int_{\omega_2} u_\infty^2 dX_2 \leq c_p^2 \int_{\omega_2} |\nabla_{X_2} u_\infty|^2 dX_2, \quad \int_{\Omega_\ell} u_\ell^2 dx \leq c_p^2 \int_{\Omega_\ell} |\nabla u_\ell|^2 dx$$

and it follows that

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u_\infty|^2 dx, \int_{\Omega_\ell} |\nabla u_\ell|^2 dx &\leq c_p^2 \int_{\Omega_\ell} f^2 dx \\ &= c_p^2 |\ell \omega_1| \int_{\omega_2} f^2(X_2) dX_2 = c_p^2 \ell^p |\omega_1| \int_{\omega_2} f^2(X_2) dX_2 \end{aligned}$$

where  $|\cdot|$  denotes the Lebesgue measure. Then (2.1) follows easily for some positive constant  $\alpha$  chosen smaller than  $\frac{1}{2} \ln(\frac{1+C}{C})$ .

**2.1. Examples.** We suppose here that  $\varphi = \varphi(X_2)$  is a nonpositive function. We define  $u_\ell$  and  $u_\infty$  as respectively the solutions to

$$\begin{cases} u_\ell \in \mathcal{K}_\ell = \left\{ v \in H_0^1(\Omega_\ell) \mid v(x) \geq \varphi(X_2) \text{ a.e. } x \in \Omega_\ell \right\}, \\ \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(v - u_\ell) dx \geq \int_{\Omega_\ell} f(X_2)(v - u_\ell) dx, \quad \forall v \in \mathcal{K}_\ell \end{cases}$$

and

$$\begin{cases} u_\infty \in \mathcal{K}_\infty = \left\{ v \in H_0^1(\omega_2) \mid v(X_2) \geq \varphi(X_2) \text{ a.e. } X_2 \in \omega_2 \right\}, \\ \int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2}(v - u_\infty) dX_2 \geq \int_{\omega_2} f(X_2)(v - u_\infty) dX_2, \quad \forall v \in \mathcal{K}_\infty. \end{cases}$$

Then the theorem 1 applies with  $K_0(X_2) = [\varphi(X_2), +\infty)$ . This is the case of the one obstacle problem. If  $\psi(X_2)$  is a nonnegative function one can introduce  $K_0(X_2) = [\varphi(X_2), \psi(X_2)]$  and get the same result for the two obstacle problem.

In the case where  $K_0(X_2) = \mathbb{R}$  for all  $X_2$  the problems (1.2) and (1.3) are respectively the weak Dirichlet problems

$$\begin{aligned} u_\ell &\in H_0^1(\Omega_\ell), & -\Delta u_\ell &= f \text{ in } \Omega_\ell, \\ u_\infty &\in H_0^1(\omega_2), & -\Delta u_\infty &= f \text{ in } \omega_2 \end{aligned}$$

and in this case the theorem 1 shows the exponential rate of convergence of  $u_\ell$  towards  $u_\infty$ .

### 3. THE CASE OF POINTWISE CONSTRAINTS ON THE GRADIENT

In this section we study the case when  $g = 0$  and  $f \geq 0$ . First we consider the case of the so called elastic-plastic torsion problem which corresponds to  $K = \mathbb{R} \times \overline{B_n(0, 1)}$ , hence  $\mathcal{K}_\ell$  and  $\mathcal{K}_\infty$  are as follows

$$\mathcal{K}_\ell = \left\{ v \in H_0^1(\Omega_\ell) \mid |\nabla v(x)| \leq 1 \text{ a.e. } x \in \Omega_\ell \right\}$$

and

$$\mathcal{K}_\infty = \left\{ v \in H_0^1(\omega_2) \mid |\nabla_{X_2} v(X_2)| \leq 1 \text{ a.e. } X_2 \in \omega_2 \right\}.$$

With this notation  $u_\ell$  and  $u_\infty$  denote respectively the solutions of the problems (1.2) and (1.3).

In this case unfortunately the function given by (2.3) does not belong to  $\mathcal{K}_\ell$  and one is forced to develop another analysis and perhaps lose the exponential rate of convergence.

Let us first prove the following:

**Theorem 2.** *Suppose that  $f \geq 0$  then  $u_\ell$  is a nondecreasing sequence in  $\ell$  converging towards  $u_\infty$  when  $\ell \rightarrow +\infty$ .*

*Proof.* The proof will be broken into different steps.

(1)  $u_\ell$  and  $u_\infty$  are nonnegative.

It is clear that  $u_\ell^+ \in \mathcal{K}_\ell$ . Thus from (1.2) we get

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(u_\ell^+ - u_\ell) dx \geq \int_{\Omega_\ell} f(X_2)(u_\ell^+ - u_\ell) dx$$

and since  $u_\ell^+ - u_\ell = u_\ell^-$  we derive

$$- \int_{\Omega_\ell} |\nabla u_\ell^-|^2 dx \geq \int_{\Omega_\ell} f(X_2) u_\ell^- dx \geq 0.$$

Now by the Poincaré inequality for the domain  $\omega_2$  we have

$$\int_{\Omega_\ell} |u_\ell^-|^2 dx \leq c_p^2 \int_{\Omega_\ell} |\nabla u_\ell^-|^2 dx = 0$$

which implies that  $u_\ell^-(x) = 0$  for a.e.  $x$  in  $\Omega_\ell$  and thus  $u_\ell \geq 0$ . Choosing  $v = u_\infty^+$  in (1.3) and doing similar estimates completes the proof.

