Anisotropic Singular Perturbations of Variational Inequalities

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Abstract We consider variational inequalities involving the \( p \)-Laplace operator with anisotropic singular perturbations where the convex set, on which the problem is defined, is also subject to perturbations. This leads to introduce a new convergence of sets, in some suitable sense, conceived from the Mosco convergence and matching well to the anisotropic singular perturbations. Convergence results and their rates are established. In order to illustrate the introduced convergence sets, obstacle and elasto-plastic perturbed problems are dealt with. This allows to go deeper in the analysis of the suggested convergence on concrete sets in Sobolev spaces.

Keywords Variational inequalities · convergence of convex sets · anisotropic singular perturbations · asymptotic behaviour · \( p \)-Laplacian.

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1 Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary. We denote by

$$x = (x_1, \ldots, x_n) = (X_1, X_2)$$

the points in $\mathbb{R}^n$ where $X_1 = (x_1, \ldots, x_q)$ and $X_2 = (x_{q+1}, \ldots, x_n)$, $n$ and $q$ are integers. We also denote

$$\nabla u := \left( \frac{\nabla X_1 u}{\nabla X_2 u} \right) = \left( \frac{\partial x_1 u, \ldots, \partial x_q u}{\partial x_{q+1} u, \ldots, \partial x_n u} \right)^T$$

and for $\varepsilon > 0$,

$$\nabla^\varepsilon u := \left( \varepsilon \nabla X_1 u \right), \quad \nabla^0 u := \left( 0 \right)_{\nabla X_2 u}.$$ 

The parameter of perturbation $\varepsilon$ appears only with the $X_1$-direction of the gradient as well as in the convex set, on which the problem is defined below. For this reason we refer to this perturbation as anisotropic.

For $1 < p < +\infty$, we consider as a model problem the following variational inequality, involving a perturbed $p$-Laplace operator,

$$\left\{ \begin{array}{l}
\int_{\Omega} |\nabla^\varepsilon u|^{p-2} \nabla^\varepsilon u \cdot \nabla^\varepsilon (v - u) \ dx \geq \langle f, v - u \rangle_{W^{1,p}_0(\Omega)}, \\
u \in K_\varepsilon,
\end{array} \right.$$  

(1)

where $\langle \cdot, \cdot \rangle_V$ denotes the duality brackets between a space $V$ and its dual $V'$, $K_\varepsilon \neq \emptyset$ is a closed convex subset of $W^{1,p}_0(\Omega)$ for all $\varepsilon > 0$. Assuming $f \in W^{-1,p'}(\Omega)$, where $p'$ is the conjugate of $p$, the above problem has a unique solution $u_\varepsilon \in K_\varepsilon$.

Many works were recently taking care of singular anisotropic perturbations for different type of problems, see for instance [4,7,9,10,11,12,14,15,16]. In particular, in [12], an abstract approach of variational inequalities is elaborated and illustrated by some applications to show that the theory covers the singular perturbations of anisotropic type as well as the isotropic ones. Although, the study, given in [12], is as general as possible, it does not include problems as (1) for two reasons. The perturbed operator in [12] has the form $\varepsilon A + B$, with nonlinear operators $A$ and $B$ defined on different Banach spaces. That is to say that the perturbed and the unperturbed parts of the operator are entirely distinct which is not the case of course in Problem (1). However, the main difference is coming from the convex sets which are also subject to perturbations. Thus some convergence definitions, fitting with the anisotropic nature of the above perturbation, have to be considered.
on the sequence of the convex sets \((K_\varepsilon)\) when \(\varepsilon \to 0\). The convergence of sets, related to perturbations, is dealt with in numerous works since the fundamental papers of Mosco \cite{21,22} where perturbations of variational inequalities of linear and nonlinear operators, with the convex sets also subject to perturbations, are considered. (See also Attouch \cite{1} and the references therein).

In the next section we establish a priori estimates and convergences of \(u_\varepsilon\), solution to quasilinear variational inequalities, provided that some boundedness assumptions and convergences of the sequence \((K_\varepsilon)\) hold. In fact without these types of assumptions there is no chance to envisage any boundedness or convergence of the solutions. We can see this clearly if for example the sets \(K_\varepsilon\) are parallel hyperplanes such \(\text{dist}(0, K_\varepsilon) = \frac{1}{\varepsilon}\). This of course leads to introduce the convergence notion on the convex sets, derived from the Mosco convergence and adapted to the present type of perturbations.

In the third and forth sections, we apply the above results to some important problems. The first one is a problem with constraints on the state where the convex set is determined by perturbed obstacles. We give sufficient conditions on the convergence of the obstacles to guarantee the suitable convergence of the convex sets. In the case of (isotropic) perturbations, an abundant literature has been devoted to this subject, (see for instance Attouch and Picard \cite{2}, Boccardo and Murat \cite{5}, Dal Maso \cite{13}, Mosco \cite{21,22} and related works). The second example is an elasto-plastic torsion problem where the convex set is determined by constraints on the gradient of the solution. There are some works about the isotropic case that give sufficient conditions on the constraints to insure the convenient convergence of the convex sets. (See for instance Azevedo and Santos \cite{3}, Kunze and Rodrigues \cite{19}, Lagnese \cite{20}). To ensure the convergence of the sequence \((K_\varepsilon)\), we are led to show some density results, then we establish the convergence of the sequence \((K_\varepsilon)\) in the case of cylindrical and some noncylindrical domains. The fifth section is devoted to investigate the rate of the convergence far from the boundary layer for cylindrical domains, i.e. \(\Omega = \omega_1 \times \omega_2\).

2 Convergence of convex sets and of solutions

2.1 Anisotropic Sobolev-type spaces

Throughout this paper the orthogonal projections of \(\Omega\) onto the space \(X_2 = 0\) and \(X_1 = 0\) are denoted by \(\Pi_1\) and \(\Pi_2\) respectively. For any \(X_1 \in \Pi_1\) we denote by \(\Omega_{X_1}\) the section of \(\Omega\) above \(X_1\) i.e.

\[
\Omega_{X_1} = \{X_2 \in \mathbb{R}^{n-q} \mid (X_1, X_2) \in \Omega\}.
\]

Then consider the following anisotropic Sobolev space

\[
\mathcal{W}(\Omega) := \left\{u \in L^p(\Omega) \mid \nabla_{X_2} u \in [L^p(\Omega)]^{n-q} \right\},
\]

equipped with the norm

\[
v \rightarrow \left(\|v\|_{L^p(\Omega)}^p + \|\nabla_{X_2} v\|_{L^p(\Omega)}^p\right)^{1/p}.
\]
It is clear that $W^{1,p}(\Omega)$ is a subspace of $W(\Omega)$. We denote by $W_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$, the space of $C^\infty$ functions with a compact support in $\Omega$, in $W(\Omega)$, i.e.

$$W_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{W(\Omega)}.$$ 

Since $\Omega$ is bounded, the following Poincaré inequality

$$|v|_{L^p(\Omega)} \leq C_p |\nabla_X v|_{L^p(\Omega)}, \quad \forall v \in W_0(\Omega)$$

holds for some constant $C_p$ depending on $\Omega$. Thus, the map

$$v \rightarrow |\nabla_X v|_{L^p(\Omega)}$$

define a norm on $W_0(\Omega)$. One can check that

$$W^{1,p}_0(\Omega) \subset W_0(\Omega) \subset L^p(\Omega) \quad \text{and} \quad L^{p'}(\Omega) \subset W'_0(\Omega) \subset W^{-1,p'}(\Omega).$$

We can easily show that the dual space $W'_0(\Omega)$ can be identified with the set of distributions such as

$$f \in W'_0(\Omega) \Leftrightarrow \exists f_0, f_i \in L^{p'}(\Omega), i = q + 1, \cdots, n, \text{ such that } f = f_0 + \sum_{i=q+1}^n \partial_{x_i} f_i.$$

More characterizations of the elements of $W_0(\Omega)$ will be given at the end of this section.

### 2.2 A perturbed variational inequality

To deal with the above model problem and more general variational inequalities, let

$$f \in W'_0(\Omega).$$

(2)

and consider the following nonlinear elliptic problem defined as

$$\left\{ \begin{array}{ll}
\int_{\Omega} a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (v_\varepsilon - u_\varepsilon) \, dx \geq \langle f, v_\varepsilon - u_\varepsilon \rangle_{W^{1,p}_0(\Omega)}, & \forall v_\varepsilon \in K_\varepsilon, \\
u_\varepsilon \in K_\varepsilon & 
\end{array} \right.$$ 

(3)

where $K_\varepsilon \neq \emptyset$ is a closed convex subset of $W^{1,p}_0(\Omega)$ depending on $\varepsilon > 0$. The function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the following standard assumptions

**Growth condition.** For $p > 1$, there exists a constant $M$ such that

$$|a(x, \xi)| \leq M \left( g(x) + |\xi|^{p-1} \right), \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega,$$

(4)

where $g \in L^{p'}(\Omega)$ and $|\cdot|$ is the usual Euclidean norm.

**Monotonicity.** For all $\xi, \eta \in \mathbb{R}^n$ and a.e. $x \in \Omega$, we have

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq 0,$$

(5)
where “·” is the scalar product in $\mathbb{R}^n$.

**Coercivity.** For a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, there exist a constant $\alpha > 0$ such that

$$a(x, \xi) \cdot \xi \geq \alpha |\xi|^p.$$  \hfill (6)

Under these assumptions, Problem (3) has a solution $u_\varepsilon \in K_\varepsilon$, which is unique if $a$ is strictly monotone, i.e. the inequality (6) is strict for $\eta \neq \xi$ (see Chipot \cite{chipot} and Kinderlehrer and Stampacchia \cite{kinderlehrer}).

2.3 A priori estimates

The first theorem below shows that the a priori estimate here looks like estimates for elementary problems as linear elliptic ones (see \cite{gel}) provided that a sequence $w_\varepsilon \in K_\varepsilon$, satisfying the same estimate, exists. In fact we are speaking about a necessary and sufficient conditions.

**Theorem 1** Under the assumption (2), assume in addition that there exists a sequence $w_\varepsilon \in K_\varepsilon$ for all $\varepsilon > 0$, such that

$$\varepsilon \nabla X_1 w_\varepsilon \text{ and } \nabla X_2 w_\varepsilon \text{ are bounded in } L^p(\Omega)$$ \hfill (7)

independently of $\varepsilon$, then

$$u_\varepsilon, \varepsilon \nabla X_1 u_\varepsilon \text{ and } \nabla X_2 u_\varepsilon \text{ are bounded in } L^p(\Omega)$$ \hfill (8)

and

$$a(\cdot, \nabla^\varepsilon u_\varepsilon) \text{ is bounded in } L^{p'}(\Omega).$$ \hfill (9)

**Proof** Taking $v_\varepsilon = w_\varepsilon$ in (3), it follows that

$$\int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \, dx \leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{W_0^1(\Omega)}$$

$$\leq |f|_{W_0^1(\Omega)} |\nabla X_2 (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}$$

$$\leq |f|_{W_0^1(\Omega)} |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} ,$$

then

$$\alpha |\nabla^\varepsilon u_\varepsilon|^p_{L^p(\Omega)} \leq \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon u_\varepsilon \, dx$$

$$\leq |f|_{W_0^1(\Omega)} \left( |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right) + \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon \, dx.$$ \hfill (10)

Using Hölder’s inequality and \cite{gel}, the last integral can be estimated as follows

$$\int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon \, dx \leq |a(x, \nabla^\varepsilon u_\varepsilon)|_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}$$

$$\leq M \left[ |g + |\nabla^\varepsilon u_\varepsilon|^{p-1} \right]_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}$$

$$= C \left( |g|_{L^{p'}(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} \right)$$
since \((p - 1)p' = p\). Throughout this paper, the positive constant \(C\) is independent of \(\varepsilon\) and may take different values at different occurrences. Then by Young’s inequality, it comes

\[
\int_{\Omega} a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon w_\varepsilon \, dx \leq \frac{\alpha}{4} |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p + C \left( |g|_{L^{p'}(\Omega)}^{p'} + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p \right).
\]

A similar inequality yields

\[
|f|_{W^{s, p'}_0(\Omega)} \left( |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p \right) \leq \alpha \frac{\varepsilon}{4} |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p + C \left( |f|_{W^{s, p'}_0(\Omega)}^p + |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p \right).
\]

Going back to (10), we deduce

\[
\frac{\alpha}{2} |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p \leq C \left( |f|_{W^{s, p'}_0(\Omega)}^p + |g|_{L^{p'}(\Omega)}^{p'} + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p \right).
\]

This means that \(|\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p\) is bounded since \(|\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p\) is assumed to be bounded and that

\[
\varepsilon |\nabla X_1 u_\varepsilon|, \quad |\nabla X_2 u_\varepsilon| \quad \text{and} \quad u_\varepsilon \quad \text{are bounded in} \quad L^p(\Omega).
\]

The boundedness of \(u_\varepsilon\) follows from \(L^p\)–Poincaré’s inequality in the \(X_2\)–direction. For the last estimate (9), one has

\[
|a(x, \nabla^\varepsilon u_\varepsilon)|_{L^{p'}(\Omega)}^{p'} \leq C \int_{\Omega} \left( g + |\nabla^\varepsilon u_\varepsilon|^{(p-1)} \right)^{p'} \, dx \leq C \left( |g|_{L^{p'}(\Omega)}^{p'} + |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^{p'} \right)
\]

which ends the proof of the theorem.

