On Inequalities of Korn’s Type

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Abstract

The goal of this note is to identify the minimal number of linear forms which is necessary to obtain some inequalities of Korn’s type. In other words we give some results allowing to estimate the $L^2$-norm of the gradient of a vector field in terms of a minimal sum of $L^2$-norms of linear forms of this gradient. We consider such inequalities either over $H^1_0(\Omega)$ or $H^1(\Omega)$.

AMS 2020 Subject Classification: 35A23, 35R45, 35Q74, 74B20

Key words: Korn Inequality, equivalent norms, linear forms.

1 Introduction and notation

Let $\Omega$ be an open set of $\mathbb{R}^n$, $n \geq 1$. We will denote by $H^1_0(\Omega)$ the space $(H^1_0(\Omega))^n$ i.e.

$$H^1_0(\Omega) = \{ v = (v_1, \cdots, v_n) \mid v_i \in H^1_0(\Omega) \ \forall i = 1, \cdots, n \} \quad (1.1)$$

Let $\nabla v = (\frac{\partial v_i}{\partial x_j})$ be the Jacobian matrix of $v$. If $\ell_i, i = 1, \cdots, k$ denote linear forms on the space of $n \times n$ matrices one is interested in inequalities of the type

$$C |\nabla v|^2 \leq \sum_{i=1}^{k} |\ell_i(\nabla v)|^2 \quad \forall v \in H^1_0(\Omega). \quad (1.2)$$

(Here $C$ is some constant, $| \cdot |_2$ stands for the usual $L^2(\Omega)$-norm, $|\nabla v|$ is the Euclidean norm of the gradient of $v$ i.e. $|\nabla v|^2 = \sum_{i,j=1}^{n}(\frac{\partial v_i}{\partial x_j})^2$). For instance, set

$$\ell_{i,j}(v) = \frac{1}{4} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad i, j = 1, \cdots, n. \quad (1.3)$$

By an easy computation, if $\mathcal{D}(\Omega)$ denotes the set of $C^\infty$-functions with compact supports in $\Omega$, we have

$$|\ell_{i,j}(v)|^2_2 = \frac{1}{4} \int_\Omega \left( \frac{\partial v_i}{\partial x_j} \right)^2 + 2 \left( \frac{\partial v_i}{\partial x_j} \right) \left( \frac{\partial v_j}{\partial x_i} \right) + \left( \frac{\partial v_j}{\partial x_i} \right)^2 \ d\Omega$$

$$= \frac{1}{4} \int_\Omega \left( \frac{\partial v_i}{\partial x_j} \right)^2 + 2 \left( \frac{\partial v_i}{\partial x_i} \right) \left( \frac{\partial v_j}{\partial x_j} \right) + \left( \frac{\partial v_j}{\partial x_i} \right)^2 \ d\Omega \quad \forall v \in (\mathcal{D}(\Omega))^n, \quad (1.4)$$
(by integration by parts). Thus, summing up for \(i, j = 1, \cdots, n\), we get, using the density of \(\mathcal{D}(\Omega)^n\) in \(H^1_0(\Omega)\),

\[
\sum_{i,j=1}^n |\ell_{i,j}(v)|_2^2 = \frac{1}{2}||\nabla v||^2_2 + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^n \left( \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \right) dx \\
= \frac{1}{2}||\nabla v||^2_2 + \frac{1}{2} \int_{\Omega} (\text{div } v)^2 dx \\
\geq \frac{1}{2}||\nabla v||^2_2 \quad \forall v \in H^1_0(\Omega) 
\]

which is often called the "first Korn Inequality".

We would like in particular determine the smallest \(k\) for which such an inequality holds, i.e. to determine the smallest number of linear forms such that the right hand side of (1.2) to the power 1/2 is an equivalent norm to the gradient norm in \(H^1_0(\Omega)\). Indeed, if \(C\) is another constant, the inequality

\[
\sum_{i=1}^k |\ell_i(\nabla v)|_2^2 \leq C||\nabla v||^2_2 \quad \forall v \in H^1_0(\Omega). 
\]

is obvious. This type of inequalities, in particular with Dirichlet conditions on some part of the boundary, plays an important role in the theory of elasticity. One can see this for instance in the remarkable books [1], [2] where in particular an historical background is given (cf. [6], [9], [10], [13]).

The paper is divided as follows. The sections 2,3,4 are devoted to Korn's inequalities in \(H^1_0(\Omega)\). Finally, in section 5 we investigate a case in \(H^1(\Omega)\) where

\[
H^1(\Omega) = \{v = (v_1, \cdots, v_n) \mid v_i \in H^1(\Omega) \quad \forall i = 1, \cdots, n\}. 
\]

To be more precise, in section 2 we show that the minimal number of forms needed to have (1.2) is independent of \(\Omega\) (Theorem 2.1). We give then some properties of this number with respect to the dimension of the space in which \(\Omega\) is considered (Theorem 2.2 and Thorem 2.3). We conclude the section by computing this number in two and three dimensions (Theorem 2.4 and proposition 2.2). In section 3, using the Fourier transform, we give a characterisation of this minimal number \(k\) using a pure algebraic property of \(\mathbb{R}^n\) (Theorem 3.2). This allows us, in the next section to discover some specific Korn inequalities. The last section is devoted to the particular \(H^1(\Omega)\)-case where the inequalities considered take the form

\[
C||\nabla v||^2_2 \leq ||v||^2_2 + \sum_{i=1}^k |\ell_i(\nabla v)|^2_2 \quad \forall v \in H^1(\Omega). 
\]

We show that the minimal number of linear forms needed to have a Korn inequality of this type is larger than the minimal number which was necessary in the case of homogeneous zero boundary conditions (Theorem 4.1). The section ends with some elementary properties of this number.

We refer to [8], [11], [3] for classical notions in particular regarding the spaces used here.
2 Preliminary results

We can first show:

**Theorem 2.1.** The smallest integer \( k \) such that (1.2) holds is independent of \( \Omega \) i.e. is the same for every open set. Similarly one can chose a constant \( C \) such that (1.2) holds for every \( \Omega \). In particular the best constant in (1.2) is the same for every \( \Omega \).

**Proof.** Let us suppose that

\[
\ell_i(\nabla v) = \sum_{m=1}^{n} a_{im} \cdot \nabla v_m
\]

so that (1.2) can be written

\[
C \int_{\Omega} \sum_{m,j=1}^{n} (\frac{\partial v_m}{\partial x_j})^2 dx \leq \sum_{i=1}^{k} \int_{\Omega} \left( \sum_{m=1}^{n} a_{im} \cdot \nabla v_m \right)^2 dx \quad \forall v \in (\mathcal{D}(\Omega))^n
\]

(2.1)

where \( a_{im} \) are vectors in \( \mathbb{R}^n \), "." denoting the usual scalar product in \( \mathbb{R}^n \). Suppose that \( x_0 \in \Omega \).

Then, \( \Omega_{x_0} = \Omega - x_0 \) is an open set containing 0. For \( v \in (\mathcal{D}(\Omega))^n \), define \( v_{x_0} \) as

\[
v_{x_0}(x) = v(x + x_0).
\]

Then, \( v \in (\mathcal{D}(\Omega))^n \) iff \( v_{x_0} \in (\mathcal{D}(\Omega_{x_0}))^n \) and (2.1) is satisfied for \( v \) iff (2.1) is satisfied for \( v_{x_0} \) with \( \Omega \) replaced by \( \Omega_{x_0} \) as one can see by a simple change of variable, the constant \( C \) being unchanged. Thus, in (2.1) one can assume that \( \Omega \) contains 0 and more precisely the ball

\[
B_r = \{ x \in \mathbb{R}^n | |x| < r \}.
\]

Then we claim that (2.1) is satisfied with the same constant \( C \) if it is satisfied for all \( v \in (\mathcal{D}(\mathbb{R}^n))^n \). Indeed, if (2.1) is satisfied it is satisfied for any \( v \in (\mathcal{D}(\mathbb{R}^n))^n \). Consider then \( v \in (\mathcal{D}(\mathbb{R}^n))^n \) and chose \( \rho \) large enough such that

\[
\text{Supp } v \subset \rho B_r
\]

where \( \text{Supp } v \) denotes the support of \( v \). Then, if one defines \( w_\rho \) by

\[
w_\rho(y) = v(\rho y),
\]

\( w_\rho \in (\mathcal{D}(B_r))^n \). Thus, assuming that the functions are extended by 0 outside of their support, one has

\[
C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} (\frac{\partial v_m}{\partial x_j}(\rho y))^2 dy \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} a_{im} \cdot \nabla v_m(\rho y) \right)^2 dy \quad \forall v \in (\mathcal{D}(\mathbb{R}^n))^n.
\]

