On Lie algebroids, L_∞ algebras, and the homotopy Poisson structure on shifted conormal bundles of coisotropic submanifolds

Semester Project

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Abstract

In this semester research project, we will quickly review Lie algebroids, L_∞ algebras and related structures. We will also elaborate general aspects of the theory of supergeometry. Finally, we will shed light on two things: the homotopy Poisson (P_∞) structure of shifted conormal bundles, and discuss a specific quotient of coisotropic manifolds in the context of Poisson algebras.

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I. PRELIMINARIES

A. Poisson geometry

In order to get acquainted with Poisson geometry, we have to discuss the setup of it. Let us therefore briefly recall the crucial definitions as well as the basic setup of Poisson geometry:

Definition 1 (Poisson manifold)**.** *Let M be a smooth manifold and $C^\infty(M)$ the algebra of smooth functions thereon. M together with a bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a* **Poisson manifold** *if the following hold:*

1. $\{f, g\} = -\{g, f\}$, *i.e.* **antisymmetry**

2. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, *i.e.* **Jacobi identity**

3. $\{fg, h\} = f\{g, h\} + g\{f, h\}$, *i.e.* **Leibniz's rule**

$\{\cdot,\cdot\}$ *is then called the* **Poisson bracket**.

Clearly, the first two conditions ensure that $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra (as will be defined later on). The last condition lets the map $\{f,\cdot\} : C^\infty(M) \to C^\infty(M)$ define a vector field X_f, called the ***Hamiltonian vector field***, in the sense that:

$$\{f, g\} = X_f g = -X_g f = dg(X_f) = -df(X_g)$$

Thus, there exists a C^∞ tensor field $w \in \Gamma(\bigwedge^2 TM)$ with $\{f, g\} = w(df, dg) = w^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$, where we made implicit use of the Einstein summation convention for the indices of the local coordinates on M. As is the case with all such tensor fields, one can define an associated homomorphism $\sharp$ in a natural way, such that the associated map $\sharp : T^*M \to TM, \alpha \longmapsto \alpha^\sharp$ is defined through $\beta(\alpha^\sharp) = w(\alpha, \beta)$ for $\alpha, \beta \in T^*M$. For further reading on Poisson geometry, please see e.g. [8] for a solid introductory set of lecture notes.

B. Lie algebroids

In order to make sense of Lie algebroids, we must shortly revisit a few notions of basic differential geometry and Lie theory, more specifically the definitions of fiber and vector bundles as well as Lie algebras. Lie algebroids can be understood as a way of generalizing Lie algebras, which are defined over a general field as follows:

Definition 2 (Lie algebra). *Let $\mathbb{K}$ be a field. A $\mathbb{K}$-vector space $\mathfrak{g}$ together with a bilinear map (called* **Lie bracket***)*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying both

1. $[X,X] = 0 \ \forall X \in \mathfrak{g}$, *i.e.* **alternating** *(and hence $[X,Y] = -[Y,X] \ \forall X,Y \in \mathfrak{g}$, i.e.* **anti-symmetric***)*

2. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \ \forall X,Y,Z$, *i.e.* **Jacobi identity**

is called **Lie algebra***.*

Later on, we will remark that any Lie algebra is a Lie algebroid, thus the notion of Lie algebroids extends Lie algebras to a more general structure.

The tangent bundle $TM := \sqcup_{m\in M} T_m M$ over a (smooth) manifold M is itself a smooth manifold whose fibers at each point $m \in M$ (by means of the natural projection $\pi : TM \to M$) are the tangent spaces $T_m M$. Vector bundles are defined in much the same way, with the subtle difference that the vector spaces obtained fiberwise - meaning that the vector spaces are the preimages of one corresponding point in M- by the map π (which must not necessarily be the natural projection as introduced above) are allowed to be arbitrary finite-dimensional vector spaces.

Formally, vector bundles are defined as follows:

Definition 3 (Vector bundle). *Given two topological spaces E and M and a continuous surjective map $\pi : E \to M$, a* **vector bundle of rank** *n is the triple (E, π, M) with the following criteria satisfied:*

1. $\forall x \in M, E_x := \pi^{-1}(x)$ *is a vector space isomorphic to $\mathbb{F}^n$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$*

2. *$\forall x \in M$ $\exists$ neighbourhood $U \subset M$ with a local trivialization $\Psi : \pi^{-1}(U) \to U \times \mathbb{F}^n$, i.e. Ψ is a homeomorphism such that $\Psi\big|_y$ is a linear isomorphism $\forall y \in U$.*

In addition, the vector bundle (E, π, M) is called **smooth** *if both E and M are smooth manifolds, π is smooth and the local trivializations Ψ are differentiable.*

As insinuated above, the most basic prototype of a smooth vector bundle is the tangent bundle of a smooth manifold M, with $\pi : TM \to M$ being the natural projection and $E_x = T_xM$ being a vector space (of the dimension as the manifold) and smoothness being given a priori. Henceforth, we will focus on the case $\mathbb{F} = \mathbb{R}$.

We now deal with the key part of this first subsection. Having introduced vector bundles, let us also define the concept of *sections*:

Definition 4 (Sections)**.** *Let A be a (smooth) vector bundle. We define the set of* **sections** *of a A to be the following:*

$$\Gamma(A) = \{\sigma : M \to A \mid \pi \circ \sigma = \mathbb{1}_M\}$$

Finally, we have elaborated on all necessary concepts to introduce Lie algebroids:

Definition 5 (Lie algebroid)**.** *Let M be a smooth manifold. A* **Lie algebroid** *on M is a vector bundle (A, p, M) with:*

1. *a vector bundle map $\rho : A \to TM$, called the* **anchor** *of A*

2. *a bracket $[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ with the properties:*

   (a) *$(\Gamma(A), [\cdot, \cdot])$ is a Lie algebra (over $\mathbb{R}$)*

   (b) *$[X, uY] = u[X, Y] + \rho(X)(u)Y$ holds $\forall X, Y \in \Gamma(A), u \in C^\infty(M)$, i.e. a generalized* **Leibniz rule** *holds*

   (c) *$\rho([X, Y]) = [\rho(X), \rho(Y)]$ holds $\forall X, Y \in \Gamma(A), u \in C^\infty(M)$, i.e. ρ is a* **homomorphism of Lie algebras**

Here, ρ being a bundle map is to be understood in the sense that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TM \\ & {}_{p}\searrow & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

Having established Lie algebroids, it is natural to introduce morphisms between them:

Definition 6 (Lie algebroid morphism)**.** *Let A, A' be two Lie algebroids with anchors ρ, ρ' over a smooth manifold M. A* **Lie algebroid morphism** *ϕ is a vector bundle map with*

1. $\rho' \circ \phi = \rho$

2. $\phi([X,Y]) = [\phi(X), \phi(Y)]\ \forall X, Y \in \Gamma(A)$

The main example of a Lie algebroid structure this paper is concerned with is the one found on the cotangent bundle T^*M of a Poisson manifold $(M, \{\cdot,\cdot\})$: The anchor map is in this case naturally given by $\sharp : T^*M \to TM$, and the Lie bracket on 1-forms $[\cdot,\cdot]$ satisfies $[df, dg] = d\{f, g\}$, with $\{\cdot,\cdot\}$ being the Poisson bracket. The Lie bracket defined this way is called ***Koszul bracket***. One can check that the axioms for a Lie algebroid are fulfilled by definition of $\{\cdot,\cdot\}$.

Similarly to Lie algebroids, there is a parallel notion of Lie *groupoids*, and of course a natural notion of how to relate the two. A comprehensive discussion of these topics would go beyond the scope of this article, however a fantastic introduction to this is given in [14] and [1], the latter one also discusses coisotropic submanifolds, which will be of greatest importance in the end of this project. There was also a "Tour through Some Examples" about groupoids in general (and the Lie case) in [21].

C. L_∞ Algebras

The structure of L_∞ algebras is built on graded mathematical objects, at the very bottom being graded vector spaces. For all that follows, we mostly will assume the base field to be $\mathbb{R}$, or at the very least have vanishing characteristic.

Definition 7 (Graded vector space ⟨1⟩)**.** *A graded* **$\mathbb{Z}$-vector space** *V is the direct sum of a family of indexed subspaces, i.e. $V = \bigoplus_{n\in\mathbb{Z}} V_n$. If $v \in V_n$, then v is said to have degree $|v| = n$.*

This definition - albeit correct - is particularly useful for the case where $|\{V_i | i \in \mathbb{Z} : V_i \neq \{0\}\}| < \infty$, since the notion of morphisms between two such vector spaces V, W is straightforward to define. Throughout this paper, refer to a graded vector space as a (possibly infinite) direct sum, however let us demonstrate how defining it using collections of vector spaces (rather than direct sums thereof) works:

Definition 8 (Graded vector space ⟨2⟩ and morphisms)**.** *A graded* **$\mathbb{Z}$-vector space** *$V_\bullet$ can be viewed as a collection of vector spaces $\{V_i\}_{i\in\mathbb{Z}}$. Consequently, a* **graded morphism** *of graded vector spaces $\phi_\bullet : V_\bullet \to W_\bullet$* **of degree** *$k$ is a collection of linear maps $\phi_i : V_i \to W_{i+k}$ $\forall i \in \mathbb{Z}$. For $k = 0$, we refer to $\phi_\bullet$ simply as a morphism rather than a graded one.*

Of course, analogous notions of $\mathbb{Z}_2$-grading, etc. exist.

Remark 9 (Tensor products of graded vector spaces)**.** *Let $V = \bigoplus_{n\in\mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vector space and $\bigotimes^m V$ be the m-fold* **graded tensor product***, it is important to point out that the interchance of tensor factors is fulfilling the following:*

$$v_1 \otimes v_2 = (-1)^{|v_1||v_2|} v_2 \otimes v_1$$

From this $m = 2$ case, all other cases follow.

