Asymptotic behaviour of some anisotropic problems

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Abstract

The goal of this paper is to explore the asymptotic behaviour of anisotropic problems governed by operators of the pseudo p-Laplacian type when the size of the domain goes to infinity in different directions.

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1 Basic notation

When Ω is a bounded open set of $\mathbb{R}^n$, we denote by $W_0^{1,r}(\Omega)$, $r > 1$, the usual Sobolev space constructed on $L^r(\Omega)$, of functions vanishing on the boundary of Ω. That is to say we set

$$W^{1,r}(\Omega) = \{v \in L^r(\Omega) \mid \partial_{x_i} v \in L^r(\Omega) \ \mid \forall i = 1, \ldots, n\}. \tag{1.1}$$

We equip this space with the norm

$$||v||_{1,r,\Omega} = \Big(\int_\Omega |v|^r + \sum_{i=1}^n |\partial_{x_i} v|^r dx \Big)^{\frac{1}{r}} \tag{1.2}$$

and we set

$$W_0^{1,r}(\Omega) = \overline{\mathcal{D}(\Omega)} = \text{ the closure of } \mathcal{D}(\Omega) \text{ in } W^{1,r}(\Omega). \tag{1.3}$$

($\mathcal{D}(\Omega)$ denotes the space of C^∞-functions with compact support in Ω). It is well known that $W_0^{1,r}(\Omega)$ is a reflexive Banach space which can be equipped with the equivalent norm

$$\big| |\nabla v| \big|_{r,\Omega} = \Big(\int_\Omega |\nabla v(x)|^r dx \Big)^{\frac{1}{r}}. \tag{1.4}$$

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(∇ denotes the usual gradient an $|\ \ |$ the euclidean norm, i.e. $|\nabla v(x)| = (\sum_1^n (\partial_{x_i} v)^2)^{\frac{1}{2}}$, $|\ \ |_{r,\Omega}$ denotes the L^r-norm on Ω). The dual of $W_0^{1,r}(\Omega)$ is denoted by $W^{-1,r'}(\Omega)$, $r' = \frac{r}{r-1}$ and consists in the distributions of the form

$$f = f_0 - \sum_{i=1}^n \partial_{x_i} f_i, \quad f_i \in L^{r'}(\Omega). \tag{1.5}$$

We use the notation

$$\langle f, v \rangle = \int_\Omega f_0 v + \sum_{i=1}^n f_i \partial_{x_i} v dx. \tag{1.6}$$

The paper is organised as follows. In the next section we address the case of a simple problem set on a rectangle with one side going to infinity. We consider all the possible values of (p, q), $p, q > 1$ for the pseudo (p, q)-operator at hand allowing to present a variety of techniques. Some of them are issued of previous works. We refer the reader to [11], [10], [4], [5], [15], for details. The section 3 relies on the experience acquired on the simple model investigated in section 1 to extend some results to more complex situations. The operators at hand are Euler equations of some anisotropic functionals of calculus of variations introduced for other reasons in [17], see also [18]. For basic notions on Sobolev spaces we refer to [2], [12], [13], [14], [16].

2 A model problem

We denote by Ω_ℓ the open subset of $\mathbb{R}^2$ defined as

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1). \tag{2.1}$$

We will set $\omega = (-1, 1)$ and $\partial\Omega_\ell$ will denote the boundary of Ω_ℓ, see the figure 2.1 below.

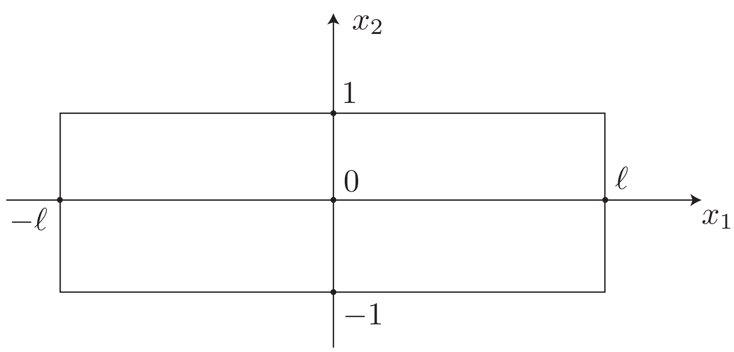

Figure 2.1: The domain Ω_ℓ

If $p, q > 1$ are two positive numbers we would like to consider u_ℓ solution to

$$\begin{cases} -\partial_{x_1}\Big(|\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell\Big) - \partial_{x_2}\Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell\Big) = f \ \text{ in } \Omega_\ell, \\ u_\ell = 0 \ \text{ on } \partial\Omega_\ell. \end{cases} \tag{2.2}$$

More precisely we are interested to the asymptotic behaviour of u_ℓ when $\ell \to +\infty$. f is a function or distribution depending only on x_2. A natural candidate for the limit of the problem is u_∞ solution to

$$\begin{cases} -\partial_{x_2}\Big(|\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\Big) = f \quad \text{in } \omega, \\ u_\infty = 0 \quad \text{on } \partial\omega, \end{cases} \tag{2.3}$$

where $\partial\omega = \{-1, 1\}$ is the boundary of ω. First let us recast these problems under their natural weak form.

We can first introduce the weak formulation of (2.3). If $f \in W^{-1,q'}(\omega)$ is given by

$$f = f(x_2) = f_0(x_2) - \partial_{x_2} f_1(x_2), \tag{2.4}$$

where $f_0, f_1 \in L^{q'}(\omega)$ then, the weak formulation to (2.3) corresponding to f reads

$$\begin{cases} u_\infty \in W_0^{1,q}(\omega), \\ \int_\omega |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\partial_{x_2}v \ dx_2 = \langle f, v\rangle = \int_\omega f_0 v + f_1\partial_{x_2}v dx_2 \quad \forall v \in W_0^{1,q}(\omega). \end{cases} \tag{2.5}$$

To arrive to a weak formulation for (2.2) one introduces

$$W^{1,p,q}(\Omega_\ell) = \{v \in L^p(\Omega_\ell) \cap L^q(\Omega_\ell) \mid \partial_{x_1}v \in L^p(\Omega_\ell),\ \partial_{x_2}v \in L^q(\Omega_\ell)\}. \tag{2.6}$$

It is a reflexive Banach space when equipped with the norm

$$||v||_{1,p,q,\Omega_\ell} = |v|_{p,\Omega_\ell} + |v|_{q,\Omega_\ell} + |\partial_{x_1}v|_{p,\Omega_\ell} + |\partial_{x_2}v|_{q,\Omega_\ell}. \tag{2.7}$$

Then we define

$$W_0^{1,p,q}(\Omega_\ell) = \overline{\mathcal{D}(\Omega_\ell)} = \text{ the closure of } \mathcal{D}(\Omega_\ell) \text{ in } W^{1,p,q}(\Omega_\ell). \tag{2.8}$$

If f is defined by (2.4) if follows easily that there exists a unique u_ℓ weak solution to (2.2) i.e. satisfying

$$\begin{cases} u_\ell \in W_0^{1,p,q}(\Omega_\ell), \\ \int_{\Omega_\ell} |\partial_{x_1}u_\ell|^{p-2}\partial_{x_1}u_\ell\partial_{x_1}v + |\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell\partial_{x_2}v \quad dx_1dx_2 \\ \qquad\qquad = \langle f, v\rangle = \int_{\Omega_\ell} f_0 v + f_1\partial_{x_2}v \ dx_1dx_2 \quad \forall v \in W_0^{1,p,q}(\Omega_\ell). \end{cases} \tag{2.9}$$

We are interested in showing that $u_\ell \to u_\infty$ when $\ell \to \infty$, but also to investigate at what speed. We will now denote dx_1dx_2 by dx.

The operators defined by (2.2), (2.3) are strictly monotone, hemicontinuous, coercive from $W_0^{1,p,q}(\Omega_\ell)$, $W_0^{1,q}(\omega)$ into their duals. Existence and uniqueness of a solution for (2.9), (2.5) follows from classical arguments (see [3], [12], [16]).

Let us first prove the following lemma.

Lemma 2.1. *Suppose that f is given by* (2.4)*. If u_ℓ is the solution to* (2.9) *there exists a constant C independent of ℓ such that*

$$\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \; dx \le C\ell. \tag{2.10}$$

Proof. Taking $v = u_\ell$ in (2.9) we get

$$\begin{aligned} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \; dx = \langle f, u_\ell \rangle &= \int_{\Omega_\ell} f_0 u_\ell + f_1 \partial_{x_2} u_\ell \; dx \\ &\le |f_0|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} + |f_1|_{q',\Omega_\ell} |\partial_{x_2} u_\ell|_{q,\Omega_\ell} \\ &\le \Big(C|f_0|_{q',\Omega_\ell} + |f_1|_{q',\Omega_\ell} \Big) |\partial_{x_2} u_\ell|_{q,\Omega_\ell} \end{aligned} \tag{2.11}$$

this by the Hölder and the Poincaré inequality. Let us recall regarding this last point an argument that we will use several times later on. If $u \in W_0^{1,p,q}(\Omega_\ell)$, let $u_n \in \mathcal{D}(\Omega_\ell)$ such that $u_n \to u$ in $W_0^{1,p,q}(\Omega_\ell)$. By the Poincaré inequality on ω one has for some constant C independent of ℓ

$$\int_\omega |u_n(x_1,x_2)|^q \; dx_2 \le C^q \int_\omega |\partial_{x_2} u_n(x_1,x_2)|^q \; dx_2, \quad \text{a.e. } x_1 \in (-\ell,\ell).$$

Integrating in x_1 we deduce

$$|u_n|_{q,\Omega_\ell} \le C |\partial_{x_2} u_n|_{q,\Omega_\ell}$$

and passing to the limit in n the same inequality holds for u or u_ℓ.

