Eidgenössische Technische Hochschule Zürich
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ALGEBRA OF CONSTRAINTS FOR THE LINEARIZED PALATINI-CARTAN THEORY ON A LIGHT-LIKE BOUNDARY

by

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RESEARCH THESIS APPROVAL

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A mamma e papà

ABSTRACT

We present the analysis of the algebra of constraints for the classical linearized Palatini-Cartan theory. In the four dimensional case, on a light-like boundary the induced metric turns out to be degenerate. Because of the degeneracy, a non commuting constraint arises, reducing the total local degrees of freedom of the theory to only one, compared to the two local degrees of freedom in ordinary General Relativity. We also analyze the global invariants for a $S^2 \times \mathbb{R}$ surface in the Kruskal and Kerr space-times and give some motivations for identifying them with the mass and the spin of the black hole.

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Chapter 1

INTRODUCTION

General Relativity is originally formulated via a linear/Koszul affine connection ∇ over the tangent bundle TM. This leads to Einstein's field equations written in terms of the Ricci tensor $R_{\mu\nu}$, function of the Christoffel symbols Γ and thus of the metric $g_{\mu\nu}$, the Ricci scalar R and the metric $g_{\mu\nu}$. Christoffel symbols are by definition symmetric since the connection is assumed to be Levi-Civita: the unique metric compatible and torsion-free affine connection. In a variational setting, this corresponds to the action functional

$$S = \int R\sqrt{-g}\, d^4x. \tag{1.0.1}$$

From another perspective, we can consider the same action (1.0.1) (in the afore case called the Einstein-Hilbert action of the theory), but let also the connection ∇ be an independent variable of the theory, so that $S = S[g, \nabla]$. This translates into letting Γ be coefficients of a generic affine connection, in particular non-symmetric. If we then assume ∇ to be metric compatible, the variational principle ensures the torsion free condition (and vice versa). This theory is known as the Palatini theory (Reference [1]).

In this setting, General Relativity is not formulated as a Lie gauge theory, like the SU(N) theories of the Standard Model, because it lacks of a principal connection over a G-principal bundle. Here it comes the theory named after Cartan and refined by Sciama and Kibble (References [2, 3, 4]), which reformulates General Relativity via the bundle isomorphism (generated by the associated soldering form) called vierbein or tetrads. The theory can be formulated taking the action functional as given by the Ricci scalar written in terms of the Cartan variables, and thus functionals of both tetrads and local connections. Since this idea looks very close to the one of Palatini, namely to let the quantities representing the metric and the connection (in the present case tetrads and local connections) be independent variables in the action, this theory is known as Palatini-Cartan theory (discussion about the name in Reference [5]).

As mentioned in [6], the reduced phase space of a theory is the space of possible initial conditions (associated to a boundary) compatible with the constraints of the theory, together with its natural symplectic structure[1]. For instance, in the case of classical mechanics on a manifold M, the reduced phase space is the cotangent bundle T^*M with its canonical symplectic structure. In the case of electromagnetism on a pseudo Riemannian manifold, the initial phase space has canonical conjugate variables A_i and $E_i = F_{0i}$, with symplectic form induced by their pairing; field equations (Maxwell's equations) are not all dynamical, but some result in a constraint. The reduced phase space of the theory is then given by the solution to the Gauss law $\nabla_i E^i = 0$ (which is not a dynamical equation as a matter of fact) modulo gauge transformations. In the case of General Relativity the initial phase space is presented as the cotangent bundle of the space of metrics on the space-like hypersurface of initial conditions. The reduced phase space is then obtained as the solutions of some field equations working as constraints modulo diffeomorphisms.
Although a useful way of treating the above cases is due to Dirac (Reference [7]), we will follow the same geometric method of Reference [6], due to Kijowski and Tulczyjew (Reference [8]), applied to the case of the Palatini-Cartan theory of gravity. Indeed we will treat both the non-degenerate and the degenerate linearized cases (non-degenerate and degenerate boundary metric g_0^∂ of a fixed background).
This thesis will start with a little review of the boundary structure of the Palatini-Cartan theory (first three chapters) and continue as an original work from Chapter 4 on.

[1]Natural in the sense that it arises from taking the differential, over the space of fields, of the boundary terms coming from the variational principle and then quotiented with its kernel.

1.1 GENERALITIES

We follow the procedure presented in Reference [6]. We refer to a classical field theory as an assignment of a space of fields $\mathcal{F}_M$ and a local functional of such fields S_M to a manifold (possibly with boundary) $(M, \partial M)$. Generally, the space $\mathcal{F}_M$ could be some mapping space, or space of sections of a vector bundle or a sheaf. Applying the variational principle on the action functional, in order to obtain Euler-Lagrange equations, carries a integration by parts and this procedure defines a 1-form $\widetilde{\alpha}_{\partial M}$ on the space of pre-boundary fields $\widetilde{\mathcal{F}}_{\partial M}$, defined to be the restriction of fields and jets of fields to the boundary. This comes with a surjective submersion given by the restriction map $\mathcal{F}_M \xrightarrow{\widetilde{\pi}_M} \widetilde{\mathcal{F}}_{\partial M}$.
Practically, one considers (and takes the limit for $\epsilon \to 0$) the collar given by $\partial M \times [0, \epsilon]$; the space of fields associated to it maps to the space of pre-boundary fields $\mathcal{F}_{\partial M \times [0,\epsilon]} \to \widetilde{\mathcal{F}}_{\partial M}$.
In this context we can define what we call the space of boundary fields $\mathcal{F}^{\partial}_{\partial M}$ as the quotient:

$$\mathcal{F}^{\partial}_{\partial M} := \widetilde{\mathcal{F}}_{\partial M} \big/ \mathrm{Ker}(\widetilde{\omega}_{\partial M}), \tag{1.1.1}$$

where the kernel of $\widetilde{\omega}_{\partial M} := \delta \widetilde{\alpha}_{\partial M}$ (the pre-symplectic form[2]) is assumed to be a subbundle of $T\widetilde{\mathcal{F}}_{\partial M}$ and the reduction to be smooth. In this way we endow the space of boundary fields with a symplectic structure $\omega_{\partial M}$, removing the degeneracy given by $\mathrm{Ker}(\widetilde{\omega}_{\partial M})$.
If we denote the vanishing locus of the action functional in the bulk as EL_M and $\pi_M : \mathcal{F}_M \to \mathcal{F}^{\partial}_{\partial M}$ as the canonical projection, we can define $L_M := \pi_M(EL_M)$; which for the collar becomes $L_{\partial M \times [0,\epsilon]}$. In order to have a well defined theory, $L_{\partial M \times [0,\epsilon]}$ is required to be a lagrangian submanifold of the space $\mathcal{F}^{\partial}_{\partial M}$. Indeed, given an appropriate choice of boundary conditions, this is necessary for the existence of solutions.
We can define the reduced phase space $C_{\partial M} \subset \mathcal{F}^{\partial}_{\partial M}$ as the space of boundary fields that have the property of possibly being completed to a pair in $L_{\partial M \times [0,\epsilon]}$; then $C_{\partial M}$ must be a coisotropic submanifold (Reference [9]).

[2] It is defined as $\widetilde{\omega}_{\partial M} := \delta \widetilde{\alpha}_{\partial M}$ and its kernel can be non-trivial in general. This is why we call it pre-symplectic.

In a recent paper by Cattaneo and Schiavina (Reference [6]), the authors analyze the nature of the submanifold $C_{\partial M}$ associated with Palatini-Cartan theory of gravity (with the condition that coframes define a non-degenerate boundary metric g^{∂}) as given by the vanishing locus of some local functionals[3] on the space of fields. They conclude that this latter is a coisotropic submanifold equivalent to the one of the Einstein-Hilbert theory[4] after a Marsden-Weinstein symplectic reduction for the action on the space of boundary fields of the internal Lorentz group.

In the following thesis, we will display the general setting of the Palatini-Cartan theory (space of fields, action functionals, boundary structure) in Chapter 2 and specifically of the boundary structure of the non-degenerate case in Chapter 3; then we will analyze in details its algebra of constraints in the linearized case. In a first analysis, this will be done by assuming a non-degenerate boundary metric g_0^{∂} (Chapter 4) and it will be shown that the reduced phase space of the linearized theory actually corresponds to a coisotropic submanifold (Theorem 4.3.8). In Chapter 5, we will relax the non-degeneracy hypothesis. This corresponds to considering a light-like hypersurface[5] and tetrads on it; the pull back along such tetrads gives a degenerate g_0^{∂}. In order to give an explicit treatment of the problem, we will consider two different cases: Minkowski-degenerate and MGD-degenerate cases (Definitions 5.1.1 and 5.3.1). We will prove that the submanifold defined by constraints of Definition 5.4.1 is no longer coisotropic (Theorem 5.4.7); in fact, the degeneracy condition of the metric will translate into the presence of a new non commuting constraint. Finally, in Chapter 7, we will give some idea about what the global invariants of the linearized theory (on the hypersurface $S^2 \times \mathbb{R}$, event horizon of a black hole) could be: we will identify two invariant functionals and motivate how these could be identifiable with the mass/area and

[3]For general details about symplectic reduction see Reference [10].

[4]If tetrads are non-degenerate (but they could still give a degenerate boundary metric), Palatini-Cartan theory is equivalent to Einstein-Hilbert formulation, i.e. equivalent critical locus of the action functional.

[5]The most immediate surface any physicist would think of is the event horizon of a black hole.

the spin of a black hole.

Chapter 2

PALATINI-CARTAN THEORY

2.1 FIELD EQUATIONS

The mathematical background of Cartan theory is well explained in References [11, 12]. Let M be a 4-dimensional smooth manifold, then one can consider the $\mathrm{SO}(3,1)$-principal bundle $P = F_O(M)$ made up of orthogonal (co-)frames for a reference metric and then can construct the associated vector bundle $\mathcal{V} = F_O(M) \times_\rho V$ over M, where $\rho : \mathrm{SO(3,1)} \rightarrow \mathrm{Aut}(V)$ is the fundamental representation and V is a metric vector space with fixed reference metric $\eta = \mathrm{diag}(1, 1, 1, -1)$; We call a manifold M with such a choice of V a V-manifold. With this choice of a reference metric, the bundle $\mathcal{V}$ is called the Minkowski bundle.

The dynamical fields of the theory are tetrads $\widetilde{\Omega}^1(M, \mathcal{V}) \ni e : TM \xrightarrow{\sim} \mathcal{V}$ ($\sim$ stays for non-degenerate[1] or bundle isomorphism) and the local connection $\omega \in \Omega^1(U, \bigwedge^2 \mathcal{V})$ associated to a principal connection $\omega_P \in \Omega^1(F_O(M), \mathfrak{so}(3, 1))$, with $U \subset M$; the space of all local connections[2] is denoted with $\mathcal{A}$. Moreover, we define the exterior covariant derivative as $d_\omega \alpha := d\alpha + [\omega, \alpha]$ where $\alpha \in \Omega^n(M, \bigwedge^p \mathcal{V})$ and the brackets denoting every Lie algebra representation[3].

In this setting, we define the theory in the following[4]:

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[1] In three dimensions this non-degeneracy can be relaxed (Reference [13]).

[2] One should not be bothered from the fact that such connections are local; indeed we will deal with local calculations when performing derivatives and the integrated quantities are always well defined global quantities, e.g. the curvature $F_\omega \in \Omega^2(M, \bigwedge^2 \mathcal{V})$ and the covariant derivative of tetrads $d_\omega e \in \Omega^2(M, \mathcal{V})$. The reasons why this holds can be found in Reference [11].

[3] We will explicitly work in the fundamental representation where, e.g., for $\alpha \in \Omega^1(M, \mathcal{V})$ it reads $(d_\omega \alpha)^a_{\mu\nu} = (d\alpha^a)_{\mu\nu} + \omega^{ab}_{[\mu} \alpha^c_{\nu]} \eta_{bc}$.

[4] In this and the following section we will not give any proofs of the results. Such proofs and detailed calculations can be found in Reference [6].

Definition 2.1.1 (Palatini-Cartan theory)**.** *The classical Palatini-Cartan theory is the assignment of the pair $(\mathcal{F}_{PC}, S_{PC})_M$ to every pseudo riemannian 4-dimensional V-manifold*[5] *with space of fields*

$$\mathcal{F}_{PC} = \widetilde{\Omega}^1(M, \mathcal{V}) \times \mathcal{A} \ni (e, \omega) \tag{2.1.1}$$

and action

$$S_{PC} = \int_M \mathrm{Tr}\big[\frac{1}{2} e \wedge e \wedge F_\omega\big]. \tag{2.1.2}$$

Remark 2.1.2. *Here the wedge product is a map $\wedge : \Omega^n(M, \bigwedge^p \mathcal{V}) \times \Omega^l(M, \bigwedge^q \mathcal{V}) \to \Omega^{n+l}(M, \bigwedge^{p+q} \mathcal{V})$ and the trace is a map $\mathrm{Tr} : \bigwedge^4 \mathcal{V} \to \mathbb{R}$ such that given a basis $\{u_j\}$ of $\mathcal{V}$ it holds that $\mathrm{Tr}[u_i \wedge u_j \wedge u_k \wedge u_l] = \varepsilon_{ijkl}$, thus the trace works as a choice of the orientation*[6]*.*

Remark 2.1.3. *In our treatment, we do not take into account the presence of the cosmological constant Λ, which would join the action with a term proportional to $e \wedge e \wedge e \wedge e$.*

Euler-Lagrange equations of Palatini-Cartan theory read:

$$d_\omega(e \wedge e) = 0 \tag{2.1.3a}$$

$$e \wedge F_\omega = 0. \tag{2.1.3b}$$

We can observe that Equation (2.1.3a) is equivalent to $d_\omega e = 0$, since $e\wedge$ is injective in the bulk.

Remark 2.1.4. *The non-degeneracy of e guarantees the dynamical equivalence to Einstein-Hilbert formalism: Equation* (2.1.3b) *describes field equations for the metric $g = e^*\eta$, whereas* (2.1.3a) *is the torsion-free condition that ensures the connection to be Levi-Civita.*

[5] The choice of V-structure is immaterial, as a change of V-structure leads to an isomorphism of the space of fields compatible with the action functionals.

