SOME REMARKS ON LIOUVILLE TYPE THEOREMS

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Abstract: The goal of this note is to present elementary proofs of statements related to the Liouville theorem.

1. Introduction
We denote by $A(x) = (a_{ij}(x))$ a $(k \times k)$-matrix where the functions $a_{ij}$, $i,j = 1, \ldots , k$ are bounded measurable functions defined on $\mathbb{R}^k$ and which satisfy, for some $\lambda, \Lambda > 0$, the usual uniform ellipticity condition:

$$\lambda |\xi|^2 \leq (A(x)\xi \cdot \xi) \leq \Lambda |\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^k, \forall \xi \in \mathbb{R}^k. \quad (1.1)$$

We address here the issue of existence of solutions to the equation:

$$-\nabla \cdot (A(x)\nabla u(x)) + a(x)u(x) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k), \quad (1.2)$$

where $a \in L^\infty_{\text{loc}}(\mathbb{R}^k)$ and $a \geq 0$. When $a \neq 0$ and $\nabla \cdot (A\nabla) = \Delta$, the usual Laplace operator, the above equation is the so called stationary Schrödinger equation for which a vast literature is available (see [16], [20]). When $a = 0$ it is well known that every bounded solution to (1.2) has to be constant (see e.g. [5], [11], [12] and also [4], [19] for some nonlinear versions). The case where $a \neq 0$, and $k \geq 3$ is very different and in this case non trivial bounded solutions might exist.

Many of the results in this paper are known in one form or another (see for instance [1], [2], [3], [10], [9], [14], [16], [17]) but we have tried to develop here simple self-contained pde techniques which do not make use of
probabilities, semigroups or potential theory as is sometimes the case (see e.g. [2], [3], [8], [17], [18]). One should note that some of our proofs extend also to elliptic systems.

This note is divided as follows. In the next section we introduce an elementary estimate which is used later. In Section 3 we present some Liouville type results, i.e., we show that under some conditions on \( a \), (1.2) does not admit nontrivial bounded solutions. Finally in the last section we give an almost sharp criterion for the existence of nontrivial solutions.

2. A preliminary estimate

Let us denote by \( \Omega \) a bounded open subset of \( \mathbb{R}^k \) with Lipschitz boundary and starshaped with respect to the origin. For any \( r \in \mathbb{R} \) we set

\[
\Omega_r = r\Omega.
\]  

(2.1)

Let us denote by \( \varrho \) a smooth function such that

\[
0 \leq \varrho \leq 1, \quad \varrho = 1 \text{ on } \Omega_{1/2}, \quad \varrho = 0 \text{ outside } \Omega,
\]

(2.2)

\[
|\nabla \varrho| \leq c_\varrho,
\]

(2.3)

where \( c_\varrho \) denotes some positive constant.

Lemma 2.1. Suppose that \( u \in H^1_{\text{loc}}(\mathbb{R}^k) \) satisfies (1.2) with \( A(x) \) satisfying (1.1). Then there exists a constant \( C \) independent of \( r \) such that

\[
\int_{\Omega_r} \{|\nabla u|^2 + au^2\} \varrho^2 \left(\frac{x}{r}\right) \, dx \\
\leq \frac{C}{r} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u|^2 \varrho^2 \left(\frac{x}{r}\right) \, dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \right\}^{\frac{1}{2}},
\]

(2.4)

where \( |\cdot| \) denotes the usual euclidean norm in \( \mathbb{R}^k \).

Proof. By (1.2) we have for every \( v \in H^1_0(\Omega_r) \)

\[
\int_{\Omega_r} A\nabla u \cdot \nabla v + auv \, dx = 0.
\]

(2.5)

Taking

\[
v = u\varrho^2 \left(\frac{x}{r}\right) = u\varrho^2
\]

(2.6)

yields

\[
\int_{\Omega_r} A\nabla u \cdot \nabla (u\varrho^2) + au^2 \varrho^2 \, dx = 0.
\]

(2.7)
Since
\[ \nabla g^2 = \frac{2\rho}{r} \nabla \rho \left( \frac{x}{r} \right) \]
we obtain
\[
\int_{\Omega_r} \{A \nabla u \cdot \nabla u\} g^2 + au^2 g^2 \, dx = -\frac{1}{r} \int_{\Omega_r \setminus \Omega_{r/2}} A \nabla u \cdot \nabla \rho \cdot 2\rho u \, dx \\
\leq \frac{C_1}{r} \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u| |u| g \, dx,
\]
where \( C_1 \) is a constant depending on \( a_{ij} \) and \( c_\rho \) only. Using the ellipticity condition (1.1) it follows easily that
\[
\min(1, \lambda) \int_{\Omega_r} \{|\nabla u|^2 + au^2\} g^2 \, dx \leq \frac{C_1}{r} \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u| |u| g \, dx.
\]
By the Cauchy–Schwarz inequality we have
\[
\int_{\Omega_r} \{|\nabla u|^2 + au^2\} g^2 \, dx \\
\leq \frac{C_1}{r \min(1, \lambda)} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u|^2 g^2 \, dx \right\}^{1/2} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \right\}^{1/2}.
\]
This completes the proof of the lemma. \( \square \)

3. Some Liouville type results

3.1. The case where the growth of \( u \) is controlled

In this case we have

**Theorem 3.1.** Under the assumptions of Lemma 2.1, let \( u \) be solution to (1.2) such that for \( r \) large,
\[
\frac{1}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \leq C'
\]  
(3.1)

where \( C' \) is a constant independent of \( r \), then \( u = \text{constant} \) and if \( a \neq 0 \) or \( k \geq 3 \) one has \( u = 0 \).

**Proof.** From (2.4) we derive that
\[
\int_{\Omega_r} |\nabla u|^2 g^2 \, dx \leq \frac{C}{r} \left\{ \int_{\Omega_r} |\nabla u|^2 g^2 \, dx \right\}^{1/2} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \right\}^{1/2}
\]
and thus
\[ \int_{\Omega_r/2} |\nabla u|^2 \, dx \leq \int_{\Omega_r} |\nabla u|^2 \, dx \leq \frac{C}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \leq CC'. \]

It follows that the nondecreasing function
\[ r \mapsto \int_{\Omega_r} |\nabla u|^2 \, dx \]

is bounded and has a limit when \( r \to +\infty \). Going back to (2.4) we have
\[ \int_{\Omega_r/2} |\nabla u|^2 \, dx \leq \frac{C}{r} \left( \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \cdot r \]

for some constant \( c \). This implies
\[ \int_{\Omega_r/2} |\nabla u|^2 \, dx \leq c \left( \int_{\Omega_r} |\nabla u|^2 \, dx - \int_{\Omega_r/2} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \to 0 \]

as \( r \to +\infty \) and the result follows. \( \square \)

**Remark 3.1.** When \( k \leq 2 \) condition (3.1) is satisfied if \( u \) is bounded, and in this case the only bounded solution of (1.2) is \( u = 0 \). Therefore we will assume throughout the rest of this paper that \( k \geq 3 \).

