ON THE ASYMPTOTIC BEHAVIOUR
OF THE EIGENMODES FOR ELLIPTIC PROBLEMS
IN DOMAINS BECOMING UNBOUNDED

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Abstract. The aim of this work is to analyze the asymptotic behaviour of the
 eigenmodes of some elliptic eigenvalue problems set on domains becoming un-
 bounded in one or several directions. In particular, in the case of a linear elliptic
 operator in divergence form, we prove that the sequence of the kth eigenvalues
 convergences to the first eigenvalue of an elliptic problems set on the section of
 the domain. Moreover, an optimal rate of convergence of this sequence is given.

1. Introduction

The goal of this note is to analyze the asymptotic behaviour of the eigenvalues and
the eigenfunctions of elliptic problems set in domains which can become unbounded
in one or several directions. Let us start with a simple example. Suppose that for
ℓ > 0 Ωℓ is the rectangle in \( \mathbb{R}^2 \) defined as
\[ Ωℓ = (−ℓ, ℓ) \times (−1, 1) = (−ℓ, ℓ) \times \omega \]
where we have set \( \omega = (−1, 1) \). We denote by \( x_1, x_2 \) the coordinates in \( \mathbb{R}^2 \) (see
figure 1.1). Let us then determine the eigenvalues and eigenfunctions of the Laplace
operator in \( Ω_ℓ \) with Dirichlet boundary conditions. To achieve that, let us consider
first the following one dimensional eigenvalue problem:

\[
\begin{aligned}
-\ddot{u} &= \lambda u & \text{in } (−a, a), \\
u(-a) &= u(a) = 0.
\end{aligned}
\]

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It is clear that the general solution of the first equation of (1.1) is given by
\[ u = A \sin \sqrt{\lambda}(x + a) + B \cos \sqrt{\lambda}(x + a) \]
\((\lambda > 0, A, B \in \mathbb{R})\). In order to match the boundary conditions one has to have
\[ B = 0, \quad \sin 2a \sqrt{\lambda} = 0. \]
\((A = 0 \text{ would give the trivial solution})\). Thus the eigenvalues of the problem (1.1) are given by
\begin{equation}
2a \sqrt{\lambda_k} = k\pi \quad \iff \quad \lambda_k = \left(\frac{k\pi}{2a}\right)^2 \quad k = 1, 2, \ldots
\end{equation}
and the corresponding eigenfunctions by
\begin{equation}
u_k = A_k \sin \frac{k\pi}{2a}(x + a).\end{equation}

If one wants to normalize \(u_k\) in such a way that its \(L^2\)-norm is equal to one, one is lead to choose \(A_k\) such that
\[ A_k^2 \int_{-a}^{a} \sin^2 \frac{k\pi}{2a}(x + a) \, dx = 1 \]
which leads to
\[ u_k = \frac{1}{\sqrt{a}} \sin \frac{k\pi}{2a}(x + a). \]

Suppose now that we want to find the eigenvalues and eigenfunctions of the problem
\begin{equation}
\begin{cases}
-\Delta u = \lambda_{\ell} u & \text{in } \Omega_{\ell}, \\
u = 0 & \text{on } \partial\Omega_{\ell}.
\end{cases}
\end{equation}
It is clear from above that
\begin{equation}
u = \frac{1}{\sqrt{\ell}} \sin \frac{k\pi}{2\ell}(x_1 + \ell) \sin \frac{m\pi}{2}(x_2 + 1)\end{equation}
is an eigenfunction for (1.4) corresponding to the eigenvalue
\begin{equation}
\left(\frac{k\pi}{2\ell}\right)^2 + \left(\frac{m\pi}{2}\right)^2 \quad k, m = 1, 2, \ldots
\end{equation}
There are no other eigenvalues. Indeed, suppose that \((v, \lambda_v)\) is another couple of eigenfunction – eigenvalue for (1.4). We have
\[-\Delta v = \lambda_v v.
\]
Multiplying by \(u\) given by (1.5) and integrating by parts we obtain
\[ \lambda_u(v, u) = \lambda_v(v, u) \]
where \(\lambda_u\) is given by (1.6). Since we have assumed \(\lambda_v \neq \lambda_u\) we get
\[ (v, u) = 0. \]
\((\cdot, \cdot)\) is the usual \(L^2(\Omega_\ell)\)-scalar product. Thus we have \((v, u) = 0\) for any \(u\) given by (1.5). By density of the span of the functions \(\sin \frac{k\pi}{2\ell}(x + \ell)\) in \(L^2(-a, a)\) we obtain
\[
\int_{\Omega_\ell} v\varphi(x_1)\psi(x_2) \, dx_1 \, dx_2 = 0 \quad \forall \varphi \in L^2(-\ell, \ell), \quad \forall \psi \in L^2(-1, 1).
\]
It follows easily that \(v = 0\) and \(\lambda_v\) cannot be an eigenvalue. This proves that (1.5), (1.6) give a complete sets of eigenfunctions and eigenvalues of (1.4). Let us examine what happens when \(\ell\) goes to plus infinity. The first eigenvalue of (1.4) is given by (1.6) for \(k = 1, m = 1\) – i.e. is given by
\[
\lambda_1^{\ell} = \left(\frac{\pi}{2\ell}\right)^2 + \left(\frac{\pi}{2}\right)^2.
\]
Clearly, it converges when \(\ell \to +\infty\) towards the first eigenvalue
\[
\mu^1 = \left(\frac{\pi}{2}\right)^2
\]
of the problem
\[
\begin{cases}
-u'' = \lambda u & \text{in } \omega, \\
u = 0 & \text{on } \partial\omega,
\end{cases}
\]
(see (1.1), (1.2)). Moreover, the rate of convergence is in \(\frac{1}{\ell^2}\). One could expect next that the \(k\)-th eigenvalue to (1.4) converges also to the \(k\)-eigenvalue to (1.8). This is not what happens. Indeed, going back to (1.6), if \(\lambda_k^\ell\) denotes the \(k\)-th eigenvalue to (1.4) one sees that
\[
\lambda_k^\ell = \left(\frac{k\pi}{2\ell}\right)^2 + \left(\frac{\pi}{2}\right)^2 \quad \text{for } \sqrt{3}\ell > k.
\]
Thus, when \(\ell \to +\infty\), one has
\[
\lambda_k^\ell \to \mu^1
\]
and all the eigenvalues of (1.4) are piling up on \(\mu^1\) when \(\ell \to +\infty\). We will see below that this phenomenon happens also in the general case. Of course if one allows \(k\) to depend on \(\ell\) the behaviour of \(\lambda_k^\ell\) could be different. For instance if \(\sqrt{3}\ell\) is an integer
\[
\lambda_{\sqrt{3}\ell}^\ell = \mu^2
\]
where \(\mu^2\) is the second eigenvalue of (1.8).

The behaviour, when \(\ell \to +\infty\), of the eigenfunctions is touchy as well. Indeed, consider the first eigenfunction of (1.4). It is given – see (1.5) – by
\[
u = \frac{1}{\sqrt{\ell}} \sin \frac{\pi}{2\ell}(x_1 + \ell) \sin \frac{\pi}{2}(x_2 + 1).
\]
When \(\ell \to +\infty\), it converges toward \(0\). To obtain something converging toward the first eigenfunction to (1.8) one has to re-scale it properly. Indeed
\[
\tilde{u} = \sin \frac{\pi}{2\ell}(x_1 + \ell) \sin \frac{\pi}{2}(x_2 + 1) \quad \to \quad \sin \frac{\pi}{2}(x_2 + 1)
\]
when \( \ell \to +\infty \), i.e. \( \bar{u} \) is such that

\[
\int_{\Omega_\ell} \bar{u}^2 \, dx_1 \, dx_2 = \ell
\]

and thus to expect convergence of the eigenfunctions from (1.4) toward the eigenfunctions to (1.8) some proper scaling is in order.

The convergence rate in the \( H^1 \) norm in any subdomain is then exactly in \( \frac{1}{\ell^2} \).

This contrasts with the non resonance case where the convergence is, in general, faster. For instance, if we consider the problem

\[
\begin{aligned}
-\Delta u_{\ell} &= f(x_2) & \text{in } \Omega_{\ell}, \\
u_{\ell} &= 0 & \text{on } \partial \Omega_{\ell},
\end{aligned}
\]

where \( f \) is some \( L^2 \) function depending only on the \( x_2 \) variable, then the decay in the \( H^1 \) norm of \( u_{\ell} - u_\infty \) in any subdomain is at least exponential (see [3], Proposition 2.1). Here, \( u_\infty \) is the solution to

\[
\begin{aligned}
-u'' &= f(x_2) & \text{in } (-1, 1), \\
u &= 0 & \text{on } \{-1, 1\}.
\end{aligned}
\]

This are these kinds of issues that we would like to analyze in the following of the paper. We will do it in a general setting. The rest of the article is divided as follow. In the next section we prove convergence of the eigenvalues in the case of a general elliptic operator. In section 3 we study the convergence of the eigenfunctions. Finally we give some applications.

