ON THE ASYMTOTIC BEHAVIOR OF ELLIPTIC, ANISOTROPIC SINGULAR PERTURBATIONS PROBLEMS

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Abstract. In this paper, we consider anitropic singular perturbations of some elliptic boundary value problems. We study the asymptotic behavior as \( \varepsilon \to 0 \) for the solution. Strong convergence in some Sobolev spaces is proved and the rate of convergence in cylindrical domains is given.

1. Introduction. The goal of this note is to analyze diffusion problems when the diffusion coefficients in certain directions are going toward zero. More precisely we are interested in determining the corresponding limit problem and the speed of convergence of the solution toward its limit.

Let us describe the class of problems that we would like to address. For \( \Omega \) a bounded open subset of \( \mathbb{R}^n \) we denote by \( x = (x_1, \ldots, x_n) = (X_1, X_2) \) the points in \( \mathbb{R}^n \) where

\[
X_1 = (x_1, \ldots, x_p) \quad \text{and} \quad X_2 = (x_{p+1}, \ldots, x_n),
\]
i.e. we split the coordinates into two parts. With this notation we set

\[
\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},
\]

where

\[
\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T, \quad \nabla_{X_2} u = (\partial_{x_{p+1}} u, \ldots, \partial_{x_n} u)^T.
\]

In all over the paper we will denote by \( \partial_{x_i} \) the partial derivative in the direction \( x_i \).

Let \( A = (a_{ij}(x)) \) be a \( n \times n \) matrix such that

\[
a_{ij} \in L^\infty(\Omega) \quad \forall i, j = 1, \ldots, n,
\]

and such that, for some \( \lambda > 0 \), we have

\[
A \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \ \text{a.e.} \ x \in \Omega.
\]

(“\( \cdot \)” denotes the canonical scalar product in \( \mathbb{R}^n \)). We decompose \( A \) into four blocks by writing

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where \( A_{11}, A_{22} \) are respectively \( p \times p \) and \((n - p) \times (n - p)\) matrices. We then set for \( \varepsilon > 0 \)

\[
A_\varepsilon = A_\varepsilon(x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}.
\]

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We have therefore, for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^n \),
\[
A_\varepsilon \xi \cdot \xi = (A_\varepsilon \xi_1, \xi_2) \geq \lambda |\xi_\varepsilon|^2 = \lambda \{ \varepsilon^2 |\xi_1|^2 + |\xi_2|^2 \},
\]
(5)
\[
A_{22} \xi_2 \cdot \xi_2 \geq \lambda |\xi_2|^2,
\]
(6)
where we have set
\[
\xi = (\xi_1, \xi_2)^T
\]
with \( \xi_1 = (\xi_1, \ldots, \xi_p)^T \), \( \xi_2 = (\xi_{p+1}, \ldots, \xi_n)^T \) and \( \xi_\varepsilon = (\varepsilon \xi_1, \xi_2) \). Thus, we have
\[
A_\varepsilon \xi \cdot \xi \geq \lambda (\varepsilon^2 \wedge 1)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega.
\]
(7)
(\( \wedge \) denotes the minimum of two numbers). It follows that \( A_\varepsilon \) and \( A_{22} \) are positive definite and for
\[
f \in L^2(\Omega),
\]
(8)
there exists a unique \( u_\varepsilon \) solution to
\[
\left\{ \begin{array}{l}
\int_{\Omega} A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega), \\
u_\varepsilon \in H^1_0(\Omega).
\end{array} \right.
\]
(9)
Let \( \Pi_{X_1} \) be the orthogonal projection from \( \mathbb{R}^n \) onto the space \( X_2 = 0 \). For any \( X_1 \in \Pi_{X_1}(\Omega) := \Pi_{\Omega} \), we denote by \( \Omega_{X_1} \) the section of \( \Omega \) above \( X_1 \) i.e.
\[
\Omega_{X_1} = \{ X_2 \mid (X_1, X_2) \in \Omega \}.
\]
Since for a.e. \( X_1 \in \Pi_{\Omega} \) we have
\[
f(X_1, \cdot) \in L^2(\Omega_{X_1}),
\]
there exists a unique \( u_0 = u_0(X_1, \cdot) \) solution to
\[
\left\{ \begin{array}{l}
\int_{\Omega_{X_1}} A_{22} \nabla X_2 u_0(X_1, X_2) \cdot \nabla X_2 v(X_2) \, dX_2 \\
= \int_{\Omega_{X_1}} f(X_1, X_2) v(X_2) \, dX_2 \quad \forall v \in H^1_0(\Omega_{X_1}), \\
u_0(X_1, \cdot) \in H^1_0(\Omega_{X_1}).
\end{array} \right.
\]
(10)
Note that \( u_0 \) is the solution of an elliptic problem set on the section \( \Omega_{X_1} \) (see the figure below).

![Figure 1](image-url)
We would like then to show that

\[ u_\varepsilon \rightarrow u_0 \text{ when } \varepsilon \rightarrow 0. \]

We have of course to precise in what sense this convergence will take place. As a preliminary remark let us notice that in the case where \( n = 2, \)

\[ \Omega = (-1, 1) \times (-1, 1), \]
\[ A = Id, \quad f = f(x_2), \]

where \( Id \) denotes the identity matrix, then \( u_\varepsilon, u_0 \) are respectively the weak solutions to

\[ \begin{cases} -\varepsilon^2 \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial \Omega, \end{cases} \]

and

\[ \begin{cases} -\partial^2_{x_2} u_0 dx = f & \text{in } \omega_2 = (-1, 1), \\ u_0 = 0 & \text{on } \partial \omega_2. \end{cases} \]

In this particular case \( u_0 \) is independent of \( x_1 \) and not identically equal to 0 if \( f \neq 0. \)

Thus the function

\[ u_0 = u_0(x_2) \notin H^1_0(\Omega) \]

and we cannot expect

\[ u_\varepsilon \rightarrow u_0 \text{ in } H^1(\Omega). \]

Our note is divided as follows. The next section is devoted to establish the convergence of \( u_\varepsilon \) towards \( u_0. \) In the third section we are concerned with the special case where

\[ \Omega = \omega_1 \times \omega_2 \]

which is the case of our above example. Then we give precise conditions which insure \( u_0 \) to belongs to \( H^1(\Omega) \) and estimate the rate of convergence of \( u_\varepsilon \) toward \( u_0 \) for different norms. Some other points of view or results can be found in [1, 2, 3, 6, 7]. For general singular perturbation problems see [8].