In graded vector spaces, it is sensible to introduce a Koszul sign, or more precisely two different kinds thereof, to deal with permutations of different (homogeneous) elements in a general manner (confer with above recalling that any arbitrary permutation can be decomposed into transpositions, for which we know the "sign rule"):

Definition 10 (Koszul sign)**.** *The* **Koszul sign** *ϵ of a permutation $\sigma \in S_n$ acting on a string of n homogeneous vectors is defined by the effect it has on the overall sign in the sense that $\sigma(v_1 \otimes ... \otimes v_n) = \epsilon(\sigma)(v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)})$. Clearly, $\epsilon \in \{1, -1\}$.*

As a sensible consequence, we can introduce the **antisymmetric Koszul sign** χ *of a permutation* $\sigma \in S_n$ *as the product of the sign of the permutation and* ϵ*, i.e.* $\chi(\sigma) := (-1)^\sigma \epsilon(\sigma)$.

Apart from the sign, we will make one further relevant characterization of permutations relevant to this paper, namely the so-called *unshuffle*:

Definition 11 (Unshuffle)**.** *A permutation* $\sigma \in S_n$ *is called a* $(p, n-p)$**-unshuffle** *(denoted by* σ_p *whenever there is no ambiguity) if*

$$\sigma(1) < \sigma(2) < ... < \sigma(p)$$
$$\sigma(p+1) < \sigma(p+2) < ... < \sigma(n)$$

holds. Equivalently, one could define $\sigma_p \in S_n$ *on the condition that* $\sigma(i) < \sigma(i+1)\ \forall i \neq p$.

Having established the relevant definitions regarding permutations and the grading of spaces, we can specify parity properties of functions between such spaces:

Definition 12 (Parity and degree of graded funtions)**.** *Let* V, W *be graded vector spaces and* $f : V \times ... \times V \to W$ *a multilinear map. We then introduce the following properties* f *can have* $\forall \sigma \in S_n$*:*

$$f(v_{\sigma(1)}, .., v_{\sigma(n)}) = \epsilon(\sigma) f(v_1, .., v_n) \Leftrightarrow f \ \textbf{symmetric}$$
$$f(v_{\sigma(1)}, .., v_{\sigma(n)}) = \chi(\sigma) f(v_1, .., v_n) \Leftrightarrow f \ \textbf{skewsymmetric}$$

What is important is that any skew symmetric map $l_n : V^{\otimes n} \to V$ *can be extended to* $\tilde{l}_n : V^{\otimes n+k} \to V^{\otimes k+1}$ *by*

$$\tilde{l}_n(v_1 \otimes ... \otimes v_{n+k}) := \sum_{\sigma_n \in S_{n+k}} \chi(\sigma_n) l_n(v_{\sigma_n(1)} \otimes ... \otimes v_{\sigma_n(n)}) \otimes v_{\sigma_n(n+1)} \otimes ... \otimes v_{\sigma_n(n+k)}$$

In addition, we can also make definition of the **degree** *of maps: A map* $l_n : V^{\otimes n} \to V$ *has degree* k *if* $l_n(v_1 \otimes ... \otimes v_n) \subset V_N$*, with* $N = k + \sum_{i=1}^n |v_i|$.

In the latter part of this paragraph, indexed families of skewsymmetric functions will be of interest, as they can be brought into relation by composition and imposing identities of the resulting function:

Definition 13 (Jacobi function and identity)**.** *Let* $(l_1, ..., l_n)$ *be a collection of skewsymmetric functions using the same notation as above. We define the* **Jacobi function** $\mathcal{J}_n : V^{\otimes n} \to V$ *to be the following:*

$$\mathcal{J}_n(v_1 \otimes ... \otimes v_n) = \sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ \tilde{l}_p(v_1 \otimes ... \otimes v_n)$$

$$= \sum_{p=1}^{n} (-1)^{p(n-p)} \sum_{\sigma_p \in S_n} \chi(\sigma_p) l_{n-p+1}(l_p(v_{\sigma_p(1)} \otimes ... \otimes v_{\sigma_p(p)})$$

$$\otimes v_{\sigma_p(p+1)} ... \otimes v_{\sigma_p(n)})$$

With this function, one can define the **generalized Jacobi identity** *given by* $\mathcal{J}_n = 0$*.*

To see why this definition indeed generalizes the Jacobi identity as it is found in Lie algebras and elsewhere, we consider $l_2(v_1 \otimes v_2) = [v_1, v_2]$, which is clearly antisymmetric, as well as $l_3 \equiv 0 \equiv l_1$. We then have

$$\begin{aligned}
\mathcal{J}_3 &= \sum_{\sigma_p \in S_3} \chi(\sigma_p) l_2(l_2(v_{\sigma_p(1)} \otimes v_{\sigma_p(2)}) \otimes v_{\sigma_p(3)} \\
&= \sum_{\sigma_p \in S_3} \chi(\sigma_p) [[v_{\sigma_p(1)}, v_{\sigma_p(2)}], v_{\sigma_p(3)}] \\
&= [[v_1, v_2], v_3] - (-1)^{|v_1||v_2|} [[v_1, v_3], v_2] \\
&\quad - (-1)^{|v_1|(|v_2|+|v_3|)} [[v_2, v_3], v_1] \\
&\stackrel{!}{=} 0,
\end{aligned}$$

which is the usual graded Jacobi identity used in graded Lie algebras:

Definition 14 (Graded Lie algebra)**.** *A* **graded Lie algebra** *is a graded vector space* V *with a (graded) skewsymmetric 2-function* $[\cdot, \cdot] : V_p \otimes V_q \to V_{p+q}$*, called the* **graded Lie bracket** *such that:*

- $[v_1, v_2] = -(-1)^{|v_1||v_2|} [v_2, v_1]$
- *The aforementioned graded Jacobi identity holds*

Finally, our tool set is complete in order to introduce what an L_∞ ***algebra*** is, namely a graded space possessing an L_∞ structure:

Definition 15 (L_m and L_∞ structure)**.** *A graded vector space* V *with a collection of skewsymmetric maps* $(l_k : V^{\otimes k} \to V, 1)$ *is said to possess a* L_m *structure if* $deg(l_k) = 2 - k$ *and* $\mathcal{J}_n = 0 \ \forall n \in \{1, ..., m\}$*.*

The notion of a L_∞ structure is obtained by considering an <u>*infinite*</u> *collection of such maps* $\{l_k\}_{k\in\mathbb{Z}}$ *together with the property that* $\mathcal{J}_n = 0\ \forall n \in \mathbb{N}$.

What will turn out to be useful (for distinguishing what will be known as P_∞ and S_∞ algebras) are the following two ways to characterize L_∞ algebras using parities rather than degrees, seeing our maps as brackets, and working on the Cartesian product rather then the tensor power (or the symmetric power, which is the reason for the following dichotomy, see [11] for a comprehensive overview):

Definition 16 (L_∞ algebra (antisymmetric sense))**.** *An L_∞ algebra* **in the antisymmetric sense** *is a vector space $V = V_0 \oplus V_1$ endowed with a multilinear n-bracket $\forall n \in \mathbb{N}_0$:*

$$[\cdot, ..., \cdot]_n : \underbrace{V \times ... \times V}_{n\ times} \to V$$

with the properties that the parity of $[\cdot, ..., \cdot]_n$ is $n \mod 2$, all brackets are antisymmetric in the $\mathbb{Z}_2$-graded sense, and $\mathcal{J}_n = 0\ \forall n \in \mathbb{N}_0$.

This definition corresponds to the original one given above. The next one is a bit more subtle:

Definition 17 (L_∞ algebra (symmetric sense))**.** *An L_∞ algebra* **in the symmetric sense** *is a vector space $V = V_0 \oplus V_1$ endowed with a multilinear n-bracket $\forall n \in \mathbb{N}_0$:*

$$[\cdot, ..., \cdot]_n : \underbrace{V \times ... \times V}_{n\ times} \to V$$

with the properties that the parity of $[\cdot, ..., \cdot]_n$ is 1 (i.e. all brackets are odd), all brackets are symmetric in the $\mathbb{Z}_2$-graded sense, and $\mathcal{J}_n = 0\ \forall n \in \mathbb{N}_0$.

For a more intuitive approach to L_∞ algebras (which are often referred to as strongly homotopy algebras), [10] gives a good overview of this.

D. Schouten-Nijenhuis bracket

Yet another bracket that will be of utmost importance for several parts of this project is the so-called *Schouten-Nijenhuis bracket*. Its main application is in the theory of multivector fields: together with the Schouten-Nijenhuis bracket it becomes a Lie superalgebra (if the change in parity, i.e. the interchanging of even and odd subspaces is taken into account). We will largely follow the introduction given in [12] with the occasional addendum to address topics more pertinent to this article.

Proposition 18 (Schouten-Nijenhuis bracket). *Given an n-dimensional smooth manifold M with $\mathcal{A}(M)$ the (exterior) algebra of multivector fields on M. Then $\exists!$ $\mathbb{R}$-bilinear map $[\cdot,\cdot] : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$, the* **Schouten-Nijenhuis bracket** *(***SN bracket** *for short), such that:*

1. $f, g \in A^0(M) = \mathcal{C}^\infty(M)$
   $\Rightarrow [f, g] = 0$

2. $X \in A^1(M) = \Gamma(TM), R \in \mathcal{A}(M)$
   $\Rightarrow [X, R] = \mathcal{L}_X R$

3. $P \in A^p(M), Q \in A^q(M)$
   $\Rightarrow [P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]$

4. $P \in A^p(M), Q \in A^q(M), R \in \mathcal{A}(M)$
   $\Rightarrow [P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]$

The existence of this bracket is by no means trivial - but is a standard proof and can be found in any textbook that deals with this structure, such as [12] (Proposition 3.1).

From our definition of the SN-bracket, some additional properties follow:

Proposition 19. $P \in A^p(M), Q \in A^q(M) \Rightarrow [P, Q] \in A^{p+q-1}(M)$

Proof. We will outline the idea of the proof: Fix $q \in \mathbb{N}_0$, proceed via induction over p. $p = 0$ is clear, $p = 1$ follows essentially from the Lie derivative property. The induction step follows from the property 4. □

Using Proposition 19 together with property 4, we can establish that for $P \in A^p(M)$, $[P, \cdot] : \mathcal{A}(M) \to \mathcal{A}(M)$ is a derivation.