2. CONVERGENCE OF THE EIGENVALUES

Let us first introduce some notation that we will use throughout the paper. We denote by \( \Omega_{\ell} \) the open subset of \( \mathbb{R}^n \) defined as

\[
\Omega_{\ell} = (-\ell, \ell)^p \times \omega.
\]

\( \ell \) is a positive number, \( 1 \leq p < n \) an integer, \( \omega \) is a bounded open subset of \( \mathbb{R}^{n-p} \). The points in \( \mathbb{R}^n \) will be denoted by

\[
x = (X_1, X_2),
\]

with

\[
X_1 = (x_1, \ldots, x_p), \quad X_2 = (x_{p+1}, \ldots, x_n).
\]

Let \( A = A(X_1, X_2) \) be a \( n \times n \)-symmetric matrix of the type

\[
A = A(X_1, X_2) = \begin{pmatrix}
A_{11}(X_1, X_2) & A_{12}(X_2) \\
A_{21}(X_2) & A_{22}(X_2)
\end{pmatrix}.
\]

In (2.1), \( A_{11} \) is a \( p \times p \) symmetric matrix. We will assume that \( A \) is uniformly bounded and uniformly positive definite on \( \mathbb{R}^p \times \omega \) that is to say that

\[
\begin{aligned}
\|A(x)\| &\leq \Lambda & \text{a.e. } x \in \mathbb{R}^p \times \omega, \\
A(x)\xi \cdot \xi &\geq \lambda |\xi|^2 & \text{a.e. } x \in \mathbb{R}^p \times \omega, \, \forall \xi \in \mathbb{R}^n.
\end{aligned}
\]
In the above formula $\| \cdot \|$ stands for a norm on matrices, $\Lambda$, $\lambda$ are some positive constants, $| \cdot |$ is the euclidean norm, the “$\cdot$” denotes the usual euclidean scalar product. If $\Omega$ is a bounded open set of $\mathbb{R}^k$ we will denote by $H^1(\Omega)$, $H^1_0(\Omega)$ the usual spaces of functions defined as

\[ H^1(\Omega) = \{ v \in L^2(\Omega) \mid \partial x_i v \in L^2(\Omega) \forall i = 1, \ldots, k \}, \]

\[ H^1_0(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial \Omega \}. \]

In the above definition $\partial x_i$ denotes the partial derivation in $x_i$, $\partial \Omega$ the boundary of $\Omega$. We will always assume that $H^1_0(\Omega)$ is equipped with the norm

\[ (\| \nabla v \|_{2, \Omega})^{1/2} = \left\{ \int_{\Omega} |\nabla v(x)|^2 \, dx \right\}^{1/2}, \]

($|\nabla v|$ denotes the euclidean norm of the gradient of $v$).

We would like to consider the eigenvalue problem

\[ \begin{cases} u_\ell \in H^1_0(\Omega_\ell), \\ \int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla v \, dx = \lambda_\ell \int_{\Omega_\ell} u_\ell v \, dx \quad \forall v \in H^1_0(\Omega_\ell). \end{cases} \]

To be more precise, we say that $\lambda_\ell$ is an eigenvalue to (2.7) if there exists $u_\ell \neq 0$ solution of the above system. When $\ell \to +\infty$, we expect that the limit eigenvalue problem (2.7) will be the eigenvalue problem on $\omega$ defined as

\[ \begin{cases} u \in H^1_0(\omega), \\ \int_{\omega} A_{22}(X_2) \nabla X_2 u \cdot \nabla X_2 v \, dX_2 = \mu \int_{\omega} uv \, dX_2 \quad \forall v \in H^1_0(\omega). \end{cases} \]

In the above system $dX_2 = dx_{p+1} \ldots dx_n$, $\nabla X_2 = (\partial x_{p+1}, \ldots, \partial x_n)$ – i.e. $\nabla X_2$ denotes the gradient in the last variables. By “limit problem” we mean that we expect that the eigenmodes of (2.7) will converge towards the eigenmodes of (2.8) when $\ell \to +\infty$. Of course this convergence has to be taken with caution as we pointed out already in our introduction.

Let us denote by $\lambda_1^\ell$ and $\mu^1$ the first eigenvalues of (2.7) and (2.8) respectively. It is well known that

\[ \lambda_1^\ell = \inf_{H^1_0(\Omega_\ell) \setminus \{0\}} \left\{ \int_{\Omega_\ell} A \nabla u \cdot \nabla u \, dx \mid u \in H^1_0(\Omega_\ell), \int_{\Omega_\ell} u^2 \, dx = 1 \right\} \]

\[ = \inf_{H^1_0(\Omega_\ell) \setminus \{0\}} \frac{\int_{\Omega_\ell} A \nabla u \cdot \nabla u \, dx}{\int_{\Omega_\ell} u^2 \, dx}, \]

and

\[ \mu^1 = \inf_{H^1_0(\omega) \setminus \{0\}} \left\{ \int_{\omega} A_{22}(X_2) \nabla X_2 u \cdot \nabla X_2 v \, dX_2 \mid u \in H^1_0(\omega), \int_{\omega} u^2 \, dX_2 = 1 \right\} \]

\[ = \inf_{H^1_0(\omega) \setminus \{0\}} \frac{\int_{\omega} A_{22}(X_2) \nabla X_2 u \cdot \nabla X_2 v \, dX_2}{\int_{\omega} u^2 \, dX_2}. \]

Moreover, it is well known that these eigenvalues are simple and the corresponding eigenfunctions do not change sign in $\Omega_\ell$ or $\omega$ – see [5], or [7].
Then we have:

**Theorem 2.1.** It holds that

\[ (2.11) \quad \mu_1^1 \leq \lambda_1^1 \leq \mu_1^1 + \frac{C}{\ell^2} \]

where \( C = p |A_{11}|_{\infty} \frac{\pi^2}{4}, \quad |A_{11}|_{\infty} = \text{esssup}_{\mathbb{R}^n} |A_{11}(x)|, \quad |\cdot| \text{ is the norm of matrices subordinated to the euclidean norm.} \)

To prove this theorem we will need the following lemma:

**Lemma 2.2.** Let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \). Let \( A_\varepsilon \) be a family of symmetric matrices such that for \( \lambda, \Lambda \) positive independent of \( \varepsilon \) we have

\[ (2.12) \quad \lambda |\xi|^2 \leq A_\varepsilon(x) \xi \cdot \xi \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \]
\[ (2.13) \quad \|A(x)\| \leq \Lambda \quad \text{a.e. } x \in \Omega. \]

Suppose that

\[ (2.14) \quad A_\varepsilon(x) \rightarrow A(x) \quad \text{a.e. } x \in \Omega \text{ as } \varepsilon \rightarrow 0. \]

Set

\[ (2.15) \quad \lambda_\varepsilon = \text{Inf}\left\{ \int_{\Omega} A_\varepsilon \nabla u \cdot \nabla u \, dx \mid u \in H_0^1(\Omega), \quad \int_{\Omega} u^2 \, dx = 1 \right\}, \]
\[ (2.16) \quad \lambda_0 = \text{Inf}\left\{ \int_{\Omega} A \nabla u \cdot \nabla u \, dx \mid u \in H_0^1(\Omega), \quad \int_{\Omega} u^2 \, dx = 1 \right\}, \]

then we have

\[ (2.17) \quad \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda_0. \]

(i.e. the first eigenvalue of the operator \( -\nabla \cdot (A_\varepsilon \nabla \cdot) \) with Dirichlet boundary conditions converges towards the first eigenvalue of \( -\nabla \cdot (A \nabla \cdot) \) with the same boundary conditions).

**Proof of the Lemma.** Let \( u_\varepsilon \) be the first eigenfunction realizing the infimum of (2.15). (It can be shown that it does not change of sign in \( \Omega \) and is unique if we assume it to be positive – see [5]). Denote by \( v \) a fixed function in \( H_0^1(\Omega) \) satisfying

\[ (2.18) \quad \int_{\Omega} v^2 \, dx = 1. \]

By (2.12), (2.13), (2.15) we have

\[ (2.19) \quad \lambda \|\nabla u_\varepsilon\|^2_{2,\Omega} \leq \int_{\Omega} A_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx = \lambda_\varepsilon \leq \int_{\Omega} A_\varepsilon(x) \nabla v \cdot \nabla v \, dx \leq C \]

where \( C \) is a constant independent of \( \varepsilon \). Thus \( \lambda_\varepsilon \) is bounded and so is \( u_\varepsilon \) in \( H_0^1(\Omega) \).