Then let us notice that for $i = 0, 1$ one has

$$|f_i|_{q',\Omega_\ell} = \Big(\int_{-\ell}^{\ell} \int_\omega |f_i(x_2)|^{q'} dx_2 dx_1 \Big)^{\frac{1}{q'}} = (2\ell)^{\frac{1}{q'}} |f_i|_{q',\omega}.$$

Thus from (2.11) we derive for some constant $C = C(q,f)$

$$|\partial_{x_2} u_\ell|^q_{q,\Omega_\ell} \le C \ell^{\frac{1}{q'}} |\partial_{x_2} u_\ell|_{q,\Omega_\ell}$$

Since $q' = \frac{q}{q-1}$ this is equivalent for some new constant to

$$|\partial_{x_2} u_\ell|_{q,\Omega_\ell} \le C \ell^{\frac{1}{q}}.$$

Going back to (2.11), the result follows. □

Somehow one can ignore f thanks to the following remark.

Lemma 2.2. *If u_ℓ is the solution to* (2.9) *and u_∞ solution to* (2.5) *one has*

$$\begin{aligned} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} v + \Big\{ |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big\} \partial_{x_2} v \; dx = 0 \\ \forall v \; \in W_0^{1,p,q}(\Omega_\ell). \end{aligned} \tag{2.12}$$

Proof. First by (2.9) if $v \in W_0^{1,p,q}(\Omega_\ell)$ one has

$$\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \ \partial_{x_1} v + |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell \ \partial_{x_2} v = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v \ dx \tag{2.13}$$

If $v \in W_0^{1,p,q}(\Omega_\ell)$ one has for almost every x_1

$$v(x_1, \cdot) \in W_0^{1,q}(\omega).$$

Thus by (2.5)

$$\int_\omega |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \ \partial_{x_2} v(x_1, x_2) \ dx_2 = \int_\omega f_0 v + f_1 \partial_{x_2} v dx_2.$$

Integrating in x_1 it comes

$$\int_{\Omega_\ell} |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \ \partial_{x_2} v \ dx = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v dx. \tag{2.14}$$

Subtracting from (2.13), (2.12) follows. □

Let us recall the following result (see [3], [6]) which garanties also the strict monotonicity of the operators at hand.

Lemma 2.3. *For any $q > 1$ there exist positive constants c_q, C_q such that*

$$||\xi|^{q-2}\xi - |\eta|^{q-2}\eta| \leq C_q |\xi - \eta| (|\xi| + |\eta|)^{q-2} \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{2.15}$$

$$(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta) \cdot (\xi - \eta) \geq c_q |\xi - \eta|^2 (|\xi| + |\eta|)^{q-2} \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{2.16}$$

Then one has :

Lemma 2.4. *Let $u_\ell = u_\ell(f)$ be the solution to (2.9) and $u_\infty = u_\infty(f)$ be the solution to (2.5). Suppose that $f_1 \geq f_2$, $f \geq 0$ then one has*

$$u_\ell(f_2) \leq u_\ell(f_1) \quad , \quad 0 \leq u_\ell(f) \leq u_\infty(f). \tag{2.17}$$

(If f is not a function, $f \geq 0$ means $\langle f, v \rangle \geq 0 \ \forall v \in W_0^{1,q}(\omega)$, $v \geq 0$).

Proof. We use the notation $u_i = u_\ell(f_i)$. From (2.9) by subtraction we get

$$\int_{\Omega_\ell} \{|\partial_{x_1} u_2|^{p-2} \partial_{x_1} u_2 - |\partial_{x_1} u_1|^{p-2} \partial_{x_1} u_1\} \partial_{x_1} v$$
$$+ \{|\partial_{x_2} u_2|^{q-2} \partial_{x_2} u_2 - |\partial_{x_2} u_1|^{q-2} \partial_{x_1} u_1\} \partial_{x_2} v = \langle f_2 - f_1, v \rangle.$$

Taking $v = (u_2 - u_1)^+$ one deduces easily using the lemma 2.3 that $(u_2 - u_1)^+ = 0$ i.e. $u_1 \geq u_2$ (see below (2.18) for a similar argument).

If $f \geq 0$ taking $f_1 = f$, $f_2 = 0$ one gets $0 \leq u_\ell(f)$.

Regarding u_∞, taking $v = u_\infty^-$ in (2.5) one gets for $f \geq 0$

$$\int_{\Omega_\ell} |\partial_{x_1} u_\infty|^{p-2}\partial_{x_1} u_\infty \ \partial_{x_1} u_\infty^- + |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \ \partial_{x_2} u_\infty^- = \int_{\Omega_\ell} f_0 u_\infty^- + f_1 \partial_{x_2} u_\infty^- \ dx \geq 0.$$

This reads also

$$\int_{\Omega_\ell} |\partial_{x_1} u_\infty^-|^{p-2}\partial_{x_1} u_\infty^- \ \partial_{x_1} u_\infty^- + |\partial_{x_2} u_\infty^-|^{q-2}\partial_{x_2} u_\infty^- \ \partial_{x_2} u_\infty^- \leq 0.$$

Thus $u_\infty^- = 0$ and $u_\infty(f) \geq 0$. Then taking $v = (u_\ell - u_\infty)^+$ in (2.12) one gets

$$\begin{aligned} \int_{\Omega_\ell} & |\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell \ \partial_{x_1}(u_\ell - u_\infty)^+ \\ & + \Big\{ |\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \Big\} \partial_{x_2}(u_\ell - u_\infty)^+ \ dx = 0 \end{aligned} \tag{2.18}$$

i.e.

$$\begin{aligned} \int_{\{u_\ell - u_\infty > 0\}} & |\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell \ \partial_{x_1} u_\ell \\ & + \Big\{ |\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \Big\} \partial_{x_2}(u_\ell - u_\infty) \ dx = 0. \end{aligned}$$

This implies that the set $\{u_\ell - u_\infty > 0\}$ is of measure 0 since $\nabla(u_\ell - u_\infty) = 0$ on this set i.e. $\nabla(u_\ell - u_\infty)^+ = 0$ (see [14]). This completes the proof of the Lemma. □

Let us now show :

Lemma 2.5. *If u_ℓ is the solution to* (2.9) *and u_∞ solution to* (2.5) *one has for every smooth functin $\varphi = \varphi(x_1)$ vanishing at $\{-\ell.\ell\}$*

$$\begin{aligned} \int_{\Omega_\ell} & \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \Big) \ \partial_{x_2}(u_\ell - u_\infty) \Big\} \varphi \ dx \\ & \leq \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-1} |\partial_{x_1}\varphi| |u_\ell - u_\infty| \ dx. \end{aligned} \tag{2.19}$$

Proof. Taking $v = (u_\ell - u_\infty)\varphi$ in (2.12) one gets

$$\begin{aligned} \int_{\Omega_\ell} & \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2}\partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2}\partial_{x_2} u_\infty \Big) \ \partial_{x_2}(u_\ell - u_\infty) \Big\} \varphi \ dx \\ & = - \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2}\partial_{x_1} u_\ell \ \partial_{x_1}\varphi \ (u_\ell - u_\infty) \ dx. \end{aligned} \tag{2.20}$$

(Recall that u_∞ is independent of x_1). Then (2.19) follows easily. □

Denote by $\rho = \rho(x_1)$ a smooth function such that

$$0 \leq \rho \leq 1, \ \ \rho = 1 \text{ on } (-\frac{1}{2}, \frac{1}{2}), \ \ \rho = 0 \text{ near } \{-1, 1\}, \ \ |\partial_{x_1}\rho| \leq C. \tag{2.21}$$

and set

$$\varphi = \rho^\alpha = \rho^\alpha(\frac{x_1}{\ell}),$$

where $\alpha > 0$. We can now prove:

Lemma 2.6. *Let $f = f_0 \in L^{q'}(\omega)$ and u_ℓ, u_∞ be the solutions to* (2.9), (2.5). *Then it holds for some constant C independent of ℓ*

$$I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big) \; \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \; dx \leq \frac{C}{\ell^{p-1}}. \tag{2.22}$$

Proof. From (2.19) one derives

$$\begin{aligned} \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big) \; \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \; dx \\ \leq \frac{\alpha C}{\ell} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-1} |u_\ell - u_\infty| \rho^{\alpha - 1} \; dx. \end{aligned} \tag{2.23}$$

Noting that $\rho^{\alpha-1} = \rho^{\frac{\alpha}{p'}} \rho^{\frac{\alpha}{p} - 1}$ and using Hölder's inequality it comes

$$\begin{aligned} I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big) \; \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \; dx \\ \leq \frac{\alpha C}{\ell} \Big(\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \; dx \Big)^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha - p} \; dx \Big)^{\frac{1}{p}}. \end{aligned} \tag{2.24}$$

Thus it follows that

$$I \leq \Big(\frac{\alpha C}{\ell} \Big)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha - p} \; dx \leq \Big(\frac{\alpha C}{\ell} \Big)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \; dx, \tag{2.25}$$

provided we chose $\alpha > p$. From the lemma 2.4 one has

$$u_\ell(f) \leq u_\ell(f^+) \leq u_\infty(f^+) \quad , \quad u_\infty(-f^-) \leq u_\ell(-f^-) \leq u_\ell(f),$$

(notice that $u_\ell(-f) = -u_\ell(f)$). Then one derives

$$|u_\ell - u_\infty| \leq |u_\ell| + |u_\infty| \leq \max\{u_\infty(f^+), u_\infty(f^-)\} + |u_\infty(f)|.$$

Since this last function is independent of x_1 one derives from (2.25)

$$I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big) \; \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \; dx \leq \frac{C}{\ell^{p-1}}$$

for some new constant C. This is (2.22). This completes the proof of the lemma.

