Asymptotic behaviour of some anisotropic problems

Michel Chipot *

Abstract

The goal of this paper is to explore the asymptotic behaviour of anisotropic problems governed by operators of the pseudo $p$-Laplacian type when the size of the domain goes to infinity in different directions.

MSC2020-Mathematics Subject Classification: 35B40, 35D30, 35J60, 35J66, 35J92.

Key words: Anisotropic operators, nonlinear elliptic operators, pseudo $p$-Laplacian, asymptotic behaviour, cylinder like domains.

1 Basic notation

When $\Omega$ is a bounded open set of $\mathbb{R}^n$, we denote by $W^{1,r}_0(\Omega)$, $r > 1$, the usual Sobolev space constructed on $L^r(\Omega)$, of functions vanishing on the boundary of $\Omega$. That is to say we set

$$W^{1,r}(\Omega) = \{v \in L^r(\Omega) \mid \forall i = 1, \ldots, n \}.$$  \hspace{1cm} (1.1)

We equip this space with the norm

$$||v||_{1,r,\Omega} = \left( \int_{\Omega} |v|^r + \sum_{i=1}^n |\partial_{x_i} v|^r dx \right)^{\frac{1}{r}}$$ \hspace{1cm} (1.2)

and we set

$$W^{1,r}_0(\Omega) = \overline{\mathcal{D}(\Omega)} = \text{the closure of } \mathcal{D}(\Omega) \text{ in } W^{1,r}(\Omega).$$ \hspace{1cm} (1.3)

($\mathcal{D}(\Omega)$ denotes the space of $C^\infty$-functions with compact support in $\Omega$). It is well known that $W^{1,r}_0(\Omega)$ is a reflexive Banach space which can be equipped with the equivalent norm

$$||\nabla v||_{r,\Omega} = \left( \int_{\Omega} |\nabla v(x)|^r dx \right)^{\frac{1}{r}}.$$ \hspace{1cm} (1.4)

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*Institute of Mathematics, University of Zürich, Winterthurerstr. 190, CH-8057 Zürich and FernUni Schweiz, Schinerstrasse 18, CH-3900 Brig-Glis, email: m.m.chipot@math.uzh.ch
(\nabla \text{ denotes the usual gradient and } | | \text{ the euclidean norm, i.e. } |\nabla v(x)| = \left( \sum_{i=1}^n (\partial x_i v)^2 \right)^{\frac{1}{2}}, | |_r,\Omega \text{ denotes the } L^r \text{-norm on } \Omega). \text{ The dual of } W^{1,r}_0(\Omega) \text{ is denoted by } W^{-1,r'}(\Omega), r' = \frac{r}{r-1} \text{ and consists in the distributions of the form }

\begin{align*}
f &= f_0 - \sum_{i=1}^n \partial x_i f_i, \quad f_i \in L^{r'}(\Omega). 
\end{align*}

We use the notation

\begin{align*}
\langle f, v \rangle &= \int_{\Omega} f_0 v + \sum_{i=1}^n f_i \partial x_i v \, dx.
\end{align*}

The paper is organised as follows. In the next section we address the case of a simple problem set on a rectangle with one side going to infinity. We consider all the possible values of \((p, q)\), \(p, q > 1\) for the pseudo \((p, q)\)-operator at hand allowing to present a variety of techniques. Some of them are issued of previous works. We refer the reader to [11], [10], [4], [5], [15], for details.

The section 3 relies on the experience acquired on the simple model investigated in section 1 to extend some results to more complex situations. The operators at hand are Euler equations of some anisotropic functionals of calculus of variations introduced for other reasons in [17], see also [18]. For basic notions on Sobolev spaces we refer to [2], [12], [13], [14], [16].

2 A model problem

We denote by \(\Omega_\ell\) the open subset of \(\mathbb{R}^2\) defined as

\begin{align*}
\Omega_\ell = (-\ell, \ell) \times (-1, 1).
\end{align*}

We will set \(\omega = (-1, 1)\) and \(\partial \Omega_\ell\) will denote the boundary of \(\Omega_\ell\), see the figure 2.1 below.

![Figure 2.1: The domain \(\Omega_\ell\)](image)

If \(p, q > 1\) are two positive numbers we would like to consider \(u_\ell\) solution to

\begin{align*}
\begin{cases}
-\partial_{x_1} \left( |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \right) - \partial_{x_2} \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell \right) = f \quad \text{in } \Omega_\ell, \\
u_\ell = 0 \quad \text{on } \partial \Omega_\ell.
\end{cases}
\end{align*}
More precisely we are interested to the asymptotic behaviour of \( u_\ell \) when \( \ell \to +\infty \). \( f \) is a function or distribution depending only on \( x_2 \). A natural candidate for the limit of the problem is \( u_\infty \) solution to
\[
\begin{aligned}
-\partial_{x_2} \left( |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) &= f \quad \text{in } \omega, \\
 u_\infty &= 0 \quad \text{on } \partial \omega,
\end{aligned}
\]
where \( \partial \omega = \{-1, 1\} \) is the boundary of \( \omega \). First let us recast these problems under their natural weak form.

We can first introduce the weak formulation of (2.3). If \( f \in W^{-1,q'}(\omega) \) is given by
\[
f = f(x_2) = f_0(x_2) - \partial_{x_2} f_1(x_2),
\]
where \( f_0, f_1 \in L^{q'}(\omega) \) then, the weak formulation to (2.3) corresponding to \( f \) reads
\[
\begin{aligned}
\int_\omega |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \partial_{x_2} v \, dx_2 &= \langle f, v \rangle = \int_\omega f_0 v + f_1 \partial_{x_2} v \, dx_2 \quad \forall v \in W^{1,q}_0(\omega).
\end{aligned}
\]

To arrive to a weak formulation for (2.2) one introduces
\[
W^{1,p,q}(\Omega_\ell) = \{ v \in L^p(\Omega_\ell) \cap L^q(\Omega_\ell) \mid \partial_{x_1} v \in L^p(\Omega_\ell), \partial_{x_2} v \in L^q(\Omega_\ell) \}.
\]
It is a reflexive Banach space when equipped with the norm
\[
||v||_{1,p,q,\Omega_\ell} = ||v||_{p,\Omega_\ell} + ||v||_{q,\Omega_\ell} + ||\partial_{x_1} v||_{p,\Omega_\ell} + ||\partial_{x_2} v||_{q,\Omega_\ell}.
\]
Then we define
\[
W^{1,p,q}_0(\Omega_\ell) = \overline{\mathcal{D}(\Omega_\ell)} = \text{the closure of } \mathcal{D}(\Omega_\ell) \text{ in } W^{1,p,q}(\Omega_\ell).
\]
If \( f \) is defined by (2.4) if follows easily that there exists a unique \( u_\ell \) weak solution to (2.2) i.e. satisfying
\[
\begin{aligned}
u_\ell \in W^{1,p,q}_0(\Omega_\ell), \\
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} v + |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell \partial_{x_2} v \, dx_1 dx_2 &= \langle f, v \rangle = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v \, dx_1 dx_2 \quad \forall v \in W^{1,p,q}_0(\Omega_\ell).
\end{aligned}
\]
We are interested in showing that \( u_\ell \to u_\infty \) when \( \ell \to \infty \), but also to investigate at what speed. We will now denote \( dx_1 dx_2 \) by \( dx \).

The operators defined by (2.2), (2.3) are strictly monotone, hemicontinuous, coercive from \( W^{1,p,q}_0(\Omega_\ell) \), \( W^{1,q}_0(\omega) \) into their duals. Existence and uniqueness of a solution for (2.9), (2.5) follows from classical arguments (see [3], [12], [16]).

Let us first prove the following lemma.
Lemma 2.1. Suppose that \( f \) is given by (2.4). If \( u_\ell \) is the solution to (2.9) there exists a constant \( C \) independent of \( \ell \) such that
\[
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \, dx \leq C\ell. \tag{2.10}
\]

Proof. Taking \( v = u_\ell \) in (2.9) we get
\[
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \, dx = \langle f, u_\ell \rangle = \int_{\Omega_\ell} f_0 u_\ell + f_1 \partial_{x_2} u_\ell \, dx \\
\leq |f_0|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} + |f_1|_{q',\Omega_\ell} |\partial_{x_2} u_\ell|_{q,\Omega_\ell} \tag{2.11}
\]
this by the Hölder and the Poincaré inequality. Let us recall regarding this last point an argument that we will use several times later on. If \( u \in W^{1,p,q}_0(\Omega_\ell) \), let \( u_n \in D(\Omega_\ell) \) such that \( u_n \rightharpoonup u \) in \( W^{1,p,q}_0(\Omega_\ell) \). By the Poincaré inequality on \( \omega \) one has for some constant \( C \) independent of \( \ell \)
\[
\int_\omega |u_n(x_1, x_2)|^q \, dx_2 \leq C^q \int_\omega |\partial_{x_2} u_n(x_1, x_2)|^q \, dx_2, \quad \text{a.e. } x_1 \in (-\ell, \ell).
\]
Integrating in \( x_1 \) we deduce
\[
|u_n|_{q,\Omega_\ell} \leq C |\partial_{x_2} u_n|_{q,\Omega_\ell}
\]
and passing to the limit in \( n \) the same inequality holds for \( u \) or \( u_\ell \).

Then let us notice that for \( i = 0, 1 \) one has
\[
|f_i|_{q',\Omega_\ell} = \left( \int_{-\ell}^{\ell} \int_\omega |f_i(x_2)|^{q'} \, dx_2 dx_1 \right)^{\frac{1}{q'}} = (2\ell)^{\frac{1}{q'}} |f_i|_{q',\omega}.
\]
Thus from (2.11) we derive for some constant \( C = C(q, f) \)
\[
|\partial_{x_2} u_\ell|_{q,\Omega_\ell} \leq C \ell^{\frac{1}{q'}} |\partial_{x_2} u_\ell|_{q,\Omega_\ell}
\]
Since \( q' = \frac{q}{q-1} \) this is equivalent for some new constant to
\[
|\partial_{x_2} u_\ell|_{q,\Omega_\ell} \leq C \ell^{\frac{1}{q'}}.
\]
Going back to (2.11), the result follows.

