Synthetic differential geometry in the Cahiers topos

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January 30, 2019

This semester thesis documents my effort to understand the formulation of scalar field theory in synthetic differential geometry given in [1]. I first became interested in this formalism while studying the phase spaces of various field theories. The phase space is the solution space of the theory in question together with an exact 2-form, called the (pre-)symplectic form, on it. Typically the solution space is carved out of a larger configuration space by the requirement that its elements satisfy certain PDE's (the field equations of the theory). For example, is a typical solution space of scalar field theory with spacetime M given by the subspace of $C^\infty(M)$, which has elements that satisfy the Klein Gordon equation. Like in this example the solution spaces are typically infinite dimensional. We woud like to understand their smooth structure for example for the sake of working with the symplectic form. For this end we first have to specify a topology on the solution space. It is not completely straightforward (but also not impossible) to see what the natural choice for this topology is. Moreover, once it is found there are other technical difficulties to be taken on. There are various ways to deal with these problems i.e to understand the differential geometry of the infinite dimensional spaces that appear in field theory. An approach, that I find particularly elegant is to formulate field theories in synthetic differential geometry [1].

Usual differential geometry relies on the possibility to introduce methods from calculus into the theory of manifolds. The idea of synthetic differential geometry is then to replace the limiting procedure from calculus by a purely algebraic construction. Basically we accomplish this by introducing an *infinitesimally short line object* D. For a given smooth manifold M we then look at the exponential M^D i.e the space of all smooth maps from D into M. The idea is that D is so short that we can identify the space M^D with the space of infinitesimal paths in M or in other words with the tangent bundle TM. As it is, this naive idea immediately encounter two fatal problems.

1. There is no manifold that has the properties we want an infinitesimally short line object to have.
2. Even if such a manifold existed the exponential M^D may fail to be a smooth manifold again. Actually this is the case to be expected. For two smooth manifolds M and N the space of smooth maps $C^\infty(M, N)$ is usually infinite dimensional.

The straightforward approach for avoiding these problems is to enlarge the category of smooth manifolds Man to a category C. This category should contain Man as a full subcategory and moreover an infinitesimal line object D. In order to avoid the second problem from above the new category C should also be closed with respect to taking exponentials. Another advantage of our new category being exponentially closed would be that this makes it possible to describe the infinite dimensional mapping spaces that appear in field theories within it and in particular understand their tangent spaces.

Of course the crucial question is now: does such a category C exist? Luckily the answer is yes- it is given by the Cahiers topos, which was first introduced and studied by Dubuc in [2]. The Cahiers topos is a category of sheaves on the site of generalized smooth spaces. What is a sheaf, a site or a generalized smooth space? When I was first introduced to the subject of synthetic differential geometry I didn't know the answers to these questions. As this thesis

[1] A standard reference for synthetic differential geometry is [5]. A very nice and short introduction to the idea of synthetic differential geometry is given in [6]

is a documentation of how I learned to formulate scalar field theory in synthetic differential geometry it will also contain the definitions, propositions and proofs that provided the answers for me.

The thesis is organized in three chapters. Chapter one is an introduction to the theories of sheaves on a site. We will first motivate the major constructions by considering the example of sheaves on the category Top of topological spaces. Then we move on and consider sheaves on more general categories. In particular we will discuss how the concept of a topology is generalized by introducing Grothendieck topologies. At the end of the first chapter we will see that categories of sheaves are closed with respect to taking small limits and colimits as well as exponential objects.

The second chapter works towards introducing the Cahiers topos as the category of sheaves on a special site- the site FMan of generalized smooth spaces. The first step is of course to introduce this site of generalized smooth spaces. For this end we discuss the category $\mathsf{C}^{\infty}\mathsf{Ring}$ of so called C^{∞}-rings. In the end our construction accomplishes that smooth manifolds can be consistently described within the Cahiers topos. Moreover it also contains an infinitesimal line object. We will show that within the Cahiers topos exponentiation of a smooth manifold with this infinitesimal object actually gives us the tangent bundle of said manifold (as an object in the Cahiers topos). Another way to say this is "the Cahiers topos is a well adapted model of synthetic differential geometry".

In chapter 3 we will use this well adapted model and formulate scalar field theory within it. We will see that all our efforts pay off and it is actually possible to understand the differential geometry of the appearing infinite dimensional mapping spaces in a mathematical rigorous way.

Everything in this thesis is rather self contained but basic knowledge of differential geometry and knowledge of the basic definitions of category theory is assumed. In the appendix some standard definitons and results from category theory which we use are collected and proved. In particular one can find there a discussion of limits and colimits, their characterization as adjoint functors, moreover exponential objects, hom functors and effective epimorphisms.

Contents

Chapter 1

Categories of Sheaves

1.1 Sheaves on the category Top

1.1.1 Representable presheaves on Top

In this section we motivate the notion of a sheaf by a consideration of the category Top of topological spaces. The advantage is, that we have a very good understanding of the morphisms in this category, namely of the contionuous maps. We will see, that it is possible to describe a topological space X by [1]

1. specifying the set $X(U)$ of continuous maps from another topological space U into X for every U and
2. establish how these sets relate to one another, i.e specify how the elements of $X(V)$ are mapped to the elements of $X(U)$ by precomposition with a continuous map $f\colon U \to V$.

The latter point maybe needs some explenation. If $f\colon U \to V$ is a continuous map we can precompose any continuous map $p\colon V \to X$ with it to obtain a continuous map from U to X. In other words does f induce a map of sets $X(V) \to X(U)$. We may call this map $X(f)$. When we have knowledge of the sets $X(U)$ for every U and also of the transformation $X(f)$ for every $f\colon X \to Y$ this is actually enough information to reconstruct the space X from it.
Lets investigate the assignments $X \mapsto X(U)$ and $f \mapsto X(f)$ a little further. We observe, that the transformation from $X(U)$ to $X(U)$ that is induced by the identity map id_U is the identity map of sets $\mathrm{id}_{X(U)}$. Moreover, if we have two continuous maps $f\colon U \to V$ and $g\colon V \to W$, then the composition of the map of sets $X(f)$, that is induced by f with the map of sets $X(g)$, that is induced by g is the same as the map of sets $X(g \circ f)$ that is induced by $g \circ f$ i.e $X(g) \circ X(f) = X(g \circ f)$. This amounts to saying that X can be identified with the functor $X\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$, that sends a topological space U to the set of continuous maps from U to X and a morphism $f\colon U \to V$ in Top to the morphism $X(f)\colon X(V) \to X(U)$ in Set.

Definition 1.1.1. *Functors from* $\mathsf{Top}^{\mathsf{op}}$ *into* Set *are called* Set*-valued presheaves on* Top.

[1] This is also called the *functor of points* perspective and was first developed by Grothendieck

One might have noticed the little "op" on the Top. It is there to indicate that the presheaf goes from the opposite category of Top into Set. This way we implement that morphisms from U to V in Top are mapped to morphisms $X(V) \to X(U)$ rather than $X(U) \to X(V)$.
So far we have seen, that to every topological space X there corresponds a Set-valued presheaf on Top. This is obtained by assigning to a topological space U the set $X(U)$ of all continuous maps from U into X and to a continuous map $f\colon U \to V$ the precomposition with f that is a map of sets $X(f)\colon X(V) \to X(U)$. The question that naturally arises next is: does in turn every Set-valued presheaf on Top come from topological space X in this way? The answer is no- we can actually construct a counterexample like the one below.

Definition 1.1.2. *To distinguish the presheafs that come from topological spaces we call them representable presheafs. If a representable presheaf corresponds to the topological space X in this way we call it represented by X and denote it as $X(-)$.*

The following provides an example of a presheaf that is not representable.

Example 1.1.3. *Consider the functor $F\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ that assigns to a topological space X the set of all bounded (not necessarily continuous) maps from X to $\mathbb{R}$ i.e $F(X) = \{f\colon X \to \mathbb{R} \mid f \text{ bounded}\}$ and that maps morphisms in $\mathsf{Top}^{\mathsf{op}}$ to morphisms in Set by the usual precomposition. This is consistent because if $f\colon X \to \mathbb{R}$ is bounded and $g\colon Y \to X$ is any continuous map, than $f \circ g$ is bounded again. We will argue by contradiction, that the so defined F cannot be representable. Assume, that F is a representable presheaf on Top. In this case it exists a topological space with an underlying set $Z = F(\{*\}) = \mathbb{R}$ and a topology such that for every topological space X, every bounded map from X to $\mathbb{R}$ is continuous if viewed as a map into Z i.e $Z(X) = F(X)$.*
Next choose any non empty closed interval in $\mathbb{R}$, equip it with the trivial topology and call it I. Then choose an open set U in Z that intersects this interval. We can look at the embedding of $\iota\colon I \to \mathbb{R}$. This is a bounded map and thus when viewed as a map from I into Z it is continuous by the assumption that F is representable. Then the preimage of U under this embedding is open and because it is non empty and I has the trivial topology it must be all of I. But this in turn means that $I \subset U$. Because our choice of I was arbitrary and we can actually replace it by any interval that intersects U we can conclude that U is all of Z. But because U was an arbitrary non empty open subset of Z we find that every open set of Z is either empty or all of Z i.e Z has the trivial topology. But this topology is way too coarse. While it makes bounded maps continuous it also does so for every other map and thus $Z(X) \neq F(X)$ in contradiction to the assumption that F is represented by Z.

1.1.2 The sheaf condition

There are two more properties of the representable presheaves $X(-)\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$, which we have not discussed yet. Suppose that we chop a topological space U up into open subspaces $\{U_i\}_{i\in I}$ that cover it. Then it holds true that

(TS1) For collections of functions $\{f_i\colon U_i \to X\}_{i\in I}$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $i, j \in I$ it always exists a unique $f\colon U \to X$ such that $f|_{U_i} = f_i$ for every $i \in I$.

(TS2) And for two functions $f, s\colon U \to X$ such that $f|_{U_i} = s|_{U_i}$ for every $i \in I$ we have $f = s$.

This has an interesting consequence in terms of commutative diagrams. Let $X(-)\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ be a presheaf represented by the topological space X. Let U be any topological space and $\{U_i \subset U\}_{i\in I}$ be any open covering of U. Then we can consider the following diagram Then we can consider the following diagrams

$$\prod_{i\in I} X(U_i) \underset{X(\iota_{jk,k})\circ p_k}{\overset{X(\iota_{jk,j})\circ p_j}{\rightrightarrows}} X(U_j \cap U_k) \ . \tag{1.1}$$

where $X(\iota_{jk,j}) \circ p_j$ means, that we first project on the j'th component of the product and then restrict the resulting function to $U_j \cap U_k$ by precomposing with the embedding of $U_j \cap U_k$ into U_j.
We can map every smooth map from U to X i.e every element in $X(U)$ to the product $\prod_{i\in I} X(U_i)$ by using the map $\prod_{i\in I}(X(\iota_i))$. Moreover, we then have

$$X(\iota_{jk,j}) \circ p_j \circ \left(\prod_{i\in I} X(\iota_i)\right) = X(\iota_{jk,k}) \circ p_k \circ \left(\prod_{i\in I} X(\iota_i)\right)$$

for every pair $j, k \in I$. This means, that $X(U)$ together with the morphism $\prod_{i\in I} X(\iota_i)$ is a cone over the diagram (1.1).
But now suppose we have another cone (A, ϕ). The morphism $\phi\colon A \to \prod_{i\in I} X(U_i)$ assigns to every element $a \in A$ a family of morphisms that we may denote by $\phi(a) = \{\phi(a)_i \colon U_i \to X\}_{i\in I}$. But because (A, ϕ) is a cone we have that the so assigned $\phi(a)_j$ and $\phi(a)_k$ agree on the overlap $U_j \cap U_k$ for every $j, k \in I$. This means, that by property (TS1) above it exists a unique continuous map $f\colon U \to X$ such that $f|_{U_i} = \phi(a)_i$ for every i. This gives us a unique map $u\colon A \to X(U)$ such that $\prod_{i\in I} X(\iota_i) \circ u = \phi$. Note that by property (TS2) this u is well defined. These findings can be summarized by the following statement.
Given a presheaf $X(-)\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ that is represented by the topological space X we have for every topological space U and ever open covering $\{U_i \subset U\}_{i\in I}$ that the set $X(U)$ together with the morphism $\prod_{i\in I} X(\iota_i)$ is the equalizer of the diagram

$$\prod_{i\in I} X(U_i) \underset{\prod j,k X(\iota_{jk,k})\circ p_k}{\overset{\prod_{j,k} X(\iota_{jk,j})\circ p_j}{\rightrightarrows}} \prod_{j,k\in I} X(U_j \cap U_k) \ . \tag{1.2}$$

We can now take this property, for which we have showed, that representable presheaves on Top always satisfy it and formulate it as a condition.

Definition 1.1.4. *If $X\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ is a presheaf, such that the diagram (1.2) is equalized by $\left(X(U), \prod_{i\in I} X(\iota_i)\right)$ for every open covering $\{\iota_i \colon U_i \to U\}$ then we call X a sheaf.*

One might think that if we restrict the presheaves at which we want to look by requirie that they are also sheaves, the non-representable ones are elliminated. But a quick look back at example (1.1.3) convinces us that this is not true.

Example 1.1.5. *Recall example (1.1.3) where we considered the functor $F\colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Set}$ given by $F(X) = \{f\colon X \to \mathbb{R} \mid f \text{ bounded}\}$. Checking the two conditions (TS1) and (TS2) convinces us that F is indeed a sheaf. But as we have already seen that it is not representable F serves also as an example for a non-respresentable* Set*-valued sheaf on* Top

1.1.3 Maps between sheaves

We know what the morphisms between two ordinary topological spaces are. But Example (1.1.5) shows, that there are also sheaves on Top that are not represented by a topological space. We want to work out how morphisms between these sheaves look. The way we do this is similar to the procedure above. We try to extract elementary properties these maps must have in order for them to be consistent with the ordinary continuous maps in the case of sheaves that are representable.
Let $X(-)$ and $Y(-)$ be two sheaves, represented by the topological spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous map. We can then compose a probe map $p\colon U \to X$ with f to obtain a probe map $f \circ p\colon U \to Y$. In other words does f map the set of probe maps $X(U)$ to the set of probe maps $Y(U)$ by postcomposition. The continuous map f thus corresponds to a collection of maps $\{f_U\}_{U \in \mathsf{Ob}(\mathsf{Top})}$ between sets of the form $X(U)$ and $Y(U)$ [2].
So what are sensible morphisms between general sheaves A and B? They should also be collections of maps of sets, one of the form $t_U\colon A(U) \to B(U)$ for every probe space U.
Lets again turn back to the representable case, where we had the collection of maps $\{f_U\}_U$ that was associated to the continuous map $f\colon U \to V$. Given another morphism between two topological spaces $\phi\colon U \to V$ we can precompose a probe map $(p\colon V \to X) \in X(V)$ with it to get $p \circ \phi\colon U \to X$. For topological spaces X and Y it is clear, that it does not matter whether we first precompose a map p with ϕ and then postcompose with f or the other way around as $(f \circ p) \circ \phi = f \circ (p \circ \phi)$. In other words do we have the following commutative diagram

$$\begin{array}{ccc} X(U) & \xleftarrow{X(\phi)} & X(V) \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f_V} \\ Y(U) & \xleftarrow[Y(\phi)]{} & Y(V). \end{array}$$

It is this condition that we want to hold true also for more general transformations between sheaves. Namely, given such a transformation $t\colon X \to Y$ in other words for every topological space U a map of sets $t_U\colon A(U) \to B(U)$ and given any continuous map between topological spaces $\phi\colon U \to V$ the following diagram shall commute

$$\begin{array}{ccc} A(U) & \xleftarrow{A(\phi)} & B(V) \\ {\scriptstyle t_U}\downarrow & & \downarrow{\scriptstyle t_V} \\ B(U) & \xleftarrow[B(\phi)]{} & B(V). \end{array} \tag{1.3}$$

To use the language of category theory, we require, that t is a natural transformation between the functors A and B.
The crucial fact, that makes our whole construction work is, that for topological spaces X and Y the set of natural transformations between $X(-)$ and $Y(-)$ actually agrees with the set of continuous maps between X and Y. This follows from a famous result called the Yoneda lemma.

[2]That this correspondence is actually a natural bijection is a consequence of the Yoneda lemma, which we proof in the next section

Proposition 1.1.6. *(Yondea lemma for* Top*) Given two topological spaces* X *and* Y *the set of natural transformations from the functor* $X(-)$ *to the functor* $Y(-)$ *is in natural bijection with* $Y(X)$*.*

We will not proof this result now as it will follow from the general Yoneda lemma in the next section.
To summarize, we have seen that it is possible to consistently describe the topological spaces within a category $\mathsf{Sh}(\mathsf{Top})$ that has as objects all the Set-valued presheaves on Top and as morphisms between two sheaves A and B all the natural transformations between A and B.

1.2 More general sheaf categories

1.2.1 Presheaves and the Yoneda lemma

For the sake of providing a motivational introduction to the subject we did the whole construction of sheaves for the category of topological spaces Top. It is quite clear how to generalize most of this and define what a presheaf on a more arbitrary category is. We will explicitly spell out this definition now.

Definition 1.2.1. *Given a category* C *its presheaf category* $\mathsf{PSh}(\mathsf{C})$ *is defined to be the category with*

1. *Objects given by the functors from the opposite category* C^{op} *to the category* Set
2. *Morphisms given by the natural transformations between these functors*
3. *The composition given by the usual composition of natural transformations.*

We can also give a generalized definition of representability of presheaves.

Definition 1.2.2. *A presheaf in the category* C *is called representable if it is isomorphic to a hom-functor* $X(-)\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$*. The hom-functor* $X(-)$ *sends an object* $U \in \mathsf{Ob}(\mathsf{C})$ *to the hom-set* $Hom_{\mathsf{C}}(U, X)$ *of* C *and a morphism* $\alpha\colon U \to U'$ *to the morphism in* Set *that maps each morphism* $U \to X$ *to the precomposition with* α *i.e* $U' \xrightarrow{\alpha} U \to X$*.*

In other words are the representable presheafs the ones that correspond to an object in C like in the preceding section the representable presheafs of probe maps with respect to a given topological space.
We will now state and proof the Yoneda lemma that was already mentioned at the end the last section.

Proposition 1.2.3. *(Yoneda Lemma)*
Let $E\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ *be a representable presheaf that is isomorphic to the hom-functor* $X(-)$ *of the object* $X \in \mathsf{Ob}(\mathsf{C})$*. Then the set of natural transformations from* E *to an arbitrary presheaf* F*, that we denote by* $\mathsf{Nat}(E, F)$ *is in bijection with the elements of* $F(X)$*.*

Proof. First we observe that we can equivalently proof the existence of a bijection $\mathsf{Nat}(X(-), F) \cong F(X)$. Let $\eta\colon X(-) \to F$ be a natural transformation. Then the naturality condition guaranties the commutativity of the following diagram for every object $B \in \mathsf{Ob}(\mathsf{C})$ and every morphism $f\colon B \to X$ in C

$$\begin{array}{ccc} X(X) & \xrightarrow{\eta_X} & F(X) \\ {\scriptstyle X(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ X(B) & \xrightarrow{\eta_B} & F(B) \end{array}$$

In particular we can chase the identity morphism $\mathrm{Id}_X \in X(X)$ around the diagram and obtain

$$\begin{array}{ccc} \mathrm{Id}_X & \longmapsto & \eta_X(\mathrm{Id}_X) = \xi \\ \downarrow & & \downarrow \\ f & \longmapsto & \eta_B(f) \end{array} .$$

Consider the morphism of sets given by

$$\begin{aligned} h\colon \mathsf{Hom}_{\mathsf{PSh(C)}}(X(-), F) &\to F(X) \\ \eta &\mapsto \eta_X(\mathrm{Id}_X). \end{aligned}$$

This map is injective because given two natural transformations $\eta, \eta'\colon X(-) \to F$ such that $\eta_X(\mathrm{Id}_X) = \eta'_X(\mathrm{Id}_X)$ we have that $\eta_B(f) = \eta'_B(f)$ for every $B \in \mathsf{Ob}(C)$ and every C-morphism $f\colon B \to X$. Thus η and η' are the same natural transformation.
The map is also surjective, because given any element $\xi \in F(X)$ we can define $\eta_B\colon X(B) \to F(B)$ to be given by $\eta_B(f) = F(f)(\xi)$. Then for $g \in \mathsf{Hom}_{\mathsf{C}}(M, N)$ we have $(F(g) \circ \eta_M)(t) = F(g)F(t)(\xi) = F(t \circ g)(\xi) = \eta_N(t \circ g) = \eta_N(X(g)(t))$ and thus η is a natural transformation. □

With the Yoneda lemma we can understand, that given an object $X \in \mathsf{Ob}(C)$ its embedding into $\mathsf{PSh(C)}$ via

$$\iota\colon \mathsf{Ob(C)} \to \mathsf{PSh(C)}$$

where

$$\iota(X) = X(-)$$

is full and faithful. By full and faithful we mean, that, that there is a bijection between the sets $\mathrm{Hom}_C(X, Y)$ and $\mathrm{Hom}_{\mathsf{PSh(C)}}(\iota(X), \iota(Y))$ for every two objects $X, Y \in \mathsf{Ob}(C)$. The Yoneda Lemma tells us that there is a bijection between the set of natural transformations from $X(-)$ to $Y(-)$ and the set $Y(X)$. But $Y(X) = \mathsf{Hom}_{\mathsf{C}}(X, Y)$. Thus the result is established.
A first consequence of all this is

Corollary 1.2.4. *If $E, F \in \mathsf{PSh(C)}$ are isomorphic representable presheaves and $E \cong X(-)$ and $F \cong Y(-)$ then we have*

$$E \cong F \quad \Leftrightarrow \quad X \cong Y.$$

in particular is the representation of a presheaf by an object unique up to isomorphism.

We conclude this section by proving a further important property of the Yoneda bijection, namely that it is natural in both X and F. What exactly this means will become clear in the following proof.

Proposition 1.2.5. *The Yoneda bijection*

$$h_{X,F}\colon \mathsf{Nat}(X(-),F) \cong F(X) \tag{1.4}$$

that has been constructed in Proposition (1.2.3) is natural in both X and F.

Proof. First we show naturality in the first argument (this is X above). Let

$$f\colon X \to Y$$

be a morphism in C. We want to show that the diagram

$$\begin{array}{ccc} \mathsf{Nat}(Y(-),F) & \xrightarrow{h_{Y,F}} & F(X) \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathsf{Nat}(X(-),F) & \xrightarrow{h_{X,F}} & F(Y) \end{array}$$

where commutes. The map $\bar{f}\colon \mathsf{Nat}(Y(-),F) \to \mathsf{Nat}(X(-),F)$ is defined as follows. For $l \in X(U)$, $\alpha \in \mathsf{Nat}(Y(-),F)$ and $U \in \mathsf{Ob}(\mathsf{C})$ we have $(\bar{f}(\alpha))_U(l) = \alpha_U(l \circ f)$. Lets chase $\alpha \in \mathsf{Nat}(Y(-),F)$ around both sides of the diagram. First we have

$$\begin{aligned} F(f) \circ h_{X,F}(\alpha) &= F(f)(\alpha_X(\mathrm{Id}_X)) \\ &= \alpha_Y(f) \end{aligned}$$

because α is a natural transformation. On the other hand

$$\begin{aligned} h_{Y,F} \circ \bar{f}(\alpha) =& (\bar{f}(\alpha))_Y(\mathrm{id}_Y) \\ =& \alpha_Y(\mathrm{id}_Y \circ f) \\ =& \alpha_Y(f) \end{aligned}$$

and so naturality in the first argument is proven.
Let $\eta\colon E \to F$ be a natural transformation between the presheaves E and F on C Naturality of h in the second argument (i.e F above in (1.4)) means, that the following diagram commutes

$$\begin{array}{ccc} \mathsf{Nat}(X(-),E) & \xrightarrow{h_{X,E}} & E(X) \\ {\scriptstyle \bar{\eta}}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \mathsf{Nat}(X(-),F) & \xrightarrow{h_{X,F}} & F(X) \end{array}$$

where $(\bar{\eta}(\alpha))_U(l) = \eta_U(\alpha_U(l))$ for $l \in X(U)$, $U \in \mathsf{Ob}(\mathsf{C})$ and $\alpha \in \mathsf{Nat}(X(-),E)$. Again we chase the element $\alpha \in \mathsf{Nat}(X(-),E)$ around the diagram.

$$\begin{aligned} h_{X,F} \circ \bar{\eta}(\alpha) &= h_{X,F}(\bar{\eta}(\alpha)) \\ &= \eta_X(\alpha_X(\mathrm{id}_X)) \end{aligned}$$

on the other hand

$$\eta_X(h_{X,E}(\alpha)) = \eta_X(\alpha_X(\mathrm{id}_X)).$$

This completes the proof of naturality of the Yoneda isomorphism. □

1.2.2 Sheaves on a site

Generalizing the sheaf condition is a little bit more tricky. When we defined under which conditions a presheaf is also a sheaf in the case of topological spaces we used the notion of an open covering of a topological space. We need a similar notion for objects of a general category.

