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BV-equivalence of 1D reparameterization invariant models with boundary

Master's Thesis

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Abstract

The present thesis studies two one-dimensional reparameterization invariant theories, namely the classically equivalent Jacobi theory and one-dimensional gravity coupled to matter (1D GR), whose treatment in a cohomological setting for field theory, named BV-BFV formalism, has highlighted a discrepancy between their boundary structures. While 1D GR is compatible with the BV-BFV axioms, the Jacobi theory manifests a singular boundary behaviour.

In order to pin down the source of this discrepancy, we suggest a notion of equivalence in the BV setting and show that the Jacobi theory and 1D GR are equivalent in this sense. In particular, we prove that their BV cohomologies are isomorphic. We then extend these results to the case with a boundary. As such, the Jacobi theory and 1D GR are an example of equivalent BV theories that yield inequivalent BV-BFV theories.

The compatibility with the BV-BFV axioms is crucial for the BV-BFV quantization program. Hence, the results of the current work suggest that 1D GR is better suited for covariant quantization on manifolds with boundary than the BV-equivalent Jacobi theory.

Contents

1 Introduction

The notion of equivalence of field theories is one that can be found throughout physics. Such a concept is relevant and useful for various reasons. At a classical level, the equations of motions of one theory might be easier to handle than the equations of motion of another, even if they yield the same predictions. Such reformulations often result in different and enlightening new interpretations of a given problem. From a quantum perspective, one theory might be better suited for quantization than another. Of course, whether two classically equivalent theories result in the same quantum theory is still an open question. With this thesis, we attempt to take another step towards the answer.

The classical physical information of a given field theory is encoded in the set EL of solutions of the Euler-Lagrange equations. In the case where the theory in question also enjoys a symmetry D, we are interested in the space of *inequivalent solutions*, namely the space of orbits EL/D. *Classical observables* are then suitable functions on EL/D. In general, such a quotient is singular and defining a sensible space of functions over it is a challenge. As such, it is convenient to find a replacement, which leads us to the Batalin-Vilkovisky (BV) formalism.

The BV formalism was first introduced in [BV83a; BV83b; BV84] as an extension of the BRST formalism [BRS76; Tyu08], named after Becchi, Rouet, Stora and Tyutin, used to quantize Lagrangian gauge theories in a way that preserves covariance. Around the same time, the Batalin-Fradkin-Vilkovisky (BFV) formalism was introduced, which deals with constrained Hamiltonian systems [BV77; BF83]. It was later noticed by various authors [McM84; Hen85; BM87; Dub87; FH90; Hen90; McC94; Sta97; Sta98] that the aforementioned formalisms enjoy a rich cohomological structure. For example, a BV theory associates a chain complex to a spacetime manifold, the *BV complex*, which is a *resolution* of the desired space of functions over the quotient EL/D. In the case of the BFV formalism, the complex at play, the *BFV complex* [Sta97; Sch09a; Sch09b], is a resolution of the space of functions over the *reduced phase space* of a given constrained Hamiltonian theory.

One can then address the topic of equivalence in the BV setting. Following the discussion above, a natural way of comparing two classical theories is through EL/D, i.e. their BV cohomologies, as done for example in [BBH95]. However this is non-trivial and the research around the notion of

"BV-equivalence" is still active, for a BV theory comes equipped with several pieces of data that one might want an equivalence relation to preserve. In [CSS18; CS19; CCS20], a *stronger* notion of BV-equivalence is implemented, which requires all data to be preserved.

The BV and BFV approaches were linked by Cattaneo, Mnev and Reshetikhin in [CMR14], where authors showed that a BV theory on the bulk induces a BFV theory on the boundary, given that some regularity conditions are met. While the presence of a boundary will typically spoil the symmetry invariance of the BV data, described through the BV cohomology, this failure will be controlled by the BFV cohomology associated to the boundary. From this perspective, the regularity conditions can be seen as a requirement for the compatibility between the BV complex on the bulk and the BFV complex on the boundary.

The BV-BFV approach is especially successful since it allows for a quantization procedure that is compatible with cutting and gluing [CMR18], which has been shown to work in various examples such as BF theory [CMR18; CMR20], split Chern-Simons theory [CMW17], 2D Yang-Mills theory [IM19] and AKSZ sigma models [CMW19].

The BV-BFV formalism was first applied to General Relativity by Schiavina in [Sch15], and this area has recently seen exciting developments.[1] For diffeomorphism invariant theories, the issue of regularity on the boundary becomes a non-trivial matter, and there are various cases where the regularity conditions necessary for the BV-BFV description fail to be met. Most notable are the examples of Palatini-Cartan gravity in (3+1) dimensions [CS17a] and the Nambu-Goto action [Mar20]. On the other hand, the respectively classically equivalent Einstein-Hilbert gravity [CS16] in (3+1) dimensions and the Polyakov action [Mar20] fulfill the BV-BFV axioms. The question of how one can go around these problems and construct a sensible BV-BFV theory for Palatini-Cartan gravity was addressed in [CS17a; CCS20].

As not all field theories are suitable for a BV-BFV description, the *lax approach* to the BV-BFV formalism was proposed by Mnev, Schiavina and Wernli in [MSW20], which gathers the data prior to the step where the regularity conditions become relevant. This setting already allows us to construct the *BV-BFV complex* [BBH95; MSW20], which is the adaptation of the BV complex to the case with boundary. Likewise, classical observables are con-

[1] Another approach to General Relativity by means of the BV formalism can be found in [Rej11; FR12b].

tained in its cohomology. As such, the lax formalism offers a sensible way of comparing two field theories on manifolds with boundary, even if one does not yield a *strict* BV-BFV theory.

In the present thesis, we deal with the classically equivalent Jacobi theory and one-dimensional gravity coupled to matter (1D GR), which can be regarded as the one-dimensional counterparts of the Nambu-Goto and Polyakov actions respectively. While 1D GR satisfies the regularity conditions of the BV-BFV formalism, the Jacobi theory produces a singular theory on the boundary [CS17b]. Even though these are one-dimensional models, they show a rich structure, and we hope that their investigation can shed some light on the source of this discrepancy for the case of higher-dimensional gravity theories.

We address this question by comparing the BV and BV-BFV cohomologies of the Jacobi theory and 1D GR, in order to probe whether the discrepancy of their boundary behaviours is reflected at the level of their classical observables. We find that the BV and BV-BFV complexes associated to the two theories on manifolds with possibly non-empty boundary are quasi-isomorphic. These results seem to suggest that, even though the Jacobi theory and 1D GR are classically equivalent, their quantizations on manifolds with boundary might not be, due to the fact that Jacobi theory fails to be compatible with the BV-BFV axioms.

The current thesis is structured as follows: In Chapter 2 we give a short review of classical Lagrangian field theory and introduce the Jacobi theory and 1D GR. This is followed by a revision of the BV, BV-BFV and lax formalisms in Chapter 3, where we also discuss the notions of equivalence that we are going to consider. In Chapter 4 we present our results for the case without a boundary and investigate the case with a boundary in Chapter 5.

Results and outlook: We show that the BV formulations of the Jacobi theory and 1D GR are equivalent, in the sense that their BV data can be interchanged in a way that preserves their cohomological structure. In particular, we show that the respective BV complexes are quasi-isomorphic: $H^\bullet(\mathfrak{BV}^\bullet_J) \simeq H^\bullet(\mathfrak{BV}^\bullet_{GR})$. As such, they associate the same set of classical observables to the bulk. This is achieved by first constructing two chain maps between the BV complexes and showing that their composition maps are either the identity or homotopic to it. We show the latter by constructing a one-parameter family of maps of the form

$$\chi_s^* = e^{s(QR+RQ)},$$

such that the identity is reproduced at $s = 0$ and the desired composition map recovered in the limit $s \to \infty$.[2] Here Q denotes the differential of the BV complex in question and R is a degree -1 derivation. This procedure could potentially be generalized or applied to other cases where one wishes to compare the BV cohomologies, examples being the first and second order formulations of Yang-Mills theory, the Nambu-Goto and Polyakov actions and Einstein-Hilbert and Palatini-Cartan gravity.

In addition, we extend these results to the case with a boundary. Recall that in this setting we can define a BV-BFV theory for 1D GR but not for the Jacobi theory. Even so, we compare their BV-BFV complexes, which are well-defined for both theories, and show that they are quasi-isomorphic, indicating that the Jacobi theory and 1D GR assign the same set of classical observables to a manifold in the presence of a boundary.

Finally, we show that the BV formulation of 1D GR can be changed in a way which preserves its cohomological structure, but changes its compatibility with the BV-BFV regularity conditions.

We thus found that the boundary discrepancy present in the BV-BFV formulations of the Jacobi theory and 1D GR is not reflected on the set of classical observables that they assign to a spacetime manifold. Nonetheless, the compatibility with the regularity conditions is crucial for BV-BFV quantization on manifolds with boundary, hence suggesting that 1D GR is better suited for covariant quantization in the presence of a boundary than the classically equivalent Jacobi theory.

[2] Note that χ_s^* is equivalent to an ordinary homotopy with parameter $\tau \in [0, 1]$ if we set $s = -\ln \tau$.

2 Lagrangian Field Theory

This chapter and the next serve as background chapters for the rest of the present work. We start the current chapter by reviewing the Lagrangian formulation of classical field theories following [And89; Del+99; BBH00] for the general discussions in Section 2.1 and [CS17b] for the specific examples presented in Sections 2.2 and 2.3.

We assume that the reader has a background on graded differential geometry [CS11; Cat15] and symplectic geometry [Sil08]. When performing computations, we will use the *total degree* $|\cdot|$, namely the sum of all degrees (internal degree, form degree and so on).

2.1 Fields, Lagrangians and symmetries

Let M be a smooth (orientable) manifold of dimension $\dim M$. In physics this usually denotes a "spacetime" manifold endowed with a Lorentzian metric. In this work we are interested in the case $\dim M = 1$, where M is an open or closed interval I on the real line $\mathbb{R}$ and is to be interpreted as the time manifold for classical mechanics or the "spacetime" for one-dimensional gravity coupled to matter (1D GR), but for now we want to keep M general.

In order to build a classical field theory on M we need to first define our fields, which in general will be smooth sections of a $\mathbb{Z}$-graded fibre bundle $E \to M$, where the fibres in non-zero degrees are assumed to be vector spaces. The space of fields will be denoted by $F = \Gamma(E)$ and the graded commutative algebra of smooth functions on F by $C^\infty(F)$. Designating the even and odd parts of F by F_{even}, F_{odd} allows us to write

$$C^\infty(F) = C^\infty(F_{\text{even}}) \otimes \bigwedge F^*_{\text{odd}},$$

where $\bigwedge F^*_{\text{odd}}$ is the exterior algebra of the dual of F_{odd}.

Example 2.1.1. In many fields theories the fields are smooth maps $\phi : M \to X$ for some auxiliary space X, hence we choose $E = X \times M$. In this setting we call M the *source* manifold and X the *target* manifold. For example in n-dimensional classical mechanics we deal with *matter fields* $q : I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$ is a time interval. The fields q are then smooth sections of the bundle $\mathbb{R}^n \times I \to I$.

Example 2.1.2. In theories of gravity over a spacetime manifold M, the field is a Lorentzian metric field g, which is given by a section of the bundle $\mathrm{Lor}(M) \to M$ of fiberwise Lorentzian bilinear forms over M. In the case $M = I$, this is simply the bundle $S^2_+ T^* I \to I$ of symmetric non-degenerate rank-$(0,2)$ tensor fields on I.

Most of the relevant objects needed to define a sensible field theory will live in a subcomplex of the de Rham complex over $F \times M$, denoted by $(\Omega^{\bullet,\bullet}(F \times M), \mathfrak{d})$. As shown in [And89] the differential can be decomposed as

$$\mathfrak{d} = \mathrm{d} + \delta,$$

where d is the *horizontal differential* along the base M and δ the *vertical differential* along the fibers F. The elements of $\Omega^{p,q}(F \times M)$ are then p-forms on F with values in the space of q-forms on M. More specifically, we are interested in the subcomplex of *local* elements $\Omega^{\bullet,\bullet}_{\mathrm{loc}}(F \times M)$. Here "local" means that evaluating a form $\alpha \in \Omega^{p,q}_{\mathrm{loc}}(F \times M)$ at $\phi \in F$ and at the vector fields $X_1, \ldots, X_p$ along F results in a q-form $\alpha(\phi, X_1, \ldots, X_p) \in \Omega^q(M)$ on M that only depends on the first k derivatives (or kth-jet, see Definition 2.1.3) of $\phi, X_1, \ldots, X_p$ at every point $x \in M$. Let us make these notions precise.

Definition 2.1.3 (Jet bundle)**.** Let M, $F = \Gamma(E)$ be as above, with local coordinates x^μ, ϕ^i respectively and denote the derivatives of ϕ^i by $\phi^i_{\mu_1 \ldots \mu_k} = \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_k}} \phi^i$. Let V^k denote the space with local coordinates

$$\{\phi^i(x), \phi^i_\mu(x), \ldots, \phi^i_{(\mu_1 \ldots \mu_k)}(x)\},$$

where we symmetrize the indices in order to account for the commuting nature of the derivatives. The *jet space of order* k [BBH00] is then the following bundle over M

$$\pi^k : J^k(E) = M \times V^k \to M.$$

Furthermore, we define the evaluation maps as [And89]

$$\begin{aligned} j^k : F \times M &\to J^k(E), \\ (\phi^i, x) &\mapsto (x, \phi^i(x), \ldots, \phi^i_{(\mu_1 \ldots \mu_k)}(x)) \end{aligned}$$

and call $j^k(\phi^i, x)$ the *kth-jet of ϕ^i at x*. The *infinite jet bundle* [And89] is then defined as the limit of the following sequence:

$$E = J^0(E) \leftarrow J^1(E) \leftarrow \cdots \leftarrow J^k(E) \leftarrow \dots$$

and π^∞, j^∞ as the limit of the maps π^k, j^k accordingly.

Definition 2.1.4 (Variational bicomplex, local forms, integrated local forms)**.** The *variational bicomplex* is the double complex [And89]

$$(\Omega^{\bullet,\bullet}(J^\infty(E)), \mathrm{d}_V, \mathrm{d}_H)\,,$$

where d_H and d_V are the differentials on $J^\infty(E)$ along M and F respectively. The complex of *local forms* over $F \times M$ is defined as

$$(\Omega^{\bullet,\bullet}_{\mathrm{loc}}(F \times M), \delta, \mathrm{d}) = (j^\infty)^* \left(\Omega^{\bullet,\bullet}(J^\infty(E)), \mathrm{d}_V, \mathrm{d}_H\right).$$

Let $\alpha \in \Omega^{\bullet,\bullet}(J^\infty(E))$. The differentials are given by

$$\begin{aligned} \mathrm{d}(j^\infty)^*\alpha &= (j^\infty)^*\mathrm{d}_H\alpha, \\ \delta(j^\infty)^*\alpha &= (j^\infty)^*\mathrm{d}_V\alpha. \end{aligned}$$

and satisfy $[\delta, \mathrm{d}] = 0$. We call an element of $\Omega^{0,\bullet}_{\mathrm{loc}}(F \times M)$ a *local functional.*

An *integrated local form* is the integral of a local form $\alpha \in \Omega^{\bullet,\mathrm{top}}_{\mathrm{loc}}(F \times M)$ on M. If α is a local functional then we call it an *integrated local functional.* We denote the resulting algebras of integrated local forms and integrated local functionals by $\Omega^{\bullet}_{\mathrm{loc}}(F)$ and $\mathfrak{C}^\infty_{\mathrm{loc}}(F) = \Omega^0_{\mathrm{loc}}(F)$ respectively.

Remark 2.1.5. If M is a closed spacetime manifold, we have the identification [BBH00]

$$\mathfrak{C}^\infty_{\mathrm{loc}}(F) \simeq \Omega^{0,\mathrm{top}}_{\mathrm{loc}}(F \times M)/\mathrm{d}\Omega^{0,\mathrm{top}-1}_{\mathrm{loc}}(F \times M).$$

To see why this is the case, let $f, g \in \Omega^{0,\mathrm{top}}_{\mathrm{loc}}(F \times M)$ be two local functionals and $\mathscr{F} = \int_M f$, $\mathscr{G} = \int_M g$ their respective integrated local functionals. Then $\mathscr{F} = \mathscr{G}$ iff the difference $f - g$ is d-exact

$$\mathscr{F} - \mathscr{G} = \int_M (f - g) = \int_M \mathrm{d}(\dots) = 0,$$

where we used that M is closed in the last step. If M has a non-empty boundary $\partial M \neq \emptyset$, this no longer holds.

It is therefore natural to consider $\mathfrak{C}^{\infty}_{\text{loc}}(F)$ when dealing with closed manifolds, as it disregards boundary terms. In the case where a boundary is present, boundary terms become relevant and working with local forms is more useful.

Remark 2.1.6. We can define two derivatives on the complex of local forms, namely the *total derivative*

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \sum_{k\geq 0} \phi^i_{(\mu\nu_1\ldots\nu_k)} \frac{\partial}{\partial \phi^i_{(\nu_1\ldots\nu_k)}},$$

and the *Euler-Lagrange derivative*

$$\frac{\delta}{\delta\phi^i} = \sum_{k\geq 0} (-1)^k \partial_{(\mu_1\ldots\mu_k)} \frac{\partial}{\partial \phi^i_{(\mu_1\ldots\mu_k)}},$$

corresponding to the differentials d and δ respectively. We have [And89]

$$\mathrm{d} = \mathrm{d}x^\mu \partial_\mu, \qquad\qquad \delta = \delta\phi^i \frac{\delta}{\delta\phi^i}.$$

Definition 2.1.7. An *evolutionary vector field* [And89] $X \in \mathfrak{X}_{\text{evo}}(F)$ on F is a vector field on $J^\infty(E)$ which is vertical with respect to $\pi^\infty : J^\infty(E) \to M$, i.e. $X \in \ker \pi^\infty_*$, such that

$$[\mathcal{L}_X, \mathrm{d}] = 0,$$

where $\mathcal{L}_X = [\iota_X, \delta]$ is the variational Lie derivative on local forms.

In Lagrangian field theories we are interested in describing the dynamics through a local functional $L \in \Omega^{0,\text{top}}_{\text{loc}}(F \times M)$, called the *Lagrangian*. In most field theories, as in the relevant cases for this work, L only depends on the 1st-jet of the fields ϕ^i, so we will only discuss the case where $L = L(\phi^i, \phi^i_\mu)$. Integrating the Lagrangian L gives us an integrated local functional, called the *action*

$$S[\phi^i] = \int_M L(\phi^i, \phi^i_\mu). \tag{1}$$

The classical solutions of a field theory are given by all the fields configurations that extremize the action, i.e. that solve the condition $\delta S = 0$. If $L = L(\phi^i, \phi^i_\mu)$ we have

$$\delta S = \int_M \frac{\delta L}{\delta\phi^i}\delta\phi^i + \int_M \partial_\mu \left(\frac{\partial L}{\partial \phi^i_\mu} \delta\phi^i \right). \tag{2}$$

Note that the last term in equation (2) is a boundary term. Let dvol_M be a volume form on M and write $L = \tilde{L}\,\mathrm{dvol}_M$. The *Noether 1-form* is then

$$\alpha_N = \frac{\partial \tilde{L}}{\partial \phi^i_\mu}\delta\phi^i\,\mathrm{dvol}_{\partial M} \in \Omega^{1,\mathrm{top}-1}(F \times M). \tag{3}$$

where $\mathrm{dvol}_{\partial M}$ is an embedded volume form on ∂M in M. It will play a crucial role when defining a field theory on the boundary.

In the case where M does not have a boundary, $\partial M = \emptyset$, the set $EL \subset F$ of classical solutions are the field configurations ϕ^i that solve the Euler-Lagrange (EL) equations

$$\frac{\delta L}{\delta \phi^i} = \frac{\partial L}{\partial \phi^i} - \partial_\mu \frac{\partial L}{\partial \phi^i_\mu} = 0. \tag{4}$$

We can then define a classical field theory as follows:

Definition 2.1.8 (Classical field theory, critical locus)**.** Let M be a space-time manifold and $E \to M$ a $\mathbb{Z}$-graded fibre bundle over M. A classical field theory on M is a pair (F, S), consisting of a space of fields $F = \Gamma(E)$ and an action S coming from a Lagrangian $L \in \Omega^{0,\mathrm{top}}_{\mathrm{loc}}(F \times M)$ as in equation (1).

The *critical locus* $EL \subset F$ of the action S is defined as the surface where the Euler-Lagrange equations (4) hold. This space is also called the "stationary surface" and quantities restricted to this surface are said to be "on-shell".

A further important aspect of field theories is the notion of *local symmetries*. These are given by a set of local automorphisms g of F, where local means that $g(\phi)$ and $g^{-1}(\phi)$ at $x \in M$ only depend on the kth-jet of ϕ at $x \in M$. We then say that g is a local symmetry if its action changes the Lagrangian by a d-exact term

$$L(g(\phi)) - L(\phi) = \mathrm{d}\alpha_g(\phi),$$

where $\alpha_g \in \Omega^{0,\mathrm{top}-1}_{\mathrm{loc}}(F \times M)$, i.e. if it changes the action S by a boundary term. As such the bulk terms in the variation of the action (equation (2)) will be unchanged and the critical locus EL will be preserved. We are now interested in describing the *infinitesimal* version of this construction.

Definition 2.1.9. An *infinitesimal local symmetry* [Del+99] of a classical field theory (F, S) is given by a vector bundle $V \to M$, such that for every $\zeta \in \Gamma(V)$ we have linear maps

$$\zeta \mapsto X_\zeta, \qquad\qquad \zeta \mapsto \alpha_\zeta,$$

where $X_\zeta \in \mathfrak{X}(F)$ is a vector field along F and $\alpha_\zeta \in \Omega^{0,\mathrm{top}-1}_{\mathrm{loc}}(F \times M)$ such that

$$\mathcal{L}_{X_\zeta} L = \mathrm{d}\alpha_\zeta \qquad \text{on } F \times M.$$

We require that X_ζ, α_ζ are local with respect to ζ. Furthermore, note that the vector fields X_ζ can be described as sections of a distribution $D \subset TF$, which we require to be *involutive* "on-shell", i.e. if $X, Y \in \Gamma(D)$, then $[X, Y] \in \Gamma(D)$ on the critical locus EL.

Remark 2.1.10. In reparameterization invariant theories, the symmetry group is given by the diffeomorphism group $\mathrm{Diff}(M)$. As such the vector bundle describing infinitesimal local symmetries is $V = TM$ and X_ζ acts as the Lie derivative of ζ. Note that in this case the distribution D is involutive everywhere on F.

Local symmetries describe redundancies in field theories on which their physical content should not depend. In this case, the space of classical solutions that we are interested in is not EL but rather the space of inequivalent classical solutions EL/D, i.e. the space of orbits of D on the critical locus EL. The space of *classical observables* is then a suitable space of functions over EL/D which we denote by $C^\infty(EL/D)$.

As a quotient, EL/D is often singular and defining $C^\infty(EL/D)$ is a non-trivial task. One way of handling this problem is to build a *resolution* of $C^\infty(EL/D)$, as is done in the BV formalism (see Chapter 3).

In this thesis we are interested in analyzing to which extend the BV formulations of the Jacobi theory and 1D GR are equivalent. Starting the discussion of equivalence in the setting of classical (non-BV) Lagrangian field theory, we say that two theories (F_i, S_i), $i \in \{1, 2\}$, describe the same physics if they share the same space of classical observables $C^\infty(EL_i/D_i)$. As the two theories in question are built over the same spacetime manifold $M_1 = M_2$ and share the same symmetries $D_1 = D_2$ (see Sections 2.2, 2.3), we only need to compare their EL equations. If these are equivalent, then both theories have the same critical locus and as such the same space of classical observables.

2.2 Jacobi theory

The starting point for the Jacobi theory [CS17b] is classical mechanics, which we now recall. The spacetime manifold in this case is taken to be an open or closed interval on the real line $M = I \subset \mathbb{R}$ with coordinate t, which we interpret as a finite time interval. The fields are chosen to be matter fields $\tilde{q} \in \Gamma(\mathbb{R}^n \times I) = C^\infty(I, \mathbb{R}^n)$ with mass m (see in Example 2.1.1).[3] We describe their kinetic energy $\tilde{T}$ as a quadratic form on the target $\mathbb{R}^n$, more specifically we take $\tilde{T}(\dot{\tilde{q}}) = \frac{m}{2}\|\dot{\tilde{q}}\|^2$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and $\dot{\tilde{q}} = \tilde{q}_t = \partial_t \tilde{q}$ is the time derivative of $\tilde{q}$. To keep the discussion general, we will be considering a potential term $V(\tilde{q})$, which will later be dropped.

The Lagrangian for classical mechanics is given by $L(\tilde{q}, \dot{\tilde{q}}) = (\tilde{T} - V)\mathrm{d}t$ and the *Hamiltonian* by its Legendre transform $H = (\|\tilde{p}\|^2/2m + V(\tilde{q}))\,\mathrm{d}t = (\tilde{T} + V)\mathrm{d}t$, where $\tilde{p} = \partial L/\partial\dot{\tilde{q}} = m\dot{\tilde{q}}$ is the canonical momentum of $\tilde{q}$. The action for classical mechanics is given by

$$S = \int_I (\tilde{p} \cdot \mathrm{d}\tilde{q} - H) = \int_I (\tilde{p} \cdot \dot{\tilde{q}}\,\mathrm{d}t - H).$$

To arrive to the Jacobi theory we then assume that the energy is constant and set $H = E\,\mathrm{d}t$, $E = \text{const.} > 0$. As such we can drop the last term in the action (Maupertuis' principle), resulting in

$$S = \int_I \tilde{p} \cdot \dot{\tilde{q}}\,\mathrm{d}t.$$

The Lagrangian we are left with is then $\tilde{p} \cdot \dot{\tilde{q}}\,\mathrm{d}t = 2\tilde{T}\,\mathrm{d}t = 2(E - V)\,\mathrm{d}t$, which can be written as $\tilde{p} \cdot \dot{\tilde{q}}\,\mathrm{d}t = 2\tilde{T}^\alpha (E - V)^\beta\,\mathrm{d}t$ with $\alpha + \beta = 1$. The geometric mean $\alpha = \beta = \frac{1}{2}$ leads to the Jacobi action

$$S_J[\tilde{q}] = \int_I 2\sqrt{(E - V)\tilde{T}}\,\mathrm{d}t.$$

To see that S_J is parameterization invariant, note that writing

$$\mathrm{d}s^2 = 2m(E - V)\,\mathrm{d}\tilde{q}^2,$$

lets us interpret the Jacobi action as the length of a path in the target space $\mathbb{R}^n$ with metric $\mathrm{d}s^2$. As such the symmetry group of the Jacobi theory is the

[3] We denote the fields of the Jacobi theory with a tilde and the fields of 1D GR without one in order to avoid confusion.

diffeomorphism group of the interval $\mathrm{Diff}(I)$, i.e. the reparameterizations of I. The critical locus of S_J is then given by the geodesics of the metric $\mathrm{d}s^2$, which are the trajectories of classical mechanics with an arbitrary parameterization [CS17b]. Imposing $E = \tilde{T}(\dot{\tilde{q}}) + V(\tilde{q})$ allows us to recover the standard parameterization.

We set $V(\tilde{q}) = 0$ for the rest of the discussion, since this is the case that we are interested in. The EL equations and the Noether 1-form can be read from the variation of the Jacobi action [CS17b]

$$\delta S_J = -\int_I \partial_t \left(\sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \right) \cdot \delta\tilde{q}\,\mathrm{d}t + \int_{\partial I} \sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \cdot \delta\tilde{q}\,\mathrm{d}t.$$

Note that the EL equations

$$\partial_t \left(\sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \right) = 0,$$

are singular for $\dot{\tilde{q}} = 0$ due to the $T^{-1/2}$ term. As such, the space of fields for the Jacobi theory is not $C^\infty(I, \mathbb{R}^n)$ but rather

$$F_J = \left\{ \tilde{q} \in C^\infty(I, \mathbb{R}^n) \mid \nexists\, t \in I : \dot{\tilde{q}}(t) = 0 \right\}.$$

We can then interpret the Jacobi theory as classical mechanics at constant energies where the solutions do not have turning points, i.e. points in which the first derivative vanishes.

2.3 One-dimensional gravity coupled to matter

In the case of 1D GR [CS17b] we also choose $M = I$, where I is to be interpreted as our 1D "spacetime". In addition to the matter field $q \in C^\infty(I, \mathbb{R}^n)$ we also consider a metric field $\mathsf{g} \in \Gamma(S^2_+ T^* I)$ as a non-vanishing section of the bundle of symmetric non-degenerate rank-$(0,2)$ tensors over I (see Example 2.1.2). For simplicity, we write $\mathsf{g} = g\,\mathrm{d}t^2$ and work with the component $g \in C^\infty(I, \mathbb{R}_{>0})$. The space of fields is then given by

$$F_{GR} = F_J \times C^\infty(I, \mathbb{R}_{>0}).$$

The condition $\dot{q} \neq 0$ in F_J is strictly speaking not necessary in the 1D GR case, but since we are ultimately interested in comparing 1D GR with the

Jacobi theory we impose it for consistency. In this picture, we can interpret 1D GR as an extension of the Jacobi theory.

We consider the action

$$S_{GR}[q,g] = \int_I \left(\frac{T}{g} + E\right)\sqrt{g}\,\mathrm{d}t = \int_I \left(\frac{T}{\sqrt{g}} + \sqrt{g}E\right)\,\mathrm{d}t.$$

Note that the Ricci tensor vanishes in 1D and hence the Einstein-Hilbert term is absent. The first term in S_{GR} is simply the matter Lagrangian for vanishing potential in the presence of a metric field and the second a cosmological term. As such we interpret the parameter E as a cosmological constant. Since we are integrating over the Riemannian density $\sqrt{g}\,\mathrm{d}t$ of the metric $\mathrm{d}s^2 = g\mathrm{d}t^2$, the symmetry group is again $\mathrm{Diff}(I)$.

Varying the action gives

$$\delta S_{GR}[q,g] = \int_I EL_g \delta g\,\mathrm{d}t - \int_I \partial_t\left(\frac{m\dot{q}}{\sqrt{g}}\right)\cdot\delta q\,\mathrm{d}t + \int_{\partial I}\frac{m\dot{q}}{\sqrt{g}}\cdot\delta q\,\mathrm{d}t,$$

where

$$EL_g = \frac{\delta S_{GR}}{\delta g} = \frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}.$$

To see that both theories give rise to the same critical locus, we integrate g out from the action $S_{GR}[q,g]$, i.e. we restrict $S_{GR}[q,g]$ to the surface where g solves the EL equations. Setting $EL_g = 0$ gives

$$g = \frac{T}{E}.$$

We can then compute

$$S_{GR}\left[q, g = \frac{T}{E}\right] = \int_I \left(\frac{TE}{T} + E\right)\sqrt{\frac{T}{E}}\,\mathrm{d}t = \int_I 2\sqrt{ET}\,\mathrm{d}t = S_J[q], \qquad (5)$$

as such both theories share the same critical locus and hence describe the same physics.

3 Batalin-Vilkovisky formalism

Together with Chapter 2, the current one serves as background for the rest of the current work. Having already presented the basic construction of classical Lagrangian field theories, we now introduce the Batalin-Vilkovisky (BV) approach to the subject.

We start this chapter with a review of the BV formalism for closed spacetime manifolds in Section 3.1 [BV83a; BV83b; BV84],[4] where we also present the BV formulations for the Jacobi theory and 1D GR [CS17b]. For a pedagogical introduction to the BV formalism we recommend [Hen90; GPS95; Mne17]. This is followed by a discussion of BV-equivalence in Section 3.2. A revision of the BV-BFV formalism for manifolds with boundary is given in Section 3.3, see [CMR14; CM19], and we exemplify this construction (or failure thereof) through 1D GR and the Jacobi theory. In Section 3.4, we use the lax formalism to rewrite the BV-BFV formalism in the language of local forms [MSW20] and in Section 3.5 we adapt our notion of equivalence to this setting.

3.1 BV formalism on the bulk

Definition 3.1.1. A *BV theory* over a closed spacetime manifold M is a quadruple $\mathfrak{F} = (\mathcal{F}, \omega, \mathcal{S}, Q)$ where

- $\mathcal{F} = \Gamma(E)$ is the space of smooth sections of a $\mathbb{Z}$-graded bundle $E \to M$ (the *BV space of fields*),
- $\omega \in \Omega^2_{\text{loc}}(\mathcal{F})$ is a symplectic integrated local 2-form of degree -1 (the *BV form*),
- $\mathcal{S} \in \mathfrak{C}^\infty_{\text{loc}}(\mathcal{F})$ is a degree 0 integrated local functional (the *BV action*),
- $Q \in \mathfrak{X}_{\text{evo}}(\mathcal{F})$ is a degree 1 evolutionary cohomological vector field, i.e. $[Q, Q] = 2Q^2 = 0$, $[\mathcal{L}_Q, \text{d}] = 0$,

such that

$$\iota_Q \omega = \delta \mathcal{S}, \tag{6}$$

[4] For a discussion of the BV formalism in the setting of non-compact spacetime manifolds see [Rej11; FR12a].

We call equation (6) the *Hamiltonian condition.* The internal degree in $\mathcal{F}$ is called the *ghost number* and will be denoted by $\mathrm{gh}(\cdot)$.