**(2)**  $u_\ell \leq u_\infty$ .

We notice, by the positivity of  $u_\infty$ , that

$$(u_\ell - u_\infty)^+ \in H_0^1(\Omega_\ell)$$

thus, since

$$u_\ell - (u_\ell - u_\infty)^+ = u_\ell \text{ or } u_\infty$$

this function is in  $\mathcal{K}_\ell$  and by (1.2) we have

$$(3.1) \quad \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (-(u_\ell - u_\infty)^+) dx \geq \int_{\Omega_\ell} f(X_2) (-(u_\ell - u_\infty)^+) dx.$$

Next for a.e.  $X_1$  it is clear that

$$u_\infty + (u_\ell(X_1, \cdot) - u_\infty)^+ \in \mathcal{K}_\infty$$

(this function is indeed equal to either  $u_\ell$  or  $u_\infty$  and  $|\nabla_{X_2} u_\ell| \leq |\nabla u_\ell| \leq 1$ ).

Thus from (1.3) it comes for a.e.  $X_1$

$$\int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} (u_\ell(X_1, \cdot) - u_\infty)^+ dX_2 \geq \int_{\omega_2} f(X_2) (u_\ell(X_1, \cdot) - u_\infty)^+ dX_2.$$

Noting that for every function  $v$ ,  $\nabla_{X_2} u_\infty \cdot \nabla_{X_2} v = \nabla u_\infty \cdot \nabla v$  and integrating in  $X_1$  we deduce

$$(3.2) \quad \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla (u_\ell - u_\infty)^+ dx \geq \int_{\Omega_\ell} f(X_2) (u_\ell - u_\infty)^+ dx.$$

Adding (3.1) and (3.2) leads to

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)^+|^2 dx = \int_{\Omega_\ell} \nabla (u_\ell - u_\infty) \cdot \nabla (u_\ell - u_\infty)^+ dx \leq 0$$

which as above by the Poincaré inequality proves (2).

**(3)** We have  $u_\ell \leq u_{\ell'}, \forall \ell' \geq \ell$ .

Since  $u_{\ell'}$  is nonnegative one has

$$(u_\ell - u_{\ell'})^+ \in H_0^1(\Omega_\ell) \subset H_0^1(\Omega_{\ell'})$$

if we assume the functions of  $H_0^1(\Omega_\ell)$  extended by 0 outside  $\Omega_\ell$ . At this point since  $\omega_1$  is starshaped with respect to 0 one should note that  $\Omega_\ell \subset \Omega_{\ell'}$ .

Then

$$u_\ell - (u_\ell - u_{\ell'})^+ \in \mathcal{K}_\ell, \quad u_{\ell'} + (u_\ell - u_{\ell'})^+ \in \mathcal{K}_{\ell'}$$

(these functions are in  $H_0^1(\Omega_\ell)$  and  $H_0^1(\Omega_{\ell'})$  respectively and they are equal to  $u_\ell$  or  $u_{\ell'}$ ). Thus by (1.2) written for  $\ell$  and  $\ell'$  we get

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (-(u_\ell - u_{\ell'})^+) dx \geq \int_{\Omega_\ell} f(X_2) (-(u_\ell - u_{\ell'})^+) dx$$

$$\int_{\Omega_\ell} \nabla u_{\ell'} \cdot \nabla (u_\ell - u_{\ell'})^+ dx \geq \int_{\Omega_\ell} f(X_2)(u_\ell - u_{\ell'})^+ dx.$$

By adding these two inequalities it comes

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_{\ell'})^+|^2 dx = \int_{\Omega_\ell} \nabla (u_\ell - u_{\ell'}) \cdot \nabla (u_\ell - u_{\ell'})^+ dx \leq 0$$

which again as above, by the Poincaré inequality proves (3).

Since we have now established that

$$0 \leq u_\ell \leq u_{\ell'} \leq u_\infty, \quad \forall \ell \leq \ell'$$

it is clear that  $u_\ell$  possesses a pointwise limit that we will denote by  $\tilde{u}_\infty$ , i.e.

$$u_\ell \rightarrow \tilde{u}_\infty \text{ on } \mathbb{R}^p \times \omega_2.$$

(4)  $\tilde{u}_\infty$  is independent of  $X_1$ .

By the assumption (1.1) for any  $h > 0$  there exists  $\ell(h) > \ell$  such that

$$(3.3) \quad \ell\omega_1 \pm he_1 \subset \ell(h)\omega_1, \quad (e_1 \text{ unit vector in } X_2 = 0).$$