[6] We assume for simplicity that M is oriented and that for $\mathcal{V}$ it is given the induced orientation (by the isomorphism with TM).

Remark 2.1.5. *Notice that the map* $e \wedge \cdot : \Omega^{\bullet}(M, \bigwedge^{\bullet} \mathcal{V}) \to \Omega^{\bullet+1}(M, \bigwedge^{\bullet+1} \mathcal{V})$ *is not an isomorphism in general even though* e *is non-degenerate. As a matter of fact, Equation* (2.1.3b) *does not imply a flat connection.*

Remark 2.1.6. *In the following we will refer to a pair* $(e, \omega) \in \mathcal{F}_{PC}$ *as a solution of the Palatini-Cartan theory of Definition 2.1.1, if* (e, ω) *is a solution of field equations* (2.1.3).

2.2 BOUNDARY STRUCTURE

Let's consider the boundary ∂M of M. The space of fields of the theory is $\mathcal{F}_{PC} = \widetilde{\Omega}^1(M, \mathcal{V}) \times \mathcal{A}$; the boundary inclusion $\iota : \partial M \to M$ allows to define $\mathcal{V}^\partial := \iota^* \mathcal{V}$ and $P^\partial := \iota^* P$. This procedure allows to define the space of pre-boundary fields $\widetilde{\mathcal{F}}_{PC} := \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial) \times \mathcal{A}^\partial$ and consequentially the space of boundary fields via symplectic reduction $\mathcal{F}^\partial_{PC}$.

Remark 2.2.1. *The pre-boundary fields,* $e \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial)$ *and* $\omega \in \mathcal{A}^\partial$*, are denoted in the same way as the ones of* $\mathcal{F}_{PC}$.

Results of this and the following chapters rely on the properties (e.g. Lemma 2.2.4) of the two fundamental maps, that we define in the following:

Definition 2.2.2. *We define the map*

$$W_e^{(p,k)} : \Omega^p\big(\partial M, \bigwedge^k \mathcal{V}^\partial\big) \longrightarrow \Omega^{p+1}\big(\partial M, \bigwedge^{k+1} \mathcal{V}^\partial\big) \tag{2.2.1}$$

as

$$W_e^{(p,k)}(\,\cdot\,) := e \wedge \cdot\ . \tag{2.2.2}$$

Definition 2.2.3. *We define the map*

$$[\,\cdot\,, e] : \Omega^1\big(\partial M, \bigwedge^2 \mathcal{V}^\partial\big) \longrightarrow \Omega^2\big(\partial M, \mathcal{V}^\partial\big) \tag{2.2.3}$$

via the action of the bracket

$$[v, e]^a_{\mu\nu} := v^{ab}_{[\mu} e^c_{\nu]} \eta_{bc}. \tag{2.2.4}$$

Lemma 2.2.4. *The map* $W_e^{(p,k)}$ *is injective for* $p = k = 1$ *and it is surjective when* $(p,k) = (1,2)$ *or* $(p,k) = (2,1)$.

The space of boundary fields, after symplectic reduction[7], is given by the following theorem:

Theorem 2.2.5. *The classical space of boundary fields for the Palatini-Cartan theory is the symplectic manifold given by the fiber bundle*

$$\mathcal{F}^{\partial}_{PC} \to \widetilde{\Omega}^1(\partial M, \mathcal{V}^{\partial}) \tag{2.2.5}$$

with fibers over $e \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^{\partial})$ *given by the reduction* $\mathcal{A}^{\partial}_{red} := \mathcal{A}^{\partial}/\sim$*, where the equivalence relation is given by* $\omega' \sim \omega \Longleftrightarrow \omega' - \omega \in \mathrm{Ker}(W_e^{(1,2)})$*; and symplectic form*

$$\omega^{\partial}_{PC} = -\int_{\partial M} \mathrm{Tr}\big[\mathbf{e} \wedge \delta\mathbf{e} \wedge \delta\boldsymbol{\omega}\big]. \tag{2.2.6}$$

The surjective submersion $\pi_{PC} : \mathcal{F}_{PC} \to \mathcal{F}^{\partial}_{PC}$ *has the explicit form:*

$$\pi_{PC} : \begin{cases} \mathbf{e} = e \\ \\ \boldsymbol{\omega} = [\omega]_e, \end{cases} \tag{2.2.7}$$

with $e \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^{\partial})$ *and* $[\omega]_e \in \mathcal{A}^{\partial}_{red}$*. Moreover there exists a symplectomorphism* $\mathcal{F}^{\partial}_{PC} \to T^*\widetilde{\Omega}^1(\partial M, \mathcal{V}^{\partial})$ *because of the identification*

$$\boldsymbol{\Theta} := \mathbf{e} \wedge \boldsymbol{\omega} \in \Omega^2(\partial M, \textstyle\bigwedge^3 \mathcal{V}^{\partial}) \simeq \mathcal{A}^{\partial}_{red} \tag{2.2.8}$$

and the symplectic form reads

$$\omega^{\partial}_{PC} = -\int_{\partial M} \mathrm{Tr}\big[\delta\mathbf{e} \wedge \delta\boldsymbol{\Theta}\big]. \tag{2.2.9}$$

[7] For the Palatini-Cartan theory, the pre-symplectic form reads $\widetilde{\omega}_{PC} = -\int_{\partial M} \mathrm{Tr}\big[e \wedge \delta e \wedge \delta\omega\big]$ since the 1-form arising from the variational principle on the action (2.1.2) is $\widetilde{\alpha}_{PC} = -\frac{1}{2}\int_{\partial M} \mathrm{Tr}\big[e \wedge e \wedge \delta\omega\big]$.

Corollary 2.2.6. *The space given by the variables* $e \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial)$ *and* $\Theta := e \wedge \omega \in \Omega^2(\partial M, \bigwedge^3 \mathcal{V}^\partial)$ *is symplectomorphic to the space of boundary fields* $\mathcal{F}^\partial_{PC}$.

Remark 2.2.7. *In the present case, result of Corollary 2.2.6 is not fundamental: in the non linear theory we cannot remove the explicit dependence on the connection* ω *(and express all in terms of* Θ*) when writing constraints. The only way through is the one of defining constraints on a particular space, called* $\mathcal{S}$*; this space will turn out to be symplectomorphic to* $\mathcal{F}^\partial_{PC}$ *and, as such, ensures the Poisson brackets of constraints to be well defined. In the linearized degenerate case, Corollary 2.2.6 will be basic instead.*

Chapter 3

NON-DEGENERATE CASE

3.1 KERNELS AND IMAGES

In this section we start assuming the non-degeneracy condition for the metric $g^{\partial} := e^*\big|_{\partial M}\eta$. So, first of all we display all the results for the non-degenerate case and later, in the next sections, we will loose our hypotheses and let the metric be degenerate.

Lemma 3.1.1. *Consider the map* $[\,\cdot\,,e] : \Omega^1\big(\partial M, \bigwedge^2 \mathcal{V}^{\partial}\big) \longrightarrow \Omega^2\big(\partial M, \mathcal{V}^{\partial}\big)$ *and the boundary metric* g^{∂}*. Let* $\mathcal{K} := \mathrm{Ker}([\,\cdot\,,e]) \cap \mathrm{Ker}(W_e^{(1,2)})$*.*
We have that

$$\dim(\mathcal{K}) = 2\dim(\mathrm{Ker} g^{\partial}). \tag{3.1.1}$$

Moreover, if g^{∂} *is non-degenerate, the map* $[\,\cdot\,,e]$ *is surjective on* $\Omega^2\big(\partial M, \mathcal{V}^{\partial}\big)$*.*

Remark 3.1.2. *This Lemma is of fundamental importance since it states that, releasing the hypothesis of the non-degeneracy of* g^{∂}*, properties of maps* W_e *and* $[\,\cdot\,,e]$ *would dramatically change; as a matter of fact,* $[\,\cdot\,,e]\big|_{\mathrm{Ker}(W_e^{(1,2)})}$ *would not be an isomorphism any longer.*

Two important corollaries also follow:

Corollary 3.1.3. *If* g^{∂} *is non-degenerate, then we have the following short exact sequence:*

$$0 \longrightarrow \mathrm{Ker}(W_e^{(1,2)}) \xrightarrow{[\,\cdot\,,e]} \Omega^2\big(\partial M, \mathcal{V}^{\partial}\big) \xrightarrow{W_e^{(2,1)}} \Omega^3\big(\partial M, \bigwedge^2 \mathcal{V}^{\partial}\big) \longrightarrow 0. \tag{3.1.2}$$

Corollary 3.1.4. *Let* g^{∂} *be non-degenerate and* $p_{(2,1)} : \Omega^2\big(\partial M, \mathcal{V}^{\partial}\big) \to \mathrm{Ker}(W_e^{(2,1)})$ *the projection to the kernel. Then, given a pair* $(e,\omega) \in \widetilde{\mathcal{F}}_{PC}$*, there exists a unique* $v \in \mathrm{Ker}(W_e^{(1,2)})$ *such that*

$$[v,e] = p_{(2,1)}(d_\omega e). \tag{3.1.3}$$

Where this latter corollary is an immediate consequence of the fact that the map $[\,\cdot\,,e]\big|_{\mathrm{Ker}W_e^{(1,2)}} : \mathrm{Ker}W_e^{(1,2)} \to \mathrm{Ker}W_e^{(2,1)}$ is an isomorphism (see Corollary 3.1.3).
The central theorem of our discussion is the following:

Theorem 3.1.5. *Let g^∂ be non-degenerate, $(e,\omega) \in \widetilde{\mathcal{F}}_{PC}$ and $p_{(2,1)} : \Omega^2\big(\partial M, \mathcal{V}^\partial\big) \to \mathrm{Ker}(W_e^{(2,1)})$ the projection to the kernel. Then it follows that there exists a unique $v = v(\omega, e) \in \mathrm{Ker}(W_e^{(1,2)})$ such that for $\omega' = \omega + v$ we have:*

$$p_{(2,1)} d_{\omega'} e = 0 \tag{3.1.4}$$

and ω' satisfies the equation

$$d_{\omega'} e = 0 \tag{3.1.5}$$

if and only if $e \wedge d_\omega e = 0$.

Remark 3.1.6. *Thanks to Theorem 2.2.5, we can state that there exists a unique representative ω' of the equivalence class $[\omega]_e \in \mathcal{A}^\partial_{red}$ such that $p_{(2,1)} d_{\omega'} e = 0$ and, given $e \wedge d_\omega e = 0$, it holds that $d_{\omega'} e = 0$, as well as the converse.*

Definition 3.1.7. *We define $\mathcal{S} \subset \widetilde{\mathcal{F}}_{PC}$ as the image of the map $\varphi : \widetilde{\mathcal{F}}_{PC} \to \widetilde{\mathcal{F}}_{PC}$ characterized by $\varphi\big((\omega, e)\big) = (\omega', e)$ with $\omega' = \omega'(\omega, e)$ such that $p_{(2,1)} d_{\omega'} e = 0$ and $d_{\omega'} e = 0$ iff $e \wedge d_\omega e = 0$.*

Proposition 3.1.8. *We have*

$$\mathcal{S} = \{(\omega', e) \in \widetilde{\mathcal{F}}_{PC} \,\big|\, p_{(2,1)}(d_{\omega'} e) = 0\}, \tag{3.1.6}$$

and the map

$$\begin{aligned} \mathcal{F}^\partial_{PC} &\longrightarrow \mathcal{S} \\ ([\omega]_e, e) &\longmapsto (\omega', e) \end{aligned} \tag{3.1.7}$$

is a symplectomorphism.

Remark 3.1.9. *One only needs to prove the backward direction of the map* (3.1.7)*, since the other one is already given by Theorem 3.1.5.*

Remark 3.1.10. *We can split $d_{\omega'} e = 0$ into two projections: the structural constraint $p_{(2,1)}(d_{\omega'} e) = 0$ and the residual constraint $e \wedge d_{\omega'} e = 0$. As shown in Proposition 3.1.8, assuming g^∂ to be non-degenerate, the imposition of the structural constraint is completely equivalent to the symplectic reduction via the kernel of the pre-symplectic form.*

Remark 3.1.11. *Assuming a non-degenerate* g^∂*, it is possible to show how the residual constraints, given by* $e \wedge d_{\omega'} e = 0$ *and* $e \wedge F_{\omega'} = 0$ *(arising from field equations), define a coisotropic submanifold. In the next chapter, we will see how this result is implemented in a linearized perspective and, after releasing the hypothesis of a non-degenerate metric* g^∂*, we will check the aftereffects of this new assumption.*

Chapter 4

LINEARIZED THEORY

4.1 LINEARIZED FIELD EQUATIONS

Consider action (2.1.2) of Definition 2.1.1 and perform the following choices of tetrads and connection:

$$\begin{aligned} e &= e_0 + b \\ \omega &= \omega_0 + a, \end{aligned} \tag{4.1.1}$$

with (e_0, ω_0) a fixed solution of the Palatini-Cartan theory 2.1.1. We retain only the quadratic terms in a, b; thus:

$$S_{LPC} = \int_M \mathrm{Tr}\big[(\frac{1}{2} b \wedge b \wedge F_{\omega_0} + e_0 \wedge b \wedge d_{\omega_0} a + \frac{1}{4} e_0 \wedge e_0 \wedge [a, a])\big]. \tag{4.1.2}$$

This produces the following Euler-Lagrange equations:

$$e_0 \wedge (d_{\omega_0} b + [a, e_0]) = 0 \Leftrightarrow d_{\omega_0} b + [a, e_0] = 0 \tag{4.1.3a}$$

$$b \wedge F_{\omega_0} + e_0 \wedge d_{\omega_0} a = 0. \tag{4.1.3b}$$

Therefore we give the following definition:

Definition 4.1.1 (Linearized Palatini-Cartan theory)**.** *Let (e_0, ω_0) be a fixed solution of the Palatini-Cartan theory 2.1.1, $\mathcal{A}_L = \Omega^1(M, \bigwedge^2 \mathcal{V})$ and $\widetilde{\Omega}^1_0(M, \mathcal{V}) \subset \Omega^1(M, \mathcal{V})$ be such that if $b \in \widetilde{\Omega}^1_0(M, \mathcal{V})$ then $e = e_0 + b \in \widetilde{\Omega}^1(M, \mathcal{V})$. The classical linearized Palatini-Cartan theory, over the background given by (e_0, ω_0), is the assignment of the pair $(\mathcal{F}_{LPC}, S_{LPC})_M$ to every pseudo riemannian 4-dimensional V-manifold with space of fields*

$$\mathcal{F}_{LPC} = \widetilde{\Omega}^1_0(M, \mathcal{V}) \times \mathcal{A}_L \ni (b, a) \tag{4.1.4}$$

and action

$$S_{LPC} = \int_M \mathrm{Tr}\big[(\frac{1}{2} b \wedge b \wedge F_{\omega_0} + e_0 \wedge b \wedge d_{\omega_0} a + \frac{1}{4} e_0 \wedge e_0 \wedge [a, a])\big]. \tag{4.1.5}$$

4.2 LINEARIZED BOUNDARY STRUCTURE

Remark 4.2.1. *In the linearized case, we can use the same machinery we developed in the previous chapters: fundamental functions, given by background tetrads, are* $W_{e_0}^{(i,j)}$ *and* $[\,\cdot\,, e_0]$ *and equations defining constraints will be the linearized ones.*

Remark 4.2.2. *The pre-boundary fields,* $(b, a) \in \widetilde{\mathcal{F}}_{LPC} := \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial) \times \mathcal{A}_L^\partial$*, are denoted in the same way as the ones of* $\mathcal{F}_{LPC}$*; same for* $(e_0, \omega_0) \in \widetilde{\mathcal{F}}_{PC}$*.*

Following from Theorem 2.2.5 we can give the following Corollary:

Corollary 4.2.3 (of Theorem 2.2.5)**.** *The classical space of boundary fields for the linearized Palatini-Cartan theory is the symplectic manifold given by the fiber bundle*

$$\mathcal{F}_{LPC}^\partial \to \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial) \tag{4.2.1}$$

with fibers over $b \in \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial)$ *given by the reduction* $\mathcal{A}_{Lred}^\partial := \mathcal{A}_L^\partial / \sim$*, where the equivalence relation is given by* $a' \sim a \iff a' - a \in \operatorname{Ker}(W_{e_0}^{(1,2)})$*; and symplectic form*

$$\omega_{LPC}^\partial = -\int_{\partial M} \operatorname{Tr}\big[e_0 \wedge \delta\mathbf{b} \wedge \delta\boldsymbol{a}\big]. \tag{4.2.2}$$

The surjective submersion $\pi_{LPC} : \mathcal{F}_{LPC} \to \mathcal{F}_{LPC}^\partial$ *has the explicit form:*

$$\pi_{LPC} : \begin{cases} \mathbf{b} = b \\ \\ \boldsymbol{a} = [a]_{e_0}, \end{cases} \tag{4.2.3}$$

with $b \in \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial)$ *and* $[a]_{e_0} \in \mathcal{A}_{Lred}^\partial$*. Moreover there exists a symplectomorphism* $\mathcal{F}_{LPC}^\partial \to T^*\widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial)$ *because of the identification*

$$\boldsymbol{\Theta} := e_0 \wedge \boldsymbol{a} \in \Omega^2(\partial M, \textstyle\bigwedge^3 \mathcal{V}^\partial) \simeq \mathcal{A}_{Lred}^\partial \tag{4.2.4}$$

and the symplectic form reads

$$\omega_{LPC}^\partial = -\int_{\partial M} \operatorname{Tr}\big[\delta\mathbf{b} \wedge \delta\boldsymbol{\Theta}\big]. \tag{4.2.5}$$

4.3 COISOTROPIC SUBMANIFOLD: CONSTRAINTS ALGEBRA

In this section we will implement the results of the non-degenerate full theory to the linearized case. Therefore we will consider background boundary tetrads giving rise to a non-degenerate pulled back metric g_0^∂. Moreover, we will perform and compute the algebra of constraints, concluding that the reduced phase space of the linearized theory is coisotropic.

Remark 4.3.1. *Following from Definition 3.1.7 and Proposition 3.1.8, we can define the space* $\mathcal{S}_L$*, which is the linear counterpart of* $\mathcal{S}$*. This will be characterized by the background projection to the kernel and by the linear structural constraint on the space of pre-boundary fields* $\widetilde{\mathcal{F}}_{LPC}$*, as*

$$\mathcal{S}_L = \{(a', b) \in \widetilde{\mathcal{F}}_{LPC} \,\big|\, p_{(2,1)_0}(d_{\omega_0} b + [a', e_0]) = 0\}, \tag{4.3.1}$$

where $p_{(2,1)_0} : \Omega^2(\partial M, \mathcal{V}^\partial) \to \mathrm{Ker}(W_{e_0}^{(2,1)})$.
Moreover, there is a symplectomorphism such that $\mathcal{F}^\partial_{LPC} \simeq \mathcal{S}_L$.

Remark 4.3.2. *In this setting, residual constraints of the theory are given by* $e_0 \wedge (d_{\omega_0} b + [a', e_0]) = 0$ *and* $b \wedge F_{\omega_0} + e_0 \wedge d_{\omega_0} a' = 0$*. Of course they also follow directly from a linearization of constraints mentioned in Remark 3.1.11.*

Following from Remarks 3.1.11 and 4.3.2, in the linearized case, we can define the constraints of the theory as functionals on $\mathcal{S}_L$ (where the symplectic structure is well defined), as given by

$$\widetilde{J}_c = \int_{\partial M} \mathrm{Tr}\big[c \wedge e_0 \wedge (d_{\omega_0} b + [a', e_0])\big] \tag{4.3.2a}$$

$$\widetilde{L}_\lambda = \int_{\partial M} \mathrm{Tr}\big[\lambda \wedge (b \wedge F_{\omega_0} + e_0 \wedge d_{\omega_0} a')\big], \tag{4.3.2b}$$

where the lagrangian multipliers

$$\begin{cases} c & \in \Gamma(\bigwedge^2 \mathcal{V}^\partial) \;\; [1] \quad \text{odd} \\ \lambda & \in \Gamma(\mathcal{V}^\partial) \;\; [1] \quad \text{even} \end{cases} \tag{4.3.3}$$

are considered as graded variables with ghost number equal to 1 and with "odd" and "even" we mean with respect to the total grade.

Remark 4.3.3. *Corollary 4.2.3 assures that the space given by the variables* $b \in \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial)$ *and* $\Theta := e_0 \wedge a \in \Omega^2(\partial M, \bigwedge^3 \mathcal{V}^\partial)$ *is symplectomorphic to the space of boundary fields* $\mathcal{F}^\partial_{LPC}$*. As we stated previously, in the non linear case, we cannot express constraints as independent of* ω*, but this is possible in the linearized one instead. Therefore we have two equivalent formulations: one on* $\mathcal{S}_L$ *and one on the space of variables* b *and* Θ*.*

Definition 4.3.4 (Constraints of the non-degenerate LPC theory)**.** *The functionals defining the constraints of the non-degenerate linearized Palatini-Cartan theory are*

$$\widetilde{J}_c = \int_{\partial M} \mathrm{Tr}\big[c \wedge e_0 \wedge d_{\omega_0} b + \Theta \wedge [c, e_0]\big] \tag{4.3.4a}$$

$$\widetilde{L}_\lambda = \int_{\partial M} \mathrm{Tr}\big[\lambda \wedge (b \wedge F_{\omega_0} + d_{\omega_0}\Theta)\big]; \tag{4.3.4b}$$

and the symplectic form reads

$$\widetilde{\omega}^\partial = -\int_{\partial M} \mathrm{Tr}\big[\delta b \wedge \delta\Theta\big], \tag{4.3.5}$$

with $b \in \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial)$ *and* $\Theta := e_0 \wedge a \in \Omega^2(\partial M, \bigwedge^3 \mathcal{V}^\partial)$*.*

Remark 4.3.5. *We prefer using variables* b *and* Θ*, instead of working with variables in* $\mathcal{S}_L$*, because, even though in the non-degenerate case the two formulations are completely equivalent, in the degenerate case we will not be able to define* $\mathcal{S}_L$ *(at least not in the same way of the non-degenerate case) and therefore we will need to work with* b *and* Θ*.*
Therefore the dynamical variables of the linearized theory are

$$\begin{cases} b & \in \widetilde{\Omega}_0^1(\partial M, \mathcal{V}^\partial) \quad \textit{even} \\ \Theta := e_0 \wedge a & \in \Omega^2(\partial M, \bigwedge^3 \mathcal{V}^\partial) \quad \textit{odd.} \end{cases} \tag{4.3.6}$$

Remark 4.3.6. *From now on, we will switch the sign in front of the symplectic form. This will not affect the final result of the algebra of constraints.*

Lemma 4.3.7. *Assume ∂M to have no boundary. The hamiltonian fields associated to the constraints of the non-degenerate linearized Palatini-Cartan theory are*

$$\widetilde{J}_c : \quad \begin{cases} \mathbb{J}_b = [c, e_0] \\ \\ \mathbb{J}_\Theta = e_0 \wedge d_{\omega_0} c \end{cases} \tag{4.3.7}$$

and

$$\widetilde{L}_\lambda : \quad \begin{cases} \mathbb{L}_b = d_{\omega_0} \lambda \\ \\ \mathbb{L}_\Theta = \lambda \wedge F_{\omega_0}. \end{cases} \tag{4.3.8}$$

Proof. We have that:

- $\widetilde{\omega}^\partial = \int_{\partial M} \mathrm{Tr}\big[\delta b \wedge \delta \Theta\big]$ is the symplectic form;
- $\mathbb{F} = \mathbb{F}_b \wedge \dfrac{\delta}{\delta b} + \mathbb{F}_\Theta \wedge \dfrac{\delta}{\delta \Theta}$ is a hamiltonian vector field over the space of fields;
- and $\widetilde{F} = \widetilde{F}[b, \Theta]$ is a functional associated to the hamiltonian vector field $\mathbb{F}$.

Then it holds that

$$\iota_{\mathbb{F}} \widetilde{\omega}^\partial - \delta \widetilde{F} = 0. \tag{4.3.9}$$

In a general case, we have that

$$\delta \widetilde{F} = \int_{\partial M} \mathrm{Tr}\big[\big(\frac{\delta \widetilde{F}}{\delta b} \wedge \delta b + \frac{\delta \widetilde{F}}{\delta \Theta} \wedge \delta \Theta\big)\big], \tag{4.3.10}$$

and

$$\iota_{\mathbb{F}} \widetilde{\omega}^\partial = \int_{\partial M} \mathrm{Tr}\big[\big(\mathbb{F}_\Theta \wedge \delta b + \mathbb{F}_b \wedge \delta \Theta\big)\big]. \tag{4.3.11}$$

It follows

$$\mathbb{F}_b = \frac{\delta \widetilde{F}}{\delta \Theta}$$
$$\mathbb{F}_\Theta = \frac{\delta \widetilde{F}}{\delta b}. \tag{4.3.12}$$

For $\widetilde{J}_c$ we have:

$$\delta \widetilde{J}_c = \int_{\partial M} \mathrm{Tr}\big[(d_{\omega_0} c \wedge e_0 \wedge \delta b + [c, e_0] \wedge \delta\Theta)\big]; \tag{4.3.13}$$

and thus

$$\mathbb{J}_b = [c, e_0]$$
$$\mathbb{J}_\Theta = e_0 \wedge d_{\omega_0} c. \tag{4.3.14}$$

Similarly, for $\widetilde{L}_\lambda$ we have:

$$\delta \widetilde{L}_\lambda = \int_{\partial M} \mathrm{Tr}\big[\lambda \wedge (F_{\omega_0} \wedge \delta b - d_{\omega_0} \delta\Theta)\big] = \int_{\partial M} \mathrm{Tr}\big[(\lambda \wedge F_{\omega_0} \wedge \delta b + d_{\omega_0}\lambda \wedge \delta\Theta)\big], \tag{4.3.15}$$

from which it easily follows

$$\mathbb{L}_b = d_{\omega_0}\lambda$$
$$\mathbb{L}_\Theta = \lambda \wedge F_{\omega_0}. \tag{4.3.16}$$

✓

Theorem 4.3.8. *Let ∂M have no boundary and g_0^∂ be non-degenerate. Then the Poisson algebra of constraints of Definition 4.3.4 is abelian and therefore the vanishing locus of such constraints defines a coisotropic submanifold. In particular*

$$\{\widetilde{J}_c, \widetilde{J}_c\} = 0 \quad \{\widetilde{L}_\lambda, \widetilde{L}_\lambda\} = 0 \quad \{\widetilde{J}_c, \widetilde{L}_\lambda\} = 0. \tag{4.3.17}$$

Proof. Since

$$\{\widetilde{F}, \widetilde{G}\} := \iota_{\mathbb{F}} \iota_{\mathbb{G}} \widetilde{\omega}^\partial = \iota_{\mathbb{F}} \delta \widetilde{G}, \tag{4.3.18}$$

for $\{\widetilde{J}_c, \widetilde{J}_c\}$ we have:

$$\{\widetilde{J}_c, \widetilde{J}_c\} = \int_{\partial M} \mathrm{Tr}\big[2[c, e_0] \wedge e_0 \wedge d_{\omega_0} c\big], \tag{4.3.19}$$

which, using

$$\begin{aligned} d_{\omega_0}([c,c]\wedge e_0\wedge e_0) &= d_{\omega_0}[c,c]\wedge e_0\wedge e_0 \quad (d_{\omega_0}e_0=0)\\ &= 2[d_{\omega_0}c,c]\wedge e_0\wedge e_0\\ &= 2d_{\omega_0}c\wedge[c,e_0\wedge e_0]\\ &= 4[c,e_0]\wedge e_0\wedge d_{\omega_0}c, \end{aligned} \tag{4.3.20}$$

reduces to the integral of a total derivative, and thus

$$\{\widetilde{J}_c,\widetilde{J}_c\} = \int_{\partial M} \mathrm{Tr}\big[\frac{1}{2}d_{\omega_0}([c,c]\wedge e_0\wedge e_0)\big] = 0. \tag{4.3.21}$$