□

We give now a very simple proof of the convergence of u_ℓ toward u_∞ which is valid for every p and q.

Theorem 2.1. *Let $f = f_0 \in L^{q'}(\omega)$ and u_ℓ, u_∞ be the solutions to* (2.9), (2.5). *Then for any ℓ_0 it holds when $\ell \to +\infty$*

$$\partial_{x_1} u_\ell \to 0 \text{ in } L^p(\Omega_{\ell_0}) \quad , \quad \partial_{x_2} u_\ell \to \partial_{x_2} u_\infty \text{ in } L^q(\Omega_{\ell_0}). \tag{2.26}$$

Proof. The first part of (2.26) follows immediately from (2.22) if one chooses $\frac{\ell}{2} > \ell_0$. For the second part let us consider a smooth function $\rho = \rho(x_1)$ such that for $\ell_0 < \ell - 1$ fixed

$$0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } (-\ell_0, \ell_0), \quad \rho \text{ has compact support in } (-\ell_0 - 1, \ell_0 + 1), \ |\partial_{x_1}\rho| \leq C.$$

Since $\rho^\alpha u_\ell \in W_0^{1,p,q}(\Omega_\ell)$ one gets from (2.9)

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \left\{|\partial_{x_1}u_\ell|^p + |\partial_{x_2}u_\ell|^q\right\}\rho^\alpha \, dx &= -\int_{\Omega_{\ell_0+1}} |\partial_{x_1}u_\ell|^{p-2}\partial_{x_1}u_\ell \partial_{x_1}\rho^\alpha u_\ell + \int_{\Omega_{\ell_0+1}} f_0 u_\ell \rho^\alpha \, dx \\ &\leq \alpha C \int_{\Omega_{\ell_0+1}} |\partial_{x_1}u_\ell|^{p-1}|u_\ell|\rho^{\alpha-1} + \int_{\Omega_{\ell_0+1}} |f_0||u_\ell| \, dx.\end{aligned}$$

Noticing that $\rho^{\alpha-1} = \rho^{\frac{\alpha}{p'}}\rho^{\frac{\alpha}{p}-1}$ and using the Young inequality $ab \leq \frac{1}{p'}a^{p'} + \frac{1}{p}b^p$, $a, b > 0$ we get for some new constant C

$$\begin{aligned}&\int_{\Omega_{\ell_0+1}} \left\{|\partial_{x_1}u_\ell|^p + |\partial_{x_2}u_\ell|^q\right\}\rho^\alpha \, dx \\ &\qquad \leq \frac{1}{p'}\int_{\Omega_{\ell_0+1}} |\partial_{x_1}u_\ell|^p\rho^\alpha \, dx + C\int_{\Omega_{\ell_0+1}} |u_\ell|^p\rho^{\alpha-p} \, dx + \int_{\Omega_{\ell_0+1}} |f_0||u_\ell| \, dx.\end{aligned}$$

Using the inequality

$$|u_\ell| \leq \max\left\{u_\infty(f_0^+), u_\infty(f_0^-)\right\}$$

which is due to

$$u_\ell(f_0) \leq u_\ell(f_0^+) \leq u_\infty(f_0^+) \quad , \quad u_\ell(-f) = -u_\ell(f),$$

we derive easily taking $\alpha > p$ that

$$\int_{\Omega_{\ell_0}} |\partial_{x_2}u_\ell|^q \, dx \leq C(\ell_0)$$

where $C(\ell_0)$ is independent of ℓ. Thus up to a subsequence there exists $v_\infty \in L^q(\Omega_{\ell_0})$, $w_\infty \in L^{q'}(\Omega_{\ell_0})$ such that

$$\partial_{x_2}u_\ell \rightharpoonup v_\infty \in L^q(\Omega_{\ell_0}) \quad , \quad |\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell \rightharpoonup w_\infty \in L^{q'}(\Omega_{\ell_0}).$$

From (2.22) one derives that up to a subsequence

$$\left\{|\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell - |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\right\}\partial_{x_2}(u_\ell - u_\infty) \to 0 \text{ a.e in } \Omega_{\ell_0}.$$

Thus, up to a subsequence, $\partial_{x_2}u_\ell \to \partial_{x_2}u_\infty$ a.e on Ω_{ℓ_0}. To see this point one notices that by (2.16) one has

$$\partial_{x_2}(u_\ell - u_\infty) \ \partial_{x_2}(u_\ell - u_\infty)\left\{|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\right\}^{q-2} \to 0 \text{ a.e in } \Omega_{\ell_0}.$$

If $\partial_{x_2}(u_\ell - u_\infty) \not\to 0$ then

$$\partial_{x_2}(u_\ell - u_\infty)\left\{|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\right\}^{q-2} \to 0 \text{ a.e in } \Omega_{\ell_0}$$

and by (2.15) it follows that

$$\left\{|\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell - |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\right\} \to 0 \ \text{ a.e in } \ \Omega_{\ell_0}$$

and again due to the strict monotonicity of the function $|x|^{q-2}x$ one has $\partial_{x_2}u_\ell \to \partial_{x_2}u_\infty$ a.e on Ω_{ℓ_0}

From this it follows (see for instance [19] lemma 8.3) for a proof that

$$\partial_{x_2}u_\ell \rightharpoonup \partial_{x_2}u_\infty \in L^q(\Omega_{\ell_0}) \quad , \quad |\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell \rightharpoonup |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty \in L^{q'}(\Omega_{\ell_0}).$$

Now from (2.22) one has

$$\int_{\Omega_{\ell_0}} \left\{|\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell - |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\right\}\partial_{x_2}(u_\ell - u_\infty) \to 0,$$

that is

$$\int_{\Omega_{\ell_0}} |\partial_{x_2}u_\ell|^q - |\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell\partial_{x_2}u_\infty - |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty\partial_{x_2}u_\ell + |\partial_{x_2}u_\infty|^q \ dx \to 0.$$

It follows that

$$\int_{\Omega_{\ell_0}} |\partial_{x_2}u_\ell|^q \ dx \to \int_{\Omega_{\ell_0}} |\partial_{x_2}u_\infty|^q \ dx$$

and the result, i.e. the strong convergence, follows. □

One can estimate the convergence rate in some situations. Indeed one has :

Theorem 2.2. *Let $f = f_0 \in L^{q'}(\omega)$ and u_ℓ, u_∞ be the solutions to* (2.9), (2.5). *Then it holds for some constant C independent of ℓ*

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1}(u_\ell - u_\infty)|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \ dx = \int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1}u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \ dx \le \frac{C}{\ell^{\frac{p}{2}q\wedge 2-1}} \quad (2.27)$$

where $q \wedge 2$ denotes the minimum of q and 2.

Proof. It follows from the lemmas 2.3 and 2.6 that

$$\tilde{I} = \int_{\Omega_\ell} \left\{|\partial_{x_1}u_\ell|^p + \Big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\Big)^{q-2} \ \partial_{x_2}(u_\ell - u_\infty)^2\right\}\rho^\alpha \ dx \le \frac{C}{\ell^{p-1}}.$$

If $q \ge 2$, since $|\partial_{x_2}(u_\ell - u_\infty)| \le |\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|$, one derives immediately (2.27), i.e.

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1}(u_\ell - u_\infty)|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \ dx = \int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1}u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \ dx \le \frac{C}{\ell^{p-1}}.$$

If $q < 2$ one has thanks to Hölder's inequality

$$\begin{aligned}
\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-q}\ dx = \\
\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^{(q-2)\frac{q}{2}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-q} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^{(2-q)\frac{q}{2}}\ dx \\
\leq \Big(\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^{(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^{(\alpha-q)\frac{2}{q}}\ dx \Big)^{\frac{q}{2}} \\
\Big(\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^q\ dx \Big)^{1-\frac{q}{2}}.
\end{aligned} \tag{2.28}$$

Chosing $(\alpha - q)\frac{2}{q} > \alpha$ and taking into account the lemmas 2.1, 2.3 we get for different constants independent of ℓ

$$\begin{aligned}
\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-q}\ dx \\
\leq \Big(\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^{(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^{\alpha}\ dx \Big)^{\frac{q}{2}} \\
\Big(\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^q\ dx \Big)^{1-\frac{q}{2}} \\
\leq C\tilde{I}^{\frac{q}{2}} \Big(\int_{\Omega_\ell} \big(|\partial_{x_2}u_\ell| + |\partial_{x_2}u_\infty|\big)^q\ dx \Big)^{1-\frac{q}{2}} \\
\leq C\tilde{I}^{\frac{q}{2}} \Big(\int_{\Omega_\ell} |\partial_{x_2}u_\ell|^q + |\partial_{x_2}u_\infty|^q\ dx \Big)^{1-\frac{q}{2}} \leq C\tilde{I}^{\frac{q}{2}} \ell^{1-\frac{q}{2}} \leq \frac{C}{\ell^{(p-1)\frac{q}{2}}} \ell^{1-\frac{q}{2}}.
\end{aligned} \tag{2.29}$$

(Note that $\{|a| + |b|\}^q \leq 2^{q-1}\{|a|^q + |b|^q\}$). Choosing also $\alpha > q$ we are ending up with

