On Morphic Actions and Integrability of $\mathcal{LA}$-Groupoids

Dissertation

zur
Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der
Mathematisch-naturwissenschaftlichen Fakultät
der
Universität Zürich

von

Luca Stefanini

aus
Italien

Promotionskomitee

Prof. Dr. Alberto S. Cattaneo (Vorsitz)
Prof. Dr. Giovanni Felder
Prof. Dr. Kirill C. H. Mackenzie

Zürich 2008

Dedicated to Motata, Rita and Toto

Synopsis

Lie theory for the integration of Lie algebroids to Lie groupoids, on the one hand, and of Poisson manifolds to symplectic groupoids, on the other, has undergone tremendous developements in the last decade, thanks to the work of Mackenzie-Xu, Moerdijk-Mrčun, Cattaneo-Felder and Crainic-Fernandes, among others.
In this thesis we study - part of - the categorified version of this story, namely the integrability of $\mathcal{LA}$-groupoids (groupoid objects in the category of Lie algebroids), to double Lie groupoids (groupoid objects in the category of Lie groupoids) providing a first set of sufficient conditions for the integration to be possible.

Mackenzie's double Lie structures arise naturally from lifting processes, such as the cotangent lift or the path prolongation, on ordinary Lie theoretic and Poisson geometric objects and we use them to study the integrability of quotient Poisson bivector fields, the relation between "local" and "global" duality of Poisson groupoids and Lie theory for Lie bialgebroids and Poisson groupoids.

In the first Chapter we prove suitable versions of Lie's 1-st and 2-nd theorem for Lie bialgebroids, that is, the integrability of subobjects (coisotropic subalgebroids) and morphisms, extending earlier results by Cattaneo and Xu, obtained using different techniques.

We develop our functorial approach to the integration of $\mathcal{LA}$-groupoids [**65**] in the second Chapter, where we also obtain partial results, within the program, proposed by Weinstein, for the integration of Poisson groupoids to symplectic double groupoids.

The task of integrating quotients of Poisson manifolds with respect to Poisson groupoid actions motivates the study we undertake in third Chapter of what we refer to as morphic actions, i.e. groupoid actions in the categories of Lie algebroids and Lie groupoids, where we obtain general reduction and integrability results.

In fact, applying suitable procedures à la Marsden-Weinstein zero level reduction to "moment morphisms", respectively of Lie bialgebroids or Poisson groupoids, canonically associated to a Poisson $\mathcal{G}$-space, we derive two approches to the integration of the quotient Poisson bivector fields.
The first, a kind of integration via symplectic double groupoids, is not always effective but reproduces the "symplectization functor" approch to Poisson actions of Lie groups, very recently developed by Fernandes-Ortega-Ratiu, from quite a different perspective. We earlier implemented this approach successfully in the special case of complete Poisson groups [**64**].
The second approach, relying both on a cotangent lift of the Poisson $\mathcal{G}$-space and on a prolongation of the original action to an action on suitable spaces of Lie algebroid homotopies, produces necessary and sufficient integrability conditions for the integration and gives a positive answer to the integrability problem under the most natural assumptions.

Zusammenfassung

Im letzten Jahrzehnt hat sich die Lie-Theorie insbesondere aufgrund ihrer Wichtigkeit für die Integration von Lie Algebroiden nach Lie Gruppoiden, einerseits, und von Poisson Mannigfaltigkeiten nach symplektischen Gruppoiden, andererseits, enorm weiterentwickelt. Arbeiten von Mackenzie-Xu, Moerdijk-Mrčun, Cattaneo-Felder und Crainic-Fernandes haben diese Entwicklung unter anderem entscheidend beeinflusst.

In der vorliegenden Dissertation interessieren wir uns diesbezüglich für den kategorientheoretischen Aspekt der Integration von $\mathcal{LA}$-Gruppoiden (nämlich von Gruppoid-Objekten in der Kategorie der Lie Algebroide) nach doppelten Lie Gruppoiden (nämlich von Gruppoid-Objekten in der Kategorie der Lie Gruppoide). Hier erhalten wir hinreichende Bedingungen für die Integration.

Mackenzies doppelte Lie-Strukturen entstehen dabei ganz natürlich aus Hochhebungsprozessen, wie der kotangentialen Hochebung oder der Pfad-Prolongation auf gewöhnlichen Lie-theoretischen oder Poisson-geometrischen Strukturen. Mit ihrer Hilfe werden wir die Integrabilität der Quotienten-Poisson-Bivektorfelder, die Beziehung zwischen lokaler und globaler Dualität der Poisson-Gruppoide sowie die Lie-Theorie der Lie-Bialgebroide und der Poisson-Gruppoide untersuchen. Im ersten Kapitel beweisen wir gewisse Varianten des ersten und zweiten Satzes von Lie über Lie-Bialgebroide, d.h. , über die Integrabilität von Unterobjekten (den ko-isotropischen Unteralgebroiden) und ihren Morphismen. Hier verallgemeinern wir frühere Resultate von Cattaneo und Xu.

Im zweiten Kapitel entwickeln wir unseren funktorialen Ansatz zur Integration der $\mathcal{LA}$-Gruppoide[**65**]. Hier erhalten wir positive Teilergebnisse zur von Weinstein vorgeschlagenen Integration der Poisson-Gruppoide nach doppelten symplektischen Gruppoiden.

Der Untersuchung der sogenannten morphischen Wirkungen, also Gruppoid-Wirkungen in der Kategorie der Lie Algebroide und der Lie Grupppoide, widmen wir uns im dritten Kapitel. Hier erhalten wir Reduktions- und Integrabilitätsresultate und wir gehen die Aufgabe an, Quotienten von Poisson-Mannigfaltigkeiten in Bezug auf Wirkungen von Poisson-Gruppoiden zu integrieren.

Tatsächlich erhalten wir durch die Anwendung geeigneter Varianten des Marsden-Weinstein-Reduktions-Verfahrens zwei Ansätze zur Integration der Quotienten von Poisson-Bivektorfeldern.

Der erste Ansatz, eine Art von Integration durch doppelte symplektische Gruppoide, ist nicht immer erfolgreich. Dennoch gibt er im speziellen Fall der Wirkungen von Lie-Gruppen Fernendes-Ortega-Ratius Symplektisierungsfunktor-Ansatz wieder. Dieser Ansatz wurde schon zuvor von uns erfolgreich im speziellen Fall der Wirkungen von Poisson-Gruppen angewendet (siehe [**64**]). Der zweite Ansatz basiert schliesslich auf einer kotangentialen Hochhebung der Wirkung und ihrer

Pfad-Prolongation nach einer Wirkung auf geeigneten Räumen von Lie-Algebroid-Homotopien. Über ihn erhalten wir notwendige sowie hinreichende Integrabilitätsbedingungen, womit wir unter kanonischen Voraussetzungen das Integrationsproblem vollständig lösen.

Acknowledgements

For his guidance and constant support I am deeply indebted to Alberto Cattaneo; most of all I wish to thank him for his persistently positive attitude, a decisive source of encouragement throughout my Ph.D. I am very grateful to Kirill Mackenzie for sharing with me many of his ideas through the years. I also wish to thank Ping Xu, Marco Zambon, Rui Loja Fernandes, Serge Parmentier and Iakovos Androulidakis for stimulating conversations, and Ping Xu, Kirill Mackenzie and Giovanni Felder for interesting comments on final drafts of this dissertation.
Special thanks go to Mauro Carfora, Claudio Dappiaggi and Sergio Pini for helping me find my way to Zürich. Nicola Kistler and to his successor Felice Manganiello deserve a big thank you for providing me in the office hours with good coffee, an essential ingredient of my diet.

Könnte jeder brave Mann
solche Glöcken finden,
seine Feinde würden dann
ohne Mühe schwinden[1] [?][2]

[1]Emanuel Schikaneder, Die Zauberflöte, 1791.
[2]Persönlische Beischrift.

Contents

Preliminary remark

The first few pages of each chapter, “section zero”s, can be read separately, as an introduction to each chapter, or together as an introduction to the whole dissertation.

Notations and conventions

Most of the notations and conventions are introduced in the main text, or should be clear from the context; we list below a few remarks:

- The symbol "$\bullet$" is used with various meanings throughout the text. It can stand for the one point manifold, an unassigned variable, the multiplication of the tangent groupoid, etc $\ldots$; it shall be clear from the context which meaning is intended.
- For any function $f \in \mathcal{C}^\infty(M)$ we denote with $X^f = \{\, f\,, \cdot\,\}$ the associated Hamiltonian vector field, when M is symplectic or Poisson. The Poisson brackets are defined as $\{f, g\} := \pi(\mathrm{d}f, \mathrm{d}g)$, respectively $\{f, g\} := \omega(X^g, X^f)$.
- The symbol $\langle\, \cdot\,, \cdot\,\rangle$ is used for the natural pairing of a vector bundle with its dual.
- $\pounds$ denotes the Lie derivative.
- All manifolds are real smooth $\mathcal{C}^\infty$ manifolds, unless otherwise stated (but not necessarily Hausdorff); $\Gamma(M, E)$ or simply $\Gamma(E)$ is the space of $\mathcal{C}^\infty$ sections of a vector bundle. We use the symbols $\mathsf{\Gamma}(f)$, respectively $\mathsf{\Gamma}(\sim)$, to denote the graph of a map f (with respect to the standard ordering) or of an equivalence relation $\sim$.
- If $X \in \Gamma(E_- \otimes E_+)$, $X^\sharp : E_-^* \to E_+$ is the associated vector bundle map, for all vector bundles $E_\pm$.
- $\oint\limits_{i,j,k} X_{ijk}$ is the sum over cyclic permutations.
- The symbol Δ is used with different meanings: to denote both singular and regular distributions, diagonal subsets and also the inclusion map of a diagonal subset in a direct product.
- The symbol φ is typically used to denote a morphism of Lie groupoids, ϕ instead to denote a morphism of Lie algebroids.

CHAPTER 1

Lie-Poisson duality

Poisson algebras first arose in the mid 19-th century from Jacobi's algebraic study of mechanical systems, embarked to understand the relation of the brackets earlier discovered by Poisson with constrained dynamical systems and conservation laws (see e.g. [**73, 76**] and references therein for an historical review). The study of the geometry of these brackets later led Lie to discover what nowadays we call Lie algebras; in fact, the dual of a Lie algebra is one of the fundamental examples of a Poisson manifold. After a very long dormancy, Poisson manifolds where rediscovered across the 1960s and 1970s in the work of Berezin, Kirillov, Kostant and Lichnerowicz [**33**], among others. After the introduction of Poisson cohomology by Lichnerowicz in 1977, the unraveling of the local structure of Poisson manifolds by Weinstein in 1983, and the discovery of symplectic groupoids towards the end of 1980s independently by Karasëv [**26**], Weinstein [**71**] and Zakrzewski [**77**], a form of "duality" between Poisson brackets and Lie brackets begun to emerge in a more general setting, linking Poisson geometry to the theory of Lie groupoids-Lie algebroids earlier started by Pradines in a series of papers [**54, 55, 56, 57**].
Instances of this duality were further investigated in the late 1980s and through the 1990s to the present day, especially via Weinstein's coisotropic calculus, in the study of Poisson group-*oids*, Poisson homogeneous spaces, symplectic realizations, generalized moment(um) maps and the integrability of Lie algebroids.

In the first two Sections of this mostly introductory Chapter we shall introduce, the preliminary material we will need in the rest of this dissertation, by reviewing well known facts about Poisson manifolds, Lie algebroids and Lie groupoids, respectively their morphisms, with emphasis on the duality issues.
In Section 1.3 we define actions of Lie groupoids and their infinitesimal counterpart.
In Section 1.4 we discuss the integrability of Lie algebroids to Lie groupoids and its dual counterpart, namely the realization of a Poisson manifold via a symplectic groupoid. We choose to follow there a logical, rather then historical, order. A brief historical account of the tremendous developments of Lie theory that took place in the last decade is nevertheless in order. The integrability of morphisms of Lie algebroids is due to Mackenzie and Xu [**48**]. After that Moerdijk and Mrčun provided in [**51**] every Lie groupoid with a source 1-connected "covering" groupoid, given by the quotient of the monodromy groupoid of the associated source foliation, and proved the integrability of Lie subgroupoids, also by foliation theoretic techniques.

About at the same time, a reduction à la Marsden-Weinstein of the phase space of a topological field theory known as the Poisson sigma model, performed by Cattaneo and Felder in [**12**], produced a topological model for the symplectic groupoid of a Poisson manifold; in the integrable case the construction yields the desired symplectic groupoid. Soon after Crainic and Fernandes adapted Cattaneo and Felder's model, following an approach foreseen by Ševera in [**68**], in order to characterize Moerdijk and Mrčun's "covering groupoid" in terms of the Lie algebroid data only and obtained, by connection theoretic means, necessary and sufficient conditions for the integrability of Lie algebroids [**16**] and Poisson manifolds [**17**]. Crainic and Fernandes' topological model, the so called *Weinstein groupoid*, was in turn recovered via the reduction of the Poisson sigma model for the dual Poisson structure in [**11**] by Cattaneo, where an integration of coisotropic submanifolds to Lagrangian subgroupoids, namely the dual phenomenon to the integrability of Lie subalgebroids and Lie algebroid morphisms, was also given.
The original contributions presented in this Chapter can be found in the last Section. After introducing Lie bialgebroids and Poisson groupoids, in a sense self-dual objects in the intersection of the Lie and Poisson worlds, we adapt Mackenzie and Xu's approach for the integration of Lie bialgebroids to Poisson groupoids [**48**] in order to prove the integrability of coisotropic subalgebroids to coisotropic subgroupoids:

Theorem (1.5.9). *Let $(\mathcal{G},\Pi)\rightrightarrows M$ be a source 1-connected Poisson groupoid with Lie bialgebroid (A,A^*) and $\mathcal{C}\rightrightarrows N$ a source 1-connected Lie subgroupoid with Lie algebroid C. Then $\mathcal{C}\subset\mathcal{G}$ is coisotropic iff so is $C\subset A$ for the dual Poisson structure induced by A^*.*

From this result it is then easy to derive the integrability of morphisms of Lie bialgebroids to morphisms of Poisson groupoids and the equivalence of the category of Lie bialgebroids to that of source 1-connected Poisson groupoids.

1.1. Poisson manifolds, Lie algebroids and Lie groupoids

In this Section we introduce the main definitions and basic facts about Poisson manifolds, Lie algebroids and Lie groupoids. The geometric study of the Poisson brackets of classical mechanics leads naturally (and led historically) to Lie algebras and Poisson manifolds. In fact, Lie algebras are dual to linear Poisson bivector fields and in turn the dual object to a general Poisson manifold can be understood as a Lie algebroid, the infinitesimal invariant of a Lie groupoid. We also discuss standard examples in some detail; a number of those actually plays a role in the general theory of Poisson geometry and Lie theory (see also, for instance, [**45, 67, 10**]).

A Poisson bracket on an associative algebra is given by a Lie bracket for which the adjoint representation takes values in the derivations of the associative product. The definition applies for any ground field, the algebra can be graded and the associative product need not be commutative; still graded commutativity is a desirable property.

Definition 1.1.1. A Poisson structure on a manifold M is a Poisson algebra on $\mathcal{C}^\infty(M)$, i.e. a Lie bracket $\{\ ,\ \}$ such that the Leibniz rule

$$\{ f , g \cdot h \} = g \cdot \{ f , h \} + \{ f , g \} \cdot h$$

holds for all $f, g, h \in \mathcal{C}^\infty(M)$.

The above Leibniz rule is equivalent to asking $\{\ ,\ \}$ to be a skewsymmetric biderivation of the pointwise product of $\mathcal{C}^\infty(M)$, therefore setting

$$\pi(\mathrm{d}f, \mathrm{d}g) := \{ f , g \} \qquad , \qquad f, g \in \mathcal{C}^\infty(M) \qquad ,$$

defines a bivector field $\pi \in \Gamma(\wedge^2 TM)$, which we shall call a Poisson bivector. A bivector field $\pi \in \mathfrak{X}^2(M)$ is the Poisson bivector of a Poisson structure iff the trivector field $[\pi , \pi] \in \Gamma(\wedge^3 TM)$, locally defined by the components

$$[\pi , \pi]^{\alpha\beta\gamma} = \sum_\mu \oint_{\alpha,\beta,\gamma} \pi^{\alpha\mu} \partial_\mu \pi^{\beta\gamma}$$

in a coordinate patch, vanishes identically on M[1]refer to the vector bundle map $\pi^\sharp : T^*M \to TM$ induced by a Poisson structure as the Poisson anchor. The image $\mathsf{Im}\,\pi^\sharp$ of the Poisson anchor of a Poisson manifold M, regarded as a submodule of the local vector fields over M, defines an integrable singular distribution in the sense of Sussmann [**66**] and Stefan [**63**]. That is, the points reached by the sequences of local flows of vector fields taking values in $\mathsf{Im}\,\pi^\sharp$, can be assembled in a partition of M in connected immersed submanifolds (not necessarily of the same dimension), the leaves of $\pi^\sharp$, which can be pasted nicely (see [**67, 3**] for details). The vector fields $X^f := \pi^\sharp \circ \mathrm{d}f$, $f \in \mathcal{C}^\infty(M)$, spanning the leaves of the distribution are called Hamiltonian vector fields in analogy with the classical Hamiltonian dynamics, since also in the Poisson case $X^f = \{ f , \cdot \}$, as a derivation of $\mathcal{C}^\infty(M)$, and the subspace of Hamiltonian vector fields is easily seen to be a Lie subalgebra of $\mathfrak{X}(M)$. Moreover, the restriction of $\pi^\sharp_{\mathcal{L}} : T^*_{\mathcal{L}}M \to T\mathcal{L}$ to a leaf $\mathcal{L}$ has maximal rank equal to $\mathsf{dim}\,\mathcal{L}$, by definition and one can check that setting

$$\omega^{\mathcal{L}}(\pi^\sharp\alpha_+, \pi^\sharp\alpha_-) := \pi(\alpha_-, \alpha_+) \qquad , \qquad \alpha_\pm \in \Omega^1(M) \qquad ,$$

yields a well defined nondegenerate closed 2-form on $\mathcal{L}$. Probably, this is the most important property of a Poisson manifold, namely, being foliated in symplectic

[1]The notation is not incidental. The graded commutative algebra $\mathfrak{X}^\bullet(M) = (\Gamma(\wedge^\bullet TM), \wedge)$ of multivector fields, carries a unique graded Lie bracket of degree -1 extending the bracket of vector fields and making it a graded Poisson algebra [**60, 61, 52**]

leaves [**27**] and might be regarded as global a manifestation of the Weinstein-Darboux local structure theorem [**70**]. Clearly, a symplectic manifold is a Poisson manifold with only one leaf or equivalently a Poisson manifold for which the sharp map has maximal rank. Poisson manifolds are quite general objects interpolating between arbitrary manifolds and symplectic manifolds.

Example 1.1.2. Poisson manifolds

i) Every manifold is a Poisson manifold for the zero bivector field, each point is a symplectic leaf.

ii) Every constant bivector field B on $\mathbb{R}^n$ is a Poisson bivector, the dimension of the symplectic leaves is the rank of the matrix representing B.

iii) The sharp map of a Poisson bivector of maximal rank is the inverse of the sharp map of a symplectic form, the whole manifold is the only symplectic leaf.

iv) Every smooth function f on $\mathbb{R}^2$, yields a Poisson structure by setting $\{ x \, , \, y \} := f(x,y)$ for the coordinate functions x and y on $\mathbb{R}^2$; the symplectic leaves are each point in the zero set $\mathsf{Z}(f)$ of f and the connected components of $\mathbb{R}^2 \backslash \mathsf{Z}(f)$. For instance, the symplectic leaves of

$$f(x,y) := \exp\left(-\frac{1}{(1-(x^2+y^2))^2} \right)$$

are the interior of the unit disk, the complement of its closure and each point on the unit circle.

Morally speaking, Poisson manifolds are nonlinear Lie algebras. In fact, the very idea of a Lie algebra emerged from Sophus Lie's [**34**] attempt to understand Poisson brackets in a geometric fashion, by examining the simplest nontrivial examples. Consider a linear Poisson structure on $\mathbb{R}^n$:

$$\{ x^i \, , \, x^j \} = \sum_k f^{ij}{}_k x^k \qquad , \qquad i,j,k = 1, \ldots, n \qquad ,$$

for some constants $\{ f^{ij}{}_k \}$, with $f^{ij}{}_k + f^{ji}{}_k = 0$. The condition for $\{ \; , \; \}$ to be a Poisson bracket bracket writes

$$\sum_{l=1}^n \oint_{i,j,k} f^{ij}{}_l f^{lk}{}_m = 0 \qquad , \qquad i,j,k,m = 1, \ldots, n \qquad ,$$

that is, of course, the requirement for the f's to be the structure constants of a Lie algebra. Replacing $\mathbb{R}^n$ with a finite dimensional vector space V, one can see that Lie algebras on V are in bijective correspondence with linear Poisson structures on V^* by the formula

$$\langle \xi \, , [\, v \, , \, w \,] \rangle = \{ F_v \, , \, F_w \}(\xi) \qquad , \qquad v, w \in V \simeq \mathcal{C}^\infty_{\mathsf{lin}}(V^*) \quad , \quad \xi \in V^* \qquad ,$$

where we identify a vector $v \in V$ with the associated linear functional F_v on V^*.

Example 1.1.3. For any Lie group G with Lie algebra $\mathfrak{g}$, the symplectic leaves of the dual Poisson structure on $\mathfrak{g}^*$ are the connected components of the coadjoint orbits of G. Recall that the coadjoint action $\mathsf{Ad}^* : G \to \mathsf{Diff}(\mathfrak{g}^*)$ is defined by

$$\langle \mathsf{Ad}^*_g \xi \,,\, x \rangle := \langle \xi \,,\, \mathsf{Ad}_{g^{-1}} x \rangle = \langle \xi \,,\, \mathrm{d}\mathcal{C}_{g^{-1}} x \rangle \qquad , \qquad g \in G \,,\, \xi \in \mathfrak{g}^* \text{ and } x \in \mathfrak{g} \quad ,$$

for the conjugation map $\mathcal{C}_g : G \to G$. It is a basic exercise in classical Lie theory to see that the induced infinitesimal action $\mathsf{ad}^* : \mathfrak{g} \to \mathfrak{X}(\mathfrak{g}^*)$ is

$$\langle \mathsf{ad}^*_x \xi \,,\, y \rangle = \langle \xi \,,\, [\, y \,,\, x \,] \rangle = \{ F_y \,,\, F_x \}(\xi) \qquad , \qquad x, y \in \mathfrak{g} \text{ and } \xi \in \mathfrak{g}^* \quad ,$$

therefore the fundamental vector fields of the action are Hamiltonian, i.e. $\mathsf{ad}^*(x)_\xi = X^{F_x}_\xi$, and span the symplectic foliation of $\mathfrak{g}^*$.

The Poisson bivector of a Poisson manifold (M, π) may be regarded as a $\mathcal{C}^\infty(M)$-bilinear operation $\wedge^2 \Omega^1(M) \to \mathcal{C}^\infty(M)$; according to the above discussion, it appears natural to ask, whether it is possible to linearize π in such a way to obtain a Lie bracket on $\Omega^1(M)$. The answer is positive: set

$$\langle [\, \theta_+ \,,\, \theta_- \,] \,,\, X \rangle = (\pounds_X \pi)(\theta_+, \theta_-) + \pi^\sharp \theta_+ (\langle \theta_- \,,\, X \rangle) - \pi^\sharp \theta_- (\langle \theta_+ \,,\, X \rangle) \qquad ,$$

for all $\theta_\pm \in \Omega^1(M)$, $X \in \mathfrak{X}(M)$ or equivalently,

$$\begin{aligned} [\, \theta_+ \,,\, \theta_- \,] &= \mathrm{d}\pi(\theta_+, \theta_-) + \iota_{\pi^\sharp \theta_+} \mathrm{d}\theta_- - \iota_{\pi^\sharp \theta_-} \mathrm{d}\theta_+ \\ &= \pounds_{\pi^\sharp \theta_+} \theta_- - \pounds_{\pi^\sharp \theta_-} \theta_+ - \mathrm{d}\pi(\theta_+, \theta_-) \qquad ; \end{aligned} \tag{1.1}$$

note that the $\mathcal{C}^\infty(M)$-bilinearity is replaced by the Leibniz rule

$$[\, \theta_+ \,,\, f \cdot \theta_- \,] = f \cdot \pi [\, \theta_+ \,,\, \theta_- \,] + \pi^\sharp \theta_+ (f) \cdot \theta_- \qquad , \qquad f \in \mathcal{C}^\infty(M) \qquad ,$$

and $[\ ,\ ]$ makes $\Omega^1(M)$ a Lie algebra over $\mathbb{R}$. This bracket we shall call the Koszul bracket, is very natural, for it is the unique extending

$$[\, \mathrm{d}f_+ \,,\, \mathrm{d}f_- \,] = \mathrm{d}\{ f_+ \,,\, f_- \} \qquad ,$$

$f_\pm \in \mathcal{C}^\infty(M)$, according to the Leibniz rule above and therefore generalizing the Lie bracket dual to a linear Poisson bracket on a vector space to the nonlinear case.
The Koszul bracket, together with the Poisson anchor endows T^*M, with a Lie algebroid structure.

Definition 1.1.4. A Lie algebroid structure on a vector bundle $A \to M$ is given by a $\mathbb{R}$-bilinear Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(A)$ and a vector bundle map $\rho : A \to TM$ such that the Leibniz rule

$$[\, a_+ \,,\, f \cdot a_- \,] = f \cdot [\, a_+ \,,\, a_- \,] + \rho(a_+)(f) \cdot a_-$$

holds for all $a_\pm \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$.

Before turning to examples and introducing the main features of Lie algebroids, we will show how the duality between Lie algebras and linear Poisson structures generalizes to a duality between Lie algebroids and fibrewise linear Poisson structures. The space of functions on A^* is the completion of the subspace of fiberwise polynomial functions, which in turn is the associative algebra generated by the space $\mathcal{C}^\infty_{\mathsf{lin}}(A^*) \oplus \mathcal{C}^\infty_{\mathsf{cst}}(A^*)$ of fiberwise linear functions $\mathcal{C}^\infty_{\mathsf{lin}}(A^*) \simeq \Gamma(A)$ and fibrewise constant functions $\mathcal{C}^\infty_{\mathsf{cst}}(A^*) \simeq \mathcal{C}^\infty(M)$. In order to endow A^* with a Poisson bracket $\{\ ,\ \}_A$ it is then sufficient to define it on $\mathcal{C}^\infty_{\mathsf{lin}}(A^*) \oplus \mathcal{C}^\infty_{\mathsf{cst}}(A^*)$ and extend it according to the Leibniz rule: setting, for all $\xi \in \Gamma(A^*)$,

$$\{ F_{a_+} , F_{a_-} \}_A(\xi) := \langle \xi , [a_+ , a_-] \rangle \qquad , \qquad a_\pm \in \Gamma(A) \qquad ,$$

in analogy with the case $M = \bullet$, where $F_{a_\pm}(\xi) = \langle a_\pm , \xi \rangle$ are the associated fibrewise linear function on A^*, $\{\ ,\ \}_A$ extends to $\mathcal{C}^\infty_{\mathsf{lin}}(A^*) \oplus \mathcal{C}^\infty_{\mathsf{cst}}(A^*)$ compatibly with the Leibniz rule is given by

$$\{ F_a , \mathrm{pr}^*_{A^*} f \}_A = \rho(a)(f) \circ \mathrm{pr}_{A^*} \qquad \text{and} \qquad \{ \mathrm{pr}^*_{A^*} f , \mathrm{pr}^*_{A^*} g \} = 0 \qquad , \qquad (1.2)$$

for all $a \in \Gamma(A)$, $f, g \in \mathcal{C}^\infty(M)$. To check that the bivector π_A on A^* defined by equations (1.2) is Poisson, for example in the local coordinates induced a choice of dual frames for A and A^*, is then straightforward. Poisson structures of this kind are called *fibrewise linear*, in the sense that so is the restriction of the corresponding Poisson bivector to the vertical subbundle of $\wedge^2 A^*$. Clearly, inverting the construction, a fibrewise linear Poisson structure on a vector bundle induces a Lie algebroid on the dual bundle.
Lie algebroids generalize various classes of differential geometric structures.

Example 1.1.5. Lie algebroids

i) Every vector bundle is a Lie algebroid for the zero bracket and anchor. A regular distribution on a manifold M, i.e. a smooth subbundle $\Delta \to M$ of TM or, equivalently, a singular distribution of constant rank, is a Lie algebroid iff it is integrable, i.e. iff the space of sections of Δ is closed under the Lie bracket of vector fields. Notably, the tangent bundle itself is a Lie algebroid.

ii) Lie algebras are Lie algebroids over the one point manifold: the anchor cannot be anything but zero. If a Lie algebroid has zero anchor, setting

$$[x_+ \overset{q}{,} x_-] := [\widetilde{x}_+ , \widetilde{x}_-]_q \qquad , \qquad x_\pm \in A_q \qquad ,$$

for any sections $\widetilde{x}_\pm \in \Gamma(A)$ with $\widetilde{x}_\pm(q) = x_\pm$, yields a bundle of Lie algebras, i.e. a smooth family of Lie algebras $\{A_q, [\ \overset{q}{,}\ ]\}_{q \in M}$ on the fibres of A, parameterized by M.

iii) Let $\sigma : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal action on a manifold M (i.e. a morphism of Lie algebras). Then the flat bundle $\mathfrak{g} \times M$ carries a canonical Lie algebroid structure, the action Lie algebroid $\mathfrak{g} \ltimes M$. Note that $\Gamma(\mathfrak{g} \times M) \simeq \mathcal{C}^\infty(M, \mathfrak{g})$ and setting

$$\langle \xi \,,\, \widetilde{X}(F) \rangle := X(\langle \xi \,,\, F \rangle) \qquad , \qquad \xi \in \mathfrak{g}^* \quad , \quad F \in \mathcal{C}^\infty(M, \mathfrak{g}) \qquad ,$$

allows extending a vector field $X \in \mathfrak{X}(M)$ to a linear endomorphism of $\mathcal{C}^\infty(M, \mathfrak{g})$. The bracket of $\mathfrak{g} \ltimes M$ is defined by

$$[F \overset{\ltimes}{,} G](q) := [\, F(q) \,,\, G(q) \,] + (\widetilde{\sigma \circ F})_q(G) - (\widetilde{\sigma \circ G})_q(F) \quad , \quad F, G \in \mathcal{C}^\infty(M, \mathfrak{g}) \quad ,$$

$q \in M$, inducing the evaluation $\rho_\ltimes$ of σ as an anchor map: $\rho_\ltimes(x, q) = \sigma(x)_q$, $(x, q) \in \mathfrak{g} \times M$.

iv) A Lie algebroid on a flat line bundle $L \simeq \mathbb{R} \times M$ is fully encoded by a vector field: one can see, by fibrewise linearity of the Lie algebroid anchor, that $\{ f \,,\, g \} = f \cdot X(g) - g \cdot X(f)$, where $X \in \mathfrak{X}(M)$ is defined by $X = \rho_L(1)$.

Analogously to the case of the Lie algebroid of a Poisson manifold, it is true in general that the image $\mathsf{Im}\rho$ of the anchor of a Lie algebroid $A \to M$, induces an integrable singular distribution. On each leaf $\mathcal{L}$ of the anchor, ρ has constant rank, thus there is a short exact sequence

$$0 \longrightarrow \mathfrak{g}_{\mathcal{L}} \longrightarrow A|_{\mathcal{L}} \longrightarrow T\mathcal{L} \longrightarrow 0$$

of vector bundles over $\mathcal{L}$, for $\mathfrak{g}_{\mathcal{L}} := \mathsf{ker}_{\mathcal{L}}\rho$. One can check that the Lie algebroid bracket on $\Gamma(M, A)$ restricts to $\Gamma(\mathcal{L}, A|_{\mathcal{L}})$ and induces a bundle of Lie algebras on $\mathfrak{g}_{\mathcal{L}}$; the fibres of $\mathfrak{g}_{\mathcal{L}}$ are called isotropy Lie algebras. Generalizing the Poisson case, the anchor foliation on the base manifold, together with the associated short exact sequences, is to be regarded as a global picture of Fernandes' local structure theorem for Lie algebroids [**20**].
We describe below the leaves of the Lie algebroids of example 1.1.5.

Example 1.1.6. Lie algebroid foliations.

i) The leaves of the distribution Δ: each point of M, respectively the whole of M, in the extreme cases;

ii) All the points of the base manifold;

iii) The orbits of the infinitesimal action;

iv) The flows of X.

Historically, Lie algebroids were discovered in the late sixties of last century by Pradines [**53, 54**]. Lie algebroids are the infinitesimal invariant of suitably smooth groupoids, just as Lie algebras are for Lie groups, and an understanding of such structures was necessary to develop a geometric study of groupoids endowed with a suitable smooth structure.

Definition 1.1.7. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid, i.e. a category all of whose arrows are invertible, such that both the total space $\mathcal{G}$ and the base space M are smooth manifolds. Moreover the the unit section $\varepsilon : M \to \mathcal{G}$, the map assigning to each object the identity arrow, the inversion map $\iota : \mathcal{G} \to \mathcal{G}$, the composition of arrows, regarded as a map $\mu : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathcal{G} \to \mathcal{G}$ and the source and target maps $\mathrm{s}, \mathrm{t} : \mathcal{G} \to M$ are required to be smooth. For $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathcal{G}$ to be a manifold and the regularity condition on μ to make sense it is further assumed that the source map be submersive.

It follows directly from the definition that the inversion map $\iota : \mathcal{G} \to \mathcal{G}$ is a diffeomorphism and the target map $\mathrm{t} : \mathcal{G} \to M$ can be written as $\mathrm{t} = \mathrm{s} \circ \iota$, therefore it is also submersive; by applying the inverse function theorem one can show that the unit section is a closed embedding.
The Lie algebroid of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is obtained through the following construction. Since the groupoid multiplication is only partially defined, so is the right translation $r_g : \mathrm{s}^{-1}(\mathrm{t}(g)) \to \mathrm{s}^{-1}(\mathrm{s}(g))$ by $g \in \mathcal{G}$ and one cannot hope to reproduce the differentiation of Lie groups to Lie algebras directly; however, mutatis mutandis, the procedure extends naturally. Being the source map submersive, its fibres are smooth submanifolds of $\mathcal{G}$, actually $(\mathcal{G}, \mathrm{s})$ is a simple foliation, and the vectors tangent to the source fibres $T^{\mathrm{s}}\mathcal{G} := \ker \mathrm{ds} \to \mathcal{G}$ form a smooth vector subbundle. The total space of the Lie algebroid A of $\mathcal{G}$ is the restriction $T^{\mathrm{s}}_M\mathcal{G}$ to M. Right translation is well defined in $T^{\mathrm{s}}\mathcal{G}$ and the vector fields which are tangent to the source fibres are self-related under right translation, called right invariant vector fields, form a $\mathrm{s}^*\mathcal{C}^\infty(M)$-submodule $\overrightarrow{\mathfrak{X}}(\mathcal{G})$ of $\mathfrak{X}(\mathcal{G})$, isomorphic to the $\mathcal{C}^\infty(M)$-module on $\Gamma(A)$. Analogously to the case of Lie groups and Lie algebras, one can easily show that $\overrightarrow{\mathfrak{X}}(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$ is a Lie subalgebra, endowing A with a Lie algebroid bracket; the anchor is the restriction of tangent target dt to A, compatibility with the Lie bracket follows from the Leibniz rule for $T\mathcal{G}$.
The orbit $\mathcal{O}_q$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ through $q \in M$ is by definition $\mathcal{O}_q = \mathrm{t}(\mathrm{s}^{-1}(q))$ and one can show that each source fibre is a smooth principal $\mathbb{G}_q$-bundle $\mathrm{t} : \mathrm{s}^{-1}(q) \to \mathcal{O}_q$ for the isotropy group $\mathbb{G}_q := \mathrm{s}^{-1}(q) \cap \mathrm{t}^{-1}(q)$. By construction of the Lie algebroid A of $\mathcal{G}$, for all $p \in \mathcal{O}_q$, $\mathrm{Im}_p\rho = T_q\mathcal{O}_p$, therefore the leaves of A are connected components of the orbits of $\mathcal{G}$. We shall denote the orbit space of the foliation induced by $\mathcal{G}$ with $M/\mathcal{G}$.
We describe below the classic examples of Lie groupoids and their Lie algebroids in some detail.

Example 1.1.8. Lie groupoids

i) Any manifold M is trivially a Lie groupoid $M \rightrightarrows M$ over itself. Since necessarily $\mathrm{s} = \mathsf{id}_M = \mathrm{t}$, the only possible multiplication is the identification $\Delta_M \to M$. On the other hand, the pair groupoid $M \times M \rightrightarrows M$, given by $\mathrm{s} = \mathrm{pr}_2$, $\mathrm{t} = \mathrm{pr}_1$, the diagonal $M \to \Delta_M$ as unit section and

$$(x, y) \cdot (y, z) = (x, z) \qquad , \qquad (x, y)^{-1} = (y, x) \qquad , \qquad x, y, z \in M \qquad ,$$

for the multiplication and inversion, is a nontrivial Lie groupoid associated with every manifold. More generally, to the graph $\Gamma(\sim)$ of any equivalence relation $\sim$ on a smooth manifold M one can associate a subcategory of the pair groupoid $M \times M$ in the obvious way and conversely every subgroupoid of $M \times M \rightrightarrows M$ with base M is the graph of an equivalence relation; $\Gamma(\sim)$ is a Lie groupoid iff $\sim$ is a *regular equivalence relation*, i.e. if $\Gamma(\sim)$ is smooth and the restriction of the first and second projection are submersive. The trivial groupoid over a manifold is the graph of the trivial equivalence relation, for the pair groupoid all points on the base are equivalent; the induced Lie algebroids are the trivial Lie algebroid and the tangent bundle, respectively.

ii) A Lie group is a Lie groupoid over the one point manifold. The anchor of a groupoid $\mathcal{G} \rightrightarrows M$ is the map $\chi = (\mathrm{t}, \mathrm{s}) : \mathcal{G} \to M \times M$. The image of χ always contains the diagonal submanifold; if $\mathsf{Im}\,\chi = \Delta_M$, $\mathcal{G}$ is a smooth bundle of Lie groups, in the sense that there are no arrows connecting different points and $\mathcal{G} = \coprod_{q \in M} \mathbb{G}_q$. Smooth bundles of Lie groups differentiate to bundles of Lie algebras.

iii) Let $\sigma : G \times M \to M$ be the action map of a Lie group action. Then the product $G \times M$ carries a (unique) Lie groupoid structure over M, the action groupoid $G \ltimes M$, whose orbits coincide with the orbits of the action. Source and target maps are $\mathrm{s}_\ltimes(g, m) = m$ and $\mathrm{t}_\ltimes(g, m) = g * m = \sigma(g, m)$, $(g, m) \in G \times M$, thus composable pairs are those of the form $[(h, g*m); (g, m)]$, $h \in G$; the multiplication $\mu_\ltimes$ is given by $(h, g * m) \circ (g, m) = (hg, m)$, hence the unit section $\varepsilon_\ltimes$ and inversion $\iota_\ltimes$ must be $\varepsilon_\ltimes(m) = (e, m)$ and $\iota_\ltimes(g, m) = (g^{-1}, g * m)$ for the unit element e of G. The Lie algebroid of $G \ltimes M$ is the action algebroid of the induced infinitesimal action. The isotropies of $(\mathfrak{g} \ltimes M)$ $G \ltimes M$, are precisely the isotropies of the (infinitesimal) action.

iv) For any left principal G-bundle $\mathrm{pr} : P \to M$ the quotient $(P \times P)/G$ for the diagonal action is a Lie groupoid over M, known as the gauge groupoid. Target and source maps are $[p_+, p_-] \mapsto \mathrm{pr}(p_\pm)$, the multiplication is

$$[p_+, p_-] \cdot [q_+, q_-] = [g * p_+, q_-] \qquad ,$$

for the unique $g \in G$ such that $p_- = g * q_+$, unit section and inversion are defined accordingly. The Lie algebroid of $(P \times P)/G$ can be identified with the Atiyah Lie algebroid $(TP)/G$ filling the short exact sequence

$$0 \longrightarrow (P \times \mathfrak{g})/G \longrightarrow (TP)/G \longrightarrow TM \longrightarrow 0 \tag{1.3}$$

for the adjoint bundle $(P\times\mathfrak{g})/G$ (see [**45**] for details). If M is a connected manifold with universal cover $\widetilde{M}$, one can apply the above construction to the covering projection $\mathrm{pr} : \widetilde{M} \to M$ for the monodromy action of the fundamental group $\pi_1(M)$ and obtain the fundamental groupoid $\Pi(M) \rightrightarrows M$. Since $\pi_1(M)$ is discrete the Atiyah sequence (1.3) yields an identification of Lie algebroids $(T\widetilde{M})/\pi_1(M) \equiv TM$. An alternative characterization of $\Pi(M)$ can be given in terms of continuous paths in M up to homotopy relative to the endpoints; in this description the groupoid multiplication is induced by path-concatenation. The construction easily extends to a disconnected manifold M.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Since the source map is submersive, For a small enough open neighbourhood $U' \subset M$ there always exists a smooth section $\Sigma' : U' \to \Sigma'(U')$ of the source map; by linear algebra, Σ' can be perturbed to a new smooth section $\Sigma : U \to \Sigma(U)$, for some smaller neighbourhood $U \subset U'$, such that $\mathrm{t} \circ \sigma'$ is a diffeomorphism. In the above construction source and target can be exchanged and we shall call a local bisection of $\mathcal{G}$ a rank s = rank t-codimensional submanifold, on which both source and target restrict to diffeomorphism to their images; when the domains of a local bisection Σ, regarded as a section either way, coincide with M, hence so do the images of $\mathrm{s} \circ \Sigma$ and $\mathrm{t} \circ \Sigma$, Σ is called a (global) bisection.

Example 1.1.9. Global bisections

i) The base manifold of any Lie groupoid is a bisection. Every section of a vector bundle $\mathrm{pr} : E \to M$ is a bisection for the abelian groupoid ($\mathrm{s} = \mathrm{pr} = \mathrm{t}$ and $\mu = +$).

ii) Bisections of a pair groupoid are in one to one correspondence to diffeomorphisms of the base.

iii) The set of a bisections of a group is the group itself.

iv) See example 2.3.13 for a groupoid with points not admitting any global bisection.

Introducing local bisections allows one to deal with left and right translation of arbitrary tangent vectors. For any local bisection Σ there are well defined diffeomorphisms

$$\mathcal{L}^{\Sigma} : \mathrm{t}^{-1}(\mathrm{s}(\Sigma)) \to \mathrm{t}^{-1}(\mathrm{t}(\Sigma)) \qquad , \qquad h \mapsto \Sigma(\mathrm{t}(h)) \cdot h \qquad ,$$

where Σ is regarded as a section of the source map, and

$$\mathcal{R}^{\Sigma} : \mathrm{s}^{-1}(\mathrm{t}(\Sigma)) \to \mathrm{s}^{-1}(\mathrm{s}(\Sigma)) \qquad , \qquad h \mapsto h \cdot \Sigma(\mathrm{s}(h)) \qquad ,$$

where Σ is regarded as a section of the target map. For any Lie groupoid $\mathcal{G} \rightrightarrows M$, there is a canonical splitting

$$T_M\mathcal{G} \simeq T^{\mathrm{s}}_M\mathcal{G} \oplus TM \qquad , \qquad \delta g = \delta g - \mathrm{ds}\delta g \oplus \mathrm{ds}\delta g \qquad ,$$

where we identify $M \simeq \varepsilon(M)$; such a splitting extends to each point $g \in \mathcal{G}$ off the base manifold, by picking a local bisection Σ through g:

$$T_g\mathcal{G} \simeq T^{\mathrm{s}}_g\mathcal{G} \oplus T_g\Sigma \qquad , \qquad \delta g = \delta^{\mathrm{s}} g \oplus \delta^{\sigma} g \qquad ,$$

where $\delta^\sigma g = \mathrm{d}\mathcal{L}^{\Sigma}{}_g \mathrm{ds}\delta g$ and $\delta^{\mathrm{s}} g := \delta g - \delta^\sigma g$. Similarly, splittings subordinated to the target map can be obtained via right translation. One can check that the composition

$$\Sigma_+ \cdot \Sigma_- := \mu(\Sigma_+ \times \Sigma_- \cap \mathcal{G}^{(2)})$$

of local bisections $\Sigma_\pm$ and the inverse bisection $\Sigma^{\text{-}1} := \iota(\Sigma)$ of a local bisection Σ are also local bisections; as a consequence left and right translations by local bisections form a pseudo group of transformations:

$$\begin{array}{rclcrcl}
\mathcal{L}^M & = & \mathsf{id}_{\mathcal{G}} & \qquad & \mathcal{R}^M & = & \mathsf{id}_{\mathcal{G}} \\
\mathcal{L}^{\Sigma_+\cdot\Sigma_-} & = & \mathcal{L}^{\Sigma_+} \circ \mathcal{L}^{\Sigma_-} & & \mathcal{R}^{\Sigma_+\cdot\Sigma_-} & = & \mathcal{R}^{\Sigma_-} \circ \mathcal{R}^{\Sigma_+} \\
\mathcal{L}^{\Sigma^{\text{-}1}} & = & (\mathcal{L}^{\Sigma})^{\text{-}1} & & \mathcal{R}^{\Sigma^{\text{-}1}} & = & (\mathcal{R}^{\Sigma})^{\text{-}1}
\end{array} \qquad ,$$

the equalities in the second row hold, provided both sides are defined. In particular, restricting to global bisections defines the group of bisections $\mathsf{Bis}(\mathcal{G})$. In the following, we shall often implicitly make use of local bisections, especially in order tocount dimensions.

A morphism of Lie groupoids is simply a smooth functor; in other words, a morphism of Lie groupoids is given by a pair of smooth maps $\varphi : \mathcal{G}^- \to \mathcal{G}^+$, $f : M^- \to M^+$, which are equivariant with respect to the groupoid structural maps in all possible ways, in particular so are the operations of left and right translations: on the domains were the expressions below are defined,

$$\phi \circ l_g = l_{\phi(g)} \circ \phi \qquad \text{and} \qquad \phi \circ r_g = r_{\phi(g)}$$

hold for all $g \in \mathcal{G}^-$. Note that, in general, morphisms of Lie groupoids do not map bisections to bisections.

Remark 1.1.10. For any Lie groupoids $\mathcal{G}^\pm \rightrightarrows M^\pm$, a smooth map $\varphi : \mathcal{G}^- \to \mathcal{G}^+$ is a morphism of Lie groupoids over the uniquely determined base map $f : M^- \to M^+$, $f = \mathrm{s}_+\varphi \circ \varepsilon_- = \mathrm{t}_+\varphi \circ \varepsilon_-$ iff

$$(\varphi \times \varphi \times \varphi)\,\Gamma(\mu_-) \subset \Gamma(\mu_+) \qquad ,$$

i.e. if φ preserves the graph of the partial multiplications.

A smooth submanifold of a Lie groupoid which is also a subcategory is a Lie subgroupoid iff the restriction of the source map remains submersive; a Lie subgroupoid $\mathcal{H}$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called a wide subgroupoid if the source map (hence the target map) of $\mathcal{G}$ is surjective onto M, i.e. if it is a Lie groupoid over M. The direct product of Lie groupoids is defined in the obvious way.

Example 1.1.11. Subgroupoids and morphisms of Lie groupoids

i) Let N be a manifold and $\mathcal{G} \rightrightarrows M$ be a (Lie) groupoid. By picking an arbitrary point $n_o \in N$, every morphism of (Lie) groupoids $\varphi : N \times N \to \mathcal{G}$ over $f : N \to M$ is of the form $\varphi(n_+, n_-) = \psi(n_+) \cdot \psi(n_-)^{-1}$, $n_\pm \in N$, where $\psi(n) = \varphi(n, n_o)$, $n \in N$. Conversely, given (smooth) maps $\psi : N \to \mathcal{G}$ and $f : N \to M$, such that $\mathrm{s} \circ \psi(n) = n_o$, $n \in N$, inverting the procedure yields a morphism of (Lie) groupoids over $\mathrm{t} \circ \psi$.

ii) The anchor $\chi : \mathcal{G} \to M \times M$ of any Lie groupoid is a morphism to the pair groupoid over the identity. When it is surjective (thus, as one can check by diagram chasing, submersive), $\mathcal{G}$ is said transitive, in the sense that the base foliation is the base manifold itself. On the other hand, when $\mathsf{Im}\chi = \Delta_M$, the base foliation consists of all the points of M and $\mathcal{G}$ is said totally intransitive.

iii) For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the manifold $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G}$ defines a wide subgroupoid $\mathcal{G} \odot \mathcal{G}$ of the pair groupoid $\mathcal{G} \times \mathcal{G} \rightrightarrows \mathcal{G}$. It is a Lie subgroupoid with source fibres $\mathrm{s}_\odot^{-1}(g) = \mathrm{s}^{-1}(\mathrm{s}(g)) \times \{g\}$, $g \in \mathcal{G}$; to see this, consider that, for all $\delta g \in T_g\mathcal{G}$ and $h \in \mathrm{s}^{-1}(\mathrm{s}(g))$, $\delta h := (\mathrm{d}\mathcal{L}'_\Sigma \circ \mathrm{d}\mathcal{L}_\Sigma^{-1})\delta g$, is a tangent vector at h such that $\mathrm{d}\mathrm{s}\delta h = \mathrm{d}\mathrm{s}\delta g$, for any choice of local bisections Σ through g and Σ' through h, therefore $(\delta h, \delta g) \in T_{(g,h)}\mathcal{G} \odot \mathcal{G}$ and $\mathrm{d}\mathrm{s}_\odot(\delta h, \delta g) = \mathrm{dpr}_2(\delta h, \delta g) = \delta g$, i.e. $\mathrm{s}_\odot$ is submersive. The division map $\delta : \mathcal{G} \odot \mathcal{G} \to \mathcal{G}$, $(g, h) \mapsto g \cdot h^{-1}$, is a morphism of Lie groupoids over the target map of $\mathcal{G}$.

iv) For any morphism of Lie groupoids $\varphi : \mathcal{G}^- \to \mathcal{G}^+$ over $f : M^- \to M^+$, the kernel groupoid $\mathsf{ker}\varphi \rightrightarrows M^-$ is $\mathsf{ker}\varphi := \varphi^{-1}(\varepsilon_+(M^+))$. It is always a wide subgroupoid of $\mathcal{G}^-$, though, in general not a *Lie* subgroupoid. The kernel of the anchor χ of a groupoid $\mathcal{G}$, is a (topological) bundle of Lie groups $\mathbb{G} = \chi^{-1}(\Delta_M) = \coprod_{q \in M} \mathbb{G}_q$. A kernel groupoid $\mathcal{N}$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a normal subgroupoid, in the sense that it is wide and, for all $g \in \mathcal{G}$

$$g \cdot n \cdot g^{-1} \in \mathbb{N}_{\mathrm{t}(g)} \quad , \quad \text{for all } n \in \mathbb{N}_{\mathrm{s}(g)} \quad ;$$

in fact, one can show that, taking the quotient of $\mathcal{G}$ by the equivalence relation $g \sim h$, if $g = h \cdot n$ for some $n \in \mathcal{N}$, $g, h \in \mathcal{G}$, induced by such a subgroupoid, yields a quotient groupoid $\mathcal{G}/\mathcal{N} \rightrightarrows M/\mathcal{N}$ over the orbit space of $\mathcal{N}$. A subgroupoid of a Lie groupoid is normal iff it is the kernel of a morphism of groupoids, since the quotient projection $\mathcal{G} \to \mathcal{G}/\mathcal{N}$ is a morphism of groupoids by construction.

Some constructions in the category of smooth manifolds extend straightforwardly to that of Lie groupoids; for an instance, consider the next example.

Example 1.1.12. The tangent prolongation groupoid. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, then so is the tangent prolongation $T\mathcal{G} \rightrightarrows TM$ for the tangent structural maps of $\mathcal{G}$. Since the construction is functorial and a groupoid structure is fully described in terms of diagrams, $T\mathcal{G} \rightrightarrows TM$ is clearly a smooth groupoid; to further check that the tangent source map is submersive is a basic exercise in differential geometry (see also remark 2.2.8). Computing the Lie algebroid of $T\mathcal{G} \rightrightarrows TM$ yields the tangent prolongation Lie algebroid $TA \to TM$ (see [**43, 40**] for the Lie algebroid anchor and bracket).

Many other interesting constructions require an understanding of morphisms in the categories of Lie algebroids and Poisson manifolds and duality therein.

1.2. Morphisms and coisotropic calculus

In the first part of this Section we first review (following [**24, 45**]) the notion of morphism and the basic constructions in the category of Lie algebroids, such as direct products and pullbacks. The second part is devoted to recall nowadays standard facts about Poisson maps and coisotropic submanifolds of Poisson manifolds; finally we discuss the dual Poisson-geometric characterization of morphisms of Lie algebroids and Lie subalgebroids in terms of coisotropic submanifolds.

1.2.1. Morphisms, pullbacks and direct products of Lie algebroids.

Consider Lie algebroids $A^{\pm} \to M^{\pm}$ over different bases and a vector bundle map $\phi : A^- \to A^+$ over $f : M^- \to M^+$. For such a map to be a morphism of Lie algebroids, the compatibility with the anchors is expressed by the natural condition $\rho_+ \circ \phi = \mathrm{d}f \circ \rho_-$; this way leaves of A^- are mapped to leaves of A^+. Whenever $M^+ = M^-$ and f is the identity map, the bracket compatibility condition is also clear: the induced map of sections $\Gamma(A^-) \to \Gamma(A^+)$ has to be a morphism of Lie algebras. The main complication encountered in defining morphisms of Lie algebroids over different bases is met in not having at disposal a natural way of mapping sections of A^- to sections of A^+. Let us first consider a special case: suppose the vector bundle fibred product $TM^- {}_{\mathrm{d}f}\times_{\rho_+} A^+$ exists; upon identifying the base manifold $M^- {}_f\times M^+ \simeq \mathsf{\Gamma}(f)$ with the graph of f, yields a vector bundle $f^{++}A^+ : TM^- {}_{\mathrm{d}f}\times_{\rho_+} A^+ \to M^-$.

Proposition 1.2.1. *Let $A^\pm \to M^\pm$ be Lie algebroids and $\phi : A^- \to A^+$ a vector bundle map over $f : M^- \to M^+$. Then there exists a unique Lie algebroid on the vector bundle $f^{++}A^+$, whose anchor ρ_{++} is the restriction of the first projection $TM^- \times A^+ \to TM^-$.*

Last proposition is a restatement of some of the results contained in [**24**]; we shall only present the idea of the proof.

Sketch of proof of proposition 1.2.1. A section a^{++} of $f^{++}A$ is given by a pair $(X, \widetilde{a}) \in \mathfrak{X}(M^-) \oplus \Gamma(f^+A)$, satisfying $\mathrm{d}f \circ X = \rho_+ \circ \widetilde{a}$; on the other hand, the space sections of the pullback bundle[2] $f^+A^+ \to M^-$ is isomorphic to $\mathcal{C}^\infty(M^-) \otimes_{\mathcal{C}^\infty(M^+)} \Gamma(A^+)$ as a left $\mathcal{C}^\infty(M^-)$-module, where the tensor product is taken for the right $\mathcal{C}^\infty(M^+)$-module on $\mathcal{C}^\infty(M^-)$ induced by precomposition with f. Then for any $a^{++}_{1,2} \in \Gamma(f^{++}A)$ there exist decompositions of the form

$$a^{++}_{1,2} = X_{1,2} \oplus \sum_{k_{1,2}} u^{k_{1,2}}_{1,2} (a^{k_{1,2}}_{1,2} \circ \phi) \qquad ,$$

in terms of $X_{1,2} \in \mathfrak{X}(M^-)$, $\{u^{k_1,k_2}_{1,2}\}_{k_1,k_2} \subset \mathcal{C}^\infty(M^-)$ and $\{a^{k_{1,2}}_{1,2}\}_{k_1,k_2} \subset \Gamma(A^+)$, which can be used to define

$$\begin{aligned} [\, a^{++}_1 , \, a^{++}_2]^{++} \; := & \;\; [\, X_1 \, , \, X_2 \,] \\ \oplus & \;\; \Bigg\{ \sum_{k_1,k_2} u^{k_1}_1 \cdot u^{k_2}_2 ([\, a^{k_1}_1 \, , \, a^{k_2}_2 \,] \circ \phi) \quad + \quad \sum_{k_2} X_1(u^{k_2}_2)(a^{k_2}_2 \circ \phi) \\ - & \;\; \sum_{k_1} X_2(u^{k_1}_1)(a^{k_1}_1 \circ \phi) \Bigg\} \qquad . \end{aligned}$$

It turns out that the above bracket does not depend on the choice of the decompositions and yields a well defined bilinear skewsymmetric operation on $\Gamma(f^{++}A)$; the definition is tailored to have the Leibniz rule satisfied and one can check that also the Jacobi identity holds. □

The Lie algebroid structure on $f^{++}A^+ \to M^-$ is that of a pullback Lie algebroid along f. When such a Lie algebroid exists, for instance when f is submersive or ρ has maximal rank, it results then natural to ask the induced map $\Gamma(A^-) \to \Gamma(f^{++}A^+)$, $a \mapsto \rho_- \circ a \oplus \phi \circ a$, to preserve the Lie brackets in order for ϕ to be a morphism of Lie algebroids over f; provided the anchor compatibility holds, this requirement amounts to

$$\rho_- \circ [\, a_1 \, , \, a_2 \,] = [\, \rho_- \circ a_1 \, , \, \rho_- \circ a_1 \,] \qquad , \tag{1.4}$$

[2]Recall that, for any smooth map $f : N \to M$ and smooth vector bundle $\mathrm{pr} : A \to M$, the space $\{(a, n) \mid \mathrm{pr}(a) = f(n)\}$ is canonically a smooth vector bundle over M^-.

for all $a_{1,2} \in \Gamma(A^-)$, on the $\mathfrak{X}(M^-)$ component and

$$\begin{aligned} \phi \circ [\, a_1 \,,\, a_2 \,] \;&=\; \sum_{k_1,k_2} u_1^{k_1} \cdot u_2^{k_2} ([\, a_1^{k_1} \,,\, a_2^{k_2} \,] \circ \phi) \quad + \quad \sum_{k_2} \rho_-(a_1)(u_2^{k_2})(a_2^{k_2} \circ \phi) \\ &- \;\; \sum_{k_1} \rho_-(a_2)(u_1^{k_1})(a_1^{k_1} \circ \phi) \end{aligned} \tag{1.5}$$

on the $\Gamma(f^+A^+)$ component, for any choice of decompositions

$$\phi \circ a_{1,2} = \sum_{k_{1,2}} u_{1,2}^{k_{1,2}} (a_{1,2}^{k_{1,2}} \circ \phi) \qquad , \tag{1.6}$$

in terms of $\{u_{1,2}^{k_1,k_2}\}_{k_1,k_2} \subset \mathcal{C}^\infty(M^-)$ and $\{a_{1,2}^{k_{1,2}}\}_{k_1,k_2} \subset \Gamma(A^+)$. Even if the pullback Lie algebroid along f does not exist, condition (1.5) still makes sense and does not depend on the choice of decompositions, provided the anchor compatibility condition holds. This leads to the general definition of a morphism of Lie algebroids.

Definition 1.2.2. Let $A^\pm \to M^\pm$ be Lie algebroids and $\phi : A^- \to A^+$ a vector bundle map over $f : M^- \to M^+$ which is compatible with the anchor maps in the sense that $\rho_+ \circ \phi = \mathrm{d}f \circ \rho_-$. Then (ϕ, f) is a morphism of Lie algebroids if the bracket compatibility (1.5) holds for all $a_{1,2} \in \Gamma(A^-)$ and any choice of decompositions (1.6) of $\phi \circ a_{1,2}$.

This notion of morphism is consistent with that of a category of Lie algebroids; even though it might seem quite intractable, this definition is sufficiently effective to deal with general constructs such as subobjects and direct products, as it was shown in [**24**].

Example 1.2.3. Morphisms of Lie algebroids. We list below some fundamental examples; the details, which can be found in [**45**], are left to the reader as an exercise.

i) If $\phi : A^- \to A^+$ is morphism of Lie algebroids over the identity map for $M^- = M^+$, the induced map of sections $\Gamma(A^-) \to \Gamma(A^+)$ is required to preserve the Lie brackets; in particular for, morphisms of Lie algebras are morphisms of Lie algebroids ($M^- = M^+ = \bullet$).

ii) For any smooth map $f\colon M \to N$, $\mathrm{d}f\colon TM \to TN$ is a morphism of Lie algebroids; this can be checked using decompositions and equating the sides bracket compatibility condition, regarding vector fields as derivations.

iii) The inductor $f^{++}\colon f^{++}A \to A$, $(X, a) \mapsto a$, for the pullback of a Lie algebroid of $A \to M$ along $f : N \to M$, is a morphism of Lie algebroids.

iv) For any morphism of Lie groupoids $\varphi : \mathcal{G}^- \to \mathcal{G}^+$ over $f : M^- \to M^+$, setting

$$\phi := \mathrm{d}\varphi|_{T^{s_-}_{M^-}\mathcal{G}^-} : A^- \to A^+$$

yields a well defined vector bundle map over f. The anchor compatibility holds, since

$$(\rho_+ \circ \phi)(a) = (\mathrm{dt}_- \circ \mathrm{d}\varphi)(a) = (\mathrm{d}f \circ \mathrm{dt}_-)(a) = (\mathrm{d}f \circ \rho_-)(a) \quad , \quad a \in \Gamma(A^-) \quad ,$$

and the bracket compatibility follows from the properties of right invariant vector fields.