Setting \( x = \rho y \) it follows that (2.1) holds for any \( v \in (\mathcal{D}(\mathbb{R}^n))^n \), the constant \( C \) being unchanged. This completes the proof of the theorem. \( \square \)
Let us denote by $N(n)$ the smallest number of linear forms needed to have an inequality of the type of (1.2). Then we have

**Theorem 2.2.**

$$N(n) \leq N(n + 1).$$  \hspace{1cm} (2.2)

**Proof.** Set $k = N(n + 1)$. We have for some vectors $a^i_m$ of $\mathbb{R}^{n+1}$, $i = 1, \ldots , k$, $m = 1, \ldots , n + 1$

$$C \int_{\mathbb{R}^{n+1}} \sum_{m,j=1}^{n+1} \left( \frac{\partial v_m}{\partial x_j} \right)^2 dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \sum_{m,j=1}^{n+1} a^i_m \frac{\partial v_m}{\partial x_j} \right)^2 dx \quad \forall v \in (D(\mathbb{R}^{n+1}))^{n+1}. \hspace{1cm} (2.3)$$

(We denoted by $a^i_{m,j}$, $j = 1, \ldots , n + 1$, the components of $a^i_m$). Let $w \in (D(\mathbb{R}^n))^n$, $\varphi \in D(\mathbb{R})$, $\varphi \neq 0$. In (2.3) let us chose

$$v(x) = (w(x')\varphi(\epsilon x_{n+1}), 0), \quad \epsilon > 0, \quad x' = (x_1, \ldots , x_n).$$

One gets

$$C \int_{\mathbb{R}^{n+1}} \sum_{m,j=1}^{n+1} \left( \frac{\partial w_m}{\partial x_j}(x')\varphi(\epsilon x_{n+1}) \right)^2 + \sum_{m=1}^{n} (w_m(x')\epsilon \varphi'(\epsilon x_{n+1}))^2 dx
\leq \sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \left( \sum_{m,j=1}^{n} a^i_{m,j} \frac{\partial w_m}{\partial x_j}(x')\varphi(\epsilon x_{n+1}) + \sum_{m=1}^{n} a^i_{m,n+1} w_m(x')\epsilon \varphi'(\epsilon x_{n+1}) \right)^2 dx. \hspace{1cm} (2.4)$$

Let us compute the right hand side integral of this inequality. One has

$$\sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \left( \varphi(\epsilon x_{n+1}) \sum_{m,j=1}^{n} a^i_{m,j} \frac{\partial w_m}{\partial x_j}(x') + \epsilon \varphi'(\epsilon x_{n+1}) \sum_{m=1}^{n} a^i_{m,n+1} w_m(x') \right)^2 dx
= \sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \varphi(\epsilon x_{n+1})^2 \left( \sum_{m,j=1}^{n} a^i_{m,j} \frac{\partial w_m}{\partial x_j}(x') \right)^2 + \epsilon^2 \varphi'(\epsilon x_{n+1})^2 \left( \sum_{m=1}^{n} a^i_{m,n+1} w_m(x') \right)^2 dx.$$ 

Indeed the double product term reads, with an obvious notation for the function $\psi$,

$$\int_{\mathbb{R}^{n+1}} \psi(x')\epsilon \varphi'(\epsilon x_{n+1}) \varphi'(\epsilon x_{n+1}) dx = 0$$

since $\varphi$ has compact support. Thus, (2.4) becomes

$$C \int_{\mathbb{R}^{n+1}} \varphi(\epsilon x_{n+1})^2 \left( \sum_{m,j=1}^{n} \frac{\partial w_m}{\partial x_j}(x') \right)^2 + \epsilon^2 \varphi'(\epsilon x_{n+1})^2 \sum_{m=1}^{n} w_m(x')^2 dx
\leq \sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \varphi(\epsilon x_{n+1})^2 \left( \sum_{m,j=1}^{n} a^i_{m,j} \frac{\partial w_m}{\partial x_j}(x') \right)^2 + \epsilon^2 \varphi'(\epsilon x_{n+1})^2 \left( \sum_{m=1}^{n} a^i_{m,n+1} w_m(x') \right)^2 dx.$$
Making the change of variable \( y = \epsilon x_{n+1} \), one gets easily that for any \( w \in (D(\mathbb{R}^n))^n \)
\[
C \int_{\mathbb{R}^{n+1}} \varphi(y)^2 \left( \sum_{m,j=1}^{n} \left( \frac{\partial w_m}{\partial x_j}(x') \right)^2 + \epsilon^2 \varphi'(y)^2 \sum_{m=1}^{n} w_m(x')^2 \right) \, dx' \, dy \\
\leq \sum_{i=1}^{k} \int_{\mathbb{R}^{n+1}} \varphi(y)^2 \left( \sum_{m,j=1}^{n} a_{m,j}^i \frac{\partial w_m}{\partial x_j}(x') \right)^2 + \epsilon^2 \varphi'(y)^2 \left( \sum_{m=1}^{n} a_{m,n+1}^i w_m(x') \right)^2 \, dx' \, dy.
\]

Letting \( \epsilon \to 0 \) we obtain, since \( \int_{\mathbb{R}} \varphi(y)^2 \, dy \neq 0 \),
\[
C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} \left( \frac{\partial w_m}{\partial x_j}(x') \right)^2 \, dx' \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m,j=1}^{n} a_{m,j}^i \frac{\partial w_m}{\partial x_j}(x') \right)^2 \, dx' \quad \forall w \in (D(\mathbb{R}^n))^n.
\]

Thus, \( k \geq N(n) \). This completes the proof of the theorem. \( \Box \)

**Theorem 2.3.** For each integer \( n \geq 1 \),

\[ N(2n) \geq 2n \text{ and } N(2n + 1) > 2n + 1. \quad (2.5) \]

**Proof.** Indeed, suppose that \( N(n) = k \) with \( k \) odd and \( k \leq n \). Note that here some of the linear forms can be allowed to vanish identically. We have for \( kn \) vectors \( a_i^m \) of \( \mathbb{R}^n \)
\[
C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} \left( \frac{\partial w_m}{\partial x_j}(x') \right)^2 \, dx' \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} a_i^m \cdot \nabla v_m \right)^2 \, dx \quad \forall v \in D(\mathbb{R}^n)^n.
\]

To lighten our notation we will allow us from now on to use \( D(\mathbb{R}^n)^n \) instead of \( (D(\mathbb{R}^n))^n \). Consider for \( \alpha = (\alpha_1, \cdots, \alpha_n) \neq 0, w \in D(\mathbb{R}^n) \)
\[ v = (\alpha_1 w, \cdots, \alpha_n w). \]

From the inequality above we derive
\[
C \int_{\mathbb{R}^n} \sum_{m=1}^{n} \alpha_m^2 \vert \nabla w \vert^2 \, dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} \alpha_m a_i^m \cdot \nabla w \right)^2 \, dx \quad \forall w \in D(\mathbb{R}^n). \quad (2.6)
\]