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_2}(u_\ell - u_\infty)|^q\ dx \leq \frac{C}{\ell^{\frac{p}{2}q-1}}.$$

Combining this with (2.22) we arrive also to (2.27). □

In the case where $p < q$ one can consider a general f and not only assume that it is in $L^{q'}(\omega_2)$. Indeed one has first :

Lemma 2.7. *Suppose that $p < q$. If ρ^α is defined by (2.21) and if α is chosen such that $\alpha\frac{q}{p} - q > \alpha$ it holds for some constant C*

$$\begin{aligned}
I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1}u_\ell|^p + \Big(|\partial_{x_2}u_\ell|^{q-2}\partial_{x_2}u_\ell - |\partial_{x_2}u_\infty|^{q-2}\partial_{x_2}u_\infty \Big)\ \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha\ dx \\
\leq \frac{C}{\ell^p} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha\ dx \Big)^{\frac{p}{q}} \ell^{1-\frac{p}{q}}.
\end{aligned} \tag{2.30}$$

Proof. Since (2.19) is valid for a general f one derives as in (2.23), (2.24)

$$\begin{aligned} I = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big)\, \partial_{x_2}(u_\ell - u_\infty) \Big\} \rho^\alpha \, dx \\ \leq \frac{\alpha C}{\ell} \Big(\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \, dx \Big)^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \Big)^{\frac{1}{p}} \\ \leq \frac{\alpha C}{\ell}\, I^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \Big)^{\frac{1}{p}}. \end{aligned} \tag{2.31}$$

From this inequality it follows since $\alpha\frac{q}{p} - q > \alpha$ for various constant C

$$\begin{aligned} I &\leq \Big(\frac{\alpha C}{\ell}\Big)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \\ &\leq \Big(\frac{\alpha C}{\ell}\Big)^p \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^{p\frac{q}{p}} \rho^{(\alpha-p)\frac{q}{p}} \, dx \Big)^{\frac{p}{q}} \Big(\int_{\Omega_\ell} 1 \, dx \Big)^{1-\frac{p}{q}} \\ &\leq \frac{C}{\ell^p} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^q \rho^\alpha \, dx \Big)^{\frac{p}{q}} \Big(\int_{\Omega_\ell} 1 \, dx \Big)^{1-\frac{p}{q}} \\ &\leq \frac{C}{\ell^p} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \Big)^{\frac{p}{q}} \ell^{1-\frac{p}{q}}. \end{aligned} \tag{2.32}$$

(In the last inequality we used the Poincaré inequality on ω). This completes the proof of the lemma. □

Then we have :

Theorem 2.3. *Suppose that $p < q$. One has*

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{pq}{q-p}-1}} \tag{2.33}$$

Proof. If $q \geq 2$ one has by (2.16)

$$(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta) \cdot (\xi - \eta) \geq c_q |\xi - \eta|^2 (|\xi| + |\eta|)^{q-2} \geq c_q |\xi - \eta|^q \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{2.34}$$

Thus from (2.30) one deduces for some constant C

$$\begin{aligned} J = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \Big\} \rho^\alpha \, dx \\ \leq \frac{C}{\ell^p} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \Big)^{\frac{p}{q}} \ell^{1-\frac{p}{q}}. \end{aligned}$$

From this it follows that

$$J \leq \frac{C}{\ell^p} J^{\frac{p}{q}} \ell^{1-\frac{p}{q}} \quad \Leftrightarrow \quad J \leq \frac{C}{\ell^{\frac{pq}{q-p}-1}},$$

and (2.33) follows by definition of ρ.

In the case when $p < q < 2$, noting $S = |\partial_{x_2}(u_\ell)| + |\partial_{x_2}(u_\infty)|$ one derives from (2.16), (2.30) for some constant

$$\tilde{I} = \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + S^{q-2} \partial_{x_2}(u_\ell - u_\infty)^2 \Big\} \rho^\alpha \, dx \\ \leq \frac{C}{\ell^p} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \Big)^{\frac{p}{q}} \ell^{1-\frac{p}{q}}. \tag{2.35}$$

Thanks to Hölder's inequality one has as in (2.28)

$$\begin{aligned} \int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx &= \int_{\Omega_\ell} S^{\frac{q}{2}(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha \frac{q}{2}} S^{\frac{q}{2}(2-q)} \rho^{\alpha(1-\frac{q}{2})} \, dx \\ &\leq \Big(\int_{\Omega_\ell} S^{q-2} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^\alpha \, dx \Big)^{\frac{q}{2}} \Big(\int_{\Omega_\ell} S^q \, dx \Big)^{1-\frac{q}{2}} \\ &\leq \tilde{I}^{\frac{q}{2}} \Big(\int_{\Omega_\ell} \{ |\partial_{x_2}(u_\ell)| + |\partial_{x_2}(u_\infty)| \}^q \, dx \Big)^{1-\frac{q}{2}}. \end{aligned}$$

It follows from (2.10) that

$$\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \leq C \tilde{I}^{\frac{q}{2}} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell)|^q + |\partial_{x_2}(u_\infty)|^q \, dx \Big)^{1-\frac{q}{2}} \leq C \tilde{I}^{\frac{q}{2}} \ell^{1-\frac{q}{2}}.$$

Going back to (2.35) we obtain

$$\tilde{I} \leq \frac{C}{\ell^p} (\tilde{I}^{\frac{q}{2}} \ell^{1-\frac{q}{2}})^{\frac{p}{q}} \ell^{1-\frac{p}{q}} = \frac{C}{\ell^p} \tilde{I}^{\frac{p}{2}} \ell^{1-\frac{p}{2}}.$$

Hence

$$\tilde{I} \leq \frac{C}{\ell^{\frac{2p}{2-p}-1}},$$

and

$$\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \leq C \frac{1}{\ell^{\frac{pq}{2-p}-1}}.$$

The inequality (2.32) follows from these two estimates. □

In the case $p \geq q \geq 2$ one can show that $u_\ell \to u_\infty$ exponentially quickly (see [6], [15] and also this issue in the next section). Indeed one has :

Theorem 2.4. *Suppose that $p \geq q \geq 2$, $f \in L^1(\omega)$. It holds for some positive constants C, α*

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq C e^{-\alpha \ell}. \tag{2.36}$$

Proof. Since $f \in L^1(\omega)$ one has

$$-\partial_{x_2}(|\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty) = f \quad \Leftrightarrow \quad |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty = -\int_0^{x_2} f(\xi) \, d\xi + C.$$

This implies that u_∞ is a C^1-function which is bounded as u_ℓ is (see the lemmas 2.4 and 2.6).

Let us set $A = |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q$. For $\sigma > 0$ consider as in [1]

$$\varphi = (e^{-\sigma|x_1|} - e^{-\sigma\ell})$$

in (2.19). Taking into account the lemma 2.3 and the fact that $\partial_{x_1}\varphi = \pm\sigma e^{-\sigma|x_1|}$ we get

$$\begin{aligned} &\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \varphi \, dx \\ &\leq C \int_{\Omega_\ell} \Big\{ |\partial_{x_1} u_\ell|^p + \Big(|\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \Big) \, \partial_{x_2}(u_\ell - u_\infty) \Big\} \varphi \, dx \qquad (2.37) \\ &\leq \sigma C \int_{\Omega_\ell} e^{-\sigma|x_1|} |\partial_{x_1} u_\ell|^{p-1} |u_\ell - u_\infty| \, dx. \end{aligned}$$

Using the Young inequality in this last integral i.e. $|a||b| \leq \frac{1}{p'}|a|^{p'} + \frac{1}{p}|b|^p$ we get for some new constants

$$\begin{aligned} &\int_{\Omega_\ell} A \, (e^{-\sigma|x_1|} - e^{-\sigma\ell}) \, dx \\ &\qquad \leq \sigma C \int_{\Omega_\ell} e^{-\sigma|x_1|} \{ |\partial_{x_1} u_\ell|^p + |u_\ell - u_\infty|^p \} \, dx. \\ &\qquad \leq \sigma C \int_{\Omega_\ell} e^{-\sigma|x_1|} \{ |\partial_{x_1} u_\ell|^p + |u_\ell - u_\infty|^q \} \, dx. \qquad (2.38) \\ &\qquad \leq \sigma C \int_{\Omega_\ell} e^{-\sigma|x_1|} \{ |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \} \, dx = \sigma C \int_{\Omega_\ell} A e^{-\sigma|x_1|} \, dx. \end{aligned}$$

(In the above, we used the fact that u_ℓ and u_∞ are uniformly bounded independently of ℓ and the Poincaré inequality on the section ω). Choosing $\sigma C = \frac{1}{2}$ it comes

$$\frac{1}{2} \int_{\Omega_\ell} A e^{-\sigma|x_1|} \, dx \leq e^{-\sigma\ell} \int_{\Omega_\ell} A \, dx \qquad (2.39)$$

that is to say

$$e^{-\sigma\frac{\ell}{2}} \int_{\Omega_{\frac{\ell}{2}}} A \, dx \leq 2e^{-\sigma\ell} \int_{\Omega_\ell} A \, dx. \qquad (2.40)$$

It follows from the lemma 2.1 that

$$\int_{\Omega_{\frac{\ell}{2}}} A \, dx = \int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq 2e^{-\sigma\frac{\ell}{2}} \int_{\Omega_\ell} A \, dx \leq C\ell e^{-\sigma\frac{\ell}{2}}. \qquad (2.41)$$

The reslt follows by choosing $\alpha < \frac{\sigma}{2}$. □

The last case to address is when $p \geq q$, $q < 2$. In this case one can prove :

