On a non-stationary fluid flow problem in an infinite periodic pipe

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words linearised non-stationary Navier-Stokes type system, periodic pipe flow, flux condition, infinite domain

MSC (2010) 35A01, 35D30, 35Q30, 76D07, 76N10

We study linearized, non-stationary Navier-Stokes type equations with the given flux in an infinite periodic (with respect to \(x_n\)-axis) pipe. The existence and uniqueness of the solution is proved. The asymptotic behavior of the solution as \(\ell \to +\infty\) is investigated.

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1 Introduction

In this paper we study the mathematical model of an incompressible, homogeneous flow in a n-dimensional infinite pipe which is periodic in the \(x_n\)-direction (\(n = 2\) or 3). For every \(x_n\) we denote by \(\sigma_{x_n}\) the section of the pipe at the level \(x_n\). \(\sigma_{x_n}\) is a one or a two dimensional open, bounded subset and we denote by \(D\) the largest diameter of \(\sigma_{x_n}\). The period of the pipe is a cell \(\Pi_L\) of width \(L\) defined as the open set

\[
\Pi_L = \{ x \in \sigma_{x_n} \times \{ x_n \}, x_n \in (0, L) \}.
\]

\(e_n\) denotes the unit vector in the \(x_n\)-direction and the pipe itself is the set

\[
\Omega = \bigcup_{z \in \mathbb{Z}} (\Pi_L \cup \sigma_0 \cup \sigma_L + zLe_n).
\]

We use the notation \(x' = (x_1)\) for the 2D problem and \(x' = (x_1, x_2)\) for the 3D problem. \(S_L\) is the lateral boundary of the cell \(\Pi_L\), i.e. \(S_L = \partial \Pi_L \setminus (\sigma_0 \cup \sigma_L) = \partial \Pi_L \cap \partial \Omega\). Furthermore we assume that the boundary of \(\Pi_L\) is of class \(C^{0,1}\).

We would like to analyze the following non-stationary problem

\[
\begin{cases}
\frac{\partial}{\partial t} \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{U} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Pi_L, \\
\text{div}(\mathbf{u}) = 0 & \text{in } \Pi_L, \\
\mathbf{u}|_{S_L} = 0 & t \in (0, T), \\
\mathbf{u}(x', 0, t) = \mathbf{u}(x', L, t) & t \in (0, T) \quad (1.1)
\end{cases}
\]

with the prescribed flux condition

\[
\int_{\sigma_{x_n}} u_{n}(x', x_n, t) \, dx' = F(t) \quad \forall x_n \in (0, L). \quad (1.2)
\]

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The research of M. Chipot, N. Kloviene, K. Pileckas and S. Zube was funding by Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No. CH-3-SMM-01/01.
In (1.1), (1.2) we have \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \) when \( n = 2 \) and \( u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \) when \( n = 3 \). \( u \) is the velocity of the fluid, \( p \) describes the pressure field and \( \nu \) is the viscosity. The given function \( U \) satisfies the following conditions:

\[
\begin{align*}
\begin{cases}
U \in L_\infty(0, T; W^1_2(\Pi_L) \cap L_\infty(\Pi_L)), & \text{div}(U) = 0, \quad U|_{S_\alpha} = 0, \\
\text{U is } L\text{-periodic in } x_n, \text{i.e. } U(x', x_n, t) = U(x', x_n + L, t). & \end{cases}
\end{align*}
\]

(1.3)

The exterior force \( f \in L_2(0, T; L_2(\Pi_L)) \) and the initial velocity \( a \in W^1_2(\Pi_L) \) is \( L \)-periodic with respect to \( x_n \), solenoidal and equal zero on the lateral boundary of the pipe.

**Remark 1.1** Let \( 0 < \alpha \leq L \). Assuming that \( \eta \) satisfy the Dirichlet homogeneous boundary condition on \( S_\alpha \) and is a solenoidal vector field. Applying Gauss-Ostrogradsky formula we get

\[
0 = \int_{\Pi_a} \text{div}(\eta) \, dx = \int_{\sigma_a} \eta_n \, dx^\prime + \int_{S_\alpha} \eta \, dS_\alpha - \int_{\sigma_a} \eta_n \, dx^\prime,
\]

therefore \( \int_{\sigma_a} \eta_n(x^\prime, 0, t) \, dx^\prime = \int_{\sigma_a} \eta_n(x^\prime, \alpha, t) \, dx^\prime \) for every \( 0 < \alpha \leq L \) where \( \sigma_a \) is a short cut for the set \( \sigma_a \times \{ \alpha \} \). We denote in what follows by \( \int_{\sigma_a} \eta_n \, dx^\prime \) the common value of this integral (cf. (1.2)).

Let us notice that the case \( U(x, t) = 0 \) coincides with the non-stationary Stokes problem.

We will prove that we can take the pressure function \( p(x, t) \) of the form

\[
p(x, t) = -q(t)x_n + p_0(t) + \tilde{p}(x, t),
\]

(1.4)

where \( p_0(t) \) is an arbitrary function and \( \tilde{p}(x, t) \) is a \( L \)-periodic function with respect to \( x_n \). To separate the pressure function term with \( q(t) \) (which will be shown is associated to the flux condition) is important for the problems in domains which have several outlets to infinity (see [20]). The Poiseuille type solution (the velocity has a prescribed flux, and the pressure function is expressed by (1.4)) for the stationary Stokes and Navier-Stokes problems in domains with periodically varying section was found in [13], [15]. The non-stationary solutions with prescribed fluxes to Stokes and Navier-Stokes problems in infinite cylinders and in unbounded domains with cylindrical outlets to infinity were studied in [9], [17], [18], [23], [24]. The existence of time periodic Poiseuille type solutions in the infinite cylinders was proved in [4], [12].

Substituting expression (1.4) into the problem (1.1), (1.2) we get:

\[
\begin{align*}
\frac{\partial}{\partial t} u - \nu \Delta u + (U \cdot \nabla) u + \nabla \tilde{p} &= q + f \quad \text{in } \Pi_L, \\
\text{div}(u) &= 0 \quad \text{in } \Pi_L, \quad u(x, 0) = a(x), \\
\underline{u}(t)|_{S_L} = 0 & \quad t \in (0, T), \\
\underline{u}(x', x_n, t) &= \underline{u}(x', L, t) \quad t \in (0, T), \\
\int_{\sigma} u_n(x', x_n, t) \, dx' &= F(t),
\end{align*}
\]

(1.5)

where \( q(t) = \begin{pmatrix} 0 \\ q(t) \end{pmatrix} \) for the 2D problem and \( q(t) = \begin{pmatrix} 0 \\ 0 \\ q(t) \end{pmatrix} \) for the 3D problem. Here we are interested in \( (u, q, \tilde{p}) \) the solution of problem (1.5). Remark the function \( q(t) \) is picked up such that to have the given flux condition satisfied (this will be detailed later). A weak solution of the above problem can be written as \( (u, q) : \mathbb{R}^3 (\text{resp. } \mathbb{R}^2) \times \mathbb{R} \rightarrow \mathbb{R}^3 (\text{resp. } \mathbb{R}^2) \) solution to

\[
\begin{align*}
\begin{cases}
\underline{u} \in L_2(0, T; H(\Pi_L)), & \frac{\partial}{\partial t} \underline{u} \in L_2(0, T; L_2(\Pi_L)), \\
\frac{\partial}{\partial t} u \big|_{t=0} = a, & u \in L_2(0, T), \\
\int_{0}^{T} \int_{\Pi_L} \frac{\partial}{\partial t} u \cdot \eta \, dx \, d\tau + \nu \int_{0}^{T} \int_{\Pi_L} \nabla u \cdot \nabla \eta \, dx \, d\tau + \int_{0}^{T} \int_{\Pi_L} (U \cdot \nabla) u \cdot \eta \, dx \, d\tau \\
= \int_{0}^{T} \int_{\Pi_L} f \cdot \eta \, dx \, d\tau + L \int_{0}^{T} q(\tau) \int_{\sigma} \eta_n \, dx' \, d\tau \quad \forall \eta \in L_2(0, T; H(\Pi_L)), & \forall t \in (0, T], \\
\int_{\sigma} u_n(x, t) \, dx' = F(t),
\end{cases}
\end{align*}
\]

(1.6)

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In this paragraph we analyze the weak formulation of the problem (1.8):

\[
\begin{align*}
\begin{cases}
F_{\tilde{\eta}} = 0, & \eta(x', 0) = \eta(x', L), & \eta|_{S_L} = 0.
\end{cases}
\end{align*}
\] (1.7)

**Remark 1.2** The right hand side of the first integral equation in (1.6) is obtained by using the fact that

\[
\int q \cdot \eta \, dx = \int q(t) \int \eta_{\sigma} \, dx' \, dx_n = \int q(t) \int \eta_{\sigma} \, dx' \, dx_n = \int q(t) \int \eta_{\sigma} \, dx'.
\]

Since the problem (1.5) is linear, we can look for the solution \( (u(x, t), q(t), \tilde{p}(x, t)) \) in the form

\[
(\mathbf{v}(x, t), q(t), \tilde{p}(x, t)) = (\mathbf{V}(x, t), 0, \tilde{p}_1(x, t)) + (\mathbf{v}(x, t), q(t), \tilde{p}_2(x, t)).
\]

(\( \mathbf{V}(x, t), \tilde{p}_1(t, x) \)) is the solution of the problem

\[
\begin{align*}
\frac{\partial}{\partial t} \mathbf{V} - \nu \Delta \mathbf{V} + (\mathbf{U} \cdot \nabla) \mathbf{V} + \nabla \tilde{p}_1 &= \mathbf{f} \quad \text{in } \Pi_L, \\
\text{div}(\mathbf{V}) &= 0 \quad \text{in } \Pi_L, \\
\mathbf{V}|_{S_L} &= 0 \quad t \in (0, T), \\
\mathbf{V}(x', 0, t) &= \mathbf{V}(x', L, t) \quad t \in (0, T),
\end{align*}
\] (2.1)

and (\( \mathbf{v}(x, t), q(t), \tilde{p}_2(x, t) \)) is the solution of the problem

\[
\begin{align*}
\frac{\partial}{\partial t} \mathbf{v} - \nu \Delta \mathbf{v} + (\mathbf{U} \cdot \nabla) \mathbf{v} + \nabla \tilde{p}_2 &= q \quad \text{in } \Pi_L, \\
\text{div}(\mathbf{v}) &= 0 \quad \text{in } \Pi_L, \\
\mathbf{v}|_{S_L} &= 0 \quad t \in (0, T), \\
\mathbf{v}(x', 0, t) &= \mathbf{v}(x', L, t) \quad t \in (0, T), \\
\int \mathbf{v}_n \, dx' &= \tilde{F}(t),
\end{align*}
\] (2.2)

where \( \tilde{F}(t) = F(t) - \int V_n(x, t) \, dx' \). We assume that the necessary compatibility condition \( \tilde{F}(0) = 0 \) holds, i.e. \( F(0) = \int V_n(x', z, 0) \, dx' \). The pressures \( \tilde{p}_1 \) and \( \tilde{p}_2 \) are \( L \)-periodic in \( x_n \). In the second section we prove existence and uniqueness of a weak solution of problem (1.5) by getting the solution of the weak formulation of problems (1.8) and (1.9). In the third section we analyze the spatial asymptotic behavior of the prescribed problem solution to the problem (3.1).