Proposition 20. *Given* $P \in \mathcal{A}(M), Q \in A^q(M) R \in A^r(M)$*, we have* $[P \wedge R, Q] = P \wedge [R, Q] + (-1)^{(q-1)r}[P, Q] \wedge R$.

Proof. After using property 3, we can subsequently use 19 and property 4 which gives us 20. □

In close analogy to 20, we state that for $Q \in A^q(M)$, $[\cdot, Q] : \mathcal{A}(M) \to \mathcal{A}(M)$ is a derivation.

Proposition 21. *Given* $P \in A^p(M), Q \in A^q(M) R \in A^r(M)$*, then the SN-bracket satisfies the graded Jacobi identity:*

$$(-1)^{(p-1)(r-1)}[P, [Q, R]] + (-1)^{(q-1)(p-1)}[Q, [R, P]] \\ +(-1)^{(r-1)(q-1)}[R, [P, Q]] = 0$$

Proof. The proof is similar to the one of Proposition 19: for $p, q, r \in \{0, 1\}$ the proposition trivially holds. We again use induction while making use of property 3 and Proposition 19 to complete the corresponding induction steps. □

With Proposition 21, we established that $(\mathcal{A}(M), [\cdot, \cdot])$ is a graded Lie algebra.

E. Supergeometry

In this project, we will cover several structures and relations found in (classical) Poisson geometry and discuss their homotopy analogues. In order for these analogues to make sense, it is crucial that we replace the manifold structure with the one of a supermanifold in different places. It therefore makes sense to review the basic notions of supergeometry and supermanifolds and to shed light on how the (imposed) $\mathbb{Z}_2$-grading changes things compared to common smooth manifolds.

We will outline a lightning review of supergeometry, as it is a fascinating topic in its own right, and also touch on topics that will not be of greater relevance for the remainder of this article. For a comprehensive introduction to the wonderful theory of supergeometry, one should consult e.g. [7].

In order to introduce supermanifolds, we have to recall some more basic concepts first:

Definition 22 ((Pre)sheaf). *Let X be a topological space and let $(U_i)_{i\in I}$ be its open sets. A* **presheaf** *$\mathcal{F}$ assigns to every open set U_i a set $\mathcal{F}(U_i)$ (which may carry additional mathematical structure such as a group or a ring) together with a restriction morphism $r_V^{U_i} : \mathcal{F}(U_i) \to \mathcal{F}(V)\ \forall V \subset U_i$ such that:*

- $r_{U_i}^{U_i} = id_{\mathcal{F}(U_i)}\ \forall i \in I$
- $r_W^V \circ r_V^{U_i} = r_W^{U_i}$ *for open sets* $W \subset V \subset U_i\ \forall i \in I$

This invites us to define some new notation: Given $f \in \mathcal{F}(U_i)$ and $V \subset U_i$ open, we denote the restriction morphism $r_V^{U_i}(f)$ on V as $f|_V$. Subsequently, a **sheaf** *is a presheaf with the two additional requirements:*

- *Let $f, g \in \mathcal{F}(U_i)$ and $(C_k)_{k\in K} \subset X$ be an open cover of U_i such that $f|_{C_k} = g|_{C_k}\ \forall k \in K$, then $f = g$ holds.*
- *Let again $(C_k)_{k\in K} \subset X$ be an open cover of U_i and $f_k \in \mathcal{F}(C_k)$ such that $f_m|_{C_m\cap C_n} = f_n|_{C_m\cap C_n}\ \forall n, m \in K$. Then $\exists f \in \mathcal{F}(U_i)$ such that $f_k = f|_{V_k}\ \forall k \in K$.*

The former condition gives uniqueness to these $(f_k)_{k\in K}$ as we started with an open cover of U_i.

Having the important definition of a sheaf at hand, we can introduce the notion of a *(locally) ringed space*, of which a supermanifold is a special case:

Definition 23 ((locally) ringed space)**.** *Let X be a topological space and let $(U_i)_{i\in I}$ be the its open sets. Let $\mathcal{F}$ be a sheaf of rings, i.e. $\mathcal{F}(U_i)$ is a ring for all $i \in I$. We then call $(X, \mathcal{F})$ a* **ringed space***. In addition, $(X, \mathcal{F})$ is called a* **locally ringed space** *if all stalks $\mathcal{F}_x := \lim_{U_i \ni x} \mathcal{F}(U_i)$ are local rings, i.e. rings with one unique maximal ideal.*

In order to bring forward the utility of these rather recondite definitions, we explain how the notion of a (non-super)smooth manifold would look like in the context of locally ringed spaces. We start with the the ringed spaces $(\mathbb{R}^n, \mathcal{C}_n^\infty)$, where for $U \subset \mathbb{R}^n$ open, $\mathcal{C}_n^\infty : U \mapsto \mathcal{C}_n^\infty(U)$ maps to the $\mathbb{R}^n$-algebra of smooth functions on U. Let now X be a (Hausdorff) topological space. An n-dimensional ***smooth manifold*** can be defined as a ringed space (X, R), where R is the sheaf-assignment from open sets in X to the $\mathbb{R}$-algebra of real functions on U, such that $\forall x \in X$ there exist an open x-neighbourghood U and a homeomorphism $h : U \to \tilde{U} \subset \mathbb{R}^n$ with the property that h is an isomorphism of $(U, R|_U)$ with $(\tilde{U}, \mathcal{C}_n^\infty|_{\tilde{U}})$.

It should now also be clear to the reader that this is a rather cumbersome way to introduce manifolds as one encounters them in a graduate lecture in differential geometry, which is why a more straight-forward is chosen. This is a shared malaise of all generalizations, however they in turn allow deeper insight into the more general structure certain defintions may entail.

Right now, we will for the first time define the *supermanifold*:

Definition 24 (Supermanifold $\langle 1 \rangle$)**.** *A* **supermanifold** *is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $(U, \mathcal{C}^\infty(U) \otimes \wedge W^*)$ for some finite-dimensional real vector space W, where the isomorphism has to respect the $\mathbb{Z}_2$-parity.*

However, following this approach in the definition - which is perfectly correct - we lose track of important features such as the dimension. Therefore we will go down another path as well in order to specify these features lost along the way. For this, we first have to introduce the concept of the *superdomain*:

Definition 25 (Superdomain)**.** *The $p|q$-dimensional* **superdomain** *$U^{p|q}$ is the (super) ringed space $(U^p, \mathcal{C}^{\infty p|q})$ where $U^p \subset \mathbb{R}^p$ open and $\mathcal{C}^{\infty p|q}$ is the sheaf of supercommuting rings defined by (for $V \subset U$ open)*

$$\mathcal{C}^{\infty p|q} : V \mapsto \mathcal{C}^\infty(V)[\theta^1, ..., \theta^q],$$

where the θ^i are the (Grassmanian) anticommuting variables satsfying the following relations:

$$(\theta^i)^2 = 0,\ \theta^i\theta^j = -\,\theta^j\theta^i\ \forall i \neq j$$
$$\Leftrightarrow \theta^i\theta^j = -\theta^j\theta^i,\ 1 \leq i, j \leq q$$

This allows us to write each element of $\mathcal{C}^{\infty p|q}(V)$ in the form of $\sum_{I\subset\{1,...,q\}} f_I\theta^I$ with the "coefficients" $f_I \in \mathcal{C}^\infty$ and $\theta^I = \theta^{i_1}\cdots\theta^{i_r}$, $I = \{i_1, ..., i_r\}$, $i_1 < ... < i_r$.

Having superdomains at our disposal, we can launch our second effort at defining a supermanifold:

Definition 26 (Supermanifold ⟨2⟩)**.** *A* **supermanifold** *of dimension $p|q$ is a (super) ringed space that is locally isomorphic to* $(\mathbb{R}^p, \mathcal{C}^{\infty p|q}) := \mathbb{R}^{p|q}$.

As a remark we note that in particular $(U^p, \mathcal{C}^{\infty p|q})$ is a supermanifold itself $\forall U \subset \mathbb{R}^p$, even for $U = \mathbb{R}^p$. In the latter case, we call $\mathbb{R}^p$ the set of *even coordinates* and θ^i the set of *odd coordinates.*

It is convenient to switch back and forth between these two definitions of a supermanifold depending on which one is more natural in the corresponding context.

The next step is to see how a supermanifold behaves in practice. How do the notions of vector fields and one-forms compare to their corresponding supergeometric counterparts? How does the linear theory on supermanifolds work? Let us first introduce a further super-generalization of an algebraic structure:

Definition 27 (graded derivations)**.** *Consider a graded commutative algebra A. An even/odd endomorphism $v \in End_{\mathbb{R}}(A)$ (in the sense of functions) is called an* **even/odd derivation** *of A if the super-Leibnitz rule holds, i.e. for $f, g \in A$*

$$v(fg) = v(f)g + (-1)^{|X||f|}fv(g)$$

The direct sum of the even and odd derivations of A $Der(A) := Der_0(A) \oplus Der_1(A)$ is the set of **graded derivations** *of A.*

In a pedantic sense, what we just defined is a *left* derivation (as the vector field is on the left of the argument). In much the same way, one could define its right counterpart.

The natural next step is to define the super-analog of differential forms and the exterior derivative, which we shall do at once:

Definition 28 (super 1-forms)**.** *Let A be a graded algebra as above. The A-dual of the module of vector fields is the module of* **1-forms** *denoted by $\Lambda^1(A) := Hom_A(D(A), A)$.*

The differential of a function $f \in A$ is the 1-form

$$df(X) = (-1)^{|X||f|}X(f),$$

where the sign is a choice of convention.

As a last subject of this subsection, we will make things more concrete: In the local theory of supermanifolds, we acknowledge the existence of odd coordinates, how does one perform calculus with these coordinates? We will illustrate this using a $p|q$-dimensional manifold.