**Remark 1**

i) Using the continuous injection \(L^p(\Omega) \subset D'(\Omega)\), with the continuity of the derivative operator in \(D'(\Omega)\), we can check that

\[
\varepsilon \nabla X_1 u_\varepsilon \rightharpoonup 0 \quad \text{in} \quad L^p(\Omega).
\]

ii) Let \(\gamma \geq 0\), then we infer from (12) that

\[
|\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma}) \quad \Rightarrow \quad |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma})
\]

iii) In particular the assumption (7) holds if \(\lim\sup_{\varepsilon \to 0} K_\varepsilon \neq \emptyset\), i.e. \(\cap_{\varepsilon < \varepsilon_0} K_\varepsilon \neq \emptyset\) for some \(\varepsilon_0 > 0\). For instance, this is the case of a monotone sequence of sets \((K_\varepsilon)\) (in the inclusion sense). Thereby, it suffices to fix \(w_\varepsilon = w_0 \in \cap_{\varepsilon < \varepsilon_0} K_\varepsilon\).
2.4 Convergence of convex sets

The existence of a sequence as $w_\varepsilon$ cannot give more than the weak convergence of subsequence of $u_\varepsilon$ without any identification of the limits. Then we are led to introduce a convergence of the closed convex sets fitting with the anisotropic singular perturbations. This beforehand serves to define the limit problem and then go deeper in the behaviour investigation of $u_\varepsilon$.

Let $(K_\varepsilon)$ be a sequence of closed convex subsets of $W_0^{1,p}(\Omega)$, we shall denote by

$$as - \lim_{\varepsilon \to 0} K_\varepsilon$$

(13)

the set of all $w$ in $W_0^0(\Omega)$, such that the strong convergence

$$\left(\varepsilon \nabla X_1 w_\varepsilon \vDash \nabla X_2 (w_\varepsilon - w)\right) \to 0 \quad \text{in } L^p(\Omega), \quad \text{as } \varepsilon \to 0,$$

holds for some sequence $w_\varepsilon \in K_\varepsilon$. We shall also denote by

$$aw - \lim_{\varepsilon \to 0} K_\varepsilon$$

(15)

the set of all $w$ in $W_0^0(\Omega)$, such that the weak convergence

$$\left(\varepsilon' \nabla X_1 w_{\varepsilon'} \vDash \nabla X_2 (w_{\varepsilon'} - w)\right) \rightharpoonup 0 \quad \text{in } L^p(\Omega), \quad \text{as } \varepsilon' \to 0,$$

holds at least for a subsequence $w_{\varepsilon'} \in K_{\varepsilon'}$.

Remark 2

i) It is clear that

$$as - \lim_{\varepsilon \to 0} K_\varepsilon \subset aw - \lim_{\varepsilon \to 0} K_\varepsilon.$$

(17)

ii) If (14) holds, it follows that

$$a(\cdot, \nabla^\varepsilon w_\varepsilon) \to a(\cdot, \nabla^0 w) \quad \text{in } L^{p'}(\Omega),$$

(18)

due to the continuity of the function $a$ in the second variable.

The limit $as - \lim_{\varepsilon \to 0} K_\varepsilon$ inherit the following proprieties.

Lemma 1 The set $as - \lim_{\varepsilon \to 0} K_\varepsilon$ is convex and closed in $W_0^0(\Omega)$.

Proof Let $v_n \in as - \lim_{\varepsilon \to 0} K_\varepsilon$ a sequence such that $v_n \to v$ in $W_0^0(\Omega)$, i.e.

$$\nabla X_2 (v_n - v) \to 0 \quad \text{in } L^p(\Omega), \quad \text{as } n \to \infty.$$

To show the closeness, one has to show that $v \in as - \lim_{\varepsilon \to 0} K_\varepsilon$. From the definition of $as - \lim_{\varepsilon \to 0} K_\varepsilon$ there exists a “sequence” $v_n^\varepsilon \in K_\varepsilon$ such that

$$\left(\frac{\varepsilon \nabla X_1 v_n^\varepsilon}{\nabla X_2 (v_n^\varepsilon - v_n)}\right) \to 0 \quad \text{in } L^p(\Omega), \quad \text{as } \varepsilon \to 0.$$

(19)

1 $a$ stands for anisotropic, $s$ for strong and $w$ for weak.
Consider then for every \( n \) a \( \varepsilon \) \((\varepsilon) > 0 \) such that for every \( \varepsilon \leq \varepsilon \) \((\varepsilon) \) it holds that

\[
\left| \varepsilon \nabla X_1 (\varepsilon^n_\alpha - v_n) \right|_{L^p(\Omega)} \leq \frac{1}{n}, \quad (20)
\]

By \((19)\) such an \( \varepsilon \) \((\varepsilon) \) exists and without loss of generality one can assume that it is chosen strictly decreasing towards 0. Let \( \varepsilon \leq \varepsilon \) \((\varepsilon) \). Denote by \( N_\varepsilon \) the integer \( n \) satisfying

\[
\varepsilon (N_\varepsilon + 1) < \varepsilon \leq \varepsilon (N_\varepsilon).
\]

Such \( N_\varepsilon \rightarrow \infty \) as \( \varepsilon \rightarrow 0 \). One has \( \varepsilon_{N_\varepsilon} \) \( \in K_\varepsilon \) and by \((20)\)

\[
\left| \varepsilon \nabla X_1 (\varepsilon_{N_\varepsilon} - v) \right|_{L^p(\Omega)} = \left| \nabla X_2 \left( \varepsilon v_{N_\varepsilon} - v \right) \right|_{L^p(\Omega)} \leq \frac{1}{N_\varepsilon} + \left| \nabla X_2 \left( v_{N_\varepsilon} - v \right) \right|_{L^p(\Omega)} \rightarrow 0,
\]

when \( \varepsilon \rightarrow 0 \). This shows that \( v \in a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon \).

To check that \( a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon \) is convex, let \( v^1, v^2 \in a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon \) then for some sequences \( v^1_\varepsilon, v^2_\varepsilon \in K_\varepsilon \) one has

\[
\left| \varepsilon \nabla X_1 (v^i_\varepsilon - v) \right|_{L^p(\Omega)} \rightarrow 0, \quad \text{for } i = 1, 2.
\]

It follows that for every \( \alpha \in [0, 1] \)

\[
\left| \varepsilon \nabla X_1 \left( (\alpha v^1_\varepsilon + (1 - \alpha) v^2_\varepsilon) - \varepsilon (\alpha v^1 + (1 - \alpha) v^2) \right) \right|_{L^p(\Omega)} \rightarrow 0, \quad \text{for } i = 1, 2
\]

and \( \alpha v^1 + (1 - \alpha) v^2 \in a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon \). This ends the proof of the lemma.

Now we introduce the desired limit set and its convergence sense.

**Definition 1** A sequence \( (K_\varepsilon) \) of subsets of \( W^{1,p}_0(\Omega) \) converges to a nonempty set \( \mathcal{K} \subset W^{1,p}_0(\Omega) \) iff

\[
a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon = a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon = \mathcal{K},
\]

and we denote \( K_\varepsilon \overset{a}{\rightarrow} \mathcal{K} \) or \( a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon = \mathcal{K} \).

We have to mention that the perturbation here is singular, i.e. \( K_\varepsilon \) and \( \mathcal{K} \) are not in the same space as it is the case in Mosco \([21,22]\), and is anisotropic since the perturbation affects only the \( X_1 \)-direction.

**Remark 3** In practice, taking into account \([74]\), the above convergence holds iff

\[
\begin{align*}
\text{i) } & K \subset a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon, \\
\text{ii) } a - \lim_{\varepsilon \rightarrow 0} K_\varepsilon \subset K.
\end{align*}
\]

The following lemma may simplify the verification of \((21)\) i).
Lemma 2 Let $D$ be a dense subset in $K$, then

$$K \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon} \iff D \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}.$$ 

Proof It suffices to note that if $D \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}$ then, due to Lemma 1, it follows that $K = D^{W(\Omega)} \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}^{W(\Omega)} = \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}$.

Let us now summarize some basic properties of the above convergence in the following lemma.

Lemma 3.i) Let $K_{\varepsilon}'$ be a “subsequence” of $K_{\varepsilon}$, then

$$\text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}' \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon},$$

(22)

$$aw - \lim_{\varepsilon \to 0} K_{\varepsilon}' \subset aw - \lim_{\varepsilon \to 0} K_{\varepsilon}.$$ 

(23)

In particular if $K_{\varepsilon}$ converges then $K_{\varepsilon}'$ also converges and we have

$$a - \lim_{\varepsilon \to 0} K_{\varepsilon} = a - \lim_{\varepsilon \to 0} K_{\varepsilon}'.$$ 

(24)

ii) If the sequence $K_{\varepsilon}$ is constant, i.e. $K_{\varepsilon} = K, \forall \varepsilon > 0$ (or it is constant for $\varepsilon$ small), then

$$K \xrightarrow{a} K \equiv K^{W(\Omega)},$$ 

(25)

and in particular, if $K_{\varepsilon} = W_{0,0}^1(\Omega)$

$$W_{0,0}^1(\Omega) \xrightarrow{a} W_0(\Omega).$$ 

(26)

Proof The two first inclusions are immediate since a sequence or a subsequence of $K_{\varepsilon}'$ is also a subsequence of $K_{\varepsilon}$ and the equality (24) can be easily deduced if we notice that

$$\text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon} \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon} \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon} \subset \text{as} - \lim_{\varepsilon \to 0} K_{\varepsilon}.$$ 

For the next point, the inclusion (21)i) follows by taking $D = K$ in Lemma 2 and the inclusion (21)ii) holds since $K_{\varepsilon} \subset K, \forall \varepsilon > 0$, and the convex $K$ is also weakly closed. This shows (25), (26) and completes the proof.

2.5 Convergence of solutions

Assuming that $K_{\varepsilon} \xrightarrow{a} K$ ($K \neq \phi$), we shall show some convergence results for the solution $u_{\varepsilon}$ when $\varepsilon \to 0$ and identify its limit. First, the candidate limit $\tilde{u}$ will be defined as a solution of the following problem

$$\begin{aligned}
\int_{\Omega} a(x, \nabla^0 u) \cdot \nabla^0 (v - u) \, dx &\geq \langle f, v - u \rangle_{W_0(\Omega)}, \quad \forall v \in K, \\
u &\in K
\end{aligned}$$

(27)

It is clear that the operator $u \to -\nabla^0 \cdot a(x, \nabla^0 u)$ is coercive, bounded, hemichnontinuous and monotone on $W_0(\Omega)$ and thanks to Lemma 1, the set $K$ is convex and closed in $W_0(\Omega)$. Thus Problem (27) has a solution $\tilde{u} \in K$.

To prove the next theorem we need the Minty Lemma (see Chipot [8]).
Lemma 4 Let $X$ be a Banach space, $T$ be a monotone hemicontinuous operator from a closed convex set $K$ in $X$ into $X'$ and $f \in X'$, then $u_0 \in K$ satisfies
\[
\langle Tu_0, v - u_0 \rangle_X \geq \langle f, v - u_0 \rangle_X, \quad \forall v \in K,
\]
if and only if
\[
\langle Tv, v - u_0 \rangle_X \geq \langle f, v - u_0 \rangle_X, \quad \forall v \in K.
\]

Now we have the following convergence results.

Theorem 2 Under the hypotheses of Theorem 1 assume in addition that $K_\varepsilon \rightharpoonup K$ as $\varepsilon \to 0$, then -up to a subsequence- we have
\[
u_\varepsilon \rightharpoonup \tilde{u}, \quad \varepsilon \nabla X_1 u_\varepsilon \rightharpoonup 0 \quad \text{and} \quad \nabla X_2 u_\varepsilon \rightharpoonup \nabla X_2 \tilde{u} \quad \text{in} \quad L^p(\Omega),
\]
where $\tilde{u}$ is a solution to the variational inequality (27). Moreover:

- if the function $a$ is strictly monotone then the above weak convergences hold for the whole sequence.