At the expense of extracting a subsequence we can assume that

\[ (2.20) \quad \lambda_\varepsilon \rightarrow \lambda_0, \]
\[ (2.21) \quad u_\varepsilon \rightharpoonup u_0 \quad \text{in } H^1_0(\Omega), \quad u_\varepsilon \rightarrow u_0 \quad \text{in } L^2(\Omega). \]
Now, by definition of the first eigenvalue one has also since $A_\varepsilon$ is symmetric
\begin{equation}
\int_\Omega A_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla v \, dx = \int_\Omega \nabla u_\varepsilon \cdot A_\varepsilon(x) \nabla v \, dx
\end{equation}
\begin{equation}
= \lambda_\varepsilon \int_\Omega u_\varepsilon v \, dx \quad \forall v \in H^1_0(\Omega).
\end{equation}
Noticing that by the Lebesgue theorem we have
\begin{equation}
A_\varepsilon \nabla v \to A \nabla v \quad \text{in} \quad L^2(\Omega),
\end{equation}
(see (2.14)) we obtain by passing to the limit in (2.22)
\begin{equation}
\int_\Omega A \nabla u_0 \cdot \nabla v \, dx = \lambda_0 \int_\Omega u_0 v \, dx \quad \forall v \in H^1_0(\Omega).
\end{equation}
Thus $\lambda_0$ is an eigenvalue of the problem (2.23). Note that $u_0 \neq 0$ since
\begin{equation}
1 = \int_\Omega u^2_\varepsilon \, dx \to \int_\Omega u^2_0 \, dx = 1.
\end{equation}
From (2.19) we derive also that it holds that
\begin{equation}
\lambda_0 \leq \int_\Omega A(x) \nabla v \cdot \nabla v \, dx \quad \forall v \in H^1_0(\Omega), \int_\Omega v^2 \, dx = 1.
\end{equation}
Thus $\lambda_0$ is the first eigenvalue of (2.23) – i.e. $\lambda_0 = \lambda_0$. By uniqueness of the possible limits this shows (2.17) and completes the proof of the Lemma.

**Remark 2.1.** Since when $\varepsilon \to 0$
\begin{equation}
\int_\Omega u^2_\varepsilon \, dx \to \int_\Omega u^2_0 \, dx, \quad \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx \to \int_\Omega A \nabla u_0 \cdot \nabla u_0 \, dx
\end{equation}
it is easy to show that $u_\varepsilon \to u_0$ in $H^1(\Omega)$ and $u_0$ is the first eigenfunction (normalized and positive) of (2.23).

We then now turn to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let us first establish the lower bound in (2.11). For this consider $A_\varepsilon$ defined by
\begin{equation}
\begin{cases}
A & \text{on} \quad \Omega_{\ell-\varepsilon}, \\
\begin{pmatrix}
\lambda & 0 & \vdots \\
0 & \ddots & \vdots \\
0 & \cdots & \lambda \\
0 & \cdots & \ddots & \ddots \\
0 & \ddots & \ddots & A_{22}
\end{pmatrix} & \text{on} \quad \Omega_\ell \setminus \Omega_{\ell-\varepsilon},
\end{cases}
\end{equation}
where $\lambda$ is the constant in the inequality (2.3). Clearly $A_\varepsilon$ satisfies all the assumptions of Lemma 2.2 with $\Omega = \Omega_\ell$. Define

$$
(2.28) \quad \lambda_{1,\ell,\varepsilon} = \inf \left\{ \int_{\Omega_\ell} A_\varepsilon \nabla u \cdot \nabla u \, dx \mid u \in H^1_0(\Omega_\ell), \int_{\Omega_\ell} u^2 \, dx = 1 \right\},
$$

the first eigenvalue of the operator $-\nabla \cdot (A_\varepsilon \nabla \cdot)$ with Dirichlet boundary conditions. Denote also by $u_\varepsilon$ the function – that we can assume $> 0$ – where the infimum of (2.28) is achieved. We have

$$
(2.29) \quad \int_{\Omega_\ell} A_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla v \, dx = \lambda_{1,\ell,\varepsilon} \int_{\Omega_\ell} u_\varepsilon v \, dx \quad \forall v \in H^1_0(\Omega_\ell).
$$

Denote then by $\varrho_\varepsilon$ a smooth function of one variable such that

$$
(2.30) \quad 0 \leq \varrho_\varepsilon \leq 1, \quad \varrho_\varepsilon = 1 \text{ on } (-\ell + \varepsilon, \ell - \varepsilon), \quad \varrho_\varepsilon(\pm \ell) = 0, \quad \varrho_\varepsilon \text{ is concave},
$$

i.e.

$$
(2.31) \quad \varrho_\varepsilon'' \leq 0.
$$

Denote by $w_1$ the positive function realizing the infimum of (2.10) and in (2.29) choose

$$
(2.32) \quad v = w_1(X_2) \prod_{i=1}^p \varrho_\varepsilon(x_i).
$$

We obtain (we decompose $A_\varepsilon$ as $A$ in (2.1) and put the $\varepsilon$ above as an upper index)

$$
(2.33) \quad \int_{\Omega_\ell} \left\{ A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 v + A_{12} \nabla X_2 u_\varepsilon \cdot \nabla X_1 v \\
+ A_{12}^T \nabla X_1 u_\varepsilon \cdot \nabla X_2 v + A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 v \right\} \, dx = \lambda_{1,\ell,\varepsilon} \int_{\Omega_\ell} u_\varepsilon v \, dx.
$$

If we denote by $\Pi$ the product in (2.32) we obtain

$$
(2.34) \quad \int_{\Omega_\ell} \left\{ A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 \Pi \cdot w_1 + A_{12} \nabla X_2 u_\varepsilon \cdot \nabla X_1 \Pi \cdot w_1 \\
+ A_{12}^T \nabla X_1 u_\varepsilon \cdot \nabla X_2 w_1 \Pi + A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 w_1 \Pi \right\} \, dx \\
= \lambda_{1,\ell,\varepsilon} \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx.
$$

First remark that

$$
(2.35) \quad \nabla X_1 \Pi = 0 \quad \text{on} \quad \Omega_{\ell-\varepsilon}
$$
so that by (2.27) the second integral in (2.34) vanishes. Next, for the first integral we have

\[
\int_{\Omega_\epsilon} A_{11}^\epsilon \nabla X_1 u_\epsilon \cdot \nabla X_1 \Pi w_1 \, dx \\
= \int_{\Omega_\epsilon \setminus \Omega_{\epsilon-\epsilon}} A_{11}^\epsilon \nabla X_1 u_\epsilon \cdot \nabla X_1 \Pi w_1 \, dx \\
= \lambda \int_{\Omega_\epsilon \setminus \Omega_{\epsilon-\epsilon}} \nabla X_1 u_\epsilon \cdot \nabla X_1 \Pi w_1 \, dx \\
= \lambda \int_{\Omega_\epsilon \setminus \Omega_{\epsilon-\epsilon}} \nabla X_1 \cdot \{ u_\epsilon \nabla X_1 \Pi w_1 \} - u_\epsilon \prod_{j \neq i} p_{x_j} g_{x_i}(x_i) w_1 \, dx \\
\geq \lambda \int_{\Omega_\epsilon \setminus \Omega_{\epsilon-\epsilon}} \nabla X_1 \cdot \{ u_\epsilon \nabla X_1 \Pi w_1 \} \, dx
\]

by (2.30), (2.31). But since \( u_\epsilon \nabla X_1 \Pi \) vanishes on the boundary of \( \Omega_\epsilon \setminus \Omega_{\epsilon-\epsilon} \) the above integral is identically equal to 0. Thus, we derive from (2.34) that it holds

\[
\int_{\Omega_\epsilon} A_{12}^\epsilon \nabla X_1 u_\epsilon \cdot \nabla X_2 w_1 \Pi \, dx + \int_{\Omega_\epsilon} A_{22}^\epsilon \nabla X_2 u_\epsilon \cdot \nabla X_2 w_1 \Pi \, dx \\
\leq \lambda_{\epsilon,\epsilon}^1 \int_{\Omega_\epsilon} u_\epsilon w_1 \Pi \, dx
\]

which is also by (2.27)

\[
\int_{\Omega_{\epsilon-\epsilon}} A_{12}^\epsilon \nabla X_1 u_\epsilon \cdot \nabla X_2 w_1 \Pi \, dx + \int_{\Omega_\epsilon} A_{22}^\epsilon \nabla X_2 u_\epsilon \cdot \nabla X_2 w_1 \Pi \, dx \\
\leq \lambda_{\epsilon,\epsilon}^1 \int_{\Omega_\epsilon} u_\epsilon w_1 \Pi \, dx.
\]

The second integral above can be written as

\[
\int_{\Omega_\epsilon} A_{22} \nabla X_2 u_\epsilon \cdot \nabla X_2 w_1 \Pi \, dx \\
= \int_{[-\ell,\ell]^p} \prod_{\omega} A_{22} \nabla X_2 u_\epsilon(X_1, X_2) \cdot \nabla w_1(X_2) \, dX_2 \, dX_1 \\
= \int_{[-\ell,\ell]^p} \prod_{\omega} \mu^1 u_\epsilon w_1 \, dX_2 \, dX_1 \\
= \mu^1 \int_{\Omega_\epsilon} u_\epsilon w_1 \Pi \, dx
\]
by definition of \( w_1 \) and since for a.e. \( X_1, u_\varepsilon(X_1, \cdot) \in H^1_0(\omega) \). It follows that (2.37) becomes

\[
\int_{\Omega_{t-\varepsilon}} A^T_{12} \nabla X_1 u_\varepsilon \cdot \nabla X_2 w_1 \Pi \, dx + \mu^1 \int_{\Omega_{t}} u_\varepsilon w_1 \Pi \, dx \leq \lambda^1_{t, \varepsilon} \int_{\Omega_{t}} u_\varepsilon w_1 \Pi, \, dx.
\]

(2.39)

Passing to the limit in \( \varepsilon \) – see also Remark 2.1 – we get (see Lemma 2.2)

\[
\int_{\Omega_{t}} A^T_{12} \nabla X_1 u_0 \cdot \nabla X_2 w_1 \, dx + \mu^1 \int_{\Omega_{t}} u_0 w_1 \, dx \leq \lambda^1_{t} \int_{\Omega_{t}} u_0 w_1 \, dx
\]