Remark 3.1.2. Since Q is Hamiltonian it preserves the BV form

$$\mathcal{L}_Q \omega = 0, \tag{7}$$

which can be checked by applying δ to equation (6) and keeping in mind that ω is by definition δ-closed.

Furthermore, the condition that Q is the Hamiltonian vector field of $\mathcal{S}$ with respect to ω together with the property that Q is cohomological imply the *Classical Master Equation* (CME) [BV84; Sch93]

$$\iota_Q \iota_Q \omega = \mathcal{L}_Q \mathcal{S} = (\mathcal{S}, \mathcal{S}) = 0, \tag{8}$$

where $(\cdot, \cdot)$ is the Poisson bracket on $\mathcal{F}$ induced by ω.

Note that changing either ω or $\mathcal{S}$ by an $\mathcal{L}_Q$-exact term does not affect equations (7,8).

Remark 3.1.3. Given a classical field theory (F, S) on a closed spacetime manifold M (possibly already including ghost fields), we can construct a BV theory by choosing [Sch93]

$$\mathcal{F} = T^*[-1]F.$$

Let Φ^i denote local coordinates on the base F and $\Phi_i^\dagger$ local coordinates on the fibers with degree $\mathrm{gh}(\Phi^\dagger) = 1 - \mathrm{gh}(\Phi)$. The BV form is then the canonical symplectic form on $\mathcal{F}$

$$\omega = \int_M \langle \delta\Phi^\dagger, \delta\Phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is a bilinear map with values on $\Omega^{\mathrm{top}}(M)$. We can then construct a BV action $\mathcal{S}$ by adding terms proportional to powers of $\Phi^\dagger$ to the action S so that it solves the CME (8) [BV84]. The cohomological vector field Q is then defined through equation (6) and we have $Q(\cdot) = (\mathcal{S}, \cdot)$, which we assume to be evolutionary.

The property that Q is cohomological lets us define the following chain complex:

Definition 3.1.4. The *BV complex* of a given BV theory $\mathfrak{F}$ is defined as the algebra of integrated local functionals on $\mathcal{F}$ endowed with the cohomological vector field Q

$$\mathfrak{BV}^\bullet = (\mathfrak{C}^\infty_{\text{loc}}(\mathcal{F}), Q),$$

where the grading is given by the ghost number. We denote its cohomology by $H^\bullet(\mathfrak{BV}^\bullet)$ and call it the *BV cohomology*.

This complex is a *resolution* of $C^\infty(EL/D)$, in the sense that the BV cohomology is given by [Hen90; Sta98; FR12a]

$$\begin{aligned} &H^{-i}(\mathfrak{BV}^\bullet) = 0 && \text{for } i > 0, \\ &H^0(\mathfrak{BV}^\bullet) \simeq C^\infty(EL/D), \end{aligned}$$

which can be interpreted as first building the resolution of $C^\infty(EL)$ via the Koszul-Tate complex and then constructing the Chevalley-Eilenberg complex in order to describe the D-invariant functions on EL. We can then interpret the cohomological vector field Q as an operator encoding the symmetries and the "on-shell" information of the theory. The fact that classical observables $\mathcal{O} \in C^\infty(EL/D)$ should be invariant under the action of the symmetries is reflected by the property that they are in the kernel of Q by construction: $Q\mathcal{O} = 0$.

Remark 3.1.5. The cohomology in degree 0, $H^0(\mathfrak{BV}^\bullet)$, is not the only degree that contains relevant physical information. For example, the cohomology in degree 1, $H^1(\mathfrak{BV}^\bullet)$, is the space of *anomalies* of a given theory [BBH00]. Roughly speaking, an anomaly describes the failure of a symmetry of a classical theory being a symmetry of any regularization scheme of the quantum theory. As such, when comparing two BV theories we want to consider the whole cohomology $H^\bullet(\mathfrak{BV}^\bullet)$ and not only the space of classical observables $H^0(\mathfrak{BV}^\bullet)$.

Remark 3.1.6. The condition $H^{-i}(\mathfrak{BV}^\bullet) = 0$ for $i > 0$ is not strictly satisfied for any BV theory, as exemplified in [Get16].

In order to illustrate how a BV theory can be constructed from a classical field theory, we now present the case where the algebra of infinitesimal local symmetries is a Lie algebra and the two relevant examples for this thesis.

Example 3.1.7 (Lie algebra case [BV84], see also [Hen90; Mne17])**.** We are going to present the finite-dimensional case, for a discussion of the infinite-dimensional case we refer to [FR12a]. This means that instead of regarding spaces of integrated local functionals, we simply consider C^∞-functions over the spaces of fields.

Given a classical field theory (F, S) over a closed spacetime manifold M and an infinitesimal local symmetry on F given by the action of a Lie algebra $\mathfrak{g}$, we can build a BRST theory [BRS76; Tyu08] over the *BRST space of fields* $F \times \mathfrak{g}[1]$, with local coordinates ϕ^i on F and degree 1 local coordinates ξ^a on $\mathfrak{g}[1]$ (the *ghosts*). The algebra of functions over $F \times \mathfrak{g}[1]$ then corresponds to the Chevalley-Eilenberg complex of $\mathfrak{g}$ twisted by $C^\infty(F)$ [CE48; AIB98]

$$C^\infty(F \times \mathfrak{g}[1]) = C^\infty(F) \otimes \bigwedge{}^\bullet \mathfrak{g}^* = C^\bullet_{CE}(\mathfrak{g}, C^\infty(F)).$$

The Chevalley-Eilenberg differential takes the form

$$\gamma = \frac{1}{2}[\xi, \xi]^c \frac{\partial}{\partial \xi^c} + \xi^a v_a^i \frac{\partial}{\partial \phi^i},$$

where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$ and v_a^i the fundamental vector fields of $\mathfrak{g}$ on F. We can then interpret γ as a cohomological vector field Q_{BRST} on $F \times \mathfrak{g}[1]$. Note that the cohomology in degree 0 of the Chevalley-Eilenberg complex is the space of functions over F which are invariant under the action of $\mathfrak{g}$. As such, $(C^\infty(F \times \mathfrak{g}[1]), Q_{BRST})$ is a resolution of the space $C^\infty(F/\mathfrak{g})$ of invariant function on F.

In order to build a resolution of the desired space of functions, namely the "on-shell" invariant functions $C^\infty(EL/\mathfrak{g})$, we need to adapt the BRST construction to the BV formalism. We start by defining the BV space of fields as

$$\mathcal{F} = T^*[-1](F \times \mathfrak{g}[1]), \tag{9}$$

with local coordinates $\Phi^j = (\phi^i, \xi^a)$ on the base and $\Phi^\dagger_j = (\phi^\dagger_i, \xi^\dagger_a)$ on the fibers. From now on we will call $\phi^\dagger_i$ the *antifields* and $\xi^\dagger_a$ the *antighosts*. The action S can then be extended as

$$\mathcal{S}[\Phi, \Phi^\dagger] = S[\phi] + \int_M \langle \Phi^\dagger, Q_{BRST} \Phi \rangle \tag{10}$$

and we set $Q(\cdot) = (\mathcal{S}, \cdot)$. Note that $Q\Phi = Q_{BRST}\Phi = \gamma\Phi$ on the fields ϕ^i and ghosts ξ^a. It can be shown that in the Lie algebra case [Sta98] Q decomposes as

$$Q = \gamma + \delta_{KT},$$

where δ_{KT} is the Koszul-Tate differential, whose action on $\mathcal{F}$ is given by

$$\begin{aligned} \delta_{KT}\phi^i &= 0, & \delta_{KT}\xi^a &= 0, \\ \delta_{KT}\phi_i^\dagger &= \frac{\delta S}{\delta \phi^i}, & \delta_{KT}\xi_a^\dagger &= v_a^i \phi_i^\dagger. \end{aligned} \tag{11}$$

The complex $(C^\infty(\mathcal{F}), Q)$ is then a resolution of $C^\infty(EL/D)$.

Remark 3.1.8. We have discussed the Lie algebra case, where the symmetry algebra is irreducible and closed, but the BV formalism is also able treat "wilder" symmetries, which can be reducible or open outside of the critical locus EL. We refer to [Hen90; GPS95; Sta98; BBH00; Mne17] for a broader discussion of the topic.

Example 3.1.9 (BV formulation of the Jacobi theory [CS17b])**.** We first consider the case where the interval $I = (a, b) \subset \mathbb{R}$ has an empty boundary $\partial I = \emptyset$. Before extending the Jacobi theory to its BV formulation, we have to introduce the ghost field. The Lie algebra of the diffeomorphism group $\mathrm{Diff}(I)$ is the algebra of vector fields $\mathfrak{g} = \mathfrak{X}(I)$ on I and thus we introduce the ghost field $\tilde{\xi}\partial_t \in \mathfrak{X}(I)[1]$, following [Pig00]. The Chevalley-Eilenberg differential (and as such the cohomological vector field Q_J) acts on the matter fields $\tilde{q} \in F_J$ as the Lie derivative with respect to the ghost and on the ghost as half of the commutator

$$\begin{aligned} Q_J \tilde{q} &= \gamma_J \tilde{q} = \mathcal{L}_{\tilde{\xi}\partial_t} \tilde{q} = \tilde{\xi}\dot{\tilde{q}}, \\ Q_J\left(\tilde{\xi}\partial_t\right) &= \gamma_J \tilde{\xi} = \frac{1}{2}\left[\tilde{\xi}\partial_t, \tilde{\xi}\partial_t\right] = \tilde{\xi}\dot{\tilde{\xi}}\partial_t. \end{aligned}$$

Following equation (9), the BV space of fields is then

$$\mathcal{F}_J = \underbrace{F_J}_{\tilde{q}\in} \times \underbrace{\Omega^{\mathrm{top}}(I) \otimes C^\infty(I, \mathbb{R}^n)[-1]}_{\tilde{q}^\dagger \in} \times \underbrace{\mathfrak{X}(I)[1]}_{\tilde{\xi}\partial_t \in} \times \underbrace{\Omega^{\mathrm{top}}(I) \otimes \Omega^{\mathrm{top}}(I)[-2]}_{\tilde{\xi}^\dagger \in}.$$

Note that we can write $\tilde{q}^\dagger$, $\tilde{\xi}^\dagger$ as

$$\tilde{q}^\dagger = \mathrm{d}t\, \tilde{q}^+, \qquad \tilde{\xi}^\dagger = \mathrm{d}t \otimes (\mathrm{d}t\, \tilde{\xi}^+),$$

where $\tilde{q}^+ \in C^\infty(I, \mathbb{R}^n)[-1]$ and $\tilde{\xi}^+ \in C^\infty(I, \mathbb{R})[-2]$. For concreteness, we define the pairings in equation (10) as

$$\begin{aligned}
&\langle \tilde{q}^\dagger, Q_J \tilde{q} \rangle = \mathrm{d}t\, \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}} = \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}}\, \mathrm{d}t, \\
&\left\langle \tilde{\xi}^\dagger, Q_J \left(\tilde{\xi}\partial_t\right)\right\rangle = \frac{1}{2}\iota_{[\tilde{\xi}\partial_t, \tilde{\xi}\partial_t]}(\mathrm{d}t \otimes \mathrm{d}t\, \tilde{\xi}^+) = \tilde{\xi}\dot{\tilde{\xi}}\, \mathrm{d}t\, \tilde{\xi}^+ = \tilde{\xi}^+\tilde{\xi}\dot{\tilde{\xi}}\, \mathrm{d}t.
\end{aligned}$$

From now on we will drop the basis vector field ∂_t and the 1-form $\mathrm{d}t$ and simply work with the components $\tilde{\xi}, \tilde{q}^+, \tilde{\xi}^+$. The canonical BV form on $\mathcal{F}_J$ and BV action are then

$$\begin{aligned}
\omega_J &= \int_I \left\{\delta\tilde{q}^+ \cdot \delta\tilde{q} + \delta\tilde{\xi}^+\delta\tilde{\xi}\right\} \mathrm{d}t, \\
\mathcal{S}_J[\tilde{q}, \tilde{q}^+, \tilde{\xi}, \tilde{\xi}^+] &= \int_I \left\{2\sqrt{E\tilde{T}} + \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}} + \tilde{\xi}^+\tilde{\xi}\dot{\tilde{\xi}}\right\} \mathrm{d}t, \qquad (12)
\end{aligned}$$

which induce the following action of Q_J on the antifield and antighost

$$\begin{aligned}
Q_J\tilde{q}^+ &= -\partial_t\left(\sqrt{\frac{E}{\tilde{T}}}m\dot{\tilde{q}} + \tilde{q}^+\tilde{\xi}\right), \\
Q_J\tilde{\xi}^+ &= -\tilde{q}^+ \cdot \dot{\tilde{q}} + \tilde{\xi}\dot{\tilde{\xi}}^+ + 2\dot{\tilde{\xi}}\tilde{\xi}^+.
\end{aligned}$$

Remark 3.1.10. Note that we can explicitly decompose the cohomological vector field Q_J into its Chevalley-Eilenberg and Koszul-Tate parts as $Q_J = \gamma_J + \delta_J$ by using equation (11) and setting $\gamma_J = Q_J - \delta_J$ on $\{\tilde{q}^+, \tilde{\xi}^+\}$. We have:

$$\begin{aligned}
\gamma_J\tilde{q}^+ &= \tilde{\xi}\dot{\tilde{q}}^+ + \dot{\tilde{\xi}}q^+, & \delta_J\tilde{q}^+ &= -\partial_t\left(\sqrt{\frac{E}{\tilde{T}}}m\dot{\tilde{q}}\tilde{\xi}\right), \\
\gamma_J\tilde{\xi}^+ &= \tilde{\xi}\dot{\tilde{\xi}}^+ + 2\dot{\tilde{\xi}}\tilde{\xi}^+, & \delta_J\tilde{\xi}^+ &= -\tilde{q}^+ \cdot \dot{\tilde{q}}.
\end{aligned}$$

Since $\tilde{q}^+$ and $\tilde{\xi}^+$ are components of a rank-$(0,1)$ and a rank-$(0,2)$ tensors over I respectively we see that $\gamma_J = \mathcal{L}_{\tilde{\xi}\partial_t}$ on $\{\tilde{q}^+, \tilde{\xi}^+\}$. A discussion of why this is the case can be found in the appendix A.1.

Example 3.1.11 (BV formulation of 1D GR [CS17b])**.** We again consider the case $I = (a, b)$. The considerations for the BV formulation of the Jacobi theory can mostly be adapted for the 1D GR case, with the addition that we also consider a metric field $g\,\mathrm{d}t^2 \in \Gamma(S^2_+T^*I)$. Since g is the component of a rank-$(0,2)$ tensor the action of Q_{GR} yields

$$Q_{GR}g = \mathcal{L}_{\xi\partial_t}g = \xi\dot{g} + 2\dot{\xi}g.$$

The BV space of fields for 1D GR is

$$\mathcal{F}_{GR} = \underbrace{\mathcal{F}_J}_{q,q^\dagger,\xi,\xi^\dagger\in} \times \underbrace{\Gamma(S^2_+T^*I)}_{g\,\mathrm{d}t^2\in} \times \underbrace{\Omega^{\mathrm{top}}(I)\otimes\Gamma[-1](S^2TI)}_{g^\dagger\in}.$$

We recall that we are keeping the condition $\dot{q}\neq 0$ for the 1D GR theory as well in order to interpret it as an extension of the Jacobi theory, even though this is not strictly necessary in this case. As before we decompose

$$g^\dagger = \mathrm{d}t \otimes (g^+\partial_t\otimes\partial_t),$$

where $g^+ \in C^\infty(I,\mathbb{R}_{>0})[-1]$. We define the pairing in equation (10) for the metric and its antifield to be

$$\langle g^\dagger, Q_{GR}(\mathrm{d}t^2 g)\rangle = g^+(\xi\dot{g} + 2\dot{\xi}g)\,\mathrm{d}t.$$

The BV form and BV action for 1D GR take the form

$$\omega_{GR} = \int_I \left\{\delta q^+\cdot\delta q + \delta\xi^+\delta\xi + \delta g^+\delta g\right\}\mathrm{d}t,$$

$$\mathcal{S}_{GR}[q,q^+,g,g^+,\xi,\xi^+] = \int_I\left\{\frac{T}{\sqrt{g}} + \sqrt{g}E + q^+\cdot\xi\dot{q} + g^+(\xi\dot{g}+2g\dot{\xi}) + \xi^+\xi\dot{\xi}\right\}\mathrm{d}t.$$

The cohomological vector field Q_{GR} then acts on the antifields and antighost as

$$\begin{aligned}
Q_{GR}q^+ &= -\partial_t\left(\frac{m\dot{q}}{\sqrt{g}} + q^+\xi\right),\\
Q_{GR}g^+ &= EL_g + \xi\dot{g}^+ - \dot{\xi}g^+,\\
Q_{GR}\xi^+ &= -q^+\cdot\dot{q} + g^+\dot{g} + 2\dot{g}^+g + \xi\dot{\xi}^+ + 2\dot{\xi}\xi^+.
\end{aligned}$$

Remark 3.1.12. As in the Jacobi theory we can decompose the cohomological vector field as $Q_{GR} = \gamma_{GR} + \delta_{GR}$. We have

$$\begin{aligned}
\gamma_{GR}q^+ &= \xi\dot{q}^+ + \dot{\xi}q^+, & \delta_{GR}q^+ &= -\partial_t\left(\frac{m\dot{q}}{\sqrt{g}}\right),\\
\gamma_{GR}g^+ &= \xi\dot{g}^+ - \dot{\xi}g^+ & \delta_{GR}g^+ &= EL_g,\\
\gamma_{GR}\xi^+ &= \xi\dot{\xi}^+ + 2\dot{\xi}\xi^+, & \delta_{GR}\xi^+ &= -q^+\cdot\dot{q} + g^+\dot{g} + 2\dot{g}^+g.
\end{aligned}$$

Since q^+, g^+ and ξ^+ are components of a rank-$(0,1)$, a rank-$(2,1)$ and a rank-$(0,2)$ tensors over I respectively we see that $\gamma_J = \mathcal{L}_{\xi\partial_t}$ on $\{q^+, g^+, \xi^+\}$. We again refer to appendix A.1 for a detailed discussion.

Remark 3.1.13. The sign convention that we use in Examples 3.1.9 and 3.1.11 differs from the one used in [CS17b]. This is does not affect the results.

3.2 Equivalence in the BV setting

We now have all the required tools to develop a notion of equivalence in the BV formalism. When comparing two BV theories $\mathfrak{F}_i$, $i \in \{1,2\}$, we want to take three objects within consideration: the BV data, the Hamiltonian conditions $\iota_{Q_i}\omega_i = \delta\mathcal{S}_i$ relating the BV data, and their BV cohomologies $H^\bullet(\mathfrak{BV}_i^\bullet)$, as it contains the set of classical observables. Following this train of thought we define:

Definition 3.2.1 (BV-equivalence)**.** Let $\mathfrak{F}_i$, $i \in \{1,2\}$, be two BV theories. We denote both de Rham differentials on $\mathcal{F}_i$ by δ. We say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are *BV-equivalent* if there exist two degree 0 maps

$$\phi : \mathcal{F}_1 \to \mathcal{F}_2, \qquad \psi : \mathcal{F}_2 \to \mathcal{F}_1,$$

such that their pullback maps ϕ^*, ψ^* are chain maps between the BV complexes $\mathfrak{BV}_1^\bullet$, $\mathfrak{BV}_2^\bullet$,

$$\phi^* \circ Q_2 = Q_1 \circ \phi^*, \qquad \psi^* \circ Q_1 = Q_2 \circ \psi^*,$$

and the composition maps

$$\begin{aligned} \lambda^* = \phi^* \circ \psi^* : \mathfrak{BV}_1^\bullet \to \mathfrak{BV}_1^\bullet, \\ \chi^* = \psi^* \circ \phi^* : \mathfrak{BV}_2^\bullet \to \mathfrak{BV}_2^\bullet, \end{aligned}$$

are the identity in the respective BV cohomologies $H^\bullet(\mathfrak{BV}_1^\bullet)$, $H^\bullet(\mathfrak{BV}_2^\bullet)$.[5]

Moreover, we require that ϕ, ψ, interchange the BV forms ω_i and BV actions $\mathcal{S}_i$ up to $\mathcal{L}_{Q_i}$-exact terms as

$$\begin{aligned} \phi^*\omega_2 &= \omega_1 + \mathcal{L}_{Q_1}\rho_1, & \psi^*\omega_1 &= \omega_2 + \mathcal{L}_{Q_2}\rho_2, \\ \phi^*\mathcal{S}_2 &= \mathcal{S}_1 + \mathcal{L}_{Q_1}\sigma_1, & \psi^*\mathcal{S}_1 &= \mathcal{S}_2 + \mathcal{L}_{Q_2}\sigma_2, \end{aligned}$$

[5] We use the pullback notation for λ^* and χ^* since they can be regarded as the pullback maps of $\lambda = \psi \circ \phi$ and $\chi = \phi \circ \psi$ respectively.

where $\rho_i \in \Omega^2_{\text{loc}}(\mathcal{F}_i)$, $\sigma_i \in \mathfrak{C}^\infty_{\text{loc}}(\mathcal{F}_i)$, and that they preserve the Hamiltonian conditions

$$\iota_{Q_1}\omega_1 = \delta\mathcal{S}_1 \underset{\phi^*}{\overset{\psi^*}{\rightleftarrows}} \iota_{Q_2}\omega_2 = \delta\mathcal{S}_2. \tag{13}$$

Remark 3.2.2. In principle, we could consider more complex transformations of the BV forms and BV actions that still preserve the Hamiltonian conditions. However, these would change their respective $\mathcal{L}_Q$-classes, which we want to preserve in our notion of equivalence. This is motivated by the fact that changing ω_i, $\mathcal{S}_i$ by $\mathcal{L}_{Q_i}$-exact terms does not change the property that the BV form is preserved by Q nor the CME, as noted in Remark 3.1.2.

Let us now explore some direct implications of Definition 3.2.1. Particularly, two BV-equivalent theories have isomorphic BV cohomologies.

Proposition 3.2.3. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two BV-equivalent theories. Using the notation of Definition 3.2.1 we have*

1. *the BV complexes of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are quasi-isomorphic*

$$H^\bullet(\mathfrak{BV}^\bullet_1) \simeq H^\bullet(\mathfrak{BV}^\bullet_2),$$

2. *the requirement that ϕ^*, ψ^* preserve the Hamiltonian conditions is equivalent to*

$$\mathcal{L}_{Q_i}(\iota_{Q_i}\rho_i + \delta\sigma_i) = 0, \tag{14}$$

3. *the theories $\mathfrak{F}_1, \lambda^*\mathfrak{F}_1 \coloneqq (\mathcal{F}_1, \lambda^*\omega_1, \lambda^*\mathcal{S}_1, Q_1)$ and $\mathfrak{F}_2, \chi^*\mathfrak{F}_2 \coloneqq (\mathcal{F}_2, \lambda^*\omega_2, \lambda^*\mathcal{S}_2, Q_2)$ are pairwise BV-equivalent.*

Proof.

1. If the the composition maps λ^*, χ^* are the identity when restricted to the respective cohomologies then ϕ^*, ψ^* are each others inverse in cohomology and in particular isomorphisms in cohomology. As such

$$H^\bullet(\mathfrak{BV}^\bullet_1) \simeq H^\bullet(\mathfrak{BV}^\bullet_2).$$

2. Applying ϕ^* to $\iota_{Q_2}\omega_2 = \delta\mathcal{S}_2$ gives

$$\begin{aligned} &\iota_{Q_1}\omega_1 + \iota_{Q_1}\mathcal{L}_{Q_1}\rho_1 = \delta\mathcal{S}_1 + \delta\mathcal{L}_{Q_1}\sigma_1 \\ \Rightarrow &\iota_{Q_1}\mathcal{L}_{Q_1}\rho_1 = \delta\mathcal{L}_{Q_1}\sigma_1, \end{aligned}$$

where we used $\iota_{Q_1}\omega_1 = \delta\mathcal{S}_1$. Noting that $[\iota_Q, \mathcal{L}_Q] = \iota_{[Q,Q]} = 0$ and $[\mathcal{L}_Q, \delta] = 0$ [Cat15] we have

$$\mathcal{L}_{Q_1}(\iota_{Q_1}\rho_1 + \delta\sigma_1) = 0.$$

It is then clear that this is a sufficient requirement for ϕ^* to preserve the Hamiltonian condition. Similarly, applying ψ^* to $\iota_{Q_2}\omega_2 = \delta\mathcal{S}_2$ gives $\mathcal{L}_{Q_2}(\iota_{Q_2}\rho_2 + \delta\sigma_2) = 0$.

3. We will only show this for the case of $\mathfrak{F}_1$ and $\lambda^*\mathfrak{F}_1$, where λ^* will assume the roles of both ϕ^* and ψ^*. Since λ^* is the composition of two chain maps it will also be a chain map

$$\lambda^* \circ Q_1 = \phi^* \circ \psi^* \circ Q_1 = \phi^* \circ Q_2 \circ \psi^* = Q_1 \circ \phi^* \circ \psi^* = Q_1 \circ \lambda^*.$$

Furthermore, $\lambda^* \circ \lambda^*$ is trivially the identity in cohomology. Since ϕ^*, ψ^* both interchange the BV forms and BV actions up to $\mathcal{L}_{Q_i}$-exact terms and preserve the Hamiltonian conditions this will also be the case for λ^*. The same argument holds in the case of $\mathfrak{F}_2$ and $\chi^*\mathfrak{F}_2$.

□

Remark 3.2.4. In the case that the BV forms are interchanged up to $\mathcal{L}_Q$-exact and δ-exact terms as

$$\phi^*\omega_2 = \omega_1 - \mathcal{L}_{Q_1}\delta\boldsymbol{\beta}_1, \qquad \psi^*\omega_1 = \omega_2 - \mathcal{L}_{Q_2}\delta\boldsymbol{\beta}_2,$$

i.e. $\rho_i = -\delta\boldsymbol{\beta}_i$, then equation (14) imposes the following transformations on the BV actions $\mathcal{S}_i$

$$\phi^*\mathcal{S}_2 = \mathcal{S}_1 + \mathcal{L}_{Q_1}\iota_{Q_1}\boldsymbol{\beta}_1, \qquad \psi^*\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{L}_{Q_2}\iota_{Q_2}\boldsymbol{\beta}_2,$$

i.e. $\sigma_i = \iota_{Q_i}\boldsymbol{\beta}_i$, since

$$\mathcal{L}_{Q_i}(\iota_{Q_i}\rho_i + \delta\sigma_i) = \mathcal{L}_{Q_i}(-\iota_{Q_i}\delta\boldsymbol{\beta}_i + \delta\iota_{Q_i}\boldsymbol{\beta}_i) = -\mathcal{L}^2_{Q_i}\boldsymbol{\beta}_i = 0,$$

where we used $\mathcal{L}_{Q_i} = [\delta, \iota_{Q_i}]$.

Remark 3.2.5. It is important to highlight how one shows that the maps λ^*, χ^* are the identity when restricted to the respective cohomologies. In order to do so, we need to find two maps $h_\lambda : \mathfrak{BV}^\bullet_J \to \mathfrak{BV}^\bullet_J$, $h_\chi : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ of degree -1 such that [Wei95]

$$\lambda^* - \mathrm{id}_2 = Q_2 h_\lambda + h_\lambda Q_2,$$
$$\chi^* - \mathrm{id}_1 = Q_1 h_\chi + h_\chi Q_1.$$

To see that this implies that the composition maps λ^*, χ^* are the identity in cohomology, note that the first term on the RHS is Q-exact, thus it corresponds to the 0 element in $H^\bullet(\mathfrak{BV}^\bullet)$. Moreover, we have $H^\bullet(\mathfrak{BV}^\bullet) \subset \ker hQ$ by construction. As such $\lambda^* - \mathrm{id}_2 = 0$ and $\chi^* - \mathrm{id}_1 = 0$ when restricted to $H^\bullet(\mathfrak{BV}^\bullet_2)$ and $H^\bullet(\mathfrak{BV}^\bullet_1)$ respectively.

Remark 3.2.6. The conditions that we imposed for BV-equivalence in Definition 3.2.1 can be either strengthened or weakened. For example in [CSS18; CCS20], the authors say that two BV theories are strongly BV-equivalent if there is symplectormorphism between the spaces $(\mathcal{F}_i, \omega_i)$ which interchanges the BV actions $\mathcal{S}_i$, i.e. that σ_i, ρ_i vanish. On the other hand, in [BBH95] the authors only compare two BV theories at the level of their BV actions and BV cohomologies.

3.3 BV-BFV formalism for manifolds with boundary

The BV formalism can be extended to the case where the underlying spacetime manifold M has a non-empty boundary $\partial M \neq \emptyset$, as presented in [CMR14]. This construction relies on the BFV formalism introduced in [BF83], see also [Sta97; Sch09a; Sch09b].

In the presence of a boundary the variation of the BV action includes boundary terms given by the Noether 1-form (equation (3)) and the Hamiltonian condition no longer holds, or rather holds up to a boundary term, which will play a crucial role when a field theory on the boundary. Let us first recall the basic definitions of the BV-BFV formalism:

Definition 3.3.1. An *exact BFV theory* over a manifold M^∂ is a quadruple $\mathfrak{F}^\partial = (\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial)$ where

- $\mathcal{F}^\partial = \Gamma(E)$ is the space of smooth sections of a $\mathbb{Z}$-graded bundle $E \to M^\partial$,
- $\omega^\partial = \delta\alpha^\partial \in \Omega^2_{\mathrm{loc}}(\mathcal{F}^\partial)$ is an exact symplectic integrated local 2-form of degree 0,

- $\mathcal{S}^\partial \in \mathfrak{C}^\infty_{\text{loc}}(\mathcal{F}^\partial)$ is a degree 1 integrated local functional,
- $Q^\partial \in \mathfrak{X}_{\text{evo}}(\mathcal{F}^\partial)$ is a degree 1 evolutionary cohomological vector field, i.e. $[Q^\partial, Q^\partial] = 0$, $[\mathcal{L}_{Q^\partial}, \mathrm{d}] = 0$,

We call $\omega^\partial, S^\partial$ the *boundary form* and *boundary action* respectively. In this case we also have that Q^∂ is the Hamiltonian vector field of the boundary action $\mathcal{S}^\partial$

$$\iota_{Q^\partial}\omega^\partial = \delta\mathcal{S}^\partial.$$

Definition 3.3.2. A *BV-BFV theory* over a spacetime manifold M with boundary ∂M is a nonuple

$$(\mathcal{F}, \omega, \mathcal{S}, Q, \mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial, \pi)$$

where $(\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial)$ is an exact BFV theory over ∂M and $\pi : \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion such that

$$\iota_Q\omega = \delta\mathcal{S} + \pi^*\alpha^\partial. \tag{15}$$

Furthermore, we have $\pi_* Q = Q^\partial$.

Remark 3.3.3. Note that equation (15) implies that Q no longer preserves ω. Instead, we have

$$\mathcal{L}_Q\omega = \pi^*\omega^\partial.$$

Moreover, the CME no longer holds, instead we have the *modified Classical Master Equation* [CMR14]

$$\iota_Q\iota_Q\omega = 2\pi^*\mathcal{S}^\partial. \tag{16}$$

Remark 3.3.4. It is important to emphasize how one can construct a boundary theory $\mathfrak{F}^\partial$ from a BV theory $\mathfrak{F}$ since there might be obstructions, as is the case with the Jacobi theory (see Example 3.3.7). Having the BV data on the bulk of a spacetime manifold M, one wants to induce an exact BFV theory on the boundary ∂M. Let $\check{\alpha} = -\int_{\partial M}\alpha_N$, where α_N is the Noether 1-form of the BV action $\mathcal{S}$. In particular we have

$$\iota_Q\omega = \delta\mathcal{S} + \check{\alpha}.$$

By restricting the fields of $\mathcal{F}$ to the boundary ∂M we can define the *space of pre-boundary fields* $\check{\mathcal{F}}^\partial$ and endow it with a *pre-boundary* 2-form $\check{\omega} = \delta\check{\alpha}$. Note that $\check{\omega}$ might be degenerate and accordingly not symplectic. In order to define a symplectic structure one then has to perform symplectic reduction [Sil08]. Let $\ker\check{\omega} = \{X \in \mathfrak{X}(\mathcal{F}) \,|\, \iota_X\check{\omega} = 0\}$. We set

$$\mathcal{F}^\partial := \check{\mathcal{F}}^\partial / \ker\check{\omega}, \tag{17}$$

which gives us the surjective submersion $\pi : \mathcal{F} \to \mathcal{F}^\partial$. Since we are taking a quotient, nothing guarantees that $\mathcal{F}^\partial$ is devoid of singularities and we might not have a surjective submersion π. It turns out that a sufficient regularity condition for $\mathcal{F}^\partial$ to be a smooth manifold is that $\ker\check{\omega}$ has locally constant dimension, i.e. it is a subbundle of $T\check{\mathcal{F}}$ [CMR14]. If everything is well-defined we can then define α^∂, ω^∂ and Q^∂ through

$$\pi^*\alpha^\partial = \check{\alpha}, \qquad \pi^*\omega^\partial = \check{\omega}, \qquad Q^\partial = \pi_* Q.$$

Let $\varphi_i^\partial \in \mathcal{F}^\partial$ denote the boundary fields and $E^\partial = \varphi_i^\partial |\varphi_i^\partial| \frac{\delta}{\delta\varphi_i^\partial}$ denote the *graded Euler vector field* [CS11] on the boundary. The boundary action can be computed through [Roy07]

$$\mathcal{S}^\partial = \iota_{E^\partial}\iota_{Q^\partial}\omega^\partial. \tag{18}$$

We then have a BFV manifold $\mathfrak{F}^\partial = (\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial)$ over the boundary ∂M. For completeness, we also define the *pre-boundary action* $\check{S} = \pi^* S^\partial$. Pulling back equation (18) via π^* yields

$$\check{\mathcal{S}} = \iota_E \iota_Q \check{\omega}, \tag{19}$$

where $E = \varphi_i |\varphi_i| \frac{\delta}{\delta\varphi_i}$, $\varphi_i \in \mathcal{F}$, is the graded Euler vector field on $\mathcal{F}$. Note that if we take equation (19) as the definition of $\check{\mathcal{S}}$, the data $(\check{\alpha}, \check{S})$ can always be defined, even if the quotient in equation (17) does not yield a smooth structure.