Let us first focus on the odd bit alone, before going to the case that involves both odd and even parameters. Let us denote the $\mathbb{R}$-algebra of the Grassmannian, anticommuting $(\theta^1, .., \theta^q)$ as Λ^q (the exterior algebra of anticommuting polynomials).

Definition 29 (Berezin integral)**.** *We define the* **Berezin integral on** Λ^q *to be the linear functional* $\int_{\Lambda^q} d^p\theta$ *satisfying:*

- $\int_{\Lambda^q} \theta^q \cdots \theta^1 d^q\theta = 1$
- $\int_{\Lambda^q} \frac{\partial f}{\partial \theta^i} d^q\theta = 0 \; \forall i,$

where $d^q\theta = d\theta^1 ... d\theta^q$ *is the (full) Grassmannian measure.*

Now, we have to put our attention to a subtlety. Consider the case $q = 2$ and $f(\theta^1, \theta^2) = \theta^1\theta^2$. What does the expression $\frac{\partial f}{\partial \theta^2}$ mean? Naively, one could say that $\frac{\partial f}{\partial \theta^2} = \theta^1$, but on the other hand $\frac{\partial f}{\partial \theta^2} = -\theta^1$ would be just as reasonable! We therefore establish the notions of left- and right-derivative.

The properties above define any odd integral uniquely. For a Berezin integral in $(\theta^1, .., \theta^q)$, we successively integrate out θ^i, starting with $i = 1$ and ending with $i = q$, where prior to each partial integration the corresponding θ^i is permuted to the very left of the expression and then the properties of $\int_{\Lambda^q} d^q\theta$ are put to use. Derivation works in the same way. It is interesting to notice that for Grassmannian variables, integration coincides with differentiaion, i.e. the integral of a function $f = f(\theta^1, .., \theta^q)$:

$$\int_{\Lambda^q} f(\theta^1, .., \theta^q) d^q\theta = \frac{\partial}{\partial \theta^q} \cdots \frac{\partial}{\partial \theta^1} f(\theta^1, .., \theta^q)$$

Let us introduce a different set of odd coordinates $(\tilde{\theta}^1, .., \tilde{\theta}^q)$ with $\theta^i = \theta^i(\tilde{\theta}^1, .., \tilde{\theta}^q)$ (whose indices we will suppress wherever there is no ambiguity), we will state how the integral changes under this change of variables. The Jacobian is the matrix $\mathcal{D} := \left\{ \frac{\partial \theta^i}{\partial \tilde{\theta}^i} \right\}_{i,j=1,...,q}$ and we have:

$$\int_{\Lambda^q} f(\theta) d^q\theta = \int_{\Lambda^q} f(\theta(\tilde{\theta})) \det \mathcal{D}^{-1} d^q\tilde{\theta}$$

Let us now consider the case where odd and even coordinates are present. When we perform integration on a $p|q$-supermanifold (rather on its odd part only), we have to replace the the exterior algebra of anticommuting polynomials by the $p|q$-***superalgebra*** $\Lambda^{p|q}$ of real commuting variables $(x^1, ..., x^p)$ and anticommuting variables $(\theta^1, ..., \theta^q)$, thus the *full* Berezin integral becomes:

$$\int_{\Lambda^{p|q}} f(x,\theta) d^q\theta d^p x = \int_{\mathbb{R}^p} d^p x \int_{\Lambda^q} f(x,\theta) d^q \theta$$

Let us now consider a change of coordinates in much the same vein as before $x^i = x^i(\tilde{x}, \tilde{\theta})$ and $\theta^i = \theta^i(\tilde{x}, \tilde{\theta})$, then we need to be careful, as naively using the notion of a Jacobian as before would not be sensible. We need to take superchanges into account:

Definition 30 (superdeterminant/Berezinian)**.** *The* **Berezin matrix** *is the following expression:*

$$\mathcal{B} = \frac{\partial(x,\theta)}{\partial(\tilde{x},\tilde{\theta})} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the entries of the block-matrices on the diagonal of $\mathcal{B}$ are $A = \frac{\partial x}{\partial \tilde{x}}, D = \frac{\partial \theta}{\partial \tilde{\theta}}$ are even and the ones of the off-diagonal block-matrices $B = \frac{\partial x}{\partial \tilde{\theta}}, C = \frac{\partial \theta}{\partial \tilde{x}}$ are odd.

The **Berezinian** *is then defined to be*

$$Ber := \det\left(A - BD^{-1}C\right) \det D^{-1}$$

Using the Berezinian, we can - as the culmination of our efforts - state the general transformation law of functions of the superalgebra:

$$\int_{\Lambda^{p|q}} f(x,\theta)\, d^q\theta d^p x = \int_{\Lambda^{p|q}} f\left(x(\tilde{x},\tilde{\theta}), \theta(\tilde{x},\tilde{\theta})\right) \times \frac{\det \frac{\partial x(\tilde{x},0)}{\partial \tilde{x}}}{\left|\det \frac{\partial x(\tilde{x},0)}{\partial \tilde{x}}\right|} \times Ber\ d^q\tilde{\theta} d^p \tilde{x},$$

where we had to introduce the additional determinant factor in order to respect the parity of the coordinate transformation.

F. Resolutions

We will now quickly introduce the notion of resolutions, which will be of importance in the very last part of this project.

Definition 31 (Resolution and Coresolution)**.** *Let A be a module over a ring R (A typically carries some additional structure, e.g. A is the algebra of functions over some (sub)manifold). A* **resolution** *is an exact sequence of the form*

$$\cdots \xrightarrow{\partial} E_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_1 \xrightarrow{\partial} E_0 \xrightarrow{\epsilon} A \to 0$$

or written more concisely

$$E_\bullet \xrightarrow{\epsilon} A \to 0$$

Similarly, we can introduce the dual notion: A **coresolution** *is an exact sequence of the form:*

$$0 \to A \xrightarrow{\epsilon} C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \xrightarrow{d} \cdots$$

or more concisely

$$0 \to A \xrightarrow{\epsilon} C^\bullet$$

We call $\epsilon : E_\bullet \xrightarrow{\epsilon} A$ the augmentation map for the case of the resolution, and analogously we call $A \xrightarrow{\epsilon} C^\bullet$ the coaugmentation map for the case of the coresolution. Where there is no ambiguity, we will use the terms resolution and coresolution interchangeably.

Given a resolution for A, we will sometimes refer to A as the "new" C^0 or E_0, respectively, seeing it as a part of the complex through ϵ.

II. INTRODUCTION

Let M be a (smooth) Poisson manifold, $\pi = \pi^{ab}\partial_a \wedge \partial_b$ the Poisson tensor and $\{f,g\}_p = \frac{\partial f}{\partial x^a}\pi^{ab}\frac{\partial g}{\partial x^b}$ the Poisson bracket. Furthermore, let $\Omega^k(M)$ denote the differential forms (of degree k) on M and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the deRham-differential. We introduce yet another differential, but on the space of multivector fields rather than forms:

Definition 32 (Lichnerowicz differential). *The* **Lichnerowicz differential** *is defined as follows:*

$$\begin{aligned} d_\pi \colon A^k(M) &\longrightarrow A^{k+1}(M) \\ \alpha &\longmapsto [\pi, \alpha], \end{aligned}$$

where $[\cdot,\cdot]$ *is the Schouten bracket introduced before.*

Our starting point for this project is a classical fact from Poisson geometry: If we recall the previously defined map $\sharp : T^*M \to TM$, we state that $\sharp$ transforms d to d_π and the Koszul bracket to the Schouten bracket, i.e.

$$\begin{array}{ccc} A^k(M) & \xrightarrow{\;d_\pi\;} & A^{k+1}(M) \\ \uparrow \sharp & \circlearrowright & \uparrow \sharp \\ \Omega^k(M) & \xrightarrow{\;d\;} & \Omega^{k+1}(M) \end{array}$$

That means we have the following equality for the Koszul backet $[\cdot,\cdot]$ and the Schouten bracket $[\![\cdot,\cdot]\!]$:

$$[\gamma_1, \gamma_2]^\sharp = [\![\gamma_1^\sharp, \gamma_2^\sharp]\!]$$

Proof. We will postpone this proof to the next chapter, where it will be handed to us on a silver plate. □

For this project, we will investigate how these relations change if we introduce a *homotopy* Poisson structure, i.e. if we introduce higher Poisson brackets with an arbitrary number of arguments. Naturally, these higher brackets $\{\cdot, ..., \cdot\}$ are still multilinear, antisymmetric, satisfy the identity in 13 for each n as well as the multiderivation property. In this context, we will introduce several new notions such as *thick morphisms*.

As mentioned in the preliminaries of this paper, our underlying manifold M will possess a superstructure. In close analogy to the non-super case, we will make the following impositions for the forms $\Omega(M)$ and multivector fields $\mathcal{A}(M)$ on M:

$$\Omega(M) := C^\infty(\Pi TM)$$
$$\mathcal{A}(M) := C^\infty(\Pi T^*M)$$

Outlining the following, we will closely follow [9].

Remark 33 ((even) Poisson brackets from functions)**.** *Given an even (fiberwise) quadratic function $\pi \in C^\infty(\Pi T^*M)$ and the Schouten bracket $[\![\cdot,\cdot]\!]$, we can define an even Poisson bracket $\{\cdot,\cdot\}$ (i.e. parity $\epsilon = 0$) on M via*

$$\{f, g\} := [\![f, [\![\pi, g]\!]]\!] = [\![[\![f, \pi]\!], g]\!],$$

where the identity in 13 is given by $[\![\pi, \pi]\!] = 0$.

