On some stationary Navier-Stokes type problems

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Abstract

The goal of this note is to present a simple proof of existence of solution to the stationary Navier-Stokes problem using a singular perturbation technique and to address some nonlocal issues related to it.

AMS Subject Classification: 35Q30, 35J92, 35J47.

Key words: Nonlinear elliptic equations, Navier Stokes, Stationary problem.

1 Introduction and notation

We will denote by $\Omega$ a smooth open set of $\mathbb{R}^n$, $n \geq 2$. If $V$ denotes a Banach space we will denote by $V$ the space of $n$ copies of it and by $V$ the space of solenoidal vectors fields belonging to it. For instance if for $r \geq 1$, $L^r(\Omega)$ denotes the usual $L^r$-space on $\Omega$ we will denote by $\mathbb{L}^r(\Omega)$ the space defined as

$$\mathbb{L}^r(\Omega) = \{v = (v_1, \ldots, v_n) \mid v_i \in L^r(\Omega) \forall i = 1, \ldots, n\}.$$ 

We denote by $\mathbb{L}^r(\Omega)$ the space defined as

$$\mathbb{L}^r(\Omega) = \{v \in L^r(\Omega) \mid \text{div} \ v = 0 \text{ in } \Omega\},$$

($\text{div} \ v = \sum_{i=1}^n \partial_{x_i} v_i$, the derivative being taken in the distributional sense). Note that $\mathbb{L}^r(\Omega)$ is a closed subspace of the Banach space $L^r(\Omega)$ that we will suppose equipped with the usual $L^r(\Omega)$-norm defined as

$$|v|_r = \left( \int_\Omega |v(x)|^r dx \right)^{\frac{1}{r}},$$

with, in the integral, $| \ |$ denoting the usual euclidean norm in $\mathbb{R}^n$. Similarly for $p \geq 1$ we will denote by $W_0^{1,p}(\Omega)$ the space

$$W_0^{1,p}(\Omega) = \{v = (v_1, \ldots, v_n) \mid v_i \in W_0^{1,p}(\Omega) \forall i = 1, \ldots, n\},$$

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and
\[ W_0^{1,p}(\Omega) = \{ v \in W_0^{1,p}(\Omega) \mid \text{div} v = 0 \text{ in } \Omega \}, \]
where \( W_0^{1,p}(\Omega) \) is the usual Sobolev space built on \( \Omega \) (Cf. [3] - [7]). These spaces will be equipped with the norm
\[ ||v||_p = \left( \int_{\Omega} |\nabla v(x)|^p dx \right)^{\frac{1}{p}}, \]
where \( |\nabla v| \) denotes the euclidean norm of the Jacobian matrix \( \nabla v \) given by
\[ |\nabla v|^2 = \sum_{i,k} (\partial_{x_i} v_k)^2. \]

We will sometimes omit the sum using the usual summation convention. In the case when \( p = 2 \) we will prefer the more classical notation \( H_0^1(\Omega) \) to denote these spaces. It is well known that the spaces defined above are reflexive Banach spaces and we will denote the dual of \( W_0^{1,p}(\Omega) \) respectively \( H_0^1(\Omega) \) by \( W_{-1}^{1,p}(\Omega) \) respectively \( H_{-1}(\Omega) \).

For \( f \in H_{-1}(\Omega) \) we would like to solve problems of the type
\[ \begin{cases}
  u \in \hat{H}_0^1(\Omega), \\
  \langle A u, v \rangle + \int_{\Omega} (u \cdot \nabla) u \cdot v dx = \langle f, v \rangle \quad \forall v \in \hat{D}(\Omega).
\end{cases} \tag{1.1} \]

Here \( \hat{D}(\Omega) = \{ v = (v_1, \ldots, v_n) \mid v_i \in D(\Omega) \forall i = 1, \ldots, n, \text{div } v = 0 \} \) where \( D(\Omega) \) denotes the usual Schwartz space of \( C^\infty(\Omega) \)-functions with compact support. \( A \) is an operator from \( \hat{H}_0^1(\Omega) \) into \( H_{-1}(\Omega) \), \( \langle , \rangle \) denoting the duality pairing between these two spaces. We will denote by \( t \) the trilinear form appearing above and defined as
\[ t(u, w, v) = \int_{\Omega} (u \cdot \nabla) w \cdot v dx = \int_{\Omega} u_k \partial_{x_k} w_i v_i dx \]
the summation convention being used in the integral above. Usually these kinds of problems are more relevant in the physical cases of \( n = 2 \) or \( 3 \) (see [11], [1], [9], [13]). We will address them in all generality using a singular perturbation technique. The paper is divided as follows. In the next section we introduce our perturbation argument. Then we conclude with some applications.