We claim that, for some \( \alpha \neq 0 \), the vectors
\[ \nu_i = \sum_{m=1}^{n} \alpha_m a_i^m, \quad i = 1, \cdots, k, \]
are linearly dependent. If \( n \) is even this is clear since \( k < n \) and any \( \alpha \) is suitable. Thus one can assume \( n \) odd and \( k = n \). One then has
\[ \det(\nu_1, \cdots, \nu_k) = \det(\sum_{m=1}^{n} \alpha_m a_1^m, \cdots, \sum_{m=1}^{n} \alpha_m a_k^m). \]
It follows that \( \det(\nu_1, \cdots, \nu_k) \) is a polynomial of degree \( n = k \) in \( \alpha_i \). If
\[
\det(a^1_1, \cdots, a^k_1) = 0.
\]
then \( \alpha = (1, \cdots, 0) \) is suitable. Else we fix \( (\alpha_2, \cdots, \alpha_n) \neq 0 \); then
\[
\det(\nu_1, \cdots, \nu_n) = C_1 \det(\nu_1, \cdots, \nu_n) + P_{n-1}(\alpha_1)
\]
where \( P_{n-1} \) is a polynomial of degree \( n - 1 \) in \( \alpha_1 \). Clearly, this polynomial of odd degree has a real root. This shows that the corresponding \( \nu_i \) are linearly dependent. So, suppose \( \alpha \) chosen in such a way that the \( \nu_i \) are linearly dependent. The inequality (2.6) becomes, with obvious notation,
\[
C|\alpha|^2 \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} (\nu_i \cdot \nabla w)^2 \, dx \quad \forall w \in \mathcal{D}(\mathbb{R}^n).
\]
(2.7)
Without loss of generality one can assume that \( \nu_n \) is a linear combination of \( \nu_1, \cdots, \nu_{n-1} \). Then clearly one is missing one direction of derivative in the right hand side of the inequality and this inequality is impossible. To see this, denote by \( \eta \) a vector of \( \mathbb{R}^n \) orthogonal to the space spanned by the \( \nu_1, \cdots, \nu_{n-1} \). Suppose that \( \nu_1, \cdots, \nu_{n-1} \) is completed in a basis of the space orthogonal to \( \eta \) that we denote by \( \tilde{\nu}_1, \cdots, \tilde{\nu}_{n-1} \). Let \( \varphi \in \mathcal{D}(\mathbb{R}) \), \( \varphi \neq 0 \), \( \epsilon > 0 \) and consider
\[
w(x) = \prod_{i=1}^{n-1} \varphi(\epsilon \cdot \tilde{\nu}_i) \varphi(x \cdot \eta) = \psi(\epsilon \cdot \tilde{\nu}_1, \cdots, \epsilon \cdot \tilde{\nu}_{n-1}) \varphi(x \cdot \eta),
\]
with an obvious notation for \( \psi \). Clearly \( w \in \mathcal{D}(\mathbb{R}^n) \). Moreover, if \( \psi_p \) denotes the partial derivative of \( \psi \) with respect to the \( p \)-variable, one has
\[
\frac{\partial w}{\partial x_j}(x) = \epsilon \varphi(x \cdot \eta) \sum_{p=1}^{n-1} \psi_p(\epsilon \cdot \tilde{\nu}) \tilde{\nu}_{p,j} + \varphi'(x \cdot \eta) \psi(\epsilon \cdot \tilde{\nu}) \eta_j.
\]
were for simplicity we denoted by \( x \cdot \tilde{\nu} \) the vector \((\epsilon \cdot \tilde{\nu}_1, \cdots, \epsilon \cdot \tilde{\nu}_{n-1})\) and by the index \( j \) the different components of the vectors. Since \( \eta \) is orthogonal to the \( \nu_i \)'s, (2.7) becomes
\[
C|\alpha|^2 \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left( \epsilon \varphi(x \cdot \eta) \sum_{p=1}^{n-1} \psi_p(\epsilon \cdot \tilde{\nu}) \tilde{\nu}_{p,j} + \varphi'(x \cdot \eta) \psi(\epsilon \cdot \tilde{\nu}) \eta_j \right)^2 \, dx
\leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{j=1}^{n} \epsilon \varphi(x \cdot \eta) \sum_{p=1}^{n-1} \psi_p(\epsilon \cdot \tilde{\nu}) \tilde{\nu}_{p,j} \nu_{i,j} \right)^2 \, dx.
\]
Making the change of variable \( y = (y', y_n) = (y_1, \cdots, y_n) = (\epsilon \cdot \tilde{\nu}_1, \cdots, \epsilon \cdot \tilde{\nu}_{n-1}, x \cdot \eta) \) one has
\[
dy = \epsilon^{n-1} \det(\tilde{\nu}_1, \cdots, \tilde{\nu}_{n-1}, \eta) \, dx
\]
and one gets
\[
C|\alpha|^2 \int_{\mathbb{R}^n} \sum_{j=1}^{n} \left( \epsilon \varphi(y_n) \sum_{p=1}^{n-1} \psi_p(y') \tilde{\nu}_{p,j} + \varphi'(y_n) \psi(y') \eta_j \right)^2 \, dy
\leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{j=1}^{n} \epsilon \varphi(y_n) \sum_{p=1}^{n-1} \psi_p(y') \tilde{\nu}_{p,j} \nu_{i,j} \right)^2 \, dy.
\]
Letting $\epsilon \to 0$ we obtain
\[ C|\alpha|^2|\eta|^2 \int_{\mathbb{R}^n} \prod_{i=1}^{n-1} \varphi(y_i)^2 \varphi'(y_n)^2 dy = 0 \]
and a contradiction since $\varphi \neq 0$. This completes the proof of the theorem. \qed

**Remark 1**: One cannot delete any term in the left hand side of the first Korn Inequality (1.5). Suppose that in (1.5) we do not have the term in $\ell_{i,j}(v) = \ell_{j,i}(v)$. Consider
\[ v = (0, \cdots, w, \cdots, 0) \]
where $w$ is at the $i^{th}$ slot. From (1.5) we deduce easily with obvious notation
\[ 2 \sum_{k \neq i} \left| \frac{\partial w}{\partial x_k} \right|^2 \geq \frac{1}{2} ||\nabla w||^2. \]
Since one derivative is missing on the left hand side we get a contradiction.

**Theorem 2.4.** It holds that
\[ N(2) = 2. \] (2.8)

**Proof.** We remark that
\[ \int_{\mathbb{R}^2} \left( \frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_1} \right)^2 + \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2} \right)^2 dx = ||\nabla v||^2 \quad \forall v \in \mathcal{D}(\mathbb{R}^2). \] (2.9)
Indeed this follows from
\[ \int_{\mathbb{R}^2} \frac{\partial v_1}{\partial x_2} \frac{\partial v_2}{\partial x_1} dx = \int_{\mathbb{R}^2} \frac{\partial v_1}{\partial x_1} \frac{\partial v_2}{\partial x_2} dx \quad \forall v \in \mathcal{D}(\mathbb{R}^2). \] (2.10)
This shows that $N(2) \leq 2$. The result then follows from Theorem 2.3. \qed

**Proposition 2.1.** One has
\[ N(n) \leq \frac{n(n-1)}{2} + 1. \] (2.11)

**Proof.** Consider the linear forms
\[ \hat{\ell}_{ij}(\nabla v) = \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right), \quad i < j. \] (2.12)
We have for $v \in \mathcal{D}(\mathbb{R}^n)^n$
\[ \int_{\mathbb{R}^n} \hat{\ell}_{ij}(\nabla v)^2 = \int_{\mathbb{R}^n} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)^2, \]
\[ = \int_{\mathbb{R}^n} \left( \frac{\partial v_i}{\partial x_j} \right)^2 - 2 \left( \frac{\partial v_i}{\partial x_j} \right) \left( \frac{\partial v_j}{\partial x_i} \right) + \left( \frac{\partial v_j}{\partial x_i} \right)^2 dx \]
\[ = \int_{\mathbb{R}^n} \left( \frac{\partial v_i}{\partial x_j} \right)^2 - 2 \left( \frac{\partial v_i}{\partial x_i} \right) \left( \frac{\partial v_j}{\partial x_j} \right) + \left( \frac{\partial v_j}{\partial x_i} \right)^2 dx, \quad i < j, \] (2.13)
(we just integrated by parts).

Summing up we obtain

\[
\sum_{i<j} |\hat{\ell}_{ij}(\nabla v)|^2 = \sum_{i\neq j} |\frac{\partial v_i}{\partial x_j}|^2 - 2 \int_{\mathbb{R}^n} \sum_{i<j} \left( \frac{\partial v_i}{\partial x_i} \right) \left( \frac{\partial v_j}{\partial x_j} \right) dx
\]

\[
= \sum_{i\neq j} |\frac{\partial v_i}{\partial x_j}|^2 + \sum_{i} |\frac{\partial v_i}{\partial x_i}|^2 - \int_{\mathbb{R}^n} (\text{div } v)^2 dx,
\]

from which it follows

\[
\sum_{i<j} |\hat{\ell}_{ij}(\nabla v)|^2 + \int_{\mathbb{R}^n} (\text{div } v)^2 dx = \|\nabla v\|^2 \quad \forall v \in \mathcal{D}(\mathbb{R}^n)^n
\]

(2.15)

and the result follows.

In dimension 3 we have

**Proposition 2.2.**

\[ N(3) = 4. \]  

(2.16)

**Proof.** By the previous proposition we have \( N(3) \leq \frac{3^2}{2} + 1 = 4 \). The result follows then for the Theorem 2.3. \qed

3 An argument via Fourier Transfom

**Theorem 3.1.** Suppose that we have found \( kn \) vectors \( a^i_m \in \mathbb{R}^n, i = 1, \cdots, k, m = 1, \cdots, n \) such that

\[ \forall x \neq 0 \quad \text{Rank}\left\{ \left( \sum_{m=1}^{n} x_m a^i_m \right)_{i=1, \cdots, k} \right\} = n \quad (3.1) \]

then \( N(n) \leq k \).