- if the function $a$ is strongly monotone in the sense that, for some constants $c > 0$,
\[
(a(x, \eta) - a(x, \xi)) \cdot (\eta - \xi) \geq c |\eta - \xi|^p, \quad \forall \eta, \xi \in \mathbb{R}^n \quad \text{and} \quad \text{a.e.} \quad x \in \Omega,
\]
then we have the strong convergences
\[
u_\varepsilon \to \tilde{u}, \quad \varepsilon \nabla X_1 u_\varepsilon \to 0 \quad \text{and} \quad \nabla X_2 u_\varepsilon \to \nabla X_2 \tilde{u} \quad \text{in} \quad L^p(\Omega).
\]

Proof First, it is clear that the assumption $K_\varepsilon \rightharpoonup K$ implies (7). Then, thanks to Theorem 1, there exists a subsequence of $u_\varepsilon$ -still labelled $u_\varepsilon$- such that
\[
u_\varepsilon \rightharpoonup \tilde{u}, \quad \nabla X_2 u_\varepsilon \rightharpoonup \nabla X_2 \tilde{u}, \quad \varepsilon \nabla X_1 u_\varepsilon \to 0 \quad \text{in} \quad L^p(\Omega).
\]
Such $\tilde{u}$ is necessarily in the set $aw \ rightharpoonup K$ by (16). Next, we choose an arbitrary $w \in as \ rightharpoonup K \subset \mathcal{K}$, and let $(w_\varepsilon)$ be a sequence satisfying (14). By the monotonicity assumption, we rewrite (3) as
\[
\int_\Omega a(x, \nabla^\varepsilon w_\varepsilon) : \nabla^\varepsilon (w_\varepsilon - u_\varepsilon) \, dx \geq \langle f, w_\varepsilon - u_\varepsilon \rangle_{W_0(\Omega)}.
\]
Passing to the limit and using the convergences (14), (18), (31), it comes
\[
\int_\Omega a(x, \nabla^0 w) : \nabla^0 (w - \tilde{u}) \, dx \geq \langle f, w - \tilde{u} \rangle_{W_0(\Omega)},
\]
for every $w \in \mathcal{K}$. Thanks to Minty’s lemma, it follows that
\[
\int_\Omega a(x, \nabla^0 \tilde{u}) : \nabla^0 (w - \tilde{u}) \, dx \geq \langle f, w - \tilde{u} \rangle_{W_0(\Omega)}, \quad \forall w \in \mathcal{K},
\]
i.e. $\tilde{u}$ is a solution to (27).

If the function $a$ is strictly monotone then the solution $\tilde{u}$ of (27) is unique and the weak convergences (28) hold for the whole sequence.
If now (29) holds, and since \( \tilde{u} \in K \), we can take \( w = \tilde{u} \) in (14), i.e.

\[
\varepsilon \nabla_{X_1} w_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} w_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in} \ L^p(\Omega),
\]

for some sequence \( w_\varepsilon \in K_\varepsilon \) and it follows that

\[
c \big| \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \big|_{L^p(\Omega)}^p \leq \int_{\Omega} (a(x, \nabla^\varepsilon u_\varepsilon) - a(x, \nabla^\varepsilon w_\varepsilon)) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \, dx
\]

\[
\leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{W_0(\Omega)} - \int_{\Omega} a(x, \nabla^\varepsilon w_\varepsilon) \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \, dx \to 0.
\]

Whence

\[
\varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{and} \quad \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \tilde{u} \quad \text{in} \ L^p(\Omega).
\]

The strong convergence \( u_\varepsilon \to \tilde{u} \) in \( L^p(\Omega) \) follows by Poincaré’s inequality in the \( X_2 \)-direction. This ends the proof.

**Remark 4** When \( K \) is a closed linear subspace of \( W_0(\Omega) \) the variational inequality (27) is reduced to the integral identity

\[
\int_{\Omega} a(x, \nabla^0 \tilde{u}) \cdot \nabla^0 w \, dx = \langle f, w \rangle_{W_0(\Omega)}, \quad \forall w \in K.
\]

2.6 Perturbed \( p \)-Laplace operator

A common example of the function \( a \) is given by

\[
a(x, \xi) = |\xi|^{p-2} \xi, \quad \forall \xi \in \mathbb{R}^n \text{ and } p > 1.
\]

This corresponds to the perturbed \( p \)-Laplace operator considered in Problem 1. We show here strong convergences, even if the \( p \)-Laplacian is not strongly monotone for \( 1 < p \leq 2 \).

The following Lemma summarizes some inequalities related to the \( p \)-Laplacian, required here and later in the last section. The proof can be found for instance in [8].

**Lemma 5** For all \( p > 1 \) and \( \xi, \eta \in \mathbb{R}^n \), it holds that for some constants \( C_i \), \( i = 1, \ldots, 4 \) depending on \( p \)

\[
C_1 \{|\xi| + |\eta|\}^{p-2} |\xi - \eta|^2 \leq \left( |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) \cdot (\xi - \eta) \quad (34)
\]

\[
|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \leq C_2 \{|\xi| + |\eta|\}^{p-2} |\xi - \eta| \quad (35)
\]

If \( p \geq 2 \), then

\[
\left( |\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) \cdot (\xi - \eta) \geq C_3 |\xi - \eta|^p \quad (36)
\]

If \( 1 < p \leq 2 \) then

\[
|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \leq C_4 |\xi - \eta|^{p-1} \quad (37)
\]
The limit of $u_\varepsilon$, solution of Problem (3), is the unique solution of the problem
\[
\left\{ \begin{array}{l}
\int_\Omega |\nabla X_2 u|^{p-2} \nabla X_2 u \cdot \nabla X_2 (v - u) \, dx \geq \langle f, v - u \rangle_{W_0^1(\Omega)}, \quad \forall v \in \mathcal{K}, \\
u \in \mathcal{K}.
\end{array} \right.
\]
(38)

The $p-$Laplacian is strictly monotone, and due to the inequality (36), it is strongly monotone for $p \geq 2$, then the strong convergences (30) are ensured by Theorem 2.

For $p > 1$ arbitrary, we have the following theorem.

**Theorem 3** Let $p > 1$ and $u_\varepsilon$ be the unique solution of Problem (1). Assume that $K_\varepsilon \rightharpoonup K$ as $\varepsilon \to 0$, then we have
\[
u_\varepsilon \to \tilde{u}, \quad \varepsilon \nabla X_2 u_\varepsilon \to 0 \quad \text{and} \quad \nabla X_2 u_\varepsilon \to \nabla X_2 \tilde{u} \quad \text{in} \ L^p(\Omega),
\]
where $\tilde{u}$ is the solution to the variational inequality (38).

**Proof** Thanks to the first assertion of Theorem 2 we have
\[
\left( \varepsilon \nabla X_2 u_\varepsilon \right) \rightharpoonup \left( 0 \right) \quad \text{in} \quad L^p(\Omega),
\]
(39)
for the whole sequence and $p > 1$ arbitrary. To get the strong convergence, one note that
\[
\int_\Omega \left( |\nabla X_2 u_\varepsilon|^{p-2} \nabla X_2 u_\varepsilon - |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon \right) \cdot \nabla X_2 (u_\varepsilon - w_\varepsilon) \, dx
\]
\[
= |\nabla X_2 u_\varepsilon|_{L^p(\Omega)}^p + |\nabla X_2 w_\varepsilon|_{L^p(\Omega)}^p - \int_\Omega |\nabla X_2 u_\varepsilon|^{p-2} \nabla X_2 u_\varepsilon \cdot \nabla X_2 w_\varepsilon \, dx
\]
\[
- \int_\Omega |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx
\]
\[
\geq |\nabla X_2 u_\varepsilon|_{L^p(\Omega)}^p + |\nabla X_2 w_\varepsilon|_{L^p(\Omega)}^p - |\nabla X_2 u_\varepsilon|_{L^p(\Omega)}^{p-1} \times |\nabla X_2 w_\varepsilon|_{L^p(\Omega)}
\]
\[
- |\nabla X_2 w_\varepsilon|_{L^p(\Omega)}^{p-1} \times |\nabla X_2 u_\varepsilon|_{L^p(\Omega)}.
\]
Hölder’s inequality is used to obtain the two last terms. The above inequality can also be written as
\[
0 \leq \left( |\nabla X_2 u_\varepsilon|_{L^p(\Omega)}^p - |\nabla X_2 w_\varepsilon|_{L^p(\Omega)}^{p-1} \right) \times \left( |\nabla X_2 u_\varepsilon|_{L^p(\Omega)} - |\nabla X_2 w_\varepsilon|_{L^p(\Omega)} \right)
\]
\[
\leq \int_\Omega \left( |\nabla X_2 u_\varepsilon|^{p-2} \nabla X_2 u_\varepsilon - |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon \right) \cdot \nabla X_2 (u_\varepsilon - w_\varepsilon) \, dx
\]
\[
\leq \langle f, u_\varepsilon - w_\varepsilon \rangle_{W_0^1(\Omega)} - \int_\Omega |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon \cdot \nabla X_2 (u_\varepsilon - w_\varepsilon) \, dx.
\]
Since $\tilde{u} \in \mathcal{K}$ choosing $w_\varepsilon \in K_\varepsilon$ such that
\[
\varepsilon \nabla X_2 w_\varepsilon \to 0 \quad \text{and} \quad \nabla X_2 w_\varepsilon \to \nabla X_2 \tilde{u} \quad \text{in} \ L^p(\Omega),
\]
and passing to the limit in the above inequality we end up with
\[
|\nabla X_2 u_\varepsilon|_{L^p(\Omega)} \to |\nabla X_2 \tilde{u}|_{L^p(\Omega)}, \quad \text{as} \quad \varepsilon \to 0.
\]
Taking into account (39) the strong convergence follows since $L^p(\Omega)$ is uniformly convex for $p > 1$. This ends the proof.
The above theorem can be used to give an equivalent definition of the space \( W_0(\Omega) \).

**Corollary 1** Let \( \Omega \) be a bounded open domain of \( \mathbb{R}^n \). Then, it holds that

\[
u \in W_0(\Omega) \text{ iff } u \in \mathcal{W}(\Omega) \text{ and } u(X_1, \cdot) \in W_{0,1}^{1,p}(\Omega_{X_1}), \quad \text{a.e. } X_1 \in \Pi_1.
\]

**Proof** For \( u \in W_0(\Omega) \) there exists a sequence \((u_n)_n \subset \mathcal{D}(\Omega)\) such that \( u_n \to u \) in \( \mathcal{W}(\Omega) \). In particular we have

\[
|\nabla X_2 (u_n - u)|_{L^p(\Omega)} \to 0.
\]

By the Lebesgue theorem we get - up to a subsequence -

\[
|\nabla X_2 (u_n (X_1, \cdot) - \nabla X_2 u(X_1, \cdot))|_{L^p(\Omega_{X_1})} \to 0, \quad \text{for a.e. } X_1 \in \Pi_1.
\]

Since \( v \to |\nabla X_2 v|_{L^p(\Omega_{X_1})} \) is a norm on \( W_{0,1}^{1,p}(\Omega_{X_1}) \), we infer that \( u(X_1, \cdot) \in W_{0,1}^{1,p}(\Omega_{X_1}) \), for a.e. \( X_1 \in \Pi_1 \), and

\[
W_0(\Omega) \subset \left\{ u \in L^p(\Omega) \mid \nabla X_2 u \in [L^p(\Omega)]^{n \times q}, \quad u(X_1, \cdot) \in W_{0,1}^{1,p}(\Omega_{X_1}), \quad \text{a.e. } X_1 \in \Pi_1 \right\}. \tag{40}
\]

For the converse inclusion, we use an anisotropic perturbation argument. Let

\[
u \in \left\{ u \in L^p(\Omega) \mid \nabla X_2 u \in [L^p(\Omega)]^{n \times q}, \quad u(X_1, \cdot) \in W_{0,1}^{1,p}(\Omega_{X_1}), \quad \text{a.e. } X_1 \in \Pi_1 \right\}. \tag{41}
\]

Since \( \nabla X_2 u \in [L^p(\Omega)]^{n \times q} \), then we can take \( \nabla X_2 \left( |\nabla X_2 u|^{p-2} \nabla X_2 u \right) \in W_0(\Omega) \) as a source term in the following quasilinear problem

\[
\int_\Omega |\nabla^\varepsilon v_\varepsilon|^{p-2} \nabla^\varepsilon v_\varepsilon \cdot \nabla^\varepsilon v_\varepsilon dx = \left< \nabla X_2 \left( |\nabla X_2 u|^{p-2} \nabla X_2 u \right), v \right>_{W_0(\Omega)}
\]

\[
= \int_\Omega |\nabla X_2 u|^{p-2} \nabla X_2 u \cdot \nabla X_2 v_\varepsilon dx, \quad \forall v \in W_{0,1}^{1,p}(\Omega)
\]

where the unique solution \( v_\varepsilon \) is in \( W_{0,1}^{1,p}(\Omega) \). Choosing \( K_\varepsilon = W_{0,1}^{1,p}(\Omega) \) and \( K = W_0(\Omega) \), then due to Lemma 3 and Theorem 3 we have

\[
\nabla^\varepsilon v_\varepsilon \to \nabla X_2 \hat{u} \text{ in } L^p(\Omega)
\]

where \( \hat{u} \in W_0(\Omega) \) is the solution of problem

\[
\int_\Omega |\nabla X_2 \hat{u}|^{p-2} \nabla X_2 \hat{u} \cdot \nabla X_2 vdx = \int_\Omega |\nabla X_2 u|^{p-2} \nabla X_2 u \cdot \nabla X_2 vdx,
\]

for every \( v \in W_{0,1}^{1,p}(\Omega) \). It remain to check that \( u = \hat{u} \) in \( \Omega \). We give here a proof for cylindrical domains, i.e.