We denote by \( \lambda_r \) the first eigenvalue of the Neumann problem associated to the operator \(- \nabla \cdot A \nabla + a\) in \( \Omega_r \setminus \Omega_{r/2} \), i.e., we set
\[ \lambda_r = \inf \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} A \nabla u \cdot \nabla u + au^2 \, dx : u \in H^1(\Omega_r \setminus \Omega_{r/2}), \right. \]
\[ \left. \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx = 1 \right\}. \quad (3.2) \]

One remarks easily that if \( u \) is a minimizer of (3.2) so is \(|u|\). One can show then that the first eigenvalue is simple. Moreover we have

**Theorem 3.2.** Under the assumptions of Lemma 2.1, suppose that for some constants \( C_0 > 0, \beta < 2 \), one has
\[ \lambda_r \geq C_0/r^\beta \quad (3.3) \]

for \( r \) sufficiently large, then the only bounded solution of (1.2) is \( u = 0 \).
Proof. From the definition of $\lambda_r$ we have
\[
\int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx \leq \frac{1}{\lambda_r} \left\{ \int_{\Omega_r \setminus \Omega_{r/2}} A \nabla u \cdot \nabla u + au^2 \, dx \right\} \quad \forall u \in H^1(\Omega_r \setminus \Omega_{r/2}).
\] (3.4)

Going back to (2.4) we find
\[
\int_{\Omega_r} (|\nabla u|^2 + au^2) \varphi^2 \left( \frac{x}{r} \right) \, dx \leq \frac{C}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx
\]
for some constant $C$ independent of $r$. Using in particular (2.2) we obtain
\[
\int_{\Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} u^2 \, dx, \quad \forall r > 0.
\] (3.5)

From (3.3) and (3.4) we derive that, for some constant $C$,
\[
\int_{\Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C}{r^2} \int_{\Omega_r \setminus \Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C}{r^{2+\beta}} \int_{\Omega_r} |\nabla u|^2 + au^2 \, dx \quad \forall r > 0.
\] (3.6)

Iterating $p$-times this formula leads to
\[
\int_{\Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C_p}{r^{(2-\beta)p+2}} \int_{\Omega_{2^{p-1}r}} |\nabla u|^2 + au^2 \, dx.
\]

By (3.5) it follows that it holds
\[
\int_{\Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C_p}{r^{(2-\beta)p+2}} \int_{\Omega_{2^pr}} u^2 \, dx,
\]
for some constant $C_p$ independent of $r$. If now $u$ is supposed to be bounded by $M$ we get
\[
\int_{\Omega_{r/2}} |\nabla u|^2 + au^2 \, dx \leq \frac{C_p}{r^{(2-\beta)p+2}} M^2 |\Omega_{2^pr}| = \frac{C_p|\Omega|M^2(2^p)^k}{r^{(2-\beta)p+2}}.
\]
($|\Omega_{2^pr}|$ denotes the Lebesgue measure of the set $\Omega_{2^pr}$). Choosing $(2-\beta)p + 2 > k$ the result follows by letting $r \to +\infty$. $\square$
Remark 3.2. Under the assumption of Theorem 3.2 we have obtained in fact that (1.2) can not admit a nontrivial solution with polynomial growth. Of course this result is optimal since $\text{Re}(e^z) = e^x \cos x_2$ is harmonic in $\mathbb{R}^k$ for any $k \geq 2$. One should note that Theorem 3.2 applies also to systems satisfying the Legendre condition when $auv$ is replaced by a nonnegative bilinear form $a(u, v)$ (see [6], [7]).

We now discuss some conditions on $a$ which imply (3.3). We have

**Theorem 3.3.** Suppose that for $|x|$ large enough

$$a(x) \geq \frac{c}{|x|^{\beta}}, \quad \beta < 2,$$

then (3.3) holds.

**Proof.** We denote by $\pi_r$ the first eigenfunction corresponding to $\lambda_r$, i.e., a minimizer of (3.2). We can assume without loss of generality that $\pi_r > 0$.

We have

$$\int_{\Omega_r \setminus \Omega_{r/2}} A \nabla \pi_r \cdot \nabla v + a \pi_r v \, dx = \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r v \, dx \quad \forall v \in H^1(\Omega_r \setminus \Omega_{r/2}).$$

Taking $v = 1$ yields

$$\int_{\Omega_r \setminus \Omega_{r/2}} a(x) \pi_r \, dx = \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \, dx.$$ 

Using (3.7) we derive, for some constant $C'$,

$$\frac{C'}{|r|^{\beta}} \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \, dx \leq \lambda_r \int_{\Omega_r \setminus \Omega_{r/2}} \pi_r \, dx$$

and the result follows. \(\square\)

We now consider other cases where (3.3) holds, in particular when no decay is imposed to $a$. We are interested for instance in the case where at infinity $a$ has enough mass locally. We start with the following lemma:

**Lemma 3.1.** Let (for instance) $Q = (0,1)^k$ be the unit cube in $\mathbb{R}^k$. For any $\varepsilon > 0$ and $\mu > 0$ there exists $\delta = \delta(\varepsilon, \mu)$ such that if the function $a$ satisfies

$$0 \leq a \leq \mu \text{ a.e. } x \in Q, \quad \int_Q a \, dx \geq \varepsilon,$$

then
then
\[ \delta \int_Q v^2 \, dx \leq \int_Q |\nabla v|^2 + av^2 \, dx \quad \forall v \in H^1(Q). \quad (3.9) \]