2. Asymptotic behavior in arbitrary domains. Clearly, \( u_0 \) is the natural candidate for the limit of \( u_\varepsilon. \) Indeed we have

**Theorem 2.1.** Under the assumptions above we have

\[ u_\varepsilon \rightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega) \quad (11) \]

where \( u_\varepsilon \) (resp. \( u_0 \)) is the solution to (9) (resp. (10)).

(In the above convergences the vectorial convergence in \( L^2(\Omega) \) means the convergence component by component).

**Proof.** Let us take \( v = u_\varepsilon \) in (9). By (5) we derive

\[ \lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \leq \langle f, u_\varepsilon \rangle \leq |f|_{L^2(\Omega)} |u_\varepsilon|_{L^2(\Omega)}. \quad (12) \]

Since \( \Omega \) is bounded, by the Poincaré inequality we have for some constant \( C \) independent of \( \varepsilon \)

\[ |v|_{L^2(\Omega)} \leq C |\nabla_{X_2} v|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega). \quad (13) \]

From (12) we then derive

\[ \lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \leq C |f|_{L^2(\Omega)} |\nabla_{X_2} u_\varepsilon|_{L^2(\Omega)}. \quad (14) \]
Dropping in the above inequality the term in $\varepsilon$ we get
\[
\lambda |\nabla X_2 u_\varepsilon|_{L^2(\Omega)}^2 \leq C |f|_{L^2(\Omega)} |\nabla X_2 u_\varepsilon|_{L^2(\Omega)}
\]
whence
\[
|\nabla X_2 u_\varepsilon|_{L^2(\Omega)} \leq \frac{C |f|_{L^2(\Omega)}}{\lambda}.
\]
Reporting this in (14) we are ending up with
\[
\int_\Omega \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + |\nabla X_2 u_\varepsilon|^2 \,dx \leq \frac{C^2 |f|_{L^2(\Omega)}^2}{\lambda^2}.
\]  
(15)
Thus – due to (13) we deduce that
\[
u_\varepsilon, \quad |\varepsilon \nabla X_1 u_\varepsilon|, \quad |\nabla X_2 u_\varepsilon|
\]
are bounded in $L^2(\Omega)$. (This of course independently of $\varepsilon$). It follows that there exist
\[
u_0 \in L^2(\Omega), \quad u_1 \in [L^2(\Omega)]^p, \quad u_2 \in [L^2(\Omega)]^{n-p}
\]
such that – up to a subsequence
\[
u_\varepsilon \rightharpoonup \nu_0, \quad \varepsilon \nabla X_1 u_\varepsilon \rightharpoonup u_1, \quad \nabla X_2 u_\varepsilon \rightharpoonup u_2 \quad \text{in} \ L^2(\Omega).
\]
(The convergence is meant component by component). Of course the convergence in $L^2(\Omega)$-weak implies the convergence in $D'(\Omega)$ and by the continuity of the derivation in $D'(\Omega)$ we deduce that
\[
u_\varepsilon \rightharpoonup \nu_0, \quad \varepsilon \nabla X_1 u_\varepsilon \rightharpoonup 0, \quad \nabla X_2 u_\varepsilon \rightharpoonup \nabla X_2 u_0 \quad \text{in} \ L^2(\Omega).
\]  
(16)
We then go back to the equation satisfied by $u_\varepsilon$ that we expand using the different blocks of $A$. This gives
\[
\int_\Omega \varepsilon^2 A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 v \,dx + \int_\Omega \varepsilon A_{12} \nabla X_2 u_\varepsilon \cdot \nabla X_1 v \,dx + \int_\Omega \varepsilon A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 v \,dx + \int_\Omega A_{22} \nabla X_2 u_\varepsilon \cdot \nabla X_2 v \,dx = \int_\Omega f \,v dx \quad \forall v \in H_0^1(\Omega).
\]
Passing to the limit in each term using (16) we get
\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 v \,dx = \int_\Omega f \,v dx \quad \forall v \in H_0^1(\Omega).
\]  
(17)
At this point we do not know yet if for a.e. $X_1 \in \Pi \Omega$ we have
\[
u_0(X_1, \cdot) \in H_0^1(\Omega_{X_1}).
\]  
(18)
To see this -and more- we remark first that taking $v = u_\varepsilon$ in (17) and passing to the limit we obtain
\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_\varepsilon \,dx = \int_\Omega f u_0 \,dx.
\]  
(19)
Next we compute
\[
I_\varepsilon = \int_\Omega A_\varepsilon \left( \nabla X_1 u_\varepsilon \right) \cdot \left( \nabla X_1 u_\varepsilon - u_0 \right) \,dx.
\]  
(20)
We get
\[
I_\varepsilon = \int_\Omega \varepsilon^2 A_{11} \nabla X_1 u_\varepsilon \cdot \nabla X_1 u_\varepsilon \,dx + \int_\Omega \varepsilon A_{12} \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_1 u_\varepsilon \,dx + \int_\Omega \varepsilon A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 (u_\varepsilon - u_0) \,dx + \int_\Omega A_{22} \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_2 (u_\varepsilon - u_0) \,dx.
\]
Using (9) it comes
\[ I_\epsilon = \int_{\Omega} f u_\epsilon \, dx - \int_{\Omega} \epsilon A_{12} \nabla X_2 u_0 \cdot \nabla X_1 u_\epsilon \, dx - \int_{\Omega} \epsilon A_{21} \nabla X_1 u_\epsilon \cdot \nabla X_2 u_0 \, dx - \int_{\Omega} A_{22} \nabla X_2 u_\epsilon \cdot \nabla X_2 u_0 \, dx. \]