## 2 The solvability of the problem

### 2.1 The analysis of problem (1.8)

In this paragraph we analyze the weak formulation of the problem (1.8):

\[
\begin{align*}
\begin{cases}
\mathbf{V} \in L_2(0, T; H(\Pi_L)), & \frac{\partial}{\partial t} \mathbf{V} \in L_2(0, T; L_2(\Pi_L)), & \mathbf{V}|_{t=0} = \mathbf{a} \\
\int_0^t \int_{\Pi_L} \mathbf{V} \cdot \eta \, dx' \, dx_n + \nu \int \nabla \mathbf{V} \cdot \nabla \eta \, dx' \, dx_n + \int \mathbf{f} \cdot \eta \, dx' \, dx_n &= \int_0^t \int_{\Pi_L} \mathbf{U} \cdot \nabla \mathbf{V} \cdot \eta \, dx' \, dx_n \\
&= \int_0^t \int_{\Pi_L} \mathbf{f} \cdot \eta \, dx' \, dx_n & \forall \eta \in L_2(0, T; H(\Pi_L)), & \forall t \in (0, T],
\end{cases}
\end{align*}
\] (2.1)

where \( H(\Pi_L) \) is defined in (1.7). Note for this problem we have no flux condition.

Let \( \lambda_\mathbf{k} \) be the eigenvalues and \( \mathbf{v}_\mathbf{k}(x) \) be the eigenfunctions of the following boundary value problem:

\[
\begin{align*}
\begin{cases}
\mathbf{v}_\mathbf{k} \in H(\Pi_L), & \nu \int \nabla \mathbf{v}_\mathbf{k}(x) \cdot \nabla \mathbf{v} \, dx = \lambda_\mathbf{k} \int \mathbf{v}_\mathbf{k}(x) \mathbf{v} \, dx & \forall \mathbf{v} \in H(\Pi_L).
\end{cases}
\end{align*}
\] (2.2)

For our set \( \Pi_L \) holds the following:
• (2.2) defines a countable set of eigenvalues $\lambda_k > 0$, $\lambda_k \to \infty$, $k = 1, 2, \ldots$; the corresponding eigenfunctions $v_k$ constitute a basis $\{v_k\}_{k \geq 1}$ in $H(\Pi_L)$.

• The eigenfunctions $v_k$ can be orthonormalized:

\[
\int_{\Pi_L} v_k \cdot v_{\ell} \, dx = \begin{cases} 1, & k = \ell, \\ 0, & k \neq \ell. \end{cases} \tag{2.3}
\]

• Moreover

\[
\int_{\Pi_L} \nabla v_k \cdot \nabla v_{\ell} \, dx = \begin{cases} \frac{\lambda_k}{\nu}, & k = \ell, \\ 0, & k \neq \ell. \end{cases} \tag{2.4}
\]

(For this result compare Lemma 8.1 in [7] or see [14].)

We look for an approximate solution $V^{(N)}(x, t)$ of the problem (2.1) in the form

\[
V^{(N)}(x, t) = \sum_{k=1}^{N} y_k^{(N)}(t)v_k(x),
\]

where the coefficients $y_k^{(N)}(t)$ are found from the differential equations

\[
\int_{\Pi_L} \frac{\partial}{\partial t} V^{(N)} \cdot v_k \, dx + \nu \int_{\Pi_L} \nabla V^{(N)} \cdot \nabla v_k \, dx + \int_{\Pi_L} (U \cdot \nabla) V^{(N)} \cdot v_k \, dx = \int_{\Pi_L} f \cdot v_k \, dx \tag{2.5}
\]

$k = 1, 2, \ldots, N$ and the initial conditions $V^{(N)}(x, 0) = \sum_{k=1}^{N} (\int_{\Pi_L} a \cdot v_k \, dx)v_k$. Using the properties (2.3) and (2.4) of the basis we then derive from the above equality the following Cauchy problem for the system of linear ordinary differential equations

\[
\begin{cases} Y^{(N)}(t) + (\mathbb{B}^{(N)}(t) + \mathcal{A}^{(N)}(t)) Y^{(N)}(t) = \mathbb{B}^{(N)}(t), \\ Y^{(N)}(0) = C^{(N)}, \end{cases} \tag{2.6}
\]

where

\[
Y^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \vdots \\ y_N^{(N)}(t) \end{pmatrix}, \quad \mathbb{B}^{(N)}(t) = \begin{pmatrix} B_1(t) \\ \vdots \\ B_N(t) \end{pmatrix}, \quad C^{(N)} = \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix},
\]

\[
\mathbb{B}^{(N)} = \text{diag}(\lambda_1, \ldots, \lambda_N), \quad B_i(t) = \int_{\Pi_L} f \cdot v_i \, dx, \quad C_i = \int_{\Pi_L} a \cdot v_i \, dx,
\]

\[
\mathcal{A}^{(N)}(t) = (\alpha_{ij}(t))_{i,j=1,\ldots,N} \text{ with } \alpha_{ij}(t) = \int_{\Pi_L} (U \cdot \nabla) v_i \cdot v_j \, dx.
\]

**Lemma 2.1** Let $f \in L_2(0, T; L_2(\Pi_L))$, $T \in (0, \infty)$ and suppose that $U$ satisfies (1.3), then there exists a unique solution $Y^{(N)} \in W_2^1(0, T)$ of the problem (2.6).

**Proof.** Let us prove that the entries $\alpha_{ij}(t)$ of the matrix $\mathcal{A}^{(N)}(t)$ are bounded. We have

\[
|\alpha_{ij}(t)| = \left| \int_{\Pi_L} (U \cdot \nabla) v_i \cdot v_j \, dx \right| \leq \|U\|_{L_\infty(\Pi_L)} \|\nabla v_i\|_{L_2(\Pi_L)} \|\nabla v_j\|_{L_2(\Pi_L)}. \tag{2.7}
\]

Thus, all the entries of the matrix $\mathcal{A}^{(N)}(t)$ are bounded functions and, therefore, the existence of the unique solution of the problem (2.6) follows from standard results for linear ordinary differential equations (see, for example, [26]).
We prove now an a priori estimate for the solution of problem (2.1).

**Theorem 2.2** Suppose that $f \in L^2(0, T; L^2(\Pi_L))$, $T \in (0, \infty)$, $a \in W^{1,2}_2(\Pi_L)$ and $U$ satisfies (1.3), then for the approximate solution $V^{(N)}(x, t)$ of the problem (2.1) holds the following estimate

$$
\sup_{t \in [0, T]} \left\| V^{(N)}(\cdot, t) \right\|_{W^{1,2}_2(\Pi_L)}^2 + \int_0^T \left\| \frac{\partial}{\partial t} V^{(N)} \right\|_{L^2(\Pi_L)}^2 d\tau \\
\leq c \left( \left\| a \right\|_{W^{1,2}_2(\Pi_L)}^2 + \int_0^T \left\| f \right\|_{L^2(\Pi_L)}^2 d\tau \right)
$$

(2.8)

where $c$ depends on $L$, $D$, $T$, $\nu$ and the function $U$, and is independent of $N$ and $t$.

**Proof.** Multiplying (2.5) by $y_k^{(N)}(t)$ and summing up for $k$ from 1 to $N$, we obtain by using Young’s and Poincaré’s inequalities

$$
\frac{1}{2} \frac{d}{dt} \int_{\Pi_L} |V^{(N)}|^2 \, dx + \nu \int_{\Pi_L} |\nabla V^{(N)}|^2 \, dx + \int_{\Pi_L} (U \cdot \nabla) V^{(N)} \cdot V^{(N)} \, dx

= \int_{\Pi_L} f \cdot V^{(N)} \, dx \leq \frac{1}{2\epsilon} \int_{\Pi_L} |f|^2 \, dx + \frac{\epsilon}{2} \int_{\Pi_L} |V^{(N)}|^2 \, dx

\leq \frac{1}{2\epsilon} \int_{\Pi_L} |f|^2 \, dx + \frac{C_p}{2} \int_{\Pi_L} |\nabla V^{(N)}|^2 \, dx,
$$

here $C_p$, depending on $D$, is the constant of the Poincaré inequality. (Remark: Since $U$ is $L$-periodic in $x_n$ and $V^{(N)}|_{S_L} = 0$, we get by applying the Divergence-theorem that $\int_{\Pi_L} (U \cdot \nabla) V^{(N)} \cdot V^{(N)} \, dx = \sum_{i,j} \int_{\Pi_L} \frac{1}{2} U^j \partial_x ((V^{(N)})^j)^2 \, dx = 0$.) Integrating in $t$ we derive

$$
\frac{1}{2} \int_{\Pi_L} |V^{(N)}(t)|^2 \, dx + \left( \nu - \frac{C_p}{2} \right) \int_0^t \left\| \nabla V^{(N)} \right\|_{L^2(\Pi_L)}^2 d\tau \leq \frac{1}{2\epsilon} \int_0^t \left\| f \right\|_{L^2(\Pi_L)}^2 d\tau + \frac{1}{2} \left\| a \right\|_{L^2(\Pi_L)}^2
$$

(2.9)

and we choose $\epsilon$ small enough such that $\epsilon < \frac{2\nu}{C_p}$.