**Proof.** If not, there exists \( \varepsilon, \mu \) and a sequence of functions \( a_n, v_n \) such that \( a_n \) satisfies (3.8) and \( v_n \in H^1(Q) \) is such that
\[ \frac{1}{n} \int_Q v_n^2 \, dx \geq \int_Q |\nabla v_n|^2 + a_n v_n^2 \, dx. \quad (3.10) \]
Dividing by \( |v_n|^2 \) the \( L^2 \)-norm of \( v_n \) we can assume without loss of generality that
\[ \int_Q v_n^2 \, dx = 1. \quad (3.11) \]
By (3.10), (3.11) we have then
\[ \int_Q |\nabla v_n|^2 \, dx \leq \frac{1}{n}, \quad \int_Q v_n^2 \, dx = 1 \quad (3.12) \]
and \( v_n \) is uniformly bounded in \( H^1(Q) \). Therefore
\[ v_n \to 1 \text{ in } H^1(Q). \quad (3.13) \]
From (3.10) we have
\[ \int_Q a_n v_n^2 \, dx \leq \frac{1}{n}. \quad (3.14) \]
Thus
\[ \varepsilon \leq \int_Q a_n \, dx = \int_Q a_n v_n^2 \, dx + \int_Q a_n (1 - v_n^2) \, dx \leq \frac{1}{n} + \mu |1 - v_n|^2 |1 + v_n|^2 \to 0 \]
when \( n \to +\infty \). Impossible. This completes the proof of the lemma. \( \square \)

With the notation of Section 2 we set
\[ \Omega = (-1,1)^k. \quad (3.15) \]
We consider the lattice generated by \( Q = (0,1)^k \) - i.e., the cubes
\[ Q_i = Q_{z_i} = z_i + Q \quad \forall z_i \in \mathbb{Z}^k. \]
Then we have
Theorem 3.4. Suppose that for \( n \) large enough,
\[
\int_{Q_i} a(x) \, dx \geq \varepsilon \quad \forall Q_i \subset \mathbb{R}^k \setminus \Omega_n,
\] (3.16)
then
\[
\lambda_{2n} \geq \delta \left( \frac{1}{\lambda} \vee 1 \right) \quad \forall n
\] (3.17)
where \( \delta \) is defined in Lemma 3.1 and \( \vee \) denotes the maximum of two numbers.

Proof. Indeed by Lemma 3.1 after a simple translation from \( Q_i \) into \( Q \) we have
\[
\delta \int_{Q_i} u^2 \, dx \leq \int_{Q_i} |\nabla u|^2 + au^2 \, dx \quad \forall Q_i \subset \mathbb{R}^k \setminus \Omega_n \quad \forall u \in H^1(Q_i).
\]
This leads clearly to
\[
\delta \int_{\Omega_{2n} \setminus \Omega_n} u^2 \, dx
\leq \int_{\Omega_{2n} \setminus \Omega_n} |\nabla u|^2 + au^2 \, dx
\leq \int_{\Omega_{2n} \setminus \Omega_n} \frac{1}{\lambda} A \nabla u \cdot \nabla u + au^2 \, dx
\leq \left( \frac{1}{\lambda} \vee 1 \right) \int_{\Omega_{2n} \setminus \Omega_n} A \nabla u \cdot \nabla u + au^2 \, dx \quad \forall u \in H^1(\Omega_{2n} \setminus \Omega_n).
\]
The result follows then from (3.2).

Remark 3.3. Combining Theorems 3.2 and 3.4 it follows that (1.2) cannot have a nontrivial bounded solution (or of polynomial growth) when (3.16) holds. This is the case when at infinity
\[
a \geq a_0 > 0
\]
or more generally
\[
a \geq a_p
\] (3.18)
where \( a_p \) is a periodic function with period \( Q \).

In the case when (3.3) holds with \( \beta = 2 \) the technique of Theorem 3.2 cannot be applied. However, we will show that the non existence of nontrivial solutions can be established in this case too – i.e., condition (3.3) is not
sharp if we impose certain growth condition on \( \{a_{ij}(x)\} \). Before turning to this let us prove some general comparison result. For simplicity we will denote also by \( \tilde{A} \) the operator \( \nabla \cdot A \nabla u = \partial_{x_i}(a_{ij}\partial_{x_j}) \).

**Proposition 3.1.** Suppose that \( O \) is a bounded open subset of \( \mathbb{R}^k \). Let \( a_1, a_2 \) be two bounded functions satisfying

\[
a_1 \geq a_2 \geq 0 \quad \text{a.e. in } O. \tag{3.19}
\]

Let \( u_1, u_2 \in H^1(O) \) be such that

\[
\begin{cases}
-\tilde{A}u_2 + a_2 u_2 \geq -\tilde{A}u_1 + a_1 u_1 \geq 0 \quad \text{in } O, \\
u_2 \geq (u_1 \vee 0) \quad \text{on } \partial O,
\end{cases} \tag{3.20}
\]

then

\[
u_2 \geq (u_1 \vee 0) \quad \text{in } O. \tag{3.21}
\]

**Proof.** The inequality

\[-\tilde{A}u + au \geq 0 \quad \text{in } O
\]

means

\[
\int_O a_{ij} \partial_{x_j} u \partial_{x_i} v + auv \, dx \geq 0 \quad \forall v \in H^1_0(O), v \geq 0.
\]

Considering \( v = u_2^+ \) and \(-\tilde{A}u_2 + a_1 u_2 \geq 0 \) leads to \( u_2 \geq 0 \). Next considering \( v = (u_1 - u_2)^+ \in H^1_0(O) \) and (3.20) we obtain

\[
\int_O a_{ij} \partial_{x_j} u_1 \partial_{x_i} (u_1 - u_2)^+ + a_1 u_1 (u_1 - u_2)^+ \, dx \\
\leq \int_O a_{ij} \partial_{x_j} u_2 \partial_{x_i} (u_1 - u_2)^+ + a_2 u_2 (u_1 - u_2)^+ \, dx.
\]

Hence

\[
\int_O a_{ij} \partial_{x_j} (u_1 - u_2) \partial_{x_i} (u_1 - u_2)^+ + (a_1 u_1 - a_2 u_2)(u_1 - u_2)^+ \, dx \leq 0.
\]

Now on \( u_1 \geq u_2 \) one has \( a_1 u_1 \geq a_1 u_2 \geq a_2 u_2 \) and it follows that \((u_1 - u_2)^+ = 0\). \( \square \)

Next we prove

**Theorem 3.5.** Assume that there exists \( R, \) large enough, such that

\[
a(x) \geq \frac{c_0}{r^2} \quad \forall |x| \geq R > 0
\]
where \( c_0 \) is a positive constant. In addition to (1.1), suppose that \( a_{ij}(x) \in C^1(\mathbb{R}^k \setminus B(0, R)) \) satisfies for some positive \( D \):
\[
\partial_{x_i} (a_{ij}(x)) x_j \leq D \quad \forall |x| > R.
\]
(In the above inequality we make the summation convention of repeated indices). Then the equation
\[
-\partial_{x_i} (a_{ij}(x)) \partial_{x_j} u + a(x) u = 0
\]
cannot have nontrivial bounded solution.