Passing to the limit in \( \epsilon \) we get
\[ \lim_{\epsilon \to 0} I_\epsilon = \int_{\Omega} f u_0 \, dx - \int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx = 0. \]

Using the coerciveness assumption we have (see (20))
\[ \lambda \int_{\Omega} \epsilon^2 |\nabla X_1 u_\epsilon|^2 + |\nabla X_2 (u_\epsilon - u_0)|^2 \, dx \leq I_\epsilon. \]

It follows that
\[ \epsilon \nabla X_1 u_\epsilon \longrightarrow 0, \quad \nabla X_2 u_\epsilon \longrightarrow \nabla X_2 u_0 \text{ in } L^2(\Omega). \]

Now we have also
\[ \int_{\Pi_{\Omega}} \int_{\Omega X_1} |\nabla X_2 (u_\epsilon - u_0)|^2 \, dX_2 \, dX_1 \longrightarrow 0. \tag{21} \]

It follows that for almost every \( X_1 \)
\[ \int_{\Omega X_1} |\nabla X_2 (u_\epsilon - u_0)|^2 \, dX_2 \longrightarrow 0. \]

Since
\[ \left\{ \int_{\Omega X_1} |\nabla X_2 v|^2 \, dX_2 \right\}^{\frac{1}{2}} \]

is a norm on \( H^1_0(\Omega X_1) \) and \( u_\epsilon(X_1, \cdot) \in H^1_0(\Omega X_1) \) we have
\[ u_0(X_1, \cdot) \in H^1_0(\Omega X_1) \]

and this for almost every \( X_1 \). Using then the Poincaré inequality we obtain
\[ \int_{\Omega X_1} |u_\epsilon - u_0|^2 \, dX_2 \leq C \int_{\Omega X_1} |\nabla X_2 (u_\epsilon - u_0)|^2 \, dX_2. \]

Integrating over \( \Pi_{\Omega} \) we get
\[ \int_{\Omega} |u_\epsilon - u_0|^2 \, dx \leq C \int_{\Omega} |\nabla X_2 (u_\epsilon - u_0)|^2 \, dx \longrightarrow 0 \]
(by (21)) and thus
\[ u_\epsilon \longrightarrow u_0 \text{ in } L^2(\Omega). \tag{22} \]

All this is up to a subsequence. If we can identify \( u_0 \) uniquely then all the convergences above will hold for the whole sequence. For this purpose recall first that
\[ u_0(X_1, \cdot) \in H^1_0(\Omega X_1). \tag{23} \]

One can cover \( \Omega \) by a countable family of open sets of the form
\[ U_i \times V_i \subset \Omega, \quad i \in \mathbb{N} \]
where $U_i$, $V_i$ are open subsets of $\mathbb{R}^p$, $\mathbb{R}^{n-p}$ respectively. One can even choose $U_i$, $V_i$ hypercubes. Then choosing $\varphi \in H^1_0(V_i)$ we derive from (17)

$$\int_{U_i} \eta(X_1) \int_{V_i} A_{22} (X_1, X_2) \nabla X_2 u_0 (X_1, X_2) \cdot \nabla X_2 \varphi (X_2) dX_2 dX_1$$

$$= \int_{U_i} \eta(X_1) \int_{V_i} f (X_1, X_2) \varphi (X_2) dX_2 dX_1 \quad \forall \eta \in D(U_i),$$

since $\eta \varphi \in H^1_0 (\Omega)$. Thus there exists a set of measure zero, $N(\varphi)$, such that

$$\int_{V_i} A_{22} (X_1, X_2) \nabla X_2 u_0 (X_1, X_2) \cdot \nabla X_2 \varphi (X_2) dX_2 = \int_{V_i} f (X_1, X_2) \varphi (X_2) dX_2$$

(24)

for all $X_1 \in U_i \setminus N(\varphi)$. Denote by $\varphi_n$ a Hilbert basis of $H^1_0 (V_i)$. Then (24) holds (replacing $\varphi$ by $\varphi_n$) for all $X_1$ such that

$$X_1 \in U_i \setminus N_i(\varphi_n)$$

where $N_i(\varphi_n)$ is a set of measure 0. Thus for

$$X_1 \in U_i \setminus \cup_i N_i(\varphi_n)$$

we have (24) for any $\varphi \in H^1_0 (V_i)$. This follows easily from the density in $H^1_0 (V_i)$ of the linear combinations of the $\varphi_n$. Let us then choose

$$X_1 \in \Pi_\Omega \setminus \cup_i \cup_n N_i(\varphi_n)$$

(note that $\cup_i \cup_n N_i(\varphi_n)$ is a set of measure 0). Let

$$\varphi \in D(\Omega_{X_1}).$$

If $K$ denotes the support of $\varphi$ we have clearly

$$K \subset \cup_i V_i$$

and thus $K$ can be covered by a finite number of $V_i$ that for simplicity we will denote by $V_1, \ldots, V_k$. Using a partition of unity there exists $\psi_i \in D(V_i)$ such that

$$\sum_{i=1}^k \psi_i = 1 \quad \text{on } K.$$

By (24) we derive

$$\int_K A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi dX_2 = \int_K A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \sum_i (\psi_i \varphi) dX_2$$

$$= \sum_i \int_{V_i} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 (\psi_i \varphi) dX_2$$

$$= \sum_i \int_{V_i} f \psi_i \varphi dX_2$$

$$= \int_K f \varphi dX_2.$$

This is also

$$\int_{\Omega_{X_1}} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi dX_2 = \int_{\Omega_{X_1}} f \varphi dX_2 \quad \forall \varphi \in D(\Omega_{X_1})$$