Somehow one can ignore \( f \) thanks to the following remark.

Lemma 2.2. If \( u_\ell \) is the solution to (2.9) and \( u_\infty \) solution to (2.5) one has
\[
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} v + \left\{ |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right\} \partial_{x_2} v \, dx = 0 \quad \forall v \in W^{1,p,q}_0(\Omega_\ell). \tag{2.12}
\]
Proof. First by \((2.9)\) if \(v \in W_0^{1,p,q}(\Omega_\ell)\) one has
\[
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \partial_{x_1} u_\ell \partial_{x_1} v + |\partial_{x_2} u_\ell|^q \partial_{x_2} u_\ell \partial_{x_2} v = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} v \, dx
\]  
(2.13)
If \(v \in W_0^{1,p,q}(\Omega_\ell)\) one has for almost every \(x_1\)
\[
v(x_1, \cdot) \in W_0^{1,q}(\omega).
\]
Thus by \((2.5)\)
\[
\int_{\omega} |\partial_{x_2} u_\infty|^q \partial_{x_2} u_\infty \partial_{x_2} v(x_1, x_2) \, dx_2 = \int_{\omega} f_0 v + f_1 \partial_{x_2} vdx_2.
\]
Integrating in \(x_1\) it comes
\[
\int_{\Omega_\ell} |\partial_{x_2} u_\infty|^q \partial_{x_2} u_\infty \partial_{x_2} v \, dx = \int_{\Omega_\ell} f_0 v + f_1 \partial_{x_2} vdx. \tag{2.14}
\]
Subtracting from \((2.13)\), \((2.12)\) follows. \(\square\)

Let us recall the following result (see [3], [6]) which garanties also the strict monotonicity of the operators at hand.

**Lemma 2.3.** For any \(q > 1\) there exist positive constants \(c_q, C_q\) such that
\[
||\xi||^q - |\eta|^q \leq C_q|\xi - \eta||(|\xi| + |\eta|)^q - 2 \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{2.15}
\]
\[
(|\xi|^q - |\eta|^q)(\xi - \eta) \geq c_q|\xi - \eta|^2(|\xi| + |\eta|)^q - 2 \quad \forall \xi, \eta \in \mathbb{R}^n. \tag{2.16}
\]

Then one has :

**Lemma 2.4.** Let \(u_\ell = u_\ell(f)\) be the solution to \((2.9)\) and \(u_\infty = u_\infty(f)\) be the solution to \((2.5)\). Suppose that \(f_1 \geq f_2, f \geq 0\) then one has
\[
u_\ell(f_2) \leq u_\ell(f_1) , \quad 0 \leq u_\ell(f) \leq u_\infty(f). \tag{2.17}
\]
(If \(f\) is not a function, \(f \geq 0\) means \((f, v) \geq 0 \quad \forall v \in W_0^{1,q}(\omega), v \geq 0\)).

**Proof.** We use the notation \(u_i = u_\ell(f_i)\). From \((2.9)\) by subtraction we get
\[
\int_{\Omega_\ell} \{|\partial_{x_1} u_2|^p \partial_{x_1} u_2 - |\partial_{x_1} u_1|^p \partial_{x_1} u_1\} \partial_{x_1} v
+ \{|\partial_{x_2} u_2|^q \partial_{x_2} u_2 - |\partial_{x_2} u_1|^q \partial_{x_2} u_1\} \partial_{x_2} v = \langle f_2 - f_1, v \rangle.
\]
Taking \(v = (u_2 - u_1)^+\) one deduces easily using the lemma 2.3 that \((u_2 - u_1)^+ = 0\) i.e. \(u_1 \geq u_2\) (see below \((2.18)\) for a similar argument).

If \(f \geq 0\) taking \(f_1 = f, f_2 = 0\) one gets \(0 \leq u_\ell(f)\).
Regarding \( u_\infty \), taking \( v = u_\infty \) in (2.5) one gets for \( f \geq 0 \)
\[
\int_{\Omega} |\partial_{x_1} u_\infty|^{p-2} \partial_{x_1} u_\infty \partial_{x_1} u_\infty + |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \partial_{x_2} u_\infty = \int_{\Omega_t} f_0 u_\infty^- + f_1 \partial_{x_2} u_\infty^- \, dx \geq 0.
\]
This reads also
\[
\int_{\Omega_t} |\partial_{x_1} u_\infty|^{p-2} \partial_{x_1} u_\infty \partial_{x_1} u_\infty + |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \partial_{x_2} u_\infty \leq 0.
\]
Thus \( u_\infty^- = 0 \) and \( u_\infty(f) \geq 0 \). Then taking \( v = (u_\ell - u_\infty)^+ \) in (2.12) one gets
\[
\int_{\Omega_t} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} (u_\ell - u_\infty)^+ + \left\{ |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right\} \partial_{x_2} (u_\ell - u_\infty)^+ \, dx = 0
\]
i.e.
\[
\int_{\{u_\ell - u_\infty > 0\}} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} u_\ell + \left\{ |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right\} \partial_{x_2} (u_\ell - u_\infty) \, dx = 0.
\]
This implies that the set \( \{u_\ell - u_\infty > 0\} \) is of measure 0 since \( \nabla (u_\ell - u_\infty) = 0 \) on this set i.e. \( \nabla (u_\ell - u_\infty)^+ = 0 \) (see [14]). This completes the proof of the Lemma.

Let us now show:

**Lemma 2.5.** If \( u_\ell \) is the solution to (2.9) and \( u_\infty \) solution to (2.5) one has for every smooth function \( \varphi = \varphi(x_1) \) vanishing at \( \{-\ell, \ell\} \)
\[
\int_{\Omega_t} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \varphi \, dx 
\leq \int_{\Omega_t} |\partial_{x_1} u_\ell|^{p-1} |\partial_{x_1} \varphi||u_\ell - u_\infty| \, dx.
\]  
(2.19)

**Proof.** Taking \( v = (u_\ell - u_\infty)\varphi \) in (2.12) one gets
\[
\int_{\Omega_t} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \varphi \, dx 
= - \int_{\Omega_t} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} \varphi (u_\ell - u_\infty) \, dx.
\]
(2.20)

(Recall that \( u_\infty \) is independent of \( x_1 \)). Then (2.19) follows easily.

Denote by \( \rho = \rho(x_1) \) a smooth function such that
\[
0 \leq \rho \leq 1, \, \rho = 1 \text{ on } (-\frac{1}{2}, \frac{1}{2}), \, \rho = 0 \text{ near } \{-1, 1\}, \, |\partial_{x_1} \rho| \leq C.
\]
and set
\[
\varphi = \rho^\alpha = \rho^\alpha \left( \frac{x_1}{\ell} \right),
\]
where \( \alpha > 0 \). We can now prove:
**Lemma 2.6.** Let $f = f_0 \in L^q(\omega)$ and $u_\ell, u_\infty$ be the solutions to (2.9), (2.5). Then it holds for some constant $C$ independent of $\ell$

$$I = \int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \leq \frac{C}{\ell^{q-p}}. \quad (2.22)$$

**Proof.** From (2.19) one derives

$$\int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \leq \frac{\alpha C}{\ell} \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p |u_\ell - u_\infty| \rho^{\alpha-1} \, dx. \quad (2.23)$$

Noting that $\rho^{\alpha-1} = \rho^{\frac{\alpha}{q-p}} \rho^{\frac{\alpha}{p}}$ and using Hölder’s inequality it comes

$$I = \int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \leq \frac{\alpha C}{\ell} \left( \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \, dx \right)^\frac{1}{p} \left( \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \right)^\frac{1}{p}. \quad (2.24)$$

Thus it follows that

$$I \leq \left( \frac{\alpha C}{\ell} \right)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \leq \left( \frac{\alpha C}{\ell} \right)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \, dx, \quad (2.25)$$

provided we chose $\alpha > p$. From the lemma 2.4 one has

$$u_\ell(f) \leq u_\ell(f^+) \leq u_\infty(f^+) \quad u_\infty(-f^-) \leq u_\ell(-f^-) \leq u_\ell(f),$$

(notice that $u_\ell(-f) = -u_\ell(f)$). Then one derives

$$|u_\ell - u_\infty| \leq |u_\ell| + |u_\infty| \leq \max\{u_\infty(f^+), u_\infty(f^-)\} + |u_\infty(f)|.$$

Since this last function is independent of $x_1$ one derives from (2.25)

$$I = \int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \leq \frac{C}{\ell^{q-p-1}}$$

for some new constant $C$. This is (2.22). This completes the proof of the lemma.

\[ \square \]

We give now a very simple proof of the convergence of $u_\ell$ toward $u_\infty$ which is valid for every $p$ and $q$.