### 2 A singular perturbation result of existence

Let \( A \) be an operator from \( \hat{H}_0^1(\Omega) \) into its dual such that
\[ A \text{ is monotone, hemicontinuous, coercive.} \tag{2.1} \]
For the latest property we will for instance assume that for some positive constant \( \nu \) one has
\[ \langle A u, u \rangle \geq \nu ||\nabla u||_2^2 \forall u \in \hat{H}_0^1(\Omega). \tag{2.2} \]
In order to solve (1.1) we are going to use a singular perturbation technique. Let us denote by \( p \) a real number such that
\[ p = 2 \text{ for } n = 2, 3 \quad \frac{3n}{n + 2} < p < n \text{ else.} \tag{2.3} \]

Then we have:
Theorem 2.1 Suppose that \( f \in \mathbb{H}^{-1}(\Omega) \). For \( \epsilon > 0 \) there exists \( u_\epsilon \) solution to
\[
\begin{aligned}
\begin{cases}
u_\epsilon &\in \tilde{\mathcal{W}}_0^{1,p}(\Omega) := \hat{\mathcal{V}}, \\
\epsilon \int_\Omega |\nabla \nu_\epsilon|^{p-2} \nabla \nu_\epsilon \cdot \nabla v dx + \langle A \nu_\epsilon, v \rangle + t(\nu_\epsilon, \nu_\epsilon, v) = \langle f, v \rangle \quad \forall \; v \in \hat{\mathcal{V}}.
\end{cases}
\end{aligned}
\tag{2.4}
\]
(\( \nabla \nu_\epsilon \cdot \nabla v \) denotes the euclidean scalar product of these two jacobian matrices considered as vectors in \( \mathbb{R}^{n^2} \))

Proof: Due to (2.3) one has \( \hat{\mathcal{V}} \subset \mathbb{H}^1(\Omega) \) and \( \mathbb{H}^{-1}(\Omega) \subset \hat{\mathcal{V}}^* \) the dual of \( \hat{\mathcal{V}} \). Moreover
\[
\mathcal{W}_0^{1,p}(\Omega) \subset \mathcal{L}^q(\Omega), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}
\]
with
\[
\frac{1}{p} < \frac{1}{3} + \frac{2}{3n} \iff \frac{3}{2p} - \frac{1}{2} < \frac{1}{n}
\]
and thus
\[
\frac{1}{q} < \frac{1}{2} - \frac{1}{2p} \iff \frac{1}{q} > \frac{2p}{p-1}.
\]
This holds also when \( p = 2 \). Note that if \( r = \frac{2p}{p-1} \) one has then \( \mathcal{W}_0^{1,p}(\Omega) \subset \mathcal{L}^r(\Omega) \) with compact embedding. One has also
\[
\frac{1}{r} + \frac{1}{r} + \frac{1}{p} = 1 
\]
and thus
\[
|t(u, w, v)| \leq |u_r| ||\nabla w||_p |v|_r. \tag{2.5}
\]