Remark 34 ((odd) Poisson brackets from functions)**.** *In much the same way as in the previous remark, one can define an odd Poisson bracket $\{\cdot,\cdot\}$ (i.e. parity $\epsilon = 1$) using an odd (fiberwise) quadratic function $\xi \in C^\infty(\Pi T^*M)$, and this time using the canonical Poisson bracket $(\cdot,\cdot)$:*

$$\{f, g\} := -(f, (\xi, g)) = -((f, \xi)), g)$$

The minus sign is pure convention. The corresponding generalized Jacobi identity is given by $(\xi, \xi) = 0$, which is sometimes called the master equation. In this context, we refer to $\xi \in C^\infty(\Pi T^*M)$ as the **master Hamiltonian** *(likewise for $\pi \in C^\infty(\Pi T^*M)$ above).*

In order to make sense of the rather cluttered way we introduced different brackets and their parities, we distinguish the two most important notions so far, *odd* and *even* Poisson manifolds, as well as their differences (see also the second section of [9]): In terms of coordinates x^i, the coordinate expressions for both even and odd Poisson brackets coincide, as does the symmetry of the brackets. However, connected with the parities, we have $\pi^{ij} = (-1)^{(|x^i|+1)(|x^j|+1)}\pi^{ji}$ and $|\pi^{ij}| = |x^i| + |x^j|$, but $\xi^{ij} = (-1)^{|x^i||x^j|}\xi^{ji}$ and $|\xi^{ij}| = |x^i| + |x^j| + 1$. So far, we have introduced several definitions about and related to Lie algebras and algebroids, however, we have not mentioned where and in what way they appear. We shall do this in the next paragraph, let us first make yet another important definition:

Definition 35 (homological vector fields and Q-manifolds)**.** *Let M be a supermanifold, Q a vector field on M. We call Q a* **homological vector field** *if Q is odd in the usual sense and $[Q,Q] = 2Q^2 = 0$. The pair (M,Q) is then referred to as a Q-manifold.*

A smooth map between two Q-manifolds $\phi : M_1 \to M_2$ is a **Q-morphism** *if Q_1 and Q_2 are ϕ-related, i.e. $D\phi(x)[Q_1(x)] = Q_2(\phi(x))$. (Equivalently this is to say that $\phi^* \circ Q_2 = Q_1 \circ \phi^*$.)*

For a Q-manifold with coordinates y^i, the condition $Q^2 = 0$ translates for $Q = Q^i(y)\frac{\partial}{\partial y^i}$ to the condition

$$Q^i \left(\frac{\partial}{\partial y^i} Q^j \right) = 0$$

An easy example for a Q-manifold is the (shifted) tangent bundle ΠTM for any supermanifold M, with $Q_{\Pi TM} = d = dy^i \frac{\partial}{\partial y^i}$ which has the wanted properties and naturally squares to zero, $d^2 = 0$.

Having now introduced a horde of definitions and objects, we now point out some equivalences and relations amongst them in order to provide a clearer view on things:

Proposition 36. *Given a $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$, we have equivalences between the following structures:*

- *L possesses a Lie superalgebra structure*
- *ΠL possesses an odd Lie superalgebra structure*
- *ΠL possesses a homological vector field of weight $+1$*
- *L^* is endowed with an even Poisson bracket of weight -1*
- *ΠL^* is endowed with an odd Poisson bracket of weight -1,*

where **weight** *refers to the degree of linear coordinates in a given coordinate expression (we will illustrate this using an example in the proof).*

Proof. Details about the definitions and properties of both the *Lie-Poisson bracket* as well as the *Lie-Schouten bracket* mentioned below can be found in e.g. [13] and [17].

- If L possesses a Lie superalgebra structure, then we may simply use the Lie-Poisson bracket $[\cdot,\cdot]^*_{LPB}$, which by definition endows L^* with an even Poisson structure of weight $+1$.

- Similarly to the first case, we may use the Lie-Schouten bracket $[\cdot,\cdot]^{\Pi*}_{LSB}$ to endow ΠL^* with an odd Poisson structure of weight -1.

- A homological vector field $Q = \frac{1}{2}\xi^i\xi^j Q^k_{ji}\frac{\partial}{\partial\xi^k}$ on ΠL immediately defines a Lie algebra via setting $\iota_{[u,v]} = (-1)^{|u|}[[Q,\iota_u],\iota_v]$ with $\iota_v = (-1)^{|v|}v^i\frac{\partial}{\partial\xi^i}$ for v^i the components of $v \in L$. The Jacobi identity holds because Q is homological, i.e. $Q^2 = 0$.

  For the basis generators e_i of L, we have that $[e_i, e_j] = (-1)^{|e_j|}Q^k_{ij}e_k$.
  We also have $[x_i, x_j]^*_{LPB} = (-1)^{|x_j|}Q^k_{ij}x_k$ and $[q_i, q_j]^{\Pi*}_{LSB} = (-1)^{|e_i|}Q^k_{ij}q_k$.

□

Similarly to the aforegoing proposition, we can also outline important equivalences when considering *homomorphisms* between Lie superalgebras rather than Lie superalgebras themselves:

Proposition 37. *Given two Lie superalgebras L_1, L_2, the following define equivalent structures:*

- *A homomorphism of Lie superalgebras $L_1 \to L_2$*

- *A homomorphism of odd Lie superalgebras $\Pi L_1 \to \Pi L_2$*

- *A Q-morphism $\Pi L_1 \to \Pi L_2$*

- *A (linear) Poisson map $L_2^* \to L_1^*$ for the Lie-Poisson brackets*

- *A (linear) Poisson map $\Pi L_2^* \to \Pi L_1^*$ for the Lie-Schouten brackets*

Proof. Using the structural equivalences above and remembering that a Poisson map between two Poisson manifolds (both odd or both even) relates the Poisson tensors, the proof follows. □

In much the same way, we can state equivalent structures for Lie *algebroids* rather than superalgebras. We omit most details of the corresponding proofs as they are very similar to the Lie superalgebra case (see [16] for comprehensive remarks). Note that the notion of Lie algebroids does not change if we define the base space to be a *super*manifold.

Proposition 38. *Given a vector bundle $\pi : E \to M$ over some manifold M, we then have the following equivalent structures:*

- *E possesses a Lie algebroid structure*
- *ΠE possesses an odd Lie algebroid structure*
- *ΠE possesses a homological vector field of weight $+1$*
- *E^* is endowed with an even Poisson bracket of weight -1*
- *ΠE^* is endowed with an odd Poisson bracket of weight -1*

Proof. We will mainly explicate the equivalence (1) $\Leftrightarrow$ (3), as it involves explicit constructions helpful to later endeavors:

- ($\Leftarrow$) Let us choose local coordinates (x^a, θ^i) on E. A given homological vector field of weight $+1$ can always be written as:

  $$Q = Q^k_{ij}\theta^i\theta^j\frac{\partial}{\partial\theta^k} + Q^a_i\theta^i\frac{\partial}{\partial x^a}$$

  We define $\rho : E \to TM$ by $\rho(X) := X^i Q^a_i \frac{\partial}{\partial x^a}$ and

  $$\begin{aligned}[X,Y] :=& X^iY^jQ^k_{ij}\frac{\partial}{\partial\theta^k} \\ &+ \rho(X)(Y^j)\frac{\partial}{\partial\theta^j} - \rho(Y)(X^i)\frac{\partial}{\partial\theta^i}\end{aligned}$$

  for $X = X^i(x)\frac{\partial}{\partial\theta^i}$ and $Y = Y^j(x)\frac{\partial}{\partial\theta^j}$. Thus we have established the Lie algebroid structure we were looking for.

- ($\Rightarrow$) If we are given a bracket $[\cdot,\cdot]$ and also a $\rho : E \to TM$, we can find functions Q^k_{ij} and Q^a_i (defined by the equations above) such that we get a vector field of the form above. One can easily see that for $([\cdot,\cdot],\rho)$ carrying a Lie algebroid structure, Q has to be homological.

$\square$

Regarding morphisms between Lie algebroids, we are able to make an analogous statement:

Proposition 39. *Given two Lie algebroids E_1, E_2 with the same (super) base space M, the following notions are equivalent:*

- *A Lie algebroid homomorphism $E_1 \to E_2$*
- *A Q-morphism $\Pi E_1 \to E_2$ that is linear in each fiber*
- *A (linear) Poisson map $E_2^* \to E_1^*$ for the Lie-Poisson brackets*
- *A (linear) Poisson map $\Pi E_2^* \to \Pi E_1^*$ for the Lie-Schouten brackets*

In the preliminaries of this project, we defined L_∞ algebras as well as Lie-algebroids. However, there exists also a notion of L_∞ algebroids, which will be of great importance for the remainder of this article:

Definition 40 (L_∞ algebroids)**.** *Let $\pi : E \to M$ be a vector bundle. Let there be a sequence of brackets of parity $n = 0, 1, 2, ...$:*

$$\underbrace{[-, ..., -]}_{nentries} : C^\infty(M, E) \times ... \times C^\infty(M, E) \to C^\infty(M, E)$$

effectively making $C^\infty(M, E)$ into a L_∞ algebra. In addition, let there also be a sequence of maps, called the **higher anchors**

$$a_n(-, ..., -) : E \times ... \times E \to TM$$

which are (multi)linear in each fiber and satisfy the following version of the Leibniz identity for functions f on M:

$$\begin{aligned}[][v_1, ..., v_n, fv_{n+1}] = a_n(v_a, ..., v_n)(f)v_{n+1} \\ + (-1)^\alpha f[v_1, ..., v_n, v_{n+1}],\end{aligned}$$

where $\alpha = |f|(n + 1 + |v_1| + ... + |v_n|)$. If the vector bundle E is endowed with such a sequence of brackets and higher anchors, we call it an L_∞ **algebroid***.*
For later convenience, we state that we can assemble the higher anchors for an L_∞ algebroid into a single (in general non-linear) bundle map $a : \Pi E \to \Pi TM$, which we call the **total anchor** *of the L_∞ algebroid.*

We now introduce the *homotopy analogs* of even and odd Poisson algebras. For the following two definitions, let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space with an even (bilinear) multiplication:

Definition 41 (P_∞ algebra)**.** *V is a P_∞* **algebra** *- sometimes called a homotopy Poisson algebra - if V is endowed with a bracket with parity n and n entries $\forall n \in \mathbb{N}_0$*

$$\{\cdot, ..., \cdot\} : V \times ... \times V \to V$$

which fulfills the (multi)derivation property and makes V into an (antisymmetric) L_∞ algebra.