Last example shows that there exist a Lie functor from the category of Lie groupoids to that of Lie algebroids extending the classical Lie functor.

Remark 1.2.4. Let $A^\pm \to M^\pm$ be Lie algebroids and $\phi : A^- \to A^+$ a vector bundle map over $f : M^- \to M^+$ which is compatible with the anchor maps in the sense of definition 1.2.2. Then one can pick any connection ∇ for $A^+ \to M^+$ to express the bracket compatibility more intrinsically. Denote with $\overline{\nabla}$ the pullback connection induced by ∇ on $f^+A^+ \to M^-$, with $\tau_\nabla \in \Gamma(A \otimes \wedge^2 A^*)$ the torsion tensor

$$\tau_\nabla(a, b) := \nabla_{\rho^+(a)} b - \nabla_{\rho^+(b)} a - [\, a \,, \, b \,] \qquad , \qquad a, b \in \Gamma(A)$$

and with $f^+\tau_\nabla$ its pullback to f^+A^+. Then (ϕ, f) satisfies the bracket compatibility condition (1.5) iff

$$\phi \circ [\, a_1 \,, \, a_2 \,] = \overline{\nabla}_{\rho^-(a_1)} \phi \circ a_2 - \overline{\nabla}_{\rho^-(a_2)} \phi \circ a_1 - f^+\tau_\nabla(a_1, a_2)$$

holds for all $a_{1,2} \in \Gamma(A^-)$ [**24**].

Let us now turn to subobjects (see [**45**] for details).

Definition 1.2.5. Let $A \to M$ be a Lie algebroid and $N \subset M$ an embedded submanifold. A Lie algebroid $B \to N$ on a vector subbundle of $A \to M$ is a Lie subalgebroid if the inclusion $B \hookrightarrow A$ is a morphism of Lie algebroids over the inclusion $N \hookrightarrow M$.

From the last definition it is clear that restrictions of morphisms of Lie algebroids to Lie subalgebroids yield morphisms of Lie algebroids.
As it is to be expected, Lie subalgebroids are precisely those vector subbundles for which the algebraic operations of the ambient Lie algebroid suitably restrict.

Lemma 1.2.6. *A vector subbundle $B \to N$ of a Lie algebroid $A \to M$ is a Lie subalgebroid iff the following conditions hold*

1. *The anchor $\rho : A \to TM$ restricts to a bundle map $B \to TN$;*
2. *For all $a_\pm \in \Gamma(A)$ such that $a_\pm|_N \in \Gamma(B)$, $[\, a_+ \,, \, a_- \,]|_N \in \Gamma(B)$;*

Remark 1.2.7. It is always possible to restrict a Lie algebroid to a open submanifold of the base. Since the conditions of lemma (1.2.6) are local, they can always be checked on some neighbourhood U, open in M, of an arbitrary point of N. Note, in particular, that for all $a_\pm$ such as in condition 2. above, it follows that $[\, a_+ \,,\, a_- \,]|_N = 0$, whenever $a_-|_N = 0$; that is, the bracket of sections of $\Gamma(B)$ can be computed using arbitrary extensions. Since for any frames $\{e^\alpha\}$, $\{e_\alpha\}$ in duality for A and A^* on (a restriction of) U, the Leibniz rule implies

$$\begin{aligned} [\, a^1 \,,\, a^2 \,] &= \sum_\alpha a^2_\alpha [\, a^1 \,,\, e^\alpha \,] + \rho(a^1)(a^2_\alpha) e^\alpha \\ &= \sum_\alpha a^2_\alpha [\, a^1 \,,\, e^\alpha \,] + \langle\, \mathrm{d}a^2_\alpha \,,\, \rho(a^1) \,\rangle \qquad , \end{aligned}$$

where $a^2_\alpha := \langle\, e_\alpha \,,\, a^2 \,\rangle \in \mathcal{C}^\infty(U)$ vanishes on $U \cap N$ for all α's, thus $\langle\, \rho(a^1) \,,\, \mathrm{d}a^2_\alpha \,\rangle$ also vanishes on $U \cap N$, since the anchor restricts to a bundle map $B|_{U \cap N} \to T(U \cap N)$.

Bearing condition (1.5) in mind the proof of last lemma is straightforward by working locally and picking extensions. As a consequence of example (1.2.3, *iv*) Lie subgroupoids differentiate to Lie subalgebroids. There is a useful corollary to lemma 1.2.6.

Corollary 1.2.8. *Given Lie algebroids A, B, C and a sequence of vector bundle inclusions $A \subset B \subset C$, if $A \subset C$ and $B \subset C$ are Lie subalgebroids, then so is $A \subset B$.*

Lie subalgebras are obviously Lie subalgebroids over the one point manifold; we list a few other examples.

Example 1.2.9. Lie subalgebroids.

i) An integrable regular distribution is a Lie subalgebroid of the tangent bundle. More generally, the tangent bundle of a submanifold is a Lie subalgebroid of the ambient manifold.

ii) Let P be a Poisson manifold and $C \subset P$ a submanifold; then the conormal bundle $N^*C \subset T^*P$ is a Lie subalgebroid iff C is coisotropic. We shall discuss this example in the next Section.

iii) For any Lie algebroid A and leaf $\mathcal{L}$ of the anchor distribution the restriction $A|_\mathcal{L} \to \mathcal{L}$ is a Lie subalgebroid.

iv) Let $A \to M$ and $B \to N$ be Lie algebroids and $\phi : A \to B$ a vector bundle map of constant rank, so that its kernel is a vector subbundle of A; then, $\mathsf{ker}\,\phi \to M$ is a Lie subalgebroid whenever ϕ is a morphism of Lie algebroids. In particular the bundle of isotropy Lie algebras $\mathfrak{g}_\mathcal{L}$ of a leaf $\mathcal{L}$ of a Lie algebroid A is a Lie subalgebroid of $A|_\mathcal{L}$.

Pullback Lie algebroids are not only useful to understand morphisms of Lie algebroids, but also play an essential role to produce general constructs, such as direct products, described below, in the category of Lie algebroids, thanks to the following universal property.

Proposition 1.2.10. *Let $A \to M$ be a Lie algebroid and $f : M' \to M$ be a smooth map. Assume that the pullback Lie algebroid $f^{++}A \to N$ exists. Then for any morphism of Lie algebroids $\phi : B \to A$ over $g : N \to M$ factoring through $h : N \to M'$ there exists a unique morphism of Lie algebroids $\psi : B \to f^{++}A$, such that $f^{++} \circ \psi = \phi$.*

Next, consider Lie algebroids A^1, A^2 and B *over the same base* M; given morphisms of Lie algebroids $\phi_{1,2} : A^{1,2} \to B$ over the identity, such that the fibred product $A^1 \,{}_{\phi_1}\!\times_{\phi_2} A^2 \to \Delta_M \simeq M$ is a vector bundle, it is possible to introduce the product Lie algebroid, over B in this case, for the vector bundle structure over M. The anchor is given by

$$\rho(a_1 \oplus a_2) = \rho_B \circ \phi_1(a_1) = \rho_B \circ \phi_2(a_2) \qquad , \qquad a_1 \oplus a_2 \in A^1 \,{}_{\phi_1}\!\times_{\phi_2} A^2 \qquad ;$$

the bracket is defined componentwise:

$$[\, a_1 \oplus a_2 \, , \, b_1 \oplus b_2 \,] = [\, a_1 \, , \, b_1 \,] \oplus [\, a_2 \, , \, b_2 \,] \quad , \quad a_1 \oplus a_2, b_1 \oplus b_2 \in \Gamma(M, A^1 \,{}_{\phi_1}\!\times_{\phi_2} A^2) \quad .$$

Last construction is a straightforward generalization of the *fibred product of Lie algebroids over the same base* in [**24**], which is recovered replacing B with TM and $\phi_{1,2}$ with $\rho_{1,2}$. General fibred products shall be studied later on in Section 2.1. Given Lie algebroids $A^{1,2} \to M^{1,2}$, denote with M^{12} the direct product $M^1 \times M^2$ and with $\mathrm{pr}_{1,2}$ the projections onto $M^{1,2}$. Since the pullback algebroids $\mathrm{pr}_{1,2}^{++}A^{1,2}$ always exist and the fibred product of manifolds $\mathrm{pr}_1^{++}A^1 \,{}_{\rho^1_{++}}\!\times_{\rho^2_{++}} \mathrm{pr}_2^{++}A^2$ is to be identified with the vector bundle $A^1 \times A^2 \to M^1 \times M^2$, there is always a fibred product Lie algebroid over TM^{12}, the direct product Lie algebroid (denoted simply as $A^1 \times A^2$) of A^1 and A^2; applying proposition 1.2.10, it is straightforward to check that it is indeed a direct product in the category of Lie algebroids, satisfying the relevant universal property.
The Lie bracket on $A^1 \times A^2$ can be described explicitly as follows. Note that $\Gamma(A^1 \times A^2) = \Gamma(\mathrm{pr}_1^{+} A^1) \oplus \Gamma(\mathrm{pr}_2^{+} A^2)$; for any choice of decompositions

$$\alpha_{1,2} = \sum_{k_{1,2}} u^{1,2}_{k_{1,2}} (a^{1,2}_{k_{1,2}} \circ \mathrm{pr}_{1,2}) \qquad \text{and} \qquad \beta_{1,2} = \sum_{l_{1,2}} v^{1,2}_{l_{1,2}} (b^{1,2}_{l_{1,2}} \circ \mathrm{pr}_{1,2})$$

of $\alpha_{1,2}, \beta_{1,2} \in \Gamma(\mathrm{pr}^+_{1,2} A^{1,2})$ with $\{u^{1,2}_{k_{1,2}}\}, \{v^{1,2}_{l_{1,2}}\} \in \mathcal{C}^\infty(M^1 \times M^2)$ and $\{a^{1,2}_{k_{1,2}}\}, \{b^{1,2}_{l_{1,2}}\} \in \Gamma(A^{1,2})$, the components of $[\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2]$ are given by $(i = 1, 2)$

$$\begin{aligned} [\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2]_i &= \sum_{k_i, l_i} u^i_{k_i} v^i_{l_i} ([\, a^i_{k_i} \,,\, b^i_{l_i} \,] \circ \mathrm{pr}_i) \\ &+ \sum_{l_i} (\rho_1(\alpha_1) \times \rho_2(\alpha_2))(v^i_{l_i})(b^i_{l_i} \circ \mathrm{pr}_i) \\ &- \sum_{k_i} (\rho_1(\beta_1) \times \rho_2(\beta_2))(u^i_{k_i})(a^i_{k_i} \circ \mathrm{pr}_i) \qquad . \end{aligned} \tag{1.7}$$

1.2.2. Poisson maps and coisotropic calculus.

Morphisms in the category of Poisson manifolds are smooth maps inducing morphisms of Poisson algebras.

Definition 1.2.11. Let $P_\pm$ be Poisson manifolds; a smooth map $P_- \to P_+$ is called a Poisson map if its pullback $\mathcal{C}^\infty(P_+) \to \mathcal{C}^\infty(P_-)$ is a morphism of Lie algebras.

Remark 1.2.12. Not every notion in the category of Poisson manifolds extends some symplectic notion: a symplectomorphism is not a Poisson map, and vice versa, unless it is a local diffeomorphism.

The notion of *Poisson submanifold* is also fully established and natural, namely a submanifold Q of a Poisson manifold P, such that the Poisson bivector field of P restricts to a Poisson bivector field on P'; equivalently, a Poisson manifold Q, $Q \subset P$ is Poisson if the inclusion is a Poisson map. However, in relation with duality issues, the role played by subobjects is often inherent to *coisotropic submanifolds*, in a sense we are to specify.

Definition 1.2.13. A submanifold C of a Poisson manifold (P, π) is called a coisotropic submanifold if $\pi^\sharp N^* C \subset TC$.

A coisotropic submanifold C of a Poisson manifold (P, π) comes equipped with a characteristic distribution Δ^C spanned by the Hamiltonian vector fields of functions in the vanishing (associative) ideal $\mathcal{I}_C = \{f \in \mathcal{C}^\infty(P) \mid f|_C = 0\}$, i.e.

$$\Delta^C_c = \mathsf{span}\{\pi^\sharp_c \mathrm{d}f \mid f \in \mathcal{I}_C\} \qquad .$$

The same symbol for Δ^C and the corresponding subset of TC shall occasionally be used. It is well known that Δ^C is always integrable and we shall denote with $\underline{C}$ the leaf space of the associated singular foliation; if $\underline{C}$ is a smooth manifold, one can show that $\mathcal{C}^\infty(\underline{C}) \simeq N\mathcal{I}_C / \mathcal{I}_C$, where

$$N\mathcal{I}_C = \{\, f \in \mathcal{C}^\infty(P) \;|\; X(f) = 0 \,,\; X \in \Delta^C \,\}$$

is the normalizer of $\mathcal{I}_C$, i.e. the smallest Poisson subalgebra of $\mathcal{C}^\infty(P)$ making $\mathcal{I}_C$ an ideal for the Poisson bracket.

Proposition 1.2.14. *Let (P, π) be a Poisson manifold and $C \subset P$ be a submanifold. Then, the following are equivalent:*

i) The conormal bundle N^*C is a Lie subalgebroid of the Koszul algebroid T^*P;

ii) The characteristic ideal $\mathcal{I}_C$ is a Poisson subalgebra of $\mathcal{C}^\infty(P)$;

iii) The characteristic distribution Δ^C is tangent to C;

iv) C is coisotropic.

PROOF. $(ii) \Rightarrow (iii) \Rightarrow (iv)$: obvious. $(i) \Rightarrow (ii)$: $f_\pm \in \mathcal{I}_C$ iff $\mathrm{d}f_\pm|_C \in \Gamma(N^*C)$, then $\{ f_+ , f_- \} = \langle \mathrm{d}f_- , \pi^\sharp \mathrm{d}f_+ \rangle$ vanishes on C, since $\pi^\sharp$ is the anchor of T^*P. $(iv) \Rightarrow (i)$: for all $\theta \in \Omega^1(P)$ and $X \in \mathfrak{X}(P)$, such that $\theta|_C \in \Gamma(N^*C)$ $X|_C \in \mathfrak{X}(C)$, $\theta(X) \in \mathcal{I}_C$ and, by Cartan's magic formula $\iota_X \mathrm{d}\theta \in \Gamma(N^*C)$; use formula (1.1), to check condition 2. of lemma 1.2.6 (see also remark (1.2.7)). □

Note that the direct product $P_1 \times P_2$ of Poisson manifolds $(P_{1,2}, \pi_{1,2})$ is canonically endowed with a product Poisson tensor $\pi_1 \times \pi_2$; the Poisson bracket on $\mathcal{C}^\infty(P_1 \times P_2)$ can be expressed as

$$\{ F , G \}_{P_1 \times P_2}(p_1, p_2) = \{ F^1_{p_2} , G^1_{p_2} \}_{P_1}(p_1) + \{ F^2_{p_1} , G^2_{p_1} \}_{P_1}(p_2) \qquad ,$$

for all $F, H \in \mathcal{C}^\infty(P_1 \times P_2)$ and $p_{1,2} \in P_{1,2}$. Here $H^{1,2}_x \in \mathcal{C}^\infty(P_{1,2})$ denotes the restriction to the $P_{1,2}$-direction of any $H \in \mathcal{C}^\infty(P_1 \times P_2)$, $x \in P_{1,2}$.
For any Poisson manifold (P, π), denote with $\overline{P}$ the Poisson manifold $(P, -\pi)$.

Corollary 1.2.15. *Let $P_\pm$ be Poisson manifolds and $\lambda : P_- \to P_+$ a smooth map. Then λ is Poisson iff $\Gamma(\lambda) \subset P_- \times \overline{P}_+$ is coisotropic.*

Example 1.2.16. For any Poisson manifold P, the diagonal $\Delta_P \subset P \times \overline{P}$ is the graph of the identity, therefore it is coisotropic, according to last corollary; on the other hand it is straightforward to check coisotropicity directly.

In fact, coisotropic submanifolds of direct products are to be thought of as generalized morphisms of Poisson manifolds.

Definition 1.2.17. [**72**] Let $P_\pm$ be Poisson manifolds. A coisotropic relation $R : P_- \to P_+$ is a coisotropic submanifold $R \subset P_- \times \overline{P}_+$. For any Poisson manifolds $P_{1,2,3}$, the composition of coisotropic relations $R_{12} : P_1 \to P_2$ and $R_{23} : P_2 \to P_3$ is

$$R_{12} \circ R_{23} = \mathrm{pr}_1 \times \mathrm{pr}_4(R_{12} \times R_{23} \cap P_1 \times \Delta_{P_2} \times P_3) \subset P_1 \times P_3 \qquad .$$

Coisotropic relations $R_{12} : P_1 \to P_2$ and $R_{23} : P_2 \to P_3$ are in very clean position if

1. $R_{123} := (R_{12} \times R_{23}$ and $\Delta_{123} := P_1 \times \Delta_{P_2} \times P_3$ intersect cleanly;
2. The restriction of the projection $\Delta_{123} \to P_1 \times P_3$ has constant rank;
3. $R_{12} \circ R_{23} \subset P_1 \times P_3$ is a smooth submanifold;
4. The projection $R_{123} \to R_{12} \circ R_{23}$ is a submersion.

Note that the composition of coisotropic relations is not, in general a smooth submanifold. The fundamental theorem of coisotropic calculus, which we give without proof, is due to Weinstein.

Theorem 1.2.18. [**72**] *For any coisotropic relation $R_{12} : P_1 \to P_2$ and $R_{23} : P_2 \to P_3$ in very clean position the composition $R_{23} : P_2 \to P_3$ is a coisotropic relation $P_1 \to P_3$.*

Corollary 1.2.19. *If $j : P_- \to P_+$ is a Poisson or anti-Poisson submersion and $C \subset P_+$ a coisotropic submanifold, then so is $j^{-1}(C) \subset P_-$.*

Proof. $j^{-1}(C) \times \bullet = \Gamma(j) \circ (C \times \bullet) \subset P_+ \times \bullet$, where $\Gamma(j)$ is a coisotropic relation either $P_- \to P_+$ or $P_- \to \overline{P}_+$ (note that C is coisotropic also in $\overline{P}_+$) and submersivity of j implies the cleanliness conditions. $\square$

Lie subalgebroids and morphisms of Lie algebroids can be characterized in the dual picture in terms of coisotropic submanifolds.

Proposition 1.2.20. [**75**] *Let $A \to M$ be a Lie algebroid and $B \to N$ a smooth vector subbundle. Then B is a Lie subalgebroid iff the annihilator B^o of B is a coisotropic submanifold of A^*.*

Proof. Suppose B is a Lie subalgebroid; then, under the identification of $\Gamma(A)$ with the space of fibrewise linear functions $\mathcal{C}^\infty_{\mathsf{lin}}(A^*)$, condition (2.) of lemma (1.2.6) is equivalent to

2^*. For all $F, G \in \mathcal{C}^\infty_{\mathsf{lin}}(A^*) \cap \mathcal{I}_{B^o} =: \mathcal{I}^{\mathsf{lin}}_{B^o}$, $\{F, G\} \in \mathcal{I}^{\mathsf{lin}}_{B^o}$,

for the induced Poisson bracket on A^* and the vanishing ideal $\mathcal{I}_{B^o}$. Since $\mathcal{I}_{B^o}$ is generated by $\mathcal{I}^{\mathsf{lin}}_{B^o}$ as a $\mathcal{C}^\infty(A^*)$-module, 2^* implies

$$\{\mathcal{I}_{B^o}, \mathcal{I}_{B^o}\} \subset \mathcal{I}_{B^o}$$

by the Leibniz rule, i.e. B^o is coisotropic. Conversely, to each $f \in \mathcal{C}^\infty(M)$, associate the fibrewise constant function $\mathrm{pr}^*_{A^*} f$, on A^*; $\mathrm{pr}^*_{A^*} f \in \mathcal{I}_{B^o}$ iff $f \in \mathcal{I}_N$. For any $a \in \Gamma(A)$, such that $a|_N \in \Gamma(B)$,

$$\{F_a, \mathrm{pr}^*_{A^*} f\} := \mathrm{pr}^*_{A^*}(\rho(a)(f)) = \langle \mathrm{d}\, f, \rho(a) \rangle \circ \mathrm{pr}_{A^*}$$

if $B^o \subset A^*$ is coisotropic, $\rho(a)(f) \in \mathcal{I}_N$, for all $f \in \mathcal{I}_N$, equivalently, $\rho(a)$ is a section of $TN = (N^*N)^o$, since the conormal bundle N^*N is spanned by $\mathrm{d}\,\mathcal{I}_N$. That is, the anchor of A restricts to B. Conditions (2.) and (1.) follow straightforwardly from the fact that $\mathcal{I}_{B^o}$ is closed of the Poisson bracket and $\mathcal{I}^{\mathsf{lin}}_{B^o} \subset \mathcal{I}_{B^o}$ a Poisson subalgebra, since the dual bracket on A^* is fibrewise linear. $\square$

Corollary 1.2.21. *Let $A^{\pm} \to M^{\pm}$ be Lie algebroids and $\phi : A^- \to A^+$ a morphism of vector bundles over $f : M^- \to M^+$. Then the following are equivalent:*

i) ϕ is a morphism of Lie algebroids;

ii) $\Gamma(\phi) \subset A^- \times A^+$ is a Lie subalgebroid;

iii) $\Gamma(\phi)^o \subset A^{-*} \times A^{+*}$ is coisotropic for the induced linear Poisson structure.

Moreover, if ϕ is base bijective, so that the transpose map ϕ^{t} is well defined the three statements above are equivalent to

iv) $\phi^{\mathsf{t}} : A^{+*} \to A^{-*}$ is Poisson.

Proof. For the equivalence of the first three statements, it suffices to show the equivalence of (i) and (ii); this can be checked by picking decompositions. When the transpose map is well defined $\Gamma(\phi)^o$ coincides with the vector bundle $(-\mathsf{id}_{A^{-*}} \times \mathsf{id}_{A^{+*}})\, \Gamma(\phi^+) \to \Gamma(f)$, up to the exchange of A^- with A^+; then $\Gamma(\phi)^o$ is coisotropic in $A^{-*} \times A^{+*}$ iff $\Gamma(\phi^+)$ is coisotropic in $A^{+*} \times \overline{A^{-*}}$, equivalently, iff ϕ^+ is Poisson. $\square$

We conclude this Subsection with a few examples of coisotropic submanifolds.

Example 1.2.22. Coisotropic submanifolds

i) Consider any Poisson bivector on $\mathbb{R}^3$; it is a straightforward exercise to show that all submanifolds of the form $\{z = c(x, y)\}$, for some smooth function $c \in \mathcal{C}^\infty(\mathbb{R}^2)$ are coisotropic.

ii) Recall that a submanifold C of a symplectic manifold (M, ω) is coisotropic if $T^\omega C \subset TC$, for the symplectic orthogonal bundle

$$T^\omega_c C = \{\delta c \in T_c M \mid \omega_c(\delta q, \delta c) = 0\,,\ \delta c \in T_c C\} \quad , \quad c \in C \qquad .$$

Since $T^\omega C = (\omega^\sharp TC)^o$, the condition is equivalent to $N^*C = T^o C \subset \omega^\sharp TC$, i.e. a submanifold of a symplectic manifold is coisotropic iff it is coisotropic for the inverse Poisson structure $\pi^\sharp = \omega^{\sharp -1}$.

iii) The symplectic leaves of a Poisson manifold are, essentially by definition, coisotropic submanifolds.

iv) Let M be a symplectic manifold and C a smooth submanifold defined by constraints $f_i = 0$, $i = 1, \ldots, n$, $f_i \in \mathcal{C}^\infty(M)$. The constraints are first class in the sense of Dirac if

$$\{\, f_i \,,\, f_j \,\} = c_{ij}{}^k f_k$$

for some smooth functions $\{c_{ij}{}^k\}$ on M. Since N^*C is spanned by the differentials $\{\mathrm{d} f_i\}$, the constraints are first class iff C is coisotropic.

1.3. Actions of Lie groupoids and action Lie groupoids

We describe in this Section how Lie group actions and infinitesimal actions can be generalized replacing Lie groups–algebras with Lie groupoids–algebroids; we also present the natural conditions for the existence in the smooth category of quotients with respect to such actions. We conclude by introducing principal bundles with structure groupoid, which prepresent a handy tool in the study of the integrability and reduction of Lie groupoids and Lie algebroids.

The main difference between the actions of Lie groups and those of Lie groupoids consists in the fact that a Lie groupoid $\mathcal{G} \rightrightarrows M$ acts on maps to M rather than on manifolds.

Definition 1.3.1. Let $\mathcal{G} \rightrightarrows M$ be a groupoid and consider a map $j : N \to M$. A left groupoid action of $\mathcal{G}$ on j is given by a map $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \to N$, $(g, n) \mapsto g * n$, such that, for all $n \in N$,

1. $j(g * n) = \mathrm{t}(g)$, whenever $g \in \mathrm{s}^{-1}(j(n))$;
2. $\varepsilon(j(n)) * n = n$;
3. $(gh) * n = g * (h * n)$, whenever $(g, h) \in \mathcal{G}^{(2)}$ and $h \in \mathrm{s}^{-1}(j(n))$.

The map j is called a moment map.

The action of a group on a set M is a groupoid action with trivial moment map $M \to \bullet$. Just like the case of group actions, to any left groupoid action one can associate an action groupoid $\mathcal{G} \ltimes N \rightrightarrows N$ with total space $\mathcal{G}\,{}_{\mathrm{s}}\!\times_j N$, the source map $\mathrm{s}_\ltimes$ is the restriction of the second projection, the target $\mathrm{t}_\ltimes$ is the action map σ itself, and the multiplication $\mu_\ltimes$ maps any composable pair $[(g, h * n); (h, n)]$, $(g, h) \in \mathcal{G}^{(2)}$ and $h \in \mathrm{s}^{-1}(j(n))$, to

$$(g, h * n) \cdot (h, n) = (gh, n) \qquad ,$$

thus unit bisection $\varepsilon_\ltimes$ and inversion $\iota_\ltimes$ must be $\varepsilon_\ltimes(n) = (\varepsilon(j(n)), n)$ and $(g, n)^{-\ltimes} = (g^{-1}, g * n)$.
In what follows we shall consider mostly actions of Lie groupoids on smooth maps; note that, in this case, the domain of the action map is always a smooth manifold and such an action is called a Lie groupoid action if the action map is also smooth. Note that the source fibres $\mathrm{s}_\ltimes^{-1}(n) = \mathrm{s}^{-1}(j(n)) \times \{n\}$, $n \in N$, of an action groupoid $\mathcal{G} \ltimes N$ have the same homotopy type as suitable source fibres of $\mathcal{G}$. The proof of the following lemma is straightforward.

Lemma 1.3.2. *The action groupoid of a Lie groupoid action is a Lie groupoid.*

Next, we shall explain the denomination "moment map"[3]. Consider an Hamiltonian action of a Lie group G (which we assume to be connected here) on a Poisson manifold (M, π), i.e. a Lie group action for which there exists a moment map $j : M \to \mathfrak{g}^*$, characterizing the induced infinitesimal action as $\sigma(x) = X^{j^* F_x}$, $x \in \mathfrak{g}$. Such a map is equivariant for the coadjoint representation on the nose iff it is Poisson or, equivalently, the diagram

$$\begin{tikzcd} & \mathcal{C}^\infty(M) \arrow[dr, "X^\bullet"] & \\ \mathfrak{g} \arrow[ur, "j^*"] \arrow[rr, "\sigma"'] & & \mathfrak{X}(M) \end{tikzcd}$$

commutes in the category of Lie algebras, where $X^\bullet$ is the map that assigns to each function on M the corresponding Hamiltonian vector field. When M is symplectic this is equivalent to the classical notion of an equivariant momentum map [**49**] for an Hamiltonian G-space (see, for example, [**9**] for more details).
Lie groupoid actions and moment maps are indeed a natural generalization of Hamiltonian actions and the momentum maps of classical mechanics. Consider the tangent prolongation groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ defined by the structure maps

$$\hat{\mathrm{s}}(\theta_g) = l_g{}^*\theta_g \qquad \hat{\mathrm{t}}(\theta_h) = r_h{}^*\theta_h \qquad \hat{\mu}(\theta_g, \theta_h) = r_{h^{-1}}{}^*\theta_g = l_{g^{-1}}\theta_h$$

(unit section and inversion are defined accordingly)[4]. An Hamiltonian action of G on M (symplectic or Poisson) can be lifted to a groupoid action of the $T^*G \rightrightarrows \mathfrak{g}^*$ along j by a trivial fibrewise extension: set $\theta_g \,\hat{*}\, m := g * m$, for all $m \in M$ and $\theta_g \in T^*G$, with $r_{g^{-1}}{}^*\theta_g = j(m)$. By equivariance,

$$\langle\, j(\theta_g \,\hat{*}\, m)\,,\, x \,\rangle = \langle\, j(m)\,,\, \mathsf{Ad}_{g^{-1}} x \,\rangle = \langle\, \hat{\mathrm{s}}(\theta_g)\,,\, \mathrm{d}r_g \circ \mathrm{d}l_{g^{-1}} x \,\rangle = \langle\, \hat{\mathrm{t}}(\theta_g)\,,\, x \,\rangle \;,\; x \in \mathfrak{g} \;,$$

the remaining property of a groupoid action are manifest.

Example 1.3.3. Groupoid actions and action groupoids.

i) For any Lie groupoid $\mathcal{G} \rightrightarrows M$ there is a left action on the target map $\mathrm{t} : \mathcal{G} \to M$ and a right action on the source map $\mathrm{s} : \mathcal{G} \to M$, the action map being the groupoid multiplication in both cases. In the latter case the corresponding action groupoid is easily identified with the groupoid $\mathcal{G} \odot \mathcal{G}$ of example 1.1.11, in the former, with the same groupoid associated with $\mathcal{G}^{\mathsf{op}}$ (i.e. the Lie groupoid corresponding to the opposite category).

ii) The action of a Lie group G on a manifold M always tangent lifts to an action of the tangent group TG on TM and $TG \ltimes TM$ is canonically isomorphic to $T(G \ltimes M)$.

[3]The "controversy" on the denomination *moment* vs *momentum* is well known. From now on we shall use mostly the term moment map.

[4]For any Lie groupoid $\mathcal{G}$ with Lie algebroid A, it is possible to define a cotangent prolongation groupoid $T^*\mathcal{G} \rightrightarrows A^*$; we shall discuss the construction in section 1.4.2.

iii) [**45**] For any closed Lie subgroup $H \subset G$ the action groupoid $G \ltimes G/H$ is canonically isomorphic to the gauge groupoid $(G \times G)/H$.

Consider a general Lie groupoid action such as above and let A be the Lie algebroid of $\mathcal{G}$. The action map induces a morphism of Lie algebras $\Gamma(A) \to \mathfrak{X}(N)$, $a \mapsto X^a$

$$X^a_n := \mathrm{d}\sigma(a_{j(n)}, 0_n) \qquad , \qquad n \in N \qquad .$$

Note that the vector bundle underlying the Lie algebroid of $\mathcal{G} \ltimes N$ is the pullback $j^+A \to N$ and the space of sections $\Gamma(j^+A) \simeq \mathcal{C}^\infty(N) \otimes_{\mathcal{C}^\infty(M)} \Gamma(A)$ can be endowed with a Lie bracket by picking decompositions $\Gamma(j^+A) \ni \alpha_{1,2} = \sum_{k_{1,2}} u_{1,2}^{k_{1,2}}(a_{1,2}^{k_{1,2}} \circ \phi)$ and setting

$$\begin{aligned} [\,\alpha_1\,,\,\alpha_2\,] \;&=\; \sum_{k_1,k_2} u_1^{k_1} \cdot u_2^{k_2}([\,a_1^{k_1}\,,\,a_2^{k_2}\,] \circ j) \;+\; \sum_{k_2} \mathrm{d}\sigma \circ (\alpha_1 \times 0)(u_2^{k_2})(a_2^{k_2} \circ j) \\ &\;-\; \sum_{k_1} \mathrm{d}\sigma \circ (\alpha_2 \times 0)(u_1^{k_1})(a_1^{k_1} \circ j) \qquad . \end{aligned} \tag{1.8}$$

One can check that the above expression is well defined and makes j^+A a Lie algebroid for the obvious anchor. Moreover such a Lie algebroid is canonically isomorphic to the Lie algebroid of $\mathcal{G} \ltimes N$.
We shall remark that it is possible to make sense of actions of Lie algebroids without making reference to actions of Lie groupoids.

Definition 1.3.4. Let $A \to M$ be a Lie algebroid and consider a map $j : N \to M$. A Lie algebroid action of A on j is given by a map $\sigma : \Gamma(A) \to \mathfrak{X}(N)$, such that

1. σ is a morphism $\mathcal{C}^\infty(M)$-modules, for the natural module on $\mathfrak{X}(N)$ induced by j via pullback;

2. σ is a morphism of Lie algebras;

Note that, thanks to condition 1., σ induces a well defined bundle map $j^+\sigma$: $\Gamma(j^+A) \to \mathfrak{X}(N)$, which is required to be compatible with j in the sense that

3. $\mathrm{d}j \circ j^+\sigma = \rho \circ j^!$

Using the data in the above definition in the same way as those produced by a Lie groupoid action one can obtain a Lie algebroid $A \ltimes N$, we shall refer to as the action Lie algebroid, with anchor $\rho^\ltimes := j^+\sigma$ on the pullback j^+A. Essentially by construction the natural map $j^! : j^+A \to A$ is a morphism of Lie algebroid.

Example 1.3.5. Action Lie algebroids

i) An infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebroid action on $M \to \bullet$ and the corresponding action Lie algebroid is the flat bundle $\mathfrak{g} \times M$ with the usual action Lie algebroid structure.

ii) The action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on $\mathrm{t} : \mathcal{G} \to M$ by *left* translation induces an action Lie algebroid on $\mathrm{t}^{+}A$; the Lie algebroid anchor $\mathrm{t}^{+}A \to T\mathcal{G}$ is easily computed as $(a_{\mathrm{t}(g)}, g) \mapsto \mathrm{d}r_g a_{\mathrm{t}(g)}$ and it is easy to identify such Lie algebroid with the action Lie algebroid $A \ltimes \mathcal{G}$ for the infinitesimal action $\Gamma(A) \to \overrightarrow{\mathfrak{X}}(\mathcal{G})$ by *right* invariant vector fields.

By definition, an equivalence relation $\sim$ on a smooth manifold M is regular if the quotient space $M/\!\sim$ carries a smooth topology (the *quotient topology*) making the quotient projection $M \to M/\!\sim$ a submersion; the graph of an equivalence relation is $\Gamma(\sim) := \{(x, y)\,|\,x \sim y\} \subset M \times M$. We shall repeatedly make use of the classical

Theorem 1.3.6 (Godement's criterion)**.** *An equivalence relation $\sim$ on a manifold M is regular iff*

1. $\Gamma(\sim)$ *is a submanifold of $M \times M$;*

2. *The restriction of the first or, equivalently, of the second projection $\Gamma(\sim) \to M$ is submersive.*

As a consequence the natural conditions for the orbit space M/G of a Lie group action to be smooth consists in asking the groupoid anchor $G \ltimes M \to M \times M$ to be an embedding, namely the action by right or left translation of $G \ltimes M$ on M to be free and proper. This motivates the following

Definition 1.3.7. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called (isotropy) free, respectively proper, if one of the following equivalent conditions is fulfilled:

i) The action of $\mathcal{G}$ on M by left translation is free, respectively proper, i.e.

$$\mathrm{t} \times \mathsf{id}_M : \mathcal{G}\,{}_{\mathrm{s}}\!\times M \to M \times M$$

is an injective immersion, respectively a proper map;

ii) The action of $\mathcal{G}$ on M by right translation is free, respectively proper, i.e.

$$\mathsf{id}_M \times \mathrm{s} : M \times_{\mathrm{t}} \mathcal{G} \to M \times M$$

is an injective immersion, respectively a proper map;

iii) The groupoid anchor $\chi : \mathcal{G} \to M \times M$ is an injective immersion, respectively a proper map.

Accordingly, a Lie groupoid action of shall be called free, respectively proper, if so is the associated action Lie groupoid. Note that the orbit space $N/\mathcal{G}$ and $N/(\mathcal{G} \ltimes N)$ coincide.

Lemma 1.3.8. *For any free and proper Lie groupoid action of $\mathcal{G} \rightrightarrows M$ on $j : N \to M$ the orbit space $N/\mathcal{G}$ carries a unique smooth topology making the quotient projection $N \to N/\mathcal{G}$ a submersion.*

PROOF. Condition 1. for of theorem 1.3.6 holds by hypothesis; to check condition 2. pick a bisection of $\mathcal{G} \ltimes N$. ☐

Since action Lie groupoids differentiate to action Lie algebroids, the connected components of a $\mathcal{G}$-orbit $\mathcal{O}^{\mathcal{G}}$ on N are the leaves of the associated action Lie algebroid, thus

$$T_n\mathcal{O}^{\mathcal{G}} = \mathrm{d}\sigma(A_{j(n)} \times 0_n) \qquad , \qquad n \in \mathcal{O}^{\mathcal{G}} \qquad ,$$

and it is straightforward to see that the quotient $N/\mathcal{G}$, when smooth, has dimension

$$\mathsf{dim}\, N/\mathcal{G} = \mathsf{dim}\, N + \mathsf{dim}\, M - \mathsf{dim}\, \mathcal{G} \qquad . \tag{1.9}$$

Example 1.3.9. For any isotropy free and proper Lie groupoid $\mathcal{G} \rightrightarrows M$, the anchor χ is a closed embedding, and so is the tangent anchor $\mathrm{d}\chi : T\mathcal{G} \to TM \times TM$. That is, the tangent prolongation groupoid is also isotropy free and proper.

Apart from Lie group actions, many other differential geometric constructions extend to Lie groupoids; in the following we shall need principal bundles with groupoid structure.

Definition 1.3.10. A left principal $\mathcal{G}$-bundle is given by a fibre bundle $\tau\colon P \to N$, a groupoid $\mathcal{G} \rightrightarrows M$ and a groupoid action on a moment map $j\colon P \to M$, such that

1. $\mathcal{G}$ acts along the fibres of τ;
2. The action is fibrewise free and transitive.

A right principal $\mathcal{G}$-bundle is defined similarly, replacing the left action with a right action.

We shall also say that a principal $\mathcal{G}$-bundle is a principal bundle with structure groupoid $\mathcal{G}$. Naturally, one can adapt this set theoretic definition to the category of topological spaces and smooth manifolds, in the case of a Lie groupoid action. In the latter case, last two conditions can be expressed by requiring that the anchor of the associated action groupoid be an isomorphism of Lie groupoids $\mathcal{G} \ltimes P \to P\,{}_\tau\!\times_\tau P$, for the subgroupoid of the pair groupoid $P\,{}_\tau\!\times_\tau P \rightrightarrows P$ defined by the equivalence relation induced by τ.

Example 1.3.11. Principal groupoid bundles

$i)$ Ordinary principal bundles are principal bundles in the sense of the definition above.

ii) For any Lie groupoid $\mathcal{G} \rightrightarrows M$, $\mathrm{s} : \mathcal{G} \to M$ is a right principal $\mathcal{G}$-bundle for the action by right translation.

There are natural conditions for a groupoid action on a Lie groupoid to be compatible with the partial multiplication.

Definition 1.3.12. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids. If $\mathcal{G}$ acts on moment maps $J : \mathcal{H} \to M$ and $j : N \to M$ in such a way that the conditions

1. $\mathrm{s}_{\mathcal{H}}(g * h) = g * \mathrm{s}_{\mathcal{H}}(h)$
2. $\mathrm{t}_{\mathcal{H}}(g * h) = g * \mathrm{t}_{\mathcal{H}}(h)$
3. $\iota_{\mathcal{H}}(g * h) = g * \iota_{\mathcal{H}}(h)$
4. $\mu_{\mathcal{H}}(g * h_+, g * h_-) = g * \mu_{\mathcal{H}}(h_+, h_-)$

hold whenever they make sense, we shall say that the action is compatible with the groupoid structure.

Quotients with respect to compatible groupoid actions are well behaved.

Lemma 1.3.13. [**51**] *Let a Lie groupoid $\mathcal{G} \rightrightarrows M$ act on a Lie groupoid $\mathcal{H} \rightrightarrows N$ compatibly with the groupoid structure; if N is a principal $\mathcal{G}$-bundle, the quotient $\mathcal{H}/\mathcal{G}$ carries a Lie groupoid structure over the quotient $N/\mathcal{H}$.*

In other words, whenever the action of $\mathcal{G}$ on the base of $\mathcal{H}$ is free and proper, so that the quotient $N/\mathcal{G}$ exists, the quotient $\mathcal{H}/\mathcal{G}$ is also smooth and it is easy to see that the the groupoid structure descends, thanks to the compatibility conditions of definition 1.3.12.
An example of last lemma plays a central role in the next Section.

Example 1.3.14. Monodromy groupoids

i) For any regular foliation $\mathcal{F}$ (i.e. corresponding to an integrable distribution of constant rank) on a manifold M, the monodromy groupoid $\mathsf{Mon}(M, \mathcal{F}) \rightrightarrows M$ is given by the classes of foliated paths (i.e. paths within the leaves of $\mathcal{F}$) with respect to foliated homotopy relative to the endpoints, for the multiplication induced by concatenation of paths with matching endpoints. The source fibre over one point is easily identified with the universal cover of that leaf, thus $\mathsf{Mon}(M, \mathcal{F}) \rightrightarrows M$ is always Lie groupoid with 1-connected source fibres.

ii) Consider the monodromy groupoid $\mathsf{Mon}(\mathcal{G}, \mathrm{s})$ for the foliation induced by the source map of a Lie groupoid with connected source fibres. On the one hand $\mathsf{Mon}(\mathcal{G}, \mathrm{s})$ is naturally acted on from the right by $\mathcal{G}$. On the other hand $\mathcal{G}$ acts on itself also by right translation and makes $\mathrm{t} : \mathcal{G} \to M \equiv \mathcal{G}/\mathcal{G}$ a principal $\mathcal{G}$-bundle. The quotient groupoid $\mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G} \rightrightarrows M$ is then a *Lie* groupoid with 1-connected source fibres.

1.4. Integrability of Lie algebroids and Poisson manifolds

We devote this Section to a review of the fundamental results in Lie theory for Lie algebroids and Lie groupoids and their dual counterparts in Poisson geometry; differently from the case of classical Lie theory, Lie algebroids (Poisson manifolds) are not always integrable to Lie groupoids (symplectic groupoids). In the first part we present the "optimal" generalizations of Lie's theorems [**34**] about the integrability of subobjects and morphisms; moreover we describe the construction of the Weinstein groupoid, a topological groupoid "integrating nonintegrable Lie algebroids". In the second part we recall the main properties of symplectic groupoids and their relation with Poisson manifolds; here we also introduce cotangent prolongation groupoids, the symplectic groupoids integrating Poisson structures which are dual to integrable Lie algebroids.

1.4.1. Integrability of Lie algebroids.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with connected source fibres[5] and Lie algebroid A; the quotient $\widetilde{\mathcal{G}} := \mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G}$ of the monodromy groupoid is a cover of $\mathcal{G}$ in a sense we are about to make precise.
Since s is a submersion, $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}}\mathcal{G} \subset \mathcal{G} \times \mathcal{G}$ is a smooth submanifold and a wide Lie subgroupoid for the pair groupoid on $\mathcal{G} \times \mathcal{G}$ (see example 1.1.11 (*iii*)). Being $\mathcal{G}$ source connected, the groupoid anchor $\chi^{\mathsf{Mon}} : \mathsf{Mon}(\mathcal{G}, \mathrm{s}) \to \mathcal{G} \times \mathcal{G}$, selecting the endpoints, is a source surjective and submersive morphism of Lie groupoids onto $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}}\mathcal{G}$; thus, there is a short exact sequence

$$\mathcal{L}(\mathcal{G}, \mathrm{s}) \hookrightarrow \mathsf{Mon}(\mathcal{G}, \mathrm{s}) \twoheadrightarrow \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}}\mathcal{G}$$

of Lie groupoids over $\mathcal{G}$, where $\mathcal{L}(\mathcal{G}, \mathrm{s}) = \ker \chi^{\mathsf{Mon}}$ is the wide normal Lie subgroupoid of homotopy classes of foliated loops in $(\mathcal{G}, \mathrm{s})$.
Note that $\mathcal{L}(\mathcal{G}, \mathrm{s})$ is stable under the natural action of $\mathcal{G}$ on $\mathsf{Mon}(\mathcal{G}, \mathrm{s})$ by right translation and $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}}\mathcal{G}$ is acted on by $\mathcal{G}$ from the right diagonally. The sequence above is equivariant for these compatible Lie groupoid actions and yields a new short exact sequence

$$\mathcal{L}(\mathcal{G}, \mathrm{s})/\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}} \xrightarrow{\kappa_{\mathcal{G}}} \mathcal{G}$$

of Lie groupoids over M by taking quotients. Computing the total space of the Lie algebroids of $\widetilde{\mathcal{G}}$ yields the same vector bundle as A and the morphism of the Lie algebroids induced by $\kappa_{\mathcal{G}}$ is the identity map; moreover, $\widetilde{\mathcal{G}}$ is indeed the "covering

[5] As it is customary, we shall call, for short, source n-connected a groupoid with n-connected source fibres. Similarly a (morphism of) groupoids are said "source-x" if the property of being "x" holds sourcewise.

groupoid" of $\mathcal{G}$, in the sense that all restrictions $\widetilde{\mathrm{s}}^{-1}(q) \to \mathrm{s}^{-1}(q)$, $q \in M$, of $\kappa_{\mathcal{G}}$ to the source fibres are covering maps. Since the source connected component of any Lie groupoid $\mathcal{G}$ is a Lie groupoid with the same Lie algebroid, we have

Theorem 1.4.1. (0th Lie theorem)[**51**] *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. Then, there exists a source 1-connected Lie groupoid $\widetilde{\mathcal{G}}$ over M, with Lie algebroid $\widetilde{A}$ and a morphism of Lie groupoids $\kappa_{\mathcal{G}}$, inducing an isomorphism $\widetilde{A} \to A$.*

As it shall be clear after theorem 1.4.3 and remark 1.4.4 below, the covering groupoid is unique up to isomorphism.

A Lie algebroid A is called integrable if it is in the image of the Lie functor; in this case, we shall say that the covering groupoid of any groupoid differentiating to A, is *the* (source 1-connected) Lie groupoid of A.

Let us consider now the integrability of Lie subalgebroids. The action Lie groupoid $\mathcal{G} \ltimes \mathcal{G}$ for the action by left translation is isomorphic to $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G}$ for the mapping $(g,h) \mapsto (gh,h)$, thus the Lie algebroid of $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G}$, with total space $T^{\mathrm{s}}\mathcal{G}$, is isomorphic to $A \ltimes \mathcal{G}$ for the bundle isomorphism $\mathcal{R}' : T^{\mathrm{s}}\mathcal{G} \to A \ltimes \mathcal{G}$ given by right translation and the right translation map $\mathcal{R} := \mathrm{t}^! \circ \mathcal{R}' : T^{\mathrm{s}}\mathcal{G} \to A$ a morphism of Lie algebroids over t, in fact it is the differential of the division map $\delta : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G} \to \mathcal{G}$. Since $\mathcal{R}$ is fibrewise a diffeomorphism any subalgebroid $B \to N$ of A can be pulled back to a vector subbundle $\mathcal{R}^{-1}(B) \to \mathrm{t}^{-1}(N)$ of $T^{\mathrm{s}}\mathcal{G}$ and the graph of $\mathrm{dt}|_{\mathcal{R}^{-1}(B)} : \mathcal{R}^{-1}(B) \to TN$ defines a *regular* distribution on $N \times_{\mathrm{t}} \mathcal{G}$. It is easy to identify the latter with the distribution spanned by the image of the infinitesimal action

$$\sigma_B : \Gamma(B) \to \mathfrak{X}(N \times_{\mathrm{t}} \mathcal{G}) \qquad , \qquad \sigma_B(b)_{(n,g)} = (\rho_n(b), \overrightarrow{b}_g) \qquad ;$$

$\mathrm{dt}|_{\mathcal{R}^{-1}(B)}$ is thus an integrable distribution, since it coincides with the anchor distribution of the action Lie algebroid $B \ltimes (N \times_{\mathrm{t}} \mathcal{G})$. Moreover, since $\mathrm{dt}|_{\mathcal{R}^{-1}(B)}$ has constant rank, the associated foliation $\mathcal{F}_B$ is regular and one can form the monodromy groupoid $\mathsf{Mon}(N \times_{\mathrm{t}} \mathcal{G}, \mathcal{F}_B)$; the latter is naturally acted on by $\mathcal{G}$ from the right, as is the projection $N \times_{\mathrm{t}} \mathcal{G} \to N$, which is a principal $\mathcal{G}$-bundle. It follows from lemma 1.3.13 that $\mathsf{Mon}(N \times_{\mathrm{t}} \mathcal{G}, \mathcal{F}_B)/\mathcal{G}$ is a source 1-connected Lie groupoid over N, whose Lie algebroid turns out to be isomorphic to B.

Theorem 1.4.2. (1st Lie's theorem)[**51**] *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid A and $B \to N$ a Lie subalgebroid of A. Then B is integrable.*

Note that the integration of B in general is not obtained as a Lie subgroupoid of $\mathcal{G}$; we shall clarify this point after discussing the integrability of morphism of Lie algebroids (see remark 1.4.4).

Consider a morphism of integrable Lie algebroids $\phi : A_- \to A_+$ over $f : M_- \to M_+$ and let $\mathcal{G}_\pm$ be Lie groupoids inducing $A_\pm$, respectively. The vector fields on $\mathcal{G}_- \times \mathcal{G}_+$ of the form

$$X_{(g_-,g_+)} = (\mathrm{d}r_{g_-} a_{m_-}, \mathrm{d}r_{g_+}\phi(a_{m_-})) \qquad , \qquad \mathrm{t}_-(g_-) = m_- \quad , \quad a \in \Gamma(A_-)$$

span an integrable distribution Δ on $(\mathrm{s}_- \times \mathrm{s}_+)^{-1}\Gamma(f) \subset \mathcal{G}_- \times \mathcal{G}_+$. Denote with $\mathcal{L}_{m_-}$ the leaf of Δ through $(\varepsilon_-(m_-), \varepsilon_+(f(m_-)))$, $m_- \in M_-$. One can check that the projection $\mathcal{L}_{m_-} \to \mathrm{s}_-^{-1}(m_-)$ is a covering map, hence a diffeomorphism, if $\mathcal{G}_-$ is source 1-connected, therefore Δ induces a family of smooth maps $\phi_{m_-} : \mathrm{s}_-^{-1}(m_-) \to \mathrm{s}_+^{-1}(f(m_-))$. One can further check that the maps of this family paste together nicely and in fact define a morphism of Lie groupoids $\varphi : \mathcal{G}_- \to \mathcal{G}_+$ over $f : M_- \to M_+$, which, by construction, induces $\phi : A_- \to A_+$ (see [**48**] for details).

Theorem 1.4.3. (2nd Lie's theorem)[**48**] *Let $\mathcal{G}_\pm \rightrightarrows M_\pm$ be Lie groupoids with Lie algebroids $A_\pm$ and $\phi\colon A_- \to A_+$ a morphism of Lie algebroids. If $\mathcal{G}_-$ is source 1-connected, there exists a unique morphism of Lie groupoids $\mathcal{G}_- \to \mathcal{G}_+$ inducing ϕ.*[6]

Uniqueness follows from the fact that the vector fields spanning the distribution are right invariant in $\mathcal{G}_- \times \mathcal{G}_+$, thus Δ, hence φ, is determined by its span at $M_- \times M_+$, which in turn is prescribed by ϕ.

Remark 1.4.4. For any Lie subalgebroid A_- of an integrable Lie algebroid A_+, the inclusion $A_- \hookrightarrow A_+$ can be integrated to a morphism of Lie groupoids $\mathcal{G}_- \to \mathcal{G}_+$, provided $\mathcal{G}_-$ is source 1-connected, and one can check by a right translation argument that this map is immersive; however it might fail to be injective. This is not the case when $A_- \simeq A_+$, since the integration $\mathcal{G}_- \to \mathcal{G}_+$ is then a bijective immersion; thus, the covering groupoid of theorem 1.4.1 is unique up to isomorphism.

To any Lie algebroid A one can associate a topological groupoid, the Weinstein groupoid $\mathcal{W}(A)$, which fails to be smooth in general. If A has an integrating groupoid $\mathcal{G}$, thanks to theorem 1.4.1 $\mathcal{G}$ can be assumed to be source 1-connected and, in this case, $\mathcal{G} \simeq \mathcal{W}(A)$. We recall below the construction [**16**] of the Weinstein groupoid.
The source 1-connected cover $\widetilde{\mathcal{G}} = \mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G}$ can be equivalently described as the quotient of the space $P\mathcal{G}$, of the so called (twice differentiable) $\mathcal{G}$-paths, namely source foliated class $\mathcal{C}^2$ paths in $\mathcal{G}$ starting at the unit section, by $\mathcal{G}$-$\mathsf{homotopy}$, i.e. $\mathcal{C}^2$ foliated homotopy relative to the endpoints restricted to $P\mathcal{G}$. This is clear, since for each class in $\mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G}$ there is a unique representative in $P\mathcal{G}$. There exists a natural map $P\mathcal{G} \to PA$ taking values in the space of class $\mathcal{C}^1$ morphisms of Lie algebroids $TI \to A$, given by the right derivative

$$g \mapsto \delta_r g \qquad , \qquad \delta_r g(u) = \mathrm{d}r^{-1}_{g(u)}\, \dot{g}(u) \quad , \quad u \in I \qquad .$$

[6]All the maps here are of class $\mathcal{C}^\infty$. The proof can be adapted to morphisms of finite regularity.

Here we identify paths $\alpha : I \to A$ with 1-forms $\alpha\,\mathrm{d}u : TI \to A$ taking values in the pullback bundle $\gamma^+A \to I$, $\gamma := \mathrm{pr}_A \circ \alpha$; such paths are morphism of Lie algebroids iff the anchor compatibility condition $\mathrm{d}\gamma = \rho_\gamma \circ \alpha$ holds, for the tangent map $\mathrm{d}\gamma \in \Omega^1(I, \gamma^+A)$, being the bracket compatibility a trivial condition. The right derivative actually takes values in PA, since, for any $\mathcal{G}$-path g, $\delta_r g$ is a vector bundle map $TI \to A$ with base map $\gamma = \mathrm{t} \circ g$ and

$$\rho_{\gamma(u)}\delta_r g(u) = \mathrm{dt}_{g(u)}\mathrm{d}r^{-1}_{g(u)}\,\dot{g}(u) = \left.\frac{\mathrm{d}}{\mathrm{d}v}\right|_{v=u} \mathrm{t}(g(v))$$

for all $u \in I$.
Since any $\mathcal{C}^1$ morphism of Lie algebroids $\alpha : TI \to A$ integrates to a unique $\mathcal{C}^2$ morphism of Lie groupoids $a : I \times I \to \mathcal{G}$, necessarily of the form

$$a(u, v) = a(u, 0) \cdot a(0, v) = g(u) \cdot g(v)^{-1}$$

for some $\mathcal{G}$-path g, and applying the right derivative to such a $\mathcal{G}$-path, returns the original A-path α, there is a bijective correspondence between the space of A-paths and that of $\mathcal{G}$-paths.
The notion of $\mathcal{G}$-homotopy translates to the following equivalence relation on the space of A-paths, the so called A-homotopy: two A-paths $\alpha_\pm \in PA$ are A-homotopic if there exists a class $\mathcal{C}^1$ morphism of Lie algebroids $h : TI^{\times 2} \to A$, such that the boundary conditions

$$\iota^*_{\partial^\pm_V} h = \alpha_\pm \qquad \text{and} \qquad \iota^*_{\partial^\pm_H} h = 0$$

hold, where $h \in \Omega^1(I^{\times 2}, X^+A)$ is regarded as a 1-form taking values in the pullback bundle for the base map X. According to remark 1.2.4, the requirement to be a Lie algebroid morphism can be expressed by the anchor compatibility condition $\mathrm{d}X = \rho_X \circ h$ and the Maurer-Cartan equation

$$\mathrm{D}\,h + \frac{1}{2}[h \overset{\wedge}{,} h] = 0 \qquad .$$

Here $\mathrm{D} : \Omega^\bullet(I^{\times 2}, X^+A) \to \Omega^{\bullet+1}(I^{\times 2}, X^+A)$ is the covariant derivative of the pullback of an arbitrary linear connection $\nabla : \mathfrak{X}(M) \otimes \Gamma(A) \to \Gamma(A)$ for the vector bundle $A \to M$ and $[\theta \overset{\wedge}{,} \theta'] = -X^+\tau^\nabla(\theta \wedge \theta')$ is the contraction with the pullback of the torsion tensor $\tau^\nabla \in \Gamma(A^* \otimes \wedge^2 A)$.

Proposition 1.4.5. [**16**] *Let $\alpha_\pm \in PA$ be two A-paths and $g_\pm \in P\mathcal{G}$ the corresponding $\mathcal{G}$-path. Then $\alpha_\pm$ are A-homotopic iff $g_\pm$ are $\mathcal{G}$-homotopic.*

We give a considerably simpler proof of this fact, than that appeared in [**16**].

PROOF. A map $H : I \times I \to \mathcal{G}$ is a $\mathcal{G}$-homotopy from $g_- := H \circ \iota^*_{\partial^-_V}$ to $g_+ := H \circ \iota^*_{\partial^+_V}$ iff its tangent map takes values in $T^{\mathrm{s}}\mathcal{G}$ and

$$\iota^*_{\partial^\pm_H}\,\mathrm{d}H = 0 \qquad .$$

For the Lie algebroid induced by $\mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}}\mathcal{G}$, $T^{\mathrm{s}}\mathcal{G} \subset T\mathcal{G}$ is a Lie subalgebroid of $T\mathcal{G}$, thus $\mathcal{R} \circ \mathrm{d}H : TI^{\times 2} \to A$ is a morphism of Lie algebroids, such that

$$\iota_\partial^* h = \mathcal{R} \circ \mathrm{d}(H \circ \iota_\partial) = \mathcal{R} \circ \iota_\partial^* \mathrm{d}H \qquad , \tag{1.10}$$

for any boundary component ∂, hence an A-homotopy from $\delta_r g_-$ to $\delta_r g_+$. Conversely, the integration $\widetilde{h}$ of any A-homotopy $h : TI \times I \to A$ from $\delta_r g_-$ to $\delta_r g_+$ induces a map $H : I \times I \to \mathcal{G}$, $H(u,v) = \widetilde{h}(u,v;0,0)$, whose image is contained in a source fibre and a morphism of Lie algebroids $\mathrm{d}H : TI^{\times 2} \to T^{\mathrm{s}}\mathcal{G}$, such that $\mathcal{R} \circ \mathrm{d}H = h$; $\mathrm{d}H$ satisfies the desired boundary condition because of (1.10), since $\mathcal{R}$ is fibrewise a diffeomorphism. ☐

Proposition (1.4.5) allows characterizing the source 1-connected cover of a source connected Lie groupoid $\mathcal{G}$, in terms of the infinitesimal data only, as the quotient

$$\widetilde{\mathcal{G}} = PA/\!\sim$$

of A-paths by A-homotopy. Since A-homotopy classes contain smooth representatives and reparameterizing such does not change the class, the composition of $\mathcal{G}$-paths in $P\mathcal{G}/\!\sim\, = \mathsf{Mon}(\mathcal{G}, \mathrm{s})/\mathcal{G}$ (induced, roughly, by right translation and concatenation) can be translated to a groupoid multiplication on $PA/\!\sim$, also given by concatenation of good representatives (see [**16**] for details). Most importantly, even when A is not integrable, $PA/\!\sim$ is always a topological groupoid, the Weinstein groupoid $\mathcal{W}(A)$.

The integrability conditions for a Lie algebroid, which we state below for completeness, are then to be understood as a characterization of the obstruction to the existence of a smooth structure on the associated Weinstein groupoid and depend on certain monodromy groups $\mathcal{N}_\bullet(A)$: for all $q \in M$, $\mathcal{N}_q(A)$ is the abelian subgroup for the isotropy algebra $\mathfrak{g}_q$ defined by the elements ζ in the center of the isotropy algebra $\mathfrak{g}_q$, such that the constant A-path $\zeta(u) \equiv \zeta$ is A-homotopic to the zero A-path $\zeta_0(u) \equiv 0_q$.

Theorem 1.4.6. (3rd Lie's theorem) [**16**] *A Lie algebroid A is integrable to a Lie groupoid iff the monodromy groups $\mathcal{N}_\bullet(A)$ are*

1. *Discrete, i.e.* $\mathsf{d}_{\mathcal{N}}(q)) > 0$ *for all* $q \in M$*,*
2. *Locally uniformly discrete, in the sense that*

$$\liminf_{\substack{p \longrightarrow q \\ p \ni \mathcal{L}_q}} \mathsf{d}_{\mathcal{N}}(p) > 0 \qquad ,$$

where $\mathsf{d}_{\mathcal{N}}(q) = D(\mathcal{N}_q, 0)$ *for the distance function D of an arbitrary norm on A.*

See [**16, 17, 12**] and references therein for the discussion of nonintegrable examples.

1.4.2. Integrability of Poisson manifolds and coisotropic submanifolds.

Symplectic groupoids are Lie groupoids with a compatible symplectic form and provide a desingularization, namely a *symplectic realization* [**15**], of the symplectic foliation of certain Poisson manifolds

Definition 1.4.7. [**26, 71, 77**] Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid endowed with a symplectic form; we shall say that the symplectic form is compatible with the groupoid structure and $(\mathcal{G}, \omega) \rightrightarrows M$ is a symplectic groupoid if the graph of the groupoid multiplication is Lagrangian in $T^*\mathcal{G} \times T^*\mathcal{G} \times \overline{T^*\mathcal{G}}$.