For simplicity we will denote by \( \hat{\mathcal{W}} \) the space \( \hat{\mathcal{L}}^r(\Omega) \). For \( v \in \hat{\mathcal{W}} \) we define an operator
\[A_\epsilon : \hat{\mathcal{W}} \to \hat{\mathcal{W}}^* \] by setting - dropping for simplicity the dependence in \( v \)
\[
\langle A_\epsilon u, w \rangle = \epsilon \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w dx + \langle A u, w \rangle - t(v, w, u).
\]
Indeed for any \( u \in \hat{\mathcal{V}} \) the linear form in \( w \) above is continuous on \( \hat{\mathcal{V}} \). We claim that \( A_\epsilon \) is strictly monotone. This follows from the strict monotonicity of the \( p \)-laplacian and from the fact that
\[
-t(v, u_1 - u_2, u_1) + t(v, u_1 - u_2, u_2) = -t(v, u_1 - u_2, u_1 - u_2) = 0. \tag{2.6}
\]
(Indeed for \( z \in \hat{\mathcal{D}}(\Omega) \) one has
\[
t(v, z, z) = \int_\Omega (v \cdot \nabla)z \cdot z dx = \int_\Omega v_k \partial_{x_k} z_i z_i dx = \int_\Omega v_k \partial_{x_k} |z|^2 dx = 0
\]
and by density (2.6 follows). Clearly \( A \) is hemicontinuous. Moreover by (2.2) one has
\[
\langle A_\epsilon u, u \rangle \geq \epsilon \int_\Omega |\nabla u|^p + \nu \|\nabla u\|^2_2 \quad \forall u \in \hat{\mathcal{W}}_0^{1,p}(\Omega)
\]
and \( A_\epsilon \) is coercive. Then, see [8], [3], for every \( v \in \hat{\mathcal{W}} \) there exists a unique \( u = S(v) \) solution to
\[ \begin{cases} u \in \hat{V}, \\ \epsilon \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla w dx + \langle A u, w \rangle - t(v, w, u) = \langle f, w \rangle & \forall w \in \hat{V}. \end{cases} \]

Taking \( w = u \) one derives easily
\[ \epsilon \int_{\Omega} |\nabla u|^{p} dx + \nu \left| \nabla u \right|^{2} \leq \langle f, u \rangle \leq \left| f \right|_{H^{-1}(\Omega)} \left| \nabla u \right|_{2} \]
and thus we get
\[ \epsilon \int_{\Omega} |\nabla u|^{p} dx + \nu \left| \nabla u \right|^{2} \leq \left| f \right|_{H^{-1}(\Omega)}^{2} / \nu \]
\( | \cdot |_{H^{-1}(\Omega)} \) being the strong dual norm in \( H^{-1}(\Omega) \). In particular
\[ \left| \nabla u \right|_{p} \leq \left( \left| f \right|_{H^{-1}(\Omega)}^{2} / \nu \epsilon \right)^{\frac{1}{p}} \]
and thus for some constant \( C \) independent of \( v \) we get
\[ |u|_{r} \leq C. \]
Thus, if \( B = \{ v \in \hat{W} \ | \ |v|_{r} \leq C \} \), it is clear that \( S \) maps \( B \) into itself and, by (2.7), \( S(B) \) is relatively compact into \( B \) - see above (2.5). The existence of \( u_{n} \) will follow from the Schauder fixed point theorem provided \( S \) is continuous. Thus, consider \( v_{n} \in \hat{W} \) such that
\[ v_{n} \to v \text{ in } \hat{W}. \]
Set \( u_{n} = S(v_{n}) \). From (2.7) one has for some constant \( C \) independent of \( n \)
\[ \left| \nabla u_{n} \right|_{p} \leq C. \]
Thus - up to a subsequence - one has for some \( u \in V \)
\[ \nabla u_{n} \to \nabla u \text{ in } L^{p}(\Omega) \cap L^{2}(\Omega), \ u_{n} \to u \text{ in } H^{1}_{0}(\Omega), \ u_{n} \to u \text{ in } L^{r}(\Omega). \]
By definition of \( u_{n} \) one has for any \( w \in \hat{V} \)
\[ \epsilon \int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla w dx + \langle A u_{n}, w \rangle - t(v_{n}, w, u_{n}) = \langle f, w \rangle. \]
By the Minty Lemma \( u_{n} \) is also the unique solution to
\[ \epsilon \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla (w - u_{n}) dx + \langle A w, w - u_{n} \rangle - t(v_{n}, w - u_{n}, w) \geq \langle f, w - u_{n} \rangle \] (2.8)
for every \( w \in \hat{V} \). One has
\[ t(v_{n}, w - u_{n}, w) = \int_{\Omega} (v_{n})_{k} \partial x_{k} (w - u_{n})_{i} w_{i} dx \]
and \((v_n)_k w_i \to (v)_k w_i\) in \(L^{p-1}(\Omega)\) thus
\[
\int_\Omega (v_n)_k \partial x_k (w - u_n)_i w_i dx \to \int_\Omega (v)_k \partial x_k (w - u)_i w_i dx.
\]
Then, passing to the limit in \(n\) in (2.8), one gets for every \(w \in \hat{\mathcal{V}}\)
\[
\epsilon \int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla (w - u) dx + \langle Aw, w - u \rangle - t(v, w - u) \geq \langle f, w - u \rangle.
\]
Using the Minty Lemma again one sees that \(u\) is solution to
\[
\epsilon \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w dx + \langle Au, w \rangle - t(v, w, u) = \langle f, w \rangle \quad \forall w \in \hat{\mathcal{V}}
\]
i.e. \(u = S(v)\). Since the limit problem has a unique solution the whole sequence \(u_n\) converges toward \(u\) and \(S\) is continuous. Thus the mapping \(S\) admits a fixed point which is solution to (2.4) since for any \(w, u \in \hat{\mathcal{V}}\) one has
\[
t(u, w, u) = -t(u, u, w).
\]
This completes the proof of Theorem 2.1.