and thus $u_0$ is the unique solution to (10) for a.e. $X_1 \in \Pi_\Omega$. This completes the proof of the theorem. \qed
3. The rate of convergence in general cylindrical domains. In this section we suppose that \( \Omega \) is of a special type namely
\[
\Omega = \omega_1 \times \omega_2
\]
where \( \omega_1, \omega_2 \) are bounded Lipschitz domains of \( \mathbb{R}^p \) and \( \mathbb{R}^{n-p} \) respectively. Then for any \( X_1 \in \Pi_0 \) one has \( \Omega_{X_1} = \omega_2 \) and the problem (10) can be written
\[
\begin{cases}
\int_{\omega_2} A_{22}(X_1, X_2) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} f(X_1, X_2) \, v \, dX_2 & \forall v \in H^1_0(\omega_2), \\
u_0(X_1, \cdot) \in H^1_0(\omega_2).
\end{cases}
\]  
(25)
As mentioned in the introduction if we want to obtain convergence in \( H^1(\Omega) \) we need to have
\[
u_0 \in H^1(\Omega). 
\]  
(26)
In order to insure that, we need to show that \( \nabla_{X_1} u_0 \) is in \( L^2(\Omega) \) which requires some assumptions since in (25) \( X_1 \) is a parameter. So we will assume in this section that
\[
\partial_{x_k} f \in L^2(\Omega), \, \partial_{x_k} A_{22} \in L^\infty(\Omega) \, \forall k = 1, \ldots, p,
\]  
(27)
(the second assumption stands for \( \partial_{x_k} u_{ij} \in L^\infty(\Omega) \, \forall i, j = p + 1, \ldots, n \)).

3.1. A regularity results.

Proposition 1. Under the assumptions (1), (2), (8) and (27) we have
\[
u_0 \in H^1(\Omega).
\]

Proof. Let \( \omega'_1 \) be an open set such that
\[
\omega'_1 \subset \subset \omega_1.
\]
For \( 0 < h < d(\omega'_1, \partial \omega_1) \), \( X_1 \in \omega'_1 \) we set
\[
\tau_h^i u_0(X_1, X_2) = u_0(X_1 + he_i, X_2), \quad i = 1, \ldots, p.
\]
For \( v \in H^1_0(\omega_2) \) we then get from (25)
\[
\int_{\omega_2} \tau_h^i (A_{22} \nabla_{X_2} u_0) \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} (\tau_h f - f) \, v \, dX_2.
\]
This implies
\[
\int_{\omega_2} \tau_h^i A_{22} \nabla_{X_2} (\tau_h^i u_0 - u_0) \cdot \nabla_{X_2} v \, dX_2 + \int_{\omega_2} (\tau_h^i A_{22} - A_{22}) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} (\tau_h f - f) \, v \, dX_2.
\]
Since \( \tau_h^i u_0 - u_0 \in H^1_0(\omega_2) \), taking \( v = \frac{\tau_h^i u_0 - u_0}{h^2} \), we obtain
\[
\int_{\omega_2} \tau_h^i A_{22} \nabla_{X_2} \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \cdot \nabla_{X_2} \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \, dX_2 = -\int_{\omega_2} (\tau_h^i A_{22} - A_{22}) \nabla_{X_2} u_0 \cdot \nabla_{X_2} \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \, dX_2
\]
\[
+ \int_{\omega_2} \left( \frac{\tau_h f - f}{h} \right) \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \, dX_2.
\]
Using the ellipticity assumption and the Cauchy-Schwarz inequality, we deduce

\[ \lambda \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|^2 \leq \left| \frac{\tau_h^i A_{22} - A_{22}}{h} \nabla X_2 u_0 \right|_{L^2(\omega_2)} \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|_{L^2(\omega_2)} + C \left| \frac{\tau_h^i f - f}{h} \right|_{L^2(\omega_2)} \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|_{L^2(\omega_2)}. \]

It follows by the Poincaré inequality that

\[ \lambda \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|^2 \leq \left| \frac{\tau_h^i A_{22} - A_{22}}{h} \right|_{L^\infty(\omega_2)} \left| \nabla X_2 u_0 \right|_{L^2(\omega_2)} \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|_{L^2(\omega_2)} + C \left| \frac{\tau_h^i f - f}{h} \right|_{L^2(\omega_2)} \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|_{L^2(\omega_2)}. \]

Then we deduce using Poincaré inequality again

\[ \left| \frac{\tau_h^i u_0 - u_0}{h} \right|_{L^2(\omega_2)} \leq C \lambda \left| \nabla X_2 \left( \frac{\tau_h^i u_0 - u_0}{h} \right) \right|_{L^2(\omega_2)} + C \left| \frac{\tau_h^i f - f}{h} \right|_{L^2(\omega_2)}, \]

where \( C' \) is dependent on \( \lambda, u_0 \) and the Poincaré constant. According to the regularity assumptions (27) (see for instance Lemma 7.23 in [5]) and integrating on \( \omega'_1 \), we get

\[ \left| \frac{\tau_h^i u_0 - u_0}{h} \right|_{L^2(\omega'_1 \times \omega_2)} \leq C''. \]

where \( C'' \) is independent of \( h \) and \( \omega'_1 \). It follows then from [5] that

\[ \nabla X_1, u_0 \in \left( L^2 (\Omega) \right)^p. \]

Since know already that \( \nabla X_2 u_0 \in \left( L^2 (\Omega) \right)^{n-p} \) the proof is complete. \( \square \)

**Remark 1.** If we consider a general domain \( \Omega \), a local regularity can be shown as in Proposition 1 above.

**Remark 2.** One should remark that \( u_0 \) vanishes in the trace sense on \( \omega_1 \times \partial \omega_2 \).

Indeed since \( u_0 \in H^1 (\Omega) \) there exists a sequence \( v_n \in C^1 (\overline{\Omega}) \) such that

\[ v_n \rightarrow u_0 \quad \text{in} \quad H^1 (\Omega). \]