**Theorem 2.1.** Let $f = f_0 \in L^q(\omega)$ and $u_\ell, u_\infty$ be the solutions to (2.9), (2.5). Then for any $\ell_0$ it holds when $\ell \to +\infty$

$$\partial_{x_1} u_\ell \to 0 \text{ in } L^p(\Omega_{\ell_0}) , \quad \partial_{x_2} u_\ell \to \partial_{x_2} u_\infty \text{ in } L^q(\Omega_{\ell_0}). \quad (2.26)$$
Proof. The first part of (2.26) follows immediately from (2.22) if one chooses \( \ell_0 > \ell \). For the second part let us consider a smooth function \( \rho = \rho(x_1) \) such that for \( \ell_0 < \ell - 1 \) fixed

\[
0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } (-\ell_0, \ell_0), \quad \rho \text{ has compact support in } (-\ell_0 - 1, \ell_0 + 1), \quad |\partial_x \rho| \leq C.
\]

Since \( \rho^a u_\ell \in W^{1,p,q}_0(\Omega_\ell) \) one gets from (2.9)

\[
\int_{\Omega_{\ell_0+1}} \left\{ |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \right\} \rho^a \, dx = - \int_{\Omega_{\ell_0+1}} |\partial_{x_1} u_\ell|^{p-2} \partial_{x_1} u_\ell \partial_{x_1} \rho^a u_\ell + \int_{\Omega_{\ell_0+1}} f_0 u_\ell \rho^a \, dx \\
\leq \alpha C \int_{\Omega_{\ell_0+1}} |\partial_{x_1} u_\ell|^{p-1} |u_\ell| \rho^{a-1} + \int_{\Omega_{\ell_0+1}} |f_0| |u_\ell| \, dx.
\]

Noticing that \( \rho^{a-1} = \rho^\frac{a}{p} \rho^{\frac{a}{p}-1} \) and using the Young inequality \( ab \leq \frac{1}{p'} a^{p'} + \frac{1}{p} b^p \), \( a, b > 0 \) we get for some new constant \( C \)

\[
\int_{\Omega_{\ell_0+1}} \left\{ |\partial_{x_1} u_\ell|^p + |\partial_{x_2} u_\ell|^q \right\} \rho^a \, dx \\
\leq \frac{1}{p'} \int_{\Omega_{\ell_0+1}} |\partial_{x_1} u_\ell|^p \rho^a \, dx + C \int_{\Omega_{\ell_0+1}} |u_\ell|^p \rho^{a-p} \, dx + \int_{\Omega_{\ell_0+1}} |f_0| |u_\ell| \, dx.
\]

Using the inequality

\[
|u_\ell| \leq \max \{ u_\infty(f_0^+), u_\infty(f_0^-) \}
\]

which is due to

\[
u_\ell(f_0) \leq u_\ell(f_0^+) \leq u_\infty(f_0^+) \quad u_\ell(-f) = -u_\ell(f),
\]

we derive easily taking \( \alpha > p \) that

\[
\int_{\Omega_{\ell_0}} |\partial_{x_2} u_\ell|^q \, dx \leq C(\ell_0)
\]

where \( C(\ell_0) \) is independent of \( \ell \). Thus up to a subsequence there exists \( v_\infty \in L^q(\Omega_{\ell_0}), \; w_\infty \in L^q(\Omega_{\ell_0}) \) such that

\[
\partial_{x_2} u_\ell \rightharpoonup v_\infty \in L^q(\Omega_{\ell_0}) \quad |\partial_{x_2} u_\ell|^q \partial_{x_2} u_\ell \rightharpoonup w_\infty \in L^q(\Omega_{\ell_0}).
\]

From (2.22) one derives that up to a subsequence

\[
\left\{ |\partial_{x_2} u_\ell|^q \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^q \partial_{x_2} u_\infty \right\} \partial_{x_2} (u_\ell - u_\infty) \rightarrow 0 \text{ a.e in } \Omega_{\ell_0}.
\]

Thus, up to a subsequence, \( \partial_{x_2} u_\ell \rightarrow \partial_{x_2} u_\infty \) a.e on \( \Omega_{\ell_0} \). To see this point one notices that by (2.16) one has

\[
\partial_{x_2} (u_\ell - u_\infty) \partial_{x_2} (u_\ell - u_\infty) \left\{ |\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty| \right\}^{q-2} \rightarrow 0 \text{ a.e in } \Omega_{\ell_0}.
\]

If \( \partial_{x_2} (u_\ell - u_\infty) \neq 0 \) then

\[
\partial_{x_2} (u_\ell - u_\infty) \left\{ |\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty| \right\}^{q-2} \rightarrow 0 \text{ a.e in } \Omega_{\ell_0}.
\]
and by (2.15) it follows that
\[ \left\{ \left| \partial x_2 u_\ell \right|^{q-2} \partial x_3 u_\ell - \left| \partial x_2 u_\infty \right|^{q-2} \partial x_3 u_\infty \right\} \to 0 \quad \text{a.e in } \Omega_{l_0} \]
and again due to the strict monotonicity of the function \(|x|^{q-2}x\) one has \(\partial x_2 u_\ell \to \partial x_2 u_\infty\) a.e on \(\Omega_{l_0}\).

From this it follows (see for instance [19] lemma 8.3) for a proof that
\[ \partial x_2 u_\ell \to \partial x_2 u_\infty \in L^q(\Omega_{l_0}) \text{ , } |\partial x_2 u_\ell|^{q-2} \partial x_2 u_\ell \to |\partial x_2 u_\infty|^{q-2} \partial x_2 u_\infty \in L^q(\Omega_{l_0}). \]

Now from (2.22) one has
\[ \int_{\Omega_{l_0}} \left\{ |\partial x_3 u_\ell|^{q-2} \partial x_3 u_\ell - |\partial x_2 u_\infty|^{q-2} \partial x_2 u_\infty \right\} \partial x_2 (u_\ell - u_\infty) \to 0, \]
that is
\[ \int_{\Omega_{l_0}} |\partial x_2 u_\ell|^q - |\partial x_2 u_\ell|^{q-2} \partial x_2 u_\ell \partial x_2 u_\infty - |\partial x_2 u_\infty|^{q-2} \partial x_2 u_\infty \partial x_2 u_\ell + |\partial x_2 u_\infty|^q dx \to 0. \]

It follows that
\[ \int_{\Omega_{l_0}} |\partial x_2 u_\ell|^q dx \to \int_{\Omega_{l_0}} |\partial x_2 u_\infty|^q dx \]
and the result, i.e. the strong convergence, follows. \(\square\)

One can estimate the convergence rate in some situations. Indeed one has:

**Theorem 2.2.** Let \(f = f_0 \in L^q(\Omega)\) and \(u_\ell, u_\infty\) be the solutions to (2.9), (2.5). Then it holds for some constant \(C\) independent of \(\ell\)
\[ \int_{\Omega_{l_{\ell}}} |\partial x_1 (u_\ell - u_\infty)|^p + |\partial x_2 (u_\ell - u_\infty)|^q dx = \int_{\Omega_{l_{\ell}}} |\partial x_1 u_\ell|^p + |\partial x_2 (u_\ell - u_\infty)|^q dx \leq \frac{C}{l_{l_{\ell}}^{q \wedge 2 - 1}} \quad (2.27) \]
where \(q \wedge 2\) denotes the minimum of \(q\) and 2.

**Proof.** It follows from the lemmas 2.3 and 2.6 that
\[ \tilde{I} = \int_{\Omega_{l_\ell}} \left\{ |\partial x_3 u_\ell|^p + \left( |\partial x_2 u_\ell| + |\partial x_2 u_\infty| \right)^{q-2} \partial x_2 (u_\ell - u_\infty)^2 \right\} \rho^\alpha dx \leq \frac{C}{l_{l_{\ell}}^{q - 1}}. \]
If \(q \geq 2\), since \(|\partial x_2 (u_\ell - u_\infty)| \leq |\partial x_2 u_\ell| + |\partial x_2 u_\infty|\), one derives immediately (2.27), i.e.
\[ \int_{\Omega_{l_{\ell}}} |\partial x_1 (u_\ell - u_\infty)|^p + |\partial x_2 (u_\ell - u_\infty)|^q dx = \int_{\Omega_{l_{\ell}}} |\partial x_1 u_\ell|^p + |\partial x_2 (u_\ell - u_\infty)|^q dx \leq \frac{C}{l_{l_{\ell}}^{q - 1}}. \]
If \( q < 2 \) one has thanks to Hölder’s inequality

\[
\int_{\Omega_{\ell}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha - q} \, dx = \\
\int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^{(q-2)\frac{q}{2}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha - q} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^{(2-q)\frac{q}{2}} \, dx \\
\leq \left( \int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^{(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^{\alpha - q} \, dx \right)^{\frac{q}{2}} \\
\leq \left( \int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^q \, dx \right)^{1-\frac{q}{2}} \tag{2.28}
\]

Choosing \((\alpha - q)\frac{q}{2} > \alpha\) and taking into account the lemmas 2.1, 2.3 we get for different constants independent of \( \ell \)

\[
\int_{\Omega_{\ell}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^{\alpha - q} \, dx \\
\leq \left( \int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^{(q-2)} |\partial_{x_2}(u_\ell - u_\infty)|^2 \rho^\alpha \, dx \right)^{\frac{q}{2}} \\
\leq \left( \int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^q \, dx \right)^{1-\frac{q}{2}} \tag{2.29}
\]

\[
\leq C \left( \int_{\Omega_{\ell}} (|\partial_{x_2} u_\ell| + |\partial_{x_2} u_\infty|)^q \, dx \right)^{1-\frac{q}{2}} \leq C \left( \frac{\ell}{\ell^p} \right)^{1-\frac{q}{2}} \leq \frac{C}{\ell^{(p-1)\frac{q}{2}}} \ell^{1-\frac{q}{2}}.
\]

(Note that \(|a| + |b|)^q \leq 2^{q-1}(|a|^q + |b|^q).\) Choosing also \( \alpha > q \) we are ending up with

\[
\int_{\Omega_{\ell}} |\partial_{x_2}(u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{q}{2}q-1}}.
\]

Combining this with (2.22) we arrive also to (2.27).

\[
\square
\]

In the case where \( p < q \) one can consider a general \( f \) and not only assume that it is in \( L^q(\omega_2) \). Indeed one has first :