\[\square\]

3 Some applications

First the result of the previous section allows us to solve the problem (1.1) in any dimension when \(A = -\nu \Delta\), \(\Delta\) being the usual Laplace operator and \(\nu\) a positive constant. Indeed we have if \(\nu > 0\):

**Theorem 3.1** Suppose that \(f \in \mathcal{H}^{-1}(\Omega)\). There exists \(u\) solution to
\[
\begin{cases}
  u \in \mathcal{H}^1_0(\Omega), \\
  \nu \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega (u \cdot \nabla) u \cdot v dx = \langle f, v \rangle \quad \forall v \in \hat{D}(\Omega).
\end{cases}
\]  
(3.1)

**Proof:** From the theorem 2.1 it is clear that there exists \(u_\epsilon\) solution to
\[
\begin{cases}
  u_\epsilon \in \mathcal{W}^{1,p}_0(\Omega) := \hat{\mathcal{V}}, \\
  \epsilon \int_\Omega |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla v dx - \nu \langle \Delta u_\epsilon, v \rangle + t(u_\epsilon, u_\epsilon, v) = \langle f, v \rangle \quad \forall v \in \hat{D}(\Omega).
\end{cases}
\]  
(3.2)

Taking in the equation above \(v = u_\epsilon\) one derives
\[
\epsilon ||\nabla u_\epsilon||_p^p + \nu ||\nabla u_\epsilon||_2^2 = \langle f, u_\epsilon \rangle \leq ||f||_{\mathcal{H}^{-1}(\Omega)} ||\nabla u_\epsilon||_2.
\]
Thus proceeding as in the previous section one gets
\[
\epsilon ||\nabla u_\epsilon||_p^p + \nu ||\nabla u_\epsilon||_2^2 \leq C
\]  
(3.3)
for some constant independent of $\epsilon$. Thus, up to a “subsequence”, there exists $u \in \hat{H}^1_0(\Omega)$ such that

$$u_\epsilon \rightharpoonup u \text{ in } \hat{H}^1_0(\Omega), \quad \nabla u_\epsilon \rightharpoonup \nabla u, \quad u_\epsilon \rightarrow u \text{ in } L^2(\Omega).$$

For $v \in \hat{D}(\Omega)$ one has then

$$(u_\epsilon)_k v_i \rightarrow u_k v_i \text{ in } L^2(\Omega)$$

and thus

$$t(u_\epsilon, u_\epsilon, v) = \int_\Omega (u_\epsilon)_k \partial_{x_k} (u_\epsilon)_i v_i dx \rightarrow \int_\Omega u_k \partial_{x_k} u_i v_i dx.$$

One has also using Hölder’s inequality and (3.3)

$$|\epsilon \int_\Omega |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla v dx| \leq \epsilon^\frac{1}{2} \int_\Omega e^{\frac{1}{p'} |\nabla u_\epsilon|^{p-1} |\nabla v|} dx \leq \epsilon^\frac{1}{2} \left( \int_\Omega e |\nabla u_\epsilon|^{p} dx \right)^\frac{1}{p'} \|\nabla v\|_p \rightarrow 0.$$

Passing to the limit in (3.2), (3.1) follows. This completes the proof of the theorem.