Remark 3.3.5. This procedure can be repeated in case that the spacetime manifold M not only has a boundary but also corners (higher strata), as presented in [CMR14].

Remark 3.3.6. The quantization program introduced in [CMR18] heavily relies on the BV-BFV structure of a given classical theory. As such, even if two theories are classically equivalent, one might be better suited for quatisation than the other, as we now explore in the example of the Jacobi theory and 1D GR.

Example 3.3.7 (Jacobi theory in the BV-BFV formalism [CS17b])**.** We now introduce a boundary on the interval and set $I = [a, b]$, i.e. $\partial I = \{a, b\}$. Computing the variation $\delta\mathcal{S}_J$ of the BV Jacobi action (c.f. equation (12)) yields

$$\check{\alpha}_J = \int_{\partial I} \left\{ \sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \cdot \delta\tilde{q} + \tilde{q}^+\tilde{\xi}\delta\tilde{q} - \tilde{\xi}^+\tilde{\xi}\delta\tilde{\xi} \right\} \mathrm{d}t.$$

As shown in [CS17b], the kernel of the corresponding pre-boundary 2-form $\check{\omega}_J = \delta\check{\alpha}_J$ is singular. As such the quotient in equation (17) fails to be smooth and we cannot define a BV-BFV theory. In [CS17b], some ways of going around this problem are discussed.

We can still compute $\check{S}$ through equation (19). Keeping in mind that $\iota_E \delta f = E_J(f) = |f| f$ for $f \in C^\infty(\mathcal{F}_J)$ we have

$$\begin{aligned}
\check{S} &= \iota_{E_J}\iota_{Q_J}\check{\omega} \\
&= \iota_{E_J}\iota_{Q_J} \int_{\partial I} \left\{ \delta\left(\sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \right) \cdot \delta\tilde{q} + \delta(\tilde{q}^+\tilde{\xi})\delta\tilde{q} - \delta(\tilde{\xi}^+\tilde{\xi})\delta\tilde{\xi} \right\} \mathrm{d}t \\
&= -\iota_{E_J}\iota_{Q_J} \int_{\partial I} \delta(\tilde{\xi}^+\tilde{\xi})\delta\tilde{\xi} \,\mathrm{d}t. = Q_J(\tilde{\xi}^+\tilde{\xi})E_J(\tilde{\xi}) - E_J(\tilde{\xi}^+\tilde{\xi})Q_J\tilde{\xi} \\
&= \tilde{\xi}^+ Q_J(\tilde{\xi})\tilde{\xi} + \tilde{\xi}^+\tilde{\xi} Q_J\tilde{\xi} = 0.
\end{aligned}$$

Example 3.3.8 (BV-BFV formulation for 1D GR [CS17b])**.** We again set $I = [a, b]$. In the case of 1D GR we have

$$\check{\alpha}_{GR} = \int_{\partial I} \left\{ \frac{m\dot{q}}{\sqrt{g}} \cdot \delta q + q^+\xi\delta q + g^+\xi\delta g - (2g^+g + \xi^+\xi)\delta\xi \right\} \mathrm{d}t.$$

The pre-boundary 2-form $\check{\omega}_{GR} = \delta\check{\alpha}_{GR}$ has a well-behaved kernel and we can define a BV-BFV theory. The space of boundary fields is

$$\mathcal{F}^\partial_{GR} = T^*(\mathbb{R}^n \times \mathbb{R}[1]),$$

at each endpoint of the interval $I = [a, b]$, with coordinates q, c on the base and p, b on the fibers. The boundary form and boundary action are

$$\begin{aligned}
\omega^\partial &= \int_{\partial I} \{\delta p \cdot \delta q + \delta b \delta c\} \,\mathrm{d}t, \\
\mathcal{S}^\partial &= \int_{\partial I} \left\{ \left(\frac{\|p\|^2}{2m} - E \right) c \right\} \mathrm{d}t
\end{aligned}$$

and the surjective subermssion π_{GR} takes the form

$$\pi_{GR}^* q = q, \qquad \pi_{GR}^* c = \sqrt{g}\xi.$$
$$\pi_{GR}^* p = \frac{m\dot{q}}{\sqrt{g}} + q^+\xi, \qquad \pi_{GR}^* b = -\frac{1}{\sqrt{g}}(2g^+g + \xi^+\xi).$$

The cohomological vector field on the boundary can then be computed through $Q^\partial = \pi_* Q$. Note that we have

$$\check{\mathcal{S}} = \pi_{GR}^* \mathcal{S}^\partial = \left(\frac{T}{g} - E\right)\sqrt{g}\xi$$

and $\pi^*\omega^\partial = \check{\omega}$.

Remark 3.3.9. As promised, Examples 3.3.7 and 3.3.8 showcase two classically (non-BV) equivalent theories (both on the bulk and on the boundary as shown in [CS17b]), which fail to reproduce equivalent BV-BFV theories.

We can then interpret 1D GR as an extension of the Jacobi theory which makes it compatible with the BV-BFV regularity conditions. For the interested reader, we refer to [CS17b] where the authors develop the BV-BFV quantization procedure for 1D GR.

3.4 Lax formalism

The Lax-formalism [MSW20] uses the complex of local forms (Definition 2.1.4) to describe the BV-BFV picture presented above at the level of local forms instead of integrated local forms as we have been doing until now. Furthermore, it gathers the data before performing the quotient in equation (17), thus avoiding potential complications.

The definitions that we work with rely on the *lax degree* $\#(\cdot)$,[6] which is a mixture of the form degree $\mathrm{fd}_M(\cdot)$ on M and the ghost number $\mathrm{gh}(\cdot)$, namely

$$\#(\cdot) = \mathrm{gh}(\cdot) - [\dim M - \mathrm{fd}_M(\cdot)].$$

We will still use the total degree for computations, which in the case of local forms is given by

$$|\cdot| = \mathrm{gh}(\cdot) + \mathrm{fd}_M(\cdot) + \mathrm{fd}_{\mathcal{F}}(\cdot),$$

where $\mathrm{fd}_{\mathcal{F}}(\cdot)$ is the form degree on $\mathcal{F}$. Note that $|\cdot|$ and $\#$ are related by

$$|\cdot| = \#(\cdot) + \mathrm{fd}_{\mathcal{F}}(\cdot) + \dim M.$$

[6] In the original paper [MSW20] the authors denote the lax degree by total degree.

Definition 3.4.1 (Lax BV-BFV theory)**.** A *lax BV-BFV theory* over a spacetime manifold M is a quadruple $\mathfrak{F}^{\text{lax}} = (\mathcal{F}^{\text{lax}}, \theta^\bullet, L^\bullet, Q)$ where

- $\mathcal{F}^{\text{lax}} = \Gamma(E)$ for some $\mathbb{Z}$-graded bundle $E \to M$,
- $\theta^\bullet \in \Omega^{1,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)$ is a local form with lax degree -1,
- $L^\bullet \in \Omega^{0,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)$ is a local functional with lax degree 0,
- $Q \in \mathfrak{X}_{\text{evo}}(\mathcal{F}^{\text{lax}})$ is an evolutionary, cohomological vector field on $\mathcal{F}^{\text{lax}}$ of degree 1, i.e. $[\mathcal{L}_Q, \mathrm{d}] = [Q, Q] = 0$,

such that

$$\iota_Q \varpi^\bullet = \delta L^\bullet + \mathrm{d}\theta^\bullet, \tag{20a}$$
$$\iota_Q \iota_Q \varpi^\bullet = 2\mathrm{d}L^\bullet, \tag{20b}$$

where $\varpi^\bullet = \delta\theta^\bullet$.

Remark 3.4.2. Equations (20) are the non-integrated version of equations (6),(15) and (8),(16) respectively in the case where we consider boundaries and corners.

Note that we can write any inhomogeneous local form $\kappa^\bullet \in \Omega^{p,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)$ as a sum over its homogeneous components, namely as $\kappa^\bullet = \sum_{k=0}^{\dim M} \kappa^k$ where $\kappa^k \in \Omega^{p,k}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)$. In this notation equations (20) read

$$\iota_Q \varpi^k = \delta L^k + \mathrm{d}\theta^{k-1},$$
$$\iota_Q \iota_Q \varpi^k = 2\mathrm{d}L^{k-1}.$$

Lemma 3.4.3 ([MSW20])**.** *Having a lax BV-BFV theory, the following equations hold:*

$$\mathcal{L}_Q \varpi^\bullet = \mathrm{d}\varpi^\bullet, \tag{21a}$$
$$\mathcal{L}_Q L^\bullet = \mathrm{d}(2L^\bullet - \iota_Q \theta^\bullet). \tag{21b}$$

Remark 3.4.4. Equations (21) describes the failure of the BV form being preserved by Q and of the CME in the language of local forms.

In the lax formalism, the relevant chain complex will no longer be $\mathfrak{BV}^\bullet$, since this is the complex of integrated local functionals over the bulk manifold and as such is better suited to describe the case without boundaries and corners. Instead, we want to consider a chain complex of (non-integrated) local forms (c.f. Remark 2.1.5).

Definition 3.4.5. The *BV-BFV complex* [BBH95; MSW20] of a lax BV-BFV theory $\mathfrak{F}^{\text{lax}}$ is defined as the space of local forms on $\mathcal{F}^{\text{lax}}$ with values in inhomogeneous forms on M, endowed with the differential $(\mathcal{L}_Q - \mathrm{d})$

$$\begin{aligned}(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet &= \Omega^\bullet_{\text{BV-BFV}}(\mathcal{F}^{\text{lax}} \times M, (\mathcal{L}_Q - \mathrm{d})) \\ &= \left(\left(\bigoplus_k \Omega^{\bullet,k}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)\right), (\mathcal{L}_Q - \mathrm{d})\right).\end{aligned}$$

We will denote its cohomology by $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet)$ and call it the *BV-BFV cohomology.*

Remark 3.4.6. Any inhomogenoeous local form $\mathcal{O}^\bullet \in \Omega^{p,\bullet}_{\text{loc}}(\mathcal{F} \times M)$ which belongs to the BV-BFV cohomology $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet)$ automatically solves the *descent equations* [Zum85; MSZ85; Wit88; MSW20]

$$(\mathcal{L}_Q - \mathrm{d})\mathcal{O}^\bullet = 0,$$

which read $\mathcal{L}_Q \mathcal{O}^k = \mathrm{d}\mathcal{O}^{k-1}$ in homogeneous components. Such equations are of interest since their solutions produce classical observables. Recall that classical observables are integrated local functionals (i.e. $p = 0$), which are in the kernel of Q. Let γ^k denote a k-dimensional closed submanifold of M. We can then construct a classical observable by integrating $\mathcal{O}^k$ over γ^k since

$$Q \int_{\gamma^k} \mathcal{O}^k = \int_{\gamma^k} \mathcal{L}_Q \mathcal{O}^k = \int_{\gamma^k} \mathrm{d}\mathcal{O}^{k-1} = 0.$$

As such, comparing the BV-BFV cohomologies of two lax theories is a natural way of comparing their physical content.

We now restrict our discussion to the one-dimensional case $M = I$, but everything can be generalized for $\dim M > 1$ (see [MSW20]). In the relevant cases for this thesis we have $\Omega^{p,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times I) = \Omega^{p,1}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times I) \times \Omega^{p,0}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times I)$ and as such we can decompose every local form $\kappa^\bullet \in \Omega^{p,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times I)$ as

$$\kappa^\bullet = \kappa^1\,\mathrm{d}t + \kappa^0.$$

Strictly speaking we should also include $\mathrm{d}t$ in κ^1 in order to be consistent with the notation in equations (20), but this notation allows us to drop the 1-form $\mathrm{d}t$ from calculations and helps us keep them cleaner. This is done by shifting $\mathrm{d}t$ to the right in every expression. Equations (20) then read

$$\iota_Q\varpi^1 = \delta L^1 - \partial_t\theta^0, \qquad \iota_Q\varpi^0 = -\delta L^0,$$
$$\iota_Q\iota_Q\varpi^1 = 2\partial_t L^0, \qquad \iota_Q\iota_Q\varpi^0 = 0,$$

where we have used $\mathrm{d} = \mathrm{d}t\,\partial_t$ and $|\mathrm{d}t| = 1$, $|\theta^0| = 1$, $|L^0| = 1$.

Remark 3.4.7. Given the lax data for a field theory on an interval I, we can attempt to define a BV-BFV theory $(\mathcal{F}, \omega, \mathcal{S}, Q, \mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial, \pi)$ as follows: the space of bulk fields $\mathcal{F}$ can be constructed by simply restricting the fields in $\mathcal{F}^{\text{lax}}$ to the bulk $M\backslash\partial M$. We then define the BV form, BV action, pre-boundary 2-form and pre-boundary action as

$$\omega = \int_I \delta\theta^1\,\mathrm{d}t, \qquad \breve{\omega} = \int_{\partial I}\delta\theta^0\,\mathrm{d}t,$$
$$\mathcal{S} = \int_I L^1\,\mathrm{d}t, \qquad \check{\mathcal{S}} = \int_{\partial I} L^0\,\mathrm{d}t$$

and set $Q(\cdot) = (\mathcal{S}, \cdot)$. $\mathcal{F}^\partial$ can then be constructed as explained in remark 3.3.4, i.e. by restricting the fields in $\mathcal{F}^{\text{lax}}$ to the boundary ∂M and performing symplectic reduction with respect to the pre-boundary 2-form $\breve{\omega}$, which (if we can perform the symplectic reduction) gives us a surjective submersion $\pi : \mathcal{F} \to \mathcal{F}^\partial$. The boundary data is then given by $\pi^*\omega^\partial = \breve{\omega}$, $\pi^*\mathcal{S}^\partial = \check{\mathcal{S}}$ and $Q^\partial = \pi_* Q$.

Example 3.4.8 (Lax formulations for the Jacobi theory)**.** The BV theory for the Jacobi theory can easily be extended to a lax theory. The space of fields $\mathcal{F}_J^{\text{lax}}$ is the space $\mathcal{F}_J$ presented in Example 3.1.9 in the case where $I = [a, b]$ has a non-empty boundary. The cohomological vector field Q_J is taken to be the same as in Example 3.1.9. The rest of the lax data for the Jacobi theory can be read from Examples 3.1.9 and 3.3.7

$$\begin{aligned}
\theta_J^1 &= \tilde{q}^+ \cdot \delta\tilde{q} + \tilde{\xi}^+\delta\tilde{\xi},\\
\theta_J^0 &= \sqrt{\frac{E}{\tilde{T}}}m\dot{\tilde{q}} \cdot \delta\tilde{q} + \tilde{q}^+\tilde{\xi} \cdot \delta\tilde{q} - \tilde{\xi}^+\tilde{\xi}\delta\tilde{\xi},\\
L_J^1 &= 2\sqrt{E\tilde{T}} + \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}} + \tilde{\xi}^+\tilde{\xi}\dot{\tilde{\xi}},\\
L_J^0 &= 0.
\end{aligned}$$

Example 3.4.9 (Lax formulations for 1D GR)**.** We can construct the lax theory $\mathfrak{F}^{\text{lax}}_{GR}$ for 1D GR in a similar way. The space of fields $\mathcal{F}^{\text{lax}}_{GR}$ is just $\mathcal{F}_{GR}$ from Example 3.1.11 in the case $I = [a, b]$ and we take Q_{GR} to be the same cohomological vector field as before. The rest of the lax data for 1D GR can be read from Examples 3.1.11 and 3.3.8. We have

$$\begin{aligned}
\theta^1_{GR} &= q^+ \cdot \delta q + \xi^+ \delta\xi + g^+ \delta g,\\
\theta^0_{GR} &= \frac{m\dot{q}}{\sqrt{g}} \cdot \delta q + q^+ \xi \cdot \delta q + g^+ \xi \delta g - (2g^+ g + \xi^+ \xi)\delta\xi,\\
L^1_{GR} &= \frac{T}{\sqrt{g}} + \sqrt{g}E + q^+ \cdot \xi\dot{q} + g^+(\xi\dot{g} + 2g\dot{\xi}) + \xi^+\xi\dot{\xi},\\
L^0_{GR} &= \left(\frac{T}{g} - E\right)\sqrt{g}\xi.
\end{aligned}$$

3.5 Equivalence in the lax setting

The notion of BV-equivalence discussed in Section 3.2 can be generalized to the case when a boundary is present. We again consider the case where M is a generic spacetime manifold of arbitrary dimension. Before defining equivalence in this setting, let us define the notion of f-transformations:

Definition 3.5.1. Let $f \in \Omega^{0,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}} \times M)$ be a local functional with $\#(f) = -1$. An f-transformation of a lax BV-BFV theory $\mathfrak{F}^{\text{lax}}$ changes $(\theta^\bullet, L^\bullet)$ as

$$\begin{aligned}
\theta^\bullet &\mapsto \theta^\bullet + \delta f^\bullet,\\
L^\bullet &\mapsto L^\bullet + \mathrm{d} f^\bullet.
\end{aligned}$$

Remark 3.5.2. Note that an f-transformation preserves equations (20) since $\varpi^\bullet = \delta\theta^\bullet$ and $\mathrm{d}L^\bullet$ are unchanged, as is the term $\delta L^\bullet + \mathrm{d}\theta^\bullet$

$$\delta L^\bullet + \mathrm{d}\theta^\bullet \mapsto \delta L^\bullet + \delta\mathrm{d}f + \mathrm{d}\theta^\bullet + \mathrm{d}\delta f = \delta L^\bullet + \mathrm{d}\theta^\bullet,$$

where we used $[\delta, \mathrm{d}] = 0$.

Definition 3.5.3 (Lax-equivalence)**.** Let $\mathfrak{F}^{\text{lax}}_i$, $i \in \{1, 2\}$, be two lax BV-BFV theories. We denote the both de Rham differentials on $\mathcal{F}^{\text{lax}}_i$ by δ and both de Rham differentials on M_i by d. We say that $\mathfrak{F}^{\text{lax}}_1$ and $\mathfrak{F}^{\text{lax}}_2$ are *lax-equivalent* if there exist two degree 0 maps

$$\phi : \mathcal{F}^{\text{lax}}_1 \to \mathcal{F}^{\text{lax}}_2, \qquad \psi : \mathcal{F}^{\text{lax}}_2 \to \mathcal{F}^{\text{lax}}_1,$$

such that the pullback maps ϕ^*, ψ^* are chain maps between the BV-BFV complexes $(\mathfrak{BV}\text{-}\mathfrak{BFV})_1^\bullet$, $(\mathfrak{BV}\text{-}\mathfrak{BFV})_2^\bullet$,

$$\phi^* \circ (\mathcal{L}_{Q_2} - \mathrm{d}) = (\mathcal{L}_{Q_1} - \mathrm{d}) \circ \phi^*, \qquad \psi^* \circ (\mathcal{L}_{Q_1} - \mathrm{d}) = (\mathcal{L}_{Q_2} - \mathrm{d}) \circ \psi^*,$$

and the composition maps

$$\begin{aligned} \lambda^* = \phi^* \circ \psi^* &: (\mathfrak{BV}\text{-}\mathfrak{BFV})_1^\bullet \to (\mathfrak{BV}\text{-}\mathfrak{BFV})_1^\bullet, \\ \chi^* = \psi^* \circ \phi^* &: (\mathfrak{BV}\text{-}\mathfrak{BFV})_2^\bullet \to (\mathfrak{BV}\text{-}\mathfrak{BFV})_2^\bullet, \end{aligned}$$

are the identity in the respective BV-BFV cohomologies $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})_1^\bullet)$, $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})_2^\bullet)$.

Moreover, we require that the maps ϕ, ψ, interchange $(\theta_i^\bullet, L_i^\bullet)$ up to $(\mathcal{L}_{Q_i} - d)$-exact terms and an f-transformation

$$\begin{aligned} \phi^*\theta_2^\bullet &= \theta_1^\bullet + (\mathcal{L}_{Q_1} - \mathrm{d})\beta_1^\bullet + \delta f_1^\bullet, \qquad & \psi^*\theta_1^\bullet &= \theta_2^\bullet + (\mathcal{L}_{Q_2} - \mathrm{d})\beta_2^\bullet + \delta f_2^\bullet, \\ \phi^* L_2^\bullet &= L_1^\bullet + (\mathcal{L}_{Q_1} - \mathrm{d})\zeta_1^\bullet + \mathrm{d} f_1^\bullet, \qquad & \psi^* L_1^\bullet &= L_2^\bullet + (\mathcal{L}_{Q_2} - \mathrm{d})\zeta_2^\bullet + \mathrm{d} f_2^\bullet, \end{aligned}$$

where $\beta_i^\bullet \in \Omega^{1,\bullet}(\mathcal{F}_i^{\text{lax}} \times M_i)$, $\zeta_i \in \Omega^{0,\bullet}(\mathcal{F}_i^{\text{lax}} \times M_i)$, $f_i^\bullet \in \Omega^{0,\bullet}(\mathcal{F}_i^{\text{lax}} \times M_i)$, and that they preserve equations (20) as

$$\iota_{Q_1}\varpi_1^\bullet = \delta L_1^\bullet + \mathrm{d}\theta_1^\bullet \underset{\phi^*}{\overset{\psi^*}{\rightleftharpoons}} \iota_{Q_2}\varpi_2^\bullet = \delta L_2^\bullet + \mathrm{d}\theta_2^\bullet, \tag{22}$$

$$\iota_{Q_1}\iota_{Q_1}\varpi_1^\bullet = 2\mathrm{d}L_1^\bullet \underset{\phi^*}{\overset{\psi^*}{\rightleftharpoons}} \iota_{Q_2}\iota_{Q_2}\varpi_2^\bullet = 2\mathrm{d}L_2^\bullet. \tag{23}$$

As in the bulk case, there are some straightforward implications of Definition 3.5.3 that are important to highlight:

Proposition 3.5.4. *Let $\mathfrak{F}_1^{\text{lax}}$ and $\mathfrak{F}_2^{\text{lax}}$ be two lax-equivalent theories. Using the notation of Definition 3.5.3 we have*

1. *the BV-BFV complexes of $\mathfrak{F}_1^{\text{lax}}$ and $\mathfrak{F}_2^{\text{lax}}$ are quasi-isomorphic*

$$H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})_1^\bullet) \simeq H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})_2^\bullet),$$

2. *the requirement that ϕ^*, ψ^* preserve equations (22,23) is equivalent to*

$$(\mathcal{L}_{Q_i} - \mathrm{d})[(\iota_{Q_i}\delta - \mathrm{d})\beta_i^\bullet - \delta\zeta_i^\bullet] = 0, \tag{24a}$$
$$(\mathcal{L}_{Q_i} - \mathrm{d})[\iota_{Q_i}\iota_{Q_i}\delta\beta_i^\bullet - 2\mathrm{d}\zeta_i^\bullet] = 0, \tag{24b}$$

3. *the theories* $\mathfrak{F}_1^{\mathrm{lax}}, \lambda^*\mathfrak{F}_1^{\mathrm{lax}} := (\mathcal{F}_1^{\mathrm{lax}}, \lambda^*\theta_1^\bullet, \lambda^* L_1^\bullet, Q_1)$ *and* $\mathfrak{F}_2^{\mathrm{lax}}, \chi^*\mathfrak{F}_2^{\mathrm{lax}} := (\mathcal{F}_2^{\mathrm{lax}}, \lambda^*\theta_2^\bullet, \lambda^* L_2^\bullet, Q_2)$ *are pairwise lax-equivalent.*

Proof. The proofs for points 1. and 3. are the same as the ones for proposition 3.2.3. For point 2., note that applying ϕ^* to $\iota_{Q_2}\varpi_2^\bullet = \delta L_2^\bullet + \mathrm{d}\theta_2^\bullet$ yields

$$\begin{aligned} &\iota_{Q_1}\delta(\mathcal{L}_{Q_1} - \mathrm{d})\beta_1^\bullet = \delta(\mathcal{L}_{Q_1} - \mathrm{d})\zeta_1^\bullet + \mathrm{d}(\mathcal{L}_{Q_1} - \mathrm{d})\beta_1^\bullet \\ \Rightarrow\quad &(\mathcal{L}_{Q_1} - \mathrm{d})\iota_{Q_1}\delta\beta_1^\bullet = (\mathcal{L}_{Q_1} - \mathrm{d})\delta\zeta_1^\bullet + (\mathcal{L}_{Q_1} - \mathrm{d})\mathrm{d}\beta_1^\bullet \\ \Rightarrow\quad &(\mathcal{L}_{Q_1} - \mathrm{d})[(\iota_{Q_1}\delta - \mathrm{d})\beta_i^\bullet - \delta\zeta_1^\bullet] = 0, \end{aligned}$$

where we used $\iota_{Q_1}\varpi_1^\bullet = \delta L_1^\bullet + \mathrm{d}\theta_1^\bullet$, $\delta^2 = 0$ and the fact that f-transformations preserve equation (22). In the same manner applying ϕ^* to $\iota_{Q_2}\iota_{Q_2}\varpi_2^\bullet = 2\mathrm{d}L_2^\bullet$ leads to

$$\begin{aligned} &\iota_{Q_2}\iota_{Q_2}\delta(\mathcal{L}_{Q_1} - \mathrm{d})\beta_1^\bullet = 2\mathrm{d}(\mathcal{L}_{Q_1} - \mathrm{d})\zeta_1^\bullet \\ \Rightarrow\quad &(\mathcal{L}_{Q_i} - \mathrm{d})[\iota_{Q_1}\iota_{Q_1}\delta\beta_1^\bullet - 2\mathrm{d}\zeta_1^\bullet] = 0. \end{aligned}$$

Similarly, applying ψ^* to $\iota_{Q_1}\varpi_1^\bullet = \delta L_1^\bullet + \mathrm{d}\theta_1^\bullet$, $\iota_{Q_1}\iota_{Q_1}\varpi_1^\bullet = 2\mathrm{d}L_1^\bullet$ gives

$$\begin{aligned} &(\mathcal{L}_{Q_2} - \mathrm{d})[(\iota_{Q_2}\delta - \mathrm{d})\beta_2^\bullet - \delta\zeta_2^\bullet] = 0, \\ &(\mathcal{L}_{Q_2} - \mathrm{d})[\iota_{Q_2}\iota_{Q_2}\delta\beta_2^\bullet - 2\mathrm{d}\zeta_2^\bullet] = 0. \end{aligned}$$

□

Remark 3.5.5. Our definition of lax-equivalence directly implies that the 2-forms $\varpi_i^\bullet$ are interchanged up to $(\mathcal{L}_Q - \mathrm{d})$-exact and δ-exact terms

$$\begin{aligned} \phi^*\varpi_2^\bullet &= \delta\phi^*\theta_2^\bullet = \varpi_1^\bullet - (\mathcal{L}_{Q_1} - \mathrm{d})\delta\beta_1^\bullet, \\ \psi^*\varpi_1^\bullet &= \delta\psi^*\theta_1^\bullet = \varpi_2^\bullet - (\mathcal{L}_{Q_2} - \mathrm{d})\delta\beta_2^\bullet. \end{aligned}$$

Similarly to the bulk case (c.f. Remark 3.2.4), choosing

$$\zeta_i^\bullet = \iota_{Q_i}\beta_i^\bullet,$$

ensures that equations (24) are satisfied. For the first we compute

$$\begin{aligned} (\mathcal{L}_{Q_i} - \mathrm{d})[(\iota_{Q_i}\delta - \mathrm{d})\beta_i^\bullet - \delta\zeta_i^\bullet] &= (\mathcal{L}_{Q_i} - \mathrm{d})[(\iota_{Q_i}\delta - \mathrm{d})\beta_i^\bullet - \delta\iota_{Q_i}\beta_i^\bullet] \\ &= (\mathcal{L}_{Q_i} - \mathrm{d})^2\beta_i^\bullet = 0. \end{aligned}$$

To prove that the second equation holds, first note that $\iota_{Q_i}\iota_{Q_i}\beta_i^\bullet = 0$, as $\beta_i^\bullet$ is a 1-form on $\mathcal{F}_i^{\mathrm{lax}}$ with values in inhomogeneous forms on M. Keeping in mind that $\mathcal{L}_{Q_i} = [\iota_{Q_i}, \delta]$ and $[\iota_{Q_i}, \mathcal{L}_{Q_i}] = 0$ we have

$$\begin{aligned}
(\mathcal{L}_{Q_i} - \mathrm{d})[\iota_{Q_i}\iota_{Q_i}\delta\beta_i^\bullet - 2\mathrm{d}\zeta_i^\bullet] &= (\mathcal{L}_{Q_i} - \mathrm{d})[\iota_{Q_i}\iota_{Q_i}\delta\beta_i^\bullet - 2\mathrm{d}\iota_{Q_i}\beta_i^\bullet] \\
&= (\mathcal{L}_{Q_i} - \mathrm{d})[\iota_{Q_i}\mathcal{L}_{Q_i}\beta_i^\bullet + \iota_{Q_i}\delta\iota_{Q_i}\beta_i^\bullet - 2\mathrm{d}\iota_{Q_i}\beta_i^\bullet] \\
&= (\mathcal{L}_{Q_i} - \mathrm{d})[\mathcal{L}_{Q_i}\iota_{Q_i}\beta_i^\bullet + \mathcal{L}_{Q_i}\iota_{Q_i}\beta_i^\bullet + \delta\iota_{Q_i}\iota_{Q_i}\beta_i^\bullet - 2\mathrm{d}\iota_{Q_i}\beta_i^\bullet] \\
&= 2(\mathcal{L}_{Q_i} - \mathrm{d})^2\iota_{Q_i}\beta_i^\bullet = 0.
\end{aligned}$$

4 BV-equivalence of the Jacobi theory and 1D GR

In this chapter we will show that the BV formulations of the Jacobi theory and 1D GR are BV-equivalent (Definition 3.2.1), by constructing two maps

$$\phi : \mathcal{F}_J \to \mathcal{F}_{GR}, \qquad \psi : \mathcal{F}_{GR} \to \mathcal{F}_J,$$

such that:

- their pullback maps ϕ^*, ψ^* are chain maps between the BV complexes $\mathfrak{BV}^\bullet_J$, $\mathfrak{BV}^\bullet_{GR}$,
- the composition maps

$$\lambda^* = \phi^* \circ \psi^* : \mathfrak{BV}^\bullet_J \to \mathfrak{BV}^\bullet_J$$
$$\chi^* = \psi^* \circ \phi^* : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR},$$

  are the identity when restricted to the respective BV cohomologies $H^\bullet(\mathfrak{BV}^\bullet_J)$, $H^\bullet(\mathfrak{BV}^\bullet_{GR})$ and
- they interchange the BV forms and BV actions up to $\mathcal{L}_Q$-exact terms and preserve the Hamiltonian conditions as in equation (13) (c.f. condition (14)).

Recall that the maps λ^*, χ^* are the identity in cohomology if there exist two maps $h_\lambda : \mathfrak{BV}^\bullet_J \to \mathfrak{BV}^\bullet_J$, $h_\chi : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ of degree -1 such that

$$\lambda^* - \mathrm{id}_J = Q_J h_\lambda + h_\lambda Q_J,$$
$$\chi^* - \mathrm{id}_{GR} = Q_{GR} h_\chi + h_\chi Q_{GR}.$$

As we will see, $\lambda^* = \mathrm{id}_J$ and as such this holds trivially for $h_\lambda = 0$. In the case of χ^*, this requires more work. Let $s \in \mathbb{R}_{\geq 0}$ and $R \in \mathfrak{X}_{\mathrm{evo}}(\mathcal{F}_{GR})$ be an evolutionary vector field of degree -1. Our approach is to first show that χ^* is homotopic to the identity id_{GR} on $\mathfrak{BV}^\bullet_{GR}$ by constructing a one-parameter family of morphisms of the form

$$\chi^*_s = e^{s(Q_{GR}R + RQ_{GR})},$$

such that $\chi^*_{s=0} = \mathrm{id}_{GR}$ and $\lim_{s\to\infty} \chi^*_s = \chi^*$. This will then allow us to construct the map h_χ.

We start by presenting the maps ϕ, ψ in Section 4.1, where we also show that ϕ^*, ψ^* fulfill the chain map conditions, interchange the BV forms, BV actions and preserve the Hamiltonian conditions in the desired way. This is followed by the analysis of the composition maps λ^* and χ^* in Section 4.2, where we show that $\lambda^* = \mathrm{id}_J$, construct the morphims χ^*_s, prove that they converge as $\lim_{s\to\infty} \chi^*_s = \chi^*$ and compute the map h_χ. The main result of this chapter is presented in Theorem 4.2.14.