**Proof.** We will be done if we can show that for some positive constant \( C \)

\[
C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} \left( \frac{\partial v_m}{\partial x_j} \right)^2 dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} a^i_m \cdot \nabla v_m \right)^2 dx \quad \forall v \in \mathcal{D}(\mathbb{R}^n)^n.
\]

(3.2)

Since everything is real the inequality above can be written

\[
C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} \left( \frac{\partial v_m}{\partial x_j} \right)^2 dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} a^i_m \cdot \nabla v_m \right)^2 dx \quad \forall v \in \mathcal{D}(\mathbb{R}^n)^n. \]

(3.3)

Using the Fourier transform defined as

\[
\hat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i y \cdot x} f(x) dx, \quad i = (-1)^{\frac{1}{2}}
\]
we obtain by the Plancherel formula that (3.3) is equivalent to
\[
C \int_{\mathbb{R}^n} \left| \sum_{m,j=1}^n a_m^i \cdot \nabla v_m \right|^2 dy \leq \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \sum_{m=1}^n \partial v_m / \partial x_j \right|^2 dy \quad \forall v \in \mathcal{D}(\mathbb{R}^n),
\]
and
\[
C \int_{\mathbb{R}^n} \left| \sum_{m,j=1}^n \partial v_m / \partial x_j \right|^2 dy \leq \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \sum_{m=1}^n a_m^i \cdot \nabla v_m \right|^2 dy \quad \forall v \in \mathcal{D}(\mathbb{R}^n). \tag{3.4}
\]
Since
\[
\partial v_m / \partial x_j = 2\pi i y_j \hat{\nu}_m(y),
\]
if we denote the components of \(a_m^i\) by \(a_m^{i,j}\) we obtain that (3.2) is equivalent to
\[
C \int_{\mathbb{R}^n} \left| y_j \hat{\nu}_m \right|^2 dy \leq \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \sum_{m,j=1}^n a_m^{i,j} y_j \hat{\nu}_m \right|^2 dy \quad \forall v \in \mathcal{D}(\mathbb{R}^n). \tag{3.5}
\]
Suppose now that (3.1) holds. Then for all \(x, y \neq 0\) one has for at least one index \(i\)
\[
\sum_{m,j=1}^n x_m a_m^{i,j} y_j \neq 0. \tag{3.6}
\]
Indeed if
\[
\sum_{m,j=1}^n x_m a_m^{i,j} y_j = 0 \quad \forall i = 1, \ldots, k
\]
then one has
\[
(\sum_{m=1}^n x_m a_m^i \cdot y) = 0 \quad \forall i = 1, \ldots, k
\]
and thus, due to (3.1), \(y = 0\) if \(x \neq 0\). This is a contradiction. It follows then that
\[
\sum_{i=1}^k \left( \sum_{m,j=1}^n a_m^{i,j} y_j x_m \right)^2 dx > 0 \quad \forall x, y \in \mathbb{R}^n \setminus \{0\}.
\]
By a compactness argument one deduces then that for some constant \(C\) it holds
\[
\sum_{i=1}^k \left( \sum_{m,j=1}^n a_m^{i,j} y_j x_m \right)^2 \geq C \sum_{j=1}^n y_j^2 \sum_{m=1}^n x_m^2 \quad \forall x, y \in \mathbb{R}^n. \tag{3.7}
\]
If this inequality holds, writing
\[
\hat{\nu}_m = \alpha_m + i\beta_m,
\]
one has for every index $i$

$$\left| \sum_{m,j=1}^{n} a_{m,j}^i y_j (\alpha_m + i \beta_m) \right|^2 = (\sum_{m,j=1}^{n} a_{m,j}^i y_j \alpha_m)^2 + (\sum_{m,j=1}^{n} a_{m,j}^i y_j \beta_m)^2$$

and thus

$$\sum_{i=1}^{k} \int_{\mathbb{R}^n} \left| \sum_{m,j=1}^{n} a_{m,j}^i y_j \hat{v}_m \right|^2 dy = \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left[ (\sum_{m,j=1}^{n} a_{m,j}^i y_j \alpha_m)^2 + (\sum_{m,j=1}^{n} a_{m,j}^i y_j \beta_m)^2 \right] dy \geq C \int_{\mathbb{R}^n} \left[ \sum_{j=1}^{n} y_j^2 \sum_{m=1}^{n} \alpha_m^2 + \sum_{j=1}^{n} y_j^2 \sum_{m=1}^{n} \beta_m^2 \right] dy = C \int_{\mathbb{R}^n} \sum_{m,j=1}^{n} |y_j^2 \hat{v}_m|^2 dy.$$

This shows that the inequality (3.5) holds for any $v \in \mathcal{D}(\mathbb{R}^n)$ and thus also (3.2). This completes the proof of the theorem.

\[\square\]

In fact we have the following characterisation of $N(n)$.

**Theorem 3.2.** $N(n)$ is the smallest integer $k$ such that there exist $kn$ vectors $a_{m}^{i} \in \mathbb{R}^n, i = 1, \cdots, k, m = 1, \cdots, n$ such that

$$\forall x \neq 0 \quad \text{Rank} \left\{ \left( \sum_{m=1}^{n} x_m a_{m}^{i} \right)_{i=1, \ldots, k} \right\} = n. \quad (3.8)$$

**Proof.** From the previous theorem it is clear that $N(n)$ is smaller or equal to this integer $k$. Suppose now that for some vectors $a_{m}^{i}, i = 1, \cdots, k, m = 1, \cdots, n$ of $\mathbb{R}^n$ we have an inequality of the type (3.2). Let $\varphi \in \mathcal{D}(\mathbb{R}), y = (y_1, \cdots, y_n) \neq 0$. Using

$$v = (y_1 \varphi, \cdots, y_n \varphi)$$

in (3.2) one gets

$$C|y|^2 \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx \leq \sum_{i=1}^{k} \int_{\mathbb{R}^n} \left( \sum_{m=1}^{n} y_m a_{m}^{i} \cdot \nabla \varphi \right)^2 dx \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

If for some $y \neq 0$ the rank of the vectors

$$\sum_{m=1}^{n} y_m a_{m}^{i}, \quad i = 1, \cdots, k$$

is not $n$, then one loses one direction of derivative in the right hand side of the inequality and this is not possible as we saw in Theorem 2.3. This completes the proof of the theorem. \[\square\]
Remark 2: The above result shows again that necessarily $N(n) \geq n$. In fact one can have an alternative proof to Theorem 3.2 by taking in (3.5) $v_m = x_m \varphi$ and showing that (3.5) and (3.7) are equivalent.

One also has:

**Theorem 3.3.** If $n = pk$ it holds

$$N(n) = N(pk) \leq N(p)k^2$$  \hspace{1cm} (3.9)

**Proof.** Let $v \in (\mathcal{D}(\mathbb{R}^p))^{pk}$. Consider the first $p$ components of $v$. One has

$$\sum_{i=1}^{p} \sum_{j=1}^{pk} \left( \frac{\partial v_i}{\partial x_j} \right)^2 = \sum_{i=1}^{p} \sum_{m=0}^{k-1} \sum_{q=1}^{p} \left( \frac{\partial v_i}{\partial x_{mp+q}} \right)^2.$$  

For $m$ fixed the function

$$\tilde{v}_m = (v_1, \ldots, v_p)(x_1, \ldots, x_{mp+1}, \ldots, x_{mp+p}, \ldots, x_n)$$

is a function of $(\mathcal{D}(\mathbb{R}^p))^{p}$ considered as a function of $(x_{mp+1}, \ldots, x_{mp+p})$ the other variables being fixed. Thus for $N(p)$ linear forms $\ell_{m,i}$ and some constant $C$ one has

$$C \int_{\mathbb{R}^p} \sum_{i=1}^{p} \sum_{q=1}^{p} \left( \frac{\partial v_i}{\partial x_{mp+q}} \right)^2 dx_{mp+1} \cdots dx_{mp+p} \leq \sum_{i=1}^{N(p)} \int_{\mathbb{R}^p} \left( \ell_{m,i}(\nabla \tilde{v}_m) \right)^2 dx_{mp+1} \cdots dx_{mp+p},$$

where $\nabla$ denote here the Jacobian matrix of $\tilde{v}_m$ with respect to the variables $(x_{mp+1}, \ldots, x_{mp+p})$. This inequality is true for any other variable kept fixed. Thus, integrating with respect to these other variables we get

$$C \int_{\mathbb{R}^{pk}} \sum_{i=1}^{p} \sum_{q=1}^{p} \left( \frac{\partial v_i}{\partial x_{mp+q}} \right)^2 dx \leq \sum_{i=1}^{N(p)} \int_{\mathbb{R}^{pk}} \left( \ell_{m,i}(\nabla \tilde{v}_m) \right)^2 dx \quad \forall m = 0, \ldots, k - 1.$$  