**Proof.** Let \( u_n \) be the solution to
\[
\begin{cases}
-\partial_{x_i} (a_{ij}(x)) \partial_{x_j} u_n + a(x) u_n = 0 & \text{in } B(0, n), \\
u_n = 1 & \text{on } \partial B(0, n),
\end{cases}
\]
where \( B(0, n) \) denotes the ball of center 0 and radius \( n \). From Proposition 3.1 we obtain that \( u_n \), solution to (3.23), is such that:
\[
-|u|_{\infty} u_n \leq u \leq |u|_{\infty} u_n,
\]
\((|u|_{\infty} \text{ denotes the } L^\infty\text{-norm of } u)\). Denote by \( v_n \) the function defined as
\[
v_n = \begin{cases}
c_1 & \text{in } B(0, R) \\
c_2 r^{\beta_1} + c_3 r^{\beta_2} & \text{in } B(0, n) \setminus B(0, R),
\end{cases}
\]
where
\[
\beta_1 = -\frac{1}{2} \{(k - 2) + \sqrt{(k - 2)^2 + 4c'}\} < 0,
\]
\[
\beta_2 = -\frac{1}{2} \{(k - 2) - \sqrt{(k - 2)^2 + 4c'}\} > 0
\]
and
\[
c_1 = c_2 R^{\beta_1} + c_3 R^{\beta_2},
\]
\[
c_2 = \frac{-\beta_2 R^{\beta_2 - 1}}{n^{\beta_2} \beta_1 R^{\beta_1 - 1} - n^{\beta_1} \beta_2 R^{\beta_2 - 1}},
\]
\[
c_3 = \frac{\beta_1 R^{\beta_1 - 1}}{n^{\beta_2} \beta_1 R^{\beta_1 - 1} - n^{\beta_1} \beta_2 R^{\beta_2 - 1}}.
\]
In the above setting, \( c' \) is a positive constant small enough that we will determine later.

We remark that \( c_2 \) and \( c_3 \) are both positive and that \( \beta_1, \beta_2 \) are the two roots to the second order equation
\[
\beta^2 + (k - 2) \beta - c' = 0.
\]
(3.26)
The choice of $c_i, i = 1, 2, 3$ is such that $v_n$ is a $C^1$ function and $v_n = 1$ on $\partial B(0, n)$.

Now we want to show that $v_n$, in fact, is a supersolution to (3.23).

It is easy to see that

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}v_n) + a(x)v_n \geq 0 \text{ in } B(0, R).$$

For any constant $\beta$ one derives also that

$$\partial_{x_i}(a_{ij}(x)\partial_{x_j}(r^\beta))$$
$$= \partial_{x_i}(a_{ij}(x)\beta r^{\beta - 2}x_j)$$
$$= \partial_{x_i}(a_{ij}(x))\beta r^{\beta - 2}x_j + a_{ij}(x)\beta(\beta - 2)r^{\beta - 4}x_ix_j$$
$$+ a_{ij}(x)\beta r^{\beta - 2}\delta_{ij}.$$ 

Therefore in $B(0, n) \setminus B(0, R)$ this leads to

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}v_n) + a(x)v_n$$
$$\geq -\partial_{x_i}(a_{ij}(x))x_j\{c_2\beta_1 r^{\beta_1 - 2} + c_3\beta_2 r^{\beta_2 - 2}\}$$
$$- a_{ij}(x)x_i x_j\{c_1\beta_1(\beta_1 - 2)r^{\beta_1 - 4} + c_3\beta_2(\beta_2 - 2)r^{\beta_2 - 4}\}$$
$$- a_{ij}(x)\delta_{ij}\{c_2\beta_1 r^{\beta_1 - 2} + c_3\beta_2 r^{\beta_2 - 2}\} + \frac{c_0}{r^4}\{c_2 r^{\beta_1} + c_3 r^{\beta_2}\}$$
$$= \{ -\partial_{x_i}(a_{ij}(x))x_j\{c_2\beta_1 r^{\beta_1 - 2} + c_3\beta_2 r^{\beta_2 - 2}\}\}$$
$$+ c_2\{ -a_{ij}(x)x_i x_j x_i(\beta_1 - 2)r^{\beta_1 - 4} - a_{ij}(x)\delta_{ij}\beta_1 r^{\beta_1 - 2} + c_0 r^{\beta_1 - 2}\}$$
$$+ c_3\{ -a_{ij}(x)x_i x_j x_i(\beta_2 - 2)r^{\beta_2 - 4} - a_{ij}(x)\delta_{ij}\beta_2 r^{\beta_2 - 2} + c_0 r^{\beta_2 - 2}\}.$$ 

We notice that

$$c_2\beta_1 r^{\beta_1 - 2} + c_3\beta_2 r^{\beta_2 - 2} = \frac{\beta_1 \beta_2 r^{\beta_1 - 2} R^{\beta_2 - 1} - c_0 r^{\beta_1 - 2}}{n^{\beta_2} \beta_1 R^{\beta_1 - 1} - n^{\beta_1} \beta_2 R^{\beta_2 - 1}} > 0,$$