Definition 1.2.6. *Let C be a category. A coverage on C is an assignment of collections of morphisms $\{f_i\colon U_i \to U\}_{i\in I}$ to every object $U \in \mathsf{Ob}(\mathsf{C})$ (called covering families of U in the following) such that if $\{U_i \to U\}_{i\in I}$ is a covering family of U and $g\colon V \to U$ a morphism, then it exists a covering family $\{h_j\colon V_i \to V\}$ of V such that every gh_j factors through some f_i*

$$\begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ {\scriptstyle h_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ V & \xrightarrow[g]{} & U \end{array}$$

A category C on which we have a coverage T is denoted (C, T) and we call it a site.

Example 1.2.7. *Clearly the coverage that assigns to a topological space all its open covers as covering families is an example. Given a topological space X and a covering of it $\{U_i \to X\}_{i\in I}$ together with any continuous map $g\colon Y \to X$ the preimage of every open set $U_i \subset U$ is open in V and furthermore the collection of all preimages $g^{-1}(U_i)$ covers V because the U_i cover U. Thus the collection $\{g^{-1}(U_i)\}_{i\in I}$ is a covering family of Y and we have*

$$\begin{array}{ccc} g^{-1}(U_i) & \xrightarrow{g|_{g^{-1}(U_i)}} & U_i \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ V & \xrightarrow[g]{} & U \end{array} .$$

With this generalized notion of a covering we can go on and define under which conditions a presheaf on a category C shall be considered a sheaf. We can not simply adapt the condition involving the equalizer that we used for presheaves on Top though. For this to work we would need to know how the overlap of two objects looks like. We can formulate another sheaf condition that works without having this knowledge though. If pullbacks exist in the category we consider this new sheaf condition turns out to be equivalent to the one that involves the equalizer as we will show below.
To formulate our new condition we first need to know what a compatible family is.

Definition 1.2.8. *Given a site (C, T), a presheaf $A \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$ and a covering family $\{p_i\colon U_i \to U\}_{i\in I}$ of the object $U \in \mathsf{Ob}(\mathsf{C})$ a compatible family of elements with respect to these is a tuple $(s_i \in A(U_i))_{i\in I}$ such that for all $j, k \in I$ and all morphisms $f\colon K \to U_j$ and $g\colon K \to U_k$ with $p_j \circ f = p_k \circ g$ we have $A(f)(s_j) = A(g)(s_k)$*

With this at hand we can formulate the announced sheaf condition.

Definition 1.2.9. *Given a site* (C, T) *a presheaf* $A \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$ *is called a sheaf with respect to the covering family* $\{p_i \colon U_i \to U\}_{i \in I}$ *if for every compatible family of elements* $(s_i \in A(U_i))_{i \in I}$ *we can find a unique element* $s \in A(U)$ *such that* $A(p_i)(s) = s_i$*. If* A *is a sheaf with respect to every covering family in* T*, then* A *is called a sheaf with respect to* T *or a* T*-sheaf.*

If the category C in question has pullbacks we can rephrase the sheaf condition as follows.

Proposition 1.2.10. *If* $F \colon \mathsf{C} \to \mathsf{Set}$ *is a presheaf on a site* (C, T) *where pullbacks exist in* C *then* F *is a sheaf if and only if for every* $U \in \mathsf{Ob}(\mathsf{C})$ *and every covering family* $\{p_i \colon U_i \to U\}$ *the object* $F(U)$ *in* Set *together with the morphism* $\prod_{i \in I} F(f_i)$ *is the equalizer of the diagrams*

$$\prod_{i \in I} F(U_i) \underset{F(\iota_{jk,k}) \circ \rho_k}{\overset{F(\iota_{jk,j}) \circ \rho_j}{\rightrightarrows}} F(U_j \times_U U_k). \tag{1.5}$$

Here the "inclusion"-morphisms $\iota_{jk,j}$ *and* $\iota_{jk,k}$ *are defined according to the following pullback diagram in* C

$$\begin{array}{ccc} U_j \times_U U_k & \overset{\iota_{jk,j}}{\dashrightarrow} & U_j \\ {\scriptstyle \iota_{jk,k}} \downarrow & & \downarrow {\scriptstyle i_j} \\ U_k & \underset{i_k}{\longrightarrow} & U \end{array}$$

and ρ_j *and* ρ_k *are the usual "projection"-morphisms in* Set*.*

Proof. "$\Leftarrow$": If $(F(U), (\prod_{i \in I} F(f_i))$ is the equalizer of (1.5) we have for every pair (A, ϕ), where A is a set and $\phi \colon A \to F(A)$ a morphism such that $(F(\iota_{jk,j}) \circ p_j) \circ \phi = (F(\iota_{jk,k}) \circ p_k) \circ \phi$ a unique morphism $u \colon A \to F(U)$ such that $(\prod_{i \in I} F(f_i)) \circ u = \phi$. Given a compatible family $\{s_i \in F(U_i)\}$ we can take a set with one element $A = \{a\}$ and define the map $\phi \colon A \to \prod_{i \in I} F(U_i)$ by $\phi(a)_i = s_i$. Because s_i is a compatible family (A, ϕ) is a cone over the diagram (1.5). We get a unique morphism $u \colon A \to F(U)$ such that $(\prod_{i \in I} F(f_i) \circ u) = \phi$. This provides a unique element $u(a)$ such that $F(f_i)(u(a)) = \phi(a)_i = s_i$

"$\Rightarrow$": On the other hand if $F \colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ is a sheaf according to definition (1.2.9) we have for every cone (A, ϕ) over the diagram (1.5) that $\phi(a)$ defines a compatible family for every $a \in A$. To see this let $K \in \mathsf{Ob}(\mathsf{C})$, $f \colon K \to U_j$ and $g \colon K \to U_k$ such that $p_j \circ f = p_k \circ g$. Then because of the universality of the pullback we have a unique morphism $m \colon K \to U_j \times_U U_k$ such that $\iota_{jk,j} \circ m = f$ and $\iota_{jk,k} \circ m = g$. But then $F(f)(\phi(a)_j) = F(\iota_{jk,j} \circ m)(\phi(a)_j) = F(m) \circ F(\iota_{jk,j})(\phi(a)_j) = F(m) \circ F(\iota_{jk,k}(\phi(a)_k) = F(g)(\phi(a)_k)$. Thus the sheaf condition guarantees us the existence of a unique $s \in F(U)$ such that $F(p_i)(s) = \phi(a)_i$ for every $i \in I$. But this in turn specifies a unique morphism $u \colon A \to F(U)$ such that $(\prod_{i \in I} F(f_i)) \circ u = \phi$ given by $u(a) = s$. □

As it will be useful later when we consider limits in the sheave categories we will quickly consider yet another way to rephrase the sheaf condition.

Proposition 1.2.11. *Let $U \in \mathsf{Ob}(\mathsf{C})$ and $S \in J(U)$ be a covering sieve. A presheaf $F\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ is a sheaf with respect to S, iff $F(U)$ together with the morphism $e\colon F(U) \to \prod_{f \in S} F(dom(f))$ given by $e(x) = \{F(f)(x)\}_f$ is the equalizer of the diagram*

$$\prod_{f \in S} F(dom(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\{f,g \mid f \in S, dom(f)=cod(g)\}} F(dom(g)) \tag{1.6}$$

where p and a are given by

$$p(x)_{f,g} = x_{f \circ g}, \quad a(x)_{f,g} = F(g)(x_f). \tag{1.7}$$

Proof. If F is a sheaf with respect to S and (A, ϕ) is any cone over the diagram we have for every $a \in A$ that $\phi(a)$ defines a compatible family. Thus there is a unique $x_a \in F(U)$ such that $F(f)(x_a) = \phi(a)_f$ for every $f \in S$. Then $u\colon A \to F(U)$ given by $u(a)) = x_a$ is the unique morphism such that $phi = e \circ u$. On the other hand if $(F(U), e)$ is the equalizer of the diagram we have that every compatible family $\{t_f \in F(dom(f)) | f \in S\}$ can be expressed as a cone over the diagram that consists of a one point set $A = \{a\}$ and a morphism $\phi\colon A \to \prod_{f \in S} F(dom(f))$ given by $\phi(a)_f = t_f$. Then the unique morphism $u\colon A \to F(U)$ that we get from the universal property of the equalizer equips us with the unique element $u(a) \in F(U)$ such that $F(f)(u(a)) = e(u(a))_f = \phi(a)_f = t_f$. □

Another thing we can do under the circumstance that pullbacks exist in C is to require an additional stability condition for our coverage.

Definition 1.2.12. *A coverage on the category C on which pullbacks exist is called pullback stable if for every covering family $\{f_i\colon U_i \to U\}$ and every morphism $g\colon V \to U$ the family $\{g^* f_i\colon g^* U_i \to V\}_{i \in I}$ is a covering family of V. Here*

$$\begin{array}{ccc} g^* U_i & \overset{g^* f_i}{\dashrightarrow} & V \\ \downarrow & & \downarrow g \\ U_i & \underset{f_i}{\longrightarrow} & U \end{array}$$

is the pullback diagram.

A pullback stable coverage is often also called a Cartesian coverage. Note, that we can add further covering families to a coverage that is not yet pullback stable to close it up under pullbacks. Doing this does not change which presheaves are sheaves with respect to the coverage. To see this note that clearly no presheaves that failed the sheaf condition before we added new covering families will become sheaves afterwards. Furthermore, if $F\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ was a sheaf before we added the new covering families we can test if it is also a sheaf with respect to the new covering families.
To show this we first need a lemma.

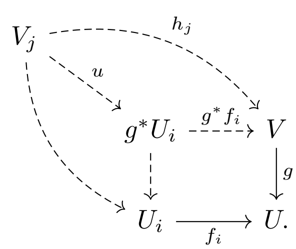
Lemma 1.2.13. *Let (C, T) be a site and let F a sheaf with respect to this site. Let $\{f_i\colon U_i \to U\}$ be a covering family in T and let $\{g_j\colon V_j \to U\}$ be any family of morphisms such that every f_i factors through at least one g_j. Then F is also a sheaf with respect to $\{g_j\colon V_j \to U\}$.*

Proof. Let $\{t_j \in F(V_j)\}$ be a compatible family of elements. Choose a factorization of the f_i say $f_i = g_j(i) \circ h_i$. Then consider the family $\{s_i = F(h_i)(t_{j(i)})\}$. Now let $f_i = g_{j'(i)} \circ h'_i$ be another factorization of f_i we have that $F(h_i)(t_{j(i)}) = F(h'_i)(t_{j'(i)})$ because $g_j \circ h'_i = g_{j'} \circ h'_i$ and the t_j form a compatible family. Moreover, if $U_r \xleftarrow{a} K \xrightarrow{b} U_p$ are two morphisms such that $f_r \circ a = f_p \circ b$ we have $F(a)(s_r) = F(a)(F(h_r)(t_{j(r)})) = F(h_r \circ a)(t_{j(r)}) = F(h_p \circ b)(t_{j(p)}) = F(b)(s_p)$, where the fourth equality is again due to the compatibility of $\{t_i\}$. Because F is a sheaf with respect to $\{f_i \colon U_i \to U\}$ it exists a unique $s \in F(U)$ such that $F(f_i)(s) = s_i$. We will show that this s is also the unique element in $F(U)$ such that $F(g_j)(s) = t_j$. For this end we note, that g_j is a morphism from V_j to U. Thus it exists for every j a covering family $\{k_{j,c} \colon V_{j,c} \to V_j\}$ such that $g_j \circ k_c$ factors through some f_i say $g_j \circ k_c = f_i \circ l$. We get

$$F(k_{j,c})F(g_j)(s) = F(l)F(f_i)(s) = F(l)(s_i) = F(l)F(h_i)(t_{j(i)}) = F(k_{j,c})(t_j). \tag{1.8}$$

But we can show that $F(k_{j,c})F(g_j)(s)$ is a compatible family. Indeed for $V_{j,r} \xleftarrow{a} K \xrightarrow{b} Vj,p$ such that $k_{j,r} \circ a = k_{j,p} \circ b$ we get $F(a)(F(k_{j,r})F(g_j)(s)) = F(b)F(k_{j,p})F(g_j)(s)$. Thus it follows, that $F(g_j)(s) = t_j$ because F is a sheaf with respect to $\{k_{j,c} \colon V_{j,c} \to V_j\}$. Clearly s is the unique such element because if s' was a different element in $F(U)$ such that $F(g_j)(s') = t_j$ we had $F(f_i)(s') = F(g_{j(i) \circ h_i})(s') = F(h_i)(t_j) = s_i$. □

With this at hand it is not hard to show that a sheaf with respect to a coverage is also a sheaf for the pulled back families. Indeed, if F is a sheaf with respect to $\{p_i \colon U_i \to U\}_{i \in I}$ and $g \colon V \to U$ is a morphism in C let $\{s_i \in F(g^*U_i)\}_{i \in I}$ be any compatible family with respect to the pulled back covering family. Then let $h_j \colon V_j \to V$ be a covering family such that every $g \circ h_j$ factors through at least one f_i. Fixing such an i for a given j gives us by universality of the pullback a unique morphism $u \colon V_j \to g^*U_i$ such that the following diagram commutes



But then every h_j factors through at least one g^*f_i and thus by lemma (1.2.13) we have that F is a sheaf with respect to $\{g^*f_i \colon g^*U_i \to V\}_{i \in I}$.

We can be even more strict with the stability conditions for a covering. Often the coverage is required to constitute a so called Grothendieck topology. To define what this is we need the notion of a sieve.

Definition 1.2.14. *Let C be a category. A sieve S on the object $U \in \mathsf{Ob}(C)$ is a subset $S \subset \mathsf{Ob}(C/U)$. That is closed under precomposition, i.e if $(V \to U) \in S$ and $(K \to V) \in \mathsf{Mor}(C)$, then the composition is in S.*

Here C/U is the slice category, that has as objects all the morphisms in C with codomain U and as morphisms from $f \colon X \to U$ to $g \colon X' \to U$ the morphisms $h \colon X \to X'$ such that

$g \circ h = f$. Note, that in particular for a given object $U \in \mathsf{Ob}(\mathsf{C})$ the presheaf $U(-)$, that is represented by U defines a sieve, namely the one which contains all the morphisms with codomain U. One can also characterize a sieves S on an object U as a subpresheave of $U(-)$. This means, that for every $X \in \mathsf{Ob}(\mathsf{C})$ it picks out a subset of $U(X)$ that we may denote by $S(X)$. With this at hand we can also define what the intersection of two sieves S and S' on U is. It is given by the subpresheaf $S \cap S'$ of $U(-)$ such that $(S \cap S')(X) = S(X) \cap S'(X)$.
We introduce yet another notation. If S is a sieve on $U \in \mathsf{Ob}(\mathsf{C})$ and $g\colon V \to U$ a morphism, then we denote by $g^*(S)$ the sieve on V that contains all the morphisms h with codomain V such that $g \circ h$ factors through at least one morphism in S. Note that this is indeed a sieve because if $h\colon K \to V$ is in $g^*(S)$, $g \circ h = m \circ n$ is a factorization through the element $m \in S$ and $f\colon J \to K$ then $g \circ h \circ f = m \circ n \circ f = m \circ (n \circ f)$ is a factorization through the element m and thus $h \circ f$ is also in $g^*(S)$. Note that the pullback family of a sieve is of this form.
We can now introduce the further stability conditions that were announced above.

Definition 1.2.15. *A Grothendieck topology J on the category C is a collection of sieves on the objects in C. Such that the four conditions listed below are met. The collection of sieves in J that are on $U \in \mathsf{C}$ is called a covering of U. A sieve in the covering of U is said to cover U. The four conditions are as follows.*

1. *If $S \in J$ covers $U \in \mathsf{Ob}(\mathsf{C})$ and $g\colon V \to U$ is any morphism, then $g^*(S)$ covers V.*
2. *The maximal sieve on U i.e the sieve that contains all morphisms with codomain U is a covering sieve for every $U \in \mathsf{Ob}(C)$.*
3. *If F and G cover U, then also their intersection $F \cap G$ covers U.*
4. *If F is a sieve on U, such that $\bigcup_V \{g\colon V \to U | g^*F \text{ covers } V\}$ is a covering sieve on U, then also F covers U.*

As with the pullback stability condition that we discussed extensively above it is possible to add further covering families to a given coverage in order to close it up to a Grothendieck topology.
Also in this case it is possible to show, that we do not change which presheaves are considered sheaves by closing up a coverage.
For example we can take a covering family $\{f_i\colon U_i \to U\}$ and generate from it a sieve as follows. Given a morphism $g\colon K \to U_i$ we add the morphism $f_i \circ g$ to the family. We do this for all morphisms g and if necessary continue with morphisms of the form $\tilde{g}\colon \tilde{K} \to K$ afterwards, then with morphisms $\tilde{\tilde{g}}\colon \tilde{\tilde{K}} \to \tilde{K}$ and so on.

Proposition 1.2.16. *Every presheaf, that is a sheaf with respect to a covering family $\{f_i\colon U_i \to U\}$ is also a sheaf with respect to the sieve that is generated by $\{f_i\colon U_i \to U\}$.*

Proof. Let $\{s_i \in F(U_i)\}$ be a compatible family with respect to $\{f_i\colon U_i \to U\}$. When we extend the covering family to a sieve we do this by means of morphisms as described above. Lets add for every such morphism $g\colon V \to U_i$ an element $F(g)(s_i)$ to the compatible family (likewise for morphisms $\tilde{K} \to K$ and so on). Then we end up with a compatible family again because the compatibility property is always reduced to the compatibility of the original elements s_i. On the other hand are all compatible families with respect to the generated sieve of this form.

Indeed, if $\{t_j\}_{j\in J}$ is any such compatible family, then we can choose some $k \in J$ such that t_k is not in $\{s_i \in F(U_i)\}$. (if this is not possible the original covering family was already a sieve and we have nothing to show). Let $K \in \mathsf{Ob}(\mathsf{C})$ s.t $t_k \in F(K)$. But then it must exist some $i \in I$ and some $g\colon K \to U_i$ such that $t_k = f_i \circ g$ because otherwise K would not be the domain of some morphism in the sieve. We have for every $K \xleftarrow{n} L \xrightarrow{m} U_i$ such that $f_i \circ m = f_i \circ g \circ n$ that $F(m)(s_i) = F(n)(f_i \circ g)$. So in particular for $m = g$ and $n = \mathrm{id}_K$ that $F(g)(s_i) = f_i \circ g = t_k$.
Now take the unique element $s \in F(U)$ that we get from the sheaf condition with respect to $\{f_i\colon U_i \to U\}$. Clearly it is also the unique element that we need for the sheaf condition with respect to the generated sieve. □

Similar proofs for the other closure properties can be found in chapter C.2.1 of [4]

1.2.3 Limits, colimits and exponential objects in $\mathsf{Sh}(\mathsf{C})$

In this section we want to show that in every category of sheaves $\mathsf{Sh}(\mathsf{C})$ that is defined on some site (C, J), small limits, small colimits and exponential objects exist. To show their existence we use that limits commute with right adjoins and and colimits with left adjoints (like is proven in Proposition (A.1.3)) together with the following proposition, which shows that limits and colimits in $\mathsf{PSh}(\mathsf{C})$ are computed pointwise.

Proposition 1.2.17. *Let* I *be a small category and* $F\colon \mathsf{I} \to \mathsf{PSh}(\mathsf{C})$ *be a diagram of shape* I *in* $\mathsf{PSh}(\mathsf{C})$*. Then the limit and the colimit of the diagram exists and can be computed pointwise i.e* $(lim_{\mathsf{PSh}(\mathsf{C})}F)(U) = lim_{\mathsf{Set}}F_i(U)$ *and* $(colim_{\mathsf{PSh}(\mathsf{C})}F)(U) = colim_{\mathsf{Set}}F_i(U)$*.*

Proof. We define the following functor

$$\begin{aligned} &F_{lim}\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set} \\ &F_{lim}(U) = \lim_{\mathsf{Set}} F_i(U) \end{aligned} \tag{1.9}$$

We also have to know how F_{lim} acts on morphisms. Let $f\colon U \to V$ be a morphism in C. Then denote the projection maps from $F_{lim}(U)$ to $F_i(U)$ and from $F_{lim}(V)$ to $F_i(V)$ by $p_{i,U}$ and $p_{i,V}$ respectively. We find

$$\begin{aligned} \phi_{ij,U} \circ F_i(f) \circ p_{i,V} &= F_j(f) \circ \phi_{ij,V} \circ p_{i,V} \\ &= F_j(f) \circ p_{j,V}. \end{aligned} \tag{1.10}$$

Thus by the universal property of limits we get a unique morphism $a(f)\colon F_{lim}(V) \to F_{lim}(U)$ such that $F_i(f) \circ p_{i,V} = p_{i,U} \circ a_{VU}$. We define $F_{lim}(f) = a(f)$. With this choice we establish, that the $p_{i,U}$ are stages of a natural transformation. Now let (G, q_i) be another cone over the diagram. Then we have for every $U \in \mathsf{Ob}(\mathsf{C})$

$$\begin{array}{ccc} G(U) & \xrightarrow{q_{i,U}} & F_i(U) \\ & {\scriptstyle q_{j,U}}\searrow & \downarrow{\scriptstyle \phi_{ij,U}} \\ & & F_j(U) \end{array} . \tag{1.11}$$

Thus we get for every U a unique morphism $z_U : G(U) \to F_{lim}(U)$ such that $q_{i,U} = p_{i,U} \circ z_U$. These morphisms are again the stages of a natural transformation because of the commutativity of the diagrams

$$\begin{array}{ccccc} & & \overset{q_{i,U}}{\frown} & & \\ G(U) & \xrightarrow{z_U} & F_{lim}(U) & \xrightarrow{p_{i,U}} & F_i(U) \\ & & & & \uparrow{\scriptstyle F_i(f)} \\ G(V) & \xrightarrow[z_V]{} & F_{lim}(V) & \xrightarrow[p_{i,V}]{} & F_i(V) \\ & & \underset{q_{i,V}}{\smile} & & \end{array} \tag{1.12}$$

and

$$\begin{array}{ccc} G(U) & \xrightarrow{q_{i,U}} & F_{i,U} \\ {\scriptstyle G(f)}\uparrow & & \uparrow{\scriptstyle F_i(f)} \\ G(V) & \xrightarrow[q_{i,V}]{} & F_{i,V} \end{array} \tag{1.13}$$

□

Our next objective is to show, that the limits and colimts of these diagrams are sheaves if the diagrams are on $\mathsf{Sh}(\mathsf{C})$. For limits this is rather easy.

Proposition 1.2.18. *Let* I *be a small category and* $F : \mathsf{I} \to \mathsf{PSh}(\mathsf{C})$ *a diagram. If* F_i *is a sheaf for every* $i \in \mathsf{Ob}(\mathsf{I})$ *then also the (pointwise) limit given by* $lim_{\mathsf{PSh}(\mathsf{C})}F$ *is a sheaf. In particular this shows that small limits exist in* $\mathsf{Sh}(\mathsf{C})$

Proof. We use that limits commute with limits, which is a special case of the fact that limits commute with right adjoints (Proposition (A.1.3) of the appendix). . For $U \in \mathsf{Ob}(\mathsf{C})$ and $S \in J(U)$ we can write down the equalizer diagram from lemma (1.2.11)

$$F_i(U) \xrightarrow{e} \prod_{f \in S} F_i(dom(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\{f,g | f \in S, dom(f)=cod(g)\}} F_i(dom(g)). \tag{1.14}$$

for every $i \in \mathsf{Ob}(\mathsf{I})I$ But because limits commute with limits and thus in particular limits commute with equalizers we get another equalizer diagram

$$\mathrm{lim}F(U) \xrightarrow{e} \prod_{f \in S} \mathrm{lim}F(dom(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\{f,g | f \in S, dom(f)=cod(g)\}} \mathrm{lim}F(dom(g)), \tag{1.15}$$

which shows, that the limit $\mathrm{lim}F$ is again a sheaf with respect to S. □

Our next objective is to show that also small colimits exist in C. This is a bit harder because colimits and equalizers do not commute in general and thus a colimit of sheaves is in general not computed as a colimit of presheaves.
TInstead we have to introduce a new functor that goes from $\mathsf{PSh}(\mathsf{C})$ to $\mathsf{Sh}(\mathsf{C})$ and that makes is possible to construct the colimit in $\mathsf{Sh}(\mathsf{C})$ by first computing it in presheaves and afterwards apply this so called *sheafification-functor*. The way we obtain this functor is called the plus construction.

The plus construction

Let (C, J) be a site, where J is a Grothendieck topology.
We will construct a left adjoint to the inclusion functor $\iota\colon \mathsf{Sh}(\mathsf{C}) \to \mathsf{PSh}(\mathsf{C})$. We do this in in two steps. First we construct from a presheaf a seperated presheaf. Then we show, that if this construction is applied again to a seperated presheaf we end up with a sheaf.