In the case when $n = 2, 3$ one can improve the result above to get:

**Theorem 3.2** Suppose that $f \in \mathcal{H}^{-1}(\Omega), n = 2, 3$. If $A$ is an operator satisfying (2.1), (2.2) there exists $u$ solution to

$$\begin{cases}
    u \in \hat{H}^1_0(\Omega), \\
    \langle Au, v \rangle + \int_\Omega (u \cdot \nabla) u \cdot v dx = \langle f, v \rangle \quad \forall v \in \hat{H}^1_0(\Omega).
\end{cases}$$

(3.4)

**Remark 1** Thus in the case where $n = 2, 3$, $p = 2$ the theorem 2.1 holds also for $\epsilon = 0$. In the case when $A$ is strictly monotone one could have followed the proof of theorem 2.1 to get the result.

**Proof:** In this case $u_\epsilon$ is solution to

$$\begin{cases}
    u_\epsilon \in \hat{H}^1_0(\Omega), \\
    \epsilon \int_\Omega \nabla u_\epsilon \cdot \nabla v dx + \langle Au_\epsilon, v \rangle + t(u_\epsilon, u_\epsilon, v) = \langle f, v \rangle \quad \forall v \in \hat{H}^1_0(\Omega).
\end{cases}$$

(3.5)

Clearly arguing as above i.e. taking $v = u_\epsilon$ one get as in (3.3)

$$\|\nabla u_\epsilon\|^2_2 \leq C.$$

It follows then from the equation in (3.5) that $Au_\epsilon$ is also bounded in the dual $\hat{H}^{-1}(\Omega)$ of $\hat{H}^1_0(\Omega)$ independently of $\epsilon$ small. Thus there exists $u \in \hat{H}^1_0(\Omega)$ and $\chi \in \hat{H}^{-1}(\Omega)$ such that

$$u_\epsilon \rightharpoonup u \text{ in } \hat{H}^1_0(\Omega), \quad u_\epsilon \rightarrow u \text{ in } L^4(\Omega), \quad Au_\epsilon \rightharpoonup \chi \text{ in } \hat{H}^{-1}(\Omega).$$
Passing to the limit in (3.5) one obtains

$$\langle \chi, v \rangle + t(u, u, v) = \langle f, v \rangle \quad \forall v \in \hat{H}^1_0(\Omega). \quad (3.6)$$

Taking $v = u$ it comes

$$\langle \chi, u \rangle = \langle f, u \rangle. \quad (3.7)$$

By the monotonicity of $A$ one has for every $v \in \hat{H}^1_0(\Omega)$

$$\langle Au, v \rangle \geq 0. \quad (3.8)$$

Taking in (3.5) $v = u_\epsilon$ one derives

$$\langle Au_\epsilon, u_\epsilon \rangle \leq \langle f, u_\epsilon \rangle$$

and thus (3.8) becomes

$$\langle f, u_\epsilon \rangle - \langle Au_\epsilon, v \rangle - \langle Av, u_\epsilon - v \rangle \geq 0 \quad \forall v \in \hat{H}^1_0(\Omega).$$

Passing to the limit in $\epsilon$ we get

$$\langle f, u \rangle - \langle \chi, v \rangle - \langle Av, u - v \rangle \geq 0 \quad \forall v \in \hat{H}^1_0(\Omega)$$

i.e. by (3.7)

$$\langle \chi - A v, u - v \rangle \geq 0 \quad \forall v \in \hat{H}^1_0(\Omega).$$

Choosing $v = u - tw$ with $w \in \hat{H}^1_0(\Omega)$ and $t > 0$ we obtain

$$\langle \chi - A(u - tw), w \rangle \geq 0 \quad \forall w \in \hat{H}^1_0(\Omega).$$