For a function  $v$  we set  $\sigma_h v(x) = v(x - he_1)$ . We claim then that

$$(3.4) \quad \sigma_h u_\ell \leq u_{\ell(h)}.$$

For that we remark, if  $\vee$  denotes the maximum of two numbers, that

$$\sigma_h u_\ell \vee u_{\ell(h)} \in \mathcal{K}_{\ell(h)}.$$

Indeed by (3.3) this is a function of  $H_0^1(\Omega_{\ell(h)})$  and the constraints on the gradient are clearly satisfied. Then noting that

$$\sigma_h u_\ell \vee u_{\ell(h)} - u_{\ell(h)} = (\sigma_h u_\ell - u_{\ell(h)})^+$$

we obtain

$$(3.5) \quad \int_{\Omega_{\ell(h)}} \nabla u_{\ell(h)} \cdot \nabla (\sigma_h u_\ell - u_{\ell(h)})^+ dx \geq \int_{\Omega_{\ell(h)}} f(X_2)(\sigma_h u_\ell - u_{\ell(h)})^+ dx.$$

Next we claim, if  $\wedge$  denotes the minimum of two numbers, that

$$\sigma_{-h}(\sigma_h u_\ell \wedge u_{\ell(h)}) \in \mathcal{K}_\ell.$$

Indeed since  $u_{\ell(h)}$  is nonnegative and  $\sigma_h u_\ell$  vanishes outside  $\Omega_\ell + he_1$ , the above function is in  $H_0^1(\Omega_\ell)$  and the constraints on the gradient are satisfied.

Noting that for any two functions with compact support  $u$  and  $v$  we have

$$\int_{\mathbb{R}^n} u \sigma_{-h} v dx = \int_{\mathbb{R}^n} \sigma_h u v dx$$

we derive from (1.2)

$$\begin{aligned} \int_{\Omega_{\ell(h)}} \nabla u_\ell \cdot \nabla (\sigma_{-h}(\sigma_h u_\ell \wedge u_{\ell(h)}) - \sigma_{-h}(\sigma_h u_\ell)) dx \\ \geq \int_{\Omega_{\ell(h)}} f(X_2)(\sigma_{-h}(\sigma_h u_\ell \wedge u_{\ell(h)}) - \sigma_{-h}(\sigma_h u_\ell)) dx \end{aligned}$$

which reads also

$$\int_{\Omega_{\ell(h)}} \nabla \sigma_h u_\ell \cdot \nabla (\sigma_h u_\ell \wedge u_{\ell(h)} - \sigma_h u_\ell) dx \geq \int_{\Omega_{\ell(h)}} f(X_2)(\sigma_h u_\ell \wedge u_{\ell(h)} - \sigma_h u_\ell) dx.$$

Noticing that

$$\sigma_h u_\ell \wedge u_{\ell(h)} - \sigma_h u_\ell = -(\sigma_h u_\ell - u_{\ell(h)})^+$$

this leads to

$$(3.6) \quad \int_{\Omega_{\ell(h)}} \nabla \sigma_h u_\ell \cdot \nabla (-(\sigma_h u_\ell - u_{\ell(h)})^+) dx \geq \int_{\Omega_{\ell(h)}} f(X_2) (-(\sigma_h u_\ell - u_{\ell(h)})^+) dx.$$

Adding then (3.5) and (3.6) we are ending up with

$$\int_{\Omega_{\ell(h)}} |\nabla(\sigma_h u_\ell - u_{\ell(h)})^+|^2 dx = \int_{\Omega_{\ell(h)}} \nabla(\sigma_h u_\ell - u_{\ell(h)}) \cdot \nabla(\sigma_h u_\ell - u_{\ell(h)})^+ dx \leq 0$$

which by the Poincaré inequality implies (3.4). Changing  $h$  into  $-h$  we are getting

$$u_\ell(x \pm h e_1) \leq u_{\ell(h)}(x).$$

Passing then to the limit in  $\ell$  we obtain

$$\tilde{u}_\infty(x \pm h e_1) \leq \tilde{u}_\infty(x)$$

i.e.

$$\tilde{u}_\infty(x + h e_1) \leq \tilde{u}_\infty(x) \leq \tilde{u}_\infty(x + h e_1)$$

which proves (4).

(5) There exists a constant  $C$  independent of  $\ell$  such that

$$(3.7) \quad \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \leq C \ell^{p-1}.$$

We consider the function

$$(3.8) \quad v_\ell = \frac{u_\ell + u_\infty}{2} = \frac{u_\ell(X_1, \cdot) + u_\infty}{2}.$$

For almost every  $X_1 \in \ell\omega_1$  this function is in  $\mathcal{K}_\infty$  and thus by (1.3) we get

$$\int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} \left\{ \frac{u_\ell(X_1, \cdot) - u_\infty}{2} \right\} dX_2 \geq \int_{\omega_2} f(X_2) \left\{ \frac{u_\ell(X_1, \cdot) - u_\infty}{2} \right\} dX_2$$

and by integration in  $X_1$  it comes

$$(3.9) \quad \int_{\Omega_\ell} \nabla u_\infty \cdot \nabla(u_\ell - u_\infty) dx \geq \int_{\Omega_\ell} f(X_2)(u_\ell - u_\infty) dx.$$