It is clear that, in the right hand side integral, $\ell_{m,i}(\nabla \tilde{v}_m)$ is in fact a linear form in $\nabla v$. Using the same estimate for the next components of $v$, grouped by set of $p$ of them, we easily get the conclusion that the $L^2$-norm of the gradient of $v$ squared can be bounded by the sum of $N(p)k^2$ $L^2$-norm of squared linear forms of $\nabla v$, that is to say that (3.9) holds. This completes the proof of the theorem.

\[\square\]

4 Applications

Suppose that $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$. We are going to construct a family of vectors in $\mathbb{R}^3$, $a_m^i$, $i = 1, \cdots, 4$, $m = 1, 2, 3$ such that for every $x \in \mathbb{R}^3, x \neq 0$ the set of vectors $\sum_{m=1}^{3} x_m a_m^i$, $i = 1, \cdots, 4$, is of rank 3. Indeed for
given by
\[
\begin{pmatrix}
  a_1^1 & a_1^2 & a_1^3 \\
  a_2^1 & a_2^2 & a_2^3 \\
  a_3^1 & a_3^2 & a_3^3 \\
  a_4^1 & a_4^2 & a_4^3 \\
\end{pmatrix}
\]
any nontrivial linear combination of these vectors is of rank 3. To see that, considering the line vectors
\[
\begin{pmatrix}
  -e_2 & -1 & 0 \\
  -e_3 & 0 & -e_1 \\
  0 & e_3 & -e_2 \\
  e_1 & e_2 & e_3 \\
\end{pmatrix}
\]
one has to see that the rank of these vectors is 3. One has
\[
\begin{vmatrix}
  -x_3 & 0 & x_1 \\
  0 & -x_3 & x_2 \\
  x_1 & x_2 & x_3 \\
\end{vmatrix}
= -x_3(-x_3^2 - x_2^2) + x_1^2 x_3 = x_3(x_1^2 + x_2^2 + x_3^2)
\]
and this is the case when \( x_3 \neq 0 \). When \( x_3 = 0 \) one has
\[
\begin{vmatrix}
  -x_2 & x_1 & 0 \\
  0 & 0 & x_1 \\
  x_1 & x_2 & 0 \\
\end{vmatrix}
= x_1(x_1^2 + x_2^2)
\]
and thus this is the case when \( x_1, x_3 \neq 0 \). When \( x_1 = x_3 = 0 \) this is the case when \( x_2 \neq 0 \). Thus the linear combinations of the above vectors are always of rank 3 when \( x \neq 0 \). When \( \{e_1, e_2, e_3\} \) is the canonical basis of \( \mathbb{R}^3 \) the corresponding splitting in squares of linear forms is the one given in paragraph 2.

We are going to consider the case \( n = 4 \). For this we will use the following lemma.

**Lemma 4.1.** Let \( A \) and \( B \) two \( n \times n \) matrices and denote by \( M \) the \( 2n \times 2n \) matrix given by
\[
M = \begin{pmatrix}
  A & B \\
  -B & A \\
\end{pmatrix}.
\]
Then one has if \( i = (-1)^{\frac{1}{2}} \)
\[
det(M) = |\det(A + iB)|^2
\]
**Proof.** Multiply the last \( n \) columns of \( M \) by \( i \) and add them to the \( n \) first. One gets
\[
det(M) = \det \begin{pmatrix}
  A + iB & B \\
  -B + iA & A \\
\end{pmatrix}.
\]
Multiply the \( n \) first lines of the new matrix by \( i \) and subtract them to the \( n \) last ones. It comes
\[
\text{det}(M) = \text{det} \begin{pmatrix} A + iB & B \\ 0 & A - iB \end{pmatrix} = \text{det}(A + iB)\text{det}(A - iB) = |\text{det}(A + iB)|^2.
\]
This completes the proof of the lemma. \( \Box \)

Then we can show :

**Theorem 4.1.** One has \( N(4) = 4 \).

**Proof.** Consider \( \{e_1, e_2, e_3, e_4\} \) a basis of \( \mathbb{R}^4 \). Then for the vectors \( a_m^i, i,m = 1, \cdots, 4 \), chose
\[
\begin{array}{cccc}
e_1 & e_2 & e_3 & e_4 \\
e_2 & -e_1 & -e_4 & e_3 \\
e_3 & e_4 & -e_1 & -e_2 \\
e_4 & -e_3 & e_2 & -e_1 \\
\end{array}
\]
For \( x \neq 0 \) any combination of these vectors is of rang 4 iff the determinant
\[
\begin{vmatrix}
x_1 & x_2 & x_3 & x_4 \\
x_2 & x_1 & x_4 & -x_3 \\
x_3 & -x_4 & x_1 & x_2 \\
x_4 & x_3 & -x_2 & x_1 \\
\end{vmatrix}
\]
is different of 0. This is the case since these vectors are 2 by 2 orthogonal. This completes the proof of the theorem. \( \Box \)

Note that one can multiply a column of the array of the \( e_i \)’s by some constant different of 0 and get another suitable set of linear forms. Note also that if \( \{e_1, e_2, e_3, e_4\} \) is the canonical basis of \( \mathbb{R}^4 \) the corresponding Korn inequality takes the form
\[
C \int_{\mathbb{R}^4} |\nabla v|^2 dx \leq \int_{\mathbb{R}^4} (\text{div } v)^2 dx + \int_{\mathbb{R}^4} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} - \frac{\partial v_3}{\partial x_4} + \frac{\partial v_4}{\partial x_3} \right)^2 dx \\
+ \int_{\mathbb{R}^4} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_2}{\partial x_4} - \frac{\partial v_3}{\partial x_1} + \frac{\partial v_4}{\partial x_2} \right)^2 dx \\
+ \int_{\mathbb{R}^4} \left( \frac{\partial v_1}{\partial x_4} - \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_4}{\partial x_1} \right)^2 dx \quad \forall v \in \mathcal{D}(\mathbb{R})^4.
\]
One can find vectors \( a_m^i, i,m = 1, \cdots, 4 \), such that any linear combination with \( x \neq 0 \) is of rang 4 but such that it is not formed of orthogonal vectors. Consider for instance
\[
\begin{array}{cccc}
e_1 & e_2 & e_3 & e_4 \\
-e_2 & e_3 & e_4 & e_1 \\
-e_3 & -e_4 & e_1 & e_2 \\
-e_4 & -e_1 & -e_2 & -e_3 \\
\end{array}
\]
For \( x \neq 0 \) any combination of these vectors is of rang 4 iff the determinant
is different of 0. To compute this determinant one multiplies the 3rd line by −1 and one applies the Lemma 4.1 to get

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 & x_4 \\
  x_4 & -x_1 & x_2 & x_3 \\
  x_3 & x_4 & -x_1 & -x_2 \\
  -x_2 & -x_3 & x_4 & -x_1 \\
\end{vmatrix} = -\det \left( \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ -x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_1 \\ x_2 \\ -x_3 \end{pmatrix}, \begin{pmatrix} x_3 \\ x_4 \\ -x_1 \\ x_4 \end{pmatrix}, \begin{pmatrix} x_4 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \right)^2
\]

Thus the determinant above is different of 0 for any \( x \neq 0 \).