and thus

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}v_n) + a(x)v_n$$
$$\geq \left\{ -D\{c_2\beta_1 r^{\beta_1 - 2} + c_3\beta_2 r^{\beta_2 - 2}\}\right\}$$
$$+ c_2\{ -a_{ij}(x)x_i x_j x_i(\beta_1 - 2)r^{\beta_1 - 4} - a_{ij}(x)\delta_{ij}\beta_1 r^{\beta_1 - 2} + c_0 r^{\beta_1 - 2}\}$$
$$+ c_3\{ -a_{ij}(x)x_i x_j x_i(\beta_2 - 2)r^{\beta_2 - 4} - a_{ij}(x)\delta_{ij}\beta_2 r^{\beta_2 - 2} + c_0 r^{\beta_2 - 2}\}. $$
Taking into account (3.26) – i.e., replacing \( \beta_i (\beta_i - 2) \) by \( c' - k \beta_i \), yields
\[
- \partial_{x_i} (a_{ij}(x) \partial_{x_j} v_n) + a(x) v_n \\
\geq c_2 r^{\beta_1 - 2} \left\{ |k a_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D| \beta_1 - c' a_{ij}(x) \frac{x_ix_j}{r^2} + c_0 \right\} \\
+ c_3 r^{\beta_2 - 2} \left\{ |k a_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D| \beta_2 - c' a_{ij}(x) \frac{x_ix_j}{r^2} + c_0 \right\} \\
\geq c_2 r^{\beta_1 - 2} \left\{ |k a_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D| \beta_1 - c' \Lambda + c_0 \right\} \\
+ c_3 r^{\beta_2 - 2} \left\{ |k a_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D| \beta_2 - c' \Lambda + c_0 \right\}.
\]

We can select a \( D \) large enough such that the term
\[
ka_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D
\]
is negative (and bounded). By noticing that \( \beta_2 \to 0^+ \) when \( c' \to 0 \) we can then always choose \( c' \) small enough such that
\[
|k a_{ij}(x) \frac{x_ix_j}{r^2} - a_{ii}(x) - D| \beta_1 - c' \Lambda + c_0 > 0.
\]
Hence we derive that
\[
- \partial_{x_i} (a_{ij}(x) \partial_{x_j} v_n) + a(x) v_n \geq 0,
\]
and by Proposition 3.1,
\[
u_n \leq v_n.
\]
For any bounded subset \( \Omega \subset B(0, d) \) in \( \mathbb{R}^k \), one has clearly
\[
0 \leq v_n \leq \text{Max}\{c_1, c_2 d^{\beta_1} + c_3 d^{\beta_2}\} \to 0 \quad \text{on} \ \Omega
\]
when \( n \to +\infty \) since
\[
n^{\beta_2} \beta_1 R^{\beta_1 - 1} - n^{\beta_1} \beta_2 R^{\beta_2 - 1} \to -\infty
\]
as \( n \to +\infty \). From (3.24) we have also on \( B(0, n) \)
\[
-|u|_\infty v_n \leq u \leq |u|_\infty v_n
\]
for any \( n \). Letting \( n \to \infty \) leads to that
\[
u = 0.
\]

**Remark 3.4.** The above result holds true for an operator in non-divergence form, i.e., under the assumption of Theorem 3.5 the equation
\[
-a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u - b_i(x) \partial_{x_i} u + a(x) u = 0
\]
with
\[(b, x) \leq D \quad \forall|x| > R\]
cannot have a nontrivial bounded solution.

4. The case of the Laplace operator

In this section we analyze the existence or nonexistence of nontrivial bounded solutions to (1.2) in the case of the Laplacian. Due to the results of the previous section it is clear that existence of nontrivial solutions will impose some kind of decay \(a\). So, let us assume
\[a(x) \in L^\infty_{\text{loc}}(\mathbb{R}^k), \quad a \geq 0, a \not\equiv 0\]
and
\[\int_{|x|>1} a(x)|x|^{-k+2} \, dx < \infty\]  \hspace{1cm} (4.1)
with
\[k \geq 3.\]
Under the above assumptions we can show

**Theorem 4.1.** (Grigor’yan [8], see also [2], [3], [15]) Assume ((4.1)). Then there exists a function \(u\) such that
\[0 < u < 1 \quad \text{in } \mathbb{R}^k\]  \hspace{1cm} (4.2)
satisfying
\[-\Delta u + au = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k).\]  \hspace{1cm} (4.3)

**Proof.** Let \(u_n\) be the solution of
\[
\begin{cases}
-\Delta u_n + au_n = 0 & \text{in } B(0, n), \\
u_n = 1 & \text{on } \partial B(0, n).
\end{cases}
\]  \hspace{1cm} (4.4)
By the maximum principle
\[0 \leq u_n \leq 1\]  \hspace{1cm} (4.5)
and
\[u_{n+1} \leq u_n \quad \text{in } B(0, n).\]  \hspace{1cm} (4.6)
Thus \(u_n \to u\) which satisfies (4.3). Moreover
\[0 \leq u \leq 1.\]
By the strong maximum principle (and since $a \not\equiv 0$) we have

$$u < 1 \quad \text{in} \quad \mathbb{R}^k.$$ 

Once more, by the strong maximum principle, it suffices to prove that

$$u \not\equiv 0.$$

Assume, by contradiction, that

$$u \equiv 0. \quad (4.7)$$

Fix a function $\zeta \in C^\infty(\mathbb{R}^k), \ 0 \leq \zeta \leq 1$ such that

$$\zeta(x) = \begin{cases} 0 & |x| < R \\ 1 & |x| \geq R + 1 \end{cases}$$

and $R$ will be determined later. Multiplying (4.4) by $\zeta(x)$ yields for $n > R + 1, \nu = \text{being the outward unit normal vector}:

$$- \int_{|x|=n} \frac{\partial u_n}{\partial \nu} \cdot \frac{1}{n^{k-2}} \ d\sigma - \int_{B(0,n)} u_n \Delta \left( \frac{\zeta}{|x|^{k-2}} \right) \ dx$$

$$+ \int_{|x|=n} \frac{\partial}{\partial \nu} \left( \frac{\zeta}{|x|^{k-2}} \right) \ d\sigma + \int_{B(0,n)} au_n \frac{\zeta}{|x|^{k-2}} \ dx = 0. \quad (4.8)$$

By (4.5)

$$\frac{\partial u_n}{\partial \nu} \geq 0 \quad \text{on} \ \partial B(0,n) \quad (4.9)$$

and

$$\frac{\partial}{\partial \nu} \left( \frac{\zeta}{|x|^{k-2}} \right) = \frac{2 - k}{n^{k-1}} \quad \text{on} \ \partial B(0,n)$$

so that

$$\int_{|x|=n} \frac{\partial}{\partial \nu} \left( \frac{\zeta}{|x|^{k-2}} \right) \ d\sigma = (2 - k)\sigma_k \quad (4.10)$$

where $\sigma_k$ denotes the area of $S^{k-1}$. From (4.8), (4.9) and (4.10) we have

$$- \int_{B(0,n)} u_n \Delta \left( \frac{\zeta}{|x|^{k-2}} \right) \ dx + \int_{B(0,n)} au_n \frac{\zeta}{|x|^{k-2}} \ dx \geq (k - 2)\sigma_k. \quad (4.11)$$