We denote by  $d_\ell(X_1)$  the function of  $X_1$  only, defined as

$$d_\ell(X_1) = \text{dist}(X_1, \mathbb{R}^p \setminus (\ell\omega_1)).$$

Due to the fact that

$$|\nabla_{X_2} u_\ell|, |\nabla_{X_2} u_\infty| \leq 1$$

and  $u_\ell = 0$  on  $(\ell\omega_1) \times \partial\omega_2$  the lateral boundary of  $\Omega_\ell$  and  $u_\infty = 0$  on  $\partial\omega_2$ , one has

$$(3.10) \quad 0 \leq u_\ell, u_\infty \leq \gamma$$

for some constant  $\gamma$ . Let us again denote by  $B_p(0, r)$  a ball of center 0 and radius  $r$  contained in  $\omega_1$ . Then for  $\tau$  such that  $\frac{\gamma}{r} \leq \tau < \ell$  one has

$$(3.11) \quad d_\ell(X_1) \geq \gamma \text{ on } \Omega_{\ell-\tau}.$$

Indeed

$$(\ell - \tau)\omega_1 + B_p(0, \tau r) = (\ell - \tau)\omega_1 + \tau B_p(0, r) = \ell \left\{ \frac{\ell - \tau}{\ell} \omega_1 + \frac{\tau}{\ell} B_p(0, r) \right\} \subset \ell\omega_1$$

i.e.

$$\text{dist}((\ell - \tau)\omega_1, \mathbb{R}^p \setminus (\ell\omega_1)) \geq \tau r \geq \gamma$$

which implies (3.11).

We define now

$$w_\ell = \frac{u_\ell + u_\infty}{2} \wedge d_\ell(X_1).$$

This function is in  $\mathcal{K}_\ell$ , moreover by fixing  $\tau \geq \frac{\gamma}{r}$  we have  $w_\ell = \frac{u_\ell + u_\infty}{2}$  on  $\Omega_{\ell-\tau}$  (see (3.10),(3.11)). Thus using this function in (1.2) we get

$$(3.12) \quad \int_{\Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla \left( \frac{u_\infty - u_\ell}{2} \right) dx + \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (w_\ell - u_\ell) dx \\ \geq \int_{\Omega_{\ell-\tau}} f(X_2) \left( \frac{u_\infty - u_\ell}{2} \right) dx + \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (w_\ell - u_\ell) dx$$

that is

$$\int_{\Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (u_\infty - u_\ell) dx + 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (w_\ell - u_\ell) dx \\ \geq \int_{\Omega_{\ell-\tau}} f(X_2) (u_\infty - u_\ell) dx + 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (w_\ell - u_\ell) dx$$

and

$$\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla (u_\infty - u_\ell) dx - \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (u_\infty - u_\ell) dx + 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (w_\ell - u_\ell) dx \\ \geq \int_{\Omega_\ell} f(X_2) (u_\infty - u_\ell) dx - \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (u_\infty - u_\ell) dx + 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (w_\ell - u_\ell) dx.$$

Adding this to (3.9) we obtain

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 dx \\ \leq - \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (u_\infty - u_\ell) dx + 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla (w_\ell - u_\ell) dx \\ + \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (u_\infty - u_\ell) dx - 2 \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} f(X_2) (w_\ell - u_\ell) dx.$$

Since  $|\nabla u_\ell|$ ,  $|\nabla u_\infty|$ ,  $|\nabla w_\ell|$  are bounded by 1 and  $u_\ell$ ,  $u_\infty$ ,  $w_\ell$  are uniformly bounded we get easily for some constants  $C_1, C_2$

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 dx \leq C_1 |\Omega_\ell \setminus \Omega_{\ell-\tau}| + C_2 |(\ell\omega_1) \setminus ((\ell-\tau)\omega_1)| \int_{\omega_2} |f| dX_2 \\ \leq C(f, \omega_2) \{|\ell\omega_1| - |(\ell-\tau)\omega_1|\} \leq C\ell^{p-1}$$

and this completes the proof of (5).

**(6)**  $\tilde{u}_\infty = u_\infty$ .

From the Poincaré inequality, see (2.7), and (3.7) we have

$$\int_{\Omega_\ell} (\tilde{u}_\infty - u_\infty)^2 dx \leq \int_{\Omega_\ell} (u_\ell - u_\infty)^2 dx \leq c_p^2 \int_{\Omega_\ell} |\nabla (u_\ell - u_\infty)|^2 dx \leq C\ell^{p-1}.$$

And we have

$$\int_{\Omega_\ell} (\tilde{u}_\infty - u_\infty)^2 dx = |\ell\omega_1| \int_{\omega_2} (\tilde{u}_\infty - u_\infty)^2 dX_2 = \ell^p |\omega_1| \int_{\omega_2} (\tilde{u}_\infty - u_\infty)^2 dX_2$$

hence

$$\int_{\omega_2} (\tilde{u}_\infty - u_\infty)^2 dX_2 \leq \frac{C}{|\omega_1|} \frac{1}{\ell}$$

and this is only possible if  $\tilde{u}_\infty = u_\infty$  and the proof is complete.  $\square$

We now generalize this result by considering a general constraint set  $K(X_2)$  which is a closed convex subset of  $\mathbb{R}^{1+n}$  such that for some  $A > 0$

$$(3.13) \quad K(X_2) \subset (-\infty, A] \times \mathbb{R}^n$$

and for some  $a > 0$

$$(3.14) \quad [0, A] \times \overline{B_p(0, a)} \times \{0\} \subset K(X_2)$$

(here the third 0 is the 0 in  $\mathbb{R}^{n-p}$ ).