The compatibility between the groupoid structure and the symplectic form of a symplectic groupoid can be expressed equivalently in terms of the multiplicativity condition

$$\mu^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$$

on the manifold $\mathcal{G}^{(2)}$ of composable elements, explicitly

$$\omega_{gh}(\delta g_+ \bullet \delta h_+, \delta g_- \bullet \delta h_-) = \omega_g(\delta g_+, \delta g_-) + \omega_h(\delta h_+, \delta h_-) \tag{1.11}$$

for the tangent multiplication and composable $\delta g_\pm \in T_g\mathcal{G}$, $\delta h_\pm \in T_h\mathcal{G}$. By repeated application of (1.11) on suitable tangent vectors and by counting dimensions, one can show

Theorem 1.4.8. [**15**] *Let $\mathcal{G} \rightrightarrows M$ be a symplectic groupoid. The following hold*

1. *The unit section $M \to \mathcal{G}$ is a Lagrangian embedding;*

2. *The inversion $\mathcal{G} \to \mathcal{G}$ is an anti-symplectic diffeomorphism;*

3. *Source and target fibres are symplectic orthogonal to each other;*

4. *The source invariant functions and the target invariant functions form mutually commuting Poisson subalgebras of $\mathcal{C}^\infty(M)$;*

as a consequence,

5. *There exists a unique Poisson structure on M making the source map Poisson and the target map anti-Poisson.*

The statements above follow by counting dimensions from the analogous statements regarding Poisson groupoids (replace symplectic with Poisson and Lagrangian with coisotropic), the proof of which shall be outlined in the next Section.

Dual Poisson structures of integrable Lie algebroids are always induced by symplectic groupoids. The following results were obtained by Coste, Dazord and Weinstein.

Proposition 1.4.9. [**15**] *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then*

i) The cotangent bundle $T^*\mathcal{G}$ carries a Lie groupoid structure over the conormal bundle N^*M making it a symplectic groupoid for the canonical symplectic form;

and, identifying N^*M with the dual bundle A^* to the Lie algebroid A of $\mathcal{G}$,

ii) The symplectic groupoid $T^*\mathcal{G} \rightrightarrows A^*$ induces the dual Poisson structure on A^* for the anticanonical symplectic form[7]*.*

The Lie groupoid of the above proposition is usually referred to as the cotangent prolongation groupoid and plays a central role throughout this dissertation. Let us define the structure maps of the groupoid structure and set some notations. Fix $A = T^{\mathrm{s}}_M\mathcal{G} \simeq NM$ as a choice of a normal bundle. Source $\hat{\mathrm{s}}$ and target $\hat{\mathrm{t}}$ are the only natural maps $T^*\mathcal{G} \to N^*M$ yielding vector bundle maps over s and t; they are defined by setting, for all $a \in A_{\mathrm{s}(g)}$, respectively $b \in A_{\mathrm{t}(g)}$,

$$\begin{array}{rclcc} \langle\, \hat{\mathrm{s}}(\theta_g)\, ,\, a\,\rangle_{\mathrm{s}(g)} & := & \langle\, \theta_g\, ,\, \mathrm{d}l_g(a - \mathrm{d}\varepsilon(\rho(a)))\,\rangle_g & , & \text{respectively} \quad , \\ \langle\, \hat{\mathrm{t}}(\theta_g)\, ,\, b\,\rangle_{\mathrm{t}(g)} & := & \langle\, \theta_g\, ,\, \mathrm{d}r_g\, b\,\rangle_g & , & \theta_g \in T^*_g\mathcal{G} \quad , \end{array}$$

for the natural pairings. The cotangent multiplication $\hat{\mu} : T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}} T^*\mathcal{G} \to T^*\mathcal{G}$ can be defined as a vector bundle map over the groupoid multiplication $\mu : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{t}} \mathcal{G} \to \mathcal{G}$, by setting

$$\Gamma(\hat{\mu}) := (\mathsf{id}_{T^*\mathcal{G}} \times \mathsf{id}_{T^*\mathcal{G}} \times -\mathsf{id}_{T^*\mathcal{G}})N^*\Gamma(\mu)$$

and checking that $(\mathsf{id}_{T^*\mathcal{G}} \times \mathsf{id}_{T^*\mathcal{G}} \times -\mathsf{id}_{T^*\mathcal{G}})N^*\Gamma(\mu)$ is indeed the graph of a map. It follows that the cotangent unit section $\hat{\varepsilon} : A^* \to T^*\mathcal{G}$ and inversion $\hat{\iota} : T^*\mathcal{G} \to T^*\mathcal{G}$ have to be the identification of A^* with N^*M and, respectively the antitranspose $-\mathrm{d}\iota^{\mathsf{t}}$ of the tangent inversion. Note that the cotangent multiplication $\theta_g \,\hat{\cdot}\, \theta_h$ of composable cotangent vectors $(\theta_g, \theta_h) \in T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}} T^*\mathcal{G}$ is completely defined by the formula

$$\langle\, \theta_g \,\hat{\cdot}\, \theta_h, \delta g \bullet \delta h\,\rangle = \langle\, \theta_g\, ,\, \delta g\,\rangle + \langle\, \theta_h\, ,\, \delta h\,\rangle \qquad , \qquad (\delta g, \delta h) \in T\mathcal{G}\,{}_{\mathrm{ds}}\!\times_{\mathrm{dt}} T\mathcal{G} \qquad ,$$

by means of which the groupoid compatibility conditions can easily be checked (see the proof of theorem 3.1.9 for analogous computations); moreover the graph of $\hat{\mu}$ is Lagrangian in $T^*\mathcal{G} \times T^*\mathcal{G} \times \overline{T^*\mathcal{G}}$ (and $\overline{T^*\mathcal{G}} \times \overline{T^*\mathcal{G}} \times T^*\mathcal{G}$) by construction. It is not hard to inspect when $T^*\mathcal{G}$ is *the* symplectic groupoid of A^*.

[7]We need to change the sign of the symplectic form since we use “Poisson-friendly” conventions: the Poisson bracket induced by the canonical symplectic has therefore opposite sign with respect to the usual Poisson bracket of symplectic geometry.

Remark 1.4.10. The source fibres of the cotangent prolongation groupoid have the same homotopy type as those of $\mathcal{G}$. To see this, note that the kernel of the cotangent source map is the annihilator $(T^{\mathrm{t}}\mathcal{G})^o$, thus $\hat{\mathrm{s}}$ has maximal rank as a bundle map and the short exact sequence

$$0 \longrightarrow \ker\hat{\mathrm{s}} \longrightarrow T^*\mathcal{G} \longrightarrow \mathrm{s}^+A^* \longrightarrow 0$$

of vector bundles over $\mathcal{G}$ admits a splitting $\sigma : \mathrm{s}^+A^* \to T^*\mathcal{G}$, which allows to describe each $\hat{\mathrm{s}}$-fibre as

$$\hat{\mathrm{s}}^{-1}(\xi) = \{\sigma_g(\xi) + \ker_g\hat{\mathrm{s}}\}_{g\in\hat{\mathrm{s}}^{-1}(\mathrm{pr}(\xi))} \qquad , \qquad \xi \in A^* \qquad ,$$

i.e. as an affine subbundle of $T^*_{\mathrm{s}^{-1}(\mathrm{pr}(\xi))}\mathcal{G}$. As a consequence all $\hat{\mathrm{s}}$-fibres homotopy retract to suitable s-fibres and, in particular $T^*\mathcal{G}$ is source 1-connected iff so is $\mathcal{G}$.

Example 1.4.11. Cotangent prolongation groupoids

i) Regard a manifold M as the trivial Lie groupoid; then the induced Lie algebroid is also trivial and the dual Poisson structure is the zero bivector field over M. The construction above produces the abelian Lie groupoid on $T^*M \to M$.

ii) Consider the cotangent prolongation of the pair groupoid $M \times M$: a direct computation shows that source and target are respectively $-\mathrm{pr}_2, \mathrm{pr}_1 : T^*M \times T^*M \to T^*M$ and

$$\begin{aligned}\langle\, (\theta_x, -\theta_y) \,\hat{\cdot}\, (\theta_y, \theta_z)\,,\, (\delta x, \delta z)\,\rangle &= \langle\, (\theta_x, -\theta_y)\,,\, (\delta x, \delta y)\,\rangle + \langle\, (\theta_y, \theta_z)\,,\, (\delta y, \delta z)\,\rangle \\ &= \langle\, (\theta_x, \theta_z)\,,\, (\delta x, \delta z)\,\rangle \end{aligned} \qquad .$$

Let $(\Lambda, \omega) \rightrightarrows P$ be a symplectic groupoid; denote with A the Lie algebroid on $T^{\mathrm{s}}_P\Lambda$ induced by the bracket of right invariant vector fields. It is well known that A is canonically isomorphic to the Koszul algebroid on T^*P. The isomorphism is provided by the sharp map of the symplectic form: thanks to theorem 1.4.8 (3.), setting

$$\langle\, \phi_\omega^{-1}(a)\,,\, \delta p\,\rangle := -\omega(a, \mathrm{d}\varepsilon\delta p) \qquad , \qquad (a, \delta p) \in A \oplus TP \qquad ,$$

yields an isomorphism of vector bundles $\phi_\omega : T^*P \to A$ over the identity map; the induced map on the subalgebra of exact 1-forms $\Omega^1_{\mathsf{exact}}(P) \to \Gamma(A)$ can be regarded as taking values in the algebra of right invariant vector fields. Define $\phi^r_\omega(\mathrm{d}f) := -X^{\mathrm{t}^*f}$, thus

$$\omega(\phi^r_\omega(\mathrm{d}f)(\varepsilon(p)), \mathrm{d}\varepsilon\mathrm{d}p) = -\langle\, (\mathrm{t}^*\mathrm{d}f)_{\varepsilon p}\,,\, \mathrm{d}\varepsilon\delta p\,\rangle = -\langle\, \mathrm{d}f\,,\, \delta p\,\rangle =: \omega(\phi_\omega(\mathrm{d}f_p), \mathrm{d}\varepsilon\delta p)$$

for all $\delta p \in TP$; by symplectic orthogonality of source and target fibres, nondegeneracy of ω implies that $\phi^r_\omega(\mathrm{d}f)$ coincides with the right invariant vector field $\overrightarrow{\phi_\omega \circ \mathrm{d}f}$. Using this fact and the Leibniz rule one can easily check that ϕ_ω preserves the canonical Lie brackets on T^*P and A.

It follows that the connected components of the orbits of a symplectic groupoid are the symplectic leaves of the induced Poisson structure; in this sense, a symplectic groupoid is to be regarded as a desingularization of the symplectic foliation on its base manifold. The discussion above motivates the following

Definition 1.4.12. A Poisson bivector is called integrable if so is the induced Koszul algebroid.

As a corollary of integrability of Lie bialgebroids, to be discussed in the next Section, one can show that the source 1-connected Lie groupoid of an integrable Poisson manifold is always a symplectic groupoid.

Theorem 1.4.13. [**48**] *A Poisson manifold* (P, π) *is integrable iff there exists a symplectic groupoid inducing* π*.*

The correspondence between integrable Poisson manifolds and source 1-connected symplectic groupoids is however not functorial. In fact, given source 1-connected symplectic groupoids $\Lambda_\pm$ over $P_\pm$ a Poisson map $f : P_- \to P_+$ does not induce a symplectomorphism $\Lambda_- \to \Lambda_+$, unless f is a diffeomorphism. Coisotropic submanifolds are instead well behaved with respect to integration; roughly speaking, there exists a correspondence between coisotropic submanifolds and Lagrangian subgroupoids. Rephrasing a result by Cattaneo [**11**],

Theorem 1.4.14. *Let* $\Lambda \rightrightarrows P$ *be a source 1-connected symplectic groupoid and* $\mathcal{L} \rightrightarrows C$ *a source 1-connected Lie subgroupoid with Lie algebroid* L*. Then* $\mathcal{L} \subset \mathcal{G}$ *is Lagrangian iff* $C \subset P$ *is coisotropic and* $L \simeq N^*C$*.*

The proof of last theorem given in [**11**] is highly non trivial and involves symplectic reduction in a suitable infinite dimensional weak symplectic manifold; an independent simple proof shall be obtained in the next Section We shall comment further on theorems 1.4.13-1.4.14, after introducing Poisson groupoids.

1.5. Poisson groupoids and Lie bialgebroids

The first part of this Section consists in a quick introduction to Lie groupoids with a compatible Poisson bivector and their infinitesimal invariant, namely Poisson groupoids and Lie bialgebroids. In the second part after reviewing Mackenzie and Xu's integration of a Lie bialgebroid to a Poisson groupoid (Lie's third theorem holds under the integrability conditions on the underlying Lie algebroid), we further develop Lie theory for Poisson groupoids. In particular, we prove (theorem 1.5.10) the integrability of morphisms of Lie bialgebroids to morphisms of Poisson groupoids via a version of Lie's first theorem (integrability of coisotropic subalgebroids, theorem 1.5.9).

Poisson groupoids are simultaneous generalizations of Poisson groups and symplectic groupoids; Weinstein's coisotropic calculus provides a powerful technique to unify the theory of these two objects.

Definition 1.5.1. [**72**] A Poisson structure on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called compatible if the graph of the groupoid multiplication

$$\Gamma(\mu) \subset \mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$$

is coisotropic. A Lie groupoid endowed with a compatible Poisson structure is called a Poisson groupoid.

A Poisson groupoid $G \rightrightarrows \bullet$ over the one point manifold is a Poisson group in the classical sense, i.e. the compatibility condition is equivalent to asking the group multiplication to be a Poisson map $G \times G \to G$; a Poisson groupoid with nondegenerate (hence symplectic) Poisson bivector is a symplectic groupoid in the sense of definition 1.4.7, by counting dimensions.

Example 1.5.2. Poisson groupoids

i) Every Lie groupoid is a Poisson groupoid for the zero Poisson structure. A Poisson manifold P is *not* a Poisson groupoid for the trivial Lie groupoid $P \rightrightarrows P$, since $\Gamma(\mu) = \{(p,p,p)\}$ is not coisotropic in $P \times P \times \overline{P}$. The pair groupoid on $\overline{P} \times P$ is a Poisson groupoid.

ii) Let P be a Poisson G-space, i.e G is a Poisson group acting on a Poisson manifold P in such a way that the action map $G \times P \to P$ is Poisson. If the action is free and proper $P \to P/G$ is a principal G bundle; it is an exercise in coisotropic calculus to check that the diagonal action of G on $\overline{P} \times P$ is Poisson and compatible with the pair groupoid. Then $(P \times \overline{P})/G$ carries a Lie groupoid over P/G, namely the gauge groupoid. It is well known [**72**] that smooth quotients of Poisson G spaces always carry a unique Poisson structure making the quotient projection a Poisson submersion. For the quotient Poisson structure $(\overline{P} \times P)/G \rightrightarrows P/G$ is a Poisson groupoid.

The main properties of a Poisson groupoid were derived in [**72**] and are listed in the following theorem. The proof we sketch uses essentially the same techniques as in [**72**].

Theorem 1.5.3. *Let $(\mathcal{G}, \Pi) \rightrightarrows M$ be a Poisson groupoid. The following hold*

1. *The unit section $M \to \mathcal{G}$ is a coisotropic embedding;*

2. *The inversion $\mathcal{G} \to \mathcal{G}$ is an anti-Poisson diffeomorphism;*

3. *The source invariant functions and the target invariant functions form Poisson subalgebras of $\mathcal{C}^\infty(M)$;*

4. *There exists a unique Poisson structure on M making the source map Poisson and the target map anti-Poisson;*

5. *The Hamiltonian vector fields of the source, respectively target, invariant functions are left, respectively right, invariant;*

6. *The subalgebras of source and target invariant functions commute.*

Sketch of proof. The first two properties follow easily from theorem 1.2.18, the very cleanliness assumptions can be shown to hold by direct inspection (1) $M \subset \mathcal{G}$ is coisotropic since

$$M \times \bullet = \mathsf{\Gamma}(\mu) \circ (\Delta_{\mathcal{G}} \times \bullet) \subset \mathcal{G} \times \bullet$$

is the composition of the coisotropic relations $\mathsf{\Gamma}(\mu) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $\Delta_{\mathcal{G}} \times \bullet : \overline{\mathcal{G}} \times \mathcal{G} \to \bullet$. (2) $\mathsf{\Gamma}(\iota) \subset \mathcal{G} \times \mathcal{G}$ is coisotropic since

$$\mathsf{\Gamma}(\iota) \times \bullet = \mathsf{\Gamma}(\mu) \circ (M \times \bullet) \subset \mathcal{G} \times \mathcal{G} \times \bullet$$

is the composition of coisotropic the relations $\mathsf{\Gamma}(\mu) : \mathcal{G} \to \overline{\mathcal{G}} \times \mathcal{G}$ and $M \times \bullet : \mathcal{G} \to \bullet$. The proof of (3) is independent from that of (1) and (4) is a straightforward consequence of (3). (3) By definition $f \in \mathcal{C}^\infty(\mathcal{G})$ is source invariant if $f = \mathrm{s}^*\varepsilon^* f$, equivalently if $f(gh) = f(h)$ for all composable g, $h \in \mathcal{G}$; that is, f is source invariant iff the associated function $\widetilde{f} \in \mathcal{C}^\infty(\mathcal{G})^{\times 3}$, $\widetilde{f}(g,h,k) := f(k) - f(h)$ vanishes on the characteristic ideal of $\mathsf{\Gamma}(\mu)$. Note that, for all $f_\pm \in \mathcal{C}^\infty(\mathcal{G})$,

$$\widetilde{\{\, f_+ \,,\, f_- \,\}_{\mathcal{G}}} = \{\, \widetilde{f_+} \,,\, \widetilde{f_-} \,\}_{\mathcal{G}\times\mathcal{G}\times\overline{\mathcal{G}}} \qquad ,$$

thus source-invariant functions form a Poisson subalgebra, by coisotropicity of $\mathsf{\Gamma}(\mu)$, and it follows from (2) that also target-invariant functions do. (5) For any source invariant $f \in \mathcal{C}^\infty(\mathcal{G})$ and composable g, $h \in \mathcal{G}$,

$$(\Pi^\sharp \times \Pi^\sharp \times -\Pi^\sharp)_{(g,h,gh)} \mathrm{d}\widetilde{f} = (0_g, -\Pi^\sharp_h \mathrm{d}f, -\Pi^\sharp_{gh} \mathrm{d}f) \in T_{(g,h,gh)} \mathsf{\Gamma}(\mu) = \mathsf{\Gamma}(\mathrm{d}\mu)_{(g,h,gh)} \quad ,$$

thus $\mathrm{d}\mathrm{t}\Pi^\sharp_h \mathrm{d}f = 0_{\mathrm{t}(h)}$ and $\Pi^\sharp_{gh}\mathrm{d}f = 0_g \bullet \Pi^\sharp_h \mathrm{d}f = \mathrm{d}l_g \Pi^\sharp_h \mathrm{d}f$ for the tangent multiplication $\bullet$; the statement for right invariant vector fields is proved analogously. (6) is a follows easily from (5). □

Let $(\mathcal{G}, \Pi) \rightrightarrows M$ be a Poisson groupoid and consider its Lie algebroid $A \equiv T^{\mathrm{s}}\mathcal{G}$. The dual bundle A^* is to be canonically identified with the conormal bundle $N^*M \subset T^*\mathcal{G}$ and, by coisotropicity of $M \subset \mathcal{G}$, it carries a natural Lie algebroid structure. The Lie algebroids on A and A^* are compatible in a way that makes sense without any reference to the Poisson groupoid they are derived from.

Definition 1.5.4. [**46, 28**] Let A be a Lie algebroid and assume A^* is also a Lie algebroid. The pair (A, A^*) is called a Lie bialgebroid if the Lie algebroid differential d_{A^*} makes $(\Gamma(\wedge^\bullet A), \wedge, [\![\,,\,]\!], \mathrm{d}_{A^*})$ a differential Gerstenhaber algebra[8].

Le us explain the above definition. For any Lie algebroid $A \to M$, the Lie bracket on $\Gamma(A)$ can be uniquely extended to a biderivation $[\![\,,\,]\!]$ of the graded commutative algebra on $\Gamma(\wedge^\bullet A) = \oplus_{i\geq 0}\Gamma(\wedge^i A)$ by setting

$$\begin{array}{rcl} [\![f\,,\,g]\!] &:=& 0 \\ [\![a\,,\,f]\!] &:=& \rho(a)(f) \\ [\![a\,,\,b]\!] &:=& [\,a\,,\,b\,] \qquad\qquad f,g\in\mathcal{C}^\infty(M)\ ,\ \ a,b\in\Gamma(A)\quad , \end{array}$$

and imposing the Leibniz rule[9]

$$[\![a\,,\,b\wedge c]\!] = [\![b\,,\,a]\!]\wedge c + (-)^{(a-1)b} b\wedge [\![a\,,\,c]\!] = 0 \tag{1.12}$$

on all homogeneous $a, b, c \in \Gamma(\wedge^\bullet A)$. Remarkably $[\![\,,\,]\!]$ makes the suspension $\Gamma(\wedge^\bullet A)[1]$ a graded Lie algebra, i.e. $[\![\,,\,]\!]$ is graded skewsymmetric,

$$[\![a\,,\,b]\!] + (-)^{(a-1)(b-1)}[\![b\,,\,a]\!] = 0$$

and the graded Jacobi identity

$$[\,a\,,\,[\,b\,,\,c\,]\,] = [\,[\,a\,,\,b\,]\,,\,c\,] + (-)^{(a-1)(b-1)}[\,a\,,\,[\,b\,,\,c\,]\,]$$

holds for all homogeneous elements $a, b, c \in \Gamma(\wedge^\bullet A)$; namely, $(\Gamma(\wedge^\bullet A), \wedge, [\![\,,\,]\!])$ is a Gerstenhaber algebra. Dually, the Lie algebroid on A induces a differential $\mathrm{d}_A : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$,

$$\begin{array}{rcl} \mathrm{d}_A\eta(a_0,\ldots,a_n) &:=& \displaystyle\sum_{i=0}^{n}(-)^i\rho(a_i)(\xi(a_1,\ldots,\hat{a}_i,\ldots,a_n)) \\ && + \displaystyle\sum_{i<j}(-)^{i+j}\xi([a_i,a_j],a_1,\ldots,a_n) \qquad , \qquad \eta\in\Gamma(\wedge^n A), \end{array}$$

which generalizes the Chevalley-Eilenberg differential and the de Rham differential. Moreover it satisfies the usual rules of Cartan calculus; in particular d_A is a derivation of the wedge product.

Remark 1.5.5. If (A, A^*) is a pair of Lie algebroids in duality the natural compatibility amounts therefore to asking the Lie algebroid differential d_{A^*} induced by A^* to be a derivation of the graded Lie bracket induced by A, i.e.

$$\mathrm{d}_{A^*}[\![a\,,\,b]\!] = [\![\mathrm{d}_{A^*}a\,,\,b]\!] + (-)^{(a-1)}[\![a\,,\,\mathrm{d}_{A^*}b]\!] \tag{1.13}$$

for all homogeneous elements $a, b \in \Gamma(\wedge^\bullet A)$. The role played by A and A^* is symmetric, in the sense that (A, A^*) is a Lie bialgebroid iff so is (A, A^*).

[8]Lie bialgebroids were discovered by Mackenzie and Xu in [**46**] as infinitesimal invariants of Poisson groupoids. The definition [**28**] we use is equivalent to that of [**46**].

[9]We sometimes denote the degree of an homogeneous $a \in \Gamma(\wedge^k A)$ with the same symbol a.

Example 1.5.6. Lie bialgebroids

i) For any Lie algebra $\mathfrak{g}$, $\mathrm{d}_{\mathfrak{g}}$ is the classical Chevalley-Eilenberg differential. If $(\mathfrak{g}, \mathfrak{g}^*)$ are Lie algebras denote with $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ the map dual to the Lie bracket on $\mathfrak{g}^*$; condition (1.13) is satisfied iff δ is a 1-cocycle in the Lie algebra cohomology with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$ for the adjoint representation, that is, if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra (see [**13**] for details).

ii) If (P, π) is a Poisson manifold the Jacobi identity for the Poisson bracket is equivalent to $[\![\pi\,,\,\pi]\!] = 0$ for the graded Lie bracket on $\mathfrak{X}^\bullet(P) := \Gamma(\wedge^\bullet TP)$ extending the Lie bracket of vector fields. By the graded Jacobi identity $\mathrm{d}_\pi := [\![\pi\,,\,\cdot\,]\!]$ makes $\mathfrak{X}^\bullet(P)$ a differential Gerstenhaber algebra and (T^*P, TP) a Lie bialgebroid.

iii) More generally, for any Lie algebroid A with a bisection $\Delta \in \Gamma(\wedge^2 A)$ such that $[\![\Delta\,,\,\Delta]\!] = 0$, the Lie bracket

$$[\,\xi_+\,,\,\xi_-\,] := \mathrm{d}_A\Delta(\xi_+, \xi_-) + \iota_{\Delta^\sharp\xi_+}\mathrm{d}_A\xi_- - \iota_{\Delta^\sharp\xi_-}\mathrm{d}_A\xi_+ \quad , \quad \xi_\pm \in \Gamma(A^*)$$

makes (A, A^*) a Lie bialgebroid, where the map $\Delta^\sharp : A^* \to A$ is induced by Δ. The bisection Δ is called an R-matrix and (A, A^*) a triangular Lie bialgebroid. If $A = \mathfrak{g}$ is a Lie algebra the condition on $\Delta \in \wedge^2\mathfrak{g}$ is the classical Yang Baxter equation.

iv) [**29**] Let (P, π) be a Poisson manifold and $\Phi : TP \to TP$ a (1,1) tensor with vanishing torsion

$$\tau^\Phi(X, Y) = [\,\Phi X\,,\,\Phi Y\,] - \Phi[\,\Phi X\,,\,Y\,] - \Phi[\,X\,,\,\Phi Y\,] + \Phi^2[\,X\,,\,Y\,] \quad , \quad X, Y \in \mathfrak{X}(P)$$

Then $[X, Y]^\Phi = [\,\Phi X\,,\,Y\,] + [\,X\,,\,\Phi Y\,] - [\,X\,,\,Y\,]$ makes TP a Lie algebroid TP_Φ with anchor Φ and (T^*P, TP_Φ) is a Lie bialgebroid iff (P, π, N) is a Poisson-Nijenhuis manifold [**5, 30, 4, 29**].

The main integrability result for Lie bialgebroids is the following theorem due to Mackenzie and Xu. We shall sketch below the idea of the proof.

Theorem 1.5.7. [**48**] *Let (A, A^*) be a Lie bialgebroid over M and $\mathcal{G} \rightrightarrows M$ a source 1-connected Lie groupoid with Lie algebroid A. Then there exists a unique Poisson structure on $\mathcal{G}$ making it a Poisson groupoid and inducing the Lie algebroid A^* on N^*M.*

First of all we shall need to introduce a certain canonical antisymplectomorphism. Note that, for any vector bundle $\mathrm{pr} : A \to M$, setting $\mathsf{r}_A : T^*A \to A^*$

$$\mathsf{r}_A(\theta)(a') := \langle\,\theta\,,\,\iota_{V\,a}(a')\,\rangle \qquad , \qquad a, a' \in A \qquad ,$$

where $\iota_V : \mathrm{pr}^+A \simeq T_VA \to TA$ is the inclusion of the vertical bundle, yields a vector bundle map over $\mathrm{pr} : A \to M$. One can check that $\mathrm{dpr} : TA \to TM$ carries a vector bundle, whose scalar multiplication and fibrewise addition are given by the tangent of the corresponding maps of A (see [**40, 44, 45**] for the details); moreover the structure map of dpr can be suitably dualized to endow also $\mathsf{r}_{A^*} : T^*A^* \to A$ with a vector bundle structure. The diagram

$$\begin{tikzcd} T^*A^* \arrow[r, "\mathsf{r}_{A^*}"] \arrow[d] & A \arrow[d] \\ A^* \arrow[r] & M \end{tikzcd} \tag{1.14}$$

is a double vector bundle in the sense of Ehresmann, i.e. a vector bundle in the category of vector bundles. Remarkably, it is possible to define an isomorphism of double vector bundles

$$\begin{tikzcd} T^*A^* \arrow[r] \arrow[d] \arrow[drr] & A \arrow[d, "\tau"] \arrow[drrr] & & \\ A^* \arrow[r] \arrow[drr] & M \arrow[drrr] & T^*A \arrow[r] \arrow[d] & A \arrow[d] \\ & & A^* \arrow[r] & M \end{tikzcd} \tag{1.15}$$
,

i.e. both (τ, id_A) and $(\tau, \mathsf{id}_{A^*})$ form a morphism of vector bundles, which is also an antisymplectomorphism for the canonical symplectic forms. The definition of τ appeared in [**46**] and generalizes Tulczyjew's canonical antisymplectomorphism $T^*T^*M \to T^*TM$, encoding the Legendre transform geometrically; let us recall the construction of τ, since it will be needed in the following. Regard the canonical pairing of A^* with A as a map $\mathbb{F} : A^* \oplus A \to \mathbb{R}$; the submanifold $L \subset T^*(A^* \times A)$ defined by those elements $(\overline{\theta}, \theta) \in T^*_{A^* \oplus A}(A^* \times A)$ such that

$$\langle\, (\overline{\theta}, \theta)\, ,\, (\delta\xi, \delta a)\, \rangle = \mathrm{d}\mathbb{F}_A(\delta\xi, \delta a) \qquad , \qquad \delta\xi, \delta a \in T(A^* \oplus A) \tag{1.16}$$

can easily be seen to be Lagrangian (choose Darboux coordinates adapted to $A^* \oplus A$). One can further show that L is the graph of a map, therefore (1.16) completely defines an antisymplectomorphism.

Whenever (A, A^*) is a Lie algebroid over M, the composition $\pi^\sharp_{A^*} \circ \tau : T^*A^* \to TA$ of the dual Poisson anchor with the canonical antisymplectomorphism is a morphism of Lie algebroids over the anchor $A^* \to TM$ for the Koszul and tangent prolongation Lie algebroids. In fact, this requirement on a pair of Lie algebroids (A, A^*) is equivalent to (A, A^*) being a Lie bialgebroid. If A is integrable and $\mathcal{G}$ its source 1-connected Lie groupoid, $\pi^\sharp_{A^*} \circ \tau$ can then be integrated to a morphism of Lie groupoids $\Pi^\sharp : T^*\mathcal{G} \to T\mathcal{G}$ for the tangent and cotangent prolongation groupoids. It turns out that $\Pi^\sharp$, as the notation suggests, is indeed the sharp map of a Poisson bivector, which, by construction indeed, is compatible with the groupoid multiplication and induces the given Lie algebroid on A^*.

Example 1.5.8. Integrability of Lie bialgebroids

i) If P is a Poisson manifold with connected components $\{P_i\}$ and $\widetilde{P}_i \to P_i$ are the associated covering principal bundles, the Poisson groupoid on

$$\Pi(P) \simeq \coprod_{i\in\pi_o(P)} (\overline{\widetilde{P}_i} \times \widetilde{P}_i)/\pi_1(P_i)$$

integrates the Lie bialgebroid on (TP, T^*P) (the Poisson bivector on $\widetilde{P}_i$ is the unique making the covering projection a Poisson submersion, see also lemma 1.5.12 below).

ii) [**35**] For any Lie groupoid $\mathcal{G} \rightrightarrows M$ with Lie algebroid A and R-matrix Δ,

$$\Pi = \overrightarrow{\Delta} - \overleftarrow{\Delta} = \overrightarrow{\Delta - \iota_*\Delta}$$

is a Poisson bivector making $(\mathcal{G}, \Pi)$ a Poisson groupoid and integrating the triangular Lie bialgebroid associated with (A, Δ).

The notion of morphism of Poisson groupoids is very natural; namely, for any Poisson groupoids $\mathcal{G}_\pm$, $\varphi : \mathcal{G}_- \to \mathcal{G}_+$ is a morphism of Poisson groupoids if it is a Poisson map and a morphism of Lie groupoids, equivalently if $\Gamma(\varphi) \subset \mathcal{G}_- \times \overline{\mathcal{G}_+}$ is a coisotropic subgroupoid. Note that, if $(A_\pm, A^*_\pm)$ are Lie bialgebroids and $\phi : A_- \to A_+$ is a morphism of Lie algebroids, there is no natural map $\Gamma(\wedge^\bullet A_-) \to \Gamma(\wedge^\bullet A_+)$. Nevertheless, by transposition, there a map $\phi^{\wedge *} : \Gamma(\wedge^\bullet A^*_+) \to \Gamma(\wedge^\bullet A^*_-)$ and to encode a compatibility condition for ϕ with the Lie algebroids on $A^*_\pm$ it is natural to ask $\phi^{\wedge *}$ to preserve the corresponding graded Lie brackets. Clearly,

$$\phi^{\wedge *}(\xi \wedge \eta) = \phi^{\wedge *}\xi \wedge \phi^{\wedge *}\eta \qquad , \qquad \xi, \eta \in \Gamma(\wedge^\bullet A^*_+) \qquad ,$$

thus, it is sufficient to check last condition on homogeneous elements of degree 0 and 1, equivalently to show that $\phi^* : \mathcal{C}^\infty(A_+) \to \mathcal{C}^\infty(A_-)$ preserves the Poisson brackets of pairs of functions which are fibrewise constant or fibrewise linear for the dual Poisson structures. It makes therefore sense to call ϕ a morphism of Lie bialgebroids if it is a Poisson map and a morphism of Lie algebroids, equivalently if $\Gamma(\phi) \subset A_- \times A_+$ is a coisotropic Lie subalgebroid; this definition clearly yields a well defined category of Lie bialgebroids.
We devote the rest of this Section to prove that the categories of integrable Lie bialgebroids and source 1-connected Lie groupoids are isomorphic; the result is a corollary of the following

Theorem 1.5.9. *Let $(\mathcal{G}, \Pi) \rightrightarrows M$ be a source 1-connected Poisson groupoid with Lie bialgebroid (A, A^*) and $\mathcal{C} \rightrightarrows N$ a source 1-connected Lie subgroupoid with Lie algebroid C. Then $\mathcal{C} \subset \mathcal{G}$ is coisotropic iff so is $C \subset A$ for the dual Poisson structure induced by A^*.*

In other words, under the connectivity assumptions, coisotropic subalgebroids of integrable Lie bialgebroids, integrate to coisotropic subgroupoids. Note, however, that the source 1-connected integration of a coisotropic subalgebroid of an integrable Lie bialgebroid A is in general only an immersed subgroupoid of the Poisson groupoid integrating A. The only iff part of theorem 1.5.9 was proved in [**75**] without any connectivity assumption, using a different approach; it is clear from our proof below that we also need no connectivity assumptions to obtain this implication.
Consider that for any morphism of Lie groupoids $\varphi : \mathcal{G}_- \to \mathcal{G}_+$, the graph $\Gamma(\varphi)$ is source 1-connected iff so is $\mathcal{G}_-$; therefore, if $\mathcal{G}_- \times \mathcal{G}_+$ is also source 1-connected, we can apply theorem 1.5.9 and conclude that, morphisms of integrable Lie bialgebroids, by the characterization in terms of graphs, integrate to morphisms of Poisson groupoids.
The connectivity assumptions on the target groupoid may be removed:

Theorem 1.5.10. *Let $\mathcal{G}_\pm \rightrightarrows M_\pm$ be Poisson groupoids and $(A_\pm, A^*_\pm)$ their Lie bialgebroids. If $\varphi : \mathcal{G}_- \to \mathcal{G}_+$ is a morphism of Lie groupoids and $\mathcal{G}_-$ is source* 1*-connected, then φ is Poisson iff so is the induced morphism of Lie algebroids $\phi : A_- \to A_+$ for the dual Poisson bivector fields induced by $A^*_\pm$.*

Let us start with the proof of theorem 1.5.9, which we divide in three steps; we emphasize that the first two steps do not involve the Poisson bivector of $\mathcal{G}$ or the Lie algebroid on A^*.

Step 1. (Proposition 1.5.11) We characterize the conormal bundle of a coisotropic subgroupoid $\mathcal{C} \subset \mathcal{G}$ as a Lagrangian subgroupoid of the cotangent prolongation $T^*\mathcal{G} \rightrightarrows A^*$ and identify its Lie algebroid with the conormal bundle N^*C^o to the annihilator of the Lie algebroid of $\mathcal{C}$;

Step 2. (Lemma 1.5.12) We identify N^*C^o with N^*C using the canonical antisymplectomorphism;

Step 3. (End of proof) The result follows by diagram chasing.

Note that for any Lie subgroupoid $\mathcal{C} \subset \mathcal{G}$ the inclusion of Lie algebroids $C \hookrightarrow A$ induces an inclusion of Lie algebroids $N^*C^o \hookrightarrow T^*A^*$. Next result depends on the underlying Lie groupoids–algebroids only.

Proposition 1.5.11. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid A and $\mathcal{C} \subset \mathcal{G}$ a Lie subgroupoid over $N \subset M$ with Lie algebroid C. Then*

i) $N^*\mathcal{C}$ is a Lagrangian subgroupoid of $T^*\mathcal{G} \rightrightarrows A^*$ with base C^o;

ii) The Lie algebroid of $N^*\mathcal{C} \rightrightarrows C^o$ is $N^*C^o \to C^o$.

PROOF. For all $\nu \in N^*_c\mathcal{C}$ and $x \in C_{\mathrm{s}(c)}$,

$$\langle\, \hat{\mathrm{s}}(\nu)\,,\, x\,\rangle = \langle\, \nu\,,\, \mathrm{d}l_c(x-\rho(x))\,\rangle = 0 \qquad ,$$

since $x-\rho(x)$ is tangent to $\mathcal{C}$ and so is its left translate by an element of $\mathcal{C}$. Denote with $\widetilde{\mathrm{s}}$ the restriction of $\hat{\mathrm{s}}$ to $N^*\mathcal{G}$. We have $\mathsf{ker}\,\widetilde{s} = (T^{\mathrm{t}}\mathcal{G})^o \cap N^*\mathcal{C} = (T^{\mathrm{t}}\mathcal{G}+T\mathcal{C})^o$, therefore $\widetilde{\mathrm{s}}$ is a vector bundle map of rank

$$\begin{aligned}\mathsf{rank}\,\widetilde{\mathrm{s}} &= (\mathsf{dim}\,\mathcal{G}-\mathsf{dim}\,\mathcal{C})-\mathsf{dim}\,\mathcal{G}+(\mathsf{rank}\mathrm{t}+\mathsf{dim}\,N)\\ &= \mathsf{rank}A-\mathsf{rank}C\\ &= \mathsf{rank}C^o \qquad ,\end{aligned}$$

that is, $\widetilde{\mathrm{s}}: N^*\mathcal{C}\to C^o$ is fibrewise surjective over a surjective submersion, hence a surjective submersion (see remark 2.2.8). (i) It remains to check that $\hat{\mu}$ restricts to $N^*\mathcal{C}$, which is clear since the tangent multiplication restricts to a fibrewise surjective multiplication on $T\mathcal{C}$ and because of (1.4.2). (ii) Let ω be the canonical symplectic form on $T^*\mathcal{G}$: we have to check that the isomorphism

$$\phi_{-\omega}: T^*A^*\to T^{\hat{\mathrm{s}}}_{A^*}T^*\mathcal{G} \qquad , \qquad \theta\to {\omega_{\hat{\varepsilon}(\theta)}}^{\sharp\text{-}1}\hat{\mathrm{t}}^*\theta \qquad ,$$

restricts to give a map $N^*C^o\to T^{\widetilde{\mathrm{s}}}N^*\mathcal{C}$. For all $f\in\mathcal{I}_{C^o}$ and $F\in\mathcal{I}_{N^*\mathcal{C}}$,

$$\overrightarrow{\phi_{-\omega}\circ\mathrm{d}f}(F) = \{\, F\,,\, \hat{\mathrm{t}}^*f\,\} = 0 \qquad ,$$

since $\hat{\mathrm{t}}^*f\in\mathcal{I}_{N^*\mathcal{C}}$ and being $N^*\mathcal{C}\subset T^*\mathcal{G}$ Lagrangian, that is, $\overrightarrow{\phi_{-\omega}\circ\mathrm{d}f}|_{N^*\mathcal{C}}\in \overrightarrow{\mathfrak{X}}(N^*\mathcal{C})$. □

Next lemma does not depend on Lie algebroids or Poisson structures.

Lemma 1.5.12. *For any vector bundle $A\to M$ and smooth subbundle $C\to N$, the canonical antisymplectomorphism $T^*A^*\to T^*A$ identifies N^*C^o with N^*C.*

PROOF. For all $\nu\in N^*_{\underline{c}}C^o$ and $\delta c\in T_cC$ with $(\underline{c},c)\in C^o\oplus C$, we have

$$\langle\, \tau(\nu_c)\,,\, \delta c\,\rangle = \mathrm{d}\mathbb{F}_A(\delta\xi,\delta c)-\langle\, \nu_c\,,\, \delta\xi\,\rangle \tag{1.17}$$

if $\mathrm{dpr}_{A^*}\delta\xi = \mathrm{dpr}_A\delta c$, $\delta\xi\in T_{\underline{c}}C^o$. On the other hand, by picking a connection for C^o, we can always find one such $\delta\xi$; for such a choice the right hand side of (1.17) vanishes, since $\mathbb{F}_A$ is constant on $C^o\oplus C$. □

It is now easy to conclude the proof.

PROOF OF THEOREM 1.5.9. Assume $C\subset A$ is a coisotropic subalgebroid. Because of lemma 1.5.12 and by coisotropicity of $C\subset A$, the diagram

$$\begin{tikzcd} T^*A^* \arrow[r, "\pi_{A^*}\circ\tau"] & TA \\ N^*C^o \arrow[u, hook] \arrow[r] & TC \arrow[u, hook] \end{tikzcd}$$

commutes in the category of Lie algebroids. Moreover, the source fibres of both $T\mathcal{C} \rightrightarrows TM$ and $N^*\mathcal{C} \rightrightarrows C^o$ have the same homotopy type as those of $\mathcal{C}$, this can be seen using a variation of the argument in remark 1.4.10. By the connectivity assumptions the diagram above integrates to

$$\begin{tikzcd} T^*\mathcal{G} \arrow[r, "\Pi^\sharp"] & T\mathcal{G} \\ N^*\mathcal{C} \arrow[u, hook] \arrow[r] & T\mathcal{C} \arrow[u, hook] \end{tikzcd} ,$$

i.e. $\mathcal{C} \subset \mathcal{G}$ is coisotropic. Conversely, if $\mathcal{C} \subset \mathcal{G}$ is coisotropic invert the argument and use the inverse antisymplectomorphism to show that $\pi^\sharp_{A^*} N^*C \subset TC$. □

If $(\Lambda, \omega) \rightrightarrows P$ is a symplectic groupoid, one can check that the isomorphism of Lie algebroids $\phi_\omega : T^*P \to T^{\mathsf{s}}_P\mathcal{G}$ is also a Poisson map for the canonical symplectic form on T^*P and the fibrewise linear Poisson structure induced by N^*P on $T^{\mathsf{s}}_P\mathcal{G} \simeq NP$ (which is then symplectic). That is, (T^*P, TP) and $(T^{\mathsf{s}}_P\mathcal{G}, N^*P)$ are isomorphic Lie bialgebroids and theorem 1.4.13 follows specializing the proof of 1.5.7. If $\mathcal{L} \subset \mathcal{G}$ is a Lagrangian subgroupoid, the Lie algebroid $L \subset T^{\mathsf{s}}_P\mathcal{G}$ of $\mathcal{L}$, by direct inspection, maps to N^*C under ϕ_ω^{-1}, thus making $N^*C \subset T^*P$ a Lagrangian subalgebroid and C coisotropic. Conversely if C is coisotropic with $\phi_\omega(N^*C) = L$, then L is Lagrangian and theorem 1.5.9 implies that so is $\mathcal{L}$, by counting dimensions; that is, we have recovered theorem 1.4.14.

Before proving theorem 1.5.10, we shall need the following easy

Lemma 1.5.13. *Let (P, π) be a Poisson manifold, $C \subset P$ a coisotropic submanifold and $\phi : M \to P$ a local diffeomorphism. Then*

1. *There exists a unique Poisson bivector $\phi^\star\pi$ on M making ϕ a Poisson map;*
2. *If $X \subset M$ is a submanifold, such that ϕ restricts to a local diffeomorphism $X \to C$, then X is coisotropic with respect to $\phi^\star\pi$.*

Proof. It is easy to see and well known that the equation

$$\phi^\star\pi(\phi^*\mathrm{d}f_+, \phi^*\mathrm{d}f_-) = \{\, f_+\, ,\, f_-\,\} \circ \phi \qquad , \qquad f_\pm \in \mathcal{C}^\infty(P)$$

defines the desired Poisson bivector. (2) By linear algebra we have $N^*_xX = \mathrm{d}\phi_x{}^{\mathsf{t}}(\mathrm{d}\phi_x T_xX)^o = \mathrm{d}\phi_x{}^{\mathsf{t}} N^*_{\phi(x)}C$ for all $x \in X$, therefore

$$\begin{aligned} (\phi^\star\pi)^\sharp N^*_xX &= (\mathrm{d}\phi_x{}^{\mathsf{t}})^{-1}\pi^\sharp_{\phi(x)} N^*_{\phi(x)}C \\ &\subset (\mathrm{d}\phi_x{}^{\mathsf{t}})^{-1} T_{\phi(x)}C \\ &= T_xX \end{aligned} .$$

by coisotropicity of C. □

Proof of theorem 1.5.10. If the target groupoid in the statement of theorem 1.5.10 is actually not source 1-connected, the integrating morphism $\varphi : \mathcal{G}_- \to \mathcal{G}_+$ can always be factored through the covering morphism $\kappa : \tilde{\mathcal{G}}_+ \to \mathcal{G}_+$ and the integration $\widetilde{\varphi} : \mathcal{G}_- \to \tilde{\mathcal{G}}_+$. Let Π_+ be the Poisson bivector on $\mathcal{G}_+$ and $\phi : \tilde{A}_+ \equiv T_M^{\tilde{\mathsf{s}}_+}\tilde{\mathcal{G}}_+ \to T_M^{\mathsf{s}_+}\mathcal{G}_+ \equiv A_+$ the isomorphism of Lie algebroids induced by the covering morphism. On the one hand, ϕ allows to induce a Lie bialgebroid on $(\widetilde{A}_+, \widetilde{A}_+^* \equiv N^*M_{\mathsf{rel}\,\tilde{\mathcal{G}}_+})$ from that on $(A_+, A_+^* \equiv N^*M_{\mathsf{rel}\,\mathcal{G}})$ via ϕ^{t} ($N^*X_{\mathsf{rel}\,Y}$ denotes the conormal bundle of X as a submanifold of Y); denote with $\tilde{\Pi}$ the Poisson bivector on $\mathcal{G}$ making it a Poisson groupoid and inducing $(\widetilde{A}_+, \widetilde{A}_+^*)$. Since both $\mathcal{G}_-$ and $\tilde{\mathcal{G}}_+$ are source 1-connected, $\widetilde{\varphi}$ is Poisson and it suffices to show that the covering morphism is also Poisson. On the other hand, because of lemma 1.5.13, $\kappa^\star\Pi_+$ is also a compatible Poisson bivector on $\tilde{\mathcal{G}}_+$, since $\kappa^{\times 3}$ restricts to a local diffeomorphism $\Gamma(\tilde{\mu}_+) \to \Gamma(\mu_+)$. We claim that the Lie bialgebroid of $(\widetilde{\mathcal{G}}_+, \kappa^\star\Pi_+)$ coincides (not only up to isomorphism!) with $(\widetilde{A}_+, \widetilde{A}_+^*)$; it follows by uniqueness (theorem 1.4.13) that $\tilde{\Pi}$ coincides with $\kappa^\star\Pi_+$ and κ is therefore Poisson. We have to show that $\phi^{\mathsf{t}} : N^*M_{\mathsf{rel}\,\mathcal{G}} \to N^*M_{\mathsf{rel}\,\tilde{\mathcal{G}}_+}$ is a morphism of Lie algebroids for the bracket on $N^*M_{\mathsf{rel}\,\tilde{\mathcal{G}}_+}$ induced by $\kappa^\star\Pi_+$. The anchor compatibility holds since

$$\kappa^\star\Pi_+^\sharp\phi^{\mathsf{t}} = \kappa^\star\Pi_+^\sharp \mathrm{d}\kappa^{\mathsf{t}}\Big|_{N^*M_{\mathsf{rel}\,\mathcal{G}}} = (\mathrm{d}\kappa^{\mathsf{t}})^{-1}\Pi^\sharp\big|_{N^*M_{\mathsf{rel}\,\mathcal{G}}} = \Pi^\sharp\big|_{N^*M_{\mathsf{rel}\,\mathcal{G}}} \qquad ,$$

where we have used coisotropicity of $M \subset (\mathcal{G}, \Pi)$ for the last equality. For sections of the form $\mathrm{d}F_{1,2}|_M \in \Gamma(A_+^*)$, $F_{1,2} \in \mathcal{I}_M \subset \mathcal{C}^\infty(\mathcal{G}_+)$, we have

$$\begin{aligned} \phi^{\mathsf{t}}[\, \mathrm{d}F_1|_M \,,\, \mathrm{d}F_2|_M \,]_{A_+^*} &= \mathrm{d}\kappa^{\mathsf{t}}\, \mathrm{d}\{\, F_1 \,,\, F_2 \,\}_{\tilde{\Pi}}|_M = \mathrm{d}\{\, \kappa^*F_1 \,,\, \kappa^*F_2 \,\}_{\kappa^\star\Pi_+}\big|_M \\ &= [\, \phi^{\mathsf{t}}\, \mathrm{d}F_1|_M \,,\, \phi^{\mathsf{t}}\, \mathrm{d}F_2|_M \,]_{\widetilde{A}_+^*} \end{aligned}$$

where the bracket in the second line is that induced by $\kappa^\star\Pi_+$. Since $\Gamma(A_+^*)$ are locally finitely generated over $\mathcal{C}^\infty(M)$ by differentials of functions in the vanishing ideal, it follows from the Leibniz rule(s) that ϕ^{t} preserves the bracket of arbitrary sections. □

CHAPTER 2

Double structures in Lie theory and Poisson geometry

Ehresmann's categorification of a groupoid is a groupoid object in the category of groupoids; this is a symmetric notion and it makes sense to regard such a structure as a "double groupoid". A double Lie groupoid is, essentially, a "Lie groupoid in the category of Lie groupoids"; one can apply the Lie functor to either object of a double Lie groupoid, to obtain an $\mathcal{LA}$-groupoid, i.e. a "Lie groupoid in the category of Lie algebroids". The application of the Lie functor can still be iterated; the result, a double Lie algebroid, is the best approximation to what one would mean as a "Lie algebroid in the category of Lie algebroids". Such double structures do arise in nature, especially from Poisson geometry and the theory of Poisson actions. Topological double groupoids arose naturally from homotopy theory [**8, 6**], while double Lie algebroids have found applications, for instance, in Lu's extension of Drinfel'd's work [**18**] on the classification of Poisson homogeneous spaces [**37**]. $\mathcal{LA}$-groupoids represent the intermediate object between double Lie groupoids and double Lie algebroids, and provide an equivalent characterization of the compatibility for Poisson bivector and group multiplication in a Poisson groupoid [**40, 42**]. Double Lie groupoids, $\mathcal{LA}$-groupoids and double Lie algebroids were introduced by Mackenzie [**40**], who also developed Lie theory "from double Lie groupoids to double Lie algebroids" [**40, 43**]. The relation between Poisson groupoids and double structures was foreseen by Weinstein in [**72**], where a program for the integration of Poisson groupoids to symplectic double groupoids was proposed, and further investigated by Mackenzie in [**42**].
Instances of Lie theory for the integration of double Lie algebroids to double Lie groupoids have appeared in literature. Karasëv noticed in [**26**] that to a 1-connected complete Poisson group one can always associate a symplectic double groupoid and Lu-Weinstein obtained an alternative construction in [**39**], which applies, under the same connectivity assumptions also in the noncomplete case. Recently, Liu and Parmentier studied in [**32**] a certain class of coboundary dynamical Poisson groupoids, introduced by Etingof and Varchenko [**19**] in relation with the classical dynamical Yang-Baxter equation, and produced an integrating symplectic double groupoid in a special case. Mackenzie and Xu's integration of Lie bialgebroids to Poisson groupoids can be equivalently regarded as the integration of the cotangent double (Lie algebroid) of a Lie bialgebroid [**41**] to a cotangent prolongation $\mathcal{LA}$-groupoid.

In this Chapter, after clarifying the notions of fibred products in the category of Lie groupoids and Lie algebroids in Section 2.1 (our techniques are borrowed from Higgins and Mackenzie [**24, 25**]), and introducing double Lie groupoids and $\mathcal{LA}$-groupoids in Section 2.2, we address the integration problem of an $\mathcal{LA}$-groupoid to a double Lie groupoid. In Section 2.3 the study of the integrability of fibred products of Lie algebroids, leads us to derive conditions for endowing the differentiable graph (always) integrating an $\mathcal{LA}$-groupoid with a further compatible multiplication making it a double Lie groupoid. Our integrability conditions are to be understood as Lie-algebroid-homotopy-lifting conditions along suitable morphisms of Lie algebroids. The main result of this Chapter is the following

Theorem (2.3.9). *Let Ω be an $\mathcal{LA}$-groupoid with integrable top Lie algebroid. If source and target of the top Lie groupoid of Ω are strongly transversal, there exists a unique vertically source 1-connected double Lie groupoid integrating Ω.*

In the final Section we specialize our results to the case of the $\mathcal{LA}$-groupoid canonically associated with a Poisson groupoid and show that, under our conditions, the integration always yields a symplectic double groupoid:

Theorem (2.4.10). *Let $(\mathcal{G}, \Pi) \rightrightarrows M$ be an integrable Poisson groupoid with weak dual Poisson groupoid $(\mathcal{G}^\star, \Pi^\star)$. If cotangent source and target map of $\mathcal{G}$ are strongly transversal, the symplectic groupoid $\mathcal{S}$ of $\mathcal{G}$ carries a further Lie groupoid making it a symplectic groupoid for $\mathcal{G}^\star$ and*

$$\begin{tikzcd}
\mathcal{S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}^\star \arrow[d, shift left] \arrow[d, shift right] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

a symplectic double groupoid.

We conclude this Chapter checking the integrability conditions in the case of complete Poisson groups, which allows for explicit computations, to extend Lu and Weinstein's result, in the complete case, by dropping all the connectivity assumptions on the Poisson group.

Theorem (2.4.12). *For any complete Poisson group G, the source 1-connected symplectic groupoid $\mathcal{S}$ of G carries a unique Lie groupoid structure over the 1-connected dual Poisson group $G^\star$ making it a symplectic groupoid for the dual Poisson structure and a double of G.*

We shall obtain further examples of integrable $\mathcal{LA}$-groupoids, arising from Poisson actions, in the next Chapter.

Notations and remarks on groupoid objects.

Let $\mathcal{G} \rightrightarrows M$ be a groupoid; the nerves of the underlying category, i.e. the strings of composable elements

$$
\begin{aligned}
\mathcal{G}^{(n)} &:= \{(g_0, \dots g_n) \in \mathcal{G}^{\times n} \mid \mathrm{t}(g_i) = \mathrm{s}(g_{i-1}) \,,\ i = 1, \dots n\} \qquad , \qquad n \geq 1, \\
\mathcal{G}^{(0)} &:= M
\end{aligned}
$$

can also be inductively defined as fibred products

$$
\mathcal{G}^{(n)} = \mathcal{G}\ {}_{\mathrm{s}}\!\times_{\mathrm{t}\circ\mathrm{p}_1} \mathcal{G}^{(n-1)} \ , \ \ n > 1 \ ,
$$

where $\mathrm{p}_1 : \mathcal{G}^{(n-1)} \to \mathcal{G}$ is the restriction of the first projection for all n. The groupoid compatibility conditions may be grouped in two sets of commuting diagrams: those for the graph compatibility

$$
\begin{tikzcd}
 & \mathcal{G} \arrow[dr, "\mathrm{s,t}"] & \\
M \arrow[ur, "\varepsilon"] \arrow[rr, "\mathsf{id}_M"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{G} \arrow[dr, "\iota"] & \\
\mathcal{G} \arrow[ur, "\iota"] \arrow[rr, "\mathsf{id}_{\mathcal{G}}"'] & & \mathcal{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M & \\
\mathcal{G} \arrow[ur, "\mathrm{s}"] \arrow[rr, "\iota"'] & & \mathcal{G} \arrow[ul, "\mathrm{t}"']
\end{tikzcd}
$$

and those for the multiplication,

$$
\begin{tikzcd}
 & \mathcal{G}^{(3)} \arrow[dl, "\mathsf{id}_{\mathcal{G}}\times\mu"'] \arrow[dr, "\mu\times\mathsf{id}_{\mathcal{G}}"] & \\
\mathcal{G}^{(2)} \arrow[dr, "\mu"'] & & \mathcal{G}^{(2)} \arrow[dl, "\mu"] \\
 & \mathcal{G} &
\end{tikzcd}
\qquad \textsf{associativity} \quad ,
$$

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathsf{id}_{\mathcal{G}}"] \arrow[d, "\Delta_{\mathcal{G}}"'] & \mathcal{G} \\
\Delta_{\mathcal{G}} \arrow[d, "\mathsf{id}_{\mathcal{G}}\times\mathrm{s}"'] & \mathcal{G}^{(2)} \arrow[u, "\mu"'] \\
\mathcal{G}\ {}_{\mathrm{s}}\!\times M \arrow[ur, "\mathsf{id}_{\mathcal{G}}\times\varepsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathsf{id}_{\mathcal{G}}"] \arrow[d, "\Delta_{\mathcal{G}}"'] & \mathcal{G} \\
\Delta_{\mathcal{G}} \arrow[d, "\mathrm{t}\times\mathsf{id}_{\mathcal{G}}"'] & \mathcal{G}^{(2)} \arrow[u, "\mu"'] \\
M \times_{\mathrm{t}} \mathcal{G} \arrow[ur, "\varepsilon\times\mathsf{id}_{\mathcal{G}}"'] &
\end{tikzcd}
\qquad \textsf{unitality} \quad ,
$$

and

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathrm{s}"] \arrow[d, "\Delta_{\mathcal{G}}"'] & M \arrow[d, "\varepsilon"] \\
\Delta_{\mathcal{G}} \arrow[d, "\iota\times\mathsf{id}_{\mathcal{G}}"'] & \mathcal{G} \\
\mathcal{G}^{(2)} \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathrm{t}"] \arrow[d, "\Delta_{\mathcal{G}}"'] & M \arrow[d, "\varepsilon"] \\
\Delta_{\mathcal{G}} \arrow[d, "\mathsf{id}_{\mathcal{G}}\times\iota"'] & \mathcal{G} \\
\mathcal{G}^{(2)} \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad \textsf{invertibility} \quad ,
$$

where we have denoted with $\Delta_{\mathcal{G}}$ both the diagonal in $\mathcal{G} \times \mathcal{G}$ and the diagonal map $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$. It makes therefore sense to consider groupoid objects in any small category with direct products and fibred products.

Definition 2.0.14. Let $\mathbf{C}$ be a small category with direct products and pullbacks. Consider objects $\boldsymbol{\mathcal{G}}, \mathbf{M} \in \mathrm{Obj}(\mathbf{C})$, such that the set underlying $\boldsymbol{\mathcal{G}}$ carries a groupoid structure over the set underlying $\mathbf{M}$, with structural maps $(\mathbf{s}, \mathbf{t}, \boldsymbol{\varepsilon}, \boldsymbol{\iota}, \boldsymbol{\mu})$; $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ is a groupoid object in $\mathbf{C}$ if all the structural maps are arrows in $\mathbf{C}$.

Note that, for any groupoid $\mathcal{G} \rightrightarrows M$, given the source map, the inversion map and the division map $\delta : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G} \to \mathcal{G}$, the remaining structural maps are recovered by considering the diagrams

$$
\begin{tikzcd}
 & \mathcal{G} \arrow[r, "\iota"] \arrow[dl, "\Delta_{\mathcal{G}}"'] & \mathcal{G} & \\
\Delta_{\mathcal{G}} \arrow[dr, "\mathrm{s}\times \mathsf{id}_{\mathcal{G}}"'] & & & \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G} \arrow[ul, "\delta"'] \\
 & M \times_{\mathrm{s}} \mathcal{G} \arrow[urr, "\varepsilon\times \mathsf{id}_{\mathcal{G}}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[rr, "\mathrm{t}"] \arrow[dr, "\iota"'] & & M \\
 & \mathcal{G} \arrow[ur, "\mathrm{s}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G}^{(2)} \arrow[rr, "\mu"] \arrow[dr, "\mathsf{id}_{\mathcal{G}}\times\iota"'] & & \mathcal{G} \\
 & \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{s}} \mathcal{G} \arrow[ur, "\delta"'] &
\end{tikzcd}
\quad . \tag{2.1}
$$

Thus, a groupoid $\mathcal{G} \rightrightarrows M$ could be alternatively defined in terms of the data $(\mathcal{G}, M, \mathrm{s}, \varepsilon, \delta)$, imposing the commutativity of the suitable diagrams; that is however lengthy and not illuminating. Nevertheless this description is quite handy when dealing with groupoid objects.
According to the above discussion, we have

Proposition 2.0.15. *Let* $\mathbf{C}$ *a small category with direct products and fibred products. For any objects* $\boldsymbol{\mathcal{G}}, \mathbf{M} \in \mathrm{Obj}(\mathbf{C})$ *a groupoid* $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ *is a groupoid object in* $\mathbf{C}$ *iff the source map, the unit section and the division map of* $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ *are arrows in* $\mathbf{C}$.