Similarly, for $\{\widetilde{L}_\lambda,\widetilde{L}_\lambda\}$ we have:

$$\{\widetilde{L}_\lambda,\widetilde{L}_\lambda\} = \int_{\partial M} \mathrm{Tr}\big[2d_{\omega_0}\lambda\wedge\lambda\wedge F_{\omega_0}\big], \tag{4.3.22}$$

which is equivalent to a total derivative as before, indeed

$$\begin{aligned} d_{\omega_0}(\lambda\wedge\lambda\wedge F_{\omega_0}) &= d_{\omega_0}(\lambda\wedge\lambda)\wedge F_{\omega_0} + \lambda\wedge\lambda\wedge d_{\omega_0}F_{\omega_0}\\ &= d_{\omega_0}(\lambda\wedge\lambda)\wedge F_{\omega_0} \quad (d_{\omega_0}F_{\omega_0}=0)\\ &= 2d_{\omega_0}\lambda\wedge\lambda\wedge F_{\omega_0}. \end{aligned} \tag{4.3.23}$$

Thus

$$\{\widetilde{L}_\lambda,\widetilde{L}_\lambda\} = \int_{\partial M} \mathrm{Tr}\big[d_{\omega_0}(\lambda\wedge\lambda\wedge F_{\omega_0})\big] = 0. \tag{4.3.24}$$

We consider the mixed Poisson bracket

$$\{\widetilde{L}_\lambda,\widetilde{J}_c\} = \int_{\partial M} \mathrm{Tr}\big[(d_{\omega_0}\lambda\wedge e_0\wedge d_{\omega_0}c + \lambda\wedge F_{\omega_0}\wedge[c,e_0])\big], \tag{4.3.25}$$

but, since

$$\begin{aligned} d_{\omega_0}(\lambda\wedge e_0\wedge d_{\omega_0}c) &= d_{\omega_0}\lambda\wedge e_0\wedge d_{\omega_0}c + \lambda\wedge e_0\wedge d^2_{\omega_0}c \qquad (d_{\omega_0}e_0=0)\\ &= d_{\omega_0}\lambda\wedge e_0\wedge d_{\omega_0}c + \lambda\wedge e_0\wedge[F_{\omega_0},c] \quad (d_{\omega_0}F_{\omega_0}=0 \text{ and } d^2_{\omega_0}\circ=[F_{\omega_0},\circ])\\ &= d_{\omega_0}\lambda\wedge e_0\wedge d_{\omega_0}c + \lambda\wedge F_{\omega_0}\wedge[c,e_0] \qquad (e_0\wedge F_{\omega_0}=0), \end{aligned}$$

$$\tag{4.3.26}$$

we have

$$\{\widetilde{L}_\lambda, \widetilde{J}_c\} = \int_{\partial M} \mathrm{Tr}\big[d_{\omega_0}(\lambda \wedge e_0 \wedge d_{\omega_0} c)\big] = 0. \tag{4.3.27}$$

✓

From the fact that $\mathcal{S}_L \simeq \mathcal{F}^\partial_{LPC}$, Theorem 4.3.8 immediately implies:

Corollary 4.3.9. *In the symplectic manifold*

$$\mathcal{F}^\partial_{LPC} \to \widetilde{\Omega}^1_0(\partial M, \mathcal{V}^\partial) \tag{4.3.28}$$

with symplectic form ω^∂_{LPC} *given by* (4.2.2)*, the vanishing locus* $C_{LPC} \subset \mathcal{F}^\partial_{LPC}$ *of functionals*

$$J_c = \int_{\partial M} \mathrm{Tr}\big[c \wedge e_0 \wedge d_{\omega_0}\mathbf{b} + \boldsymbol{\Theta} \wedge [c, e_0]\big]; \quad L_\lambda = \int_{\partial M} \mathrm{Tr}\big[\lambda \wedge (\mathbf{b} \wedge F_{\omega_0} + d_{\omega_0}\boldsymbol{\Theta})\big], \tag{4.3.29}$$

with $\boldsymbol{\Theta} := e_0 \wedge \mathbf{a}$*, is coisotropic.*

This proves that the reduced phase space of the non-degenerate linearized Palatini-Cartan theory is coisotropic. This of course also follows from the linearization of the result of Reference [6] on the non-degenerate Palatini-Cartan theory. However, this important sanity check serves as a basic starting point for the degenerate theory (which has not been studied in the non linearized case).

Chapter 5

DEGENERATE CASES

5.1 MINKOWSKI-DEGENERATE CASE

Here we release the hypothesis of a non-degenerate boundary metric. The case is implemented by considering a light-like boundary and therefore by some sort of background boundary tetrads which give a degenerate g_0^∂.

As a toy model, we consider the easiest possible degenerate tetrads, the one of the Minkowski background. We will refer to this case as Minkowski-degenerate.

These tetrads are defined in the following:

Definition 5.1.1 (Minkowski-degenerate tetrads)**.** *Let M be a pseudo Riemannian 4-dimensional manifold with coordinates $x = (x_1, x_2, x_3, x_4)$ and Minkowski metric $g_0 = \mathrm{diag}(1, 1, 1, -1)$ (Minkowski-degenerate case). We define coordinates $x_+ = x_3 + x_4$ and $x_- = x_3 - x_4$. Light-like boundary tetrads for the boundary given by $x_- = 0$ are*

$$e_0 : \begin{cases} e_+^a = \delta_3^a + \delta_4^a \\ \\ e_1^a = \delta_1^a \\ \\ e_2^a = \delta_2^a \,. \end{cases} \tag{5.1.1}$$

The pulled back boundary metric is therefore $g_0^\partial = \mathrm{diag}(0, 1, 1)$.

Within this set up, kernels and images of the functions W_{e_0} and $[\,\cdot\,, e_0]$ will be characterized by different relations, in particular Lemma 3.1.1 assures that, in the present case[1], $\dim(\mathcal{K}_0) = 2$. We need to characterize these spaces.

Proposition 5.1.2. *In the Minkowski-degenerate case, the kernel of the map $W_{e_0}^{(1,2)}$ is*

[1] $\mathcal{K}_0 := \mathrm{Ker}([\,\cdot\,, e_0]) \cap \mathrm{Ker}(W_{e_0}^{(1,2)})$.

characterized by the following relations on $v \in \Omega^1(\partial M, \bigwedge^2 \mathcal{V}^\partial)$*:*

$$\begin{aligned}
&v_1^{12} = v_+^{23} && v_1^{12} = v_+^{24}\\
&v_2^{12} = -v_+^{13} && v_2^{12} = -v_+^{14}\\
&v_2^{23} = -v_1^{13} && v_2^{24} = -v_1^{14}\\
&v_1^{23} = v_1^{24} && v_2^{13} = v_2^{14}\\
&v_1^{34} = 0 && v_2^{34} = 0\\
&v_+^{34} = v_1^{13} - v_1^{14} && v_+^{34} = v_2^{23} - v_2^{24}.
\end{aligned} \tag{5.1.2}$$

Thus $\dim(\mathrm{Ker}(W_{e_0}^{(1,2)})) = 18 - 12 = 6$ *as in the non-degenerate case.*

Proof. The proof is a straightforward calculation following from

$$\varepsilon_{abcd} e_0{}_{[\mu}^a v_{\nu]}^{bc} = 0. \tag{5.1.3}$$

✓

Proposition 5.1.3. *In the Minkowski-degenerate case, the kernel of* $W_{e_0}^{(2,1)}$ *is characterized by the following relations on* $\alpha \in \Omega^2(\partial M, \mathcal{V}^\partial)$*:*

$$\begin{aligned}
&\alpha_{12}^4 = \alpha_{12}^3 && \alpha_{12}^2 = \alpha_{+1}^4\\
&\alpha_{12}^2 = \alpha_{+1}^3 && \alpha_{12}^1 = \alpha_{2+}^4\\
&\alpha_{12}^1 = \alpha_{2+}^3 && \alpha_{+1}^1 = \alpha_{2+}^2.
\end{aligned} \tag{5.1.4}$$

Thus $\dim(\mathrm{Ker}(W_{e_0}^{(2,1)})) = 12 - 6 = 6$ *as in the non-degenerate case.*

Proof. The proof is a straightforward calculation following from

$$\varepsilon^{\mu\nu\sigma} e_0{}_\mu^{[a} \alpha_{\nu\sigma}^{b]} = e_0{}_+^{[a} \alpha_{12}^{b]} + e_0{}_2^{[a} \alpha_{+1}^{b]} + e_0{}_1^{[a} \alpha_{2+}^{b]} = 0. \tag{5.1.5}$$

✓

Proposition 5.1.4. *In the Minkowski-degenerate case, the image of the restriction of the map* $[\,\cdot\,, e_0]$ *to* $\mathrm{Ker}(W_{e_0}^{(1,2)})$ *is characterized by the following relations:*

$$\begin{aligned} &[v,e_0]^3_{12} = [v,e_0]^4_{12} \qquad &&[v,e_0]^2_{12} = [v,e_0]^3_{+1} = [v,e_0]^4_{+1} \\ &[v,e_0]^2_{+1} = [v,e_0]^1_{2+} \qquad &&[v,e_0]^1_{12} = [v,e_0]^3_{2+} = [v,e_0]^4_{2+} \\ &[v,e_0]^1_{+1} = [v,e_0]^2_{2+} \qquad &&[v,e_0]^1_{+1} = -[v,e_0]^2_{2+}, \end{aligned} \tag{5.1.6}$$

with $v \in \mathrm{Ker}(W_{e_0}^{(1,2)})$.

Proof. With a simple calculation following from computing

$$[v,e_0]^a_{\mu\nu} = v^{ae}_{[\mu} {e_0}^f_{\nu]} \eta_{ef} \quad \text{such that} \quad e_0 \wedge v = 0, \tag{5.1.7}$$

we get:

$$[v,e_0]^a_{12} = \begin{cases} v^{12}_1 & a=1 \\ v^{12}_2 & a=2 \\ v^{32}_1 - v^{31}_2 & a=3 \\ v^{32}_1 - v^{31}_2 & a=4, \end{cases} \tag{5.1.8}$$

$$[v,e_0]^a_{1+} = \begin{cases} v^{13}_1 - v^{14}_1 = 0 & a=1 \\ v^{12}_+ & a=2 \\ v^{21}_2 & a=3 \\ v^{21}_2 & a=4 \end{cases} \tag{5.1.9}$$

and

$$[v,e_0]^a_{2+} = \begin{cases} v^{21}_{+} & a=1 \\ v^{23}_2 - v^{24}_2 = 0 & a=2 \\ v^{12}_1 & a=3 \\ v^{12}_1 & a=4, \end{cases} \tag{5.1.10}$$

from which it easily follows the assertion. ✓

5.2 A NEW CONSTRAINT

Remark 5.2.1 (Recap). *In the non-degenerate linear case, we have:*

- $[\,\cdot\,, e_0]\big|_{\mathrm{Ker}W^{(1,2)}_{e_0}}$ *is injective;*
- $\mathrm{Im}([\,\cdot\,, e_0]\big|_{\mathrm{Ker}W^{(1,2)}_{e_0}}) = \mathrm{Ker}W^{(2,1)}_{e_0}$*;*
- *thus the map* $[\,\cdot\,, e_0]\big|_{\mathrm{Ker}W^{(1,2)}_{e_0}} : \mathrm{Ker}W^{(1,2)}_{e_0} \to \mathrm{Ker}W^{(2,1)}_{e_0}$ *is an isomorphism;*
- *therefore, given a pair* $(a,b) \in \widetilde{\mathcal{F}}_{LPC}$*, there is a unique* $v \in \mathrm{Ker}(W^{(1,2)}_{e_0})$ *such that* $[v,e_0] = p_{(2,1)0}(d_{\omega_0}b + [a,e_0]) = 0$*;*
- *and thus, given* $(a,b) \in \widetilde{\mathcal{F}}_{LPC}$*, there is a unique* $v \in \mathrm{Ker}(W^{(1,2)}_{e_0})$ *such that* $a' = a + v$ *satisfies:* $p_{(2,1)_0}(d_{\omega_0}b + [a', e_0]) = 0$ *and* $d_{\omega_0}b + [a', e_0] = 0$ *iff* $e_0 \wedge (d_{\omega_0}b + [a, e_0]) = 0$*.*

These results allowed to define the space $\mathcal{S}_L$ *and assert that it is symplectomorphic to the space of boundary fields* $\mathcal{F}^{\partial}_{LPC}$*.*

Remark 5.2.2. *In the degenerate case at hand, we notice that the image of the map* $[\,\cdot\,, e_0]\big|_{\mathrm{Ker}W^{(1,2)}_{e_0}}$ *does not coincide with the kernel of* $W^{(2,1)}_{e_0}$*, indeed, following from Propositions 5.1.3 and 5.1.4, there are two conditions left in the present Minkowski case:*

$$[v,e_0]^2_{+1} = [v,e_0]^1_{2+} \qquad [v,e_0]^1_{+1} = -[v,e_0]^2_{2+}. \tag{5.2.1}$$

This is expected from the first isomorphisms theorem:

$$\dim(\mathrm{Im}([\,\cdot\,,e_0]\big|_{\mathrm{Ker}W_{e_0}^{(1,2)}})) + \dim(\mathrm{Ker}([\,\cdot\,,e_0]\big|_{\mathrm{Ker}W_{e_0}^{(1,2)}})) = \dim(\mathrm{Ker}(W_{e_0}^{(1,2)})); \tag{5.2.2}$$

set that

- $\dim(\mathrm{Ker}(W_{e_0}^{(1,2)})) = 6$,
- $\dim(\mathrm{Ker}([\,\cdot\,,e_0]\big|_{\mathrm{Ker}W_{e_0}^{(1,2)}})) = 2$ *(because of Lemma 3.1.1),*

we get $\dim(\mathrm{Im}([\,\cdot\,,e_0]\big|_{\mathrm{Ker}W_{e_0}^{(1,2)}})) = 4$. *Thus, since we have* $\mathrm{Im}([\,\cdot\,,e_0]\big|_{\mathrm{Ker}W_{e_0}^{(1,2)}}) \subset \Omega^2(\partial M, \mathcal{V}^\partial)$ *we need* $12-4=8$ *conditions for characterizing this image, therefore* 2 *more conditions that the ones needed for* $\mathrm{Ker}(W_{e_0}^{(2,1)})$*, which are* $12-6=6$ *(given in Proposition 5.1.3).*