\[
\Omega = \omega_1 \times \omega_2, \quad \text{where } \omega_1 \subset \mathbb{R}^q \text{ and } \omega_2 \subset \mathbb{R}^{n-q}.
\]

\[
\tag{43}
\]
For general domains, i.e. not necessarily cylindrical, we can argue as in [10]. Let \( \varphi_1 \in \mathcal{D}(\omega_1) \) and \( \varphi_2 \in W^{1,p}_0(\omega_2) \), then we derive from (42)

\[
\int_{\omega_1} \varphi_1(X_1) \int_{\omega_2} |\nabla X_2 \tilde{u}|^{p-2} \nabla X_2 \tilde{u}(X_1, X_2) \cdot \nabla X_2 \varphi(X_2) \, dX_2 \, dX_1
= \int_{\omega_1} \varphi_1(X_1) \int_{\omega_2} |\nabla X_2 u|^{p-2} \nabla X_2 u(X_1, X_2) \cdot \nabla X_2 \varphi(X_2) \, dX_2 \, dX_1, \quad \forall \varphi_1 \in \mathcal{D}(\omega_1),
\]

since \( \varphi_1 \varphi_2 \in W^{1,p}_0(\Omega) \). Thus, for a.e. \( X_1 \in \omega_1 \), we have

\[
\int_{\omega_2} |\nabla X_2 \tilde{u}|^{p-2} \nabla X_2 \tilde{u}(X_1, X_2) \cdot \nabla X_2 \varphi_2(X_2) \, dX_2
= \int_{\omega_2} |\nabla X_2 u|^{p-2} \nabla X_2 u(X_1, X_2) \cdot \nabla X_2 \varphi_2(X_2) \, dX_2, \quad \forall \varphi_2 \in W^{1,p}_0(\omega_2). \quad (44)
\]

By (40) and assumption (41) we have

\[
\tilde{u}(X_1, \cdot) \in W^{1,p}_0(\omega_2) \text{ and } u(X_1, \cdot) \in W^{1,p}_0(\omega_2), \text{ for a.e. } X_1 \in II_1.
\]

Thus we can take \( \tilde{u}(X_1, \cdot) - u(X_1, \cdot) \) as a test function in (44) and it comes that

\[
\int_{\omega_2} \left( |\nabla X_2 \tilde{u}|^{p-2} \nabla X_2 \tilde{u}(X_1, X_2) - |\nabla X_2 u|^{p-2} \nabla X_2 u(X_1, X_2) \right) \cdot \nabla X_2 (\tilde{u} - u) (X_1, X_2) \, dX_2 = 0.
\]

Due to the strict monotonicity, which follows from (34) by taking \( \xi = \nabla^0 \tilde{u} \) and \( \eta = \nabla^0 u \), we infer that

\[
(\tilde{u} - u) (X_1, \cdot) = 0 \text{ a.e. in } \omega_2,
\]

for a.e. \( X_1 \in \omega_1 \). This implies that \( \tilde{u} = u \) a.e. in \( \Omega \) and the corollary follows.

3 Convex sets with perturbed obstacle constraints

In this section and in the next one, we give some examples of convex sets where the convergence in the sense of Definition 1 can be ensured. We refer the reader to Chipot [6] and Kinderlehrer and Stampacchia [18] for applications and more mathematical background about related problems.

For \( \varepsilon > 0 \), we consider a sequence of obstacles \( \psi_\varepsilon \in W^{1,p}(\Omega) \) and the associated sequence of convex sets

\[
K_\varepsilon = K_{\psi_\varepsilon} := \left\{ v \in W^{1,p}_0(\Omega) \mid \forall \varepsilon \psi_\varepsilon \text{ a.e. in } \Omega \right\}.
\]

The set \( K_\varepsilon \) is not empty provided that \( \psi_\varepsilon^+ := \max \{ \psi_\varepsilon, 0 \} \in W^{1,p}_0(\Omega) \). Assuming that, in some sense, \( \psi_\varepsilon \) converges to some \( \psi_0 \in W(\Omega) \) such that \( \psi_0^+ \in W_0^0(\Omega) \), one expects that the limit of \( K_{\psi_\varepsilon} \) is

\[
K = K_{\psi_0} := \left\{ v \in W_0^0(\Omega) \mid \forall \psi_0 \text{ a.e. in } \Omega \right\}.
\]

To be more precise we have the following result.
Theorem 4 Under the above assumptions, if
\[ \psi_\varepsilon \rightharpoonup \psi_0 \quad \text{in} \ L^p (\Omega) \quad \text{as} \ \varepsilon \to 0, \]
then we have \( a w - \lim_{\varepsilon \to 0} K_{\psi_\varepsilon} \subset K_{\psi_0}. \)
Moreover if
\[ \nabla^\varepsilon \psi_\varepsilon \to \nabla^0 \psi_0 \quad \text{in} \ L^p (\Omega) \quad \text{as} \ \varepsilon \to 0, \]
then it holds that \( a - \lim_{\varepsilon \to 0} K_{\psi_\varepsilon} = K_{\psi_0}. \)

Proof Let \( v \in a w - \lim_{\varepsilon \to 0} K_{\psi_\varepsilon}. \) Then there exists a subsequence \( v_\varepsilon \in K_{\psi_\varepsilon}, \) such that
\[ \nabla^\varepsilon v_\varepsilon \rightharpoonup \nabla^0 v \quad \text{in} \ L^p (\Omega). \] Since \( v_\varepsilon \geq \psi_\varepsilon \) a.e. in \( \Omega, \) it comes that
\[ \int_\Omega (v_\varepsilon - \psi_\varepsilon) \varphi \, dx \geq 0, \quad \forall \varphi \in D (\Omega), \varphi \geq 0. \]
Due to Poincaré’s inequality, in the \( X_2 - \text{direction}, \) \( v_\varepsilon \) is bounded in \( L^p (\Omega). \) We have then –up to a new subsequence– \( v_\varepsilon \rightharpoonup v \) in \( L^p (\Omega). \) and using (45), it comes that
\[ \int_\Omega (v_\varepsilon - \psi_\varepsilon) \varphi \, dx \to \int_\Omega (v - \psi_0) \varphi \, dx \geq 0, \quad \forall \varphi \in D (\Omega), \varphi \geq 0. \]
Thus we have \( v \geq \psi_0, \) a.e. in \( \Omega, \) i.e. \( v \in K_{\psi_0}. \)
To establish the last assertion, we have just to show that \( K_{\psi_0} \subset a s - \lim_{\varepsilon \to 0} K_{\psi_\varepsilon}. \)

Let \( v \in K_{\psi_0} \) then, due to the last assertion of Lemma 3 there exists a sequence \( w_\varepsilon \in W^{1,p}_0 (\Omega) \) such that
\[ \nabla^\varepsilon w_\varepsilon \to \nabla^0 v, \quad \text{in} \ L^p (\Omega) \quad \text{as} \ \varepsilon \to 0. \]
But \( w_\varepsilon \) may not belong to \( K_\varepsilon, \) hence we consider the sequence
\[ v_\varepsilon = \max \{ \psi_\varepsilon, w_\varepsilon \} = (\psi_\varepsilon - w_\varepsilon)^+ + w_\varepsilon \in W^{1,p}_0 (\Omega). \]
It is clear that \( v_\varepsilon \geq \psi_\varepsilon, \) i.e. \( v_\varepsilon \in K_{\psi_\varepsilon}. \) This sequence converges to \( v \) in the sense of (46). Indeed, due to (46), (47) and the boundedness of the positive part as an operator on \( W^{1,p}_0 (\Omega) \) (see for instance Heinonen et al. [18, Lemma 1.22]) we have
\[ \nabla^\varepsilon (\psi_\varepsilon - w_\varepsilon)^+ \to \nabla^0 (\psi_0 - v)^+ \quad \text{in} \ L^p (\Omega) \]
and it follows that
\[ \nabla^\varepsilon v_\varepsilon \to \nabla^0 ((\psi_0 - v)^+ + v) = \nabla^0 v \quad \text{in} \ L^p (\Omega) \quad \text{as} \ \varepsilon \to 0. \]
This means that \( v \in a s - \lim_{\varepsilon \to 0} K_{\psi_\varepsilon} \) and the theorem is proved.
4 Convex sets with perturbed gradient constraints

In this section, the convex set is determined by a constraint on the gradient of the solution. Consider a sequence of nonnegative functions \( \beta_\varepsilon \in L^\infty(\Omega) \) and consider the set

\[
K_\varepsilon = K_{\beta_\varepsilon} := \left\{ v \in W_0^{1,p}(\Omega) \mid |\nabla^\varepsilon v| \leq \beta_\varepsilon \text{ a.e. in } \Omega \right\},
\]

which is a nonempty closed convex set of \( W_0^{1,p}(\Omega) \). The problem (3) with the constraint set \( K_{\beta_\varepsilon} \) admits a solution \( u_\varepsilon \in K_{\beta_\varepsilon}, \forall \varepsilon > 0 \).

Assuming that \( \beta_\varepsilon \) converges to some nonnegative function \( \beta_0 \) in \( L^\infty(\Omega) \), i.e.

\[
|\beta_\varepsilon - \beta_0|_{L^\infty(\Omega)} \to 0, \text{ as } \varepsilon \to 0,
\]

one expects that the limit of \( K_{\beta_\varepsilon} \) is

\[
K = K_{\beta_0} := \left\{ v \in W_0(\Omega) \mid |\nabla X_2 v| \leq \beta_0 \text{ a.e. in } \Omega \right\}.
\]

4.1 Preliminary results

Let us start by the first inclusion in (21).

**Theorem 5** Assume that (49) holds, then \( aw - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \subset K_{\beta_0} \).

**Proof** For \( \delta > 0 \), let

\[
M_\delta := \left\{ v \in W_0(\Omega) \mid |\nabla X_2 v| \leq \beta_0 + \delta \text{ a.e. in } \Omega \right\},
\]

which is a convex and closed set in \( W_0(\Omega) \), hence it is also a weakly closed set. Let \( v \in aw - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \). Then there exists a subsequence \( v_\varepsilon \in K_{\beta_\varepsilon} \) (still labeled \( v_\varepsilon \)) such that \( \nabla^\varepsilon v_\varepsilon \rightharpoonup \nabla^0 v \) in \( L^p(\Omega) \). As \( \beta_\varepsilon \to \beta_0 \) in \( L^\infty(\Omega) \), then \( \beta_\varepsilon \leq \beta_0 + \delta, \text{ a.e. in } \Omega, \) for \( \varepsilon \) small enough, and it comes that

\[
|\nabla X_2 v| \leq |\nabla^\varepsilon v| \leq \beta_\varepsilon \leq \beta_0 + \delta, \text{ a.e. in } \Omega.
\]

Thus \( v_\varepsilon \in M_\delta \), for \( \varepsilon \) small enough, which implies that its weak limit \( v \) also belongs to \( M_\delta \). Since this holds for arbitrary \( \delta \) we deduce that \( v \in \bigcap_{\delta > 0} M_\delta = K_{\beta_0} \).

To show the other inclusion, i.e. \( K_{\beta_0} \subset as - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \), we have first to check that \( K_{\beta_0} \subset L^\infty(\Omega) \).

**Lemma 6** Let \( \Omega \) be a domain of \( \mathbb{R}^n \), bounded in the \( X_2 \)-direction, then

\[
\{ v \mid v \in W_0(\Omega), |\nabla X_2 v| \in L^\infty(\Omega) \} \subset L^\infty(\Omega).
\]

**Proof** As \( \mathcal{D}(\Omega) \) is dense in \( W_0(\Omega) \) by definition, the canonical extension \( u \to \tilde{u}(x) \), where

\[
\tilde{u}(x) := \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}
\]

define a continuous mapping from \( W_0(\Omega) \) to \( \mathcal{W}(\mathbb{R}^n) \) and it holds, in the distributional sense, that

\[
\nabla X_2 \tilde{u} = \nabla X_2 u \chi_\Omega.
\]
Let $u \in W_0(\Omega)$ such that $|\nabla_{X_2} u| \in L^\infty(\Omega)$. We consider the ball $B_R \subset \mathbb{R}^{n-q}$ such that $\Omega \subset \subset \mathbb{R}^q \times B_R$ and we denote by $u_\varepsilon$ the convolution of $\tilde{u}$ defined as

$$u_\varepsilon (x) = \rho_\varepsilon \ast \tilde{u} (x) = \int_{\mathbb{R}^n} \rho_\varepsilon (y) \tilde{u} (x-y) \, dy$$

(51)

for a.e. $x \in \mathbb{R}^n$, where $\rho_\varepsilon$ is the usual mollifier in $\mathbb{R}^n$. We can show that

$$u_\varepsilon \rightarrow \tilde{u} \text{ in } L^p(\mathbb{R}^n) \quad \text{and} \quad |u_\varepsilon|_{L^p} \leq |\tilde{u}|_{L^p}.$$ 

In particular

$$u_\varepsilon \rightarrow \tilde{u} \text{ in } L^p(\mathbb{R}^q \times B_R).$$

(52)

Going back to (51), one has

$$\partial_{x_i} u_\varepsilon (x) = \int_{\mathbb{R}^n} \rho_\varepsilon (y) \partial_{x_i} \tilde{u} (x-y) \, dy, \text{ for } i = q+1, \ldots, n$$

and thus

$$|\nabla_{X_2} u_\varepsilon (x)|^2 = \sum_{i=q+1}^n |\partial_{x_i} u_\varepsilon (x)|^2 \leq \int_{\mathbb{R}^n} \rho_\varepsilon (y) |\nabla_{X_2} \tilde{u} (x-y)|^2 \, dy$$

by Jensen’s inequality. Thanks to (50) and $|\nabla_{X_2} u| \in L^\infty(\Omega)$, it holds that $|\nabla_{X_2} \tilde{u}| \in L^\infty(\mathbb{R}^n)$ and thus

$$|\nabla_{X_2} u_\varepsilon| \in L^\infty(\mathbb{R}^n).$$

For $\varepsilon$ small enough, $\text{supp}(u_\varepsilon) \subset \mathbb{R}^q \times B_R$ and by the mean value theorem, applied for the smooth function $u_\varepsilon$, it follows that

$$|u_\varepsilon| \leq CR \text{ a.e. in } \mathbb{R}^q \times B_R,$$

for some constant $C$. Finally, since the convergence (52) preserves boundedness in $L^\infty(\mathbb{R}^q \times B_R)$, we infer that

$$|\tilde{u}| \leq CR \text{ a.e. in } \mathbb{R}^q \times B_R,$$

which means that $u \in L^\infty(\Omega)$ as claimed.