(\( \Pi = \Pi_{\varepsilon} \to 1 \) as \( \varepsilon \to 0 \)). But since \( u_0 \in H^1_0(\Omega_{t}) \) and \( A^T_{12}, w_1 \) depend on \( X_2 \) only we have

\[
\int_{\Omega_{t}} A^T_{12} \nabla X_1 u_0 \cdot \nabla X_2 w_1 \, dx = \int_{\Omega_{t}} \nabla X_1 \cdot (u_0 A_{12} \nabla X_2 w_1) \, dx = 0.
\]

We derive then

\[
\mu^1 \int_{\Omega_{t}} u_0 w_1 \, dx \leq \lambda^1_{t} \int_{\Omega_{t}} u_0 w_1 \, dx
\]

(2.40)

which implies the first inequality of (2.11) since \( u_0, w_1 \) are positive (see Remark 2.1). It remains to show the second inequality of (2.11). For that let \( v \) be a smooth function in \( H^1_0([\ell, \ell]) \) such that

\[
\int_{[-\ell, \ell]} v^2 \, dX_1 = 1
\]

(2.42)

and \( w_1 \) as above. It is clear that

\[
u = vw_1 = v(X_1)w_1(X_2) \in H^1_0(\Omega_{t}) \quad \text{and} \quad \int_{\Omega_{t}} u^2 \, dx = 1.
\]

Thus, by (2.9) we have

\[
\lambda^1_{t} \leq \int_{\Omega_{t}} A \nabla (vw_1) \cdot \nabla (vw_1) \, dx
\]

\[
= \int_{\Omega_{t}} A_{11} \nabla X_1 v \cdot \nabla X_1 v w_1^2 \, dx + \int_{\Omega_{t}} A_{12} \nabla X_2 w_1 \cdot \nabla X_1 v (vw_1) \, dx
\]

\[
+ \int_{\Omega_{t}} A^T_{12} \nabla X_1 v \cdot \nabla X_2 w_1 (vw_1) \, dx
\]

\[
+ \int_{\Omega_{t}} A_{22} \nabla X_2 w_1 \cdot \nabla X_2 w_1 v^2 \, dx
\]

\[
= \int_{\Omega_{t}} A_{11} \nabla X_1 v \cdot \nabla X_1 v w_1^2 \, dx + 2 \int_{\Omega_{t}} A_{12} \nabla X_2 w_1 \nabla X_1 v (vw_1) \, dx
\]

\[
+ \int_{\Omega_{t}} A_{22} \nabla X_2 w_1 \cdot \nabla X_2 w_1 v^2 \, dx.
\]

(2.44)
One remarks that the second integral is equal to 0 since \( v \in H^1_0([-\ell, \ell]^p) \). Indeed – recall that \( A_{12}, w_1 \) are depending on \( X_2 \) only – and thus we have

\[
2 \int_{\Omega_\ell} A_{12} \nabla X_2 w_1 \nabla X_1 v \cdot vw_1 \, dx
\]

\[
= \int_\omega \int_{[-\ell, \ell]^p} \nabla X_1 \cdot \{v^2 A_{12} \nabla X_2 w_1\} \, dX_1 \, dX_2 = 0.
\]

Next, due to the definition of \( w_1 \) we have – see (2.42)

\[
\int_{\Omega_\ell} A_{22} \nabla X_2 w_1 \cdot \nabla X_2 w_1 v^2 \, dx
\]

\[
= \int_{[-\ell, \ell]^p} v^2 \int_\omega A_{22} \nabla X_2 w_1 \cdot \nabla X_2 w_1 \, dX_2 \, dX_1
\]

\[
= \int_{[-\ell, \ell]^p} v^2 \mu_1 = \mu_1.
\]

Thus, from (2.44) we derive

\[
\lambda_1^\ell \leq \mu_1 + \int_{\Omega_\ell} A_{11} \nabla X_1 v \cdot \nabla X_1 vw_1^2 \, dx
\]

\[
\forall v \in H^1_0([-\ell, \ell]), \quad \int_{[-\ell, \ell]^p} v^2 \, dX_1 = 1.
\]

It follows that

\[
\lambda_1^\ell \leq \mu_1 + \int_{[-\ell, \ell]^p} |A_{11}| \|\nabla X_1 v\|^2 \, dX_1 \int_\omega w_1^2 \, dX_2
\]

\[
\leq \mu_1 + |A_{11}|_\infty \int_{[-\ell, \ell]^p} \|\nabla X_1 v\|^2 \, dX_1
\]

\[
\forall v \in H^1_0([-\ell, \ell]^p), \quad \int_{[-\ell, \ell]^p} v^2 \, dX_1 = 1.
\]

It is clear – see Section 1 – that the first eigenvalue for the Laplace operator with Dirichlet boundary conditions in \([-\ell, \ell]^p\) is given by

\[
p \left( \frac{\pi}{2\ell} \right)^2.
\]

Thus taking the infimum on \( v \) in (2.48) we obtain

\[
\lambda_1^\ell \leq \mu_1 + \frac{p \pi^2 |A_{11}|_\infty}{4\ell^2}.
\]

This completes the proof of Theorem 2.1. \(\square\)

As a corollary we have

**Theorem 2.3** (Convergence of the first eigenvalue). *Let \( \Omega_\ell' \) denote a domain in \( \mathbb{R}^n \) such that\n
\[
\Omega_\ell \subset \Omega_\ell' \subset \Omega_{\ell'}
\]
for some \( \ell' \). Let \( \lambda_{\ell}^{1} \) be the first eigenvalue of \( -\nabla \cdot (A \nabla \cdot) \) with Dirichlet boundary conditions on \( \Omega_{\ell} \) – i.e.

\[
\lambda_{\ell}^{1} = \inf \left\{ \int_{\Omega_{\ell}} A \nabla u \cdot \nabla u \, dx \mid u \in H_{0}^{1}(\Omega_{\ell}), \int_{\Omega_{\ell}} u^{2} \, dx = 1 \right\}
\]

then it holds

\[
\lim_{\ell \to +\infty} \lambda_{\ell}^{1} = \mu^{1}.
\]

**Proof.** It is easy to see from the definition – see for instance (2.52) – that the first eigenvalue of the operator \( -\nabla \cdot (A \nabla \cdot) \) with Dirichlet boundary conditions on \( \Omega \) is nonincreasing when the size of \( \Omega \) is increasing. Thus we have by (2.51)

\[
\mu^{1} \leq \lambda_{\ell}^{1} \leq \lambda_{\ell}^{1} \leq \mu^{1} + \frac{c}{\ell^{2}}
\]

and the result follows. \( \square \)

As we explained in our introduction it is reasonable to expect the \( \lambda_{\ell}^{k} \) – the \( k \)th eigenvalue of the operator \( -\nabla \cdot (A \nabla \cdot) \) with Dirichlet boundary conditions on \( \Omega_{\ell} \) converges towards \( \mu^{1} \) as \( \ell \) goes to plus infinity. Indeed we have

**Theorem 2.4 (Convergence of the \( k \)th eigenvalue \( \lambda_{\ell}^{k} \)).** With the above notation there exists a constant \( C_{k} = C(k, p, |A_{11}|_{\infty}) \) such that

\[
\mu^{1} \leq \lambda_{\ell}^{k} \leq \mu^{1} + \frac{C_{k}}{\ell^{2}}.
\]

**Proof.** Of course we can assume w.l.o.g. that \( k \geq 2 \). Let us split the domain \( \Omega_{\ell} \) in \( k \), subdomain \( Q_{i} \) in the \( x_{1} \)-direction – i.e. let us set

\[
Q_{i} = \left(-\ell + (i - 1) \frac{2\ell}{k}, -\ell + i \frac{2\ell}{k}\right) \times (-\ell, \ell)^{p-1} \times \omega \quad i = 1, \ldots, k.
\]

Moreover let us denote by \( \lambda_{Q_{i}}^{1} \) the first eigenvalue defined by

\[
\lambda_{Q_{i}}^{1} = \lambda_{Q_{i}}^{1}(A)
\]

\[
= \inf \left\{ \int_{Q_{i}} A \nabla u \cdot \nabla u \, dx \mid u \in H_{0}^{1}(Q_{i}), \int_{Q_{i}} u^{2} \, dx = 1 \right\}
\]

and by \( u_{i} \) the first eigenfunction – i.e. the only positive function achieving the infimum above. Set now

\[
Q = \left(-\ell \frac{\ell}{k}, \frac{\ell}{k}\right) \times (-\ell, \ell)^{p-1} \times \omega.
\]

Denote by \( \lambda_{Q}^{1} = \lambda_{Q}^{1}(A) \) the first eigenvalue defined by (2.57) where \( Q_{i} \) is replaced by \( Q \). Performing a translation in \( x_{1} \) in the integral

\[
\int_{Q_{i}} A \nabla u \cdot \nabla u \, dx
\]

it is easy to see that

\[
\lambda_{Q_{i}}^{1}(A) = \lambda_{Q}^{1}(\tilde{A})
\]
where $\tilde{A}(x) = A(x_1 + (2i - 1)\frac{\ell}{k} - \ell, x_2, \ldots, x_n)$. Since

$$\Omega_{\ell/k} \subset Q \subset \Omega_{\ell}$$

it follows from (2.54) that it holds

$$\mu^1 \leq \lambda^1_{Q, j}(A) = \lambda^1_{Q}(\tilde{A}) \leq \lambda^1_{\ell/k} \leq \mu^1 + \frac{pk^2\pi^2|A_{11}|_\infty}{4\ell^2}.$$  