Definition 1.2.19. *A presheaf F on the site (C, J) is called seperated with respect to a covering sieve in $R \in J(U)$ if for every compatible family $\{s_f \in F(dom(f)) | f \in R\}$ the element $s \in F(U)$ such that $F(f)((s)) = s_f$ is unique if it exists.*

Given a presheaf $F \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$, we define a functor $F^+\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ as follows. It sends $u \in \mathsf{Ob}(\mathsf{C})$ to the equivalence class of pairs (R, s), where $R \in J(U)$ and $s = \{s_f \in F(dom(f)) | f \in R\}$ is a compatible family with respect to R and the equivalence relation is defined as follows. The pairs (R, s) and (R', s') get identified iff there is a $R'' \in J(U)$ contained in the intersection $R \cap R'$ on which the restrictions of s and s' agree. Moreover for a given morphism $g\colon V \to U$ we obtain a well defined map

$$\begin{aligned} F^+(g)\colon F^+(U) &\to F^+(V) \\ [(R, s)] &\mapsto [(g^*(R), (s_{g\circ h} | h \in g^*(R))))]. \end{aligned} \tag{1.16}$$

Here we used the first closure property of a Grothendiek topology. Moreover it is clear, that with this definition F^+ is a functor.

Lemma 1.2.20. *For every presheaf $F\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ the functor F^+ like defined above is a separated presheaf.*

Proof. Let $R \in J(U)$ be a covering sieve and $\{s_f \in F^+(dom(f)) | f \in R\}$ be a compatible family with respect to R. Clearly every s_f is an equivalence class of pairs (L, t) with $L \in J(dom(f))$ and $t_g = (t_g \in F(dom(g)) | g \in L)$. Now assume we have two pairs (A, t) and (A', t') such that $F^+(f)[(A, t)] = F^+(f)[(A', t')] = [(L, t)]$. This means it exists an $P \in J(U)$ such that $P \subset f^*(A) \cap f^*(A')$ and the restrictions of $(s_{f\circ g} | g \in f^*(A))$ and $(s'_{f\circ g} | g \in f^*(A'))$ to P agree. We want to show that it exists an $A'' \subset A \cap A'$ such that the restrictions of t and t' to A'' agree. But from property (4) in the definition of a Grothendieck topology (definition 1.2.15) we know that the sieve S of all morphisms of the form $f \circ g$ where $f \in R$ and $g \in A''$ is in $J(U)$ because $l^*(S)$ is in $J(dom(l))$ for every $l \in R$. But we have that $S \subset A \cap A'$ and clearly t and t' agree when restricted to S. □

Next we want to show, that when F is already a separated presheaf, F^+ is a sheaf.

Lemma 1.2.21. *If $F\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ is a separated presheaf on the site (C, J), then F^+ is a sheaf on this site.*

Proof. First we observe that if F is separated and (R, s), (R', s') are two representatives of the same element in $F^+(U)$, then s and s' agree on all of $R \cap R'$ rather than only on some sieve contained in this intersection. Indeed, we already know that there is some covering sieve $\tilde{R} \subset R \cap R'$ on which s and s' agree. For $f \in R \cap R'$ we have that $\{F(g)(s_f) | g \in \tilde{R}\}$ and $\{F(g)(s'_f) | g \in \tilde{R}\}$ define the same compatible families. Thus it follows that $s_f = s'_f$ because

F is separable.
Now let R be a covering sieve on $U \in \mathsf{Ob}(\mathsf{C})$ and $(t_f \in F^+(dom(f))|f \in R)$ a compatible family of elements. We denote by (S_f, s_f) the representative of t_f that is obtained by taking the union of all representatives of t_f. This is well defined because the compatible families in any two representatives agree on the intersection of their respective sieves like was shown above. Next we can define the sieve $\{f \circ g | f \in R, g \in S_f\}$. Because of property (4) in definition (1.2.15) this is again a covering sieve. Moreover, we can label by it all the element in a compatible family that is given by the union $\cap_{f \in R} s_f$. We thus have an element $t = [(\{f \circ g | f \in R, g \in S_f\}, \cup_{f \in R} s_f)]$ of $F^+(U)$ and because clearly S_f is contained in $f^*\{f \circ g | f \in R, g \in S_f\}$ we have that $F(f)(t) = t_f$ for every $f \in R$. Moreover t is the unique such element because A^+ is separated due to lemma (1.2.20). Thus F^+ is a sheaf. □

We can define the natural transformation $\eta \colon F \to F^+$ with the stages

$$\eta_X(x \in F(X)) = [(M_X, \{F(f)(x) | f \in M_X\})] \tag{1.17}$$

where M_X is the maximal sieve on $X \in \mathsf{Ob}(\mathsf{C})$. This is really a natural transformation because for $g \colon Y \to X$ a morphism in C we have

$$\eta_Y \circ F(g)(x \in F(X)) = [(M_Y, \{(F(g \circ f)(x)) | f \in M_Y\})] \tag{1.18}$$

and on the other hand

$$F^+(g) \circ \eta_X(x) = F^+(g)([(M_X, \{F(f)(x) | f \in M_X\})]) = [(g^*(M_X), \{F(g \circ h)(x) | h \in g^*(M_X)\})]. \tag{1.19}$$

But we find, that the compatible families that we get from computing x through either compositions agree when restricted to $g^*(M_X) \subset M_Y$. Thus both equivalence classes are equal and we have indeed a natural transformation.

Definition 1.2.22. *Define the functor* $a \colon \mathsf{PSh}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C})$ *through*

$$a(F) = (F^+)^+ \tag{1.20}$$

In order to show that we constructed a left adjoint of the inclusion we need the following lemma.

Lemma 1.2.23. *Given any morphism* ϕ *in* $\mathsf{PSh}(\mathsf{C})$ *between the presheaf* F *and the sheaf* A *it factors uniquely through* η *as* $\phi = \tilde{\phi} \circ \eta$.

Proof. We start with an element $[(R, \{x_f | f \in R\})] \in F^+(U)$. From the definition of η we know that for $h \colon V \to U$ we have $\eta_V(x_h) = \{F(k)(x_h) | k \in M_X\}$, where M_V is the maximal sieve on V. But because $P^+(h)[(R, \{(x_f) | f \in R\})] = [(h^*(R), \{x_{h \circ f' | f' \in h^*(R)}\})]$, moreover $h^*(R) = M_V$ for $h \in R$ and finally $P(k)x_h = P(\mathrm{id}_K)(x_{h \circ k})$ for every $k \colon K \to V$ because $\{x_f | f \in R\}$ is a matching family we have that $\eta_V(x_h) = P^+(h)(\{x_f | f \in R\})$ for every $(h \colon V \to U) \in R$. Thus if the map $\tilde{\phi}$ from the assertion exists $\tilde{\phi}(\{x_f | f \in R\})$ is given by the element $y \in F(U)$ such that

$$F(h)(y) = F(h)(\tilde{\phi}(\{x_f | f \in R\})) = \tilde{\phi}(P^+(h)(\{x_f | f \in R\})) = \tilde{\phi}(\eta_V(x_h)) = \phi(x_h). \tag{1.21}$$

Where the second equality uses that $\tilde{\phi}$ is a natural transformation. But such a $y \in F(U)$ indeed exists and it is even unique, since F is a sheaf.

□

But with this at hand we can understand, that given an object $X \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$ and an object $Y \in \mathsf{Ob}(\mathsf{Sh}(\mathsf{C}))$ for every element in $\phi \in \mathsf{Hom}_{\mathsf{PSh}(\mathsf{C})}(X, \iota(Y))$ we get a unique $\tilde{\phi} \in \mathsf{Hom}_{\mathsf{Sh}(\mathsf{C})}(a(X), Y)$ such that $\tilde{\phi} \circ \eta_X \circ \eta_{X^+} = \phi$. Thus we can define

$$\begin{aligned} \alpha_{XY} \colon \mathsf{Hom}_{\mathsf{PSh}(\mathsf{C})}(X, \iota(Y)) &\to \mathsf{Hom}_{\mathsf{Sh}(\mathsf{C})}(a(X), Y) \\ \phi &\mapsto \tilde{\phi}. \end{aligned} \tag{1.22}$$

But α_{XY} clearly is a bijection and moreover because it is constructed by means of the natural transformation η this bijection is natural in X and Y. Thus it is an adjunction between the functors ι and a or in other words a is the right adjoint of ι. Lets collect this result in a proposition.

Proposition 1.2.24. *There is a left adjoint to the inclusion $\iota \colon \mathsf{Sh}(\mathsf{C}) \to \mathsf{PSh}(\mathsf{C})$ given by the functor $a \colon \mathsf{PSh}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C})$.*

This is a very useful result. Recall for instance, that left adjoints preserve colimits (appendix). We only need one final ingredient to show that small colimits exist in $\mathsf{Sh}(\mathsf{C})$. This is the following lemma.

Lemma 1.2.25. *$F \in \mathsf{PSh}(\mathsf{C})$ is a sheaf iff $\eta \colon F \to F^+$ is an isomorphism.*

Proof. Let $x, y \in F(U)$. Then if $\eta_U(x) = \eta_U(y)$ this means that $F(f)(x) = F(f)(y)$ for every $f \in M_U$. Thus $x = y$ because F is a sheaf. Moreover we have for every element $[(R, s)] \in F^+(U)$ that it exists a unique $\tilde{s} \in F(U)$ such that $F(f)(\tilde{s}) = s_f$ because F is a sheaf. But then we have that $M_U \cap R = R$ and thus $\eta_U(\tilde{s}) = [(M_U, \{F(f)(\tilde{s}|f \in M_U)\})] = [(R, s)]$. □

In particular does this lemma show, that $a \circ \iota$ is naturally isomorphic to the identity. Now let $F \colon I \to \mathsf{Sh}(\mathsf{C})$ be any small diagram of shape I in $\mathsf{Sh}(\mathsf{C})$. We have for the colimit

$$\mathrm{colim}(F) = \mathrm{colim}((a \circ \iota)F) \cong a(\mathrm{colim}(\iota(F))) \tag{1.23}$$

But we already know how to calculate small colimits in $\mathsf{PSh}(\mathsf{C})$ and thus we have shown, that also small colimits exist in $\mathsf{Sh}(\mathsf{C})$. The last thing we want to do in this section is to show that also exponential objects exist in $\mathsf{Sh}(\mathsf{C})$. Like with limits, this exponential objects can be computed in $\mathsf{PSh}(\mathsf{C})$. So we first prove that exponential objects exist in $\mathsf{PSh}(\mathsf{C})$ and then demonstrate, that X^Y is a sheaf for every $Y \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$ if X is a sheaf.
If the presheaf exponential object X^Y exists it has to satisfy $\mathsf{Hom}(Z \times Y, X) \cong \mathsf{Hom}(Z, X^Y)$ for all $Z \in \mathsf{Ob}(\mathsf{PSh}(\mathsf{C}))$. In particular for a representable presheaf $\iota(C)$, where $C \in \mathsf{C}$ we get

$$X^Y(C) \overset{\heartsuit}{\cong} \mathsf{Hom}(\iota(C), X^Y) \cong \mathsf{Hom}(\iota(C) \times Y, X)$$

where $\heartsuit$ is just a direct application of the Yoneda lemma. This motivates

Proposition 1.2.26. *The exponential object X^Y in* $\mathsf{PSh}(\mathsf{C})$ *is given by the functor*

$$X^Y(C) = \mathsf{Hom}(\iota(C) \times Y, X).$$

together with the evaluation map ev: $X^Y \times Y \to X$ *that has the components*

$$ev_C(\theta, y) = \theta_C(id_C, y) \in X(C) \tag{1.24}$$

where $\theta \in X^Y(C)$ *and* $y \in Y$. *In particular do exponential objects in* $\mathsf{PSh}(\mathsf{C})$ *exist.*

Proof. Let $A \in \mathsf{PSh}(\mathsf{C})$ and $\phi\colon A \times Y \to X$ a natural transformation. To prove the claim of the proposition we have to show that there is a unique natural transformation $\psi\colon A \to X^Y$ such that

$$\begin{array}{ccc} A \times Y & & \\ {\scriptstyle \psi\times \mathrm{id}_Y}\downarrow & \searrow^{\phi} & \\ X^Y \times Y & \xrightarrow{\mathrm{ev}} & Q \end{array} \tag{1.25}$$

commutes. The components of such a ψ thus necessarily have to satisfy

$$\mathrm{ev}_C(\psi_C(a) \times y) = \psi_C(a)(\mathrm{id}_C, y) = \phi_C(a, y) \tag{1.26}$$

for $a \in A(C)$ and $y \in Y(C)$. Moreover naturality is the requirement that for every morphism $f\colon D \to C$ the diagram

$$\begin{array}{ccc} A(D) & \xrightarrow{\psi_D} & X^Y(D) \\ {\scriptstyle B(f)}\uparrow & & \uparrow{\scriptstyle X^Y(f)} \\ A(C) & \xrightarrow[\psi_C]{} & X^Y(C) \end{array}$$

commutes. Thus a ψ also has to satisfy

$$\psi_D(A(f)b) = X^Y(f)(\psi_C(b)). \tag{1.27}$$

Combining the two conditions (1.26) and (1.27) we obtain

$$\begin{aligned} (\psi_C(b))_D(f, y) =& (\psi_C(b))_D(f \circ \mathrm{id}_D, y) \\ =& (X^Y(f)\psi_C(b))_D(\mathrm{id}_D, y) \\ =& (\psi_D(A(f)b))_D(\mathrm{id}_D, y) \\ =& \phi_D(A(f)b, y). \end{aligned}$$

for arbitrary $f\colon D \to C$ and $y \in Y(D)$. On the other hand it is immediately shown, that if $\psi\colon A \to X^Y$ is defined to have the components

$$(\psi_C(b))_D(f, y) = \phi_D(A(f)b, y)$$

it is a natural transformation, that makes the diagram (1.25) commute. □

We want to show, that if $X, Y \in \mathsf{Ob}(\mathsf{Sh}(\mathsf{C}))$ i.e they are sheaves, then also X^Y is in $\mathsf{Ob}(\mathsf{Sh}(\mathsf{C}))$. If this holds true we have proved existence of exponential objects in $\mathsf{Sh}(\mathsf{C})$ and also that we can compute them in the category of presheaves on C. Actually the sheaf property of X^Y already follows if X is a sheaf.

Proposition 1.2.27. *If $X, Y \in \mathsf{Ob}(\mathsf{PSh})(\mathsf{C})$ and X is a sheaf, then also the exponetial object X^Y (computed in $\mathsf{PSh}(\mathsf{C})$ like in Proposition (1.2.26)) is a sheaf.*

Proof. First we demonstrate, that X^Y is seperated if X is. Let $C \in \mathsf{Ob}(\mathsf{C})$ and $S \in J(C)$ a covering sieve of C. Moreover let $\phi, \varphi \in X^Y(C)$ such that

$$X^Y(s)(\phi) = X^Y(s)(\varphi).$$

This means, that if $g \colon D \to \mathrm{dom}(S)$ and $y \in Y(D)$ for $D \in \mathsf{Ob}(\mathsf{C})$ we have

$$\phi(s \circ g, y) = \varphi(s \circ g, y)$$

and in particular for $g = \mathrm{id}_{\mathrm{dom}(s)}$

$$\phi(s, y) = \varphi(s, y). \tag{1.28}$$

Now let $D \in \mathsf{Ob}(\mathsf{C})$, $f \colon D \to C$, $y \in Y(C)$ and $(\tilde{g} \colon E \to D) \in k^*S$ be arbitrary. Then

$$\begin{aligned} X(\tilde{g})\phi_D(k, y) &\overset{\spadesuit}{=} \phi_E(k \circ \tilde{g}, Y(\tilde{g})y) \\ &\overset{\heartsuit}{=} \varphi_E(k \circ \tilde{g}, Y(\tilde{g})y) \\ &\overset{\clubsuit}{=} X(\tilde{g})\varphi_D(k, y) \end{aligned}$$

where $\spadesuit$ and $\clubsuit$ is a consequence of the naturality of ϕ and respectively φ and $\heartsuit$ uses (1.28). But due to condition (1) in the definition of a Grothendieck topology we know that k^*S covers D. If we assume that X is seperated it follows that $\phi(k, y) = \varphi(k, y)$ and thus $\phi = \varphi$ because k and y were arbitrary.
Next we show that X^Y is a sheaf if X is a sheaf. Again, let $S \in J(C)$ be a covering sieve of $C \in \mathsf{Ob}(\mathsf{C})$. Moreover let $\{\phi_s \colon \iota(D) \times Y \to X | (s \colon D \to C) \in S\}$ be a compatible family of elements. Our strategy to obtain the natural transformation $\phi \colon \iota(C) \times Y \to X$ such that $(\iota(s) \times \mathrm{id}_Y)\phi = \phi_s$ for every $s \in S$, which is required by the sheaf consition is as follows. We first construct a natural transformation $\tilde{\phi} \colon \iota(C) \times Y \to X^+$ such that the diagram

$$\begin{array}{ccc} \iota(D) \times Y & \xrightarrow{\phi_s} & X \\ {\scriptstyle \iota(s)\times \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \iota(C) \times Y & \xrightarrow{\tilde{\phi}} & X^+, \end{array} \tag{1.29}$$

where η_X is the natural transformation (2.1.9), commutes. Because X is a sheaf we know that η_X is an isomorphism and thus once we constructed $\tilde{\phi}$ the proposition is proved by setting $\phi = \eta_X^{-1} \circ \tilde{\phi}$.
$\tilde{\phi}$ is defined to have the components

$$\tilde{\phi}_E(k, y) = \{(\phi_{k \circ h})_{dom(h)}(\mathrm{id}_{dom(M)}, Y(h)y) | h \in k^*S\}$$

where $k\colon E \to C$ and $y \in Y(E)$. We have to check that this indeed defines a natural transformation into X^+. Recalling the definition of X^+ we have to check, that $\{(\phi_{k\circ h})_{dom(h)}(\mathrm{id}_{dom(h)}, Y(h)y)|h \in k^*S\}$ is a compatible family of elements. This means, that for $m\colon F \to dom(h)$ we have to prove that $X(m)(\phi_{k\circ h})_{dom(h)}(\mathrm{id}_{dom(h)}, Y(h)y) = (\phi_{k\circ h\circ m})_F(\mathrm{id}_F, Y(h\circ m)y)$. This is accomplished by the following calculation

$$\begin{aligned} X(m)(\phi_{k\circ h})_{dom(h)}(\mathrm{id}_{dom(h)}, Y(h)y) =& (\phi_{k\circ h})_F(m, Y(h\circ m)y) \\ =& (\phi_{k\circ h\circ m})_F(\mathrm{id}_F, Y(h\circ m)y) \end{aligned}$$

where for the last equation we used that $(\phi_{k\circ h})_F(m, Y(h\circ m)y) = (X^Y(m)\phi_{k\circ h})_F(1_F, Y(h\circ m)y) = (\phi_{k\circ h\circ m})_F(1_F, Y(h\circ m)y)$. Last we have to check, that this choice of $\tilde{\phi}$ makes the above diagram (1.29) commute. First we have that for $(k\colon B \to D) \in f^*S$ and $y \in Y(B)$ that

$$\begin{aligned} \tilde{\phi}_B \circ (\iota(f) \times \mathrm{id}_Y)_B(k, y) =& \tilde{\phi}_B(f\circ k, y) \\ =& \{(\phi_{f\circ k\circ h})_{dom(h)}(\mathrm{id}_{dom(h)}, Y(h)y)|h \in k^*f^*S\} \end{aligned}$$

But clearly, because $f \in S$ we have $k^*f^*S = M_B$ where M_B is the maximal sieve on B. Second we find by the definition of η (see (2.1.9)) that

$$\begin{aligned} \eta_B((\phi_f)_B(k, y)) =& \eta_B((\phi_{f\circ k})_B(\mathrm{id}_B, y)) \\ =& \{(\phi_{f\circ k\circ h})_{dom(h)}(\mathrm{id}_{dom(h)}, Y(h)y)|h \in M_B\}. \end{aligned}$$

This completes the proof of the proposition. □

Chapter 2

The Cahiers Topos

This chapter will be concerned with introducing the well-adapted model for synthetic differential geometry, with which we want to formulate scalar field theory later on. For this end we first introduce the category $\mathsf{C^\infty Ring}$ and show that it is possible to full and faithfull embedd the category Man of smooth manifolds into it (or rather into its opposite $\mathsf{C^\infty Ring^{op}}$). Our main tool to show that such an embedding exists is a result called Milnor's exercise. It is a statement about *paracompact* smooth manifolds. We take as our definition of a smooth manifold that it is a second countable, Hausdorff space with a maximal smooth atlas. Such a manifold is in particular always paracompact.
After we understood how to describe smooth manifolds within $\mathsf{C^\infty Ring}$ we will have a look at Weil algebras. For our purpose there is one particulary interesting Weil algebra- the dual numbers. These can intuitively be understood as the space of smooth maps from an infinitesimally short line to $\mathbb{R}$. We will show that also Weil algebras have a canonical C^∞-ring structure. The Cahiers topos will finally be constructed as a category of Set-valued sheaves on the opposite category of a specific subcategory of $\mathsf{C^\infty Ring}$. This subcategory is choosen in a way that we can describe smooth manifolds within the Cahiers topos but also have an infinitesimal line object (given by the Yoneda embedding of the dual numbers). This infinitesimal line objects allows us to make synthetic constructions. At the end of this section we will actually be able to proof that the Cahiers topos are a well adapted model for synthetic differntial geometry.

2.1 Construction of the Cahiers topos

2.1.1 The category $\mathsf{C^\infty Ring}$

Definition 2.1.1. *A C^∞-ring is given by a set A and for every smooth function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ with $m, n \geq 0$ a map*

$$A_f\colon \underbrace{A \times \cdots \times A}_{m\ times} \to \underbrace{A \times \cdots \times A}_{n\ times}$$

such that the assignment $f \mapsto A_f$ is compatible with composition, identity and projections i.e the following conditions are satisfied.

1. *For every two smooth maps* $f\colon \mathbb{R}^m \to \mathbb{R}^n$ *and* $g\colon \mathbb{R}^n \to \mathbb{R}^d$ *we have* $A_g \circ A_f = A_{g\circ f}$
2. *For every* $n \geq 0$ *we have* $A_{id_{\mathbb{R}^n}} = id_{A^n}$.
3. *For the projection* $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ *we have* $A_{\pi_i}\colon A^n \to A$, $(a_1, \dots, a_n) \mapsto a_i$

It is evident, that a C^∞-ring structure in particular induces an $\mathbb{R}$-algebra structure on the set A. Scalar multiplication with $\lambda \in \mathbb{R}$ is given by A_{f_λ} where $f_\lambda\colon \mathbb{R} \to \mathbb{R}$ is multiplication by λ. Addition is given by $A_{f_{ad}}$ where $f_{ad}\colon \mathbb{R}^2 \to \mathbb{R}$ is addition in $\mathbb{R}$ and multiplication by $A_{f_{mult}}$ where $f_{mult}\colon \mathbb{R}^2 \to \mathbb{R}$ is multiplication in $\mathbb{R}$. If A was already equipped with an $\mathbb{R}$-algebra structure we may have the possibility to find a C^∞-ring structure on A such that the original algebra structure is induced by this. Then we also say that the respective C^∞-ring structure extends the algebra structure of A. Evidently the $\mathsf{C^\infty Ring}$-morphisms also respect the induced algebra structure. We will often denote $A_{f_\lambda}(a) = \lambda a$ and $A_{f_{mult}}(a_1, a_2)) = a_1 a_2$ and $A_{f_{ad}(a_1,a_2)} = a_1 + a_2$ in the following.

The C^∞-rings are the objects of the category $\mathsf{C^\infty Rings}$. Its morphisms are given by maps between sets of the form $\kappa\colon A \to B$ such that

$$\begin{array}{ccc} A^n & \xrightarrow{\kappa^n} & B^n \\ {\scriptstyle A_f}\downarrow & & \downarrow{\scriptstyle B_f} \\ A^m & \xrightarrow{\kappa^m} & B^m \end{array}$$

commutes for every $n, m \geq 0$ and $f\colon \mathbb{R}^m \to \mathbb{R}^n$ smooth. Finally the composition of morphisms is given by the usual composition of maps between sets.

2.1.2 The full and faithful embedding of Man into $\mathsf{C^\infty Ring}$

Let N be a smooth manifold. We denote the set of smooth functions from N to $\mathbb{R}$ by $C^\infty(N)$. This set together with the assignment $f \mapsto C^\infty(N)_f$ given by

$$C^\infty(N)_f\colon C^\infty(N)^m \to C^\infty(N)^n, \quad (h_1, \dots, h_m) \mapsto f \circ (h_1, \dots, h_m)$$

for every smooth $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is a C^∞-ring. To show this one can check the three defining conditions above in a completely straightforward manner.
For getting used to this definitions and also because we will need it later, let us proof the following proposition.