**Remark 5** If we assume $p > n - q$, the above lemma follows by using Sobolev’s injections. In fact, for a.e. $X_1$ we have $u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}) \subset L^\infty(\Omega_{X_1})$ and

$$|u(X_1, \cdot)|_{L^\infty(\Omega_{X_1})} \leq C |\nabla_{X_2} u(X_1, \cdot)|_{L^p(\Omega_{X_1})} \leq C |\nabla_{X_2} u(X_1, \cdot)|_{L^\infty(\Omega_{X_1})}$$

since $\Omega$ is bounded in the $X_2$-direction, whence

$$|u(X_1, \cdot)|_{L^\infty(\Omega_{X_1})} \leq C |\nabla_{X_2} u|_{L^\infty(\Omega)} \quad \text{for a.e. } X_1,$$

where the last $C$ is independent of $X_1$. Thus $|u|_{L^\infty(\Omega)} \leq C |\nabla_{X_2} u|_{L^\infty(\Omega)}$.  


4.2 Convergence for cylindrical domains

Let now \( v \in K_{\beta_0} \), due to the last assertion of Lemma 3, there exists a sequence \( v_\varepsilon \in W^{1,p}_0(\Omega) \) such that \( \nabla v_\varepsilon \to \nabla v \) in \( L^p(\Omega) \). To insure that \( v \in as - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \), each \( v_\varepsilon \) must be chosen in \( \bar{K}_{\beta_\varepsilon} \), i.e. \( v_\varepsilon \) satisfies a constraint on the gradient \( |\nabla v_\varepsilon| \leq \beta_\varepsilon \) a.e. in \( \Omega \). Such a sequence \( v_\varepsilon \) can be explicitly constructed for some types of domains as it will be described in the remainder of this section.

We assume in this subsection that \( \Omega \) is cylindrical, i.e.

\[
\Omega = \omega_1 \times \omega_2, \quad \omega_1 \subset \mathbb{R}^q \text{ and } \omega_2 \subset \mathbb{R}^{n-q}
\]

where \( \omega_1 \) is a regular bounded domain.

**Theorem 6** Assume that (10) and (53) hold. In addition suppose that

\[
\beta_\varepsilon \in C(\bar{\Omega}) \quad \text{and} \quad \beta_\varepsilon \geq \sigma > 0 \text{ a.e. in } \Omega,
\]

for some constant \( \sigma \). Then \( K_{\beta_0} \subset as - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \), i.e. \( K_{\beta_0} = a - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \). \( C(\bar{\Omega}) \) is the space of restrictions of \( C(\mathbb{R}^n) \)-functions to \( \Omega \).

**Proof** First, we proceed by truncation to show that the set

\[
D_{\beta_0} := \{ v \in K_{\beta_0} \mid \text{supp } v \subset \omega_1 \times \bar{\omega}_2 \}
\]

is dense in \( K_{\beta_0} \). Then the support of the functions of \( D_{\beta_0} \) can be kept in \( \omega_1 \times \bar{\omega}_2 \) after a partial regularization in \( X_1 \). This leads to define, for each function in \( D_{\beta_0} \), a converging sequence in \( K_{\beta_\varepsilon} \) towards this function. Due to Lemma 2, this is sufficient to conclude that \( K_{\beta_0} \subset as - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \).

i) **Truncation.** Let \( v \in K_{\beta_0} \) and for a small \( d_1 > 0 \), consider the set

\[
\omega_1' := \{ X_1 \in \omega_1 \mid \text{dist } (X_1, \mathbb{R}^q \setminus \omega_1) > d_1 \}
\]

and the truncated function

\[
v' := v \chi_{\omega_1'}
\]

where \( \chi_{\omega_1'} \) is the indicator function of the set \( \omega_1' \). We still have

\[
|\nabla_{X_2} v'| \leq \beta_0 \text{ a.e. in } \Omega,
\]

i.e. \( v' \in K_{\beta_0} \). Moreover, for any \( \delta > 0 \), there exists \( d_1 > 0 \), small enough, such that

\[
|\nabla_{X_2} (v - v')|_{L^p(\Omega)} = |\nabla_{X_2} v \chi_{\omega_1 \setminus \omega_1'}|_{L^p(\Omega)}
\leq |\beta_0|_{\infty} |\chi_{\omega_1 \setminus \omega_1'}|_{L^p(\Omega)}
\leq |\beta_0|_{\infty} \left[ \text{meas } (\omega_1 \setminus \omega_1') \times \omega_2 \right]^{\frac{1}{p}} \leq \delta,
\]

which is nothing else than the density of the set of functions, supported far from the boundary \( \partial \omega_1 \times \omega_2 \), in \( \bar{K}_{\beta_0} \).

ii) **Regularization in the \( X_1 \) direction.** For \( \alpha \in (0,1) \), consider the (positive) \( \varepsilon^\alpha \)-mollifier sequence

\[
p_\varepsilon (X_1) := \frac{1}{\varepsilon^{q\alpha}} \rho \left( \frac{X_1}{\varepsilon^{\alpha}} \right).
\]
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Let \( v \in D_{\beta_\varepsilon} \), then for \( \varepsilon \) small enough we still have \( \text{supp}(\rho_\varepsilon * v) \subset \omega_1 \times \omega_2 \), which in particular implies that

\[
\rho_\varepsilon * v \in W^{1,p}_{0}(\Omega).
\]

Extending \( v \) by 0 outside \( \Omega \), we can show that for a.e. \( X_2 \in \omega_2 \)

\[
\rho_\varepsilon * v (\cdot, X_2) \to v (\cdot, X_2) \text{ in } L^p(\mathbb{R}^q)
\]

\[
\nabla_{X_2} (\rho_\varepsilon * v (\cdot, X_2)) = \rho_\varepsilon * \nabla_{X_2} v (\cdot, X_2) \to \nabla_{X_2} v \text{ in } L^p(\mathbb{R}^q).
\]

Since

\[
|\nabla_{X_2} (\rho_\varepsilon * v (\cdot, X_2))|_{L^p(\mathbb{R}^q)} \leq |\rho|_{L^{1}(\mathbb{R}^q)} |\nabla_{X_2} v (\cdot, X_2)|_{L^p(\mathbb{R}^q)} = |\nabla_{X_2} v (\cdot, X_2)|_{L^p(\mathbb{R}^q)},
\]

by Lebesgue’s theorem in \( \mathbb{R}^{n-q} \) we derive

\[
\nabla_{X_2} (\rho_\varepsilon * v) \to \nabla_{X_2} v \text{ in } L^p(\mathbb{R}^n).
\] (57)

On the other hand, we have

\[
\partial_{x_i} (\rho_\varepsilon * v) (x) = (\partial_{x_i} \rho_\varepsilon) * v (x)
\]

\[
= \int_{\mathbb{R}^q} \partial_{x_i} \rho_\varepsilon (Y_1) v (X_1 - Y_1, X_2) dY_1
\]

\[
= \frac{1}{\varepsilon^\alpha} \int_{\mathbb{R}^q} \frac{1}{\varepsilon^{q\alpha}} \partial_{x_i} \rho \left( \frac{Y_1}{\varepsilon^\alpha} \right) v (X_1 - Y_1, X_2) dY_1, \text{ for } i = 1, \cdots, q.
\]

Then by the property of the convolution product in \( L^p(\mathbb{R}^q) \) one gets

\[
|\varepsilon \partial_{x_i} (\rho_\varepsilon * v) (\cdot, X_2)|_{L^p(\mathbb{R}^q)} \leq \varepsilon^{1-\alpha} |\partial_{x_i} \rho|_{L^q(\mathbb{R}^q)} |v (\cdot, X_2)|_{L^p(\mathbb{R}^q)}
\]

\[
\leq C \varepsilon^{1-\alpha} |v (\cdot, X_2)|_{L^p(\mathbb{R}^q)}.
\]

Elevating to the power \( p \) and integrating on \( \mathbb{R}^{n-q} \), it comes

\[
|\varepsilon \partial_{x_i} (\rho_\varepsilon * v)|_{L^p(\mathbb{R}^n)} \leq C \varepsilon^{1-\alpha} |v|_{L^p(\mathbb{R}^n)}, \text{ for } i = 1, \cdots, q,
\]

i.e.,

\[
|\varepsilon \nabla_{X_i} (\rho_\varepsilon * v)|_{L^p(\mathbb{R}^n)} \to 0 \text{ as } \varepsilon \to 0.
\]

Thus, taking into account (57), we deduce that

\[
\nabla^\varepsilon (\rho_\varepsilon * v) \to \nabla^0 v \text{ in } L^p(\Omega) \text{ as } \varepsilon \to 0.
\] (58)

Until now, the convenient sequence is not yet defined since \( \rho_\varepsilon * v \) may not belong to \( K_\varepsilon \). Due to Lemma 8 it holds that \( K_{\beta_\varepsilon} \subset L^\infty (\Omega) \) and thus

\[
|\nabla^\varepsilon (\rho_\varepsilon * v)| \leq |\varepsilon (\nabla_{X_1} \rho_\varepsilon) * v| + |\rho_\varepsilon * \nabla_{X_2} v|
\]

\[
\leq C \varepsilon^{1-\alpha} |v|_{\infty} + \rho_\varepsilon * |\nabla_{X_2} v|
\]

\[
\leq C \varepsilon^{1-\alpha} |v|_{\infty} + \rho_\varepsilon * \beta_0
\]

\[
\leq C \varepsilon^{1-\alpha} |v|_{\infty} + |\rho_\varepsilon * \beta_0 - \beta_0| + |\beta_0 - \beta_\varepsilon| + \beta_\varepsilon,
\]

hence

\[
|\nabla^\varepsilon (\rho_\varepsilon * v)| \leq C \varepsilon + \beta_\varepsilon
\] (59)
where $C_\varepsilon \to 0$, as $\varepsilon \to 0$. This implies that (recall that $\beta_\varepsilon \geq \sigma$

$$|\nabla^\varepsilon (\rho_\varepsilon * v)| \leq \frac{C_\varepsilon}{\sigma} \sigma + \beta_\varepsilon \leq (1 + C_\varepsilon/\sigma) \beta_\varepsilon,$$

i.e.

$$v_\varepsilon := \frac{1}{1 + \rho_\varepsilon * v} v \in K_{\beta_\varepsilon}$$

and

$$\nabla^\varepsilon v_\varepsilon \to \nabla^0 v \quad \text{in} \quad L^p (\Omega) \quad \text{as} \quad \varepsilon \to 0.$$ (In the above estimates we used the fact $\rho_\varepsilon * \beta_0, \beta_\varepsilon \to \beta_0$ uniformly in $\mathbb{R}^q \times \omega_2$, $\beta_0 (\cdot, X_2)$ assumed to be extended outside $\omega_1$ for a.e. $X_2 \in \omega_2$). This completes the proof.

4.3 Convergence for some noncylindrical domains

We consider now two classes of domains satisfying some star-shapeness type properties.