Contrary to the case \( n = 4 \), in the case \( n = 6 \) the vectors \( a_m^i, i, m = 1, \cdots, 6 \) cannot be chosen of the type \( \pm \epsilon_i \). Indeed, consider the following matrix with possibly \(-1\) in front of some of the \( x_i \):

\[
M = \begin{pmatrix}
  x_1, x_2, x_3, x_4, x_5, x_6 \\
  x_6, x_1, x_2, x_3, x_4, x_5 \\
  x_5, x_6, x_1, x_2, x_3, x_4 \\
  x_4, x_5, x_6, x_1, x_2, x_3 \\
  x_3, x_4, x_5, x_6, x_1, x_2 \\
  x_2, x_3, x_4, x_5, x_6, x_1 \\
\end{pmatrix}.
\]

Take \( x_1, x_2, x_3, x_5 = 0 \) and compute for \( \epsilon_i = \pm 1, \forall i = 1, \cdots, 12 \) the determinant

\[
D = \begin{vmatrix}
  0, 0, 0, \epsilon_1 x_4, 0, \epsilon_2 x_6 \\
  \epsilon_3 x_6, 0, 0, 0, \epsilon_4 x_4, 0 \\
  0, \epsilon_5 x_6, 0, 0, 0, \epsilon_6 x_4 \\
  \epsilon_7 x_4, 0, \epsilon_8 x_6, 0, 0, 0 \\
  0, \epsilon_9 x_4, 0, \epsilon_{10} x_6, 0, 0 \\
  0, 0, \epsilon_{11} x_4, 0, \epsilon_{12} x_6, 0 \\
\end{vmatrix}.
\]

One gets easily

\[
D = -\{\epsilon_1 \epsilon_4 \epsilon_9 x_4^3 + \epsilon_2 \epsilon_5 \epsilon_{10} x_6^3\}\{\epsilon_4 \epsilon_7 \epsilon_{11} x_4^3 + \epsilon_3 \epsilon_8 \epsilon_{12} x_6^3\}.
\]
which would vanish for $x_4 = \pm x_6$ i.e. for some $x \neq 0$.

**Remark 3**: The decomposition technique of Theorem 3.3 allows to improve the Korn inequality (2.15) in terms of the minimal number of linear forms used. Indeed for instance for $n = 8 = 2 \times 4$ one has

$$N(8) \leq \min\{N(2)4^2, N(4)2^2\} = 16$$

when in Korn’s inequality one uses $n(n-1)/2 + 1 = 29$ linear forms.

## 5 The case of $H^1(\Omega)$

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $n \geq 1$. We recall that $H^1(\Omega)$ is the space $(H^1(\Omega))^n$ i.e.

$$H^1(\Omega) = \{ v = (v_1, \cdots, v_n) \mid v_i \in H^1(\Omega) \; \forall i = 1, \cdots, n \}. \tag{5.1}$$

If $\ell_i, i = 1, \cdots k$ denote linear forms on the space of $n \times n$ matrices one is interested in inequalities of the type

$$C \|\nabla v\|_2^2 \leq \|v\|_2^2 + \sum_{i=1}^{k} |\ell_i(\nabla v)|_2^2 \quad \forall v \in H^1(\Omega). \tag{5.2}$$

One example of such an inequality is the so called second Korn inequality (see [1], [13], [10], [4]). It reads when $\Omega$ is connected and has a Lipschitz-continuous boundary in the sense of [12]:

$$C \|\nabla v\|_2^2 \leq \|v\|_2^2 + \sum_{i,j=1}^{n} |\ell_{i,j}(v)|_2^2 \quad \forall v \in H^1(\Omega), \tag{5.3}$$

where

$$\ell_{i,j}(v) = \frac{1}{2} (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), \quad i, j = 1, \cdots, n. \tag{5.4}$$

Then we can ask ourselves what is the smallest integer $k$ for which we have an inequality of the type (5.2). We will denote it by $N'(n)$. We have seen previously that $N(2) = 2$. One could expect the same value for $N'(2)$. This is not true; one has instead:

**Theorem 5.1.** It holds

$$N'(2) = 3. \tag{5.5}$$

To prove this theorem we will need two lemmas.

**Lemma 5.1.** Let $\nu$ be a unit vector of $\mathbb{R}^2$. One cannot find a constant $C$ such that

$$C \|\nabla v\|_2^2 \leq \|v\|_2^2 + \left|\frac{\partial v_i}{\partial \nu}\right|_2^2 \quad \forall v \in H^1(\Omega). \tag{5.6}$$

**Proof.** Changing possibly the coordinates we can assume that

$$0 \in \Omega, \quad \nu = e_1$$
where \( e_1 \) is the first vector of the canonical basis of \( \mathbb{R}^2 \). Then, we choose \( \delta \) small enough such that \((-\delta, \delta)^2 \subset \Omega\) and consider

\[
\psi, \varphi \in D((-\delta, \delta)), \psi, \varphi \neq 0.
\]

Set

\[
v = \psi(x_1)\varphi\left(\frac{x_2}{\epsilon}\right), \quad \epsilon < 1.
\]

Dropping the measures of integration, a simple computation leads to

\[
|\frac{\partial v}{\partial x_1}|_2^2 = \epsilon \int_{-\delta}^{\delta} \psi'^2 \int_{-\delta}^{\delta} \varphi^2, \quad |\frac{\partial v}{\partial x_2}|_2^2 = \frac{1}{\epsilon} \int_{-\delta}^{\delta} \psi^2 \int_{-\delta}^{\delta} \varphi'^2, \quad |v|_2^2 = \epsilon \int_{-\delta}^{\delta} \psi^2 \int_{-\delta}^{\delta} \varphi^2.
\]

Thus for this \( v \) the inequality (5.6) would become with obvious notation

\[
\epsilon |\psi'|_2^2 |\varphi|^2 + \epsilon |\psi|_2^2 |\varphi'|_2^2 \geq C\{\epsilon |\psi'|_2^2 |\varphi|^2 + \frac{1}{\epsilon} |\psi|_2^2 |\varphi'|_2^2\}
\]

which is impossible for \( \epsilon \) small enough. This completes the proof of the lemma.

**Remark 4**: With a similar proof, when \( \Omega \) is an open set of \( \mathbb{R}^n \), one could show that one cannot have an inequality of the type

\[
C|\nabla v|_2^2 \leq |v|_2^2 + \sum_{i=1}^{k} |\frac{\partial v}{\partial \nu_i}|_2^2 \quad \forall v \in H^1(\Omega)
\]

if the space spanned by the \( \nu_i \) has a dimension less than \( n \) (see also (2.7)). As a consequence one can show that one cannot drop a term in the right hand side of the second Korn Inequality (5.3). Suppose that in (5.3) we do not have the term in \( \ell_{i,j}(v) = \ell_{j,i}(v) \). Consider

\[
v = (0, \ldots, w, \ldots, 0)
\]

where \( w \) is at the \( i^{th} \) position. From (5.3), we easily deduce

\[
C|\nabla w|_2^2 \leq |w|_2^2 + 2 \sum_{k \neq i} |\frac{\partial w}{\partial x_k}|_2^2 \quad \forall w \in H^1(\Omega)
\]

which is impossible since it implies an inequality of the type above.

**Lemma 5.2.** Let \( A \) be a second order elliptic operator with constant coefficients in \( \mathbb{R}^2 \) i.e.

\[
A = a_1 \frac{\partial^2}{\partial x_1^2} + a_2 \frac{\partial^2}{\partial x_2^2} + b \frac{\partial^2}{\partial x_1 \partial x_2}
\]

with \( a_1, a_2 > 0, b^2 - 4a_1a_2 < 0 \). One cannot find a constant \( C \) such that

\[
C\{|\nabla v|_2^2 + |D^2 v|_2^2\} \leq |\nabla v|_2^2 + |Av|_2^2 \quad \forall v \in H^2(\Omega).
\]

In the formula above \( |D^2 v|^2 = (\frac{\partial^2 v}{\partial x_1^2})^2 + 2(\frac{\partial^2 v}{\partial x_1 \partial x_2})^2 + (\frac{\partial^2 v}{\partial x_2^2})^2 \), \( H^2(\Omega) \) denotes the space defined by

\[
H^2(\Omega) = \{ v \in L^2(\Omega) \mid |\nabla v|, |D^2 v| \in L^2(\Omega) \},
\]