Next we multiply the equality (2.5) by $\frac{d}{dt} y_k^{(N)}(t)$, then we sum up and so we obtain

$$
\int_{\Pi_L} \frac{\partial}{\partial t} V^{(N)} \, dx + \nu \frac{d}{dt} \int_{\Pi_L} |\nabla V^{(N)}|^2 \, dx + \int_{\Pi_L} (U \cdot \nabla) V^{(N)} \cdot \frac{\partial}{\partial t} V^{(N)} \, dx

= \int_{\Pi_L} f \cdot \frac{\partial}{\partial t} V^{(N)} \, dx \leq \frac{1}{2\delta} \left\| f \right\|_{L^2(\Pi_L)}^2 + \frac{\delta}{2} \left\| \nabla V^{(N)} \right\|_{L^2(\Pi_L)}^2
$$

which leads to

$$
(1 - \frac{\delta}{2}) \int_{\Pi_L} \frac{\partial}{\partial t} V^{(N)} \, dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Pi_L} |\nabla V^{(N)}|^2 \, dx

\leq \int_{\Pi_L} (U \cdot \nabla) V^{(N)} \cdot \frac{\partial}{\partial t} V^{(N)} \, dx + \frac{1}{2\delta} \left\| f \right\|_{L^2(\Pi_L)}^2

\leq \frac{1}{2} \sup_{x \in \Pi_L} |U|^2 \int_{\Pi_L} |\nabla V^{(N)}|^2 \, dx + \frac{1}{2} \int_{\Pi_L} \left| \frac{\partial}{\partial t} V^{(N)} \right|^2 \, dx + \frac{1}{2\delta} \left\| f \right\|_{L^2(\Pi_L)}^2.
$$

(2.10)
Let us choose \( \delta \) small enough such that \( \delta < 1 \), hence it holds

\[
\frac{d}{dt} \int_{\Pi_L} |\nabla \mathbf{V}^{(N)}|^2 \, dx \leq \frac{1}{\nu} m(t) \int_{\Pi_L} |\nabla \mathbf{V}^{(N)}|^2 \, dx + \frac{1}{\nu \delta} |f|^2_{L^2(\Pi_L)}
\]

with

\[
m(t) = \sup_{x \in \Pi_L} |U|^2.
\]

By integrating in \( t \) and using Gronwall’s inequality, we get

\[
||\nabla \mathbf{V}^{(N)}||^2_{L^2(\Pi_L)} \leq e^{\frac{1}{\nu} \int_0^r m(\tau) \, d\tau} \frac{1}{\nu \delta} \int_0^r |f|^2_{L^2(\Pi_L)} \, d\tau + ||\nabla a||^2_{L^2(\Pi_L)}
\]

\[
\leq e^{\frac{1}{\nu} \int_0^r m(\tau) \, d\tau} \left( \frac{1}{\nu \delta} \int_0^r |f|^2_{L^2(\Pi_L)} \, d\tau + ||\nabla a||^2_{L^2(\Pi_L)} \right) \leq c_1 \int_0^r |f|^2_{L^2(\Pi_L)} \, d\tau + c_2 ||\nabla a||^2_{L^2(\Pi_L)}
\]

Integrating (2.10) in \( t \) and using the above estimate, we derive

\[
(1 - \delta) \int_0^t \left( \frac{\partial}{\partial t} \mathbf{V}^{(N)} \right)^2_{L^2(\Pi_L)} \, dt + \nu \int_0^t |\nabla \mathbf{V}^{(N)}|^2 \, dx \leq \int_0^t m(\tau) \left( c_1 \left( \int_0^\tau |f|^2_{L^2(\Pi_L)} \, d\tau \right) + c_2 \int T_0 \right) \left| \nabla a \right|_{L^2(\Pi_L)}^2 \, d\tau
\]

which completes the proof. Remark that constants depend on \( D, \nu, L, T \) and \( U \). The inequality of the theorem is a combination of the previous inequality and (2.9). \( \square \)

**Theorem 2.3** Suppose that \( f \in L^2(R_0, T; L^2(\Pi_L)) \), \( T \in (0, \infty) \), \( a \in W^1_2(\Pi_L) \) and \( U \) satisfies (1.3). Then the problem (1.8) admits a unique weak solution \( \mathbf{V}(x,t) \in L^2(0,T; H(\Pi_L)) \) and the following estimate

\[
\sup_{t \in [0,T]} ||\mathbf{V}||^2_{W^2_2(\Pi_L)} + \int_0^T \left| \frac{\partial}{\partial t} \mathbf{V} \right|_{L^2(\Pi_L)}^2 \, d\tau \leq c(||a||^2_{W^2_2(\Pi_L)} + \int_0^T ||f||^2_{L^2(\Pi_L)} \, d\tau)
\]

is valid, where the constant \( c \) is depending on \( \nu, T, L, D \) and the function \( U \).

**Proof.** From the previous theorem we get that \( \{ \mathbf{V}^{(N)} \} \) resp. \( \{ \frac{\partial}{\partial t} \mathbf{V}^{(N)} \} \) are bounded in the space \( L^2(0,T; W^1_2(\Pi_L)) \) resp. \( L^2(0,T; L^2(\Pi_L)) \) hence there exist subsequences \( \{ \mathbf{V}^{(N_k)} \} \) resp. \( \{ \frac{\partial}{\partial t} \mathbf{V}^{(N_k)} \} \) which converge weakly to \( \mathbf{V} \) resp. \( \frac{\partial}{\partial t} \mathbf{V} \) in the space \( L^2(0,T; W^1_2(\Pi_L)) \) resp. \( L^2(0,T; L^2(\Pi_L)) \).

For the approximate solution we have for \( k \) large enough, i.e. \( N_k \geq M \)

\[
\int_0^t \int_{\Pi_L} \frac{\partial}{\partial t} \mathbf{V}^{(N_k)} \cdot \eta \, dxd\tau + \nu \int_0^t \int_{\Pi_L} \nabla \mathbf{V}^{(N_k)} \cdot \nabla \eta \, dxd\tau
\]

\[
+ \int_0^t \int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{V}^{(N_k)} \cdot \eta \, dxd\tau = \int_0^t \int_{\Pi_L} f \cdot \eta \, dxd\tau
\]

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for $\eta \in L_2(0,T; [v_1, \ldots, v_M])$, where $[v_1, \ldots, v_M]$ denotes the span of $v_1, \ldots, v_M$. Passing to the limit as $N_k \to \infty$ we get

$$\int_0^t \int_{\Omega_L} \frac{\partial}{\partial \tau} \mathbf{v} \cdot \eta \, dx \, d\tau + \nu \int_0^t \int_{\Omega_L} \nabla \mathbf{v} \cdot \nabla \eta \, dx \, d\tau + \int_0^t \int_{\Omega_L} (\mathbf{U} \cdot \nabla) \mathbf{v} \cdot \eta \, dx \, d\tau = \frac{1}{2} \int_{\Omega_L} |\mathbf{V}_1 - \mathbf{V}_2|^2 \, dx + \nu \int_0^t \int_{\Omega_L} \nabla (\mathbf{V}_1 - \mathbf{V}_2) \cdot \nabla (\mathbf{V}_1 - \mathbf{V}_2) \, d\tau + \int_0^t \int_{\Omega_L} (\mathbf{U} \cdot \nabla) (\mathbf{V}_1 - \mathbf{V}_2) \cdot (\mathbf{V}_1 - \mathbf{V}_2) \, dx \, d\tau = 0,$$

(2.14)

So the existence of the weak solution of the problem (1.8) is shown, since $L_2(0,T; [v_1, \ldots, v_M])$ is dense in $L_2(0,T; H(\Omega_L))$ when $M \to \infty$.

Now let us show the uniqueness. For this we assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are two solutions of the problem (2.1) and then the following equality holds

$$\int_0^t \int_{\Omega_L} \frac{\partial}{\partial \tau} (\mathbf{V}_1 - \mathbf{V}_2) \cdot \eta \, dx \, d\tau + \nu \int_0^t \int_{\Omega_L} \nabla (\mathbf{V}_1 - \mathbf{V}_2) \cdot \nabla \eta \, dx \, d\tau + \int_0^t \int_{\Omega_L} (\mathbf{U} \cdot \nabla) (\mathbf{V}_1 - \mathbf{V}_2) \cdot (\mathbf{V}_1 - \mathbf{V}_2) \, dx \, d\tau = 0.$$

Let us take $\eta = \mathbf{V}_1 - \mathbf{V}_2$ and so we get

$$\frac{1}{2} \int_{\Omega_L} |\mathbf{V}_1 - \mathbf{V}_2|^2 \, dx + \nu \int_0^t \int_{\Omega_L} \nabla (\mathbf{V}_1 - \mathbf{V}_2) \cdot \nabla (\mathbf{V}_1 - \mathbf{V}_2) \, d\tau + \int_0^t \int_{\Omega_L} (\mathbf{U} \cdot \nabla) (\mathbf{V}_1 - \mathbf{V}_2) \cdot (\mathbf{V}_1 - \mathbf{V}_2) \, dx \, d\tau = 0,$$

i.e.

$$\frac{1}{2} \int_{\Omega_L} |\mathbf{V}_1 - \mathbf{V}_2|^2 \, dx + \nu \int_0^t \int_{\Omega_L} |\nabla (\mathbf{V}_1 - \mathbf{V}_2)|^2 \, d\tau = 0$$

and therefore the uniqueness follows. For the proof of the estimate see Theorem 2.2. \(\square\)

### 2.2 The analysis of problem (1.9)

This paragraph is devoted to the analysis of the weak formulation of the problem (1.9):

$$\begin{cases}
\mathbf{v} \in L_2(0,T; H(\Omega_L)), & \frac{\partial}{\partial \tau} \mathbf{v} \in L_2(0,T; L_2(\Omega_L)), \quad \mathbf{v} |_{\tau=0} = 0, \quad q \in L_2(0,T), \\
\int_0^t \int_{\Omega_L} \frac{\partial}{\partial \tau} \mathbf{v} \cdot \eta \, dx \, d\tau + \nu \int_0^t \int_{\Omega_L} \nabla \mathbf{v} \cdot \nabla \eta \, dx \, d\tau + \int_0^t \int_{\Omega_L} (\mathbf{U} \cdot \nabla) \mathbf{v} \cdot \eta \, dx \, d\tau = L \int_0^T q(\tau) \int_0^\sigma \eta_v \, dx \, d\tau \quad \forall \eta \in L_2(0,T; H(\Omega_L)), \quad \forall t \in (0,T], \\
\int_0^\sigma \eta_v \, dx \, d\tau = \bar{F}(t),
\end{cases}$$

(2.15)

where $\bar{F}(t) = F(t) - \int_0^\sigma V_n \, dx'$ and $\mathbf{V}$ is the solution of the problem (2.1) and the compatibility condition $\bar{F}(0) = 0$ holds.
We are looking for an approximate solution \((\mathbf{v}^{(N)}(x,t), q^{(N)}(t))\) of the problem (2.15) in the form
\[
\mathbf{v}^{(N)}(x,t) = \sum_{k=1}^{N} y_k^{(N)}(t) \mathbf{v}_k(x),
\]
where \(\mathbf{v}_k(x)\) are the eigenfunctions of the problem (2.2) (see paragraph 2.1) and the coefficients \(y_k^{(N)}(t)\) are found from the differential equation
\[
\int_{\Pi_L} \frac{\partial}{\partial t} \mathbf{v}^{(N)} \cdot \mathbf{v}_k \, dx + \nu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla \mathbf{v}_k \, dx + \int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{v}^{(N)} \cdot \mathbf{v}_k \, dx = L q^{(N)}(t) \int_{\sigma} v_{kn} \, dx',
\]
k = 1, ..., N and the initial condition \(\mathbf{v}^{(N)}(x,0) = 0\). In (2.17) \(v_{kn}\) means the last component of the vector \(\mathbf{v}_k\). Function \(q^{(N)}(t)\) is picked up in order to satisfy the flux condition
\[
\int_{\sigma} v^{(N)}_{kn}(x,t) \, dx' = \tilde{F}(t).
\]

Using the properties (2.3) and (2.4) of the basis combining with (2.17) we derive the following Cauchy problem for the system of linear ordinary differential equations
\[
\begin{cases}
\mathbf{Y}^{(N)'}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t)) \mathbf{Y}^{(N)}(t) = L \mathbf{B}^{(N)} q^{(N)}(t), \\
\mathbf{Y}^{(N)}(0) = 0,
\end{cases}
\]
where
\[
\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \vdots \\ y_N^{(N)}(t) \end{pmatrix}, \quad \mathbf{B}^{(N)} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},
\]
\(\mathbb{J}^{(N)} = \text{diag}(\lambda_1, ..., \lambda_N)\) - diagonal matrix, \(\mathbb{A}^{(N)}(t)\) is \((N \times N)\) - matrix with the elements \(\alpha_{ij}(t) = \int_{\Pi_L} (\mathbf{U} \cdot \nabla) v_i \cdot v_j \, dx\) and \(\beta_k = \int_{\sigma} v_{kn} \, dx'\) with \(i,j = 1, ..., N\).