4.3.1 Star shaped domains in the $X_2-$direction

We assume that each section $\Omega_{X_1}$ is regular and intersects the hyperplane $X_2 = 0$. The domain $\Omega$ is said to be a star shaped domain in the $X_2-$direction if

$$\Omega_\lambda \subset \Omega, \quad \forall \lambda > 1,$$

where

$$\Omega_\lambda := \{(X_1, X_2) \in \mathbb{R}^n \mid (X_1, \lambda X_2) \in \Omega\}.$$

For $v \in W_0^0 (\Omega)$ (resp. $v \in W_0^{1,p} (\Omega)$), we set

$$v_\lambda (X_1, X_2) := v(X_1, \lambda X_2),$$

and it is clear that $v_\lambda \in W_0^0 (\Omega_\lambda)$ (resp. $v_\lambda \in W_0^{1,p} (\Omega_\lambda)$), for $\lambda > 1$, and so it can be extended by 0 to have $v_\lambda \in W_0^0 (\Omega)$ (resp. $v_\lambda \in W_0^{1,p} (\Omega)$ or $v_\lambda \in W^{1,p} (\mathbb{R}^n)$).

It is also easy to check that

$$\nabla_{X_2} v_\lambda = \lambda (\nabla X_2 v)_\lambda.$$

Due to the mean continuity of the $L^p-$functions, (see Nečas [23, Page 51]), we have

$$\lim_{\lambda \to 1} v_\lambda = v \quad \text{and} \quad \lim_{\lambda \to 1} \nabla_{X_2} v_\lambda = \nabla_{X_2} v \quad \text{in} \quad L^p (\Omega).$$

Moreover, we can easily see that $|v_\lambda|_\infty = |v|_\infty$ and if $v \in K_{\beta_0}$ then

$$I_\lambda v_\lambda \in K_{\beta_0} \quad \text{where} \quad I_\lambda = \frac{\sigma}{\lambda (\sigma + 2 |(\beta_0)_\lambda - \beta_0|_\infty)}. $$
Indeed, without loss of generality we can assume \( \beta_0 \geq \sigma/2 \) and for \( v \in K_{\beta_0} \) we can write

\[
|\nabla X_2 v_\lambda| = \lambda \left| (\nabla X_2 v)_{\lambda} \right| \leq \lambda \left( (\beta_0)_{\lambda} - \beta_0 \right) + \beta_0 \\
\leq \lambda \frac{\left| (\beta_0)_{\lambda} - \beta_0 \right|}{\beta_0} + 1 \leq \beta_0/1, \text{ a.e. in } \Omega_{\lambda}.
\]

So we end up with \( I_\lambda v_\lambda \in K_{\beta_0} \). We have to mention that the function \( I_\lambda v_\lambda \) will play an essential role in the following to show that \( a - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} = K_{\beta_0} \) since

\[
\lim_{\lambda \to 1} \nabla X_2 (I_\lambda v_\lambda) = \nabla X_2 v \quad \text{in } L^p(\Omega).
\]

The last limit is an immediate consequence of \( \lim_{\lambda \to 1} I_\lambda = 1 \) which is fulfilled thanks to the uniform continuity of \( \beta_0 \) on \( \bar{\Omega} \).

On the other hand, if \( \text{dist} (\text{supp} (v), \partial \Omega) = 0 \), we may also have

\[
\text{dist} (\text{supp} (v_\lambda), \partial \Omega) = 0.
\]

The set of points that keeps \( \text{supp} (v_\lambda) \) and \( \partial \Omega \) adhesive, is included in \( \partial \Omega \cap \partial \Omega_{\lambda} \), for \( \lambda \) sufficiently close to 1, see Figure 1. To be more precise, we consider the set \( \mathcal{M} \) defined as follows

\[
(X_1, X_2) \in \mathcal{M} \Rightarrow \exists \lambda_0 (X_1) \geq 1, \text{ such that } (X_1, X_2) \in \partial \Omega_{\lambda}, \forall \lambda \in [1, \lambda_0 (X_1)],
\]

\[
\Rightarrow \exists \lambda_0 (X_1) \geq 1, \text{ such that } (X_1, \lambda X_2) \in \partial \Omega, \forall \lambda \in [1, \lambda_0 (X_1)].
\]

The set \( \mathcal{M} \) is not empty. In fact, since the intersection of every section \( \Omega_{X_1} \) with the hyperplane \( X_2 = 0 \) is not empty, then

\[
\forall (X_1, 0) \in \partial \Omega, \text{ we have } (X_1, 0) \in \partial \Omega_{\lambda}, \forall \lambda \geq 1,
\]

i.e. \( (X_1, 0) \in \mathcal{M} \). Note that (63) implies, for a fixed \( X_1 \), that the whole segment \( \{(X_1, \lambda X_2) | \forall \lambda \in [1, \lambda_0 (X_1)]\} \) is included in \( \partial \Omega \).

Define \( M \) as the projection of \( \mathcal{M} \) on the hyperplane \( X_2 = 0 \) and for \( s > 0 \) we denote by \( M_s \) its \( s - \text{neighborhood} \) (see Figure 4), then we have the following theorem.

**Theorem 7** Assume that (49) and (54) hold. In addition suppose that \( \Omega \) satisfies (60) and

\[
\text{meas} (M_s) \to 0, \quad s \to 0,
\]

then \( K_{\beta_0} \subset a - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \), i.e. \( K_{\beta_0} = a - \lim_{\varepsilon \to 0} K_{\beta_\varepsilon} \).

**Proof** Let \( v \in K_{\beta_\varepsilon} \). As above, we proceed by truncation (which is more delicate in this case) and regularization in the \( X_1 - \text{direction} \) of the function \( v_\lambda \), defined by (63).

1) **Truncation.** The function \( v_\lambda \) is not necessarily supported inside \( \Omega \) because of the points of the set \( \mathcal{M} \). To recover this property and ensure it certainly, we need to avoid this kind of points by considering the truncated function

\[
\hat{v} := v \chi_{\Omega \setminus (M_s \times \mathbb{R}^n)},
\]
for which we have $I_{\lambda}v_{\lambda} \in K_{\beta_0}$, with a support strictly included in $\Omega$. Then under the assumption \((64)\), we have

$$
|\nabla_{X_2} (I_{\lambda}v_{\lambda} - I_{\lambda}\hat{v}_{\lambda})|_{L^p(\Omega)} = I_{\lambda} \left|\nabla_{X_2} v_{\lambda} \right|_{\chi_{\Omega \cap (M_s \times \mathbb{R}^n - q)} L^p(\Omega)} \leq |\beta_0|_{\infty} \left|\nabla_{X_2} \chi_{\Omega \cap (M_s \times \mathbb{R}^n - q)}\right|_{L^p(\Omega)} \leq C |\beta_0|_{\infty} (\text{meas} (M_s))^{\frac{1}{p}} \rightarrow 0 \quad \text{(as } s \rightarrow 0).$

Thanks to Lemma 2, it is sufficient, in the following, to consider functions $v_{\lambda}$ such that $\text{dist} (\text{supp} (v_{\lambda}), \partial \Omega) > 0$.

ii) Regularization in the $X_1$–direction. Let $\rho_e (X_1)$ be the mollifier sequence defined in \((56)\) and for every $X_1 \in \Pi_1$ extend $v_{\lambda}$ by 0 outside $\Omega_{X_1}$. Note that for $\varepsilon$ small enough, it holds that $\rho_\varepsilon * v_{\lambda} \in W^{1,p}_0 (\Omega)$. Since $|I_{\lambda}v_{\lambda}|_{\infty} \leq |v_{\lambda}|_{\infty} = |v|_{\infty}$ then arguing as above, this time with the function $I_{\lambda}v_{\lambda}$, we deduce

$$
\nabla^\varepsilon (\rho_e * I_{\lambda}v_{\lambda}) \rightarrow I_{\lambda} \nabla^0 v_{\lambda} \quad \text{in } L^p (\Omega), \quad \text{as } \varepsilon \rightarrow 0. \quad (65)
$$

The desired sequence is given by

$$
v_{\varepsilon,\lambda} := \frac{1}{1 + C_\varepsilon / \sigma} \rho_\varepsilon * (I_{\lambda}v_{\lambda}) \in W^{1,p}_0 (\Omega), \quad (66)
$$

where $C_\varepsilon$ is still defined by \((59)\). It holds that $|\nabla^\varepsilon v_{\varepsilon,\lambda}| \leq \beta_\varepsilon$ a.e. in $\Omega$, which means that $v_{\varepsilon,\lambda} \in K_{\beta_\varepsilon}$. Due to \((65)\), it follows that

$$
|\nabla^\varepsilon v_{\varepsilon,\lambda} - I_{\lambda} \nabla^0 v_{\lambda}|_{L^p(\Omega)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0
$$

and thus $I_{\lambda}v_{\lambda} \in as - \lim_{\varepsilon \rightarrow 0} K_{\beta_\varepsilon}$. Taking now \((62)\) with Lemma 1 into account, we infer that $v \in as - \lim_{\varepsilon \rightarrow 0} K_{\beta_\varepsilon}$, and the theorem is proved.
4.3.2 Star-shaped domains

We consider now that \( \Omega \) is a star shaped domain, i.e.
\[
\Omega_\lambda \subset \Omega, \quad \forall \lambda > 1,
\]
where this time \( \Omega_\lambda := \{ x \in \mathbb{R}^n \mid \lambda x \in \Omega \} \). The star-shapeness is strict if the above inclusion is strict, i.e. \( \Omega_\lambda \subset \subset \Omega, \forall \lambda > 1 \).

Remark 6 Star-shapeness in \( X_2 \)-direction does not imply star-shapeness (in all directions), or vice versa. This can be easily verified in dimension two (see Figures 1 and 2).

For \( v \in W^0_0(\Omega) \) (resp. \( \in W^{1,p}_0(\Omega_L) \)) and as in the previous case we can check that,
\[
\lim_{\lambda \to 1} I_\lambda v_\lambda = v \quad \text{and} \quad \lim_{\lambda \to 1} \nabla_{X_2} I_\lambda v_\lambda = \nabla_{X_2} v \quad \text{in} \quad L^p(\Omega).
\]
Moreover if \( v \in K_{\beta_0} \), then \( I_\lambda v_\lambda \in K_{\beta_0} \). In this case we have also

Theorem 8 Assume that (49) and (54) holds and that \( \Omega \) is strictly star-shaped, then
\[ K_{\beta_0} \subset \text{as} - \lim_{\epsilon \to 0} K_{\beta_\epsilon}, \quad \text{i.e.} \quad K_{\beta_0} = a - \lim_{\epsilon \to 0} K_{\beta_\epsilon}. \]

Proof We follow the same argument as in the proof of Theorem 7. The truncation step is not needed.

Remark 7 i) It is also possible to assume that \( \Omega \) is a strictly star-shaped domain with respect to another point \( a \) in \( \Omega \) else the origin, i.e. \( \Omega - a \) is a strictly star-shaped domain. Of course assuming this allows to keep the result of the above theorem.

ii) If the star-shapeness is not strict, then the argument of truncation (in the \( X_1 \)-direction), to avoid the points of \( M \) defined above, may not work. In dimension two, this is illustrated in Figure 3 where the right domain is not strictly star-shaped. The assumption (64) cannot hold because \( \text{meas}(M) > 0 \).

5 Problems with convex sets defined on the sections

In this section we suppose that \( \Omega \) is cylindrical namely
\[
\Omega = \omega_1 \times \omega_2,
\]
where \( \omega_1, \omega_2 \) are bounded open subsets in \( \mathbb{R}^p \) and \( \mathbb{R}^{n-p} \) respectively.

5.1 Problems setting

For \( \epsilon > 0 \), let \( K_{\epsilon}(X_1) \) be a family of closed convex sets of \( W^{1,p}_0(\omega_2) \) depending on \( X_1 \). Then we consider two problems having constraints involving this family.
5.1.1 Problem with perturbed operator

We consider the set

\[ G_\varepsilon := \left\{ v \in W^{1,p}_0(\Omega) \mid v(X_1,\cdot) \in K_\varepsilon(X_1), \text{ for a.e. } X_1 \in \omega_1 \right\} \]

for which we have the following assertion.

**Proposition 1** The set \( G_\varepsilon \) is closed in \( W^{1,p}_0(\Omega) \) and convex.

**Proof** It is easy to see that \( G_\varepsilon \) defined above is convex. Let then \( v_n \in G_\varepsilon \) be a converging sequence in \( W^{1,p}_0(\Omega) \). Denote \( v \) its limit. Applying the inverse Lebesgue theorem in \( L^p(\omega_1) \) we deduce that -up to a subsequence-

\[ \int_{\omega_2} |v_n(X_1,\cdot) - v(X_1,\cdot)|^p \, dX_2 \to 0 \text{ a.e. in } \omega_1 \]

and thus \( v_n(X_1,\cdot) \to v(X_1,\cdot) \text{ a.e. in } \omega_2 \) (up to a subsequence). This means that \( v(X_1,\cdot) \in K_\varepsilon(X_1) \), for a.e. \( X_1 \in \omega_1 \), since \( K_\varepsilon(X_1) \) is closed which ends the proof.