With this constraint set and the boundary condition  $g = 0$ ,  $\mathcal{K}_\ell$  and  $\mathcal{K}_\infty$  are as follows

$$\mathcal{K}_\ell = \left\{ v \in H_0^1(\Omega_\ell) \mid (v(x), \nabla v(x)) \in K(X_2) \text{ a.e. } x \in \Omega_\ell \right\}$$

and

$$\mathcal{K}_\infty = \left\{ v \in H_0^1(\omega_2) \mid (v(X_2), 0, \nabla_{X_2} v(X_2)) \in K(X_2) \text{ a.e. } X_2 \in \omega_2 \right\}.$$

As before with this notation  $u_\ell$  and  $u_\infty$  denote respectively the solutions of the problems (1.2) and (1.3).

Now defining

$$\tilde{\mathcal{K}}_\ell = \left\{ v \in H^1(\Omega_\ell) \mid \begin{aligned} &\exists g_v \in H_0^1(\omega_2) \text{ s.t. } v = g_v \text{ on } \partial\Omega_\ell, \\ &(v(x), \nabla v(x)) \in K(X_2) \text{ a.e. } x \in \Omega_\ell \end{aligned} \right\}$$

note that if  $v \in \tilde{\mathcal{K}}_\ell$  then it is independent of  $X_1$  on  $\partial(\ell\omega_1) \times \omega_2$ . We have

**Lemma 1.** *For all  $v \in \tilde{\mathcal{K}}_\ell$  the following inequality holds*

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla(v - u_\infty) dx \geq \int_{\Omega_\ell} f(X_2)(v - u_\infty) dx.$$

*Proof.* Let  $v \in \tilde{\mathcal{K}}_\ell$  then we define

$$\tilde{v}(X_2) = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} v(X_1, X_2) dX_1.$$

Because  $v \in H^1(\Omega_\ell)$  and  $v = 0$  on  $(\ell\omega_1) \times \partial\omega_2$  we have  $\tilde{v} \in H_0^1(\omega_2)$ .

Let  $g_v \in H_0^1(\omega_2)$  such that  $v - g_v \in H_0^1(\Omega_\ell)$  and let us consider  $w_k \in C_c^1(\Omega_\ell)$  such that  $w_k \rightarrow v - g_v$  in  $H^1(\Omega_\ell)$ . Then for any  $\eta \in C^1(\omega_2)$  and  $i = 1, \dots, p$  we have

$$\begin{aligned} \int_{\Omega_\ell} \eta(X_2) \partial_{x_i} v dx &= \lim_{k \rightarrow +\infty} \int_{\Omega_\ell} \eta(X_2) \partial_{x_i} (w_k + g_v) dx \\ &= \lim_{k \rightarrow +\infty} \int_{\Omega_\ell} \eta(X_2) \partial_{x_i} w_k dx = \lim_{k \rightarrow +\infty} \int_{\Omega_\ell} \partial_{x_i} (\eta(X_2) w_k) dx = 0 \end{aligned}$$

the last equality holds because for a function  $w \in C_c^1(\Omega)$  we have  $\int_\Omega \partial_{x_i} w dx = 0$ . Now since the equality above holds for all  $\eta \in C^1(\omega_2)$  we obtain

$$0 = \int_{\ell\omega_1} \nabla_{X_1} v dX_1 \text{ a.e. } X_2 \in \omega_2.$$

Hence by the convexity of  $K(X_2)$  we have

$$(\tilde{v}, 0, \nabla_{X_2} \tilde{v}) = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} (v, \nabla_{X_1} v, \nabla_{X_2} v) dX_1 = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} (v, \nabla v) dX_1 \in K(X_2)$$

which shows that  $\tilde{v} \in \mathcal{K}_\infty$ .

Using the inequality satisfied by  $u_\infty$  with  $\tilde{v}$  we obtain

$$\begin{aligned} \int_{\omega_2} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} \left( \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} v dX_1 - u_\infty \right) dX_2 \\ \geq \int_{\omega_2} f(X_2) \left( \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} v dX_1 - u_\infty \right) dX_2 \end{aligned}$$

which after some computations leads to

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla (v - u_\infty) dx \geq \int_{\Omega_\ell} f(X_2) (v - u_\infty) dx.$$

□

Then we have

**Theorem 3.** *Suppose that  $f \geq 0$  and all the above mentioned conditions for  $K(X_2)$  hold. Then  $u_\ell$  is a nondecreasing sequence in  $\ell$  converging towards  $u_\infty$  when  $\ell \rightarrow +\infty$ .*

*Proof.* It is enough to follow the different steps of the proof of theorem 2.

(1) Since  $u_\ell^+$  is equal to 0 or  $u_\ell$  it is clear that  $u_\ell^+ \in \mathcal{K}_\ell$ , similarly  $u_\infty^+ \in \mathcal{K}_\infty$  and the derivation of  $u_\ell \geq 0$  and  $u_\infty \geq 0$  is then identical to the one in theorem 2.

(2) Since  $u_\ell - (u_\ell - u_\infty)^+$  is equal to  $u_\ell$  or  $u_\infty$  it is clear that this function belongs to  $\mathcal{K}_\ell$ . We have that  $u_\infty + (u_\ell - u_\infty)^+ \in \tilde{\mathcal{K}}_\ell$  because this function is equal to  $u_\infty(X_2)$  on  $\partial\Omega_\ell$  and everywhere it is either equal to  $u_\ell$  or equal to  $u_\infty$ , hence by the previous lemma we obtain the inequality (3.2) and the derivation of  $u_\ell \leq u_\infty$  is then identical to the one in theorem 2.