Letting $t \to 0$ and using the hemicontinuity of $A$ we get

$$\langle \chi - Au, w \rangle \geq 0 \quad \forall w \in \hat{H}^1_0(\Omega)$$

and thus - changing $w$ into $-w$

$$\chi = Au.$$

Going back to (3.6) we see that $u$ is solution to (3.4). This completes the proof of the theorem. \(\square\)

We conclude by an example of application in the framework of nonlocal problems. Let $a$ be a real function such that

$$a$$

is nonnegative, continuous, nondecreasing on $[0, +\infty)$. \quad (3.9)

Let us set

$$F(z) = \int_0^z a(s)ds.$$\n
Clearly $F$ is a convex function on $[0, +\infty)$. Then we have:
Proposition 3.1 The operator $A u = -a(|\nabla u|^2) \Delta u$ is hemicontinuous and monotone from $H^1_0(\Omega)$ into its dual.

Proof: The functional

$$J(u) = F(|\nabla u|^2)$$

is convex. This follows from the convexity of $u \to |\nabla u|^2$ and $F$. Indeed for $t \in (0,1)$ one has

$$J(v + t(u - v)) = F(|\nabla(v + t(u - v))|^2)$$

$$\leq F(t|\nabla u|^2 + (1-t)|\nabla v|^2)$$

$$\leq tF(|\nabla u|^2) + (1-t)F(|\nabla v|^2)$$

$$= tJ(u) + (1-t)J(v).$$

Then one has

$$J(v + t(u - v)) - J(v) \leq t(J(u) - J(v)).$$

Exchanging $u$ and $v$ one gets

$$J(u + t(v - u)) - J(u) \leq t(J(v) - J(u)).$$

Summing up we derive

$$\frac{J(v + t(u - v)) - J(v)}{t} + \frac{J(u + t(v - u)) - J(u)}{t} \leq 0.$$

Letting $t \to 0$ it comes

$$\frac{d}{dt}J(v + t(u - v)) \bigg|_0 + \frac{d}{dt}J(u + t(v - u)) \bigg|_0 \leq 0. \quad (3.10)$$

By the chain rule

$$\frac{d}{dt}J(v + t(u - v)) \bigg|_0 = \frac{d}{dt}F(|\nabla(v + t(u - v))|^2) \bigg|_0$$

$$= 2a(|\nabla v|^2) \int_{\Omega} \nabla v \cdot \nabla (u - v) dx.$$

Exchanging the role of $u$ and $v$ one derives from (3.10)

$$\int_{\Omega} a(|\nabla v|^2) \nabla v \cdot \nabla (u - v) dx + \int_{\Omega} a(|\nabla u|^2) \nabla u \cdot \nabla (v - u) dx \leq 0.$$

which is also

$$\langle -a(|\nabla u|^2) \Delta u + a(|\nabla v|^2) \Delta v, u - v \rangle \geq 0 \quad \forall \ u, v \in H^1_0(\Omega).$$

This is the monotonicity of the operator. The hemicontinuity follows from the fact that

$$\langle -a(|\nabla (u + tv)|^2) \Delta (u + tv), w \rangle = \int_{\Omega} a(|\nabla (u + tv)|^2) \nabla (u + tv) \cdot \nabla w dx$$
and if \( t \to t_0 \)
\[
\nabla(u + tv) \to \nabla(u + t_0v) \quad \text{in} \quad L^2(\Omega).
\]

Then we have:

**Theorem 3.3** Suppose \( n = 2, 3 \). Let \( a \) be a function satisfying (3.9). If in addition
\[
a(0) > 0 \tag{3.11}
\]
for \( f \in H^{-1}(\Omega) \) there exists a solution to the nonlocal stationary Navier-Stokes problem

\[
\begin{cases}
u \in H^{1,0}_0(\Omega), \\
a(||\nabla u||^2_2) \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega (u \cdot \nabla)u \cdot v dx = \langle f, v \rangle \quad \forall v \in \hat{H}^1_0(\Omega). 
\end{cases} \tag{3.12}
\]

Moreover, denote by \( C_S \) the Sobolev constant such that
\[
|u|^4 \leq C_S||\nabla u||^2_2 \quad \forall u \in \mathbb{H}^1_0(\Omega) \tag{3.13}
\]
and by \( G \) the increasing function defined as
\[
G(z) = a(z^2)z.
\]

Then if
\[
\frac{G^{-1}(|f|_{H^{-1}(\Omega)})C_S}{a(0)} < 1. \tag{3.14}
\]
the solution to (3.12) is unique.