(we refer again the reader to [5], [7], [11] for basic notions on Sobolev spaces).
Proof. Let us consider
\[ u_n = \sin(n x_1) e^{n(\alpha x_1 + \beta x_2)}. \]
We claim that we can find \( \alpha, \beta \) such that
\[ Au_n = 0. \] \hspace{1cm} (5.8)
Indeed, by an easy computation, setting 
\[ e = e^{n(\alpha x_1 + \beta x_2)}, \quad S = \sin(n x_1), \quad C = \cos(n x_1), \]
we have
\[ \frac{\partial u_n}{\partial x_1} = n \cos(n x_1)e + n \alpha \sin(n x_1)e, \quad \frac{\partial u_n}{\partial x_2} = n \beta \sin(n x_1)e, \]
\[ \frac{\partial^2 u_n}{\partial x_1^2} = n^2 e(-S + 2 \alpha C + \alpha^2 S), \quad \frac{\partial^2 u_n}{\partial x_1 \partial x_2} = n^2 e \{ \beta C + \alpha \beta S \}, \quad \frac{\partial^2 u_n}{\partial x_2^2} = n^2 e \beta^2 S, \]
\[ Au_n = n^2 e \{ -(a_1 + \alpha^2 a_1 + b a \beta + a_2 \beta^2) S + (2 a a_1 + b \beta) C \}. \]
Note that in the formulas above \( C \) is a notation for the cosine and not the constant in (5.7).
Now we will have (5.8) provided we can choose \( \alpha, \beta \) such that
\[ 2aa_1 + b \beta = 0, \quad -a_1 + \alpha^2 a_1 + b \alpha \beta + a_2 \beta^2 = 0. \] \hspace{1cm} (5.9)
The first equation of (5.9) is satisfied when
\[ \alpha = -\frac{b \beta}{2a_1}. \]
To satisfy the second equation one has to find \( \beta \) such that
\[ -a_1 + \frac{b^2 \beta^2}{4a_1^2} a_1 - \frac{b^2 \beta^2}{2a_1} + a_2 \beta^2 = 0 \]
i.e. \( \beta \) such that
\[ \beta^2 \{ -b^2 + 4a_1 a_2 \} = 4a_1^2 \]
which is possible since \( b^2 - 4a_1 a_2 < 0 \). Suppose that (5.7) holds and that \( \alpha, \beta \) are fixed such that (5.8) holds. For any \( \epsilon \leq 1 \) we have
\[ C\{ ||\nabla u_n||_2^2 + ||D^2 u_n||_2^2 \} \leq ||\nabla u_n||_2^2 + ||Au_n||_2^2 \] \hspace{1cm} (5.10)
where we have denoted by \( |D^2 u_n| \) the quantity defined as
\[ |D^2 u_n|^2 = (\frac{\partial^2 u_n}{\partial x_1^2})^2 + \epsilon(\frac{\partial^2 u_n}{\partial x_1 \partial x_2})^2 + (\frac{\partial^2 u_n}{\partial x_2^2})^2. \] \hspace{1cm} (5.11)
Let us compute (5.11). From the computations above, it follows that
\[ |D^2 u_n|^2 = n^4 e^2 \{ ((\alpha^2 - 1)S + 2 \alpha C)^2 + \epsilon \beta C + \alpha \beta S)^2 + \beta^4 S^2 \} \]
\[ = n^4 e^2 \{ ((\alpha^2 - 1)^2 + \beta^4) S^2 + 4 \alpha (\alpha^2 - 1) SC + 4 \alpha^2 C^2 + \epsilon \beta C + \alpha \beta S)^2 \}. \] \hspace{1cm} (5.12)
Consider the quadratic form
\[ Q(S, C) = ((\alpha^2 - 1)^2 + \beta^4)S^2 + 4\alpha(\alpha^2 - 1)SC + 4\alpha^2C^2. \]
The discriminant of \( Q \) is given by
\[ 16\alpha^2(\alpha^2 - 1)^2 - 16\alpha^2((\alpha^2 - 1)^2 + \beta^4) = -16\alpha^2\beta^4 < 0 \]
if \( b \neq 0 \). Thus the quadratic form \( Q \) is positive definite and, for some constant \( \gamma \), one has
\[ Q(S, C) \geq \gamma(S^2 + C^2). \]
It follows that for \( \epsilon \) small enough, one has
\[ Q(S, C) + \epsilon\{\beta C + \alpha BS\}^2 \geq \frac{\gamma}{2}(S^2 + C^2). \]
Since \( S, C \) are sine and cosine we have \( S^2 + C^2 = 1 \) and from (5.12), we derive that for \( \epsilon \) small enough
\[ |D_\epsilon^2 u_n|^2 \geq n^4e^{2\gamma}. \]
Without loss of generality we can assume that the constant \( C \) in (5.7) is such that \( C < 1 \). Then combining (5.8) and (5.10) we get
\[ C||D_\epsilon^2 u_n||_2^2 - (1 - C)||\nabla u_n||_2^2 \leq 0. \]
From the definition of \( u_n \) one, easily derives
\[ ||\nabla u_n||_2^2 = \int_\Omega e^{2n(\alpha x_1 + \beta x_2)}n^2\{(\cos(nx_1) + \alpha \sin(nx_1))^2 + \beta^2 \sin(nx_1)^2\}dx_1dx_2 \]
\[ \leq \int_\Omega e^{2n(\alpha x_1 + \beta x_2)}n^2\delta dx_1dx_2 \]
for some constant \( \delta \) independent of \( n \). Going back to (5.13), (5.14) we derive
\[ \int_\Omega \{n^4C^2 - (1 - C)\delta n^2\}e^{2n(\alpha x_1 + \beta x_2)}dx_1dx_2 \leq 0 \]
which is clearly impossible for \( n \) large enough. This completes the proof of the lemma when \( b \neq 0 \).

When \( b = 0 \), i.e. \( \alpha = 0 \) (5.12) becomes
\[ |D_\epsilon^2 u_n|^2 = n^4e^{2\gamma}\{2 + \beta^4\}S^2 + \epsilon\beta^2C^2 \]
\[ \geq n^4e^{2\gamma}/2 \]
for \( \epsilon = 1 \) and \( \gamma = \min\{(1 + \beta^4), \beta^2\} \). This is (5.13) and one can conclude as before. This completes the proof of the lemma.
Remark 5: With the same proof one would show the impossibility of the inequalities
\[ C\|v\|_2^2 + \|\nabla v\|_2^2 + \|D^2 v\|_2^2 \leq \|v\|_2^2 + \|\nabla v\|_2^2 + \|Av\|_2^2 \quad \forall v \in H^2(\Omega), \]
\[ C\|D^2 v\|_2^2 \leq \|v\|_2^2 + \|Av\|_2^2 \quad \forall v \in H^2(\Omega). \]
We can now complete the proof of the theorem 5.1.

Proof. First due to the second Korn inequality above we have \( N'(2) \leq 3 \). Recall that a linear form on the \( 2 \times 2 \) Jacobian matrices can be written as
\[ \ell_i(\nabla v) = a_i^1 \cdot \nabla v_1 + a_i^2 \cdot \nabla v_2 \]
where \( a_i^1, a_i^2 \) are vectors in \( \mathbb{R}^2 \).

Then we have \( N'(2) > 1 \). Indeed else we would have for some constant \( C \)
\[ C\|\nabla v\|_2^2 \leq \|v\|_2^2 + \|a_1^1 \cdot \nabla v_1 + a_1^2 \cdot \nabla v_2\|_2^2 \quad \forall v \in \mathbb{H}^1(\Omega). \]
Taking \( v = (w, 0) \) we would get an inequality of the type (5.6) which is impossible (see also the next theorem). Let us now show that one cannot have either \( N'(2) = 2 \). Indeed in this case we would have for some constant \( C' \)
\[ C\|\nabla v\|_2^2 \leq \|v\|_2^2 + \|a_1^1 \cdot \nabla v_1 + a_1^2 \cdot \nabla v_2\|_2^2 + \|a_2^1 \cdot \nabla v_1 + a_2^2 \cdot \nabla v_2\|_2^2 \quad \forall v \in \mathbb{H}^1(\Omega). \] (5.15)
where \( a_1^1, a_1^2, a_2^1, a_2^2 \) are vectors in \( \mathbb{R}^2 \). We claim then first that for any \((\alpha, \beta) \neq (0, 0)\) we necessarily have
\[ \alpha a_1^1 + \beta a_2^1, \quad \alpha a_1^2 + \beta a_2^2 \quad \text{are linearly independent.} \] (5.16)
Indeed, taking \( v = (\alpha w, \beta w) \) in (5.15), we get that, for some other constant \( C' \),
\[ C'\|\nabla w\|_2^2 \leq \|w\|_2^2 + \|{(\alpha a_1^1 + \beta a_2^1) \cdot \nabla w}\|_2^2 + \|{(\alpha a_1^2 + \beta a_2^2) \cdot \nabla w}\|_2^2 \quad \forall w \in H^1(\Omega). \]

If the two vectors in (5.16) are linearly dependent we get a contradiction with the lemma 5.1. In particular we cannot have a linear combination \( \alpha a_1^1 + \beta a_2^1 = 0 \) with \( (\alpha, \beta) \neq (0, 0) \). Thus, \( a_1^1, a_2^1 \) are linearly independent.