By consequence, for \( f \in L^{p'}(\Omega) \), there exists a unique \( u_\varepsilon \) in \( G_\varepsilon \), solution to the following perturbed problem

\[
\begin{align*}
\int_{\Omega} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla v - u_\varepsilon \cdot \nabla v &\geq \int_{\Omega} f(v - u_\varepsilon) \, dx, \\
u_\varepsilon &\in G_\varepsilon.
\end{align*}
\]

(68)

5.1.2 Problem with unperturbed operator

In order to see how the behaviour of \( u_\varepsilon \), solution to (68), looks like as \( \varepsilon \to 0 \), we will consider two problems with unperturbed operators defined as natural limits of the perturbed \( p \)-Laplacian defined in (68). In fact since there is a reduction in the dimension we are, of course, led to consider a limit problem defined on \( \omega_2 \) and...
for technical reasons a possible equivalent one defined on the whole \( \Omega \). For a.e. \( X_1 \) in \( \omega_1 \), let us consider

\[
\begin{cases}
\int_{\omega_2} |\nabla X_2 w (X_1, \cdot)|^{p-2} \nabla X_2 w (X_1, \cdot) \cdot \nabla X_2 (v - w (X_1, \cdot)) \, dX_2 \\
\geq \int_{\omega_2} f (v - w (X_1, \cdot)) \, dX_2, \quad \forall v \in K_\varepsilon (X_1), \\
w (X_1, \cdot) \in K_\varepsilon (X_1).
\end{cases}
\]

(69)

The solution \( w \) of the above problem exists and is unique since the operator is strictly monotone and it can be considered as a function on the whole \( \Omega \). Then set

\[ G_\varepsilon := \{ v \in W_0 (\Omega) \mid v (X_1, \cdot) \in K_\varepsilon (X_1) \text{ for a.e. } X_1 \in \omega_1 \}, \]

where the \( W_0 (\Omega) \) is defined above and can be written here as

\[ W_0 (\Omega) = L^p (\omega_1; W^{1,p}_0 (w_2)). \]

Arguing as in Proposition 1, we can show that the set \( G_\varepsilon \) is closed in \( W_0 (\Omega) \) and convex.

(70)

Thus the second problem, having \( G_\varepsilon \) as constraints sets,

\[
\begin{cases}
\int_\Omega |\nabla X_2 w_\varepsilon |^{p-2} \nabla X_2 w_\varepsilon \cdot \nabla X_2 (v - w_\varepsilon) \, dx \\
\geq \int_\Omega f (v - w_\varepsilon) \, dx, \quad \forall v \in G_\varepsilon, \\
w_\varepsilon \in G_\varepsilon.
\end{cases}
\]

(71)

has a unique solution \( w_\varepsilon \).

Remark 8

i) It is clear that \( G_\varepsilon \subset G_\varepsilon \).

ii) The regularity of \( w \), solution to Problem (69), in \( X_1 \)-direction depends on the regularity of \( f \) and on the convex sets \( G_\varepsilon \).

In the sequel we need the following convexity type lemma.

Lemma 7 Let \( \varphi \) be a smooth function in \( \mathcal{C}^\infty (\omega_1) \) such that

\[ 0 \leq \varphi (X_1) \leq 1, \quad \forall X_1 \in \omega_1. \]

If \( v_1, v_2 \in G_\varepsilon \) then \( \varphi v_1 + (1 - \varphi) v_2 \in G_\varepsilon \).

Proof If \( v_1, v_2 \in G_\varepsilon \) then for a.e. \( X_1, v_1 (X_1, \cdot), v_2 (X_1, \cdot) \in K_\varepsilon (X_1) \) and

\[ \varphi (X_1) v_1 (X_1, \cdot) + (1 - \varphi (X_1)) v_2 (X_1, \cdot) \in K_\varepsilon (X_1). \]

This completes the proof since \( \varphi v_1 + (1 - \varphi) v_2 \in W_0 (\Omega) \).

Proposition 2 Assume, for a.e. \( X_1 \in \omega_1 \), that

the set of restrictions of functions from \( G_\varepsilon \) on \( \Omega_{X_1} \) is dense in \( K_\varepsilon (X_1) \),

(72)

then Problem (69) and Problem (71) have the same unique solution.
Assume that there exists a sequence \( \{ \phi_n \} \) such that for all \( \phi \in G \), \( \phi_n \to \phi \) in \( \mathcal{D}(\omega_1) \), we get

\[
\int_{\omega_1} \phi \int_{\omega_2} |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon \cdot \nabla X_2 (v - w_\varepsilon) \, dX_2 \, dX_1 \geq \int_{\omega_1} \phi \int_{\omega_2} (v - w_\varepsilon) \, dX_2 \, dX_1.
\]

Since \( \phi \geq 0 \) is arbitrary in \( \mathcal{D}(\omega_1) \) we can rewrite the above inequality, for a.e. \( X_1 \), as

\[
\int_{\omega_2} |\nabla X_2 w_\varepsilon|^{p-2} \nabla X_2 w_\varepsilon (X_1, \cdot) \cdot \nabla X_2 (v(X_1, \cdot) - w_\varepsilon (X_1, \cdot)) \, dX_2 \geq \int_{\omega_2} f(X_1, \cdot) (v(X_1, \cdot) - w_\varepsilon (X_1, \cdot)) \, dX_2, \quad \forall v \in G.
\]

By the density assumption (72), this means that \( w_\varepsilon (X_1, \cdot) \) is also a solution to (69) for a.e. \( X_1 \). This ends the proof.

Remark 9 Here is some cases where the assumption (72) holds.

i) The set of restrictions of functions from \( G \) on \( \Omega_{X_1} \) is equal to \( K \) if the convex set \( K (X_1) = K \) is independent of \( X_1 \). To see this, we can consider \( K \) as a subset of \( G \), since for all \( v_2 = v_2 (X_2) \in K \) we have \( v(x) = v_2 (X_2) \in G \).

ii) We get the same conclusion if the family of convex sets \( K (X_1) \) is defined by an obstacle, i.e. \( K (X_1) = K \psi (X_1, \cdot) \) where \( \psi \in \mathcal{W} (\Omega) \), such that \( \psi_\varepsilon \in \mathcal{W}_0 (\Omega) \). Indeed, for a fixed \( X_1 \in \omega_1 \) and any \( v_2 \in K (X_1) \), i.e. \( v_2 \geq \psi (X_1, \cdot) \), we consider the function \( v \in \mathcal{W}_0 (\Omega) \) defined as

\[
v(X_1, X_2) = \max \{ \psi (X_1, X_2), v_2 (X_2) \}.
\]

Clearly \( v(X_1, \cdot) = v_2 \) and \( v \in G \) since \( v(X_1, \cdot) \geq \psi (X_1, \cdot) \) for a.e. \( X_1 \in \omega_1 \).

5.2 Rate of convergence

Let \( \omega'_1 \subseteq \omega_1 \) and denote

\[
\Omega' = \omega'_1 \times \omega_2.
\]

The following theorem ties the behaviour of the solution of problem (68) to the behaviour of the solution of problem (71).

Theorem 9 Assume that there exists a sequence \( v_\varepsilon \in G \) for all \( \varepsilon > 0 \), satisfying (7). Then

\[
\nabla^p u_\varepsilon \text{ and } \nabla X_2 w_\varepsilon \text{ are bounded in } L^p (\Omega)
\]

where \( u_\varepsilon \) (resp. \( w_\varepsilon \)) is the solution of problem (68) (resp. (71)).

Assume in addition that \( w_\varepsilon \in W^{1,p} (\Omega) \), then :
If $p \geq 2$

\[
|u_\varepsilon - w_\varepsilon|_{L^p(\Omega)}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C_\varepsilon \frac{1}{\varepsilon^{m_2 + 1}} \left( \left| \nabla X_1, w_\varepsilon \right|_{L^p(\Omega)} + 1 \right)^{p-2} \left| \nabla X_1, w_\varepsilon \right|_{L^p(\Omega)} + 1 \right)^{\frac{1}{p-1}}.
\]

If $1 < p < 2$

\[
|u_\varepsilon - w_\varepsilon|_{L^p(\Omega)}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C_\varepsilon \varepsilon^{p-1} \left| \nabla X_1, w_\varepsilon \right|_{L^p(\Omega)}^{p-1} \varepsilon^{-p} \left| \nabla X_1, w_\varepsilon \right|_{L^p(\Omega)} + 1 \right)^{\frac{2-p}{p}}.
\]

In particular if $p = 2$

\[
|u_\varepsilon - w_\varepsilon|_{L^2(\Omega)}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^2(\Omega)} \leq C_\varepsilon \left| \nabla X_1, w_\varepsilon \right|_{L^2(\Omega)} + 1 \right).
\]

Proof First, the boundedness of $\nabla^\varepsilon u_\varepsilon$ follows directly from Theorem 1 and the boundedness of $\nabla X_1, w_\varepsilon \varepsilon$ follows also by the same argument used to show Theorem 1.

In the sequel of the proof, we shall use the following smooth cut-off functions depending on $X_1$ and satisfying

\[0 \leq \rho \leq 1, \ \text{supp} (\rho) \subset \omega_1, \rho = 1 \text{ on } \omega_1'.\]

Let $\alpha = \max \{2, p\}$. Then we test Problems (68) and (71) by

\[v = u_\varepsilon + \rho^\alpha (w_\varepsilon - u_\varepsilon) \in G_\varepsilon, \quad v = w_\varepsilon + \rho^\alpha (u_\varepsilon - w_\varepsilon) \in G_\varepsilon \]

respectively we get

\[
\int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (\rho^\alpha (u_\varepsilon - w_\varepsilon)) \, dx \leq \int_\Omega f \rho^\alpha (u_\varepsilon - w_\varepsilon) \, dx,
\]

\[
\int_\Omega |\nabla X_1, w_\varepsilon|^{p-2} \nabla X_1, w_\varepsilon \cdot \nabla X_1 (\rho^\alpha (u_\varepsilon - w_\varepsilon)) \, dx \geq \int_\Omega f \rho^\alpha (u_\varepsilon - w_\varepsilon) \, dx.
\]

Combining the above inequalities yields

\[
\int_\Omega \rho^\alpha |\nabla X_1, w_\varepsilon|^{p-2} \nabla X_1, w_\varepsilon \cdot \nabla X_1 (\rho^\alpha (u_\varepsilon - w_\varepsilon)) \, dx \leq \int_\Omega \rho^\alpha |\nabla X_1, w_\varepsilon|^{p-2} \nabla X_1, w_\varepsilon \cdot \nabla X_1 (u_\varepsilon - w_\varepsilon) \, dx,
\]

then

\[
\int_\Omega \rho^\alpha |\nabla X_1, w_\varepsilon|^{p-2} \nabla X_1, w_\varepsilon \cdot \nabla X_1 (u_\varepsilon - w_\varepsilon) \, dx
\]

\[
\leq \int_\Omega \rho^\alpha |\nabla X_1, w_\varepsilon|^{p-2} \nabla X_1, w_\varepsilon \cdot \nabla X_1 (u_\varepsilon - w_\varepsilon) \, dx
\]

\[
- \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} (u_\varepsilon - w_\varepsilon) |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla X_1, u_\varepsilon \cdot \nabla X_1, \rho \, dx.
\]
Setting

\[ I_\varepsilon := \int_\Omega \rho^\alpha \left\{ |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon - |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon \right\} \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \, dx, \]

then the above inequality can be written as

\[
I_\varepsilon \leq \int_\Omega \rho^\alpha \left\{ |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla^\varepsilon w_\varepsilon - |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla^\varepsilon u_\varepsilon \right\} \cdot \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \, dx \\
- \varepsilon^2 \int_\Omega \rho^\alpha |\nabla^\varepsilon w_\varepsilon|^{p-2} \nabla X_1 w_\varepsilon \cdot \nabla X_1 (u_\varepsilon - w_\varepsilon) \, dx \\
- \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} (u_\varepsilon - w_\varepsilon) |\nabla^\varepsilon u_\varepsilon|^{p-2} \nabla X_1 u_\varepsilon \cdot \nabla X_1 \rho \, dx. \quad (74)
\]

Applying inequality [35] we get

\[
I_\varepsilon \leq C \int_\Omega \rho^\alpha \left\{ |\nabla^\varepsilon w_\varepsilon| + |\nabla X_2 w_\varepsilon| \right\}^{p-2} \left| \nabla^\varepsilon w_\varepsilon - \nabla^\varepsilon u_\varepsilon \right| \left| \nabla^\varepsilon (u_\varepsilon - w_\varepsilon) \right| \, dx \\
+ \varepsilon^2 \int_\Omega \rho^\alpha |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla X_1 w_\varepsilon| |\nabla X_1 (u_\varepsilon - w_\varepsilon)| \, dx \\
+ \alpha \varepsilon^2 \int_\Omega \rho^{\alpha-1} |\nabla X_1 \rho| |u_\varepsilon - w_\varepsilon| |\nabla^\varepsilon u_\varepsilon|^{p-2} |\nabla X_1 u_\varepsilon| \, dx, \quad (75)
\]

and by consequence we derive

\[
I_\varepsilon \leq C \varepsilon \int_\Omega |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla X_1 w_\varepsilon| |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)| \, dx \\
+ C \varepsilon^2 \int_\Omega |\nabla^\varepsilon u_\varepsilon|^{p-2} |\nabla X_1 u_\varepsilon| |\rho (u_\varepsilon - w_\varepsilon)| \, dx. \quad (76)
\]

Then we distinguish two cases according to the values of \( p \).