**Lemma 2.4** Let \(q^{(N)} \in L_2(0,T), T \in (0, \infty)\) and suppose that \(\mathbf{U}\) satisfies (1.3), then there exists a unique solution \(\mathbf{Y}^{(N)} \in W_2^1(0,T)\) of the problem (2.19).

**Proof.** The proof of this lemma is similar to the one of Lemma 2.1. \(\square\)

The elements \(\alpha_{ij}(t)\) of the matrix \(\mathbb{A}^{(N)}(t)\) are bounded (see the proof of Lemma 2.1). The fundamental matrix \(\mathbf{Z}^{(N)}(t)\) of the problem (2.19) is the solution of the matrix Cauchy problem
\[
\begin{cases}
\mathbf{Z}^{(N)'}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t)) \mathbf{Z}^{(N)}(t) = \mathbf{0}, \\
\mathbf{Z}^{(N)}(0) = \mathbb{E}^{(N)},
\end{cases}
\]
where \(\mathbb{E}^{(N)}\) is the unit matrix and \(\mathbf{0}\) is the zero matrix. This problem is equivalent to the integral equation of the type:
\[
\exp(\mathbb{J}^{(N)}(t)) \mathbf{Z}^{(N)}(t) = \mathbb{E}^{(N)} - \int_{0}^{t} \mathbb{A}^{(N)}(\tau) \exp(\mathbb{J}^{(N)}(\tau)) \mathbf{Z}^{(N)}(\tau) \, d\tau.
\]
\(\| \cdot \|\) denotes an operator norm for the matrices and we deduce
\[
\| \exp(\mathbb{J}^{(N)}(t)) \mathbf{Z}^{(N)}(t) \| \leq \| \mathbb{E}^{(N)} \| + \int_{0}^{t} \| \mathbb{A}^{(N)}(\tau) \| \| \exp(\mathbb{J}^{(N)}(\tau)) \mathbf{Z}^{(N)}(\tau) \| \, d\tau,
\]
i.e.,
\[
\| \exp(\mathbb{J}^N(t))Z^N(t) \| \leq 1 + \int_0^t \| \mathbb{A}^N(\tau) \| \exp(\mathbb{J}^N(\tau))Z^N(\tau) \| \, d\tau
\]
and by the Gronwall inequality
\[
\| \exp(\mathbb{J}^N(t))Z^N(t) \| \leq \exp(\int_0^t \| \mathbb{A}^N(\tau) \| \, d\tau).
\]
Thus for \( t \in (0, T) \)
\[
\| Z^N(t) \| \leq \exp(-\lambda_1 t + \int_0^t \| \mathbb{A}^N(\tau) \| \, d\tau) \leq C(T).
\]

From the theory of ordinary differential equations we get that
\[
\det(Z^N(t)) = \det(Z^N(0)) \exp(-\int_0^t \text{tr}(\mathbb{J}^N + \mathbb{A}^N(\tau)) \, d\tau) = \exp(-\int_0^t \text{tr} \mathbb{J}^N \, d\tau).
\]
Since \( \mathbb{J}^N \) is formed from the eigenvalues, it follows that \( \det(Z^N(t)) \geq C_N(T) \). Since for a fixed \( N \) the matrix \( Z^N \) is bounded, we see easily using the form of \( (Z^N)^{-1} \) that \( \| (Z^N)^{-1}(t) \| \leq C_N(T) \).

The solution \( Y^N(t) \) of the problem (2.19) is given by
\[
Y^N(t) = L \int_0^t Z^N(t)(Z^N(\tau))^{-1} \beta^N(q^N(\tau)) \, d\tau.
\]

We find the functions \( q^N(t) \) by using the flux condition (2.18). Substituting \( v^N(x, t) \) into (2.18) gives
\[
F(t) = \int \sigma_k y_k(t) \, dx' = L \beta^N \cdot \int_0^t Z^N(t)(Z^N(\tau))^{-1} \beta^N q^N(\tau) \, d\tau.
\]
Thus, \( q^N(t) \) has to be found as the solution of the Volterra integral equation of the first kind
\[
L \int_0^t \beta^N \cdot Z^N(t)(Z^N(\tau))^{-1} \beta^N q^N(\tau) \, d\tau = \tilde{F}(t).
\]
Since both functions vanish at 0, differentiating and using (2.20), we reduce the equation above to a Volterra integral equation of the second kind
\[
q^N(t) - \int_0^t K^N(t, \tau)q^N(\tau) \, d\tau = \vartheta(t)
\]
with the kernel
\[
K^N(t, \tau) = \frac{\beta^N}{\kappa_N} \cdot (\mathbb{J}^N + \mathbb{A}^N(t))Z^N(t)(Z^N(\tau))^{-1} \beta^N,
\]
where \( \kappa_N = |\beta^N|^2 \) and \( \vartheta(t) = \frac{1}{L\kappa_N} \frac{d}{dt} \tilde{F}(t) \).
In virtue of (2.21) we have that for any fixed $N$ the kernel $K(t, \tau)$ is bounded for all $0 \leq \tau \leq t$ and hence $K(t, \tau) \in L_2(Q^T)$ with $Q^T = (0, T) \times (0, T)$. Therefore, for any $\frac{d}{dt} F \in L_2(0, T)$ there exists a unique solution $q(N) \in L_2(0, T)$ of the integral equation (2.23) and the following estimate

$$\|q(N)\|_{L_2(0, T)} \leq C_N \|\frac{d}{dt} F\|_{L_2(0, T)} \leq C_N \left( \|\frac{d}{dt} F\|_{L_2(0, T)} + \|\frac{\partial}{\partial t} \tilde{V}\|_{L_2(0, T)} \right) \quad (2.24)$$

holds (see, for example, [27]). The constant $C_N$ in (2.24) depends on the kernel $K(t, \tau)$ and we cannot say in advance that $C_N$ stays bounded as $N \to \infty$.

**Remark 2.5** In the estimate (2.24) we use the function $\frac{d}{dt} F = \frac{d}{dt} F - \frac{\partial}{\partial t} V_n$. But $\frac{\partial}{\partial t} V$ is only a $L^2$-function, and we cannot speak about the usual trace estimate of it. However, the trace of the normal component $\frac{\partial}{\partial t} V_n$ is defined on $\sigma$ as an element of the dual space $W^{-1/2}_2(\sigma)$ and the corresponding estimate holds true (see [25], Ch.1).

So we have now shown the existence of a unique approximate weak solution $(v(N)(x, t), q(N)(t))$. Next we will obtain an a priori estimate for the approximate solution.

**Theorem 2.6** Suppose that $\tilde{F}(t) \in W^1_2(0, T)$, $\tilde{F}(0) = 0$, $T \in (0, \infty)$ and $U$ satisfies (1.3). Then for the approximate weak solution $(v(N)(x, t), q(N)(t))$ of the problem (1.9) the following estimate

$$\sup_{t \in [0, T]} \|v(N)(\cdot, t)\|_{W^2_2(\Pi_L)} + \int_0^T \|\frac{\partial}{\partial t} v(N)\|_{L_2(\Pi_L)}^2 d\tau + \|q(N)\|_{L_2(0, T)}^2 \leq c \int v_n(N) dx \quad (2.25)$$

holds. The constant $c$ is depending on $L, D, T, \nu$ and the function $U$ and is independent of $N$ and $t$.

**Proof.** Multiplying the equality (2.17) by $y_k(N)(t)$ and summing up by $k$ from 1 to $N$ we get:

$$\frac{1}{2} \frac{d}{dt} \int_{\Pi_L} |v(N)|^2 dx + \int_{\Pi_L} |\nabla v(N)|^2 dx + \int_{\Pi_L} (U \cdot \nabla) v(N) \cdot v(N) dx = L_q(N) \int_{\sigma} v_n(N) dx. \quad (2.26)$$

Having in mind the flux condition (2.18), we derive the following estimate

$$\frac{1}{2} \frac{d}{dt} \int_{\Pi_L} |v(N)|^2 dx + \int_{\Pi_L} |\nabla v(N)|^2 dx \leq \varepsilon |q(N)|^2 + \frac{L^2}{\varepsilon} \int \tilde{F}^2 dx. \quad (2.27)$$

Integrating the inequality above in $t$ we get

$$\frac{1}{2} \int_{\Pi_L} |v(N)|^2 dx + \frac{\varepsilon}{2} \int_0^t \int_{\Pi_L} |\nabla v(N)|^2 dx d\tau \leq \int_0^t |q(N)|^2 d\tau + \frac{L^2}{\varepsilon} \int_0^\tau \int \tilde{F}^2 d\tau. \quad (2.28)$$

Let us now multiply (2.17) by $\frac{d}{dt} y_k(N)(t)$ and sum up by $k$ from 1 to $N$ then we obtain

$$\int_{\Pi_L} \frac{\partial}{\partial t} v(N)^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Pi_L} |\nabla v(N)|^2 dx + \int_{\Pi_L} (U \cdot \nabla) v(N) \cdot \frac{\partial}{\partial t} v(N) dx = L_q(N) \int_{\sigma} \frac{\partial}{\partial t} v_n(N) dx. \quad (2.29)$$

Using Cauchy-Schwarz and Young inequalities we can get the estimate

$$\int_{\Pi_N} \frac{\partial}{\partial t} v(N)^2 dx + \frac{\nu}{2} \frac{d}{dt} \int_{\Pi_L} |\nabla v(N)|^2 dx \leq \int \left( |U \cdot \nabla v(N)| \right)^2 dx + L_q(N) \frac{d}{dt} \tilde{F} \leq \frac{1}{2} \left( \sup_{x \in \Pi_L} |U|^2 \right) \int_{\Pi_L} |\nabla v(N)|^2 dx + \frac{1}{2} \int_{\Pi_L} \frac{\partial}{\partial t} v(N)^2 dx + \varepsilon |q(N)|^2 \frac{d}{dt} \tilde{F}. \quad (2.29)$$
Hence
\[ \frac{d}{dt} \int_{\Pi_L} |\nabla v(N)|^2 dx \leq \frac{m(t)}{\nu} \int_{\Pi_L} |\nabla v(N)|^2 dx + \frac{2\varepsilon}{\nu} |q(N)|^2 + \frac{2L^2}{\varepsilon\nu} |d_t \tilde{F}|^2 \]