4.1 Chain maps

Before presenting the maps ϕ, ψ, we note that we are interested on how ϕ^*, ψ^* act on the respective BV complexes. Since ϕ^*, ψ^* are morphisms of the algebras of local functionals $\mathfrak{C}^\infty_{\text{loc}}(\mathcal{F}_i)$, $i \in \{J, GR\}$, we solely need to define ϕ^*, ψ^* on $\Phi, \Phi^\dagger \in \mathcal{F}_i$ for them to be defined for any integrated local functional. In the same manner, it is sufficient to check that the chain map conditions hold for $\Phi, \Phi^\dagger \in \mathcal{F}_i$.

The first map that we consider carries out the procedure of integrating out fields in the BV setting. In our case, it will restrict the metric field to the surface where it solves the EL equations $g = T/E$ and its antifield to the surface $g^+ = 0$.

Lemma 4.1.1. *Let $\phi : \mathcal{F}_J \to \mathcal{F}_{GR}$ be the map defined through*

$$\phi^* q = \tilde{q}, \qquad \phi^* \xi = \tilde{\xi}, \qquad \phi^* g = \frac{\tilde{T}}{E},$$
$$\phi^* q^+ = \tilde{q}^+, \qquad \phi^* \xi^+ = \tilde{\xi}^+, \qquad \phi^* g^+ = 0.$$

Its pullback map ϕ^* is a chain map between $\mathfrak{BV}^\bullet_{GR}$, $\mathfrak{BV}^\bullet_J$,

$$\phi^* \circ Q_{GR} = Q_J \circ \phi^*,$$

and maps the BV form and BV action of 1D GR as

$$\phi^* \omega_{GR} = \omega_J,$$
$$\phi^* \mathcal{S}_{GR} = \mathcal{S}_J.$$

Furthermore, it preserves the Hamiltonian condition

$$\iota_{Q_{GR}} \omega_{GR} = \delta \mathcal{S}_{GR} \xrightarrow{\phi^*} \iota_{Q_J} \omega_J = \delta \mathcal{S}_J.$$

Proof. The proof for the chain map condition $\phi^* \circ Q_{GR} = Q_J \circ \phi^*$ is a matter of straightforward computations

$$\begin{aligned}
\phi^* Q_{GR} q &= \phi^*(\xi \dot{q}) = \tilde{\xi}\dot{\tilde{q}} = Q_J \tilde{q} = Q_J \phi^* q, \\
\phi^* Q_{GR} \xi &= \phi^*(\xi \dot{\xi}) = \tilde{\xi}\dot{\tilde{\xi}} = Q_J \tilde{\xi} = Q_J \phi^* \xi, \\
\phi^* Q_{GR} g &= \phi^*(\tilde{\xi}\dot{g} + 2\dot{\tilde{\xi}} g) = \tilde{\xi}\frac{\dot{\tilde{T}}}{E} + 2\dot{\tilde{\xi}}\frac{\tilde{T}}{E} = Q_J \frac{\tilde{T}}{E} = Q_J \phi^* g, \\
\phi^* Q_{GR} q^+ &= \phi^* \left(-\partial_t \left(\frac{m\dot{q}}{\sqrt{g}} \right) + \xi \dot{q}^+ + \dot{\xi} q^+ \right) = -\partial_t \left(\sqrt{\frac{E}{\tilde{T}}} m \dot{\tilde{q}} \right) + \tilde{\xi}\dot{\tilde{q}}^+ + \dot{\tilde{\xi}}\tilde{q}^+ \\
&= Q_J \tilde{q}^+ = Q_J \phi^* q^+, \\
\phi^* Q_{GR} \xi^+ &= \phi^*(-q^+ \cdot \dot{q} + g^+ \dot{g} + 2\dot{g}^+ g + \xi \dot{\xi}^+ + 2\dot{\xi}\xi^+) = -\tilde{q}^+ \cdot \dot{\tilde{q}} + \tilde{\xi}\dot{\tilde{\xi}}^+ + 2\dot{\tilde{\xi}}\tilde{\xi}^+ \\
&= Q_J \tilde{\xi}^+ = Q_J \phi^* \xi^+, \\
\phi^* Q_{GR} g^+ &= \phi^* \left(\frac{1}{2\sqrt{g}} \left(E - \frac{T}{g} \right) + \xi \dot{g}^+ - \dot{\xi} g^+ \right) = 0 = Q_J 0 = Q_J \phi^* g^+.
\end{aligned}$$

For the pullback of the BV form and BV action we then compute

$$\begin{aligned}
\phi^* \omega_{GR} &= \phi^* \left(\int_I \left\{ \delta q^+ \cdot \delta q + \delta \xi^+ \delta \xi + \delta g^+ \delta g \right\} \mathrm{d}t \right) \\
&= \int_I \left\{ \delta \tilde{q}^+ \cdot \delta \tilde{q} + \delta \tilde{\xi}^+ \delta \tilde{\xi} \right\} \mathrm{d}t = \omega_J,
\end{aligned}$$

and

$$\begin{aligned}
\phi^* \mathcal{S}_{GR} &= \phi^* S_{GR}[q, g] + \int_I \left\{ q^+ \cdot \xi \dot{q} + g^+ (\xi \dot{g} + 2g\dot{\xi}) + \xi^+ \xi \dot{\xi} \right\} \mathrm{d}t \\
&= S_{GR} \left[\tilde{q}, g = \frac{\tilde{T}}{E} \right] + \int_I \left\{ \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}} + \tilde{\xi}^+ \tilde{\xi}\dot{\tilde{\xi}} \right\} \mathrm{d}t = \mathcal{S}_J,
\end{aligned}$$

since $S_{GR}\left[\tilde{q}, g = \frac{\tilde{T}}{E}\right] = S_J[\tilde{q}]$ as shown in equation (5). □

Remark 4.1.2. The map $\phi : \mathcal{F}_J \to \mathcal{F}_{GR}$ displayed above is the *Batalin-Vilkovisky-Legendre transform*, see [CCS20]. This transform plays the role of the *BV-pushforward* [Cos11; CMR18] in the classical BV formalism.

Before presenting the second chain map we need to introduce some useful notation. Let $v \in C^\infty(I, \mathbb{R}^n)$ be a $\mathbb{R}^n$-valued field of any degree and let

$u = \dot{q}/\|\dot{q}\|$ denote the normalized velocity of q. Note that u is always well-defined because we assume $\dot{q} \neq 0$. We can then decompose $v = v_\parallel + v_\perp$ into its parallel $v_\parallel$ and perpendicular $v_\perp$ components with respect to u as

$$v_\parallel = (u \cdot v)u = (\dot{q} \cdot v)\frac{\dot{q}}{\|\dot{q}\|^2} = (\dot{q} \cdot v)\frac{m\dot{q}}{2T},$$
$$v_\perp = v - (u \cdot v)u = v - (\dot{q} \cdot v)\frac{m\dot{q}}{2T},$$

where we used that $T = \frac{m}{2}\|\dot{q}\|^2$.

Lemma 4.1.3. *Let $\psi : \mathcal{F}_{GR} \to \mathcal{F}_J$ be the map defined through*

$$\psi^*\tilde{q} = q,$$
$$\psi^*\tilde{\xi} = \xi,$$
$$\psi^*\tilde{q}^+_\parallel = \eta^{3/2}\left(q^+_\parallel - \left[g^+\dot{g} + 2\dot{g}^+ g\right]\frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E}\left[\dot{EL_g g^+} - EL_g\dot{g}^+\right]\frac{m\dot{q}}{2T}\right),$$
$$\psi^*\tilde{q}^+_\perp = \eta^{3/2}\left(q^+_\perp + \frac{2m}{E}g^+\ddot{q}_\perp\right),$$
$$\psi^*\tilde{\xi}^+ = \eta^{3/2}\left(\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+g^+\right),$$

where $\eta = gE/T$. Its pullback map ψ^* is a chain map between $\mathfrak{BV}^\bullet_J$, $\mathfrak{BV}^\bullet_{GR}$,

$$\psi^* \circ Q_J = Q_{GR} \circ \psi^*,$$

and maps the BV form and BV action of the Jacobi as

$$\psi^*\omega_J = \omega_{GR} - \mathcal{L}_{Q_{GR}}\delta\boldsymbol{\beta},$$
$$\psi^*\mathcal{S}_J = \mathcal{S}_{GR} + \mathcal{L}_{Q_{GR}}\iota_{Q_{GR}}\boldsymbol{\beta},$$

where $\boldsymbol{\beta} \in \Omega^1_{\text{loc}}(\mathcal{F}_{GR})$ is given by

$$\boldsymbol{\beta} = \int_I \Bigg\{-\frac{4g^{7/2}}{\Omega^2}Tg^+\delta g^+ + \left(\frac{2g^2}{\Omega} + \eta^{3/2}\frac{2\sqrt{g}}{E}\right)g^+q^+_\perp \cdot \delta q$$
$$+ \left(\frac{4g^{7/2}}{\Omega^2}T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\dot{g}^+g^+\frac{m\dot{q}}{2T}\cdot\delta q - (\eta^{3/2}-1)\xi^+\frac{m\dot{q}}{2T}\cdot\delta q\Bigg\}\,\mathrm{d}t,$$

with $\Omega = \sqrt{g}T + g\sqrt{TE}$.

Furthermore, ψ^* preserves the Hamiltonian condition

$$\iota_{Q_J}\omega_J = \delta\mathcal{S}_J \xrightarrow{\psi^*} \iota_{Q_{GR}}\omega_{GR} = \delta\mathcal{S}_{GR}.$$

Proof. To show that ψ^* fulfills the chain map condition we again have to check it for every $\tilde{\Phi}, \tilde{\Phi}^\dagger \in \mathcal{F}_J$. The computations can be found in the appendix A.2.

In the case of $\psi^*\omega_J$ and $\psi^*\mathcal{S}_J$ we have

$$\psi^*\omega_J = \psi^* \int_I \delta\theta_J^1 \,\mathrm{d}t, = \int_I \delta\psi^*\theta_J^1 \,\mathrm{d}t,$$
$$\psi^*\mathcal{S}_J = \psi^* \int_I L_J^1 \,\mathrm{d}t, = \int_I \psi^* L_J^1 \,\mathrm{d}t,$$

using the non-integrated local forms $(\theta_J^\bullet, L_J^\bullet)$ from Example 3.4.8. The local forms $\psi^*\theta_J^1$, $\psi^*L_J^1$ are computed in the appendix A.5, we have

$$\delta\psi^*\theta_J^1 \,\mathrm{d}t = \delta\theta_{GR}^1 \,\mathrm{d}t - \mathcal{L}_{Q_{GR}}\delta\beta^1 \,\mathrm{d}t + \mathrm{d}(\dots),$$
$$\psi^* L_J^1 \,\mathrm{d}t = L_{GR}^1 \,\mathrm{d}t + \mathcal{L}_{Q_{GR}}\iota_{Q_{GR}}\beta^1 \,\mathrm{d}t + \mathrm{d}(\dots),$$

where $(\theta_{GR}^\bullet, L_{GR}^\bullet)$ are the non-integrated local forms from Example 3.4.9 and

$$\beta^1 = -\frac{4g^{7/2}}{\Omega^2} T g^+ \delta g^+ + \left(\frac{2g^2}{\Omega} + \eta^{3/2}\frac{2\sqrt{g}}{E}\right) g^+ q_\perp^+ \cdot \delta q$$
$$+ \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2}\frac{g^{3/2}}{E}\right) \dot{g}^+ g^+ \frac{m\dot{q}}{2T} \cdot \delta q - (\eta^{3/2} - 1)\xi^+ \frac{m\dot{q}}{2T} \cdot \delta q,$$

with $\eta = gE/T$ and $\Omega = \sqrt{g}T + g\sqrt{TE}$. Integrating the expressions for $\delta\psi^*\theta_J^1$, $\psi^*L_J^1$ and noting that we are considering $\partial I = \emptyset$ gives the desired results with $\boldsymbol{\beta} = \int_I \beta^1 \,\mathrm{d}t$ due to the vanishing of the boundary terms.

As ψ^* transforms $\omega_J, \mathcal{S}_J$ in the way presented in Remark 3.2.4, it automatically preserves the Hamiltonian condition. $\square$

Remark 4.1.4. The map $\psi : \mathcal{F}_{GR} \to \mathcal{F}_J$ acts on the matter and ghost fields $\{\tilde{q}, \tilde{\xi}\}$ by interchanging them with their counterparts in the 1D GR theory $\{q, \xi\}$. At the level of the antifields $\{\tilde{q}^+, \tilde{\xi}^+\}$, it shifts their counterparts $\{q^+, \xi^+\}$ with terms that are proportional to g^+, $\dot{g}^+$ and rescales the result with a term which is equal to 1 when restricted to the surface where g solves the EL equations $g = T/E$.

Note that we can interpret the theory $\psi^*\mathfrak{F}_J := (\mathcal{F}_{GR}, \psi^*\omega_J, \psi^*\mathcal{S}_J, Q_{GR})$ as a version of the Jacobi theory that is "embedded" in 1D GR.

4.2 Quasi-isomorphism

Having the two chain maps ϕ^*, ψ^*, we are now ready to check if their composition maps λ^*, χ^* are the identity in the respective BV cohomologies, thus showing that all conditions for the BV-equivalence of the Jacoby theory and 1D GR are satisfied. We start with λ^*:

Lemma 4.2.1. *The composition map* $\lambda^* : \mathfrak{BV}^\bullet_J \to \mathfrak{BV}^\bullet_J$ *is the identity*

$$\lambda^*\tilde{q} = \tilde{q}, \qquad \lambda^*\tilde{\xi} = \tilde{\xi},$$
$$\lambda^*\tilde{q}^+ = \tilde{q}^+, \qquad \lambda^*\tilde{\xi}^+ = \tilde{\xi}^+.$$

Proof. On the matter and ghost fields $\{\tilde{q}, \tilde{\xi}\}$ this is trivial, since at this level both ψ^* and ϕ^* simply interchange $\tilde{q}$ with q and $\tilde{\xi}$ with ξ

$$\lambda^*\tilde{q} = (\phi^* \circ \psi^*)\tilde{q} = \phi^* q = \tilde{q},$$
$$\lambda^*\tilde{\xi} = (\phi^* \circ \psi^*)\tilde{\xi} = \phi^* \xi = \tilde{\xi}.$$

In order to compute the action of λ^* on the antifield and antighost, first recall that $\phi^* g^+ = 0$ and note that $\phi^* g = \tilde{T}/E$ implies $\phi^*(\eta^{3/2}) = (\phi^*(g)E/\tilde{T})^{3/2} = 1$. We then have

$$\begin{aligned}
\lambda^*\tilde{q}^+_\parallel &= (\phi^* \circ \psi^*)\tilde{q}^+_\parallel \\
&= \phi^*(\eta^{3/2})\phi^* \left(q^+_\parallel - \left[g^+\dot{g} + 2\dot{g}^+ g\right] \frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E} \left[\dot{EL_g} g^+ - EL_g \dot{g}^+\right] \frac{m\dot{q}}{2T} \right) \\
&= \phi^* q^+_\parallel = \tilde{q}^+_\parallel,
\end{aligned}$$
$$\lambda^*\tilde{q}^+_\perp = (\phi^* \circ \psi^*)\tilde{q}^+_\perp = \phi^*(\eta^{3/2})\phi^* \left(q^+_\perp + \frac{2m}{E} g^+ \ddot{q}_\perp \right) = \phi^* q^+_\perp = \tilde{q}^+_\perp,$$
$$\lambda^*\tilde{\xi}^+ = (\phi^* \circ \psi^*)\tilde{\xi}^+ = \phi^*(\eta^{3/2})\phi^* \left(\xi^+ + \frac{g^{3/2}}{E} \dot{g}^+ g^+ \right) = \phi^* \xi^+ = \tilde{\xi}^+,$$

thus showing $\lambda = \mathrm{id}_J$. □

Moving now to χ^*, we first determine its explicit expression through direct computations:

Corollary 4.2.2. *The composition map* $\chi^* : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ *acts as*

$$\begin{aligned}
&\chi^* q = q,\\
&\chi^* \xi = \xi,\\
&\chi^* g = \frac{T}{E},\\
&\chi^* q^+_\parallel = \eta^{3/2}\left(q^+_\parallel - \left[g^+\dot{g} + 2\dot{g}^+ g\right]\frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E}\left[\dot{E L_g} g^+ - E L_g \dot{g}^+\right]\frac{m\dot{q}}{2T}\right),\\
&\chi^* q^+_\perp = \eta^{3/2}\left(q^+_\perp + \frac{2m}{E} g^+ \ddot{q}_\perp\right),\\
&\chi^* \xi^+ = \eta^{3/2}\left(\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+ g^+\right),\\
&\chi^* g^+ = 0.
\end{aligned} \tag{25}$$

Proof. Recall that $\chi^* = \psi^* \circ \phi^*$ and let $\varphi_i \in \{q, q^+, \xi, \xi^+\}$. In this case we have $\phi^* \varphi_i = \tilde{\varphi}_i$ and as such $\chi^* \varphi_i = \psi^* \tilde{\varphi}_i$, which reproduces the expressions above due to the explicit form of ψ^*. For $\{g, g^+\}$ we compute

$$\chi^* g = (\psi^* \circ \phi^*) g = \psi^*\left(\frac{\tilde{T}}{E}\right) = \frac{T}{E},$$
$$\chi^* g^+ = (\psi^* \circ \phi^*) g^+ = \psi^*(0) = 0.$$

□

In order to prove that χ^* is the identity in the BV cohomology, we are going to build an one-parameter family of maps $\chi^*_s = e^{sD}$, where $s \in \mathbb{R}_{\geq 0}$, which reproduces the identity map id_{GR} at $s = 0$ and χ^* when $s \to \infty$. Here, $D \in \mathfrak{X}(\mathcal{F}_{GR})$ is chosen to be a degree 0 vector field defined as

$$D = Q_{GR} R + R Q_{GR} = [Q_{GR}, R],$$

where $R \in \mathfrak{X}_{\mathrm{evo}}(\mathcal{F}_{GR})$ is a degree -1 vector field which we assume to be evolutionary

$$[\mathcal{L}_R, \mathrm{d}] = 0.$$

Note that this condition reduces to $[R, \mathrm{d}] = 0$ on $\mathfrak{BV}^\bullet_{GR}$, since $\mathcal{L}_R f = Rf$ for any integrated local functional $f \in \mathfrak{C}^\infty_{\mathrm{loc}}(\mathcal{F}_{GR})$. In particular, this means that R commutes with the time derivative ∂_t when acting on $\Phi, \Phi^\dagger \in \mathcal{F}_{GR}$. This is done such that the extension of R to higher jets is straightforward. Since Q_{GR} is an evolutionary vector field, we then have that $D \in \mathfrak{X}_{\text{evo}}(\mathcal{F}_{GR})$ is also evolutionary, i.e. $[\mathcal{L}_D, \mathrm{d}] = 0$.

Moreover, we will assume that R preserves the tensor rank on I

$$R : \mathcal{T}^n_m(I) \to \mathcal{T}^n_m(I),$$

where $\mathcal{T}^n_m(I)$ denotes the space of rank-(n, m) tensors fields over I. The usefulness of this property will be made clear in Lemma 4.2.5, where we will show that this feature, together with the action of R on the ghost ξ, imply that R commutes with the Chevalley-Eilenberg differential $[R, \gamma_{GR}] = 0$, which will greatly simplify the calculations.

Remark 4.2.3. In principle, the vector field R is neither required to be evolutionary nor to preserve the tensor rank on the spacetime manifold in order for us to be able to construct an one-parameter family of morphisms $\chi^*_s = e^{sD}$ that has the map χ^* as its limit, but these are useful properties, which produce the desired results in our case.

The computation of the morphisms $\chi^*_s = e^{sD}$ will be split into smaller Lemmas, starting with their action on the matter and the ghost fields $\{q, \xi\}$ in Lemma 4.2.4, followed by their action on the metric field g in Lemma 4.2.6 and on the antifields and antighost $\{g^+, \xi^+, q^+\}$ in Lemmas 4.2.8, 4.2.9 and 4.2.11 respectively. In Lemma 4.2.12, we prove that χ^* is homotopic to the identity by showing that the morphisms χ^*_s converges to χ^* in the $s \to \infty$ limit.

Lemma 4.2.4. *The assigment $Rq = 0$, $R\xi = 0$ gives*

$$\chi^*_s q = q, \qquad\qquad \chi^*_s \xi = \xi.$$

Proof. We start with the computation for the ghost ξ. We have

$$\begin{aligned} D\xi &= (Q_{GR}R + RQ_{GR})\xi = R\left(\xi\dot{\xi}\right) = 0, \\ \Rightarrow \quad \chi^*_s \xi &= e^{sD}\xi = \xi, \end{aligned}$$

where we used $R\dot{\xi} = \partial_t(R\xi) = 0$. The calculation for q is analogous:

$$\begin{aligned} Dq &= (Q_{GR}R + RQ_{GR})q = R(\xi\dot{q}) = 0, \\ \Rightarrow \quad \chi^*_s q &= e^{sD}q = q. \end{aligned}$$

□

We now prove that R commutes with the Chevalley-Eilenberg differential γ_{GR}. Effectively, this means that we can ignore the Chevalley-Eilenberg part of Q_{GR} in $D = [Q_{GR}, R]$ and only have to regard the Koszul-Tate differential when explicitly computing the action of D. Recall that the Chevalley-Eilenberg differential acts as $\gamma_{GR} = \mathcal{L}_{\xi\partial_t}$ on $\{q, g, q^+, g^+, \xi^+\}$ and as $\gamma_{GR} = \frac{1}{2}\mathcal{L}_{\xi\partial_t}$ on $\{\xi\}$.

Lemma 4.2.5. *Let $R \in \mathfrak{X}_{\text{evo}}(\mathcal{F}_{GR})$ be an evolutionary vector field on $\mathcal{F}_{GR}$ with the following properties*

- *R vanishes on $\mathfrak{X}[1](I)$,*
- *R preserves the tensor rank on I,*

and let γ_{GR} be the Chevalley-Eilenberg differential of the 1D GR theory. Then $[R, \gamma_{GR}] = 0$.

Proof. Recall that all the fields we are considering are components of tensor fields over I. In particular, note that the property that R vanishes on $\mathfrak{X}[1](I)$ implies $R\xi = 0$, since $\xi\partial_t \in \mathfrak{X}[1](I)$. As $\gamma_{GR} \sim \mathcal{L}_{\xi\partial_t}$, it suffices to show that $[R, \mathcal{L}_{\xi\partial_t}] = 0$ on functions, 1-forms and vector fields over I. We assume that all objects have an internal grading throughout the proof in order to account for the ghost number.

We will first show $[R, \gamma_{GR}] = 0$ for functions and 1-forms. Since $\mathcal{L}_{\xi\partial_t} = [\iota_{\xi\partial_t}, \mathrm{d}]$ on $\Omega^\bullet(I)$ we have

$$[R, \mathcal{L}_{\xi\partial_t}] = [R, [\iota_{\xi\partial_t}, \mathrm{d}]] = [\mathrm{d}, [R, \iota_{\xi\partial_t}]] + [\iota_{\xi\partial_t}, [\mathrm{d}, R]] = [\mathrm{d}, [R, \iota_{\xi\partial_t}]],$$

where we used that R is evolutionary. As such it is sufficient to show that $\Omega^\bullet(I) \subset \ker[R, \iota_{\xi\partial_t}]$. By definition all functions on I are in the kernel of $\iota_{\xi\partial t}$: $C^\infty(I) = \Omega^0(I) \subset \ker \iota_{\xi\partial_t}$. Let $f \in \Omega^0(I)$. Since we assume that R preserves the tensor rank we also have $Rf \in \Omega^0(I)$, then

$$R\,\iota_{\xi\partial_t} f = 0, \qquad \iota_{\xi\partial_t} Rf = 0,$$

and as such $[R, \iota_{\xi\partial_t}]f = 0$. Let now $\varpi = f\mathrm{d}t \in \Omega^1(I)$ be a 1-form. Taking into account that $|\mathrm{d}t| = 1$, we have

$$\begin{aligned} R\,\iota_{\xi\partial_t}\varpi &= R\,\iota_{\xi\partial_t}(f\mathrm{d}t) = R(f\mathrm{d}t(\xi\partial_t)) = -R(f\xi) = -Rf\xi, \\ \iota_{\xi\partial_t}R\varpi &= \iota_{\xi\partial_t}R(f\mathrm{d}t) = \iota_{\xi\partial_t}[Rf\mathrm{d}t] = Rf\mathrm{d}t(\xi\partial_t) = -Rf\xi, \\ \Rightarrow [R, \iota_{\xi\partial_t}]\varpi &= R\,\iota_{\xi\partial_t}\varpi - \iota_{\xi\partial_t}R\varpi = -Rf\xi + Rf\xi = 0, \end{aligned}$$

thus showing that $\Omega^0(I) \times \Omega^1(I) \subset \ker[R, \iota_{\xi\partial_t}]$. This implies

$$[\mathrm{d}, [R, \iota_{\xi\partial_t}]]f = 0, \qquad [\mathrm{d}, [R, \iota_{\xi\partial_t}]]\varpi = \mathrm{d}[R, \iota_{\xi\partial_t}]\varpi = 0,$$

where we used $f \in \Omega^0(I) \subset \ker[R, \iota_{\xi\partial_t}]$, $\mathrm{d}f \in \Omega^1(I) \subset \ker[R, \iota_{\xi\partial_t}]$ and $\mathrm{d}\varpi = 0$, since $\Omega^1(I) = \Omega^{\mathrm{top}}(I)$.

Consider now a vector field $X = f\partial_t \in \mathfrak{X}(I)$ of degree n. In this case we have:

$$\begin{aligned} &R\mathcal{L}_{\xi\partial_t}X = R[\xi\partial_t, f\partial_t] = R(\xi\dot{f} - (-1)^n f\dot{\xi})\partial_t = -(\xi R\dot{f} + (-1)^n Rf\dot{\xi})\partial_t, \\ &\mathcal{L}_{\xi\partial_t}RX = [\xi\partial_t, Rf\partial_t] = (\xi R\dot{f}\partial_t - (-1)^{n-1}Rf\dot{\xi})\partial_t, \\ \Rightarrow &[R, \mathcal{L}_{\xi\partial_t}]X = R\mathcal{L}_{\xi\partial_t}X + \mathcal{L}_{\xi\partial_t}RX = 0, \end{aligned}$$

where we used $\partial_t(Rf) = R\dot{f}$. Since $[R, \mathcal{L}_{\xi\partial_t}] = 0$ on functions, 1-forms and vector fields it holds for all tensors. As such we have $[R, \gamma_{GR}] = 0$ on $\mathcal{F}_{GR}$. $\square$

A direct implication of Lemma 4.2.5 is that the vector field D reduces to

$$D = [Q_{GR}, R] = [\delta_{GR}, R].$$

With this result in hand, we now proceed with the computation of the action of $\chi_s^* = e^{sD}$ on the metric field g.

Lemma 4.2.6. *The assigment*

$$Rg = \frac{-2g^{3/2}}{E}g^+,$$

results in

$$\chi_s^* g = e^{-s}g + (1 - e^{-s})\frac{T}{E}. \tag{26}$$

Proof. Keeping in mind that $D = [\delta_{GR}, R]$, $\delta_{GR}g = 0$ and

$$\delta_{GR}g^+ = EL_g = \frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}},$$

(c.f. Remark 3.1.12), we compute Dg to be

$$\begin{aligned} Dg &= (\delta_{GR}R + R\delta_{GR})\, g = \delta_{GR}\left(\frac{-2g^{3/2}}{E}g^+\right) \\ &= \frac{-2g^{3/2}}{E}\left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}\right) = -\left(g - \frac{T}{E}\right). \end{aligned}$$

We can then show that

$$D^k g = (-1)^k \left(g - \frac{T}{E} \right) \quad \text{for } k \geq 1, \tag{27}$$

using induction. As we have computed, this holds for $k = 1$. Assuming that equation (27) holds for an arbitrary k, we then have the following for $k + 1$

$$D^{k+1} g = (-1)^k \left(Dg - D\frac{T}{E} \right) = (-1)^{k+1} \left(g - \frac{T}{E} \right),$$

where we used $DT = m\dot{q}D\dot{q} = m\dot{q}\partial_t(Dq) = 0$. This results in

$$\begin{aligned} \chi_s^* g = e^{sD} g &= g + \sum_{k \geq 1} \frac{s^k}{k!} (-1)^k \left(g - \frac{T}{E} \right) \\ &= g + (e^{-s} - 1) \left(g - \frac{T}{E} \right) \\ &= e^{-s} g + (1 - e^{-s}) \frac{T}{E}, \end{aligned}$$

which completes the proof. □

Remark 4.2.7. It is useful to keep in mind that Dg is proportional to EL_g

$$Dg = \delta_{GR} \left(\frac{-2g^{3/2}}{E} g^+ \right) = -\frac{2g^{3/2}}{E} EL_g.$$

Having $\chi_s^* \Phi$ for $\Phi \in \{q, \xi, g\}$, we now proceed with the computation of $\chi_s^* \Phi^+ = e^{sD} \Phi^+$ for $\Phi^+ \in \{q^+, \xi^+, g^+\}$. As we will shortly see, finding a recursive formula for $D^k \Phi^+$ is quite challenging. Instead, it turns out that it is easier to take a slight detour: we will first compute $\chi_s^*(g^{3/2}\Phi^+)$ through $D^k(g^{3/2}\Phi^+)$ and then use the property that $\chi_s^* = e^{sD}$ is a morphism in order to recover $\chi_s^* \Phi^+$

$$\begin{aligned} & \chi_s^*(g^{3/2}\Phi^+) = (\chi_s^* g)^{3/2} \, \chi_s^* \Phi^+ \\ \Leftrightarrow \quad & \chi_s^* \Phi^+ = \frac{\chi_s^*(g^{3/2}\Phi^+)}{(\chi_s^* g)^{3/2}} = \frac{\chi_s^*(g^{3/2}\Phi^+)}{\left[e^{-s} g + (1 - e^{-s}) \frac{T}{E} \right]^{3/2}}. \end{aligned} \tag{28}$$

Since $\chi_s^* g$ is nowhere vanishing for any $s \in \mathbb{R}_{\geq 0}$, this expression is well-defined $\forall s \in \mathbb{R}_{\geq 0}$ iff $\chi_s^*(g^{3/2}\Phi^+)$ is well-defined $\forall s \in \mathbb{R}_{\geq 0}$ as well.

Let us start with the antifield of the metric field g^+, where we set $Rg^+ = 0$ (see Lemma 4.2.8). In order to see where the aforementioned problem arises we compute

$$\begin{aligned} Dg^+ &= (\delta_{GR}R + R\delta_{GR})\, g^+ = R(EL_g) = \frac{\delta EL_g}{\delta g} Rg \\ &= \left(-\frac{E}{4g^{3/2}} + \frac{3T}{4g^{5/2}}\right) \frac{-2g^{3/2}}{E} g^+ = \left(\frac{1}{2} - \frac{3}{2}\frac{T}{Eg}\right) g^+. \end{aligned}$$

One can then proceed with the calculation of $D^k g^+$ for higher k's and notice that the expressions become quite lengthy, mostly due to the g^{-1} term which will always gives higher powers of itself. The idea to avoid this complication is to find an n such that the recursive formula for $D^k(g^n g^+)$ becomes apparent. We have

$$\begin{aligned} D(g^n g^+) &= n g^{n-1} Dg g^+ + g^n Dg^+ \\ &= n g^{n-1} \left(\frac{T}{E} - g\right) g^+ + g^n \left(\frac{1}{2} - \frac{3}{2}\frac{T}{Eg}\right) g^+ \\ &= \left(n - \frac{3}{2}\right) \frac{T}{E} g^{n-1} g^+ + \left(\frac{1}{2} - n\right) g^n g^+. \end{aligned} \tag{29}$$

To have a recursive expression for $D^k(g^n g^+)$, we want to make the first term vanish while keeping the second, so we choose $n = 3/2$. These considerations lead to the following Lemma:

Lemma 4.2.8. *The assigment $Rg^+ = 0$ leads to*

$$\chi_s^*(g^{3/2} g^+) = e^{-s} g^{3/2} g^+.$$

Proof. Setting $n = 3/2$ in equation (29) yields

$$D(g^{3/2} g^+) = -g^{3/2} g^+.$$

It is then straightforward to see that

$$D^k(g^{3/2} g^+) = (-1)^k g^{3/2} g^+ \quad \text{for } k \geq 0,$$

and as such

$$\begin{aligned} \chi_s^*(g^{3/2} g^+) = e^{sD}(g^{3/2} g^+) &= \sum_{k \geq 0} \frac{s^k}{k!} (-1)^k g^{3/2} g^+ \\ &= e^{-s} g^{3/2} g^+. \end{aligned}$$

□

Having seen that we can compute $\chi_s^* g^+$ by first determining $\chi_s^*(g^{3/2}g^+)$, we now apply the same method to the computation of $\chi_s^*\xi^+$ and subsequently to the computation of $\chi_s^* q^+$.

Lemma 4.2.9. *The assigment*

$$R\xi^+ = 0, \qquad R(q^+ \cdot \dot{q}) = -\frac{3\sqrt{g}}{E} EL_g \xi^+, \tag{30}$$

results in

$$\chi^* \xi_s^+ (g^{3/2}\xi^+) = g^{3/2}\xi^+ - (e^{-2s} - 1)\frac{g^{3/2}}{E}\dot{g}^+ g^+.$$

Remark 4.2.10. Defining $R(q^+ \cdot \dot{q})$ is equivalent to defining $Rq_\parallel^+$ since

$$Rq_\parallel^+ = R\left(q^+ \cdot \dot{q}\frac{m\dot{q}}{2T}\right) = R(q^+ \cdot \dot{q})\frac{m\dot{q}}{2T},$$

as $m\dot{q}/(2T)$ never vanishes, due to the condition $\dot{q} \neq 0$.