Let us denote by \( \Gamma_0 \) the trace operator on \( \omega_1 \times \partial \omega_2 \), \( \gamma_0 \) the trace operator on \( \partial \omega_2 \). One has by the continuity of the trace operator

\[ \Gamma_0 v_n \rightarrow \Gamma_0 u_0 \quad \text{in} \quad L^2 (\omega_1 \times \partial \omega_2), \]
and thus for a.e. \( X_1 \in \omega_1 \) – up to a subsequence
\[
\Gamma_0 v_n (X_1, \cdot) \rightarrow \Gamma_0 u_0 (X_1, \cdot) \quad \text{in } L^2 (\partial \omega_2). 
\]

Since for a.e. \( X_1 \in \omega_1 \) one has also
\[
v_n (X_1, \cdot) \rightarrow u_0 (X_1, \cdot) \quad \text{in } H^1 (\omega_2),
\]
it follows that
\[
\gamma_0 v_n (X_1, \cdot) = \Gamma_0 v_n (X_1, \cdot) \rightarrow \gamma_0 u_0 (X_1, \cdot) = 0 \quad \text{in } L^2 (\partial \omega_2).
\]

This shows that \( \Gamma_0 u_0 = 0 \).

3.2. **The convergence theorems.** It is clear that for any open set \( \omega'_1 \) satisfying \( \omega'_1 \subset \subset \omega_1 \), we can find an other open set \( \omega''_1 \) such that \( \omega'_1 \subset \subset \omega''_1 \subset \subset \omega_1 \) and a smooth function \( \rho \) satisfies
\[
\text{supp } \rho \subset \omega''_1, \quad \rho = 1 \text{ on } \omega'_1. \tag{28}
\]

In addition, we suppose that
\[
\partial_x a_{ij}, \partial_x p_{ij} \in L^\infty (\omega''_1 \times \omega_2) \quad i = 1, \ldots, p, \quad j = p + 1, \ldots, n. \tag{29}
\]

According to the proposition above, it follows that
\[
\rho^2 (u_\varepsilon - u_0) \in H^1_0 (\Omega).
\]

For \( v \in H^1_0 (\Omega) \), integrating (25) over \( \omega_1 \) and subtracting it from (9) yield
\[
\varepsilon^2 \int_\Omega A_{11} \nabla X_i u_\varepsilon \cdot \nabla X_i v \, dx + \varepsilon \int_\Omega A_{12} \nabla X_j u_\varepsilon \cdot \nabla X_i v \, dx + \varepsilon \int_\Omega A_{21} \nabla X_1 u_\varepsilon \cdot \nabla X_2 v \, dx + \varepsilon \int_\Omega A_{22} \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_2 v \, dx = 0. \tag{30}
\]

Then testing with \( v = \rho^2 (u_\varepsilon - u_0) \) we obtain
\[
\varepsilon^2 \int_\Omega A_{11} \nabla X_i (u_\varepsilon - u_0) \cdot \nabla X_i \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx \\
+ \varepsilon \int_\Omega A_{12} \nabla X_j (u_\varepsilon - u_0) \cdot \nabla X_i \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx \\
+ \varepsilon \int_\Omega A_{21} \nabla X_1 (u_\varepsilon - u_0) \cdot \nabla X_2 \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx \\
+ \varepsilon \int_\Omega A_{22} \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_2 \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx \\
= - \varepsilon^2 \int_\Omega A_{11} \nabla X_i u_0 \cdot \nabla X_i \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx - \varepsilon \int_\Omega A_{12} \nabla X_j u_0 \cdot \nabla X_i \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx \\
- \varepsilon \int_\Omega A_{21} \nabla X_1 u_0 \cdot \nabla X_2 \left( \rho^2 (u_\varepsilon - u_0) \right) \, dx,
\]
We apply the ellipticity assumption on the left hand side, for the first two terms right hand side. First, we decompose this term, using the density of $L^2$:

$$
\rho^2 A_2 \nabla (u_\varepsilon - u_0) \cdot \nabla (u_\varepsilon - u_0) \, dx = 
$$

$$
- 2 \varepsilon^2 \int_\Omega \rho (u_\varepsilon - u_0) A_{11} \nabla X_i u_0 \cdot \nabla X_i \rho \, dx 
- \varepsilon \int_\Omega \rho^2 A_{12} \nabla X_i u_0 \cdot \nabla X_i (u_\varepsilon - u_0) \, dx 
- \varepsilon \int_\Omega (u_\varepsilon - u_0) \nabla X_i (u_\varepsilon - u_0) \, dx.
$$

We apply the ellipticity assumption on the left hand side, for the first two terms and the last four terms of the right hand side we use the Cauchy-Schwarz, Young and Poincaré inequalities, we get

$$
\lambda^2 \left| \rho \nabla X_1 (u_\varepsilon - u_0) \right|_{L^2(\Omega)}^2 + \lambda \left| \rho \nabla X_2 (u_\varepsilon - u_0) \right|_{L^2(\Omega)}^2 
\leq \varepsilon \int_\Omega \rho^2 A_{12} \nabla X_i u_0 \cdot \nabla X_i (u_\varepsilon - u_0) \, dx 
+ C \varepsilon^2 \left( \left| u_\varepsilon - u_0 \right|_{L^2(\omega_1 \times \omega_2)}^2 + \left| \nabla X_1 u_0 \right|_{L^2(\omega_1 \times \omega_2)}^2 \right) 
+ \frac{\lambda \varepsilon}{2} \left| \rho \nabla X_2 (u_\varepsilon - u_0) \right|_{L^2(\Omega)}^2.
$$