(recall that \( m(t) = \sup_{x \in \Pi_L} |U|^2 \)) and so integrating in \( t \)
\[ \int_{\Pi_L} |\nabla v(N)|^2 dx \leq e^{\frac{1}{\nu} \int_0^t m(\tau) d\tau} \left( \frac{2\varepsilon}{\nu} \int_{\Pi_L} |q(N)|^2 + \frac{2L^2}{\varepsilon\nu} |d_t \tilde{F}|^2 \right) e^{-\frac{1}{\nu} \int_0^s m(s) ds} \]
\[ \leq c_1 \varepsilon \int_0^t |q(N)|^2 d\tau + \frac{c_2}{\varepsilon} \int_0^t |d_t \tilde{F}|^2 d\tau. \]

With this estimate and going back to (2.29) we derive by integrating in \( t \)
\[ \frac{1}{2} \int_0^t \int_{\Pi_L} \left| \frac{\partial}{\partial \tau} v(N) \right|^2 dx d\tau + \nu \int_{\Pi_L} \left| \nabla v(N) \right|^2 dx \]
\[ \leq \frac{1}{2} \int_0^t m(\tau) d\tau (c_1 \varepsilon \int_0^\tau |q(N)|^2 d\tau + \frac{c_2}{\varepsilon} \int_0^\tau |d_t \tilde{F}|^2 d\tau) + \varepsilon \int_0^t |q(N)|^2 d\tau + \frac{L^2}{\varepsilon} \int_0^t |d_t \tilde{F}|^2 d\tau \]
\[ \leq c_1 (\varepsilon \int_0^t |q(N)|^2 d\tau + \frac{1}{\varepsilon} \int_0^t |d_t \tilde{F}|^2 d\tau). \] (2.30)

Our next aim is to get an estimate for \( q(N)(t) \). Let \( \omega \) be the solution of the problem:
\[ \begin{cases} \omega \in H(\Pi_L), \\
\nu \int_{\Pi_L} \nabla \omega \cdot \nabla \eta dx = L \int_{\sigma} \eta_n dx' \forall \eta \in H(\Pi_L). \end{cases} \] (2.31)

The existence and uniqueness of the solution of the problem (2.31) follows from the Lax-Milgram theorem (see [5]). We claim that
\[ L \int_{\sigma} \omega_n dx' = \xi_0 > 0. \]
Indeed if we take \( \eta = \omega \) in (2.31) we obtain
\[ \nu \int_{\Pi_L} |\nabla \omega|^2 dx = L \int_{\sigma} \omega_n dx' = \xi_0 \geq 0. \]

From this estimate follows that the integral in the right hand side does not depend on \( x_n \) and therefore
\[ \int_{\Pi_L} \omega_n dx = L \int_{\sigma} \omega_n dx'. \]

If \( \xi_0 = 0 \) then we derive \( \int_{\Pi_L} |\nabla \omega|^2 dx = 0 \), so \( \nabla \omega = 0 \) and then from (2.31) we get \( \int_{\sigma} \eta_n dx' = 0 \forall \eta \in H(\Pi_L) \) which is impossible.
We have also summing up from 1 to $N$, we obtain

$$
\int_{\Omega} \frac{\partial}{\partial t} \mathbf{v}(t) \cdot \omega(t) \, dx + \nu \int_{\Omega} \nabla \mathbf{v}(t) \cdot \nabla \omega(t) \, dx + \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{v}(t) \cdot \omega(t) \, dx
= Lq(t) \int_{\sigma} \omega_n(t) \, dx'
$$

(2.32)

with $\omega(t) = \sum_{k=1}^{N} \gamma_k \mathbf{v}_k$. In virtue of (2.31) and the flux condition (2.18) we have

$$
\nu \int_{\Omega} \nabla \omega(t) \cdot \nabla \omega(t) \, dx = \nu \int_{\Omega} \nabla \omega(t) \cdot \nabla \omega(t) \, dx + \nu \int_{\Omega} \nabla \omega(t) \cdot \nabla (\omega(t) - \omega(t)) \, dx
= L \mathcal{F}(t) + \nu \int_{\Omega} \nabla \omega(t) \cdot \nabla (\omega(t) - \omega(t)) \, dx.
$$

We have also

$$
L \int_{\sigma} \omega_n(t) \, dx' = L \int_{\sigma} \omega_n(t) \, dx' + L \int_{\sigma} (\omega_n(t) - \omega_n(t)) \, dx' = \xi_0 + L \int_{\sigma} (\omega_n(t) - \omega_n(t)) \, dx'.
$$

Therefore the equation (2.32) can be rewritten as

$$
\int_{\Omega} \frac{\partial}{\partial t} \mathbf{v}(t) \cdot \omega(t) \, dx + L \mathcal{F}(t) + \nu \int_{\Omega} \nabla \omega(t) \cdot \nabla (\omega(t) - \omega(t)) \, dx
+ \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{v}(t) \cdot \omega(t) \, dx = q(t) \xi_0 + Lq(t) \int_{\sigma} (\omega_n(t) - \omega_n(t)) \, dx'.
$$

Next we look for an estimate of the function $q(t)$. From the last equality it follows that

$$
\xi_0 q(t) = Lq(t) \int_{\sigma} (\omega_n(t) - \omega_n(t)) \, dx' + \nu \int_{\Omega} \nabla \omega(t) \cdot \nabla (\omega(t) - \omega(t)) \, dx
+ \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{v}(t) \cdot \omega(t) \, dx + L \mathcal{F}(t).
$$

Because of the orthogonality of the eigenfunctions $\mathbf{v}_k$ and since $\omega(t) - \omega(t) = -\sum_{k=N+1}^{\infty} \gamma_k \mathbf{v}_k$ we derive

$$
\int_{\Omega} \nabla \mathbf{v}(t) \cdot \nabla (\omega(t) - \omega(t)) \, dx = 0.
$$

So we have, by using Young’s and Hölder’s inequalities

$$
\int_{0}^{t} \xi_0^2 |q(t)|^2 \, d\tau \leq \int_{0}^{t} \left\{ L |q(t)| \int_{\sigma} (\omega_n(t) - \omega_n(t)) \, dx' + \| \frac{\partial}{\partial \tau} \mathbf{v}(t) \|_{L^2(\Omega)} \| \omega(t) \|_{L^2(\Omega)} \right\}^2 \, d\tau
+ \| \mathbf{U} \|_{L^4(\Omega)} \| \nabla \mathbf{v}(t) \|_{L^2(\Omega)} \| \omega(t) \|_{L^4(\Omega)} + L \| \mathcal{F} \|^2 \, d\tau
\leq c \int_{0}^{t} |q(t)|^2 \, d\tau \int_{\Omega} |\omega(t) - \omega(t)|^2 \, dx + \int_{0}^{t} \left\{ \int_{\Omega} \left( \frac{\partial}{\partial \tau} \mathbf{v}(t) \right) \, dx \right\}^2 \, d\tau \int_{0}^{t} |\omega(t)|^2 \, dx
+ \sup_{\tau \in [0,t]} \left( \int_{\Omega} |\nabla \mathbf{U}|^2 \, dx \right) \int_{0}^{t} \int_{\Omega} |\nabla \mathbf{v}(t)|^2 \, dx \int_{0}^{t} |\nabla \omega(t)|^2 \, dx \, d\tau + \int_{0}^{t} \frac{\mathcal{F}^2}{\beta} \, d\tau.
$$
Since
\[ \int_{\Omega_L} |\omega^{(N)}|^2 \, dx \leq \int_{\Omega_L} |\omega|^2 \, dx, \quad \int_{\Omega_L} |\nabla \omega^{(N)}|^2 \, dx \leq \int_{\Omega_L} |\nabla \omega|^2 \, dx \]
and \( \|\omega - \omega^{(N)}\|_{L^2(\Omega_L)} \to 0 \) when \( N \to \infty \) we get
\[ \int_0^t |q^{(N)}(\tau)|^2 \, d\tau \leq c \int_0^t \left( \|\frac{\partial}{\partial \tau} \omega^{(N)}\|_{L^2(\Omega_L)}^2 + \|\nabla \omega^{(N)}\|_{L^2(\Omega_L)}^2 + |\tilde{F}|^2 \right) \, d\tau. \] (2.33)

On the other hand from (2.28) and (2.30) we establish
\[ \int_{\Omega_L} |\omega^{(N)}|^2 \, dx + \int_0^t \int_{\Omega_L} |\nabla \omega^{(N)}|^2 \, dx \, d\tau + \int_0^t \int_{\Omega_L} \frac{\partial}{\partial \tau} \omega^{(N)} \, dx \, d\tau + \int_{\Omega_L} |\nabla \omega^{(N)}|^2 \, dx \]
\[ \leq c \left( \frac{1}{\varepsilon} \|\tilde{F}\|_{W^1_2(0,t)}^2 + \varepsilon \int_0^t |q^{(N)}|^2 \, d\tau \right). \] (2.34)

For sufficiently small \( \varepsilon \) we get from (2.33) and (2.34) that
\[ \int_0^t |q^{(N)}(\tau)|^2 \, d\tau \leq c \int_0^t \left( |\tilde{F}(\tau)|^2 + \left| \frac{d}{d\tau} \tilde{F}(\tau) \right|^2 \right) \, d\tau. \] (2.35)

The estimate (2.25) is a consequence of (2.34) and (2.35).

**Theorem 2.7** Suppose that \( \tilde{F} \in W^1_2(0,T), \tilde{F}(0) = 0, T \in (0,\infty) \) and \( U \) satisfies (1.3). Then the problem (2.15) admits a unique solution \( (\mathbf{v}, q) \in L^2(0,T;H(\Omega_L)) \times L^2(0,T) \) and the following estimate
\[ \sup_{t \in [0,T]} \|\mathbf{v}\|_{W^1_2(\Omega_L)} + \int_0^T \|\frac{\partial}{\partial \tau} \mathbf{v}\|_{L^2(\Omega_L)}^2 \, d\tau + \|q\|_{L^2(0,T)}^2 \leq c \|\tilde{F}\|_{W^1_2(0,T)}^2 \] (2.36)
is valid. Here the constant \( c \) depends on \( L, D, T, \nu \) and the function \( U \).