PROOF. If $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ is a groupoid object in $\mathbf{C}$, the division map $\boldsymbol{\delta} := \boldsymbol{\mu} \circ (\mathsf{id}_{\boldsymbol{\mathcal{G}}} \times \boldsymbol{\iota})$ is also an arrow. For the opposite implication use the diagrams (2.1) in sequence to recover $\boldsymbol{\iota}$, $\mathbf{t}$ and $\boldsymbol{\mu}$ in terms of compositions of arrows. $\square$

If $\mathbf{C}$ does not possess general fibred products, the same idea applies, provided the relevant fibred products exist.

Example 2.0.16. A Poisson groupoid $(\mathcal{G}, \Pi) \rightrightarrows M$ is *not* a groupoid object in the category of Poisson manifolds, for many reasons. First of all the target map is anti-Poisson for the induced Poisson structure on the base manifold and the inversion map is always anti-Poisson. Enlarging the category of Poisson manifolds by adding the anti-Poisson maps to the space of arrows does not solve the problem, since the domain of the groupoid multiplication is not canonically endowed with a Poisson structure.

Example 2.0.17. For any group G the total space of the fundamental groupoid $\Pi(G) \rightrightarrows G$ also carries a group structure induced by pointwise multiplication of paths in G. For such structures it is a groupoid object in the category of groups, since it is also possible to pointwise multiply homotopies, therefore pointwise multiplication of homotopy classes is well defined and commutes with concatenation up to reparametrization. Note that the space of composable pairs $\Pi(G)^{(2)}$ is described by all triples $([l], l(0) = g = r(1), [r])$, where $[l]$ and $[r]$ are homotopy classes of paths in G; setting for all $([l_\pm], g_\pm, [r_\pm]) \in \Pi(G)^{(2)}$

$$([l_+], g_+, [r_+]) \cdot ([l_-], g_-, [r_-]) := ([l_+ \cdot l_-], g_+ g_-, [r_+ \cdot r_-]) \qquad ,$$

where homotopy classes are multiplied pointwise for any choice of representatives, yields a well defined group multiplication. Similarly are defined the groups on the relevant fibred products.

2.1. Fibred products of Lie groupoids and Lie algebroids

In this technical Section we clarify the notions of fibred products in the category of Lie groupoids and of Lie algebroids, showing that they exist under the natural transversality conditions. In the case of Lie groupoids, the proof of existence is essentially independent of the groupoid multiplication. For this reason, and for later purposes, we introduce here differentiable graphs, namely "Lie groupoids without a multiplication" and study the construction of fibred products in their category.

Definition 2.1.1. A (differentiable) graph on a pair of manifolds (Γ, M) is given by an immersion $\varepsilon : M \to \Gamma$ and a pair of submersions $\mathrm{s}, \mathrm{t} : \Gamma \to M$, which are both left inverses to ε: $\mathrm{s} \circ \varepsilon = \mathsf{id}_M$ and $\mathrm{t} \circ \varepsilon = \mathsf{id}_M$[1].

Clearly Lie groupoids are differentiable graphs in the sense of the above definition, hence so are vector bundles and Lie algebroids for the underlying abelian groupoids; thus we shall use the same terminology for graphs as for Lie groupoids: for any differentiable graph (Γ, M) we shall call $\varepsilon : M \to \Gamma$ the unit section and $\mathrm{s} : \Gamma \to M$, respectively $\mathrm{t} : \Gamma \to M$, the source map, respectively the target map. The natural notion of a morphism of graphs $(\Gamma', M') \to (\Gamma, M)$, namely a pair of smooth maps $\varphi : \Gamma' \to \Gamma$ and $f : M' \to M$ such that

$$\varepsilon \circ f = \varphi \circ \varepsilon' \qquad \mathrm{s} \circ \varphi = f \circ \mathrm{s}' \qquad \mathrm{t} \circ \varphi = f \circ \mathrm{t}' \qquad ,$$

makes the set of differentiable graphs a category with direct products. For any differentiable graph (Γ, M), a pair of submanifolds $\Gamma^o \subset \Gamma$ and $M^o \subset M$ is called a

[1] Pradines [**58**] uses the term *differentiable graph*, when only a pair of submersions is given; similar objects are called bisubmersions in [**2**]. We chose to use "*graph*" for the sake of economy.

(differentiable) subgraph if source and target of Γ restrict to submersions $\Gamma^o \to M^o$; a subgraph shall be called wide if $M^o = M$.
Fibred products of differentiable graphs and preimages of differentiable subgraphs under morphisms of graphs (in particular kernels, to be defined below) are not differentiable graphs in general; however they are, under natural transversality conditions.
Recall (for example, from [**1, 31, 14**]) that a smooth map $f : N \to M$ is transversal to a submanifold $M^o \subset M$ if the tangent map induces a surjection $T_pN \to T_{f(p)}M/T_{f(p)}M^o$ for all $p \in N$, equivalently if

$$\mathrm{d}fT_pN + T_{f(p)}M^o = T_{f(q)}M \qquad , \qquad p \in N \qquad .$$

If $f : N \to M$ is transversal to $M^o \subset M$, we shall write $f \pitchfork M^o$; in that case $f^{-1}(M^o) \subset N$ is a submanifold and

$$T_{p_o}f^{-1}(M^o) = (\mathrm{d}f)^{-1}_{p_o}T_{f(p_o)}M^o \qquad , \qquad p_o \in f^{-1}(M^o) \qquad .$$

Note that transversality is not a necessary condition for $f^{-1}(M^o)$ to be a submanifold; however, last formula holds true whenever $f^{-1}(M^o)$ is a smooth submanifold.

Example 2.1.2. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (xy, 0)$ and $M^o = \{x \neq 0, y = 0\}$: $f^{-1}(M^o) = \mathbb{R}^2\backslash\{xy = 0\} \subset \mathbb{R}^2$ is a smooth submanifold even if f is not transversal to M^o,

$$\mathrm{d}f_{(x,y)}T_{(x,y)}\mathbb{R}^2 = T_{xy}\mathbb{R} \oplus \{0_y\} = T_{xy}M^0 \neq \mathbb{R}^2 \qquad ,$$

and $Tf^{-1}(M^o) = T_{f^{-1}(M^o)}\mathbb{R}^2 = (\mathrm{d}f)^{-1}TM^0$.

Two smooth maps $f_{1,2} : M^{1,2} \to M$ are said transversal, and we shall write $f_1 \pitchfork f_2$, if $f_1 \times f_2 : M^1 \times M^2 \to M \times M$ is transversal to the diagonal $\Delta_M \subset M \times M$, equivalently

$$\mathrm{d}f_1T_{q_1}M^1 + \mathrm{d}f_2T_{q_2}M^2 = T_qM \qquad ,$$

for all $(q_1, q_2) \in M^{12}$, $q = f_{1,2}(q_{1,2})$. In that case the fibred product $M^{12} := M^1 {}_{f_1}\!\times_{f_2} M^2 \equiv (f_1 \times f_2)^{-1}(\Delta_M) \subset M^1 \times M^2$ is a smooth submanifold and

$$T_{(q_1,q_2)}M^{12} = T_{q_1}M^1 {}_{\mathrm{d}f_1}\!\times_{\mathrm{d}f_2} T_{q_1}M^1 \qquad , \qquad (q_1, q_2) \in M^{12} \qquad ,$$

where the fibred product on the right hand side is taken in the category of vector spaces.

Proposition 2.1.3. *Let $\varphi_{1,2} : \Gamma^{1,2} \to \Gamma$ be morphisms of graphs over $f_{1,2} : M^{1,2} \to M$ such that $\varphi_1 \pitchfork \varphi_2$ and $f_1 \pitchfork f_2$ (so that the fibred products $\Gamma^{12} := \Gamma^1 {}_{\varphi_1}\!\times_{\varphi_2} \Gamma^2 \subset \Gamma^1 \times \Gamma^2$ and $M^{12} := M^1 {}_{\varphi_1}\!\times_{\varphi_2} M^2 \subset M^1 \times M^2$ are smooth submanifolds). Then (Γ^{12}, M^{12}) is a subgraph of the direct product $(\Gamma^1 \times \Gamma^2, M^1 \times M^2)$ iff the* source transversality condition

$$\mathrm{d}\varphi_1T^{\mathrm{s}_1}_{x_1}\Gamma^1 + \mathrm{d}\varphi_2T^{\mathrm{s}_2}_{x_2}\Gamma^2 = T^{\mathrm{s}}_q\Gamma \tag{2.2}$$

holds for all $(x_1, x_2) \in \Gamma^{12}$ and $x = \varphi_{1,2}(x_{1,2})$.

Note that condition (2.2) above is equivalent to asking all the restrictions of ϕ_1 and ϕ_2 to the source fibres to be transversal, whenever they map to the same source fibre of Γ.

Proof. Since $\mathrm{s}_{12} : (\mathrm{s}_1, \mathrm{s}_2)|_{\Gamma^{12}}$ takes values in M^{12} it suffices to check its submersivity, the same argument applies to t_{12}. Applying the snake lemma (see, for instance, [**69**]) to the exact commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & T^{\mathrm{s}_1}_{x_1}\Gamma^1 \oplus T^{\mathrm{s}_2}_{x_2}\Gamma^2 \arrow[r] \arrow[d] & T_{x_1}\Gamma^1 \oplus T_{x_2}\Gamma^2 \arrow[r, "\mathrm{ds}_1 \oplus \mathrm{ds}_2"] \arrow[d, "\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2"'] & T_{q_1}M^1 \oplus T_{q_2}M^2 \arrow[r] \arrow[d, "\mathrm{d}f_1 - \mathrm{d}f_2"'] & 0 \\
0 \arrow[r] & T^{\mathrm{s}}_x M \arrow[r] & T_x\Gamma \arrow[r, "\mathrm{ds}"] & T_q M \arrow[r] & 0
\end{tikzcd}$$

yields the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{ker}\ (\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)|_{T^{\mathrm{s}_1}_{x_1}\Gamma^1 \oplus T^{\mathrm{s}_2}_{x_2}\Gamma^2} \arrow[r] & T_{(g_1,g_2)}\Gamma^{12} \arrow[r] & T_{(q_1,q_2)}M^{12} \arrow[dll] & \\
& \mathsf{coker}\ (\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)|_{T^{\mathrm{s}_1}_{x_1}\Gamma^1 \oplus T^{\mathrm{s}_2}_{x_2}\Gamma^2} \arrow[r] & C_\Gamma \arrow[r] & C_M \arrow[r] & 0\,,
\end{tikzcd} \tag{2.3}$$

where the rightmost arrow on the top row is the restriction of $\mathrm{ds}_1 \times \mathrm{ds}_2$ and the cokernels $C_\Gamma = \mathsf{coker}\ (\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)$ and $C_M = \mathsf{coker}\ (\mathrm{d}f_1 - \mathrm{d}f_2)$ vanish. $\square$

By specializing last result we obtain transversality conditions for preimages of differentiable subgraphs under morphisms of graphs to be also differentiable subgraphs.

Corollary 2.1.4. *Let $\varphi : \Gamma' \to \Gamma$ be a morphism of differentiable graphs over $f : M' \to M$ and $(\Gamma^o, M^o) \subset (\Gamma, M)$ a subgraph such that $f \pitchfork M^o$ and $\phi \pitchfork \Gamma^o$ (so that $f^{-1}(M^o) \subset M'$ and $\phi^{-1}(\Gamma^o) \subset \Gamma'$ are smooth submanifolds). Then $(\phi^{-1}(\Gamma^o), f^{-1}(M^o)) \subset (\Gamma', M')$ is a differentiable subgraph iff*

$$\mathrm{d}\varphi T^{\mathrm{s}'}_{q'}\Gamma' + T^{\mathrm{s}_o}_{f(q')}\Gamma^o = T^{\mathrm{s}}_{f(q')}\Gamma$$

for all $q' \in f^{-1}(M^o)$.

Proof. Identify $\varphi^{-1}(\Gamma^o)$ with the fibred product $\Gamma^1 {}_{\varphi}\!\times_{\iota_o} \Gamma^o$ for the inclusion $\iota_o : \Gamma^o \to \Gamma$. $\square$

For any morphism $\varphi : \Gamma' \to \Gamma$ of differentiable graphs over $f : M' \to M$, we shall say that the preimage $\mathsf{ker}\,\varphi := (\varphi^{-1}(\varepsilon(M)), M)$ of the trivial wide subgraph of (Γ, M) is the kernel of (φ, f). The transversality condition of last corollary on f is trivial and the source transversality condition reduces to

$$\mathrm{d}\varphi T^{\mathrm{s}'}_{q'}\Gamma' = T^{\mathrm{s}}_{f(q')}\Gamma \qquad , \qquad q' \in M' \qquad . \tag{2.4}$$

We shall say that a morphism of graphs satisfying (2.4) is $\mathsf{source\ submersive}$. If φ is *not* transversal to $\varepsilon(M)$ but $\mathsf{ker}\,\varphi$ is however smooth, it could still happen to be a subgraph.

Lemma 2.1.5. *Let $\varphi : \Gamma' \to \Gamma$ be a morphism of differentiable graphs over $f : M' \to M$, such that $\varphi^{-1}(\varepsilon(M)) \subset \Gamma'$ is a smooth submanifold, then $\mathsf{ker}\,\varphi$ is a smooth graph iff*

$$\mathrm{d}\varphi T^{\mathrm{s}'}_{q'}\Gamma \simeq \mathrm{d}\varphi T_{f(q)}\Gamma / \mathrm{d}\varepsilon \mathrm{d}f(T_{q'}M') \tag{2.5}$$

for all $q' \in \varphi^{-1}(\varepsilon(M))$, $f(q') = q$.

Proof. The long exact sequence in the proof of proposition 2.1.3 takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{ker}\ \mathrm{d}\varphi|_{T^{\mathrm{s}'}_{q'}\Gamma'} \arrow[r] & T_{g'}\mathsf{ker}\ \varphi \arrow[r] & T_{q'}M' \arrow[dll] & \\
& \mathsf{coker}\ \mathrm{d}\varphi|_{T^{\mathrm{s}'}_{q'}\Gamma'} \arrow[r] & C_\Gamma \arrow[r] & C_M \arrow[r] & 0
\end{tikzcd} \quad ,$$

where the cokernels now are $C_M = 0$, $C_\Gamma = T_{f(q)}\Gamma/(\mathrm{d}\varphi T_{q'}\Gamma' + \mathrm{d}\varepsilon(T_qM)) \simeq T^{\mathrm{s}}_{f(q)}\Gamma/(\mathrm{d}\varphi T_{q'}\Gamma'/\mathrm{d}\varepsilon\mathrm{d}f(T'_qM'))$ and $\mathsf{coker}\ \mathrm{d}\varphi|_{T^{\mathrm{s}'}_{q'}\Gamma'} = T^{\mathrm{s}}_{f(q)}\Gamma/\mathrm{d}\varphi T^{\mathrm{s}'}_{q'}\Gamma'$. ☐

A similar argument could be applied to the long exact sequence for fibred products of arbitrary graphs, but the resulting condition is not illuminating, since the cokernels do not have, in general, a nice description.

Example 2.1.6. Consider the action of S^1 on $\mathbb{R}^2$ by rotation. The anchor of the action groupoid $S^1 \ltimes \mathbb{R}^2$ is a morphism of differentiable graphs to the pair groupoid $\mathbb{R}^2 \times \mathbb{R}^2$, which is not source submersive. Its kernel is the isotropy groupoid, whose fibres are S^1 over $(x, y) = 0$ and $\{1\}$ over $(x, y) \neq 0$, hence not a differentiable subgraph.

Example 2.1.7. Consider the action of $\mathbb{R}$ on the cylinder $S^1 \times \mathbb{R}$ by rotation. The action groupoid $\mathbb{R} \ltimes (S^1 \times \mathbb{R})$ is a graph with differentiable kernel $\cup_{k\in\mathbb{Z}}\{2\pi k\} \times S^1 \times \mathbb{R}$ for the groupoid anchor $\chi : \mathbb{R} \ltimes (S^1 \times \mathbb{R}) \to S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}$, $(x; \theta, h) \mapsto (x+\theta, h; \theta, h)$. The anchor χ is not transversal to the diagonal, nor it is source submersive, but

$$\begin{array}{rcl}
\mathrm{d}\chi\, T_{(2\pi k,\theta,h)}\mathbb{R} \ltimes (S^1 \times \mathbb{R}) & \simeq & \mathbb{R} \oplus 0 \oplus 0 \oplus 0 + \Delta_{\mathbb{R}\oplus\mathbb{R}} \\
& = & \mathrm{d}\chi T^{\mathrm{s}}_{(2\pi k,\theta,h)}\mathbb{R} \ltimes (S^1 \times \mathbb{R}) + \Delta_{T_{(\theta,h)}S^1\times\mathbb{R}}
\end{array} \quad ,$$

that is, condition (2.5) is satisfied.

Fibred products of Lie groupoids are naturally endowed with a groupoid structure, therefore they stay in the category, provided the transversality conditions for the underlying graphs hold. The groupoid multiplication allows reducing the source transversality condition to a requirement on the induced Lie algebroids.

Theorem 2.1.8. *Let $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{G}$ be morphisms of Lie groupoids over $f_{1,2} : M^{1,2} \to M$ such that $\varphi_1 \pitchfork \varphi_2$ and $f_1 \pitchfork f_2$ (so that the fibred products $\mathcal{G}^{12} := \mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2 \subset \mathcal{G}^1 \times \mathcal{G}^2$ and $M^{12} := M^1 {}_{\varphi_1}\!\times_{\varphi_2} M^2 \subset M^1 \times M^2$ are smooth submanifolds). Then $(\mathcal{G}^{12}, M^{12})$ is a Lie subgroupoid of the direct product $(\mathcal{G}^1 \times \mathcal{G}^2, M^1 \times M^2)$ iff the source transversality condition*

$$\mathrm{d}\varphi_1 T^{\mathrm{s}_1}_{g_1}\mathcal{G}^1 + \mathrm{d}\varphi_2 T^{\mathrm{s}_2}_{g_2}\mathcal{G}^2 = T^{\mathrm{s}}_g\mathcal{G} \tag{2.6}$$

holds for all $(g_1, g_2) \in \mathcal{G}^{12}$ and $g = \varphi_{1,2}(g_{1,2})$, equivalently iff

$$\phi_1 A^1_{q_1} + \phi_2 A^2_{q_2} = A_q \tag{2.7}$$

for all $(q_1, q_2) \in M^{12}$ and $q = \varphi_{1,2}(q_{1,2})$, where $A^{1,2}$, A are the Lie algebroids of $\mathcal{G}^{1,2}$, $\mathcal{G}$ and $\phi_{1,2} : A^{1,2} \to A$ the morphisms of Lie algebroids induced by $\varphi_{1,2}$.

Proof. It remains to show that the two transversality conditions are equivalent. Evaluating (2.6) on $g_{1,2} = \varepsilon_{1,2}(q_{1,2})$ yields (2.7). Conversely, by equivariance under right translation, we have, for all $(g_1, g_2) \in \mathcal{G}^{12}$,

$$\begin{aligned} \mathrm{d}\varphi_1 T^{\mathrm{s}_1}_{g_1}\mathcal{G}^1 + \mathrm{d}\varphi_2 T^{\mathrm{s}_2}_{g_2}\mathcal{G}^2 &= \mathrm{d}r^{-1}_{\varphi_1(g_1)}\phi_1 A^1_{\mathrm{t}_1(g_1)} + \mathrm{d}r^{-1}_{\varphi_2(g_2)}\phi_2 A^2_{\mathrm{t}_2(g_2)} \\ &= \mathrm{d}r^{-1}_g(\phi_1 A^1_{\mathrm{t}_1(g_1)} + \phi_2 A^2_{\mathrm{t}_2(g_2)}) \end{aligned}$$

and $T^{\mathrm{s}}_g\mathcal{G} = \mathrm{d}r^{-1}_g A_{\mathrm{t}(g)}$, for $g = \varphi_{1,2}(g_{1,2})$. Then (2.6) follows from (2.7). ☐

Stronger transversality conditions for the existence of fibred products were given in [**45**]. Note that the source transversality condition (2.6) and the infinitesimal condition (2.7) are equivalent, even dropping the transversality conditions on $\varphi_{1,2}$ and $f_{1,2}$. We state a special case of last result for further reference.

Corollary 2.1.9. *Let $\varphi : \mathcal{G}' \to \mathcal{G}$ be a morphism of Lie groupoids over $f : M' \to M$ and $(\mathcal{G}^o, M^o) \subset (\mathcal{G}, M)$ a Lie subgroupoid such that $f \pitchfork M^o$ and $\phi \pitchfork \mathcal{G}^o$ (so that $f^{-1}(M^o) \subset M'$ and $\phi^{-1}(\mathcal{G}^o) \subset \mathcal{G}'$ are smooth submanifolds). Then $(\phi^{-1}(\mathcal{G}^o), f^{-1}(M^o)) \subset (\mathcal{G}', M')$ is a Lie subgroupoid iff*

$$\mathrm{d}\varphi T^{\mathrm{s}'}_{g'}\mathcal{G}' + T^{\mathrm{s}_o}_{f(g')}\mathcal{G}^o = T^{\mathrm{s}}_{f(q')}\mathcal{G} \tag{2.8}$$

for all $g' \in \varphi^{-1}(\mathcal{G}^o)$, equivalently

$$\phi A'_{q'} + A^o_{f(q')} = A_{f(q')}$$

for all $q' \in f^{-1}(M^o)$, where A, A' and A^0 are the Lie algebroids of $\mathcal{G}'$, $\mathcal{G}$ and $\mathcal{G}^o$.

In view of theorem 2.1.8 and corollary 2.1.9 we shall say that a morphism of Lie groupoids $\varphi : \mathcal{G}' \to \mathcal{G}$ over $f : M' \to M$ is transversal to a Lie subgroupoid $\mathcal{G}^o \rightrightarrows M^o$ of $\mathcal{G} \rightrightarrows M$ if $\phi \pitchfork \mathcal{G}^o$, $f \pitchfork M^o$ and the source transversality condition (2.8) holds. Similarly, we shall say that $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{G}$ are transversal morphisms of Lie groupoids over $f_{1,2} : M^{1,2} \to M$ if $\varphi_1 \times \varphi_2 : \mathcal{G}^1 \times \mathcal{G}^2 \to \mathcal{G} \times \mathcal{G}$ is transversal to the diagonal subgroupoid $\Delta_{\mathcal{G}} \rightrightarrows \Delta_M$ of the direct product groupoid $\mathcal{G} \times \mathcal{G}$, equivalently if $\varphi_1 \pitchfork \varphi_2$, $f_1 \pitchfork f_2$ and the source transversality condition (2.6) holds.

Remark 2.1.10. If a morphism of Lie groupoids $\varphi : \mathcal{G}' \to \mathcal{G}$ over $f : M' \to M$ is transversal to the trivial subgroupoid $\varepsilon(M) \subset \mathcal{G}$, its kernel $\mathsf{ker}\,\varphi = \varphi^{-1}(\varepsilon(M)) \subset \mathcal{G}$ is Lie subgroupoid iff ϕ is source submersive in the sense of (2.4). If φ is *not* transversal to $\varepsilon(M)$, $\mathsf{ker}\,\varphi$ could be a Lie subgroupoid even if φ is not source submersive: e.g. for any Lie group G consider $\varphi : G \to G$, $g \mapsto e$. In this case lemma 2.1.5 gives a necessary and sufficient condition for a smooth kernel groupoid to be a Lie subgroupoid.

Next we shall consider preimages and fibred products in the category of Lie algebroids. Differently from the case of Lie groupoids, preimages of Lie subalgebroids under morphisms always carry a Lie algebroid structure, provided they are smooth vector bundles.

Proposition 2.1.11. *Let $A \to M$ and $A' \to M'$ be Lie algebroids and $B \to N$ a Lie subalgebroid of $A \to M$. Consider a morphism of Lie algebroids $\phi : A' \to A$ over $f : M' \to M$ such that $f \pitchfork N$ and $\phi^{-1}(B) \to f^{-1}(N)$ is a smooth vector subbundle. Then $\phi^{-1}(B) \subset A'$ is a Lie subalgebroid.*

Proof. First of all note that the anchor $\rho' : A' \to TM'$ restricts to a bundle map $\phi^{-1}(B) \to Tf^{-1}(N)$, since, for all $b' \in \phi^{-1}(B)$, $\mathrm{d}f\rho'(b') = \rho_B(\phi(b')) \in TN$, for the anchor ρ_B on B induced by the anchor of A; we can then check the condition 2. of lemma 1.2.6 (see also remark 1.2.7) to show that $\phi^{-1}(B) \subset A'$ is a Lie subalgebroid. The condition is local and we can restrict to coordinate charts U of M' and V of M, such that $f(U) \subset V$; setting $U_o := U \cap f^{-1}(N)$, $V_o := V \cap N$ and $f_o := f|_{U_o}$ we can always assume that $U_o \subset U$ and $V_o \subset V$ are smooth submanifolds, since $f \pitchfork N$. Upon restriction, fix trivializing frames in duality $\{e^\alpha\}$ for A and $\{e_\alpha\}$ for A^* over U. For any $\beta \in \Gamma(U, f^+A)$ such that $\beta|_{U_o} = o$,

$$\beta = \sum_\alpha \langle\, e_\alpha \circ f\,,\, \beta\,\rangle (e^\alpha \circ f) = \sum_\alpha \beta_\alpha (e^\alpha \circ f)$$

yields a decomposition $\{\beta_\alpha \otimes e^\alpha\}$ in $\mathcal{C}^\infty(U) \otimes_{\mathcal{C}^\infty(V)} \Gamma(V, A)$ with $\beta_\alpha|_{U_o} = 0$. Let $a^{1,2} \in \Gamma(U, A)$ be local sections such that $a^{1,2}|_{U_o} \in \Gamma(U, \phi^{-1}(B))$, i.e. $\phi \circ a^{1,2}|_{U_o} \in \Gamma(U_o, f_o^+(B))$ and pick decompositions

$$\phi \circ a^{1,2}|_{U_o} = \sum_{i_{1,2}} u^{1,2}_{i_{1,2}} (b^{1,2}_{i_{1,2}} \circ f_o)$$

with $\{u^{1,2}_{i_{1,2}}\} \subset \mathcal{C}^\infty(U_o)$ and $\{b^{1,2}_{i_{1,2}}\} \subset \Gamma(V_o, B)$; for any choice of extensions $\{\widetilde{u}^{1,2}_{i_{1,2}}\} \subset \mathcal{C}^\infty(U)$ of $\{u^{1,2}_{i_{1,2}}\} \subset \mathcal{C}^\infty(U_o)$ and $\{\widetilde{b}^{1,2}_{i_{1,2}}\} \subset \Gamma(V, A)$ of $\{b^{1,2}_{i_{1,2}}\} \subset \Gamma(V_o, B)$,

$$\left.\left(\phi \circ a^{1,2} - \sum_{i_{1,2}} \widetilde{u}^{1,2}_{i_{1,2}} (\widetilde{b}^{1,2}_{i_{1,2}} \circ f)\right)\right|_{U_o} = 0$$

therefore there exist decompositions

$$\phi \circ a^{1,2} = \sum_{i_{1,2}} \widetilde{u}^{1,2}_{i_{1,2}} (\widetilde{b}^{1,2}_{i_{1,2}} \circ f) + \sum_{l_{1,2}} v^{1,2}_{l_{1,2}} (d^{1,2}_{l_{1,2}} \circ f) \tag{2.9}$$

with $\{d^{1,2}_{l_{1,2}}\} \subset \Gamma(V, A)$ and $\{v^{1,2}_{l_{1,2}}\} \subset \mathcal{C}^\infty(U)$, with the property that $v^{1,2}_{l_{1,2}}|_{U_o} = 0$ for all $l_{1,2}$. Evaluating the bracket compatibility condition for ϕ on the decompositions (2.9), yields

$$\begin{aligned}\phi \circ [\, a^1 \,,\, a^2 \,] = & \sum_{i_1, i_2} \widetilde{u}^1_{i_1} \widetilde{u}^2_{i_2} ([\, \widetilde{b}^1_{i_1}, \widetilde{b}^2_{i_2} \,] \circ f) + \sum_{i_2} \rho'(a^1)(\widetilde{u}^{i_2}_2)(\widetilde{b}^2_{i_2} \circ f) - \sum_{i_1} \rho'(a^2)(\widetilde{u}^{i_1}_1)(\widetilde{b}^1_{i_1} \circ f) \\ & + \sum_{l_1, l_2} v^1_{l_1} v^2_{l_2} ([\, d^1_{l_1}, d^2_{l_2} \,] \circ f) + \sum_{i_1, l_2} \widetilde{u}^1_{i_1} v^2_{l_2} ([\, \widetilde{b}^1_{i_1}, d^2_{l_2} \,] \circ f) + \sum_{l_1, i_2} v^1_{l_1} \widetilde{u}^2_{i_2} ([\, d^1_{l_1}, \widetilde{b}^2_{i_2} \,] \circ f) \\ & + \sum_{l_2} \rho'(a^1)(v^2_{l_2})(d^2_{l_2} \circ f) - \sum_{l_1} \rho'(a^2)(v^1_{l_1})(d^1_{l_1} \circ f) \qquad .\end{aligned}$$

Restricting last expression to U_o the terms in the second line vanish, since so do the v's, those on the third line also vanish since $\rho'(a^{1,2})|_{U_o}$ is tangent to U_o and $\mathrm{d}v^{l_{1,2}}_{1,2} \in \Gamma(N^* U_o)$; for the same reason the terms of the form $\rho(a)(\widetilde{u})$ do not depend on the choice of extension. Then

$$\begin{aligned}\phi \circ [\, a^1 \,,\, a^2 \,]|_{U_o} \;\; = & \;\; \sum_{i_1, i_2} u^1_{i_1} u^2_{i_2} ([\, \widetilde{b}^1_{i_1} \,,\, \widetilde{b}^2_{i_2} \,]|_{U_o} \circ f_o) + \sum_{i_2} \rho'(a_1)(u^{i_2}_2)(b^{i_2}_2 \circ f_o) \\ - & \;\; \sum_{i_1} \rho'(a_2)(u^{i_1}_1)(b^{i_1}_1 \circ f_o)\end{aligned}$$

takes values in $B|_{U_o}$, since the local sections $\{\widetilde{b}^1_{i_1}\}$ extend sections of $B|_{U_o}$ and $B \subset A$ is a Lie subalgebroid, thus $[\, \widetilde{b}^1_{i_1}, \widetilde{b}^2_{i_2} \,]|_{U_o} \in \Gamma(U_o, B)$. $\square$

Remark 2.1.12. Specializing last proposition to the trivial subalgebroid $B = M'$, we obtain that the kernel of a morphism of Lie algebroids $\phi : A' \to A$ is always a Lie algebroid, provided ϕ has constant rank.

The existence of fibred products of Lie algebroids under natural transversality conditions was stated without proof by Higgins and Mackenzie in [**24**] and can now be deduced from proposition 2.1.11.

Theorem 2.1.13. *Let $\phi_{1,2} : A^{1,2} \to A$ be morphisms of Lie algebroids over $f_{1,2} : M^{1,2} \to M$ such that $f_1 \pitchfork f_2$ and the fibred product $A^{12} := A^1 {}_{\varphi_1}\!\times_{\varphi_2} A^2 \to M^{12} := M^1 {}_{\varphi_1}\!\times_{\varphi_2} M^2$ is a smooth subbundle of the direct product $A^1 \times A^2 \to M^1 \times M^2$. Then $A^{12} \subset A^1 \times A^2$ is a Lie subalgebroid, hence a fibred product in the category of Lie algebroids.*

Proof. $(\phi_1 \times \phi_2) : A^1 \times A^2 \to B^{\times 2}$ is a morphism of Lie algebroids. To see this, consider that $\psi_{1,2} : A^1 \times A^2 \to B$, $\psi_{1,2} := \phi_{1,2} \circ \mathrm{pr}_{1,2}$ are morphisms of Lie algebroids, for the projections $\mathrm{pr}_{1,2} : A^1 \times A^2 \to A^{1,2}$ and $\phi_1 \times \phi_2$ is the unique morphism of vector bundles whose composition with $\mathrm{pr}_{1,2}$ give $\psi_{1,2}$. Since $B^{\times 2}$ is a direct product in the category of Lie algebroids there exists a unique morphism of Lie algebroids ϕ such that $\mathrm{pr}_{1,2} \circ \phi = \psi_{1,2}$, since it is in the category of vector bundles, $\phi = \phi_1 \times \phi_2$. The diagonal subbundle $\Delta_B \subset B^{\times 2}$ is the graph of the identity, it follows from proposition 2.1.11 and corollary 1.2.21 that the vector bundle $A^{12} = (\phi_1 \times \phi_2)^{-1}\Delta_B$ is a Lie subalgebroid of the direct product $A^1 \times A^2$. For the induced Lie algebroid structure, A^{12} is indeed a fibred product: let $\chi_{1,2} : C \to A^{1,2}$ be morphisms of Lie algebroids such that $\phi_1 \circ \chi_1 = \phi_2 \circ \chi_2$, then there exist a unique morphism of Lie algebroids $C \to A^1 \times A^2$ such that $\mathrm{pr}_{1,2} \circ \chi = \chi_{1,2}$; since $(\phi_1 \times \phi_2) \circ \chi = (\phi_1 \circ \chi, \phi_2 \circ \chi)$ takes values in Δ_B, χ takes values in A^{12}. That is, $\chi : C \to A^{12}$ lifts χ_1 and χ_2. Regarding any other $\chi' : C \to A^{12}$ with the same property as a morphism $C \to A^1 \times A^2$, shows that $\chi = \chi'$. ∎

Typically, the fibred product Lie algebroid A^{12} exists when $\phi_1 \pitchfork \phi_2$, $f_1 \pitchfork f_2$ and the infinitesimal linear transversality condition (2.7) holds. In that case we shall say that ϕ_1 and ϕ_2 are transversal morphism of Lie algebroids.

Example 2.1.14. For any morphism of Lie algebroids $\phi : A' \to A$ over $f : M' \to M$, the (ordinary) graph $\Gamma(\phi) \equiv A' {}_{\phi}\!\times_{\mathrm{id}_A} A$ always carries a Lie algebroid structure over $\Gamma(f) \equiv M' {}_{f}\!\times_{\mathrm{id}_M} M$, isomorphic to $A' \to M'$ by the projection $A' {}_{\phi}\!\times_{\mathrm{id}_A} A \to A'$; the identity $A \to A$ is transversal to any morphism of Lie algebroids $A' \to A$.

Example 2.1.15. For any map $f : N \to M$ the pullback algebroid $f^{++}A \to N$ of a Lie algebroid $A \to M$, provided it exists, is, by construction indeed, the fibred product $TN {}_{\mathrm{d}f}\!\times_{\rho} A$.

Remark 2.1.16. From the two equivalent transversality conditions of theorem 2.1.8 it is clear that a fibred product of Lie groupoids $\mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2$, when it is Lie, differentiates to a fibred product of Lie algebroids. In fact, the source transversality condition (2.6), implies that the induced morphisms $\phi_{1,2}$ of Lie algebroids are transversal in the sense of the definition above; therefore the associated fibred product of Lie algebroids $A^1 {}_{\phi_1}\!\times_{\phi_2} A^2$ exists and can be easily seen to coincide with the Lie algebroid of $\mathcal{G}^1 {}_{\varphi_1}\times_{\varphi_2} \mathcal{G}^2$ as a vector bundle. It also does as a Lie algebroid by uniqueness.

2.2. Double Lie groupoids and $\mathcal{LA}$-groupoids

We define in the first part of this Section double Lie groupoids, their morphisms and derive conditions for kernels of morphisms to be double Lie groupoids. Characterizing double Lie groupoids as suitably smooth groupoid objects in the category of Lie groupoids, leads us directly in the second part to their infinitesimal invariant, namely $\mathcal{LA}$-groupoids, i.e. suitably smooth groupoid objects in the category of Lie algebroids; we further obtain conditions for kernels of morphisms of $\mathcal{LA}$-groupoids to stay in the category.

2.2.1. Double Lie groupoids.

A double groupoid in the sense of Ehresmann

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{2.10}$$

is a groupoid object in the category of groupoids. The definition is symmetric. Assume that $\mathcal{D} \rightrightarrows \mathcal{V}$, is a groupoid object in the category of groupoids with structure maps $(\mathrm{s}_H,\ \mathrm{t}_H,\ \varepsilon_H,\ \iota_H,\ \mu_H)$. Then $\mathcal{D} \rightrightarrows \mathcal{H}$ and $\mathcal{V} \rightrightarrows M$ are groupoids, with structure maps $(\mathrm{s}_V,\ \mathrm{t}_V,\ \varepsilon_V,\ \iota_V,\ \mu_V)$ and $(\mathrm{s}_v,\ \mathrm{t}_v,\ \varepsilon_v,\ \iota_v,\ \mu_v)$, making $(\mathrm{s}_H,\ \mathrm{t}_H,\ \varepsilon_H,\ \iota_H,\ \mu_H)$ morphisms of groupoids. The base diagrams for the groupoid structure of $\mathcal{D} \rightrightarrows \mathcal{V}$ define a groupoid on $\mathcal{H}$ over M given by the base maps $(\mathrm{s}_h,\ \mathrm{t}_h,\ \varepsilon_h,\ \iota_h,\ \mu_h)$. The compatibility conditions for the structure maps of $\mathcal{D} \rightrightarrows \mathcal{V}$ to be morphisms of groupoids over those of $\mathcal{H} \rightrightarrows M$ with respect to the vertical groupoids are equivalent to the compatibility conditions for the structure maps of $\mathcal{D} \rightrightarrows \mathcal{H}$ to be morphisms over those of $\mathcal{V} \rightrightarrows M$ with respect to the vertical groupoids. For example the compatibility of $\mathrm{s}_H : \mathcal{D} \to \mathcal{V}$ with the vertical multiplications, namely $\mathrm{s}_H \circ \mu_V = (\mathrm{s}_H \times \mathrm{s}_H) \circ \mu_v$, is the compatibility of $\mu_V : \mathcal{D} {}_{\mathrm{s}_V}\times_{\mathrm{t}_V} \mathcal{D} \to \mathcal{D}$ with the horizontal source map; the other compatibility conditions can be read similarly in two ways. Then $\mathcal{D} \rightrightarrows \mathcal{H}$ is also a groupoid object in the category of groupoids.
We shall refer to the groupoid structures of a general double groupoid such as (2.10) as *top horizontal*, *top vertical*, *side horizontal* and *side vertical*, with the obvious meaning.

Definition 2.2.1. [**40**] A double Lie groupoid $\mathsf{D} := (\mathcal{D}, \mathcal{H}, \mathcal{V}, M)$ such as (2.10) is a double groupoid, such that $\mathcal{D} \rightrightarrows \mathcal{V}$, $\mathcal{D} \rightrightarrows \mathcal{H}$, $\mathcal{H} \rightrightarrows M$, $\mathcal{V} \rightrightarrows M$ are Lie groupoids and the double source map

$$\mathbb{S} \doteq (\mathrm{s}_V, \mathrm{s}_H) : \mathcal{D} \to \mathcal{H} \,{}_{\mathrm{s}_h}\!\times_{\mathrm{s}_v} \mathcal{V}$$

is submersive[2].

We remark that the double source submersivity is required to make the domains of the top multiplications, $\mathcal{D}^{(2)}_H \rightrightarrows \mathcal{V}^{(2)}$ and $\mathcal{D}^{(2)}_V \rightrightarrows \mathcal{H}^{(2)}$, Lie groupoids; thus, in particular, a double Lie groupoid is a groupoid object in the category of Lie groupoids. Let us introduce a special class of morphisms of Lie groupoids.

Definition 2.2.2. A morphism of Lie groupoids $\varphi : \mathcal{G}' \to \mathcal{G}$ over $f : M' \to M$ is called an $\mathcal{LG}$-fibration if

1. It is submersive and base submersive;

2. The characteristic map $(\varphi, \mathrm{s}') : \mathcal{G}' \longrightarrow \mathcal{G} \,{}_{\mathrm{s}}\!\times_f M'$ is submersive, equivalently φ is source submersive.

A strong $\mathcal{LG}$-fibration is an $\mathcal{LG}$-fibration such that

1'. It is surjective and base surjective;

2'. The characteristic map is surjective, equivalently φ is source surjective.

A strong $\mathcal{LG}$-fibration is a fibration of Lie groupoids in the sense of Higgins and Mackenzie [**25, 45**].
For a double groupoid (2.10) all of whose side and top groupoids are Lie groupoids, the following conditions are easily seen to be are equivalent

i) The double source map $\mathbb{S} \doteq (\mathrm{s}_V, \mathrm{s}_H) : \mathcal{D} \to \mathcal{H} \,{}_{\mathrm{s}_h}\!\times_{\mathrm{s}_v} \mathcal{V}$ is submersive,

ii) The top horizontal source map $\mathrm{s}_H : \mathcal{D} \to \mathcal{V}$ is an $\mathcal{LG}$-fibration,

iii) The top vertical source map $\mathrm{s}_V : \mathcal{D} \to \mathcal{H}$ is an $\mathcal{LG}$-fibration.

In particular, the top horizontal and vertical source maps of a double Lie groupoid are $\mathcal{LG}$-fibrations and the transversality conditions of theorem 2.1.8 to make the fibred products $\mathcal{D} \,{}_{\mathrm{s}_H}\!\times_{\mathrm{t}_H} \mathcal{D}$ and $\mathcal{D} \,{}_{\mathrm{s}_V}\!\times_{\mathrm{t}_V} \mathcal{D}$ Lie groupoids are met.
We list below the typical examples; more interesting ones shall be studied throughout the rest of this dissertation.

[2] In [**40**] the double source map is required to be also surjective; this condition does not really play a role in the study of the internal structure of a double Lie groupoid and the descent to double Lie algebroids. Moreover, there are interesting examples, such as Lu and Weinstein's double of a Poisson group (2.4.11) for instance, which do not fulfill the double source surjectivity condition.

Example 2.2.3. Double Lie groupoids

i) Any Lie groupoid $\mathcal{G} \rightrightarrows M$ is the top horizontal groupoid of a double Lie groupoid for the trivial groupoids on $\mathcal{G}$ and M as vertical groupoids.

ii) The pair double Lie groupoid

$$\begin{tikzcd} \mathcal{G}\times\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M\times M \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

of any Lie groupoid $\mathcal{G} \rightrightarrows M$ is defined by the pair groupoid on the top vertical edge and the direct product groupoid on the top horizontal edge.

iii) A double vector bundle in the sense of Ehresmann, i.e. a vector bundle in the category of vector bundles, is a double Lie groupoid for the abelian structures on the four sides.

iv) For any Lie group G,

$$\begin{tikzcd} \Pi(G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \bullet \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

is a double Lie groupoid, for the pointwise multiplication of homotopy classes of paths described in example 2.0.17.

Morphisms and subobjects of double Lie groupoids are defined in the obvious way.

Definition 2.2.4. Let $\mathsf{D}^{\pm} = (\mathcal{D}^{\pm}, \mathcal{H}^{\pm}, \mathcal{V}^{\pm}, M^{\pm})$ be double Lie groupoids; a morphism of double Lie groupoids $\Phi : \mathsf{D}^- \to \mathsf{D}^+$

$$\begin{tikzcd}[row sep=small] \mathcal{D}^- \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "\Phi"] & \mathcal{V}^- \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "\varphi_v"] & & \\ \mathcal{H}^- \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, "\varphi_h"'] & M^- \arrow[drr, "f"] & \mathcal{D}^+ \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V}^+ \arrow[d, shift left] \arrow[d, shift right] \\ & & \mathcal{H}^+ \arrow[r, shift left] \arrow[r, shift right] & M^+ \end{tikzcd} \tag{2.11}$$

is given by four maps $(\Phi, \varphi_h, \varphi_v, f)$ such that all sides of the diagram above are morphisms of Lie groupoids. D^- is called a double Lie subgroupoid if all of its side groupoids are Lie subgroupoids of the corresponding sides of D^+.

It is straightforward to see that, with this notion of morphism, double Lie groupoids form a category. We shall say that (Φ, φ_v), respectively (Φ, φ_h), is the top horizontal, respectively vertical, component of Φ and, similarly, that (ϕ_h, f), respectively (ϕ_v, f), is the side horizontal, respectively vertical, component. The following lemma gives a characterization of the kernel of a morphism of double Lie groupoids; it is not, in general, a double Lie groupoid. However, when the top components are suitably well behaved, the double Lie groupoid structure is inherited by the kernel.

Lemma 2.2.5. *For any morphism* (2.11) *of double Lie groupoids:*

i) The top vertical kernel $\mathcal{K}_V := \mathsf{ker}\,(\Phi, \varphi_h)$ *has a natural groupoid structure over the side vertical kernel* $\mathcal{K}_v := \mathsf{ker}\,(\varphi_v, f)$*, making*

$$\begin{array}{ccc} \mathcal{K}_V & \rightrightarrows & \mathcal{K}_v \\ \downdownarrows & & \downdownarrows \\ \mathcal{H} & \rightrightarrows & M \end{array} \tag{2.12}$$

a double subgroupoid;

ii) Assume that Φ *and* φ_v *are* $\mathcal{LG}$*-fibrations, respectively over* φ_h *and* f *(so that* $\mathcal{K}_V$ *and* $\mathcal{K}_v$ *are Lie subgroupoids for the vertical structures of* D*); then,* $\mathcal{K}_V \rightrightarrows \mathcal{K}_v$ *is a Lie groupoid iff*

$$(\Phi,\, \mathrm{s}_H{}^-) : \mathcal{D}^- \to \mathcal{D}^+ {}_{\mathrm{s}_{H^+}}\!\times_{\varphi_v} \mathcal{V}^- \tag{2.13}$$

is submersive;

iii) Under the hypotheses of (*ii*)*, the double groupoid* (2.12) *is a double Lie subgroupoid of* D *iff*

$$(\Phi,\, \mathrm{s}_V{}^-) : \mathcal{D}^- \to \mathcal{D}^+ {}_{\mathrm{s}_{V^+}}\!\times_{\varphi_h} \mathcal{H}^- \tag{2.14}$$

is an $\mathcal{LG}$*-fibration over*

$$(\varphi_h,\, \mathrm{s}_v) : \mathcal{V}^- \to \mathcal{V}^+ {}_{\mathrm{s}_{V^+}}\!\times_f M^- \tag{2.15}$$

Note that the fibred product Lie groupoid in (*iii*) is always well defined, since the top vertical source $\mathrm{s}_V{}^+$is an $\mathcal{LG}$-fibration, and $(\Phi,\, \mathrm{s}_V{}^-)$ is always a morphism of Lie groupoids over $(\varphi_h,\, \mathrm{s}_v^-)$.

Proof. (*i*) For any horizontally composable $k_{1,2} \in \mathcal{K}_V$

$$\begin{aligned} \Phi(\mu_H{}^-(k_1,k_2)) &= \mu_H{}^+(\Phi(k_1),\Phi(k_2)) \\ &= \mu_H{}^+(\varepsilon_V{}^+(\varphi_h(\mathrm{s}_V{}^-(k_1))),\varepsilon_V{}^+(\varphi_h(\mathrm{s}_V{}^-(k_1)))) \\ &= \varepsilon_V{}^+(\mu_h^+(\varphi_h(\mathrm{s}_V{}^-(k_1)),\varphi_h(\mathrm{s}_V{}^-(k_1)))) \qquad , \end{aligned}$$

thus the top horizontal multiplication restricts to $\mathcal{K}_V$; in the same way, it is easy to see that all the top horizontal maps of $\mathcal{D}^-$ restrict to $\mathcal{K}_V \rightrightarrows K_v$. (*ii*) Taking the long exact sequence for the diagram

$$\begin{tikzcd}
0 \arrow[r] & T_k\mathcal{K}_V \arrow[r] \arrow[d] & T_k\mathcal{D}^- \arrow[r, "\mathrm{d}\Phi"] \arrow[d, "\mathrm{d}\,\mathrm{s}_H{}^-"'] & T_{\Phi(k)}\mathcal{D}^+ \arrow[r] \arrow[d, "\mathrm{d}\,\mathrm{s}_H{}^+"'] & 0 \\
0 \arrow[r] & T_{\mathrm{s}_H{}^-(k)}\mathcal{K}_v \arrow[r] & T_{\mathrm{s}_H{}^-(k)}\mathcal{V}^- \arrow[r, "\mathrm{d}\varphi_v"] & T_{\varphi_v(\mathrm{s}_H{}^-(k))}\mathcal{V}^+ \arrow[r] & 0
\end{tikzcd}$$

yields $\mathsf{coker}\,\mathrm{d}\,\mathrm{s}_H{}^-|_{T\mathcal{K}_V} = 0$ iff $\mathsf{coker}\,\mathrm{d}\Phi|_{T^{\mathrm{s}_H{}^-}\mathcal{D}^-} = 0$, that is $\mathrm{s}_H : \mathcal{K}_V \to \mathcal{K}_v$ is submersive iff so is $(\Phi, \mathrm{s}_H{}^-)$. *(iii)* Consider that, for all $k \in \mathcal{K}_V$,

$$T_k^{\mathrm{s}_V{}^-}\mathcal{K}_V = T_k^{(\Phi,\,\mathrm{s}_V{}^-)}\mathcal{D}^- \qquad \text{and} \qquad T^{\mathrm{s}_v^-}_{\mathrm{s}_H{}^-(k)}\mathcal{K}_v = T^{(\varphi_v,\,\mathrm{s}_v^-)}_{\mathrm{s}_H{}^-(k)}\mathcal{V}^- \qquad ,$$

therefore the source submersivity condition on $(\Phi, \mathrm{s}_H{}^-)$ is equivalent to the submersivity condition on the double source map of (2.12). □

2.2.2. $\mathcal{LA}$-groupoids.

$\mathcal{LA}$-groupoids are groupoid objects in the category of Lie algebroids and the first order infinitesimal invariant of double Lie groupoids.

Definition 2.2.6. [**40**] An $\mathcal{LA}$-groupoid $\Omega := (\Omega, A, \mathcal{G}, M)$

$$\begin{tikzcd}
\Omega \arrow[r, shift left, "\widehat{}"] \arrow[r, shift right] \arrow[d] & A \arrow[d] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd} \tag{2.16}$$

is a groupoid object in the category of Lie algebroids, such that $\Omega \rightrightarrows A$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids and the double source map

$$\$ \doteq (\hat{\mathrm{s}}, \mathrm{Pr}) : \Omega \to A\,{}_{\mathrm{pr}}\!\times_{\mathrm{s}} \mathcal{G}$$

is surjective[3].

We shall refer to the Lie groupoid and Lie algebroid structures of an $\mathcal{LA}$-groupoid as top and side, with the obvious meaning and denote with $(\hat{\mathrm{s}}, \hat{\mathrm{t}}, \hat{\varepsilon}, \hat{\iota}, \hat{\mu})$ the top groupoid structural maps of a typical $\mathcal{LA}$-groupoid such as (2.16). Note that an $\mathcal{LA}$-groupoid is a double groupoid for the abelian groupoids on the side Lie algebroids and the double source map \$ should be understood with respect to this structure.
Before explaining the surjectivity condition on \$, which ensures that the domain of the top multiplication be a Lie algebroid indeed, let us introduce the infinitesimal analog of $\mathcal{LG}$-fibrations.

[3] Then it is also submersive, as it shall be clear from example 2.2.8 below.

Definition 2.2.7. A morphism of Lie algebroids $\phi : A' \to A$ over $f : M' \to M$ is called an $\mathcal{LA}$-fibration if

1. It is submersive and base submersive;

2. The characteristic map $(\phi, \mathrm{pr}') : A' \longrightarrow A\,{}_{\mathrm{pr}}\!\times_f M'$ is surjective, equivalently ϕ is fibrewise surjective.

A strong $\mathcal{LA}$-fibration is an $\mathcal{LA}$-fibration which is also base surjective, hence surjective.

A strong $\mathcal{LA}$-fibration is a fibration of Lie algebroids in the sense of Higgins and Mackenzie [**24, 45**]. For any vector bundles E and E', we shall say that a vector bundle map $E' \to E$ satisfying the conditions above is a (strong) $\mathcal{VB}$-fibration; such maps are $\mathcal{LA}$-fibrations for the zero algebroid structures.
Let us describe an obvious example for further reference.

Remark 2.2.8. A base submersive and fibrewise surjective vector bundle map ϕ over f is a $\mathcal{VB}$-fibration, since for any choice of trivializations the Jacobian $\mathsf{J}(\phi)$ has the form

$$\mathsf{J}(\phi) = \begin{pmatrix} \mathbb{F} & * \\ 0 & \mathsf{J}(f) \end{pmatrix} \tag{2.17}$$

for the matrix $\mathbb{F}$ representing Φ and therefore has maximal rank. From (2.17) is clear that a $\mathcal{VB}$-fibration is also fibrewise submersive.

The source surjectivity condition on the double source map of a $\mathcal{LA}$-groupoid is equivalent to asking the top source map $\hat{\mathrm{s}}$ to be fibrewise surjective; since it is also base submersive and base surjective, the top source map is a strong $\mathcal{LA}$-fibration and so is the top target map $\hat{\mathrm{t}} = \hat{\mathrm{s}} \circ \hat{\iota}$. In fact the following are equivalent:

i) The double source map $\$ \doteq : \Omega \to A\,{}_{\mathrm{pr}}\!\times_{\mathrm{s}} \mathcal{G}$ is surjective,

ii) The top source map $\hat{\mathrm{s}} : \Omega \to A$ is a strong $\mathcal{LA}$-fibration,

iii) The top vector bundle projection $\mathrm{Pr} : \Omega \to \mathcal{G}$ is a strong $\mathcal{LG}$-fibration.

It is then clear that condition (ii) above makes the domain of the top multiplication $\Omega\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}} \Omega$ of an $\mathcal{LA}$-groupoid (2.16) a Lie algebroid.

Example 2.2.9. $\mathcal{LA}$-groupoids

i) Any Lie groupoid $\mathcal{G} \rightrightarrows M$ is an $\mathcal{LA}$-groupoid for the rank zero algebroids on $\mathcal{G}$ and M. Any Lie algebroid $A \to M$ is an $\mathcal{LA}$-groupoid for the trivial Lie groupoid on A and M.

ii) The prototypical example of an $\mathcal{LA}$-groupoid is the tangent prolongation $\mathcal{LA}$-groupoid

$$\begin{tikzcd} T\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

of a Lie groupoid $\mathcal{G} \rightrightarrows M$. Tangent maps are always morphisms of Lie algebroids, then the tangent prolongation is a groupoid object. It is sufficient to check that the double source map is a surjective submersion; surjectivity is clear (pick a bisection) and submersivity follows from example (2.2.8).

iii) Any double vector bundle is an $\mathcal{LA}$-groupoid for the horizontal abelian groupoids and the vertical algebroids with zero anchor and bracket. In, general, an $\mathcal{LA}$-groupoid is a double Lie groupoid, replacing the Lie algebroid structures with abelian groupoids.

iv) Consider example (*ii*) in the special case of a Lie group G; the Lie algebroid $TG \to G$ of $\Pi(G) \rightrightarrows G$ is an $\mathcal{LA}$-*group*, for the tangent group on the top vertical side and the tangent Lie algebroid on the top vertical side.

To see that applying the Lie functor to a double Lie groupoid yields an $\mathcal{LA}$-groupoid, consider the following obvious lemma.

Lemma 2.2.10. *Let $\varphi : \mathcal{G}' \to \mathcal{G}$ over $f : M' \to M$ be an $\mathcal{LG}$-fibration, then the induced morphism of Lie algebroids $\phi : A' \to A$ is an $\mathcal{LA}$-fibration.*

Given any double Lie groupoid (2.10) the top horizontal source map $\mathrm{s}_H : \mathcal{D} \to \mathcal{V}$ differentiates to an $\mathcal{LA}$-fibration $\hat{\mathrm{s}} : A_V(\mathcal{D}) \to V$ from the Lie algebroid $A_V(\mathcal{D})$ of $\mathcal{D} \rightrightarrows \mathcal{H}$ to the Lie algebroid V of $\mathcal{V}$. Then differentiating the diagrams for the top horizontal groupoid $\mathcal{D} \rightrightarrows \mathcal{V}$, yields the Lie groupoid $A_V(\mathcal{D}) \rightrightarrows V$, which is also a groupoid object in the the category of Lie algebroids with surjective double source map.

Also morphisms of $\mathcal{LA}$-groupoids are defined in the obvious way.

Definition 2.2.11. Let $\Omega^\pm = (\Omega^\pm, \mathcal{G}^\pm, A^\pm, M^\pm)$ be $\mathcal{LA}$-groupoids; a morphism of $\mathcal{LA}$-groupoids $\Phi : \Omega^- \to \Omega^+$

$$\begin{tikzcd} \Omega^- \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[drr, "\hat{\phi}"] & A^- \arrow[d] \arrow[drr, "\phi"] & & \\ \mathcal{G}^- \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, "\varphi"] & M^- \arrow[drr, "f"] & \Omega^+ \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A^+ \arrow[d] \\ & & \mathcal{G}^+ \arrow[r, shift left] \arrow[r, shift right] & M^+ \end{tikzcd} \tag{2.18}$$

is given by four maps $(\hat{\phi}, \varphi, \phi, f)$ such that the vertical faces of the diagram above are morphisms of Lie algebroids, while the horizontal faces are morphisms of Lie algebroids.

The same remarks as after definition (2.2.4) apply, mutatis mutandis, and we shall use the analogous nomenclature for the components of a morphism of $\mathcal{LA}$-groupoids, as for morphisms of double Lie groupoids.

Lemma 2.2.12. *For any morphism* (2.18) *of $\mathcal{LA}$-groupoids, such that both $\hat{\phi}$ and φ are $\mathcal{LA}$-fibrations, respectively over φ and over f (so that the kernel Lie algebroids $\hat{K} := \mathsf{ker}\,\hat{\phi} \to \mathcal{G}$ and $K := \mathsf{ker}\,\phi \to M$ exist),*

i) $\hat{K}$ has a natural groupoid structure over K, making

$$\begin{tikzcd} \hat{K} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & K \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{2.19}$$

a sub-groupoid-object of Ω^-;

ii) $\hat{K} \rightrightarrows K$ is a Lie groupoid iff

$$(\hat{\phi}, \hat{\mathrm{s}}^-) : \Omega^- \to \Omega^+ {}_{\hat{\mathrm{s}}^+}\!\times_{\phi} A^- \tag{2.20}$$

is submersive;

iii) In addition, (2.19) *is an $\mathcal{LA}$-groupoid iff* (2.20) *is surjective.*

Proof. The proof of (i) is a simple exercise in diagram chasing and the proof of (ii) is the same as the analogous statement in lemma (2.2.5). (iii) By applying the snake lemma fibrewise to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \hat{K} \arrow[r] \arrow[d] & \Omega^- \arrow[r, "\hat{\phi}"] \arrow[d, "\hat{\mathrm{s}}^-"'] & \Omega^+ \arrow[r] \arrow[d, "\hat{\mathrm{s}}^+"'] & 0 \\ 0 \arrow[r] & K \arrow[r] & A^- \arrow[r, "\phi"] & A^+ \arrow[r] & 0 \end{tikzcd}$$

the restriction of the top source map of Ω^- to $\hat{K}$ is seen to be fibrewise surjective onto K iff the restriction of $\hat{\phi}$ to $\mathsf{ker}\,\hat{\mathrm{s}}^-$ is fibrewise surjective onto $\mathsf{ker}\,\hat{\mathrm{s}}^+$; last condition is equivalent to surjectivity of 2.19 by right translation in the abelian groupoids. $\square$

2.3. Integrability of $\mathcal{LA}$-groupoids

> In this Section we develop our functorial approach [**65**] to the integrability of $\mathcal{LA}$-groupoids. In the first part we obtain the natural integrability result for fibred products of Lie algebroids to fibred products of Lie groupoids and moreover we produce necessary and sufficient conditions for an integrating fibred product to be source (1-)connected. This allows us to derive easily integrability conditions for $\mathcal{LA}$-groupoids and morphisms of $\mathcal{LA}$-groupoids by diagrammatics in the second part; a major part of the technical work required to this aim has already been done in the first Section of this Chapter. The conditions presented here are slightly more general than those obtained in [**65**].

Unlike the case of Lie groupoids, the vertically source connected component of a double Lie groupoid need not be a double Lie subgroupoid and, even if it is, it might not have a vertically source 1-connected cover. This problem makes the integration of $\mathcal{LA}$-groupoids no straightforward corollary to the integration of Lie algebroids; namely, the top groupoid of an $\mathcal{LA}$-groupoid does not, in general, induce a groupoid on the Weinstein groupoids of its vertical Lie algebroids. In fact there are examples of $\mathcal{LA}$-groupoids which are integrable to double Lie groupoids and do not admit any vertically source 1-connected integration (see example 2.3.13).
We provide below (theorem 2.3.9) a criterion, which is often computable in the examples, for the integration of an $\mathcal{LA}$-groupoid Ω, with integrable top and side Lie algebroids, to a vertically source 1-connected double Lie groupoid, depending on the infinitesimal data only. Our conditions can be understood as $\mathcal{LA}$-homotopy lifting properties for the top source and target map of Ω.
The idea is the following. Source, target and unit section of the top groupoid of an $\mathcal{LA}$-groupoid are integrable to morphisms of Lie groupoids making the graph compatibility diagrams (2.21) commute. Moreover, the regularity condition on the double source map of Ω can also be "integrated" to the submersivity condition on the double source of the resulting graph Γ, whose (vertical) nerves are then Lie groupoids integrating the top vertical nerves of Ω. To obtain a double Lie groupoid, it is then sufficient to integrate the top multiplication of Ω to a compatible multiplication for Γ; this can be done provided suitable vertical nerves of Γ are source 1-connected.