Therefore, if we want to adjust Theorem 3.1.5 to the Minkowski-degenerate case, we need to impose that $d_{\omega_0}b+[a,e_0]$ (such that $e_0\wedge(d_{\omega_0}b+[a,e_0])=0$) is compatible with the extra two conditions (5.2.1); we do this via another constraint of the theory:

$$\widetilde{R}_\tau := \int_{\partial M} \mathrm{Tr}\big[\tau\wedge(d_{\omega_0}b+[a,e_0])\big], \tag{5.2.3}$$

where

$$\tau\in\Omega^1(\partial M, \bigwedge^3 \mathcal{V}^\partial) \quad [1] \quad \text{odd} \tag{5.2.4}$$

and $\widetilde{R}_\tau[b,a+v]=\widetilde{R}_\tau[b,a]$ for $v\in\mathrm{Ker}(W_{e_0}^{(1,2)})$ (see Claim 5.2.4).
Therefore we can write

$$\begin{aligned}\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau_\mu^{abc}(d_{\omega_0}b+[a,e_0])_{\nu\sigma}^d = \varepsilon_{abcd}\big(&\tau_+^{abc}(d_{\omega_0}b+[a,e_0])_{12}^d+\tau_2^{abc}(d_{\omega_0}b+[a,e_0])_{+1}^d+\\ &+\tau_1^{abc}(d_{\omega_0}b+[a,e_0])_{2+}^d\big),\end{aligned} \tag{5.2.5}$$

and impose that it gives only the components (5.2.1) and thus cancels the others, by setting $\tau_+^{abc}=0\ \forall a,b,c$ and $\tau_\alpha^{123}=\tau_\alpha^{124}=0$ $(\alpha=1,2)$, obtaining

$$\begin{aligned}\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau_\mu^{abc}(d_{\omega_0}b+[a,e_0])_{\nu\sigma}^d = &\tau_2^{134}(d_{\omega_0}b+[a,e_0])_{+1}^2-\tau_1^{234}(d_{\omega_0}b+[a,e_0])_{2+}^1+\\ &+\tau_1^{134}(d_{\omega_0}b+[a,e_0])_{2+}^2+\tau_2^{234}(d_{\omega_0}b+[a,e_0])_{1+}^1.\end{aligned} \tag{5.2.6}$$

This gives other two constraints on τ that are summarized together with the other ones in the following 10 equations:

$$\begin{aligned}
&\tau_+^{abc} = 0 \quad \forall a, b, c \\
&\tau_\alpha^{123} = 0 \quad \alpha = 1, 2 \\
&\tau_\alpha^{124} = 0 \quad \alpha = 1, 2 \\
&\tau_1^{234} = \tau_2^{134} \\
&\tau_1^{134} = -\tau_2^{234}.
\end{aligned} \tag{5.2.7}$$

Remark 5.2.3. *The two conditions* (5.2.1) *are equivalent to taking* $[v, e]^2_{+1} = [v, e]^1_{2+}$ *and* $[v, e]^2_{2+} = 0$*; therefore we can state that constraints will not be univocally determined, but there might be a function* γ *(*$\neq -1$*) into last constraint, i.e.* $\tau_1^{134} = -\gamma\tau_2^{234}$*. Our reasonings still hold, one just needs to specify such function.*

Claim 5.2.4. *It is easy to prove that, given constraints* (5.2.7)*,* $\tau \wedge [v, e_0] = 0$ *for* $v \in \mathrm{Ker}(W_{e_0}^{(1,2)})$ *and thus* $\widetilde{R}_\tau$ *is independent on such a shift by* v*.*

We are only left with calculating the algebra for this new constraint. This will be done in the following section, where we will consider the most general diagonal case (MGD case). We will show that this new algebra is not abelian and, in particular, that $\widetilde{R}_\tau$ is not first class, unless τ is constant.

5.3 MOST GENERAL DIAGONAL DEGENERATE CASE

Definition 5.3.1 (MGD-degenerate tetrads)**.** *Let* M *be a pseudo Riemannian 4-dimensional manifold with coordinates* $x = (x_1, x_2, x_3, x_4)$ *and metric* $g_0 = \mathrm{diag}((g_{(1)})^2, (g_{(2)})^2, f^2, -f^2)$ *(MGD-degenerate case) with* $f = f(x)$*,* $g_{(\alpha)} = g_{(\alpha)}(x)$*. We define coordinates* $x_+ = x_3 + x_4$ *and* $x_- = x_3 - x_4$*. Light-like boundary tetrads for the* *boundary given by* $x_- = 0$ *are*

$$e_0 : \begin{cases} e^a_+ & = f(\delta^a_3 + \delta^a_4) \\ \\ e^a_\alpha & = g_{(\alpha)}\delta^a_\alpha \quad \alpha = 1, 2. \end{cases} \tag{5.3.1}$$

The pulled back metric is therefore $g^\partial_0 = \mathrm{diag}(0, (g_{(1)})^2, (g_{(2)})^2)$.

Within this set up, kernels and images of functions W_{e_0} and $[\,\cdot\,, e_0]$ will be characterized by different relations compared to the Minkowski case, in particular the conditions left, that in the previous case were given by (5.2.1), now are

$$[v, e]^2_{+1} = [v, e]^1_{2+} g_{(1)}/g_{(2)} \qquad [v, e]^1_{+1} = -[v, e]^2_{2+} g_{(1)}/g_{(2)}. \tag{5.3.2}$$

Remark 5.3.2. *The choice of the function* $g_{(1)}/g_{(2)}$ *in condition* $[v, e]^1_{+1} = -[v, e]^2_{2+} g_{(1)}/g_{(2)}$ *has been fixed to have only a sign symmetry within constraints* (5.3.3)*, but, as mentioned previously, it could also be fixed differently, as long as different from* $-g_{(1)}/g_{(2)}$.

Analogously to the Minkowski-degenerate case, constraints on τ are given by

$$\begin{aligned} &\tau^{abc}_+ = 0 \quad \forall a, b, c \\ &\tau^{123}_\alpha = 0 \quad \alpha = 1, 2 \\ &\tau^{124}_\alpha = 0 \quad \alpha = 1, 2 \\ &\tau^{234}_1 = \tau^{134}_2 g_{(1)}/g_{(2)} := -A \\ &\tau^{134}_1 = -\tau^{234}_2 g_{(1)}/g_{(2)} := -B. \end{aligned} \tag{5.3.3}$$

Detailed calculations are given in the Appendix A.

5.4 NON-COISOTROPIC SUBMANIFOLD: CONSTRAINTS ALGEBRA

Definition 5.4.1 (Constraints of the MGD-degenerate LPC theory)**.** *The functionals defining the constraints of the MGD-degenerate linearized Palatini-Cartan theory are*

$$\widetilde{J}_c = \int_{\partial M} \mathrm{Tr}\big[c \wedge e_0 \wedge d_{\omega_0} b + \Theta \wedge [c, e_0]\big] \tag{5.4.1a}$$

$$\widetilde{L}_\lambda = \int_{\partial M} \mathrm{Tr}\big[\lambda \wedge (b \wedge F_{\omega_0} + d_{\omega_0}\Theta)\big]; \tag{5.4.1b}$$

$$\widetilde{R}_\tau = \int_{\partial M} \mathrm{Tr}\big[\tau(A, B) \wedge (d_{\omega_0} b + [a, e_0])\big], \tag{5.4.1c}$$

where constraints on τ are given by (5.3.3)*.*

Remark 5.4.2. *Here we stress that $\widetilde{J}_c$ and $\widetilde{L}_\lambda$ depend on the 'right' variables, namely b and Θ, whereas $\widetilde{R}_\tau$ presents still some explicit dependence on a. We will prove in the following Lemma that this dependence is spurious and that $\widetilde{R}_\tau$ is a functional of only b and Θ. This is fundamental, since only the space given by variables b and Θ would give a well defined (thus non-degenerate) symplectic form.*

Lemma 5.4.3. *Let $\tau \in \Omega^1(\partial M, \bigwedge^3 \mathcal{V}^\partial)$ respect constraints* (5.3.3)*, $e_0 \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial)$ be of the kind* (5.3.1) *and $s \in \Omega^1(\partial M, \bigwedge^2 \mathcal{V}^\partial)$. Then we have*

$$\int_{\partial M} \mathrm{Tr}\big[\tau \wedge [s, e_0]\big] = \int_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}\Theta^{234}_{+1} + \Theta^{134}_{+2}) + B(\frac{g_{(2)}}{g_{(1)}}\Theta^{134}_{+1} - \Theta^{234}_{+2})\big) d^3x, \tag{5.4.2}$$

where $\Theta := e_0 \wedge s \in \Omega^2(\partial M, \bigwedge^3 \mathcal{V}^\partial)$ and A, B defined in (5.3.3)*.*

Proof. In the following we will denote e_0 with e for the sake of simplicity.

We see that

$$
\begin{aligned}
\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau_{\mu}^{abc}s_{\nu}^{de}e_{\sigma}^{f}\eta_{ef} &= \varepsilon^{\mu\nu\sigma}\big((\tau_{\mu}^{134}(s_{\nu}^{21}e_{\sigma}^{1}+s_{\nu}^{23}e_{\sigma}^{3}-s_{\nu}^{24}e_{\sigma}^{4})- \\
&\quad -\tau_{\mu}^{234}(s_{\nu}^{12}e_{\sigma}^{1}+s_{\nu}^{13}e_{\sigma}^{3}-s_{\nu}^{14}e_{\sigma}^{4})\big) \\
&= \varepsilon^{\mu\nu\sigma}(-\tau_{\mu}^{134}s_{\nu}^{12}e_{\sigma}^{1}+\tau_{\mu}^{234}s_{\nu}^{21}e_{\sigma}^{2}-\tau_{\mu}^{134}s_{\nu}^{32}e_{\sigma}^{3} \\
&\quad +\tau_{\mu}^{234}s_{\nu}^{31}e_{\sigma}^{3}+\tau_{\mu}^{134}s_{\nu}^{42}e_{\sigma}^{4}-\tau_{\mu}^{234}s_{\nu}^{41}e_{\sigma}^{4}) \\
&= -\tau_{1}^{134}s_{[2}^{12}e_{+]}^{1}-\tau_{2}^{134}s_{[+}^{12}e_{1]}^{1}+\tau_{1}^{234}s_{[2}^{21}e_{+]}^{2}+\tau_{2}^{234}s_{[+}^{21}e_{1]}^{2} \\
&\quad -\tau_{1}^{134}s_{[2}^{32}e_{+]}^{3}-\tau_{2}^{134}s_{[+}^{32}e_{1]}^{3}+\tau_{1}^{234}s_{[2}^{31}e_{+]}^{3}+\tau_{2}^{234}s_{[+}^{31}e_{1]}^{3} \\
&\quad +\tau_{1}^{134}s_{[2}^{42}e_{+]}^{4}+\tau_{2}^{134}s_{[+}^{42}e_{1]}^{4}-\tau_{1}^{234}s_{[2}^{41}e_{+]}^{4}-\tau_{2}^{234}s_{[+}^{41}e_{1]}^{4} \\
&= -\tau_{2}^{134}s_{+}^{12}e_{1}^{1}+\tau_{2}^{234}s_{+}^{12}e_{2}^{2}+\tau_{1}^{134}s_{2}^{23}e_{+}^{3}-\tau_{2}^{134}s_{1}^{23}e_{+}^{3} \\
&\quad -\tau_{1}^{234}s_{2}^{13}e_{+}^{3}+\tau_{2}^{234}s_{1}^{13}e_{+}^{3}-\tau_{1}^{134}s_{2}^{24}e_{+}^{4}+\tau_{2}^{134}s_{1}^{24}e_{+}^{4} \\
&\quad +\tau_{1}^{234}s_{2}^{14}e_{+}^{4}-\tau_{2}^{234}s_{1}^{14}e_{+}^{4},
\end{aligned} \tag{5.4.3}
$$

where in last equality we use the fact that our tetrads are diagonal.

We now insert their explicit form (5.3.1)

$$
\begin{aligned}
\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau_{\mu}^{abc}s_{\nu}^{de}e_{\sigma}^{f}\eta_{ef} &= -\tau_{2}^{134}s_{+}^{12}g_{(1)}+\tau_{2}^{234}s_{+}^{12}g_{(2)}+\tau_{1}^{134}s_{2}^{23}f-\tau_{2}^{134}s_{1}^{23}f \\
&\quad -\tau_{1}^{234}s_{2}^{13}f+\tau_{2}^{234}s_{1}^{13}f-\tau_{1}^{134}s_{2}^{24}f+\tau_{2}^{134}s_{1}^{24}f \\
&\quad +\tau_{1}^{234}s_{2}^{14}f-\tau_{2}^{234}s_{+}^{14}f \\
&= -Bfs_{2}^{23}+Af\frac{g_{(2)}}{g_{(1)}}s_{1}^{23}+Afs_{2}^{13}+Bf\frac{g_{(2)}}{g_{(1)}}s_{1}^{13} \\
&\quad +Bfs_{2}^{24}-Af\frac{g_{(2)}}{g_{(1)}}s_{1}^{14}-Afs_{2}^{14}-Bf\frac{g_{(2)}}{g_{(1)}}s_{1}^{14} \\
&= A\big(\frac{g_{(2)}}{g_{(1)}}f(s_{1}^{23}-s_{1}^{24})+f(s_{2}^{13}-s_{2}^{14})\big) \\
&\quad +B\big(\frac{g_{(2)}}{g_{(1)}}(f(s_{1}^{13}-s_{1}^{14})-s_{+}^{34}g_{(1)})-(-s_{+}^{34}g_{(2)}+f(s_{2}^{23}-s_{2}^{24}))\big).
\end{aligned} \tag{5.4.4}
$$