Definition 42 (S_∞ algebra)**.** *V is a S_∞* **algebra** *- sometimes called a homotopy Schouten algebra - if V is endowed with an odd bracket with n entries $\forall n \in \mathbb{N}_0$*

$$\{\cdot, ..., \cdot\} : V \times ... \times V \to V$$

which fulfills the (multi)derivation property and makes V into a (symmetric) L_∞ algebra.

If we are given a supermanifold M, the prototype of such a graded vector space which is endowed with a suitable algebra structure is $V = C^\infty(M)$. In this case, homotopy Poisson and Schouten structures on M are defined by a special kind of functions, so-called *Master Hamiltonians.* For later convenience, we redefine the sign of the Schouten bracket $[\![\cdot, \cdot]\!]_S$ to $[\![R_1, R_2]\!] := (-1)^{|R_1|+1}[\![R_1, R_2]\!]$ on $C^\infty(\Pi T^* M)$, and thus in particular for a set of coordinates we have $[\![x^*_i, x^j]\!] = (-1)^{|x^i|}\delta^j_i$.

Definition 43 (P_∞ Master Hamiltonian)**.** *Let $P \in C^\infty(\Pi T^* M)$ be an even function, which we address in terms of a formal expansion around the the zero-section in $\Pi T^* M$ using base and fiber coordinates x, x^* (note that all components naturally depend on x):*

$$\begin{aligned} P &= P_0 + P_1 + P_2 + P_3 + ... \\ &= P_0 + P^i x^*_i + \frac{1}{2!} P^{ij} x^*_j x^*_i + \frac{1}{3!} P^{ijk} x^*_k x^*_j x^*_i \end{aligned}$$

We call P a P_∞ **master Hamiltonian** *iff it satisfies $[\![P, P]\!] = 0$.*

Using this Master Hamiltonian, we can now establish a P_∞ structure by defining suitable (higher) brackets $\{\!|\cdot, ..., \cdot|\!\}$ in the following way:

Definition 44 (Higher Poisson brackets)**.** *Given P as above, we define* **higher Poisson brackets** *to be the following ($n \in \mathbb{N}_0$):*

$$\{\!| f_1, ..., f_n |\!\} := [...[P, f_1], ..., f_n]$$

We then make the following proposition:

Proposition 45. *The condition that* $[\![P,P]\!] = 0$ *is fulfilled is equivalent to* $\{\!|f_1,...,f_n|\!\}$ *satisfying the generalized Jacobi identities.*

Proof. We refer the reader to [18] for the full proof as well as further applications of the above equivalence, however we stress some key points:

- As the brackets $\{\!|\cdot,...,\cdot|\!\}$ are (anti)symmetric in the $\mathbb{Z}_2$-graded sense, it suffices to study their behaviour on identical even arguments such as $\{\!|a,...,a|\!\}$, if linearity and multiplication by odd constants is taken into account. This will allow for the convenient use of generating functions (thought of in degrees as number of identical entries in $\{\!|\cdot,...,\cdot|\!\}$).
- Using the fact above, the identity is shown by restricting the degree of $[\![P,P]\!]$ in the coordinates x^* and proceeding step by step.

□

Thus, we have fulfilled all the axioms necessary to establish a P_∞ structure.

There is an analogous construction for establishing a homotopy Schouten structure using the canonical Poisson bracket, however we will not delve into the details of this as the P_∞ structure is what will be our main interest when we are going to investigate coisotropic submanifolds in the latter part of this project.

At this point, we have to readdress some specific types of vector fields we introduced earlier on in this project: homological and the Hamiltonian vector fields. We will show how homological vector fields can be used to describe L_∞ algebras and algebroids in an efficient way, and how the associated (lifted) Hamiltonian vector fields can be used to furnish a homotopy algebra. As mentioned before, we will focus on the P_∞ case.

If V is a $\mathbb{Z}_2$-graded vector space as above, and $Q \in \mathfrak{X}(V)$ a formal odd vector field which we may write as a formal expansion in the degree of weights (c.f. [20], *(Thm. 3.4)*)

$$\begin{aligned} Q &= Q_{-1} + Q_0 + Q_1 + Q_2 + ... \\ &= Q^\alpha(\xi)\frac{\partial}{\partial \xi^\alpha} \\ &= \left(Q_0^\alpha + \xi^i Q_i^\alpha + \frac{1}{2!}\xi^i\xi^j Q_{ji}^\alpha + \frac{1}{3!}\xi^i\xi^j\xi^k Q_{kji}^\alpha + ...\right)\frac{\partial}{\partial \xi^\alpha} \end{aligned}$$

If we see $v \in V$ as a vector field $i(v) = v^i\frac{\partial}{\partial \xi}$ and likewise for $u \in \Pi V$ through $\iota(u) = (-1)^{|u|}u^i\frac{\partial}{\partial \xi}$, Q defines higher brackets on V via

$$[v_1, ..v_n] := [...[Q, v_1], ..., v_n](0)$$

and on ΠV via

$$\iota([u_1,...,u_n]) = (-1)^{\Sigma_k |u_k|(n-k)}[...[Q,\iota(u_1)],...,\iota(u_n)](0)$$

In analogy to 45, $(V,[\cdot,..\cdot])$ and $(\Pi V,\iota([\cdot,..\cdot]))$ define an L_∞ algebra in the symmetric and anti-symmetric sense respectively if and only if $Q^2=0$, i.e. if Q is homological.

Rather than L_∞ algebras, one can also proceed similarly to find L_∞ algebroids: Consider a vector bundle $\pi : E \to M$ and its shifted bundle over the same base $\tilde{\pi} : \Pi E \to M$. Let $Q \in \mathfrak{X}(\Pi E) = \mathfrak{X}_{-1}(\Pi E) + \mathfrak{X}_0(\Pi E) + \mathfrak{X}_1(\Pi E) + ...$ be again a formal odd vector field which we may write as a formal expansion in the degree of weights (with x being the base coordinates and ξ the coordinates along the fiber)

$$\begin{aligned} Q =& Q_{-1} + Q_0 + Q_1 + Q_2 + ... \\ =& Q^\alpha(x,\xi)\frac{\partial}{\partial \xi^\alpha} \\ =& \underbrace{\left(Q^\alpha(x)\frac{\partial}{\partial\xi^\alpha}\right)}_{\in\mathfrak{X}_{-1}(\Pi E)} + \underbrace{\left(Q^\beta(x)\frac{\partial}{\partial x^\beta} + \xi^i Q_i^\alpha(x)\frac{\partial}{\partial \xi^\alpha}\right)}_{\in\mathfrak{X}_0(\Pi E)} \\ &+ \underbrace{\left(\xi^i Q_i^\beta(x)\frac{\partial}{\partial x^\beta} + \frac{1}{2!}\xi^i\xi^j Q_{ji}^\alpha(x)\frac{\partial}{\partial\xi^\alpha}\right)}_{\in\mathfrak{X}_1(\Pi E)} \\ &+ \underbrace{\left(\xi^i\xi^j Q_{ji}^\beta(x)\frac{\partial}{\partial x^\beta} + \frac{1}{3!}\xi^i\xi^j\xi^k Q_{kji}^\alpha(x)\frac{\partial}{\partial\xi^\alpha}\right)}_{\in\mathfrak{X}_2(\Pi E)} + ... \end{aligned}$$

We have the following proposition:

Proposition 46. *Let E be a vector bundle as above. We can define higher brackets by*

$$\iota([u_1,...,u_n]) := (-1)^{\Sigma_k |u_k|(n-k)}[...[Q,\iota(u_1)],...,\iota(u_n)]_{-1}$$

and (higher) anchors by

$$a_n([u_1,...,u_n])(f) := (-1)^{\Sigma_k |u_k|(n-k)}[...[Q,\iota(u_1)],...,\iota(u_n)]_0(f) \ ,$$

where the subscripts $0,-1$ denote the implicit projection on the subspaces of weight $0,-1$ respectively.

These two definitions furnish an L_∞ algebroid structure if and only if $Q^2=0$, i.e. Q is homological.

Proof. The proof is very similar to the one above where we had to show the equivalence of $[P, P] = 0$ with the generalized Jacobi identities, again we refer the reader to [18] for the proof in all its glory. □

An interesting viewpoint on this matter in the context of generating functions is given in [18].

III. HIGHER BRACKETS AND THICK MORPHISMS

A. Higher brackets

We will now introduce what will be the centerpiece of this project. Before moving on the thick morphisms, we define so called *higher* Koszul brackets. The initial introduction might seem rather inexpedient, which is why we will later try to find a geometrical setting where this structure arises in a natural way.

Again, for this introduction we will closely follow [9] in addition to [20] (*see Chapter 4.2*).

Definition 47 (Higher Koszul brackets)**.** *Let* $\{\!|\cdot, ..., \cdot|\!\}$ *be the higher Poisson brackets induced by a* P *as above. We then define the* **higher Koszul brackets** $[\![\cdot, ..., \cdot]\!]$ *on* $\Omega(M)$ *for functions and differentials as follows (and then extended by the Leibniz rule), where* $\alpha_n = (-1)^{(n-1)|f_1|+(n-2)|f_2|+...|f_{n-1}|+n}$*:*

$$\begin{aligned}
[\![f]\!] &:= \{\!|f|\!\} \\
[\![f_1, ..., f_k]\!] &:= 0 \text{ for } k > 1 \\
[\![f_1, df_2, ..., df_k]\!] &:= \alpha_n \{\!|f_1, ..., f_n|\!\} \\
[\![df_1, df_2, ..., df_k]\!] &:= -\alpha_n d\{\!|f_1, ..., f_n|\!\} \ ,
\end{aligned}$$

where all other brackets between functions and differentials vanish. Note that these relations induce an L_∞ *algebroid structure on* T^*M *immediately.*

We are about to have a look back at where we started: We state that the *Lichnerowicz differential* (for a manifold M possessing a P_∞ structure induced by some Master Hamiltonian P) has the following coordinate expression:

$$d_P = (-1)^{|x^a|}\left(\frac{\partial P}{\partial x^*_a}\frac{\partial}{\partial x^a} + \frac{\partial P}{\partial x^a}\frac{\partial}{\partial x^*_a}\right)$$

Recalling how we defined the anchor map in our classical case in the beginning (where no P_∞ structure was present), it is straight-forward to see that the first term of the coordinate expression of d_P corresponds to this map exactly. More concisely, we have the following map:

$$\begin{aligned}
a_P &: \Pi T^*M \to \Pi TM \\
(x^a, x^*_a) &\mapsto \left(x^a, \underbrace{(-1)^{|x^a|}\frac{\partial P}{\partial x^*_a}}_{dx^a}\right) \ ,
\end{aligned}$$

which is linear in each fiber (for quadratic P, as in the classical case, that is). As every L_∞ anchor is an L_∞ morphism, we see that the following diagram is commutative (c.f. classical fact)

$$\begin{array}{ccc}
\mathcal{A}(M) & \xrightarrow{d_P} & \mathcal{A}(M) \\
{\scriptstyle a_P^*}\uparrow & \circlearrowright & \uparrow{\scriptstyle a_P^*} \\
\Omega(M) & \xrightarrow{d} & \Omega(M)
\end{array}$$

If we now make use of an arbitrary (non-quadratic) P in the context of higher Koszul brackets on $\Omega(M)$ (while $\mathcal{A}(M)$ is still endowed with the canonical Schouten bracket), clearly $a_P : \Pi T^*M \to \Pi TM$ fails to be a linear map! It is therefore unclear what the adjoint of such a map should be.