Theorem 2.5. *Suppose that $p \geq q$, $q < 2$, $f \in L^1(\omega)$. It holds for some positive constants C*

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_1} u_\ell|^p + |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{pq}{2-q}-1}}. \tag{2.42}$$

Proof. Choosing ρ as in (2.21) one has - see (2.16), (2.23)

$$\begin{aligned}
\tilde{I} = \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + S^{q-2} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^\alpha \, dx \\
\leq \frac{C}{\ell} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-1} |u_\ell - u_\infty| \rho^{\alpha-1} \, dx \\
\leq \frac{C}{\ell} \Big(\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \, dx \Big)^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \Big)^{\frac{1}{p}} \\
\leq \frac{C}{\ell} \tilde{I}^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^q \rho^{\alpha-p} \, dx \Big)^{\frac{1}{p}} \leq \frac{C}{\ell} \tilde{I}^{\frac{1}{p'}} \Big(\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-p} \, dx \Big)^{\frac{1}{p}}.
\end{aligned} \tag{2.43}$$

Recall that $S = |\partial_{x_2}(u_\ell)| + |\partial_{x_2}(u_\infty)|$. In the two last inequalities we used the fact that u_ℓ and u_∞ are uniformly bounded and the Poincaré inequality. Arguing as before we have

$$\begin{aligned}
\int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-p} \, dx &= \int_{\Omega_\ell} S^{(q-2)\frac{q}{2}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha\frac{q}{2}} S^{(2-q)\frac{q}{2}} \rho^{\alpha(1-\frac{q}{2})-p} \, dx \\
&\leq \Big(\int_{\Omega_\ell} S^{(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^\alpha \Big)^{\frac{q}{2}} \Big(\int_{\Omega_\ell} S^q \, dx \Big)^{1-\frac{q}{2}} \\
&\leq C \tilde{I}^{\frac{q}{2}} \ell^{1-\frac{q}{2}}.
\end{aligned} \tag{2.44}$$

provided $\alpha(1 - \frac{q}{2}) - p > 0$. Thus from (2.43) we derive

$$\tilde{I} \leq \frac{C}{\ell} \tilde{I}^{\frac{1}{p'}} (\tilde{I}^{\frac{q}{2}} \ell^{1-\frac{q}{2}})^{\frac{1}{p}} \quad \Leftrightarrow \quad \tilde{I}^{\frac{1}{p}-\frac{q}{2p}} \leq \frac{C}{\ell} \ell^{\frac{1}{p}-\frac{q}{2p}} \quad \Leftrightarrow \quad \tilde{I} \leq \frac{C}{\ell^{\frac{2p}{2-q}-1}}. \tag{2.45}$$

Going back to (2.44) one has if $\alpha > p$

$$\int_{\Omega_{\frac{\ell}{2}}} |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \int_{\Omega_\ell} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha-p} \, dx \leq \frac{C}{(\ell^{\frac{2p}{2-q}-1})^{\frac{q}{2}}} \ell^{1-\frac{q}{2}} = \frac{C}{\ell^{\frac{pq}{2-q}-1}}.$$

Combining this with (2.45) leads easily to (2.42) since $pq < 2p$. This completes the proof. □

3 Some generalisations

Let us denote by ω_1 a bounded convex domain of $\mathbb{R}^m$ containing 0 and by ω_2 a bounded domain in $\mathbb{R}^{n-m}, m \geq 1$. Let us set for $\ell > 0$

$$\Omega_\ell = \ell\omega_1 \times \omega_2.$$

We will denote the points in Ω_ℓ by

$$x = (X_1, X_2)$$

where $X_1 = x_1, \dots, x_m$, $X_2 = x_{m+1}, \dots, x_n$. If $p_i, i = 1, \dots, m$, $q_j, j = m+1, \dots, n$ are numbers larger than 1 set

$$\vec{p} = (p_1, \dots, p_m) \quad , \quad \vec{q} = (q_{m+1}, \dots, q_n).$$

Then we define

$$\begin{gathered} W^{1,\vec{p},\vec{q}}(\Omega_\ell) = \{v \in L^{p_i}(\Omega_\ell) \cap L^{q_j}(\Omega_\ell) \mid \partial_{x_i} v \in L^{p_i}(\Omega_\ell),\ \partial_{x_j} v \in L^{q_j}(\Omega_\ell),\ \forall i, j\}, \\ W^{1,\vec{q}}(\omega_2) = \{v \in L^{q_j}(\omega_2) \mid \partial_{x_j} v \in L^{q_j}(\omega_2),\ \forall j\}. \end{gathered} \tag{3.1}$$

In the definition above the indices i are running from 1 to m and the indices j from $m+1$ to n. Clearly $W^{1,\vec{p},\vec{q}}(\Omega_\ell)$, $W^{1,\vec{q}}(\omega_2)$ are reflexive Banach spaces when equipped with the norms

$$\begin{gathered} ||v||_{1,\vec{p},\vec{q}} = \sum_{i=1}^{m} \big(|v|_{p_i,\Omega_\ell} + |\partial_{x_i} v|_{p_i,\Omega_\ell}\big) + \sum_{j=m+1}^{n} \big(|v|_{q_j,\Omega_\ell} + |\partial_{x_j} v|_{q_j,\Omega_\ell}\big) \\ ||v||_{1,\vec{q}} = \sum_{j=m+1}^{n} \big(|v|_{q_j,\Omega_\ell} + |\partial_{x_j} v|_{q_j,\Omega_\ell}\big). \end{gathered} \tag{3.2}$$

One denotes by $W_0^{1,\vec{p},\vec{q}}(\Omega_\ell)$ (respectively $W_0^{1,\vec{q}}(\omega_2)$) the closure of $\mathcal{D}(\Omega_\ell)$ (respectively $\mathcal{D}(\omega_2)$) in these spaces and by u_ℓ the solution to

$$\begin{cases} u_\ell \in W_0^{1,\vec{p},\vec{q}}(\Omega_\ell), \\ \int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i-2} \partial_{x_i} u_\ell \ \partial_{x_i} v + |\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell \ \partial_{x_j} v \ dx = \langle f, v \rangle \quad \forall v \in W_0^{1,\vec{p},\vec{q}}(\Omega_\ell). \end{cases} \tag{3.3}$$

In the formula above we make the summation convention, i.e. we are summing in i and j. $f = f(X_2)$ is a continuous linear form on $W_0^{1,\vec{q}}(\omega_2)$ defined as

$$\begin{gathered} f = f_0 - \sum_{j=m+1}^{n} \partial_{x_j} f_j, \quad f_0 \in \cap_j L^{q_j'}(\omega_2), \quad f_j \in L^{q_j'}(\omega_2), \\ \langle f, v \rangle = \sum_{\omega_2} f_0 v + \sum_{j=m+1}^{n} f_j \partial_{x_j} v \ dx. \end{gathered} \tag{3.4}$$

We would like to sketch some behaviour of u_ℓ when $\ell \to \infty$, in particular to show that $u_\ell \to u_\infty$ where u_∞ is the solution to

$$\begin{cases} u_\infty \in W_0^{1,\vec{q}}(\omega_2), \\ \int_{\omega_2} |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \ \partial_{x_j} v \ dX_2 = \langle f, v \rangle \quad \forall v \in W_0^{1,\vec{q}}(\omega_2). \end{cases} \tag{3.5}$$

Note that by the same arguments as in section 2 the problems (3.3), (3.5) admit a unique solution.

The analogue of lemma 2.1 is the following.

Lemma 3.1. *Let u_ℓ be the solution of* (3.3) *for f given by* (3.4). *There exists a constant C independent of ℓ such that*

$$\int_{\Omega_\ell} \sum_{i=1}^{m} |\partial_{x_i} u_\ell|^{p_i} + \sum_{j=m+1}^{n} |\partial_{x_j} u_\ell|^{q_j} \ dx \leq C\ell^m. \tag{3.6}$$

Proof. Let $q = max(q_j) = q_{j_0}$ for some j_0. Taking $v = u_\ell$ in (3.3) we get with the summation convention in i, j

$$\begin{aligned}\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx &= \int_{\Omega_\ell} \Big\{ f_0 u_\ell + f_{j_0} \partial_{x_{j_0}} u_\ell + \sum_{j\neq j_0} f_j \partial_{x_j} u_\ell \Big\} \ dx \\ &\leq \int_{\Omega_\ell} \Big\{ |f_0||u_\ell| + |f_{j_0}||\partial_{x_{j_0}} u_\ell| + \sum_{j\neq j_0} |f_j||\partial_{x_j} u_\ell| \Big\} \ dx \\ &\leq |f_0|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} + |f_{j_0}|_{q'_{j_0},\Omega_\ell} |\partial_{x_{j_0}} u_\ell|_{q_{j_0},\Omega_\ell} + \sum_{j\neq j_0} |f_j|q'_j, \Omega_\ell |\partial_{x_j} u_\ell| q_j, \Omega_\ell.\end{aligned} \tag{3.7}$$

Using the Poincaré inequality

$$|u_\ell|_{q,\Omega_\ell} = |u_\ell|_{q_{j_0},\Omega_\ell} \leq C |\partial_{x_{j_0}} u_\ell|_{q_{j_0},\Omega_\ell}$$

we derive

$$\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx \leq \{C|f_0|_{q'_{j_0},\Omega_\ell} + |f_{j_0}|_{q'_{j_0},\Omega_\ell}\} |\partial_{x_{j_0}} u_\ell|_{q_{j_0},\Omega_\ell} + \sum_{j\neq j_0} |f_j|_{q'_j,\Omega_\ell} |\partial_{x_j} u_\ell|_{q_j,\Omega_\ell}.$$