We give now some non-vanishing components of Θ:

$$\begin{cases} \Theta^{234}_{+1} & = e^2_{[+}s^{34}_{1]} + e^4_{[+}s^{23}_{1]} + e^3_{[+}s^{42}_{1]} = f(s^{23}_1 - s^{24}_1) \\ \\ \Theta^{134}_{+2} & = e^1_{[+}s^{34}_{2]} + e^4_{[+}s^{13}_{2]} + e^3_{[+}s^{41}_{2]} = f(s^{13}_2 - s^{14}_2) \\ \\ \Theta^{134}_{+1} & = e^1_{[+}s^{34}_{1]} + e^4_{[+}s^{13}_{1]} + e^3_{[+}s^{41}_{1]} = f(s^{13}_1 - s^{14}_1) - s^{34}_+ g_{(1)} \\ \\ \Theta^{234}_{+2} & = e^2_{[+}s^{34}_{2]} + e^4_{[+}s^{23}_{2]} + e^3_{[+}s^{42}_{2]} = f(s^{23}_2 - s^{24}_2) - s^{34}_+ g_{(2)} \end{cases} \tag{5.4.5}$$

and we can thus identify them with last equality in (5.4.4). Therefore we finally get

$$\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau^{abc}_{\mu}s^{de}_{\nu}e^f_{\sigma}\eta_{ef} = A(\frac{g_{(2)}}{g_{(1)}}\Theta^{234}_{+1} + \Theta^{134}_{+2}) + B(\frac{g_{(2)}}{g_{(1)}}\Theta^{134}_{+1} - \Theta^{234}_{+2}) \tag{5.4.6}$$

and thus the assertion. ✓

Lemma 5.4.4. *The hamiltonian field associated to constraint $\widetilde{R}_\tau$ of Definition 5.4.1 is given by*

$$\widetilde{R}_\tau : \quad \begin{cases} \mathbb{R}_\Theta = d_{\omega_0}\tau \\ \\ (\mathbb{R}_b)^1_2 = -\frac{g_{(2)}}{g_{(1)}}A \\ \\ (\mathbb{R}_b)^2_1 = -A \\ \\ (\mathbb{R}_b)^2_2 = -\frac{g_{(2)}}{g_{(1)}}B \\ \\ (\mathbb{R}_b)^1_1 = B. \end{cases} \tag{5.4.7}$$

Proof. We can write such constraint using Lemma 5.4.3 and take the variation

$$\delta\widetilde{R}_\tau = \int_{\partial M} \mathrm{Tr}\big[d_{\omega_0}\tau \wedge \delta b\big] - \int_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}\delta\Theta^{234}_{+1} + \delta\Theta^{134}_{+2}) + B(\frac{g_{(2)}}{g_{(1)}}\delta\Theta^{134}_{+1} - \delta\Theta^{234}_{+2})\big)d^3x, \tag{5.4.8}$$

from which it easily follows the assertion. ✓

Proposition 5.4.5. *Constraint $\widetilde{R}_\tau$ of Definition 5.4.1 is not first class, unless τ is constant.*

Proof. We only need to calculate the Poisson bracket of $\widetilde{R}_\tau$ with itself:

$$\begin{aligned}
\{\widetilde{R}_\tau, \widetilde{R}_\tau\} &= \int\limits_{\partial M} -2\big(A(\frac{g_{(2)}}{g_{(1)}}(d_{\omega_0}\tau)^{234}_{+1} + (d_{\omega_0}\tau)^{134}_{+2}) \\
&\qquad + B(\frac{g_{(2)}}{g_{(1)}}(d_{\omega_0}\tau)^{134}_{+1} - (d_{\omega_0}\tau)^{234}_{+2})\big)d^3x \\
&= \int\limits_{\partial M} -2\big(A(\frac{g_{(2)}}{g_{(1)}}d_{\omega_0[+}\tau^{234}_{1]} + d_{\omega_0[+}\tau^{134}_{2]}) \\
&\qquad + B(\frac{g_{(2)}}{g_{(1)}}d_{\omega_0[+}\tau^{134}_{1]} - d_{\omega_0[+}\tau^{234}_{2]})\big)d^3x \\
&= \int\limits_{\partial M} 2\big(A(\frac{g_{(2)}}{g_{(1)}}\partial_+ A + \partial_+(\frac{g_{(2)}}{g_{(1)}}A)) \\
&\qquad + B(\frac{g_{(2)}}{g_{(1)}}\partial_+ B + \partial_+(\frac{g_{(2)}}{g_{(1)}}B))\big)d^3x \\
&\neq 0 \qquad \text{in general.}
\end{aligned} \tag{5.4.9}$$

If τ is constant, this reduces to zero. ✓

We want to introduce a condition which will simplify a little bit the computations, in particular it will ensure that the first class constraints $\widetilde{J}_c$ and $\widetilde{L}_\lambda$ will commute with the new constraint $\widetilde{R}_\tau$.

Lemma 5.4.6. *Let $\tau \in \Omega^1(\partial M, \bigwedge^3 \mathcal{V}^\partial)$ respect constraints* (5.3.3) *and $e_0 \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial)$ of the kind* (5.3.1)*. Then if $\partial_+ e^a_{0\mu} = 0\ \forall \mu, a$ we have*

$$\begin{aligned}
\int\limits_{\partial M} \mathrm{Tr}\big[[\tau, F_{\omega_0}] \wedge \lambda\big] = \int\limits_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{234}_{+1} + (\lambda \wedge F_{\omega_0})^{134}_{+2}) + \\
+ B(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{134}_{+1} - (\lambda \wedge F_{\omega_0})^{234}_{+2})\big)d^3x.
\end{aligned} \tag{5.4.10}$$

Proof. The proof is rather similar to the one of Lemma 5.4.3.

We have

$$\varepsilon_{abcd}\varepsilon^{\mu\nu\sigma}\tau_\mu^{abe}F^{fc}_{\omega_0\,\nu\sigma}\eta_{ef}\lambda^d = -\tau_1^{134}F^{12}_{\omega_0\,2+}\lambda^1 - \tau_2^{134}F^{12}_{\omega_0\,+1}\lambda^1 + \tau_1^{234}F^{21}_{\omega_0\,2+}\lambda^2 + \tau_2^{234}F^{21}_{\omega_0\,+1}\lambda^2$$
$$- \tau_1^{134}F^{32}_{\omega_0\,2+}\lambda^3 - \tau_2^{134}F^{32}_{\omega_0\,+1}\lambda^3 + \tau_1^{234}F^{31}_{\omega_0\,2+}\lambda^3 + \tau_2^{234}F^{31}_{\omega_0\,+1}\lambda^3$$
$$+ \tau_1^{134}F^{42}_{\omega_0\,2+}\lambda^4 + \tau_2^{134}F^{42}_{\omega_0\,+1}\lambda^4 - \tau_1^{234}F^{41}_{\omega_0\,2+}\lambda^4 - \tau_2^{234}F^{41}_{\omega_0\,+1}\lambda^4. \tag{5.4.11}$$

We give now the following components of $\lambda \wedge F_{\omega_0}$:

$$\begin{cases} (\lambda \wedge F_{\omega_0})^{234}_{+1} &= \lambda^2 F^{34}_{\omega_0\,+1} + \lambda^4 F^{23}_{\omega_0\,+1} + \lambda^3 F^{42}_{\omega_0\,+1} \\ (\lambda \wedge F_{\omega_0})^{134}_{+2} &= \lambda^1 F^{34}_{\omega_0\,+2} + \lambda^4 F^{13}_{\omega_0\,+2} + \lambda^3 F^{41}_{\omega_0\,+2} \\ (\lambda \wedge F_{\omega_0})^{134}_{+1} &= \lambda^1 F^{34}_{\omega_0\,+1} + \lambda^4 F^{13}_{\omega_0\,+1} + \lambda^3 F^{41}_{\omega_0\,+1} \\ (\lambda \wedge F_{\omega_0})^{234}_{+2} &= \lambda^2 F^{34}_{\omega_0\,+2} + \lambda^4 F^{23}_{\omega_0\,+2} + \lambda^3 F^{42}_{\omega_0\,+2} \end{cases} \tag{5.4.12}$$

We calculate F_{ω_0} using the equation

$$F^{ab}_{\mu\nu} = e^a_\rho R^\rho_{\mu\nu\sigma}\bar{e}^\sigma_c \eta^{cb}, \tag{5.4.13}$$

which relates it with the Riemann tensor.

We start calculating Riemann tensor from Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\tau}(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}), \tag{5.4.14}$$

for the metric

$$(g^\partial)_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (g_{(1)})^2 & 0 \\ 0 & 0 & (g_{(2)})^2 \end{pmatrix}. \tag{5.4.15}$$

This gives

$$\begin{aligned}
\Gamma^1_{+1} &= \partial_+ \ln\big(g_{(1)}\big) & \Gamma^2_{+2} &= \partial_+ \ln\big(g_{(2)}\big) \\
\Gamma^1_{12} &= \partial_2 \ln\big(g_{(1)}\big) & \Gamma^2_{12} &= \partial_1 \ln\big(g_{(2)}\big) \\
\Gamma^1_{22} &= -\partial_1 \ln\big(g_{(2)}\big) & \Gamma^2_{22} &= \partial_2 \ln\big(g_{(2)}\big) \\
\Gamma^1_{11} &= \partial_1 \ln\big(g_{(1)}\big) & \Gamma^2_{11} &= -\partial_2 \ln\big(g_{(1)}\big).
\end{aligned} \tag{5.4.16}$$

We now use

$$R^\rho{}_{\mu\nu\sigma} = \partial_{[\mu}\Gamma^\rho{}_{\nu]\sigma} + \Gamma^\rho{}_{\lambda[\mu}\Gamma^\lambda{}_{\nu]\sigma}, \tag{5.4.17}$$

for calculating Riemann tensor. We give some components

$$\begin{aligned}
&R^+{}_{+1+} = 0 \quad R^+{}_{+2+} = 0 \\
&R^1{}_{+12} = \partial_+\partial_2 \ln\big(g_{(1)}\big) + \partial_2 \ln\big(g_{(1)}\big)\partial_+ \ln\big(g_{(1)}/g_{(2)}\big) \\
&R^1{}_{+22} = -\partial_+\partial_1 \ln\big(g_{(2)}\big) + \partial_1 \ln\big(g_{(2)}\big)\partial_+ \ln\big(g_{(2)}/g_{(1)}\big).
\end{aligned} \tag{5.4.18}$$

Using (5.4.13) we can calculate F_{ω_0}, this will be expressed by the components of the Riemann tensor; here we give some components:

$$\begin{aligned}
&F^{34}_{\omega_0 +1} = 0 & &F^{34}_{\omega_0 +2} = 0 \\
&F^{12}_{\omega_0 +1} = g_{(1)}/g_{(2)} R^1{}_{+12} & &F^{12}_{\omega_0 +2} = g_{(1)}/g_{(2)} R^1{}_{+22}.
\end{aligned} \tag{5.4.19}$$

plus other in general non-vanishing components, because of a non-vanishing Riemann tensor. However, in case of $\partial_+ e^a_{0\mu} = 0$, it is easy to see that

$$F^{ab}_{\omega_0 +\mu} = 0 \quad \text{for all } \mu, a, b. \tag{5.4.20}$$

This proves the Lemma trivially. ✓

Theorem 5.4.7. *Let ∂M have no boundary and the components of the tetrad $e_0 \in \widetilde{\Omega}^1(\partial M, \mathcal{V}^\partial)$ of the kind* (5.3.1) *be such that $\partial_+ e^a_{0\mu} = 0\ \forall \mu, a$. Then the algebra of constraints 5.4.1 is given by*

$$\{\widetilde{R}_\tau, \widetilde{R}_\tau\} = \int_{\partial M} 2\Big(A\big(\frac{g_{(2)}}{g_{(1)}}\partial_+ A + \partial_+\big(\frac{g_{(2)}}{g_{(1)}}A\big)\big) + B\big(\frac{g_{(2)}}{g_{(1)}}\partial_+ B + \partial_+\big(\frac{g_{(2)}}{g_{(1)}}B\big)\big)\Big)d^3x, \tag{5.4.21}$$

whereas all other brackets are vanishing.

Therefore, the vanishing locus of such constraints defines a non-coisotropic submanifold for non-constant τ.

Proof. The submanifold is non-coisotropic because we already showed the existence of a non commuting constraint. We just need to perform brackets of $\widetilde{J}_c$ and $\widetilde{L}_\lambda$ with $\widetilde{R}_\tau$ because other brackets are given by Theorem 4.3.8 and Proposition 5.4.5.

The Poisson bracket $\{\widetilde{J}_c, \widetilde{R}_\tau\}$ is given by

$$\begin{aligned}\{\widetilde{J}_c, \widetilde{R}_\tau\} = \int_M \mathrm{Tr}\big[d_{\omega_0}\tau \wedge [c, e_0]\big] - \int_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}(e_0 \wedge d_{\omega_0} c)^{234}_{+1} + (e_0 \wedge d_{\omega_0} c)^{134}_{+2}) + \\ + B(\frac{g_{(2)}}{g_{(1)}}(e_0 \wedge d_{\omega_0} c)^{134}_{+1} - (e_0 \wedge d_{\omega_0} c)^{234}_{+2})\big) d^3x.\end{aligned} \tag{5.4.22}$$

We notice that

$$\begin{aligned}\int_{\partial M} \mathrm{Tr}\big[d_{\omega_0} \wedge \tau[c, e_0]\big] &= \int_{\partial M} \mathrm{Tr}\big[\tau \wedge d_{\omega_0}[c, e_0]\big] \\ &= \int_{\partial M} \mathrm{Tr}\big[\tau \wedge [d_{\omega_0} c, e_0]\big] + \int_{\partial M} \mathrm{Tr}\big[\cancel{\tau \wedge [c, d_{\omega_0} e_0]}\big] \\ &= \int_{\partial M} \mathrm{Tr}\big[\tau \wedge [d_{\omega_0} c, e_0]\big]\end{aligned} \tag{5.4.23}$$

and because of Lemma 5.4.3

$$\begin{aligned}\int_{\partial M} \mathrm{Tr}\big[\tau \wedge [d_{\omega_0} c, e_0]\big] = \int_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}(e_0 \wedge d_{\omega_0} c)^{234}_{+1} + (e_0 \wedge d_{\omega_0} c)^{134}_{+2}) + \\ + B(\frac{g_{(2)}}{g_{(1)}}(e_0 \wedge d_{\omega_0} c)^{134}_{+1} - (e_0 \wedge d_{\omega_0} c)^{234}_{+2})\big) d^3x.\end{aligned} \tag{5.4.24}$$

Therefore

$$\{\widetilde{J}_c, \widetilde{R}_\tau\} = 0. \tag{5.4.25}$$

On the other hand, the Poisson bracket $\{\widetilde{L}_\lambda, \widetilde{R}_\tau\}$ is given by

$$\begin{aligned}\{\widetilde{L}_\lambda, \widetilde{R}_\tau\} &= \int\limits_{\partial M} \mathrm{Tr}\big[d_{\omega_0}\lambda \wedge d_{\omega_0}\tau\big] - \int\limits_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{234}_{+1} + (\lambda \wedge F_{\omega_0})^{134}_{+2})+ \\ &\qquad + B(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{134}_{+1} - (\lambda \wedge F_{\omega_0})^{234}_{+2})\big) d^3x \\ &= \int\limits_{\partial M} \mathrm{Tr}\big[[\tau, F_{\omega_0}] \wedge \lambda\big] - \int\limits_{\partial M} \big(A(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{234}_{+1} + (\lambda \wedge F_{\omega_0})^{134}_{+2})+ \\ &\qquad + B(\frac{g_{(2)}}{g_{(1)}}(\lambda \wedge F_{\omega_0})^{134}_{+1} - (\lambda \wedge F_{\omega_0})^{234}_{+2})\big) d^3x,\end{aligned} \tag{5.4.26}$$

where in last equality we used $D_0^2 \circ = [F_{\omega_0}, \circ]$. This Poisson bracket is vanishing because of the Lemma 5.4.6. ✓

Chapter 6

PHYSICAL INTERPRETATIONS

6.1 SCHWARZSCHILD/KRUSKAL CASE

We consider boundary tetrads of Schwarzschild space-time in Kruskal coordinates (References [14, 15]), where the boundary is the $S^2 \times \mathbb{R}$ surface (event horizon) given by $r = 2m$:

$$e_0 : \begin{cases} e_+^a & = 4me^{-\frac{1}{2}}(\delta_3^a + \delta_4^a) \\ e_1^a & = 2m\delta_1^a \\ e_2^a & = 2m\sin\theta\delta_2^a. \end{cases} \tag{6.1.1}$$

In the Appendix B one can find all the details leading to these tetrads.