**Proof:** The existence of a solution is an immediate consequence of Theorem 3.2 and proposition 3.1, (2.2) being satisfied with \( \nu = a(0) \). For uniqueness, first notice that if \( u \) is solution to (3.12), taking \( v = u \) in the equation leads to
\[
a(||\nabla u||^2_2) \int_\Omega \nabla u \cdot \nabla v dx + t(u, u, v) = a(||\nabla u||^2_2) \int_\Omega \nabla u \cdot \nabla v dx + t(u, u, v) \quad \forall v \in \hat{H}^1_0(\Omega).
\]

We derive then
\[
a(||\nabla u||^2_2)||\nabla u||^2_2 \leq \langle f, u \rangle \leq |f|_{H^{-1}(\Omega)}||\nabla u||^2_2
\]
and by our definition of \( G \)
\[
||\nabla u||^2_2 \leq G^{-1}(|f|_{H^{-1}(\Omega)}). \tag{3.15}
\]

Let \( u_1, u_2 \) be two solutions to (3.12). One has
\[
a(||\nabla u_1||^2_2) \int_\Omega \nabla u_1 \cdot \nabla v dx + t(u_1, u_1, v) = a(||\nabla u_2||^2_2) \int_\Omega \nabla u_2 \cdot \nabla v dx + t(u_2, u_2, v) \quad \forall v \in \hat{H}^1_0(\Omega).
\]
This is also
\[ a(0) \int_{\Omega} \nabla(u_1 - u_2) \cdot \nabla v dx + \]
\[ (a(||\nabla u_1||_2^2 - a(0)) \int_{\Omega} \nabla u_1 \cdot \nabla v dx - (a(||\nabla u_2||_2^2 - a(0)) \int_{\Omega} \nabla u_2 \cdot \nabla v dx \]
\[ = t(u_2, u_2, v) - t(u_1, u_1, v) \quad \forall v \in H_0^1(\Omega). \]

Taking \( v = u_1 - u_2 \) and using the monotonicity of the operator \(-a(||\nabla u||_2^2 - a(0))\Delta u\) - see proposition 3.1 - one obtains
\[ a(0) \int_{\Omega} |\nabla(u_1 - u_2)|^2 dx \leq t(u_2, u_2, u_1 - u_2) - t(u_1, u_1, u_1 - u_2). \]
Since
\[ t(u_2, u_2, u_1 - u_2) = t(u_2, (u_2 - u_1) + u_1, u_1 - u_2) \]
\[ = t(u_2, u_1, u_1 - u_2), \]
we obtain
\[ a(0) \int_{\Omega} |\nabla(u_1 - u_2)|^2 dx \leq t(u_2, u_1, u_1 - u_2) - t(u_1, u_1, u_1 - u_2) \]
\[ = t(u_2 - u_1, u_1, u_1 - u_2). \]
Thus it comes
\[ a(0)||\nabla(u_1 - u_2)||_2^2 \leq |u_1 - u_2|^2||\nabla u_1||_2. \]
By (3.13), (3.15) this implies
\[ ||\nabla(u_1 - u_2)||_2^2 \leq \frac{G^{-1}(|f|_{H^{-1}(\Omega)})C_S}{a(0)}||\nabla(u_1 - u_2)||_2^2 \]
and (3.14) implies that \( ||\nabla(u_1 - u_2)||_2 = 0 \) i.e. that the solution to (3.12) is unique. Note that
(3.14) corresponds as in the classical case where \( a \) - the viscosity of the fluid - is a constant to have \( f \) small or \( a(0) \) large.

\[ \square \]