This implies

$$
\frac{\lambda \varepsilon}{2} \left| \rho \nabla X_1 (u_\varepsilon - u_0) \right|_{L^2(\Omega)}^2 + \frac{\lambda \varepsilon}{2} \left| \rho \nabla X_2 (u_\varepsilon - u_0) \right|_{L^2(\Omega)}^2 
\leq \varepsilon \int_\Omega \rho^2 A_{12} \nabla X_i u_0 \cdot \nabla X_i (u_\varepsilon - u_0) \, dx 
+ C \varepsilon^2 \left( \left| u_\varepsilon - u_0 \right|_{L^2(\omega_1 \times \omega_2)}^2 + \left| \nabla X_1 u_0 \right|_{L^2(\omega_1 \times \omega_2)}^2 \right),
$$

where the constant $C$ is independent of $\varepsilon$. Next, we estimate the first term of the right hand side. First, we decompose this term, using the density of $D(\omega_1 \times \omega_2)$ in $L^2(\omega_1 \times \omega_2)$ and due to (29), it follows that

$$
\int_\Omega \rho^2 A_{12} \nabla X_i u_0 \cdot \nabla X_i (u_\varepsilon - u_0) \, dx
$$

$$
= \sum_{i=1}^p \sum_{j=p+1}^n \int_\Omega \rho^2 a_{ij} \partial x_i u_0 \partial x_j (u_\varepsilon - u_0) \, dx
$$

$$
= \sum_{i=1}^p \sum_{j=p+1}^n \int_\Omega \partial x_i \left( \rho^2 a_{ij} (u_\varepsilon - u_0) \right) \partial x_j u_0 \, dx 
- \sum_{i=1}^p \sum_{j=p+1}^n \int_\Omega \partial x_i \left( \rho^2 a_{ij} \right) \partial x_j u_0 (u_\varepsilon - u_0) \, dx
$$

$$
- \sum_{i=1}^p \sum_{j=p+1}^n \int_\Omega \partial x_i \left( \rho^2 a_{ij} \right) \partial x_j u_0 (u_\varepsilon - u_0) \, dx.
$$

(33)
With this decomposition, we are able to easily estimate this term, then using Cauchy-Schwarz, Young and Poincaré inequalities, we obtain
\[
\varepsilon^2 (\rho \nabla X_1 (u_\varepsilon - u_0))^2 + |\rho \nabla X_2 (u_\varepsilon - u_0)|^2 
\leq C \varepsilon^2 \left( |u_\varepsilon - u_0|^2_{L^2(\omega_1' \times \omega_2)} + |\nabla X_1 u_0|^2_{L^2(\omega_1'' \times \omega_2)} + |\nabla X_2 u_0|^2_{L^2(\omega_1'' \times \omega_2)} \right).
\]

Thus, due to (15), (28) and the Poincaré inequality, it comes
\[
|u_\varepsilon - u_0|_{L^2(\omega_1' \times \omega_2)}, \quad |\nabla X_1 (u_\varepsilon - u_0)|_{L^2(\omega_1' \times \omega_2)} \leq C \varepsilon, \quad (34)
\]
\[
|\nabla X_1 (u_\varepsilon - u_0)|_{L^2(\omega_1' \times \omega_2)} \leq C. \quad (35)
\]

Then we deduce
\[
\nabla X_1 (u_\varepsilon - u_0) \rightharpoonup 0 \text{ weakly in } L^2(\omega_1' \times \omega_2).
\]

Indeed, since \(|\nabla X_1 (u_\varepsilon - u_0)|\) is bounded in \(L^2(\omega_1' \times \omega_2)\) and by the density of \(D(\omega_1' \times \omega_2)\) in \(L^2(\omega_1' \times \omega_2)\), it is enough to check that
\[
\int_{\omega_1' \times \omega_2} \nabla X_1 (u_\varepsilon - u_0) \cdot \varphi \, dx \to 0 \quad \forall \varphi \in (D(\omega_1' \times \omega_2))^p,
\]
which easily follows if we use the convergence
\[
u_\varepsilon \to u_0 \text{ in } L^2(\Omega).
\]

Finally, we can state the following theorem.

**Theorem 3.1.** Under the assumptions of Proposition 1 and if we assume that we have (29) then for any
\[
\omega_1' \subset \subset \omega_1,
\]
there exists a constant \(C > 0\) independent of \(\varepsilon\) such that
\[
|u_\varepsilon - u_0|_{L^2(\omega_1' \times \omega_2)}, \quad |\nabla X_2 (u_\varepsilon - u_0)|_{L^2(\omega_1' \times \omega_2)} \leq C \varepsilon, \quad (36)
\]

and
\[
\nabla X_1 u_\varepsilon \rightharpoonup \nabla X_1 u_0 \text{ weakly in } L^2(\omega_1' \times \omega_2).
\]

On the way to prove the convergence of \(u_\varepsilon\) toward \(u_0\) we slightly improve the above result by having some information on \(\frac{1}{\varepsilon} (u_\varepsilon - u_0)\). More precisely we have

**Theorem 3.2.** Given the assumptions of Theorem 3.1 the following claims are equivalent
\[
i (i) \quad \frac{1}{\varepsilon} (u_\varepsilon - u_0), \quad \frac{1}{\varepsilon} |\nabla X_2 (u_\varepsilon - u_0)| \to 0 \text{ strongly in } L^2(\omega_1' \times \omega_2), \forall \omega_1' \subset \subset \omega_1,
\]
\[
(ii) \quad \frac{1}{\varepsilon} (u_\varepsilon - u_0) \to 0 \text{ weakly in } L^2(\omega_1' \times \omega_2), \forall \omega_1' \subset \subset \omega_1.
\]

**Proof.** Of course (i) \(\Rightarrow\) (ii). To see the converse we derive from (31) and (33)
\[
|\rho \nabla X_1 (u_\varepsilon - u_0)|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^2} |\rho \nabla X_2 (u_\varepsilon - u_0)|^2_{L^2(\Omega)}
\leq \int_{\Omega} g_e \cdot \nabla X_1 (u_\varepsilon - u_0) \, dx + \frac{1}{\varepsilon} \int_{\Omega} H_e \cdot \nabla X_2 (u_\varepsilon - u_0) \, dx + \frac{1}{\varepsilon} \int_{\Omega} g_e \cdot (u_\varepsilon - u_0) \, dx.
\]