**Proof.** According to the estimate of Theorem 2.6, the proof of the existence is equivalent to the proof of Theorem 2.3. Next we would like to prove the uniqueness of the solution. For this we consider \( \mathbf{v}(x,t) = \mathbf{v}_1(x,t) - \mathbf{v}_2(x,t) \) and \( q(t) = \mathbf{q}_1(t) - \mathbf{q}_2(t) \), where \( \mathbf{v}_i, \mathbf{q}_i \) are solutions of the problem (2.15). Since both solutions have to fulfill the flux condition, we obtain that
\[ \int_\sigma v_n \, dx' = \int_\sigma (v_{1n} - v_{2n}) \, dx' = \tilde{F}(t) - \tilde{F}(t) = 0 \]
and so by subtracting the equations satisfied by \( (\mathbf{v}_1, q_1) \) and \( (\mathbf{v}_2, q_2) \), and taking \( \mathbf{v} \) as a test function we get
\[ \frac{1}{2} \int_{\Omega_L} |\mathbf{v}|^2 \, dx + \nu \int_0^t \int_{\Omega_L} |\nabla \mathbf{v}|^2 \, dx \, d\tau = L \int_0^t q \int_\sigma v_n \, dx' \, d\tau = 0. \]

This implies that \( \mathbf{v}(x,t) = 0 \) and therefore \( \mathbf{v}_1 = \mathbf{v}_2 \). Going back to (2.15) we obtain that
\[ 0 = L \int_0^t q(\tau) \int_\sigma \eta_n \, dx' \, d\tau \]
for all possible \( \eta_n \), which implies \( q = 0 \) and so \( q_1 = q_2 \). This completes the proof of the uniqueness. \( \square \)
2.3 The analysis of problem (1.5)

Combining the obtained results of the problems (1.8) and (1.9) (see subsections 2.1 and 2.2) we can state the following:

**Theorem 2.8** Suppose that \( f \in L_2(0,T;L_2(\Pi_L)), a \in W^1_2(\Pi_L), F \in W^1_2(0,T), T \in (0,\infty), F(0) = \int a_n \, dx' \) and \( U \) satisfies (1.3). Then the problem (1.5) admits a unique weak solution \((u,q) \in L_2(0,T;H(\Pi_L)) \times L_2(0,T)\) and the following estimate

\[
\sup_{t \in [0,T]} \|u(t)\|_{W^1_2(\Pi_L)}^2 + \int_0^T \|\partial_t u\|_{L_2(\Pi_L)}^2 \, d\tau + \|q\|_{L_2(0,T)}^2 \\
\leq c(\|F\|_{W^1_2(0,T)}^2 + \int_0^T \|f\|_{L_2(\Pi_L)}^2 \, d\tau + \|a\|_{W^1_2(\Pi_L)}^2)
\]

is valid. Here the constant \( c \) depends on \( L, D, T, \nu \) and the function \( U \).

**Proof.** For the proof of this theorem see Theorem 2.3 and Theorem 2.7. \( \square \)

3 Asymptotic behavior

In this section we will analyze the asymptotic behavior in space between the problem (1.5), defined in one periodicity cell \( \Pi_L \) and the analogous problem, defined in the bigger domain

\[
\Pi_{\ell L} = \{x \in \sigma_x \times \{x_n\}, x_n \in (-\ell L,\ell L)\} = \bigcup_{z=-\ell L}^{\ell L-1} (\Pi_L \cup \sigma_0 \cup \sigma_L + z L e_n).
\]

Let us define the lateral boundary of this domain by \( S_{\ell L} = \partial \Pi_{\ell L} \setminus (\sigma_{-\ell L} \cup \sigma_{\ell L}) \).

We will compare the weak solution \((u,q)\) of the problem (1.5) to the solution \((u_\ell,q_\ell)\) of the problem

\[
\begin{align*}
\begin{cases}
\quad u_\ell \in L_2(0,T;H(\Pi_\ell)), \quad \frac{\partial}{\partial t} u_\ell \in L_2(0,T;H(\Pi_\ell)), \quad u_\ell |_{t=0} = a, \quad q_\ell \in L_2(0,T), \\
\int_0^T \int_{\Pi_\ell} f \cdot \eta \, dx \, d\tau + \nu \int_0^T \int_{\Pi_\ell} \nabla u_\ell \cdot \nabla \eta \, dx \, d\tau + \int_0^T \int_{\Pi_\ell} (U \cdot \nabla) u_\ell \cdot \eta \, dx \, d\tau \\
\int_0^T \int_{\Pi_\ell} u_\ell \eta \, dx \, d\tau + 2 \ell \int_0^T \int_{\Pi_\ell} q_\ell \eta \, dx \, d\tau + \int_\sigma \int_{x' \in \Pi_{\ell L}} f(x',t) \, dx' \eta(x',t) \forall \eta \in L_2(0,T;H(\Pi_\ell)), \forall t \in [0,T],
\end{cases}
\end{align*}
\]

where

\( H(\Pi_\ell) = \{\eta \in W^1_2(\Pi_\ell) \mid \text{div}(\eta) = 0, \eta(x',-\ell L) = \eta(x',\ell L), \eta |_{S_{\ell L}} = 0\} \).

\( f, a \) and \( U \) are \( L \)-periodically extended functions in \( x_n \) and the given function \( U \) satisfies (1.3). For the existence of the solution of the problem (3.1) compare section 2. If \( \ell \in \mathbb{N}^+ \) then it is obvious that we get the existence and uniqueness from the previous section. If \( \ell \in \mathbb{R}^+ \setminus \mathbb{N}^+ \) we need to assume as well that \( \Pi_L \) is a symmetric set with respect to the hiperplane \( x_n = \frac{L}{2} \) and that \( U \) and \( a \) are \( L \)-periodic symmetric functions with respect to \( x_n = \frac{L}{2} \) in the set \( \Pi_{\ell L} \). Note that we then have \( U(x',-\ell L) = U(x',\ell L) \) and \( a(x',-\ell L) = a(x',\ell L) \).

We will analyze two different cases of the asymptotic behavior. First we will assume that \( \ell \in \mathbb{N}^+ \), i.e. the domain \( \Pi_{\ell L} \) consists exactly of \( 2\ell \) periodicity cells \( \Pi_L \). Under this assumption we can prove the equality between the weak solution \((u,q)\) of the problem (1.5) to the solution \((u_\ell,q_\ell)\) of the problem (3.1). Secondly we will take \( \ell \in \mathbb{R}^+ \) and we will prove that \( u_\ell \) of the problem (3.1) converges exponentially to \( u \) of the problem (1.5).
First let us prove an auxiliary result for the weak solution \((u, q)\) of the problem (1.5).

**Lemma 3.1** Let \(\ell \in \mathbb{N}^+\), \(T \in (0, \infty)\), \(f \in L_2(0, T; L_2(\Pi_L))\), \(a \in W_2(\Pi_L)\) and \(U\) satisfies (1.3). Let \((u, q)\) be the weak solution of the problem (1.5) and \(u, a, U\) and \(f\) in the domain \(\Pi_L\) are \(L\)-periodically extended with respect to \(x_n\). Then for every \(v \in L_2(0, T; H(\Pi_L))\) the following identity

\[
\int_{0}^{t} \int_{\Pi_L} \frac{\partial}{\partial \tau} u \cdot v \, dx d\tau + \nu \int_{0}^{t} \int_{\Pi_L} \nabla u \cdot \nabla v \, dx d\tau + \int_{0}^{t} \int_{\Pi_L} (U \cdot \nabla) u \cdot v \, dx d\tau
\]

holds.

**Proof.** Let us extend the functions \(u\) and \(f\) \(L\)-periodically with respect to \(x_n\) in the domain \(\Pi_L\). Having in mind that the weak solution of the problem (1.5) is unique (see Theorem 2.8) and the function \(q\) does not depend on \(x_n\), it is obvious that the \(L\)-periodically extended function \(q\), with respect to \(x_n\), is the same in \(\Pi_L\) as in the domain \(\Pi_L\).

Let us take in the integral identity (1.6) a test function \(v \in L_2(0, T; H(\Pi_L))\). Using the periodicity condition of the functions \(u\) and \(U\), then we obtain on the left side of the integral identity:

\[
\int_{0}^{t} \int_{\Pi_L} \frac{\partial}{\partial \tau} u \cdot v \, dx d\tau + \nu \int_{0}^{t} \int_{\Pi_L} \nabla u \cdot \nabla v \, dx d\tau + \int_{0}^{t} \int_{\Pi_L} (U \cdot \nabla) u \cdot v \, dx d\tau
\]

\[
= \sum_{\ell=1}^{\ell-1} \left( \int_{0}^{t} \int_{\Pi_L} \frac{\partial}{\partial \tau} u(x, \tau) \cdot v(x + Lz e_n, \tau) \, dx d\tau + \nu \int_{0}^{t} \int_{\Pi_L} \nabla u(x, \tau) \cdot \nabla v(x + Lz e_n, \tau) \, dx d\tau + \int_{0}^{t} \int_{\Pi_L} (U(x, \tau) \cdot \nabla) u(x, \tau) \cdot v(x + Lz e_n, \tau) \, dx d\tau \right)
\]

Note that in the above calculations it is important that \(\ell \in \mathbb{N}^+\), otherwise the domain \(\Pi_L\) could not be split into an integer number of periodicity cells \(\Pi_L\).

Let us set \(\bar{v} = \sum_{\ell=1}^{\ell-1} v(x + Lz e_n, t)\). Since \(v|_{S_L} = 0\) it is clear that \(\bar{v}|_{S_L} = 0\). Moreover, we have

\[
\text{div}(\bar{v}) = \sum_{\ell=1}^{\ell-1} \text{div}(v(x + zL e_n, t)) = 0
\]

and

\[
\bar{v}(x', 0, t) = \sum_{\ell=1}^{\ell-1} v(x', 0 + zL, t) = \sum_{\ell=1}^{\ell-1} v(x', zL, t) = \sum_{\ell=1}^{\ell-1} v(x', L + zL, t) = \bar{v}(x', L, t).
\]
Above, we used the periodicity condition $v(x', -\ell L) = v(x', \ell L)$. So we get $\tilde{v} \in L_2(0, T; H(\Pi_L))$ and therefore

$$
\int_0^t \int_{\Pi_L} \frac{\partial}{\partial \tau} u \cdot v \, dxd\tau + \nu \int_0^t \int_{\Pi_L} \nabla u \cdot \nabla v \, dxd\tau + \int_0^t \int_{\Pi_L} (U \cdot \nabla) u \cdot \tilde{v} \, dxd\tau = 0
$$

This identity completes the proof of the lemma. $\Box$

From the Theorem 2.8 we have that the solutions of the problems (1.6) and (3.1) are unique, therefore combining with the result of Lemma 3.1 we derive that $(u, q) = (u_\ell, \tilde{q}_\ell)$.

### 3.2 Case $\ell \in \mathbb{R}^+ \setminus \mathbb{N}^+$

The assumption that $\ell \in \mathbb{R}^+ \setminus \mathbb{N}^+$ requires more restrictions. For the existence of $(u_\ell, \tilde{q}_\ell)$ (see section 2) we need that the domain $\Pi_L$ and the functions $U$ and $a$ are symmetric with respect to $x_n = \frac{\ell}{2}$. We are able to prove the convergence of $u_\ell$ to $u$, when we take the test functions in (1.6) in addition equal to zero on the whole boundary of $\Pi_{\ell L}$ (see next lemma).