Proposition 2.1.2. *$C^\infty(\mathbb{R}^m)$ is the free C^∞-ring on m generators.*

Proof. We will show that a $\mathsf{C^\infty Ring}$ morphism $\phi\colon C^\infty(\mathbb{R}^m) \to A$ is uniquely determined once we know the images of the projections $\pi_1, \dots, \pi_m\colon \mathbb{R}^m \to \mathbb{R}$. We know that the diagram

$$\begin{array}{ccc} (C^\infty(\mathbb{R}^m))^m & \xrightarrow{\phi} & A^m \\ {\scriptstyle C^\infty(\mathbb{R}^m)_f}\downarrow & & \downarrow{\scriptstyle A_f} \\ C^\infty(\mathbb{R}^m) & \xrightarrow[\phi]{} & A \end{array}$$

commutes for every $f \in C^\infty(\mathbb{R}^m)$. In particular we can chase $(\pi_1, \ldots, \pi_m) \in (C^\infty(\mathbb{R}^m))^m$ around it to get

$$\phi(f \circ (\pi_1, \ldots, \pi_m)) = \phi(f) = A_f(\phi^m((\pi_1, \ldots, \pi_m)))$$

which proves the result. □

It is actually possible to extend the above assignment $N \mapsto C^\infty(N)$ to a full and faithful embedding of Man into $\mathsf{C^\infty Ring^{op}}$. This is accomplished by assigning to a smooth map $f\colon N \to M$ the $\mathsf{C^\infty Ring}$-morphism in $\mathsf{Hom}_{\mathsf{C^\infty}}(C^\infty(M), C^\infty(N))$ given by precomposition with f. Proving that the so defined functor is full and faithful relies on a proposition that is known as "Milnor's exercise", for which we first need a lemma.

Lemma 2.1.3. *For every closed subset C of a smooth manifold M one can find a smooth map $\phi\colon M \to \mathbb{R}$ such that $\phi^{-1}(0) = C$*

Proof.

1. Let (a, b) be an open interval in $\mathbb{R}$. We can define the smooth function $\tilde{\varphi}\colon \mathbb{R} \to [0, \infty)$ to be given by

$$\tilde{\varphi} = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } x \in (-1, 1) \\ 0 & \text{else} \end{cases}$$

   We can precompose it with the translation and rescaling

$$\begin{aligned} t\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \left(x - \frac{b+a}{2}\right)\frac{2}{b-a} = \frac{2x-a-b}{b-a} \end{aligned}$$

   to obtain a smooth function $\varphi = \tilde{\varphi} \circ t$, which has the property that $\varphi^{-1}(0, \infty) = (a, b)$ or other words vanishes outside of the interval (a, b).

2. Next we remark, that as an easy consequence of (1) we can construct a smooth function $\varphi\colon \mathbb{R}^n \to [0, \infty)$ that vanishes outside a box

$$(a_1, b_1) \times \ldots \times (a_n, b_n). \tag{2.1}$$

3. We can write an open subset $U \subset \mathbb{R}^n$ as a countable union of open boxes like (2.1). Then we chose for every such box a function like in (2) and call the function for the m-th box ϕ_m. We then take the following sum

$$\phi = \sum_{m=1}^{\infty} \phi_m \epsilon_m.$$

   We can choose the constants $\epsilon_m \in \mathbb{R}$ such that

$$\partial_\alpha(\phi_m \epsilon_m) \leq 2^{-m}$$

for all multiindices in n letters with $|\alpha| \leq m$. This is possible because the ϕ_m have compact support. But then the partial sums

$$\sum_{m_1}^{p} \partial_\alpha(\phi_m \epsilon_m)$$

are dominated by

$$\sum_{m=1}^{p} 2^{-m}$$

from one m on. Thus the so defined $\phi\colon \mathbb{R}^m \to \mathbb{R}$ is again a smooth function (that clearly vanishes outide U).

4. Last, let M be a smooth manifold. We can use the Whitney embedding theorem to get an embedding $E\colon M \hookrightarrow \mathbb{R}^n$. Let $C \subset M$ be a closed subset of M. Then it exists a closed set $\tilde{C} \subset \mathbb{R}^n$ such that $\tilde{C} \cap F(M) = F(C)$. We know that $\mathbb{R}^n \backslash \tilde{C}$ is open and thus there exists due to (3) a smooth map $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ that has the property $\varphi^{-1}(0,\infty) = M\backslash\tilde{C}$. But this means, that

$$\phi := \varphi|_{E(M)} \circ E\colon M \to R$$

is the desired smooth function.

□

Proposition 2.1.4. *(Milnor's exercise) Let M be a smooth manifold and $C^\infty(M)$ the algebra of smooth functions from M to $\mathbb{R}$. There exists for every algebra homomorphism $\varphi \in \mathsf{Hom}(C^\infty(M),\mathbb{R})$ a unique point $x \in M$ such that $\varphi(f) = f(x)$ for every $f \in C^\infty(M)$ i.e φ is evaluation at the point x.*

Proof. Let $\varphi \in \mathsf{Hom}_{\mathbb{R}\mathsf{Alg}}(C^\infty(M),\mathbb{R})$ and define $Z = \{Z_f = f^{-1}(\{0\})|f \in \ker(\varphi)\}$. Clearly all the elements in Z are closed subsets of M and moreover Z is a filter of closed subsets of M. Indeed, the zero map is in the kernel of φ and thus Z is not empty. Moreover, if f and g are in the kernel of φ, then f^2+g^2 is also in the kernel and $Z_{f^2+g^2} = Z_f \cap Z_g$. Last we have to show, that if $Z_f \in Z$ and $Z_f \subset K$ with K closed, there exists a $g \in C^\infty(M)$ such that $Z_g = K$. For this end we pick any smooth function $\tilde{g}\colon M \to \mathbb{R}$ such that $\tilde{g}^{-1}(0) = K$. This is possible due to the preceding lemma. Then $g = \tilde{g}f$ is in the kernel of φ and $g^{-1}(0) = K$.
Now consider any smooth function $\tilde{k} \in C^\infty(M)$ that is unbounded on every non-compact closed subset of M. An example for such a function is constructed as follows. We pick a $x \in M$ and then define the function that measures the distance to x with respect to a Riemannian metric on M. To ensure that we get an element in the kernel of φ we replace $\tilde{k}$ by $k = \tilde{k} - \varphi(k)\cdot 1$. Clearly k still has the property of being unbounded on every non-compact closed subset. If $K \subset \mathbb{R}$ is compact we know that $k^{-1}(K)$ is closed and thus it must also be compact. In particular $Z_k = k^{-1}(0)$ is compact. Assume that $\bigcap_{f\in\ker(\varphi)} Z_f$ is empty. Then the complements of all the Z_f cover Z_k. Choose a finite sub covering $Z^c_{f_1}, \dots Z^c_{f_n}$. Then there exists some $x \in M$ that is in the intersection $(\bigcap_{i=1}^n Z_{f_i}) \cap Z_k$ as Z has the finite intersection property because it is a filter. This is a contradiction. Thus $\cap_{f\in\ker(\varphi)} Z_f$ is non empty. Choose $x_0 \in \cap_{f\in\ker(\varphi)} Z_f$. Then for

an arbitrary $f \in C^\infty(M)$ we have that $f - \varphi(f) \cdot 1$ is in the kernel of φ. Thus it vanishes on x_0 and we have $\varphi(f) = f(x_0)$. Uniqueness follows from the fact that for two different points $x_1, x_2 \in M$ we can always find a smooth function $f\colon M \to \mathbb{R}$ such that $f(x_1) \neq f(x_2)$. □

With this at hand it is rather straightforward to establish the result that was announced above.

Corollary 2.1.5. *The functor* $C^\infty(-)\colon \mathsf{Mfd} \to \mathsf{C}^\infty\mathsf{Ring}^{\mathsf{op}}$ *given by*

$$M \mapsto C^\infty(M)$$
$$(f\colon M \to N) \mapsto ((-) \circ f\colon C^\infty(N) \to C^\infty(M)$$

is full and faithful.

Proof. Let M, N be smooth paracompact manifolds. We first assign to each $\mathbb{R}$-algebra homomorphism from $C^\infty(N)$ to $C^\infty(M)$ a unique set theoretic map from M to N. Subsequently we show, that this map is actually smooth.
Let $\phi\colon C^\infty(N) \to C^\infty(M)$ be any $\mathbb{R}$-algebra homomorphism. Define $P\colon C^\infty(M) \to \mathbb{R}$ to be evaluation at the point $p \in M$. Then $P \circ \phi$ is an $\mathbb{R}$-algebra homomorphism from $C^\infty(N)$ to $\mathbb{R}$. Due to Milnors exercise it exists some point $q \in N$ such that $P \circ \phi$ is evaluation at q. We define the map $f\colon M \to N$ such that p is sent to $f(p) = q$. Now we want to show that this f is smooth. For this end we show, that $g \circ f$ is smooth for any smooth map g with domain N. Indeed for $p \in M$ we have $g(f(p)) = \phi(g)(p)$. But $\phi(g)$ is smooth.
We thus have a full and faithful functor from Mf to $\mathbb{R}\mathsf{Alg}^{\mathsf{op}}$. Because the forgetful functor $\mathsf{C}^\infty\mathsf{Ring} \to \mathbb{R}\mathsf{Alg}$ is faithful we have that $C^\infty(-)$ is full and faithful also when it is considered as a functor into $\mathsf{C}^\infty\mathsf{Ring}^{\mathsf{op}}$. □

The lesson of all this is of course, that we can consistently describe smooth manifolds in the category $\mathsf{C}^\infty\mathsf{Ring}^{\mathsf{op}}$.
Lets continue with some further investigation of these C^∞-rings of the form $C^\infty(M)$. In particular the final result of the following discussion, namely that the coproduct in $\mathsf{C}^\infty\mathsf{Ring}$ of $C^\infty(M)$ with $C^\infty(N)$, for M and N smooth manifolds, is given by $C^\infty(M \times N)$ will be very useful for our purpose.

The first step is Hadamards lemma. A classical result, that will prove useful multiple times below.

Lemma 2.1.6. *(Hadamard's lemma) Let* $f\colon U \to \mathbb{R}$ *be a smooth real valued function, that is defined on some star-shaped open neighborhood of a point* a *in* $\mathbb{R}^n$*. Then we can find unique smooth functions* $\{g_i\colon U \to \mathbb{R}\}_{i=1}^n$ *such that*

$$f(x) = f(a) + \sum_{i=1}^{n}(x_i - a_i)g_i(x)$$

Proof. Let $x \in U$ be some point in U. Define the smooth function $h\colon [0,1] \to \mathbb{R}$ by

$$h(t) = f(a + t(x - a)).$$

Then we have that

$$h'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i).$$

But then

$$\begin{aligned} h(1) - h(0) &= \int_0^1 h'(t)dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial f_i}(a + t(x-a))(x_i - a_i)dt \\ &= \sum_{i=1}^{n}(x_i - a_i) \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))dt. \end{aligned}$$

With $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))dt$ the statement follows. □

With Hadamards lemma at hand we are in a position to start the announced investigation of C^∞-rings.

Proposition 2.1.7. *Let A be a C^∞-ring and I an algebra ideal in A. The canonical projection $p\colon A \to A/I$ induces a C^∞-ring structure on A/I making p into a $\mathsf{C^\infty Ring}$-morphism.*

Proof. Let $a, b \in A^m$ such that $a_i = b_i \mathrm{mod}\, I$. What we want to show is that for every smooth $f\colon \mathbb{R}^m \to \mathbb{R}$ it holds true that

$$A_f(a) = A_f(b) \text{ mod } I.$$

To see this we use that by Hadamard's lemma we can write

$$f(x) - f(y) = \sum_{i=1}^{n}(x_i - y_i)g_i(x, y)$$

for unique smooth functions $g_i\colon \mathbb{R}^{2m} \to \mathbb{R}$. Thus we get

$$A_f(a) - A_f(b) = \sum_{i=1}^{m}(a_i - b_i)A_{g_i}(a_1, \ldots, a_n, b_1, \ldots, b_n)$$

where the right hand side is in I because the $(a_i - b_i)$ are. This concludes the proof. □

In particular we can write the C^∞-ring $C^\infty(X)$ of smooth functions from a closed subset $X \subset \mathbb{R}^n$ into $\mathbb{R}$ as

$$C^\infty(X) \cong C^\infty(\mathbb{R}^n)/m_X^0,$$

where m_X^0 is the ideal of functions in $C^\infty(\mathbb{R}^n)$ that vanish on X. This works, because we can extend every smooth function on X to a smooth function on $\mathbb{R}^n$- not necessarily uniquely- but by taking the above quotient we identify different possibilities.
The above argument does not apply to the case of $U \subset \mathbb{R}^n$ open though because there are in general smooth functions in $C^\infty(U)$ that can not be extended to all of $\mathbb{R}^n$. Luckily we can

always find a closed subset $\tilde{U} \subset \mathbb{R}^{n+1}$ that is diffeomorphic to U. It is constructed as follows. By the proof of Lemma (2.1.3) we know that it exists a smooth function $f\colon \mathbb{R}^m \to \mathbb{R}$ such that $U = f^{-1}(\mathbb{R} - \{0\})$ i.e a smooth function that vanishes outside of U. Then we define

$$\tilde{U} = \{(x, y) \in \mathbb{R}^m \times \mathbb{R} | y \cdot f(x) = 1\} \subset \mathbb{R}^{m+1}.$$

which is clearly diffeomorphic to U.

Definition 2.1.8. *Let A be a C^∞-ring and Σ a set of elements of A. We call*

$$A \xrightarrow{\eta} A\{\Sigma^{-1}\}$$

where $A\{\Sigma^{-1}\}$ and η are a C^∞-object and a C^∞-morphism respectively, the universal solution to inverting the elements of Σ if

1. *for each $a \in \Sigma$ the element $\eta(a) \in A\{\Sigma^{-1}\}$ is invertible and*
2. *for every $\mathsf{C}^\infty\mathsf{Ring}$-morphism $\phi\colon A \to B$ such that for every $a \in \Sigma$ the element $\phi(a)$ is invertible we find a unique C^∞-ring morphims $\psi\colon A\{\Sigma^{-1}\} \to B$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A\{\Sigma^{-1}\} \\ & {\scriptstyle\phi}\searrow & \downarrow{\scriptstyle\psi} \\ & & B \end{array}$$

A useful consequence of the fact that η is defined to be a $\mathbb{C}^\infty$-ring-morphism is that

$$A\{\{a, b\}^{-1}\} \cong A\{(a \cdot b)^{-1}\}. \tag{2.2}$$

for two given elements a and b of A. Moreover we have the following results.

Lemma 2.1.9. *The universal solution to inverting $a \in C^\infty(\mathbb{R}^n)$ is given by*

$$C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1)$$

Proof. Define $\eta\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1)$ to be given by the composition of the obvious projection morphisms. Then $\eta(a)$ is invertible- simply because its inverse is explicitly given by $[y]$. We want to check the universal property for this η. For this end let $\phi\colon C^\infty(\mathbb{R}^n) \to B$ be a $\mathsf{C}^\infty\mathsf{Ring}$-morphism such that $\phi(a)$ is invertible. Then we define

$$\bar{\psi}\colon C^\infty(\mathbb{R}^{n+1}) \to B$$

as the $\mathsf{C}^\infty\mathsf{Ring}$-morphism such that $\bar{\psi}(f) = \phi(f)$ for every $f \in C^\infty(\mathbb{R}^{n+1})$ that does not depend on y and $\bar{\psi}(y) = \phi(a)^{-1}$. Clearly we have

$$\bar{\psi}(y \cdot a(x)) = \bar{\psi}(y) \cdot \bar{\psi}(a(x)) = 1$$

and thus $\bar{\psi}$ induces a morphism

$$\psi\colon C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1) \to B$$

and this is the unique morphism such that

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \xrightarrow{\eta} & C^\infty(\mathbb{R}^{n+1})/(y \cdot a(x) - 1) \\ & \overset{\phi}{\searrow} & \downarrow \psi \\ & & B \end{array}$$

commutes. □

Lemma 2.1.10. *Let $a \in C^\infty(\mathbb{R}^n)$ be a characteristic function for the open subset $U \subset R^n$ i.e $U = a^{-1}(\mathbb{R} - \{0\})$. Then*

$$C^\infty(\mathbb{R}^n)\{a^{-1}\} \cong C^\infty(U)$$

Proof. From the discussion above we know that it always exists a closed set $\tilde{U} \subset \mathbb{R}^{n+1}$ that is diffeomorphic to U. But then

$$C^\infty(U) \cong C^\infty(\tilde{U}) \cong C^\infty(\mathbb{R}^{n+1})/m^0_{\tilde{U}}$$

Now we can apply Lemma (2.2) from page 24 of [7] from which immediately follows that

$$m^0_{\tilde{U}} = (y \cdot a(x) - 1).$$

Now the statement follows with Lemma (2.1.9). □

We want to proof some results about certain coproducts in $\mathsf{C}^\infty\mathsf{Ring}$ and will denote the the coproduct of two $\mathsf{C}^\infty\mathsf{Ring}$-objects A and B as $A \otimes_\infty B$ if it exists.

Lemma 2.1.11. *Let A and B be two $\mathsf{C}^\infty\mathsf{Ring}$-objects such that their coproduct exists and $I \subset A$ and $J \subset B$ ideals. If we denote by (I, J) the ideal that is generated by $i_A(I) \cup i_B(J)$, where $i_A \colon A \to A \otimes_\infty B$ and $i_B \colon B \to A \otimes_\infty B$ are the canonical inclusions we have that also the coproduct $A/I \otimes_\infty B/J$ exists and that it is given by*

$$A \otimes_\infty B/(I, J)$$

Proof. Let

$$\begin{array}{ccc} A/I & & B/J \\ & \overset{\phi_A}{\searrow} \quad \overset{\phi_B}{\swarrow} & \\ & C & \end{array}$$

be another cone over the coequalizer diagram for $A/I \otimes_\infty B/J$. Then the compositions

$$A \xrightarrow{p_I} A/I \xrightarrow{\phi_A} C$$

and

$$B \xrightarrow{p_J} B/J \xrightarrow{\phi_B} C$$

give by the universal property of $A \otimes_\infty B$ a unique morphism $h\colon A \otimes_\infty B \to C$ such that the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_\infty B & \longleftarrow & B \\ {\scriptstyle p_I}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p_J} \\ A/I & \xrightarrow{\phi_A} & C & \xleftarrow{\phi_B} & B/J \end{array}$$

commutes. But because clearly $h(\iota_A(I)) = \phi_A(p_I) = 0$ and $h(\iota_B(J)) = \phi_B(p_J) = 0$ we get from this a unique morphism $\tilde{h}\colon A \otimes_\infty B/(I, J) \to C$ such that

$$\begin{array}{ccccc} A/I & \longrightarrow & A \otimes_\infty B/(I,J) & \longleftarrow & B/J \\ & {\scriptstyle \phi_A}\searrow & \downarrow{\scriptstyle \tilde{h}} & \swarrow{\scriptstyle \phi_B} & \\ & & C & & \end{array}$$

commutes. □

As an immediate consequence of the fact that $C^\infty(\mathbb{R}^m)$ and $C^\infty(\mathbb{R}^n)$ are the free C^∞-rings on m and respectively n generators we have that

$$C^\infty(\mathbb{R}^m) \otimes_\infty C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^m \times \mathbb{R}^n).$$

Using this and the previous Lemma we can conclude.

Corollary 2.1.12. *If I is an ideal in $C^\infty(\mathbb{R}^m)$ and J an ideal in $C^\infty(\mathbb{R}^n)$ we have that*

$$C^\infty(\mathbb{R}^m)/I \otimes_\infty C^\infty(\mathbb{R}^n)/J \cong C^\infty(\mathbb{R}^m \times \mathbb{R}^n)/(I, J)$$

Next we want to show, that given two smooth manifolds M and N the coproduct of the C^∞-ring $C^\infty(M)$ with $C^\infty(N)$ is given by $C^\infty(M \times N)$. First we will prove this in the case where $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are open subsets of a Cartesian space. To do so we exploit some of the results we proved above. Afterwards we generalize to arbitrary smooth manifolds.

Lemma 2.1.13. *For $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ the coproduct in $\mathsf{C^\infty Ring}$ of $C^\infty(U)$ with $C^\infty(V)$- which we denote by $C^\infty(U) \otimes_\infty C^\infty(V)$- is given by $C^\infty(U \times V)$*

Proof. Let $a\colon \mathbb{R}^m \to \mathbb{R}$ and $b\colon \mathbb{R}^n \to \mathbb{R}$ be characteristic functions for U and V respectively. Then we can make the following manipulations

$$\begin{aligned} & C^\infty(U) \otimes_\infty C^\infty(V) \\ \overset{\heartsuit}{\cong} & C^\infty(\mathbb{R}^{m+1})/(y_1 \cdot a(x_1) - 1) \otimes_\infty C^\infty(\mathbb{R}^{n+1})/(y_2 \cdot b(x_2) - 1) \\ \overset{\spadesuit}{\cong} & C^\infty(\mathbb{R}^{m+n+2})/((y_1 \cdot a(x_1) - 1, (y_2 \cdot b(x_2) - 1) \\ \overset{\clubsuit}{\cong} & C^\infty(\mathbb{R}^{m+n})\{\{a, b\}^{-1}\} \\ \overset{\diamondsuit}{\cong} & C^\infty(\mathbb{R}^{m+n})\{(a \cdot b)^{-1}\} \\ \overset{\square}{\cong} & C^\infty(U \times V). \end{aligned}$$

Lets go through each equation seperately.

♡ This follows from Lemma (2.1.10) and the fact, that $(y_1 \cdot a(x_1) - 1) = m_{\tilde{U}}$ (respectively $(y_2 \cdot b(x_2) - 1) = m_{\tilde{V}}$), shown in above Lemma's proof.

♠ Is a direct consequence of Corollary (2.1.12).

♣ Uses Lemma (2.1.9).

♢ Is a simple application of (2.2).

□ Finally follows from Lemma (2.1.10) because clearly $a \cdot b$ is a characteristic function of $U \times V$.

□

Proposition 2.1.14. *Let M and N be two smooth manifolds. The coproduct in* $\mathsf{C^\infty Ring}$ *of $C^\infty(M)$ with $C^\infty(N)$ is given by $C^\infty(M \times N)$.*

Proof. We realize M and N as submanifolds of some $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then we can always find open neighborhoods $M \subset U \subset \mathbb{R}^m$ and $N \subset V \subset \mathbb{R}^n$ such that there are smooth retractions $r_U : U \to M$ and $r_V : V \to N$. A proof that these open neighborhoods and retractions actually exist can be found on page (69-70) of [3]. The above retractions induce a retraction $\rho \colon U \times V \to M \times N$. We denote the relevant inclusions by $i_M : M \to U$ and $i_N : N \to V$ and $\eta \colon M \times N \to U \times V$. We can take the dual morphisms in $\mathsf{C^\infty Ring}$ and summarize our findings in the following commutative diagram

$$\begin{array}{ccccc}
C^\infty(U) & \xrightarrow{p_U^*} & C^\infty(U \times V) & \xleftarrow{p_V^*} & C^\infty(V) \\
{\scriptstyle i_M^*}\downarrow\uparrow{\scriptstyle r_U^*} & & {\scriptstyle \eta^*}\downarrow\uparrow{\scriptstyle \rho^*} & & {\scriptstyle i_N^*}\downarrow\uparrow{\scriptstyle r_V^*} \\
C^\infty(M) & \xrightarrow[p_M^*]{} & C^\infty(M \times N) & \xleftarrow[p_N^*]{} & C^\infty(N)
\end{array}$$

We will now show that the lower horizontal part of the diagram is a coproduct. Let $\phi_M : C^\infty(M) \to C^\infty(M \times N)$ and $\phi_N : C^\infty(N) \to C^\infty(M \times N)$ be $\mathsf{C^\infty Ring}$-morphisms. Then because the upper horizontal part is a coproduct due to Lemma (2.1.13) we get a unique morphism $\xi \colon C^\infty(U \times V) \to A$ such that

$$\phi_M \circ i_M^* = \xi \circ p_U^*$$

and

$$\phi_N \circ i_N^* = \xi \circ p_V^*.$$

But then by defining $\omega := \xi \circ \rho^*$ we get

$$\omega \circ p_M^* = \xi \circ \rho^* \circ p_M^* = \xi \circ p_U^* \circ r_U^* = \phi_M \circ i_M^* \circ r_U^* = \phi_M$$

and similar

$$\omega \circ p_N^* = \xi \circ \rho^* \circ p_N^* = \xi \circ p_V^* \circ r_V^* = \phi_N \circ i_N^* \circ r_V^* = \phi_N.$$

Moreover we can easily show uniqueness. If $\tilde{\omega}\colon C^\infty(M \times N) \to A$ is another morphism such that $\tilde{\omega} \circ p_M^* = \phi_M$ and $\tilde{\omega} \circ p_N^* = \phi_N$ we have that $\tilde{\omega} \circ \eta^*$ satisfies

$$\tilde{\omega} \circ \eta^* \circ p_U^* = \tilde{\omega} \circ p_M^* \circ i_M^* = \phi_M \circ i_M^*$$

and

$$\tilde{\omega} \circ \eta^* \circ p_V^* = \tilde{\omega} \circ p_N^* \circ i_N^* = \phi_N \circ i_N^*.$$

Thus we have $\tilde{\omega} \circ \eta^* = \xi$ and

$$\omega = \xi \circ \rho^* = \tilde{\omega} \circ \eta^* \circ \rho^* = \tilde{\omega}$$

□

2.1.3 Weil algebras

Definition 2.1.15. *A Weil algebra is defined to be a real, unital and commutative algebra W that is local with nilpotent, maximal ideal I such that $W/I \cong \mathbb{R}$. Moreover it is required that W is finite dimensional as a vector space.*

From this definition one directly sees that we can write $W = \mathbb{R} \oplus I$. Thus every element $w \in W$ may be written as $w = \underline{w} + \hat{w}$ where $\underline{w} \in \mathbb{R}$ and $\hat{w}$ is nilpotent.