These tetrads are a special case of (5.3.1) with $f = 4me^{-\frac{1}{2}}$, $g_{(1)} = 2m$ and $g_{(2)} = 2m\sin\theta$. Therefore relations for τ become

$$\begin{aligned} &\tau_+^{abc} = 0 \quad \forall a, b, c \\ &\tau_\alpha^{123} = 0 \quad \alpha = 1, 2 \\ &\tau_\alpha^{124} = 0 \quad \alpha = 1, 2 \\ &\tau_1^{234} = \tau_2^{134} / \sin\theta \\ &\tau_1^{134} = -\tau_2^{234} / \sin\theta. \end{aligned} \tag{6.1.2}$$

Remark 6.1.1. *Following from the results of previous section (all hypotheses are compatible with the present case), we can state that constraints over an event horizon of a static black hole define a non-coisotropic submanifold, in particular, since the non-first class constraint* $\widetilde{R}_\tau$ *does not generate any gauge transformation and thus counts as "-1" local degree of freedom (Reference [16]), local degrees of freedom in the symplectic quotient are reduced to only one. Since the linearized case corresponds to gravitational waves, this translates in the loss of one polarization mode on the event horizon.*

6.2 PLANE WAVES ON S^1

We can find an analogy with a pretty simple example, given in the following. We start in a Minkowski background with M defined by coordinates $x = (x_+, x_-, x_1, x_2)$, then we perform a compactification of x_+ on S^1. The boundary will be given by setting $x_- = 0$. Now consider two fields π and ϕ on S^1, with symplectic form given by $\int \delta\pi\delta\phi$. The constraint

$$R_\tau = \int (\tau'\pi + \tau\phi)\,\mathrm{d}\theta \tag{6.2.1}$$

gives the following Poisson bracket:

$$\{R_\tau, R_\tau\} = \int \tau\tau'\,\mathrm{d}\theta. \tag{6.2.2}$$

We can expand τ in Fourier series, obtaining

$$\tau = a_0 + \sum_{n>0} \big(a_n \cos(n\theta) + b_n \sin(n\theta)\big); \tag{6.2.3}$$

and therefore $\{R_\tau, R_{\tilde{\tau}}\}$ is proportional to $\sum_{n>0}(a_n\tilde{b}_n - \tilde{a}_n b_n)$.

The constant part of τ, i.e. a_0, gives a first class constraint.

One can see the analogy with Proposition 5.4.5. This analogy also suggests that $\widetilde{R}_\tau$ could be second class, indeed $(a_n\tilde{b}_n - \tilde{a}_n b_n)$ is non degenerate for each n.

Chapter 7

GLOBAL INVARIANTS

In the case of a $S^2 \times \mathbb{R}$ hypersurface, we would like to look for invariant functionals (we call them mass and spin even though we do not know their actual nature yet). The ultimate intention would be the one of demonstrating that such invariant functionals are unique and identifiable with the mass and the spin. In this way, for a Kerr metric (Reference [17]), we would have a new understanding of the no-hair theorem. In this Chapter we present some partial results towards this goal. In particular we propose two functionals and prove that they satisfy a certain list of requirements. In order to properly conclude our argument, we should answer all the remaining questions of Remark 7.2.1. This part will be postponed to some future work.

7.1 MASS

We start with looking for a functional M of b or/and Θ such that:

(i) it is invariant under diffeomorphisms and local $\mathrm{SO}(3,1)$;

(ii) it is invariant under the first class hamiltonian fields of the theory (Reference [16]) and[1] $\partial_+ M = 0$;

(iii) given $b = \alpha e_0$, then $M[b] \propto \alpha$.

Our ansatz for such a functional is

$$M[b] = \frac{1}{\kappa} \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, b_\nu) \sqrt{|\det(g_0^\partial)|}\, d^2x, \tag{7.1.1}$$

where $(e_{0\mu}, b_\nu) = e^a_{0\mu} b^b_\nu \eta_{ab}$, g_0^∂ is the non-degenerate part of the boundary metric, $g_0^{\partial\mu\nu}$ its inverse and $\kappa = 4\pi^{\frac{1}{2}}(\int_{S^2} \sqrt{|\det(g_0^\partial)|}\, d^2x)^{\frac{1}{2}}$ a constant.

[1]Here $M = M[b]$ is seen as a zero form on $\mathbb{R}$. We want this to be independent of the choice of the point in $\mathbb{R}$ on which we calculate the integral on S^2. The coordinate of $\mathbb{R}$ is denoted with "+".

This ansatz is chosen such that *(i)* and *(iii)* are automatically satisfied, whereas *(ii)* must be explicitly checked.

For this purpose we consider the hamiltonian fields $\mathbb{J}$ and $\mathbb{L}$, and perform the calculations. For $\mathbb{J}$ we have

$$\begin{aligned}\kappa\{\widetilde{J}_c, M\} &= \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, (\mathbb{J}_b)_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x \\ &= \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, ([c, e_0])_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x,\end{aligned} \tag{7.1.2}$$

which, using

$$(e_{0\mu}, ([c, e_0])_\nu) = -(e_{0\nu}, ([c, e_0])_\mu) \tag{7.1.3}$$

reduces to zero, showing the invariance.

For $\mathbb{L}$ we have

$$\begin{aligned}\kappa\{\widetilde{L}_\lambda, M\} &= \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, (\mathbb{L}_b)_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x \\ &= \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, (d_{\omega_0}\lambda)_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x \\ &= \int_{S^2} g_0^{\partial\mu\nu} e_{0\mu}^a (d_{\omega_0}\lambda)_\nu^b \eta_{ab}\sqrt{|\det(g_0^\partial)|}\, d^2x.\end{aligned} \tag{7.1.4}$$

We consider part of the integrand, and we notice that

$$\begin{aligned}
e^a_{0\mu}(d_{\omega_0}\lambda)^b_\nu\eta_{ab} &= e^a_{0\mu}(\partial_\nu\lambda^b + \omega^{bc}_\nu\lambda^d\eta_{cd})\eta_{ab} \\
&= (e^a_{0\mu}\partial_\nu\lambda^b + e^a_{0\mu}\omega^{bc}_\nu\lambda^d\eta_{cd} + \partial_\nu e^a_{0\mu}\lambda^b - \partial_\nu e^a_{0\mu}\lambda^b)\eta_{ab} \\
&= (e^a_{0\mu}\partial_\nu\lambda^b + \partial_\nu e^a_{0\mu}\lambda^b - (\partial_\nu e^a_{0\mu} + \omega^{ac}_\nu e^d_{0\mu}\eta_{cd})\lambda^b)\eta_{ab} \\
&= (e^a_{0\mu}\partial_\nu\lambda^b + \partial_\nu e^a_{0\mu}\lambda^b - D_\nu e^a_{0\mu}\lambda^b)\eta_{ab} \\
&= (e^a_{0\mu}\partial_\nu\lambda^b + \partial_\nu e^a_{0\mu}\lambda^b - e^a_{0\rho}\Gamma^\rho_{\nu\mu}\lambda^b)\eta_{ab} \\
&= (e^a_{0\mu}\partial_\nu\lambda^b + \nabla_\nu e^a_{0\mu}\lambda^b)\eta_{ab} \\
&= (e^a_{0\mu}\nabla_\nu\lambda^b + \nabla_\nu e^a_{0\mu}\lambda^b)\eta_{ab} \\
&= \nabla_\nu(e^a_{0\mu}\lambda^b)\eta_{ab},
\end{aligned} \tag{7.1.5}$$

where $D_\mu\alpha^a_{\alpha_1\ldots\alpha_n} := \partial_\mu\alpha^a_{\alpha_1\ldots\alpha_n} + \omega^{ab}_\mu\alpha^c_{\alpha_1\ldots\alpha_n}\eta_{bc}$ with $\alpha \in \Omega^n(\partial M, \mathcal{V}^\partial)$.

This implies

$$\begin{aligned}
\kappa\{\widetilde{L}_\lambda, M\} &= \int_{S^2} g_0^{\partial\mu\nu}\nabla_\mu(e^a_{0\nu}\lambda^b)\eta_{ab}\sqrt{|\det(g_0^\partial)|}\,d^2x \\
&= 0 \quad \text{(Gauss theorem)},
\end{aligned} \tag{7.1.6}$$

where ∇ is the associated linear connection.

For last hamiltonian field, $\mathbb{R}$, we have only to consider its first class part; for this, M is trivially invariant, unless τ is constant. If, for instance, we restrict to a constant τ in the Kruskal case we have:

$$\begin{aligned}
\kappa\{R_\tau, M\} &= \int_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, (\mathbb{R}_b)_\nu)\sqrt{|\det(g_0^\partial)|}\,d^2x \\
&= \int_{S^2} g_0^{\partial\mu\nu}e^a_{0\mu}(\mathbb{R}_b)^b_\nu\eta_{ab}\sqrt{|\det(g_0^\partial)|}\,d^2x
\end{aligned} \tag{7.1.7}$$

which, using

$$\frac{e^a_{o\mu}(\mathbb{R}_b)^b_\nu\eta_{ab}}{2m} = \begin{pmatrix} B & -A\sin\theta \\ -A\sin\theta & -B\sin^2\theta \end{pmatrix}, \tag{7.1.8}$$

reduces to zero when taking the trace with $g_0^{\partial\mu\nu}$.

We need to show that $\partial_+ M = 0$:

$$\begin{aligned}\partial_+ M[b] &= \frac{1}{\kappa}\partial_+ \int\limits_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, b_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x \\ &= \frac{1}{\kappa}\int\limits_{S^2} g_0^{\partial\mu\nu}\nabla_+(e_{0\mu}, b_\nu)\sqrt{|\det(g_0^\partial)|}\, d^2x.\end{aligned} \tag{7.1.9}$$

Now we use the following identity[2]:

$$\begin{aligned}\nabla_+(e_\mu, b_\nu) &= \nabla_+(e_\mu, b_\nu) - \nabla_\nu(e_\mu, b_+) + \nabla_\nu(e_\mu, b_+) \\ &= (\nabla_+ e_\mu^a b_\nu^b + e_\mu^a \nabla_+ b_\nu^b - \nabla_\nu e_\mu^a b_+^b - e_\mu^a \nabla_\nu b_+^b)\eta_{ab} + \nabla_\nu(e_\mu, b_+) \\ &= (e_\mu^a \omega_+^{bc} b_\nu^d \eta_{cd} + e_\mu^a(\partial_+ b_\nu^b - \Gamma^\alpha_{+\nu} b_\alpha^b) - e_\mu^a \omega_\nu^{bc} b_+^d \eta_{cd} + \\ &\quad - e_\mu^a(\partial_\nu b_+^b - \Gamma^\alpha_{+\nu} b_\alpha^b))\eta_{ab} + \nabla_\nu(e_\mu, b_+) \\ &= (e_\mu^a (db)_{+\nu}^b + e_\mu^a \omega_{[+}^{bc} b_{\nu]}^d \eta_{cd})\eta_{ab} + \nabla_\nu(e_\mu, b_+) \\ &= e_\mu^a((db)_{+\nu}^b + \omega_{[+}^{bc} b_{\nu]}^d \eta_{cd})\eta_{ab} + \nabla_\nu(e_\mu, b_+) \\ &= e_\mu^a (d_\omega b)_{+\nu}^b \eta_{ab} + \nabla_\nu(e_\mu, b_+) \\ &= (e_\mu, (d_\omega b)_{+\nu}) + \nabla_\nu(e_\mu, b_+).\end{aligned} \tag{7.1.10}$$

Then, imposing the structural and the residual constraints, we obtain:

$$\partial_+ M[b] = -\frac{1}{\kappa}\int\limits_{S^2} g_0^{\partial\mu\nu}(e_{0\mu}, [a, e_0]_{+\nu})\sqrt{|\det(g_0^\partial)|}\, d^2x, \tag{7.1.11}$$

where we used Gauss theorem for getting rid of the $\nabla_\nu(e_\mu, b_+)$ term.

Finally we can use the identity

$$(e_{0\mu}, [a, e_0]_{+\nu}) = ({e_0}_\mu^a a_+^{bc} e_{0\nu}^d \eta_{cd} - {e_0}_\mu^a a_\nu^{bc} e_{0+}^d \eta_{cd})\eta_{ab}, \tag{7.1.12}$$

where we note that ${e_0}_\mu^a a_+^{bc} e_{0\nu}^d \eta_{cd}\eta_{ab} = -{e_0}_\nu^a a_+^{bc} e_{0\mu}^d \eta_{cd}\eta_{ab}$.