B. Thick morphisms

This is the point where we are going to make use of a new notion of morphisms. In order to find an adjoint map to the above stated $a_P : \Pi T^*M \to \Pi TM$, we have to enlarge the class of morphisms to incorporate so-called ***thick morphisms*** as they were introduced in [19], which can be viewed as generalized pullbacks for functions on supermanifolds.
Given two supermanifolds M_1, M_2 with local coordinates x^a, y^i and fiber coordinates p_a, q_i, respectively. We know that T^*M_1 and T^*M_2 are symplectic manifolds with symplectic forms $\omega_1 = dp_a dx^a$ and $\omega_2 = dq_i dy^i$.

Definition 48 (Thick morphism). *A* **thick morphism** $\Phi : M_1 \Rrightarrow M_2$ *is a formal Lagrangian submanifold* $\Phi \subset T^*M_2 \times (-T^*M_1)$ *with respect to the form* $\omega_2 - \omega_1$*, which is locally specified by a so-called generating function of the form* $S(x, q)$*:*

$$d(y^i q_i - S) = q_i dy^i - p_a dx^a$$

The generating function $S(x, q)$ *of a thick morphism* Φ *can be formally expanded in terms of the fiber coordinates on* M_2 *as follows:*

$$S(x, q) = S_0(x) + S^i(x) q_i + \frac{1}{2!} S^{ij}(x) q_i q_j + ...$$

In this context we refer the reader to [2], [4] and [3], where the notion of formal canonical relations was studied. Using nomenclature related this, thick morphisms are also called *microformal morphisms.*

Thick morphisms might seem rather abstract and unfamiliar at first, however there are a lot of sensible generalizations of notions from more ordinary maps that translate to thick morphisms in a logical way. One example that is going to accompany us throughout the remainder of this project is the pullback:

Definition 49 (Pullbacks of thick morphisms). *Let* Φ *be a thick morphism with a generating function* $S(x, q)$ *(as above). The* **pullback** Φ^* *is a formal mapping* $\Phi^* : C^\infty(M_2) \to C^\infty(M_1)$ *(for* $g \in C^\infty(M_2)$*) is defined as*

$$\Phi^*[g](x) = g(y) + S(x, q) - y^i q_i \ ,$$

where q_i, y^i *are determined by the equations*

$$y^i = (-1)^{|q^i|} \frac{\partial S(x, q)}{\partial q_i}, \ \ q_i = \frac{\partial g(y)}{y^i}$$

Definition 50 (P_∞ thick morphisms)**.** *Let there be two manifolds with M_1, M_2 with Master Hamiltonians $P_k \in C^\infty(\Pi T^* M_k), k = 1, 2$. We call an odd thick morphism Φ between the manifolds*

$$\Phi : M_1 \Rrightarrow M_2$$

P_∞ thick if Φ fulfills the following important property for the two π_k, which are the restrictions on Φ of the projection maps $\Pi T^* M_1 \times \Pi T^* M_2 \to \Pi T^* M_k, k = 1, 2$:

$$\pi_1^* P_1 = \pi_2^* P_2$$

Moreover, for a P_∞ thick morphism $\Phi : M_1 \Rrightarrow M_2$, the pullback $\Psi^* : \Pi C^\infty(M_2) \to \Pi C^\infty(M_1)$ is an L_∞ morphism of the homotopy Poisson brackets.

Having introduced pullbacks of thick morphisms, we would like to introduce the notion of *adjoint* thick morphisms which will also shed light on the peculiar onomastic choice. To achieve this, we will state the following theorem without proof:

Proposition 51. *Let E_1, E_2 be two vector bundles and $\Phi : E_1 \to E_2$ a fiberwise map, which does not necessarily have to be linear (i.e. a vector bundle homomorphism). There is the* **fiberwise adjoint** *$\Phi^* : E_2^* \to E_1^*$ which is thick. For fiberwise linear Φ, it reduces to the usual notion of the adjoint map (hence is a thickening of the classical case). Moreover, for two such fiberwise maps Φ_1, Φ_2, we have $(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*$.*
Explicitely, it is given by the following (where $\kappa : T^*E \to T^*E^*$ is the Mackenzie-Xu diffeomorphism):

$$\Phi^* := (\kappa \times \kappa)(\Phi)^{op} \subset T^*E_1^* \times (-T^*E_2^*)$$

Furthermore, having adjoints at our disposal, we can introduce the notion of **pushforwards** *(as pullbacks of the corresponding adjoints):*

$$\Phi_* := (\Phi^*)^* : C^\infty(E_1^*) \to C^\infty(E_2^*) \ ,$$

with the property that for the space $C^\infty(M, E_1) \subset C^\infty(E_1^*)$, we have that $\Phi_*(C^\infty(M, E_1)) \subset C^\infty(M, E_2)$. Again, we did not need to assume that Φ is a fiberwise linear map. However, if given so, it conveniently reduces to the usual pushforward $e \mapsto \Phi \circ e$.

So far, our digression on thick morphisms and related features thereof seems to be rather unconnected to our initial problem of finding a map $a_P^* : \Omega(M) \to \mathcal{A}(M)$ with the desired properties related to the higher Koszul brackets on $\Omega(M)$. Luckily, we are only two important claims away from connecting these dots.

Proposition 52. *The (total) anchor for an L_∞ algebroid $E \to M$ induced L_∞ morphisms*

- *… of the higher Schouten brackets:* $C^\infty(\Pi E^*) \to C^\infty(\Pi T^*M)$
- *… of the higher Poisson brackets:* $\Pi C^\infty(E^*) \to \Pi C^\infty(T^*M)$

Proof. See [20], Prop. 4.1, Thms. 4.1 and 4.10. □

Finally, we have all ingredients ready to present our remedy to the situation:

Proposition 53. *Let M be a manifold endowed with a P_∞ structure. Then there exists an L_∞ morphism*

$$a_P^* : \Omega(M) = C^\infty(\Pi TM) \to C^\infty(\Pi T^*M) = \mathcal{A}(M)$$

between the higher Koszul brackets on $\Omega(M)$ (induced by some P) and the Schouten-Nijenhuis bracket on $\mathcal{A}(M)$.

IV. P_∞ STRUCTURE ON COISOTROPIC SUBMANIFOLDS

In this last part of the present project, we investigate where the homotopy Poisson structure appears in a natural way.

Definition 54 (Coisotropic submanifolds). *Let M be a Poisson manifold and $C \subset M$ a submanifold. We define the (on C)* **vanishing ideal** *in $C^\infty(M)$:*

$$I_C := \{f \in C^\infty(M) \mid f|_C = 0\}$$

We call C a **coisotropic** *submanifold iff $\{I_C, I_C\} \subset I_C$, i.e. I_C is closed under the Poisson bracket.*

However, it is worth pointing out that I_C does not necessarily have to be a Poisson ideal.

Note that if C is a coistropic submanifold of a Poisson manifold, this does not imply that C is a Poisson submanifold. For further reading about coisotropic submanifolds (including examples) and further applications, we refer the reader to [5] and particularly [6], which also deals with the reduced phase space $N(I_C)/I_C$ and many other notions that will be crucial later on.

We now show that the Poisson tensor has a particular form when expressed in terms of coordinates adapted to a coisotropic submanifold $C \subset M$. Let $\dim C = r, \dim M = r + s$, and $x^i, i \in \{1, ..., r\}$ the adapted coordinates on C (which will carry i, j indices) and $y^\mu, \mu \in \{1, ..., s\}$ residual coordinates on M (which will carry μ, ν indices). Then we can express the Poisson tensor π in the following way:

$$\pi = \begin{bmatrix} \pi^{ij} & \pi^{i\mu} \\ \pi^{\nu j} & \pi^{\mu\nu} \end{bmatrix}$$

Naturally, the relation of $\pi^{i\mu}$ and $\pi^{\nu j}$ is given by the usual (anti)symmetry of the Poisson tensor. However, we now show that the bottom right block vanishes:

Proposition 55. *In the above setting, we have $\pi^{\mu\nu} = 0$.*

Proof. We consider the Poisson bracket restricted to C, where we know $\{f,g\}\big|_C = 0$:

$$\begin{aligned} 0 &= \{f,g\}\big|_C \\ &= \left(\pi^{ab}\partial_a f \partial_b g\right)\big|_C \\ &= \underbrace{\left(\pi^{ij}\partial_i f \partial_j g + \pi^{i\mu}\partial_i f \partial_\mu g + \pi^{\nu j}\partial_\nu f \partial_j g\right)\big|_C}_{=0,\text{ as } f,g|_C = 0 \text{ and } \partial_{i,j} \text{ is w.r.t. } x^{i,j} \text{ only}} \\ &+ \left(\pi^{\mu\nu}\partial_\mu f \partial_\nu g\right)\big|_C \end{aligned}$$