**Lemma 2.7.** Suppose that \( p < q \). If \( \rho^\alpha \) is defined by (2.21) and if \( \alpha \) is chosen such that \( \alpha \frac{q}{p} - q > \alpha \) it holds for some constant \( C \)

\[
I = \int_{\Omega_{\ell}} \left\{ |\partial_{x_1} u_\ell|^p + (|\partial_{x_2} u_\ell|^{q-2} |\partial_{x_2} u_\ell|^q - |\partial_{x_2} u_\infty|^{q-2} |\partial_{x_2} u_\infty|^q) \partial_{x_2}(u_\ell - u_\infty) \right\} \rho^\alpha \, dx \\
\leq \frac{C}{\ell^p} \left( \int_{\Omega_{\ell}} |\partial_{x_2}(u_\ell - u_\infty)|^q \rho^\alpha \, dx \right)^\frac{p}{q} \ell^{1-\frac{q}{2}} \tag{2.30}
\]
Proof. Since (2.19) is valid for a general $f$ one derives as in (2.23), (2.24)

$$
I = \int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + \left( |\partial_{x_2} u_\ell|^{q-2} \partial_{x_2} u_\ell - |\partial_{x_2} u_\infty|^{q-2} \partial_{x_2} u_\infty \right) \partial_{x_2} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx
\leq \frac{\alpha C}{\ell} \left( \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p \rho^\alpha \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \right)^{\frac{1}{p}}
\leq \frac{\alpha C}{\ell} I^\frac{1}{p} \left( \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx \right)^{\frac{1}{p}}.
$$

(2.31)

From this inequality it follows since $\alpha \frac{q}{p} - q > \alpha$ for various constant $C$

$$
I \leq \left( \frac{\alpha C}{\ell} \right)^p \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^{\alpha-p} \, dx
\leq \left( \frac{\alpha C}{\ell} \right)^p \left( \int_{\Omega_\ell} |u_\ell - u_\infty|^p \rho^2 \rho^{(\alpha-p)\frac{q}{p}} \, dx \right)^{\frac{p}{q}} \left( \int_{\Omega_\ell} 1 \, dx \right)^{1-\frac{q}{q}}
\leq \frac{C}{\ell^p} \left( \int_{\Omega_\ell} |u_\ell - u_\infty|^q \rho^\alpha \, dx \right)^{\frac{2}{q}} \left( \int_{\Omega_\ell} 1 \, dx \right)^{1-\frac{q}{q}}
\leq \frac{C}{\ell^p} \left( \int_{\Omega_\ell} |\partial_{x_2} (u_\ell - u_\infty)|^q \rho^\alpha \, dx \right)^{\frac{2}{q}} \ell^{1-\frac{q}{q}}.
$$

(2.32)

(In the last inequality we used the Poincaré inequality on $\omega$). This completes the proof of the lemma.

Then we have:

**Theorem 2.3.** Suppose that $p < q$. One has

$$
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} (u_\ell - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{mp}{q-p}-1}}
$$

(2.33)

Proof. If $q \geq 2$ one has by (2.16)

$$
(\xi^{q-2} \xi - |\eta|^{q-2} \eta) \cdot (\xi - \eta) \geq c_q |\xi - \eta|^2 (|\xi| + |\eta|)^{q-2} \geq c_q |\xi - \eta|^q \quad \forall \xi, \eta \in \mathbb{R}^n.
$$

(2.34)

Thus from (2.30) one deduces for some constant $C$

$$
J = \int_{\Omega_\ell} \left\{ |\partial_{x_1} u_\ell|^p + |\partial_{x_2} (u_\ell - u_\infty)|^q \right\} \rho^\alpha \, dx
\leq \frac{C}{\ell^p} \left( \int_{\Omega_\ell} |\partial_{x_2} (u_\ell - u_\infty)|^q \rho^\alpha \, dx \right)^{\frac{2}{q}} \ell^{1-\frac{q}{q}}.
$$

From this it follows that

$$
J \leq \frac{C}{\ell^p} J^\frac{2}{q} \ell^{1-\frac{q}{q}} \iff J \leq \frac{C}{\ell^{\frac{mp}{q-p}-1}}
$$

and (2.33) follows by definition of $\rho$. 

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In the case when \( p < q < 2 \), noting \( S = |\partial x_2(u_\ell)| + |\partial x_2(u_\infty)| \) one derives from (2.16), (2.30) for some constant
\[
\tilde{I} = \int_{\Omega_\ell} \left\{ |\partial x_1 u_\ell|^p + S^{q-2} |\partial x_2(u_\ell - u_\infty)|^2 \right\} \rho^\alpha \, dx \\
\leq \frac{C}{\ell^p} \left( \int_{\Omega_\ell} |\partial x_2(u_\ell - u_\infty)|^q \rho^\alpha \, dx \right)^{\frac{p}{q}} \ell^{1 - \frac{q}{p}}.
\] (2.35)

Thanks to Hölder’s inequality one has as in (2.28)
\[
\int_{\Omega_\ell} |\partial x_2(u_\ell - u_\infty)|^q \rho^\alpha \, dx = \int_{\Omega_\ell} S_{\frac{q}{2}}^{\frac{q}{2}(q-2)} |\partial x_2(u_\ell - u_\infty)|^q \rho^\alpha S_{\frac{q}{2}}^{\frac{q}{2}(2-q)} \rho^{\alpha(1-\frac{q}{2})} \, dx \\
\leq \left( \int_{\Omega_\ell} S_{\frac{q}{2}}^{q-2} |\partial x_2(u_\ell - u_\infty)|^2 \rho^\alpha \, dx \right)^{\frac{q}{2}} \left( \int_{\Omega_\ell} S_q \, dx \right)^{1 - \frac{q}{2}} \\
\leq \tilde{I}_{\frac{q}{2}} \left( \int_{\Omega_\ell} \{|\partial x_2(u_\ell)| + |\partial x_2(u_\infty)|\}^q \, dx \right)^{1 - \frac{q}{2}}.
\]

It follows from (2.10) that
\[
\int_{\Omega_\ell} |\partial x_2(u_\ell - u_\infty)|^q \rho^\alpha \, dx \leq C \tilde{I}_{\frac{q}{2}} \left( \int_{\Omega_\ell} |\partial x_2(u_\ell)|^q + |\partial x_2(u_\infty)|^q \, dx \right)^{1 - \frac{q}{2}} \leq C \tilde{I}_{\frac{q}{2}} \ell^{1 - \frac{q}{2}}.
\]

Going back to (2.35) we obtain
\[
\tilde{I} \leq \frac{C}{\ell^p} \left( \tilde{I}_{\frac{q}{2}} \ell^{1 - \frac{q}{2}} \right)^{\frac{p}{q}} \ell^{1 - \frac{q}{p}} = \frac{C}{\ell^p} \tilde{I}_{\frac{q}{2}} \ell^{1 - \frac{q}{p}}.
\]

Hence
\[
\tilde{I} \leq \frac{C}{\ell^{\frac{2p}{2p-1}}},
\]
and
\[
\int_{\Omega_\ell} |\partial x_2(u_\ell - u_\infty)|^q \rho^\alpha \, dx \leq C \frac{1}{\ell^{\frac{2p}{2p-1}}}.
\]

The inequality (2.32) follows from these two estimates.

In the case \( p \geq q \geq 2 \) one can show that \( u_\ell \to u_\infty \) exponentially quickly (see [6], [15] and also this issue in the next section). Indeed one has :

**Theorem 2.4.** Suppose that \( p \geq q \geq 2 \), \( f \in L^1(\Omega) \). It holds for some positive constants \( C, \alpha \)
\[
\int_{\Omega_\ell} |\partial x_1 u_\ell|^p + |\partial x_2(u_\ell - u_\infty)|^q \, dx \leq Ce^{-\alpha \ell}.
\] (2.36)

**Proof.** Since \( f \in L^1(\Omega) \) one has
\[
-\partial x_2(|\partial x_2 u_\infty|^{q-2} \partial x_2 u_\infty) = f \Leftrightarrow |\partial x_2 u_\infty|^{q-2} \partial x_2 u_\infty = - \int_0^x f(\xi) \, d\xi + C.
\]
This implies that \( u_\infty \) is a \( C^1 \)-function which is bounded as \( u_\ell \) is (see the lemmas 2.4 and 2.6).

Let us set \( A = |\partial_{x_1} u_\ell|^p + |\partial_{x_2} (u_\ell - u_\infty)|^q \). For \( \sigma > 0 \) consider as in [1]

\[
\varphi = (e^{-\sigma|x_1|} - e^{-\sigma \ell})
\]

in (2.19). Taking into account the lemma 2.3 and the fact that \( \partial_{x_1} \varphi = \pm \sigma e^{-\sigma|x_1|} \) we get

\[
\int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} (u_\ell - u_\infty)|^q \varphi \, dx \leq C \int_{\Omega_\ell} e^{-\sigma|x_1|} |\partial_{x_1} u_\ell|^p - |\partial_{x_2} (u_\ell - u_\infty)|^q \partial_{x_2} (u_\ell - u_\infty) \varphi \, dx
\]

(2.37)

Using the Young inequality in this last integral i.e. \( |a||b| \leq \frac{1}{p'} |a|^{p'} + \frac{1}{q'} |b|^{q'} \) we get for some new constants

\[
\int_{\Omega_\ell} A (e^{-\sigma|x_1|} - e^{-\sigma \ell}) \, dx \leq \sigma C \int_{\Omega_\ell} e^{-\sigma|x_1|} \{ |\partial_{x_1} u_\ell|^p + |u_\ell - u_\infty|^q \} \, dx.
\]

(2.38)

(2.39)

(2.40)

That is to say

\[
e^{-\frac{\sigma \ell}{2}} \int_{\Omega_\ell} A \, dx \leq 2e^{-\frac{\sigma \ell}{2}} \int_{\Omega_\ell} A \, dx.
\]

(2.41)

It follows from the lemma 2.1 that

\[
\int_{\Omega_\ell} A \, dx = \int_{\Omega_\ell} |\partial_{x_1} u_\ell|^p + |\partial_{x_2} (u_\ell - u_\infty)|^q \, dx \leq 2e^{-\frac{\sigma \ell}{2}} \int_{\Omega_\ell} A \, dx \leq C e^{-\frac{\sigma \ell}{2}}.
\]