Note now that for $f \in L^{q'}(\omega_2)$ one has for some constant C independent of ℓ

$$|f|_{q',\Omega_\ell} = \Big(\int_{\ell\omega_1} \int_{\omega_2} |f(X_2)|^{q'} dX_2 dX_1 \Big)^{\frac{1}{q'}} \leq C \ell^{\frac{m}{q'}}. \tag{3.8}$$

Thus we get

$$\begin{aligned}\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx &\leq C \sum_j \ell^{\frac{m}{q'_j}} |\partial_{x_j} u_\ell|_{q_j,\Omega_\ell} \\ &\leq \epsilon \sum_j |\partial_{x_j} u_\ell|^{q_j}_{q_j,\Omega_\ell} + C_\epsilon \ell^m,\end{aligned} \tag{3.9}$$

using the Young inequality $|ab| \leq \epsilon |a|^q + C_\epsilon |b|^{q'}$. The result follows by choosing $\epsilon = \frac{1}{2}$. □

With the same proofs we have the analogues of Lemmas 2.2 and 2.4 namely with the summation convention

$$\begin{aligned}\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i - 2} \partial_{x_i} u_\ell \partial_{x_i} v + \Big\{ |\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \Big\} \partial_{x_j} v \ dx = 0 \\ \forall v \ \in W_0^{1,\vec{p},\vec{q}}(\Omega_\ell).\end{aligned} \tag{3.10}$$

Similarly if $u_\ell = u_\ell(f)$ is the solution to (3.3) and $u_\infty = u_\infty(f)$ the solution to (3.5) and if $f_1 \geq f_2$, $f \geq 0$ then one has

$$u_\ell(f_2) \leq u_\ell(f_1) \ \ , \ \ 0 \leq u_\ell(f) \leq u_\infty(f). \tag{3.11}$$

Remark 1. *Note that* (3.10) *allows a perhaps simpler proof of* (3.6) *where, however, the dependence in f is lost. Indeed taking $v = u_\ell$ in* (3.10) *we get with the summation convention in i and j*

$$\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx = \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \partial_{x_j} u_\ell \ dx$$
$$\leq \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j-1} |\partial_{x_j} u_\ell| \ dx.$$

Using the Young inequality $|ab| \leq \frac{1}{q'_j}|a|^{q'_j} + \frac{1}{q_j}|b|^{q_j}$ it comes

$$\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx \leq \int_{\Omega_\ell} \frac{1}{q'_j} |\partial_{x_j} u_\infty|^{q_j} + \frac{1}{q_j} |\partial_{x_j} u_\ell|^{q_j} \ dx$$
$$\leq \int_{\Omega_\ell} \frac{1}{\min_j q_j} |\partial_{x_j} u_\ell|^{q_j} + \frac{1}{\min_j q'_j} |\partial_{x_j} u_\infty|^{q_j} \ dx$$

and thus for some constant C since $\min_j q_j > 1$

$$\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \ dx \leq C \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j} \ dx \leq C\ell^m.$$

Then we can turn to the generalisation of lemma 2.6. Denote by $\rho = \rho(X_1)$ a smooth function such that

$$0 \leq \rho \leq 1, \ \rho = 1 \text{ on } \frac{1}{2}\omega_1, \ \rho = 0 \text{ near } \partial\omega_1, \quad |\nabla_{X_1}\rho| \leq C, \tag{3.12}$$

where $\nabla_{X_1}\rho$ denotes the gradient of ρ in X_1, i.e. $\nabla_{X_1}\rho = (\partial_{x_1}\rho, \dots, \partial_{x_m}\rho)$.

We can show :

Lemma 3.2. *Let $f = f_0 \in L^{q'}(\omega_2)$, $q = \max q_j$ and u_ℓ, u_∞ be the solutions to* (3.3), (3.5). *Then it holds for some constant C independent of ℓ*

$$I = \int_{\Omega_\ell} \Big\{ |\partial_{x_i} u_\ell|^{p_i} + \Big(|\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \Big) \ \partial_{x_j}(u_\ell - u_\infty) \Big\} \rho^\alpha \ dx$$
$$\leq \frac{C}{\ell^{p-m}}. \tag{3.13}$$

($p = \min_i(p_i)$ denotes the smallest p_i)

Proof. From (3.10) taking $v = \rho^\alpha(\frac{X_1}{\ell})(u_\ell - u_\infty)$ one derives easily with the summation convention in i and j

$$\int_{\Omega_\ell} \Big\{ |\partial_{x_i} u_\ell|^{p_i} + \Big(|\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \Big) \ \partial_{x_j}(u_\ell - u_\infty) \Big\} \rho^\alpha \ dx$$
$$\leq \frac{\alpha C}{\ell} \int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i-1} |u_\ell - u_\infty| \rho^{\alpha-1} \ dx. \tag{3.14}$$

Noting that $\rho^{\alpha-1} = \rho^{\frac{\alpha}{p_i'}}\rho^{\frac{\alpha}{p_i}-1}$ and using Young's inequality it comes

$$I = \int_{\Omega_\ell} \Big\{|\partial_{x_i} u_\ell|^{p_i} + \Big(|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty\Big)\ \partial_{x_j}(u_\ell - u_\infty)\Big\}\rho^\alpha\ dx$$
$$\leq \frac{1}{p_i'}\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i}\rho^\alpha\ dx + \frac{C}{\ell^{p_i}}\int_{\Omega_\ell} |u_\ell - u_\infty|^{p_i}\rho^{\alpha-p_i}\ dx. \tag{3.15}$$

Recalling our summation in i and the fact that $p_i' > 1$ it follows that for some constant C

$$I \leq C\sum_i \frac{1}{\ell^{p_i}}\int_{\Omega_\ell} |u_\ell - u_\infty|^{p_i}\rho^{\alpha-p_i}\ dx \leq C\sum_i \frac{1}{\ell^{p_i}}\int_{\Omega_\ell} |u_\ell - u_\infty|^{p_i}\ dx, \tag{3.16}$$

provided we chose α large enough. Note that at this point we did not use the assumption $f = f_0 \in L^{q'}(\omega_2)$, $q = \max q_j$.

Arguing now like in Lemma 2.6 one can bound $|u_\ell - u_\infty|$ by something depending only on X_2 to get

$$I \leq C\sum_i \frac{1}{\ell^{p_i - m}}, \tag{3.17}$$

This completes the proof of the lemma. □

The convergence of u_ℓ toward u_∞ is insured for general p_i, q_j by the following result.

Theorem 3.1. *Let $f = f_0 \in L^{q'}(\omega_2)$, $q = \max q_j$. Let u_ℓ, u_∞ be the solutions to* (3.3), (3.5) *respectively. If $p_i > m$ $\forall i$ one has for every $\ell_0 > 0$ when $\ell \to +\infty$*

$$\partial_{x_i} u_\ell \to 0 \text{ in } L^{p_i}(\Omega_{\ell_0}) \quad , \quad \partial_{x_j} u_\ell \to \partial_{x_j} u_\infty \text{ in } L^{q_j}(\Omega_{\ell_0}). \tag{3.18}$$

Proof. The first part of (3.18) follows directly from (3.13). For the second part let us consider a smooth function ρ such that for $\ell_0 < \ell - 1$

$$0 \leq \rho \leq 1 \quad , \quad \rho = 1 \text{ on } \ell_0\omega_1 \quad , \quad \rho \text{ has a support in } (\ell_0+1)\omega_1 \quad , \quad |\nabla_{X_1}\rho| \leq C.$$

Then $\rho^\alpha u_\ell$ is a test function for (3.3) and one has

$$\int_{\Omega_{\ell_0+1}} \Big\{|\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j}\Big\}\rho^\alpha\ dx$$
$$= -\alpha\int_{\Omega_{\ell_0+1}} |\partial_{x_i} u_\ell|^{p_i-2}\partial_{x_i} u_\ell \partial_{x_i}\rho\ u_\ell\rho^{\alpha-1}\ dx + \int_{\Omega_{\ell_0+1}} f_0 u_\ell\ dx$$
$$\leq \alpha C\int_{\Omega_{\ell_0+1}} |\partial_{x_i} u_\ell|^{p_i-1}|u_\ell|\rho^{\alpha-1}\ dx + \int_{\Omega_{\ell_0+1}} |f_0||u_\ell|\ dx.$$

Using the fact that $\rho^{\alpha-1} = \rho^{\frac{1}{p_i'}}\rho^{\frac{1}{p_i}-1}$ we get by Young's inequality for some constant C

$$\int_{\Omega_{\ell_0+1}} \Big\{|\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j}\Big\}\rho^\alpha\ dx$$
$$\leq \frac{1}{p_i'}\int_{\Omega_{\ell_0+1}} |\partial_{x_i} u_\ell|^{p_i}\rho^\alpha\ dx + C\int_{\Omega_{\ell_0+1}} |u_\ell|^{p_i}\rho^{\alpha-p_i}\ dx + \int_{\Omega_{\ell_0+1}} |f_0||u_\ell|\ dx.$$

We assume here that $\alpha > p_i$. We know that by (3.11)

$$|u_\ell| \le \max\{u_\infty(f_0^+), u_\infty(f_0^-)\}$$

and since this bound is independent of ℓ we get

$$|\partial_{x_j} u_\ell|_{q_j,\Omega_{\ell_0}} \le C(\ell_0).$$

The rest of the proof follows as in Theorem 2.1 since by (3.13) we have for every j

$$\int_{\Omega_{\ell_0}} \left(|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty\right) \partial_{x_j}(u_\ell - u_\infty)\; dx. \to 0.$$

This completes the proof of the theorem. □

We have then an analogue to theorem 2.3 for a general f.