Example 2.1.16. *An example for a Weil algebra are the so called dual numbers. $\mathbb{R}[\epsilon]/\epsilon = \mathbb{R} \oplus \epsilon\mathbb{R}$ with the product $(a + \epsilon b)(a' + \epsilon b') = aa' + \epsilon(ab' + ba')$. Or in other words $\epsilon^2 = 0$.*

The algebra structure of a Weil algebra can be extended to that of a C^∞-Ring. We will prove the following even stronger statement here.

Proposition 2.1.17. *Let W be a Weil algebra and B a C^∞-Ring. Then it exists a unique C^∞-Ring structure on $B \otimes_\mathbb{R} W$ that extends its $\mathbb{R}$-algebra structure such that the canonical inclusion $j\colon B \to B \otimes_\mathbb{R} W$ becomes a $\mathsf{C^\infty Ring}$ morphism.*

Objects of the form $B \otimes_\mathbb{R} W$ will be interesting for us when we construct the category of generalized smooth spaces and the Cahiers topos in the following section. It is clear, that by taking $B = \mathbb{R}$ we obtain a C^∞-ring structure on the Weil algebra W.
To prove the above statement we again employ Hadamards lemma that we already discussed before. Explicitely we use it to show the following statement.

Lemma 2.1.18. *Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $k \geq 0$ a positive integer. Then there exist unique smooth functions $\phi_\alpha \in C^\infty(\mathbb{R}^n)$ and $\psi_\beta(\mathbb{R}^{2n})$ such that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have*

$$\phi(x + y) = \sum_{|\alpha| \leq k} \phi_\alpha(x) y^\alpha + \sum_{|\beta| = k+1} \psi_\beta(x, y) y^\beta. \tag{2.3}$$

Here α and β are multi indices in n letters.

Proof. The proof is by an iterated application of Hadamards lemma. First we use it to write

$$\phi(x+y) = \phi(x) + \sum_{i=1}^{n} y^i g_i(x+y). \tag{2.4}$$

If k was equal to 0 we are done because with $\phi_0(x) = \phi(x)$ and with identifying g_i as the ψ_β with a 1 in the i-th letter of the multi index and zeros elsewhere (2.4) is the desired expansion. Else we have to apply Hadamards lemma to the g_i. We have to iterate this k times to arrive at the desired expansion. □

We want to understand the functions $\phi_\alpha \in C^\infty(\mathbb{R}^n)$ and $\psi_\beta(\mathbb{R}^{2n})$ better. For this end we will discuss the special case of $n = 1$ first.
We start by simply defining $g_0 := \phi$. The first application of Hadmards lemma to the smooth $\phi\colon \mathbb{R} \to \mathbb{R}$ gives us

$$\phi(x) = g_0(a) + (x-a)g_1(x)$$

for

$$g_1(x) = \int_0^1 \phi'(a + t_1(x-a))dt_1$$

where $\phi'(x) = \frac{d}{dt}\phi(x)$. Next we apply Hadmards lemma to g. This gives us

$$\phi(x) = g_0(a) + (x-a)(g_1(a) + (x-a)g_2(x)).$$

We can extrapolate this and find for an arbitrary positive integer k that

$$\phi(x) = \sum_i (x-a)^i g_i(a) + (x-a)^{k+1} g_{k+1}(x). \tag{2.5}$$

Lets try to understand g_i for the next i's.

$$\begin{aligned} g_1(x) =& \int_0^1 \frac{d}{ds}\left(\int_0^1 \phi'(a + t_1(s-a))dt_1\right)\bigg|_{s=a+t_1(x-a)} dt_2 \\ =& \int_0^1 \int_0^1 t_1 \phi^{(2)}(a + t_1(a + t_2(x-a) - a))dt_1 dt_2. \end{aligned}$$

In particular we have

$$\begin{aligned} g_1(a) =& \int_0^1 \int_0^1 t_1 \phi^(2)(a + t_1(a + t_2(a-a) - a))dt_1 dt_2 \\ =& \int_0^1 \int_0^1 t_1 \phi^{(2)}(a))dt_1 dt_2 \\ =& \frac{1}{2}\phi^{(2)}(a). \end{aligned}$$

Similar

$$g_3(x) = \int_0^1 \frac{d}{ds}\left(\int_0^1 \int_0^1 t_1\phi^{(2)}(a + t_1(a + t_2(s-a) - a))dt_1 dt_2\right)\Bigg|_{s=a+t_3(x-a)} dt_3$$
$$= \int_0^1 \int_0^1 \int_0^1 t_1^2 t_2 \phi^{(3)}(a + t_0(a + t_2(a + t_3(x-a) - a) - a))$$

and in particular

$$g_3(a) = \int_0^1 \int_0^1 \int_0^1 t_1^2 t_2 \phi^{(3)}(a) dt_1 dt_2 dt_3$$
$$= \frac{1}{2 \cdot 3}\phi^{(3)}(a).$$

We now see where this leads to. The expansion (2.5) becomes

$$\phi(x) = \sum_{i=0}^{k} \phi^{(i)}(a)(x-a)^i + g^{k+1}(x)(x-a)^{k+1}$$

or renaming x to y and a to x we can plug $x+y$ into the above expansion and get

$$\phi(x+y) = \sum_{i=0}^{k} \phi^{(i)}(x)y^i + g^{k+1}(x)y^{k+1}.$$

The above can be generalized to the case of $n \geq 1$ easily. Due to Schwarz's theorem multiple equal terms appear as soon as derivatives with respect to different variables are involved. More concretely there are $\frac{|\alpha|!}{\alpha!}$ terms with $\partial_\alpha \phi$, where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$ and $|\alpha| = \alpha_1 + \ldots \alpha_n$. Thus the total prefactor of this derivative is

$$\frac{|\alpha|!}{\alpha!}\frac{1}{|\alpha|!} = \frac{1}{\alpha!}.$$

Lemma (2.1.18) is sufficient to proof Proposition (2.1.17).

Proof. (of Proposition 2.1.17) We can write the Weil algebra W as $W = \mathbb{R} \oplus W'$ where every element in W' is nilpotent. Then there exists some $k \geq 0$ such that the product of any $k+1$ elements in W' is zero.
We equip the $\mathbb{R}$-tensor product $B \otimes_{\mathbb{R}} W$, where B is a C^∞-ring with its canonical R-algebra structure and make the following calculation

$$B \otimes_{\mathbb{R}} W = B \otimes_{\mathbb{R}} (\mathbb{R} \oplus W')$$
$$= B \oplus (B \otimes_{\mathbb{R}} W').$$

Note that also the product of any $k+1$ elements in $B \otimes_{\mathbb{R}} W'$ is zero.
Now let $\phi\colon \mathbb{R}^m \to \mathbb{R}^n$. Due to Lemma (2.1.31) we can write

$$(\phi(x+y))^i = \sum_{|\alpha| \leq k} (\phi_\alpha(x))^i y^\alpha + \sum_{|\beta|=k+1} (\psi_\beta(x,y))^i y^\beta$$

for unique $\phi^i_\alpha \in C^\infty(\mathbb{R}^m)$ and $\psi^i_\beta \in C^\infty(\mathbb{R}^{2m})$. Let $r_i = j(x_i) + y_i$ be an element in $T := B \otimes_{\mathbb{R}} W = B \oplus (B \otimes_{\mathbb{R}} W')$, where $x_i \in B$ and $y_i \in B \otimes_{\mathbb{R}} W'$ and $j \colon B \to B \otimes W$ is the canonical morphism that sends $b \in B$ to $j(b) = b \otimes 1$.
Now assume that T indeed has a C^∞-structure that extends its $\mathbb{R}$-algebra structure . Then it is of the form

$$T_\phi(r) = T_\phi(j(x) + y) = \sum_{|\alpha| \le k} T_{\phi_\alpha}(j(x)) y^\alpha + \sum_{|\beta| = k+1} T_{\psi_\beta}(j(x), y)) y^\beta$$

We can use the nilpotency of the $y_i's$ to get

$$T_\phi(r) = \sum_{|\alpha| \le k} T_{\phi_\alpha}(j(x)) y^\alpha = \sum_{|\alpha| \le k} j(B_{\phi_\alpha}(x)) y^\alpha, \tag{2.6}$$

where B_{ϕ_α} is the C^∞-ring operation of ϕ_α in B. Thus we have uniqueness of the extension if it exists.
We still have to check if this really defines a C^∞-structure. Indeed, if $f \colon \mathbb{R}^m \to \mathbb{R}^n$ and $g \colon \mathbb{R}^n \to \mathbb{R}^d$ are smooth we get

$$\begin{aligned}(T_g(T_f(j(x) + y)))_i =& T_g \left(\sum_{|\alpha| \le k} j(B_{f^t_\alpha}(x)) y^\alpha \right) \\ =& \sum_{|\alpha'| \le k} j(B_{g^i_{\alpha'}}(B_{f^t}(x))) \left(\sum_{1 \le |\alpha| \le k} j(B_{f^{\alpha'}_\alpha}(x)) y^\alpha \right) \\ =& \sum_{|\alpha'| \le k} j(B_{(g \circ f)^i_\alpha}) y^\alpha,\end{aligned}$$

where in the last step we used that the $g_{\alpha'}$ and the f_α correspond to partial derivatives like discussed above, the chain rule, that $j(x) \cdot j(x') = j(x \cdot x')$ and finally that we already know, that B is a C^∞-Ring.
For $n \ge 0$ we have

$$\begin{aligned}(T_{\mathrm{id}_{\mathbb{R}^n}}(j(x) + y))^i =& \sum_{|\alpha| \le k} j(B_{(\mathrm{id}^i_{\mathbb{R}^n})_\alpha}(x)) y^\alpha \\ =& j(x^i) + y^i,\end{aligned}$$

because $(\mathrm{id}^i_{\mathbb{R}^n})_0 = \mathrm{id}^i_{\mathbb{R}^n}$, $(\mathrm{id}^i_{\mathbb{R}^n})_{(0,\ldots,0,1,0,\ldots,0)} = 1$ where the 1 is in the i'th entry of the multi index, all the other summands in the expansion are zero and also do we already know that B is a C^∞-Ring. Similar we have

$$T_{\pi_i}(j(x) + y) = \sum_{|\alpha| \le k} j(B_{(\pi_i)_\alpha}(x)) y^\alpha = j(x^i) + y^i.$$

This concludes the proof. □

With this at hand we can in particular understand the C^∞-ring structure of a Weil algebra.

Example 2.1.19. *Given any Weil algebra W it admits a unique extension of its algebra structure to a C^∞-ring structure. Like already mentioned above this is a special case of proposition (2.1.17) where we just take $B = \mathbb{R}$ and get $B \otimes_{\mathbb{R}} W \cong W$. But then for $f\colon \mathbb{R}^m \to \mathbb{R}^n$ from (2.6) we find that*

$$\begin{aligned} W_f(j(x)+y) &= \sum_{|\alpha|\leq k} j(\mathbb{R}_{f_\alpha}(x))y^\alpha \\ &= \sum_{|\alpha|\leq k} \frac{1}{\alpha!} j(\partial^\alpha f(x))y^\alpha \end{aligned} \tag{2.7}$$

In other words is $W_f(j(x)+y)$ a Taylor expansion of f at x in the nilpotents y up to order k. As a special case of this we have the dual numbers from example (2.1.16). For them (2.7) terminates at the first order.

Furthermore we have the following result, concerning the role that $B \otimes_{\mathbb{R}} W$ plays in the category $\mathsf{C^\infty Ring}$.

Proposition 2.1.20. *If B is any C^∞-Ring and W is a Weil algebra that is equipped with a C^∞-structure like in example (2.1.19) then the coproduct $B \otimes_\infty W$ in $\mathsf{C^\infty Ring}$ is given by $B \otimes_{\mathbb{R}} W$ together with the morphisms*

$$\begin{aligned} j\colon B &\to B \otimes_{\mathbb{R}} W \\ b &\mapsto b \otimes 1 \end{aligned}$$

and

$$\begin{aligned} k\colon W &\to B \otimes_{\mathbb{R}} W \\ w &\mapsto 1 \otimes w \end{aligned}$$

Proof. Let (A, ψ_B, ψ_W) be another cocone over the coproduct diagram. Let $\sum_{i=1}^n b_i \otimes w_i$ be any element in $B \otimes_{\mathbb{R}} W$ and assume that there is a morphisms $u\colon B \otimes_{\mathbb{R}} W \to A$ such that $\psi_B = u \circ j$ and $\psi_W = u \circ k$. We have

$$\begin{aligned} u(\sum_{i=1}^n b_i \otimes w_i) &= u\left(\sum_{i=1}^n j(b_i)k(w_i)\right) \\ &= \sum_{i=1}^n u(j(b_i))u(k(w_i)) \\ &= \sum_{i=1}^n \psi_B(b_i)\psi_W(w_i). \end{aligned}$$

If we thus define

$$\begin{aligned} u\colon B \otimes_{\mathbb{R}} W &\to A \\ \sum_{i=1}^n b_i \otimes w_i &\mapsto \sum_{i=1}^n \psi_B(b_i)\psi_W(w_i), \end{aligned}$$

we found the unique $\mathsf{C^\infty Ring}$-morphism that was demanded by the universal property of the coproduct. □

Let us prove some further results about Weil algebras that we will need later on.

Definition 2.1.21. *Let M be a smooth manifold. An $\mathbb{R}$-algebra morphism ϕ with domain $C^\infty(M)$ has local character at $p \in M$ if for every $f \in C^\infty(M)$ that vanishes on a neighborhood of p we have $\phi(f) = 0$.*

Proposition 2.1.22. *Let M be a paracompact smooth manifold and $W = \mathbb{R} \oplus W'$ a Weil algebra. Then any $\mathbb{R}$-algebra morphism $\phi\colon C^\infty(M) \to W$ is*

1. *Of the form $f \mapsto f(p) + \phi_1(f)$ for a unique $p \in M$ and $\phi_1\colon C^\infty(R^n) \to W'$.*
2. *Of local character at p.*

Proof.

1. Denote by $\pi_0\colon W \to \mathbb{R}$ the projection onto the first summand in $W = \mathbb{R} \oplus W'$. Then due to Milnor's exercise the composition $\pi_0 \circ \phi\colon C^\infty \to \mathbb{R}$ is evaluation at a unique point $p \in M$. By defining

$$\phi_1(f) = \phi(f) - f(p)$$

we get $\phi(f) = f(p) + \phi_1(f)$.

2. Let $f, g \in C^\infty(M)$. Then

$$\phi(fg) = \phi(f)\phi(g) = f(p)g(p) + \phi_1(f)g(p) + \phi_1(g)f(p) + \phi_1(f)\phi_1(g)$$

and thus

$$\phi_1(fg) = \phi_1(f)g(p) + \phi_1(g)f(p) + \phi_1(f)\phi_1(g).$$

In particular if $f(p) = g(p) = 0$ we find

$$\phi_1(fg) = \phi_1(f)\phi_1(g)$$

and moreover

$$\phi_1(g^k) = \phi_1(g)^k$$

for every $k \in \mathbb{N}$. Now let $f \in C^\infty(M)$ such that it vanishes on a neighborhood $U \subset M$ of p. We can chose $g \in C^\infty(M)$ such that $g(p) = 0$ and $g|_{M\setminus U} = 1$. This is simply the smooth cutoff function for the closed set $M\setminus U$ and the open set $M\setminus\{p\}$ whose existence is guaranteed by a partition of unity argument. With this choices we have $f = fg^k$ and thus

$$\phi_1(f) = \phi_1(f)\phi_1(g)^k$$

which is zero for k big enough due to the nilpotency of $\phi_1(g)$ which is after all in W'.

□

Corollary 2.1.23. *For $U \subset \mathbb{R}^n$ an open subset and W a Weil algebra of dimension k the set of $\mathbb{R}$-algebra morphisms $C^\infty(U) \to W$ is in bijective correspondence with*

$$\{(x_1 + \hat{x_1}, \dots, x_n + \hat{x_n}) \in \mathbb{R}^{n(k+1)} | (x_1, \dots x_n) \in U\}.$$

Proof. An element of $w \in W$ is given by specifying the $k+1$ coordinates $a_0, \dots, a_k$. We can then write $w = a_0 + a_1 w_1 + \dots + a_k w_k$ where $w_1, \dots, w_k$ is a linear basis of W' and $W + \mathbb{R} \oplus W'$. We will denote the element $(a_0, \dots a_k)$ that corresponds to $w \in W$ by $\overline{w}$.

Let $\phi\colon C^\infty(U) \to W$ be a $\mathbb{R}$-algebra morphism. We assign to it the point

$$(\overline{\phi_1(p_1)}, \dots, \overline{\phi_n(p_n)}) \in \mathbb{R}^{n(k+1)}.$$

Due to the previous proposition ϕ is of the form $\phi\colon f \mapsto f(q) + \phi_1(f)$ for a unique $q \in U$ and thus $(\overline{\phi_1(p_1)}, \dots, \overline{\phi_n(p_n)})$ is in $\{(x_1 + \hat{x_1}, \dots, x_n + \hat{x_n}) \in \mathbb{R}^{n(k+1)} | (x_1, \dots x_n) \in U\}$. On the other hand, given a point $(x_1 + \hat{x_1}, \dots, x_n + \hat{x_n}) \in \mathbb{R}^{n(k+1)}$ we can use that $C^\infty(\mathbb{R}^n)$ is the free C^∞-ring on n generators to define a unique $\mathbb{R}$-algebra morphism $\phi\colon C^\infty(\mathbb{R}^n) \to W$. For this end we set $\phi(p_i) = x_1 + \hat{x_1}^1 w_1 + \dots \hat{x_1}^k w_k$. We can extend ϕ to a morphism $\varphi\colon C^\infty(U) \to W$ as follows. Let $V \subset U$ be an open neighborhood of the point $(x_1, \dots, x_n)$ such that the closure $\overline{V}$ is also contained in U. Then we can define a cutoff function $c\colon \mathbb{R}^n \to \mathbb{R}$ such that $c(x) = 1$ for all $x \in V$ and $\operatorname{supp}(c) = 1$. We define

$$\begin{aligned} \tilde{\phi}\colon C^\infty(U) &\to W \\ \tilde{\phi}(f) &= \phi(cf) \end{aligned}$$

where cf is to be understood as the extension of $(c|_U)f$ to $\mathbb{R}^n$ that is zero outside of U. This definition does not depend on he choice of cutoff function because ϕ is of local character at $(x_1, \dots, x_n)$ due to the previous proposition. □

We will need all this results about C^∞-rings and Weil algebras in the next section to construct our well adapted model for synthetic differential geometry. One can think of the following proposition as a motivation why this might actually work out.

Proposition 2.1.24. *$Hom_{\mathbb{R}\mathsf{Alg}}(C^\infty(M), \mathbb{R}_\epsilon)$ and TM are equal to each other as sets.*

Proof. Given an $\mathbb{R}$-algebra morphism $\varphi\colon C^\infty(M) \to \mathbb{R}_\epsilon$ we can compose it with the projection

$$\pi\colon \mathbb{R}_\epsilon \to \mathbb{R} \tag{2.8}$$

$$(x + y\epsilon) \mapsto x \tag{2.9}$$

to get a morphism from $C^\infty(M)$ to $\mathbb{R}$. By Milnor's exercise this is evaluation at a point $x \in M$ and thus

$$\varphi(f) - f(x) \cdot 1 = X(f) \cdot \epsilon.$$

But

$$\begin{aligned} X(fg) \cdot \epsilon =& \varphi(fg) - f(x)g(x) \cdot 1 \\ =& f(x)g(x) + X(f)g(x) \cdot \epsilon + f(x)X(g) \cdot \epsilon + f(x)g(x) - f(x)g(x) \\ =& (X(f)g(x) - f(x)X(g)) \cdot \epsilon. \end{aligned}$$

Thus X is a derivation at the point $x \in M$. □

This is promising. In the end we want to obtain a category that includes both the smooth manifolds but also mapping spaces between them and their respective tangent spaces as objects. The construction of this category is subject to the next section and will be based on what we have learned above about $\mathsf{C^\infty Ring}$

2.1.4 The Cahiers topos

The objects of $\mathsf{C^\infty Ring}$ that we are interested in are the $C^\infty(M)$ for a smooth manifold M and the dual numbers $\mathbb{R}_\epsilon$ that we will need to construct the tangent bundle like was teased above in proposition (2.1.24). To include all these we will consider the full subcategory of $\mathsf{C^\infty Ring}$ that consists of objects of the form $C^\infty(M) \otimes_\infty W = C^\infty(M) \otimes_\mathbb{R} W$, where M is a smooth manifold and W is a Weil algebra. Clearly these objects are a special case of the objects $B \otimes_\mathbb{R} W$ for an arbitrary C^∞-ring B that we discussed extensively above. Because $C^\infty(*) \cong \mathbb{R}$ we can choose M to be a manifold with one point and obtain the Weil algebra W as the object $W \cong C^\infty(*) \otimes_\infty W$. On the other hand is $\mathbb{R}$ a Weil algebra and thus we also have $C^\infty(M) \cong C^\infty(M) \otimes_\infty \mathbb{R}$.
We have to account for the fact that manifolds embed by corollary (2.1.5) full and faithful into $\mathsf{C^\infty Ring^{op}}$ rather than $\mathsf{C^\infty Ring}$. Thus we will consider the opposite category of the full subcategory from above.

Definition 2.1.25. *Take the full subcategory of* $\mathsf{C^\infty Ring}$ *that has objects of the form* $C^\infty(M) \otimes_\infty W$*, where* M *is a smooth manifold and* W *is a Weil algebra. The opposite category of this subcategory is denoted as* FMan*. We call it the category of generalized smooth spaces or also the category of formal manifolds.*

We will denote the objects of FMan by $M \times lW$. This is consistent as the coproducts in $\mathsf{C^\infty Ring}$ become products in FMan and the manifold M is identified with $C^\infty(M) \in \mathsf{Ob}(\mathsf{C^\infty Ring})$. We write ℓW for W as an object in $\mathsf{C^\infty Ring^{op}}$. This ℓW is also called the locus of the Weil algebra W.

The Cahiers topos will be defined as a Grothendieck topos on FMan. Of course the definition of a Grothendieck topos requires that we equip FMan with a Grothendieck topology. This is the next step in our construction. Denote by $\{\rho_i \colon U_i \to M\}$ a covering family of M in the coverage with open sets. We can define a coverage on FMan by declaring

$$\{\rho_i \times \mathrm{id} \colon U_i \times lW \to M \times lW\} \tag{2.10}$$

to be a covering family of $M \times lW$ for every covering family $\{\rho_i \colon U_i \to M\}$ of M. We should specify what exactly we mean with this. By the full and faithful embedding $\mathsf{Mf} \to \mathsf{C^\infty Ring^{op}}$ each of the ρ_i correspond to precisely one morphism from $C^\infty(M)$ to $C^\infty(U_i)$ in $\mathsf{C^\infty Ring}$ or in other words one morphism from $C^\infty(U_i)$ to $C^\infty(M)$ in $\mathsf{C^\infty Ring^{op}}$. We denote this morphism in $\mathsf{C^\infty Ring^{op}}$ also by ρ_i. Then $\rho_i \times \mathrm{id}$ is the morphism in FMan that corresponds to the morphism $C^\infty(\rho_i) \otimes \mathrm{Id}_W \colon C^\infty(M) \otimes_\infty W \to C^\infty(U_i) \otimes_\infty W$ in $\mathsf{C^\infty Ring}$.
We still have to show that this actually defines a coverage.