The result follows by choosing \( \alpha < \frac{\sigma}{2} \). \( \Box \)

The last case to address is when \( p \geq q, q < 2 \). In this case one can prove:
Theorem 2.5. Suppose that \( p \geq q, q < 2, f \in L^1(\omega) \). It holds for some positive constants \( C \)
\[
\int_{\Omega_{\frac{1}{2}}} |\partial x_1 u_{\ell}|^p + |\partial x_2 (u_{\ell} - u_\infty)|^q \, dx \leq \frac{C}{\ell^{\frac{pq}{2} - 1}}.
\] (2.42)

Proof. Choosing \( \rho \) as in (2.21) one has - see (2.16), (2.23)
\[
\tilde{I} = \int_{\Omega_{\ell}} |\partial x_1 u_{\ell}|^p + S^{q-2} |\partial x_2 (u_{\ell} - u_\infty)|^2 \rho^\alpha \, dx
\]
\[
\leq \frac{C}{\ell} \int_{\Omega_{\ell}} |\partial x_1 u_{\ell}|^{p-1} |u_{\ell} - u_\infty| \rho^{\alpha-1} \, dx
\]
\[
\leq \frac{C}{\ell} \left( \int_{\Omega_{\ell}} |\partial x_1 u_{\ell}|^p \rho^\alpha \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega_{\ell}} |u_{\ell} - u_\infty|^p \rho^{p-q} \, dx \right)^{\frac{1}{p}}
\]
\[
\leq \frac{C}{\ell} \tilde{I}^\frac{1}{p} \left( \int_{\Omega_{\ell}} |u_{\ell} - u_\infty|^q \rho^{p-q} \, dx \right)^{\frac{1}{p}} \leq \frac{C}{\ell} \tilde{I}^\frac{1}{p} \left( \int_{\Omega_{\ell}} |\partial x_2 (u_{\ell} - u_\infty)|^q \rho^{p-q} \, dx \right)^{\frac{1}{p}}.
\] (2.43)

Recall that \( S = |\partial x_2 (u_{\ell})| + |\partial x_1 (u_{\infty})| \). In the two last inequalities we used the fact that \( u_{\ell} \) and \( u_\infty \) are uniformly bounded and the Poincaré inequality. Arguing as before we have
\[
\int_{\Omega_{\ell}} |\partial x_2 (u_{\ell} - u_\infty)|^q \rho^{p-q} \, dx = \int_{\Omega_{\ell}} S^{q-2} |\partial x_2 (u_{\ell} - u_\infty)|^q \rho^{\alpha \frac{q}{2} - (2-q)\frac{q}{2} \rho^{p-q}} \, dx
\]
\[
\leq \left( \int_{\Omega_{\ell}} S^{q-2} |\partial x_2 (u_{\ell} - u_\infty)|^2 \rho^\alpha \, dx \right)^{\frac{q}{2}} \left( \int_{\Omega_{\ell}} S^q \, dx \right)^{1 - \frac{q}{2}}
\]
\[
\leq C \tilde{I}^{\frac{q}{2}} \ell^{1 - \frac{q}{2}}.
\] (2.44)

provided \( \alpha (1 - \frac{q}{2}) - p > 0 \). Thus from (2.43) we derive
\[
\tilde{I} \leq \frac{C}{\ell} \tilde{I}^\frac{1}{p} \left( \tilde{I}^{\frac{q}{2}} \ell^{1 - \frac{q}{2}} \right)^{\frac{1}{p}} \iff \tilde{I}^{\frac{1}{p} - \frac{q}{p}} \leq \frac{C}{\ell} \ell^{\frac{1}{2p} - \frac{q}{2p}} \iff \tilde{I} \leq \frac{C}{\ell^{\frac{pq}{2} - 1}}.
\] (2.45)

Going back to (2.44) one has if \( \alpha > p \)
\[
\int_{\Omega_{\ell}} |\partial x_2 (u_{\ell} - u_\infty)|^q \, dx \leq \int_{\Omega_{\ell}} |\partial x_2 (u_{\ell} - u_\infty)|^q \rho^{p-q} \, dx \leq \frac{C}{\left( \ell^{\frac{2p}{2q} - 1} \right)^\frac{1}{2}} \ell^{1 - \frac{q}{2}} = \frac{C}{\ell^{\frac{pq}{2} - 1}}.
\]

Combining this with (2.45) leads easily to (2.42) since \( pq < 2p \). This completes the proof. \( \square \)

3 Some generalisations

Let us denote by \( \omega_1 \) a bounded convex domain of \( \mathbb{R}^m \) containing 0 and by \( \omega_2 \) a bounded domain in \( \mathbb{R}^{n-m}, m \geq 1 \). Let us set for \( \ell > 0 \)
\[
\Omega_{\ell} = \ell \omega_1 \times \omega_2.
\]
We will denote the points in $\Omega_\ell$ by
\[ x = (X_1, X_2) \]
where $X_1 = x_1, \ldots, x_m$, $X_2 = x_{m+1}, \ldots, x_n$. If $p_i, i = 1, \ldots, m$, $q_j, j = m+1, \ldots, n$ are numbers larger than 1 set
\[ \bar{p} = (p_1, \ldots, p_m), \quad \bar{q} = (q_{m+1}, \ldots, q_n). \]

Then we define
\[ W^{1, \bar{p}, \bar{q}}(\Omega_\ell) = \{ v \in L^{p_i(\Omega_\ell)} \cap L^{q_j(\Omega_\ell)} \mid \partial_{x_i} v \in L^{p_i(\Omega_\ell)}, \partial_{x_j} v \in L^{q_j(\Omega_\ell)}, \forall i, j \}, \]
\[ W^{1, \bar{q}}(\omega_2) = \{ v \in L^{q_j(\omega_2)} \mid \partial_{x_j} v \in L^{q_j(\omega_2)}, \forall j \}. \tag{3.1} \]

In the definition above the indices $i$ are running from 1 to $m$ and the indices $j$ from $m+1$ to $n$. Clearly $W^{1, \bar{p}, \bar{q}}(\Omega_\ell)$, $W^{1, \bar{q}}(\omega_2)$ are reflexive Banach spaces when equipped with the norms
\[ ||v||_{1, \bar{p}, \bar{q}} = \sum_{i=1}^m (|v|_{p_i, \Omega_\ell} + |\partial_{x_i} v|_{p_i, \Omega_\ell}) + \sum_{j=m+1}^n (|v|_{q_j, \Omega_\ell} + |\partial_{x_j} v|_{q_j, \Omega_\ell}) \tag{3.2} \]
\[ ||v||_{1, \bar{q}} = \sum_{j=m+1}^n (|v|_{q_j, \Omega_\ell} + |\partial_{x_j} v|_{q_j, \Omega_\ell}). \]

One denotes by $W^{1, \bar{p}, \bar{q}}(\Omega_\ell)$ (respectively $W^{1, \bar{q}}(\omega_2)$) the closure of $D(\Omega_\ell)$ (respectively $D(\omega_2)$) in these spaces and by $u_\ell$ the solution to
\[ \begin{cases} u_\ell \in W^{1, \bar{p}, \bar{q}}(\Omega_\ell), \\ \int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i-2} \partial_{x_i} u_\ell \partial_{x_i} v + |\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell \partial_{x_j} v \, dx = \langle f, v \rangle \quad \forall v \in W^{1, \bar{p}, \bar{q}}(\Omega_\ell). \tag{3.3} \end{cases} \]

In the formula above we make the summation convention, i.e. we are summing in $i$ and $j$. $f = f(X_2)$ is a continuous linear form on $W^{1, \bar{q}}(\omega_2)$ defined as
\[ f = f_0 - \sum_{j=m+1}^n \partial_{x_j} f_j, \quad f_0 \in \cap_j L^{q_j}(\omega_2), \quad f_j \in L^{q_j}(\omega_2), \tag{3.4} \]
\[ \langle f, v \rangle = \sum_{\omega_2} f_0 v + \sum_{j=m+1}^n f_j \partial_{x_j} v \, dx. \]

We would like to sketch some behaviour of $u_\ell$ when $\ell \to \infty$, in particular to show that $u_\ell \to u_\infty$ where $u_\infty$ is the solution to
\[ \begin{cases} u_\infty \in W^{1, \bar{q}}(\omega_2), \\ \int_{\omega_2} |\partial_{x_j} u_\infty|^{q_j-2} \partial_{x_j} u_\infty \partial_{x_j} v \, dX_2 = \langle f, v \rangle \quad \forall v \in W^{1, \bar{q}}(\omega_2). \tag{3.5} \end{cases} \]

Note that by the same arguments as in section 2 the problems (3.3), (3.5) admit a unique solution.