2.3.1. Integrability of fibred products.

Under the natural transversality conditions, fibred products of integrable Lie algebroids integrate to fibred products of Lie groupoids. Next result is a corollary of theorem 2.1.8.

Theorem 2.3.1. *Let $\phi_{1,2} : A^{1,2} \to A$ be transversal morphisms of Lie algebroids over $f_{1,2} : M^{1,2} \to M$ (so that the fibred product Lie algebroid $A^{12} := A^1 {}_{\varphi_1}\times_{\varphi_2} A^2 \to M^{12} := M^1 {}_{\varphi_1}\times_{\varphi_2} M^2$ exists). Assume that $A^{1,2}$, A are integrable; then any integrations $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{G}$ of $\phi_{1,2}$, are transversal morphisms of Lie groupoids and the fibred product Lie groupoid $\mathcal{G}^1 {}_{\varphi_1}\times_{\varphi} \mathcal{G}^2 \rightrightarrows M^1 {}_{f_1}\times_{f_2} M^2$ exists.*

Proof. In the long exact sequence (2.3) for the fibred product of the graphs underlying $\mathcal{G}^{12}$ the cokernels of $\mathrm{d}f_1 - \mathrm{d}f_2$ and $(\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2)|_{T^{s_1}\mathcal{G}^1 \oplus T^{s_2}\mathcal{G}^2}$ vanish everywhere, the first one by hypothesis, the second by right translation and the linear transversality condition (2.7); then the cokernel of $\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2$ also vanishes everywhere. $\square$

Note that a fibred product $\mathcal{G}^1 {}_{\varphi_1}\times_{\varphi_2} \mathcal{G}^2$ of source (1-)connected Lie groupoids might fail to be source (1-)connected; however, it is possible to encode the source (1-)connectivity of a fibred product in terms of the induced morphisms of Lie algebroids $\phi_{1,2}$.

Theorem 2.3.2. *Let $\mathcal{G}^{1,2}$ and $\mathcal{G}$ be source 1-connected Lie groupoids and $\varphi_{1,2} : \mathcal{G}^{1,2} \to \mathcal{G}$ be morphisms of Lie groupoids over $f_{1,2} : M^{1,2} \to M$ such that the fibred product Lie groupoid $\mathcal{G}^1 {}_{\varphi_1}\times_{\varphi_2} \mathcal{G}^2 \rightrightarrows M^1 {}_{f_1}\times_{f_2} M^2$ exists; denote with $\phi_{1,2} : A^{1,2} \to A$ the induced morphisms of Lie algebroids. Then $\mathcal{G}^1 {}_{\varphi_1}\times_{\varphi_2} \mathcal{G}^2 \rightrightarrows M^1 {}_{f_1}\times_{f_2} M^2$ is source connected iff*

0. *For any $A^{1,2}$-paths $\alpha^-_{1,2}$ such that $\phi_1 \circ \alpha^-_1$ is A-homotopic to $\phi_2 \circ \alpha^-_2$, there exist $A^{1,2}$-paths $\alpha^+_{1,2}$, which are $A^{1,2}$-homotopic to $\alpha^-_{1,2}$, such that $\phi_1 \circ \alpha^-_1 = \phi_2 \circ \alpha^-_2$;*

furthermore it is source 1-connected iff

1. *For any $A^{1,2}$-paths $\alpha_{1,2}$, which are $A^{1,2}$-homotopic to the constant $A^{1,2}$-paths $\alpha^o_{1,2} \equiv 0^{A^{1,2}}_{\mathrm{pr}_{1,2}(\alpha_{1,2}(0))}$ and such that $\phi_1 \circ \alpha_1 = \phi_2 \circ \alpha_2$, there exist $A^{1,2}$-homotopies $h_{1,2}$ from $\alpha_{1,2}$ to the constant $A^{1,2}$-paths $\alpha^o_{1,2}$, such that $\phi_1 \circ h_1 = \phi_2 \circ h_2$.*

Proof. First of all note that for any $\mathcal{G}^{1,2}$-paths $\gamma^-_{1,2}$ and $u \in I$

$$\begin{aligned} \delta_r(\varphi_{1,2} \circ \gamma_{1,2})(u) &= \mathrm{d}r^{-1}_{\varphi_{1,2}(g_{1,2}(u))} \mathrm{d}\varphi_{1,2} \dot\gamma_{1,2}(u) = \mathrm{d}\varphi_{1,2} \mathrm{d}r^{-1}_{\gamma_{1,2}(u)} \dot\gamma_{1,2}(u) \\ &= (\phi_{1,2} \circ \delta_r \gamma_{1,2})(u) . \end{aligned}$$

Assume $\mathcal{G}^{12}$ is source connected, let $\alpha^-_{1,2}$ be $A^{1,2}$-paths such that $\phi_1 \circ \alpha^-_1$ is A-homotopic to $\phi_2 \circ \alpha^-_2$ and denote with $\gamma^-_{1,2}$ the unique corresponding $\mathcal{G}^{1,2}$-paths; we have $\delta_r(\varphi_{1,2} \circ \gamma^-_{1,2}) = \phi_{1,2} \circ \alpha^-_{1,2}$, thus $\varphi_{1,2} \circ \gamma^-_{1,2}$ are $\mathcal{G}$-homotopic $\mathcal{G}$-paths and $\varphi_1(\gamma^-_1(1)) = \varphi_2(\gamma^-_2(1))$. Since $\mathcal{G}^{12}$ is source connected, one can find $\mathcal{G}^{1,2}$-paths $\gamma^+_{1,2}$ such that $\gamma^+_{1,2}(1) = \gamma^-_{1,2}(1)$, (i.e. $\mathcal{G}^{1,2}$-homotopic to $\gamma^-_{1,2}$) with $\varphi_1 \circ \gamma^+_1 = \varphi_2 \circ \gamma^+_2$; the unique corresponding $A^{1,2}$-paths $\alpha^+_{1,2} := \delta_r \gamma^+_{1,2}$ are then homotopic to $\alpha^-_{1,2}$ and satisfy $\phi_1 \circ \alpha^-_1 = \phi_2 \circ \alpha^-_2$.

Conversely, assume that (0.) holds, let $(g_1, g_2) \in \mathcal{G}^{12}$ and pick $A^{1,2}$-paths $\alpha^-_{1,2}$ with $A^{1,2}$-homotopy classes $[\alpha^-_{1,2}] = g_{1,2}$. Then $[\phi_1 \circ \alpha^-_1] = \varphi_1(g_1) = \varphi_2(g_2) = [\phi_2 \circ \alpha^-_2]$, i.e. $\phi_1 \circ \alpha_1$ is A-homotopic to $\phi_2 \circ \alpha_2$ and one can find $A^{1,2}$-paths $\alpha^+_{1,2}$, which are $A^{1,2}$-homotopic to $\alpha^-_{1,2}$ and satisfy $\phi_1 \circ \alpha^-_1 = \phi_2 \circ \alpha^-_2$. The unique corresponding $\mathcal{G}^{1,2}$-paths $\gamma_{1,2}$ satisfy $\gamma_{1,2}(1) = g_{1,2}$ and $\delta_r(\varphi_1 \circ \gamma_1) = \delta_r(\varphi_2 \circ \gamma_2)$, therefore $\varphi_1 \circ \gamma_1 = \varphi_2 \circ \gamma_2$, by uniqueness, and the pair (γ_1, γ_2) form a $\mathcal{G}^{12}$-path starting from the unit section and reaching (g_1, g_2); that is, $\mathcal{G}^{12}$ is source connected.
Assume now that $\mathcal{G}^{12}$ is source 1-connected and let $\alpha_{1,2}$ be $A^{1,2}$-paths, which are $A^{1,2}$-homotopic to the constant $A^{1,2}$-paths $\alpha^o_{1,2}$, with $\phi_1 \circ \alpha_1 = \phi_2 \circ \alpha_2$. The unique corresponding $\mathcal{G}^{1,2}$-paths are in fact loops $\lambda_{1,2}$ in the source fibres of $\mathcal{G}^{1,2}$ starting form $\varepsilon_{1,2}(\mathrm{pr}_{1,2}(\alpha_{1,2}(0)))$, such that $\varphi_1 \circ \lambda_1 = \varphi_2 \circ \lambda_2$, i.e. the pair (λ_1, λ_2) forms a $\mathcal{G}^{12}$-loop. Since $\mathcal{G}^{12}$ is source 1-connected, one can find homotopies $H^{1,2}$ within the source fibres of $\mathrm{pr}_{1,2}(\alpha_{1,2}(0)$ from $\lambda_{1,2}$ to the constant $\mathcal{G}^{1,2}$ loops $\lambda^o_{1,2} \equiv \varepsilon_{1,2}(\mathrm{pr}_{1,2}(\alpha_{1,2}(0)))$ such that $\varphi_1 \circ H^1 = \varphi_2 \circ H^2$. The unique corresponding $A^{1,2}$-homotopies $h^{1,2}$ satisfy the boundary conditions

$$\begin{array}{rcl} \iota^*_{\partial^-_H} h^{1,2}(u) & = & \delta_r H^{1,2}(u,0) = \delta_r \lambda_{1,2}(u) = \alpha_{1,2} \\ \iota^*_{\partial^+_H} h^{1,2}(u) & = & \delta_r H^{1,2}(u,1) = \delta_r \lambda^o_{1,2}(u) = 0 \end{array} \qquad ,$$

thus are $A^{1,2}$-homotopies from $\alpha_{1,2}$ to the constant $A^{1,2}$-paths $\alpha^o_{1,2}$; moreover

$$\begin{array}{rcl} \phi_{1,2} \circ h^{1,2} & = & \phi_{1,2} \delta^u_r H^{1,2}(u,\varepsilon) \cdot \mathrm{d}u + \phi_{1,2} \delta^\varepsilon_r H^{1,2}(u,\varepsilon) \cdot \mathrm{d}\varepsilon \\ & = & \mathrm{d}r^{-1}_{\varphi_{1,2}(H^{1,2}(u,\varepsilon))} \mathrm{d}\varphi_{1,2} \dfrac{\partial}{\partial u} H^{1,2}(u,\varepsilon) \cdot \mathrm{d}u \\ & + & \mathrm{d}r^{-1}_{\varphi_{1,2}(H^{1,2}(u,\varepsilon))} \mathrm{d}\varphi_{1,2} \dfrac{\partial}{\partial \varepsilon} H^{1,2}(u,\varepsilon) \mathrm{d}\varepsilon \\ & = & \delta^u_r(\varphi_{1,2} \circ H^{1,2})(u,\varepsilon) \cdot \mathrm{d}u + \delta^\varepsilon_r(\varphi_{1,2} \circ H^{1,2})(u,\varepsilon) \cdot \mathrm{d}\varepsilon \end{array} \qquad ,$$

for the partial right derivatives $\delta^{\varepsilon,u}_r$, i.e. $\phi_1 \circ h^1 = \phi_2 \circ h^2$,
Conversely, assume that (1.) holds and $\mathcal{G}^{1,2}$-loops $\lambda_{1,2}$ such that $\varphi_1 \circ \lambda_1 = \varphi_2 \circ \lambda_2$ are assigned; we have to find homotopies $H^{1,2}$ from $\lambda_{1,2}$ to the constant loops $\lambda^o_{1,2}$ within the source fibres, such that $\varphi_1 \circ H^1 = \varphi_2 \circ H^2$. The unique $A^{1,2}$-paths $\alpha_{1,2}$ corresponding to $\lambda_{1,2}$ are $A^{1,2}$-homotopic to the constant $A^{1,2}$-paths $\alpha^o_{1,2}$, since $\lambda_{1,2}$ are $\mathcal{G}^{1,2}$-homotopic to the constant $\mathcal{G}^{1,2}$-loops $\lambda^o_{1,2}$; moreover $\phi_1 \circ \alpha_1 = \phi_2 \circ \alpha_2$, thus we can find $A^{1,2}$-homotopies $h^{1,2}$ from $\alpha_{1,2}$ to $\alpha^o_{1,2}$ with $\phi_1 \circ h_1 = \phi_2 \circ h_2$, which we shall regard as morphisms of Lie algebroids $h^{1,2} : TI^{\times 2} \to A^{1,2}$. The unique corresponding $\mathcal{G}^{1,2}$-homotopies $H^{1,2}$ are given by $H^{1,2}(u,\varepsilon) = \widetilde{h}^{1,2}(u,\varepsilon;0,0)$, where $\widetilde{h}^{1,2} : I^{\times 2} \times I^{\times 2} \to \mathcal{G}^{1,2}$ are the unique morphisms of Lie groupoids integrating $h^{1,2}$. Then $\varphi_{1,2} \circ \widetilde{h}^{1,2}$ are the unique morphisms of Lie groupoids integrating $\phi_{1,2} \circ h^{1,2}$ and $\varphi_1 \circ H^1 = \varphi_2 \circ H^2$. $\square$

Last result motivates the following

Definition 2.3.3. Two morphisms of Lie algebroids $\phi_{1,2} : A^{1,2} \to A$ are strongly transversal iff they are transversal morphisms of Lie algebroids and the lifting conditions (0., 1.) of theorem 2.3.2 hold.

It straightforward to translate conditions 0. and 1. above to the case of the kernel groupoid of of a morphism of Lie groupoids; it turns out that condition 1. is equivalent to the vanishing of suitable loop groups. Let $\phi : A \to A'$ be a morphism of Lie algebroids over $f : M \to M'$ and assume $\mathsf{ker}\,\phi \subset A$ is a Lie subalgebroid. For any $q \in M$, consider the space of class $\mathcal{C}^\infty$ A-loops based in q, taking values in $\mathsf{ker}\,\phi$ and A-homotopic to the null constant path, modulo $\mathcal{C}^\infty$ A-homotopy taking values in $\mathsf{ker}\,\phi$, namely $\mathcal{C}^\infty$ $\mathsf{ker}\,\phi$-loops based in q modulo $\mathcal{C}^\infty$ $\mathsf{ker}\,\phi$-homotopy:

$$\mathbb{K}_q(\phi) := \{\alpha \in \mathsf{ker}\,\phi\text{-}\mathit{paths} | \alpha(0) = \alpha(1) = 0_q,\ \alpha \sim_A \alpha^o\}/\mathsf{ker}\,\phi\text{-}\mathit{homotopy}$$

Note that, since $\mathsf{ker}\,\phi \subset A$ is a Lie subalgebroid the composition of smooth $\mathsf{ker}\,\phi$-paths is well defined up to smooth $\mathsf{ker}\,\phi$-homotopy and induces a group multiplication on the loop spaces above (we shall not need this group multiplication here or in the following).

Corollary 2.3.4. *Let $\varphi : \mathcal{G} \to \mathcal{G}'$ be a morphism of Lie groupoids over $f : M' \to M$ such that the kernel groupoid $\mathsf{ker}\,\varphi \subset \mathcal{G}'$ is a Lie subgroupoid and denote with $\phi : A \to A'$ the induced morphism of Lie algebroids. If $\mathcal{G}$ and $\mathcal{G}'$ are source 1-connected, then $\mathsf{ker}\,\varphi$ is source connected iff*

0. *For any A-path α^- such that $\phi \circ \alpha^-$ is A'-homotopic to the constant A'-path $(\phi \circ \alpha^-)^o \equiv 0_{\mathrm{pr}'(\phi(\alpha^-(0)))}$, there exists an A-path α^+, which is A-homotopic to α^-, such that $\phi \circ \alpha^+ = (\phi \circ \alpha^-)^o$;*

furthermore it is source 1-connected iff

1. *For any A-path α, which is A-homotopic to the constant A-path $\alpha^o \equiv 0_{\mathrm{pr}(\alpha(0))}$ and such that $\phi \circ \alpha = (\phi \circ \alpha)^o$ is the constant A'-path, there exists an A-homotopy h form α to the constant A-path $\alpha^o \equiv 0_{\mathrm{pr}(\alpha(0))}$, such that $\phi \circ h = 0$*

equivalently iff

$1'$. *The loop groups $\mathbb{K}_q(\phi)$ are trivial for all $q \in M$.*

Proof. Set $K := \mathsf{ker}\,\phi$. Conditions (0.) and (1.) are just restatements of the corresponding conditions in theorem 2.3.2 for the case $\varphi_1 = \varphi : \mathcal{G} \to \mathcal{G}'$ and $\varphi_2 = \varepsilon' : M' \to \varepsilon'(M')$. The K-loops representing elements of $\mathbb{K}_q(\phi)$ are in particular A-paths satisfying the hypothesis of (1.), thus (1.) implies ($1'$.). For the opposite implication, consider that any K-path is K-homotopic to a smooth K-paths and reparametrization of smooth K-paths does not change the K-homotopy class. Thus condition 1. holds, for all smooth α with compact support, which in particular represent elements of $\mathbb{K}_q(\phi)$. $\square$

We conclude this Subsection introducing a special class of morphisms of Lie algebroids.

Definition 2.3.5. [**65**] A morphism $\phi : A \to B$ of Lie algebroids has the

l_0) 0-$\mathcal{LA}$-homotopy lifting property if, for any A-path α_- and B-path β_+, which is B-homotopic to $\beta_- := \phi \circ \alpha_-$, there exists an A-path α_-, which is A-homotopic to α_+ and satisfies $\phi \circ \alpha_+ = \beta_+$;

l_1) 1-$\mathcal{LA}$-homotopy lifting property if, for any A-path α, which is A-homotopic to the constant A-path $\alpha_o \equiv 0_{\mathrm{pr}_A(\alpha(0))}$, and B-homotopy h from $\beta := \phi \circ \alpha$ to the constant B-path $\beta_o \equiv 0_{\mathrm{pr}_B(\beta(0))}$, there exists an A-homotopy $\hat{h}$ from α to α_o, such that $\phi \circ \hat{h} = h$.

The $\mathcal{LA}$-homotopy lifting conditions above on a morphism of integrable Lie algebroids $A \to B$, translate to the infinitesimal level path lifting conditions in the source fibres of the source 1-connected Lie groupoids $\mathcal{A}$ and $\mathcal{B}$ of A and B along the integration $\mathcal{A} \to \mathcal{B}$; therefore, with the same notations and assumptions of theorem 2.3.2 we have

Lemma 2.3.6. *The fibred product $\mathcal{G}^1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{G}^2 \rightrightarrows M^1 {}_{f_1}\!\times_{f_2} M^2$ is*

i) source connected if $\phi_1 : A^1 \to A$ has the 0-$\mathcal{LA}$-homotopy lifting property

furthermore, it is

ii) source 1-connected if $\phi_1 : A^1 \to A$ has the 1-$\mathcal{LA}$-homotopy lifting property

Clearly, he statement holds true replacing ϕ_1 with ϕ_2.

Proof. Condition 0. of theorem 2.3.2 follows from (i) since one can set $\alpha_2^+ = \alpha_2^-$ and lift $\phi_2 \circ \alpha_2^+$ along ϕ_1. Similarly, condition 1. of 2.3.2 follows from (ii). □

2.3.2. Integrability of $\mathcal{LA}$-groupoids.

Here and in the following we shall repeatedly use of next easy

Lemma 2.3.7. *Let $\varphi : \mathcal{G}' \to \mathcal{G}$ be a morphism of Lie groupoids over $f : M' \to M$ integrating $\phi : A' \to A$. Then φ is an $\mathcal{LG}$-fibration iff ϕ is an $\mathcal{LA}$-fibration.*

Proof. By right translation, fibrewise surjectivity of ϕ is equivalent to source submersivity of φ. Then the implication to the right is clear and the implication to the left follows by submersivity of f and the 5-lemma. □

Let us consider the integrability of the graph underlying an $\mathcal{LA}$-groupoid. We shall say that a differentiable graph $(\Gamma, M; \mathrm{s}, \mathrm{t})$ is an invertible graph if its total space is endowed with an automorphism ι such that $\iota^2 = 1$ and $\mathrm{t} \circ \iota = \mathrm{s}$.

Proposition 2.3.8. *The top differentiable graph of any $\mathcal{LA}$-groupoid with integrable top Lie algebroid integrates to an invertible differentiable graph in the category of source 1-connected Lie groupoids.*

Proof. Since A is a Lie subalgebroid of Ω and Ω is integrable (theorem 1.4.2), A is also integrable. Denote with $\mathcal{A}$ and Γ the source 1-connected integrations of A and Ω, respectively. The graph compatibility diagrams

$$\begin{tikzcd} & \Omega \arrow[dr, "\hat{\mathrm{s}},\hat{\mathrm{t}}"] & \\ A \arrow[ur, "\hat{\varepsilon}"] \arrow[rr, "\mathrm{id}_A"'] & & A \end{tikzcd} \qquad \begin{tikzcd} & \Omega \arrow[dr, "\hat{\iota}"] & \\ \Omega \arrow[ur, "\hat{\iota}"] \arrow[rr, "\mathrm{id}_\Omega"'] & & \Omega \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} & \\ \Omega \arrow[ur, "\hat{\mathrm{s}}"] \arrow[rr, "\hat{\iota}"'] & & \Omega \arrow[ul, "\hat{\mathrm{t}}"'] \end{tikzcd}$$

integrate to commuting diagrams

$$\begin{tikzcd} & \Gamma \arrow[dr, "\mathrm{s}_H,\mathrm{t}_H"] & \\ \mathcal{A} \arrow[ur, "\varepsilon_H"] \arrow[rr, "\mathrm{id}_{\mathcal{A}}"'] & & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} & \Gamma \arrow[dr, "\iota_H"] & \\ \Gamma \arrow[ur, "\iota_H"] \arrow[rr, "\mathrm{id}_\Gamma"'] & & \Gamma \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} & \\ \Gamma \arrow[ur, "\mathrm{s}_H"] \arrow[rr, "\iota_H"'] & & \Gamma \arrow[ul, "\mathrm{t}_H"'] \end{tikzcd} \tag{2.21}$$

in the category of Lie groupoids; thus, ι_H is a diffeomorphism, being the inverse to itself, ε_H is injective, having left inverses and s_H, t_H are surjective, being left inverses to ε_H. Moreover the tangent diagrams

$$\begin{tikzcd} & T_{\varepsilon_H(a)}\Gamma \arrow[dr, "\mathrm{d}\,\mathrm{s}_H{,}\mathrm{d}\,\mathrm{t}_H"] & \\ T_a\mathcal{A} \arrow[ur, "\mathrm{d}\varepsilon_H"] \arrow[rr, "\mathrm{id}_{T_a\mathcal{A}}"'] & & T_a\mathcal{A} \end{tikzcd}$$

commute for all $a \in \mathcal{A}$, therefore ε_H is immersive. It remains to show that s_H and t_H are submersive, which follows from lemma 2.3.7, since both $\hat{\mathrm{s}}$ and $\hat{\mathrm{t}}$ are $\mathcal{LA}$-fibrations. $\square$

Consider the top nerves of an $\mathcal{LA}$-groupoid $\Omega = (\Omega, \mathcal{G}; A, M)$

$$\begin{array}{lll} \Omega^{(0)} & := & A \\ \Omega^{(1)} & := & \Omega \\ \Omega^{(n+1)} & := & \Omega\, {}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}\mathrm{op}_1} \Omega^{(n)} \qquad , \qquad n \geq 1 \qquad , \end{array}$$

where $\mathrm{p}_1 : \Omega^{(n)} \to \Omega$ denotes the restriction of the first projection $\Omega^{\times n} \to \Omega$. Since the top source map $\hat{\mathrm{s}}$ is an $\mathcal{LA}$-fibration each top nerve $\Omega^{(n)}$ carries a Lie algebroid structure over the corresponding side nerve $\mathcal{G}^{(n)}$ making it a Lie subalgebroid of the direct product $\Omega^{\times n} \to \mathcal{G}^{\times n}$, $n \geq 1$ (theorem 2.1.13). Assume now that the top Lie algebroid of Ω is integrable and consider the integrating graph $(\Gamma, \mathcal{A})$ of lemma

2.3.8 given by the source 1-connected integrations $\Gamma \rightrightarrows \mathcal{G}$ and $\mathcal{A} \rightrightarrows M$ of the top and side Lie algebroids $\Omega \to \mathcal{G}$ and $A \to M$. Each nerve,

$$\begin{array}{lcl} \Gamma^{(0)} & := & \mathcal{A} \\ \Gamma^{(1)} & := & \Gamma \\ \Gamma^{(n+1)} & := & \Gamma \ {}_{\mathrm{s}_H}\!\times_{\mathrm{t}_H \circ \mathrm{p}_1} \Gamma^{(n)} \subset \Gamma^{\times n} \qquad , \qquad n \geq 1 \qquad , \end{array}$$

of Γ is a smooth submanifold of the corresponding direct product, where $\mathrm{s}_H, \mathrm{t}_H : \Gamma \to \mathcal{A}$ are the integrations of top source and target of Γ and $\mathrm{p}_1 : \Gamma^{(n)} \to \Gamma$ denotes the restriction of the first projection $\Gamma^{\times n} \to \Gamma$. Moreover, since $\mathcal{LA}$-fibrations integrate to $\mathcal{LG}$-fibrations (lemma 2.3.7), the transversality conditions of theorem 2.3.2 are met and $\Gamma^{(n)}$ carries a Lie groupoid structure over the corresponding side nerve $\mathcal{G}^{(n)}$ making it a Lie subgroupoid of the direct product $\Gamma^{\times n} \to \mathcal{G}^{\times n}$, $n \geq 1$. Note that if s_H has the 0- and 1-homotopy lifting properties, it follows from lemma 2.3.6 that *all* the nerves $\Gamma^{(\bullet)} \rightrightarrows \mathcal{G}^{(\bullet)}$ are source 1-connected.

Theorem 2.3.9. *Let Ω be an $\mathcal{LA}$-groupoid with integrable top Lie algebroid. If source and target of the top Lie groupoid of Ω are strongly transversal, there exists a unique vertically source 1-connected double Lie groupoid integrating Ω.*

Remark 2.3.10. In particular, the transversality condition holds when the top source, equivalently target, map of Ω enjoys the $\mathcal{LA}$-homotopy lifting conditions of definition 2.3.5.

PROOF OF THEOREM 2.3.9. With the same notations as above, $\Gamma \ {}_{\mathrm{s}_H}\!\times M \simeq M \times_{\mathrm{t}_H} \Gamma \simeq \Delta_\Gamma \simeq \Gamma$ and $\Gamma^{(2)}$ are source 1-connected Lie groupoids, then the unitality diagrams

$$\begin{tikzcd}
& \Gamma \arrow[r, "\mathrm{id}_\Gamma"] \arrow[dl, "\Delta_\Gamma"'] & \Gamma & \\
\Delta_\Gamma \arrow[dr, "\mathrm{id}_\Gamma \times \mathrm{s}_H"'] & & & \Gamma^{(2)} \arrow[ul, "\mu_H"'] \\
& \Gamma \ {}_{\mathrm{s}_H}\!\times M \arrow[urr, "\mathrm{id}_\Gamma \times \varepsilon_H"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Gamma \arrow[r, "\mathrm{id}_\Gamma"] \arrow[dl, "\Delta_\Gamma"'] & \Gamma & \\
\Delta_\Gamma \arrow[dr, "\mathrm{t}_H \times \mathrm{id}_\Gamma"'] & & & \Gamma^{(2)} \arrow[ul, "\mu_H"'] \\
& M \times_{\mathrm{t}_H} \Gamma \arrow[urr, "\varepsilon_H \times \mathrm{id}_\Gamma"'] & &
\end{tikzcd}$$

and invertibility diagrams

$$\begin{tikzcd}
& \Gamma \arrow[r, "\mathrm{s}_H"] \arrow[dl, "\Delta_\Gamma"'] & \mathcal{A} \arrow[dr, "\varepsilon_H"] & \\
\Delta_\Gamma \arrow[dr, "\iota_H \times \mathrm{id}_\Gamma"'] & & & \Gamma \\
& \Gamma^{(2)} \arrow[urr, "\mu_H"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Gamma \arrow[r, "\mathrm{t}_H"] \arrow[dl, "\Delta_\Gamma"'] & \mathcal{A} \arrow[dr, "\varepsilon_H"] & \\
\Delta_\Gamma \arrow[dr, "\mathrm{id}_\Gamma \times \iota_H"'] & & & \Gamma \\
& \Gamma^{(2)} \arrow[urr, "\mu_H"'] & &
\end{tikzcd}$$

commute, since the integrate the corresponding diagrams for the top Lie groupoid of Ω. The third nerve $\Gamma^{(3)}$ is also source 1-connected, hence the associativity diagram

$$\begin{tikzcd} & \Gamma^{(3)} \arrow[dl, "\mathsf{id}_\Gamma\times\mu_H"'] \arrow[dr, "\mu_H\times\mathsf{id}_\Gamma"] & \\ \Gamma^{(2)} \arrow[dr, "\mu_H"'] & & \Gamma^{(2)} \arrow[dl, "\mu_H"] \\ & \Gamma & \end{tikzcd}$$

commutes. Then $(\Gamma, \mathcal{G}; \mathcal{A}, M)$ is a groupoid object in the category of Lie groupoids, thanks to proposition 2.3.8; the source submersivity condition follows from lemma 2.3.7. □

The transversality conditions for integrability imply that all top horizontal nerves of the integrating double Lie groupoid are source 1-connected; however, only the role of the second and third nerve is essential.

Remark 2.3.11. In the proof of last theorem, even if $\Gamma^{(3)}$ is only source connected, last diagram still commute since

$$\begin{tikzcd} & \Gamma^{(3)} \arrow[dl, "\mathsf{id}_\Gamma\times\mu_H"'] \arrow[dr, "\mu_H\times\mathsf{id}_\Gamma"] & \\ \Gamma^{(2)} \arrow[dr, "\mu_H"'] & X \arrow[u, "\kappa"] \arrow[l, "\mu_l"] \arrow[r, "\mu_r"'] & \Gamma^{(2)} \arrow[dl, "\mu_H"] \\ & \Gamma & \end{tikzcd}$$

commutes for the covering morphism $\kappa : X \to \Gamma^{(3)}$ and the morphisms $\mu_{l,r} : X \to \Gamma^{(2)}$ from the source 1-connected cover X of $\Gamma^{(3)}$ integrating $\mathsf{id}_\Gamma \times \mu_H$ and $\mu_H \times \mathsf{id}_\Gamma$ respectively. Thus to in order to integrate an $\mathcal{LA}$-groupoid with integrable top algebroid to a double Lie groupoid, it is sufficient to have $\Gamma^{(2)}$ source 1-connected and $\Gamma^{(3)}$ source connected.

In principle, an integrable $\mathcal{LA}$-groupoid could admit a vertically source 1-connected integration which does not possess source (1-)connected top horizontal nerves; we could not find examples of this kind.

Example 2.3.12. Consider the tangent prolongation $\mathcal{LA}$-groupoid

$$\mathsf{T}\mathbb{T}^2 = (T\mathbb{T}^2, \mathbb{T}^2; TS^1, S^1)$$

of the pair groupoid on the torus $\mathbb{T}^2$: $S^1 \times S^1 \rightrightarrows S^1$; then the fundamental groupoid $\Pi(\mathbb{T}^2) = (\mathbb{R}^2 \times \mathbb{R}^2)/\mathbb{Z}^2$ carries a further natural Lie groupoid (induced by the direct product of pair groupoids $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$) over $\Pi(S^1) = (\mathbb{R} \times \mathbb{R})/\mathbb{Z}$, making $(\Pi(\mathbb{T}^2), \mathbb{T}^2; \Pi(S^1), S^1)$ a vertically source 1-connected Lie groupoid integrating $\mathsf{T}\mathbb{T}^2$. The top source map of $\mathsf{T}\mathbb{T}^2$ is the second projection $TS^1 \times TS^1 \to TS^1$, which clearly satisfies the $\mathcal{LA}$-homotopy lifting conditions $l_{0,1}$.

Let us consider an example of a tangent prolongation $\mathcal{LA}$-groupoid, which has no vertically source 1-connected integration.

Example 2.3.13. Consider the Lie groupoid given by $\mathcal{G} = \mathbb{R}^2 \times \mathbb{Z}$, $M = \mathbb{R}^2 \equiv \{0\} \times \mathbb{R}^2$, $\mathrm{s}(k; x, y) = (0, x - k, y)$, $\mathrm{t}(k; x, y) = (0, x + k, y)$,

$$\mu((l; x + k + l, y), (k; x, y)) = (l + k; x + l, y) \quad \text{ and } \quad \iota(k; x, y) = (-k; x, y) \quad .$$

Removing lattices from all the layers of $\mathcal{G}$ but the base yields a Lie subgroupoid $\mathcal{H} \rightrightarrows M$,

$$\mathcal{H} := \mathbb{R}^2 \times \{0\} \coprod (\coprod_{k \neq 0} \mathbb{R}^2 \backslash \mathbb{Z}^2 \times \{k\})$$

such that the differentiable graph $(\Pi(\mathcal{H}), \Pi(M))$ (source, target, unit section and inversion being induced by composition) carries no compatible groupoid multiplication, therefore $(\Pi(\mathcal{H}), \mathcal{H}; \Pi(M), M)$ is never a double Lie groupoid. To see this, consider the the equivalence classes in $\Pi(\mathcal{H})$ of the paths

$$\gamma_{\pm}(t) = (1; \pm 1 \pm \sin(\pi t)/2, 1/2 + t) \qquad , \qquad t \in I \qquad :$$

$\mathrm{s} \circ \gamma_-(t) = (0; -\sin(\pi t)/2, 1/2 + t)$ and $\mathrm{t} \circ \gamma_+(t) = (0; \sin(\pi t), 1/2 + t)$ define homotopic paths in M, but there is no representative in the homotopy class of γ_+ which is pointwise composable with some representative of the class of γ_-; the source map of $\Pi(\mathcal{H})$ does not satisfy the $\mathcal{LA}$-homotopy lifting condition l_0.

Integrability of morphisms of $\mathcal{LA}$-groupoids follows by diagrammatics in the same fashion.

Corollary 2.3.14. *Let* $\Phi : \Omega^- \to \Omega^+$ *be a morphism of* $\mathcal{LA}$*-groupoids* (2.18). *Assume that* $\Omega^{\pm}$ *are integrable to double Lie groupoids* $\Gamma^{\pm}$*; moreover assume that top horizontal source and target of* Ω^- *are strongly transversal. Then there exist a unique morphism of double Lie groupoids* $\widetilde{\Phi}$ *integrating* Φ.

PROOF. The integration $\widetilde{\Phi} : \Gamma^- \to \Gamma^+$ over $\varphi : \mathcal{G}^- \to \mathcal{G}_+$ of the top vertical component $\Phi : \Omega^- \to \Omega^+$ makes the diagram

$$\begin{tikzcd}
(\Gamma^-)^{(2)} \arrow[r, "{\mu_H}^-"] \arrow[d, "\widetilde{\Phi}\times\widetilde{\Phi}"'] & \Gamma^- \arrow[d, "\widetilde{\Phi}"] \\
(\Gamma^+)^{(2)} \arrow[r, "{\mu_H}^+"'] & \Gamma^+)
\end{tikzcd} \tag{2.22}$$

commute. □

In the same spirit as in remark 2.3.11, we shall comment on the connectivity requirements on the top horizontal nerves.

Remark 2.3.15. In order to integrate a morphism of $\mathcal{LA}$-groupoids such as above it is sufficient to have the second top horizontal nerve of Γ^- source connected. Commutativity of diagram (2.22) is induced by that of the similar diagram for the covering groupoid.

2.4. Double structures, duality and integrability of Poisson groupoids

Here we specialize the integrability results of last Section to the $\mathcal{LA}$-groupoid canonically associated with a Poisson groupoid. We show that when the Poisson bivector is integrable and the strong transversality conditions on the associated $\mathcal{LA}$-groupoid are met, the integration produces a symplectic double groupoid realizing a strong duality between the given Poisson groupoid and its unique source 1-connected weak dual. Finally we study a class of examples where the transversality conditions can be computed explicitly and show that all complete Poisson groups are integrable to symplectic double groupoids.

Recall from Section 1.5 that (A, A^*) is a Lie bialgebroid iff so is (A^*, A). Moreover changing the signs of the anchor and bracket of a Lie algebroid A, yields another Lie algebroid $-A$; it is then easy to see that (A, A^*) is a Lie bialgebroid iff so is the flip $(A^*, -A)$.

Definition 2.4.1. Two Poisson groupoids $\mathcal{G}_\pm \rightrightarrows M$ with Lie algebroids $(A_\pm, A^*_\pm)$ are in weak duality if (A_-, A^*_-) and the flip $(A^*_+, -A_+)$ are isomorphic Lie bialgebroids[4].

This notion of duality is a symmetric relation: if $\phi : A^*_+ \to A_-$ induces an isomorphism of Lie bialgebroids $(A^*_+, -A_+) \to (A_-, A^*_-)$, $-\phi^{\mathsf{t}}$ induces an isomorphism $(A^*_-, -A_-) \to (A_+, A^*_+)$. However, a Poisson groupoid might not have any weak dual and weak duals are not unique (for instance $-\phi$ makes (A^*_+, A_+) weakly dual to $(-A_-, -A^*_-)$

Example 2.4.2. For any Poisson manifold (P, π), the Lie bialgebroid of $\overline{P} \times P \rightrightarrows P$ is (canonically) $(TP, -T^*P)$ and there is no Poisson groupoid integrating its flip $(-T^*P, -TP)$ when π is not integrable. If π is integrable to a symplectic groupoid $\Lambda \rightrightarrows P$, $\overline{\Lambda}$ is the canonical choice of a weak dual to $\overline{P} \times P$.

[4]This notion was introduced by Weinstein in [**72**] simply as "duality" for Poisson groupoids

Remark 2.4.3. The convention of declaring Poisson groupoids $\mathcal{G}_\pm$ in weak duality if $(A_+, A_+^*) \simeq (A_-^*, A_-)$ is also frequent in the literature, especially for Poisson groups. To recover this notion of duality one has to change the sign of a Poisson bivector, that is $\mathcal{G}_\pm$ are dual to each other in this sense iff $\mathcal{G}_+$ is dual to $\overline{\mathcal{G}_-}$ in the sense of definition 2.4.1.

A stronger notion of duality for Poisson groupoids arises from symplectic double groupoids.

Definition 2.4.4. A symplectic double groupoid is a double Lie groupoid

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}_- \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G}_+ \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \tag{2.23}$$

endowed with a symplectic form which is compatible with both top groupoid structures.

Next we shall compute the $\mathcal{LA}$-groupoid of a symplectic double groupoid. Before doing that we shall need the following

Proposition 2.4.5. *For any Poisson groupoid $\mathcal{G} \rightrightarrows M$, the cotangent prolongation groupoid induces an $\mathcal{LA}$-groupoid*

$$\begin{tikzcd} T^*\mathcal{G} \arrow[r, shift left, "\widehat{}"] \arrow[r, shift right] \arrow[d] & A^* \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \quad . \tag{2.24}$$

Proof. By definition, the graph of the cotangent multiplication is a Lie subalgebroid of $T^*\mathcal{G}^{\times 3}$; moreover it is easy to see that $\Gamma(\hat{\iota}) = N^*\Gamma(\iota) \subset T^*\mathcal{G}^{\times 2}$ is also a Lie subalgebroid, since $\iota : \mathcal{G} \to \overline{\mathcal{G}}$ is Poisson. The graph of the unit section $\Gamma(\hat{\varepsilon}) \subset A^* \times T^*\mathcal{G}$ is a smooth subbundle and $A^* \times T^*\mathcal{G} \equiv N^*M \times T^*\mathcal{G} \subset T^*\mathcal{G}^{\times 2}$ a Lie subalgebroid; on the other hand we can identify $\Gamma(\hat{\varepsilon})$ with the diagonal $\Delta_{N^*M} \subset T^*\mathcal{G}^{\times 2}$ as a subbundle, hence with a Lie subalgebroid. It follows then from lemma 1.2.8 that $\Gamma(\hat{\varepsilon}) \subset A^* \times T^*\mathcal{G}$ is a Lie subalgebroid. Note that one can identify $\Gamma(\hat{\mathrm{t}})$ with the subbundle $\{\hat{\varepsilon}(\hat{\mathrm{t}}(\theta_g)), \theta_g, \theta_g\}$ of $\Gamma(\hat{\mu})$, i.e.

$$\Gamma(\hat{\mathrm{t}}) \simeq (\mathrm{pr}_2 \times \mathrm{pr}_3)|_{\Gamma(\hat{\mu})}^{-1}(\Delta_{T^*\mathcal{G}}) \qquad ,$$

thus with a Lie subalgebroid, due to proposition 2.1.11. Then

$$\Gamma(\hat{\mathrm{t}}) \subset \Gamma(\hat{\mu}) \subset T^*\mathcal{G} \times T^*\mathcal{G} \times T^*\mathcal{G}$$

is a sequence of Lie subalgebroids; since

$$T^*\mathcal{G} \times A^* \simeq A^* \times T^*\mathcal{G} \simeq N^*M \times \Delta_{T^*\mathcal{G}} \subset T^*\mathcal{G} \times T^*\mathcal{G} \times T^*\mathcal{G}$$

is a Lie subalgebroid, it also follows from lemma 1.2.8 that $\Gamma(\hat{\mathrm{t}}) \subset T^*\mathcal{G}\times A^*$ is a Lie subalgebroid. Analogously one shows that $\Gamma(\hat{\mathrm{s}}) \subset T^*\mathcal{G}\times A^*$ is a Lie subalgebroid. Finally note that $\mathsf{ker}_g\,\hat{\mathrm{s}} = N^*_g\mathrm{t}^{-1}(g)$, for all $g\in\mathcal{G}$, then $\hat{\mathrm{s}}$ is a bundle map of rank

$$\mathsf{rank}_g\,\hat{\mathrm{s}} = \mathsf{dim}\,\mathcal{G} - (\mathsf{dim}\,\mathcal{G} - \mathsf{rank}\,\mathrm{t}) = \mathsf{rank}_{\mathrm{t}(g)}A^*$$

over s, thus an $\mathcal{LA}$-fibration. □

Applying the Lie functor vertically to a symplectic double groupoid (2.23) yields an $\mathcal{LA}$-groupoid, which is to be canonically identified with the cotangent prolongation $\mathcal{LA}$-groupoid associated with its side horizontal Poisson groupoid $\mathcal{G}_+$. Let $A_V(\mathcal{S}) := T^{\mathrm{s}_V}_{\mathcal{G}_+}\mathcal{S}$ and $A_- := T^{\mathrm{s}_v}_M\mathcal{G}_-$ denote the Lie algebroids of the vertical groupoids and Ω be the symplectic form on $\mathcal{S}$. Note that the isomorphism of Lie algebroids $\phi_\Omega : T^*\mathcal{G}_+ \to A_V(\mathcal{S})$ identifies $A_+^* \equiv N^*M \subset T^*\mathcal{G}_+$ with A_-: in fact, for all $(\delta g_-, \delta q) \in T^{\mathrm{s}_v}_M\mathcal{G}_- \oplus TM$, we have

$$\begin{aligned}\langle\, \phi_\Omega^{-1}(\mathrm{d}\varepsilon_H\delta g_-)\,,\, \mathrm{d}\varepsilon_h\delta q\,\rangle &= -\Omega(\mathrm{d}\varepsilon_H\delta g_-, \mathrm{d}\varepsilon_V\mathrm{d}\varepsilon_h\delta q)\\ &= -\Omega(\mathrm{d}\varepsilon_H\delta g_-, \mathrm{d}\varepsilon_H\mathrm{d}\varepsilon_v\delta q)\\ &= 0 \qquad ,\end{aligned}$$

since $\mathcal{G}_- \subset \mathcal{S}$ is Lagrangian. Moreover ϕ_Ω is an isomorphism of Lie groupoids from $A_V(\mathcal{S}) \rightrightarrows A_-$ to $T^*\mathcal{G}_+ \rightrightarrows A_+^*$: for all composable pairs $(\delta x, \delta y) \in A_V(\mathcal{S})$ and $(\delta g_-, \delta h_-) \in T\mathcal{G}_-$,

$$\begin{aligned}\langle\, \phi_\Omega^{-1}(\delta x \cdot_{A_V(\mathcal{S})} \delta y))\,,\, \delta g_+ \cdot_{T\mathcal{G}_+} \delta h_+)\,\rangle &= -\Omega(\delta x \cdot_{A_V(\mathcal{S})} \delta y, \mathrm{d}\varepsilon_V(\delta g_+ \cdot_{T\mathcal{G}_+} \delta h_+))\\ &= -\Omega(\delta x \cdot_{A_V(\mathcal{S})} \delta y, \mathrm{d}\varepsilon_V(\delta g_+) \cdot_{A_V(\mathcal{S})} \mathrm{d}\varepsilon_V(\delta h_+))\\ &= -\Omega(\delta x, \mathrm{d}\varepsilon_V(\delta g_+)) - \Omega(\delta y, \mathrm{d}\varepsilon_V(\delta h_+))\\ &= \langle\, \phi_\Omega^{-1}(\delta x)\,,\, \delta g_+\,\rangle + \langle\, \phi_\Omega^{-1}(\delta y)\,,\, \delta h_+\,\rangle\\ &= \langle\, \phi_\Omega^{-1}(\delta x) \cdot_{T^*\mathcal{G}_+} \phi_\Omega^{-1}(\delta y)\,,\, \delta g_+ \cdot_{T\mathcal{G}_+} \delta h_+\,\rangle \quad .\end{aligned}$$

In other words $\phi_\Omega : T^*\mathcal{G} \to A_V(\mathcal{S})$ induces an isomorphism of $\mathcal{LA}$-groupoids. Note that ϕ_Ω is a Poisson map for the fibrewise linear structure induced by $A_V(\mathcal{S})^* \equiv N^*\mathcal{G}_+$ and the *canonical* symplectic form on $T^*\mathcal{G}_+$; therefore the restriction $A_+^* \to A_-$ is Poisson for the Poisson structure induced by $-A_+$ on A_+, i.e. an isomorphism of Lie bialgebroids $(A_+^*, -A_+) \to (A_-, A_-^*)$.
Therefore we have

Proposition 2.4.6. [**42**] *The side groupoids of a symplectic double groupoid are weakly dual Poisson groupoids.*

Even if a Poisson groupoid might not admit a weak dual, it is easy to see that any Poisson groupoid, which is integrable as a Poisson manifold, has a unique source 1-connected dual.

Lemma 2.4.7. *Any integrable Poisson groupoid $\mathcal{G}$ has a unique canonical weakly dual source 1-connected Poisson groupoid $\mathcal{G}^\star$.*

PROOF. It follows from proposition 2.4.5 that the unit section $A^* \to T^*\mathcal{G}$ of the cotangent prolongation groupoid is a closed embedding of Lie algebroids. Thus A^* is integrable, since so is $T^*\mathcal{G}$ (theorem 1.4.2), and the source 1-connected integration $\mathcal{G}^\star$ of A^* carries a unique compatible Poisson bivector inducing the Lie bialgebroid $(A^*, -A)$, thanks to theorem 1.5.7. $\square$

In the following we shall refer to the Poisson groupoid $\mathcal{G}^\star$ of last lemma as *the* weak dual of $\mathcal{G}$. The above arguments motivate the following

Definition 2.4.8. Two Poisson groupoids $(\mathcal{G}_\pm, \Pi_\pm) \rightrightarrows M$ are in strong duality if there exists a symplectic double groupoid inducing the given Poisson structures $\Pi_\pm$; a double of an integrable Poisson groupoid $\mathcal{G}$ is a symplectic double groupoid realizing a strong duality between $\mathcal{G}$ and its unique weak dual $\mathcal{G}^\star$.[5]

Example 2.4.9. For any symplectic groupoid $\Lambda \rightrightarrows P$, $(\overline{\Lambda} \times \Lambda; \Lambda, \overline{P} \times P; P)$ is a symplectic double groupoid. The weak duality is canonically realized by the isomorphism of Lie bialgebroids $(T^{\mathrm{s}}_P\Lambda, N^*P) \to (T^*P, TP)$, since the target is the flip of the Lie bialgebroid of $\overline{P} \times P \rightrightarrows P$.

Last example shows that symplectic groupoids always admit strongly dual Poisson groupoids (canonically). Specializing theorem 2.3.9 to the case of the cotangent prolongation $\mathcal{LA}$-groupoid associated with a Poisson groupoid we obtain a criterion for integrability of Poisson groupoids to symplectic double groupoids and for weak duality to imply strong duality.

Theorem 2.4.10. *Let $(\mathcal{G}, \Pi) \rightrightarrows M$ be any integrable Poisson groupoid with weak dual Poisson groupoid $(\mathcal{G}^\star, \Pi^\star)$. If cotangent source and target map of $\mathcal{G}$ are strongly transversal, the symplectic groupoid $\mathcal{S}$ of $\mathcal{G}$ carries a further Lie groupoid making it a symplectic groupoid for $\mathcal{G}^\star$ and*

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}^\star \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

a symplectic double groupoid.

Note that we make no source connectivity assumptions on $\mathcal{G}$.

PROOF. It remains to show (1) that the integrating double Lie groupoid is symplectic for the top horizontal multiplication and (2) the side vertical groupoid is indeed the canonical weakly dual groupoid. (1) The symplectic form ω of $\mathcal{S}$ induces an isomorphism of Lie bialgebroids

$$\Phi := (\phi_\omega \times \phi_\omega \times \phi_{-\omega})^{-1} : A_V(\mathcal{S}) \times A_V(\mathcal{S}) \times A_V(\mathcal{S}) \xrightarrow{\sim} T^*\mathcal{G} \times T^*\mathcal{G} \times T^*\mathcal{G}$$

[5]This notion was suggested to the author by K. Mackenzie in a private conversation (2005).

for the Lie bialgebroids $(A_V(\mathcal{S}), A_V(\mathcal{S})^*)$, respectively $(T^*\mathcal{G}, T\mathcal{G})$, on the first two components and $(A_V(\mathcal{S}), -A_V(\mathcal{S})^*)$, respectively $(-T^*\mathcal{G}, T\mathcal{G})$ on the last (note that the first is the Lie bialgebroid of the symplectic groupoid $\overline{\mathcal{S}} \rightrightarrows \mathcal{G}$ and the second is the Lie bialgebroid associated with the Poisson manifold $\overline{\mathcal{G}}$). Let $\mathsf{L}(\mu_H)$ be the graph of the top multiplication of the $\mathcal{LA}$-groupoid of $\mathcal{S}$, namely the Lie algebroid of the graph of the top horizontal multiplication μ_H of $\mathcal{S}$, and regard it as a Lie groupoid over N^*M. For all composable $a_\pm \in \mathsf{L}(\mu_H)$ and $\delta g_\pm \in T\mathcal{G}$, we have

$$\begin{aligned}\langle\, \phi^{-1}_{-\omega}(a_+ \cdot_{A_V(\mathcal{S})} a_-)\,,\, \delta g_+ \cdot_{T\mathcal{G}} \delta g_-\,\rangle &= \omega(a_+ \cdot_{A_V(\mathcal{S})} a_-, \mathrm{d}\varepsilon_V(\delta g_+ \cdot_{T\mathcal{G}} \delta g_-)) \\ &= \omega(a_+, \mathrm{d}\varepsilon_V \delta g_+) + \omega(a_-, \mathrm{d}\varepsilon_V \delta g_-) \\ &= -\langle\, \phi^{-1}_{\omega}(a_+)\,,\, \delta g_+\,\rangle - \langle\, \phi^{-1}_{\omega}(a_-)\,,\, \delta g_-\,\rangle \qquad ,\end{aligned}$$

that is, Φ identifies $\mathsf{L}(\mu_H) \subset A_V(\mathcal{S} \times \mathcal{S} \times \overline{\mathcal{S}})$ with $(T\Gamma(\mu))^o \equiv N^*\Gamma(\mu) \subset T^*(\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}})$. Therefore, by coisotropicity of $\Gamma(\mu)$ (use theorem 1.4.14, the connectivity requirements are implied by the strong transversality assumption), $\Gamma(\mu_H) \subset \mathcal{S} \times \mathcal{S} \times \overline{\mathcal{S}}$ is also Lagrangian and $\mathcal{S}$ is a symplectic double groupoid. The Poisson structure induced on (the groupoid underlying) $\mathcal{G}^\star$ makes it weakly dual to $\mathcal{G}$ by proposition 2.4.6, thus it must coincide with $\Pi^\star$ by uniqueness (theorem 1.5.7). □

The symplectic double groupoid of last theorem, when it exists, is a vertically source 1-connected double of its side horizontal Poisson groupoid; we shall discuss below a class of examples which allow for checking the transversality conditions of theorem 2.4.10.

2.4.1. The case of complete Poisson groups.

Let (G, Π) be a Poisson group. The (left) dressing action of $\mathfrak{g}^*$ on G is the infinitesimal action

$$\Upsilon : \mathfrak{g}^* \to \mathfrak{X}(G) \qquad , \qquad \Upsilon(\xi) := \Pi^\sharp \overleftarrow{\xi} \qquad ,$$

where $\overleftarrow{\xi}$ is the left invariant 1-form on G associated with $\xi \in \mathfrak{g}^*$: $\overleftarrow{\xi}_g = l^*_{g^{-1}}\xi$, $g \in G$. The (left) dressing vector fields on G are those in the image of $\mathfrak{g}$ under the left dressing action map. Left dressing vector fields do not have, in general, complete flows; when they have, e.g. when G is compact, G is called a complete Poisson group.

The left trivialization $T^*G \to \mathfrak{g}^* \ltimes G$ is always an isomorphism of Lie algebroids to the action Lie algebroid. When G is complete, the infinitesimal dressing action integrates to a global action, $\widetilde{\Upsilon} : G^\star \times G \to G$ of the dual 1-connected Poisson group $G^\star$ on G and the action groupoid $G^\star \ltimes G \rightrightarrows G$ is the source 1-connected integration of $\mathfrak{g} \ltimes G$. Provided G is 1-connected (in this case $G^\star$ is also complete [**36**]) the same argument applies to the integration of the infinitesimal right dressing action

$\mathfrak{g}^* \to \mathfrak{X}(G)$, defined using $\Pi^\star$, to a global action $G^\star \times G \to G^\star$. One can show that the 1-connected integration D of the Drinfel'd double $\mathfrak{d}$ is isomorphic to bitwisted product $G^\star \bowtie G$ carrying two further symplectic groupoid structures over G and $G^\star$ [**26, 36**] and making $(G^\star \bowtie G, G; G^\star, \bullet)$ a double of the Poisson group on G. A construction of a double in the noncomplete case, for a 1-connected G was given in [**39**] by Lu and Weinstein.

Example 2.4.11. Lu and Weinstein's double [**39**]. When G, $G^\star$ and D are 1-connected there exist integrations $\lambda : G \hookrightarrow D$, respectively $\rho : G^\star \hookrightarrow D$, of $\mathfrak{g} \hookrightarrow \mathfrak{d}$, respectively $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$. Moreover, one can show that D has a compatible Poisson structure π_D, which happens to be nondegenerate on the submanifold of elements d, admitting a decomposition $d = \lambda(g_+)\rho(g_+^\star) = \rho(g_-^\star)\lambda(g_-)$, $g_\pm \in G$ and $g_\pm^\star \in G^\star$. Moreover there is also a natural double Lie groupoid

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G^\star \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd} \qquad \mathcal{D} = G \times G^\star \times G^\star \times G \quad .$$

It turns out that the double subgroupoid, whose total space is

$$\mathcal{S} = \{(g_+, g_+^\star, g_-^\star, g_-) \mid \lambda(g_+)\rho(g_+^\star) = \rho(g_-^\star)\lambda(g_-)\}$$

carries a compatible symplectic form, inducing the Poisson structures on G and $G^\star$, which is the inverse of the pullback of π_D, under the natural local diffeomorphism $\mathcal{S} \to D$.

We remark that the total space of Lu-Weinstein's double is only locally diffeomorphic to D, unless G is complete, therefore it is in general neither source (1-)connected over G, nor over $G^\star$. Moreover, the 1-connectivity of G is essential in both constructions.
In the complete case it is possible to drop the connectivity assumptions on G and, nevertheless, to obtain vertically source 1-connected double.