**(36)**
where $G_{\varepsilon}$, $H_{\varepsilon}$ and $g_{\varepsilon}$ are supported in $\omega''_1 \times \omega_2$ and converging strongly in $L^2 (\omega''_1 \times \omega_2)$ (they are combination of $\rho$, $u_{\varepsilon}$, $u_0$ and their derivatives). From Theorem 3.1 we have also
\[
\nabla_{X_1} (u_{\varepsilon} - u_0) \rightharpoonup 0 \quad \text{in} \quad L^2 (\omega'_1 \times \omega_2),
\]
\[
\frac{1}{\varepsilon} [\nabla_{X_2} (u_{\varepsilon} - u_0)] \rightharpoonup 0 \quad \text{in} \quad L^2 (\omega''_1 \times \omega_2).
\]
(The latest sequence is bounded and the only possible limit in $L^2 (\omega''_1 \times \omega_2)$ is 0 by $(ii)$). This completes the proof of the theorem since $\rho = 1$ on $\omega'_1$. □

**Remark 3.** Note that $(ii)$ implies also the convergence $u_{\varepsilon}$ toward $u_0$ in $H^1 (\omega'_1 \times \omega_2)$ (see (36)).

**Remark 4.** If we fixed $\omega'_1 \subset \subset \omega_1$, the equivalent claims of Theorem 3.2 can be rephrased as
\[
(i) \quad \frac{1}{\varepsilon} (u_{\varepsilon} - u_0), \quad \frac{1}{\varepsilon} [\nabla_{X_2} (u_{\varepsilon} - u_0)] \rightharpoonup 0 \quad \text{strongly in} \quad L^2 (\omega_1 \times \omega_2), \forall \omega_1 \subset \subset \omega'_1,
\]
\[
(ii) \quad \frac{1}{\varepsilon} (u_{\varepsilon} - u_0) \rightharpoonup 0 \quad \text{weakly in} \quad L^2 (\omega'_1 \times \omega_2).
\]

**Remark 5.** A necessary condition to get the weak convergence of $\frac{1}{\varepsilon} (u_{\varepsilon} - u_0)$ toward 0 in $L^2 (\omega'_1 \times \omega_2)$ is
\[
\int_{\omega'_1 \times \omega_2} A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} v \, dx + \int_{\omega'_1 \times \omega_2} A_{21} \nabla_{X_1} u_0 \cdot \nabla_{X_2} v \, dx = 0 \quad \forall v \in D(\omega'_1 \times \omega_2).
\]
Indeed, rewriting (30) as
\[
\varepsilon \int_{\Omega} A_{11} \nabla_{X_1} u_{\varepsilon} \cdot \nabla_{X_1} v \, dx + \int_{\Omega} A_{12} \nabla_{X_2} u_{\varepsilon} \cdot \nabla_{X_1} v \, dx + \int_{\Omega} A_{21} \nabla_{X_1} u_{\varepsilon} \cdot \nabla_{X_2} v \, dx + \frac{1}{\varepsilon} \int_{\Omega} A_{22} \nabla_{X_2} (u_{\varepsilon} - u_0) \cdot \nabla_{X_2} v \, dx = 0.
\]
Passing to the limit in each term and using Theorem 3.1 to deduce (37).

### 3.3. Block diagonal structure.

The next theorems improve the convergence rate when the matrix $A$ has a diagonal structure.

**Theorem 3.3.** Under the assumptions of Theorem 3.1 and in addition assume that
\[
A_{12} = A_{21} = 0,
\]
then we have
\[
|u_{\varepsilon} - u_0|_{L^2 (\omega'_1 \times \omega_2)} \cdot |\nabla_{X_2} (u_{\varepsilon} - u_0)|_{L^2 (\omega'_1 \times \omega_2)} = o (\varepsilon),
\]
and
\[
|\nabla_{X_1} (u_{\varepsilon} - u_0)|_{L^2 (\omega'_1 \times \omega_2)} = o (1).
\]

**Proof.** According to the diagonal structure of $A$, we can rewrite (31) as
\[
\lambda |\rho \nabla_{X_1} (u_{\varepsilon} - u_0)|^2_{L^2 (\Omega)} + \frac{\lambda}{\varepsilon^2} |\rho \nabla_{X_2} (u_{\varepsilon} - u_0)|^2_{L^2 (\Omega)}
\]
\[
\leq \int_{\Omega} (u_{\varepsilon} - u_0) A_{11} \nabla_{X_1} (u_{\varepsilon} - u_0) \cdot \nabla_{X_1} \rho^2 \, dx - \int_{\Omega} (u_{\varepsilon} - u_0) A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1} \rho^2 \, dx
\]
\[
- \int_{\Omega} \rho^2 A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1} (u_{\varepsilon} - u_0) \, dx.
\]
Since we can replace $\omega'_1$ by $\omega''_1$ in Theorem 3.1, it can be shown that all the terms of the right hand side in (38) go to zero, which completes the proof of the theorem since $\rho = 1$ on $\omega'_1 \times \omega_2$.

**Theorem 3.4.** Under the assumptions of Theorem 3.3 and if, moreover, the limit $u_0$ and the matrix $A_{11}$ are smooth enough in $X_1$ directions i.e. they satisfy

$$\nabla_{X_1}^2 u_0 \in L^2(\Omega) \quad \text{and} \quad \nabla_{X_1} A_{11} \in L^\infty(\Omega),$$

we have

$$|u_\varepsilon - u_0|_{L^2(\omega'_1 \times \omega_2)}, \quad |\nabla_{X_2} (u_\varepsilon - u_0)|_{L^2(\omega'_1 \times \omega_2)} = O(\varepsilon^2),$$

and

$$|\nabla_{X_1} (u_\varepsilon - u_0)|_{L^2(\omega'_1 \times \omega_2)} = O(\varepsilon).$$

$$\nabla_{X_1}^2 u_0 = \left(\partial_{x_i} \partial_{x_j} u_0\right)_{i,j=1,\ldots,p} \text{ is the Hessian matrix in the directions } X_1. \)$$

**Remark 6.** As we have seen in Proposition 1, the regularity of $u_0$ in the directions $X_1$ depends on the regularity of $A_{22}$ and $f$ in the same directions.