The analogue of lemma 2.1 is the following.
Lemma 3.1. Let $u_\ell$ be the solution of (3.3) for $f$ given by (3.4). There exists a constant $C$ independent of $\ell$ such that

$$
\int_{\Omega_\ell} \sum_{i=1}^m |\partial_{x_i} u_\ell|^{p_i} + \sum_{j=m+1}^n |\partial_{x_j} u_\ell|^{q_j} \, dx \leq C\ell^m. \tag{3.6}
$$

Proof. Let $q = \max(q_j) = q_{j_0}$ for some $j_0$. Taking $v = u_\ell$ in (3.3) we get with the summation convention in $i,j$

$$
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx = \int_{\Omega_\ell} \{ f_0 u_\ell + f_{j_0} \partial_{x_{j_0}} u_\ell + \sum_{j \neq j_0} f_j \partial_{x_j} u_\ell \} \, dx \leq \int_{\Omega_\ell} \{ |f_0| |u_\ell| + |f_{j_0}| \|\partial_{x_{j_0}} u_\ell\| + \sum_{j \neq j_0} |f_j| |\partial_{x_j} u_\ell| \} \, dx \leq |f_0| q_{q', \Omega_\ell} |u_\ell|_{q, \Omega_\ell} + |f_{j_0}| q_{q', \Omega_\ell} \|\partial_{x_{j_0}} u_\ell\|_{q_{j_0}, \Omega_\ell} + \sum_{j \neq j_0} |f_j| q_{q', \Omega_\ell} \|\partial_{x_j} u_\ell\|_{q_j, \Omega_\ell}. \tag{3.7}
$$

Using the Poincaré inequality

$$
|u_\ell|_{q, \Omega_\ell} = |u_\ell|_{q_{j_0}, \Omega_\ell} \leq C |\partial_{x_{j_0}} u_\ell|_{q_{j_0}, \Omega_\ell}
$$

we derive

$$
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx \leq \{ C|f_0| q_{q', \Omega_\ell} + |f_{j_0}| q_{q', \Omega_\ell} \} |\partial_{x_{j_0}} u_\ell|_{q_{j_0}, \Omega_\ell} + \sum_{j \neq j_0} |f_j| q_{q', \Omega_\ell} \|\partial_{x_j} u_\ell\|_{q_j, \Omega_\ell}. \tag{3.8}
$$

Note now that for $f \in L^q(\omega_2)$ one has for some constant $C$ independent of $\ell$

$$
|f|_{q', \Omega_\ell} = \left( \int_{\omega_1} \int_{\omega_2} |f(X_2)|^{q'} dX_2 dX_1 \right)^{\frac{1}{q'}} \leq C\ell^m. \tag{3.9}
$$

Thus we get

$$
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx \leq C \sum_{j} \ell_{q_j}^m |\partial_{x_j} u_\ell|_{q_j, \Omega_\ell} \leq \epsilon \sum_{j} |\partial_{x_j} u_\ell|_{q_j, \Omega_\ell} + C_\epsilon \ell^m, \tag{3.10}
$$

using the Young inequality $|ab| \leq \epsilon|a|^q + C_\epsilon|b|^{q'}$. The result follows by choosing $\epsilon = \frac{1}{2}$. \hfill \square

With the same proofs we have the analogues of Lemmas 2.2 and 2.4 namely with the summation convention

$$
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i-2} \partial_{x_i} u_\ell \partial_{x_i} v + \{ |\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\ell - |\partial_{x_j} u_\ell|^{q_j-2} \partial_{x_j} u_\infty \} \partial_{x_j} v \, dx = 0 \tag{3.11}
$$

for $v \in W^{1, q, \ell}_{0}(\Omega_\ell)$.

Similarly if $u_\ell = u_\ell(f)$ is the solution to (3.3) and $u_\infty = u_\infty(f)$ the solution to (3.5) and if $f_1 \geq f_2$, $f \geq 0$ then one has

$$
u_\ell(f_2) \leq u_\ell(f_1),\quad 0 \leq u_\ell(f) \leq u_\infty(f). \tag{3.12}
$$
Remark 1. Note that (3.10) allows a perhaps simpler proof of (3.6) where, however, the dependence in \( f \) is lost. Indeed taking \( v = u_\ell \) in (3.10) we get with the summation convention in \( i \) and \( j \)

\[
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx = \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty \partial_{x_j} u_\ell \, dx \\
\leq \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j-1}|\partial_{x_j} u_\ell| \, dx.
\]

Using the Young inequality \(|ab| \leq \frac{1}{q_j} |a|^{q_j} + \frac{1}{q_j} |b|^{q_j}\) it comes

\[
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx \leq \int_{\Omega_\ell} \frac{1}{q_j} |\partial_{x_j} u_\infty|^{q_j} + \frac{1}{q_j} |\partial_{x_j} u_\ell|^{q_j} \, dx \\
\leq \int_{\Omega_\ell} \frac{1}{\min_j q_j} |\partial_{x_j} u_\ell|^{q_j} + \frac{1}{\min_j q_j} |\partial_{x_j} u_\infty|^{q_j} \, dx
\]

and thus for some constant \( C \) since \( \min_j q_j > 1 \)

\[
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \, dx \leq C \int_{\Omega_\ell} |\partial_{x_j} u_\infty|^{q_j} \, dx \leq C \ell^m.
\]

Then we can turn to the generalisation of lemma 2.6. Denote by \( \rho = \rho(X_1) \) a smooth function such that

\[
0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } \frac{1}{2}\omega_1, \quad \rho = 0 \text{ near } \partial\omega_1, \quad |\nabla X_1 \rho| \leq C, \quad (3.12)
\]

where \( \nabla X_1 \rho \) denotes the gradient of \( \rho \) in \( X_1 \), i.e. \( \nabla X_1 \rho = (\partial_{x_1} \rho, \ldots, \partial_{x_m} \rho) \).

We can show:

Lemma 3.2. Let \( f = f_0 \in L^q(\omega_2), \quad q = \max q_j \) and \( u_\ell, u_\infty \) be the solutions to (3.3), (3.5). Then it holds for some constant \( C \) independent of \( \ell \)

\[
I = \int_{\Omega_\ell} \left\{ |\partial_{x_i} u_\ell|^{p_i} + \left( |\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty \right) \partial_{x_j} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \\
\leq \frac{C}{\ell^{p-m}}. \quad (3.13)
\]

(\( p = \min_i (p_i) \) denotes the smallest \( p_i \))

Proof. From (3.10) taking \( v = \rho^\alpha (\frac{X_1}{\ell}) (u_\ell - u_\infty) \) one derives easily with the summation convention in \( i \) and \( j \)

\[
\int_{\Omega_\ell} \left\{ |\partial_{x_i} u_\ell|^{p_i} + \left( |\partial_{x_j} u_\ell|^{q_j-2}\partial_{x_j} u_\ell - |\partial_{x_j} u_\infty|^{q_j-2}\partial_{x_j} u_\infty \right) \partial_{x_j} (u_\ell - u_\infty) \right\} \rho^\alpha \, dx \\
\leq \frac{C}{\ell} \int_{\Omega_\ell} |\partial_{x_j} u_\ell|^{p_j-1}|u_\ell - u_\infty|\rho^{\alpha-1} \, dx. \quad (3.14)
\]
Recalling our summation in $i$ and the fact that $p_i > 1$ it follows that for some constant $C$

$$I \leq C \sum_{i} \frac{1}{p_i} \int_{\Omega} |u_\ell - u_\infty|^{p_i} \rho^{\alpha - p_i} \, dx \leq C \sum_{i} \frac{1}{p_i} \int_{\Omega} |u_\ell - u_\infty|^{p_i} \, dx,$$

provided we chose $\alpha$ large enough. Note that at this point we did not use the assumption $f = f_0 \in L^q(\omega_2)$, $q = \max q_j$.

Arguing now like in Lemma 2.6 one can bound $|u_\ell - u_\infty|$ by something depending only on $X_2$ to get

$$I \leq C \sum_{i} \frac{1}{p_i - m},$$

(3.17)

This completes the proof of the lemma. □

The convergence of $u_\ell$ toward $u_\infty$ is insured for general $p_i, q_j$ by the following result.

**Theorem 3.1.** Let $f = f_0 \in L^q(\omega_2)$, $q = \max q_j$. Let $u_\ell, u_\infty$ be the solutions to (3.3), (3.5) respectively. If $p_i > m \forall i$ one has for every $\ell_0 > 0$ when $\ell \to +\infty$

$$\partial_{x_i} u_\ell \to 0 \text{ in } L^{p_i}(\Omega_{\ell_0}) , \quad \partial_{x_j} u_\ell \to \partial_{x_j} u_\infty \text{ in } L^{q_j}(\Omega_{\ell_0}).$$

(3.18)

**Proof.** The first part of (3.18) follows directly from (3.13). For the second part let us consider a smooth function $\rho$ such that for $\ell_0 < \ell - 1$

$$0 \leq \rho \leq 1 , \quad \rho = 1 \text{ on } \ell_0 \omega_1 , \quad \rho \text{ has a support in } (\ell_0 + 1) \omega_1 , \quad |\nabla_{x_i} \rho| \leq C.$$

Then $\rho^{\alpha} u_\ell$ is a test function for (3.3) and one has

$$\int_{\Omega_{\ell_0 + 1}} \left\{ |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \right\} \rho^{\alpha} \, dx$$

$$= -\alpha \int_{\Omega_{\ell_0 + 1}} |\partial_{x_i} u_\ell|^{p_i - 2} \partial_{x_i} u_\ell \partial_{x_j} \rho \, u_\ell \rho^{\alpha - 1} \, dx + \int_{\Omega_{\ell_0 + 1}} f_0 u_\ell \, dx$$

$$\leq \alpha C \int_{\Omega_{\ell_0 + 1}} |\partial_{x_i} u_\ell|^{p_i - 1} u_\ell \rho^{\alpha - 1} \, dx + \int_{\Omega_{\ell_0 + 1}} |f_0| u_\ell \, dx.$$

Using the fact that $\rho^{\alpha - 1} = \rho^{\frac{1}{p_i}} \rho^{\frac{1}{p_i} - 1}$ we get by Young’s inequality for some constant $C$

$$\int_{\Omega_{\ell_0 + 1}} \left\{ |\partial_{x_i} u_\ell|^{p_i} + |\partial_{x_j} u_\ell|^{q_j} \right\} \rho^{\alpha} \, dx$$

$$\leq \frac{1}{p_i} \int_{\Omega_{\ell_0 + 1}} |\partial_{x_i} u_\ell|^{p_i} \rho^{\alpha} \, dx + C \int_{\Omega_{\ell_0 + 1}} |u_\ell|^{p_i} \rho^{\alpha - p_i} \, dx + \int_{\Omega_{\ell_0 + 1}} |f_0| u_\ell \, dx.$$
We assume here that $\alpha > p_i$. We know that by (3.11)

$$|u_\ell| \leq \max\{u_\infty(f_0^+, u_\infty(f_0^-)\}$$

and since this bound is independent of $\ell$ we get

$$|\partial_x u_\ell|_{q_j, \Omega_0} \leq C(\ell_0).$$

The rest of the proof follows as in Theorem 2.1 since by (3.13) we have for every $j$

$$\int_{\Omega_0} \left( |\partial_x u_\ell|_{q_j}^{-2} \partial_x u_\ell - |\partial_x u_\infty|_{q_j}^{-2} \partial_x u_\infty \right) \partial_x (u_\ell - u_\infty) \, dx \to 0.$$ 

This completes the proof of the theorem. 