Theorem 2.4.12. [**64**] *For any complete Poisson group G, the source 1-connected symplectic groupoid $\mathcal{S}$ of G carries a unique Lie groupoid structure over the 1-connected dual Poisson group $G^\star$ making it a double of G.*

The transversality conditions of theorem 2.4.10 are met if the top source map of the cotangent prolongation groupoid has the $\mathcal{LA}$-homotopy lifting properties 2.3.5, therefore theorem 2.4.12 is a consequence of the following

Lemma 2.4.13. *For any complete Poisson group G, the cotangent source map $\hat{\mathrm{s}} : T^*G \to \mathfrak{g}^*$ has the 0- and 1-$\mathcal{LA}$-homotopy lifting properties.*

PROOF. Note that the diagram

$$\begin{tikzcd} T^*G \arrow[r, "\sim"] \arrow[dr, "\hat{s}"'] & \mathfrak{g}^* \ltimes G \arrow[d, "\mathrm{pr}"] \\ & \mathfrak{g}^* \end{tikzcd}$$

commutes in the category of Lie algebroids for the left trivialization on the top edge: it suffices to prove the statement for the projection $\mathfrak{g}^* \ltimes G \to \mathfrak{g}^*$. $\mathfrak{g}^* \ltimes G$-paths are pairs (ξ, γ) of $\mathfrak{g}^*$-paths and paths in G, such that $\Upsilon(\xi(u))_{\gamma(u)} = \mathrm{d}\gamma_u$, $u \in I$. A Lie algebroid homotopy $h_\ltimes : TI^{\times 2} \to \mathfrak{g}^* \ltimes G$ is a pair (h, X) for which h is a $\mathfrak{g}^*$-homotopy and $X : I \to G$ satisfies $\mathrm{d}X = \Upsilon \circ h$. To see this, note that $h_\ltimes$ takes values in the pullback bundle $X^+(\mathfrak{g}^* \ltimes G) \simeq \mathfrak{g}^* \times I^{\times 2}$. The bracket compatibility for $h_\ltimes$ can be written choosing the de Rham differential on G with coefficients in $\mathfrak{g}^*$ as a linear connection ∇ for $\mathfrak{g}^* \ltimes G$; this way the covariant derivative for the pullback connection is simply the de Rham differential on $I^{\times 2}$ with coefficients in $\mathfrak{g}^*$. The torsion tensor of ∇ is

$$\begin{aligned} \tau^\nabla(F_+, F_-) &= \mathrm{d}F_{-g}(\rho^\ltimes(F_+)) - \mathrm{d}F_{+g}(\rho^\ltimes(F_-)) - [F_+ \overset{\ltimes}{;} F_-]_g \\ &= -[F_+(g) \overset{*}{;} F_-(g)] \quad , \end{aligned}$$

$F_\pm \in \mathcal{C}^\infty(G, \mathfrak{g}^*)$, thus the pullback of τ^∇ induces the canonical graded Lie bracket, also denoted by $[\,\overset{*}{;}\,]$, on $\Omega^\bullet(I^{\times 2}, \mathfrak{g}^*)$ and the bracket compatibility condition for $h_\ltimes$ reduces to the classical Maurer-Cartan equation

$$\mathrm{d}h + \frac{1}{2}[\, h \overset{*}{;} h \,] = 0$$

for $h \in \Omega^1(I^{\times 2}, \mathfrak{g}^*)$. Suppose now (ξ_-, γ_-) is a fixed $\mathfrak{g}^* \ltimes G$-path, let h be a $\mathfrak{g}^*$-homotopy form ξ_- to some other $\mathfrak{g}^*$-path ξ_+ and $H : I^{\times 2} \to G^\star$ the unique $G^\star$-homotopy integrating h, i.e. such that

$$h = \delta_r^u H \cdot du + \delta_r^\varepsilon H \cdot d\varepsilon$$

for the partial right derivatives. We claim that $h_\ltimes := (h, X)$, where

$$\begin{aligned} X(u, \varepsilon) &= (H(u, \varepsilon) \cdot H(u, 0)^{-1}) * \gamma_-(u) \\ &= H(u, \varepsilon) * (H(u, 0)^{-1} * \gamma_-(u)) \quad , \end{aligned}$$

$u, \varepsilon \in I$, is a $\mathfrak{g}^* \ltimes G$-homotopy; here $*$ denotes the integrated dressing action map. We postpone to the end of the proof this straightforward but lengthy check. The lifting conditions follow: by construction

$$\begin{aligned} \iota^*_{\partial^\pm_H}(h, X) &= (\iota^*_{\partial^\pm_H} h, X \circ \iota_{\partial^\pm_H}) = 0 \\ \iota^*_{\partial^\pm_V}(h, X) &= (\xi_\pm, \gamma_\pm) \end{aligned}$$

for some path γ_+ in G, (l_0) If h is a fixed $\mathfrak{g}^*$-homotopy to some fixed $\mathfrak{g}^*$-path ξ_+, (ξ_+, γ_+) is the desired lift. (l_1) In particular if ξ_- is $\mathfrak{g}^* \ltimes G$-homotopic to the constant $\mathfrak{g}^* \ltimes G$-path $\xi_o \equiv 0$ and h a $\mathfrak{g}^*$-homotopy from ξ_- to the constant $\mathfrak{g}^*$-path $\xi_+ \equiv 0$, we have

$$\mathrm{d}\gamma_+|_u = \Upsilon(\xi_+(u))_{\gamma_+(u)} = 0_{\gamma_+(u)} \qquad ,$$

hence $\gamma_+(u) \equiv \gamma_+(0) = \gamma_-(0)$, since the base paths of homotopic Lie algebroid paths are homotopic relatively to the endpoints; (h, X) is then the desired homotopy. In order to prove our claim it remains to check the anchor compatibility condition for (h, X); set $a = \delta_r^u H$ and $b = \delta_r^\varepsilon H$. The derivative of X in the ε-direction is thus

$$\begin{aligned} \partial_\varepsilon X(u,\varepsilon) &= \Upsilon(\partial_\varepsilon H(u,\varepsilon))_{H(u,0)^{-1}*\gamma_-(u)} = \Upsilon(\mathrm{d}r_{H(u,\varepsilon)}b(u,\varepsilon))_{H(u,0)^{-1}*\gamma_-(u)} \\ &= \Upsilon(b(u,\varepsilon))_{X(u,\varepsilon)} \qquad , \end{aligned}$$

since for all $g \in G$, $h^\star \in G^\star$, $\xi = \dot{g}^\star(o) \in \mathfrak{g}^*$ and path $g^\star$ in $G^\star$

$$\Upsilon(\mathrm{d}r_{h^\star}\xi) = \left.\frac{d}{d\alpha}\right|_{\alpha=o} \widetilde{\Upsilon}(g^\star(\alpha)\cdot h^\star, g) = \left.\frac{d}{d\alpha}\right|_{\alpha=o} \widetilde{\Upsilon}(g^\star(\alpha), h^\star * g) = \Upsilon(\xi)_{h^\star * g} \quad .$$

The computation of the derivative in the u-direction is more involved. We have

$$\partial_u X(u,\varepsilon) = \mathrm{d}\widetilde{\Upsilon}_{(H(u,\varepsilon)\cdot H(u,0)^{-1},\gamma_-(u))}(\delta H, \dot{\gamma}_-(u)) \qquad ,$$

where $\quad \dot{\gamma}_-(u) = \Upsilon(\xi_-(u))_{\gamma_-(u)} = \Upsilon((\iota^*_{\partial_H^-} h)(u)) = \Upsilon(a(u,0))_{X(u,0)} \quad$ and

$$\begin{aligned} \delta H &= \mathrm{d}\mu^\star_{(H(u,\varepsilon),H(u,0)^{-1})}(\partial_u H(u,\varepsilon), \mathrm{d}\iota_\star \partial_u H(u,0)) \\ &= \mathrm{d}\mu^\star_{(H(u,\varepsilon),H(u,0)^{-1})}(\mathrm{d}r_{H(u,\varepsilon)}a(u,\varepsilon), \mathrm{d}\iota_\star \mathrm{d}r_{H(u,0)}a(u,0)) \\ &= \mathrm{d}\mu^\star_{(H(u,\varepsilon),H(u,0)^{-1})}(0_{H(u,\varepsilon)}, \mathrm{d}\iota_\star \mathrm{d}r_{H(u,0)}a(u,0)) \\ &+ \mathrm{d}\mu^\star_{(H(u,\varepsilon),H(u,0)^{-1})}(\mathrm{d}r_{H(u,\varepsilon)}a(u,\varepsilon), 0_{H(u,0)^{-1}}) \\ &= \delta H_+ + \delta H_- \qquad , \end{aligned}$$

since $\mathrm{d}\mu^\star : TG^\star \times TG^\star \to TG^\star$ is fibrewise linear, with

$$\begin{aligned} \delta H_+ &= \mathrm{d}l_{H(u,\varepsilon)}\mathrm{d}\iota_\star \mathrm{d}r_{H(u,0)}a(u,0) = \mathrm{d}l_{H(u,\varepsilon)}\mathrm{d}l_{H(u,0)^{-1}}\mathrm{d}\iota_\star a(u,0) \\ &= \mathrm{d}l_{H(u,\varepsilon)\cdot H(u,0)^{-1}}\mathrm{d}\iota_\star a(u,0) \\ \delta H_- &= \mathrm{d}r^{-1}_{H(u,0)}\mathrm{d}r_{H(u,\varepsilon)}a(u,\varepsilon) = \mathrm{d}r_{H(u,\varepsilon)\cdot H(u,0)^{-1}}a(u,\varepsilon) \qquad . \end{aligned}$$

The tangent action map $\mathrm{d}\widetilde{\Upsilon} : TG^\star \times TG \to TG$ is also fibrewise linear, hence

$$\begin{aligned} \partial_u X(u,\varepsilon) &= \mathrm{d}\widetilde{\Upsilon}_{(H(u,\varepsilon)\cdot H(u,0)^{-1},\gamma_-(u))}(\delta H_+, \Upsilon(a(u,0))_{\gamma_-(u)}) \\ &+ \mathrm{d}\widetilde{\Upsilon}_{((H(u,\varepsilon)\cdot H(u,0),\gamma_-(u))}(\delta H_-, 0_{\gamma_-(u)}); \end{aligned}$$

the first term of last expression vanishes, since it can be rewritten as

$$
\begin{aligned}
&\mathrm{d}\widetilde{\Upsilon}_{@}(\mathrm{d}l_{(H(u,\varepsilon)\cdot H(u,0)^{-1})}\mathrm{d}\iota_{\star}a(u,0),\Upsilon(a(u,0))_{\gamma_-(u)}\\
=&\mathrm{d}\widetilde{\Upsilon}_{@}(\mathrm{d}l_{H(u,\varepsilon)\cdot H(u,0)^{-1}}\mathrm{d}\iota_{\star}a(u,0),\mathrm{d}\widetilde{\Upsilon}_{(e_\star,\gamma_-(u))}(a(u,0),0_{\gamma_-(u)}))\\
=&\mathrm{d}\widetilde{\Upsilon}_{@}(\mathrm{d}\mu^{\star}_{(H(u,\varepsilon)\cdot H(u,0)^{-1},e_\star)}(\mathrm{d}l_{(H(u,\varepsilon)\cdot H(u,0)^{-1})}\mathrm{d}\iota_{\star}a(u,0),a(u,0)),0_{\gamma_-(u)})\\
=&\mathrm{d}\widetilde{\Upsilon}_{@}(0_{H(u,\varepsilon)\cdot H(u,0)^{-1}},0_{\gamma_-(u)})=0
\end{aligned}
$$

by equivariance, where we have set $@=(H(u,\varepsilon)\cdot H(u,0)^{-1},\gamma_-(u))$ to simplify the expressions; therefore

$$
\begin{aligned}
\partial_u X(u,\varepsilon) &= \mathrm{d}\widetilde{\Upsilon}_{((H(u,\varepsilon)\cdot H(u,0),\gamma_-(u))}(\mathrm{d}r_{H(u,\varepsilon)\cdot H(u,0)^{-1}}a(u,\varepsilon),0_{\gamma_-(u)})\\
&= \mathrm{d}\widetilde{\Upsilon}_{(e_\star,H(u,\varepsilon)\cdot H(u,0)^{-1}*\gamma_-(u))}(a(u,\varepsilon),0_{H(u,\varepsilon)\cdot H(u,0)^{-1}*\gamma_-(u)})\\
&= \Upsilon(a(u,\varepsilon))_{X(u,\varepsilon)} \qquad .
\end{aligned}
$$

We just have shown that

$$
\begin{aligned}
\mathrm{d}X_{(u,\varepsilon)} &= \Upsilon(\delta_r^{\varepsilon}H(u,\varepsilon))_{X(u,\varepsilon)}\cdot d\varepsilon+\Upsilon(\delta_r^{u}H(u,\varepsilon))_{X(u,\varepsilon)}\cdot du\\
&= \Upsilon(h(u,\varepsilon))_{X(u,\varepsilon)}
\end{aligned}
$$

and this concludes the proof. $\square$

CHAPTER 3

Morphic actions

On the one hand Poisson manifolds behave well under reduction for actions as general as those of Poisson groupoids: a Poisson bivector always descends to the quotient by a free and proper compatible action. On the other hand, Poisson manifolds are not always integrable to symplectic groupoids; the natural question is thus: *Are quotients of integrable Poisson manifolds also integrable?*
A positive answer was recently given by Fernandes-Ortega-Ratiu [**22**] in the case of Lie group actions and Lu [**38**] gave a construction of symplectic groupoids for certain Poisson homogeneous spaces. In the case of Poisson group actions with a complete moment map, Xu described in [**74**] a reduction procedure on a lifted moment map; when the reduced space of the latter is smooth, it is a symplectic groupoid for the quotient Poisson structure.
In fact, lifting processes naturally produce stronger symmetries out of weaker ones, often associated with suitable moment maps which can be used to compute information on the original action. Consider for instance the action of a Lie group G on an ordinary manifold M. It was already remarked by Smale [**62**] that it is possible to *cotangent lift* the action to a symplectic action of G on T^*M; such a lift is always endowed with an equivariant moment map $\hat{j} : T^*M \to \mathfrak{g}^*$ in the sense of Marsden-Weinstein [**49**], morally the dual

$$\langle \, \hat{j}\,(\theta_q) \, , \, \xi \, \rangle = \langle \, \theta_g \, , \, \sigma(\xi) \, \rangle \tag{3.1}$$

of the infinitesimal action [**50**]. *Path prolongation* is another lifting procedure which generates moment maps. Let a Lie group G act on a symplectic manifold (M, ω) by symplectic diffeomorphisms. The space $\Pi(M)$ of paths in M up to homotopy relative to the endpoints carries a natural symplectic form and a G-action. Mikami and Weinstein observed in [**50**] that there always exist an equivariant moment map $J : \Pi(M) \to \mathfrak{g}^*$

$$\langle \, J([\gamma]) \, , \, \xi \, \rangle = \int_\gamma \iota_{\sigma(\xi)}\omega$$

associated with this action. An analogous lifting procedure via Lie algebroid paths can be applied to a Poisson action of a Lie group on an integrable Poisson manifold P, with symplectic groupoid Λ, to obtain a symplectic action on Λ, with equivariant moment map $\Lambda \to \mathfrak{g}^*$ [**21**]; this generalization of Mikami and Weinstein's moment map, was used in [**22**] to produce an integration of P/G.

It is thus quite a general phenomenon that lifting processes tend to enhance symmetries; this effect is more transparent in the general setting of Poisson geometry and when more structure is around. Indeed cotangent lifting and path prolongation are best understood as instances of "duality" between Poisson geometric objects and Lie algebroids-Lie groupoids, and lead to interesting examples of double Lie structures.
Suppose now a Poisson group G acts on a Poisson manifold P. The cotangent lifted moment map $\hat{j} : T^*P \to \mathfrak{g}^*$ (3.1) preserves the Poisson structures, in the sense that it is a morphism of Lie bialgebroids (fig. 2), and it is still equivariant, provided the suitable cotangent lift of G is correctly identified. In fact $\hat{j}$ is equivariant for the coadjoint action on the codomain, but, on the domain, the classical cotangent lift needs to be replaced with an action, which is symplectic in the sense of Mikami and Weinstein [**50**], of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$. Remarkably, the compatibility of the Poisson group action with the Poisson bivector fields is recovered in the dual description in terms of Lie algebroids.

$$\begin{tikzcd} T^*G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathfrak{g}^* \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

Figure 1

$$\begin{tikzcd} T^*P \arrow[r, "\hat{j}"] \arrow[d] & \mathfrak{g}^* \arrow[d] \\ P \arrow[r] & \bullet \end{tikzcd}$$

Figure 2

$$\begin{tikzcd} T^*G \ltimes T^*P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T^*P \arrow[d] \\ G \ltimes P \arrow[r, shift left] \arrow[r, shift right] & P \end{tikzcd}$$

Figure 3

In fact, $T^*G \rightrightarrows \mathfrak{g}^*$ extends to the cotangent prolongation $\mathcal{LA}$-groupoid (Fig. 1), and the cotangent lifted action induces the $\mathcal{LA}$-groupoid in figure 3, obtained by Mackenzie in [**44**], which completely encodes a Poisson action dually in the category of Lie algebroids and, roughly, lifts it to the category of symplectic manifolds. It was also observed in [**44**] that the zero level reduction of $\hat{j}$ can be used to compute the Lie bialgebroid of the quotient Poisson bivector on P/G.

The study of a Poisson action of a Poisson groupoid on a Poisson manifold, as we shall see in Section 3.1, naturally leads to – and is equivalent to – a compatible action of the cotangent lifted $\mathcal{LA}$-groupoid, respectively, under integrability conditions, a compatible action of a symplectic double groupoid, on a canonical "moment morphism" of Lie algebroids, respectively of Lie groupoids. This motivates the study we carry on in this Chapter of the reduction of morphic actions in the categories of Lie algebroids and Lie groupoids, namely groupoid actions in these categories for which the associated action groupoid is an object.
In Section 3.1 we introduce morphic actions of $\mathcal{LA}$-groupoids and perform the cotangent lift of a Poisson groupoid action to a morphic action in the category of Lie algebroids (theorem 3.8); by specializing to the Poisson group case, we obtain

an alternative construction for Mackenzie's lift of [**44**][1].
In Section 3.2 we study the reduction of morphic actions of $\mathcal{LA}$-groupoids and prove the following general result:

Theorem (3.2.1). *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid acting morphically on a morphism of Lie algebroids $\hat{j} : B \to A$ over $j : N \to M$. If the action is free and proper (so that B/Ω and $N/\mathcal{G}$ are smooth manifolds) then there exists a unique Lie algebroid on $B/\Omega \to N/\mathcal{G}$ making the quotient projection $B \to B/\Omega$ a strong $\mathcal{LA}$-fibration over $N \to N/\mathcal{G}$.*

Thereafter we consider the reduction of the cotangent lifted moment morphism and give a characterization of the Koszul algebroid associated with a quotient Poisson manifold as a quotient Lie algebroid (proposition 3.2.26).
In Section 3.3 we consider the reduction of morphic actions of double Lie groupoids and the integrability of morphic actions of $\mathcal{LA}$-groupoids. The symplectic case is dealt with in Section 3.4, where we obtain the following

Theorem (3.4.1). *Let $(\mathcal{S}, \mathcal{G}; \mathcal{G}^\bullet, M)$ be a symplectic double groupoid acting morphically on a morphism of Lie groupoids $\mathcal{J} : \Lambda \to \mathcal{G}^\bullet$ over $j : P \to M$, where $\Lambda \rightrightarrows P$ is a symplectic groupoid, in such a way that Λ is a symplectic $\mathcal{S}$-space. If $\mathcal{J}$ is source submersive and the side action is free and proper, then*

i) The reduced kernel groupoid $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G} \rightrightarrows P/\mathcal{G}$ carries a unique symplectic form making the projection $\mathcal{J}^{-1}(\varepsilon_\bullet(M)) \to \mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G}$ a Poisson submersion;

ii) $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G} \rightrightarrows P/\mathcal{G}$ is a symplectic groupoid for the quotient Poisson manifold $P/\mathcal{G}$.

Last result can be applied to an integrable Poisson $\mathcal{G}$-space P, provided the Poisson groupoid $\mathcal{G}$ is integrable to a symplectic double groupoid, in order to produce a symplectic groupoid for the quotient Poisson bivector; this approach is effective, for instance, in the case of complete Poisson group actions, discussed at the end of this Chapter. We obtain Xu's reduction of [**74**] and Fernandes-Ortega-Ratiu integration of [**22**] as special cases of our approach to the integration of quotients of complete Poisson group actions, developed in [**64**].
In the last Section we also use an $\mathcal{LA}$-path-prolongation of the cotangent lifted action associated with a a Poisson $\mathcal{G}$-space P to derive our main application

Theorem (3.4.4). *Let a Poisson groupoid $\mathcal{G} \rightrightarrows M$ act freely and properly on $j : P \to M$. If the action is Poisson, then $P/\mathcal{G}$ is integrable to a symplectic groupoid iff $\ker \hat{j}$ is an integrable Lie algebroid.*

[1]The chance of extending the results of [**44**] to Poisson actions of Poisson groupoids and the method used below for this generalization emerged from private conversations with Kirill Mackenzie (January 2007).

Here $\hat{j} : T^*P \to A^*$ is a lifted moment map canonically associated with the original action, where A is the Lie algebroid of $\mathcal{G}$ and $\ker \hat{j} \subset T^*P$ is the vertical subbundle for the infinitesimal action of $\Gamma(A)$.
In particular, we give a positive answer to the above question under most natural assumptions.

Corollary (3.4.5). *If P is integrable, then so is $P/\mathcal{G}$.*

Definitions and remarks on groupoid actions in a category.

Let $\mathcal{G} \rightrightarrows M$ be a groupoid acting on a moment map $j : N \to M$; denote with $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \to N$ the action map. The action can be fully described in terms of diagrams

$$
\begin{tikzcd}
 & \mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\sigma"] & \\
\mathcal{G} \arrow[dr, "\mathrm{t}"'] & & N \arrow[dl, "j"] \\
 & M &
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[r, "\mathsf{id}_N"] \arrow[d, "\Delta_N"'] & N \\
\Delta_N \arrow[dr, "j\times\mathsf{id}_N"'] & \mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \arrow[u, "\sigma"'] \\
 & M \times_j N \arrow[u, "\varepsilon\times\mathsf{id}_N"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{G}^{(2)}\,{}_{\mathrm{s}\circ\mathrm{pr}_1}\!\times_j N = \mathcal{G}\,{}_{\mathrm{s}}\!\times_\sigma (\mathcal{G}\,{}_{\mathrm{s}}\!\times_j N) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\sigma"] & \\
\mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \arrow[dr, "\mathrm{t}"'] & & \mathcal{G}\,{}_{\mathrm{s}}\!\times_j N \arrow[dl, "j"] \\
 & M &
\end{tikzcd}
. \tag{3.2}
$$

compatibility with the moment map — unitality — multiplicativity

This leads the abstract notion of a morphic action, namely a groupoid action in a category.

Definition 3.0.14. Let $\mathbf{C}$ be a small category with direct and fibered products, $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ a groupoid object in $\mathbf{C}$ and $\mathbf{j} : \mathbf{N} \to \mathbf{M}$ an arrow. A groupoid action of $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ on $\mathbf{j}$ is a morphic action if the action map $\boldsymbol{\sigma} : \boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\!\times_{\mathbf{j}} \mathbf{N} \to \mathbf{N}$ is an arrow.

Morphic actions are precisely those for which the associated action groupoid stays in the category.

Proposition 3.0.15. *Let* $\mathbf{C}$ *be a category with fibred products. A groupoid action of a groupoid object* $\boldsymbol{\mathcal{G}} \rightrightarrows \mathbf{M}$ *on an arrow* $\mathbf{j} : \mathbf{N} \to \mathbf{M}$ *is morphic iff the action groupoid* $\boldsymbol{\mathcal{G}} \ltimes \mathbf{N} \rightrightarrows \mathbf{N}$ *is a groupoid object.*

Proof. The implication to the left is clear, since the action map is the target of the action groupoid. For the implication to the right consider that the source map $\mathbf{s}^{\ltimes} : \boldsymbol{\mathcal{G}} \ltimes \mathbf{N} \to \mathbf{N}$ is the projection to the second factor of the fibered product $\boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\!\times_{\mathbf{j}} \mathbf{N}$, hence an arrow. Moreover, the unit section $\boldsymbol{\varepsilon}^{\ltimes} : \mathbf{N} \to \boldsymbol{\mathcal{G}} \ltimes \mathbf{N}$ admits a factorization

$$
\begin{tikzcd}
\mathbf{N} \arrow[r, "\boldsymbol{\varepsilon}^{\ltimes}"] \arrow[d, "\Delta"] & \boldsymbol{\mathcal{G}} \ltimes \mathbf{N} \\
\Delta_{\mathbf{N}} \arrow[r, "\mathsf{s}\times\mathsf{id}_{\mathbf{N}}"'] & \mathbf{M} \times_{\mathbf{j}} \mathbf{N} \arrow[u, "\varepsilon\times\mathsf{id}_{\mathbf{N}}"']
\end{tikzcd}
$$

in terms of arrows, therefore it is also an arrow. According to proposition 2.0.15, it now suffices to show that the division map $\boldsymbol{\delta}^{\ltimes} : \boldsymbol{\mathcal{G}} \ltimes \mathbf{N}\,{}_{\mathbf{s}^{\ltimes}}\!\times_{\mathbf{s}^{\ltimes}} \boldsymbol{\mathcal{G}} \ltimes \mathbf{N} \to \boldsymbol{\mathcal{G}} \ltimes \mathbf{N}$,

$$
\boldsymbol{\delta}^{\ltimes}((g_+, n), (g_-, n)) = (g_+ \cdot g_-^{-1}, g_- * n) = (\boldsymbol{\delta}(g_+, g_-), \boldsymbol{\sigma}(g_-, n)) \qquad ,
$$

is an arrow; this holds true, since there is a factorization

$$\begin{tikzcd}
 & \boldsymbol{\mathcal{G}} \ltimes \mathbf{N}\,{}_{\mathbf{s}^{\ltimes}}\times_{\mathbf{s}^{\ltimes}} \boldsymbol{\mathcal{G}} \ltimes \mathbf{N} \arrow[rr, "\boldsymbol{\delta}^{\ltimes}"] \arrow[dl, "\sim"'] & & \boldsymbol{\mathcal{G}} \ltimes \mathbf{N} & \\
(\boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\times_{\mathbf{s}} \boldsymbol{\mathcal{G}})\,{}_{\mathbf{s}\times\mathbf{s}}\times_{\mathbf{j}\times\mathbf{j}} \Delta_{\mathbf{N}} \arrow[dr, "\sim"'] & & & & (\boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\times_{\mathbf{s}} \boldsymbol{\mathcal{G}})\,{}_{\mathbf{s}\circ\mathrm{pr}_2}\times_{\mathbf{s}\circ\mathrm{pr}_1} \boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\times_{\mathbf{j}} \mathbf{N} \arrow[ul, "\boldsymbol{\delta}\times\boldsymbol{\sigma}"'] \\
 & (\boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\times_{\mathbf{s}} \boldsymbol{\mathcal{G}})\,{}_{\mathbf{s}\circ\mathrm{pr}_2}\times_{\mathbf{j}} \mathbf{N} \arrow[rr, "\sim"'] & & (\boldsymbol{\mathcal{G}}\,{}_{\mathbf{s}}\times_{\mathbf{s}\circ\mathrm{pr}_1} \Delta_{\boldsymbol{\mathcal{G}}})\,{}_{\mathbf{s}\circ\mathrm{pr}_3}\times_{\mathbf{j}} \mathbf{N} \arrow[ur, "\sim"'] &
\end{tikzcd} \tag{3.3}$$

in terms of arrows. ☐

The same remark as in the introduction to Chapter 2 applies: if $\mathbf{C}$ does not posses general fibered products, the statement of last proposition still holds true, up to restricting to a suitable class of groupoid objects for which the relevant fibered products exist. As we shall see in Sections 3.1 and 3.3, this is the case for the categories of Lie algebroids and Lie groupoids, restricting to $\mathcal{LA}$-groupoids and double Lie groupoids.

3.1. Morphic actions in the category of Lie algebroids

In this Section we introduce morphic actions in the category of Lie algebroids, namely compatible actions of $\mathcal{LA}$-groupoids on morphisms of Lie algebroids. Furthermore, we produce our main example, the cotangent lift of a Poisson groupoid action. As it was remarked by He-Liu-Zhong [**23**], to a Poisson $\mathcal{G}$-space P for a Poisson groupoid $\mathcal{G}$ one can always associate a canonical morphism of Lie bialgebroids $\hat{j}$, which is to be regarded as a moment map for an action of the Lie bialgebroid of $\mathcal{G}$. We show that $\hat{j}$ is indeed a moment map for an action of the cotangent prolongation groupoid and such cotangent lifted action determines a morphic action of the cotangent prolongation $\mathcal{LA}$-groupoid of $\mathcal{G}$ on the Koszul algebroid of P. As a consequence to a Poisson $\mathcal{G}$-space we can associate an action $\mathcal{LA}$-groupoid, fully encoding the compatibility of the original action, in the category of Lie algebroids.

3.1.1. Morphic actions of Lie algebroids and action $\mathcal{LA}$-groupoids.

Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid and j a morphism of Lie algebroids

$$\mathsf{\Omega} := \begin{tikzcd} \Omega \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\widehat{\ }" above] \arrow[d] & A \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \mathsf{j} := \begin{tikzcd} B \arrow[r, "\hat{j}"] \arrow[d] & A \arrow[d] \\ N \arrow[r, "j"'] & M \end{tikzcd} \quad .$$

Suppose that $\Omega \rightrightarrows A$ acts on $\hat{j}$ and $\mathcal{G} \rightrightarrows M$ acts on j in such a way that the corresponding action maps

$$\boldsymbol{\sigma} := \begin{tikzcd} \Omega \,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} B \arrow[r, "\hat{\sigma}"] \arrow[d] & B \arrow[d] \\ \mathcal{G} \,{}_{\mathrm{s}}\!\times_{j} N \arrow[r, "\sigma"'] & M \end{tikzcd}$$

define a vector bundle map. Note that the vector bundle on the left in the last diagram is always well defined, since $\hat{\mathrm{s}}$ is an $\mathcal{LA}$-fibration over s, and it carries a fibered product Lie algebroid making it a Lie subalgebroid of the direct product Lie algebroid $\Omega \times A \to \mathcal{G} \times N$, since both $\hat{\mathrm{s}}$ and $\hat{j}$ are morphisms of Lie algebroids. Moreover, it is always possible to form a diagram

$$\mathsf{\Omega} \ltimes \mathsf{B} := \begin{tikzcd} \Omega \ltimes B \arrow[r, shift left, "\ltimes"] \arrow[r, shift right] \arrow[d] & B \arrow[d] \\ \mathcal{G} \ltimes N \arrow[r, shift left] \arrow[r, shift right] & N \end{tikzcd}$$

for the action Lie groupoids $\Omega \ltimes B \rightrightarrows B$ and $\mathcal{G} \ltimes N \rightrightarrows N$.

Proposition 3.1.1. $\mathsf{\Omega} \ltimes \mathsf{B}$ *is an $\mathcal{LA}$-groupoid iff $\boldsymbol{\sigma}$ is morphic.*

Proof. The implication to the right is clear, since σ is the target top map of $\mathsf{\Omega} \ltimes \mathsf{B}$. Suppose that $\boldsymbol{\sigma}$ is morphic. According to proposition (3.0.15), to have $\mathsf{\Omega} \ltimes \mathsf{B}$ a groupoid object, it suffices to show that the fibered product Lie algebroid $\Omega \ltimes B \,{}_{\hat{\mathrm{s}}^{\ltimes}}\!\times_{\hat{\mathrm{s}}^{\ltimes}} \Omega \ltimes B$ exists; in that case the diagram (3.3) commutes in the category of Lie algebroids. By proving that $\hat{\mathrm{s}}^{\ltimes}$ is an $\mathcal{LA}$-fibration over $\mathrm{s}^{\ltimes}$, we also show that $\mathsf{\Omega} \ltimes \mathsf{B}$ is an $\mathcal{LA}$-groupoid, i.e. $\hat{\mathrm{s}}^{\ltimes}$ is fibrewise surjective. In order to check fibrewise submersivity of $\hat{\mathrm{s}}^{\ltimes}$ one can first fix $(g, n) \in \mathcal{G} \ltimes N$, $\omega \in \Omega_g$ and $b \in B_n$ with $\hat{\mathrm{s}}(\omega) = \hat{j}\,(b)$, then repeat the argument above for the tangent maps. $\square$

We shall call $\mathsf{\Omega} \ltimes \mathsf{B}$ the action $\mathcal{LA}$-groupoid associated with the morphic action. An $\mathcal{LA}$-groupoid shall be called (isotropy) free, respectively proper, if so are its top and side groupoids.

Remark 3.1.2. For any morphic action of an $\mathcal{LA}$-groupoid the associated action $\mathcal{LA}$-groupoid is free, respectively proper, iff so are the top and side groupoid actions.

The following corollary, which we get for free from proposition 3.1.1, is important in the applications.

Corollary 3.1.3. *Any morphic action* $\boldsymbol{\sigma} : \Omega \ltimes \mathsf{B} \to \mathsf{B}$ *is a strong* $\mathcal{LA}$*-fibration.*

Proof. $\boldsymbol{\sigma}$ is the top target morphism of an $\mathcal{LA}$-groupoid. $\square$

Let us describe the prototypical example of an action $\mathcal{LA}$-groupoid.

Example 3.1.4. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, $j : N \to M$ a smooth map and $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{j} N \to N$ a smooth groupoid action. Then the tangent action map $\mathrm{d}\sigma : T(\mathcal{G}\,{}_{\mathrm{s}}\!\times_{j} N) = T\mathcal{G}\,{}_{\mathrm{ds}}\!\times_{\mathrm{d}j} TN \to TN$ is a morphic action of the tangent prolongation groupoid $T\mathcal{G} \rightrightarrows TM$ on $\mathrm{d}j : TN \to TM$. The associated action $\mathcal{LA}$-groupoid

$$\begin{tikzcd}
T\mathcal{G} \ltimes TN \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TN \arrow[d] \\
\mathcal{G} \ltimes N \arrow[r, shift left] \arrow[r, shift right] & N
\end{tikzcd} \quad ,$$

is also the tangent prolongation $\mathcal{LA}$-groupoid of the action groupoid $\mathcal{G} \ltimes N$.

Remark 3.1.5. Corollary 3.1.3, implies that the tangent lifted action map is fibrewise surjective; we shall use this fact in the next Subsection to perform the cotangent lift of a Lie groupoid action. This fact can be also checked directly: for all $\delta x \in T_{g*p}P$, pick any bisection Σ_g through g, thus $\delta x = \delta g * \delta p$, with $\delta g := \mathrm{d}\mathcal{R}^{\Sigma_g}\mathrm{d}j(\delta x)$ and $\delta p = \delta g^{-1} * \delta x$.

Next we shall introduce a characteristic morphism associated with action $\mathcal{LA}$-groupoids, the moment morphism $\mathsf{J} : \Omega \ltimes \mathsf{B} \to \Omega$,

$$\begin{tikzcd}
\Omega \ltimes B \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[drr, "\mathrm{pr}_\Omega"] & B \arrow[d] \arrow[drr, "\hat{j}"] & & \\
\mathcal{G} \ltimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[drr, "\mathrm{pr}_\mathcal{G}"'] & N \arrow[drr, "j"] & \Omega \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A \arrow[d] \\
& & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd} \quad , \tag{3.4}$$

where pr_Ω and $\mathrm{pr}_\mathcal{G}$ denote the projections to the first components. According to lemma 2.2.12, the regularity of the kernel $\mathsf{K}^{\ltimes}$ of J is controlled by that of

$$(\mathrm{pr}_\Omega, \hat{j}\,) : \Omega \ltimes B \to \Omega\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} B \qquad ,$$

which is the identity map in this case. Therefore, whenever $\hat{j}$ is an $\mathcal{LA}$-fibration,

$$\mathsf{K}^{\ltimes} := \begin{array}{ccc} \hat{K} & \rightrightarrows & K \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array} \qquad , \tag{3.5}$$

is a sub-$\mathcal{LA}$-groupoid of $\Omega \ltimes \mathsf{B}$, where $\hat{K}$ and K are the wide subalgebroids on $\mathrm{Pr}_K : \mathsf{ker}\ \mathrm{pr}_\Omega \to \mathcal{G} \ltimes N$ and $\mathrm{pr}_K : \mathsf{ker}\ \hat{j} \to N$ and it is sufficient to have $\hat{j}$ a map of constant rank for $\mathsf{K}^{\ltimes}$ to be an $\mathcal{LA}$-groupoid. Actually, $\hat{K}$ is to be identified with the pullback bundle $\mathrm{s}_{\ltimes}^{+}\mathsf{ker}\ \hat{j} \to \mathcal{G} \ltimes N$ and, as a Lie groupoid, it is to be thought of as an action groupoid $\mathcal{G} \ltimes K$. In fact, the action of Ω on B restricts to an action $\sigma_K : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{\mathrm{pr}\circ\hat{j}} K = \mathcal{G}\,{}_{\mathrm{s}}\!\times_{j\circ\mathrm{pr}_K} K \to K$, $(g, k) \mapsto g *_K k$,

$$g *_K k := 0^\Omega_g * k = \hat{\sigma}(0^\Omega_g, k) \qquad ,$$

of $\mathcal{G}$ on K; note that σ_K is well defined, since $\hat{j}\,(g *_K k) = \hat{\mathrm{t}}(0^\Omega_g)$ vanishes by fibrewise linearity of $\hat{\mathrm{t}}$. The action map σ_K is indeed the restriction of $\hat{\sigma}$ to the wide Lie subalgebroid $0^\Omega_{\mathcal{G}}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} K \subset \Omega\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} B$ and, in this sense, the action of $\mathcal{G}$ on K is fibrewise linear.

3.1.2. The cotangent lift of a Poisson groupoid action.

If a Poisson groupoid acts on a Poisson manifold, the domain of the action map does not carry any natural Poisson bivector and the compatibility of the groupoid action with the Poisson structures cannot be formulated in the requirement that the action map be Poisson. There is however a natural replacement for this condition.

Definition 3.1.6. Let $\mathcal{G} \rightrightarrows M$ be a Poisson groupoid act on a moment map $j : P \to M$, where P is a Poisson manifold. A Poisson action of $\mathcal{G}$ on j is a groupoid action $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_j P \to P$, such that

$$\Gamma(\sigma) \subset \mathcal{G} \times P \times \overline{P} \qquad \text{is coisotropic.} \tag{3.6}$$

For any Poisson action such as above $j : P \to M$ is called a Poisson $\mathcal{G}$-space.

Any Lie groupoid action is a Poisson action for the zero Poisson structures. When $\mathcal{G}$ is a Lie group G and $j = P \to \bullet$ the compatibility conditions amounts to asking the action map $\sigma : G \times P \to P$ to be Poisson; a Poisson action of a symplectic groupoid on a symplectic manifold is a *symplectic groupoid action* in the sense of Mikami and Weinstein [**50**], since the graph of the action map must be Lagrangian by dimensional constraints.

Remark 3.1.7. It follows from the definition and from proposition 3.1.11 that the moment map $j : P \to M$ has to be anti-Poisson for the Poisson bivector induced by $\mathcal{G}$ on M. Note that this is the case, e.g. for the action of a Poisson groupoid on itself by left translation: $j \equiv \mathrm{t} : \mathcal{G} \to M$ and $\sigma \equiv \mu : \mathcal{G}^{(2)} \to \mathcal{G}$.

To any Lie groupoid action of $\mathcal{G} \rightrightarrows M$ on $j : N \to M$, where N is any manifold, one can associate a Poisson groupoid action of the cotangent prolongation groupoid $T^*\mathcal{G} \rightrightarrows A^*$ on the moment map $\hat{j} : T^*N \to A^*$, defined by

$$\langle \hat{j}\,(\alpha_n)\,,\, a_{j(n)} \,\rangle := \langle\, \alpha_n\,,\, \mathrm{d}\sigma(a_{j(n)}, 0_n)\,\rangle \quad , \quad \alpha_n \in T^*_n N \quad , \quad a_{j(n)} \in A_{j(n)} \quad , \tag{3.7}$$

i.e. dualizing the infinitesimal action of A; note that for all sections $a \in \Gamma(A)$ and $\alpha \in \Omega^1(N)$,

$$\langle\, \hat{j} \circ \alpha\,,\, a \circ j\,\rangle := \langle\, \alpha\,,\, X^a\,\rangle$$

for the infinitesimal action $X^\bullet : \Gamma(A) \to \mathfrak{X}(N)$, or, in terms of fibrewise linear functions, $\hat{j}^{\,*} F_a = F_{X^a}$.

Lemma 3.1.8. *The moment map $\hat{j} : T^*N \to A^*$ defined above is a Poisson map for the dual Poisson structures; moreover, if the action is locally free $\hat{j}$ has maximal rank.*

Proof. The pullback map $\hat{j}^{\,*} : \mathcal{C}^\infty(A^*) \to \mathcal{C}^\infty(T^*N)$ maps fibrewise linear, respectively fibrewise constant, functions to fibrewise linear, respectively fibrewise constant, functions. It suffices to show that $\hat{j}^{\,*}\{\,F\,,\,G\,\} = \{\,\hat{j}^{\,*}F\,,\,\hat{j}^{\,*}G\,\}$ in the cases

(i) $F = F_{a_+}$ and $G = F_{a_-}$, $a_\pm \in \Gamma(A)$,

(ii) $F = F_a$ and $G = \mathrm{pr}^* f$, $a \in \Gamma(A)$, $f \in \mathcal{C}^\infty(M)$,

(iii) $F = \mathrm{pr}^* f_+$ and $G = \mathrm{pr}^* f_-$, $f_\pm \in \mathcal{C}^\infty(M)$,

the result follows by the Leibniz rule. (i) We have

$$\begin{aligned} \{\,\hat{j}^{\,*}F_{a_+}\,,\,\hat{j}^{\,*}F_{a_-}\,\} &= \{\,F_{X^{a_+}}\,,\,F_{X^{a_-}}\,\} = F_{[\,X^{a_+}\,,\,X^{a_-}\,]} = \hat{j}^{\,*}F_{[\,a_+\,,\,a_-\,]} \\ &= \hat{j}^{\,*}\{\,F_{a_+}\,,\,F_{a_-}\,\} \end{aligned} \quad .$$

Since for all $n \in N$,

$$X^a_n(j^* f) = \langle\, \mathrm{d}f_{j(n)}\,,\, \mathrm{d}j\mathrm{d}\sigma(a_{j(n)}, 0_n)\,\rangle = \rho_{j(n)}(a)(f) \quad ,$$

(ii) follows:

$$\begin{aligned} \{\,\hat{j}^{\,*}F_a\,,\,\hat{j}^{\,*}\mathrm{pr}^* f\,\} &= \{\,F_{X^a}\,,\,\mathrm{pr}^* j^* f\,\} = \mathrm{pr}^*(X^a(j^* f)) = j^*(\rho(a)(f)) \\ &= \hat{j}^{\,*}\{\,F_a\,,\,\mathrm{pr}^* f\,\} \end{aligned} \quad .$$

Condition (iii) holds trivially, since both sides vanish. The second part of the statement is a direct consequence of the definition (3.7) of $\hat{j}$. ∎

We are now ready to dualize the tangent lift of a groupoid action.

Theorem 3.1.9. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_{j} N \to N$ a Lie groupoid action on $j : N \to M$. Then σ lifts to a Poisson action of $T^*\mathcal{G} \rightrightarrows A^*$ on $\hat{j} : T^*N \to A^*$ with action map $\hat{\sigma} : T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*N \to T^*N$, $(\theta_g, \alpha_p) \mapsto \theta_g \,\hat{*}\, \alpha_p$. The cotangent lifted action is uniquely determined in terms of the tangent lifted action $*$ by the formula*

$$\langle\, \theta_g \,\hat{*}\, \alpha_p, \delta g * \delta p \,\rangle = \langle\, \theta_g \,,\, \delta g \,\rangle + \langle\, \alpha_p \,,\, \delta p \,\rangle \tag{3.8}$$

for all $(\theta_g, \alpha_p) \in T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*N$ and $(\delta g, \delta n) \in T\mathcal{G}\,{}_{\mathrm{ds}}\!\times_{\mathrm{d}j} TN$.

Proof. First of all note that $N^*\mathsf{\Gamma}(\sigma)$ is the graph of a map. Consider that the restriction of the first projection $(T^*\mathcal{G} \times T^*N) \times T^*N \to T^*\mathcal{G} \times T^*N$ to $N^*\mathsf{\Gamma}(\sigma)$ takes values in $T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*N$. For all $a_m \in A$ with $m = \mathrm{s}(g) = j(n)$, the pair

$$\begin{aligned} \delta g &:= \mathrm{d}l_g(a_m - \mathrm{d}\varepsilon\rho(a_m)) \\ \delta n &:= \mathrm{d}\sigma(-a_m, 0_m) \end{aligned}$$

belongs to the domain of the tangent action map,

$$\begin{aligned} \mathrm{ds}\,\delta g &= \mathrm{ds}(a_m - \mathrm{d}\varepsilon\rho(a_m)) \\ &= -\rho(a_m) \equiv \mathrm{dt}(-a_m) \\ &= \mathrm{d}j\,\mathrm{d}\sigma(-a_m, 0_m) \qquad , \end{aligned}$$

and

$$\begin{aligned} \delta g * \delta n &= [0_g \bullet (a_m - \mathrm{d}\varepsilon\rho(a_m))] * [(-a_m) * 0_n)] \\ &= 0_g * [(a_m - \mathrm{d}\varepsilon\rho(a_m)) \bullet (0_m - a_m)] * 0_n \\ &= 0_g * [a_m \bullet 0_m - \mathrm{d}\varepsilon\rho(a_m) \bullet a_m] * 0_n \\ &= 0_g * [a_m - a_m] * 0_n \\ &= 0_{g*n} \qquad . \end{aligned}$$

Therefore, for all $(\theta_g, \alpha_p, \beta_{g*n}) \in N^*\mathsf{\Gamma}(\sigma)$, and $a_n \in A$

$$\begin{aligned} \langle\, \hat{\mathrm{s}}(\theta_g) - \hat{j}\,(\alpha_n) \,,\, a_m \,\rangle &= \langle\, \theta_g \,,\, l_g(a_m - \mathrm{d}\varepsilon\rho(a_m) \,\rangle - \langle\, \alpha_p \,,\, \mathrm{d}\sigma(-a_m, 0_m) \,\rangle \\ &= \langle\, \theta_g \,,\, \delta g \,\rangle + \langle\, \alpha_n \,,\, \delta n \,\rangle \\ &= -\langle\, \beta_{g*n} \,,\, \delta g * \delta n \,\rangle \\ &= 0 \qquad . \end{aligned}$$

The restriction $\lambda : N^*\mathsf{\Gamma}(\sigma) \to T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*N$ of the projection above is an isomorphism of vector bundles: by counting dimensions

$$\mathsf{rank}\, N^*\mathsf{\Gamma}(\sigma) = \mathsf{dim}\, N + \mathsf{dim}\, M = \mathsf{rank}\, T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*N \qquad ,$$

moreover

$$\mathsf{ker}_{(g,p,g*n)}\lambda = \{(0_g, 0_n, \alpha_{g*n}) \mid \alpha_{g*n}(\delta g * \delta n) = 0\,,\, (\delta g, \delta n) \in T(\mathcal{G} \ltimes N)\} = \{0\} \quad ,$$

by fibrewise surjectivity of the tangent lifted action map. Hence, one can set

$$\mathsf{\Gamma}(\hat\sigma) := (\mathsf{id}_{T^*\mathcal{G}} \times \mathsf{id}_{T^*N} \times -\mathsf{id}_{T^*N})\, N^*\mathsf{\Gamma}(\sigma) \subset T^*\mathcal{G} \times T^*N \times T^*N$$

to define a bundle map $\hat\sigma$ over σ satisfying (3.8); the graph of $\hat\sigma$ is Lagrangian in $T^*\mathcal{G} \times T^*N \times \overline{T^*N}$ by construction. The properties of an action map follow: the following expressions are to be understood whenever they make sense

Compatibility with the moment map :

$$\begin{aligned}
\langle\, \hat{j}\,(\theta_g \mathbin{\hat{*}} \alpha_n)\,,\, a_{j(g*n)} \,\rangle &= \langle\, \theta_g \mathbin{\hat{*}} \alpha_n\,,\, \mathrm{d}\sigma(a_{j(g*n)}, 0_{g*n}) \,\rangle \\
&= \langle\, \theta_g \mathbin{\hat{*}} \alpha_n\,,\, (\mathrm{d}\, r_g\, a_{j(g*n)}) * 0_n \,\rangle \\
&= \langle\, \theta_g\,,\, \mathrm{d}\, r_g\, a_{j(g*n)} \,\rangle + \langle\, \alpha_n\,,\, 0_n \,\rangle \\
&= \langle\, \hat{\mathrm{t}}(\theta_g)\,,\, a_{j(g*n)} \,\rangle \qquad ,
\end{aligned}$$

where we regarded $a_{j(n)} \in A_{j(n)}$ as a vector tangent to $\mathcal{G}$ in the second and third step;

Unitality : since $\hat\varepsilon$ is the identification $A^* \simeq N^*M$,

$$\begin{aligned}
\langle\, \hat\varepsilon(\hat{j}\,(\alpha_n)) \mathbin{\hat{*}} \alpha_n\,,\, \delta n \,\rangle &= \langle\, \hat\varepsilon(\hat{j}\,(\alpha_n)) \mathbin{\hat{*}} \alpha_n\,,\, \mathrm{d}\varepsilon(\mathrm{d}j(\delta n)) * \delta n \,\rangle \\
&= \langle\, \hat\varepsilon(\hat{j}\,(\alpha_n))\,,\, \mathrm{d}\varepsilon(\mathrm{d}j(\delta n)) \,\rangle + \langle\, \alpha_n\,,\, \delta n \,\rangle \\
&= \langle\, \alpha_n\,,\, \delta n \,\rangle \quad ;
\end{aligned}$$

Multiplicativity :

$$\begin{aligned}
\langle\, \theta_g \mathbin{\hat{*}} (\theta_h * \alpha_n)\,,\, \delta g * \delta h * \delta n \,\rangle &= \langle\, \theta_g\,,\, \delta g \,\rangle + \langle\, \theta_h\,,\, \delta h \,\rangle + \langle\, \alpha_p\,,\, \delta p \,\rangle \\
&= \langle\, \theta_g \mathbin{\hat{\cdot}} \theta_h\,,\, \delta g \bullet \delta h \,\rangle + \langle\, \alpha_p\,,\, \delta n \,\rangle \\
&= \langle\, (\theta_g \mathbin{\hat{\cdot}} \theta_h) \mathbin{\hat{*}} \alpha_n\,,\, \delta g \bullet \delta h * \delta n \,\rangle \\
&= \langle\, (\theta_g \mathbin{\hat{\cdot}} \theta_h) \mathbin{\hat{*}} \alpha_n\,,\, \delta g * \delta h * \delta n \,\rangle \qquad ,
\end{aligned}$$

then $(\theta_g \mathbin{\hat{\cdot}} \theta_h, \alpha_p, (\theta_g \mathbin{\hat{\cdot}} \theta_h) * \alpha_p)$ and $(\theta_g \mathbin{\hat{\cdot}} \theta_h, \alpha_p, \theta_g * (\theta_h * \alpha_p))$ are the same covector, since the tangent lifted action is fibrewise surjective. □

Remark 3.1.10. The canonical symplectic form on $T^*\mathcal{G}$ induces the opposite dual Poisson structure on A^*; then $\hat{j}$ is indeed anti-Poisson as a moment map for the cotangent lifted action.

Note that the moment map $\hat{j}$ is a bundle map over j and, by construction, the cotangent lifted action map is linear over the original action map; that is, one can regard $\hat\sigma$ as a morphic action over σ, for the abelian Lie algebroids on $T^*\mathcal{G}$, A^* and T^*P. When the cotangent lift is applied to a Poisson groupoid action this fact still holds true.

Proposition 3.1.11. *Let $\mathcal{G} \rightrightarrows M$ be a Poisson groupoid acting on $j : P \to M$ with action map $\sigma : \mathcal{G}\,{}_{\mathrm{s}}\!\times_j P \to P$. Let $\hat{j} : T^*P \to A^*$ denote the moment map of the cotangent lifted action. Then, if σ is a Poisson action*

i) The moment map $\hat{j}$ is a morphism of algebroids $T^*P \to A^*$;

ii) The cotangent lifted action $\hat{\sigma}$ is morphic over σ.

(i) was proved in [**23**] and together with lemma 3.1.8 implies that $\hat{j}$ is a morphism of Lie bialgebroids (this was also remarked in [**23**]). We shall give a simpler proof of this fact.

Proof. By construction $\Gamma(\hat{\sigma}) \subset T^*\mathcal{G} \times T^*P \times T^*P$ is a Lie subalgebroid whenever $\Gamma(\sigma) \subset \mathcal{G} \times P \times \overline{P}$ is coisotropic. We may identify $\Gamma(\hat{j})$ with the subbundle $\{\hat{\varepsilon}(\hat{j}(\alpha)), \alpha, \alpha)\} \subset \Gamma(\hat{\sigma})$, which is the preimage of the Lie subalgebroid $A^* \times T^*P$ of $T^*\mathcal{G} \times T^*P$ under the projection on the first two factors $\Gamma(\hat{\sigma}) \to T^*\mathcal{G} \times T^*P$. Thus it follows from corollary 1.2.8 that $\Gamma(\hat{j})$ is canonically isomorphic to a Lie subalgebroid of $\Gamma(\hat{\sigma})$, therefore to a Lie subalgebroid of $T^*\mathcal{G} \times T^*P \times T^*P$. Apply corollary 1.2.21 to conclude that $\hat{j}$ is a morphism of Lie algebroids. It follows that $T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*P$ carries a fibred product Lie algebroid making $\hat{\sigma}$ a morphism of Lie algebroids. $\square$

As a consequence, to every Poisson action of a Poisson groupoid one can associate an action $\mathcal{LA}$-groupoid

$$\begin{array}{ccc} T^*\mathcal{G} \ltimes T^*P & \rightrightarrows & T^*P \\ \downarrow & & \downarrow \\ \mathcal{G} \ltimes P & \rightrightarrows & P \end{array} \qquad . \tag{3.9}$$

Example 3.1.12. In the case of a Poisson action of a Poisson group G, formulas (3.7) and (3.8) allow to compute $\hat{j}$ and $\hat{\sigma}$ explicitly:

$$\hat{j}(\alpha_p) = {\sigma_p}^*\alpha_p \qquad \text{and} \qquad \theta_g \,\hat{*}\, \alpha_p = {\sigma_{g^{-1}}}^*\alpha_p \qquad ,$$

for all $\alpha_p \in T^*P$ and $\theta_g \in T^*_gG$ with $r^*_g\theta_g = \sigma^{p*}\alpha_p$. The associated action $\mathcal{LA}$-groupoid was constructed in [**44**].

Remark 3.1.13. Note that the action of a Poisson groupoid $\mathcal{G}$ on $j : P \to M$ is Poisson iff the fibred product $T^*\mathcal{G}\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} T^*P$ carries a Lie groupoid over T^*P making (3.9) an $\mathcal{LA}$-groupoid.

3.2. Reduction of morphic actions of $\mathcal{LA}$-groupoids

This Section is devoted to the study of the reduction of morphic actions of $\mathcal{LA}$-groupoids. We prove (theorem 3.2.1) that quotients with respect to suitably free and proper such actions are always Lie algebroids. Thereafter we discuss the reduction, in the spirit of a categorification of Marsden-Weinstein zero level reduction, of the moment morphism (of $\mathcal{LA}$-groupoids) canonically associated with a morphic action (theorem 3.2.24). In the special case of the cotangent lift of a Poisson groupoid action, the reduction of the moment morphism produces the Koszul algebroid of the quotient Poisson bivector (proposition 3.2.26).

The main purpose of this Section is to prove the following general reduction result:

Theorem 3.2.1. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid acting morphically on a morphism of Lie algebroids $\hat{j} : B \to A$ over $j : N \to M$. If the action is free and proper (so that B/Ω and $N/\mathcal{G}$ are smooth manifolds), then there exists a unique Lie algebroid on $B/\Omega \to N/\mathcal{G}$ making the quotient projection $B \to B/\Omega$ a strong $\mathcal{LA}$-fibration over $N \to N/\mathcal{G}$.*

We split the proof in a topological part (Subsection 3.2.1), where we push the vector bundle forward over the quotient, and in an algebraic part (Subsection 3.2.2), where we push down the Lie brackets and anchor. In order to achieve the second goal we characterize the Lie-Rinehart algebra on the quotient bundle, equivalent to a Lie algebroid, as a Lie-Rinehart algebra naturally associated with the $\mathcal{LA}$-groupoid $\Omega \ltimes \mathsf{B}$ encoding the action. Namely, sections of the top Lie algebroid of $\Omega \ltimes \mathsf{B}$, which are also functors for the horizontal groupoids (morphic sections), form a Lie-Rinehart algebra which descends to the quotient by the natural equivalence relation given by (categorical) natural transformations. The space of morphic sections modulo equivalence is isomorphic to the space of projectable sections of $B \to N$ modulo equivalence, therefore, under the regularity assumptions on the top and side actions, sections of the quotient inherit a Lie-Rinehart algebra. Even when the top and side actions are neither free nor proper, but in particular in the non free case, when the quotient bundle lives in the category of stratified manifolds, one can regard the Lie-Rinehart algebra of morphic sections modulo equivalence as a desingularization or a model of a Lie algebroid over the pathological quotient bundle.

Provided only the side action is free and proper, the kernel of the moment morphism behaves however well with respect to reduction under natural regularity assumptions on the top moment map; in fact the restriction of the top action to the kernel $\mathcal{LA}$-groupoid is essentially a fibrewise linear lift of the side action.

Remarkably, the moment morphism associated with the cotangent lift of a Poisson $\mathcal{G}$-space always satisfies the regularity requirement and its "kernel reduction" procedure yields a quotient Lie algebroid which is canonically isomorphic to the Koszul algebroid of the quotient Poisson bivector. This fact shall be used in the last Section to derive applications of the reduction procedure developed here to the integrability of quotient Poisson manifolds.

3.2.1. Free and proper morphic actions: topological reduction.

For any morphic action of an $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$ on a morphism of Lie algebroids $\hat{j} : B \to A$ over $j : N \to M$, by linearity of the top action map, the vector bundle projection $\mathrm{pr}_B : B \to N$ is equivariant under the actions of $\mathcal{G}$ and Ω, i.e. $\mathrm{pr}_B(\omega_g * b_n) = g * n$ for all $(\omega_g, b_n) \in \Omega\,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} B$. Then, whenever the quotients B/Ω and $N/\mathcal{G}$ are smooth, pr_B descends to a smooth map $\underline{\mathrm{pr}}_B : B/\Omega \to N/\mathcal{G}$. We shall show that $\underline{\mathrm{pr}}_B$ is actually a vector bundle under the natural hypotheses.

Proposition 3.2.2. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid acting morphically on a morphism of Lie algebroids $\hat{j} : B \to A$ over $j : N \to M$. If both the top and side actions are free and proper, there exists a unique vector bundle of rank* $\mathsf{rank}B + \mathsf{rank}A - \mathsf{rank}\Omega$ *on the induced map $\underline{\mathrm{pr}}_B : B/\Omega \to N/\mathcal{G}$ making the quotient projection $B \to B/\Omega$ a strong $\mathcal{VB}$-fibration over $N \to N/\mathcal{G}$.*

According to proposition 3.1.1 to a morphic action such as that in the statement we can associate a free and proper action $\mathcal{LA}$-groupoid $(\Omega \ltimes B, \mathcal{G} \ltimes N; B, N)$. Thus we have to show that for any free and proper $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$ the induced map $\underline{\mathrm{pr}}_A : A/\Omega \to M/\mathcal{G}$ carries a vector bundle making the quotient projection $A \to A/\Omega$ a $\mathcal{VB}$-fibration, where the quotients are taken for the actions by left translation.
We split the proof in three parts:

Step 1. (Lemma 3.2.3) We mod out by the action of Ω along the fibres of A, which is governed by a wide normal subgroupoid Ω^{v} of $\Omega \rightrightarrows A$;

Step 2. (Lemma 3.2.5) We show that $A \to M$ and $\Omega \to \mathcal{G}$ descend to vector bundles $A/\Omega^{\mathrm{v}} \to M$ and $\Omega/\Omega^{\mathrm{v}} \to \mathcal{G}$;

Step 3. (Conclusion of the proof) We use the groupoid $\Omega/\Omega^{\mathrm{v}} \rightrightarrows A/\Omega^{\mathrm{v}}$ to push $A/\Omega^{\mathrm{v}} \to M$ forward to a vector bundle over $M/\mathcal{G}$ by identifying the fibres along the orbits of $\mathcal{G}$.

The quotient vector bundle is then essentially obtained via a reduction by stages

$$A/\Omega = \frac{A/\Omega^{\mathrm{v}}}{\Omega/\Omega^{\mathrm{v}}} \qquad ;$$

we shall make this precise in the conclusion of the proof.

Lemma 3.2.3. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid and $\Omega^{\mathrm{v}} := \mathrm{Pr}^{-1}(\varepsilon(M))$ be the restriction of $\Omega \to \mathcal{G}$ over M. Then*

i) $\Omega^{\mathrm{v}} \rightrightarrows A$ is a wide normal Lie subgroupoid of Ω

and, if $\Omega \rightrightarrows A$ is isotropy free

ii) The quotient groupoid $\Omega/\Omega^{\mathrm{v}} \rightrightarrows A/\Omega^{\mathrm{v}}$ is an isotropy free Lie groupoid.

Note that the quotient is taken by the equivalence relation induced by a normal subgroupoid, *not* by the action of Ω^{v} on Ω by left translation. On the other hand the quotient $\Omega/\Omega^{\mathrm{v}}$ coincides with the orbit space of the natural action $(\omega^{\mathrm{v}}, \overline{\omega}^{\mathrm{v}}; \omega) \mapsto \omega^{\mathrm{v}} \cdot \omega \cdot \overline{\omega}^{\mathrm{v}}$, of $\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}} \rightrightarrows A \times A$ on $\hat{\chi} : \Omega \to A \times A$; it turns out that this action is free and proper, whenever $\Omega \rightrightarrows A$ is isotropy free and proper.

Proof. (i) $\Omega^{\mathrm{v}} \rightrightarrows A$ is the kernel groupoid of a strong $\mathcal{LG}$-fibration, apply corollary 2.1.9. (ii) The action of Ω on A restricts to a free and proper action of Ω^{v}, thus A/Ω^{v} is smooth. First we show that $\Omega/\Omega^{\mathrm{v}}$ is also smooth. The action of $\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}}$is free: if $\omega^{\mathrm{v}}_+ \cdot \omega \cdot \overline{\omega}^{\mathrm{v}}_+ = \omega^{\mathrm{v}}_- \cdot \omega \cdot \overline{\omega}^{\mathrm{v}}_-$, $\hat{\mathrm{s}}(\omega^{\mathrm{v}}_+) = \hat{\mathrm{t}}(\omega) = \hat{\mathrm{s}}(\omega^{\mathrm{v}}_-)$ and $\hat{\mathrm{t}}(\omega^{\mathrm{v}}_+) = \hat{\mathrm{t}}(\omega^{\mathrm{v}}_-)$, thus $\omega^{\mathrm{v}}_+ = \omega^{\mathrm{v}}_-$, by injectivity of $\hat{\chi}$. The same argument applies for immersivity of the anchor of $(\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}}) \ltimes \Omega$; that is, the action is free. To see that the action of $\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}}$ is proper we have to show that for all sequences $\{(\omega^{\mathrm{v}}_k, \overline{\omega}^{\mathrm{v}}_k; \omega_k)\} \in (\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}})\,{}_{\hat{\mathrm{s}} \times {}^{\mathrm{op}}\hat{\mathrm{s}}} \times_{\hat{\chi}} \Omega$ such that $\{\omega_k\}$ converges to some ω_∞ and $\{\omega^{\mathrm{v}}_k \cdot \omega_k \cdot \overline{\omega}^{\mathrm{v}}_k\}$ converges to some $\widetilde{\omega}_\infty$, both $\{\omega^{\mathrm{v}}_k\}$ and $\{\overline{\omega}^{\mathrm{v}}_k\}$ have convergent subsequences; since

$$\begin{aligned}
\hat{\chi}(\overline{\omega}^{\mathrm{v}}_k) &= (\hat{\mathrm{s}}(\omega_k), \hat{\mathrm{s}}(\omega^{\mathrm{v}}_k \cdot \omega_k \cdot \omega^{\mathrm{v}}_k)) \longrightarrow (\hat{\mathrm{s}}(\omega_\infty), \hat{\mathrm{s}}(\widetilde{\omega}_\infty)) \text{ and} \\
\hat{\chi}(\omega^{\mathrm{v}}_k) &= (\hat{\mathrm{t}}(\omega^{\mathrm{v}}_k \cdot \omega_k \cdot \omega^{\mathrm{v}}_k), \hat{\mathrm{t}}(\omega_k)) \longrightarrow (\hat{\mathrm{t}}(\widetilde{\omega}_\infty), \hat{\mathrm{t}}(\omega_\infty)) \qquad ,
\end{aligned}$$

that is true, by properness of $\hat{\chi}$. The quotient groupoid by a normal subgroupoid is always well defined; the induced source map $\Omega/\Omega^{\mathrm{v}} \to A/\Omega^{\mathrm{v}}$ is a surjective submersion since so are the quotient projection $A \to A/\Omega^{\mathrm{v}}$ and the source map $\Omega \to A$, thus $\Omega/\Omega^{\mathrm{v}} \rightrightarrows A/\Omega^{\mathrm{v}}$ is a Lie groupoid. Next we show that $\Omega/\Omega^{\mathrm{v}} \rightrightarrows A/\Omega^{\mathrm{v}}$ is isotropy free. Assume $\underline{\chi}(\underline{\omega}) = \underline{\chi}(\underline{\widetilde{\omega}})$, where $\underline{\chi} : \Omega/\Omega^{\mathrm{v}} \to A/\Omega^{\mathrm{v}} \times A/\Omega^{\mathrm{v}}$ is the induced groupoid anchor, and pick any representatives $\omega \in \underline{\omega}$, $\widetilde{\omega} \in \underline{\widetilde{\omega}}$. Then

$$\hat{\mathrm{t}}(\widetilde{\omega}) = \omega^{\mathrm{v}}_{\mathrm{t}} * \hat{\mathrm{t}}(\omega) = \hat{\mathrm{t}}(\omega^{\mathrm{v}}_{\mathrm{t}}) \qquad\qquad \hat{\mathrm{s}}(\widetilde{\omega}) = \omega^{\mathrm{v}}_{\mathrm{s}} * \hat{\mathrm{s}}(\omega) = \hat{\mathrm{t}}(\omega^{\mathrm{v}}_{\mathrm{s}})$$

for some $\omega^{\mathrm{v}}_{\mathrm{t}}$,$\omega^{\mathrm{v}}_{\mathrm{s}} \in \Omega^{\mathrm{v}}$ and we can form the composition $x = \omega^{\mathrm{v}\text{-}1}_{\mathrm{t}} \cdot \widetilde{\omega} \cdot \omega^{\mathrm{v}}_{\mathrm{s}} \cdot \omega^{-1}$; since $\hat{\mathrm{t}}(x) = \hat{\mathrm{s}}(\omega^{\mathrm{v}}_{\mathrm{t}}) = \hat{\mathrm{t}}(\omega) = \hat{\mathrm{s}}(x)$, x must be a unit, thus $\widetilde{\omega} \cdot \omega^{\mathrm{v}}_{\mathrm{s}} = \omega^{\mathrm{v}}_{\mathrm{t}} \cdot \omega$, i.e. $\underline{\widetilde{\omega}} = \underline{\omega}$. $\square$

Note that, if Ω is an isotropy free $\mathcal{LA}$-groupoid, $\hat{\chi}(\Omega^{\mathrm{v}})$ is a vector subbundle of $A \oplus A$, since, for all $\omega_g \in \Omega_g$ $\hat{\mathrm{t}}(\omega_g) = \hat{\mathrm{s}}(\omega_g)$ implies that g is an isotropy, therefore a unit. Moreover, the graph of the equivalence relation induced by Ω^{v} on Ω is a subbundle of $\Omega \oplus \Omega$: $\mathrm{Pr}(\underline{\omega}^{\mathrm{v}} \cdot \omega \cdot \omega^{\mathrm{v}}) = \mathrm{Pr}(\omega)$, for all composable $\underline{\omega}^{\mathrm{v}}$, $\omega^{\mathrm{v}} \in \Omega^{\mathrm{v}}$ and $\omega \in \Omega$. That is, both actions of Ω^{v} on A and of $\Omega^{\mathrm{v}} \times {}^{\mathrm{op}}\Omega^{\mathrm{v}}$ on Ω are fibre preserving.