**Proof.** By (28) and (39) we have for a.e. $X_2 \in \omega_2$

$$\rho^2 A_{11} \nabla_{X_1} u_0(\cdot, X_2) \in H^1_0(\omega_1).$$

Thus, by Green’s formula it follows that

$$\int_\Omega \rho^2 A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1} (u_\varepsilon - u_0) \, dx = \int_\Omega \int_{\omega_1} \rho^2 A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1} (u_\varepsilon - u_0) \, dX_1 dX_2$$

$$= -\int_\Omega \left[\nabla_{X_1} \cdot (\rho^2 A_{11} \nabla_{X_1} u_0)\right] (u_\varepsilon - u_0) \, dX_1 dX_2.$$ 

Using this in (38), applying Poincaré and Young inequalities in the last three integrals we derive

$$\lambda |\rho \nabla_{X_1} (u_\varepsilon - u_0)|^2_{L^2(\Omega)} + \frac{\lambda}{2 \varepsilon^2} |\rho \nabla_{X_2} (u_\varepsilon - u_0)|^2_{L^2(\Omega)}$$

$$\leq \frac{\lambda}{2 \varepsilon^2} |\rho \nabla_{X_2} (u_\varepsilon - u_0)|^2_{L^2(\Omega)} + \varepsilon^2 C |\nabla_{X_1} (u_\varepsilon - u_0)|^2_{L^2(\omega'_1 \times \omega_2)}$$

$$+ \varepsilon^2 C |\nabla_{X_1} u_0|^2_{L^2(\Omega)} + \varepsilon^2 C |\nabla_{X_1}^2 u_0|^2_{L^2(\Omega)},$$

where $C$ represents constants independent of $\varepsilon$. This implies, taking into account (35), that

$$\lambda |\rho \nabla_{X_1} (u_\varepsilon - u_0)|^2_{L^2(\Omega)} + \frac{\lambda}{2 \varepsilon^2} |\rho \nabla_{X_2} (u_\varepsilon - u_0)|^2_{L^2(\Omega)} \leq C \varepsilon^2.$$ 

This completes the proof of the theorem since $\rho = 1$ on $\omega'_1 \times \omega_2$. □

**Remark 7.** We can show as in Theorem 3.3, using Remark 5, that a necessary and sufficient condition to get the weak convergence

$$\frac{1}{\varepsilon} (u_\varepsilon - u_0) \rightharpoonup 0 \quad \text{in} \quad L^2(\omega'_1 \times \omega_2),$$

for every $\omega'_1 \subset \subset \omega_1$, is

$$\int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} v \, dx + \int_\Omega A_{21} \nabla_{X_1} u_0 \cdot \nabla_{X_2} v \, dx = 0 \quad \forall v \in H^1_0(\Omega). \quad (40)$$

For example, if $A_{12}$ and $A_{21}$ are constants, this is the case when

$$A_{12} = -A_{21}^T.$$
To conclude this section, we give the following example to clarify the previous situation.

**Example 1.** We take 

\[ \Omega = (0, 1) \times (0, 1) \]

and 

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

In this case, we do not have the weak convergence 

\[ \frac{1}{\varepsilon} (u_\varepsilon - u_0) \rightharpoonup 0 \text{ in } L^2(\Omega). \]

Indeed, if we combine (9) and (25), we get 

\[
\int_\Omega \varepsilon \partial_{x_1} u_\varepsilon \partial_{x_1} v \, dx + \int_\Omega \partial_{x_2} u_\varepsilon \partial_{x_1} v \, dx + \int_\Omega \frac{1}{\varepsilon} (u_\varepsilon - u_0) \partial_{x_2}^2 v \, dx = 0 \quad \forall \, v \in D(\Omega).
\]

Supposing that we have \( \frac{1}{\varepsilon} (u_\varepsilon - u_0) \rightharpoonup 0 \) in \( L^2(\Omega) \) and letting \( \varepsilon \to 0 \), then we deduce 

\[
\int_\Omega \partial_{x_2} u_0 \partial_{x_1} v \, dx = 0.
\]

Replacing \( v \) by \( \partial_{x_2} v \) we derive 

\[
\int_\Omega \partial_{x_2}^2 u_0 \partial_{x_1} v \, dx = - \int_\Omega \partial_{x_2} u_0 \partial_{x_1} \partial_{x_2} v \, dx = 0.
\]

We have therefore, from (25), that 

\[
\int_\Omega \partial_{x_1} f v \, dx = - \int_\Omega f \partial_{x_1} v \, dx = 0,
\]

which implies that \( f \) and \( u_0 \) are independent of \( x_1 \).

**Remark 8.** In the case when the limit solution \( u_0 \) is independent of \( X_1 \), as in the example above, and in addition if the hypothesis (40), which is reduced to 

\[
\int_\Omega A_{12} \nabla X_2 u_0 \cdot \nabla X_1 v \, dx = 0 \quad \forall v \in H^1_0(\Omega),
\]

is true, we can show, using the iteration technique given in [4], that \( u_\varepsilon \) converges towards \( u_0 \) in \( H^1(\omega'_1 \times \omega_2) \) with an exponential rate of convergence, for any \( \omega'_1 \subset \subset \omega_1 \), i.e.

\[
|u_\varepsilon - u_0|_{H^1(\omega'_1 \times \omega_2)} \leq C e^{-\alpha \varepsilon},
\]

where the constants \( \alpha, C > 0 \) are independent of \( \varepsilon \).

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