- If \( p \geq 2 \), then thanks to the Hölder inequality (where \( \frac{p-2}{p} + \frac{1}{p} = 1 \) and \( \frac{p-1}{p} + \frac{1}{p} = 1 \)), it follows that

\[
I_\varepsilon \leq C \varepsilon |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla X_1 w_\varepsilon|_{L^p(\Omega)} |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \\
+ C \varepsilon |\nabla^\varepsilon u_\varepsilon|^{p-1} |\rho (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}.
\]

The \( L^p \)-Poincaré inequality on \( \omega_2 \) in the last term yields

\[
|\rho (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}.
\]

Thus

\[
I_\varepsilon \leq C \varepsilon \left\{ |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla X_1 w_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon u_\varepsilon|^{p-1} |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \right\}. \quad (77)
\]

Due to the uniform monotonicity of the \( p \)-Laplacian [36], we derive since \( \alpha = p \)

\[
C \varepsilon |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}^p \leq I_\varepsilon \\
\leq C \varepsilon \left\{ |\nabla^\varepsilon w_\varepsilon|^{p-2} |\nabla X_1 w_\varepsilon|_{L^p(\Omega)} + |\nabla^\varepsilon u_\varepsilon|^{p-1} |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \right\} |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)}.
\]
Thanks to the H"older inequality (where \( p = \frac{p-1}{p} + \frac{1}{p} = 1 \)), we end up with
\[
|\rho \nabla^\xi (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \\
\leq C\varepsilon^{1/(p-1)} \left\{ \left| \nabla X_1 w_\varepsilon \right|_{L^p(\Omega)}^{p-2} \left| \nabla X_1 u_\varepsilon \right|_{L^p(\Omega)} + \left| \nabla^\xi u_\varepsilon \right|_{L^p(\Omega)}^{p-1} \right\}^{1/(p-1)}
\] (78)
and taking into account \( (73) \), we end up with
\[
|\rho \nabla^\xi (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \\
\leq C\varepsilon^{1/(p-1)} \left\{ \left( \varepsilon \left| \nabla X_1 w_\varepsilon \right|_{L^p(\Omega)} + 1 \right)^{p-2} \left| \nabla X_1 u_\varepsilon \right|_{L^p(\Omega)} + 1 \right\}^{1/(p-1)}.
\]

- If \( 1 < p < 2 \), then the inequalities \( (76) \) and \( |\nabla^\xi \xi|^{p-2} \leq |\nabla X_1 \xi|^{p-2} \) imply
\[
I_\varepsilon \leq C\varepsilon^{p-1} \int_{\Omega} \left| \nabla X_1 w_\varepsilon \right|_{L^p(\Omega)}^{p-1} \left| \rho \nabla^\xi (u_\varepsilon - w_\varepsilon) \right| \, dx \\
+ C\varepsilon \int_{\Omega} \left| \varepsilon \nabla X_1 u_\varepsilon \right|_{L^p(\Omega)}^{p-1} \left| \rho (u_\varepsilon - w_\varepsilon) \right| \, dx.
\]

Thanks to the Hölder inequality (where \( \frac{p-1}{p} + \frac{1}{p} = 1 \)), it follows that
\[
I_\varepsilon \leq C\varepsilon^{p-1} \left\{ \left| \nabla X_1 w_\varepsilon \right|_{L^p(\Omega)}^{p-1} \left| \rho \nabla^\xi (u_\varepsilon - w_\varepsilon) \right|_{L^p(\Omega)} + \right. \\
+ C\varepsilon \left. \left| \varepsilon \nabla X_1 u_\varepsilon \right|_{L^p(\Omega)}^{p-1} \left| \rho (u_\varepsilon - w_\varepsilon) \right|_{L^p(\Omega)} \right\}.
\] (79)

Since \( \alpha = 2 \), we rewrite \( (74) \) for \( \xi = \nabla^\xi u_\varepsilon \) and \( \eta = \nabla^\xi w_\varepsilon \) as
\[
\left| \rho \nabla^\xi (u_\varepsilon - w_\varepsilon) \right|^p \\
\leq C \left\{ \rho^p \left| \nabla^\xi u_\varepsilon \right|^{p-2} \nabla^\xi u_\varepsilon - \left| \nabla^\xi w_\varepsilon \right|^{p-2} \nabla^\xi w_\varepsilon \right\} \cdot \nabla^\xi (u_\varepsilon - w_\varepsilon) \\
\times \left\{ \left| \nabla^\xi u_\varepsilon \right| + \left| \nabla^\xi w_\varepsilon \right| \right\}^{\frac{p(2-p)}{2}}.
\]

Integrating on \( \Omega \) and applying Hölder’s inequality (where \( \frac{2}{p} + \frac{2-p}{2} = 1 \)) in the right side, we get
\[
\left| \rho \nabla^\xi (u_\varepsilon - w_\varepsilon) \right|^{p}_{L^p(\Omega)} \leq C \left( I_\varepsilon \right)^{\frac{p}{2}} \times \left\{ \int_{\Omega} \left( \left| \nabla^\xi u_\varepsilon \right| + \left| \nabla^\xi w_\varepsilon \right| \right)^p \, dx \right\}^{\frac{2-p}{2}}.
\]

Taking into account \( (70) \), it follows that
\[
\left| \rho \nabla^\xi (u_\varepsilon - w_\varepsilon) \right|^{\frac{p}{2}}_{L^p(\Omega)} \\
\leq C\varepsilon^{\frac{p}{2}(p-1)} \left\{ \left| \nabla X_1 w_\varepsilon \right|_{L^p(\Omega)}^{p-1} + \varepsilon^{2-p} \left| \varepsilon \nabla X_1 u_\varepsilon \right|_{L^p(\Omega)}^{p-1} \right\}^{\frac{p}{2}} \\
\times \left\{ \left| \nabla^\xi u_\varepsilon \right|_{L^p(\Omega)}^{p-1} + \left| \nabla^\xi w_\varepsilon \right|_{L^p(\Omega)}^{p-1} \right\}^{\frac{2-p}{2}}.
\]
Assume in addition that

\[ |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C \varepsilon^{p-1} \left\{ |\nabla X_i w_\varepsilon|_{L^p(\Omega)}^{p-1} + \varepsilon^{2-p} |\varepsilon \nabla X_i u_\varepsilon|_{L^p(\Omega)}^{p-1} \right\} \times \left\{ |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)}^p + |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p \right\}^{\frac{(2-p)}{p}}. \]  

(80)

Taking into account (73) and

\[ |\nabla^\varepsilon w_\varepsilon|_{L^p(\Omega)}^p \leq 2^{p-1} \left\{ |\varepsilon \nabla X_i w_\varepsilon|_{L^p(\Omega)}^p + |\nabla X_i w_\varepsilon|_{L^p(\Omega)}^p \right\}, \]

we end up with

\[ |\rho \nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C \varepsilon^{p-1} \left\{ |\nabla X_i w_\varepsilon|_{L^p(\Omega)}^{p-1} + \varepsilon^{2-p} \right\} \times \left\{ |\varepsilon \nabla X_i w_\varepsilon|_{L^p(\Omega)}^p + 1 \right\}^{\frac{(2-p)}{p}}. \]

This ends the proof of the theorem.

The behaviour of \( u_\varepsilon \) depends essentially on the behaviour of \( \nabla X_i w_\varepsilon \). This is more emphasized in the following corollary.

**Corollary 2** Under the assumption of Theorem 9 we have

Assuming that \( \varepsilon \nabla X_i w_\varepsilon \) is bounded in \( L^p(\Omega) \), \( 1 < p < +\infty \), then

\[ |u_\varepsilon - w_\varepsilon|_{L^p(\Omega)}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \] are bounded. \hspace{1cm} (81)

Assuming that \( \nabla X_i w_\varepsilon \) is bounded in \( L^p(\Omega) \) then :

- If \( p \geq 2 \)

\[ |u_\varepsilon - w_\varepsilon|_{L^p(\Omega')}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')} \leq C \varepsilon^{\frac{1}{p-1}}. \]

- If \( 1 < p < 2 \)

\[ |u_\varepsilon - w_\varepsilon|_{L^p(\Omega')}, \ |\nabla^\varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')} \leq C \varepsilon^{p-1}. \]

- In particular if \( p = 2 \) then

\[ |\nabla X_i (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega')} \leq C \] and \( (u_\varepsilon - w_\varepsilon) \to 0 \) in \( H^1(\Omega') \).

**Proof** We need only to show the last weak convergence. If \( p = 2 \) then \( \nabla (u_\varepsilon - w_\varepsilon) \) is bounded in \( L^2(\Omega') \) and we can extract a weakly converging sequence in \( L^2(\Omega') \) that should have the same distributional limit, which is 0 since \( (u_\varepsilon - w_\varepsilon) \to 0 \) in \( L^2(\Omega') \). By the uniqueness of the limit, the whole sequence \( (u_\varepsilon - w_\varepsilon) \to 0 \) in \( H^1(\Omega') \). This ends the proof.

In accordance with Remark 4 ii, we can state the following generalization of Theorem 9.

**Theorem 10** Assume that there exists a sequence \( v_\varepsilon \in G_\varepsilon \) for all \( \varepsilon > 0 \), such that \( |\nabla^\varepsilon v_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma}) \), for some \( \gamma \geq 0 \), then

\[ |\nabla^\varepsilon u_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma}) \] and \( |\nabla X_i w_\varepsilon|_{L^p(\Omega)} = O(\varepsilon^{-\gamma}) \). \hspace{1cm} (82)

Assume in addition that \( w_\varepsilon \in W^{1,p}(\Omega) \), then:
If $p \geq 2$, 
\[
|\nabla \varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C \left\{ \left( |\varepsilon \nabla X_1 w_\varepsilon|_{L^p(\Omega)} + \varepsilon^{-\gamma} \right)^{p-2} |\varepsilon \nabla X_1 w_\varepsilon|_{L^p(\Omega)} + \varepsilon^{1-\gamma(p-1)} \right\}^{\frac{1}{p-1}}. 
\]  
(83)

If $1 < p < 2$, 
\[
|\nabla \varepsilon (u_\varepsilon - w_\varepsilon)|_{L^p(\Omega)} \leq C \left( |\varepsilon \nabla X_1 w_\varepsilon|_{L^p(\Omega)}^{p-1} + \varepsilon^{1-\gamma(p-1)} \right) \left( |\varepsilon \nabla X_1 w_\varepsilon|_{L^p(\Omega)}^p + \varepsilon^{-\gamma(p-1)} \right)^{\frac{p-1}{p}}. 
\]  
(84)

In particular if $p = 2$ then 
\[
|\nabla \varepsilon (u_\varepsilon - w_\varepsilon)|_{L^2(\Omega')} \leq C \left( |\varepsilon \nabla X_1 w_\varepsilon|_{L^2(\Omega')} + \varepsilon^{1-\gamma} \right). 
\]  
(85)

**Proof** The estimation of $\nabla \varepsilon u_\varepsilon$ is already stated in Remark iv and the estimation of $\nabla X_2 w_\varepsilon$ can be obtained by an analogous argument. The inequalities (83) and (84) are direct consequences of (78) and (80) respectively.

**Example.** The above theorem extends earlier results, obtained in [10], to nonlinear problems and variational inequalities. Let us give another example illustrating the above results. Let $\phi_\varepsilon \in W^{1, p}_0(\Omega)$, $\varphi_\varepsilon \in W^{1, p}_0(\omega_1)$ be smooth functions such that 
\[
\phi_\varepsilon^+ \in W^{1, p}_0(\Omega), \quad \varphi_\varepsilon > 0 \text{ on } \Omega \quad \text{and} \quad \psi_\varepsilon = \frac{\phi_\varepsilon}{\varphi_\varepsilon} \in W^{1, p}_0(\Omega).
\]

Then consider the following non empty convex set related to $\varphi_\varepsilon$
\[
G_\varepsilon = G_\psi = \left\{ \varphi_\varepsilon \varphi \in W^{1, p}_0(\Omega) \bigg| \begin{array}{l}
\varphi \in W^{1, p}_0(\omega_2), \\
\varphi_\varepsilon(X_1) \varphi(X_2) \geq \phi_\varepsilon(x) \text{ a.e. } x \in \Omega
\end{array} \right\}.
\]

For $X_1 \in \omega_1$, we define the convex set $K_\varepsilon(X_1)$ as 
\[
K_\varepsilon(X_1) = K_{\psi_\varepsilon}(X_1, \cdot) = \left\{ \varphi_\varepsilon(X_1) \varphi \in W^{1, p}_0(\omega_2) \big| \varphi(X_2) \geq \psi_\varepsilon(X_1, X_2) \text{ a.e. } X_2 \in \omega_2 \right\}.
\]

It is clear that $K_\varepsilon(X_1)$ is not empty since $\psi_\varepsilon^+(X_1, \cdot) = \frac{\phi_\varepsilon^+(X_1, \cdot)}{\psi_\varepsilon(X_1)} \in W^{1, p}_0(\omega_1)$.

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