Suppose now that $u_i$ is extended by 0 in $\Omega_{\ell} \setminus Q_i$. Denote by $u^j_i$ the $j$-th eigenfunction corresponding to $\lambda^j_{\ell}$ (our numbering of the eigenvalues is such that the $\lambda^j_{\ell}$ are not necessarily pairwise distinct). Since one has more unknowns than equations it is possible to find $\alpha_1, \ldots, \alpha_k$ not all equal to zero such that

$$u = \sum_{i=1}^k \alpha_i u_i$$

satisfies

$$\langle u, u^j_i \rangle = \sum_{i=1}^k \alpha_i \langle u_i, u^j_i \rangle = 0 \quad \forall j = 1, \ldots, k - 1.$$  

$\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega_{\ell})$. It is well known – see [9] – that

$$\lambda^1_\ell \leq \lambda^k_\ell = \inf_{v \in H^1_0(\Omega_{\ell}) \setminus \{0\}} \left\{ \frac{\int_{\Omega_{\ell}} A\nabla v \cdot \nabla v \, dx}{\int_{\Omega_{\ell}} v^2 \, dx} \right\}.$$  

Thus from (2.62) we get

$$\mu^1 \leq \lambda^k_\ell \leq \int_{\Omega_{\ell}} A\nabla u \cdot \nabla u \, dx / \int_{\Omega_{\ell}} u^2 \, dx.$$  

Since the $u_i$ have disjoint supports it follows

$$\mu^1 \leq \lambda^k_\ell \leq \sum_{i=1}^k \alpha_i^2 \int_{Q_i} A\nabla u_i \cdot \nabla u_i \, dx / \sum_{i=1}^k \alpha_i^2 \int_{Q_i} u_i^2 \, dx$$

$$= \sum_{i=1}^k \alpha_i^2 \lambda^1_{Q_i}(A) / \sum_{i=1}^k \alpha_i^2$$

(recall that $u_i$ is such that $\int_{Q_i} u_i^2 \, dx = 1$). Going back to (2.60) we obtain

$$\mu^1 \leq \lambda^k_\ell \leq \mu^1 + \frac{pk^2\pi^2|A_{11}|_\infty}{4\ell^2}$$

which completes the proof of the theorem. \hfill \square
3. Convergence of the eigenfunctions

As we have seen in the introduction the convergence of the eigenfunctions depends strongly on the scaling of them – recall that they are defined up to a multiplicative constant. To stress further this point we consider the case where the eigenfunctions are from separated variables i.e. the case when

\[ A = \begin{pmatrix} A_{11}(X_1) & 0 \\ 0 & A_{22}(X_2) \end{pmatrix}. \]

A satisfying the assumptions of the preceding section, in particular for some constants \( \lambda, \Lambda > 0 \) we have

\[ \lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^n, \ \forall \xi \in \mathbb{R}^n. \]

Let us set

\[ Q_\ell = (-\ell, \ell)^p \]

and denote by \( \lambda_\ell \) the first eigenvalue of the operator \(-\nabla \cdot (A_{11}(X_1)\nabla X_1 \cdot \cdot)\) on \( Q_\ell \) with Dirichlet boundary conditions – i.e.

\[ \lambda_\ell = \text{Inf} \left\{ \int_{Q_\ell} A_{11}(X_1)\nabla X_1 u \cdot \nabla X_1 u dX_1 \mid u \in H^1_0(Q_\ell), \int_{Q_\ell} u^2 dX_1 = 1 \right\}. \]

Due to (3.2) one has for \( u \in H^1_0(Q_\ell) \)

\[ \lambda \int_{Q_\ell} |\nabla X_1 u|^2 dX_1 \leq \int_{Q_\ell} A_{11}(X_1)\nabla X_1 u \cdot \nabla X_1 u dX_1 \]

\[ \leq \Lambda \int_{Q_\ell} |\nabla X_1 u|^2 dX_1. \]

Taking the infimum in \( u \) this leads (see (2.49)) to

\[ p\frac{\lambda \pi^2}{4\ell^2} \leq \lambda_\ell \leq p\frac{\Lambda \pi^2}{4\ell^2}. \]

We denote by \( u_\ell \) the first eigenfunction corresponding to \( \lambda_\ell \), positive and such that

\[ \int_{Q_\ell} u_\ell^2 dX_1 = \ell^p. \]

We introduce \( v_\ell \) as

\[ v_\ell(y) = u_\ell(\ell y) \quad \text{a.e. } y \in Q_1 = (-1,1)^p. \]

Then we have:

**Proposition 3.1.** \( \ell^2 \lambda_\ell \) is the first eigenvalue of the operator

\[ -\nabla \cdot (A_{11}(\ell y)\nabla \cdot) \]

on \( Q_1 \) with Dirichlet boundary conditions and \( v_\ell \) is the first normalized eigenfunction, i.e. we have in particular

\[ \int_{Q_1} A_{11}(\ell y)\nabla v \cdot \nabla v dy = \ell^2 \lambda_\ell \int_{Q_1} v^2 dy \quad \forall v \in H^1_0(Q_1), \]

\[ v_\ell \in H^1_0(Q_1), \int_{Q_1} v^2 dy = 1. \]
\((\nabla \text{ is the gradient in the } y\text{-variable}).\)

**Proof.** Clearly \(v_\ell \in H^1_0(Q_1)\). Moreover we have

\[
\int_{Q_1} v^2_\ell(y) \, dy = \int_{Q_1} u^2_\ell(\ell y) \, dy = \int_{Q_\ell} u^2_\ell(X_1) \frac{dX_1}{\ell^p} = 1.
\]

By definition of \(\lambda_\ell\) and \(u_\ell\) we have

\[
\lambda_\ell = \frac{\int_{Q_\ell} A_{11}(X_1) \nabla X_1 \cdot \nabla X_1 u_\ell \, dX_1}{\int_{Q_\ell} u^2_\ell \, dX_1} = \frac{\int_{Q_\ell} A_{11}(X_1) \nabla X_1 \cdot \nabla X_1 u_\ell \, dX_1}{\int_{Q_\ell} u^2_\ell \, dX_1} \leq \frac{\int_{Q_\ell} A_{11}(X_1) \nabla v \cdot \nabla v \, dX_1}{\int_{Q_\ell} v^2 \, dX_1} = \ell^p.
\]

One notices that by (3.8)

\[
\nabla v_\ell(y) = \ell \nabla X_1 u_\ell(\ell y).
\]

Moreover for every \(w \in H^1_0(Q_1)\) with \(\int_{Q_1} w^2 \, dy = 1\) we have

\[
v(X_1) = w \left( \frac{X_1}{\ell} \right) \in H^1_0(Q_\ell), \quad \nabla X_1 v(X_1) = \frac{1}{\ell} \nabla w \left( \frac{X_1}{\ell} \right)
\]

and

\[
\int_{Q_\ell} v^2(X_1) \, dX_1 = \ell p.
\]

From these remarks we derive for every \(w \in H^1_0(Q_1)\), \(\int_{Q_1} w^2 \, dy = 1\):

\[
\lambda_\ell = \frac{1}{\ell^2} \int_{Q_1} A_{11}(\ell y) \nabla v_\ell \cdot \nabla v_\ell \, dy \leq \frac{1}{\ell^2} \int_{Q_1} A_{11}(\ell y) \nabla w \cdot \nabla w \, dy.
\]

This shows that

\[
\ell^2 \lambda_\ell \text{ is the first eigenvalue of } -\nabla \cdot (A_{11}(\ell y) \nabla \cdot ) \text{ on } Q_1,
\]

\[
\ell^2 \lambda_\ell \text{ is the first eigenfunction of } -\nabla \cdot (A_{11}(\ell y) \nabla \cdot ) \text{ on } Q_1,
\]

for the Dirichlet boundary conditions and this completes the proof of the proposition. \(\square\)

We now study the asymptotic behaviour of \(v_\ell\). For this we suppose that

\[
\lim_{|X_1| \to +\infty} A_{11}(X_1) = A_\infty
\]

where \(A_\infty\) is a constant \(p \times p\) matrix. Then we have

**Proposition 3.2.** Under the assumption (3.18) it holds that

\[
\ell^2 \lambda_\ell \to \lambda_\infty, \quad v_\ell \to v_\infty \text{ in } H^1_0(Q_1)
\]
where \( \lambda_\infty \) is the first eigenvalue, \( v_\infty \) the first normalized eigenfunction of the operator
\(-\nabla \cdot (A_\infty \nabla \cdot)\) with Dirichlet boundary conditions on \( Q_1 \) — i.e. in particular

\[
\int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v \, dy = \lambda_\infty \int_{Q_1} v_\infty v \, dy \quad \forall \, v \in H^1_0(Q_1),
\]

\[
v_\infty \in H^1_0(Q_1), \int_{Q_1} v_\infty^2 \, dy = 1.
\]