Lemma 3.2.4. *Let* $\mathrm{pr} : E \to M$ *be a vector bundle of rank* e *and* $\sim$ *a regular* linear equivalence relation *on* E*, in the sense that* $\Gamma(\sim) \subset E \oplus E$ *is a vector subbundle. Then the induced projection* $\underline{\mathrm{pr}} : E/{\sim} \to M$ *carries a unique (up to bundle isomorphisms) vector bundle making the quotient projection* $\pi : E \to E/{\sim}$ *a* $\mathcal{VB}$*-fibration.*

Proof. Fix any trivializing atlas $\{U_\alpha, \tau^\alpha\}$ for E. For all $u \in U_\alpha$, setting

$$[e_+] + \lambda \cdot [e_-] = [e_+ + \lambda \cdot e_-] \qquad , \qquad e_\pm \in \mathrm{pr}^{-1}(u) \quad , \quad \lambda \in \mathbb{R} \quad ,$$

endows $\underline{\mathrm{pr}}^{-1}(u)$ with a well defined linear structure. The submersion $\psi := \pi \circ \tau^{\alpha\text{-}1}$ induces a regular equivalence relation on $U_\alpha \times \mathbb{R}^e$, also denoted by $\sim$, and it is easy to see that $(U_\alpha \times \mathbb{R}^e)/\sim = U_\alpha \times \mathbb{R}^e/\ker_u \psi = U_\alpha \times \mathbb{R}^{e-k}$ for some $k \leq e$.

$$\begin{tikzcd}
\mathrm{pr}^{-1}(U_\alpha) \arrow[r, "\tau^\alpha"] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{R}^e \arrow[d] \\
\underline{\mathrm{pr}}^{-1}(U_\alpha) \equiv \mathrm{pr}^{-1}(U_\alpha)/\sim \arrow[r, "\underline{\tau}^\alpha"'] & U_\alpha \times \mathbb{R}^n/\sim\ \simeq U_\alpha \times \mathbb{R}^{e-k}
\end{tikzcd}$$

One can then endow $\mathrm{pr} : E/{\sim} \to M$ with a vector bundle structure by considering the maximal trivializing atlas containing $\{U_\alpha, \underline{\tau}^\alpha\}$, where $\underline{\tau}^\alpha : \underline{\mathrm{pr}}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{e-k}$, $\underline{\tau}^\alpha([e]_{E/\sim}) := [\tau^\alpha(e)]_{(U_\alpha \times \mathbb{R}^e)/\sim}$ is a well defined fibrewise linear diffeomorphism. Uniqueness is clear. $\square$

The equivalence relations on A and Ω induced by Ω^{v} are both linear in the sense of last lemma, due to linearity of the top groupoid multiplication of Ω; we shall denote with q and $\hat{\mathrm{q}}$ the projections $A/\Omega^{\mathrm{v}} \to M$ and $\Omega/\Omega^{\mathrm{v}} \to \mathcal{G}$, respectively. Then we have

Lemma 3.2.5. *For any isotropy free and proper* $\mathcal{LA}$*-groupoid* $(\Omega, \mathcal{G}; A, M)$*, the following statements hold:*

i) The projection $q : A/\Omega^{\mathrm{v}} \to M$ *carries a vector bundle making the quotient map* $A \to A/\Omega^{\mathrm{v}}$ *a* $\mathcal{VB}$*-fibration;*

ii) The projection $\hat{\mathrm{q}} : \Omega/\Omega^{\mathrm{v}} \to \mathcal{G}$ *carries a vector bundle making the quotient map* $\Omega \to \Omega/\Omega^{\mathrm{v}}$ *a* $\mathcal{VB}$*-fibration;*

iii) Let $\underline{\mathrm{s}}$ *and* $\underline{\mathrm{t}}$ *denote source and target of* $\Omega/\Omega^{\mathrm{v}} \rightrightarrows A/\Omega^{\mathrm{v}}$*, then the induced maps* $\underline{\mathrm{s}}^! : \Omega/\Omega^{\mathrm{v}} \to \mathrm{s}^+(A/\Omega^{\mathrm{v}})$ *and* $\underline{\mathrm{t}}^! : \Omega/\Omega^{\mathrm{v}} \to \mathrm{t}^+(A/\Omega^{\mathrm{v}})$ *are vector bundle isomorphisms.*

Proof. It remains to prove the last statement. Note that $\underline{\mathrm{s}}^!$ is fibrewise surjective, since so are $\hat{\mathrm{s}}$ and the quotient projections. For ω_g, $\widetilde{\omega}_g \in \Omega$ such that $\underline{\mathrm{s}}(\underline{\omega}_g) = \underline{\mathrm{s}}(\underline{\widetilde{\omega}}_g)$, there is some $\omega^{\mathrm{v}} \in \Omega^{\mathrm{v}}$ such that $x = \widetilde{\omega}_g \cdot \omega^{\mathrm{v}\text{-}1} \cdot \omega_g^{\text{-}1}$ is defined; since $\mathrm{Pr}(x) = g \cdot g^{\text{-}1}$, $x \in \Omega^{\mathrm{v}}$ and $\underline{\omega}_g = \underline{\widetilde{\omega}}_g$, that is, $\underline{\mathrm{s}}^!$ is fibrewise a linear isomorphism. The same reasoning shows that $\underline{\mathrm{t}}$ is a bundle isomorphism. $\square$

We are ready to conclude the proof of proposition 3.2.2.

End of proof of proposition 3.2.2. It is always possible to define a section $\theta \in \Gamma(\mathcal{G}, \mathrm{t}^{+}(A/\Omega^{\mathrm{v}}) \oplus \mathrm{s}^{+}(A/\Omega^{\mathrm{v}})^{*})$ by setting

$$\langle \theta, \xi^{\mathrm{t}} \oplus \underline{a} \rangle := \langle \xi^{\mathrm{t}}, \underline{\mathrm{t}}^{!} \circ \underline{s}^{!-1} \circ \underline{a} \rangle \quad , \quad \xi^{\mathrm{t}} \oplus \underline{a} \in \Gamma(\mathcal{G}, \mathrm{t}^{+}(A/\Omega^{\mathrm{v}})^{*} \oplus \mathrm{s}^{+}(A/\Omega^{\mathrm{v}})^{*}) \quad .$$

In other words $\{\theta_g^{\sharp}\}_{g \in \mathcal{G}}$ is a smooth family of linear isomorphisms

$$\theta_g^{\sharp} : (A/\Omega^{\mathrm{v}})_{\mathrm{s}(g)} \xrightarrow{\sim} (A/\Omega^{\mathrm{v}})_{\mathrm{t}(g)}$$

that can be characterized by the following property: for all $g \in \mathcal{G}$ and $[a]_{A/\Omega^{\mathrm{v}}}$,

$$\theta_g^{\sharp}([a]_{A/\Omega^{\mathrm{v}}}) = [\omega_g]_{\Omega/\Omega^{\mathrm{v}}} * [a]_{A/\Omega^{\mathrm{v}}} \quad , \tag{3.10}$$

where $[\omega_g]_{\Omega/\Omega^{\mathrm{v}}}$ is the unique element in $\underline{\mathrm{s}}^{-1}([a]_{A/\Omega^{\mathrm{v}}})$ with $\underline{\mathrm{Pr}}([\omega_g]_{\Omega/\Omega^{\mathrm{v}}}) = g$. From equation (3.10) is easy to see that $\theta_{\bullet}^{\sharp}$ enjoys the pseudo-group properties

$$\theta_{gh}^{\sharp} = \theta_g^{\sharp} \cdot \theta_h^{\sharp} \qquad \text{and} \qquad \theta_{\varepsilon(q)}^{\sharp} = \mathsf{id}_{(A/\Omega^{\mathrm{v}})_q}$$

for all composable $g, h \in \mathcal{G}$ and $q \in M$. In the language of [**45**], $\theta_{\bullet}^{\sharp}$ is a linear action of $\mathsf{Im}\chi$ on $A/\Omega_V \to M$; as a consequence [**45**]. There exists a unique smooth vector bundle $X \to M/\mathcal{G}$, such that $\mathrm{q}^{!}X = A/\Omega_V$. Note that $\mathrm{q}^{!} : A/\Omega_V \to X$ is fibrewise a diffeomorphism and

$$\begin{aligned}
\mathsf{dim}\, X &= \mathsf{dim}\, M/\mathcal{G} + \mathsf{dim}\, A/\Omega_V - \mathsf{dim}\, M \\
&= \mathsf{dim}\, M - \mathsf{dim}\, \mathcal{G} + 2\mathsf{dim}\, A - \mathsf{dim}\, \Omega^{\mathrm{v}} \\
&= 2\mathsf{dim}\, A - \mathsf{dim}\, \Omega \\
&= \mathsf{dim}\, A/\Omega \qquad .
\end{aligned}$$

Concretely, X is the quotient of A/Ω^{v} for the regular equivalence relation $\sim_{\theta}$ induced by $\theta_{\bullet}$,

$$[a_+]_{A/\Omega_V} \sim [a_-]_{A/\Omega_V} \quad \text{iff} \quad \theta_g([a_+]_{A/\Omega_V}) = [a_-]_{A/\Omega_V} \quad \text{for some (unique) } g \in \mathcal{G} \quad ,$$

which allows identifying the fibres over the same $\mathcal{G}$-orbit. Since the quotient projection $A \to A/\Omega$ is Ω_V-invariant, it descends to a surjective submersion $A/\Omega^{\mathrm{v}} \to A/\Omega$; last map is stable under $\sim_{\theta}$, thus it descend to a surjective submersion $X \to A/\Omega$. To conclude the proof, it suffices to show that last map is injective, which is a straightforward check. □

3.2.2. The Lie-Rinehart algebras of morphic and pseudoinvariant sections: algebraic reduction.

Before endowing the quotient vector bundle of proposition 3.2.2 with a Lie algebroid structure we need an algebraic digression; in fact we are going to obtain the quotient Lie algebroid via a reduction procedure on Lie-Rinehart algebras arising from the associated action $\mathcal{LA}$-groupoid.

Lie-Rinehart algebras provide a minimal algebraic model for Lie algebroids.

Definition 3.2.6. [**59**] Let $\mathbb{K}$ be a commutative ring and $\mathcal{A}$ a $\mathbb{K}$-algebra. A Lie-Rinehart algebra over $\mathcal{A}$ is given by

1. A $\mathbb{K}$-Lie algebra $\mathfrak{L}$
2. A left $\mathcal{A}$-module $\diamond : \mathcal{A} \otimes_{\mathbb{K}} \mathfrak{L} \to \mathfrak{L}$;
3. A representation $\mathsf{r} : \mathfrak{L} \to \mathsf{Der}\mathcal{A}$, in the Lie algebra of derivations of $\mathcal{A}$, which is compatible with the natural $\mathcal{A}$-module $\cdot : \mathcal{A} \otimes_{\mathbb{K}} \mathsf{Der}\mathcal{A} \to \mathsf{Der}\mathcal{A}$, in the sense that

$$\mathsf{r}(a \diamond l) = a \cdot \mathsf{r}(l) \qquad , \qquad a \in \mathcal{A} \quad \text{and} \quad l \in \mathfrak{L} \qquad ,$$

and characterizies the defect for the Lie bracket $[\ ,\ ]$ of $\mathfrak{L}$ to be $\mathcal{A}$-bilinear via the Leibniz rule

$$[\, l_+ \,,\, a \diamond l_- \,] - a \diamond [\, l_- \,,\, l_+ \,] = \mathsf{r}(l_+)(a) \diamond l_- \qquad , \qquad l_\pm \in \mathfrak{L} \quad \text{and} \quad a \in \mathcal{A} \qquad .$$

We shall say that the data above constitute an $\mathcal{A}$-Lie-Rinehart algebra on a $\mathbb{K}$-module $\mathfrak{L}$, for short an $\mathcal{A}$-LR-algebra and call r the Lie-Rinehart anchor, respectively $\mathcal{A}$ the base (algebra) of a Lie-Rinehart algebra. Typically, we shall consider examples where $\mathcal{A}$ is a ring of functions and call it base ring accordingly.

Remark 3.2.7. Let $E \to M$ be a vector bundle. By replacing $\mathbb{K}$ with $\mathbb{R}$, $\mathfrak{L}$ with $\Gamma(E)$ and $\mathcal{A}$ with $\mathcal{C}^\infty(M)$, one can see that a Lie Algebroid on E is equivalent to a $\mathcal{C}^\infty(M)$-LR-algebra; in fact, since the LR-anchor takes values in $\mathsf{Der}\,\mathcal{C}^\infty(M) \simeq \mathfrak{X}(M)$, thus it induces a vector bundle map $\rho : E \to TM$ by setting $\rho(e_q) := \mathsf{r}(e)(q)$, for any section $e \in \Gamma(E)$, $q \in M$ (here e_q denotes the value of e in q).

Let us discuss in some detail a typical example of a Lie-Rinehart algebra.

Example 3.2.8. The Lie-Rinehart algebra of multiplicative vector fields
A vector field X on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is multiplicative if

$$X_{gh} = X_g \bullet X_h \qquad , \qquad (g,h) \in \mathcal{G}^{(2)} \qquad ,$$

for the cotangent multiplication $\bullet$ on $T\mathcal{G} \rightrightarrows TM$. The multiplicativity condition is equivalent to asking a vector field X to be a morphism of groupoids from $\mathcal{G} \rightrightarrows M$ to the tangent prolongation $T\mathcal{G} \rightrightarrows TM$, over the base map X^M defined by setting

$$\mathrm{d}\varepsilon X^M_m = X_{\varepsilon(m)} \qquad , \qquad m \in M \qquad ; \tag{3.11}$$

it is straightforward to see that X^M is then a smooth vector field over M, which is s- and t-related to X. Multiplicative vector fields are precisely those, whose local flows are multiplicative [**47**], in the sense that the equation $\phi_u(gh) = \phi_u(g) \cdot \phi_(h)$ holds for all $(g, h) \in \mathcal{G}^{(2)}$, provided both sides are defined. As a consequence, the space $\mathfrak{X}_\mu(\mathcal{G})$ of multiplicative vector fields is easily seen to be a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$ and the base map of a bracket of multiplicative vector fields is

$$[X, Y]^M = [X^M, Y^M] \qquad , \qquad X, Y \in \mathfrak{X}_\mu(\mathcal{G}) \qquad .$$

Moreover, $\mathfrak{X}_\mu(\mathcal{G})$ is naturally endowed with a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-(bi)module structure, for the subring of $\mathcal{G}$-invariant functions, i.e. those functions which are constant along the orbits of $\mathcal{G}$: $F \in \mathcal{C}^\infty(M)^{\mathcal{G}}$ iff $\mathrm{s}^*F = \mathrm{t}^*F$. The module is given by

$$F \diamond X := (\mathrm{s}^*F) \cdot X \qquad , \qquad F \in \mathcal{C}^\infty(M)^{\mathcal{G}} \, , \; X \in \mathfrak{X}_\mu(\mathcal{G})$$

The map $\rho : \mathfrak{X}_\mu(\mathcal{G}) \to \mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}}$, $X \mapsto X^M$ is a Lie algebra homomorphism; note that ρ is takes values in $\mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}}$ indeed,

$$\begin{aligned} X^M_{\mathrm{s}(g)}(F) &= \langle\, \mathrm{d}F_{\mathrm{s}(g)} \, , \, \mathrm{ds}X_{\varepsilon(\mathrm{s}(g))} \,\rangle = X_g(\mathrm{s}^*F) = X_g(\mathrm{t}^*F) = \langle\, \mathrm{d}F_{\mathrm{t}(g)} \, , \, \mathrm{ds}X_{\varepsilon(\mathrm{t}(g))} \,\rangle \\ &= X^M_{\mathrm{t}(g)}(F) \qquad , \end{aligned}$$

for all $g \in \mathcal{G}$ and $F \in \mathcal{C}^\infty(M)^{\mathcal{G}}$. It is straightforward to check that the compatibility between the $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module and the Lie algebra on $\mathfrak{X}_\mu(\mathcal{G})$ is encoded by the usual Leibniz rule; in other words, $\mathfrak{X}_\mu(\mathcal{G})$ is a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-Lie-Rinehart algebra over $\mathbb{R}$.

In a general $\mathcal{LA}$-groupoid multiplicative sections play a role analogous to that of multiplicative vector fields in the tangent prolongation $\mathcal{LA}$-groupoid. Multiplicative sections have no natural characterization in terms of flows; nevertheless, one can show that they form a LR-algebra by using the dual description, in terms of fibrewise linear functions.

Definition 3.2.9. Let $(\Omega, A; \mathcal{G}, M)$ be an $\mathcal{LA}$-groupoid. We shall call $\omega \in \Gamma(\mathcal{G}, \Omega)$ a morphic section if

$$\omega(gh) = \omega(g) \,\hat{\cdot}\, \omega(h) \qquad , \qquad (g, h) \in \mathcal{G}^{(2)} \qquad , \tag{3.12}$$

equivalently if ω a morphism of Lie groupoids over the (uniquely determined) base section

$$a = \hat{\mathrm{s}} \circ \omega \circ \varepsilon = \hat{\mathrm{t}} \circ \omega \circ \varepsilon : M \to A \qquad .$$

Note that the base section of a morphic section is equivariant for the actions of $\mathcal{G}$ on M and of Ω on A by left translation: for all $g \in \mathrm{s}^{-1}(m)$, $m \in M$.

$$a(g * m) = a(\mathrm{t}(g)) = \hat{\mathrm{t}}(\omega(g)) = \omega(g) * \hat{\mathrm{s}}(\omega(g)) = \omega(g) * a(\mathrm{s}(g)) = \omega(g) * a(m) \quad .$$

Remark 3.2.10. For any $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$ the zero section 0^Ω is a morphic section over the zero section 0^A. Morphic sections are precisely the smooth functors $\mathcal{G} \to \Omega$ which are left inverses to the (bundle) projection functor $\Omega \to \mathcal{G}$.

The following easy lemma gives an effective characterization of morphic sections.

Lemma 3.2.11. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid and ω be a section of the top Lie algebroid. Then the following are equivalent:*

i) ω is morphic;

ii) The fiberwise linear function $\widetilde{\omega} \in \mathcal{C}^\infty(\Omega^* \times \Omega^* \times \Omega^*)$,

$$\widetilde{\omega}(\xi^1, \xi^2, \xi^3) := \omega(\xi^1) + \omega(\xi^2) + \omega(\xi^3) \qquad , \qquad \xi^1, \xi^2, \xi^3 \in \Omega^* \qquad ,$$

vanishes on $\Gamma(\hat{\mu})^o$.

Proof. ω is morphic iff $\omega \times \omega \times \omega : \Gamma(\mu) \to \Gamma(\hat{\mu}) = \Gamma(\hat{\mu})^{oo}$, where $\Gamma(\hat{\mu})^o \subset \Omega^* \times \Omega^* \times \Omega^*$ is the annihilator of $\Gamma(\hat{\mu})$ and $\Gamma(\hat{\mu})^{oo} = (\Gamma(\hat{\mu})^o)^o$. $\square$

From now on we shall denote with $\widehat{\Gamma}(\mathcal{G}, \Omega)$ the space of morphic sections of an $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$. It is sometimes convenient to regard a morphic section ω with base section a as a morphic pair (a, ω); this point of view allows computing the base section of a morphic section in the dual picture using the following tautological lemma, which can be proved in the same way as lemma 3.2.11.

Lemma 3.2.12. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid,and consider sections $\omega \in \widehat{\Gamma}(\mathcal{G}, \Omega)$ and $a \in \Gamma(M, A)$. Then,*

i) (a, ω) is a morphic pair;

ii) The fiberwise linear function $E(a, \omega) \in \mathcal{C}^\infty(A^* \times \Omega^*)$,

$$E(a, \omega)(\alpha, \xi) = a(\alpha) + \omega(\xi) \qquad , \qquad \alpha \in A^* \quad , \xi \in \Omega^* \qquad ,$$

vanishes on $\Gamma(\hat{\varepsilon})^o$.

Morphic sections $\widehat{\Gamma}(\mathcal{G}, \Omega) \subset \Gamma(\mathcal{G}, \Omega)$ form an $\mathbb{R}$-linear subspace[2] naturally endowed with a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-(bi)module:

$$F \diamond \omega = (\mathrm{s}^* F) \cdot \omega \qquad , \qquad F \in \mathcal{C}^\infty(M)^{\mathcal{G}} \ , \ \omega \in \widehat{\Gamma}(\mathcal{G}, \Omega) \qquad ;$$

if a is the base section of ω, $F \cdot a$ is the base section of $F \diamond \omega$. If for all $g \in \mathcal{G}$, $(\Omega, \mathcal{G}; A, M)$ admits a morphic section (ω, a) with $a(\mathrm{s}(g)) \neq 0 \neq a(\mathrm{t}(g))$, it is easy to see, using the defining condition (3.12), that $\mathcal{C}^\infty(M)^{\mathcal{G}} \subset \mathcal{C}^\infty(\mathcal{G})$ is the largest subring for which the $\mathcal{C}^\infty(\mathcal{G})$-module on $\Gamma(\mathcal{G}, \Omega)$ restricts to $\widehat{\Gamma}(\mathcal{G}, \Omega)$. For any morphic pair (ω, a)

$$\mathrm{ds} \circ \hat{\rho} \circ \omega = \rho \circ a \circ \mathrm{s} \qquad \text{and} \qquad \mathrm{dt} \circ \hat{\rho} \circ \omega = \rho \circ a \circ \mathrm{t} \qquad ,$$

[2]Note that the top multiplication $\hat{\mu}_{(g,h)} : \Omega^{(2)}_{(g,h)} \to \Omega_{gh}$ is fibrewise linear for the linear structure on $\Omega^{(2)}_{(g,h)}$ by $\Omega_g \times \Omega_h$. It is *not* linear in the separate components.

thus the mapping $\tilde{\rho} : \widehat{\Gamma}(\mathcal{G}, \Omega) \to \mathfrak{X}(M)$, $\omega \longmapsto \rho \circ a$ takes values in $\mathsf{Der}\, \mathcal{C}^\infty(M)^{\mathcal{G}}$: for all $f \in \mathcal{C}^\infty(M)$, $\rho(a)_{\mathrm{t}(g)}(f) = [\mathrm{dt} \circ (\hat{\rho}(\omega))]_{\mathrm{t}(g)}(f) = (\hat{\rho}(\omega))_g(\mathrm{t}^* f)$, and, analogously, $\rho(a)_{\mathrm{s}(g)}(f) = (\hat{\rho}(\omega))_g(\mathrm{s}^* f)$; thus, if f is a $\mathcal{G}$-invariant function, $\rho(a)_{\mathrm{t}(g)}(f) = \rho(a)_{\mathrm{s}(g)}(f)$ and $\rho(a)(f)$ is also $\mathcal{G}$-invariant.

Theorem 3.2.13. *The space of morphic sections $\widehat{\Gamma}(\mathcal{G}, \Omega)$ of an $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$ is a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-Lie-Rinehart algebra for the Lie bracket induced from $\Gamma(\mathcal{G}, \Omega)$ and the anchor $\tilde{\rho} : \widehat{\Gamma}(\mathcal{G}, \Omega) \to \mathsf{Der}\, \mathcal{C}^\infty(M)^{\mathcal{G}}$ defined above. In particular, for any morphic pairs $(\omega_\pm, a_\pm)$, $([\omega_+, \omega_-], [a_+, a_-])$ is also a morphic pair.*

Proof. If the second statement holds $\widehat{\Gamma}(\mathcal{G}, \Omega)$ is an $\mathbb{R}$-Lie subalgebra and a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-submodule of $\Gamma(\mathcal{G}, \Omega)$; moreover $\tilde{\rho}$ is a morphism of $\mathbb{R}$-Lie algebras. The top multiplication $\hat{\mu} : \Omega \,{}_{\hat{\mathrm{s}}}\!\times_{\hat{\mathrm{t}}} \Omega \to \Omega$ is a morphism of Lie algebroids, hence $\Gamma(\hat{\mu}) \subset \Omega^{\times 3}$ is a Lie subalgebroid and $\Gamma(\hat{\mu})^o \subset \Omega^{* \times 3}$ a coisotropic submanifold. Since $\tilde{\omega}_\pm \in \mathcal{I}_{\Gamma(\hat{\mu})^o}$, $\{\tilde{\omega}_+, \tilde{\omega}_-\} \in \mathcal{I}_{\Gamma(\hat{\mu})^o}$; moreover, for all ξ^1, ξ^2, $\xi^3 \in \Omega^*$,

$$\begin{aligned}
\{\widetilde{\omega}_+, \widetilde{\omega}_-\}(\xi_1, \xi_2, \xi_3) &= \sum_{i=1}^{3} \{\omega_+, \omega_-\}(\xi_i) = \sum_{i=1}^{3} \langle [\omega_+, \omega_-], \xi_i \rangle \\
&= \widetilde{[\omega_+, \omega_-]}(\xi_1, \xi_2, \xi_3) \qquad ,
\end{aligned}$$

then $[\omega_+, \omega_-]$ is morphic. Let us check the second statement. Since $(a_\pm, \omega_\pm)$ are morphic pairs $\{E(a_+, \omega_-), E(a_+, \omega_-)\}$ vanishes on $\Gamma(\hat{\varepsilon})^o$; being $\Gamma(\hat{\varepsilon})$ coisotropic and, for all $(\alpha, \xi) \in A^* \times \Omega^*$,

$$\begin{aligned}
\{E(a_+, \omega_+), E(a_-, \omega_-)\}(\alpha, \xi) &= \{a_+, a_-\}(\alpha) + \{\omega_+, \omega_-\}(\xi) \\
&= \langle [a_+, a_-], \alpha \rangle + \langle [\omega_+, \omega_-], \xi \rangle \\
&= E([a_+, a_-], [\omega_+, \omega_-])(\alpha, \xi) \qquad ,
\end{aligned}$$

$E([a_+, a_-], [\omega_+, \omega_-]) \in \mathcal{I}_{\Gamma(\hat{\varepsilon})^o}$, i.e. $[a_+, a_-]$ is the base section of $[\omega_+, \omega_-]$. The Leibniz rule follows:

$$\begin{aligned}
[\omega_+, f \diamond \omega_-]_g &= [\omega_+, \mathrm{s}^* f \cdot \omega_-]_g = (\mathrm{s}^* f \cdot [\omega_+, \omega_-])_g + [(\hat{\rho}(\omega_+))]_g(\mathrm{s}^* f) \cdot \omega_- \\
&= (f \diamond [\omega_+, f \diamond \omega_-])_g + \mathrm{s}^* \tilde{\rho}(\omega_+)(g) \omega_-(g) \\
&= (f \diamond [\omega_+, \omega_-] + \tilde{\rho}(\omega_+) \diamond \omega_-)_g
\end{aligned}$$

holds, for all $f \in \mathcal{C}^\infty(M)^{\mathcal{G}}$ and $g \in \mathcal{G}$. □

Apart from multiplicative vector fields, other examples of morphic sections have already appeared in literature.

Example 3.2.14. Morphic sections of the cotangent prolongation $\mathcal{LA}$-groupoid of a Lie groupoid are the multiplicative 1-forms considered in [**47**].

Morphic sections are biinvariant in a sense we are about to specify. A section $\omega \in \Gamma(\mathcal{G}, \Omega)$ is **right pseudoinvariant**, respectively **left pseudoinvariant**, if, for all $(g, h) \in \mathcal{G}^{(2)}$

$$\omega(gh) = \omega(g) \,\hat{\cdot}\, R^\omega(h) \quad , \quad \text{respectively} \quad , \quad \omega(gh) = L^\omega(g) \,\hat{\cdot}\, \omega(h) \quad , \tag{3.13}$$

for some sections $R^\omega, L^\omega \in \Gamma(\mathcal{G}, \Omega)$. The formulas (3.13) determine R^ω and L^ω uniquely:

$$R^\omega(g) = \omega(\varepsilon(\mathrm{t}(g)))^{-1} \,\hat{\cdot}\, \omega(g) \quad , \quad \text{respectively} \quad , \quad L^\omega(g) = \omega(g) \,\hat{\cdot}\, \omega(\varepsilon(\mathrm{s}(g)))^{-1}$$

and it is easy to show that R^ω and L^ω are both morphic: for example

$$R^\omega(gh) = \omega(\varepsilon(\mathrm{t}(h)))^{-1} \,\hat{\cdot}\, \omega(gh) = \omega(\varepsilon(\mathrm{t}(h)))^{-1} \,\hat{\cdot}\, \omega(g) \,\hat{\cdot}\, R^\omega(h) = R^\omega(g) \,\hat{\cdot}\, R^\omega(h) \quad .$$

The base section r^ω, respectively l^ω, of the morphic section associated with a right, respectively left, pseudoinvariant section is

$$r^\omega(m) = \hat{\mathrm{s}}(\omega(\varepsilon(m))) \quad , \quad \text{respectively} \quad , \quad l^\omega(m) = \hat{\mathrm{t}}(\omega(\varepsilon(m))) \quad ,$$

$m \in M$. If ω is a pseudoinvariant section with associated morphic section X, we shall say that (ω, X) is an **invariant pair**; we shall also say that X is the **morphic component** of ω.

Example 3.2.15. Right, respectively left, invariant vector fields on a Lie groupoid are pseudoinvariant sections of the tangent prolongation $\mathcal{LA}$-groupoid with zero morphic component. Morphic sections of an $\mathcal{LA}$-groupoid are right and left pseudoinvariant sections, coinciding with their invariant components.

In the same way as morphic sections, right and left pseudoinvariant sections admit a dual geometric characterization.

Lemma 3.2.16. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid and $\omega, R \in \Gamma(\mathcal{G}, \Omega)$. Then the following are equivalent:*

(i) (ω, R) is a right invariant pair;

(ii) The fiberwise linear function $\overrightarrow{\omega} \in \mathcal{C}^\infty_{\mathsf{lin}}(\Omega^* \times \Omega^* \times \Omega^*)$

$$\overrightarrow{\omega}(\xi^1, \xi^2, \xi^3) := \omega(\xi^1) + R^\omega(\xi^2) + \omega(\xi^3)$$

vanishes on $\Gamma(\hat{\mu})^o$.

The analogous statements hold for left invariant functions.

PROOF. Adapt the proof of theorem 3.2.13. □

We shall denote with $\overrightarrow{\Gamma}(\mathcal{G}, \Omega)$ and $\overleftarrow{\Gamma}(\mathcal{G}, \Omega)$ the $\mathbb{R}$-linear spaces of right and left invariant sections of an $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$; the natural $\mathbb{R}$-linear structure of $\Gamma(\mathcal{G}, \Omega)$ restricts to the invariant sections, due to linearity of the top multiplication

$\hat{\mu}$. A $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module can be defined both on $\overrightarrow{\Gamma}(\mathcal{G},\Omega)$, respectively $\overleftarrow{\Gamma}(\mathcal{G},\Omega)$, precisely in the same way as for morphic sections and the anchor

$$\overrightarrow{\rho} : \overrightarrow{\Gamma}(\mathcal{G},\Omega) \longrightarrow \mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}} \quad , \quad \omega \longmapsto \rho(r^\omega) \qquad ,$$

respectively

$$\overleftarrow{\rho} : \overleftarrow{\Gamma}(\mathcal{G},\Omega) \longrightarrow \mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}} \quad , \quad \omega \longmapsto \rho(l^\omega) \qquad ,$$

is compatible with the multiplication by $\mathcal{G}$-invariant functions; the images of $\overrightarrow{\rho}$ and $\overleftarrow{\rho}$ actually lay in $\mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}}$, since, for any right, respectively left, invariant section ω, r^ω, respectively l^ω is the base section of a morphic section.
Note that $\overrightarrow{\Gamma}(\mathcal{G},\Omega)$ and $\overleftarrow{\Gamma}(\mathcal{G},\Omega)$ are linearly isomorphic over $\mathcal{C}^\infty(M)^{\mathcal{G}}$, the isomorphism being given (either way) by composition with the top inversion map of $(\Omega,\mathcal{G};A,M)$. More precisely, under the mapping

$$\overline{\cdot} : \overrightarrow{\Gamma}(\mathcal{G},\Omega) \longrightarrow \overleftarrow{\Gamma}(\mathcal{G},\Omega) \qquad , \qquad \omega \longmapsto \overline{\omega} := \hat{\iota}\circ\omega\circ\iota \qquad ,$$

we have

$$\begin{aligned} \overline{\omega}(gh) &= \omega(h^{-1}g^{-1})^{-1} = (\omega(h^{-1})\,\hat{\cdot}\,R^\omega(g^{-1}))^{-1} = (R^\omega(g^{-1}))^{-1}\,\hat{\cdot}\,(\omega(h^{-1}))^{-1} \\ &= (R^\omega(g^{-1}))^{-1}\,\hat{\cdot}\,\overline{\omega}(h) = R^\omega(g)\,\hat{\cdot}\,\overline{\omega}(h) \qquad , \end{aligned}$$

that is, $\overline{\omega}$ is left pseudoinvariant indeed and $L^{\overline{\omega}} = \overline{R^\omega} = R^\omega$, since morphic sections are $\overline{\cdot}$-stable. Then $\overline{\cdot}$ is compatible with the anchors, in the sense that $\overleftarrow{\rho}\circ\overline{\cdot} = \overrightarrow{\rho}$:

$$\overrightarrow{\rho}(\omega) = \widetilde{\rho}(R^\omega) = \widetilde{\rho}(L^{\overline{\omega}}) = \overleftarrow{\rho}(\overline{\omega}) \qquad , \qquad \omega \in \overrightarrow{\Gamma}(\mathcal{G},\Omega) \qquad .$$

Thinking in terms of linear functions on Ω^*,

$$\overline{\omega} = (\hat{\iota}^{\mathsf{t}})^*\omega \qquad\qquad \text{and} \qquad\qquad L^{\overline{\omega}} = (\hat{\iota}^{\mathsf{t}})^* R^\omega = R^\omega$$

for the Poisson automorphism $\hat{\iota}^{\mathsf{t}} : \Omega^* \to \Omega^*$. The isomorphism inverse to $\overline{\cdot}$ enjoys the same properties as $\overline{\cdot}$ and we shall denote it also by the same symbol.

Proposition 3.2.17. *For any $\mathcal{LA}$-groupoid $(\Omega,\mathcal{G};A,M)$ the space of right, respectively left, invariant sections $\overrightarrow{\Gamma}(\mathcal{G},\Omega)$, respectively $\overleftarrow{\Gamma}(\mathcal{G},\Omega)$, is a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-Lie-Rinehart algebra over $\mathbb{R}$ for the Lie bracket induced from $\Gamma(\mathcal{G},\Omega)$ and the anchor $\overrightarrow{\rho} : \overrightarrow{\Gamma}(\mathcal{G},\Omega) \to \mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}}$, respectively $\overleftarrow{\rho} : \overleftarrow{\Gamma}(\mathcal{G},\Omega) \to \mathsf{Der}\,\mathcal{C}^\infty(M)^{\mathcal{G}}$ defined above. Moreover, $\overline{\cdot} : \overleftarrow{\Gamma}(\mathcal{G},\Omega) \to \overrightarrow{\Gamma}(\mathcal{G},\Omega)$ is an isomorphism of $\mathcal{C}^\infty(M)^{\mathcal{G}}$-Lie-Rinehart algebras.*

Corollary 3.2.18. *For any $\mathcal{LA}$-groupoid $(\Omega,\mathcal{G};A,M)$,*

$$[\overrightarrow{\Gamma}(\mathcal{G},\Omega),\widehat{\Gamma}(\mathcal{G},\Omega)] \subset \overrightarrow{\Gamma}(\mathcal{G},\Omega) \quad \textit{and} \quad [\overleftarrow{\Gamma}(\mathcal{G},\Omega),\widehat{\Gamma}(\mathcal{G},\Omega)] \subset \overleftarrow{\Gamma}(\mathcal{G},\Omega)\,.$$

Proof. The proof goes on the same lines of that of theorem 3.2.13. The following facts hold

(i) For any $\omega_\pm \in \overrightarrow{\Gamma}(\mathcal{G},\Omega)$ with associated morphic sections $R^\omega_\pm \in \widehat{\Gamma}(\mathcal{G},\Omega)$, $[\omega_+,\omega_-]$

is right invariant with associated morphic section section $[R^{\omega_+}, R^{\omega_-}]$;

(ii) The anchor map $\overrightarrow{\rho}$ is a morphism of Lie algebras;

As in theorem 3.2.13, (i) can be proved using lemma 3.2.16, (ii) follows and the check of the Leibniz rule amounts to a straightforward computation. The similar statements for $\overleftarrow{\Gamma}(\mathcal{G},\Omega)$ can be shown using the top groupoid inversion map, at the same time proving the isomorphism $\overleftarrow{\Gamma}(\mathcal{G},\Omega) \simeq \overrightarrow{\Gamma}(\mathcal{G},\Omega)$. For any $\omega_\pm \in \overleftarrow{\Gamma}(\mathcal{G},\Omega)$ and $\xi \in \Omega^*$

$$\begin{aligned}\langle[\omega_+,\omega_-],\xi\rangle &= \{\omega_+,\omega_-\}(\xi) = \{(\hat{\iota}^{\mathrm{t}})^*\overline{\omega}_+,(\hat{\iota}^{\mathrm{t}})^*\overline{\omega}_-\}(\xi) = (\hat{\iota}^{\mathrm{t}})^*\{\overline{\omega}_+,\overline{\omega}_-\}(\xi)\\ &= \langle\overline{[\overline{\omega}_+,\overline{\omega}_-]},\xi\rangle\end{aligned}$$

and

$$L^{[\omega_+,\omega_-]} = L^{\overline{[\overline{\omega}_+,\overline{\omega}_-]}} = R^{[\overline{\omega}_+,\overline{\omega}_-]} = [R^{\overline{\omega}_+},R^{\overline{\omega}_-}] = [L^{\omega_+},L^{\omega_-}]$$

as it follows from the remarks above. □

Morphic sections of an $\mathcal{LA}$-groupoid $(\Omega,\mathcal{G};A,M)$ are smooth functors $\mathcal{G}\to\Omega$, then the obvious notion of equivalence on $\widehat{\Gamma}(\mathcal{G},\Omega)$ is described by natural transformations. However, we shall say that two morphic sections $\omega_\pm \in \widehat{\Gamma}(\mathcal{G},\Omega)$ are equivalent if there exists a smooth natural transformation η from ω_- to ω_+, which is moreover compatible with the projection functor $\mathrm{Pr}:\Omega\to\mathcal{G}$, i.e. a smooth map $\eta: M\to\Omega$, $m\to\eta_m$, such that $\mathrm{Pr}\circ\eta=\varepsilon$ and

$$\omega_+(g)\,\hat{\cdot}\,\eta_{\mathrm{s}(g)} = \eta_{\mathrm{t}(g)}\,\hat{\cdot}\,\omega_-(g) \qquad , \qquad g\in\mathcal{G} \qquad . \tag{3.14}$$

An equivalence η from ω_- to ω_+ shall be denoted also by $\eta:\omega_-\Rightarrow\omega_+$. Note that, for any equivalence of morphic sections as above, the base sections $a_\pm\in\Gamma(M,A)$ are related by the formula

$$a_+(m) = \eta_m * a_-(m) \qquad , \qquad m\in M \qquad ,$$

for the action of Ω on A by left translation.
The extra compatibility condition with the projection functor makes an equivalence of morphic sections a section of the restriction of Ω to M and, technically, it is required to make the groupoid $\mathsf{Nat}(\mathcal{G},\Omega)\rightrightarrows\widehat{\Gamma}(\mathcal{G},\Omega)$ on the set of equivalences of morphic sections a groupoid object in the category of $\mathcal{C}^\infty(M)^{\mathcal{G}}$-modules. For any pair of equivalences $\eta^{1,2}:\omega_-^{1,2}\Rightarrow\omega_+^{1,2}$ of morphic sections and $F\in\mathcal{C}^\infty(M)^{\mathcal{G}}$, setting

$$\eta_m := \eta^1_m +_{\varepsilon(m)} F(m)\cdot\eta^2_m$$

yields an equivalence $\eta=(\eta^1+F\diamond\eta^2):\omega^1_-+F\diamond\omega^2_-\Rightarrow\omega^1_++F\diamond\omega^2_+$, where the sum is taken in the fibre of Ω over $\varepsilon(m)$. The algebraic properties of a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module can be easily proved using the commutation relation (3.14).

There are interesting cases in which the extra compatibility condition is fulfilled by all natural transformations:

Remark 3.2.19. Let $\eta : \omega_- \Rightarrow \omega_+$ be a natural transformation of morphic sections. One can compute, for all $g \in \mathcal{G}$,

$$\mathrm{t}(\mathrm{Pr}\, \eta_{\mathrm{s}(g)}) = \mathrm{pr}\, \hat{\mathrm{t}}(\eta_{\mathrm{s}(g)}) = \mathrm{pr}\, \hat{\mathrm{s}}(\omega_+(g)) = \mathrm{s}(g)$$

and similarly $\mathrm{s}(\mathrm{Pr}\, \eta_{\mathrm{t}(g)}) = \mathrm{t}(g)$; thus for $g = \varepsilon(q)$, $\mathrm{s}(\mathrm{Pr}\, \eta_q) = q = \mathrm{t}(\mathrm{Pr}\, \eta_q)$. That is, whenever all the isotropy groups of $\mathcal{G}$ are trivial, the compatibility with the projection functor is automatically satisfied.

Theorem 3.2.20. *For any $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$, the $\mathcal{C}^\infty(M)^{\mathcal{G}}$-Lie-Rinehart algebra on $\widehat{\Gamma}(\mathcal{G}, \Omega)$ descends to the quotient $\underline{\widehat{\Gamma}(\mathcal{G}, \Omega)} := \widehat{\Gamma}(\mathcal{G}, \Omega)/\!\sim$.*

Proof. The $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module descends to the quotient since $\mathsf{Nat}(\mathcal{G}, \Omega)$ is a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module. The anchors of equivalent morphic sections coincide as derivations of $\mathcal{C}^\infty(M)^{\mathcal{G}}$: for any equivalence $\eta : \omega_- \Rightarrow \omega_+$ of morphic sections and $F \in \mathcal{C}^\infty(M)$

$$\widetilde{\rho}_m(\omega_+)(f) = \rho_m(\hat{\mathrm{s}}_{\varepsilon(m)}\omega_+)(f) = \langle \mathrm{d}\, f, (\rho \circ \hat{\mathrm{t}})_{\varepsilon(m)}\eta_m \rangle = \langle \mathrm{d}\, \mathrm{t}^* f, \hat{\rho}_{\varepsilon(m)}\eta_m \rangle$$

and, similarly, $\widetilde{\rho}_m(\omega_-)(f) = \langle \mathrm{d}\, \mathrm{s}^* f, \hat{\rho}_{\varepsilon(m)}\eta_m \rangle$; then, $\widetilde{\rho}_m(\omega_+)(f) = \widetilde{\rho}_m(\omega_-)(f)$, whenever f is $\mathcal{G}$-invariant. Then the anchor descends to a morphism of $\mathcal{C}^\infty(M)^{\mathcal{G}}$-modules $\underline{\widehat{\Gamma}(\mathcal{G}, \Omega)} \to \mathsf{Der}\, \mathcal{C}^\infty(M)^{\mathcal{G}}$; if the Lie bracket descends, the anchor is then automatically a morphism of Lie algebras and the Leibniz rule holds. Let $\eta^{1,2} : \omega_-^{1,2} \Rightarrow \omega_+^{1,2}$ be a pair of equivalences of morphic sections. Then, setting

$$(\omega_+^{1,2} \star \eta^{1,2})(g) := \omega_+^{1,2}(g) \,\hat{\cdot}\, \eta^{1,2}_{\varepsilon(\mathrm{s}(g))} = \eta^{1,2}_{\varepsilon(\mathrm{t}(g))} \,\hat{\cdot}\, \omega_-^{1,2}(g) \qquad , \qquad g \in \mathcal{G} \qquad ,$$

defines biinvariant sections with right and left morphic components

$$L^{\omega_+^{1,2} \star \eta^{1,2}} = \omega_+^{1,2} \qquad \text{and} \qquad R^{\omega_+^{1,2} \star \eta^{1,2}} = \omega_-^{1,2} \qquad .$$

According to the proof of theorem 3.2.17, $[\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]$ is a biinvariant section with morphic components

$$L^{[\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]} = [\omega_+^1, \omega_+^2] \qquad \text{and} \qquad R^{[\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]} = [\omega_-^1, \omega_-^2] \qquad ,$$

as a consequence $[\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]_g = [\omega_+^1, \omega_+^2]_g \,\hat{\cdot}\, [\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]_{\varepsilon(\mathrm{s}(g))}$, by left invariance, on the other hand, $[\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]_g = [\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2]_{\varepsilon(\mathrm{t}(g))} \,\hat{\cdot}\, [\omega_-^1, \omega_-^2]_g$, by right invariance. Note that $\eta^{12} := [\omega_+^1 \star \eta^1, \omega_+^2 \star \eta^2] \circ \varepsilon$ is, by construction compatible with the projection functor, therefore we have found an equivalence of morphic sections $\eta^{12} : [\omega_-^1, \omega_-^2] \Rightarrow [\omega_+^1, \omega_+^2]$. $\square$

Next we shall see that, for any $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$, $\widehat{\Gamma}(\mathcal{G}, \Omega)/\!\sim$ is isomorphic to the Lie-Rinehart algebra of $\Gamma(M/\mathcal{G}, A/\Omega)$ for the groupoid actions by left translation, whenever the quotient Lie algebroid exists. Note that the construction of $\widehat{\Gamma}(\mathcal{G},\Omega)/\sim$ is independent of any assumption on $(\Omega,\mathcal{G};A,M)$, beside the $\mathcal{LA}$-groupoid structure; hence, even when the quotient Lie algebroid on the topological bundle $A/\Omega \to A/\mathcal{G}$ does not exist, $\widehat{\Gamma}(\mathcal{G},\Omega)/\sim$ is to be regarded as a model or a desingularization thereof.

From now on we shall assume that $(\Omega,\mathcal{G};A,M)$ is an isotropy free and proper $\mathcal{LA}$-groupoid, so that the quotient vector bundle $A/\Omega \to M/\mathcal{G}$ exists. A section $a \in \Gamma(M,A)$ is projectable if there exists a section $\underline{a} \in \Gamma(M/\mathcal{G}, A/\Omega)$, such that

$$\begin{tikzcd} A \arrow[r] & A/\Omega \\ M \arrow[u, "a"] \arrow[r] & M/\mathcal{G} \arrow[u, "\underline{a}"'] \end{tikzcd}$$

commutes for the quotient projections on the horizontal edges. Clearly, a section is projectable iff it is equivariant for the groupoid actions by left translation, that is, for all $m \in M$ and $g \in \mathrm{s}^{-1}(m)$, there exists an $\omega_g^a \in \Omega_g$, such that

$$a(g * m) = \omega_g^a * a(m) \qquad . \tag{3.15}$$

Since $\Omega \rightrightarrows A$ is free, ω_g^a is uniquely determined by g; since it is proper, the factorization $\omega^a = \hat{\chi}^{-1} \circ (a \times a) \circ \chi$ defines a smooth section $\omega^a \in \Gamma(\mathcal{G},\Omega)$. It turns out that $\omega^a \in \widehat{\Gamma}(\mathcal{G},\Omega)$; recall that the base section of a morphic section is always equivariant. Due to fibrewise linearity of the top groupoid action, projectable sections form an $\mathbb{R}$-linear space $\Gamma^{\downarrow}(M,A)$; it is easy to see that $\Gamma^{\downarrow}(M,A)$ is also naturally a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-module.

Proposition 3.2.21. *For any free and proper $\mathcal{LA}$-groupoid $(\Omega,\mathcal{G};A,M)$,*

i) The projection $\widehat{\Gamma}(\mathcal{G},\Omega) \to \Gamma^{\downarrow}(M,A)$ is an isomorphism of $\mathcal{C}^\infty(M)^{\mathcal{G}}$-modules;

ii) $\Gamma^{\downarrow}(M,A) \subset \Gamma(M,A)$ is a Lie-Rinehart subalgebra.

Note that $\Gamma^{\downarrow}(M,A)$ is a $\mathcal{C}^\infty(M)^{\mathcal{G}}$-LR algebra, thus a LR-subalgebra with change of base.

PROOF. (i) We only have to prove that, for any $a \in \Gamma^{\downarrow}(M,A)$, the section ω^a defined by (3.15) is morphic: for any $(g,h) \in \mathcal{G}^{(2)}$, applying (3.15) and associativity of the groupoid actions,

$$\begin{aligned} \omega^a(gh) * a(m) &= a(gh * m) = a(g * h * m) = \omega^a(g) * \omega^a(h) * a(m) \\ &= \omega^a(g) \,\hat{\cdot}\, \omega^a(h) * a(m) \qquad ; \end{aligned}$$

thus $\omega^a(gh) * a(m) = \omega^a(g) \,\hat{\cdot}\, \omega^a(h)$, since $\Omega \rightrightarrows A$ is isotropy free. (ii) Straightforward consequence of (i) and theorem 3.2.13. $\square$

The natural notion of equivalence for projectable sections is $a_+ \sim a_-$, $a_\pm \in \Gamma^{\downarrow}(M, A)$, if $\underline{a_+} = \underline{a_-}$; this is equivalent to requiring that, for all $m \in M$, there exists an $\eta_m \in \Omega$ such that

$$a_+(m) = \eta_m * a_-(m) \qquad , \tag{3.16}$$

by equivariance. Once again, since $\Omega \rightrightarrows A$ is free and proper, equation (3.16) determines a smooth map $\eta : M \to \Omega$, actually a natural transformation $\eta : \omega^a_- \Rightarrow \omega^a_+$. To see this, consider that, for all $m \in M$ and $g \in \mathrm{s}^{-1}(m)$,

$$a_+(g*m) = \omega^{a_+}(g) * a_+(m) = \omega^{a_+}(g) * \eta_{\mathrm{s}(g)} * a_-(m) = (\omega^{a_+}(g) \mathbin{\hat{\cdot}} \eta_{\mathrm{s}(g)}) * a_+(m)$$

and

$$a_+(g*m) = \eta_{\mathrm{t}(g)} * a_-(g*m) = \eta_{\mathrm{t}(g)} * \omega^{a_-}(g) * a_-(m) = (\eta_{\mathrm{t}(g)} \mathbin{\hat{\cdot}} \omega^{a_-}(g)) * a_-(m) \quad .$$

It follows then from remark 3.2.19 that η is an equivalence of morphic sections, since $\mathcal{G} \rightrightarrows M$ is free.

We are ready to conclude the proof of theorem 3.2.1.

Proof of theorem 3.2.1. Note that the base sections of equivalent morphic sections are equivalent in the sense of (3.16). Then the isomorphism $\hat{\Gamma}(\mathcal{G} \ltimes N, \Omega \ltimes B) \to \Gamma^{\downarrow}(N, B)$ descends to an isomorphism of $\mathcal{C}^\infty(N)^{\mathcal{G}}$-modules

$$\underline{\hat{\Gamma}(\mathcal{G} \ltimes N, \Omega \ltimes B)} \xrightarrow{\sim} \Gamma^{\downarrow}(N, B)/{\sim} \equiv \Gamma(N/\mathcal{G}, B/\Omega) \qquad ,$$

endowing the quotient vector bundle $B/\Omega \to N/\mathcal{G}$ with a Lie algebroid structure. For all $\beta \in \Gamma(N/\mathcal{G}, B/\Omega)$ and $b \in \Gamma^{\downarrow}(N, B)$ with $\hat{\mathrm{q}} \circ b = \beta \circ \mathrm{q}$, the equality

$$\rho_{B/\Omega} \circ \beta = \rho_B \circ b$$

holds in the space of derivations on $\mathcal{C}^\infty(N)^{\mathcal{G}}$, and similarly, for all $\beta' \in \Gamma(N/\mathcal{G}, B/\Omega)$ and $b' \in \Gamma^{\downarrow}(N, B)$ with $\hat{\mathrm{q}} \circ b' = \beta' \circ \mathrm{q}$, by definition,

$$[\,\beta\,,\,\beta'\,]_{B/\Omega}([n]_{N/\mathcal{G}}) = [[\,b\,,\,b'\,]_B(n)]_{B/\Omega} \qquad , \qquad n \in N \qquad ,$$

that is, the Lie bracket $[\,b\,,\,b'\,]$ is $(\hat{\mathrm{q}}, \mathrm{q})$-related to the Lie bracket $[\,\beta\,,\,\beta'\,]$; the quotient projection is then by construction a fibrewise surjective, base submersive and surjective morphism of Lie algebroids hence a strong $\mathcal{LA}$-fibration. Uniqueness is clear. ∎

Example 3.2.22. Consider the tangent prolongation $\mathcal{LA}$-groupoid, and the actions of $T\mathcal{G}$ on TM, respectively $\mathcal{G}$ on M by left translation. It easy to see that the reduction of the (tangent lift of) the action of $\mathcal{G}$ on M by left translation yields $TM/T\mathcal{G} \simeq T(M/\mathcal{G})$, as a vector bundle. Since the quotient projection is submersive for any $\underline{\delta m}_{\underline{m}} \in (TM/T\mathcal{G})_{\underline{m}}$, one can pick a representative $\delta m \in T_m M$, for all $m \in \underline{m}$; define $TM/T\mathcal{G} \to T(M/\mathcal{G})$, $\underline{\delta m}_{\underline{m}} \mapsto [\delta m_m]$. Such a map does not depend on the choice of representatives since, for all $\delta m_\pm \in T_m M$, with $\delta m_+ \sim \delta m_-$, $\delta m_+ = \eta_m * \delta m_-$, for some $\eta \in T_{\varepsilon(m)}\mathcal{G}$; for any class C^1 path $\gamma^\pm$, such that

$\dot\gamma^{\pm}(0) = \eta_m$, set $\gamma^+ := \mathrm{t} \circ \gamma^{\pm}$ and $\gamma^- := \mathrm{s} \circ \gamma^{\pm}$. Thus $\dot\gamma^{+,-}(0) = \delta m_{+,-}$, $[\gamma^{+,-}]$ is a C^1 path with tangent vector $[\delta m_{+,-}]$ at 0 and by definition $[\gamma^+] = [\gamma^-]$. The mapping $TM/T\mathcal{G} \to T(M/\mathcal{G})$ is fibrewise linear and injective, since, if $[\delta m] = 0$, then δm is tangent to the orbits of $\mathcal{G}$, i.e. $\delta m = \mathrm{d}\,\mathrm{t}\,\delta g = \delta g * 0_m$ for some $T^{\mathrm{s}}_{\varepsilon(m)}\mathcal{G}$, and $\underline{\delta m} = 0$. Then $TM/T\mathcal{G} \simeq T(M/\mathcal{G})$, since both bundles have the same rank. Moreover, consider that the quotient projection $M \to M/\mathcal{G}$ is invariant for the action of $\mathcal{G}$ on M, then so is the tangent map $TM \to T(M/\mathcal{G})$ under the action of $T\mathcal{G}$; the induced map $TM/T\mathcal{G} \to T(M/\mathcal{G})$ is precisely the map defined above, which is then a smooth isomorphism of vector bundles. Up to this identification, the fiberwise linear map on the quotient associated with the Lie-Rinehart anchor is the identity and the induced Lie bracket, namely that of multiplicative vector fields, is the canonical bracket on $\mathfrak{X}(M/\mathcal{G})$.

Example 3.2.23. From last example and the remarks in example 3.1.4, one can see that, for any free and proper action of a Lie group G on a manifold M, $T(M/G)$ is to be canonically identified with TM/TG for the tangent lifted action.

3.2.3. Reduction of the moment morphism: quotient Lie algebroids and Poisson structures.

The kernel of the moment morphism (3.4) associated with a morphic action is well behaved under reduction under mild regularity assumptions; in particular, in the case of the cotangent lift of a Poisson groupoid action, these assumptions are met whenever the action is free and proper.

Theorem 3.2.24. *Let $(\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid acting morphically on morphism of Lie algebroids $\hat{j} : B \to A$ over $j : N \to M$ of maximal rank. Then*

i) The kernel of the moment morphism is an $\mathcal{LA}$-groupoid of the form

$$\begin{tikzcd} \mathcal{G} \ltimes K \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & K \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} ,$$

where $K = \operatorname{\mathsf{ker}} \hat{j}$;

ii) If $\mathcal{G}$ acts freely and properly on N, the action on K is also free and proper and there exists a unique Lie algebroid of rank

$$\mathsf{rank} K/\mathcal{G} = \mathsf{rank} K = \mathsf{rank} B - \mathsf{rank} A \tag{3.17}$$

on $K/\mathcal{G} \to N/\mathcal{G}$ making the quotient projection $K \to K/\mathcal{G}$ an $\mathcal{LA}$-fibration.

Proof. (i) was already remarked in Subsection 3.1.1. (ii) For all $(g_\pm, k) \in \mathcal{G} \ltimes K$, by fibrewise linearity of the top action, we have

$$g_+ * k = g_- * k \quad \Leftrightarrow \quad g_+^{-1} g_- * \mathrm{pr}(k) = \mathrm{pr}(k) \quad \Rightarrow g_+ = g_-$$

since the action of $\mathcal{G}$ on N is free. For all sequences $\{(g_n, k_n)\} \subset \mathcal{G} \ltimes K$, such that k_n converges to some k_∞ and $g_n * k_n$ converges to some $\widetilde{k}_\infty$, we have

$$g_n * \mathrm{pr}(k_n) = \mathrm{pr}(g_n * k_n) \longrightarrow \mathrm{pr}(\widetilde{k}_\infty) \qquad \text{and} \qquad \mathrm{pr}(k_n) \longrightarrow \mathrm{pr}(k_\infty) \qquad ,$$

thus g_n has a convergent subsequence, due to properness of the action of $\mathcal{G}$ on N; (ii) follows specializing theorem 3.2.1. $\square$

The quotient Lie algebroid $K/\mathcal{G} \to N/\mathcal{G}$ is, in a sense, the push forward of K under the quotient projection $\mathrm{q} : N \to N/\mathcal{G}$. In fact the action σ_K, in the language of [**45**], defines a linear action of q on K and $K/\mathcal{G}$ is the unique vector bundle $\underline{K}$ over $N/\mathcal{G}$ such that $K = \mathrm{q}^+\underline{K}$. Last proposition shows that such a vector bundle is canonically endowed with a compatible Lie algebroid structure.
Note that the restriction of a morphic action to the kernel $\mathcal{LA}$-groupoid can be free and proper even when the top action is not. We shall present below an example of this phenomenon arising from Poisson groupoid actions.

Let us briefly review the basic facts about quotient Poisson structures. Consider a Poisson action $\sigma : G \times P \to P$ of a Poisson group G. If the action is free and proper P/G, is a smooth manifold that can be easily endowed with a Poisson structure for which the quotient projection is a Poisson submersion. For any $F \in \mathcal{C}^\infty(G \times P)$, denote with $F_p^G \in \mathcal{C}^\infty(G)$ and $F_g^P \in \mathcal{C}^\infty(P)$, $(g,p) \in G \times P$,

$$(F_p^G)(g) := F(g,p) =: (F_g^P)(p) \qquad ,$$

the restrictions to the P- and G-direction(s). The Poisson bracket $\{\,,\}_{G\times P}$ of $G \times P$ can be expressed as

$$\{F, H\}_{G\times P}(g,p) = \{F_p^G, H_p^G\}_G(g) + \{F_g^P, H_g^P\}_P(p) \qquad , \qquad (g,p) \in G \times P \quad ,$$

in terms of the Poisson brackets $\{\,,\}_G$ of G and $\{\,,\}_P$ of P. Smooth functions on P/G are to be identified with G-invariant functions on P; for any $f \in \mathcal{C}^\infty(P/G)$ and $(g,p) \in G \times P$, $f(g * p) = f(p)$; then $(\sigma^* f)_p^G$ is constant on G for all p and $(\sigma^* f)_g^P = \mathrm{pr}_B^* f$ for all g. Since σ is Poisson, for all $f, h \in \mathcal{C}^\infty(P)^G$ and $(g,p) \in G \times P$,

$$\begin{aligned} \{f,h\}_P(g*p) &= \{(\sigma^* f)_g^P, (\sigma^* h)_g^P\}_P(p) + \{(\sigma^* f)_p^G, (\sigma^* h)_p^G\}_G(g) \\ &= \{f,h\}_P(p) \quad , \end{aligned}$$

i.e. $\{\,,\}_P$ restricts to a biderivation of $\mathcal{C}^\infty(P)^G$. Thus, upon identifying $\mathcal{C}^\infty(P/G)$ with $\mathcal{C}^\infty(P)^G$, $\{\,,\}_P$ yields a Poisson bracket $\{\,,\}_{P/G}$ on P/G; note that Jacobi identity for $\{\,,\}_P$ implies the same property for $\{\,,\}_{P/G}$ and the quotient map is Poisson by construction.