Notice that $\Delta \left( \frac{\zeta}{|x|^{k-2}} \right)$ has compact support (in $R < |x| < R + 1$) since $\Delta \left( \frac{1}{|x|^{k-2}} \right) = c\delta_0$ where $\delta_0$ denotes the Dirac measure at 0. Therefore one has

$$\lim_{n \to \infty} \int_{B(0,n)} u_n \Delta \left( \frac{\zeta}{|x|^{k-2}} \right) \ dx = 0 \quad (4.12)$$
by assumption (4.7). On the other hand
\[
\int_{B(0,n)} au_n \frac{\zeta}{|x|^{k-2}} \, dx \leq \int_{R<|x|} \frac{a(x)}{|x|^{k-2}} \, dx. 
\] (4.13)

By (4.11), (4.12) and (4.13) we have
\[
\int_{R<|x|} \frac{a(x)}{|x|^{k-2}} \, dx \geq (k-2)\sigma_k. 
\]

Choosing $R$ sufficiently large and using assumption (4.1) yields a contradiction. This completes the proof of $u \not\equiv 0$.

**Remark 4.1.** With the same proof and under the assumption (4.1) one can show that (1.2) admits a non trivial solution provided
\[
\lim_{R \to \infty} \int_{|x| \geq R} \partial_i (a_{ij}(x) \partial_j |x|^{2-k}) < \lambda (k-2)\sigma_k. 
\]
This is in particular the case when $a_{ij}(x) = \delta_{ij}$ for $|x|$ large.

In the radially symmetric case we can say more.

**Theorem 4.2.** Suppose that the solutions to (1.2) are radially symmetric, then they do not change sign and are multiple of each other.

**Proof.** Let $u$ be a radially symmetric solution to (1.2). Let us first prove that $u$ does not change sign. Changing $u$ into $-u$ we can suppose
\[
u(0) \geq 0. 
\]

We argue by contradiction and assume that $u$ changes sign. If $u(0) > 0$, there exists a $r_0$ such that
\[
u(r_0) = 0. 
\]

Then
\[
\int_{B(0,r_0)} \nabla u \cdot \nabla v + au v \, dx = 0 \quad \forall v \in H_0^1(B(0,r_0)). 
\] (4.14)

Taking $v = u$ we obtain that $u \equiv 0$ in $B(0,r_0)$ and a contradiction. If $u(0) = 0$ then changing $u$ in $-u$ there is a component of the set
\[
\{x|u(x) > 0\} 
\]
which is an annulus $A$. But then we get (4.14) with $B(0,r_0)$ replaced by $A$ and a contradiction as above.
Consider now $u, v$ two solutions to (1.2). If $u \equiv 0$, $u = 0 \cdot v$, or else we have by the first part of the theorem (after changing $u$ into $-u$ if needed):

$$u(0) > 0.$$ 

Then $w = v - \frac{v(0)}{u(0)} u$ is a solution to (1.2) such that $w(0) = 0$. Since it does not change sign, 0 is a minimum or a maximum and by the maximum principle $w \equiv 0$. This completes the proof of the theorem.

In the case of radially symmetric solutions we also have:

**Proposition 4.1.** Suppose that $a = a(r)$, $r = |x|$ and let $u$ be a bounded positive radially symmetric solution to

$$-\Delta u + au = 0 \quad \text{in } D'(\mathbb{R}^k).$$

We have

$u(0) > 0$, $u = u(r)$ is nondecreasing on $(0, +\infty)$, $\lim_{r \to \infty} u(r) = u(\infty) < +\infty$.

**Proof.** $u(0) > 0$ results from the previous theorem. In addition we have

$$-u'' - \frac{k-1}{r} u' + au = 0$$

$$\implies ra = r u'' + (k-1)u' = \frac{(r^{k-1}u')'}{r^{k-2}} \geq 0. \quad (4.15)$$

Thus $r^{k-1} u'$ is nondecreasing. Since it vanishes at 0 we have $u' \geq 0$ and $u$ is nondecreasing. Hence $u$ has a limit at $\infty$ since $u$ is bounded.

As a consequence we have the following property for the solution $u$ that we constructed in the Theorem 4.1.

**Theorem 4.3.** Suppose that for $|x| \geq R_0$

$$a(x) \leq a_0(|x|) \quad \text{with} \quad \int_{R}^{+\infty} r a(r) \, dr < +\infty. \quad (4.16)$$

Then the solution $u$ constructed in Theorem 4.1 verifies

$$\lim_{|x| \to \infty} u(x) = 1.$$
Proof. We introduce
\[ \tilde{a} = \begin{cases} |a|_\infty & \text{for } |x| < R_0, \\ a_0(r) & \text{for } |x| \geq R_0. \end{cases} \]
Let \( \tilde{u}_n \) be the solution to
\[
\begin{cases}
-\Delta \tilde{u}_n + \tilde{a} \tilde{u}_n = 0 & \text{in } B(0, n), \\
\tilde{u}_n = 1 & \text{on } \partial B(0, n).
\end{cases}
\]
By Proposition 3.1 we have
\[ 0 \leq \tilde{u}_{n+1} \leq \tilde{u}_n \leq u_n \leq 1. \] (4.17)
Of course since \( \tilde{a} \) is radially symmetric, so is \( \tilde{u}_n \), and it converges to a radially symmetric function \( \tilde{u} \) which is a nontrivial solution to (see Theorem 4.1)
\[ -\Delta \tilde{u} + \tilde{a} \tilde{u} = 0 \quad \text{in } D'(\mathbb{R}^k). \]
From Proposition 4.1 we have
\[ 0 < \lim_{|x| \to \infty} \tilde{u} = \tilde{u}(\infty) \leq 1. \]
Suppose that \( \tilde{u}(\infty) < 1 \). Consider \( \tilde{v}_n \) the solution of
\[
\begin{cases}
-\Delta \tilde{v}_n + \tilde{a} \tilde{v}_n = 0 & \text{in } B(0, n) \\
\tilde{v}_n = 1 - \tilde{u}(\infty) & \text{on } \partial B(0, n).
\end{cases}
\]
One has
\[
\begin{cases}
-\Delta (\tilde{v}_n + \tilde{u}) + \tilde{a}(\tilde{v}_n + \tilde{u}) = 0 & \text{in } B(0, n), \\
\tilde{v}_n + \tilde{u} = 1 + \tilde{u} - \tilde{u}(\infty) \leq 1 & \text{on } \partial B(0, n).
\end{cases}
\]
Thus, by the maximum principle,
\[ \tilde{v}_n + \tilde{u} \leq \tilde{u}_n \quad \text{in } B(0, n). \]
Now clearly \( \tilde{u}_n = (1 - \tilde{u}(\infty)) \tilde{u}_n \) and thus
\[ (1 - \tilde{u}(\infty)) \tilde{u}_n + \tilde{u} \leq \tilde{u}_n. \]
Passing to the limit in \( n \) we obtain
\[ (1 - \tilde{u}(\infty)) \tilde{u} + \tilde{u} \leq \tilde{u} \]
which contradicts \( \tilde{u}(\infty) < 1 \). Thus we have \( \tilde{u}(\infty) = 1 \).
Now from (4.17) we derive, passing to the limit,
\[ \tilde{u} \leq u \leq 1. \]
Since, we already know that \( \lim_{|x| \to \infty} \tilde{u}(x) = 1 \) the result follows. This completes the proof of the theorem. \( \square \)