**Proof.** Taking \( v = v_\ell \) in the first equation of (3.10) and using (3.6), (3.2) we get

\[
\lambda \int_{Q_1} |\nabla v_\ell|^2 \, dy \leq \int_{Q_1} A_{11}(\ell y) \nabla v_\ell \cdot \nabla v_\ell \, dy = \ell^2 \lambda_\ell \leq \frac{\Lambda}{4 \pi^2}.
\]

Thus \( \ell^2 \lambda_\ell \) is bounded and \( v_\ell \) is bounded in \( H^1_0(Q_1) \) and up to a subsequence we can assume that for some \( \lambda_\infty, v_\infty \) it holds that

\[
\ell^2 \lambda_\ell \to \lambda_\infty,
\]

\[
v_\ell \to v_\infty \quad \text{in} \quad H^1_0(Q_1), \quad v_\ell \to v_\infty \quad \text{in} \quad L^2(Q_1).
\]

We would like to identify these limits. First by (3.18) we have

\[
A_{11}(\ell y) \nabla v \to A_\infty \nabla v \quad \text{in} \quad L^2(Q_1)
\]

for any \( v \in H^1_0(Q_1) \). Thus passing to the limit in (3.10) — recall that \( A_{11} \) is symmetric — we obtain

\[
\int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v_\ell \, dy = \lambda_\infty \int_{Q_1} v_\infty v_\ell \, dy \quad \forall \, v \in H^1_0(Q_1),
\]

\[
v_\infty \in H^1_0(Q_1), \int_{Q_1} v_\infty^2 \, dy = 1.
\]

Thus \( v_\infty \) is an eigenfunction of \(-\nabla \cdot (A_\infty \nabla \cdot)\). We also have by (3.10)

\[
\int_{Q_1} A_{11}(\ell y) \nabla v_\ell \cdot \nabla v_\ell \, dy = \ell^2 \lambda_\ell \to \lambda_\infty.
\]

It follows that

\[
\int_{Q_1} A_{11}(\ell y) \nabla (v_\ell - v_\infty) \cdot \nabla (v_\ell - v_\infty) \, dy
\]

\[
= \int_{Q_1} A_{11}(\ell y) \nabla v_\ell \cdot \nabla v_\ell \, dy - 2 \int_{Q_1} A_{11}(\ell y) \nabla v_\infty \cdot \nabla v_\ell
\]

\[
+ \int_{Q_1} A_{11}(\ell y) \nabla v_\infty \cdot \nabla v_\infty \, dy
\]

\[
\to \lambda_\infty - 2 \int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v_\infty \, dy + \int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v_\infty \, dy
\]

\[
= \lambda_\infty - \int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v_\infty.
\]

It results from (3.25) — with \( v = v_\infty \) — that this last quantity is 0. It follows then that

\[
v_\ell \to v_\infty \quad \text{in} \quad H^1_0(Q_1).
\]
Going back to (3.15) we have

\[(3.29) \quad \ell^2 \lambda_\ell \leq \int_{Q_1} A_{11}(\ell y) \nabla w \cdot \nabla w \, dy \quad \forall w \in H^1_0(Q_1), \int_{Q_1} w^2 \, dy = 1.\]

Passing to the limit we get

\[(3.30) \quad \lambda_\infty = \int_{Q_1} A_\infty \nabla v_\infty \cdot \nabla v_\infty \, dy \leq \int_{Q_1} A_\infty w \cdot \nabla w \, dy \quad \forall w \in H^1_0(Q_1), \int_{Q_1} w^2 \, dy = 1.\]

This shows that \(\lambda_\infty\) is the first eigenvalue of the operator \(-\nabla \cdot (A_\infty \nabla \cdot )\) and \(v_\infty\) its first normalized (positive) eigenfunction. Since \(\lambda_\infty, v_\infty\) are uniquely determined we have obtained that (3.28), (3.22) hold for every subsequence and thus the whole sequences converge. This completes the proof of the proposition.

In addition we can show:

**Proposition 3.3.** Under the assumptions of Proposition 3.2 it holds that

\[(3.31) \quad |v_\ell - v_\infty|_{\infty, Q_1} \to 0\]

when \(\ell \to +\infty\). \(| \cdot |_{\infty, Q_1}\) denotes the usual \(L^\infty(Q_1)\)-norm.

**Proof.** First it is clear from the usual regularity results for elliptic equations that \(v_\infty\) is a smooth function. Moreover, combining (3.10) and (3.20) we have in a weak sense

\[-\nabla \cdot (A_{11}(\ell y) \nabla v_\ell) - \ell^2 \lambda_\ell v_\ell = 0 = -\nabla \cdot (A_\infty \nabla v_\infty) - \lambda_\infty v_\infty.\]

This implies

\[-\nabla \cdot (A_{11}(\ell y) \nabla (v_\ell - v_\infty)) - \ell^2 \lambda_\ell (v_\ell - v_\infty) = \nabla \cdot ([A_{11}(\ell y) - A_\infty] \nabla v_\infty) - (\lambda_\infty - \ell^2 \lambda_\ell) v_\infty.\]

Since \(\ell^2 \lambda_\ell\) is bounded from below independently of \(\ell\), using the well known \(L^\infty\)-estimates of Stampacchia (see [8] or [10]) we obtain

\[|v_\ell - v_\infty|_{\infty, Q_1} \leq C\left\{||A_{11}(\ell y) - A_\infty||_{p, Q_1} |\nabla v_\infty|_{p, Q_1} + |\lambda_\infty - \ell^2 \lambda_\ell||v_\infty|_{p, Q_1} + |v_\ell - v_\infty|_{2, Q_1}\right\}\]

where \(C\) is a constant independent of \(\ell\), \(| \cdot |_{p, Q_1}\) denotes the usual \(L^p(Q_1)\)-norm, \(p > n\). The results follow then from the Lebesgue theorem and Proposition 3.2.

We can now state a convergence theorem for the eigenfunctions. We have

**Theorem 3.4** (Convergence of the eigenfunction). Let \(A\) be a symmetric matrix satisfying (3.1), (3.2), (3.18). Let \(u_{1, \ell}\) be the first eigenfunction (positive) and normalized by

\[(3.32) \quad \int_{\Omega_\ell} u_{1, \ell}^2 \, dx = \ell^p,\]
corresponding to $\lambda_1^\ell$ (see (2.9)). Let $w_1$ be the first positive eigenfunction realizing the infimum of (2.10). Then for any $\ell_0 > 0$ we have
\begin{equation}
|u_{1,\ell} - v_\infty(0)w_1|_{\infty, \Omega_{\ell_0}} \to 0
\end{equation}
when $\ell \to +\infty$. $\cdot |_{\infty, \Omega_{\ell_0}}$ is the usual $L^\infty(\Omega_{\ell_0})$-norm, $v_\infty$ has been defined in (3.20).

Proof. We claim that
\begin{equation}
(3.34)
\end{equation}
$u_{1,\ell} = u_\ell w_1$
where $u_\ell$ has been defined above (3.7). It is indeed clear from (3.7) that
\begin{equation}
(3.35)
u_\ell w_1 \in H^1(\Omega_\ell), \quad \int_{\Omega_\ell} (u_\ell w_1)^2 \, dx = \ell^p.
\end{equation}
Moreover for $v \in H^1_0(\Omega_\ell)$ it holds
\begin{equation}
\int_{\Omega_\ell} A\nabla v \cdot \nabla v \, dx = \int_{\Omega_\ell} A_{11}\nabla X_1 v \cdot \nabla X_1 v \, dx + \int_{\Omega_\ell} A_{22}\nabla X_2 v \cdot \nabla X_2 v \, dx
= \int_\omega \int_{Q_\ell} A_{11}(X_1) \nabla X_1 v \cdot \nabla X_1 v \, dX_1 \, dX_2
+ \int_{Q_\ell} \int_\omega A_{22}(X_2) \nabla X_2 v \cdot \nabla X_2 v \, dX_2 \, dX_1.
\end{equation}
Using the definitions of $\lambda_\ell$, $\mu_1$ respectively (and the fact that $v(\cdot, X_2) \in H^1_0(Q_\ell)$ a.e. $X_2$) we get
\begin{equation}
(3.36)\int_{\Omega_\ell} A\nabla v \cdot \nabla v \, dx \geq \int_\omega \int_{Q_\ell} \lambda_\ell v^2 \, dX_1 \, dX_2 + \int_{Q_\ell} \mu_1 \int_\omega v^2 \, dX_2 \, dX_1
= (\lambda_\ell + \mu_1) \int_{Q_\ell} v^2 \, dX_1 \, dX_2.
\end{equation}
Now one has also
\begin{equation}
\int_{Q_\ell} A\nabla (u_\ell w_1) \cdot \nabla (u_\ell w_1) \, dx = \int_\omega \int_{Q_\ell} A_{11}(X_1) \nabla X_1 u_\ell \cdot \nabla X_1 u_\ell \cdot w_1^2 \, dx + \int_{Q_\ell} \int_\omega A_{22}(X_2) \nabla X_2 w_1 \cdot \nabla X_2 w_1 u_\ell^2 \, dx
= (\lambda_\ell + \mu_1) \int_{\Omega_\ell} (u_\ell w_1)^2 \, dx.
\end{equation}
This shows that
\begin{equation}
\lambda_1^\ell = \lambda_\ell + \mu_1, \quad u_{1,\ell} = u_\ell w_1.
\end{equation}
Using now (3.8) we have
\begin{equation}
(3.34)
u_{1,\ell}(x) - v_\infty(0)w_1 = u_\ell(X_1)w_1(X_2) - v_\infty(0)w_1(X_2)
= \left\{ v_\ell \left( \frac{X_1}{\ell} \right) - v_\infty(0) \right\} w_1(X_2)
= \left\{ v_\ell \left( \frac{X_1}{\ell} \right) - v_\infty \left( \frac{X_1}{\ell} \right) + v_\infty \left( \frac{X_1}{\ell} \right) - v_\infty(0) \right\} w_1(X_2)
\end{equation}
and thus

\[ |u_{1,\ell}(x) - v_\infty(0)w_1|_{\infty, \Omega_0} \leq |w_1|_{\infty, \Omega_0} \left\{ |v_\ell - v_\infty|_{\infty, Q_1} + \left| v_\infty \left( \frac{X_1}{\ell} \right) - v_\infty(0) \right|_{\infty, Q_{\ell_0}} \right\}. \]