(3) For  $\ell' \geq \ell$  since both  $u_\ell - (u_\ell - u_{\ell'})^+$  and  $u_{\ell'} + (u_\ell - u_{\ell'})^+$  are either equal to  $u_\ell$  or  $u_{\ell'}$ , they are respectively in  $\mathcal{K}_\ell$  and  $\mathcal{K}_{\ell'}$ . Then the derivation of  $u_\ell \leq u_{\ell'}$  is identical to the one in theorem 2.

(4) Since  $\sigma_h u_\ell \vee u_{\ell(h)}$  is either equal to  $\sigma_h u_\ell$  or  $u_{\ell(h)}$ , it satisfies the constraints and is in  $\mathcal{K}_{\ell(h)}$ . Similarly since  $\sigma_{-h}(\sigma_h u_\ell \wedge u_{\ell(h)}) = u_\ell \wedge \sigma_{-h} u_{\ell(h)}$  it is either equal to  $u_\ell$  or  $\sigma_{-h} u_{\ell(h)}$ , hence it satisfies the constraints and is in  $\mathcal{K}_\ell$ . Then to show that  $\tilde{u}_\infty$  is independent of  $X_1$  is identical to the one in theorem 2.

(5) The function  $v_\ell$  defined in (3.8) is in  $\tilde{\mathcal{K}}_\ell$  because  $v_\ell = \frac{1}{2}u_\infty(X_2)$  on  $\partial\Omega_\ell$  and it is a convex combination of  $u_\ell$  and  $u_\infty$ . Hence by the previous lemma we obtain the inequality (3.9).

Due to (3.13) and part (1) we have

$$0 \leq u_\ell, u_\infty \leq A.$$

Then for  $\tau$  such that  $\frac{A}{ar} < \tau < \ell$  we have

$$ad_\ell(X_1) \geq A \quad \text{on } \Omega_{\ell-\tau}$$

(same proof as above).

Defining

$$w_\ell = \frac{u_\ell + u_\infty}{2} \wedge ((ad_\ell(X_1)) \wedge A)$$

this function is in  $\mathcal{K}_\ell$ . Indeed  $w_\ell$  is 0 on  $\partial(\ell\omega_1) \times \omega_2$  because  $d_\ell = 0$  there and  $w_\ell$  is 0 on  $(\ell\omega_1) \times \partial\omega_2$  because  $u_\ell$  and  $u_\infty$  are 0 there. Also  $w_\ell$  is either equal to  $\frac{1}{2}(u_\ell + u_\infty)$  or  $(ad_\ell(X_1)) \wedge A$ . We have that  $\frac{1}{2}(u_\ell + u_\infty)$  satisfies the constraint because it is a convex combination of  $u_\ell$  and  $u_\infty$ , we have  $(ad_\ell(X_1)) \wedge A \in [0, A]$  and  $|\nabla_{X_1}((ad_\ell(X_1)) \wedge A)| \leq a|\nabla_{X_1}d_\ell(X_1)| \leq a$  hence by (3.14),  $w_\ell$  satisfies the constraints and we obtain the inequality (3.12). The rest of the proof of the inequality (3.7) is identical to the one in theorem 2.

(6) The proof of  $\tilde{u}_\infty = u_\infty$  is identical to the one of theorem 2.  $\square$

**Remark 1.** *The same result holds if the Laplace operator is replaced by any elliptic operator and  $f$  by a nonnegative distribution in  $H^{-1}(\Omega_\ell)$ . In the case where  $K(X_2) = K_0(X_2) \times \mathbb{R}^n$  one gets pointwise constraints on the value of the function. Also in the case where  $K(X_2) = \mathbb{R} \times K'(X_2)$  the constraints are on the gradient only and if there exist  $a > 0$ ,  $c > 0$  such that*

$$\overline{B_p(0, a)} \times \{0\} \subset K'(X_2) \subset \mathbb{R}^p \times B_{n-p}(0, c)$$

then because in our problem there is a 0 boundary condition on  $(\ell\omega_1) \times \partial\omega_2$  and  $\omega_2$  is bounded, the functions are uniformly bounded and hence the problem is equivalent to the case when  $K(X_2) = [-A, A] \times K'(X_2)$  for large enough  $A$ , which is dealt with in the previous theorem.

#### 4. THE CASE WHERE $f = 0$

In this section we consider the case when  $f = 0$  and the constraints are on the gradient only i.e.  $K$  is of the type  $K = \mathbb{R} \times K'$  where  $K'$  is a closed convex subset of  $\mathbb{R}^n$  such that for some  $a > 0$

$$(4.1) \quad \overline{B_p(0, a)} \times \{0\} \subset K'$$

(the second 0 is the 0 in  $\mathbb{R}^{n-p}$ ).

This case relates to the Saint-Venant Principle (see [T], [HK], [K]).