It is well known that quotient structures also exist for free and proper Poisson groupoid actions.

Theorem 3.2.25. *Let a Poisson groupoid $\mathcal{G} \rightrightarrows M$ act on $P \to M$ freely and properly (so that the quotient manifold $P/\mathcal{G}$ is smooth). If the action is Poisson, there exists a unique Poisson structure on $P/\mathcal{G}$, such that the quotient projection is a Poisson submersion.*

PROOF. For any $f \in \mathcal{C}^\infty(M)$, define $F \in \mathcal{C}^\infty(\mathcal{G} \times P \times \overline{P})$ setting $F(g,p,q) := f(p) - f(q)$; then $f \in \mathcal{C}^\infty(P)^{\mathcal{G}}$ iff $F \in \mathcal{I}_{\Gamma(\sigma)}$. For all $f_\pm \in \mathcal{C}^\infty(P)^{\mathcal{G}}$,

$$\{\, F_+\,,\, F_- \,\}_{\mathcal{G}\times P\times\overline{P}}(g,p,q) = \{\, f_+\,,\, f_- \,\}_P(p) - \{\, f_+\,,\, f_- \,\}_P(q) \qquad ,$$

since $\Gamma(\sigma)$ is coisotropic both sides of last equation vanish identically on the graph of the action map, i.e. $\{\, f_+\,,\, f_- \,\} \in \mathcal{C}^\infty(P)^{\mathcal{G}}$. $\square$

Next we shall describe quotient Poisson structures within the framework of reduction of $\mathcal{LA}$-groupoids. Consider the moment morphism

$$\begin{tikzcd}
T^*\mathcal{G} \ltimes T^*P \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[drr] & T^*P \arrow[d] \arrow[drr, "\hat{j}"] & & \\
\mathcal{G} \ltimes P \arrow[r, shift left] \arrow[r, shift right] \arrow[drr] & P \arrow[drr, "j"] & T^*\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & A^* \arrow[d] \\
& & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

associated with the cotangent lift of a Poisson $\mathcal{G}$-space $j : P \to M$.

Proposition 3.2.26. *Let $P \to M$ a Poisson $\mathcal{G}$-space. If $\mathcal{G}$ acts freely and properly, the reduced kernel $K/\mathcal{G}$ of the associated action $\mathcal{LA}$-groupoid is a Lie algebroid canonically isomorphic to the Koszul algebroid on $T^*(P/\mathcal{G})$.*

Note that for all $(g,\kappa) \in \mathcal{G} \ltimes K_{(g,p)}$ and $(\delta g, \delta p) \in T\mathcal{G} \ltimes TP_{(g,p)}$

$$\begin{aligned} \langle\, g *_K \kappa\,,\, \delta g * \delta p \,\rangle &= \langle\, 0_g\,,\, \delta g \,\rangle + \langle\, \kappa\,,\, \delta p \,\rangle \\ &= \langle\, \kappa\,,\, \delta p \,\rangle \qquad , \end{aligned}$$

i.e the canonical pairing of T^*P with TP restricts to a $(\mathcal{G}, T\mathcal{G})$-invariant pairing of K with TP, which might be degenerate. However, passing to the quotients, we obtain a nondegenerate pairing $\langle\langle\ ,\ \rangle\rangle$,

$$\langle\langle\, [\kappa]_{K/\mathcal{G}}\,,\, [\delta p]_{T(P/\mathcal{G})} \,\rangle\rangle := \langle\, \kappa\,,\, \delta p \,\rangle \qquad , \qquad (\kappa, \delta p) \in K \oplus TP \qquad ,$$

of $K/\mathcal{G}$ with $T(P/\mathcal{G}) \equiv TP/T\mathcal{G}$. The corresponding sharp map $K/\mathcal{G} \to T^*(P/\mathcal{G})$ is clearly injective and according to formula (3.17)

$$\mathsf{rank} K/\mathcal{G} = \mathsf{dim}\, P - (\mathsf{dim}\, \mathcal{G} - \mathsf{dim}\, M) = \mathsf{dim}\, P/\mathcal{G} \qquad ,$$

thus it is an isomorphism. It is now easy to prove proposition 3.2.26.

Proof. For all $f \in \mathcal{C}^\infty(P/\mathcal{G}) \equiv \mathcal{C}^\infty(P)^{\mathcal{G}}$ and $a \in \Gamma(A)$,

$$\langle \hat{j}\,(\mathrm{d}f)\,,\,a\,\rangle = \langle\,\mathrm{d}f\,,\,X^a\,\rangle = 0 \qquad ,$$

for the infinitesimal action $X^\bullet$. Thus $\mathrm{d}f \in \Gamma(K) \subset \Omega^1(P)$ represents the class $\mathrm{d}_{P/\mathcal{G}}f \in \Omega^1(P/\mathcal{G}) \equiv \Gamma^\downarrow(P,K)/\sim$ and for all $f_\pm \in \mathcal{C}^\infty(P)^{\mathcal{G}}$, $\{\,f_+\,,\,f_-\,\}_{P/\mathcal{G}}$ is represented by $\{\,f_+\,,\,f_-\,\}$. We have

$$\begin{aligned}\rho_{K/\mathcal{G}}\mathrm{d}_{P/\mathcal{G}}f &= [\rho_K\mathrm{d}f]_{T(P/\mathcal{G})} = [\pi^\sharp\mathrm{d}f]_{T(P/\mathcal{G})}\\ &= \pi^\sharp_{P/\mathcal{G}}\mathrm{d}_{P/\mathcal{G}}f\end{aligned}$$

and for all $p \in P$

$$\begin{aligned}[\,\mathrm{d}_{P/\mathcal{G}}f_+\,,\,\mathrm{d}_{P/\mathcal{G}}f_-\,]_{K/\mathcal{G}}([p]_{P/\mathcal{G}}) &= [[\,\mathrm{d}f_+\,,\,\mathrm{d}f_-\,]_K(p)]_{K/\mathcal{G}}\\ &= [\mathrm{d}\{\,f_+\,,\,f_-\,\}_p]_{K/\mathcal{G}}\\ &= \mathrm{d}_{P/\mathcal{G}}(\{\,f_+\,,\,f_-\,\}_{P/\mathcal{G}})_{[p]_{P/\mathcal{G}}} \qquad .\end{aligned}$$

Then, up to the identification of $K/\mathcal{G}$ with $T^*(P/\mathcal{G})$ provided by $\langle\langle\;,\;\rangle\rangle$, $K/\mathcal{G}$ coincides with the Koszul algebroid. □

3.3. Integrability of morphic actions

We discuss in this Section morphic actions in the category of Lie groupoids, introduced in [**7**], and develop an integrated version of the reduction procedures studied in the last Section. To obtain a quotient Lie groupoid from a morphic action of a double Lie groupoid, beside the natural requirements for the quotients to be smooth, one has to further assume that the double source map of the double Lie groupoid which is acting be surjective (theorem 3.3.2). Nevertheless a kernel reduction procedure on moment morphisms (of double Lie groupoids) is effective under natural assumptions (proposition 3.3.5). On the one hand, a free and proper morphic action of a double Lie groupoid always differentiates to a morphic action of the associated $\mathcal{LA}$-group-oid and we further show that the quotient Lie groupoid of the original action is a Lie groupoid integrating the quotient Lie algebroid of the induced morphic action of $\mathcal{LA}$-groupoid (proposition 3.3.3). On the other hand, under a suitable completeness condition in terms of the $\mathcal{LA}$-homotopy lifting conditions of Chapter 2, a morphic action of an integrable $\mathcal{LA}$-groupoid Ω on an integrable Lie algebroid can be integrated to a morphic action in the category of Lie groupoids (theorem 3.3.4).

Consider a double Lie groupoid $\boldsymbol{\mathcal{D}}$ and a morphism of Lie groupoids $\boldsymbol{\mathcal{J}}$

$$\mathsf{D} := \begin{array}{ccc} \mathcal{D} & \rightrightarrows & \mathcal{V} \\ \downdownarrows & & \downdownarrows \\ \mathcal{H} & \rightrightarrows & M \end{array} \qquad \boldsymbol{\mathcal{J}} := \begin{array}{ccc} \mathcal{G} & \xrightarrow{\mathcal{J}} & \mathcal{V} \\ \downdownarrows & & \downdownarrows \\ N & \xrightarrow[j]{} & M \end{array} \quad .$$

It is easy to see that, if actions of $\mathcal{D} \rightrightarrows \mathcal{V}$ on $\mathcal{J}$ and of $\mathcal{H}$ on j are given, the diagram

$$\begin{array}{ccc} \mathcal{D} \ltimes \mathcal{G} & \rightrightarrows & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ \mathcal{H} \ltimes N & \rightrightarrows & N \end{array} \tag{3.18}$$

is a double Lie groupoid iff the action maps, respectively $\widetilde{\sigma}$ and σ, form a morphism of Lie groupoids

$$\boldsymbol{\sigma} := \begin{array}{ccc} \mathcal{D} \,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ \mathcal{H} \,{}_{\mathrm{s}_h}\!\times_{j} N & \longrightarrow & N \end{array} \quad .$$

In particular the top horizontal source map $\mathrm{s}_H{}^{\ltimes}$ of (3.18) is clearly source submersive since

$$\mathrm{dt}_H{}^{\ltimes} : T^{\mathrm{s}_V{}^{\ltimes}} \mathcal{D} \ltimes \mathcal{G} = T^{\mathrm{s}_V} \mathcal{D} \,{}_{\mathrm{ds}_H}\!\times_{\mathrm{d}\mathcal{J}} T^{\mathrm{s}} \mathcal{G} \longrightarrow T^{\mathrm{s}} \mathcal{G}$$

is the restriction of the first projection and s_H is an $\mathcal{LG}$-fibration. Then a statement analogous to proposition 3.1.1 holds true also in the case of a morphic actions of double Lie groupoids.

Proposition 3.3.1. *With the above notations diagram (*3.18*) is a double Lie groupoid iff $\boldsymbol{\sigma}$ is morphic in the category of Lie groupoids. In that case $\boldsymbol{\sigma}$ is an $\mathcal{LG}$-fibration.*

For any morphic action $\boldsymbol{\sigma}$ such as described above, we shall say that $\boldsymbol{\mathcal{D}} \ltimes \boldsymbol{\mathcal{G}}$ is the associated action double groupoid. If both top and side actions are free and proper and provided a suitable regularity condition on the double source map of D is met, the top reduced space carries a natural Lie groupoid over the side reduced space.

Theorem 3.3.2. *Let $(\mathcal{D}, \mathcal{H}; \mathcal{V}, N)$ a double Lie groupoid with surjective double source map act morphically on a morphism of Lie groupoids $\mathcal{J} : \mathcal{G} \to \mathcal{V}$ over $j : N \to M$. If the action is free and proper (so that $\mathcal{G}/\mathcal{D}$ and $N/\mathcal{H}$ are smooth manifolds), then there exists a unique Lie groupoid $\mathcal{G}/\mathcal{D} \rightrightarrows N/\mathcal{H}$ making the quotient projection $\mathcal{G} \to \mathcal{G}/\mathcal{D}$ a strong $\mathcal{LG}$-fibration over $N \to N/\mathcal{H}$.*

Proof. Thanks to proposition 3.3.1 it is sufficient to prove the statement for the morphic action of $\mathcal{D}$ on its side vertical groupoid by left translation when $\mathcal{D} \rightrightarrows \mathcal{V}$ and $\mathcal{H} \rightrightarrows N$ are free and proper groupoids; the general case follows considering the action groupoid associated with the morphic action. Note that the double source map $\mathbb{S}^{\ltimes} : \mathcal{D} \ltimes \mathcal{G} \to (\mathcal{H} \ltimes N)\;{}_{\mathrm{s}_h^{\ltimes}} \times_{\mathcal{J}} \mathcal{G}$ is also a surjective, since it admits the factorization

$$\begin{tikzcd}
\mathcal{D}\;{}_{\mathrm{s}_H} \times_{\mathcal{J}} \mathcal{G} \arrow[r, "\mathbb{S}^{\ltimes}"] \arrow[d, "\mathbb{S}\times\mathrm{id}_{\mathcal{G}}"', "\simeq"] & (\mathcal{H} \ltimes N)\;{}_{\mathrm{pr}_2} \times_{\mathcal{J}} \mathcal{G} \\
(\mathcal{H}\;{}_{\mathrm{s}_h} \times_{\mathrm{s}_v} \mathcal{V})\;{}_{\mathrm{pr}_2} \times_{\mathcal{J}} \mathcal{G} \arrow[r, "\simeq"'] & \mathcal{H}\;{}_{\mathrm{s}_h} \times_{\mathrm{s}_v \circ \mathrm{pr}_1} \Gamma(\mathcal{J}) \arrow[u, "\simeq"']
\end{tikzcd} \quad .$$

For all $(d, v) \in \mathcal{D}\;{}_{\mathrm{s}_H} \times \mathcal{V}$,

$$\begin{aligned}
\mathrm{s}_v(d * v) &= \mathrm{s}_v(\mathrm{t}_H(d)) = \mathrm{t}_h(\mathrm{s}_V(d)) = \mathrm{s}_V(d) * \mathrm{s}_h(\mathrm{s}_V(d)) = \mathrm{s}_V(d) * \mathrm{s}_v(\mathrm{s}_H(d)) \\
&= \mathrm{s}_V(d) * \mathrm{s}_v(v) \quad ,
\end{aligned}$$

that is, the source map of $\mathcal{V}$ descends to the quotient; by similar arguments one can show that target and inversion also descend. Note that for all $v^{\pm} \in \mathcal{V}$, with $\mathrm{s}_v(v^+) = h * \mathrm{t}_v(v^-)$, for some $h \in \mathcal{H}$, there always exist composable representatives: since there exist an element $d \in \mathcal{D}$ such that $\mathrm{s}_V(d) = h^{-1}$ and $\mathrm{s}_H(d) = v$, thanks to the regularity condition on the double source map, $d * v^+ \equiv \mathrm{t}_H(d)$ is composable with v^-. Then one can define a multiplication on the graph $(\mathcal{G}/\mathcal{D}, M/\mathcal{H})$ by picking composable representatives $w^{\pm} \in \underline{v^{\pm}}$ and setting, $\underline{v^+} \cdot \underline{v^-} := \underline{w^+ \cdot_v w^-}$ where $\underline{v}$ denotes the class of $v \in \mathcal{V}$. Moreover, for all pairs $w^{\pm}_{1,2} \in \underline{v^{\pm}}$ of composable representatives such that $w^{\pm}_2 = d^{\pm} * w^{\pm}_1$, one has

$$\mathrm{s}_v(w_2^+) = \mathrm{s}_V(d^+) * \mathrm{s}_v(w_1^+) \qquad \text{and} \qquad \mathrm{t}_v(w_2^-) = \mathrm{t}_V(d^-) * \mathrm{t}_v(w_1^-) \qquad ,$$

therefore d^+ and d^- are vertically composable elements, since $\mathcal{H}$ acts freely on N, and

$$\begin{aligned}
\underline{w_2^+} \cdot \underline{w_2^+} &= \underline{\mathrm{t}_H(d^+) \cdot_v \mathrm{t}_H(d^-)} = \underline{\mathrm{t}_H(d^+ \cdot_V d^-)} = \underline{(d^+ \cdot_V d^-) * (w_1^+ \cdot_v w_1^-)} \\
&= \underline{w_1^-} \cdot \underline{w_1^-} \qquad ;
\end{aligned}$$

that is, the multiplication on $\mathcal{G}/\mathcal{D}$ does not depend on the choice of composable representatives. It is straightforward to check that the induced source map is submersive. By the snake lemma one can see that the quotient projection is an $\mathcal{LG}$-fibration iff the top vertical source map of $\mathcal{D}$ is orbitwise submersive, i.e. if $\mathrm{ds}_V : T_v\mathcal{O}_v^{\mathcal{D}} \to T_{\mathrm{s}_v(v)}\mathcal{O}^{\mathcal{H}}_{\mathrm{s}_v(v)}$ is onto; this can be checked easily using submersivity of the double source map, since $\mathcal{O}_v^{\mathcal{D}}$ and $\mathcal{O}^{\mathcal{H}}_{\mathrm{s}_v(v)}$ are locally diffeomorphic respectively to ${\mathrm{s}_H}^{-1}(v)$ and $\mathrm{s}_h^{-1}(\mathrm{s}_v(v))$. $\square$

By functoriality, a morphic action of a double Lie groupoid differentiates to a a morphic action on the vertically induced $\mathcal{LA}$-groupoid; in particular, all the fibred products relevant to the definition of the induced morphic action exist, since $\mathcal{LG}$-fibrations differentiate to $\mathcal{LA}$-fibrations. Namely, when a morphic action as above is given, the action morphism differentiates to a morphism of Lie algebroids

$$\begin{tikzcd} A_V(\mathcal{D})\,{}_{\hat{\mathrm{s}}}\times_{\hat{j}}\, A(\mathcal{G}) \arrow[r] \arrow[d] & A(\mathcal{G}) \arrow[d] \\ \mathcal{H}\,{}_{\mathrm{s}}\times_{j}\, N \arrow[r] & N \end{tikzcd},$$

where we identify the Lie algebroid $A_V(\mathcal{D} \ltimes \mathcal{G})$ of $\mathcal{D} \ltimes \mathcal{G} \rightrightarrows \mathcal{H} \ltimes N$ with the fibred product $A_V(\mathcal{D})\,{}_{\hat{\mathrm{s}}}\times_{\hat{j}}\, A(\mathcal{G})$ for the morphisms $\hat{\mathrm{s}}\colon A_V(\mathcal{D}) \to A(\mathcal{V})$ and $\hat{j} : A(\mathcal{G}) \to A(\mathcal{V})$, differentiating the top horizontal source map and the moment map of the top action. It is immediate to check that, the associated action double Lie groupoid differentiates to the action $\mathcal{LA}$-groupoid for the induced morphic action of Lie algebroids.

Proposition 3.3.3. *Let $(\mathcal{D}, \mathcal{H}; \mathcal{V}, N)$ be a double Lie groupoid acting morphically on a morphism of Lie algebroids $\mathcal{J} : \mathcal{G} \to \mathcal{V}$ over $j : N \to M$ and assume that the quotient Lie groupoid $\mathcal{G}/\mathcal{D} \rightrightarrows N/\mathcal{H}$ exists and makes the quotient projection a strong $\mathcal{LG}$-fibration. Then the quotient Lie algebroid $A(\mathcal{G})/A_V(\mathcal{D}) \to N/\mathcal{H}$ for the induced morphic action of Lie algebroids exists and is canonically isomorphic to the Lie algebroid of $\mathcal{G}/\mathcal{D} \rightrightarrows N/\mathcal{H}$.*

Proof. As in the proof of theorem 3.3.2 it is sufficient to consider the case of horizontally free and proper double Lie groupoids for the action on the side vertical groupoid by left translation. Since the tangent prolongation groupoid $T\mathcal{D} \rightrightarrows T\mathcal{V}$ is also free and proper, $A_V(\mathcal{D}) \subset T\mathcal{D}$ and $A(\mathcal{V}) \subset T\mathcal{V}$ are embedded as normal bundles and the induced groupoid anchor $\overline{\chi} : A_V(\mathcal{D}) \to A(\mathcal{V}) \times A(\mathcal{V})$ is the restriction of the top horizontal tangent anchor $\mathrm{d}\chi_H : T\mathcal{D} \to T\mathcal{V} \times T\mathcal{V}$, the Lie groupoid $A_V(\mathcal{D}) \rightrightarrows A(\mathcal{V})$ is a free and proper and the quotient Lie algebroid $A(\mathcal{V})/A_V(\mathcal{D}) \to M/\mathcal{H}$ exists. The quotient projection $\mathcal{V} \to \mathcal{V}/\mathcal{D}$ is a strong $\mathcal{LG}$-fibration over $M \to M/\mathcal{H}$, thus it differentiates to a strong $\mathcal{LA}$-fibration $\wp$: $A(\mathcal{V}) \to A(\mathcal{V}/\mathcal{D})$. For all $q \in M$, define

$$\psi_q : (A(\mathcal{V})/A_V(\mathcal{D}))_{\underline{q}} \to A(\mathcal{V}/\mathcal{D})_{\underline{q}} \tag{3.19}$$

as $\psi_q([a_q]) = \wp_q(a_q)$, by picking any $a_q \in A(\mathcal{V})_q$ such that $a_q \in [a_q]$; ψ_q is well defined since the commuting diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{D} \,{}_{\mathrm{s}_H}\!\times \mathcal{V} \arrow[dl, "\mathrm{pr}_2"'] \arrow[dr, "\sigma"] & \\ \mathcal{V} \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & \mathcal{V}/\mathcal{D} & \end{tikzcd} \quad \text{differentiates to} \quad \begin{tikzcd}[column sep=small] & A_V(\mathcal{D}) \,{}_{\hat{\mathrm{s}}}\!\times A(\mathcal{V}) \arrow[dl, "\mathrm{pr}_2"'] \arrow[dr, "\hat{\sigma}"] & \\ A(\mathcal{V}) \arrow[dr, "\wp"'] & & A(\mathcal{V}) \arrow[dl, "\wp"] \\ & A(\mathcal{V}/\mathcal{D}) & \end{tikzcd}$$

Note that for all elements $h * q$ in the $\mathcal{H}$-orbit through q,

$$\psi_{h*q}([a_{h*q}]) = \wp(a_{h*q}) = \wp(\omega_h * a_q) = \psi_q([a_q]) = \psi_q([a_{h*q}])$$

for the unique $\omega_h \in A_V(\mathcal{D})_h$, such that $\overline{\chi}(\omega_h) = ((a_{h*q}), a_q)$. Then ψ_q does not depend on the choice of q and it is surjective, since $\wp$ is a strong $\mathcal{LA}$-fibration; by counting dimensions, one can see that it is actually a linear isomorphism, inducing a bundle isomorphism $A(\mathcal{V}/\mathcal{D}) \to A(\mathcal{V})/A_V(\mathcal{D})$ over the identity of $M/\mathcal{H}$; up to this identification, the Lie algebroid on $A(\mathcal{V}/\mathcal{D})$ coincides with that on $A(\mathcal{V})/A_V(\mathcal{D})$ by uniqueness (theorem 3.2.1). $\square$

Next we shall consider the integrability of morphic actions of $\mathcal{LA}$-groupoids. Under natural conditions for an $\mathcal{LA}$-groupoid $(\Omega, \mathcal{G}; A, M)$ to have a source 1-connected integration $\mathsf{\Gamma}$, all of its morphic actions on morphisms of integrable Lie algebroids also integrate to morphic actions of $\mathsf{\Gamma}$.

Theorem 3.3.4. *Let $\Omega := (\Omega, \mathcal{G}; A, M)$ be an $\mathcal{LA}$-groupoid and $\hat{j} : B \to A$ a morphism of Lie algebroids over $j : N \to M$. Assume that $(\Omega, \mathcal{G}; A, M)$ has integrable top Lie algebroid and B is also integrable with source 1-connected Lie groupoid $\mathcal{B}$. Then, if the top source map of $(\Omega, \mathcal{G}; A, M)$ satisfies the $\mathcal{LA}$-homotopy lifting conditions of definition 2.3.5, any morphic action of Ω on $\hat{j}$ integrates to a morphic action of the vertically source 1-connected double Lie groupoid $\mathsf{\Gamma} := (\Gamma; \mathcal{G}, \mathcal{A}, M)$ on the integration $\mathcal{J} : \mathcal{B} \to \mathcal{A}$.*

Proof. Note that, under the assumptions, the top source map $\mathrm{s}_H : \Gamma \to \mathcal{A}$ of γ is a $\mathcal{LG}$-fibration and the top source map $\hat{\mathrm{s}}$ of Ω and $\hat{j}$ are strongly transversal; therefore the fibered products Lie groupoids

$$\Gamma \,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} \mathcal{B} \qquad \text{and} \qquad \Gamma \,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} (\Gamma \,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} \mathcal{B}) \simeq \Gamma_H^{(2)} \,{}_{\mathrm{s}_H \circ \mathrm{pr}_1}\!\times_{\mathcal{J}} \mathcal{B}$$

are well defined and source 1-connected for the integration $\widetilde{\sigma} : \Gamma \,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} \mathcal{B} \to \mathcal{B}$ of the top action map $\hat{\sigma} : \Omega \,{}_{\hat{\mathrm{s}}}\!\times_{\hat{j}} B \to B$. The compatibility diagrams 3.2 for $\widetilde{\sigma}$ to be an action map compatible with $\mathcal{J}$ commute then by functoriality. $\square$

It follows from the last theorem and proposition 3.3.3 above that the reduction $\mathcal{B}/\Gamma \rightrightarrows N/\mathcal{G}$, provided it exists, is an integration of the reduction $B/\Omega \to N/\mathcal{G}$.

As in the case of morphic actions of $\mathcal{LA}$-groupoids there is a moment morphism associated with each morphic action of a double Lie groupoid whose kernel is well behaved under reduction.
According to lemma 2.2.12 the moment morphism

$$\begin{tikzcd}
\mathcal{D} \ltimes \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "\mathrm{pr}_{\mathcal{D}}"] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \arrow[drr, "\mathcal{J}"] & & \\
\mathcal{H} \ltimes N \arrow[r, shift left] \arrow[r, shift right] \arrow[drr] & N \arrow[drr, "j"] & \mathcal{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\
 & & \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

has a kernel double Lie groupoid iff $\mathcal{J}$ is source submersive, typically if it is an $\mathcal{LG}$-fibration. In that case it is easy to identify the kernel with an action double Lie groupoid

$$\begin{tikzcd}
\mathcal{H} \ltimes \mathcal{K} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{K} \arrow[d, shift left] \arrow[d, shift right] \\
\mathcal{H} \ltimes N \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

for the restriction $\sigma_{\mathcal{K}}$ of the top action,

$$h *_{\mathcal{K}} \kappa \equiv \sigma_{\mathcal{K}}(h, \kappa) := \widetilde{\sigma}(\varepsilon_V(h), \kappa) \quad , \quad \varepsilon_v(\,\mathrm{s}_h(h)) = \mathcal{J}(\kappa) \quad , \quad (h, \kappa) \in \mathcal{H} \times \mathcal{K} \quad ,$$

to the kernel groupoid $\mathcal{K} := \mathsf{ker}\,\mathcal{J}$. Remarkably, not only is $\sigma_{\mathcal{K}}$ a morphic action of the double Lie groupoid trivially associated to $\mathcal{H}$ over σ along $\mathcal{J}_{\mathcal{K}} := \mathrm{s}_v \circ J|_{\mathcal{K}} = \mathrm{t}_v \circ J|_{\mathcal{K}}$, but it is also a compatible action in the sense of definition 1.3.12. As a consequence the kernel double Lie groupoid always has a quotient, provided the side action is free and proper.

Proposition 3.3.5. *Let $(\mathcal{D}, \mathcal{H}; \mathcal{V}, N)$ be a double Lie groupoid acting morphically on a morphism of Lie algebroids $\mathcal{J} : \mathcal{G} \to \mathcal{V}$ over $j : N \to M$. If $\mathcal{J}$ is source submersive (so that the kernel double Lie groupoid exists), and the side action is free and proper, then:*

i) The induced action of $\mathcal{H}$ on $\mathcal{K} = \mathsf{ker}\,\mathcal{J}$ is a compatible groupoid action in the sense of definition 1.3.12;

ii) The quotient $\mathcal{K}/\mathcal{H}$ carries a unique Lie groupoid structure over $N/\mathcal{H}$ making the quotient projection a strong $\mathcal{LG}$-fibration.

iii) The source fibres of $\mathcal{K}/\mathcal{H}$ have the same homotopy type as those of $\mathcal{K}$

Remark 3.3.6. Note that no surjectivity condition on the double source map is necessary. Moreover, one can see from the proof below that *the second and third statements remain true when only a compatible groupoid action, in the sense of definition 1.3.12, of $\mathcal{H}$ on $\mathcal{K}$ is given.*

Remark 3.3.7. It follows from proposition 3.3.3 that the Lie algebroid of $\mathcal{K}/\mathcal{H}$ is canonically isomorphic to the reduction $\ker \hat{j}\,/\mathcal{H}$ associated with the moment morphism of the induced action of $\mathcal{LA}$-groupoids.

Proof. (i) Clearly the induced action is compatible with source, target and inversion of $\mathcal{K}$. Moreover, for all composable $\kappa_\pm \in \mathcal{K}$,

$$\mathcal{J}_\mathcal{K}(\kappa_+) = j(\mathrm{s}_\mathcal{K}(\kappa_+)) = j(\mathrm{t}_\mathcal{K}(\kappa_-)) = \mathcal{J}_\mathcal{K}(\kappa_-) \qquad ,$$

thus

$$\begin{aligned} h *_\mathcal{K} (\kappa_+ \cdot \kappa_-) &= (\varepsilon_v(h) \cdot_V \varepsilon_v(h)) * (\kappa_+ \cdot \kappa_-) \\ &= (\varepsilon_v(h) * \kappa_+) \cdot_V (\varepsilon_v(h) * \kappa_-) \\ &= (h *_\mathcal{K} \kappa_+) \cdot (h *_\mathcal{K} \kappa_-) \end{aligned} \qquad .$$

(ii) Since the side action is free and proper, $N \to N/\mathcal{H}$ is a principal $\mathcal{H}$ bundle; apply lemma 1.3.13. Alternatively the statement can be proved using a nonlinear version of the proof of lemma 3.2.5. For all $n \in N$ and $h \in \mathrm{s}_h^{-1}(j(h))$, set

$$\theta_h : \mathrm{s}_\mathcal{K}^{-1}(n) \to \mathrm{s}_\mathcal{K}^{-1}(h * n) \qquad , \qquad \kappa \mapsto h *_\mathcal{K} \kappa \qquad ,$$

this yields a smooth family of diffeomorphisms enjoying the usual pseudo-group property. It is easy to see that the obvious equivalence relation $\sim_\theta$ induced by $\theta_\bullet$ is regular (properness of $\sigma_\mathcal{K}$ is implied by properness of the base action map). Being $\sigma_\mathcal{K}$ an action map, one can check that actually $\Gamma(\sim_\theta) \subset \mathcal{K}\times\mathcal{K}$ is a Lie subgroupoid of the pair groupoid, as well as the graph $\Gamma(\sim) \subset N \times N$ of the equivalence relation associated with the action on N and $\Gamma(\sim_\theta) \rightrightarrows \Gamma(\sim)$ a Lie subgroupoid of the direct product $\mathcal{K}^{\times 2}$. That is, in the language [**45**], $(\Gamma(\sim_\theta), \Gamma(\sim))$ defines a congruence on $\mathcal{K} \rightrightarrows N$, which therefore provides the descent data to push the Lie groupoid forward on $N/\mathcal{H}$ along the orbits of the side action [**45**][3]. (iii) Note that for all $n \in N$ and $\kappa_\pm \in \mathrm{s}_\mathcal{K}^{-1}(n)$, $h *_\mathcal{K} \kappa_- = \kappa_+$ implies that $(h, n) \in \mathcal{H} \ltimes N$ is an isotropy, therefore a unit; that is, $\mathcal{H}$ acts transversally to the source fibres and the quotient projection $\mathcal{K} \to \mathcal{K}/\mathcal{H}$ is sourcewise a diffeomorphism. □

In complete analogy with the reduction of the moment morphism of a morphic action in the category of Lie algebroids, the reduced kernel groupoid is really a "pushforward" of the kernel groupoid, obtained by identifying the source fibres along the $\mathcal{G}$-orbits on P.

[3]A congruence in [**45**] is further required to fulfill certain surjectivity requirements which are not met here; the extra condition is only needed to make the quotient projection a *strong* $\mathcal{LG}$-fibration, i.e. a fibration of Lie groupoids in the language of Mackenzie.

3.4. Integration of quotient Poisson structures

We derive here two approaches to the integration of quotient Poisson structures for compatible Poisson groupoid actions. The first consist in the integration of the action $\mathcal{LA}$-groupoid associated with the cotangent lifted action and in the kernel reduction of the corresponding moment morphism (theorem 3.4.1). This approach is not always effective but, when it is, it has the advantage of providing an explicit description of the symplectic form on an integration of the quotient Poisson bivector and control on the connectivity of its source fibres (corollary 3.4.2). The second approach allows us to obtain necessary and sufficient conditions for the integrability of quotient Poisson bivector fields (theorem 3.4.4), but no explicit description of the symplectic form on the integrations. The result is obtained roughly by a prolongation of the cotangent lifted action to a compatible action on the space of Lie algebroid homotopies for the kernel of the cotangent lifted moment morphism. We finally consider a class of examples, i.e. the case of complete Poisson group actions, where both approaches apply.

Let us consider the reduction of the moment morphism $\boldsymbol{\mathcal{J}}$ associated with a compatible morphic action of a symplectic double groupoid $\boldsymbol{\mathcal{S}}$

$$\boldsymbol{\mathcal{S}} = \begin{array}{ccc} \mathcal{S} & \rightrightarrows & \mathcal{G}^\bullet \\ \downdownarrows & & \downdownarrows \\ \mathcal{G} & \rightrightarrows & M \end{array} \qquad \boldsymbol{\mathcal{J}} = \begin{array}{ccc} \Lambda & \xrightarrow{\mathcal{J}} & \mathcal{G}^\bullet \\ \downdownarrows & & \downdownarrows \\ P & \xrightarrow[j]{} & M \end{array} \quad ,$$

where $\Lambda \rightrightarrows P$ is a symplectic groupoid, $\mathcal{J}$ an anti-Poisson map (hence a morphism of Poisson groupoids $\Lambda \to \overline{\mathcal{G}^\bullet}$) and both the top and side actions are Poisson; in particular, since both $\mathcal{S}$ and Λ are symplectic, the graph of the top action is a Lagrangian subgroupoid of $\mathcal{S} \times \Lambda \times \overline{\Lambda}$ for the vertical groupoid of $\mathcal{S}$. Let us denote with Ω the symplectic form of $\mathcal{S}$ and with ω that of Λ, the last requirement is equivalent to the multiplicativity condition

$$\widetilde{\sigma}^*\omega = \mathrm{pr}_{\mathcal{S}}^*\Omega + \mathrm{pr}_{\Lambda}^*\omega \tag{3.20}$$

on $\mathcal{S} \ltimes \Lambda$. Explicitly, (3.20) reads

$$\omega_{s*\lambda}(\delta s_+ * \delta\lambda_+, \delta s_+ * \delta\lambda_+) = \Omega_s(\delta s_+, \delta s_+) + \omega_\lambda(\delta\lambda_+, \delta\lambda_+) \qquad ,$$

where we have used the symbol $*$ for the tangent lift of $\widetilde{\sigma}$, for all composable $\delta s_\pm \in T\mathcal{S}$ and $\delta\lambda_\pm \in T\Lambda$. In this setting the kernel double groupoid of the moment morphism takes the form

$$\begin{tikzcd} \mathcal{G} \ltimes \mathcal{J}^{-1}(\varepsilon_\bullet(M)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{J}^{-1}(\varepsilon_\bullet(M)) \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G} \ltimes P \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad ,$$

where $\varepsilon_\bullet : M \to \mathcal{G}^\bullet$ is the unit section (here we assume that $\mathcal{J}$ is regular enough). Under the hypothesis of theorem 3.3.5 for the underlying morphic action the reduction of the $\mathcal{LA}$-groupoid above produces a Lie groupoid $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G} \rightrightarrows P/\mathcal{G}$; the reduction procedure is compatible with the side Poisson action in the sense of the following

Theorem 3.4.1. *Let $(\mathcal{S}, \mathcal{G}; \mathcal{G}^\bullet, M)$ be a symplectic double groupoid acting morphically on a morphism of Lie groupoids $\mathcal{J} : \Lambda \to \mathcal{G}^\bullet$ over $j : P \to M$, where $\Lambda \rightrightarrows P$ is a symplectic groupoid, in such a way that Λ is a symplectic $\mathcal{S}$-space. If $\mathcal{J}$ is source submersive and the side action is free and proper, then*

i) The reduced kernel groupoid $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G} \rightrightarrows P/\mathcal{G}$ carries a unique symplectic form making the quotient projection $\mathrm{pr} : \mathcal{J}^{-1}(\varepsilon_\bullet(M)) \to \mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G}$ *a Poisson submersion;*

ii) $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G} \rightrightarrows P/\mathcal{G}$ is a symplectic groupoid for the quotient Poisson manifold $P/\mathcal{G}$.

Proof. The quotient $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G}$ is smooth and carries a Lie groupoid on $P/\mathcal{G}$ thanks to proposition 3.3.5. Set $\mathcal{K} := \mathcal{J}^{-1}(\varepsilon_\bullet(M))$. ($i$) On the one hand, by coisotropicity of $\varepsilon_\bullet(M) \subset \mathcal{G}^\bullet$, $\mathcal{K} \subset \mathcal{S}$ is also coisotropic, since $\mathcal{J}$ is anti-Poisson, and evaluating the multiplicativity condition 3.20 on $\mathcal{G} \ltimes \mathcal{K}$ yields

$$\omega_{g*\kappa}(\delta g_+ * \delta\kappa_+, \delta g_+ * \delta\kappa_+) = \omega_\kappa(\delta\kappa_+, \delta\kappa_+) \qquad ,$$

being $\mathcal{G} \subset \mathcal{S}$ Lagrangian, for all composable $\delta g_\pm \in T\mathcal{G}$ and $\delta\kappa_\pm \in T\mathcal{K}$. Setting

$$\underline{\omega}_{[\kappa]}([\delta\kappa_+], [\delta\kappa_-]) := \omega_\kappa(\delta\kappa_+, \delta\kappa_-) \tag{3.21}$$

for any representatives $\delta\kappa_\pm \in [\delta\kappa_\pm] \in T_{[k]}\mathcal{K}/\mathcal{G}$ over the same $\kappa \in [\kappa]$, yields a 2-form $\underline{\omega}$ on $\mathcal{K}/\mathcal{G}$. The left hand side of (3.21) does not depend on the choice of $\delta\kappa_\pm$, provided they are tangent to $\mathcal{K}$ at the same κ (in fact there are unique such representatives) and, by changing the representative of $[\kappa]$, we have

$$\omega_{g*\kappa}(\kappa'_+, \delta\kappa'_+) = \omega_\kappa(\delta g_+^{-1} * \delta\kappa'_+, \delta g_-^{-1} * \delta\kappa'_+) \qquad ,$$

for all $\delta g_\pm$ such that $\mathrm{d}\varepsilon_\bullet(\mathrm{dt}(\delta g_\pm)) = \mathrm{d}\mathcal{J}(\mathrm{d}\kappa'_\pm)$. Then $\underline{\omega}$ is well defined and multiplicative; by construction $\mathrm{pr}^*\underline{\omega} = \iota^*\omega$, thus it is closed. We claim that the characteristic distribution Δ of $\mathcal{K}$ spans $T_\kappa\mathcal{O}$ to the $\mathcal{G}$-orbit $\mathcal{O}$ through κ at each $\kappa \in \mathcal{K}$;

it follows that $\underline{\omega}$ is nondegenerate. To see this, note that all vectors $\delta o \in T_\kappa\mathcal{O}$ are those of the form $a * 0_k$ with $a \in T^{\mathrm{s}}_{\varepsilon(j(k))}\mathcal{G}$, thus

$$\omega_\kappa(\delta o, \delta\kappa) = \omega_\kappa(a * 0_\kappa, 0_{\varepsilon(j(\kappa))} * \delta\kappa) = 0 \qquad , \quad \text{for all} \quad \delta\kappa \in T_\kappa\mathcal{K},$$

that is, $T_\kappa\mathcal{O}$ is contained in the symplectic orthogonal $T^\omega_\kappa\mathcal{K} = \Delta_\kappa$; the two spaces coincide since

$$\begin{aligned} \mathsf{dim}\,\mathcal{O} &= \mathsf{dim}\,\mathcal{G} - \mathsf{dim}\,M \quad = \quad \mathsf{dim}\,\Lambda - (\mathsf{dim}\,M + \mathsf{dim}\,\Lambda - \mathsf{dim}\,\mathcal{G}^\bullet) \\ &= \mathsf{rank} T^\omega\mathcal{K}. \end{aligned}$$

(ii) Since the characteristic leaves of $\mathcal{K}$ are the connected components of the $\mathcal{G}$-orbits, $\mathcal{C}^\infty(\mathcal{K})^{\mathcal{G}} \subset \mathcal{C}^\infty(\mathcal{K})^\Delta$; thus all extensions $F_\pm \in \mathcal{C}^\infty(\Lambda)$ of $f_\pm \in \mathcal{C}^\infty(\mathcal{K})^{\mathcal{G}}$ are in the normalizer of $\mathcal{I}_{\mathcal{K})}$ and the restriction of the Hamiltonian vector fields $X^{F_\pm}$ are tangent to $\mathcal{C}^\infty(\mathcal{K})^{\mathcal{G}}$. Let $X^F_\pm = \mathrm{d}\iota Y^{F_\pm}$ for some (in general non smooth) vector fields $Y^{F_\pm}$ on $\mathcal{K}$; we have

$$(\mathrm{dpr}^{\mathsf{t}}_{\mathrm{pr}(\kappa)} \circ \omega^\sharp_{\mathrm{pr}(\kappa)} \circ \mathrm{dpr}_k)Y^{F_\pm} = (\mathrm{d}\iota^{\mathsf{t}}_{\iota(\kappa)} \circ \omega^\sharp_k)X^{F_\pm}_{\iota(\kappa)} = \mathrm{d}_P f_\pm = \mathrm{dpr}^{\mathsf{t}}_{\mathrm{pr}(\kappa)}\mathrm{d}_{P/\mathcal{G}} f_\pm \qquad ,$$

i.e. the Hamiltonian vector fields $\underline{X}^{f_\pm}$ of $f_\pm \in \mathcal{C}^\infty(\mathcal{K}/\mathcal{G})$ are given by $\underline{X}^{f_\pm}_{\mathrm{pr}(\kappa)} = \mathrm{dpr}Y^{F_\pm}_\kappa$ and the Poisson bracket $\{\ ,\ \}_{\mathcal{K}/\mathcal{G}}$ associated with $\underline{\omega}$ can be computed using extensions:

$$\begin{aligned} \mathrm{pr}^*\{\, f_+\,,\, f_-\,\}_{\mathcal{K}/\mathcal{G}} &:= \underline{\omega}(\underline{X}^{f_-}, \underline{X}^{f_+}) \circ \mathrm{pr} = \mathrm{pr}^*\underline{\omega}(Y^{F_-}, Y^{F_+}) = \omega(X^{F_-}, X^{F_+}) \circ \iota \\ &=: \iota^*\{\, F_+\,,\, F_-\,\}_\Lambda \qquad . \end{aligned}$$

Let now $\{\ ,\ \}'$ be the Poisson bracket induced by the symplectic groupoid of (i) on $P/\mathcal{G}$ and $u_\pm \in \mathcal{C}^\infty(P)^{\mathcal{G}}$, since

$$\begin{aligned} \{\, u_+\,,\, u_-\,\}'([p]_{P/\mathcal{G}}) &:= \{\, \mathrm{s}^*_{\mathcal{K}/\mathcal{G}}u_+\,,\, \mathrm{s}^*_{\mathcal{K}/\mathcal{G}}u_-\,\}_{\mathcal{K}/\mathcal{G}}(\varepsilon_{\mathcal{K}/\mathcal{G}}([p]_{P/\mathcal{G}})) \\ &= \{\, \mathrm{s}^*_\Lambda u_+\,,\, \mathrm{s}^*_\Lambda u_-\,\}_\Lambda(\varepsilon_\Lambda(p))) \\ &= \{\, u_+\,,\, u_-\,\}_P(p) \\ &=: \{\, u_+\,,\, u_-\,\}_{P/\mathcal{G}}([p]_{P/\mathcal{G}}) \qquad , \end{aligned}$$

for all $p \in P$, (ii) follows by uniqueness (theorem 3.2.25). ☐

It follows from proposition 3.3.5 and the proof above that the source fibres of $\mathcal{J}^{-1}(\varepsilon_\bullet(M))/\mathcal{G}$ have the same homotopy type as those of $\mathcal{J}^{-1}(\varepsilon_\bullet(M))$. Therefore we obtain a condition for $\mathcal{J}^{-1}(\varepsilon_\bullet(M))$ to be source 1-connected in terms of the infinitesimal data only, i.e. a condition for integration to commute with reduction, as an application of corollary 2.3.4.

Corollary 3.4.2. *Assume that Λ and $\mathcal{G}^\bullet \equiv \mathcal{G}^\star$ are source 1-connected. Then the source connected component of $\mathcal{J}^{-1}(\varepsilon_\star(M))/\mathcal{G}$ is the source 1-connected integration of $P/\mathcal{G}$ iff the loop groups $\mathbb{K}_\bullet(\hat{j}\,)$ are trivial.*

The condition of lemma 3.4.2 was recently considered in [**22**], along the lines of [**21**], in the special case of Poisson actions of Lie groups and in [**64**], in the case of Poisson actions of Poisson groups.

Remark 3.4.3. Assume that a Poisson groupoid $\mathcal{G}$ induces a top source map satisfying the $\mathcal{LA}$-homotopy lifting conditions of definition 2.3.5 on the cotangent prolongation $\mathcal{LA}$-groupoid, and therefore is integrable to a symplectic double groupoid. Then last result, together with propositions 3.2.26, 3.3.4 implies that, for any integrable Poisson $\mathcal{G}$-space P, the quotient Poisson bivector on $P/\mathcal{G}$ is also integrable, since the integrated action map defines a symplectic action on the integration $\mathcal{J}$ of the cotangent lifted moment map $\hat{j} : T^*P \to A^*$ (this follows reasoning in the same way as in the proof of theorem 2.4.10). In fact an integration – generally not source (1-)connected – is given by the quotient $\mathcal{J}^{-1}(\varepsilon_\star(M))/\mathcal{G}$.

By a suitable path-lifting procedure of the cotangent lifted action of $T^*\mathcal{G} \rightrightarrows A^*$ we can prove integrability of quotient Poisson manifolds arising from Poisson $\mathcal{G}$-spaces, independently of the existence of a double (or any integration) of $\mathcal{G}$.

Theorem 3.4.4. *Let a Poisson groupoid $\mathcal{G} \rightrightarrows M$ act freely and properly on $j : P \to M$. If the action is Poisson, then $P/\mathcal{G}$ is integrable to a symplectic groupoid iff $\ker \hat{j}$ is an integrable Lie algebroid.*

In particular, we give a positive answer to the above question under most natural assumptions: since a Lie subalgebroid of an integrable Lie algebroid is also integrable (theorem 1.4.2), we have:

Corollary 3.4.5. *If P is integrable, then so is $P/\mathcal{G}$.*

We show in two steps that integrability of $\ker \hat{j}$ is a sufficient condition:

Step 1. Through the cotangent lift of the $\mathcal{G}$-action on j, we obtain a compatible groupoid action of $\mathcal{G}$ on the Weinstein groupoid $\mathcal{W}(K)$ of the kernel Lie algebroid $K := \ker \hat{j}$ of the moment map $\hat{j} : T^*P \to A^*$

Step 2. We identify the quotient $\mathcal{W}(K)/\mathcal{G}$ with the symplectic groupoid of $P/\mathcal{G}$

Finally we explain why the integrability condition is also necessary.

Let us fix some notations. For $i = 1, 2$, denote with Δ_i the diagonal morphism $TI^{\times i} \to TI^{\times i} \times TI^{\times i}$ and regard $0_g \in T^*\mathcal{G}$ as the constant morphism of Lie algebroids $TI^{\times i} \to T^*\mathcal{G}$, denoted $g_{(i)}$.

Proof of theorem 3.4.4. (*Step 1.*) Note that, for all morphisms of Lie algebroids $h_i : TI^{\times i} \to K$ over $\gamma_i : I^{\times i} \to \mathcal{G}$,

$$\mathrm{d}j \circ \mathrm{d}\gamma_i = \mathrm{d}j \circ \pi^\sharp \circ h_i = \rho_{A^*} \circ \hat{j} \circ h_i = 0 \qquad ,$$

where π is the Poisson bivector of P, thus the image of the base map γ_i is contained in some j-fibre. If $j \circ \gamma_i \equiv \mathrm{s}(g)$, $g \in \mathcal{G}$, setting

$$g * h_i := \hat{\sigma} \circ (g_{(i)} \times h_i) \circ \Delta_i$$

yields a morphism of Lie algebroids $TI^{\times i} \to K$, since $\hat{j} \circ (g * h_i) = \hat{\mathrm{t}} \circ g_{(i)} \equiv 0_{\mathrm{t}(g)}$. The moment map $J : \mathcal{W}(K) \to M$ for the lifted action is induced by post composition with $\hat{j}$ of representatives of classes of K-paths, equivalently

$$J([\kappa]) := j(\mathrm{pr}_K \kappa(0)) \qquad ,$$

$[\kappa] \in \mathcal{W}(K)$; the action map $\sigma_{\mathcal{W}(K)} : \mathcal{G}\,{}_{\mathrm{s}}\times_J \mathcal{W}(K) \to \mathcal{W}(K)$ is given by

$$\begin{aligned} \sigma_{\mathcal{W}(K)}(g, [\kappa]) &:= [g_{(1)} *_K \kappa]_{\mathcal{W}(K)} \\ &=: g \circledast [\kappa] \qquad . \end{aligned} \tag{3.22}$$

For all representatives $\kappa_\pm$ of the same class and K-homotopy h from κ_- to κ_+, $g_{(2)} * h$ is a K-homotopy from $g_{(1)} * \kappa_-$ to $g_{(1)} * \kappa_+$, thus $\sigma_{\mathcal{W}(K)}$ is well defined by (3.22); that it is an action map compatible with J, follows straightforwardly from the cotangent lifted action being morphic. Whenever $g \circledast [\kappa]$ is defined, we have

$$\begin{aligned} \mathrm{s}_{\mathcal{W}(K)}(g \circledast [\kappa]) &= \mathrm{pr}_K(g * \kappa(0)) = g * \mathrm{pr}_K(\kappa(0)) \\ &= g * \mathrm{s}_{\mathcal{W}(K)}([\kappa]) \end{aligned}$$

and similarly $\mathrm{t}_{\mathcal{W}(K)}(g \circledast [\kappa]) = g * \mathrm{t}_{\mathcal{W}(K)}([\kappa])$. For all composable classes $[\kappa_\pm] \in \mathcal{W}(K)$ we may choose compactly supported smooth representatives to compute $g \circledast ([\kappa_+] \cdot [\kappa_+])$:

$$g \circledast ([\kappa_+] \cdot [\kappa_-]) = \begin{bmatrix} 2 \cdot g * \kappa_-(2u) & 0 \le u \le 1/2 \\ 2 \cdot g * \kappa_+(2u-1) & 1/2 \le u \le 1 \end{bmatrix} = (g \circledast [\kappa_+]) \cdot (g \circledast [\kappa_-]) \quad ,$$

that is, the lifted action is compatible with the concatenation of K-paths.
(*Step 2.*) Under the integrability assumptions $\mathcal{W}(K)$ is the source 1-connected integration of K; thus it is possible to form an action $\mathcal{LA}$-groupoid

$$\begin{array}{ccc} \mathcal{G} \ltimes \mathcal{W}(K) & \rightrightarrows & \mathcal{W}(K) \\ \downdownarrows & & \downdownarrows \\ \mathcal{G} \ltimes P & \rightrightarrows & P \end{array}$$

and, according to theorem 3.3.2 (with $\mathcal{D} = \mathcal{G} \ltimes \mathcal{W}(K)$, $\mathcal{J} = \mathsf{id}_{\mathcal{W}}(K)$, for the action by left translation), to push $\mathcal{W}(K) \rightrightarrows P$ forward to a source 1-connected Lie groupoid $\mathcal{W}(K)/\mathcal{G} \equiv \mathcal{W}(K)/(\mathcal{G} \ltimes \mathcal{W}(K))$ over $P/\mathcal{G}$. Since the Lie algebroid of $\mathcal{W}(K)/\mathcal{G}$ is $K/\mathcal{G}$, thanks to proposition 3.3.3, which in turn is isomorphic to $T^*(P/\mathcal{G})$ (proposition 3.2.26), $\mathcal{W}(K)/\mathcal{G}$ is a *Lie* groupoid integrating the Lie algebroid of the quotient Poisson structure on $P/\mathcal{G}$. It follows by the uniqueness of quotient Poisson bivector fields (theorem 1.4.13) that the quotient Poisson manifold is integrable to a symplectic groupoid.

Conversely, assume now that $P/\mathcal{G}$ is integrable to some symplectic groupoid Λ. The quotient projection $\mathrm{pr}: P \to P/\mathcal{G}$ is asurjective submersion, therefore the pullback Lie algebroid $\mathrm{pr}^{++}T^*(P/\mathcal{G})$ always exists and can be identified (as a Lie algebroid) with K by factoring the projection $K \to T^*(P/\mathcal{G})$ along the identity of P (proposition 1.2.10). Note that the Lie algebroid on $\mathrm{pr}^{++}T^*(P/\mathcal{G})$ coincides with the fibred product $TP\,{}_{\mathrm{dpr}}\!\times_{\underline{\pi}^\sharp} T^*(P/\mathcal{G})$ The pullback Lie groupoid $\mathrm{pr}^{++}\Lambda \rightrightarrows P$, namely the fibred product $(P \times P)\,{}_{\mathrm{pr}\times\mathrm{pr}}\!\times_{\chi_\Lambda} \Lambda$, also exists thanks to the regularity of pr and integrates $TP\,{}_{\mathrm{dpr}}\!\times_{\underline{\pi}^\sharp} T^*(P/\mathcal{G})$. □

Example 3.4.6. Consider a Poisson groupoid $\mathcal{G} \rightrightarrows M$ and the Poisson action of $\mathcal{G}$ on itself by left translation. The quotient Poisson manifold $\mathcal{G}/\mathcal{G} \simeq M$ exists and the quotient projection is given by the source map. Therefore *for any Poisson groupoid $\mathcal{G} \rightrightarrows M$ with integrable Poisson structure, the Poisson structure induced on M is integrable.* Note that $T^*M \subset T^*\mathcal{G}$ is a Lie subalgebroid, being the core Lie algebroid of the cotangent prolongation $\mathcal{LA}$-groupoid; thus its integrability follows straightforwardly from that of $\mathcal{G}$.

Our proof of the integrability of quotient Poisson bivector fields, even though constructive, does not produce explicitly a *symplectic groupoid* for the quotient Poisson structure. In fact we obtain a *Lie groupoid* integrating the corresponding Koszul algebroid, which is in general source 1-connected (the projection $\mathcal{W}(K) \to \mathcal{W}(K)/\mathcal{G}$ is sourcewise a diffeomorphism), thus it carries a compatible symplectic form, thanks to Mackenzie and Xu's theorem 1.4.13. Nevertheless we have no explicit characterization of the symplectic form. On the other hand the integration of quotient Poisson bivector fields "via symplectic double groupoids" (theorem 3.4.1) produces an integrating Lie groupoid *and* a canonical symplectic form.
We conclude by discussing a class of examples where the latter method is effective.

3.4.1. The case of complete Poisson group actions.

The functorial approach of Chapter 2 to the integration of $\mathcal{LA}$-groupoids can be applied to a wide class of $\mathcal{LA}$-groupoids, namely action $\mathcal{LA}$-groupoids associated with the cotangent lift of a Poisson action of a Poisson group (G, Π) (including all actions of Lie groups by Poisson diffeomorphisms in the case $\Pi = 0$).

Proposition 3.4.7. [**64**] *Let G be a complete Poisson group and P an integrable Poisson manifold with source 1-connected symplectic groupoid Λ. If P is a Poisson G-space, then*

i) $\mathcal{J} : \Lambda \to G^\star$ is a symplectic $\mathcal{S}$-space for the top horizontal groupoid $\mathcal{S} \rightrightarrows G^\star$ of the vertically source 1-connected double of G;

ii) The action map $\widetilde{\sigma} : \mathcal{S}\,{}_{\mathrm{s}_H}\!\times_{\mathcal{J}} \Lambda \to \Lambda$ is a morphism of Lie groupoids over the action map $\sigma : G \times P \to P$;

iii) The action double groupoid of the integrated action

$$\begin{array}{ccc} \mathcal{S} \ltimes \Lambda & \rightrightarrows & \Lambda \\ \Downarrow & & \Downarrow \\ G \ltimes P & \rightrightarrows & P \end{array} \tag{3.23}$$

is the vertically source 1-connected double Lie groupoid integrating the $\mathcal{LA}$-groupoid (3.9) *associated with the Poisson action.*

Proof. It follows from corollary 1.5.10 that $\mathcal{J}$ is anti-Poisson, since it integrates a morphism of Lie bialgebroids $(T^*P, TP) \to (A^*, A)$; theorem 3.3.4 and lemma 2.4.13 imply that the cotangent lift of the given action integrates to a morphic action of the vertically 1-connected double of G. (i) It remains to show that the graph $\Gamma(\widetilde{\sigma})$ of the integrated action map is coisotropic, by the same reasoning as in the proof of theorem 2.4.10 one can show that it is actually Lagrangian; (ii) holds by construction and (iii) is now obvious. □

We are therefore in the condition to apply theorem 3.4.1 and corollary 3.4.2 to obtain an integrating symplectic groupoid for quotients of Poisson group actions

Corollary 3.4.8. *Under the hypotheses of proposition 3.4.7, if G acts freely and properly on P, then*

i) $\mathcal{J}^{-1}(e_\star)/G \rightrightarrows P/G$ is a symplectic groupoid for the quotient Poisson structure;

i) The source connected component of $\mathcal{J}^{-1}(e_\star)/G \rightrightarrows P/G$ is source 1-connected iff the groups $\mathbb{K}_\bullet(\hat{j}\,)$ are trivial.

Remark 3.4.9. Theorem 3.4.8 generalizes a result by Xu ([**74**], theorem 4.2), regarding Poisson actions with a complete moment map. A moment map [**36**] for a Poisson G-space (P, π) is a Poisson map $j : P \to G^\star$ such that $\sigma(x) = \pi^\sharp j^* \overleftarrow{x}$, for the infinitesimal action $\sigma(\cdot) : \mathfrak{g} \to \mathfrak{X}(P)$ and the left invariant 1-form $\overleftarrow{x}$ on $G^\star$ associated with $x \in \mathfrak{g} \simeq \mathfrak{g}^{**}$; such a Poisson map is called complete, when the Hamiltonian vector field of j^*f is complete for all compactly supported $f \in \mathcal{C}^\infty(G^\star)$. If G is 1-connected, so that the right dressing action of G on $G^\star$ is globally defined, j is always equivariant. When the action admits a complete moment map j, defefine $J : \Lambda \to G^\star$ and $J(\lambda) = j(\mathrm{t}(\lambda)) \cdot j(\mathrm{s}(\lambda))^{-1}$ (this space carries a natural action of G). Assuming that $J^{-1}(e_\star)/G$ is smooth, the construction of [**74**], produces a groupoid $J^{-1}(e_\star)/G \rightrightarrows P/G$, when G is complete and 1-connected. Note that J is by construction a morphism of Lie groupoids and one can check [**75**] that it differentiates to $\hat{j}$, therefore J coincides with our $\mathcal{J}$; it is easy to see that the G action on $J^{-1}(e_\star)$ of [**74**] is the same as that induced by the $\mathcal{S} \rightrightarrows G^\star$-action on Λ ($\mathcal{S} \simeq G^\star \bowtie G$, under the assumptions). Specializing last theorem to the case considered in [**74**], shows that Xu's quotient is always smooth.

Remark 3.4.10. A Lie group G is trivially ($\Pi = 0$) a complete Poisson group, with the abelian group $\overline{\mathfrak{g}^*}$ as a dual Poisson group; in this case a Poisson group action is an action by Poisson diffeomorphisms and our approach reproduces the "symplectization functor" treatment of Fernandes [**21**] and Fernandes-Ortega-Ratiu [**22**] from the viewpoint of double structures. The construction given in [**22**] (proposition 4.6) of a symplectic groupoid for the quotient Poisson manifold is precisely the construction of theorem 3.4.1 in the special case $\Pi = 0$.

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Curriculum Vitae

Stefanini Luca

Geboren am 9. May 1978 in Bergamo (Italien)

Ausbildung

Juli 1997: *Diploma di Maturità Scientifica*, Liceo Lorenzo Mascheroni, Bergamo (Italien)
1997-2002: Studium der Physik, Università di Pavia, Pavia (Italien)
September 2002: *Laurea in Fisica Teorica*
Tesi di Laurea: "Tecniche Algebriche nelle Teorie di Campo Topologiche"
Leitung: Chiar.mo Prof. Dr. Mauro Carfora (Università di Pavia)

Dissertation

Titel: "On Morphic Actions and Integrability of $\mathcal{LA}$-Groupoids"
Leitung: Prof. Dr. Alberto S. Cattaneo

Gegenwärtige Stellung

Seit März 2003 Doktorand und Assistent am Institut für Mathematik der Universität Zürich.