We have then an analogue to theorem 2.3 for a general $f$.

**Theorem 3.2.** Let $u_\ell, u_\infty$ be the solutions to (3.3), (3.5). Suppose that

$$\forall i = 1, \ldots, m, \exists j_i \in \{m + 1, \ldots, n\} \text{ such that } p_i < q_{j_i}. \quad (3.19)$$

Then there exists a constant $C$ such that

$$I = \int_{\Omega_0} \left\{ |\partial_x u_\ell|_{p_i}^{p_i} + \left( |\partial_x u_\ell|_{q_j}^{-2} \partial_x u_\ell - |\partial_x u_\infty|_{q_j}^{-2} \partial_x u_\infty \right) \partial_x (u_\ell - u_\infty) \right\} \rho^\alpha \, dx$$

$$\leq C \sum_{q_i \geq 2} \frac{1}{q_j \ell_0^{q_j - p_i}} + C \sum_{q_i < 2} \frac{1}{q_j \ell_0^{q_j - p_i}} \quad (3.20)$$

$\rho^\alpha = \rho^\alpha(\frac{\ell_0}{\ell})$ is as in Lemma 3.2.

**Proof.** Going back to (3.16) one has if $(\alpha - p_i) \frac{q_j}{p_i} > \alpha \forall i$

$$I \leq C \sum_i \frac{1}{\ell_0^p} \int_{\Omega_0} |u_\ell - u_\infty|_{p_i} \rho^{\alpha - p_i} \, dx$$

$$\leq C \sum_i \frac{1}{\ell_0^p} \left( \int_{\Omega_0} |u_\ell - u_\infty|_{q_i} \rho^{(\alpha - p_i) \frac{q_j}{p_i}} \, dx \right) \frac{p_i}{q_j} \left( \int_{\Omega_0} 1 \, dx \right)^{1 - \frac{p_i}{q_j}}$$

$$\leq C \sum_i \frac{1}{\ell_0^p} \left( \int_{\Omega_0} |u_\ell - u_\infty|_{q_i} \rho^{\alpha} \, dx \right) \frac{p_i}{q_j} \ell_0^{m(1 - \frac{p_i}{q_j})}. \quad (3.21)$$

Using the Poincaré inequality we get

$$I \leq C \sum_i \frac{1}{\ell_0^p} \left( \int_{\Omega_0} |\partial_x u_\ell|_{q_j}^{-2} \partial_x u_\ell - |\partial_x u_\infty|_{q_j}^{-2} \partial_x u_\infty \right) \partial_x (u_\ell - u_\infty) \geq C_{q_j} |\partial_x (u_\ell - u_\infty|_{q_j}) \quad (3.22)$$

If $q_{j_i} \geq 2$ one has

$$\{ |\partial_x u_\ell|_{q_j}^{-2} \partial_x u_\ell - |\partial_x u_\infty|_{q_j}^{-2} \partial_x u_\infty \} \partial_x (u_\ell - u_\infty) \geq C_{q_j} |\partial_x (u_\ell - u_\infty|_{q_j}) \quad (3.22)$$
and thus for some constant
\[ \int_{\Omega} |\partial_{x_j}(u_\ell - u_\infty)|^{q_j} \rho^\alpha \, dx \leq CI. \tag{3.23} \]

If \( p_i < q_{j_i} < 2 \) one has (see the theorem 2.3)
\[ \int_{\Omega} |\partial_{x_j}(u_\ell - u_\infty)|^{q_j} \rho^\alpha \, dx \]
\[ \leq CI \frac{q_j}{2} \left( \int_{\Omega} \partial_{x_j} u_\ell \partial_{x_j} u_\infty \right)^{1-q_j} \]
\[ \leq CI \frac{q_j}{2} \ell^{m(1-\frac{q_j}{2})}. \tag{3.24} \]

Thus from (3.21) we derive replacing \( p_i \) by \( p \) and \( q_{j_i} \) by \( q \)
\[ I \leq C \sum_i \frac{1}{\ell^p} \ell^{m(1-\frac{p}{4})} + C \sum_i \frac{1}{\ell^p} \ell^{m(1-\frac{p}{4})} \ell^{m(1-\frac{p}{4})} \]
\[ \leq C \sum_i \frac{1}{\ell^p} \ell^{m(1-\frac{p}{4})} + C \sum_i \frac{1}{\ell^p} \ell^{m(1-\frac{p}{4})}. \tag{3.25} \]

The first sum is for \( i \) such that \( q_{j_i} \geq 2 \), the second one for the \( i \)'s such that \( q_{j_i} < 2 \). Using the Young inequality with \( \epsilon \) we get
\[ I \leq \epsilon I + C \epsilon \sum_i \left( \frac{\ell^{m(1-\frac{p}{4})}}{\ell^p} \right)^{\frac{1}{1-p}} + \epsilon I + C \epsilon \sum_i \left( \frac{\ell^{m(1-\frac{p}{4})}}{\ell^p} \right)^{\frac{1}{1-p}}. \tag{3.26} \]

Choosing \( \epsilon \) small enough we get
\[ I \leq C \epsilon \sum_i \frac{1}{\ell^{p-m}} + C \epsilon \sum_i \frac{1}{\ell^{2p-m}}. \tag{3.27} \]

Coming back to our notation in \( p_i, q_{j_i} \) (3.19) follows. This completes the proof of the theorem. \( \square \)

**Theorem 3.3.** We suppose that \( p_i \geq 2, \forall i = 1, \ldots, m \). In addition we assume that
\[ \forall i = 1, \ldots, m, \exists j_i \in \{ m+1, \ldots, n \} \text{ such that } p_i = q_{j_i}. \tag{3.28} \]

Then there exists constants \( C, \alpha \) independent of \( \ell \) such that
\[ \int_{\Omega} \sum_{i=1}^m \left| \partial_{x_i}(u_\ell - u_\infty) \right|^{p_i} + \left| \partial_{x_{j_i}}(u_\ell - u_\infty) \right|^{p_i} \]
\[ + \sum_{j \neq j_i} \left\{ \left| \partial_{x_j} u_\ell \right|^{q_j-2} \partial_{x_j} u_\ell - \left| \partial_{x_j} u_\infty \right|^{q_j-2} \partial_{x_j} u_\infty \right\} \partial_{x_j}(u_\ell - u_\infty) \, dx \leq Ce^{-\alpha \ell}. \tag{3.29} \]

Note that when \( j \neq j_i \), \( q_j \) is only assumed to be greater than 1 and that the \( q_{j_i} \) are not necessarily distinct as the \( p_i \).
\textbf{Proof.} For }\ell_1 \leq \ell - 1\text{ we denote by }\rho_{\ell_1} = \rho_{\ell_1}(X_1)\text{ a smooth function satisfying }

\begin{equation}
0 \leq \rho_{\ell_1} \leq 1 \ , \ \rho_{\ell_1} = 1 \text{ on } \ell_1 \omega_1 \ , \ \rho_{\ell_1} = 0 \text{ outside } (\ell_1 + 1) \omega_1 \ , \ |\nabla_{X_1} \rho_{\ell_1}| \leq C \tag{3.30}
\end{equation}

where }C\text{ is some positive constant. Taking } v = \rho_{\ell_1}(u_\ell - u_\infty)\text{ as test function in (3.10) we get }

\begin{align}
\int_{\Omega_{\ell_1+1}} \left\{ \sum_{i=1}^{m} |\partial_x u_\ell|^{p_i} \right. & \left. + \sum_{j=m+1}^{n} \{ |\partial_x^2 u_\ell|^{q_j-2} \partial_x u_\ell - |\partial_x u_\infty|^{q_j-2} \partial_x u_\infty \} \partial_x (u_\ell - u_\infty) \right\} \rho_{\ell_1} \ dx \\
& \leq \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} \sum_{i=1}^{m} |\partial_x u_\ell|^{p_i} \partial_x (u_\ell - u_\infty) \ dx \\
& \leq C \sum_{i=1}^{m} \left( \int_{D_{\ell_1}} |\partial_x u_\ell|^{p_i} \ dx \right)^{\frac{1}{p_i}} \left( \int_{D_{\ell_1}} |u_\ell - u_\infty|^{q_i} \ dx \right)^{\frac{1}{q_i}} \tag{3.31}
\end{align}

where we have set }D_{\ell_1} = \Omega_{\ell_1+1} \setminus \Omega_{\ell_1}.\text{ Let us define }A\text{ as }

\begin{align}
A &= \sum_{i=1}^{m} |\partial_x (u_\ell - u_\infty)|^{p_i} + |\partial_x^2 (u_\ell - u_\infty)|^{q_i} \\
& \quad + \sum_{j \neq j_i} \{ |\partial_x^2 u_\ell|^{q_j-2} \partial_x u_\ell - |\partial_x u_\infty|^{q_j-2} \partial_x u_\infty \} \partial_x (u_\ell - u_\infty). \tag{3.32}
\end{align}

Using the lemma 2.3, (3.32) and the Poincaré inequality on the section of the domain one deduces from (3.31)

\begin{align}
\int_{\Omega_{\ell_1}} A \ dx & \leq \int_{\Omega_{\ell_1+1}} A \rho_{\ell_1} \ dx \\
& \leq C \sum_{i=1}^{m} \left( \int_{D_{\ell_1}} |\partial_x u_\ell|^{p_i} \ dx \right)^{\frac{1}{p_i}} \left( \int_{D_{\ell_1}} |\partial_x^2 (u_\ell - u_\infty)|^{q_i} \ dx \right)^{\frac{1}{q_i}} \tag{3.33}
\end{align}