Proof. We start by computing $D\xi^+$

$$\begin{aligned}
D\xi^+ &= R\delta_{GR}\xi^+ = R\left(-q^+ \cdot \dot{q} + g^+\dot{g} + 2\dot{g}^+ g\right) \\
&= -R(q^+ \cdot \dot{q}) - g^+\partial_t\left(\frac{-2g^{3/2}}{E}g^+\right) - \dot{g}^+ 2\frac{(-2)g^{3/2}}{E}g^+ \\
&= -R(q^+ \cdot \dot{q}) + \frac{2g^{3/2}}{E}g^+\dot{g}^+ + \frac{4g^{3/2}}{E}\dot{g}^+ g^+ \\
&= -R(q^+ \cdot \dot{q}) + \frac{2g^{3/2}}{E}\dot{g}^+ g^+,
\end{aligned}$$

where we used that $g^+g^+ = 0$ in the second line and $g^+\dot{g}^+ = -\dot{g}^+ g^+$ in the third. With this in hand we can proceed with the calculation of $D(g^{3/2}\xi^+)$

$$\begin{aligned}
D(g^{3/2}\xi^+) &= \frac{3}{2}\sqrt{g}Dg\xi^+ + g^{3/2}D\xi^+ \\
&= -\frac{3g^2}{E}EL_g\xi^+ - g^{3/2}R(q^+ \cdot \dot{q}) + \frac{2g^3}{E}\dot{g}^+ g^+ \\
&= \frac{2}{E}\partial_t(g^{3/2}g^+)g^{3/2}g^+.
\end{aligned}$$

It should be now clear why we chose the specific form for $Rq^+ \cdot \dot{q}$: the first two terms in the second line cancel and we are left with a term for which we know how to compute D^k. Using induction we can then prove

$$D^k(g^{3/2}\xi^+) = -\frac{(-2)^k}{E}\partial_t(g^{3/2}g^+)g^{3/2}g^+ \quad \text{for } k \geq 1.$$

We have already shown that it holds for $k = 1$. Assuming that it is true for k, the expression for $k+1$ reads:

$$\begin{aligned} D^{k+1}(g^{3/2}\xi^+) &= -\frac{(-2)^k}{E}\partial_t(Dg^{3/2}g^+)g^{3/2}g^+ - \frac{(-2)^k}{E}\partial_t(g^{3/2}g^+)Dg^{3/2}g^+ \\ &= 2\frac{(-2)^k}{E}\partial_t(g^{3/2}g^+)g^{3/2}g^+ \\ &= -\frac{(-2)^{k+1}}{E}\partial_t(g^{3/2}g^+)g^{3/2}g^+. \end{aligned}$$

Since $\partial_t(g^{3/2}g^+)g^{3/2}g^+ = g^3\dot{g}^+g^+$ due to $g^+g^+ = 0$ we then have

$$\begin{aligned} \chi_s^*(g^{3/2}\xi^+) = e^{sD}(g^{3/2}\xi^+) &= \sum_{k\geq 0}\frac{s^k}{k!}D^k(g^{3/2}\xi^+) \\ &= g^{3/2}\xi^+ - \left(\sum_{k\geq 1}\frac{(-2s)^k}{k!}\right)\frac{g^3}{E}\dot{g}^+g^+ \\ &= g^{3/2}\xi^+ - (e^{-2s}-1)\frac{g^3}{E}\dot{g}^+g^+. \end{aligned}$$

□

Lemma 4.2.11. *The assigment*

$$Rq_\parallel^+ = -\frac{3\sqrt{g}}{E}EL_g\xi^+\frac{m\dot{q}}{2T}, \qquad Rq_\perp^+ = \frac{3\sqrt{g}}{E}g^+q_\perp^+, \tag{31}$$

yields

$$\begin{aligned} \chi_s^*(g^{3/2}q_\parallel^+) &= g^{3/2}q_\parallel^+ + (e^{-s}-1)2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} + (e^{-2s}-1)\frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}, \\ \chi_s^*(g^{3/2}q_\perp^+) &= g^{3/2}q_\perp^+ - (e^{-s}-1)\frac{2m}{E}\ddot{q}_\perp g^{3/2}g^+, \end{aligned}$$

where $\sigma(\varphi) = \varphi\partial_t(g^{3/2}g^+) - \dot{\varphi}g^{3/2}g^+$.

Proof. The strategy for this proof is the same as for the proofs of Lemmas 4.2.8 and 4.2.9, but the computations are much more cumbersome and as such we present them in the appendix A.3. □

Summarizing, we have shown that defining the vector field R as

$$\begin{aligned}
&Rq = 0, && R\xi = 0, && Rg = \frac{-2g^{3/2}}{E}g^+,\\
&Rq^+_\parallel = -\frac{3\sqrt{g}}{E}EL_g\xi^+\frac{m\dot{q}}{2T}, && R\xi^+ = 0, && Rg^+ = 0,\\
&Rq^+_\perp = \frac{3\sqrt{g}}{E}g^+q^+_\perp,
\end{aligned} \tag{32}$$

leads to a one-parameter family of morphisms $\chi^*_s = e^{sD}$, whose action on $\Phi, g^{3/2}\Phi^+ \in \mathcal{F}_{GR}$ is given by

$$\begin{aligned}
&\chi^*_s q = q,\\
&\chi^*_s \xi = \xi,\\
&\chi^*_s g = e^{-s}g + (1-e^{-s})\frac{T}{E},\\
&\chi^*_s(g^{3/2}q^+_\parallel) = g^{3/2}q^+_\parallel + (e^{-s}-1)2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} + (e^{-2s}-1)\frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T},\\
&\chi^*_s(g^{3/2}q^+_\perp) = g^{3/2}q^+_\perp - (e^{-s}-1)\frac{2m}{E}\ddot{q}_\perp g^{3/2}g^+,\\
&\chi^*_s(g^{3/2}\xi^+) = g^{3/2}\xi^+ - (e^{-2s}-1)\frac{g^3}{E}\dot{g}^+g^+,\\
&\chi^*_s(g^{3/2}g^+) = e^{-s}g^{3/2}g^+.
\end{aligned} \tag{33}$$

We are now ready to show that it converges to the map χ^* in the $s \to \infty$ limit, i.e. that χ^* is homotopic to the identity.

Lemma 4.2.12. *The composition map* $\chi^* : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ *from Corollary 4.2.2 is homotopic to the identity* id_{GR} *on* $\mathfrak{BV}^\bullet_{GR}$.

Proof. Since $\chi^*_{s=0} = e^{0\cdot D} = \mathrm{id}_{GR}$, it is left to show that

$$\lim_{s\to\infty}\chi^*_s = \chi^*,$$

on $\mathcal{F}_{GR}$. For the fields $\{q, \xi, g\}$, we compute

$$\begin{aligned}
\lim_{s\to\infty} \chi_s^* q &= \lim_{s\to\infty} q = q = \chi^* q, \\
\lim_{s\to\infty} \chi_s^* \xi &= \lim_{s\to\infty} \xi = \xi = \chi^* \xi, \\
\lim_{s\to\infty} \chi_s^* g &= \lim_{s\to\infty} \left(e^{-s} g + (1-e^{-s}) \frac{T}{E} \right) = \frac{T}{E} = \chi^* g.
\end{aligned}$$

For the antifields, note that taking the $s \to \infty$ limit of equation (28) yields

$$\lim_{s\to\infty} \chi_s^* \Phi^+ = \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \chi_s^* (g^{3/2} \Phi^+). \tag{34}$$

Recall that we set $\eta = gE/T$. In the case of $\{q_\perp^+, \xi^+, g^+\}$ we have

$$\begin{aligned}
\lim_{s\to\infty} \chi_s^* q_\perp^+ &= \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \chi_s^* (g^{3/2} q_\perp^+) \\
&= \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \left(g^{3/2} q_\perp^+ - (e^{-s}-1) \frac{2m}{E} \ddot{q}_\perp g^{3/2} g^+ \right) \\
&= \eta^{3/2} \left(q_\perp^+ + \frac{2m}{E} \ddot{q}_\perp g^+ \right) = \chi^* q_\perp^+, \\
\lim_{s\to\infty} \chi_s^* \xi^+ &= \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \chi_s^* (g^{3/2} \xi^+) \\
&= \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \left(g^{3/2} \xi^+ - (e^{-2s}-1) \frac{g^3}{E} \dot{g}^+ g^+ \right) \\
&= \eta^{3/2} \left(\xi^+ + \frac{g^{3/2}}{E} \dot{g}^+ g^+ \right) = \chi^* \xi^+, \\
\lim_{s\to\infty} \chi_s^* g^+ &= \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \chi_s^* (g^{3/2} g^+) = \left(\frac{E}{T}\right)^{3/2} \lim_{s\to\infty} \left(e^{-s} g^{3/2} g^+ \right) \\
&= 0 = \chi^* g^+.
\end{aligned}$$

The computations for $\lim_{s\to\infty} \chi_s^* q_\parallel^+$ are again quite cumbersome and are presented in the appendix A.3. The result however is

$$\begin{aligned}
\lim_{s\to\infty} \chi_s^* q_\parallel^+ &= \eta^{3/2} \left(q_\parallel^+ - \left[g^+ \dot{g} + 2\dot{g}^+ g \right] \frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E} \left[E\dot{L}_g g^+ - EL_g \dot{g}^+ \right] \frac{m\dot{q}}{2T} \right) \\
&= \chi^* q_\parallel^+.
\end{aligned}$$

□

We now show how this result leads to χ^* being the identity when restricted to the BV cohomology $H^\bullet(\mathfrak{BV}^\bullet_{GR})$.

Lemma 4.2.13. *The map $\chi^* : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ is the identity on $H^\bullet(\mathfrak{BV}^\bullet_{GR})$.*

Proof. We need to find a well-defined operator $h_\chi : \mathfrak{BV}^\bullet_{GR} \to \mathfrak{BV}^\bullet_{GR}$ of degree -1 such that

$$\chi^* - \mathrm{id}_{GR} = Q_{GR} h_\chi + h_\chi Q_{GR}, \tag{35}$$

on $\mathfrak{BV}^\bullet_{GR}$.

Note first that D commutes with Q_{GR}, which can be shown by using the property that Q_{GR} is cohomological, i.e. that $Q_{GR}^2 = 0$

$$\begin{aligned} [D, Q_{GR}] &= (Q_{GR}R + RQ_{GR})Q_{GR} - Q_{GR}(Q_{GR}R + RQ_{GR}) \\ &= Q_{GR}RQ_{GR} - Q_{GR}RQ_{GR} = 0. \end{aligned} \tag{36}$$

With this in mind we compute

$$\begin{aligned} \chi^* - \mathrm{id}_{GR} &= \int_0^\infty \frac{\mathrm{d}}{ds} e^{sD} \mathrm{d}s = \int_0^\infty e^{sD} D \mathrm{d}s = \int_0^\infty e^{sD}(Q_{GR}R + RQ_{GR})\mathrm{d}s \\ &= Q_{GR}\left(\int_0^\infty e^{sD} R \mathrm{d}s\right) + \left(\int_0^\infty e^{sD} R \mathrm{d}s\right) Q_{GR}. \end{aligned}$$

Hence defining $h_\chi f$ for $f \in \mathfrak{BV}^\bullet_{GR}$ as

$$h_\chi f = \int_0^\infty e^{sD} R f \mathrm{d}s,$$

reproduces equation (35). We now need to show that $h_\chi f$ is well-defined. We start with the action of h_χ on $\Phi, \Phi^\dagger \in \mathcal{F}_{GR}$. Note that $\{q, \xi, \xi^+, g^+\} \in \ker R$ and as such

$$h_\chi q = 0, \qquad h_\chi \xi = 0, \qquad h_\chi \xi^+ = 0, \qquad h_\chi g^+ = 0.$$

In the case of the metric field g we compute

$$\begin{aligned} h_\chi g &= \int_0^\infty e^{sD} R g \,\mathrm{d}s = -\frac{2}{E}\int_0^\infty e^{sD}(g^{3/2} g^+)\mathrm{d}s \\ &= -\frac{2}{E}\int_0^\infty e^{-s} g^{3/2} g^+ \mathrm{d}s = -\frac{2g^{3/2}}{E} g^+. \end{aligned}$$

For $q_\parallel^+$ and $q_\perp^+$ we only present the results, the explicit computation can be found in the appendix A.4. We have

$$h_\chi q_\parallel^+ = (1-\eta^{3/2})\left[\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+ g^+\right]\frac{m\dot{q}}{2T} + \frac{(3\sqrt{\eta}+1)(1-\sqrt{\eta})}{(\sqrt{\eta}+1)^2}\frac{g^{3/2}}{E}\dot{g}^+ g^+ \frac{m\dot{q}}{2T},$$
$$h_\chi q_\perp^+ = \frac{2}{T}\left(\frac{\eta}{\sqrt{\eta}+1}+1\right) g^{3/2} g^+ q_\perp^+.$$

Having proven that the action of h_χ is well-defined for $\Phi, \Phi^\dagger \in \mathcal{F}_{GR}$, we can now show that it converges for all $f \in \mathfrak{BV}^\bullet_{GR}$, but first it will be useful to redefine $s = -\ln\tau$ with $\tau \in [0,1]$, such that we consider integrals over a compact interval instead of over $\mathbb{R}_{\geq 0}$. Performing this transformation results in

$$h_\chi f = \int_0^\infty e^{sD} R f \mathrm{d}s = \int_1^0 e^{-\ln(\tau)D} R f \mathrm{d}(-\ln\tau) = \int_0^1 \frac{e^{-\ln(\tau)D}}{\tau} R f \mathrm{d}\tau.$$

Let φ_i denote all $\Phi, \Phi^\dagger \in \mathcal{F}_{GR}$ and write $R = R\varphi_i \frac{\delta}{\delta\varphi_i}$. The action of h_χ on any function $f \in \mathfrak{BV}^\bullet_{GR}$ is then given by

$$\begin{aligned} h_\chi f &= \int_0^1 \frac{e^{-\ln(\tau)D}}{\tau} R f \mathrm{d}\tau = \int_0^1 \frac{e^{-\ln(\tau)D}}{\tau}\left(R\varphi_i \frac{\delta f}{\delta\varphi_i}\right)\mathrm{d}\tau \\ &= \int_0^1 \left(\frac{e^{-\ln(\tau)D}}{\tau} R\varphi_i\right)\left(e^{-\ln(\tau)D}\frac{\delta f}{\delta\varphi_i}\right)\mathrm{d}\tau, \end{aligned}$$

where we used that $e^{sD} = e^{-\ln(\tau)D}$ is a morphism in the last equality. The integral over the first integrand is finite (this is simply $h_\chi\varphi_i$ in the new parameter τ) and the second integrand $e^{-\ln(\tau)D}\frac{\delta f}{\delta\varphi_i} = e^{sD}\frac{\delta f}{\delta\varphi_i}$ is nowhere divergent $\forall\tau \in [0,1]$, since $\chi_s^* = e^{sD}$ is well-defined on $\mathfrak{BV}^\bullet_{GR}$. Thus, the integral converges, finishing the proof that χ^* is the identity on $H^\bullet(\mathfrak{BV}^\bullet_{GR})$. □

Thus we have finally proven the main theorem of this chapter:

Theorem 4.2.14. *The BV theories $\mathfrak{F}_J$ and $\mathfrak{F}_{GR}$ of the Jacobi theory and 1D GR are BV-equivalent.*

Proof. Recall that two BV theories are BV-equivalent if there exist two maps ϕ, ψ such that:

- ϕ^*, ψ^* are chain maps between the BV complexes,

- the maps $\lambda^* = \phi^* \circ \psi^*$, $\chi^* = \psi^* \circ \phi^*$ are the identity in the respective BV cohomologies,

- ϕ^*, ψ^* interchange the BV forms and BV actions up to $\mathcal{L}_Q$-exact terms and preserve the Hamiltonian conditions.

The maps ϕ, ψ for the present case were constructed in Lemmas 4.1.1 and 4.1.3 respectively. In particular, ϕ^*, ψ^* were shown to be chain maps between the BV complexes, to interchange the BV forms, BV actions and preserve the Hamiltonian condition in the desired way. In addition, in Lemmas 4.2.1 and 4.2.13 we proved that the maps λ^* and χ^* act as the identity when restricted to the respective BV cohomologis $H^\bullet(\mathfrak{BV}^\bullet_J)$ and $H^\bullet(\mathfrak{BV}^\bullet_{GR})$, thus showing that the BV complexes $\mathfrak{BV}^\bullet_J$ and $\mathfrak{BV}^\bullet_{GR}$ are quasi-isomorphic

$$H^\bullet(\mathfrak{BV}^\bullet_J) \simeq H^\bullet(\mathfrak{BV}^\bullet_{GR}).$$

□

We have thus seen that the BV formulations of the Jacobi theory and 1D GR are quasi-isomorphic. As such, the discrepancy of their BV-BFV formulations is not captured by their bulk BV theories. The question is then whether this result changes when a boundary on I is introduced. We devote the next chapter to the study of this scenario, and show that the result presented in Theorem 4.2.14 can be extended to the case with a boundary.

5 Equivalence in the presence of a boundary

In Chapter 4 we showed that the BV formulations of the Jacobi theory and 1D GR are BV-equivalent on the bulk. We now want to see how this result changes when we introduce a boundary $\partial I = \{a, b\}$ on the interval I. As we will see in Section 5.1, the lax formulations of both theories are lax-equivalent in the sense of Definition 3.5.3, even though one of them yields a BV-BFV theory and while the other does not (see Examples 3.3.7,3.3.8).

Furthermore, we are interested in the effect of the composition maps λ^*, χ^* on the boundary structure, which we explore in Section 5.2. As showed in Proposition 3.5.4, the theories $\mathfrak{F}_J^{\text{lax}}, \lambda^*\mathfrak{F}_J^{\text{lax}}$ and $\mathfrak{F}_{GR}^{\text{lax}}, \chi^*\mathfrak{F}_{GR}^{\text{lax}}$ are pairwise lax-equivalent. The question now is whether λ^*, χ^* also preserve the BV-BFV structure. More specifically, we want to investigate how they change the kernel of the pre-boundary forms $\ker\check{\omega}$ and as such the quotient $\mathcal{F}^\partial = \check{\mathcal{F}}^\partial / \ker\check{\omega}$.

In the case of λ^* this is trivial since it is the identity. Regarding χ^*, we argue that $\ker\chi^*\check{\omega}_{GR}$ has a singular behaviour and that we cannot construct a BV-BFV theory from the data $\chi^*\mathfrak{F}_{GR}^{\text{lax}}$. Thus, although χ^* is the identity in both the BV and BV-BFV cohomologies, it spoils the BV-BFV structure of 1D GR.

5.1 Lax-equivalence of the Jacobi theory and 1D GR

The strategy to show that $\mathfrak{F}_J^{\text{lax}}$ and $\mathfrak{F}_{GR}^{\text{lax}}$ are lax-equivalent will be to use the same maps ϕ, ψ as in the bulk case (c.f. Lemmas 4.1.1 and 4.1.3). Their pullback maps ϕ^*, ψ^* will naturally extend to the complexes of local forms $\Omega_{\text{loc}}^{\bullet,\bullet}(\mathcal{F}_i^{\text{lax}} \times I)$, $i \in \{J, GR\}$, and as such to the BV-BFV complexes $(\mathfrak{BV}\text{-}\mathfrak{BFV})_i^\bullet$.

Lemma 5.1.1. *Let $\phi : \mathcal{F}_J^{\text{lax}} \to \mathcal{F}_{GR}^{\text{lax}}$ be the map from Lemma 4.1.1. Its pullback map ϕ^* is chain map between the BV-BFV complexes $(\mathfrak{BV}\text{-}\mathfrak{BFV})_{GR}^\bullet$, $(\mathfrak{BV}\text{-}\mathfrak{BFV})_J^\bullet$,*

$$\phi^* \circ (\mathcal{L}_{Q_{GR}} - \mathrm{d}) = (\mathcal{L}_{Q_J} - \mathrm{d}) \circ \phi^*.$$

Furthermore ϕ^* maps $(\theta_{GR}^\bullet, L_{GR}^\bullet)$ as

$$\phi^*\theta_{GR}^\bullet = \theta_J^\bullet,$$
$$\phi^* L_{GR}^\bullet = L_J^\bullet,$$

and preserves equations (20)

$$\iota_{Q_{GR}}\varpi^\bullet_{GR} = \delta L^\bullet_{GR} + \mathrm{d}\theta^\bullet_{GR} \xrightarrow{\phi^*} \iota_{Q_J}\varpi^\bullet_J = \delta L^\bullet_J + \mathrm{d}\theta^\bullet_J,$$

$$\iota_{Q_{GR}}\iota_{Q_{GR}}\varpi^\bullet_{GR} = 2\mathrm{d}L^\bullet_{GR} \xrightarrow{\phi^*} \iota_{Q_J}\iota_{Q_J}\varpi^\bullet = 2\mathrm{d}L^\bullet_J.$$

Proof. Since ϕ^* is a pullback it is automatically a chain map w.r.t. de Rham differentials δ, d: $\phi^*\circ\delta = \delta\circ\phi^*$, $\phi^*\circ \mathrm{d} = \phi^*\circ \mathrm{d}$ [Cat15]. Furthermore, the property that ϕ^* is also a chain map w.r.t. to the cohomological vector fields Q_i implies $\phi^*\circ\iota_{Q_{GR}} = \iota_{Q_J}\circ\phi^*$. Thus

$$\phi^*\circ\mathcal{L}_{Q_{GR}} = \phi^*\circ[\iota_{Q_{GR}},\delta] = [\iota_{Q_J},\delta]\circ\phi^* = \mathcal{L}_{Q_J}\circ\phi^*,$$

proving that ϕ^* is a chain map w.r.t. $(\mathcal{L}_{Q_i} - \mathrm{d})$.

Applying ϕ^* to $(\theta^\bullet_{GR}, L^\bullet_{GR})$ gives

$$\begin{aligned}
\phi^*\theta^1_{GR} &= \phi^*\left(q^+\cdot\delta q + \xi^+\delta\xi + g^+\delta g\right) = \tilde{q}^+\cdot\delta\tilde{q} + \tilde{\xi}^+\delta\tilde{\xi} = \theta^1_J,\\
\phi^*\theta^0_{GR} &= \phi^*\left(\frac{m\dot{q}}{\sqrt{g}}\cdot\delta q + q^+\xi\delta q + g^+\xi\delta g - (2g^+g + \xi^+\xi)\delta\xi\right)\\
&= \sqrt{\frac{E}{\tilde{T}}}m\dot{\tilde{q}}\cdot\delta\tilde{q} + \tilde{q}^+\tilde{\xi}\delta\tilde{q} + -\tilde{\xi}^+\tilde{\xi}\delta\tilde{\xi} = \theta^0_J,\\
\phi^*L^1_{GR} &= \phi^*\left(\frac{T}{\sqrt{g}} + \sqrt{g}E + q^+\cdot\xi\dot{q} + g^+(\xi\dot{g} + 2g\dot{\xi}) + \xi^+\xi\dot{\xi}\right)\\
&= 2\sqrt{E\tilde{T}} + \tilde{q}^+\cdot\tilde{\xi}\dot{\tilde{q}} + \tilde{\xi}^+\tilde{\xi}\dot{\tilde{\xi}} = L^1_J,\\
\phi^*L^0_{GR} &= \phi^*\left(\left(\frac{T}{g} - E\right)\sqrt{g}\xi\right) = 0 = L^0_J.
\end{aligned}$$

As such ϕ^* automatically preserves equations (20). $\square$

Lemma 5.1.2. *Let $\psi : \mathcal{F}^{\text{lax}}_{GR} \to \mathcal{F}^{\text{lax}}_J$ be the map from Lemma 4.1.3. Its pullback map ψ^* is chain map between the BV-BFV complexes $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_J$, $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$,*

$$\psi^*\circ(\mathcal{L}_{Q_J} - \mathrm{d}) = (\mathcal{L}_{Q_{GR}} - \mathrm{d})\circ\psi^*.$$

Furthermore, ψ^* maps $(\theta^\bullet_J, L^\bullet_J)$ as

$$\begin{aligned}
\psi^*\theta^\bullet_J &= \theta^\bullet_{GR} + (\mathcal{L}_{Q_{GR}} - \mathrm{d})\beta^\bullet + \delta f^\bullet,\\
\psi^*L^\bullet_J &= L^\bullet_{GR} + (\mathcal{L}_{Q_{GR}} - \mathrm{d})\iota_{Q_{GR}}\beta^\bullet + \mathrm{d}f^\bullet,
\end{aligned}$$

where $\beta^\bullet = \beta^1 \mathrm{d}t + \beta^0 \in \Omega^{1,\bullet}_{\mathrm{loc}}(\mathcal{F}^{\mathrm{lax}}_{GR} \times I)$ *and* $f^\bullet = f^1 \mathrm{d}t + f^0 \in \Omega^{0,\bullet}_{\mathrm{loc}}(\mathcal{F}^{\mathrm{lax}}_{GR} \times I)$ *with*

$$\begin{aligned}
\beta^1 =& -\frac{4g^{7/2}}{\Omega^2} T g^+ \delta g^+ + \left(\frac{2g^2}{\Omega} + \eta^{3/2} \frac{2\sqrt{g}}{E} \right) g^+ q_\perp^+ \cdot \delta q \\
&+ \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \dot{g}^+ g^+ \frac{m\dot{q}}{2T} \cdot \delta q - (\eta^{3/2} - 1) \xi^+ \frac{m\dot{q}}{2T} \cdot \delta q, \\
\beta^0 =& \, \xi \beta^1 + \frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q, \\
f^1 =& \, 2g^+ \left(g - \frac{2g^{3/2}}{\Omega} T \right), \\
f^0 =& \, \xi f^1,
\end{aligned}$$

and $\eta = gE/T$, $\Omega = \sqrt{g}T + g\sqrt{ET}$. *Moreover,* ψ^* *preserves equations (20)*

$$\iota_{Q_J} \varpi^\bullet_J = \delta L^\bullet_J + \mathrm{d}\theta^\bullet_J \xrightarrow{\psi^*} \iota_{Q_{GR}} \varpi^\bullet_{GR} = \delta L^\bullet_{GR} + \mathrm{d}\theta^\bullet_{GR},$$

$$\iota_{Q_J} \iota_{Q_J} \varpi^\bullet_J = 2\mathrm{d}L^\bullet_J \xrightarrow{\psi^*} \iota_{Q_{GR}} \iota_{Q_{GR}} \varpi^\bullet = 2\mathrm{d}L^\bullet_{GR}.$$

Proof. The argument to prove that ψ^* is a chain map w.r.t. $(\mathcal{L}_{Q_i} - \mathrm{d})$ is the same as for ϕ^* in Lemma 5.1.1.

The computations for $\psi^* \theta^\bullet_J$, $\psi^* L^\bullet_J$ are quite lengthy and are presented in the appendix A.5. As ψ^* transforms $\theta^\bullet_J, L^\bullet_J$ in the way presented in Remark 3.5.5, it automatically preserves equations (20). □

The result that the maps λ^*, χ^* are the identity in the respective BV cohomologies presented in Section 4.2 can be extended to the respective BV-BFV cohomology. For λ^*, this is again trivial:

Lemma 5.1.3. *The composition map* $\lambda^* : (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_J \to (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_J$ *is the identity.*

Proof. This follows directly from the bulk result presented in Lemma 4.2.1. □

In the case of χ^*, we need to define an extension of the one-parameter family of morphisms $\chi^*_s = e^{sD}$ and of the map h_χ, so that they are defined on the whole complex of local forms $\Omega^{\bullet,\bullet}_{\mathrm{loc}}(\mathcal{F}^{\mathrm{lax}}_{GR} \times I)$ and not only on integrated local functionals.

Lemma 5.1.4. *The composition map* $\chi^* : (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR} \to (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$*, whose action on* $\Phi, \Phi^+ \in \mathcal{F}^{\text{lax}}_{GR}$ *is given in Corollary (4.2.2), is homotopic to the identity on* $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$*.*

Proof. We consider an one-parameter family of morphisms $\boldsymbol{\chi}^*_s = e^{sD_{\text{lax}}}$, where $D_{\text{lax}} = [(\mathcal{L}_{Q_{GR}} - \mathrm{d}), \mathcal{L}_R]$ and $s \in \mathbb{R}_{\geq 0}$. Note that

$$D_{\text{lax}} = [(\mathcal{L}_{Q_{GR}} - \mathrm{d}), \mathcal{L}_R] = [\mathcal{L}_{Q_{GR}}, \mathcal{L}_R] = \mathcal{L}_{[Q_{GR}, R]} = \mathcal{L}_D,$$

so the morphisms $\boldsymbol{\chi}^*_s = e^{sD_{\text{lax}}}$ are just the natural extension of $\chi^*_s = e^{sD}$ to local forms.

Before proceeding, recall that the morphisms $\chi^*_s = e^{sD}$ are well-defined and converge to χ^* on $\Phi, \Phi^+ \in \mathcal{F}^{\text{lax}}_{GR}$. As such, the same holds for the space of local functionals $\Omega^{0,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}}_{GR} \times I)$, since these objects are functions on $\mathcal{F}^{\text{lax}}_{GR}$ with values on forms on I.