First let us define the new function space

$$
\widetilde{H}(\Pi_{\ell L}) = \{ \eta \in W^1_2(\Pi_{\ell L}) \mid \text{div}(\eta) = 0, \eta |_{\partial \Pi_{\ell L}} = 0 \}. \quad (3.3)
$$

**Lemma 3.2** Let $\ell \in \mathbb{R}^+ \setminus \mathbb{N}^+$, $T \in (0, \infty)$, $f \in L_2(0, T; L_2(\Pi_{\ell L}))$, $a \in W^1_2(\Pi_L)$ and $U$ satisfies (1.5). Let $(u, q)$ (resp. $(u_\ell, \tilde{q}_\ell)$) be the weak solution of the problem (1.5) (resp. (3.1)) and $u, a, U$ and $f$ are extended $L$-periodically with respect to $x_n$. Then for every $v \in L_2(0, T; \widetilde{H}(\Pi_{\ell L}))$ the following integral identity

$$
\int_0^t \int_{\Pi_{\ell L}} \frac{\partial}{\partial \tau} u \cdot v \, dxd\tau + \nu \int_0^t \int_{\Pi_{\ell L}} \nabla u \cdot \nabla v \, dxd\tau + \int_0^t \int_{\Pi_{\ell L}} (U \cdot \nabla) u \cdot v \, dxd\tau = 0
$$

is valid.

**Proof.** Let us extend the function $v$ by zero outside the domain $\Pi_{\ell L}$. By (3.2) and since $\int_{\sigma} v_n \, dx' = 0$ one has

$$
\int_0^t \int_{\Pi_{\ell L}} \frac{\partial}{\partial \tau} u \cdot v \, dxd\tau + \nu \int_0^t \int_{\Pi_{\ell L}} \nabla u \cdot \nabla v \, dxd\tau + \int_0^t \int_{\Pi_{\ell L}} (U \cdot \nabla) u \cdot v \, dxd\tau = 0.
$$

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Note since now the test functions are zero on the whole boundary we loose the term including \( q \), i.e.
\[
\int_0^t q \int v_n \, dx \, d\tau = 0.
\]

**Theorem 3.3** Let \( f \in L_2(0,T; L_2(\Pi_L)) \), \( a \in W^1_2(\Pi_L) \), \( T \in (0, \infty) \), \( U \) satisfies (1.3) and in addition we assume \( \sup_{t \in [0,T]} \sup_{x \in \Pi_L} |U|^2 < 4\nu \). Let \((u,q)\) (resp. \((\ell,t)\)) be the weak solution of the problem (1.5) (resp. (3.1)) and \( u, a, U \) and \( f \) are extended \( L \)-periodically with respect to \( x_n \). Then the following estimate
\[
\sup_{t \in [0,T]} \|(u_\ell - u)(t, \cdot, \cdot)\|_{W^2_2(\Pi_{2 \ell})}^2 + \int_0^T \| \frac{\partial}{\partial t}(u_\ell - u) \|_{L^2_2(\Pi_{2 \ell})}^2 \, dt \leq C e^{-t \alpha}
\]
holds for some constants \( C, \alpha \) independent of \( \ell \).

**Proof.** Subtracting the integral identity (3.4) from (3.1) we get
\[
\begin{align*}
\int_0^t \int_{\Pi_{2 \ell}} & \frac{\partial}{\partial t}(u_\ell - u) \cdot v \, dx \, d\tau + \nu \int_0^t \int_{\Pi_{2 \ell}} \nabla(u_\ell - u) \cdot \nabla v \, dx \, d\tau \\
& + \int_0^t \int_{\Pi_{2 \ell}} (U \cdot \nabla)(u_\ell - u) \cdot v \, dx \, d\tau = 0 \quad \forall v \in L_2(0,T; \hat{H}(\Pi_{2 \ell})).
\end{align*}
\]

Note that in the above identity it is not possible to take \( w_\ell = u_\ell - u \) as a test function since \( u_\ell - u \notin \hat{H}(\Pi_{2 \ell}) \).

Next, let us introduce the function \( \delta \) such that
\[
\delta(x_n) = \begin{cases} 
1 & \text{on } [-\ell_1 L, \ell_1 L], \\
0 & \text{on } \mathbb{R} \setminus ((-\ell_1 - 1) L, (\ell_1 + 1) L), \\
((\ell_1 + 1) L - |x_n|) / L & \text{on } ((-\ell_1 - 1) L, -\ell_1 L) \cup (\ell_1 L, (\ell_1 + 1) L),
\end{cases}
\]
where \( 0 < \ell_1 \leq \ell - 1 \). Further we have for almost ever \( t \)
\[
\text{div}(\delta w_\ell) = \begin{cases} 
0 & \text{on } \Pi_{2 \ell} \setminus D_{\ell_1}, \\
\frac{\partial \delta}{\partial x_n} w_{\ell \cdot n} & \text{on } D_{\ell_1}
\end{cases}
\]
with \( D_{\ell_1} = \Pi_{(\ell_1 + 1) L} \setminus \Pi_{\ell L} \). Using the flux condition i.e. the fact that
\[
0 = \int_{\sigma} (u_\ell - u_n) \, dx' = \int_{\sigma} w_{\ell \cdot n} \, dx'
\]
and Remark 1.1, we get
\[
\begin{align*}
\int_{D_{\ell_1}} \frac{\partial \delta}{\partial x_n} w_{\ell \cdot n} \, dx &= \int_{-\ell_1 L}^{-(\ell_1 + 1) L} \int_{\sigma} \frac{1}{L} w_{\ell \cdot n} \, dx' \, dx_n = 0, \\
\int_{D_{\ell_1}} \frac{\partial \delta}{\partial x_n} w_{\ell \cdot n} \, dx &= \int_{\ell_1 L}^{(\ell_1 + 1) L} \int_{\sigma} \frac{1}{L} w_{\ell \cdot n} \, dx' \, dx_n = 0,
\end{align*}
\]
where

$D^+_n = D_{\ell_1} \cap \{ x_n > 0 \}$, \quad $D^-_n = D_{\ell_1} \cap \{ x_n < 0 \}$.

Therefore there exists a function $\beta$ (see [2] and [10]) such that

\[
\begin{aligned}
\text{div}(\beta) &= \frac{\partial \delta}{\partial x_n} w_{\ell,n} \quad \text{in } D_{\ell_1}, \\
||\nabla \beta||_{L^2(D_{\ell_1})} &\leq \tilde{c} ||w_{\ell,n}||_{L^2(D_{\ell_1})}
\end{aligned}
\]

(3.8)

with $\beta \in L_2(0, T; W^{1,2}_0(D_{\ell_1}))$ and the constant $\tilde{c}$ depends on $L$ and $D$. Note $\beta$ is measurable with respect to $t$.

Indeed, let us define $G := \frac{\partial \delta}{\partial x_n} w_{\ell,n} = \mp \frac{1}{2} w_{\ell,n}$ in $D^+_1$. It holds $\int_{D^+_1} G \, dx = 0$ and $w_{\ell,n} \in L_2(0, T; H(\Pi_{\ell_1}))$.

Let $\{G_m\} \subset D(D^+_1 \times (0, T))$ be a sequence approximating $G$ in $L_2(0, T; L_2(\Pi_{\ell_1}))$ and let us choose

$G^*_m = G_m - \phi \int_{D^+_1} G_m \, dx, \quad m \in \mathbb{N}$

with

$\phi \in D(D^+_1), \quad \int_{D^+_1} \phi \, dx = 1.$

Then $\{G^*_m\} \subset D(D^+_1 \times (0, T))$ still approximates $G$ in $L_2(0, T; L_2(\Pi_{\ell_1}))$ and at the same time the condition $\int_{D^+_1} G^*_m \, dx = 0$ is fulfilled. Therefore we can find a measurable $\beta_m$ (see equation III.3.8 and Lem. III.3.1 in [10]) with

\[
\begin{aligned}
\text{div}(\beta_m) &= G^*_m \quad \text{in } D^+_1 \\
||\nabla \beta_m||_{L^2(D^+_1)} &\leq \tilde{c} ||G^*_m||_{L^2(D^+_1)}
\end{aligned}
\]

(3.9)

and $\beta_m \in L_2(0, T; W^{1,2}_0(D^+_1))$. Passing to the limit we obtain that $\beta$ is measurable with respect to $t$.

Next, we extend $\beta$ by zero outside $D_{\ell_1}$. We take now $v = \delta w_\ell - \beta \in L_2(0, T; \hat{H}(\Pi_{\ell_1}))$ in (3.6) and obtain

$$
\int_0^t \int_{\Omega_{\ell_1}} \frac{\partial}{\partial \tau} w_\ell \cdot (\delta w_\ell - \beta) \, dx \, d\tau + \nu \int_0^t \int_{\Omega_{\ell_1}} \nabla w_\ell \cdot \nabla (\delta w_\ell - \beta) \, dx \, d\tau + \int_0^t \int_{\Omega_{\ell_1}} (U \cdot \nabla) w_\ell \cdot (\delta w_\ell - \beta) \, dx \, d\tau = 0.
$$

Since $w_\ell = \delta w_\ell + (1 - \delta) w_\ell$ and $\int_0^t \int_{\Omega_{\ell_1}} (U \cdot \nabla) \delta w_\ell \cdot \delta w_\ell \, dx \, d\tau = 0$ we get

$$
\int_0^t \int_{\Omega_{\ell_1}} \frac{\partial}{\partial \tau} w_\ell \cdot w_\ell \, dx \, d\tau + \nu \int_0^t \int_{\Omega_{\ell_1}} |\nabla w_\ell|^2 \, dx \, d\tau = \int_0^t \int_{D_{\ell_1}} \frac{\partial}{\partial \tau} w_\ell \cdot (\delta w_\ell - \beta) \, dx \, d\tau
$$

$$
- \nu \int_0^t \int_{D_{\ell_1}} \nabla w_\ell \cdot \nabla (\delta w_\ell - \beta) \, dx \, d\tau - \int_0^t \int_{D_{\ell_1}} (U \cdot \nabla) (1 - \delta) w_\ell \cdot \delta w_\ell \, dx \, d\tau
$$

$$
+ \int_0^t \int_{D_{\ell_1}} (U \cdot \nabla) w_\ell \cdot \beta \, dx \, d\tau.
$$