It follows that

\begin{equation}
\int_{\Omega_{\ell_1}} A \ dx \leq \frac{C}{C+1} \int_{\Omega_{\ell_1+1}} A \tag{3.34}
\end{equation}

Denote by }\lfloor \frac{\ell}{2} \rfloor\text{ the integer part of }\frac{\ell}{2}.\text{ Setting } a = \frac{C}{C+1} \text{ and iterating this formula }\lfloor \frac{\ell}{2} \rfloor\text{ times starting from }\frac{\ell}{2} \text{ we obtain easily taking into account the inequality }\frac{\ell}{2} - 1 < \lfloor \frac{\ell}{2} \rfloor \leq \frac{\ell}{2}\text{ }

\begin{align}
\int_{\Omega_{\frac{\ell}{2}}} A \ dx & \leq a \lfloor \frac{\ell}{2} \rfloor \int_{\Omega_{\frac{\ell}{2} + \lfloor \frac{\ell}{2} \rfloor}} A \ dx \leq a^{\frac{\ell}{2} - 1} \int_{\Omega_{\ell}} A \ dx. \tag{3.35}
\end{align}
To evaluate this last integral one relies on the lemma 3.1. Indeed using the lemma 2.3 one has

\[ A \leq \sum_{i=1}^{m} (|\partial_{x_i} u| + |\partial_{x_i} u_{\infty}|)^{p_i} + (|\partial_{x_j} u_{\ell}| + |\partial_{x_j} u_{\infty}|)^{p_i} + \sum_{j \neq j_i} C_{q_j} (|\partial_{x_j} u_{\ell}| + |\partial_{x_j} u_{\infty}|)^{q_j} \]

Using again the formula \(|a| + |b| \leq 2^{q-1}(|a| + |b|)\) one derives for some constant

\[ A \leq C \left\{ \sum_{i=1}^{m} |\partial_{x_i} u|^{p_i} + |\partial_{x_i} u_{\infty}|^{p_i} + \sum_{j=m+1}^{n} |\partial_{x_j} u_{\ell}|^{q_j} + |\partial_{x_j} u_{\infty}|^{q_j} \right\} \]

Since \(u_{\infty}\) is independent of \(X_1\) it follows from (3.6) that

\[ \int_{\Omega_{\ell}} A \, dx \leq C_{\ell}^{m} \]

and from (3.35) one derives

\[ \int_{\Omega_{\ell}} A \, dx \leq C e^{-\frac{1}{2} \ln \frac{1}{a} \ell^{m}}. \quad (3.36) \]

This leads to (3.29) provided we chose \(\alpha < \frac{1}{2} \ln \frac{1}{a}\).

In the case where \(f \in L^{\infty}(\omega_2)\) one can show the following.

**Theorem 3.4.** We suppose that \(f \in L^{\infty}(\omega_2)\) and \(p_i \geq 2, \forall i = 1, \cdots, m\). In addition we assume that

\[ \forall i = 1, \cdots, m, \exists j_i \in \{m + 1, \ldots, n\} \text{ such that } p_i \geq q_{j_i} \geq 2. \quad (3.37) \]

Then there exists constants \(C, \alpha\) independent of \(\ell\) such that

\[ \int_{\Omega_{\ell}} \sum_{i=1}^{m} |\partial_{x_i} (u_{\ell} - u_{\infty})|^{p_i} + |\partial_{x_j} (u_{\ell} - u_{\infty})|^{p_i} \]

\[ + \sum_{j \neq j_i} \{ |\partial_{x_j} u_{\ell}|^{q_{j_i} - 2} \partial_{x_j} u_{\ell} - |\partial_{x_j} u_{\infty}|^{q_{j_i} - 2} \partial_{x_j} u_{\infty} \} \partial_{x_j} (u_{\ell} - u_{\infty}) \, dx \leq C e^{-\alpha \ell}. \quad (3.38) \]

**Proof.** As in Theorem 3.3 we derive (3.31) that is

\[ \int_{\Omega_{\ell+1}} \left\{ \sum_{i=1}^{m} |\partial_{x_i} u_{\ell}|^{p_i} \right\}

\[ + \sum_{j=m+1}^{n} \{ |\partial_{x_j} u_{\ell}|^{q_{j_i} - 2} \partial_{x_j} u_{\ell} - |\partial_{x_j} u_{\infty}|^{q_{j_i} - 2} \partial_{x_j} u_{\infty} \} \partial_{x_j} (u_{\ell} - u_{\infty}) \} \rho_{\ell_i} \, dx \]

\[ \leq C \sum_{i=1}^{m} \left( \int_{D_{\ell_i}} |\partial_{x_i} u_{\ell}|^{p_i} \, dx \right)^{\frac{1}{p_i}} \left( \int_{D_{\ell_i}} |u_{\ell} - u_{\infty}|^{p_i} \, dx \right)^{\frac{1}{p_i}}, \]

where \(D_{\ell_1} = \Omega_{\ell_1+1} \setminus \Omega_{\ell_1}\).
One claims that for some constant $C$ independent of $\ell$ one has
\[ |u_\ell - u_\infty| \leq C. \quad (3.40) \]

Then, we derive that
\[
\left( \int_{D_{\ell_1}} |u_\ell - u_\infty|^{\frac{p_i}{q_i}} \, dx \right)^{\frac{1}{p_i}} \leq C \left( \int_{D_{\ell_1}} |u_\ell - u_\infty|^{\frac{q_i}{q_i}} \, dx \right)^{\frac{1}{p_i}}.
\]

Recalling the notation (3.32), (3.33) follows easily and the rest of the proof as well.

To prove (3.40), suppose that $\omega_2$ is contained in the strip
\[ \{ X_2 \mid |x_n| \leq s \} \]
for some positive $s$ (recall that $\omega_2$ is supposed to be bounded in $\mathbb{R}^{n-m}$). Then set
\[ \beta = 1 + \frac{1}{q_n - 1}, \quad g = s^\beta - |x_n|^\beta. \]

One finds easily since $(\beta - 1)(q_n - 1) = 1$ that
\[ \partial_{x_n} g = -\beta |x_n|^\beta - 2 x_n, \quad |\partial_{x_n} g|^{q_n - 2} \partial_{x_n} g = -\beta q_n - 1 x_n, \quad -\partial_{x_n} (|\partial_{x_n} g|^{q_n - 2} \partial_{x_n} g) = \beta q_n - 1. \]

If $|f|_\infty$ denotes the $L^\infty$-norm of $f$ setting
\[ h = \left( |f|_\infty \right)^{\frac{1}{q_n - 1}} \frac{g}{\beta} \]

one has (see (3.3))
\[
- \sum_{i=1}^m \partial_{x_i} (|\partial_{x_i} u_\ell|^{p_i - 2} \partial_{x_i} u_\ell) - \sum_{j=m+1}^n \partial_{x_j} (|\partial_{x_j} u_\ell|^{q_j - 2} \partial_{x_j} u_\ell) = f \leq |f|_\infty
\]
\[ = -\partial_{x_n} (|\partial_{x_n} h|^{q_n - 2} \partial_{x_n} h) = -\sum_{i=1}^m \partial_{x_i} (|\partial_{x_i} h|^{p_i - 2} \partial_{x_i} h) - \sum_{j=m+1}^n \partial_{x_j} (|\partial_{x_j} h|^{q_j - 2} \partial_{x_j} h). \]

Using in the weak formulation $v = (u_\ell - h)^+$ one deduces easily that
\[ u_\ell(f) \leq h \leq (|f|_\infty)^{\frac{1}{q_n - 1}} \frac{s^\beta}{\beta}. \]

Since $-u_\ell(f) = u_\ell(-f)$, (3.40) follows easily. This completes the proof of the theorem. \qed

**Remark 2.** One could try to mix assumptions of the type of Theorem 3.2 and 3.4 however it will make the result regarding the speed of convergence messy, the convergence being insured by the theorem 3.1 for the $p_i$’s large enough. In the case of theorems 3.3, 3.4 one can take advantage of the exponential speed of convergence to get existence results in unbounded domains in the spirit of [9], [7].
Remark 3. The operators that we have considered here are the sum of $p$-Laplacians in one dimension. One can consider also operators sums of $p$-Laplacians in larger dimensions. For instance, with the notation of this section, if $u_\ell$ is the weak solution to
\[
\begin{cases}
-\nabla X_1 \cdot (|\nabla X_1 u_\ell|^{p-2}\nabla X_1 u_\ell) - \nabla X_2 \cdot (|\nabla X_2 u_\ell|^{q-2}\nabla X_2 u_\ell) = f & \text{in } \Omega_\ell \\
u_\ell = 0 & \text{on } \partial \Omega_\ell,
\end{cases}
\]
one can show if $p = q \geq 2$, using the technique of theorem 3.3, that $u_\ell$ converges exponentially quickly toward the solution to
\[
\begin{cases}
-\nabla X_2 \cdot (|\nabla X_2 u_\infty|^{q-2}\nabla X_2 u_\infty) = f & \text{in } \omega_2 \\
u_\infty = 0 & \text{on } \partial \omega_2.
\end{cases}
\]
($\nabla X_i$ denotes the divergence in $\mathbb{R}^m$ or $\mathbb{R}^{n-m}$). Similarly one can consider operators sums of $p$-Laplacians of different dimensions i.e. problems of the type
\[
\begin{cases}
-\sum_i \nabla Y_i \cdot (|\nabla Y_i u_\ell|^{p_i-2}\nabla Y_i u_\ell) = f & \text{in } \Omega_\ell \\
u_\ell = 0 & \text{on } \partial \Omega_\ell,
\end{cases}
\]
where $Y_i$ denotes some subset of the coordinates, and develop results similar to the ones of this note. The case of the sum of $n$-dimensional $p$-Laplacians was considered in [8].

References