We prove now that condition (4.16) is sharp within the class of radial functions. This was observed in [3] with a different technique (see also [10], [9]). More recently (R. Pinsky [18]) established the sharpness of condition (4.16) in the class of functions satisfying the additional assumption
\[
a(x) \leq \frac{C}{(1 + |x|)^2}. \tag{4.18}
\]

So, let \( a(r) \) be a function such that
\[
\int_{0}^{+\infty} ra(r) \, dr = +\infty. \tag{4.19}
\]

**Lemma 4.1.** Under the assumption (4.19) there does not exist a bounded nontrivial radially symmetric solution to
\[-\Delta u + a(r)u = 0 \quad \text{in} \quad D'(\mathbb{R}^k). \tag{4.20}\]

**Proof.** Suppose that (4.20) admits a nontrivial bounded positive solution \( u(r) \) (see Theorem 4.3). Integrating the first equality of (4.15) we find
\[
\int_{0}^{r} sa(s)u(s) \, ds = \int_{0}^{r} su''(s) \, ds + (k - 1) \int_{0}^{r} u'(s) \, ds
\]
\[
= ru'(r) + (k - 2)\{u(r) - u(0)\} = (ru)' + (k - 3)u(r) - (k - 2)u(0).
\]

Integrating again in \( r \) yields
\[
ru(r) = \int_{0}^{r} \left( \int_{0}^{s} \xi a(\xi)u(\xi) \, d\xi \right) ds - (k - 3) \int_{0}^{r} u(s) \, ds + (k - 2)u(0)r \tag{4.21}
\]
For \( s \geq \frac{r}{2} \) we have
\[
\int_{0}^{s} \xi a(\xi)u(\xi) \, d\xi \geq u(0) \int_{0}^{\frac{r}{2}} \xi a(\xi) \, d\xi.
\]

Thus from (4.21) we easily obtain
\[
u(r) \geq \frac{1}{r} \int_{\frac{r}{2}}^{r} \left( \int_{0}^{s} \xi a(\xi)u(\xi) \, d\xi \right) ds - (k - 3)u(\infty) + (k - 2)u(0)
\]
\[\geq \frac{1}{2} \int_{0}^{\frac{r}{2}} \xi a(\xi) \, d\xi \cdot u(0) - (k - 3)u(\infty) + (k - 2)u(0).
\]

By (4.19) the left-hand side of this inequality goes to \(+\infty\) with \( r \). This contradicts the boundedness of \( u \) and completes the proof of the lemma. \( \square \)
As a consequence we can now show:

**Theorem 4.4.** Suppose that for $|x|$ large

$$a(x) \geq \bar{a}(r) = \bar{a}(|x|)$$  \hspace{1cm} (4.22)

where $\bar{a}$ satisfies (4.19) then the problem

$$-\Delta u + au = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k)$$  \hspace{1cm} (4.23)

cannot admit nontrivial bounded solutions.

**Proof.** Suppose that (4.22) holds for $|x| \geq R$. Then define

$$\tilde{a} = \begin{cases} 0 & \text{when } |x| \leq R, \\ \bar{a}(r) & \text{when } |x| > R. \end{cases}$$

$\tilde{a}$ is a radially symmetric function satisfying (4.19). Let $u$ be a bounded solution to (4.23). Let $u_n, v_n$ be the solution of

$$-\Delta u_n + \tilde{a}u_n = 0 \quad \text{in } B(0, n), \quad u_n = |u|_\infty \quad \text{on } \partial B(0, n),$$  \hspace{1cm} (4.24)

$$-\Delta v_n + av_n = 0 \quad \text{in } B(0, n), \quad v_n = |u|_\infty \quad \text{on } \partial B(0, n),$$  \hspace{1cm} (4.25)

where $|u|_\infty$ denotes the $L^\infty$-norm of $u$. It follows from Proposition 3.1 and the maximum principle that

$$u < v_n \leq u_n, \quad 0 \leq u_{n+1} \leq u_n \leq |u|_\infty.$$  \hspace{1cm} (4.26)

Changing $u$ into $-u$ one if needed, can assume that the set

$$\{u > 0\} = \{x \in \mathbb{R}^k \mid u(x) > 0\}$$

has a positive measure. Now, clearly, by the uniqueness of the solution to (4.24), $u_n$ is radially symmetric. By (4.26) $u_n$ converges to $u_\infty$ solution of

$$-\Delta u_\infty + \tilde{a}u_\infty = 0 \quad \text{in } \mathbb{R}^k$$

and $u_\infty$ is radially symmetric. By Lemma 4.1 this implies that $u_\infty = 0$. Hence from (4.26) we get

$$u \leq 0$$

which contradicts the fact that $\{u > 0\}$ is of positive measure. \hfill \square