In fact the result above holds also for \( n = 4 \). Taking for \( n = 4 \ \ p = \frac{3n}{n+2} = 2 \) will make us loose compactness. One can however proceed in the following way to get a solution to (3.5). Consider \( \{w_i\} \) a Hilbert basis of \( \hat{H}^1(\Omega) \) and denote by \( \hat{V}_n \) the finite dimensional space spanned by \( w_1, \ldots, w_n \). For \( v \in \hat{V}_n \), by the arguments used before, there exists a unique \( u_n \) solution to
\[ \left\{ \begin{array}{l}
u_n \in \hat{V}_n, \\
e \int_{\Omega} \nabla u_n \cdot \nabla w dx + \langle Au_n, w \rangle - t(v, w, u_n) = \langle f, w \rangle \ \forall w \in \hat{V}_n.
\right\} \]
Taking \( w = u_n \) one deduces easily
\[ |u_n|_4 \leq C ||\nabla u_n||_2 \leq C' \] (3.16)
where $C'$ is independent of $v$.

Arguing again as above and assuming $\hat{V}_n$ equipped with the $L^4(\Omega)$-norm, one sees that the mapping $v \to S(v) = u_n$ maps $B_n = \{v \in \hat{V}_n | |v|_4 \leq C'\}$ into itself. Moreover this mapping is clearly continuous since it boils out to solve a finite dimensional linear system. By the Brower fixed point theorem $S$ has a fixed point i.e. there exists a solution to

$$
\begin{cases}
  u_n \in \hat{V}_n, \\
  \epsilon \int_\Omega \nabla u_n \cdot \nabla w dx + \langle Au_n, w \rangle - t(u_n, w, u_n) = \langle f, w \rangle \quad \forall \, w \in \hat{V}_n.
\end{cases}
$$

Clearly from (3.16) there exists $u \in \hat{H}^1_0(\Omega)$ such that - up to a subsequence - when $n \to \infty$

$$
\nabla u_n \rightharpoonup \nabla u, \quad u_n \to u \text{ in } L^2(\Omega), \quad u_n \to u \text{ a.e.}
$$

Moreover if $A$ is bounded, which is the case for $Au = -a(\|\nabla u\|_2^2)\Delta u$, the sequence $-\epsilon \Delta u_n + Au_n$ is bounded in $\hat{V}'$ and for some $\chi \in \hat{V}'$ one has

$$
-\epsilon \Delta u_n + Au_n \rightharpoonup \chi \text{ in } \hat{V}'.
$$

One has $u_{n,k}u_{n,i} \to u_k u_i$ a.e. and $|u_{n,k}u_{n,i}|_2 \leq C$. Thus, $u_{n,k}u_{n,i} \to u_k u_i$ in $L^2(\Omega)$. Passing to the limit in (3.17) one gets

$$
\langle \chi, u \rangle - t(u, w, u) = \langle f, w \rangle \quad \forall \, w \in \hat{V}.
$$

Taking $w = u$ leads to

$$
\langle \chi, u \rangle = \langle f, u \rangle.
$$

From the monotonicity of the operator $-\epsilon \Delta + A$ one has

$$
\langle (-\epsilon \Delta + A)u_n - (-\epsilon \Delta + A)v, u_n - v \rangle \geq 0.
$$

Now, taking $w = u_n$ in (3.17) leads to

$$
\langle (-\epsilon \Delta + A)u_n, u_n \rangle = \langle f, u_n \rangle.
$$

Passing to the limit in $n$ and arguing like in the end of the proof of theorem 3.2 leads to $\chi = -\epsilon \Delta u + Au$ i.e. by (3.18) to the solution to (3.5).

One refers to [2] for the particular case of an affine function $a(s)$. See also [10] where the evolution problem seems to have been investigated with this kind of nonlocal affine function $a$ for the first time in the case of the Navier-Stokes problem (Cf. also [12]).

Acknowledgement: This work was performed when the author was visiting the USTC in Hefei and during a part time employment at the S. M. Nikolskii Mathematical Institute of RUDN University, 6 Miklukho-Maklay St, Moscow, 117198, supported by the Ministry of Education and Science of the Russian Federation. He is grateful to these institutions for their support.
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