Theorem 3.2. *Let u_ℓ, u_∞ be the solutions to* (3.3), (3.5). *Suppose that*

$$\forall i = 1, \cdots, m, \quad \exists j_i \in \{m+1, \dots, n\} \quad \text{such that} \quad p_i < q_{j_i}. \tag{3.19}$$

Then there exists a constant C such that

$$\begin{aligned} I = \int_{\Omega_\ell} \Big\{ |\partial_{x_i} u_\ell|^{p_i} + \left(|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty\right) \partial_{x_j}(u_\ell - u_\infty)\Big\}\rho^\alpha\; dx \\ \le C \sum_{q_{j_i} \ge 2} \frac{1}{\ell^{\frac{p_i q_{j_i}}{q_{j_i}-p_i} - m}} + C \sum_{q_{j_i} < 2} \frac{1}{\ell^{\frac{2p_i}{2-p_i} - m}}. \end{aligned} \tag{3.20}$$

$\rho^\alpha = \rho^\alpha(\frac{X_1}{\ell})$ *is as in Lemma 3.2.*

Proof. Going back to (3.16) one has if $(\alpha - p_i)\frac{q_{j_i}}{p_i} > \alpha \;\; \forall i$

$$\begin{aligned} I &\le C \sum_i \frac{1}{\ell^{p_i}} \int_{\Omega_\ell} |u_\ell - u_\infty|^{p_i} \rho^{\alpha - p_i}\; dx \\ &\le C \sum_i \frac{1}{\ell^{p_i}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^{q_{j_i}} \rho^{(\alpha-p_i)\frac{q_{j_i}}{p_i}}\; dx\Big)^{\frac{p_i}{q_{j_i}}} \Big(\int_{\Omega_\ell} 1\; dx\Big)^{1-\frac{p_i}{q_{j_i}}} \\ &\le C \sum_i \frac{1}{\ell^{p_i}} \Big(\int_{\Omega_\ell} |u_\ell - u_\infty|^{q_{j_i}} \rho^{\alpha}\; dx\Big)^{\frac{p_i}{q_{j_i}}} \ell^{m(1-\frac{p_i}{q_{j_i}})}. \end{aligned}$$

Using the Poincaré inequality we get

$$I \le C \sum_i \frac{1}{\ell^{p_i}} \Big(\int_{\Omega_\ell} |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{q_{j_i}} \rho^{\alpha}\; dx\Big)^{\frac{p_i}{q_{j_i}}} \ell^{m(1-\frac{p_i}{q_{j_i}})}. \tag{3.21}$$

If $q_{j_i} \ge 2$ one has

$$\{|\partial_{x_{j_i}} u_\ell|^{q_{j_i}-2}\partial_{x_{j_i}} u_\ell - |\partial_{x_{j_i}} u_\infty|^{q_{j_i}-2}\partial_{x_{j_i}} u_\infty\}\partial_{x_{j_i}}(u_\ell - u_\infty) \ge C_{q_{j_i}} |\partial_{x_{j_i}}(u_\ell - u_\infty|^{q_{j_i}} \tag{3.22}$$

and thus for some constant

$$\int_{\Omega_\ell} |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{q_{j_i}} \rho^\alpha \; dx \le CI. \tag{3.23}$$

If $p_i < q_{j_i} < 2$ one has (see the theorem 2.3)

$$\begin{aligned} &\int_{\Omega_\ell} |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{q_{j_i}} \rho^\alpha \; dx \\ &\qquad \le CI^{\frac{q_{j_i}}{2}} \Big(\int_{\Omega_\ell} |\partial_{x_{j_i}} u_\ell|^{q_{j_i}} + |\partial_{x_{j_i}} u_\infty)|^{q_{j_i}} \; dx \Big)^{1-\frac{q_{j_i}}{2}} \\ &\le CI^{\frac{q_{j_i}}{2}} \ell^{m(1-\frac{q_{j_i}}{2})}. \end{aligned} \tag{3.24}$$

Thus from (3.21) we derive replacing p_i by p and q_{j_i} by q

$$\begin{aligned} I &\le C \sum_i \frac{1}{\ell^p} I^{\frac{p}{q}} \ell^{m(1-\frac{p}{q})} + C \sum_i \frac{1}{\ell^p} I^{\frac{q}{2}\frac{p}{q}} \ell^{m(1-\frac{q}{2})\frac{p}{q}} \ell^{m(1-\frac{p}{q})} \\ &\le C \sum_i \frac{1}{\ell^p} I^{\frac{p}{q}} \ell^{m(1-\frac{p}{q})} + C \sum_i \frac{1}{\ell^p} I^{\frac{p}{2}} \ell^{m(1-\frac{p}{2})}. \end{aligned} \tag{3.25}$$

The first sum is for i such that $q_{j_i} \ge 2$, the second one for the i's such that $q_{j_i} < 2$. Using the Young inequality with ϵ we get

$$I \le \epsilon I + C_\epsilon \sum_i \Big(\frac{\ell^{m(1-\frac{p}{q})}}{\ell^p} \Big)^{\frac{1}{1-\frac{p}{q}}} + \epsilon I + C_\epsilon \sum_i \Big(\frac{\ell^{m(1-\frac{p}{2})}}{\ell^p} \Big)^{\frac{1}{1-\frac{p}{2}}}. \tag{3.26}$$

Choosing ϵ small enough we get

$$I \le C_\epsilon \sum_i \frac{1}{\ell^{\frac{pq}{q-p}-m}} + C_\epsilon \sum_i \frac{1}{\ell^{\frac{2p}{2-p}-m}}. \tag{3.27}$$

Coming back to our notation in p_i, q_{j_i} (3.19) follows. This completes the proof of he theorem. □

Theorem 3.3. *We suppose that $p_i \ge 2$, $\forall i = 1, \cdots, m$. In addition we assume that*

$$\forall i = 1, \cdots, m, \quad \exists j_i \in \{m+1, \ldots, n\} \quad \textit{such that} \quad p_i = q_{j_i}. \tag{3.28}$$

Then there exists constants C, α independent of ℓ such that

$$\begin{aligned} &\int_{\Omega_{\frac{\ell}{2}}} \sum_{i=1}^m |\partial_{x_i}(u_\ell - u_\infty)|^{p_i} + |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{p_i} \\ &\qquad + \sum_{j \ne j_i} \{ |\partial_{x_j} u_\ell|^{q_j - 2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \} \partial_{x_j}(u_\ell - u_\infty) \; dx \le Ce^{-\alpha \ell}. \end{aligned} \tag{3.29}$$

Note that when $j \ne j_i$, q_j is only assumed to be greater than 1 and that the q_{j_i} are not necessarily distinct as the p_i.

Proof. For $\ell_1 \le \ell - 1$ we denote by $\rho_{\ell_1} = \rho_{\ell_1}(X_1)$ a smooth function satisfying

$$0 \le \rho_{\ell_1} \le 1 \quad , \quad \rho_{\ell_1} = 1 \text{ on } \ell_1\omega_1 \quad , \quad \rho_{\ell_1} = 0 \text{ outside } (\ell_1+1)\omega_1 \quad , \quad |\nabla_{X_1}\rho_{\ell_1}| \le C \tag{3.30}$$

where C is some positive constant. Taking $v = \rho_{\ell_1}(u_\ell - u_\infty)$ as test function in (3.10) we get

$$\begin{aligned}
&\int_{\Omega_{\ell_1+1}} \Big\{ \sum_{i=1}^m |\partial_{x_i} u_\ell|^{p_i} \\
&\qquad + \sum_{j=m+1}^n \{|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty\}\partial_{x_j}(u_\ell - u_\infty)\Big\}\rho_{\ell_1}\, dx \\
&\qquad = -\int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} \sum_{i=1}^m |\partial_{x_i} u_\ell|^{p_i-2}\partial_{x_i} u_\ell \partial_{x_i}\rho_{\ell_1}(u_\ell - u_\infty)\, dx \\
&\qquad \le C\sum_{i=1}^m \Big(\int_{D_{\ell_1}} |\partial_{x_i} u_\ell|^{p_i}\, dx\Big)^{\frac{1}{p_i'}} \Big(\int_{D_{\ell_1}} |u_\ell - u_\infty|^{p_i}\, dx\Big)^{\frac{1}{p_i}}
\end{aligned} \tag{3.31}$$

where we have set $D_{\ell_1} = \Omega_{\ell_1+1}\setminus\Omega_{\ell_1}$. Let us define A as

$$\begin{aligned}
A = \sum_{i=1}^m |\partial_{x_i}(u_\ell - u_\infty)|^{p_i} + |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{q_{j_i}} \\
+ \sum_{j\ne j_i} \{|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty\}\partial_{x_j}(u_\ell - u_\infty).
\end{aligned} \tag{3.32}$$

Using the lemma 2.3, (3.32) and the Poincaré inequality on the section of the domain one deduces from (3.31)

$$\begin{aligned}
\int_{\Omega_{\ell_1}} A\, dx &\le \int_{\Omega_{\ell_1+1}} A\rho_{\ell_1}\, dx \\
&\le C\sum_{i=1}^m \Big(\int_{D_{\ell_1}} |\partial_{x_i} u_\ell|^{p_i}\, dx\Big)^{\frac{1}{p_i'}} \Big(\int_{D_{\ell_1}} |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{q_{j_i}}\, dx\Big)^{\frac{1}{p_i}} \\
&\le C\sum_{i=1}^m \Big(\int_{D_{\ell_1}} A\, dx\Big)^{\frac{1}{p_i'}} \Big(\int_{D_{\ell_1}} A\, dx\Big)^{\frac{1}{p_i}} = C\int_{D_{\ell_1}} A\, dx.
\end{aligned} \tag{3.33}$$