The result follows then from (3.31) and from the continuity of \( v_\infty \). \( \square \)

**Remark 3.1.** The scaling of the eigenfunctions is a very delicate issue. For instance (3.32) is not enough to insure the convergence of \( u_{1,\ell} \) toward \( w_1 \), one has to further scale by \( v_\infty(0) \) a constant depending on the behaviour of the operator at \( \infty \). From (3.34) we always have

\[ u_{1,\ell} = v_\ell \left( \frac{X_1}{\ell} \right) w_1(X_2). \]

Now the behaviour of \( v_\ell \) is not so easy to control. For instance (see (3.10)) if \( A_{11} \) is periodic then \( v_\ell \) is the solution of a homogenization problem for which only weak convergence is known (see [4]).

We consider now the case of \( k \)-th eigenfunctions. For that we denote by \( \lambda_{k,\ell} \) the \( k \)-th eigenvalue of the operator \(-\nabla_{X_1} \cdot (A_{11}(X_1)\nabla_{X_1})\) on \( Q_\ell \) with Dirichlet boundary conditions. We denote by \( u_k^\ell \) a corresponding eigenfunction chosen such that

\[ u_k^\ell \perp u_j^\ell \quad \forall \ j < k, \quad \int_{Q_\ell} (u_k^\ell)^2 dX_1 = \ell^p. \]

Arguing as in the introduction it is not difficult to see that the eigenvalues of the operator \(-\nabla \cdot (A\nabla \cdot)\) with Dirichlet boundary conditions on \( \Omega_\ell \) are given by

\[ \lambda_{k,\ell} + \mu^m \]

where the \( \mu^m \) are the eigenvalues of the operator \(-\nabla_{X_2} (A_{22} \nabla_{X_2})\) with Dirichlet boundary conditions on \( \omega \).

We can first show:

**Proposition 3.5.** Assuming (3.2) we have

\[ \frac{\pi^2}{4\ell^2} \leq \lambda_{k,\ell} \leq \frac{\Lambda k^2 \pi^2}{4ell^2}. \]

**Proof.** The proof is similar to the proof of Theorem 2.4 and we only sketch it. We split the domain \( Q_\ell \) in \( k \) subdomain \( Q_i \) in the \( x_1 \)-direction by setting

\[ Q_i = \left( -\ell + (i - 1) \frac{2\ell}{k}, -\ell + i \frac{2\ell}{k} \right) \times (-\ell, \ell)^{p-1} \quad i = 1, \ldots, k. \]

We set

\[ \lambda_{1, Q_i}(A_{11}) = \inf \left\{ \int_{Q_i} A_{11} \nabla_{X_1} u \cdot \nabla_{X_1} u \ dx \mid u \in H_0^1(Q_i), \int_{Q_i} u^2 dX_1 = 1 \right\}. \]

We also define

\[ Q = \left( -\ell \frac{k}{k}, \ell \frac{k}{k} \right) \times (-\ell, \ell)^{p-1} \]
and denote by $\lambda_{1,Q}(A_{11})$ the first eigenvalue defined by (3.41) where $Q_i$ is replaced by $Q$. It is easy to show (compare to (2.59)) that

\[(3.43)\]  
$\lambda_{1,Q_i}(A_{11}) = \lambda_{1,Q}(\tilde{A}_{11})$

where $\tilde{A}_{11} = A_{11}(x_1 + (2i - 1)\ell - i, x_2, \ldots, x_p)$. Since

\[(3.44)\]  
$Q_{\ell} \subset Q$

it holds that (see (3.6))

\[(3.45)\]  
$\lambda_{1,Q_i}(A_{11}) = \lambda_{1,Q}(\tilde{A}_{11}) \leq \frac{pAk^2\pi^2}{4\ell^2}$.

Then one proceeds as in below (2.60) to get (3.39).

As a consequence of (3.38), (3.39), for $\ell$ large enough, a $k$-th eigenvalue of the operator $-\nabla \cdot (A\nabla \cdot)$ with Dirichlet boundary condition on $\Omega_\ell$ is given by

\[(3.46)\]  
$\lambda_k^\ell = \lambda_{k,\ell} + \mu^1$

and a $k$-th normalized eigenfunction is given by

\[(3.47)\]  
$u_{k,\ell} = u_k^\ell(X_1) \cdot w_1(X_2)$,

recall definition (3.37).

Let us set

\[(3.48)\]  
$v_k^\ell(y) = u_k^\ell(\ell y), \quad y \in Q_1$.

Then we have

**Proposition 3.6.** $\ell^2 \lambda_{k,\ell}$ is a $k$-th eigenvalue of the operator

\[(3.49)\]  
$-\nabla \cdot (A_{11}(\ell y)\nabla \cdot)$

on $Q_1$ with Dirichlet boundary conditions and $v_k^\ell$ is a $k$-th normalized eigenfunction.

**Proof.** This follows from the fact that $v_k^\ell$ satisfies (3.10) with $\lambda_\ell$ replaced by $\lambda_{k,\ell}$ and from the equalities

\[(3.50)\]  
$\int_{Q_1} v_k^\ell(y)v_j^\ell(y)dy = \int_{Q_1} u_k^\ell(\ell y)u_j^\ell(\ell y)dy$

$= \int_{Q_\ell} u_k^\ell(X_1)u_j^\ell(X_1)\frac{dX_1}{\ell^p} = \delta_{k,j}$.

**Proposition 3.7.** Under the assumption (3.18) there is a subsequence of $\ell$ such that

\[(3.51)\]  
$\ell^2 \lambda_{k,\ell} \to \lambda_{k,\infty}, \quad v_k^\ell \to v_k^\infty \text{ in } H^1_0(Q_1)$

where $\lambda_{k,\infty}$ is a $k$-th eigenvalue, $v_k^\infty$ a $k$-normalized eigenfunction of the operator $-\nabla (A_\infty \nabla \cdot)$ with Dirichlet boundary conditions on $Q_1$. 

Proof. One proceeds by induction on $k$. For $k = 1$ the result has already been proven. Assuming this holds for $k - 1$ we have to establish it for $k$. First it follows from (3.39) that $\ell^2\lambda_{k,\ell}$ and $v_k^\ell$ are bounded. One can then extract a subsequence and pass to the limit as in Proposition 3.2. The fact that $v_\infty^k$ is a $k$-th eigenfunction follows from (3.50) by passing to the limit in $\ell$. □

Remark 3.2. If $\lambda_{k,\infty}$ is simple, the whole sequence of eigenvalues or eigenfunctions converges.

Then we have

**Theorem 3.8** (Convergence of the $k$-th eigenfunctions). *Under the above assumptions for any $\ell_0 > 0$ there exists a subsequence of $\ell$ such that*

$$|u_{k,\ell} - v_\infty^k(0)|_{2,\Omega_{\ell_0}} \to 0.$$  

*Proof.* This follows the steps of Proposition 3.3 and Theorem 3.4. □

4. A non resonance case

The estimate (2.14) of Theorem 2.1 will allow us to prove a convergence result for invertible linear elliptic operators with zero order terms. More precisely, let the matrix $A = A(x)$ be given by (2.4) and satisfying (2.5), (2.6). For $O = \Omega_\ell$ or $\omega$, we denote by $H^{-1}(O)$ the dual space of $H_0^1(O)$. Let also $a_\infty \in L^\infty(\omega)$, $f \in H^{-1}(\omega)$ and, for all $\ell > 0$, $a_\ell \in L^\infty(\omega)$. We consider the problems

$$\begin{align*}
\begin{cases}
  u_\ell \in H_0^1(\Omega_\ell), \\
  -\nabla \cdot (A(x)\nabla u_\ell) = a_\ell(X_2)u_\ell + f \quad \text{in } H^{-1}(\Omega_\ell),
\end{cases}
\end{align*}$$

and

$$\begin{align*}
\begin{cases}
  u_\infty \in H_0^1(\omega), \\
  -\nabla \cdot (A_{22}(X_2)\nabla u_\infty) = a_\infty(X_2)u_\infty + f \quad \text{in } H^{-1}(\omega).
\end{cases}
\end{align*}$$