Let $\mathfrak{w} = f\delta\varphi_J \mathrm{d}t^K \in \Omega^{p,q}_{\text{loc}}(\mathcal{F}^{\text{lax}}_{GR} \times I)$ be a generic local form, where φ_i are local coordinates on $\mathcal{F}^{\text{lax}}_{GR}$, t the local coordinate on I, $f \in \Omega^{0,0}_{\text{loc}}(\mathcal{F}^{\text{lax}}_{GR} \times I)$ and $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$ multi-indices. Note that $q = 0, 1$. We then have

$$\begin{aligned}\boldsymbol{\chi}^*_s \mathfrak{w} &= e^{sD_{\text{lax}}} \mathfrak{w} = e^{s\mathcal{L}_D}(f\delta\varphi_J \, \mathrm{d}t^K) \\ &= (e^{s\mathcal{L}_D} f)\delta(e^{s\mathcal{L}_D}\varphi_J) \, \mathrm{d}t^K = (e^{sD} f)\delta(e^{sD}\varphi_J) \, \mathrm{d}t^K,\end{aligned}$$

which in turn implies

$$\begin{aligned}\lim_{s\to\infty} \boldsymbol{\chi}^*_s \mathfrak{w} &= \lim_{s\to\infty} (e^{sD} f)\delta(e^{sD}\varphi_J) \, \mathrm{d}t^K = (\chi^* f)\delta(\chi^*\varphi_J) \, \mathrm{d}t^K \\ &= \chi^*(f\delta\varphi_J \, \mathrm{d}t^K) = \chi^* \mathfrak{w},\end{aligned}$$

where we used that $\lim_{s\to\infty} \chi^*_s = \chi^*$ on $\Omega^{0,0}_{\text{loc}}(\mathcal{F}^{\text{lax}}_{GR} \times I)$. Since $\boldsymbol{\chi}^*_{s=0} = \mathrm{id}_{GR}$, the map χ^* is homotopic to the identity on the complex of local forms $\Omega^{\bullet,\bullet}_{\text{loc}}(\mathcal{F}^{\text{lax}}_{GR} \times I)$ and as such also on the BV-BFV complex $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$. □

Lemma 5.1.5. *The composition map* $\chi^* : (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR} \to (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$ *is the identity on* $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR})$*.*

Proof. This proof follows the same train of thought as the proof in the bulk case (Lemma 4.2.13). In order to show that χ^* is the identity on $H^\bullet((\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR})$, we need to find a map

$$\boldsymbol{h}_\chi : (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR} \to (\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR},$$

of degree -1 such that

$$\chi^* - \mathrm{id}_{GR} = (\mathcal{L}_{Q_{GR}} - \mathrm{d})\boldsymbol{h}_\chi + \boldsymbol{h}_\chi(\mathcal{L}_{Q_{GR}} - \mathrm{d}). \tag{37}$$

Similarly as before, we have $[D_{\mathrm{lax}}, (\mathcal{L}_{Q_{GR}} - \mathrm{d})] = 0$ since

$$[D_{\mathrm{lax}}, (\mathcal{L}_{Q_{GR}} - \mathrm{d})] = [\mathcal{L}_D, (\mathcal{L}_{Q_{GR}} - \mathrm{d})] = \mathcal{L}_{[D, Q_{QR}]} = 0,$$

where we used that D is evolutionary and $[D, Q_{GR}] = 0$ (c.f. equation (36)). As such

$$\begin{aligned}
\chi^* - \mathrm{id}_{GR} &= \int_0^\infty \frac{\mathrm{d}}{ds} e^{sD_{\mathrm{lax}}} \mathrm{d}s = \int_0^\infty e^{sD_{\mathrm{lax}}} D_{\mathrm{lax}} \mathrm{d}s \\
&= \int_0^\infty e^{sD_{\mathrm{lax}}}((\mathcal{L}_{Q_{GR}} - \mathrm{d})\mathcal{L}_R + \mathcal{L}_R(\mathcal{L}_{Q_{GR}} - \mathrm{d}))\mathrm{d}s \\
&= (\mathcal{L}_{Q_{GR}} - \mathrm{d}) \left(\int_0^\infty e^{sD_{\mathrm{lax}}} \mathcal{L}_R \mathrm{d}s \right) + \left(\int_0^\infty e^{sD_{\mathrm{lax}}} \mathcal{L}_R \mathrm{d}s \right) (\mathcal{L}_{Q_{GR}} - \mathrm{d}).
\end{aligned}$$

Let $\mathfrak{w} = f\delta\varphi_J \mathrm{d}t^K \in \Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}_{GR} \times I)$. Since $D_{\mathrm{lax}} = \mathcal{L}_D$ we define $\boldsymbol{h}_\chi \mathfrak{w}$ as

$$\boldsymbol{h}_\chi \mathfrak{w} = \int_0^\infty e^{s\mathcal{L}_D} \mathcal{L}_R \mathfrak{w} \mathrm{d}s.$$

Note that for $f \in \Omega^{0,0}_{\mathrm{loc}}(\mathcal{F}^{\mathrm{lax}}_{GR} \times I)$ we have $\boldsymbol{h}_\chi f = h_\chi f$, where h_χ is the map constructed in Lemma 4.2.13. All that is left to show now is that $\boldsymbol{h}_\chi$ converges on $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$. As before we reparameterize $s = -\ln\tau$ with $\tau \in [0, 1]$. We then have

$$\begin{aligned}
\boldsymbol{h}_\chi \mathfrak{w} &= \int_0^\infty e^{s\mathcal{L}_D} \mathcal{L}_R \mathfrak{w} \mathrm{d}s = \int_1^0 e^{-\ln(\tau)\mathcal{L}_D} \mathcal{L}_R \mathfrak{w} \mathrm{d}(-\ln(\tau)) \\
&= \int_0^1 \frac{e^{-\ln(\tau)\mathcal{L}_D}}{\tau} \mathcal{L}_R \mathfrak{w} \mathrm{d}\tau.
\end{aligned}$$

Setting $\mathfrak{w} = f\delta\varphi_I \mathrm{d}t^K$ yields

$$\begin{aligned}
\boldsymbol{h}_\chi \mathfrak{w} &= \int_0^1 \frac{e^{-\ln(\tau)\mathcal{L}_D}}{\tau} \mathcal{L}_R[f\delta\varphi_I \mathrm{d}t^K] \mathrm{d}\tau = \int_0^1 \frac{e^{-\ln(\tau)\mathcal{L}_D}}{\tau} [\mathcal{L}_R f \delta\varphi_I \pm f \mathcal{L}_R \delta\varphi_I] \mathrm{d}t^K \mathrm{d}\tau \\
&= \int_0^1 \left[\left\{ \frac{e^{-\ln(\tau)D}}{\tau} Rf \right\} \delta e^{-\ln(\tau)D} \varphi_I \mp e^{-\ln(\tau)D} f \delta \left\{ \frac{e^{-\ln(\tau)D}}{\tau} R\varphi_I \right\} \right] \mathrm{d}t^K \mathrm{d}\tau.
\end{aligned}$$

The terms in the brackets $\{\cdot\}$ are just the integrands of $h_\chi f$ and $h_\chi \varphi_I$, which converge as shown in Lemma 4.2.13. Since the other terms $e^{-\ln(\tau)D}\varphi_I = e^{sD}\varphi_I$, $e^{-\ln(\tau)D}f = e^{sD}f$ are well-defined $\forall \tau \in [0,1]$, and we are integrating over a compact interval, the integral converges and $\boldsymbol{h}_\chi$ is well-defined on $(\mathfrak{BV}\text{-}\mathfrak{BFV})^\bullet_{GR}$. □

We have now gathered all the ingredients we need in order to show the following:

Theorem 5.1.6. *The lax theories $\mathfrak{F}^{\mathrm{lax}}_J$ and $\mathfrak{F}^{\mathrm{lax}}_{GR}$ of the Jacobi theory and 1D GR are lax-equivalent.*

Proof. Recall that two lax theories $\mathfrak{F}^{\mathrm{lax}}_J$, $\mathfrak{F}^{\mathrm{lax}}_{GR}$ are lax-equivalent if there exist two maps ϕ, ψ such that

- ϕ^*, ψ^* are chain maps between the BV-BFV complexes,
- the composition maps $\lambda^* = \phi^* \circ \psi^*$, $\chi^* = \psi^* \circ \phi^*$ are the identity in the BV-BFV cohomologies,
- ϕ^*, ψ^* interchange $(\theta^\bullet_i, L^\bullet_i)$ up to $(\mathcal{L}_{Q_i} - \mathrm{d})$-exact terms and an f-transformation and preserve equations (20),

The maps ϕ, ψ with the desired properties were constructed in Lemmas 5.1.1 and 5.1.2, where we also show that these maps interchange $(\theta^\bullet_i, L^\bullet_i)$ and preserve equations (20) in the desired way. Furthermore, in Lemmas 5.1.3 and 5.1.5 we showed that the composition maps λ^*, χ^* are the identity when restricted to respective BV-BFV cohomologies. □

5.2 Effect on the BV-BFV structure of 1D GR

We now address the question of what happens to the BV-BFV structure of the lax 1D GR theory $\mathfrak{F}^{\mathrm{lax}}_{GR}$ when pull back the data via χ^*.

Theorem 5.2.1. *The lax-theory $\chi^*\mathfrak{F}^{\mathrm{lax}}_{GR} := (\mathcal{F}^{\mathrm{lax}}_{GR}, \chi^*\theta^\bullet_{GR}, \chi^* L^\bullet_{GR}, Q_{GR})$ does not yield a BV-BFV theory.*

Proof. Recall that pulling back $(\theta^\bullet_{GR}, L^\bullet_{GR})$ with ϕ^* gives $\phi^*\theta^\bullet_{GR} = \theta^\bullet_J$ and $\phi^* L^\bullet_{GR} = L^\bullet_J$ (see Lemma 5.1.1). Applying the map $\chi^* = \psi^* \circ \phi^*$ to $(\theta^\bullet_{GR}, L^\bullet_{GR})$ then yields

$$\chi^*\theta^\bullet_{GR} = (\psi^* \circ \phi^*)\theta^\bullet_{GR} = \psi^*\theta^\bullet_J = \theta^\bullet_J[\psi^*\tilde{q}, \psi^*\tilde{q}^+, \psi^*\tilde{\xi}, \psi^*\tilde{\xi}^+],$$
$$\chi^* L^\bullet_{GR} = (\psi^* \circ \phi^*)L^\bullet_{GR} = \psi^* L^\bullet_J = L^\bullet_J[\psi^*\tilde{q}, \psi^*\tilde{q}^+, \psi^*\tilde{\xi}, \psi^*\tilde{\xi}^+].$$

Thus, the lax data of $\chi^*\mathfrak{F}_{GR}^{\text{lax}}$ has the same form as the lax data for the Jacobi theory presented in Example 3.4.8 on the submanifold of $\mathcal{F}_{GR}^{\text{lax}}$ with local coordinates $\{\psi^*\tilde{q}, \psi^*\tilde{q}^+, \psi^*\tilde{\xi}, \psi^*\tilde{\xi}^+\}$, since

$$\begin{aligned}
\chi^*\theta_{GR}^1 &= (\psi^*\tilde{q}^+)\cdot\delta(\psi^*\tilde{q}) + (\psi^*\tilde{\xi}^+)\delta(\psi^*\tilde{\xi}),\\
\chi^*\theta_{GR}^0 &= \psi^*\left(\sqrt{\frac{E}{\tilde{T}}}m\dot{\tilde{q}}\right)\cdot\delta(\psi^*\tilde{q}) + (\psi^*\tilde{q}^+)(\psi^*\tilde{\xi})\delta(\psi^*\tilde{q}) - (\psi^*\tilde{\xi}^+)(\psi^*\tilde{\xi})\delta(\psi^*\tilde{\xi}),\\
\chi^*L_{GR}^1 &= 2\psi^*\left(\sqrt{E\tilde{T}}\right) + (\psi^*\tilde{q}^+)\cdot(\psi^*\tilde{\xi})(\psi^*\dot{\tilde{q}}) + (\psi^*\tilde{\xi}^+)(\psi^*\tilde{\xi})(\psi^*\dot{\tilde{\xi}}),\\
\chi^*L_{GR}^1 &= 0.
\end{aligned}$$

Furthermore, applying χ^* to equations (20) for the lax formulation of 1D GR yields

$$\begin{aligned}
\iota_{Q_{GR}}\psi^*\varpi_J^\bullet &= \delta\psi^*L_J^\bullet + \mathrm{d}\psi^*\theta_J^\bullet,\\
\iota_{Q_{GR}}\iota_{Q_{GR}}\psi^*\varpi_J^\bullet &= 2\,\mathrm{d}\psi^*L_J^\bullet.
\end{aligned}$$

This means that the theory $\chi^*\mathfrak{F}_{GR}^{\text{lax}}$ is just a version of Jacobi theory in the 1D GR space, which is defined on a submanifold of $\mathcal{F}_{GR}^{\text{lax}}$ with local coordinates $\{\psi^*\tilde{q}, \psi^*\tilde{q}^+, \psi^*\tilde{\xi}, \psi^*\tilde{\xi}^+\}$. This theory will have the same behaviour as the original Jacobi theory and as such the kernel of the pre-boundary 2-form

$$\begin{aligned}
\chi^*\check{\omega}_{GR} &= \int_{\partial I}\delta\chi^*\theta_{GR}^1\mathrm{d}t = \int_{\partial I}\delta\psi^*\theta_J^1\mathrm{d}t = \psi^*\check{\omega}_J\\
&= \check{\omega}_J[\psi^*\tilde{q}, \psi^*\tilde{q}^+, \psi^*\tilde{\xi}, \psi^*\tilde{\xi}^+],
\end{aligned}$$

will be singular, just as the kernel of the pre-boundary 2-form $\check{\omega}_J$ of the Jacobi theory [CS17b]. As such, the data $\chi^*\mathfrak{F}_{GR}^{\text{lax}}$ does not yield a BV-BFV theory. □

In this chapter we have seen that the BV-equivalence of the Jacobi theory and 1D GR extends to lax-equivalence in the presence of a boundary. Furthermore, we have shown that the 1D GR data can be transformed in a way such that its cohomological structure is preserved, while the behaviour of the kernel of the pre-boundary 2-form is altered. After the transformation, $\ker\chi^*\check{\omega}_{GR}$ is singular and as such the transformed version of 1D GR is not compatible with the BV-BFV regularity conditions.

We have thus presented two pairs of theories, $(\mathfrak{F}_J^{\text{lax}}, \mathfrak{F}_{GR}^{\text{lax}})$ and $(\mathfrak{F}_{GR}^{\text{lax}}, \chi^*\mathfrak{F}_{GR}^{\text{lax}})$, which have isomorphic BV-BFV cohomologies, but differ in terms of their compatibility with the BV-BFV axioms. As the BV-BFV quantization program is only possible when these regularity conditions are met, these results suggest that theories with equivalent classical BV and lax structures might yield different quantum theories on manifolds with boundary.

Conclusions

In this thesis we addressed the topic of equivalence of field theories in the BV and BV-BFV setting. More specifically, we investigated the examples of the Jacobi theory and one-dimensional gravity coupled to matter (1D GR), which although classically equivalent, assign different structures to boundaries when studied under the lens of the BV-BFV formalism. While the Jacobi theory produces a singular boundary theory, 1D GR fulfills the regularity conditions and as such is compatible with the BV-BFV procedure. This kind of behaviour is also present in higher dimensional parameterization invariant models, such as the Nambu-Goto and Polyakov actions [Mar20] and Einstein-Hilbert gravity and Palatini-Cartan gravity in (3+1) dimensions [CS16; CS17a].

The question then is whether this kind of discrepancy is reflected in the bulk BV theories or even in the pre-boundary data, which already entails relevant information about the boundary theory and can be defined prior to symplectic reduction.

Motivated by the fact that the classical observables of a BV theory are encoded in its BV cohomology, we tackled this problem on the bulk by proposing a notion of BV-equivalence which aims at putting the BV data and BV cohomology in the foreground. More precisely, we said that two BV theories are BV-equivalent (c.f. Definition 3.2.1) if there exist two maps which interchange the BV forms ω_i and BV actions $\mathcal{S}_i$, $i \in \{1, 2\}$, in a way that preserves the Hamiltonian conditions $\iota_{Q_i}\omega_i = \delta\mathcal{S}_i$. Furthermore, we required that the pullback maps of these maps are chain maps between the BV complexes, such that their composition maps are the identity in the respective BV cohomologies. If these requirements are met, then the two theories have isomorphic cohomologies $H^\bullet(\mathfrak{BV}_1^\bullet) \simeq H^\bullet(\mathfrak{BV}_2^\bullet)$ and thus share the same set of classical observables.

We then proceeded to show that the BV formulations of the Jacobi theory and 1D GR are BV-equivalent (Theorem 4.2.14) by explicitly constructing the maps in question and showing that they fulfill the desired requirements. In our case it turned out that one of the aforementioned composition maps was already the identity at the level of the BV complexes and as such also in the BV cohomology. For the other composition map χ^*, we found a degree -1 evolutionary vector field R such that the one-parameter family of maps

$$\chi_s^* = e^{s(Q_{GR}R + RQ_{GR})},$$

reproduces the identity at $s = 0$ and χ^* in the limit $s \to \infty$, thus showing that χ^* is homotopic to the identity and as such the identity in cohomology. This procedure could potentially be generalized or applied to other cases where one wishes to compare two BV theories, for example the already mentioned Nambu-Goto and Polyakov actions, Einstein-Hilbert and Palatini-Cartan gravity or the first and second order formulations of Yang-Mills theory.

With the results for bulk equivalence in hand, we analyzed the case with a boundary by using the lax formalism (c.f. Section 3.4), which allows us to collect the boundary data prior to symplectic reduction. As such, it let us consider the boundary structure of a given theory even if it is not compatible with the BV-BFV procedure, as in the case of the Jacobi theory. In this setting, the classical observables are encoded in the BV-BFV cohomology. The notion of equivalence in the presence of a boundary (Definition 3.5.3) is very similar to the one used in the bulk case, and we were able to show that the maps used for BV-equivalence also give us the desired conditions for equivalence in the lax setting (Theorem 5.1.6). Most notably, they also induce an isomorphism between the BV-BFV cohomologies through the composition maps of their pullback maps. We then showed that pulling back the lax 1D GR data via the composition map χ^* changes the boundary data in such a way, that the resulting theory is not compatible with the BV-BFV procedure (Theorem 5.2.1).

Taking all of these considerations into account, the results of the present work seem to imply that, in the case of the Jacobi theory and 1D GR, their compatibility with the BV-BFV regularity conditions on the boundary is not captured by on their sets of classical observables. On the other hand, the BV-BFV quantization program is only possible if these regularity conditions are met, thus suggesting that 1D GR is better suited for covariant quantization on manifolds with boundary than the classically equivalent Jacobi theory.

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A Lengthy Calculations

A.1 Preliminaries for calculations - tensor number

This appendix has two purposes. It serves as a preliminary for the computations in the rest of the appendix, by presenting a straight-forward way to compute the action of the Chevalley-Eilenberg differentials γ_J, γ_{GR} and explains why they act as $\mathcal{L}_{\xi\partial_t}$ on the antifields and antighosts (see Remarks 3.1.10 and 3.1.12). We will be using the 1D GR theory in this discussion but all considerations hold for the Jacobi theory as well.

Let M be a manifold of arbitrary dimension and $X = X^\sigma \partial_\sigma \in \mathfrak{X}(M)$. Recall that the Lie derivative $\mathcal{L}_X$ acts on the components of a tensor field $\mathbf{A} \in \mathcal{T}^n_m(M)$ of rank-(n,m) as

$$\begin{aligned}\mathcal{L}_X \mathbf{A}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = X^\sigma \partial_\sigma \mathbf{A}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} - \partial_\sigma X^{\mu_1} \mathbf{A}^{\sigma\ldots\mu_n}_{\nu_1\ldots\nu_m} - \cdots - \partial_\sigma X^{\mu_n} \mathbf{A}^{\mu_1\ldots\sigma}_{\nu_1\ldots\nu_m} \\ + \partial_{\nu_1} X^\sigma \mathbf{A}^{\mu_1\ldots\mu_n}_{\sigma\ldots\nu_m} + \cdots + \partial_{\nu_n} X^\sigma \mathbf{A}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\sigma}.\end{aligned} \tag{38}$$

Let now $M = I$ denote an interval (i.e. $\dim M = 1$) and $X = \xi\partial_t \in \mathfrak{X}(I)[I]$ be the ghost field. In this setting Equation (38) is greatly simplified since $\mu_i = \nu_i = t$, where t is the coordinate on I. Let $\mathbf{A} \in \mathcal{T}^n_m(I)$ and denote its component by A. We define *tensor number* as $t(A) = (m-n)$. We then have

$$\begin{aligned}\mathcal{L}_{\xi\partial_t} A &= \xi\partial_t A - n\partial_t\xi A + m\partial_t\xi A \\ &= \xi\dot{A} + t(A)\dot{\xi}A.\end{aligned} \tag{39}$$

As an example we list the tensor number for the fields, ghosts, antifields and antighosts of the 1D GR theory (see Example 3.1.11)

$$\begin{aligned}&t(q) = 0-0 = 0, \qquad t(g) = 2-0 = 2, \qquad t(\xi) = 0-1 = 0, \\ &t(q^+) = 1-0 = 0, \qquad t(g^+) = 1-2 = 0, \qquad t(\xi^+) = 2-0 = 2,\end{aligned} \tag{40}$$

which explains why we claimed that the Chevalley-Eilenberg differential γ_{GR} acts as $\mathcal{L}_{\xi\partial_t}$ on the antifields and antighosts in Remarks 3.1.10 and 3.1.12. As such we have $\gamma_{GR} = \mathcal{L}_{\xi\partial_t}$ on all the functions on $\{q, g, q^+, g^+, \xi^+\}$ and $\gamma_{GR} = \frac{1}{2}\mathcal{L}_{\xi\partial_t}$ on the ghost. Since the ghost is a special case, we assume that the tensor fields only depend on $\{q, g, q^+, g^+, \xi^+\}$ for the rest of the discussion. When computing $\gamma_{GR}(\cdot)$, we then consider the parts with ghosts and without separately.

The discussion until now only holds for tensor fields that only depend on the 0th-jets of $\{q, g, q^+, g^+, \xi^+\}$. The action of γ_{GR} is then naturally extended to all jets since we assume that Q, and as such γ_{GR}, is evolutionary, i.e. $[\gamma_{GR}, \mathcal{L}_{\mathrm{d}}] = 0$. For example, if A only depends on 0th-jets then

$$\begin{aligned} \gamma_{GR}\dot{A} &= \partial_t \gamma_{GR} A = \partial_t[\xi \dot{A} + t(A)\dot{\xi} A] \\ &= \xi \ddot{A} + [1 + t(A)]\dot{\xi}\dot{A} + t(A)\ddot{\xi} A. \end{aligned} \tag{41}$$

For a general tensor field $\mathcal{A}$ which depends on arbitrary jets of the fields we have

$$\gamma_{GR}\mathcal{A} = \xi \dot{\mathcal{A}} + \sum_{n \geq 1} t_n(\mathcal{A}) \partial_t^n \xi a_n, \tag{42}$$

for some real scalars $t_n(\mathcal{A})$ and some functions a_n that depend on the jets of $\Phi, \Phi^+ \in \mathcal{F}_{GR}$. In order to extend the notion of tensor number to such objects we define

Definition A.1.1. Let $\mathcal{A}$ be a tensor field that depends on arbitrary jets of $\Phi, \Phi^+ \in \mathcal{F}_{GR}$. The tensor number $t(\mathcal{A})$ of $\mathcal{A}$ is defined as the scalar $t_1(\mathcal{A})$ in Equation (42).

Note that in order to compute γ_{GR} we only have to find out what the $t_n(\mathcal{A})$ are. For most of the computations we are only going to encounter tensor fields that depend on the 0-th jets, and they will atmost include 2nd-jets. As such we want to find a pragmatic way of computing $t(\mathcal{A})$. If necessary, we then look at higher $t_n(\mathcal{A})$, for example by following Equation (41). We list some useful properties of $t(\cdot)$, since they immensely simplify the explicit computations of $\gamma_{GR}(\mathcal{A})$.

Proposition A.1.2. *Let $\mathcal{A}, \mathcal{B}$ be two tensor fields that depend on an arbitrary number of jets of $\Phi, \Phi^+ \in \mathcal{F}_{GR}$. The tensor number has the following properties:*

1. $t(\mathcal{A}\mathcal{B}) = t(\mathcal{A}) + t(\mathcal{B})$,
2. $t(\mathcal{A}^n) = nt(\mathcal{A})$,
3. $t(\dot{\mathcal{A}}) = 1 + t(\mathcal{A})$.

Proof. Note that the only two terms from Equation (42) that can contribute to these properties are the first two. Therefore we will only show the computations for two tensor fields A, B that only depend on the 0th-jets, but they extend to the general case in a straightforward way.

1. Let $gh(A) = a$. We compute

$$\begin{aligned}\gamma_{GR}(AB) &= (\mathcal{L}_{\xi\partial_t}A)B + (-1)^a A(\mathcal{L}_{\xi\partial_t}B)\\ &= (\xi\dot{A} + t(A)\dot{\xi}A)B + (-1)^a A(\xi\dot{B} + t(B)\dot{\xi}B)\\ &= \xi\partial_t(AB) + [t(A) + t(B)]\dot{\xi}(AB).\end{aligned}$$

2. Using that γ_{GR} is a derivative we see that

$$\gamma_{GR}A^n = nA^{n-1}\gamma_{GR}A = nA^{n-1}[\xi\dot{A} + t(A)\dot{\xi}A] = \xi\partial_t A^n + nt(A)\dot{\xi}A^n.$$

3. This equality follows directly from Equation (41).

□

We finish this section by presenting the action of γ_{GR} tensor numbers for some relevant quantities

$$\begin{aligned}&t(\dot{q}) = 1 + t(q) = 1,\\ &t(T) = t(\|\dot{q}\|^2) = 2t(\dot{q}) = 2,\\ &t(u) = t\left(\frac{\dot{q}}{\|q\|}\right) = t(\dot{q}) - t(\|\dot{q}\|) = 0,\\ &t(EL_g) = t\left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}\right) = t\left(\frac{1}{\sqrt{g}}\right) = -\frac{1}{2}t(g) = -1.\end{aligned} \tag{43}$$

We exemplify this method of calculating $\gamma_{GR}(\mathcal{A})$ with the computation of $\mathcal{A} = T = \frac{m}{2}\|\dot{q}\|^2$. Recall that $\gamma_{GR}q = \xi\dot{q}$ and as such $\gamma_{GR}\dot{q} = \xi\ddot{q} + \dot{\xi}\dot{q}$. $\gamma_{GR}T$ could potentially have terms proportional to $\ddot{\xi}$ since it depends on the derivative $\dot{q}$, but since there are no such terms in $\gamma_{GR}\dot{q}$ there won't be any in $\gamma_{GR}T$. As such we have

$$\gamma_{GR}T = \xi\dot{T} + t(T)\dot{\xi}T = \xi\dot{T} + 2\dot{\xi}T.$$

A.2 Chain map condition for ψ^*

In this section we present the computations that complete the proof of Lemma 4.1.3, namely we show that the pullback map of the map $\psi : \mathcal{F}_{GR} \to$

$\mathcal{F}_J$ given by

$$\begin{aligned}
&\psi^* \tilde{q} = q,\\
&\psi^* \tilde{\xi} = \xi,\\
&\psi^* \tilde{q}_\parallel^+ = \eta^{3/2} \left(q_\parallel^+ - \left[g^+ \dot{g} + 2\dot{g}^+ g \right] \frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E} \left[E\dot{L_g} g^+ - EL_g \dot{g}^+ \right] \frac{m\dot{q}}{2T} \right),\\
&\psi^* \tilde{q}_\perp^+ = \eta^{3/2} \left(q_\perp^+ + \frac{2m}{E} g^+ \ddot{q}_\perp \right),\\
&\psi^* \tilde{\xi}^+ = \eta^{3/2} \left(\xi^+ + \frac{g^{3/2}}{E} \dot{g}^+ g^+ \right),
\end{aligned}$$

fulfills the chain map condition

$$\psi^* \circ Q_J = Q_{GR} \circ \psi^*.$$

In the case of the fields $\{\tilde{q}, \tilde{\xi}\}$ we simply compute

$$\begin{aligned}
\psi^* Q_J \tilde{q} &= \psi^*(\tilde{\xi}\dot{\tilde{q}}) = \xi \dot{q} = Q_{GR} q = Q_{GR} \psi^* \tilde{q},\\
\psi^* Q_J \tilde{\xi} &= \psi^*(\tilde{\xi}\dot{\tilde{\xi}}) = \xi \dot{\xi} = Q_{GR} \xi = Q_{GR} \psi^* \tilde{\xi}.
\end{aligned}$$

When dealing with the antifields $\{\tilde{q}^+, \tilde{\xi}^+\}$ it is useful to first show that ψ^* is a chain map w.r.t. to the Chevalley-Eilenberg differentials and then proceed to show that is also fulfills this condition w.r.t. the Koszul-Tate differentials.

In the case of the Chevalley-Eilenberg differentials γ_J, γ_{GR} it is sufficient to investigate how ψ^* changes the tensorial properties of the fields, i.e. to analyse the tensor number introduced Section A.1. Indeed using $\psi^* \tilde{\xi} = \xi$ we can compute

$$\begin{aligned}
\psi^* \gamma_J \tilde{\Phi}^+ &= \psi^*(\tilde{\xi}\dot{\tilde{\Phi}}^+ + t(\tilde{\Phi}^+)\dot{\tilde{\xi}}\tilde{\Phi}^+)\\
&= \xi \partial_t(\psi^* \tilde{\Phi}^+) + t(\tilde{\Phi}^+)\dot{\xi}\psi^* \tilde{\Phi}^+.
\end{aligned} \tag{44}$$

The most general form of the other side of the chain map condition $\gamma_{GR}\psi^*\tilde{\Phi}^+$ is given by

$$\gamma_{GR}(\psi^* \tilde{\Phi}^+) = \xi \partial_t(\psi^* \tilde{\Phi}^+) + t(\psi^* \tilde{\Phi}^+)\dot{\xi}\psi^* \tilde{\Phi}^+ + \sum_{n\geq 2} \partial_t^n \xi a_n, \tag{45}$$

where the field dependent coefficients a_n do not vanish trivially since the expressions for $\psi^*\tilde{\Phi}^+$ depend on derivative terms such as $\dot{g}, \dot{g}^+$, and $\dot{EL}_g$. In order to show that the two sides of the chain map condition given in Equations (44) and (45) are equal we need prove that ψ^* preserves the tensor number $t(\cdot)$ and that the coefficients a_n vanish

$$t(\tilde{\Phi}^+) = t(\psi^*\tilde{\Phi}^+), \qquad\qquad a_n = 0.$$

Recalling that $t(g) = t(T) = 2$, we see that the rescaling factor $\eta^{3/2}$ has vanishing tensor number

$$t(\eta) = t\left(\frac{g}{T}\right) = t(g) - t(T) = 0.$$

Furthermore, since $t_{n\geq 2}(g) = t_{n\geq 2}(T) = 0$, we have $\gamma_{GR}\eta = \xi\dot{\eta}$ and thus it can be ignored, since it neither changes $t(\cdot)$ nor $t_{n\geq 2}(\cdot)$.

We start by showing $\psi^* \circ \gamma_J = \gamma_{GR} \circ \psi^*$ on the antifield $\tilde{q}^+$. Using Equations (40), (41) and (43) it then follows that all the terms in $\psi^*\tilde{q}_\parallel^+$ have tensor number 1

$$\begin{aligned}
&t(q_\parallel^+) = t(u(u\cdot q^+)) = t(q^+) + 2t(u) = 1,\\
&t\left(g^+\dot{g}\frac{m\dot{q}}{2T}\right) = -1 + (2+1) + (0+1-2) = 1,\\
&t\left(g^+\frac{g^{3/2}}{E}\dot{EL}_g\frac{m\dot{q}}{2T}\right) = -1 + \frac{3}{2}\cdot 2 + (-1+1) + (0+1-2) = 1,\\
&t\left(\dot{g}^+ g\frac{m\dot{q}}{2T}\right) = (-1+1) + 2 + (0+1-2) = 1,\\
&t\left(\dot{g}^+\frac{g^{3/2}}{E}EL_g\frac{m\dot{q}}{2T}\right) = (-1+1) + \frac{3}{2}\cdot 2 - 1 + (0+1-2) = 1,
\end{aligned}$$

showing that $t(\psi^*\tilde{q}_\parallel^+) = 1 = t(\tilde{q}_\parallel^+)$. We still need to check what happens with the terms in $\gamma_{GR}\psi^*\tilde{q}_\parallel^+$ that are proportional to $\ddot{\xi}$. Using Equation (41) we can see that they take the form

$$\ddot{\xi}\left(-\left[g^+(2g) - 2g^+g\right]\frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E}\left[-EL_g g^+ + EL_g g^+\right]\right) = 0,$$

and thus $\psi^*\gamma_J\tilde{q}_\parallel^+ = \gamma_{GR}\psi^*\tilde{q}_\parallel^+$.

In the case of $\tilde{q}_\perp^+$ we have $t(q_\perp^+) = t(q^+) + 2t(u) = t(q^+)$ since $t(u) = 0$. The same reasoning applies to $\ddot{q}_\perp$, but here we need to consider terms which are proportional to higher derivatives of the ghost since

$$\gamma_{GR}\ddot{q} = \partial_t^2(\gamma_{GR} q) = \partial_t^2(\xi\dot{q}) = \xi\dddot{q} + 2\dot{\xi}\ddot{q} + \ddot{\xi}\dot{q}.$$

The terms proportional to $\ddot{\xi}$ in $\gamma_{GR}\psi^*\tilde{q}_\perp$ come from $\gamma_{GR}\ddot{q}$ and $u(u\cdot\gamma_{GR}\ddot{q})$. As such they take the form

$$\ddot{\xi}\,(\dot{q} - u(\dot{q}\cdot u)) = \ddot{\xi}\,(\dot{q} - \dot{q}) = 0,$$

hence showing that there are no terms proportional to $\ddot{\xi}$ in $\gamma_{GR}\psi^*\tilde{q}_\perp^+$. Furthermore, $t(g^+\ddot{q}_\perp) = -1 + 2 = 1$ and as such $\psi^*\gamma_J\tilde{q}_\perp^+ = \gamma_{GR}\psi^*\tilde{q}_\perp^+$. In order to check the chain map condition for $\tilde{\xi}^+$ first note that

$$\gamma_{GR}\dot{g}^+ = \partial_t(\xi\dot{g}^+ - \dot{\xi}g^+) = \xi\ddot{g}^+ - \ddot{\xi}g^+,$$

since $\gamma_{GR} g^+ = \xi\dot{g}^+ - \dot{\xi}g^+$. This in turn implies that

$$\begin{aligned}\gamma_{GR}(\dot{g}^+g^+) &= \gamma_{GR}\dot{g}^+g^+ - \dot{g}^+\gamma_{GR}g^+ = \xi\ddot{g}^+g^+ - \dot{\xi}\dot{g}^+g^+\\ &= \xi\partial_t(\dot{g}^+g^+) - \dot{\xi}\dot{g}^+g^+.\end{aligned}$$

As such $t(\dot{g}^+g^+) = -1$ and $t(g^{3/2}\dot{g}^+g^+) = \frac{3}{2}\cdot 2 - 1 = 2$. Furthermore, $t(\xi^+) = 2$ then means that $t(\psi^*\tilde{\xi}^+) = t(\tilde{\xi}^+) = 2$ and since there are no other derivative terms in $\psi^*\tilde{\xi}^+$ we have $a_n = 0$, which completes the proof for

$$\psi^*\circ\gamma_J = \gamma_{GR}\circ\psi^*.$$

We now move to the Koszul-Tate differentials. In the case of $\tilde{q}_\parallel^+$ we first note that

$$\psi^*\delta_J\tilde{q}_\parallel^+ = \psi^*\left(\delta_J(\tilde{q}^+\cdot\dot{\tilde{q}})\frac{m\dot{\tilde{q}}}{2\tilde{T}}\right) = -\psi^*\left(\delta_J^2\tilde{\xi}^+\frac{m\dot{\tilde{q}}}{2\tilde{T}}\right) = \psi^*(0) = 0,$$

since $\delta_J\tilde{\xi}^+ = -\tilde{q}^+\cdot\dot{\tilde{q}}$ and $\delta_J^2 = 0$. The term $\delta_{GR}\psi^*\tilde{q}_\parallel^+$ vanishes for a similar reason

$$\begin{aligned}\delta_{GR}\psi^*(\tilde{q}_\parallel^+) &= \delta_{GR}\left\{\eta^{3/2}\left(q_\parallel^+ - \left[g^+\dot{g} + 2\dot{g}^+g\right]\frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\frac{m\dot{q}}{2T}\right)\right\}\\ &= -\eta^{3/2}\delta_{GR}^2\left\{\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+g^+\right\}\frac{m\dot{q}}{2T} = 0,\end{aligned}$$

where we used that $\delta_{GR}\xi^+ = -q^+ \cdot \dot{q} + g^+ \dot{g} + 2\dot{g}^+ g$ and $\delta_{GR}(\dot{g}^+ g^+) = \dot{EL}_g g^+ - \dot{g}^+ EL_g$.

