SEMINAR ON THE CALDERÓN INVERSE PROBLEM (WS 2024/25)

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1. Introduction

Inverse problems are concerned with determining interior characteristics of a medium from boundary measurements and are often motivated by practical problems, e.g. medical and geophysical imaging, where the medium is patient's body or Earth's interior (see [5]). The medium may be thought of as a "black box", whose properties we would like to determine using its interactions with waves and other external stimuli. One of the most influential inverse problems [11] is the long-standing Calderón problem (going back to 1980), which asks if one can recover an anisotropic electrical conductivity from current-to-voltage measurements on the boundary, and is at the core of the imaging technique called Electrical Impedance Tomography. Intimately related are integral geometry questions (e.g. X-ray transforms), where we determine a quantity from its integrals along boundary joining curves. This seminar will serve as an introduction to the Calderón problem and to the tools of Mathematical Analysis and Partial Differential Equations.

2. Mathematical overview

Mathematically, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary, and let $q \in C^\infty(\Omega)$ be a smooth potential. Denote by $-\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ the Laplacian. Given $f \in C^\infty(\partial\Omega)$, we may solve the following boundary value problem for $u \in C^\infty(\Omega)$ (assuming 0 is not the Dirichlet eigenvalue of $-\Delta + q$)

$$(-\Delta + q)u = 0, \quad u|_{\partial\Omega} = f. \tag{2.1}$$

The map $\Lambda_q : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), f \mapsto \partial_\nu u$ is called the *Dirichlet-to-Neumann (DN) map*. Here, ∂_ν denotes the boundary normal derivative. Similarly, for $\gamma \in C^\infty(\Omega)$ positive everywhere, we may define $\Lambda_\gamma f := \partial_\nu y$, where

$$\nabla \cdot (\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

The maps Λ_γ and Λ_q are intimately related, and the unique determination of q from Λ_q or γ from Λ_γ is in many cases equivalent.

The DN map encodes the current-to-voltage measurements, and the Calderón problem is asking whether the map Λ_q uniquely determines q. Intimately related are the questions of *stable determination*, i.e. whether q_1 and q_2 are close if Λ_{q_1} and Λ_{q_2} are close in suitable norms, and *partial data*, i.e. whether Λ_q restricted to functions with support in a subset $S \subset \partial\Omega$ determine q.

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3. Topics

We will distribute the talks on the following topics in the preliminary meeting. More topics can be made available upon request (including: Calderón problem on Riemannian manifolds [9] or Carleman estimates for second order elliptic PDE [6]).

1-3. Sobolev spaces. References: Chapter 2 and Section 4.2 of [8]; Sections 5.1, 5.2, 5.3, and 5.5 in [4]. To include a discussion of trace (in particular the space $H^{\frac{1}{2}}(\partial\Omega)$) and extension operators. Time-permitting Sobolev emedding theorem. Energy estimates approach to well-posedness of elliptic PDEs. Reference: Sections 6.1, 6.2, and optionally 6.3 in [4]. (Note: Section 6.3 proves the elliptic regularity theorem.)

4. Definition of Dirichlet-to-Neumann map. Reduction of the conductivity equation to (2.1). Reference: Section 3.1 in [8]. Note: builds also on understanding of Lectures 1-3.

5-6. Construction of *Complex Geometric Optics* solutions to (2.1) and the proof of uniqueness theorem. These solutions behave like $e^{ix\cdot\zeta}$ where $\zeta \in \mathbb{C}^n$ satisfies $\zeta \cdot \zeta = 0$. Uniqueness means that Λ_q determines q uniquely for $q \in L^\infty(\Omega)$. References: Sections 3.2 and 3.3 in [8].

7-8. Stability in the Calderón problem. This means that Λ_{γ_1} and Λ_{γ_2} close implies that γ_1 and γ_2 are close (in a suitable sense). Includes an example in the unit disk [8, p. 33], as well as the proof of the logarithmic stability result [8, Theorem 4.2] under apriori estimates on q. References: Sections 4.1 and 4.3 in [8].

9-10. Partial data in the Calderón problem: for $\Gamma \subset \partial\Omega$, we have that $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ for all f, implies $\gamma_1 = \gamma_2$. References: Chapter 5 in [8]. (Note: the proof involves Carleman estimates.)

11. Boundary determination: prove that Λ_γ determines γ on $\partial\Omega$ (possibly with derivatives of γ on $\partial\Omega$). References: Section 5 of [10] or alternatively [12].

12. Calderón problem on surfaces. This deals with the case of dimension $n = 2$ in the setting of either Riemannian surfaces or alternatively domains in $\mathbb{R}^2$. References: Section 11.6 of [7] or alternatively [1].

13. Counterexamples to the Calderón problem for disjoint partial data. Reference: Sections 3.3, 3.4, and optionally 4.4 in [3].

References

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