Proposition 2.1.26. *The assignment of covering families*

$$M \times \ell W \mapsto \{\{\phi_i \times id \colon U_i \times \ell W \to M \times \ell W\} | \{\phi_i \colon U_i \to M\} \textit{is an open covering}\}$$

defines a coverage on FMan*.*

Proof. Let $\{\phi_i \times \mathrm{id}\colon U_i \times \ell W \to M \times \ell W\}$ be a covering family and $g\colon N \times \ell V \to M \times \ell W$ a morphism of generalized smooth spaces. The composite

$$C^\infty(M) \to C^\infty(M) \otimes W \to C^\infty(N) \times V \to C^\infty(N),$$

where the last morphism is induced by $\pi_0\colon W \to \mathbb{R}$, is equivalently a smooth map $f\colon N \to M$. But by definition is $\{\phi_i\colon U_i \to M\}$ an open covering of M and thus it exists an open covering $\{\psi_j\colon S_j \to N\}$ such that for every S_j it exists an U_i and a morphism $u_j\colon S_j \to U_i$ that makes the diagram

$$\begin{array}{ccc} S_j & \xrightarrow{u_j} & U_i \\ {\scriptstyle\psi_j}\downarrow & & \downarrow{\scriptstyle\phi_i} \\ N & \xrightarrow[f]{} & M \end{array} \tag{2.11}$$

commute. Again by the definition of the covering families on FMan we have that $\{\psi_j \times \mathrm{id}\colon S_j \times \ell V \to N \times \ell V\}$ is a covering family of $N \times \ell V$. We claim that it exists a dashed morphism that makes

$$\begin{array}{ccc} S_j \times \ell V & \dashrightarrow & U_i \times \ell W \\ {\scriptstyle\psi_j \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle\phi_i} \\ N \times \ell V & \xrightarrow[g]{} & M \times \ell W \end{array}$$

commute. We can equivalently show that it exists a dashed morphism of C^∞-rings that makes

$$\begin{array}{ccc} C^\infty(M) \otimes W & \xrightarrow{g^*} & C^\infty(N) \otimes V \\ {\scriptstyle\phi_i^* \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle\psi_j^* \otimes \mathrm{id}} \\ C^\infty(U_i) \otimes W & \dashrightarrow & C^\infty(S_j) \otimes V \end{array}$$

commute.

To show this we choose a linear basis $1_W, w_1, \ldots, w_k$ of W and likewise a linear basis $1_V, v_1, \ldots, v_l$ of V. Then each of the w_t is send to

$$b_t = b_{t,0} \otimes 1_V + \sum_{p=1}^{l} b_{t,p} \otimes v_p \in C^\infty(S_j) \otimes V$$

by the composite $(\psi_j^* \otimes \mathrm{id}) \circ g^*$. Note that the smooth functions $b_{t,0}, \ldots, b_{t,l}$ are obtained by restricting smooth functions $B_{t,0}, \ldots, B_{t,l} \in C^\infty(N)$ along ψ_j because

$$(\psi_j^* \otimes \mathrm{id}) \left(B_{t,0} \otimes 1_V + \sum_{p=1}^{l} B_{t,p} \otimes v_p \right) = \psi_j^*(B_{t,0}) \otimes 1_V + \sum_{p=1}^{l} \psi_j^*(B_{t,p}) \otimes v_p.$$

The C^∞-ring $C^\infty(S_j)$ together with the morphism

$$b\colon W \to C^\infty(S_j) \otimes V$$
$$b(w_p) = b_p$$

and the composite

$$(\mathrm{id} \otimes 1_V) \circ u_j * \colon C^\infty(U_i) \to C^\infty(S_j) \otimes V$$

define a cocone over the coproduct diagram and thus by the universal property of coproducts we get the unique morphism

$$g_j^* \colon C^\infty(U_i) \otimes W \to C^\infty(S_j) \otimes V.$$

But for $a_0 \otimes 1_W + \sum_{p=1}^{l} a_p \otimes w_p \in C^\infty(M) \otimes W$ we get

$$\begin{aligned}
& g_j^* \left((\phi_i^* \otimes \mathrm{id}) \left(a_0 \otimes 1_W + \sum_{p=1}^{l} a_p \otimes w_p \right) \right) \\
=& g_j^* \left(\phi_i^*(a_0) \otimes 1_W + \sum_{p=1}^{k} \phi_i^*(a_p) \otimes w_p \right) \\
=& g_j^* \left(i_{C^\infty(U_i)}(\phi_i^*(a_0)) i_W(1_W) + \sum_{p=1}^{k} i_{C^\infty(U_i)}(\phi_i^*(a_p)) i_W(w_p) \right) \\
\overset{\heartsuit}{=}& ((\mathrm{id} \otimes 1_V) \circ u_j^*)(\phi_i^*(a_0))(((\psi_i^* \otimes \mathrm{id}) \circ g^*)(1_W)) \\
& + \sum_{p=1}^{k} ((\mathrm{id} \otimes 1_V) \circ u_j^*)(\phi_i^*(a_p))(((\psi_i^* \otimes \mathrm{id}) \circ g^*)(w_p)) \\
\overset{\spadesuit}{=}& (\psi_i^* \otimes \mathrm{id}) \circ g^*) \left(a_0 \otimes 1_W + \sum_{p=1}^{l} a_p \otimes w_p \right),
\end{aligned}$$

where $i_{C^\infty(U_i)}$ and i_W are the canonical inclusions of $C^\infty(U_i)$ and W into $C^\infty(U_i) \otimes_{\mathbb{R}} W$ respectively. For $\heartsuit$ we used that g_j^* comes is induced via the universal property of the coproduct and $\spadesuit$ follows from the commutativity of (2.11). □

Definition 2.1.27. *The Cahiers topos* C *is defined to be the category of sheaves on* FMan *with respect to this coverage.*

$$\mathsf{C} = \mathsf{Sh}(\mathsf{FMan})$$

The next proposition is evidently crucial if we want to describe smooth manifolds and Weil algebras in the Cahiers topos.

Proposition 2.1.28. *The so defined site is subcanonical or in other words every representable preheaf is a sheaf with respect to the coverage defined above.*

Proof. We want to show that given some $t = M \times lW \in \mathsf{Ob}(\mathsf{FMan})$ the presheaf represented by it $\iota(t) = \mathsf{Hom}_{\mathsf{FMan}}((-), t) = \mathsf{Hom}_{\mathsf{C^\infty Ring}}(C^\infty \otimes_{\mathbb{R}} W, (-))$ is already a sheaf. We thus have to show that given a covering family

$$\{\rho_i \times \mathrm{Id}_{lV} \colon U_i \times lV \to U \times lV\}$$

the diagram

$$\mathsf{Hom}(t, C^\infty(U)\otimes_{\mathbb{R}} V) \longrightarrow \prod_{i\in I}\mathsf{Hom}(t, C^\infty(U_i)\otimes_{\mathbb{R}} V) \rightrightarrows \prod_{i,j\in I}\mathsf{Hom}(t, C^\infty(U_i\cap U_j)\otimes_{\mathbb{R}} V)$$

is an equalizer. But because the hom-functor preserves limits in the second argument (see Proposition (A.1.2) of the Appendix) it suffices to show that

$$C^\infty(U)\otimes_{\mathbb{R}} V \longrightarrow \prod_{i\in I} C^\infty(U_i)\otimes_{\mathbb{R}} V \rightrightarrows \prod_{i,j\in I} C^\infty(U_i\cap U_j)\otimes_{\mathbb{R}} V$$

is an equalizer. So lets proof this. Let $a\colon \bar{t}\to\prod_{i\in I} C^\infty(U_i)\otimes_{\mathbb{R}} W'$ be another cone. A $\mathsf{C^\infty Ring}$-morphism $h\colon \bar{t}\to C^\infty(U)\otimes_{\mathbb{R}} V$ such that $\left(\prod_{i\in I} C^\infty(\rho_i)\otimes_\infty \mathrm{Id}_W\right)\circ h = a$ has to satisfy

$$\left(\prod_{i\in I} C^\infty(\rho_i)\otimes_\infty \mathrm{Id}_W\right)(h(\bar{f}\in\bar{t})) = a(\bar{f}). \tag{2.12}$$

But choosing a linear basis $(1_V, v_1,\ldots,v_k)$ of V we can write $h(\bar{f}) = h_0(\bar{f})\otimes 1_V + h_1(\bar{f})\otimes v_1 + \ldots + h_k(\bar{f})\otimes v_k$ for some $h_0,\ldots,h_k\colon \bar{t}\to C^\infty(U)$. Similary we get for ever $i\in I$ an expansion $a_i = a_{i,0}\otimes 1_V + a_{i,1}\otimes v_1 + \ldots + a_{i,k}\otimes v_k$. The condition (2.12) becomes the requirement, that for ever $i\in I$ it holds true that

$$h_j(\bar{f})\circ\rho_i = a_{i,j}(\bar{f}). \tag{2.13}$$

But clearly the smooth functions $a_{i,j}(\bar{f})$ and $a_{k,j}(\bar{f})$ agree when they are restricted to $U_i\cap U_k$ and thus we get by the sheaf property of smooth functions a unique $h_j(\bar{f})$. Finally we have to show that the so obtained h is a $\mathsf{C^\infty Ring}$-morphism. Indeed, for $f\colon\mathbb{R}^m\to\mathbb{R}^n$ and $(\alpha_1,\ldots,\alpha_m)\in A^m$ we get

$$\begin{aligned}&\left(\prod_{i\in I} C^\infty(\rho_i)\otimes_\infty \mathrm{Id}_W\right)^n (C^\infty(U)\otimes_{\mathbb{R}} V)_f((h(\alpha_i))_i)\\ &= (C^\infty(U_i)\otimes_{\mathbb{R}} W')_f((a(\alpha_i))_i)\end{aligned}$$

on the other hand

$$\begin{aligned}&\left(\prod_{i\in I} C^\infty(\rho_i)\otimes_\infty \mathrm{Id}_W\right)^n (h^n(A_f((\alpha_i)_i))\\ &= a^n(A_f((\alpha_i)_i)) = \left(\prod_{i\in I} C^\infty(U_i)\otimes_{\mathbb{R}} V\right)_f (a(\alpha_i)_i).\end{aligned}$$

Using once again, that $U\mapsto C^\infty(U)$ defines a sheaf, we can conclude that

$$C^\infty(U)\otimes_{\mathbb{R}} V)_f((h(\alpha_i))_i = A_f((\alpha_i)_i$$

and thus that h is a C^∞-ring morphism. □

In the chapter on sheaves it was explained how we can close up a given coverage to a Grothendieck topology and that a presheaf which is a sheaf with respect to the coverage is also a sheaf with respect to the so obtained Grothendieck topology. Thus we can close the coverage up in this way now, which makes the Cahiers topos into a category of sheaves with respect to a Grothendieck topology i.e a Grothendieck topos.
We have now completed our construction of a category of sheaves in which we have as objects the usual manifolds, that are embedded full and faithful into $\mathsf{C}^\infty\mathsf{Ring}^{\mathsf{op}}$ by Corollary (2.1.5) and then full and faithful by the Yoneda embedding into the Cahiers topos C. Moreover the Cahiers topos has an infinitesimal line object given by the Yoneda embedding of the dual numbers $l\mathbb{R}_\epsilon$. The intuition here is, that like the space of smooth functions from a smooth manifold M into $\mathbb{R}$ is given by the formal dual to M i.e the C^∞-ring $C^\infty(M)$, the C^∞-ring of smooth functions from $D = \ell\mathbb{R}_\epsilon$ into $\mathbb{R}$ is given by $\mathbb{R}_\epsilon$. But elements of $\mathbb{R}_\epsilon$ have the form $a + \epsilon b$ for $a, b \in \mathbb{R}$. We interpret this as follows. All smooth functions from $f: D \to \mathbb{R}$ are completely determined by their value at at a point (given by a) and their first derivative at this point (given by b).
The Cahiers topos has the advantage of being closed under taking small limits, colimits and exponentials. This is a consequence of them being defined as a Grothendieck topos. Another useful result is that open coverings of manifolds are mapped to effective epimorphisms in the Cahiers topos by the inclusion. What this means is made more formal in the following proposition.

Proposition 2.1.29. *If $\{\rho_i : U_i \to M\}$ is an open covering of the manifolds M The induced map $\pi: \coprod_\alpha \iota(U_\alpha) \to \iota(M)$ is an effective epimorphism. In other words the following diagram is a coequalizer*

$$\textstyle\coprod_\alpha \iota(U_\alpha) \times_{\iota(M)} \coprod_\beta \iota(U_\beta) \rightrightarrows \coprod_\alpha \iota(U_\alpha) \longrightarrow \iota(M). \tag{2.14}$$

To prove it we need two lemmas concerning the consistency of taking open submanifolds and intersections of submanifolds with the description of manilfolds within the Cahier topos.

Lemma 2.1.30. *For M a smooth manifold and $i: U \hookrightarrow M$ an open submanifold, $(\iota(U), \iota(i))$ is a subobject of $\iota(M)$ in the Cahiers topos.*

Proof. We have to show that $\iota(i)$ is a monomorphism. In other words that every stage of $\iota(i)$ is an injective map of sets. Let $t = N \times \ell W$ be a generalized smooth space. Then $\iota(i)_t$ maps $f \in \iota(U)(t)$ to $\iota(M)(f)(i) = i \circ f$ according to the Yoneda lemma (1.2.3). But if $f, g \in \iota(U)(t)$ are such that

$$\iota(M)(f)(i) = i \circ f = i \circ g = \iota(M)(g)(i)$$

it dirctly follows that $f = g$ and thus $\iota(i)_t$ is injective and as t was arbitrary $\iota(i)$ monic. □

Lemma 2.1.31. *If M is a smooth manifold and $U, V \hookrightarrow M$ are two open submanifolds, the fiber product $\iota(U) \times_{\iota(M)} \iota(V)$ is naturally isomorphic to $\iota(U \cap V)$.*

Proof. The first step in the proof is to find a candidate for the natural isomorphism. Let $t = N \times \ell W$ be a generalized smooth space. The inclusions $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$ correspond to the maps

$$\iota(i_U)_t : \mathsf{Hom}(C^\infty(U \cap V), C^\infty(N) \otimes W) \to \mathsf{Hom}(C^\infty(U), C^\infty(N) \otimes W)$$

and

$$\iota(i_V)_t \colon \mathsf{Hom}(C^\infty(U\cap V), C^\infty(N)\otimes W) \to \mathsf{Hom}(C^\infty(V), C^\infty(N)\otimes W)$$

explicitely given by the assignments

$$\mathsf{Hom}(C^\infty(U\cap V), C^\infty(N)\otimes W) \ni \phi \mapsto \phi((-)\circ i_U)$$

and

$$\mathsf{Hom}(C^\infty(U\cap V), C^\infty(N)\otimes W) \ni \phi \mapsto \phi((-)\circ i_V)$$

respectively. These clearly define a cone over the following diagram of sets

$$\begin{array}{ccc} & & \mathsf{Hom}(C^\infty(U), C^\infty(N)\otimes W) \\ & & \big\downarrow \\ \mathsf{Hom}(C^\infty(V), C^\infty(N)\otimes W) & \longrightarrow & \mathsf{Hom}(C^\infty(M), C^\infty(N)\otimes W) \end{array} \tag{2.15}$$

The unique map we get from the universality of the pullback is

$$\eta_{N\times\ell W} \colon \mathsf{Hom}(C^\infty(U\cap V), C^\infty(N)\otimes W) \to \iota(U)\times_{\iota(M)}\iota(V)(N\times\ell W).$$

These will be the components of our candidate for the required natural isomorphism.
Next we want to show injectivity and surjectivity of this map. Injectivity is a consequence of the previous lemma. Indeed, we now that $U\cap V$ is an open submanifold of U. This shows that the composition $p_{U,t}\circ\eta_t = \iota(i_U)_t$ is injective due to the lemma and thus also η_t has to be injective.
Showing surjectivity is a bit harder. We start with an element of $\psi \in \iota(U)\times_{\iota(M)}\iota(V)(N\times\ell W)$ and want to construct its preimage under $\eta_{N\times\ell W}$. The images of ψ under $p_{U,t}$ and $p_{V,t}$ are assignments

- $C^\infty(U) \ni f \mapsto \psi_U(f) \in C^\infty(N)\otimes W$; and
- $C^\infty(V) \ni f \mapsto \psi_V(f) \in C^\infty(N)\otimes W$

such that for $f \in C^\infty(M)$ we have

$$\psi_U(f\circ i_U) = \psi_V(f\circ i_V).$$

We can compose these maps with the $C^\infty(N)\otimes W \to C^\infty(N)$ that is induced by the projection $\pi_0 \colon W \to \mathbb{R}$. These compositions corresponds according to Corollary (2.1.5) to the two maps $\bar\psi_U \colon N \to U$ and $\bar\psi_V \colon N \to V$ such that

$$\begin{array}{ccc} N & \xrightarrow{\bar\psi_U} & U \\ {\scriptstyle\bar\psi_V}\big\downarrow & & \big\downarrow \\ V & \longrightarrow & M \end{array}$$

commutes. Thus by universality of the limit $U \cap V = U \times_M V$ we get a smooth map $\tilde{\psi}\colon N \to U \cap V$ such that $\tilde{\psi_U} = i_{U\cap V,U} \circ \tilde{\psi}$ and $\tilde{\psi_V} = i_{U\cap V,V} \circ \tilde{\psi}$.
Now we pick a $\mathbb{R}$-linear basis $1, w_1, \ldots, w_k$ of W. Then we can write with $f \in C^\infty(U)$ and $g \in C^\infty(V)$

$$\psi_U(f) = \tilde{\psi}_U^*(f) + \sum_{i=1}^{k} f_i w_i$$

and

$$\psi_V(g) = \tilde{\psi}_V^*(g) + \sum_{i=1}^{k} g_i w_i.$$

where $f_i, g_i \in C^\infty(N)$. We can evaluate the $\psi_U(f)$ and $\psi_V(g)$ at a point $n \in N$. Then we get from Proposition (2.1.22) that $f \mapsto \sum_{i=1}^k f_i(n) w_i$ and $g \mapsto \sum_{i=1}^k g_i(n) w_i$ are of local character at $\tilde{\psi}_U(n) = \tilde{\psi}_V(n) = \tilde{\psi}(n)$. It follows, that if f and g agree on an open neighborhood of $\tilde{\psi}(n)$ we have that $f_i(n) = g_i(n)$.
Now let $h \in C^\infty(U \cap V)$. We define

$$\phi\colon C^\infty(U \cap V) \to C^\infty(N) \otimes W$$

by setting

$$\phi(h) = \tilde{\psi}^*(h) + \sum_{i=1}^{k} h_i w_i.$$

The elements $h_i \in C^\infty(N)$ are given by setting $h_i(n) = f_i(n) = g_i(n)$. The smooth functions $f \in C^\infty(U)$ and $g \in C^\infty(V)$ are constructed like in the proof of Corollary (2.1.3) such that they agree with h on an open neighborhood of $\tilde{\phi}(n)$. Because $f \mapsto \sum_{i=1}^k f_i(n) w_i$ and $g \mapsto \sum_{i=1}^k g_i(n) w_i$ are of local character at $\tilde{\psi}(n)$ this definition does not depend on the choice of f and g.
Finally we have to check, that the so defined ϕ is indeed the preimage of ψ under $\eta_{N\times \ell W}$. For this end we observe that

$$\iota(i_U)_t(\phi) = \phi((-) \circ i_U) = \psi_U$$

and

$$\iota(i_V)_t(\phi) = \phi((-) \circ i_V) = \psi_V.$$

Thus we must have that $\eta_{N\times \ell W}(\phi) = \psi$ because otherwise the assignment

$$\psi \mapsto \phi$$

would violate the uniqueness requirement in the universal property of the pullback diagram (2.15).
This concludes the proof. □

Proof. (of Proposition (2.1.29)) Using Lemma (2.1.31) we find the following natural isomorphisms

$$\left(\coprod_{\alpha} \iota(U_\alpha)\right) \times_{\iota(M)} \left(\coprod_{\beta} \iota(U_\beta)\right) \cong \coprod_{\alpha,\beta} \iota(U_\alpha) \times_{\iota(M)} \iota(U_\beta) \cong \coprod_{\alpha,\beta} \iota(U_\alpha \cap U_\beta).$$

The kernel pair in the diagram (2.14) is thus naturally isomorphic to

$$F := \ \textstyle\coprod_{\alpha,\beta} \iota(U_\alpha \cap U_\beta) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \coprod_{\alpha} \iota(U_\alpha).$$

To prove the proposition we have to show that $\mathrm{colim}(F) \cong \iota(M)$. For this end consider an arbitrary object of the Cahiers topos, say X. Then we have the following isomorphisms, which are all natural in X

$$\begin{aligned} \mathsf{Nat}(\mathrm{colim}F, X) &\cong \lim \mathsf{Nat}(F, X) \\ &\cong \lim \left[\prod_i \mathsf{Nat}(\iota(U_i), X) \rightrightarrows \prod_{i,j} \mathsf{Nat}(\iota(U_i \cap U_j), X)\right] \\ &\cong \lim \left[\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \cap U_j)\right] \\ &\cong X(M). \end{aligned} \tag{2.16}$$

The first two isomorphisms are by preservation of limits by the Hom-functor (see (A.1.2) of the Appendix), the third is the Yoneda lemma and the last isomorphism is by the sheaf property of X. Now we look at two particular choices for X.

- First we set $X = \iota M$. Then there is a unique natural transformation $a\colon \mathrm{colim}F \to \iota M$ that is mapped to the identity on ιM under (2.16).
- Second we set $X = \mathrm{colim}F$. Then the image of the identity on $\mathrm{colim}F$ under (2.16) is a unique natural transformation $b\colon \iota(M) \to \mathrm{colim}F$.

Now we can use that the isomorphism (2.16) is natural in X. This gives to us the commutative diagram

$$\begin{array}{ccc} \mathsf{Nat}(\mathrm{colim}F, \iota(M)) & \longrightarrow & \mathsf{Nat}(\iota(M), \iota(M)) \\ {\scriptstyle \mathsf{Nat}(\mathrm{colim}F, b)}\downarrow & & \downarrow{\scriptstyle \mathsf{Nat}(\iota(M), b)} \\ \mathsf{Nat}(\mathrm{colim}F, \mathrm{colim}F) & \longrightarrow & \mathsf{Nat}(\iota(M), \mathrm{colim}F). \end{array}$$

Chasing the element $a \in \mathsf{Nat}(\mathrm{colim}F, \iota(M))$ defined above around this diagram yields

$$\begin{array}{ccc} a & \longmapsto & \mathrm{id}_{\iota(M)} \\ \downarrow & & \downarrow \\ b \circ a & \longmapsto & b. \end{array}$$

Due to the definition of b as being the image of the identity under (2.16) we find that $b \circ a = \mathrm{id}_{\mathrm{colim}F}$. In a similar way we show that $a \circ b = \mathrm{id}_{\iota(M)}$ and thus $\iota(M) \cong \mathrm{colim}F$. The proposition is proved. □

2.2 Synthetic differential geometry in the Cahier Topos

With the Cahiers topos we have constructed a category of sheaves. We have seen, that in a category of sheaves all small limits, colimits and exponetial objects exist. Moreover we investigated how to explicitely compute these. Let us review some important examples for the case of the Cahiers topos.

One kind of limit is the product of two objects. We can specify how the product $X, Y \in \mathsf{Ob}(\mathsf{C})$ explicitly looks. As a limit of sheaves it is computed pointwise. Thus it is the sheaf $X \times Y \colon \mathsf{FMan}^{\mathsf{op}} \to \mathsf{Set}$ that acts on an object $t \in \mathsf{Ob}(\mathsf{FMan})$ as

$$(X \times Y)(t) = X(t) \times Y(t), \tag{2.17}$$

where the right hand side is the product of $X(t)$ and $Y(t)$ in the category Set. Another limit is the terminal object. It is the limit of the diagram in C with empty index category and is denoted $\{*\}$. It is easily characterized as the sheaf that acts on objects in C as

$$\{*\}(t) = \{*\}, \tag{2.18}$$

where the right hand side is the terminal object in Set, i.e a singleton.
Let us finally consider the exponential object in the Cahiers topos. For $X, Y \in \mathsf{Ob}(\mathsf{C})$ it is given by the sheaf $Y^X \colon \mathsf{FMan}^{\mathsf{op}} \to \mathsf{Set}$ that acts on objects as

$$Y^X(t) = \mathrm{Hom}_{\mathsf{C}}(\iota(t) \times X, Y). \tag{2.19}$$

We can regard both $(-)^X \colon \mathsf{C} \to \mathsf{C}$ and $Y^{(-)} \colon \mathsf{C}^{\mathsf{op}} \to \mathsf{C}$ as functors. Concretely, if $f \colon Y \to Z$ is any morphism in C the natural transformation $f^X \colon Y^X \to Z^X$ has components

$$\begin{aligned}(f^X)_t \colon \mathrm{Hom}_{\mathsf{C}}(\iota(t) \times X, Y) &\to \mathrm{Hom}_{\mathsf{C}}(\iota(t) \times X, Z),\\ (h \colon \iota(t) \times X \to Y) &\mapsto (f \circ h \colon \iota(t) \times X \to Z).\end{aligned} \tag{2.20}$$

Also if $g \colon X \to Z$ is any morphism in C, then $Y^g \colon Y^Z \to Y^X$ is specified by the components

$$\begin{aligned}(Y^g)_t \colon \mathrm{Hom}_{\mathsf{C}}(\iota(t) \times Z, Y) &\to \mathrm{Hom}_{\mathsf{C}}(\iota(t) \times X, Y),\\ (h \colon \iota(t) \times Z \to Y) &\mapsto (h \circ (id_t \times g) \colon t \times X \to Y).\end{aligned} \tag{2.21}$$

We are now in a position to define the *synthetic tangent bundle* of an object $X \in \mathsf{Ob}(\mathsf{C})$ in the Cahiers topos. It shall be a sheaf TX that has the generalized points $TX(\{*\}) = \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(X, \mathbb{R}_\epsilon)$. This is motivated by Proposition (2.1.24) which showed that for a smooth manifold M the set $\mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(M), \mathbb{R}_\epsilon)$ is in bijection with the underlying set of the usual tangent bundle TM. The natural choice for the synthetic tangent bundle of X is thus the exponential object $X^{\iota(D)}$, which is after all to be thought of as the space of morphisms from ιD into X.