The computations for the perpendicular part of $\tilde{q}^+$ go as follows

$$\begin{aligned}
\psi^* \delta_J \tilde{q}_\perp^+ &= \psi^* \left(-\partial_t \left(\sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \right) + \tilde{u}\partial_t \left(\sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \right) \cdot \tilde{u} \right) \\
&= \psi^* \left(\frac{1}{2} \left(\frac{E}{\tilde{T}} \right)^{3/2} \frac{\dot{\tilde{T}}}{E} m\dot{\tilde{q}} - \sqrt{\frac{E}{\tilde{T}}} m\ddot{\tilde{q}} + \tilde{u}\sqrt{2mE} \underbrace{\dot{\tilde{u}} \cdot \tilde{u}}_{=0} \right) \\
&= \left(\frac{E}{T} \right)^{3/2} \left(\frac{\dot{T}}{2E} m\dot{q} - \frac{T}{E} m\ddot{q} \right),
\end{aligned}$$

the other side of the equation reads

$$\begin{aligned}
\delta_{GR}\psi^* \tilde{q}_\perp^+ &= \eta^{3/2} \delta_{GR} \left(q_\perp^+ + \frac{2m}{E} \ddot{q}_\perp g^+ \right) \\
&= \eta^{3/2} \left(-\partial_t \left(\frac{m\dot{q}}{\sqrt{g}} \right) + u\partial_t \left(\frac{m\dot{q}}{\sqrt{g}} \right) \cdot u + \frac{2m}{E} \ddot{q}_\perp EL_g \right) \\
&= \eta^{3/2} \left(-\frac{m\ddot{q}}{\sqrt{g}} - \cancel{\partial_t \left(\frac{1}{\sqrt{g}} \right)} m\dot{q} + u \frac{m\ddot{q} \cdot u}{\sqrt{g}} + \cancel{\partial_t \left(\frac{1}{\sqrt{g}} \right)} m\dot{q} + \frac{2m}{E} \ddot{q}_\perp EL_g \right) \\
&= \eta^{3/2} \left(-\frac{m\ddot{q}}{\sqrt{g}} + \frac{m\dot{q}}{2T} \frac{m\ddot{q} \cdot \dot{q}}{\sqrt{g}} + \frac{2m}{E} \ddot{q}_\perp EL_g \right) \\
&= \eta^{3/2} \left(-\frac{m\ddot{q}}{\sqrt{g}} + \frac{\dot{T}}{2T\sqrt{g}} m\dot{q} + \frac{2m}{E} \ddot{q}_\perp EL_g \right),
\end{aligned}$$

where we have used that $T = m\|\dot{q}\|/2$ and $\dot{T} = m\ddot{q} \cdot \dot{q}$. The last term can be expanded to give

$$\begin{aligned}
\frac{2m}{E} \ddot{q}_\perp EL_g &= \frac{2m}{E} \ddot{q} \left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}} \right) - \frac{2m}{E} \dot{q} \frac{\ddot{q} \cdot \dot{q}}{\|\dot{q}\|^2} \left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}} \right) \\
&= \frac{m\ddot{q}}{\sqrt{g}} - \frac{mT}{Eg^{3/2}} \ddot{q} - \frac{\dot{T}}{2T\sqrt{g}} m\dot{q} + \frac{\dot{T}}{2Eg^{3/2}} m\dot{q}.
\end{aligned}$$

Putting everything together results in

$$\delta_{GR}\psi^*\tilde{q}_\perp^+ = \eta^{3/2}\left(\frac{\dot{T}}{2Eg^{3/2}}m\dot{q} - \frac{T}{Eg^{3/2}}m\ddot{q}\right)$$
$$= \left(\frac{E}{T}\right)^{3/2}\left(\frac{\dot{T}}{2E}m\dot{q} - \frac{T}{E}m\ddot{q}\right),$$

which shows that $\psi^*\delta_J\tilde{q}_\perp^+ = \delta_{GR}\psi^*\tilde{q}_\perp^+$. Finally we show that ψ^* acts as a chain map w.r.t. δ_{KT} on ξ^+. We have

$$\psi^*\delta_J\tilde{\xi}^+ = \psi^*(-\tilde{q}^+\cdot\dot{\tilde{q}}) = \psi^*(-\tilde{q}^+)\cdot\dot{q}$$
$$= \eta^{3/2}\left(-q^+\cdot\dot{q} + \left[g^+\dot{g} + 2\dot{g}^+g\right] + \frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\right),$$
$$\delta_{GR}\psi^*\tilde{\xi}^+ = \eta^{3/2}\delta_{GR}\left(\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+g^+\right)$$
$$= \eta^{3/2}\left(-q^+\cdot\dot{q} + g^+\dot{g} + 2\dot{g}^+g + \frac{g^{3/2}}{E}\dot{EL_g}g^+ - \frac{g^{3/2}}{E}\dot{g}^+EL_g\right)$$
$$= \eta^{3/2}\left(-q^+\cdot\dot{q} + \left[g^+\dot{g} + 2\dot{g}^+g\right] + \frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\right),$$

finally showing that

$$\psi^*\circ\delta_J = \delta_{GR}\circ\psi^*,$$

which completes the proof.

A.3 Computation of $\chi_s^*q^+$, χ^*q^+

In this section we present the calculations for $\chi_s^*(g^{3/2}q^+) = e^{sD}(g^{3/2}q^+)$ and $\lim_{s\to\infty}\chi_s^*q_\parallel^+$, which complete the proofs of Lemmas 4.2.11 and 4.2.12. Recall that $D = [Q_{GR}, R] = [\delta_{GR}, R]$ (c.f. Lemma 4.2.5) and that

$$Rq_\parallel^+ = -\frac{3\sqrt{g}}{E}EL_g\xi^+\frac{m\dot{q}}{2T}, \qquad Rq_\perp^+ = \frac{3\sqrt{g}}{E}g^+q_\perp^+.$$

As before the strategy is to first compute $\chi_s^*q^+$ through $\chi_s^*(g^{3/2}q^+)$. We start with the computations for the parallel part $D(g^{3/2}q_\parallel^+)$. Due to

$$D\left(\frac{m\dot{q}}{2T}\right) = 0,$$

we have

$$D(g^{3/2}q_{\|}^{+}) = D\left(g^{3/2}q^{+}\cdot\frac{m\dot{q}}{2T}\right) = D(q^{+}\cdot\dot{q})\frac{m\dot{q}}{2T}.$$

We will therefore omit the term $m\dot{q}/(2T)$ from the computations in order to keep them cleaner. We start by calculating

$$\begin{aligned}
D(q^{+}\cdot\dot{q}) &= \left(\delta_{GR}R + R\delta_{GR}\right)q^{+}\cdot\dot{q} \\
&= -\delta_{GR}\left(\frac{3\sqrt{g}}{E}EL_g\xi^{+}\right) - R\left(\partial_t\left(\frac{m\dot{q}}{\sqrt{g}}\right)\right)\cdot\dot{q} \\
&= \frac{3\sqrt{g}}{E}EL_g(q^{+}\cdot\dot{q} - g^{+}\dot{g} - 2\dot{g}^{+}g) - \partial_t\left(\frac{m\dot{q}}{2g^{3/2}}\frac{2g^{3/2}}{E}g^{+}\right)\cdot\dot{q} \\
&= \frac{3}{2}\left(1-\frac{T}{Eg}\right)q^{+}\cdot\dot{q} - \frac{3\sqrt{g}}{E}EL_g(g^{+}\dot{g} + 2\dot{g}^{+}g) - \frac{m\ddot{q}\cdot q}{E}g^{+} - \frac{m\|\dot{q}\|^2}{E}\dot{g}^{+} \\
&= \frac{3}{2}\left(1-\frac{T}{Eg}\right)q^{+}\cdot\dot{q} - \frac{3\sqrt{g}}{E}EL_g(g^{+}\dot{g} + 2\dot{g}^{+}g) - \frac{\dot{T}}{E}g^{+} - \frac{2T}{E}\dot{g}^{+}.
\end{aligned}$$

Let $\sigma(\varphi) = \varphi\partial_t(g^{3/2}g^+) - \dot{\varphi}g^{3/2}g^+$. With the result for $D(q^+ \cdot \dot{q})$ we compute the following

$$\begin{aligned}
D(g^{3/2}q^+ \cdot q) &= \frac{3\sqrt{g}}{2}Dgq^+ \cdot \dot{q} + g^{3/2}D(q^+ \cdot \dot{q}) \\
&= \frac{3\sqrt{g}}{2}\left(\frac{T}{E} - g\right) q^+ \cdot \dot{q} + g^{3/2}D(q^+ \cdot \dot{q}) \\
&= -\frac{3g^2}{E}EL_g(g^+\dot{g} + 2\dot{g}^+g) - g^{3/2}\frac{\dot{T}}{E}g^+ - g^{3/2}\frac{2T}{E}\dot{g}^+ \\
&= \left(\sqrt{g}\frac{3T}{2E} - \frac{3}{2}g^{3/2}\right)(g^+\dot{g} + 2\dot{g}^+g) - g^{3/2}\frac{\dot{T}}{E}g^+ - g^{3/2}\frac{2T}{E}\dot{g}^+ \\
&= \frac{T}{E}\partial_t(g^{3/2})g^+ + \frac{3T}{E}g^{3/2}\dot{g}^+ - \frac{3}{2}g^{3/2}\dot{g}g^+ - 3gg^{3/2}\dot{g}^+ - g^{3/2}\frac{\dot{T}}{E}g^+ - g^{3/2}\frac{2T}{E}\dot{g}^+ \\
&= \frac{T}{E}\partial_t(g^{3/2}g^+) - \frac{\dot{T}}{E}g^{3/2}g^+ - 3\left[g\partial_t(g^{3/2}g^+) - \dot{g}(g^{3/2}g^+)\right] \\
&= \sigma\left(\frac{T}{E}\right) - 3\sigma(g) \\
&= -2\sigma\left(\frac{T}{E}\right) - 3\sigma\left(g - \frac{T}{E}\right) \\
&= -2\sigma\left(\frac{T}{E}\right) - 2\frac{3}{E}\sigma(g^{3/2}EL_g).
\end{aligned}$$

Reintroducing $m\dot{q}/(2T)$ gives

$$D(g^{3/2}q_{\parallel}^+) = -2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} - 2 \cdot \frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}. \tag{46}$$

Using induction it is then possible to show that

$$D^k(g^{3/2}q_{\parallel}^+) = (-1)^k 2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} + (-2)^k\frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}, \tag{47}$$

for $k \geq 1$. The case $k = 1$ is presented in Equation (46). To see how the case $k+1$ follows from the case k, note that

$$D\sigma\left(\frac{T}{E}\right) = D\left(\frac{T}{E}\partial_t(g^{3/2}g^+) - \frac{\dot{T}}{E}g^{3/2}g^+\right) = -\sigma\left(\frac{T}{E}\right),$$

where we used $DT = 0$ and $D(g^{3/2}g^+) = -g^{3/2}g^+$. Before computing $D\sigma(g^{3/2}EL_g)$ note that

$$\begin{aligned} g^{3/2}EL_g &= \frac{E}{2}\left(g - \frac{T}{E}\right) = -\frac{E}{2}Dg \\ \Rightarrow \quad D(g^{3/2}EL_g) &= -\frac{E}{2}D^2 g = \frac{E}{2}Dg = -g^{3/2}EL_g. \end{aligned}$$

The action of D on $\sigma(g^{3/2}EL_g)$ is then

$$\begin{aligned} D(\sigma(g^{3/2}EL_g)) &= D\left(g^{3/2}EL_g\partial_t(g^{3/2}g^+) - \partial_t(g^{3/2}EL_g)g^{3/2}g^+\right) \\ &= -2\left(g^{3/2}EL_g\partial_t(g^{3/2}g^+) - \partial_t(g^{3/2}EL_g)g^{3/2}g^+\right) \\ &= -2\sigma(g^{3/2}EL_g), \end{aligned}$$

which proves the Equation (47). Having $D^k(g^{3/2}q_\parallel^+)$ we can now write

$$\begin{aligned} \chi_s^* q_\parallel^+ &= e^{sD}(g^{3/2}q_\parallel^+) = \sum_{k\geq 0}\frac{s^k}{k!}D^k(g^{3/2}q_\parallel^+) \\ &= g^{3/2}q_\parallel^+ + \left(\sum_{k\geq 1}\frac{(-s)^k}{k!}\right)2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} + \left(\sum_{k\geq 1}\frac{(-2s)^k}{k!}\right)\frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T} \\ &= g^{3/2}q_\parallel^+ + (e^{-s}-1)2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} + (e^{-s}-1)\frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}, \end{aligned}$$

taking the $s \to \infty$ limit then yields

$$\lim_{s\to\infty}\chi_s^*(g^{3/2}q_\parallel^+) = g^{3/2}q_\parallel^+ - 2\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} - \frac{3}{E}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}.$$

as desired. We can then extract $\lim_{s\to\infty}\chi_s^* q_\parallel^+$ from this expression using

$$\lim_{s\to\infty}\chi_s^* q_\parallel^+ = (E/T)^{3/2}\lim_{s\to\infty}\chi_s^*(g^{3/2}q_\parallel^+),$$

see Equation (34). We have

$$\begin{aligned} \lim_{s\to\infty}\chi_s^* q_\parallel^+ &= \eta^{3/2}\left(q_\parallel^+ - 2g^{-3/2}\sigma\left(\frac{T}{E}\right)\frac{m\dot{q}}{2T} - \frac{3}{E}g^{-3/2}\sigma(g^{3/2}EL_g)\frac{m\dot{q}}{2T}\right) \\ &= \eta^{3/2}\left(q^+\cdot\dot{q} - 2g^{-3/2}\sigma\left(\frac{T}{E}\right) - \frac{3}{E}g^{-3/2}\sigma(g^{3/2}EL_g)\right)\frac{m\dot{q}}{2T}. \end{aligned} \tag{48}$$

This expression can be further simplified, but first note that

$$E\dot{L}_g = -\frac{E}{4g^{3/2}}\dot{g} + \frac{3T}{4g^{5/2}}\dot{g} - \frac{\dot{T}}{2g^{3/2}}$$
$$\Rightarrow \frac{3T}{Eg}\dot{g} - \frac{2\dot{T}}{E} = \frac{4g^{3/2}}{E}E\dot{L}_g + \dot{g},$$

and

$$EL_g = \frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}$$
$$\Rightarrow \frac{2T}{E} = 2g - \frac{4g^{3/2}}{E}EL_g.$$

Using these two identities the last two terms in Equation (48) yield

$$\begin{aligned}
&2g^{-3/2}\sigma\left(\frac{T}{E}\right) + \frac{3}{E}g^{-3/2}\sigma(g^{3/2}EL_g) \\
&= \frac{2T}{E}g^{-3/2}\partial_t(g^{3/2}g^+) - \frac{2\dot{T}}{E}g^+ + \frac{3}{E}EL_g\partial_t(g^{3/2}g^+) - \frac{3}{E}\partial_t(g^{3/2}EL_g)g^+ \\
&= g^+\left[\frac{3T}{Eg}\dot{g} - \frac{2\dot{T}}{E} + \frac{3}{E}EL_g\partial_t(g^{3/2}) - \frac{3}{E}\partial_t(g^{3/2}EL_g)\right] + \dot{g}^+\left[\frac{2T}{E} + \frac{3}{E}g^{3/2}EL_g\right], \\
&= g^+\left[\dot{g} + \frac{g^{3/2}}{E}E\dot{L}_g\right] + \dot{g}^+\left[2g - \frac{g^{3/2}}{E}EL_g\right], \\
&= \left[g^+\dot{g} + 2\dot{g}^+g\right] + \frac{g^{3/2}}{E}\left[E\dot{L}_g g^+ - EL_g\dot{g}^+\right],
\end{aligned}$$

and as such

$$\begin{aligned}
\lim_{s\to\infty}\chi_s^* q_\parallel^+ &= \eta^{3/2}\left(q_\parallel^+ - \left[g^+\dot{g} + 2\dot{g}^+g\right]\frac{m\dot{q}}{2T} - \frac{g^{3/2}}{E}\left[E\dot{L}_g g^+ - EL_g\dot{g}^+\right]\frac{m\dot{q}}{2T}\right) \\
&= \chi^* q^+.
\end{aligned}$$

We now move to the computation of $\chi_s^* q_\perp^+$. As before our strategy will be to compute $\chi_s^* q_\perp^+$ via $D^k(g^{3/2} q_\perp^+)$. First note that we have

$$\begin{aligned}
\delta_{GR} q_\perp^+ &= -\partial_t \left(\frac{m\dot{q}}{\sqrt{g}} \right) + \frac{m\dot{q}}{2T} \partial_t \left(\frac{m\dot{q}}{\sqrt{g}} \right) \cdot \dot{q} \\
&= -\frac{m\ddot{q}}{\sqrt{g}} - \cancel{\partial_t \left(\frac{1}{\sqrt{g}} \right)} m\dot{q} + \frac{m\dot{q}}{2T} \frac{m\ddot{q} \cdot \dot{q}}{\sqrt{g}} + \cancel{\partial_t \left(\frac{1}{\sqrt{g}} \right)} m\dot{q} \\
&= -\frac{m\ddot{q}_\perp}{\sqrt{g}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
D q_\perp^+ = (\delta_{GR} R + R \delta_{GR}) q_\perp^+ &= \delta_{GR} R q_\perp^+ - R \left(\frac{m\ddot{q}_\perp}{\sqrt{g}} \right) \\
&= \delta_{GR} R q_\perp^+ + \frac{m\ddot{q}_\perp}{2g^{3/2}} \frac{(-2) g^{3/2}}{E} g^+ \\
&= \delta_{GR} R q_\perp^+ - \frac{m\ddot{q}_\perp}{E} g^+,
\end{aligned}$$

and thus

$$\begin{aligned}
D(g^{3/2} q_\perp^+) &= \frac{3\sqrt{g}}{2} D g q_\perp^+ + g^{3/2} D q_\perp^+ \\
&= -\frac{3g^2}{E} E L_g q_\perp^+ + g^{3/2} \delta_{GR} R q_\perp^+ - g^{3/2} \frac{m\ddot{q}_\perp}{E} g^+ \\
&= -\frac{3g^2}{E} \delta_{GR} g^+ q_\perp^+ + \frac{3g^2}{E} \delta_{GR} (g^+ q_\perp^+) - g^{3/2} \frac{m\ddot{q}_\perp}{E} g^+ \\
&= -\frac{3g^2}{E} g^+ \delta_{GR} q_\perp^+ - g^{3/2} \frac{m\ddot{q}_\perp}{E} g^+ \\
&= \frac{3g^2}{E} g^+ \frac{m\ddot{q}_\perp}{\sqrt{g}} - g^{3/2} \frac{m\ddot{q}_\perp}{E} g^+ \\
&= \frac{2m}{E} \ddot{q}_\perp g^{3/2} g^+.
\end{aligned}$$

Since $D\ddot{q}_\perp = 0$, the computation of the higher powers of $D^k(g^{3/2} q_\perp^+)$ becomes quite straightforward. We have

$$D^k(g^{3/2} q_\perp^+) = -(-1)^k \frac{2m}{E} \ddot{q}_\perp g^{3/2} g^+ \quad \text{for } k \geq 1,$$

which results in

$$\begin{aligned}
\chi_s^*(g^{3/2}q_\perp^+) = e^{sD}(g^{3/2}q_\perp^+) &= \sum_{k\geq 0}\frac{s^k}{k!}D^k(g^{3/2}q_\perp^+) \\
&= g^{3/2}q_\perp^+ - \left(\sum_{k\geq 1}\frac{(-s)^k}{k!}\right)\frac{2m}{E}\ddot{q}_\perp g^{3/2}g^+ \\
&= g^{3/2}q_\perp^+ - (e^{-s}-1)\frac{2m}{E}\ddot{q}_\perp g^{3/2}g^+,
\end{aligned}$$

as desired.

A.4 Computation of $h_\chi q^+$

In order to complete the proof of Lemma 4.2.13 we need to compute the action of the map h_χ on $q_\parallel^+$ and $q_\perp^+$. For the perpendicular part of q^+ we have

$$\begin{aligned}
h_\chi q_\perp^+ &= \int_0^\infty e^{sD}Rq_\perp^+\mathrm{d}s = \int_0^\infty e^{sD}\left(\frac{3}{E}\sqrt{g}g^+q_\perp^+\right)\mathrm{d}s \\
&= \frac{3}{E}\int_0^\infty (e^{sD}g)^{1/2}e^{sD}(g^{3/2}g^+)e^{sD}(g^{3/2}q_\perp^+)(e^{sD}g)^{-3}\mathrm{d}s \\
&= \frac{3}{E}\int_0^\infty (e^{sD}g)^{-5/2}e^{-s}g^{3/2}g^+\left(g^{3/2}q_\perp^+ - (e^{-s}-1)\cancel{\frac{2m}{E}\ddot{q}_\perp g^{3/2}g^+}\right)\mathrm{d}s \\
&= \frac{3}{E}g^3g^+q_\perp^+\int_0^\infty \frac{e^{-s}}{\left[e^{-s}g+(1-e^{-s})\frac{T}{E}\right]^{5/2}}\mathrm{d}s.
\end{aligned}$$

The integral yields

$$\begin{aligned}
I_\perp &= \int_0^\infty \frac{e^{-s}}{\left[e^{-s}g+(1-e^{-s})\frac{T}{E}\right]^{5/2}}\mathrm{d}s = \frac{2}{3}\left(g-\frac{T}{E}\right)^{-1}\left[e^{-s}g+(1-e^{-s})\frac{T}{E}\right]^{-3/2}\Bigg|_0^\infty \\
&= \frac{2}{3}\left(g-\frac{T}{E}\right)^{-1}\left[\left(\frac{E}{T}\right)^{3/2}-\frac{1}{g^{3/2}}\right] = \frac{2}{3}\frac{E}{Tg^{3/2}}\frac{\eta^{3/2}-1}{\eta-1} \\
&= \frac{2}{3}\frac{E}{Tg^{3/2}}\left(\frac{\eta}{\sqrt{\eta}+1}+1\right).
\end{aligned}$$

Which results in

$$h_\chi q_\perp^+ = \frac{2}{T}\left(\frac{\eta}{\sqrt{\eta}+1}+1\right)g^{3/2}g^+q_\perp^+.$$

Similarly we have

$$
\begin{aligned}
h_\chi q_\parallel^+ &= \int_0^\infty e^{sD} R q_\parallel^+ \mathrm{d}s = -\frac{3}{E}\int_0^\infty e^{sD}\left(\sqrt{g}\left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}\right)\xi^+ \frac{m\dot{q}}{2T}\right)\mathrm{d}s \\
&= \frac{3}{2}\frac{m\dot{q}}{2T}\int_0^\infty e^{sD}\left(\left(\frac{T}{Eg} - 1\right)\xi^+\right)\mathrm{d}s \\
&= \frac{3}{2}\frac{m\dot{q}}{2T}\int_0^\infty \left(\frac{T}{E}(e^{sD}g)^{-1} - 1\right)(e^{sD}g)^{-3/2}\left[g^{3/2}\xi^+ - (e^{-2s}-1)\frac{g^3}{E}\dot{g}^+ g^+\right]\mathrm{d}s \\
&= \frac{3}{2}\frac{m\dot{q}}{2T}\int_0^\infty \left(\frac{T}{E} - e^{sD}g\right)(e^{sD}g)^{-5/2}\left[g^{3/2}\xi^+ + \frac{g^3}{E}\dot{g}^+ g^+ - e^{-2s}\frac{g^3}{E}\dot{g}^+ g^+\right]\mathrm{d}s \\
&= \frac{3}{2}\frac{m\dot{q}}{2T}\left[g^{3/2}\xi^+ + \frac{g^3}{E}\dot{g}^+ g^+\right] I_1 - \frac{3}{2}\frac{m\dot{q}}{2T}\frac{g^3}{E}\dot{g}^+ g^+ I_2.
\end{aligned}
$$

The integrals that we need to consider are

$$
\begin{aligned}
I_1 &= \int_0^\infty \left(\frac{T}{E} - e^{-s}g - (1-e^{-s})\frac{T}{E}\right)\left[e^{-s}g + (1-e^{-s})\frac{T}{E}\right]^{-5/2}\mathrm{d}s \\
&= \left(\frac{T}{E} - g\right)\int_0^\infty \frac{e^{-s}}{\left[e^{-s}g + (1-e^{-s})\frac{T}{E}\right]^{5/2}}\mathrm{d}s \\
&= \frac{T}{E}(1-\eta)I_\perp = -\frac{2}{3}g^{-3/2}(\eta^{3/2}-1),
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^\infty \left(\frac{T}{E} - e^{-s}g - (1-e^{-s})\frac{T}{E}\right)\frac{e^{-2s}}{\left[e^{-s}g + (1-e^{-s})\frac{T}{E}\right]^{5/2}}\mathrm{d}s \\
&= \left(\frac{T}{E} - g\right)\int_0^\infty \frac{e^{-3s}}{\left[e^{-s}g + (1-e^{-s})\frac{T}{E}\right]^{5/2}}\mathrm{d}s = \dots \\
&= \frac{2}{3}\left(\frac{E}{T}\right)^{5/2}\left(g - \frac{T}{E}\right)\frac{(3\sqrt{\eta}+1)}{(\sqrt{\eta}+1)^3\eta^{3/2}} \\
&= \frac{2}{3}g^{-3/2}\frac{(3\sqrt{\eta}+1)(\sqrt{\eta}-1)}{(\sqrt{\eta}+1)^2}.
\end{aligned}
$$

Gathering everything results in

$$
h_\chi q_\parallel^+ = (1-\eta^{3/2})\left[\xi^+ + \frac{g^{3/2}}{E}\dot{g}^+ g^+\right]\frac{m\dot{q}}{2T} + \frac{(3\sqrt{\eta}+1)(1-\sqrt{\eta})}{(\sqrt{\eta}+1)^2}\frac{g^{3/2}}{E}\dot{g}^+ g^+ \frac{m\dot{q}}{2T}.
$$

A.5 Computation of $(\psi^*\theta_J^\bullet, \psi^* L_J^\bullet)$

In this section we will present the calculations for $(\psi^*\theta_J^\bullet, \psi^* L_J^\bullet)$, which were used in the proofs of Lemmas 4.1.3 and 5.1.2. Recall that $\theta_J^\bullet = \theta_J^1\,\mathrm{d}t + \theta_J^0$, $L_J^\bullet = L_J^1\,\mathrm{d}t + L_J^0$ with

$$\theta_J^1 = \tilde{q}^+ \cdot \delta\tilde{q} + \tilde{\xi}^+\delta\tilde{\xi}, \qquad \theta_J^0 = \sqrt{\frac{E}{\tilde{T}}} m\dot{\tilde{q}} \cdot \delta\tilde{q} + \tilde{q}^+\tilde{\xi}\delta\tilde{q} - \tilde{\xi}^+\tilde{\xi}\delta\tilde{\xi},$$

$$L_J^1 = 2\sqrt{E\tilde{T}} + \tilde{q}^+ \cdot \tilde{\xi}\dot{\tilde{q}} + \tilde{\xi}^+\tilde{\xi}\dot{\tilde{\xi}}, \qquad L_J^0 = 0.$$

as in Example 3.4.8. We want to show that

$$\begin{aligned} \psi^*\theta_J^\bullet &= \theta_{GR}^\bullet + (\mathcal{L}_{Q_{GR}} - \mathrm{d})\beta^\bullet + \delta f^\bullet, \\ \psi^* L_J^\bullet &= L_{GR}^\bullet + (\mathcal{L}_{Q_{GR}} - \mathrm{d})\iota_{Q_{GR}}\beta^\bullet + \mathrm{d}f^\bullet, \end{aligned}$$

with $\beta^\bullet = \beta_J^1\,\mathrm{d}t + \beta^0$, $f^\bullet = f^1\,\mathrm{d}t + f^0$, where

$$\begin{aligned} \beta^1 =& -\frac{4g^{7/2}}{\Omega^2} T g^+ \delta g^+ + \left(\frac{2g^2}{\Omega} + \eta^{3/2}\frac{2\sqrt{g}}{E}\right) g^+ q_\perp^+ \cdot \delta q, \\ &+ \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\dot{g}^+ g^+ \frac{m\dot{q}}{2T} \cdot \delta q - (\eta^{3/2} - 1)\xi^+ \frac{m\dot{q}}{2T} \cdot \delta q \\ \beta^0 =& \xi\beta^1 + \frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q, \\ f^1 =& 2g^+\left(g - \frac{2g^{3/2}}{\Omega} T\right), \\ f^0 =& \xi f^1. \end{aligned}$$

and $\eta = gE/T$, $\Omega = \sqrt{g}T + g\sqrt{TE}$. Specifically we will compute

$$\begin{aligned} \Delta\theta^1\mathrm{d}t &= \psi^*\theta_J^1\mathrm{d}t - \theta_{GR}^1\mathrm{d}t = \mathcal{L}_{Q_{GR}}\beta^1\mathrm{d}t - \mathrm{d}\beta^0 + \delta f^1\mathrm{d}t, \\ \Delta\theta^0 &= \psi^*\theta_J^0 - \theta_{GR}^0 = \mathcal{L}_{Q_{GR}}\beta^0 + \delta f^0\mathrm{d}t, \\ \Delta L^1\mathrm{d}t &= \psi^* L_J^1\mathrm{d}t - L_{GR}^1\mathrm{d}t = \mathcal{L}_{Q_{GR}}\iota_{Q_{GR}}\beta^1\mathrm{d}t - \mathrm{d}\iota_{Q_{GR}}\beta^0 + \mathrm{d}f^0, \\ \Delta L^0 &= \psi^* L_J^0 - L_{GR}^0 = \mathcal{L}_{Q_{GR}}\iota_{Q_{GR}}\beta^0. \end{aligned}$$

We will from now on drop the label GR in order to keep the calculations cleaner. The terms on the left hand sides can be computed explicitly by using the form of ψ^* (see Lemma 4.1.3). We get:

$$\begin{aligned}
\Delta\theta^1 =& (\eta^{3/2}-1)q^+\cdot\delta q + (\eta^{3/2}-1)\xi^+\delta\xi - g^+\delta g \\
& - \eta^{3/2}\left[g^+\dot{g} + 2\dot{g}^+ g\right]\frac{m\dot{q}}{2T}\cdot\delta q - \eta^{3/2}\frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\frac{m\dot{q}}{2T}\cdot\delta q \\
& + \eta^{3/2}\frac{2m}{E}g^+\ddot{q}_\perp\cdot\delta q + \eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\delta\xi, \\
\Delta\theta^0 =& \left[\sqrt{\frac{E}{T}}m\dot{q} - \frac{m\dot{q}}{\sqrt{g}}\right]\cdot\delta q \\
& + (\eta^{3/2}-1)q^+\xi\delta q - (\eta^{3/2}-1)\xi^+\xi\delta\xi - g^+\xi\delta g + 2g^+g\delta\xi \\
& - \eta^{3/2}\left[g^+\dot{g} + 2\dot{g}^+ g\right]\frac{m\dot{q}}{2T}\cdot\xi\delta q - \eta^{3/2}\frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\frac{m\dot{q}}{2T}\cdot\xi\delta q \\
& + \eta^{3/2}\frac{2m}{E}g^+\xi\ddot{q}_\perp\cdot\delta q + \eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\xi\delta\xi \\
=& \left[\sqrt{\frac{E}{T}}m\dot{q} - \frac{m\dot{q}}{\sqrt{g}}\right]\cdot\delta q + 2g^+g\delta\xi - \xi\Delta\theta^0, \\
\Delta L^1 =& 2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g}E \\
& + (\eta^{3/2}-1)q^+\cdot\xi\dot{q} + (\eta^{3/2}-1)\xi^+\xi\dot{\xi} - g^+(\xi\dot{g} + 2\dot{\xi}g) \\
& - \eta^{3/2}\left[g^+\dot{g} + 2\dot{g}^+g\right]\xi - \eta^{3/2}\frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - EL_g\dot{g}^+\right]\xi \\
& + \eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\xi\dot{\xi}, \\
\Delta L^0 =& -\sqrt{g}\xi\left(\frac{T}{g} - E\right).
\end{aligned}$$

The following identities are going to be used throughout the calculations:

$$\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}} = \frac{2g^{3/2}}{\Omega}EL_g, \tag{49}$$

$$2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g}E = -\frac{4g^{7/2}}{\Omega^2}TEL_g^2, \tag{50}$$

where $\Omega = \sqrt{g}T + g\sqrt{TE}$ with $t(\Omega) = 3$. To see that the first Equation (49) holds we compute

$$\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}} = \frac{1}{\sqrt{gT}}\left(\sqrt{gE} - \sqrt{T}\right) = \frac{gE - T}{\sqrt{gT}\left[\sqrt{T} + \sqrt{gE}\right]}$$
$$= \frac{2g^{3/2}}{\Omega}\left(\frac{E}{2\sqrt{g}} - \frac{T}{2g^{3/2}}\right) = \frac{2g^{3/2}}{\Omega}EL_g.$$

For the second one (50) we have

$$2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g}E = -\left(g^{1/4}\sqrt{E} - \frac{\sqrt{T}}{g^{1/4}}\right)^2.$$