Defining $\lambda_\ell^1$ and $\mu^1$ as in (2.12), (2.13), we have the

**Theorem 4.1.** *We assume that, for some $\mu$ satisfying*

$$\mu < \mu^1,$$

*there holds*

$$a_\ell \leq \mu, \quad a_\infty \leq \mu \quad \text{a.e. in } \omega, \forall \ell > 0.$$  

*Then for all $\ell_0, r > 0$, there exists a constant $C$ such that for all $\ell > 0$,

$$||\nabla(u_\ell - u_\infty)||_{2,\Omega_{\ell_0}} \leq C|a_\ell - a_\infty|_{\infty,\omega} + C\frac{1}{\ell^r}.$$  

We first prove the

**Lemma 4.2.** *Under the above assumptions, the problems (4.1) and (4.2) admit a unique solution.*
Proof. We only prove the lemma for (4.1). By (4.4) and (2.14), for all \( v \in H^1_0(\Omega) \), it holds that
\[
\int_{\Omega_\ell} a_\ell(X_2)v^2 \, dx \leq \mu \int_{\Omega_\ell} v^2 \, dx \leq \frac{\mu}{\lambda_1} \int_{\Omega_\ell} A \nabla v \cdot \nabla v \, dx
\]
(4.6)
\[
\leq \frac{\mu}{\mu_1} \int_{\Omega_\ell} A \nabla v \cdot \nabla v \, dx.
\]
Hence with (4.3), (2.6),
\[
\int_{\Omega_\ell} A \nabla v \cdot \nabla v \, dx - \int_{\Omega_\ell} a_\ell(X_2)v^2 \, dx \geq (1 - \frac{\mu}{\mu_1}) \int_{\Omega_\ell} A \nabla v \cdot \nabla v \, dx
\]
\[
\geq c_1 \int_{\Omega_\ell} |\nabla v|^2 \, dx
\]
where \( c_1 > 0 \). We conclude using the Lax Milgram Theorem. \( \square \)

Proof of Theorem 4.1. Let \( v_\ell \) be the solution of
\[
\begin{align*}
\left\{ v_\ell \in H^1_0(\omega), \\
-\nabla \cdot (A_{22}(X_2)\nabla v_\ell) = a_\ell(X_2)v_\ell + f \quad \text{in } H^{-1}(\omega).
\end{align*}
\]
(4.7)
For all \( \varphi \in H^1_0(\Omega_\ell) \), we have clearly
\[
\int_{\Omega_\ell} A \nabla v_\ell \cdot \nabla \varphi \, dx = \int_{\Omega_\ell} a_\ell(X_2)v_\ell \varphi \, dx + \langle f, \varphi \rangle.
\]
Also, the solution \( u_\ell \) of (4.1) satisfies
\[
\int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla \varphi \, dx = \int_{\Omega_\ell} a_\ell(X_2)u_\ell \varphi \, dx + \langle f, \varphi \rangle.
\]
Subtracting this two identities and choosing \( \varphi = (u_\ell - v_\ell)\varrho^2(\frac{X_1}{\ell_1}) \) where \( 0 < \ell_1 < \ell \) and \( \varrho \) is a smooth function satisfying
\[
\varrho = 1 \quad \text{on } \left( -\frac{1}{2}, \frac{1}{2} \right)^p, \quad \varrho = 0 \quad \text{outside } (-1,1)^p, \quad |\nabla X_1 \varrho| \leq C,
\]
where \( C \) is some positive constant, we derive using also (4.6),
\[
\int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla \{ (u_\ell - v_\ell)\varrho^2 \} \, dx
\]
\[
= \int_{\Omega_\ell} a_\ell(X_2)(u_\ell - v_\ell)^2 \varrho^2 \, dx
\]
\[
\leq \frac{\mu}{\mu_1} \int_{\Omega_\ell} A \nabla \{ (u_\ell - v_\ell)\varrho \} \cdot \nabla \{ (u_\ell - v_\ell)\varrho \} \, dx.
\]
ON THE ASYMPTOTIC BEHAVIOUR OF THE EIGENMODES

(We denote, for simplicity $\varrho(\frac{X}{\ell_1})$ by $\varrho$.) Now this latter integral equals to

$$\int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla (u_\ell - v_\ell) \varrho^2 \, dx$$

$$+ 2 \int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla \varrho (u_\ell - v_\ell) \varrho \, dx + \int_{\Omega_\ell} A \nabla \varrho \cdot \nabla \varrho (u_\ell - v_\ell)^2 \, dx.$$

Therefore, denoting now $\nabla \varrho(\frac{X}{\ell_1})$ by $\nabla \varrho$,

$$\left(1 - \frac{\mu}{\mu^1}\right) \int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla (u_\ell - v_\ell) \varrho^2 \, dx \leq$$

$$- \frac{2}{\ell_1} \int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla \varrho (u_\ell - v_\ell) \varrho \, dx +$$

$$\frac{2\mu}{\mu^1 \ell_1} \int_{\Omega_\ell} A \nabla (u_\ell - v_\ell) \cdot \nabla \varrho (u_\ell - v_\ell) \varrho \, dx +$$

$$\frac{\mu}{\mu^1 \ell_1^2} \int_{\Omega_\ell} A \nabla \varrho \cdot \nabla \varrho (u_\ell - v_\ell)^2 \varrho^2 \, dx.$$

With (2.6), (2.5) and the Cauchy-Schwarz inequality, we then obtain, for a new positive constant $C$,

$$\int_{\Omega_\ell} |\nabla (u_\ell - v_\ell)|^2 \varrho^2 \, dx \leq \frac{C}{\ell_1} \left( \int_{\Omega_\ell} |\nabla (u_\ell - v_\ell)|^2 \varrho^2 \, dx \right)^{1/2} \left( \int_{\Omega_\ell} (u_\ell - v_\ell)^2 \, dx \right)^{1/2}$$

$$+ \frac{C}{\ell_1^2} \int_{\Omega_\ell} (u_\ell - v_\ell)^2 \varrho^2 \, dx.$$

At this stage, we use the following “anisotropic” Poincaré inequality: there exists a constant $C(\omega)$ depending only on $\omega$ such that, for every $v \in H^1_0(\Omega_\ell)$,

$$\int_{\Omega_\ell} v^2 \, dx \leq C(\omega) \int_{\Omega_\ell} |\nabla X_2 v|^2 \, dx$$

(see [2] or [3] for a proof). Then we get

$$\int_{\Omega_{\ell_{1/2}}} |\nabla (u_\ell - v_\ell)|^2 \, dx \leq \frac{C}{\ell_1^2} \int_{\Omega_{\ell_{1/2}}} |\nabla (u_\ell - v_\ell)|^2 \, dx,$$

from which we deduce arguing as in [2],

$$||\nabla (u_\ell - v_\ell)||_{2,\Omega_{\ell_0}} \leq \frac{C}{\ell^r}.$$

Let us now approxime $u_\infty$ by $v_\ell$. Subtracting (4.2) from (4.7), we obtain, for all $\varphi \in H^1_0(\omega)$,

$$\int_\omega A_{22}(X_2) \nabla_2 (v_\ell - u_\infty) \cdot \nabla_2 \varphi \, dX_2 = \int_\omega (a_\ell v_\ell - a_\infty u_\infty) \varphi \, dX_2$$

$$= \int_\omega (a_\ell - a_\infty) u_\infty \varphi + \int_\omega a_\ell (v_\ell - u_\infty) \varphi \, dX_2.$$
Choosing \( \varphi = v_\ell - u_\infty \), we infer with the Cauchy-Schwarz inequality,

\[
\int_\omega A_{22}(X_2) \nabla_2 (v_\ell - u_\infty) \cdot \nabla_2 (v_\ell - u_\infty) \, dX_2 \leq |a_\ell - a_\infty|_{\infty, \omega} |u_\infty|_{2, \omega} |v_\ell - u_\infty|_{2, \omega} + \int_\omega a_\ell (v_\ell - u_\infty)^2 \, dX_2.
\]

We estimate this latter integral by

\[
\frac{\mu}{\mu^1} \int_\omega A_{22}(X_2) \nabla_2 (v_\ell - u_\infty) \cdot \nabla_2 (v_\ell - u_\infty) \, dX_2
\]

(see (4.6)) and use the Poincaré inequality, to obtain

\[
||\nabla (v_\ell - u_\infty)||_{2, \omega} \leq C |a_\ell - a_\infty|_{\infty, \omega},
\]

which implies

\[
(4.9) \quad ||\nabla (v_\ell - u_\infty)||_{2, \Omega_{\ell 0}} \leq C_1 |a_\ell - a_\infty|_{\infty, \omega}.
\]

The theorem follows combining (4.8) and (4.9). \( \square \)

We deduce immediately from Theorem 4.1 the

**Corollary 4.3.** Under the assumption of Theorem 4.1, assume, in addition, that

\[
a_\ell (\cdot) \equiv a_\infty (\cdot) \leq \mu < \mu^1 \quad a.e \text{ in } \omega, \forall \ell > 0.
\]

Then for every \( \ell_0, r > 0 \), there exists a constant \( C \) such that for all \( \ell > 0 \),

\[
||\nabla (u_\ell - u_\infty)||_{2, \Omega_{\ell_0}} \leq \frac{C}{\ell^r}.
\]

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