**Remark 4.2.** Theorem 4.4 applies for instance when

$$a(x) = \frac{C_0}{|x|^2}$$
for a constant $C_0$ and $|x|$ large enough. Let $\lambda_r$ be given by (3.2). Then for $r$ large enough we have

\[
\frac{c}{r^2} \leq \lambda_r \leq \frac{C}{r^2} \quad (4.27)
\]

for some constants $c, C$. In other words the technique of Theorem 3.2 cannot work in this case. To show (4.27), recall the definition (3.2) and use the constant function

\[
u = \frac{1}{|\Omega_r \setminus \Omega_r/2|^{1/2}} \in H^1(\Omega_r \setminus \Omega_r/2)
\]

($| \cdot |$ is the Lebesgue measure); we obtain

\[
\lambda_r \leq \frac{1}{|\Omega_r \setminus \Omega_r/2|} \int_{\Omega_r \setminus \Omega_r/2} a(x) \, dx = \frac{C_0}{|\Omega_r \setminus \Omega_r/2|} \int_{\Omega_r \setminus \Omega_r/2} \frac{dx}{|x|^2} \leq C
\]

for $r$ large enough. To obtain the left-hand side inequality of (4.27) we remark (see Theorem 3.3) that for $r$ large enough

\[
\lambda_r = \int_{\Omega_r \setminus \Omega_r/2} a \pi_r \, dx / \int_{\Omega_r \setminus \Omega_r/2} \pi_r \, dx
\]

\[
= \int_{\Omega_r \setminus \Omega_r/2} C_0 \pi_r / |x|^2 \, dx / \int_{\Omega_r \setminus \Omega_r/2} \pi_r \, dx
\]

\[
\geq \frac{c}{r^2} \int_{\Omega_r \setminus \Omega_r/2} \pi_r \, dx / \int_{\Omega_r \setminus \Omega_r/2} \pi_r \, dx = \frac{c}{r^2},
\]

with $c = C_0$ for $\Omega_r = B(0, r)$. This completes the proof of (4.27).

We conclude this note with the following result.

**Theorem 4.5.**

Suppose that (4.3) admits a bounded solution, then it admits a positive solution.

If

\[
\int_{|x| > 1} a(x) |x|^{-k+2} \, dx = \infty \quad (4.28)
\]

then (4.3) cannot admit nontrivial bounded solution such that

\[
0 < c \leq u. \quad (4.29)
\]

**Proof.** We first prove the existence of a positive solution. If $u < 0$, $-u$ is a positive solution. So, we can assume that $u$ changes sign. Then introduce $u_n$ solution of

\[
-\Delta u_n + au_n = 0 \quad \text{in } B(0, n), u_n = |u|_\infty \quad \text{on } \partial B(0, n). \quad (4.30)
\]
One has

\[ 0 < u_{n+1} \leq u_n \leq |u|_{\infty} \quad (4.31) \]

and \( u_n \) converges to some function \( u_\infty \) for instance in \( L^1_{\text{loc}}(\mathbb{R}^k) \). Then \( u_\infty \) is a solution of (4.3). Moreover by the maximum principle one has \( u \leq u_n \) on \( B(0,n) \) and thus \( u \leq u_\infty \). \( u_\infty \) cannot vanish identically and is the positive solution we are looking for.

Suppose now that \( u \) is a nonnegative bounded solution to

\[-\Delta u = -au.\]

Set \( U(r) = \int_{\partial B_1} u(r \sigma) \, d\sigma \) where \( B_1 \) denotes the unit ball of \( \mathbb{R}^k \). Then

\[-(r^{k-1}U')' = -r^{k-1} \int_{\partial B_1} a(r \sigma) u(r \sigma) \, d\sigma \quad (4.32)\]

hence

\[-r^{k-1}U' = - \int_0^r s^{k-1} \int_{\partial B_1} a(s \sigma) u(s \sigma) \, d\sigma \, ds \quad (4.33)\]

and \( U(r) = \int_{\partial B_1} u(r \sigma) \, d\sigma \) is nondecreasing. Moreover \( U \) is a solution of the second order differential equation (4.32). A particular solution is given (see (4.33)) by

\[ U = \int_0^r \frac{1}{s^{k-1}} \int_0^s t^{k-1} \int_{\partial B_1} a(t \sigma) u(t \sigma) \, d\sigma \, dt \, ds. \quad (4.34) \]

The solution of the homogeneous equation is given by

\[ \frac{A}{r^{k-2}} + B. \]

Thus we have

\[ U(r) = \frac{A}{r^{k-2}} + B + \int_0^r \frac{1}{s^{k-1}} \int_0^s t^{k-1} \int_{\partial B_1} a(t \sigma) u(t \sigma) \, d\sigma \, dt \, ds. \quad (4.34) \]

Since \( u \) is bounded, so is \( U \) and necessarily \( A = 0, B \geq 0 \). From (4.34) we derive

\[ U(r) = B + \int_0^r \frac{1}{s^{k-1}} \int_0^s t^{k-1} \int_{\partial B_1} a(t \sigma) u(t \sigma) \, d\sigma \, dt \, ds. \quad (4.35) \]
Integrating by parts we get

\[
U(r) = B - \frac{1}{(k-2)r^{k-2}} \int_0^r t^{k-1} \int_{\partial B_1} a(t\sigma) u(t\sigma) \, d\sigma \, dt \\
+ \int_0^r \frac{1}{(k-2)r^{k-2}} t^{k-1} \int_{\partial B_1} a(t\sigma) u(t\sigma) \, d\sigma \, dt \\
= B + \frac{1}{k-2} \int_0^r \int_{\partial B_1} ta(t\sigma)(1 - \frac{t^{k-2}}{r^{k-2}})u(t\sigma) \, d\sigma \, dt.
\]

(4.36)

When

\[
\int_{|x|>1} \frac{a(x)}{|x|^{k-2}} \, dx = \int_1^{+\infty} \int_{\partial B_1} ta(t\sigma) \, d\sigma \, dt = +\infty,
\]

then the equation (4.3) cannot have a solution such that

\[
0 < c \leq u \leq C.
\]

Indeed from (4.36) we would get

\[
U(r) \geq B + \frac{1}{k-2} \int_1^{+\infty} \int_{\partial B_1} ta(t\sigma)(1 - \frac{1}{r^{k-2}})c \to +\infty
\]

which contradicts the fact that \( u \) and \( U \) are bounded.

\[ \square \]

**Remark 4.3.** Using this result one recovers easily the Lemma 4.1.

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