Therefore $\pi^{\mu\nu} = 0$ must hold. □

We can write our Poisson tensor in the following way: $\pi = \pi^{ij}(x,y)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \pi^{i\mu}(x,y)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^\mu}$. We now introduce a new set of 1-***shifted coordinates*** on $NC \subset M$:

$$\frac{\partial}{\partial y^\mu} \leftrightarrow \theta_\mu$$
$$y^\mu \leftrightarrow \frac{\partial}{\partial \theta_\mu}$$

Using the coordinates x^i, θ_μ on $N^*[1]C$, we are able to formally expand π around the $(y=0)$ zero section in the following was, seeing $\hat{\pi} \in C^\infty(\Pi T^* N^*[1]C)$:

$$\hat{\pi} = \sum_I \frac{\partial \pi^{ij}}{\partial y^I}\Big|_{y=0} \frac{\partial}{\partial \theta_I}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_I \frac{\partial \pi^{i\mu}}{\partial y^I}\Big|_{y=0}\theta_\mu \frac{\partial}{\partial \theta_I}\frac{\partial}{\partial x^i} \ ,$$

where we omitted the wedge product indicators $(\wedge)$ and I is a suitable multiindex.
In the first part of this project we introduced the Schouten-Nijenhuis on M (or more concisely on $\mathcal{A}(M)$ with $[\pi,\pi] = 0$. This is fundamentally equivalent to our formal $\hat{\pi}$ satisfying $[\hat{\pi},\hat{\pi}] = 0$ on $N^*[1]C$, which has further ramifications as we will shortly see.

$\hat{\pi}$ fulfills all necessary requirements to be a suitable master Hamiltonian (as was introduced earlier. Therefore we can see that $\hat{\pi}$ equips the shifted conormal bundle $N^*[1]C$ (which is a supermanifold) with a P_∞ structure!
In our formal expansion of $\hat{\pi}$, we group the appearing terms with respect to their weights (written below the corresponding expression):

$$\begin{aligned} \hat{\pi} &= \underbrace{\text{vector field}}_{\text{weight } 1} + \underbrace{\text{bivector field}}_{\text{weight } 0} + (\text{lower weights } -1, ...) \\ &\equiv \quad Q \quad + \quad P \quad + \quad G \quad + \quad ... \end{aligned}$$

From $[\hat{\pi}, \hat{\pi}] = 0$, we arrive at the following relations amongst Q, P, G:

$$\underbrace{[Q,Q]=0}_{①}, \underbrace{[Q,P]=0}_{②}, \underbrace{[P,P]+2[Q,G]=0}_{③}$$

These relations will be of crucial importance for the ensuing consideration. Recall the ("raising indices") map induced by π - now seen as acting on the tangent bundle rather:

$$\sharp : T^*M \to TM$$

In this context, we can consider the restriction of the image of the off-diagonal blocks (on C) of our Poisson tensor π, Im(off-diagonal) $:= D \subset TC$. D is the *characteristic distribution* spanned by

$$Q = \theta_\mu \pi^{i\mu}(x,0)\frac{\partial}{\partial x^i}$$

Naturally, the quotient C/D does not need to be a smooth manifold, or even a manifold at all! In general, the analysis of C/D is tedious, and very few general statements can be made about the (smooth) structure of it. Were one to assume that C/D is indeed a smooth quotient, one can immediately see that C/D inherits a Poisson structure: In this quotient, the off-diagonal blocks $\pi^{i\mu}, \pi^{\nu j}$ vanish (precisely because of how we defined D), which leads to π having the form of a Poisson tensor on C/D.

In the general case of C/D not (necessarily) being smooth, we are able to find quite literally a resolution for this:

Definition 56 (Poisson normalizer)**.** *Let M be a Poisson manifold, $C \subset M$ a coisotropic submanifold and I_C the vanishing ideal. We define the* **Poisson normalizer***:*

$$N(I_C) := \{f \in C^\infty(M) \mid \{I_C, f\} \subset I_C\}$$

Therefore, I_C is a true Poisson ideal in $N(I_C)$ and evidently we have the Poisson algebra $N(I_C)/I_C$.

In the case where C/D is indeed a smooth manifold, we have the following correspondence (as Poisson algebras):

$$C^\infty\left(C/D\right) = N(I_C)/I_C$$

Therefore, it is sensible to consider $N(I_C)/I_C$ for all further intents and purposes. We will also suppress the subscript $_C$ where no ambiguity is present.

Now we make *cohomological* considerations for the first time: We will shortly after prove that Q acts as a differential in the cohomological sense. Also, P descends to the cohomology, which we shall call $H_Q^\bullet$ and f-representatives by $\bar{f}_Q$, by the way of defining the corresponding $\bar{P}$:

$$\bar{P}\left(\bar{f}_Q, \bar{g}_Q\right) := \overline{P(df, dg)}_Q \ \left(= \overline{[[P, f], g]}_Q\right)$$

For the sake of brevity, let us introduce the abbreviation $C^\infty(\Pi T^* N^*[1]C) \equiv C^\infty(\tilde{C})$. We can now state and prove the following interesting proposition:

Proposition 57. *For the cohomology $H_Q^\bullet$, we have:*

- *Q acts as a differential:*

$$0 \xrightarrow{Q} C^\infty(\tilde{C})^0 \xrightarrow{Q} C^\infty(\tilde{C})^1 \xrightarrow{Q} \cdots \xrightarrow{Q} C^\infty(\tilde{C})^n \xrightarrow{Q} \cdots$$

  where $C^\infty(\tilde{C})^n \subset C^\infty(\tilde{C})$ are the functions of degree n in θ.

- *$H_Q^\bullet$ is a graded Poisson algebra*
- *H_Q^0 is a Poisson algebra*

Proof. We prove this in the following way:

- Consider $(C^\infty(\Pi T^* N^*[1]C, \hat{\pi})$, the vector field Q indeed acts as a *differential*: For $f \in C^\infty(\Pi T^* N^*[1]C)$, we have that $Q^2 = \frac{1}{2}[Q, Q] = 0$ because of ① and has the desired weight.
- Because of ②, fortunately we have for f, g with $Qf = Qg = 0$ that $QP(f, g) = [Q, P(f, g)] = [Q, [[P, f], g]] = [[Q, [P, f]], g] + [[P, f], \underbrace{Qg}_{=0}] = [[\underbrace{[Q, P]}_{=0}, f], g] + [[P, f], \underbrace{Qg}_{=0}] = 0$.
- Because of ③, we can state that $\bar{P}$ is Poisson in the sense that on the level of cohomology, we have $[P, P] = 0$. Indeed, the second term in ③ is zero in cohomology:

  Since $[P, P]$ amounts to a trivector field, we now need to consider three functions as arguments. Let f_1, f_2, f_3 be such that $Qf_i = 0$ for $i = \{1, 2, 3\}$, then

$$\begin{aligned}
[P, P](f_1, f_2, f_3) =& 2[Q, G](f_1, f_2, f_3) \\
=& G(\underbrace{Qf_1, f_2, f_3}_{=0}) + G(f_1, \underbrace{Qf_2}_{=0}, f_3) \\
&+ G(f_1, f_2, \underbrace{Qf_3}_{=0}) + QG(f_1, f_2, f_3)
\end{aligned}$$

and since $QG(f_1, f_2, f_3) \in \operatorname{Im} Q$, the last term vanishes in the cohomology $H_Q^\bullet = {}^{\ker Q}/_{\operatorname{Im} Q}$, we indeed have $[P, P] = 0$ on the level of cohomology. (This also also precisely why we refer to this structure as *homotopy analogue*: It is a Poisson structure up to homotopy.) Consequently, $\left(H_Q^\bullet, \bar{P}\right)$ is a *graded* Poisson algebra!

- If $H_Q^\bullet$ is a graded Poisson algebra, naturally in its zeroth degree, it is Poisson in the usual sense. Thus H_Q^0 is a Poisson algebra.

□

Furthermore, it was shown in e.g. [15] *(Lemma 2.24)* that

$$H_Q^0 = {}^{N(I)}\!/_{I}$$

We have gone full circle: After having established the P_∞ structure on $N^*[1]C$, we considered the quotient ${}^{C}\!/_{D}$, which unfortunately does not necessarily have to be a (smooth) Poisson manifold. Looking then at the relation $C^\infty\left({}^{C}/_{D}\right) = {}^{N(I_C)}/_{I_C}$, we ended up finding a (co)resolution with $H_Q^0 = {}^{N(I)}/_{I}$ being a true Poisson algebra.

V. OUTLOOK

As a next step, it would make sense to first illustrate a specific aspect of the theory of P_∞/S_∞ and thick morphisms, as well as to look into more general themes where the insights outlined in this project might prove themselves useful.
Regarding the former point, in this article we were mostly concerned with the existence and properties of thick morphisms and their adjoints. In order to shed light on the concrete mathematical structure of things, it might be helpful - albeit slightly technical and perhaps messy - to find a presentable way to express the relevant thick morphisms and their adjoints in an explicit way. This would be the hands-on application of the proposition from two sections before.
In the context of the latter point, we might be able to make use of our construction explained in the section before. We started with a (non-super)manifold M, and while making use of adapted coordinates on some coisotropic submanifold C, we arrived at the [1]-shifted conormal bundle $N^*[1]C$ which is a supermanifold.

It is now natural to ask ourselves the following important question: If we are now given an arbitrary supermanifold $\tilde{M}$ endowed with some P_∞ structure P, does there exist a certain set of conditions on P and $\tilde{M}$ which we could outline here such that we are able to make some conjecture of the following form:

Conjecture. *Let $\tilde{M}$ be a supermanifold and P be a P_∞ structure on $\tilde{M}$. If the above conditions are fulfilled, then there exists a Poisson manifold (M, π) and a coisotropic submanifold $C \subset M$ such that we have $N^*[1]C = \tilde{M}$ and the P_∞ structures coincide ($P = \hat{\pi}$).*

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