Let us consider as boundary condition  $g$ , a function such that

$$(4.2) \quad g \in L^\infty(\omega_2) \cap H_0^1(\omega_2), \quad (0, \nabla_{X_2}g(X_2)) \in K' \quad \text{a.e. } X_2 \in \omega_2.$$

In this case  $\mathcal{K}_\ell$  is

$$\mathcal{K}_\ell = \left\{ v \in H^1(\Omega_\ell) \mid v = g \text{ on } \partial\Omega_\ell, \nabla v(x) \in K' \text{ a.e. } x \in \Omega_\ell \right\}$$

and  $u_\ell$  is the unique solution to

$$(4.3) \quad \begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla(v - u_\ell) dx \geq 0, \quad \forall v \in \mathcal{K}_\ell. \end{cases}$$

Then we have

**Theorem 4.** *There exist positive constants  $C$  and  $\alpha$  independent of  $\ell$  such that*

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla u_\ell|^2 dx \leq C e^{-\alpha\ell}.$$

*Proof.* Due to the assumption that  $g$  is bounded, for some constant  $\gamma$  one has

$$(4.4) \quad |g(X_2)| \leq \gamma \text{ a.e. } X_2 \in \omega_2.$$

One derives then easily that one has also

$$(4.5) \quad |u_\ell(x)| \leq \gamma \text{ a.e. } x \in \Omega_\ell.$$

Indeed the functions

$$u_\ell - (u_\ell - \gamma)^+, \quad u_\ell + (-\gamma - u_\ell)^+$$

are in  $\mathcal{K}_\ell$  (they are equal to  $u_\ell$  on  $\partial\Omega_\ell$  by (4.4) and are equal to  $u_\ell, \gamma$  or  $-\gamma$ ). Thus using them as test functions in (4.3) one gets

$$\int_{\Omega_\ell} |\nabla(u_\ell - \gamma)^+|^2 dx = \int_{\Omega_\ell} \nabla(u_\ell - \gamma) \cdot \nabla(u_\ell - \gamma)^+ dx \leq 0$$

and

$$\int_{\Omega_\ell} |\nabla(u_\ell + \gamma)^-|^2 dx = - \int_{\Omega_\ell} \nabla(u_\ell + \gamma) \cdot \nabla(u_\ell + \gamma)^- dx \leq 0$$

which together with the Poincaré inequality for the domain  $\omega_2$  imply (4.5).

Let us introduce the constant  $M$  given by

$$M = \frac{\gamma + 1}{a}$$

and the functions  $\delta_{\ell_1}$  defined as

$$\delta_{\ell_1}(X_1) = \text{dist}(X_1, \ell_1\omega_1).$$

Let  $r > 0$  such that  $B_p(0, r) \subset \omega_1$  and let us fix  $\tau$  such that  $\tau \geq \frac{M}{r}$  and  $\ell_1 + \tau < \ell$ . We claim then that

$$(4.6) \quad \delta_{\ell_1}(X_1) \geq M \text{ on } \Omega_\ell \setminus \Omega_{\ell_1 + \tau}.$$

Indeed by the convexity of  $\omega_1$  we have

$$\begin{aligned} \ell_1\omega_1 + B_p(0, \tau r) &= \ell_1\omega_1 + \tau B_p(0, r) \\ &= (\ell_1 + \tau) \left\{ \frac{\ell_1}{\ell_1 + \tau} \omega_1 + \frac{\tau}{\ell_1 + \tau} B_p(0, r) \right\} \subset (\ell_1 + \tau)\omega_1 \end{aligned}$$

i.e.

$$\text{dist}(\ell_1\omega_1, \mathbb{R}^p \setminus (\ell_1 + \tau)\omega_1) \geq \tau r \geq M$$

and this clearly implies (4.6).

Let us define

$$(4.7) \quad v = u_\ell \left\{ 1 \wedge \frac{a\delta_{\ell_1}(X_1)}{1 + |u_\ell|} \right\}.$$

On the domain  $\Omega_\ell \setminus \Omega_{\ell_1 + \tau}$  by (4.6) we have

$$a\delta_{\ell_1}(X_1) \geq \gamma + 1 \geq |u_\ell| + 1$$

thus  $v = u_\ell$  on this domain and it is in particular equal to  $g$  on  $\partial\Omega_\ell$ . Moreover  $\nabla v = \nabla u_\ell \in K'$ .

If now  $x \in \Omega_{\ell_1}$  one has  $v = 0$  hence  $\nabla v = 0 \in K'$ .

Finally for  $x \in \Omega_{\ell_1 + \tau} \setminus \Omega_{\ell_1}$  if

$$a\delta_{\ell_1}(X_1) \geq 1 + |u_\ell|$$

then  $v = u_\ell$  and  $\nabla v \in K'$ . If

$$(4.8) \quad a\delta_{\ell_1}(X_1) < 1 + |u_\ell|$$

one has

$$v = \frac{au_\ell \delta_{\ell_1}(X_1)}{1 + |u_\ell|}$$

and

$$\begin{aligned} \nabla v &= \frac{u_\ell}{1 + |u_\ell|} a \nabla \delta_{\ell_1}(X_1) + \frac{a \delta_{\ell_1}(X_1)}{(1 + |u_\ell|)^2} \nabla u_\ell \\ &= \frac{|u_\ell|}{1 + |u_\ell|} a \operatorname{sign}(u_\ell) \nabla \delta_{\ell_1}(X_1) + \frac{1}{1 + |u_\ell|} \frac{a \delta_{\ell_1}(X_1)}{1 + |u_\ell|} \nabla u_\ell. \end{aligned}$$

By (4.1) and (4.8) one has  $a \operatorname{sign}(u_\ell) \nabla \delta_{\ell_1} \in K'$  and  $\frac{a \delta_{\ell_1}}{1 + |u_\ell|} \nabla u_\ell \in K'$  thus  $\nabla v \in K'$ . This implies that  $v \in \mathcal{K}_\ell$ .