Definition 2.2.1. *The synthetic tangent bundle of an object $X \in \mathsf{Ob}(\mathsf{C})$ is defined to be the object $TX = X^{\iota(D)}$. The projection $\pi\colon TX \to X$ is given by the the image of the morphism $0\colon \{*\} \to \iota(D)$ under the functor $X^{(-)}$ from above.*

Note, that the following diagram commutes

$$\begin{array}{ccc} TX & \xrightarrow{Tf:=f^{\iota(D)}} & TY \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \tag{2.22}$$

for every $f\colon X \to Y$ as is easily seen by computing the image of $(h\colon t \times D \to X)$, where $t = N \times \ell W$, along both sides of the diagram. Moreover one can show that for every object M in Man we get $T\iota(M) \cong \iota(TM)$, where TM on the right hand side is the ordinary tangent bundle of M.

Theorem 2.2.2. *If M a smooth manifold the sheaves $\iota(TM)$ and $T\iota(M) = \iota(M)^{\iota(D)}$ are naturally isomorphic.*

Proof. We want to show that there is a natural bijection between the sets $\iota(M)^{\iota(D)}(t = N \times lV)$ and $\iota(TM)(t) = \mathsf{Hom}_{\mathsf{FMan}}(t, TM)$ for all $t \in \mathsf{Ob}(\mathsf{FMan})$. We have

$$\iota(M)^{\iota(D)}(t) = \mathsf{Hom}_{\mathsf{C}}(\iota(t) \times \iota(D), \iota(M)).$$

Due to the Yoneda lemma and because the Yoneda embedding preserves products this is isomorphic to

$$\mathsf{Hom}_{\mathsf{FMan}}(t \times D, M) \cong \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(M), t \otimes \mathbb{R}_\epsilon).$$

Our objective is thus to construct a natural bijection between the sets

$$\mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(M), t \otimes \mathbb{R}_\epsilon)$$

and

$$\mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TM), t).$$

We will first show the result for for $M = \mathbb{R}^m$. Afterwards we can use the fact that every smooth manifold has a good open cover together with Proposition (2.1.29) and Proposition (C.0.4) from the appendix to conclude the proof for general smooth manifolds.

Let $\chi \in \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(T\mathbb{R}^m \cong \mathbb{R}^{2m}), t)$. Then

$$\begin{aligned} \xi\colon C^\infty(\mathbb{R}^m) &\to t \otimes \mathbb{R}_\epsilon \\ \xi(f) &= \chi(\pi_0 \circ df) + \epsilon\chi(\pi_1 \circ df), \end{aligned}$$

where $\pi_0, \pi_1\colon T\mathbb{R} \cong \mathbb{R}^2 \to \mathbb{R}$ are the projections onto the first and second factor. We call this assignment $F_t\colon \chi \mapsto \xi$. We have to show that ξ is again a $\mathsf{C}^\infty\mathsf{Ring}$-morphism, that it is natural in t and that it is a bijective map

$$T_t\colon \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(T\mathbb{R}^m), t) \to \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(T\mathbb{R}^m), t \otimes \mathbb{R}_\epsilon).$$

First we show that ξ is again a $\mathsf{C}^\infty\mathsf{Ring}$-morphism. To see this let $k\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. We want to show that for every $(f_1, \dots f_n) \in (C^\infty(M))^n$ it holds true that $\xi((C^\infty(M))_k((f_1, \dots, f_n))) = (t \otimes \mathbb{R}_\epsilon)_k(\xi^n(f_1, \dots, f_n))$ i.e that the following diagram commutes

$$\begin{array}{ccc} C^\infty(M)^n & \xrightarrow{\xi^n} & (t \otimes \mathbb{R}_\epsilon)^n \\ {\scriptstyle (C^\infty(M))_k}\downarrow & & \downarrow{\scriptstyle (t\otimes\mathbb{R}([\epsilon])_k} \\ C^\infty(M) & \xrightarrow{\xi} & (t \otimes \mathbb{R}_\epsilon) \end{array}$$

Indeed

$$\begin{aligned} &\xi((C^\infty(M))_l((f_1, \dots, f_n))) \\ =&\xi(l \circ (f_1, \dots, f_n)) \\ =&\chi(\pi_0 \circ d(l \circ (f_1, \dots, f_n))) + \epsilon\chi(\pi_1 \circ d(l \circ (f_1, \dots, f_l))) \end{aligned}$$

and on the other hand

$$\begin{aligned} &(t \otimes \mathbb{R}_\epsilon)_k(\xi^n(f_1, \dots, f_n)) \\ =&(t \otimes \mathbb{R}_\epsilon)_k(\chi^n(\pi_0 \circ df_1, \dots, \pi_0 \circ df_n)) + \epsilon\chi^n(\pi_1 \circ df_1, \dots, \pi_1 \circ df_n)) \\ =&t_k(\chi^n(\pi_0 \circ df_1, \dots, \pi_0 \circ df_n)) + \sum_{i=1}^n t_{\partial_i k}(\chi^l(\pi_0 \circ df_1, \dots, \pi_0 \circ df_n))\epsilon\chi(\pi_1 \circ df_i) \\ \overset{\spadesuit}{=}&\xi(k \circ (f_1 \dots, f_n)) + \epsilon\xi((\sum_{i=1}^n \partial_i k \circ (f_1, \dots f_n))(\pi_1 \circ df_i)) \\ =&\xi(k \circ (f_1 \dots, f_n)) + \epsilon\chi(\pi_1 \circ d(l \circ (f_1, \dots, f_l))) \end{aligned}$$

where for $\spadesuit$ the chain rule and that χ is a $\mathsf{C}^\infty\mathsf{Ring}$-morphism is used.
Moreover the assignment

$$T_t\colon \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(T\mathbb{R}^m), t) \to \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(T\mathbb{R}^m), t \otimes \mathbb{R}_\epsilon)$$

is natural in t. Indeed we have for every $\mathsf{C}^\infty\mathsf{Ring}$-morphism $a\colon t' \to t$ that

$$\begin{aligned} T_{t'}(\iota(TM)(a)(\chi))(f) =&T_{t'}(a \circ \chi)(f) \\ =&a \circ \chi(\pi_0 \circ df) + \epsilon a \circ \chi(\pi_1 \circ df). \end{aligned}$$

on the other hand

$$\begin{aligned} \iota(M)^{\iota(D)}(a)(T_t(\chi))(f) &= \iota(M)^{\iota(D)}(\chi(\pi_0 \circ d(-)) + \epsilon\chi(\pi_1 \circ d(-)))(f) \\ &= a \circ \chi(\pi_0 \circ df) + \epsilon a \circ \chi(\pi_1 \circ df) \end{aligned}$$

Last we want to show that F_t is a bijection for every $t \in \mathsf{Ob}(\mathsf{FMan})$. We have that $C^\infty(\mathbb{R}^m)$ and $C^\infty(\mathbb{R}^{2m})$ are the free $\mathsf{C}^\infty\mathsf{Ring}$ objects on m respectively $2m$ generators. Thus a given $\mathsf{C}^\infty\mathsf{Ring}$ morphism $\xi\colon C^\infty(\mathbb{R}^m) \to t\otimes\mathbb{R}_\epsilon$ is uniquely determined by specifying the images of the generators of $C^\infty(\mathbb{R}^m)$ namely $\xi(\pi_i)$ for $1 \leq i \leq m$. But then we know that a $\chi\colon C^\infty(\mathbb{R}^{2m}) \to t$ such that $F_t(\chi) = \xi$ has to satisfy $\xi(\pi_i) = \chi(\tilde{\pi_0} \circ d\pi_i) + \chi(\tilde{\pi_1} \circ d\pi_i) = \chi(\pi_i) + \epsilon\chi(\pi_{i+m})$. Thus

such a χ exists and is unique.
With all this we have established the result for $M = \mathbb{R}^m$. Like announced we will generalize to arbitrary smooth manifolds M.
Let M be a smooth manifold and $\{U_i\}$ a good open cover of M. We know from proposition (2.1.29) that

$$\coprod_i \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(U_i), t \otimes_\infty D) \to \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(M), t \otimes_\infty D)$$

and

$$\coprod_i \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TU_i), t) \to \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TM), t)$$

are effective epimorphisms. Moreover we already know that $\mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(U_i \cong \mathbb{R}^m), t \otimes_\infty D) \cong \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TU_i \cong \mathbb{R}^{2m}), t)$ by the discussion above. This induces an isomorphism $\coprod_i \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(U_i \cong \mathbb{R}^m), t \otimes_\infty D) \cong \coprod_i \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TU_i \cong \mathbb{R}^{2m}), t)$. When we look again at how the above isomorphism is constructed it is evident, that it induces an isomorphism between the respective kernel pairs. We can thus apply Proposition (C.0.4) and get an isomorphism

$$\mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(M), t \otimes_\infty D) \cong \mathsf{Hom}_{\mathsf{C}^\infty\mathsf{Ring}}(C^\infty(TM), t)$$

This concludes the proof. □

Chapter 3

Scalar Field Theory in the Cahiers topos

In this section we will finally formulate scalar field theory in synthethetic differential geometry by using the Cahiers topos. We will first introduce some notions in the context of usual differential geometry though to get an impression what is to be transferred to the synthetic setting later on.

3.1 Scalar field theory

Definition 3.1.1. *Let (M, g) be a connected and smooth Lorentzian manifold. A Cauchy surface in M is an embedded submanifold $\Sigma \hookrightarrow M$ such that every timelike curve in M may be extended to a timelike curve that intesects Σ precisely once.*

The intuition behind this definition is that a Cauchy surface behaves like all of space at a given instance in time.

Definition 3.1.2. *A connected and smooth Lorentzian manifold (M, g) is called globally hyperbolic if it admits a Cauchy surface.*

An example for a globally hyperbolic manifold with Cauchy surface Σ is given by $\Sigma \times I$, where I is some interval in $\mathbb{R}$.. We will consider scalar field configurations $\phi \colon M \to \mathbb{R}$ where M is of the above form i.e $M = \Sigma \times I$. Our field configurations are elements of the inifinite dimensional smooth space $C^\infty(M)$. It is for this reason that we call $C^\infty(M)$ the configuration space of the theory. Another reasonable choice for a configuration space would have been $\Gamma(E)$- the space of smooth sections of a one dimensional vector bundle E on M. In the case where E is trivial we could simply reduce to $C^\infty(M)$ by choosing a section as basis for $\Gamma(E)$. In the case of a non trivial vector bundle the analysis would be further complicated by the need to work locally. Here we will not be concerned with this matter and just consider $C^\infty(M)$ as our configuration space from the start.
The next step is to introduce differential equations and distinguish those elements of the

configuration space which satisfy them. An example for such a differential equation is

$$\Box_M \phi = 0. \tag{3.1}$$

Here $\Box_M$ is defined in local coordinates by

$$\Box_M \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi,$$

where $\phi \in C^\infty(M)$ and ∇ is the Levi Civita connection on M. Like the notation suggests is this definition of $\Box_M$ motivated by the d'Alembert operator $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$ on Minkowski space. It is possible to express $\Box_M$ completely in terms of partial derivatives. First we may write it like

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\phi =& \nabla_\mu(g^{\mu\nu}\nabla_\nu\phi) \\ =& \nabla_\mu(g^{\mu\nu}\partial_\nu\phi). \end{aligned} \tag{3.2}$$

Moreover, given any vector field V on M we can make the following calculation

$$\begin{aligned} \nabla_\mu V^\mu =& \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \\ =& \partial_\mu V^\mu + \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\lambda\mu}) \\ =& \partial_\mu V^\mu + \frac{1}{2} g^{\mu\rho}(\partial_\lambda g_{\mu\rho})V^\lambda \\ \overset{\spadesuit}{=}& \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}}(\partial_\lambda \sqrt{|g|}) \\ =& \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu). \end{aligned} \tag{3.3}$$

For the equality $\spadesuit$ we used that with the help of the chain rule and Jacobis formula the following derivation can be made

$$\begin{aligned} \partial_\lambda\sqrt{-g} =& \frac{1}{2\sqrt{|g|}}\partial_\lambda |g| \\ =& \frac{1}{2\sqrt{|g|}}|g|(g^{\mu\nu}\partial_\lambda g_{\mu\nu}) \\ =& \frac{1}{2}\sqrt{|g|}(g^{\mu\nu}\partial_\lambda g_{\mu\nu}). \end{aligned}$$

Last we can insert (3.3) into (3.2) to get

$$\Box_M \phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\partial^\nu\phi). \tag{3.4}$$

Thus we can write equation (3.1) as

$$\partial_\mu(\sqrt{|g|}\partial^\nu\phi) = 0. \tag{3.5}$$

This is also called the free scalar field equation. We can add further terms to it. A first example is to simply add a mass term, which leads to the so called Klein-Gordon equation.

$$(\Box_M + m^2)\phi = 0$$

But we can also add more complicated non linear terms like

$$\Box_M \phi + \lambda\phi^3 = 0 \tag{3.6}$$

or even

$$\Box_M \phi + \sin(\phi) = 0. \tag{3.7}$$

Equation (3.6) is the field equation of ϕ^4-theory and (3.7) is the Sine-Gordon equation. The subspace of elements in $C^\infty(M)$ that satisfy such an equation is called the solution space of the respective equation. It may be denoted by $\mathfrak{Sol}(M)$, where we leave implicit with respect to which equation this is the solution space.

Next we discuss the idea of deriving the above differential equations via a variation principle from a Lagrangian density. For the free scalar field equation (3.5) the Lagrangian density is given by the differential form

$$\mathcal{L}_{free} = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\sqrt{|g|}dx_0 \wedge \ldots \wedge dx_n, \tag{3.8}$$

where n is the dimension of M. We can integrate (3.8) over all of M which gives to us the so called action functional

$$S_{free}[\phi] = \int_M g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\sqrt{|g|}dx_0 \wedge \ldots \wedge dx_n. \tag{3.9}$$

We know that the globally hyperbolic manifold M is diffeomorphic to $\Sigma \times I$. We will simplify our analysis by requiring that M has the form $M = \Sigma \times [a, b]$ i.e that the interval I is closed and bounded. The idea is that we just look at the fields between a starting time a and a final time b. Moreover we require the field configurations we look at have compact support. A proper variation of a field configuration $\phi \in C_0^\infty(M)$ is a smooth map $\sigma \colon M \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that $\sigma((p, a), s) = \phi((p, a))$, $\sigma((p, b), s) = \phi((p, b))$ and $\sigma((p, t), 0) = \phi((p, t))$ for every $(p, t) \in \Sigma \times [a, b]$ and $s \in (-\epsilon, \epsilon)$. The idea of the variation principle is that an element $\phi \in C^\infty(M)$ is in $\mathfrak{Sol}(M)$ if it is a stationary point of the action functional. In other words do we want to find field configurations $\phi \in C_0^\infty(M)$ such that for every proper variation σ of ϕ with compact support we have

$$\left.\frac{d}{ds}\right|_{s=0} S[\sigma(\cdot, s)] = 0 \tag{3.10}$$

Let us calculate that variation of S. To simplify notation we define

$$\delta\phi(p) = \left.\frac{1}{ds}\sigma(p, s)\right|_{s=0}.$$

Then we get

$$\begin{aligned}
0 \overset{!}{=} \frac{d}{ds} S[\sigma(\cdot, s)]\Big|_{s=0} &= \int_M g^{\mu\nu} \frac{d}{ds} \partial_\mu \sigma \partial_\nu \sigma \sqrt{|g|} dx_0 \wedge \ldots \wedge dx_n \Big|_{s=0} \\
&= \int_M g^{\mu\nu} \partial_\mu (\delta\phi \partial_\nu \phi \sqrt{|g|}) dx_0 \wedge \ldots \wedge dx_n - \int_M g^{\mu\nu} \delta\phi \partial_\mu (\partial_\nu \phi \sqrt{|g|}) dx_0 \wedge \ldots \wedge dx_n \\
&\overset{\spadesuit}{=} \int_M d(g^{\mu\nu} (\partial_\nu \phi) \delta\phi \sqrt{|g|} d\hat{x}_\mu) - \int_M g^{\mu\nu} \delta\phi \partial_\mu (\partial_\nu \phi \sqrt{|g|}) dx_0 \wedge \ldots \wedge dx_n \\
&\overset{\heartsuit}{=} \int_\Sigma g^{0\nu} (\partial_\nu \phi) \delta\phi \sqrt{|g|} dx_1 \wedge \ldots \wedge dx_n - \int_M g^{\mu\nu} \delta\phi \partial_\mu (\partial_\nu \phi \sqrt{|g|}) dx_0 \wedge \ldots \wedge dx_n
\end{aligned}$$

For $\spadesuit$ we defined $d\hat{x}_\mu = dx_0 \wedge \ldots \wedge dx_{\mu-1} \wedge dx_{\mu+1} \wedge \ldots \wedge dx_n$ and for $\heartsuit$ we used the Stokes theorem and that both ϕ and σ have compact support. Finally, because the variation is proper the boundary term vanishes and the condition (3.10) amounts to

$$0 \overset{!}{=} \int_M g^{\mu\nu} \delta\phi \partial_\mu (\partial_\nu \phi \sqrt{|g|}) dx_0 \wedge \ldots . \wedge dx_n \tag{3.11}$$

But this holds true for every proper variation if and only if ϕ satisfies the free scalar field equation (3.5).

The boundary term is very interesting for itself if we also allow for non proper variations. For simplicity let us assume that near the boundary Σ the metric splits as

$$g = -(dx_0)^2 + h_{ij} dx_i dx_j$$

for some eucledian metric h_{ij}. Then the boundary term becomes

$$a_\Sigma = \int_\Sigma \partial_0 \phi \delta\phi \sqrt{|h|} dx_1 \wedge \ldots \wedge dx_n.$$

We can see this as a 1-form on the infinite dimensional solution space of the free scalar field equation that is parametrized by the initial values of $\phi(x)$ and $(\partial_0\phi)(x)$ for $x \in \Sigma$. Taking the exterior derivative δ of this one form gives us

$$\omega_\Sigma := \delta a_\Sigma = \int_\Sigma \delta(\partial_0 \phi) \delta\phi \sqrt{|h|} dx_1 \wedge x_1 \ldots \wedge x_n.$$

This is called the symplectic form. The solution space $\mathfrak{Sol}(M)$ together with the symplectic form ω_Σ on it is called the phase space of the theory. It can be shown, that the symplectic form is a closed and non-degenerate 2-form. The latter condition ensures, that the map

$$\begin{aligned}
\omega_\Sigma^\# \colon T\mathfrak{Sol}(M) &\to T^*\mathfrak{Sol}(M) \\
X &\to \iota_X \omega
\end{aligned}$$

where $\iota_X \omega_\Sigma$ denotes the contraction of ω_Σ with X is an isomorphism. A fact which is very useful because it ensures, that for a smooth map $H \colon \mathfrak{Sol}(M) \to \mathbb{R}$ there exists a unique vector field X_H such that

$$\iota_{X_H} \omega_\Sigma = -dH.$$

This X_H is called the Hamiltionan vector field for H.

3.2 The synthetic formulation

3.2.1 General idea

We want to construct scalar field theory within the Cahiers topos. This way we can hope to replace the classical constructions that were described above by synthetic ones. An advantage of this approach will for example be, that every space that appears has a tangent space that is an object of the same category as the original space itself. Namely they are all objects of the Cahiers topos.
Objects in the Cahiers topos are sheaves that go from the category of generalized smooth spaces FMan to Set. Our strategy to obtain a synthetic scalar field theory is to replace the spaces A that appear in the classical construction by objects $B \in \mathsf{Ob}(\mathsf{C})$ such that $B(\{*\})$ equals the underlying set of A. Let us invent some new notation to avoid the inflationary use of ι's for equations that are fomulated between objects that are embedded into the Cahiers topos. In this section we will denote the image of a generalized smooth space $t = M \times \ell W$ under the Yoneda embedding by $\underline{t}$. In particular we can consider a smooth manifold M as an object in FMan and the corresponding object in the Cahiers topos will be denoted $\underline{M}$. Moreover we denote the dual numbers $\mathbb{R}_\epsilon$ by D when we consider them as an object in the Cahiers topos and the real numbers $\mathbb{R}$ are denoted by R when considered inside the Cahiers topos.

3.2.2 The configuration space and the solution space

First we set up a configuration space. In the classical theory this is given by $C^\infty(M)$ for the respective space time manifold M as we have seen above.

Definition 3.2.1. *Let M be a finite dimensional manifold. The configuration space of the synthetic scalar field theory is defined to be the exponential object $R^{\underline{M}}$ inside the Cahiers topos C.*

We can immediately make the following calculation.

$$\begin{aligned} R^{\underline{M}}(t = N \times \ell W) &\overset{\heartsuit}{=} \mathrm{Hom}_{\mathsf{C}}(\underline{t} \times \underline{M}, R) \\ &= \mathrm{Hom}_{\mathsf{FMan}}(t \times M, \mathbb{R}) \\ &\overset{\clubsuit}{\cong} \mathrm{Hom}_{\mathsf{C^\infty Ring}}(C^\infty(\mathbb{R}), C^\infty(N \times M) \otimes_\infty W) \\ &\overset{\spadesuit}{\cong} C^\infty(N \times M) \otimes_\infty W \\ &\overset{\diamondsuit}{\cong} C^\infty(N \times M) \otimes_{\mathbb{R}} W. \end{aligned} \tag{3.12}$$

Here ♥ uses the fact that we can calculate exponential objects of sheaves as exponential ojects of presheaves like was proved in Proposition (1.2.27). ♣ uses Proposition (2.1.14). ♠ uses that $C^\infty(\mathbb{R})$ is the free C^∞-ring on one generator and thus a $\mathsf{C^\infty Ring}$-morphism in $\mathrm{Hom}_{\mathsf{C^\infty Ring}}(C^\infty(\mathbb{R}), C^\infty(N \times M) \otimes_\infty W)$ is uniquely determined by specifying the image of the generator $\mathrm{id}_{\mathbb{R}}$. Last ♦ uses Proposition (2.1.20). In particular we obtain

$$R^{\underline{M}}(*) = C^\infty(M). \tag{3.13}$$

The tangent space of $R^{\underline{M}}$ is given by

$$T(R^{\underline{M}}) = (R^{\underline{M}})^D \cong R^{\underline{M}\times D} \cong (R^D)^{\underline{M}} \cong (TR)^{\underline{M}}.$$

where we used Proposition (B.0.2) from the Appendix two times. By Theorem (2.2.2) we know that $TR \cong \underline{T\mathbb{R}}$ where $\underline{T\mathbb{R}}$ is the Yoneda embedding of the usual tangent bundle of $\mathbb{R}$ as a generalized smooth space. In other words is the total space of the tangent bundle of our configuration space given by the space of $T\mathbb{R}$-valued fields on M. We can make a calculation similar to (3.12).

$$\begin{aligned} T(R^{\underline{M}})(t = N \times \ell W)) =& \mathsf{Hom}_{\mathsf{C}}(\underline{t} \times \underline{M} \times D, R) \\ =& \mathsf{Hom}_{\mathsf{C^\infty Ring}}(C^\infty(\mathbb{R}), C^\infty(N \times M) \otimes_\infty W \otimes_\infty \mathbb{R}[\epsilon]) \\ =& C^\infty(N \times M) \otimes_{\mathbb{R}} W \otimes_{\mathbb{R}} \mathbb{R}[\epsilon]. \end{aligned} \tag{3.14}$$

We can thus write a generalized point of $T(R^{\underline{M}})$ as $\phi + \epsilon\psi$ where ϕ and ψ are generalized points of the configuration space $R^{\underline{M}}$. An example that comes to mind is the variation of a scalar field $\phi \in R^{\underline{M}}(\{*\}) = C^\infty(M)$. Then the expression

$$\psi = \frac{d}{ds}\sigma(\cdot, s)\bigg|_{s=0}$$

is a again an element of $C^\infty(M)$ and like was pointed our above we can interpret $\phi + \epsilon\psi$ as a generalized element of the tangent bundle $T(R^{\underline{M}})$. The meaning of ψ is that of a tangent vector with base point ϕ.