In (5.15) let us now change \((v_1, v_2)\) into \( w = (w_1, w_2) = (\alpha_1 v_1 + \alpha_2 v_2, \beta_1 v_1 + \beta_2 v_2, ) \) where the matrix
\[ M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \]
is invertible. We then obtain that, for some constant \( c \) and for every \( v \in \mathbb{H}^1(\Omega) \),
\[ c\|\nabla v\|_2^2 \leq \|v\|_2^2 + \|{(\alpha_1 a_1^1 + \beta_1 a_2^1) \cdot \nabla v_1 + (\alpha_2 a_1^1 + \beta_2 a_2^1) \cdot \nabla v_2}\|_2^2 \]
\[ + \|{(\alpha_1 a_1^2 + \beta_1 a_2^2) \cdot \nabla v_1 + (\alpha_2 a_1^2 + \beta_2 a_2^2) \cdot \nabla v_2}\|_2^2. \]
(We used the equality \( \nabla w = M\nabla v \) and that, if \( M \) is an invertible matrix and \( \| \| \) is a matrix norm, then
\[ \|X\|_M = \|MX\| \]
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is a norm equivalent to the Euclidean norm on the space of $2 \times 2$-matrices).

Since $a_1^1$ and $a_2^1$ are linearly independent one can choose $\alpha_i, \beta_i$ such that

$$\alpha_1 a_1^1 + \beta_1 a_1^2 = e_2, \quad \alpha_2 a_1^1 + \beta_2 a_2^1 = -e_1.$$ 

Then the above inequality becomes, for some new constant $c$,

$$c \| \nabla v \|_2^2 \leq \| v \|_2^2 + \frac{\partial v_1}{\partial x_2} \frac{\partial v_2}{\partial x_1}^2 + \| \nu_1 \cdot \nabla v_1 + \nu_2 \cdot \nabla v_2 \|_2^2 \quad \forall v \in H^1(\Omega) \quad (5.17)$$

where we have set

$$\nu_1 = \alpha_1 a_1^2 + \beta_1 a_2^2, \quad \nu_2 = \alpha_2 a_1^2 + \beta_2 a_2^2.$$ 

Since (5.17) coincides with (5.15), with

$$a_1^1 = e_2, \quad a_1^2 = -e_1, \quad a_2^1 = \nu_1, \quad a_2^2 = \nu_2$$

by (5.16), we can assume without loss of generality that, for any $(\alpha, \beta) \neq (0, 0)$,

$$\alpha e_2 - \beta e_1, \quad \alpha \nu_1 + \beta \nu_2 \quad \text{are linearly independent.} \quad (5.18)$$

Suppose that

$$\nu_1 = \nu_{11} e_1 + \nu_{12} e_2, \quad \nu_2 = \nu_{21} e_1 + \nu_{22} e_2. \quad (5.19)$$

Then we have

$$\alpha \nu_1 + \beta \nu_2 = (\alpha \nu_{11} + \beta \nu_{21}) e_1 + (\alpha \nu_{12} + \beta \nu_{22}) e_2$$

and (5.18) reads in terms of determinant

$$\begin{vmatrix} -\beta & \alpha \\ \alpha \nu_{11} + \beta \nu_{21} & \alpha \nu_{12} + \beta \nu_{22} \end{vmatrix} \neq 0 \quad \forall (\alpha, \beta) \neq (0, 0)$$

i.e.

$$-\beta^2 \nu_{22} - \alpha \beta (\nu_{12} + \nu_{21}) - \alpha^2 \nu_{11} \neq 0 \quad \forall (\alpha, \beta) \neq (0, 0).$$

This implies that

$$(\nu_{12} + \nu_{21})^2 - 4 \nu_{11} \nu_{22} < 0 \quad (5.20)$$

(i.e. the quadratic form above cannot have real roots). Thus we can assume that (5.20) holds.

Then, if (5.17) holds, taking $v = \nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})$ for $u \in H^2(\Omega)$, we get that the inequality above becomes, for some new constant $c$,

$$c \| D^2 u \|_2^2 \leq \| \nabla u \|_2^2 + \| \nabla \cdot \frac{\partial u}{\partial x_1} + \nu_1 \cdot \nabla \frac{\partial u}{\partial x_2} \|_2^2 \quad \forall u \in H^2(\Omega).$$

Adding $c \| \nabla u \|_2^2$ to both sides of this inequality one easily derives that

$$\frac{c}{1 + c} \| \nabla u \|_2^2 + \| D^2 u \|_2^2 \leq \| \nabla u \|_2^2 + \| Au \|_2^2 \quad \forall u \in H^2(\Omega) \quad (5.21)$$
where $A$ is the operator defined by (cf. (5.19))

$$A = \nu_{11} \frac{\partial^2}{\partial x_1^2} + (\nu_{12} + \nu_{21}) \frac{\partial^2}{\partial x_1 \partial x_2} + \nu_{22} \frac{\partial^2}{\partial x_2^2}.$$  

By (5.20) $\nu_{11}, \nu_{22}$ do not vanish and have the same sign which, without loss of generality (changing possibly $A$ into $-A$), we can assume positive. But then (5.20) implies that $A$ is an elliptic operator and the impossibility of (5.21) results from lemma 5.2. This completes the proof of the theorem. 

Note that by Theorem 5.1 we know that $N'(2)$ is the same for any $\Omega$. We do not know if the result holds for any dimension. However we have some complementary results valid for arbitrary $\Omega$'s:

**Theorem 5.2.** One has

$$N'(n) \geq N(n). \quad (5.22)$$

**Proof.** Let $k = N'(n)$. For some vectors $a_{m}^{i}$, $i = 1, \cdots, k$, $m = 1, \cdots, n$, of $\mathbb{R}^{n}$ one has for some constant $C$

$$C||\nabla v||_{2}^{2} \leq ||v||_{2}^{2} + \sum_{i=1}^{k} \left| \sum_{m=1}^{n} a_{m}^{i} \cdot \nabla v_{m}(\frac{x}{\epsilon}) \right|_{2}^{2} \forall v \in \mathbb{H}^{1}(\Omega).$$

Without loss of generality one can assume $0 \in \Omega$. Let then consider $v \in \mathcal{D}(\mathbb{R}^{n})^{n}$. For $\epsilon$ small enough one has

$$v(\frac{x}{\epsilon}) \in \mathbb{H}^{1}(\Omega)$$

and for this function the inequality above reads

$$C \int_{\Omega} \frac{1}{\epsilon^2} |\nabla v(\frac{x}{\epsilon})|^{2} dx \leq \int_{\Omega} |v(\frac{x}{\epsilon})|^{2} dx + \sum_{i=1}^{k} \int_{\Omega} \frac{1}{\epsilon^2} \left| \sum_{m=1}^{n} a_{m}^{i} \cdot \nabla v_{m}(\frac{x}{\epsilon}) \right|^{2} dx.$$

Making the change of variable $y = \frac{x}{\epsilon}$, one easily gets

$$C \int_{\mathbb{R}^{n}} |\nabla v(y)|^{2} dy \leq \epsilon^2 \int_{\mathbb{R}^{n}} |v(y)|^{2} dy + \sum_{i=1}^{k} \int_{\mathbb{R}^{n}} \left| \sum_{m=1}^{n} a_{m}^{i} \cdot \nabla v_{m}(y) \right|^{2} dy.$$

Passing to the limit when $\epsilon \to 0$ one obtains

$$C||\nabla v||_{2}^{2} \leq \sum_{i=1}^{k} \left| \sum_{m=1}^{n} a_{m}^{i} \cdot \nabla v_{m} \right|_{2}^{2} \forall v \in \mathcal{D}(\mathbb{R}^{n})^{n}.$$ 

This shows that $N(n) \leq k$ and completes the proof of the theorem. 

**Remark 5:** Note that theorem 5.1 shows that the inequality (5.22) can be strict. 

One can also show:
Theorem 5.3. One has

\[ N'(2n) \geq 2n \text{ and } N'(2n + 1) > 2n + 1. \]  \hfill (5.23)

\textbf{Proof.} This follows from Theorem 2.3 and Theorem 5.2. \hfill \Box

\textbf{Acknowledgement} : Part of this work was performed when I was visiting the University of Science and Technology of China (USTC) in Hefei and during a part time employment at the S. M. Nikolskii Mathematical Institute of RUDN University, 6 Miklukho-Maklay St, Moscow, 117198, supported by the Ministry of Education and Science of the Russian Federation. I am grateful to these institutions for their support.

\section*{References}


Sur des inégalités du type inégalité de Korn : On se propose dans cette note de déterminer le nombre minimal de formes linéaires qui sont nécessaires pour obtenir des inégalités de type Korn. En d’autres termes nous montrons comment obtenir des estimations de la norme $L^2$ du gradient d’un champ de vecteurs en fonction d’une somme minimale de normes $L^2$ de formes linéaires de ce gradient. On considère ce type d’inégalités sur $H^1_0(\Omega)$ ou $H^1(\Omega)$. 

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