It follows that

$$\int_{\Omega_{\ell_1}} A\, dx \le \frac{C}{C+1}\int_{\Omega_{\ell_1+1}} A \tag{3.34}$$

Denote by $[\frac{\ell}{2}]$ the integer part of $\frac{\ell}{2}$. Setting $a = \frac{C}{C+1}$ and iterating this formula $[\frac{\ell}{2}]$ times starting from $\frac{\ell}{2}$ we obtain easily taking into account the inequality $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \le \frac{\ell}{2}$

$$\int_{\Omega_{\frac{\ell}{2}}} A\, dx \le a^{[\frac{\ell}{2}]}\int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} A\, dx \le a^{\frac{\ell}{2}-1}\int_{\Omega_\ell} A\, dx. \tag{3.35}$$

To evaluate this last integral one relies on the lemma 3.1. Indeed using the lemma 2.3 one has

$$A \leq \sum_{i=1}^{m} (|\partial_{x_i} u| + |\partial_{x_i} u_\infty|)^{p_i} + (|\partial_{x_{j_i}} u_\ell| + |\partial_{x_{j_i}} u_\infty|)^{p_i} + \sum_{j \neq j_i} C_{q_j} (|\partial_{x_j} u_\ell| + |\partial_{x_j} u_\infty|)^{q_j}$$

Using again the formula $(|a| + |b|)^q \leq 2^{q-1}(|a|^q + |b|^q)$ one derives for some constant

$$A \leq C\Big\{ \sum_{i=1}^{m} |\partial_{x_i} u|^{p_i} + |\partial_{x_i} u_\infty|^{p_i} + \sum_{j=m+1}^{n} |\partial_{x_j} u_\ell|^{q_j} + |\partial_{x_j} u_\infty|^{q_j} \Big\}$$

Since u_∞ is independent of X_1 it follows from (3.6) that

$$\int_{\Omega_\ell} A \; dx \leq C\ell^m$$

and from (3.35) one derives

$$\int_{\Omega_{\frac{\ell}{2}}} A \; dx \leq C e^{-\ell \frac{1}{2} \ln \frac{1}{a}} \ell^m. \tag{3.36}$$

This leads to (3.29) provided we chose $\alpha < \frac{1}{2} \ln \frac{1}{a}$. □

In the case where $f \in L^\infty(\omega_2)$ one can show the following.

Theorem 3.4. *We suppose that $f \in L^\infty(\omega_2)$ and $p_i \geq 2$, $\forall i = 1, \cdots, m$. In addition we assume that*

$$\forall i = 1, \cdots, m, \quad \exists j_i \in \{m+1, \ldots, n\} \quad \text{such that} \quad p_i \geq q_{j_i} \geq 2. \tag{3.37}$$

Then there exists constants C, α independent of ℓ such that

$$\begin{aligned} &\int_{\Omega_{\frac{\ell}{2}}} \sum_{i=1}^{m} |\partial_{x_i}(u_\ell - u_\infty)|^{p_i} + |\partial_{x_{j_i}}(u_\ell - u_\infty)|^{p_i} \\ &\qquad + \sum_{j \neq j_i} \{|\partial_{x_j} u_\ell|^{q_j - 2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j - 2} \partial_{x_j} u_\infty\} \partial_{x_j}(u_\ell - u_\infty) \; dx \leq C e^{-\alpha \ell}. \end{aligned} \tag{3.38}$$

Proof. As in Theorem 3.3 we derive (3.31) that is

$$\begin{aligned} &\int_{\Omega_{\ell_1+1}} \Big\{ \sum_{i=1}^{m} |\partial_{x_i} u_\ell|^{p_i} \\ &\qquad + \sum_{j=m+1}^{n} \{|\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty\} \partial_{x_j}(u_\ell - u_\infty) \Big\} \rho_{\ell_1} \; dx \\ &\qquad \leq C \sum_{i=1}^{m} \Big(\int_{D_{\ell_1}} |\partial_{x_i} u_\ell|^{p_i} \; dx \Big)^{\frac{1}{p_i'}} \Big(\int_{D_{\ell_1}} |u_\ell - u_\infty|^{p_i} \; dx \Big)^{\frac{1}{p_i}}, \end{aligned} \tag{3.39}$$

where $D_{\ell_1} = \Omega_{\ell_1+1} \backslash \Omega_{\ell_1}$.

One claims that for some constant C independent of ℓ one has

$$|u_\ell - u_\infty| \leq C. \tag{3.40}$$

Then, we derive that

$$\Big(\int_{D_{\ell_1}} |u_\ell - u_\infty|^{p_i}\ dx\Big)^{\frac{1}{p_i}} \leq C\Big(\int_{D_{\ell_1}} |u_\ell - u_\infty|^{q_{j_1}}\ dx\Big)^{\frac{1}{p_i}}.$$

Recalling the notation (3.32), (3.33) follows easily and the rest of the proof as well.

To prove (3.40), suppose that ω_2 is contained in the strip

$$\{X_2 \ \mid\ |x_n| \leq s\}$$

for some positive s (recall that ω_2 is supposed to be bounded in $\mathbb{R}^{n-m}$). Then set

$$\beta = 1 + \frac{1}{q_n - 1} \quad , \quad g = s^\beta - |x_n|^\beta.$$

One finds easily since $(\beta - 1)(q_n - 1) = 1$ that

$$\partial_{x_n} g = -\beta |x_n|^{\beta-2} x_n, \quad |\partial_{x_n} g|^{q_n-2}\partial_{x_n} g = -\beta^{q_n-1} x_n, \quad -\partial_{x_n}\big(|\partial_{x_n} g|^{q_n-2}\partial_{x_n} g\big) = \beta^{q_n-1}.$$

If $|f|_\infty$ denotes the L^∞-norm of f setting

$$h = (|f|_\infty)^{\frac{1}{q_n-1}} \frac{g}{\beta}$$

one has (see (3.3))

$$-\sum_{i=1}^m \partial_{x_i}\big(|\partial_{x_i} u_\ell|^{p_i-2}\partial_{x_i} u_\ell\big) - \sum_{j=m+1}^n \partial_{x_j}\big(|\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell\big) = f \leq |f|_\infty$$
$$= -\partial_{x_n}\big(|\partial_{x_n} h|^{q_n-2}\partial_{x_n} h\big) = -\sum_{i=1}^m \partial_{x_i}\big(|\partial_{x_i} h|^{p_i-2}\partial_{x_i} h\big) - \sum_{j=m+1}^n \partial_{x_j}\big(|\partial_{x_j} h|^{q_j-2}\partial_{x_j} h\big).$$

Using in the weak formulation $v = (u_\ell - h)^+$ one deduces easily that

$$u_\ell(f) \leq h \leq (|f|_\infty)^{\frac{1}{q_n-1}} \frac{s^\beta}{\beta}.$$

Since $-u_\ell(f) = u_\ell(-f)$, (3.40) follows easily. This completes the proof of the theorem. □

Remark 2. *One could try to mix assumptions of the type of Theorem 3.2 and 3.4 however it will make the result regarding the speed of convergence messy, the convergence being insured by the theorem 3.1 for the p_i's large enough. In the case of theorems 3.3, 3.4 one can take advantage of the exponential speed of convergence to get existence results in unbounded domains in the spirit of [9], [7].*

Remark 3. *The operators that we have considered here are the sum of p-Laplacians in one dimension. One can consider also operators sums of p-Laplacians in larger dimensions. For instance, with the notation of this section, if u_ℓ is the weak solution to*

$$\begin{cases} -\nabla_{X_1} \cdot \Big(|\nabla_{X_1} u_\ell|^{p-2} \nabla_{X_1} u_\ell \Big) - \nabla_{X_2} \cdot \Big(|\nabla_{X_2} u_\ell|^{q-2} \nabla_{X_2} u_\ell \Big) = f \quad in \quad \Omega_\ell \\ u_\ell = 0 \quad on \quad \partial\Omega_\ell, \end{cases}$$

one can show if $p = q \geq 2$, using the technique of theorem 3.3, that u_ℓ converges exponentially quickly toward the solution to.

$$\begin{cases} -\nabla_{X_2} \cdot \Big(|\nabla_{X_2} u_\infty|^{q-2} \nabla_{X_2} u_\infty \Big) = f \quad in \quad \omega_2 \\ u_\infty = 0 \quad on \quad \partial\omega_2. \end{cases}$$

($\nabla_{X_i} \cdot$ denotes the divergence in $\mathbb{R}^m$ or $\mathbb{R}^{n-m}$). Similarly one can consider operators sums of p-Laplacians of different dimensions i.e. problems of the type

$$\begin{cases} -\sum_i \nabla_{Y_i} \cdot \Big(|\nabla_{Y_i} u_\ell|^{p_i-2} \nabla_{Y_i} u_\ell \Big) = f \quad in \quad \Omega_\ell \\ u_\ell = 0 \quad on \quad \partial\Omega_\ell, \end{cases}$$

where Y_i denotes some subset of the coordinates, and develop results similar to the ones of this note. The case of the sum of n-dimensional p-Laplacians was considered in [8].

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