Moreover it follows from above that when  $\nabla v$  is not vanishing one has

$$|\nabla v| \leq a|u_\ell| + |\nabla u_\ell|.$$

Thus using  $v$  in (4.3) we derive easily

$$\int_{\Omega_{\ell_1+\tau}} |\nabla u_\ell|^2 dx \leq \int_{\Omega_{\ell_1+\tau} \setminus \Omega_{\ell_1}} \nabla u_\ell \cdot \nabla v dx \leq \int_{\Omega_{\ell_1+\tau} \setminus \Omega_{\ell_1}} \{a|u_\ell| |\nabla u_\ell| + |\nabla u_\ell|^2\} dx.$$

Using the Poincaré inequality it follows that for some constant  $C$

$$\int_{\Omega_{\ell_1}} |\nabla u_\ell|^2 dx \leq C \int_{\Omega_{\ell_1+\tau} \setminus \Omega_{\ell_1}} |\nabla u_\ell|^2 dx$$

i.e.

$$\int_{\Omega_{\ell_1}} |\nabla u_\ell|^2 dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_1+\tau}} |\nabla u_\ell|^2 dx.$$

Iterating this formula starting for instance from  $\frac{\ell}{2}$  with a step  $\tau$  we get

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla u_\ell|^2 dx \leq \left(\frac{C}{1+C}\right)^N \int_{\Omega_{\frac{\ell}{2}+N\tau}} |\nabla u_\ell|^2 dx.$$

Choosing  $N$  as the largest number such that

$$N\tau \leq \frac{\ell}{2}$$

we obtain since  $(N+1)\tau > \frac{\ell}{2}$

$$\begin{aligned} (4.9) \quad \int_{\Omega_{\frac{\ell}{2}}} |\nabla u_\ell|^2 dx &\leq \left(\frac{C}{1+C}\right)^{\frac{\ell}{2\tau}-1} \int_{\Omega_\ell} |\nabla u_\ell|^2 dx \\ &= \frac{1+C}{C} e^{-\left(\frac{1}{2\tau} \ln\left(\frac{1+C}{C}\right)\right)\ell} \int_{\Omega_\ell} |\nabla u_\ell|^2 dx. \end{aligned}$$

Remains to estimate the last integral above. For that we consider

$$v = g \left\{ 1 \wedge \frac{a \delta_{\ell-\tau}(X_1)}{1 + |g|} \right\}$$

(i.e. we replace in (4.7)  $u_\ell$  by  $g$  and  $\ell_1$  by  $\ell - \tau$ ). It is easy to check that  $v \in \mathcal{K}_\ell$  and that from (4.3) we get

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} \nabla u_\ell \cdot \nabla v dx \leq \left( \int_{\Omega_\ell} |\nabla u_\ell|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} |\nabla v|^2 dx \right)^{\frac{1}{2}}$$

which implies that

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u_\ell|^2 dx &\leq \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} |\nabla v|^2 dx \leq \int_{\Omega_\ell \setminus \Omega_{\ell-\tau}} (a|g| + |\nabla_{X_2} g|)^2 dx \\ &\leq C(g, a) |(\ell\omega_1) \setminus ((\ell - \tau)\omega_1)| \end{aligned}$$

here  $C(g, a)$  denotes a constant depending on  $g$  and  $a$ . Also  $|\cdot|$  is the Lebesgue measure.

It follows that

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq C(g, a) \{\ell^p - (\ell - \tau)^p\} |\omega_1|.$$

Combining this with (4.9) the result follows by choosing  $\alpha$  a positive constant smaller than  $\frac{1}{2\tau} \ln(\frac{1+C}{C})$ .  $\square$

**Remark 2.** *The assumption (4.1) cannot be dropped. Indeed suppose that  $n = 2$ ,  $\Omega_\ell = \ell(-1, 1) \times (-1, 1)$ ,  $K' = [0, +\infty) \times \mathbb{R}$ ,  $g \in H_0^1(-1, 1)$ . In this case if  $u \in \mathcal{K}_\ell$  one has*

$$u = g \text{ on } \{-\ell, \ell\} \times (-1, 1), \quad 0 \leq \partial_{x_1} u \text{ in } \Omega_\ell$$

*this leaves as the only possibility that  $u = g$ , i.e.  $\mathcal{K}_\ell$  reduces to  $\{g\}$  for any  $\ell$ , hence  $u_\ell = g$ . Now if  $g \neq 0$  then  $u_\ell$  does not converge towards 0.*

**Remark 3.** *The theorem 4 can be extended for rather more general constraint  $K(x) = K_0(x) \times K'(x)$  with mild assumptions. Note that in this case the convex set  $K$  might also depend on  $X_1$ . Also more general operators or domains  $\Omega_\ell$  can be considered.*

**Acknowledgments:** The first author has been supported by the Swiss National Science Foundation under the contracts #20-117614/1, # 21-129807/1

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