3.2.3 Solution space of field equations

The next step is to understand how we can formulate field equations in our settings. For this end we introduce the d'Alembert operator on the synthetic configuration space R^M. It is a morphism in C and as such a natural transformation $\Box_M : R^M \to R^M$. We define it by specifying its components. For a generic $t = N \times \ell W$ they are

$$(\Box_M)_t = \Box_M^{\text{vert}} \otimes_{\mathbb{R}} \text{id}_W : C^\infty(N \times M) \otimes_{\mathbb{R}} W \to C^\infty(N \times M) \otimes_{\mathbb{R}} W. \tag{3.15}$$

The vertical d'Alembert operator $\Box_M^{\text{vert}}$ is defined with respect to the Lorentzian geometry of M. Concretely this means that if we understand $N \times M$ as a trivial bundle over M, then we can use local coordinates x and ξ and locally understand an element of $C^\infty(N \times M)$ as a smooth function in this coordinates. The vertical d'Alembert operator then acts on $\phi \in C^\infty(N \times M)$ as

$$\Box_M^{\text{vert}}\phi(\xi, x) = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial}{\partial x_\nu}\phi(\xi, x)\right) \tag{3.16}$$

where $g^{\mu\nu}$ are the components of the Lorentzian metric on M. With this at hand we can explain the definition of $\Box_M$. If $\{e_i\}$ is a linear basis of W and $\phi \in C^\infty(N \times M) \otimes_{\mathbb{R}} W$ we can expand $\phi = \sum_i \phi^i e_i$, where $\phi^i \in C^\infty(N \times M)$. Then like the notation suggests the action of $\Box_M$ on ϕ is by acting with $\Box_M^{\text{vert}}$ on the ϕ^i without mixing components. With $\Box_M^{\text{vert}}$ at

hand we can already construct the solution space of the free scalar field equation (3.1). For equations like (3.6), (3.7) or the Klein Gordon equation i.e equations that contain further terms in addition to the d'Alembert operator we have a little more work to do.
Concretely we want to turn a smooth map $\rho\colon \mathbb{R} \to \mathbb{R}$ into a natural transformation that goes from $R^{\underline{M}}$ to $R^{\underline{M}}$ and in particular sends the generalized points $\phi \in R^{\underline{M}}(\{*\})$ to $\rho \circ \phi$. This is accomplished by defining $(R^{\underline{M}})_\rho$ to be the natural transformation with components

$$(R^{\underline{M}})_{\rho,t} = (C^\infty(N \times M) \otimes_{\mathbb{R}} W)_\rho$$

where again $t = N \times \ell W$ is a generic generalized smooth space. Given a $\phi \in (C^\infty(N \times M) \otimes_{\mathbb{R}} W$ we can write it as $\underline{\phi} + \hat{\phi}$, where $\underline{\phi} \in C^\infty(N \times M)$ and $\hat{\phi} \in C^\infty(N \times M) \otimes_{\mathbb{R}} W'$ for $W = \mathbb{R} \otimes W'$ and W' is nilpotent (of degree k) and then from (2.1.17) we get

$$(R^{\underline{M}})_{\rho,t}(\phi) = (C^\infty(N \times M) \otimes_{\mathbb{R}} W)_\rho(\phi) = \sum_{i=0}^{k} (\rho \circ \underline{\phi})\hat{\phi}.$$

This is exactly what we were after.

We can add our two natural transformations together to obtain

$$P_M = \Box_M + (R^{\underline{M}})_\rho \colon R^{\underline{M}} \to R^{\underline{M}}. \tag{3.17}$$

Adding together has to be understood at the level of stages. Given $\phi \in \mathsf{Hom}_{\mathsf{C}}(\underline{t} \times \underline{M}, \underline{R})$ its image under $(P_M)_t$ is given by $(\Box_M)_t(\phi) + ((R^{\underline{M}})_{\rho,t}(\phi)$.

Definition 3.2.2. *The solution space $\mathfrak{Sol}(M)$ is defined to be the pullback of the diagram*

$$\begin{array}{ccc} & & R^{\underline{M}} \\ & & \big\downarrow P_M \\ \underline{\{*\}} \cong \underline{\{*\}}^{\underline{M}} & \xrightarrow[0^{\underline{M}}]{} & R^{\underline{M}} \end{array}$$

To define $0^{\underline{M}}$ we used that $(-)^{\underline{M}}$ defines a functor like is described in the section on the exponential object in section (B) of the Appendix. The idea is, that because $\underline{\{*\}}(t) = \mathsf{Hom}_{\mathsf{FMan}}(t, \underline{\{*\}}) = \{*\}$ we can understand $\mathfrak{Sol}(P_M)$ as a subset of $R^{\underline{M}}(t)$ for every $t \in \mathsf{Ob}(\mathsf{FMan})$. From commutativity of the pullback diagram we get for every $\phi \in \mathfrak{Sol}(t)$ that $(0^{\underline{M}})_t(*) = 0_{R^{\underline{M}}(t)} = (P_M)_t(\phi)$. In other words: $\mathfrak{Sol}(P_M)$ has the generalized points

$$\mathfrak{Sol}(P_M)(t) = \{\phi \in R^{\underline{M}}(t) = C^\infty(N \times M) \otimes_{\mathbb{R}} W| \quad (P_M)_t(\phi) = 0\} \tag{3.18}$$

In particular for $t = \{*\}$ we get

$$\mathfrak{Sol}(P_M)(\{*\}) = \{\phi \in C^\infty(M)| \quad (P_M)_t(\phi) = \Box_M(\phi) + \rho(\phi) = 0\}.$$

To understand the structure of the tangent bundle of the solution space we use that the tangent functor $(-)^D$ is a right adjoint (Appendix (B)) and as such preserves limits. Thus we get the following commutative diagram

$$\begin{array}{ccc} T(\mathfrak{Sol}(M)) & \longrightarrow & T(R^{\underline{M}}) \\ \big\downarrow & & \big\downarrow {\scriptstyle T(P_M)} \\ \underline{\{*\}} \cong T(\underline{\{*\}}^{\underline{M}}) & \underset{T(0^{\underline{M}})}{\longrightarrow} & T(R^{\underline{M}}) \end{array}$$

Note that with $T(0^{\underline{M}})$ and $T(P_M)$ we mean $(0^{\underline{M}})^D$ and $(P_M)^D$ respectively. Thus we have for every $t \in \mathsf{Ob}(\mathsf{FMan})$ that $T(\mathfrak{Sol}(M))(t)$ is a subset of $T(R^{\underline{M}})(t)$ such that for every of its elements $\phi \in T(\mathfrak{Sol}(M))(t)$ the equation

$$(T(P_M))_t(\phi) = 0$$

holds. But looking at (2.20) we quickly realize that

$$(T(P_M))_t(\phi) = (P_M)_{t \times \ell \mathbb{R}_\epsilon}(\phi).$$

Let us investigate what the right hand side evaluates to if $\phi = \varphi + \epsilon\psi$.

$$\begin{aligned} &(P_M)_{t \times \ell \mathbb{R}_\epsilon}(\phi) = ((\Box_M)_{t \times \ell \mathbb{R}_\epsilon} + (R^{\underline{M}})_{t \times \ell \mathbb{R}_\epsilon}(\varphi + \epsilon\psi) \\ =&(\Box_M)_t(\phi) + \epsilon(\Box_M)_t(\psi) + (C^\infty(N \times M) \otimes_{\mathbb{R}} W \otimes_{\mathbb{R}} \mathbb{R}_\epsilon)_\rho(\varphi + \epsilon\psi) \\ =&(\Box_M)_t(\phi) + \epsilon(\Box_M)_t(\psi) + \rho \circ \varphi + (\rho' \circ \varphi)\epsilon\psi. \end{aligned}$$

But this expression is zero precisely when the two conditions

$$P_M(\phi) = 0$$

and

$$P^{lin}_{M,\varphi}(\psi) = \Box_M \phi + (\rho' \circ \varphi)\psi$$

are satisfied. We have obtained a more elementary description of the generalized point $T\mathfrak{Sol}(M)(t)$ namely

$$T\mathfrak{Sol}(M)(t) = \{\phi = \varphi + \epsilon\psi \in C^\infty(N \times M) \otimes_{\mathbb{R}} W \otimes_{\mathbb{R}} \mathbb{R}_\epsilon | P_M(\varphi) = 0, P^{lin}_{M,\varphi}(\psi) = 0\}$$

Appendices

Appendix A

Limits and colimits

Various mathematical constructions may be generalized in category theory by saying, that they are limits or colimits of specific diagrams.
First we need to define what a diagram is.

Definition A.0.1. *Let* J *and* C *be categories. A diagram of shape* J *in* C *is a functor* $F\colon \mathsf{J} \to \mathsf{C}$. J *is called index category and a diagram is called small if its index category is a small category.*

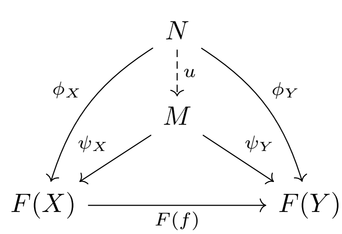
This makes it possible to define what a cone to a diagram is.

Definition A.0.2. *Let* $F\colon \mathsf{J} \to \mathsf{C}$ *be a diagram in* C *of shape* J*. A cone to* F *is an object* N *in* C *together with a family of morphisms* $\psi_X\colon N \to F(X)$*, one for every* $X \in \mathsf{Ob}(\mathsf{J})$ *such that for every morphism* $f\colon X \to Y$ *in* J *we have* $F(f) \circ \psi_X = \psi_Y$*.*

We can single out cones that fulfill a certain universal property.

Definition A.0.3. *A limit of the diagram* $F\colon \mathsf{J} \to \mathsf{C}$ *is a cone* (M, ψ) *to* F*, such that for every other cone* (M, ϕ) *to* F *it exists a unique morphism* $u\colon N \to M$ *such that* $\phi_X \circ u = \psi_X$ *for every* $X \in \mathsf{Ob}(\mathsf{C})$*.*

This defining property of a limit is summarized by the following commuting diagram which can be drawn for every two objects X and Y in J and morphism $f\colon X \to Y$



(A.1)

An important remark is, that the limit of a diagram is always unique up to isomorphisms if it exists. To show this is a very straightforward application of the defining universal property. It is often possible to show, that limits exist in a given category for diagrams of very general shape.

Example A.0.4. *An important example for a limit is the product of objects in a category. It is given as the limit of a diagram with discrete index category, i.e a category which has only identity morphisms. For example is the product of two objects in C given by the following limit.*

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle \phi_X}\swarrow & & \searrow{\scriptstyle \phi_Y} \\ X & & Y \end{array} \tag{A.2}$$

The morphisms ϕ_X and ϕ_Y are called projection morphisms.

Example A.0.5. *Another example of a limit is the pullback. It is the limit of the following diagram*

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{A.3}$$

and is often denoted as $X \times_Z Y$.

A colimit is the dual notion to that of a limit. To define it we first introduce what a cocone is. It is the same as a cone but where the direction of all the morphisms ψ_X is reversed. A colimit is then a cocone (M, ψ), which fulfills the universal property that for every other cocone (N, ϕ) it exists a unique morphism $u \colon M \to N$ such that the following diagram commutes for every two objects X, Y in J and morphism $f \colon X \to Y$ between them.

$$\begin{array}{ccc} & N & \\ {\scriptstyle \phi_X}\nearrow & \uparrow{\scriptstyle u} & \nwarrow{\scriptstyle \phi_Y} \\ & M & \\ {\scriptstyle \psi_X}\nearrow & & \nwarrow{\scriptstyle \psi_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array} \tag{A.4}$$

Example A.0.6. *An example is the coproduct. It is the colimit of a diagram with discrete index category. For example if the index category has two objects we denote the coproduct*

$$\begin{array}{ccc} & X \coprod Y & \\ {\scriptstyle \phi_X}\nearrow & & \nwarrow{\scriptstyle \phi_Y} \\ X & & Y. \end{array} \tag{A.5}$$

In the case where our diagram is in Set *the coproduct agrees with the well known disjoint union. I any case ϕ_X and ϕ_Y are called inclusion morphisms.*

A.1 Adjoint functors

If $F, F' \colon C \to D$ are two functors, then a natural transformation t from F to F' is called a natural equivalence if $t_X \colon F(X) \to F'(X)$ is an isomorphism in D for each X in C.

Moreover, given a functor $F\colon C \to D$ we call it an equivalence of categories if it exists a functor $G\colon D \to C$ such that FG and GD are related to the identity functor by a natural equivalence. Next we want to define what it means for two functors to be adjoint to each other. For this end let $F\colon C \to D$ and $G\colon D \to C$ denote a pair of functors. We call a collection of isomorphisms

$$\alpha_{X,Y}\colon \mathrm{Hom}_D(F(X), Y) \cong \mathrm{Hom}_C(X, G(Y))$$

that is natural in $X \in \mathrm{Ob}(C)$ and $Y \in \mathrm{Ob}(D)$ an adjunction from F to G. Note that an adjunction is the same as a natural equivalence between the two functors $\mathrm{Hom}_D(F(-),(-))\colon C^{op} \times D \to \mathsf{Set}$ and $\mathrm{Hom}_C((-), G(-))$. If such an adjunction between F and G exists one writes

$$F : C \Leftrightarrow D : G$$

and calls F and G adjoint functors, where G is the right adjoint of F and F is the left adjoint of G.

Definition A.1.1. *A functor $F\colon \mathsf{C} \to \mathsf{D}$ is said to preserve limits (respectively colimits) if it sends limits (respectively colimits) in C to limits (respectively colimits) in D.*

We want to show that right adjoint functors preserve limits and left adjoint functors preserve colimits. For the proof we need the following result first, which is also very interesting on its own.

Proposition A.1.2. *Let C be a category. For a fixed $a \in \mathsf{C}^{\mathsf{op}}$ the functor*

$$\mathsf{Hom}(a, (-))\colon \mathsf{C} \to \mathsf{Set}$$

preserves limits and for a fixed $b \in \mathsf{C}$ the functor

$$\mathsf{Hom}((-), b)\colon \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$$

preserves limits or in other words sends colimits in C to limits in Set.

Proof. The proof is basically a careful investigation of the definition of limits and colimits. We will only prove the first statement here as the second one is formally dual to it.

1. First we observe, that for every cone with tip Y over a diagram $X_\bullet$ of shape I in C we get a unique morphism from Y into $\lim X_\bullet$. On the other hand does every such morphism uniquely determine the respective cone it is induced by and thus we have the following bijection

$$\mathsf{Hom}(Y, \lim X_\bullet) \cong \mathrm{Cones}(Y, X_\bullet)$$

   where $\mathrm{Cones}(Y, X_\bullet)$ is the set of cones over $X_\bullet$ with tip Y.

2. We can associate to every element in $\lim \mathsf{Hom}(Y, X_\bullet)$ a unique collection of morphisms $\{p_i \in \mathsf{Hom}(Y, X_i)\}_{i \in \mathsf{Ob}(\mathsf{I})}$ such that for each $\alpha \in \mathsf{Hom}_{\mathsf{I}}(i,j)$ we have $p_j = X_\alpha \circ p_i$. But this is the very definition of a cone over $X_\bullet$ with tip Y. This shows

$$\mathrm{Cone}(Y, X_\bullet) \cong \lim(\mathsf{Hom}(Y, X_\bullet)).$$

Thus we have in total

$$\mathsf{Hom}(Y, \lim X_\bullet) \cong \lim(\mathsf{Hom}(Y, X_\bullet))$$

□

With this at hand we can show

Proposition A.1.3. *Right adjoints preserve limits and left adjoints prserve colimits.*

Proof. Let

$$F : \mathsf{C} \Leftrightarrow \mathsf{D} : G$$

be adjoint functors. We will show that if the limit of the diagram $X\colon \mathsf{I} \to \mathsf{C}$ exists, then it is preserved by G. The assertion, that F preserves colimits can be proven in a formal dual way. First note that as F and G are adjoints we have

$$\mathsf{Hom}_\mathsf{D}(F(X), Y) \cong \mathsf{Hom}_\mathsf{C}(X, G(Y))$$

for every $X \in \mathsf{Ob}(\mathsf{C})$ and $Y \in \mathsf{Ob}(\mathsf{D})$. Using this and the previous proposition about the hom-functor we get

$$\begin{aligned}\mathsf{Hom}_\mathsf{C}(Y, G(\lim(X_\bullet))) \cong& \mathsf{Hom}_\mathsf{D}(F(Y), \lim(X_\bullet))\\ \cong& \lim(\mathsf{Hom}_\mathsf{D}(F(Y), X_\bullet)\\ \cong& \lim(\mathsf{Hom}_\mathsf{C}(Y, G(X_\bullet))\\ \cong& \mathsf{Hom}_\mathsf{C}(Y, \lim(G(X_\bullet))).\end{aligned}$$

Now we can simply apply Corollary (1.2.4) and conclude

$$G(\lim(X_\bullet)) \cong \lim(G(X_\bullet)),$$

which is what we wanted to prove. □

Next we discuss how it is possible to, characterize limits and colimits as certain right adjoints and respectively left adjoints. For this end we consider a small category D and an arbitrary category C. We define the diagram category C^D to have as objects all the functors from D to C i.e the diagrams of shape D in C and as objects the natural transformations between them. Then we can define the functor

$$Const\colon \mathsf{C} \to \mathsf{C}^\mathsf{D}$$

that sends every object in C to the constant diagram and every morphism in C to the unique choice of natural transformation between two such constant diagrams. We assert that the right adjoint of this functor is if it exists a functor

$$Lim\colon C^D \to C$$

that sends every diagram to its limit. Dually is the left adjoint of $Const$ if it exist the functor $Colim\colon C^D \to C$ that sends every diagram to its colimit.

To see this recall, that if *Lim* is the right adjoint of *Const*, then it exists for every two objects $X \in \mathsf{C}$ and $Y \in \mathsf{C}^{\mathsf{D}}$ a natural isomorphism

$$\alpha_{XY} \colon \mathrm{Hom}_{\mathsf{C}^{\mathsf{D}}}(\mathrm{const}(X), Y) \to \mathrm{Hom}_{\mathsf{C}}(X, \lim(Y)).$$

Thus we get for every natural transformation from $\mathrm{const}(X)$ to Y i.e for every cone over the diagram Y one unique morphism in C from X to $Lim(Y)$. Because the isomorphism α_{XY} is natural in X and Y this is precisely the morphism we get from the universal property of the limit. This argument works dually for colimits.

Appendix B

The exponential object

Definition B.0.1. *Let C be a category in which all binary products with the object $Y \in \mathsf{Ob}(\mathsf{C})$ exist. Let $X \in \mathsf{Ob}(\mathsf{C})$ be another object. The pair (X^Y, ev), where $X^Y \in \mathsf{Ob}(\mathsf{C})$ and $ev\colon X^Y \times Y \to X$ is called exponential object if for every other pair $(Z \in \mathsf{Ob}(\mathsf{C}), e\colon Z \times Y \to X)$ it exists a unique $u\colon Z \to X^Y$ such that*

$$\begin{array}{ccc} T \times Y & \xrightarrow{u \times id_Y} & X^Y \times Y \\ & \searrow^{e} & \downarrow{\scriptstyle ev} \\ & & X \end{array}$$

commutes. The morphism ev is called evaluation map

The first thing we can observe is that the exponential object gives to us a functor, namely

$$\begin{aligned} (-)^Y &\colon \mathsf{C} \to \mathsf{C}, \\ & X \mapsto X^Y \\ & (f\colon X_1 \to X_2) \mapsto (f^Y : X_1^Y to X_2^Y) \end{aligned}$$

where f^Y is the unique morphism induced by the universal property of the exponential object X_2^Y. Also by this universal property we find a bijection between hom-sets

$$\mathsf{Hom}_{\mathsf{C}}(Z, X^Y) \cong \mathsf{Hom}_{\mathsf{C}}(Z \times Y, X) \tag{B.1}$$

which is called the currying-isomorphism. It assigns to every $u\colon Z \to X^Y$ the composite $\mathrm{ev} \circ u \times id_Y$. But by the universal property it exists for every $e\colon Z \times Y \to X$ a $u\colon Z \to X^Y$ such that $e = \mathrm{ev} \circ u \times id_Y$ and thus it is indeed a bijection.
Moreover it is easily checked, that the currying-isomorphism is natural in Z and X. This shows, that it is an adjunction between the left canonical left adjoint functor given by

$$(-) \times Y \colon \mathsf{C} \to \mathsf{C}$$

and the right adjoint functor $(-)^Y$ defined above.
We haven't really discussed what the idea behind the definition of the exponential object is

yet. Lets catch up on this. If our category in question has a terminal object we can choose $Z = 1$ in (B.1) which gives to us the isomorphism

$$\mathsf{Hom}(1, X^Y) \cong \mathsf{Hom}_{\mathsf{C}}(Y, X)$$

which shows, that in Set we can identify X^Y as the set of functions from Y to X. The exponential object in an arbitrary category is thus a generalization of the notion of a function set. Lets finally collect some important properties involving the exponential object

Proposition B.0.2. *Let $X, Y, Z \in \mathsf{Ob}(\mathsf{C})$ and let 1 be the terminal object of C. Then the identieties hold true*

1. $(X^Y)^Z = X^{Y \times Z}$
2. $X^1 = X$
3. $1^Y = X$
4. $(X \times Y)^Z = X^Z \times Y^Z$

Proof. The proof of (1) is just a simple calculation, which becomes possible by the isomorphism (B.1). Let L be another object in C.

$$\begin{aligned}\mathsf{Hom}(L, (X^Y)^Z) &\cong \mathsf{Hom}(L \times Z, X^Y) \\ &\cong \mathsf{Hom}(L \times Z \times Y, X) \\ &\cong \mathsf{Hom}(L, X^{(Z \times Y)}).\end{aligned}$$

Then we can use the Corollary (1.2.4) to conclude $(X^Y)^Z \cong X^{(Z \times Y)}$.
Similar but even more easy we can prove (2) by making the observation that $\mathsf{Hom}(Z, X^1) \cong \mathsf{Hom}(Z \times 1, X) \cong \mathsf{Hom}(Z, X)$.
(3) is proved by simply using $\mathsf{Hom}(Z, 1^X) \cong \mathsf{Hom}(Z \times X) \cong 1$ to see that there is precisely one morphism from Y into X^1 and thus X^1 is again the terminal object.
Finally is (4) a consequence of $(-)^Z$ being a right adjoint and thus product preserving. □

Appendix C

Epimorphisms and effective epimorphisms

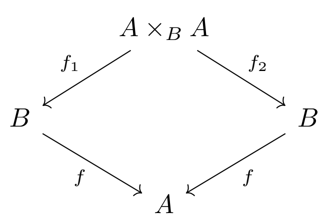
Definition C.0.1. *A morphism $f\colon A \to B$ in a category C is called an epimorphism if for every other two morphisms $g_1, g_2\colon B \to C$ such that*

$$g_1 \circ f = g_2 \circ f$$

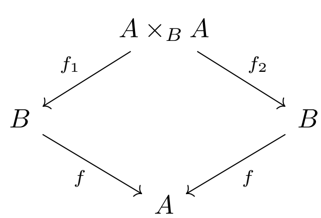
we have that $g_1 = g_2$.

Clearly the epimorhisms in the category Set are the surjective maps. For other categories however this analogy might not hold.

Definition C.0.2. *A kernel pair of the morphism $f\colon A \to B$ is given by the triple $(A \times_B A, f_1, f_2)$ that is the pullback of the following diagram*



Definition C.0.3. *An effective epimorphism is a morphism $f\colon A \to B$ with kernel pair*



such that

$$A \times_B A \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \xrightarrow{f} B$$

is a coequalizer diagram.

Proposition C.0.4. *Let $f\colon U \to A$ and $g\colon V \to B$ be two effective epimorphisms in a category C. Every isomorphism $i\colon U \cong V$ that induces an isomorphism $U \times_A U \cong V \times_B V$ induces also an isomorphism $A \cong B$.*

Proof. Denote the kernel pair diagrams by

$$\begin{array}{ccc} U \times_A U & \xrightarrow{\pi_0} & U \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ U & \xrightarrow[f]{} & A \end{array} \tag{C.1}$$

and

$$\begin{array}{ccc} V \times_B V & \xrightarrow{\bar\pi_0} & V \\ {\scriptstyle \bar\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ V & \xrightarrow[g]{} & B \end{array} \tag{C.2}$$

That an isomorphism $i\colon U \to V$ induces an isomorphism $U \times_A U \cong V \times_B V$ means that $g \circ i \circ \pi_0 = g \circ i \circ \pi_1$ and $f \circ i^{-1} \circ \bar\pi_0 = f \circ i^{-1} \circ \bar\pi_1$. Because f is an effective epimorphism we have that

$$U \times_A U \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} U \xrightarrow{f} A \tag{C.3}$$

is a coequalizer diagram. Thus there is a unique morphism $h\colon A \to B$ such that

$$h \circ f = g \circ i. \tag{C.4}$$

On the other hand

$$V \times_B V \underset{\bar\pi_1}{\overset{\bar\pi_0}{\rightrightarrows}} V \xrightarrow{g} B \tag{C.5}$$

is a coequalizer diagram because also g is an effective epimorphism. Thus we also have a unique morphism $\bar h\colon B \to A$ such that

$$\bar h \circ g = f \circ i^{-1}. \tag{C.6}$$

Combining (C.4) and (C.6) we get that $\bar h \circ h \circ g = g$ and $h \circ \bar h \circ f = f$. Because f and g are epimorphism this shows that $\bar h$ is the inverse morphism to h and thus the latter is the isomorphism we wanted to find. ∎

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