Therefore (7.1.11) reduces to

$$\partial_+ M[b] = \frac{1}{\kappa}\int\limits_{S^2} g_0^{\partial\mu\nu} {e_0}_\mu^a a_\nu^{bc} e_{0+}^d \eta_{cd}\eta_{ab}\sqrt{|\det(g_0^\partial)|}\, d^2x. \tag{7.1.13}$$

[2] Writing (e_0, ω_0) as (e, ω).

We would be left with showing that this integral is vanishing for the class of tetrads for which we expect the functional M to be physically meaningful[3].

We have shown that our ansatz (7.1.1) respects also condition *(ii)* (partially).

Considering Kruskal background, it follows that $M[e_0]$ corresponds to the mass of a static black hole, in natural units where $r_s = 2m = 2M$ (M mass).

A similar result occurs in the case of a Kerr black hole, since the determinant of the non degenerate part of the metric is

$$\begin{aligned} \det g_o^{\partial} &= \det \begin{pmatrix} \Sigma & 0 \\ 0 & \frac{A}{\Sigma}\sin^2\theta \end{pmatrix} \\ &= A\sin^2\theta, \end{aligned} \tag{7.1.14}$$

where $\Sigma = (m + \sqrt{m^2 - a^2})^2 + a^2\cos^2\theta$, $A = (2m(m + \sqrt{m^2 - a^2}))^2$ and a is the spin per unit mass of the black hole.

In such a case, we also get the spin contribution to M, in particular we have

$$M[e_0] = \frac{1}{2}\sqrt{2M(M + \sqrt{M^2 - a^2}\,)} \tag{7.1.15}$$

in natural units, which reduces to the previous case for $a = 0$.

Following from Reference [18], we can identify function (7.1.15) with the area of the black hole.

7.2 SPIN

Let's consider another functional,

$$S[\Theta] = \int_{S^2\times\mathbb{R}} \mathrm{Tr}\big[e_0 \wedge \Theta\big], \tag{7.2.1}$$

[3]This is one of the open questions we mentioned in the beginning.

which is our ansatz for the spin[4].

If we set $\Theta \to \Theta_0 := e_0 \wedge \omega_0$ in the Kruskal background, we obtain

$$\int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[e_0 \wedge \Theta_0\big] = 0, \tag{7.2.2}$$

since the integrand

$$\mathrm{Tr}\big[e_0 \wedge \Theta_0\big] \propto -e_0^{[3}{}_{+}e_0^{4]}{}_1 \cos\theta = 0, \tag{7.2.3}$$

where we used that the only non-vanishing component of the connection in Kruskal coordinates is $\omega_2^{12} = -\cos\theta$. Result (7.2.2) makes sense because for a static black hole there is no angolar momentum.

We need, as before, to check the invariance under the hamiltonian fields associated to the first class constraints.

We start with $\mathbb{J}$ and we get

$$\{\widetilde{J}_c, S\} = \int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[e_0 \wedge d_{\omega_0} c \wedge e_0\big] = \int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[d_{\omega_0}(e_0 \wedge c \wedge e_0)\big] = 0. \tag{7.2.4}$$

For $\mathbb{L}$ we have

$$\{\widetilde{L}_\lambda, S\} = \int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[\lambda \wedge e_0 \wedge F_{\omega_0}\big] = 0. \tag{7.2.5}$$

Finally, for $\mathbb{R}$ we have

$$\{\widetilde{R}_\tau, S\} = \int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[e_0 \wedge d_{\omega_0}\tau\big] = \int\limits_{S^2\times\mathbb{R}} \mathrm{Tr}\big[d_{\omega_0}(e_0 \wedge \tau)\big] = 0, \tag{7.2.6}$$

Remark 7.2.1. *There are four things left to prove:*

- *the invariance of the functional M with respect to $\widetilde{R}_\tau$ for general tetrads (or at least in the Kerr case) for a constant τ;*
- *the functional $M = M[b]$, modulo the constraints, is independent of the choice of points on $\mathbb{R}$ on which we calculate the integral over S^2 (we did this job partially up to* (7.1.13)*);*

[4] Without normalization constants.

– *one should then calculate the connection for the Kerr metric (after regularizing the metric all over the event horizon like in Reference [19] page 690) and obtain* Θ *or* Θ_0 *in such a case. Then it will be possible to check whether this gives contribution to the functional* (7.2.1) *and whether that could be regarded as the spin of the black hole;*

– *these functionals are the unique global invariants of the theory.*

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APPENDICES

Appendix A

MGD-DEGENERATE CASE COMPUTATIONS

Boundary tetrads we consider in the MGD-degenerate case are of the kind

$$e_0 : \begin{cases} e_+^a & = f(\delta_3^a + \delta_4^a) \\ \\ e_\alpha^a & = g_{(\alpha)}\delta_\alpha^a \quad \alpha = 1, 2. \end{cases} \tag{A.1}$$

For these tetrads we have different relations for $v \in \Omega^1(\partial M, \bigwedge^2 \mathcal{V}^\partial)$ such that $e_0 \wedge v = 0$:

$$\begin{aligned} \beta^{(1)} v_1^{12} &= v_+^{23} & \beta^{(1)} v_1^{12} &= v_+^{24} \\ \beta^{(2)} v_2^{12} &= -v_+^{13} & \beta^{(2)} v_2^{12} &= -v_+^{14} \\ v_2^{23} &= -v_1^{13} g_{(2)}/g_{(1)} & v_2^{24} &= -v_1^{14} g_{(2)}/g_{(1)} \\ v_1^{23} &= v_1^{24} & v_2^{13} &= v_2^{14} \\ v_1^{34} &= 0 & v_2^{34} &= 0 \\ v_+^{34} &= \beta^{(1)}(v_1^{13} - v_1^{14}) & v_+^{34} &= \beta^{(2)}(v_2^{23} - v_2^{24}), \end{aligned} \tag{A.2}$$

where $\beta^{(1)} = f/g_{(1)}$ and $\beta^{(2)} = f/g_{(2)}$. These of course still imply $v_+^{34} = 0$. We then characterize the image of the map $v \mapsto [v, e_0]$ such that $e_0 \wedge v = 0$ with $v \in \Omega^1(\partial M, \bigwedge^2 \mathcal{V}^\partial)$:

$$[v, e_0]_{\mu\nu}^a = v_{[\mu}^{ae} e_{0}{}_{\nu]}^{f} \eta_{ef}, \tag{A.3}$$

then

$$[v,e_0]^a_{12} = \begin{cases} g_{(2)}v_1^{12} & a=1 \\ g_{(1)}v_2^{12} & a=2 \\ g_{(2)}v_1^{32} - g_{(1)}v_2^{31} & a=3 \\ g_{(2)}v_1^{32} - g_{(1)}v_2^{31} & a=4 \end{cases}$$

$$[v,e_0]^a_{1+} = \begin{cases} f(v_1^{13} - v_1^{14}) = 0 & a=1 \\ g_{(1)}v_+^{12} & a=2 \\ \beta^{(2)}v_2^{21} & a=3 \\ \beta^{(2)}v_2^{21} & a=4 \end{cases} \tag{A.4}$$

$$[v,e_0]^a_{2+} = \begin{cases} g_{(2)}v_+^{21} & a=1 \\ f(v_2^{23} - v_2^{24}) = 0 & a=2 \\ \beta^{(1)}v_1^{12} & a=3 \\ \beta^{(1)}v_1^{12} & a=4, \end{cases}$$

from which follows

$$\begin{aligned} &[v,e_0]^3_{12} = [v,e_0]^4_{12} && \beta^{(2)}[v,e_0]^2_{12} = [v,e_0]^3_{+1}g_{(1)} = [v,e]^4_{+1}g_{(1)} \\ &[v,e_0]^2_{+1} = [v,e_0]^1_{2+}g_{(1)}/g_{(2)} && \beta^{(1)}[v,e_0]^1_{12} = [v,_0 e]^3_{2+}g_{(2)} = [v,e_0]^4_{2+}g_{(2)} \\ &[v,e_0]^1_{+1} = [v,e_0]^2_{2+}g_{(1)}/g_{(2)} && [v,e_0]^1_{1+} = [v,e_0]^2_{2+}g_{(1)}/g_{(2)}. \end{aligned} \tag{A.5}$$

We then characterize the kernel of $W^{(2,1)}_{e_0}$:

$$\varepsilon^{\mu\nu\sigma}e_0{}^{[a}_{\mu}\alpha^{b]}_{\nu\sigma} = e_0{}^{[a}_{+}\alpha^{b]}_{12} + e_0{}^{[a}_{2}\alpha^{b]}_{+1} + e_0{}^{[a}_{1}\alpha^{b]}_{2+} = 0, \tag{A.6}$$

giving

$$\begin{aligned} \alpha^4_{12} &= \alpha^3_{12} & \beta^{(2)}\alpha^2_{12} &= \alpha^4_{+1}g_{(1)} \\ \beta^{(2)}\alpha^2_{12} &= \alpha^3_{+1}g_{(1)} & \beta^{(1)}\alpha^1_{12} &= \alpha^4_{2+}g_{(2)} \\ \beta^{(1)}\alpha^1_{12} &= \alpha^3_{2+}g_{(2)} & \alpha^1_{+1} &= \alpha^2_{2+}g_{(1)}/g_{(2)}. \end{aligned} \tag{A.7}$$

The two conditions left are

$$[v, e_0]^2_{+1} = [v, e_0]^1_{2+}g_{(1)}/g_{(2)} \qquad [v, e_0]^1_{1+} = [v, e_0]^2_{2+}g_{(1)}/g_{(2)}. \tag{A.8}$$

Appendix B

KRUSKAL TETRADS

Given the metric[1]

$$g = e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - e^{2A(r)}dt^2, \tag{B.1}$$

we can see, by inspection, that

$$\begin{aligned} e^t &= e^A dt \qquad e^r = e^B dr \\ e^\theta &= r d\theta \qquad e^\varphi = r\sin\theta d\varphi, \end{aligned} \tag{B.2}$$

which satisfy $\eta_{ab}e^a e^b = g$, are our vierbein[2].

The vanishing torsion condition completely determines the connection ω, through the equation

$$(d_\omega e)^a = de^a + \omega^a{}_b \wedge e^b = 0. \tag{B.3}$$

Taking the exterior derivative of (B.2), yields

$$\begin{aligned} de^r &= 0 \\ de^t &= \frac{\mathrm{d}A}{\mathrm{d}r} e^A dr \wedge dt \\ de^\theta &= dr \wedge d\theta \\ de^\varphi &= \sin\theta dr \wedge d\varphi + r\cos\theta d\theta \wedge d\varphi. \end{aligned} \tag{B.4}$$

We can write $(d\theta, d\varphi, dr, dt)$ in terms of tetrads and therefore (B.4) becomes

$$\begin{aligned} de^r &= 0 \\ de^t &= (-\frac{\mathrm{d}A}{\mathrm{d}r} e^{-B}) e^t \wedge e^r \\ de^\theta &= (r^{-1}e^{-B}) e^r \wedge e^\theta \\ de^\varphi &= (r^{-1}e^{-B}) e^r \wedge e^\varphi + (r^{-1}\cot\theta) e^\theta \wedge e^\varphi, \end{aligned} \tag{B.5}$$

[1]This is the Schwarzschild metric, properly setting A and B.

[2]We denote here $(1,2,3,4)$ as (θ, φ, r, t).

which can be used in (B.3) for identifying the connection ω, for which we have the six independent components

$$\begin{aligned}
{\omega^t}_r &= (\frac{\mathrm{d}A}{\mathrm{d}r}e^{-B})e^t \\
{\omega^r}_\theta &= (-r^{-1}e^{-B})e^\theta \\
{\omega^\theta}_\varphi &= (-r^{-1}\cot\theta)e^\varphi \\
{\omega^r}_\varphi &= (-r^{-1}e^{-B})e^\varphi \\
{\omega^t}_\theta &= 0 \\
{\omega^t}_\varphi &= 0,
\end{aligned} \tag{B.6}$$

where the other components are given by the antisymmetry ${\omega^i}_j = -{\omega_j}^i$.

We observe that e^t in (B.2) is degenerate, since we know that $A(r) = -\infty$ when restricted to the event horizon.

To overcome the problem of the degeneracy, consider tetrads (B.2) under the change of coordinates, where θ and φ do not change and $r_s = 2m$ is the Schwarzschild radius with m the mass in natural units, given by (for $r > r_s$)

$$\begin{aligned}
T &= (\frac{r}{r_s} - 1\big)^{1/2}e^{r/2r_s}\cosh\left(\frac{t}{2r_s}\right) \\
X &= (\frac{r}{r_s} - 1\big)^{1/2}e^{r/2r_s}\sinh\left(\frac{t}{2r_s}\right);
\end{aligned} \tag{B.7}$$

we have set $A(r) = \log\big(1 - \frac{r_s}{r}\big)$ and $B(r) = -\log\big(1 - \frac{r_s}{r}\big)$. Here r is a function of T and X.

This gives the following tetrads (Kruskal)

$$\begin{aligned}
e^T &= 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}}e^{-r/2r_s}dT \\
e^X &= 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}}e^{-r/2r_s}dR \\
e^\theta &= rd\theta \\
e^\varphi &= r\sin\theta d\varphi.
\end{aligned} \tag{B.8}$$

We take the components of (B.8) and restrict them to the $S^2 \times \mathbb{R}$ light-like boundary with fixed r and coordinates $(+, 1, 2)$. (with notation $a = 1, 2, 3, 4$):

$$\begin{aligned} e_1^a &= r\delta_1^a \\ e_2^a &= r \sin\theta \delta_2^a \\ e_+^a &= 2\frac{r_s^{\frac{3}{2}}}{\sqrt{r}} e^{-r/2r_s}(\delta_3^a + \delta_4^a). \end{aligned} \tag{B.9}$$

These tetrads are no longer degenerate in the present case.

The degeneracy of the boundary metric is given by

$$g_0^\partial = e^*\eta = \mathrm{diag}(0, r^2, r^2 \sin^2\theta), \tag{B.10}$$

where e is given by boundary tetrads in (B.9).