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Using the Young inequality \( a \cdot b \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2 \) we obtain

\[
\frac{t}{2} \int_0^t \int_{\Omega_{\ell}^L} \frac{\partial}{\partial \tau} \mathbf{w}_\ell \cdot \mathbf{w}_\ell \, dxd\tau + \nu \int_0^t \int_{\Omega_{\ell}^L} |\nabla \mathbf{w}_\ell|^2 \, dxd\tau \leq \frac{1}{2} \int_0^t \int_{D_{\ell}^1} |\nabla \mathbf{w}_\ell|^2 \, dxd\tau \\
+ \frac{1}{2} \int_0^t \int_{D_{\ell}^1} |\delta \mathbf{w}_\ell - \beta|^2 \, dxd\tau + \frac{\nu}{2} \int_0^t \int_{D_{\ell}^1} |\nabla \mathbf{w}_\ell|^2 \, dxd\tau + \frac{\nu}{2} \int_0^t \int_{D_{\ell}^1} |\nabla(\delta \mathbf{w}_\ell - \beta)|^2 \, dxd\tau \\
+ \frac{1}{2} \int_0^t \int_{D_{\ell}^1} |\beta|^2 \, dxd\tau
\]

where \( C \) is a constant. Next we will find an estimate for \( |\frac{\partial}{\partial \tau} \mathbf{w}_\ell| \). Let \( \delta \) be the function defined in (3.7), then we have that

\[
\text{div}(\delta \frac{\partial}{\partial \tau} \mathbf{w}_\ell) = \begin{cases} 0 & \text{on } \Omega_{\ell \ell}^L \setminus D_{\ell}, \\ \frac{\partial \delta}{\partial x_n} \frac{\partial}{\partial t} \mathbf{w}_{\ell,n,n} & \text{on } D_{\ell}. \end{cases}
\]

Using the flux condition, it holds

\[
\int_{D_{\ell}^2} \frac{\partial \delta}{\partial x_n} \frac{\partial}{\partial t} \mathbf{w}_{\ell,n,n} \, dx = \frac{\partial}{\partial t} \int_{\Omega_{\ell}^L} \mathbf{w}_{\ell,n,n} \, dx = 0.
\]

Hence there exists a function \( \gamma \) (see [2] and [10]) such that

\[
\begin{aligned}
\text{div}(\gamma) &= \frac{\partial \delta}{\partial x_n} \frac{\partial}{\partial t} \mathbf{w}_{\ell,n,n} \quad \text{in } D_{\ell}, \\
||\nabla \gamma||_{L^2(D_{\ell})} &\leq \hat{c} ||\frac{\partial}{\partial t} \mathbf{w}_{\ell,n,n}||_{L^2(D_{\ell})}
\end{aligned}
\tag{3.11}
\]

with \( \gamma \in L^2(0, T; W^{1,2}_0(D_{\ell})) \) and the constant \( \hat{c} \) is depends on \( L \) and \( D \). We extend \( \gamma \) by zero outside \( D_{\ell} \). We take now \( v = \delta \frac{\partial}{\partial \tau} \mathbf{w}_\ell - \gamma \in L^2(0, T; \tilde{H}(\Omega_{\ell}^L)) \) in (3.6) and obtain

\[
\begin{align*}
\int_0^t \int_{\Omega_{\ell}^L} \frac{\partial}{\partial \tau} \mathbf{w}_\ell \cdot (\delta \frac{\partial}{\partial \tau} \mathbf{w}_\ell - \gamma) \, dxd\tau + \nu \int_0^t \int_{\Omega_{\ell}^L} \nabla \mathbf{w}_\ell \cdot \nabla (\delta \frac{\partial}{\partial \tau} \mathbf{w}_\ell - \gamma) \, dxd\tau \\
+ \int_0^t \int_{\Omega_{\ell}^L} (U \cdot \nabla) \mathbf{w}_\ell \cdot (\delta \frac{\partial}{\partial \tau} \mathbf{w}_\ell - \gamma) \, dxd\tau = 0.
\end{align*}
\]
This leads to

\[
\int_0^t \int_{\Pi_{1L}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + \nu \int_0^t \int_{\Pi_{1L}} \nabla w_{\ell} \cdot \nabla \frac{\partial}{\partial \tau} (\delta w_{\ell}) \, dx \, d\tau = - \int_0^t \int_{D_{\ell_1}} \frac{\partial}{\partial \tau} w_{\ell} \cdot (\delta \frac{\partial}{\partial \tau} w_{\ell} - \gamma) \, dx \, d\tau \\
- \nu \int_0^t \int_{D_{\ell_1}} \nabla w_{\ell} \cdot \nabla \gamma \, dx \, d\tau - \int_0^t \int_{\Pi_{1L}} (U \cdot \nabla) w_{\ell} \cdot \frac{\partial}{\partial \tau} w_{\ell} \, dx \, d\tau \\
- \int_0^t \int_{D_{\ell_1}} (U \cdot \nabla) w_{\ell} \cdot (\delta \frac{\partial}{\partial \tau} w_{\ell} - \gamma) \, dx \, d\tau.
\]

Using the equality

\[
\int_0^t \int_{\Pi_{1L}} \nabla w_{\ell} \cdot \nabla \frac{\partial}{\partial \tau} (\delta w_{\ell}) \, dx \, d\tau = \int_0^t \int_{\Pi_{1L}} \delta \nabla w_{\ell} \cdot \nabla \frac{\partial}{\partial \tau} w_{\ell} \, dx \, d\tau + \int_0^t \int_{D_{\ell_1}} \frac{\partial \delta}{\partial x_n} \frac{\partial}{\partial \tau} w_{\ell} \cdot \frac{\partial}{\partial \tau} w_{\ell} \, dx \, d\tau
\]

and the Young inequality we derive

\[
\int_0^t \int_{\Pi_{1L}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + \nu \int_0^t \int_{\Pi_{1L}} \delta |\nabla w_{\ell}|^2 \, dx \, d\tau \leq \frac{1}{2} \int_0^t \int_{D_{\ell_1}} |\frac{\partial \delta}{\partial x_n} \frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau \\
+ \frac{1}{2} \int_0^t \int_{D_{\ell_1}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + \frac{1}{2} \int_0^t \int_{D_{\ell_1}} |\frac{\partial}{\partial \tau} w_{\ell} - \gamma|^2 \, dx \, d\tau \\
+ \nu \int_0^t \int_{D_{\ell_1}} |\nabla w_{\ell}|^2 \, dx \, d\tau + \nu \int_0^t \int_{D_{\ell_1}} |\nabla \gamma|^2 \, dx \, d\tau + \frac{|U|_{\infty}}{2\epsilon} \int_0^t \int_{\Pi_{1L}} |\nabla w_{\ell}|^2 \, dx \, d\tau \\
+ \frac{\epsilon}{2} \int_0^t \int_{\Pi_{1L}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + \frac{|U|_{\infty}}{2}\epsilon \int_0^t \int_{D_{\ell_1}} |\nabla w_{\ell}|^2 \, dx \, d\tau + \frac{1}{2} \int_0^t \int_{D_{\ell_1}} |\delta \frac{\partial}{\partial \tau} w_{\ell} - \gamma|^2 \, dx \, d\tau.
\]

Using the properties of \( \gamma \) we get

\[
\int_0^t \int_{\Pi_{1L}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + \nu \int_{\Pi_{1L}} |\nabla w_{\ell}|^2 \, dx \leq C \int_0^t \int_{D_{\ell_1}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau + C \int_0^t \int_{\Pi_{1L}} |\nabla w_{\ell}|^2 \, dx \, d\tau \\
+ \frac{|U|_{\infty}}{2\epsilon} \int_0^t \int_{\Pi_{1L}} |\nabla w_{\ell}|^2 \, dx \, d\tau + \frac{\epsilon}{2} \int_0^t \int_{\Pi_{1L}} |\frac{\partial}{\partial \tau} w_{\ell}|^2 \, dx \, d\tau.
\]

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Combining this with (3.10) we obtain for some constant $C$

$$\frac{1}{2} \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\tau + \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau + \frac{\nu}{2} \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau + \frac{\nu}{2} \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau$$

$$\leq C \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\tau + C \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau + \frac{|U|^2}{2c} \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau$$

$$+ \epsilon \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau.$$  

Since we supposed $|U|^2 < 4\nu$ we can choose $\epsilon$ such that $\frac{|U|^2}{2c} < \epsilon < 2$ to get for some constant $c > 0$

$$\frac{1}{2} \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\tau + \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau + c \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau$$

$$\leq C \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\tau + C \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau.$$  

Let us define

$$B(\ell_1) = \frac{1}{2} \int_0^t \int_{\Omega} |\nabla u|^2 \, dx \, d\tau + \frac{\nu}{2} \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau + \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau$$

$$+ (c + C) \int_0^t \int_{\Omega} |\nabla \nabla u|^2 \, dx \, d\tau,$$

$$0 < b = \frac{c}{c + C} < 1.$$  

(3.12)

(3.13)

It is clear that

$$B(\ell_1) \leq bB(\ell_1 + 1) \quad \forall \ell_1 \leq \ell - 1.$$  

(3.14)

We denote by $[\cdot]$ the integer part of numbers. Starting from $\ell_1 = \frac{\ell}{2}$ and iterating this inequality $[\frac{\ell}{2}]$-times we get

$$B\left(\frac{\ell}{2}\right) \leq b\left[\frac{\ell}{2}\right] B\left(\frac{\ell}{2} + \left[\frac{\ell}{2}\right]\right).$$

Since $\frac{\ell}{2} - 1 < \left[\frac{\ell}{2}\right] \leq \frac{\ell}{2}$ we have

$$B\left(\frac{\ell}{2}\right) \leq \frac{1}{b^{\left[\frac{\ell}{2}\right]}} B\left(\frac{\ell}{2} + \frac{\ell}{2}\right) = \frac{1}{b} \exp(-\frac{\ell}{2} \ln(\frac{1}{b})) B(\ell).$$

(3.15)

(Since $0 < b < 1$ it holds $\ln(\frac{1}{b}) > 0$.) Next we would like to find an estimate for $B(\ell)$. We know from Theorem 2.8 that

$$\sup_{t \in [0,T]} \|u_t\|_{W^2_2(\Omega)}^2 + \int_0^T \|\nabla u_t\|_{L^2(\Omega)}^2 \, dt \leq c(\|F\|_{W^2_2(\Omega, T)}^2 + \int_0^T \|\nabla u_t\|_{L^2(\Omega)}^2 \, dt + \|a\|_{W^2_2(\Omega)}^2).$$

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and we have a similar estimate for $u$. Thus we obtain easily

$$B(\ell) \leq c \left( \|F\|_{W_2^2(0,T)}^2 + \int_0^T \|f\|_{L_2(\Pi_{\ell L})}^2 \, dt + \|a\|_{W_2^2(0,T)}^2 \right)$$

and therefore due to the periodicity of our data

$$B(\ell) \leq C \ell.$$

(3.16)

Going back to (3.15) we derive

$$B(\frac{\ell}{2}) \leq C e^{-\alpha \ell} \leq C e^{-\alpha}$$

(3.17)

for any $0 < \alpha < \alpha' = \frac{1}{2} \ln(\frac{1}{b})$. Hence

$$\sup_{t \in [0,T]} \|u_{\ell} - u\|_{W_2^2(\Pi_{\ell L})}^2 + \int_0^T \|\partial_t (u_{\ell} - u)\|_{L_2(\Pi_{\ell L})}^2 \, dt \leq C e^{-\alpha}$$

(3.18)

which completes the proof.

References