The term in the brackets can be changed to

$$g^{1/4}\sqrt{E} - \frac{\sqrt{T}}{g^{1/4}} = g^{1/4}\left(\sqrt{E} - \sqrt{\frac{T}{g}}\right) = \frac{g^{1/4}}{\sqrt{E} + \sqrt{\frac{T}{g}}}\left(E - \frac{T}{g}\right)$$
$$= \frac{g^{5/4}\sqrt{T}}{\sqrt{g}T + g\sqrt{TE}}(2\sqrt{g}EL_g) = \frac{2g^{7/4}\sqrt{T}}{\Omega}EL_g,$$

and as such

$$2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g}E = -\frac{4g^{7/2}}{\Omega^2}TEL_g^2.$$

Computation of $\Delta\theta^1$: We want to show

$$\Delta\theta^1 \mathrm{d}t = \mathcal{L}_Q\beta^1\mathrm{d}t - \mathrm{d}\beta^0 + \delta f^1\mathrm{d}t.$$

In order to compute $\mathcal{L}_Q\beta^1$ we decompose the cohomological vector field as $Q = \gamma + \delta_{KT}$. Starting with $\mathcal{L}_\gamma$, we write $\beta^1 = A_1\delta g^+ + A_\perp \cdot \delta q + A_2 \cdot \delta q$ where

$$\begin{aligned} A_1 &= -\frac{4g^{7/2}}{\Omega^2}Tg^+, \\ A_\perp &= \left(\frac{2g^2}{\Omega} + \eta^{3/2}\frac{2\sqrt{g}}{E}\right)g^+q_\perp^+, \\ A_2 &= \left(\frac{4g^{7/2}}{\Omega^2}T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\dot{g}^+g^+\frac{m\dot{q}}{2T} - (\eta^{3/2} - 1)\xi^+\frac{m\dot{q}}{2T}, \end{aligned} \tag{51}$$

with the following tensor and ghost numbers:

$$
\begin{aligned}
t(A_1) &= \frac{7}{2}\cdot 2 - 2\cdot 3 + 2 - 1 = 2, & |A_1| &= -1,\\
t(A_\perp) &= 2\cdot 2 - 3 - 1 + 1 = 1, & |A_\perp| &= 1,\\
t(A_2) &= \frac{7}{2}\cdot 2 - 2\cdot 3 + 2 - 1 + 1 - 2 = 1, & |A_2| &= -2.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\mathcal{L}_\gamma(A_1\delta g^+) &= \gamma A_1 \delta g^+ + A_1\delta\gamma g^+ = \xi\dot{A}_1\delta g^+ + 2\dot{\xi}A_1\delta g^+ + A_1\delta(\xi\dot{g}^+ - \dot{\xi}g^+)\\
&= \xi\dot{A}_1\delta g^+ + \cancel{2}\dot{\xi}A_1\delta g^+ + A_1\delta\xi\dot{g}^+ - A_1\xi\delta\dot{g}^+ + \cancel{A_1\dot{\xi}\delta g^+}\\
&= \partial_t\left(\xi A_1\delta g^+\right) + A_1\dot{g}^+\delta\xi,\\
\mathcal{L}_\gamma(A_\perp\delta q) &= \gamma A_\perp\cdot\delta q - A_\perp\cdot\delta\gamma q = \partial_t(\xi A_\perp)\cdot\delta q - A_\perp\cdot\delta(\xi\dot{q})\\
&= \partial_t(\xi A_\perp)\cdot\delta q - \cancel{A_\perp\cdot\dot{q}}\delta\xi + A_\perp\xi\delta\dot{q}\\
&= \partial_t(\xi A_\perp\cdot\delta q)\\
\mathcal{L}_\gamma(A_2\cdot\delta q) &= \gamma A_2\cdot\delta q - A_2\cdot\delta\gamma q = \partial_t(\xi A_2)\cdot\delta q - A_2\cdot\delta(\xi\dot{q})\\
&= \partial_t(\xi A_2)\cdot\delta q - A_2\cdot\dot{q}\delta\xi + A_2\xi\delta\dot{q}\\
&= \partial_t(\xi A_2\cdot\delta q) - A_2\cdot\dot{q}\delta\xi,
\end{aligned}
$$

which shows

$$
\begin{aligned}
\mathcal{L}_\gamma\beta^1 = \partial_t(\xi\beta^1) - \frac{4g^{7/2}}{\Omega^2}Tg^+\dot{g}^+ - \left(\frac{4g^{7/2}}{\Omega^2}T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\dot{g}^+g^+\delta\xi + (\eta^{3/2}-1)\xi^+\delta\xi\\
= \partial_t(\xi\beta^1) + \eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\delta\xi + (\eta^{3/2}-1)\xi^+\delta\xi.
\end{aligned}
$$

Before addressing the computation of $\mathcal{L}_{\delta_{KT}}\beta^1$ we note that

$$\begin{aligned}
\delta EL_g &= \left(-\frac{E}{4g^{3/2}} + \frac{3T}{4g^{5/2}}\right)\delta g - \frac{\delta T}{2g^{3/2}}, \\
\delta\Omega &= \delta\left(\sqrt{g}T + g\sqrt{ET}\right) = \left(\frac{T}{2\sqrt{g}} + \sqrt{ET}\right)\delta g + \left(\sqrt{g} + \frac{g\sqrt{E}}{2\sqrt{T}}\right)\delta T \\
&= \left(\Omega - \frac{\sqrt{g}T}{2}\right)\frac{\delta g}{g} + \left(\Omega - \frac{g\sqrt{ET}}{2}\right)\frac{\delta T}{T}, \\
\Omega^2 &= gT^2 + 2g^{3/2}T^{3/2}\sqrt{E} + g^2 ET, \\
\frac{4g^{7/2}}{\Omega^2}TEL_g &= \frac{2g^2}{\Omega}T\left(\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}}\right) = \frac{2}{\Omega}\left(g^2\sqrt{ET} - g^{3/2}T\right) \\
&= \frac{2}{\Omega}\left(g\Omega - 2g^{3/2}T\right) = 2g - \frac{4g^{3/2}}{\Omega}T, \\
\delta\left(\frac{g^{3/2}}{\Omega}\right) &= \frac{3\sqrt{g}}{2\Omega}\delta g - \frac{g^{3/2}}{\Omega^2}\left(\Omega - \frac{\sqrt{g}T}{2}\right)\frac{\delta g}{g} - \frac{g^{3/2}}{\Omega^2}\left(\Omega - \frac{g\sqrt{ET}}{2}\right)\frac{\delta T}{T} \\
&= \frac{1}{2}\left(\frac{\sqrt{g}}{\Omega} + \frac{gT}{\Omega^2}\right)\delta g - \left(\frac{g^{3/2}}{T\Omega} - \frac{g^{5/2}\sqrt{E}}{2\sqrt{T}\Omega^2}\right)\delta T.
\end{aligned}$$

The calculation for the first term in $\mathcal{L}_{\delta_{KT}}\beta^1$ goes as follows

$$\begin{aligned}
&\mathcal{L}_{\delta_{KT}}\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\delta g^+\right) = -\frac{4g^{7/2}}{\Omega^2}TEL_g\delta g^+ - \frac{4g^{7/2}}{\Omega^2}Tg^+\delta EL_g \\
&\quad = -\delta\left(\frac{4g^{7/2}}{\Omega^2}TEL_g g^+\right) + \delta\left(\frac{4g^{7/2}}{\Omega^2}TEL_g\right)g^+ - \frac{4g^{7/2}}{\Omega^2}Tg^+\delta EL_g \\
&\quad = \delta\left(\frac{4g^{3/2}}{\Omega}Tg^+ - 2gg^+\right) + 2\delta g g^+ - \delta\left(\frac{4g^{3/2}}{\Omega}\right)Tg^+ - \frac{4g^{3/2}}{\Omega}\delta T g^+ \\
&\qquad - \frac{4g^{7/2}}{\Omega^2}Tg^+\left(-\frac{E}{4g^{3/2}} + \frac{3T}{4g^{5/2}}\right)\delta g + \frac{4g^{7/2}}{\Omega^2}Tg^+\frac{\delta T}{2g^{3/2}}.
\end{aligned}$$

Gathering everything in front of δg we have

$$\begin{aligned}
& g^{+}\left[-2+\left(\frac{2\sqrt{g}}{\Omega}+\frac{2gT}{\Omega^2}\right)T+\frac{g^2ET-3gT^2}{\Omega^2}\right]\delta g \\
&= g^{+}\left[-2+\frac{2\sqrt{g}T}{\Omega}+\frac{g^2ET-gT^2}{\Omega^2}\right]\delta g \\
&= g^{+}\left[-2+\frac{2\Omega-2g\sqrt{ET}}{\Omega}+\frac{\Omega^2-2g^{3/2}T^{3/2}\sqrt{E}-2gT^2}{\Omega^2}\right]\delta g \\
&= g^{+}\left[1-\frac{2g\sqrt{ET}\Omega+2g^{3/2}T^{3/2}\sqrt{E}+2gT^2}{\Omega^2}\right]\delta g \\
&= g^{+}\left[1-\frac{2g^2ET+4g^{3/2}T^{3/2}\sqrt{E}+2gT^2}{\Omega^2}\right]\delta g \\
&= g^{+}\left[1-\frac{2\Omega^2}{\Omega^2}\right]\delta g = -g^{+}\delta g.
\end{aligned}$$

The terms in front of δT simplify to

$$\begin{aligned}
& g^{+}\left[-\left(\cancel{\frac{4g^{3/2}}{\Omega}}-\frac{2g^{5/2}\sqrt{ET}}{\Omega^2}\right)+\cancel{\frac{4g^{3/2}}{\Omega}}+\frac{2g^2T}{\Omega^2}\right]\delta T \\
&= g^{+}\frac{2g^{3/2}}{\Omega}\left[\frac{g\sqrt{ET}+\sqrt{g}T}{\Omega}\right]\delta T = g^{+}\frac{2g^{3/2}}{\Omega}\delta T,
\end{aligned}$$

and as such:

$$\mathcal{L}_{\delta_{KT}}\left(-\frac{4g^{7/2}}{\Omega^2}Tg^{+}\delta g^{+}\right)=\delta\left(\frac{4g^{3/2}}{\Omega}Tg^{+}-2gg^{+}\right)+g^{+}\left(\frac{2g^{3/2}}{\Omega}\delta T-\delta g\right).$$

The rest of $\mathcal{L}_{\delta_{KT}}\beta^1$ yields

$$\begin{aligned}
\mathcal{L}_{\delta_{KT}}\left(\frac{2g^2}{\Omega}g^+q_\perp^+\cdot\delta q,\right) &= \frac{2g^2}{\Omega}\left(EL_g q_\perp^+ + g^+\frac{m\ddot{q}_\perp}{\sqrt{g}}\right)\cdot\delta q \\
&= \sqrt{g}\left(\sqrt{\frac{E}{T}}-\frac{1}{\sqrt{g}}\right)q_\perp^+\cdot\delta q + g^+\frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q \\
&= (\eta^{1/2}-1)q_\perp^+\cdot\delta q + g^+\frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q, \\
\mathcal{L}_{\delta_{KT}}\left(\eta^{3/2}\frac{2\sqrt{g}}{E}g^+q_\perp^+\cdot\delta q\right) &= \eta^{3/2}\frac{2\sqrt{g}}{E}\left[\left(\frac{E}{2\sqrt{g}}-\frac{T}{2g^{3/2}}\right)q_\perp^+ + g^+\frac{m\ddot{q}_\perp}{\sqrt{g}}\right]\cdot\delta q \\
&= (\eta^{3/2}-\eta^{1/2})q_\perp^+\cdot\delta q + \eta^{3/2}\frac{2}{E}g^+m\ddot{q}_\perp\cdot\delta q, \\
\mathcal{L}_{\delta_{KT}}\left(\frac{4g^{7/2}}{\Omega^2}T\dot{g}^+g^+\frac{m\dot{q}}{2T}\cdot\delta q\right) &= \frac{4g^{7/2}}{\Omega^2}T\left[\dot{EL_g}g^+ - \dot{g}^+EL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q, \\
\mathcal{L}_{\delta_{KT}}\left(-\eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\frac{m\dot{q}}{2T}\cdot\delta q\right) &= -\eta^{3/2}\frac{g^{3/2}}{E}\left[\dot{EL_g}g^+ - \dot{g}^+EL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q, \\
\mathcal{L}_{\delta_{KT}}\left(-(\eta^{3/2}-1)\xi^+\frac{m\dot{q}}{2T}\cdot\delta q\right) &= (\eta^{3/2}-1)\left[q^+\cdot\dot{q} - g^+\dot{g} - 2\dot{g}^+g\right]\frac{m\dot{q}}{2T}\cdot\delta q \\
&= (\eta^{3/2}-1)q_\parallel^+\cdot\delta q - (\eta^{3/2}-1)\left[g^+\dot{g}+2\dot{g}^+g\right]\frac{m\dot{q}}{2T}\cdot\delta q.
\end{aligned}$$

Gathering everything gives

$$
\begin{aligned}
\mathcal{L}_Q\beta^1 &= \partial_t(\xi\beta^1) + \eta^{3/2}\frac{g^{3/2}}{E}\dot{g}^+g^+\delta\xi + (\eta^{3/2}-1)\xi^+\delta\xi \\
&\quad + \delta\left(\frac{4g^{3/2}}{\Omega}Tg^+ - 2gg^+\right) + g^+\left(\frac{2g^{3/2}}{\Omega}\delta T - \delta g\right) \\
&\quad + (\eta^{1/2}-1)q_\perp^+\cdot\delta q + g^+\frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q \\
&\quad + (\eta^{3/2}-\eta^{1/2})q_\perp^+\cdot\delta q + \eta^{3/2}\frac{2}{E}g^+m\ddot{q}_\perp\cdot\delta q \\
&\quad + \left(\frac{4g^{7/2}}{\Omega^2}T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\left[E\dot{L}_g g^+ - \dot{g}^+EL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q \\
&\quad + (\eta^{3/2}-1)q_\parallel^+\cdot\delta q - (\eta^{3/2}-1)\left[g^+\dot{g} + 2\dot{g}^+g\right]\frac{m\dot{q}}{2T}\cdot\delta q \\
&= \Delta\theta^1 + \partial_t(\xi\beta^1) - \delta f^1 + \frac{2g^{3/2}}{\Omega}g^+\delta T + g^+\frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q \\
&\quad + \frac{4g^{7/2}}{\Omega^2}T\left[E\dot{L}_g g^+ - \dot{g}^+EL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q + \left[g^+\dot{g} + 2\dot{g}^+g\right]\frac{m\dot{q}}{2T}\cdot\delta q,
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
\Delta\theta^1 &= \mathcal{L}_Q\beta^1 - \partial_t\left(\xi\beta^1 + \frac{2g^{3/2}}{\Omega}g^+m\dot{q}\cdot\delta q\right) + \delta f^1 + \partial_t\left(\frac{2g^{3/2}}{\Omega}g^+m\dot{q}\cdot\delta q\right) \\
&\quad - \frac{2g^{3/2}}{\Omega}g^+\delta T - g^+\frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q \\
&\quad - \frac{4g^{7/2}}{\Omega^2}T\left[E\dot{L}_g g^+ - \dot{g}^+EL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q - \left[g^+\dot{g} + 2\dot{g}^+g\right]\frac{m\dot{q}}{2T}\cdot\delta q \\
&= \mathcal{L}_Q\beta^1 - \partial_t\beta^0 + \delta f^1 \\
&\quad + g^+\left[\partial_t\left(\frac{2g^{3/2}}{\Omega}\right)m\dot{q}\cdot\delta q + \frac{2g^{3/2}}{\Omega}m\ddot{q}\cdot\delta q + \frac{2g^{3/2}}{\Omega}m\dot{q}\cdot\delta\dot{q} - \frac{2g^{3/2}}{\Omega}\delta T\right. \\
&\quad \left. - \frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q - \frac{4g^{7/2}}{\Omega^2}TE\dot{L}_g\frac{m\dot{q}}{2T}\cdot\delta q - \dot{g}\frac{m\dot{q}}{2T}\cdot\delta q\right] \\
&\quad + \dot{g}^+\left[\left(\frac{4g^{3/2}}{\Omega}T - 2g\right) + \frac{4g^{7/2}}{\Omega^2}TEL_g\right]\frac{m\dot{q}}{2T}\cdot\delta q.
\end{aligned}
\tag{52}
$$

The last term vanishes due to Equation (50). In order to show that the term proportional to g^+ vanishes as well we note:

$$
\begin{aligned}
\delta T &= m\dot{q}\cdot\delta\dot{q},\\
\ddot{q}_\perp &= \ddot{q} - \dot{q}\frac{\dot{q}\cdot\ddot{q}}{\|\dot{q}\|^2} = \ddot{q} - \dot{q}\frac{\dot{T}}{2T},\\
\dot{EL_g} &= \left(-\frac{E}{4g^{3/2}} + \frac{3T}{4g^{5/2}}\right)\dot{g} - \frac{\dot{T}}{2g^{3/2}},\\
\partial_t\left(\frac{2g^{3/2}}{\Omega}\right) &= \left(\frac{\sqrt{g}}{\Omega} + \frac{gT}{\Omega^2}\right)\dot{g} - \left(\frac{2g^{3/2}}{T\Omega} - \frac{g^{5/2}\sqrt{E}}{\sqrt{T}\Omega^2}\right)\dot{T}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& g^+\left[\partial_t\left(\frac{2g^{3/2}}{\Omega}\right)m\dot{q}\cdot\delta q + \frac{2g^{3/2}}{\Omega}m\ddot{q}\cdot\delta q + \cancel{\frac{2g^{3/2}}{\Omega}m\dot{q}\cdot\delta\dot{q}} - \cancel{\frac{2g^{3/2}}{\Omega}\delta T}\right.\\
&\left. - \frac{2g^{3/2}}{\Omega}m\ddot{q}_\perp\cdot\delta q - \frac{4g^{7/2}}{\Omega^2}T\dot{EL_g}\frac{m\dot{q}}{2T}\cdot\delta q - \dot{g}\frac{m\dot{q}}{2T}\cdot\delta q\right]\\
&= g^+\left[\left(\frac{\sqrt{g}}{\Omega} + \frac{gT}{\Omega^2}\right)\dot{g}m\dot{q} - \left(\frac{2g^{3/2}}{T\Omega} - \frac{g^{5/2}\sqrt{E}}{\sqrt{T}\Omega^2}\right)\dot{T}m\dot{q}\right.\\
&\quad + \cancel{\frac{2g^{3/2}}{\Omega}m\ddot{q}}\cdot\delta q - \frac{2g^{3/2}}{\Omega}\left(\cancel{m\ddot{q}} - m\dot{q}\frac{\dot{T}}{2T}\right)\\
&\quad\left. - \frac{4g^{7/2}}{\Omega^2}T\left(\left(\left(-\frac{E}{4g^{3/2}} + \frac{3T}{4g^{5/2}}\right)\dot{g} - \frac{\dot{T}}{2g^{3/2}}\right)\frac{m\dot{q}}{2T} - \dot{g}\frac{m\dot{q}}{2T}\right]\cdot\delta q\\
&= g^+\frac{\dot{g}}{\Omega^2}\left[2\sqrt{g}T\Omega + 2gT^2 + g^2TE - 3gT^2 - \Omega^2\right]\frac{m\dot{q}}{2T}\cdot\delta q\\
&\quad + g^+\frac{\dot{T}}{\Omega^2}\left[-4g^{3/2}\Omega + 2g^{5/2}\sqrt{ET} + 2g^{3/2}\Omega + 2g^2T\right]\frac{m\dot{q}}{2T}\cdot\delta q\\
&= g^+\frac{\dot{g}}{\Omega^2}\left[\cancel{2}gT^2 + 2g^{3/2}T^{3/2}\sqrt{E} + g^2TE - \cancel{gT^2} - \Omega^2\right]\frac{m\dot{q}}{2T}\cdot\delta q\\
&\quad + g^+\frac{\dot{T}}{\Omega^2}\left[-2g^{3/2}\Omega + 2g^{3/2}(\sqrt{g}T + g\sqrt{ET})\right]\frac{m\dot{q}}{2T}\cdot\delta q\\
&= 0.
\end{aligned}
$$

Taking this into account and introducing $\mathrm{d}t$ in Equation (52) yields

$$
\Delta\theta^1\mathrm{d}t = \mathcal{L}_Q\beta^1\mathrm{d}t - \partial_t\beta^0\mathrm{d}t + \delta f^1\mathrm{d}t = \mathcal{L}_Q\beta^1\mathrm{d}t - \mathrm{d}\beta^0 + \delta f^1\mathrm{d}t, \tag{53}
$$

since $|\beta^0| = 0$.

Computation of $\Delta\theta^0$: We want to compute

$$\Delta\theta^0 = \mathcal{L}_Q \beta^0 + \delta f^0.$$

First note that Equation (53) implies $\mathcal{L}_Q \beta^1 = \Delta\theta^1 + \partial_t \beta^0 - \delta f^1$, which we use to compute $\mathcal{L}_Q \beta^0$. Since

$$\beta^0 = \xi \beta^1 + \frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q,$$

we have

$$\mathcal{L}_Q \beta^0 = \xi \dot{\xi} \beta^1 - \xi \left(\Delta\theta^1 + \partial_t \beta^0 - \delta f^1 \right) + \mathcal{L}_Q \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q \right).$$

Keeping in mind that

$$t \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \right) = \frac{3}{2} \cdot 2 - 3 - 1 + 1 = 0,$$

the last term reads

$$\begin{aligned}
\mathcal{L}_\gamma \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q \right) &= \xi \partial_t \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \right) \cdot \delta q + \frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta(\xi \dot{q}) \\
&= \xi \partial_t \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \right) \cdot \delta q + \frac{4g^{3/2}}{\Omega} g^+ T \delta\xi \\
&\quad + \xi \frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta\dot{q}, \\
\mathcal{L}_{\delta_{KT}} \left(\frac{2g^{3/2}}{\Omega} g^+ m\dot{q} \cdot \delta q \right) &= \frac{2g^{3/2}}{\Omega} E L_g m\dot{q} \cdot \delta q = \left[\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}} \right] m\dot{q} \cdot \delta q,
\end{aligned}$$

which results in

$$
\begin{aligned}
\mathcal{L}_Q\beta^0 &= \xi\dot{\xi}\beta^1 - \xi\Delta\theta^1 - \xi\partial_t\left(\xi\beta^1 + \cancel{\frac{2g^{3/2}}{\Omega}g^+ m\dot{q}\cdot\delta q}\right) + \xi\delta f^1 \\
&\quad + \cancel{\xi\partial_t\left(\frac{2g^{3/2}}{\Omega}g^+ m\dot{q}\right)}\cdot\delta q + \frac{4g^{3/2}}{\Omega}g^+T\delta\xi \\
&\quad + \cancel{\xi\frac{2g^{3/2}}{\Omega}g^+ m\dot{q}\cdot\delta\dot{q}} + \left[\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}}\right]m\dot{q}\cdot\delta q \\
&= \left[\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}}\right]m\dot{q}\cdot\delta q - \xi\Delta\theta^1 + \cancel{\xi\dot{\xi}\beta^1} - \cancel{\xi\dot{\xi}\beta^1} \\
&\quad - \delta(\xi f^1) + \delta\xi 2g^+\left(g - \cancel{\frac{2g^{3/2}}{\Omega}T}\right) + \cancel{\frac{4g^{3/2}}{\Omega}g^+T\delta\xi} \\
&= \Delta\theta^0 - \delta f^0.
\end{aligned}
$$

Showing $\Delta\theta^0 = \mathcal{L}_Q\beta^0 + \delta f^0$ as desired.

Computation of ΔL^1: We want to show

$$\Delta L^1 = \mathcal{L}_Q\iota_Q\beta^1 - \mathrm{d}\iota_Q\beta^0 + \mathrm{d}f^0.$$

Note that β^1 is of the form $\beta^1 = a_i\delta b_i$, where we sum over i. We have

$$
\begin{aligned}
\mathcal{L}_Q\iota_Q(a_i\delta b_i) &= \mathcal{L}_Q(a_iQb_i) = Qa_iQb_i \\
&= \gamma a_i\gamma b_i + \delta_{KT}a_i\gamma b_i + \gamma a_i\delta_{KT}b_i + \delta_{KT}a_i\delta_{KT}b_i.
\end{aligned}
$$

Starting with the term $A_1\delta g^+$ (see Equation (51)) we have

$$\begin{aligned}
\gamma\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)\gamma g^+ &= \left[\xi\partial_t\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)-2\dot{\xi}\frac{4g^{7/2}}{\Omega^2}Tg^+\right]\cdot\left[\xi\dot{g}^+-\dot{\xi}g^+\right]\\
&= \xi\frac{4g^{7/2}}{\Omega^2}T\dot{g}^+\dot{\xi}g^+ - 2\dot{\xi}\frac{4g^{7/2}}{\Omega^2}Tg^+\xi\dot{g}^+\\
&= \frac{4g^{7/2}}{\Omega^2}T\dot{g}^+g^+\xi\dot{\xi},\\
\delta_{KT}\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)\gamma g^+ &= -\frac{4g^{7/2}}{\Omega^2}TEL_g(\xi\dot{g}^+-\dot{\xi}g^+),\\
\gamma\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)\delta_{KT}g^+ &= \left[\xi\partial_t\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)-2\dot{\xi}\frac{4g^{7/2}}{\Omega^2}Tg^+\right]EL_g,\\
\delta_{KT}\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\right)\delta_{KT}g^+ &= -\frac{4g^{7/2}}{\Omega^2}TEL_g^2 = 2\sqrt{ET}-\frac{T}{\sqrt{g}}-\sqrt{g},
\end{aligned}$$

where we used Equation (50). As such

$$\begin{aligned}
\mathcal{L}_{Q\iota_Q}\left(-\frac{4g^{7/2}}{\Omega^2}Tg^+\delta g^+\right) &= \frac{4g^{7/2}}{\Omega^2}T\dot{g}^+g^+\xi\dot{\xi}-\partial_t\left(\xi\frac{4g^{7/2}}{\Omega^2}Tg^+\right)EL_g\\
&\quad -\frac{4g^{7/2}}{\Omega^2}TEL_g\xi\dot{g}^+ + 2\sqrt{ET}-\frac{T}{\sqrt{g}}-\sqrt{g}\\
&= \frac{4g^{7/2}}{\Omega^2}T\dot{g}^+g^+\xi\dot{\xi}-\partial_t\left(\xi\frac{4g^{7/2}}{\Omega^2}Tg^+EL_g\right)\\
&\quad +\xi\frac{4g^{7/2}}{\Omega^2}Tg^+E\dot{L}_g\\
&\quad -\frac{4g^{7/2}}{\Omega^2}TEL_g\xi\dot{g}^+ + 2\sqrt{ET}-\frac{T}{\sqrt{g}}-\sqrt{g}.
\end{aligned}$$

For the term $A_\perp\cdot\delta q$ we have

$$\begin{aligned}
\mathcal{L}_{Q\iota_Q}(A_\perp\delta q) = QA_\perp Qq &= \gamma A_\perp\cdot\xi\dot{q}+\delta_{KT}(A_\perp)\cdot\xi\dot{q}\\
&= \dot{\xi}A_\perp\cdot\xi\dot{q}+\delta_{KT}(A_\perp\cdot\xi\dot{q}) = 0.
\end{aligned}$$

For the computation w.r.t. the term

$$\mathcal{L}_{Q\iota_Q}\left[\left(\frac{4g^{7/2}}{\Omega^2}T-\eta^{3/2}\frac{g^{3/2}}{E}\right)\frac{m\dot{q}}{2T}\dot{g}^+g^+\cdot\delta q\right]$$

first define

$$B = \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \frac{m\dot{q}}{2T},$$

with $t(B) = \frac{7}{2} \cdot 2 - 2 \cdot 3 + 2 + 1 - 2 = 2$ and note

$$t(B\dot{g}^+ g^+) = 2 + 1 + t(g^+) + t(g^+) = 1.$$

As such

$$\begin{aligned}
\mathcal{L}_{Q} \iota_{Q} \left[B\dot{g}^+ g^+ \cdot \delta q \right] &= Q \left[B\dot{g}^+ g^+ \right] \xi \dot{q} \\
&= \partial_t \left[\xi B\dot{g}^+ g^+ \right] \cdot \xi \dot{q} + B \left[E\dot{L}_g g^+ - EL_g \dot{g}^+ \right] \cdot \xi \dot{q} \\
&= \dot{\xi} B\dot{g}^+ g^+ \cdot \xi \dot{q} + B \left[E\dot{L}_g g^+ - EL_g \dot{g}^+ \right] \cdot \xi \dot{q} \\
&= \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \dot{g}^+ g^+ \dot{\xi} \xi \\
&\quad + \left(\frac{4g^{7/2}}{\Omega^2} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \left[E\dot{L}_g g^+ - EL_g \dot{g}^+ \right] \xi.
\end{aligned}$$

For the last term in β^1

$$-(\eta^{3/2} - 1)\xi^+ \frac{m\dot{q}}{2T} \cdot \delta q,$$

we have

$$\begin{aligned}
\mathcal{L}_{Q} \iota_{Q} \left[-(\eta^{3/2} - 1)\xi^+ \frac{m\dot{q}}{2T} \cdot \delta q \right] =& - \gamma \left[(\eta^{3/2} - 1)\xi^+ \frac{m\dot{q}}{2T} \right] \cdot \xi \dot{q} \\
& - (\eta^{3/2} - 1) \left[-q^+ \cdot \dot{q} + g^+ \dot{g} + 2\dot{g}^+ g \right] \frac{m\dot{q}}{2T} \cdot \xi \dot{q} \\
=& (\eta^{3/2} - 1)\xi^+ \xi \dot{\xi} - (\eta^{3/2} - 1) \left[-q^+ \cdot \dot{q} + g^+ \dot{g} + 2\dot{g}^+ g \right] \xi.
\end{aligned}$$

All in all the expression for $\mathcal{L}_Q \iota_Q \beta^1$ is

$$
\begin{aligned}
\mathcal{L}_Q \iota_Q \beta^1 = {} & \cancel{\frac{4g^{7/2}}{\Omega^2} T \dot{g}^+ g^+ \xi \dot{\xi}} - \partial_t \left(\xi \frac{4g^{7/2}}{\Omega^2} T g^+ E L_g \right) + \xi \cancel{\frac{4g^{7/2}}{\Omega^2} T g^+ \dot{EL}_g} \\
& - \cancel{\frac{4g^{7/2}}{\Omega^2} T E L_g \xi \dot{g}^+} + 2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g} \\
& + \left(\cancel{\frac{4g^{7/2}}{\Omega^2}} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \dot{g}^+ g^+ \dot{\xi} \xi + \left(\cancel{\frac{4g^{7/2}}{\Omega^2}} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) \dot{EL}_g g^+ \xi \\
& - \left(\cancel{\frac{4g^{7/2}}{\Omega^2}} T - \eta^{3/2} \frac{g^{3/2}}{E} \right) E L_g \dot{g}^+ \xi \\
& + (\eta^{3/2} - 1) \xi^+ \xi \dot{\xi} - (\eta^{3/2} - 1) \left[-q^+ \cdot \dot{q} + g^+ \dot{g} + 2 \dot{g}^+ g \right] \xi \\
= {} & 2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g} E \\
& + (\eta^{3/2} - 1) q^+ \cdot \xi \dot{q} + (\eta^{3/2} - 1) \xi^+ \xi \dot{\xi} - g^+ (\xi \dot{g} + 2 \dot{\xi} g) \\
& - \eta^{3/2} \left[g^+ \dot{g} + 2 \dot{g}^+ g \right] \xi - \eta^{3/2} \frac{g^{3/2}}{E} \left[\dot{EL}_g g^+ - E L_g \dot{g}^+ \right] \xi \\
& - \partial_t \left(\xi \frac{4g^{7/2}}{\Omega^2} T g^+ E L_g \right) + \partial_t \left(2 g^+ g \xi \right) \\
= {} & \Delta L^1 + \partial_t \left(2 g^+ g \xi + \frac{4g^{7/2}}{\Omega^2} T E L_g g^+ \xi \right),
\end{aligned}
\tag{54}
$$

where we used that

$$
\begin{aligned}
g^+ \dot{g} \xi + 2 \dot{g}^+ g \xi &= g^+ \dot{g} \xi + \partial_t \left(2 g^+ g \xi \right) - 2 g^+ \dot{g} \xi - 2 g^+ \dot{g} \xi \\
&= \partial_t \left(2 g^+ g \xi \right) - 2 g^+ \dot{g} \xi - g^+ \dot{g} \xi.
\end{aligned}
$$

Recall that we are want to show $\Delta L^1 \, \mathrm{d}t = \mathcal{L}_Q \iota_Q \beta^1 \, \mathrm{d}t - \mathrm{d}\iota_Q \beta^0 + \mathrm{d}f^0$. The last two terms read

$$
\mathrm{d}\iota_Q \beta^0 - \mathrm{d}f^0 = \partial_t \left[\xi \iota_Q \beta^1 + \frac{2g^{3/2}}{\Omega} g^+ m \dot{q} \cdot \iota_Q \delta q - 2 \xi g^+ \left(g - \frac{2g^{3/2}}{\Omega} T \right) \right] \mathrm{d}t.
$$

which can be simplified by noting that

$$\begin{aligned}
\xi\iota_Q\beta^1 =& -\xi\frac{4g^{7/2}}{\Omega^2}Tg^+Qg^+ + \xi\left(\frac{2g^2}{\Omega} + \eta^{3/2}\frac{2\sqrt{g}}{E}\right)g^+q_\perp^+\cdot Qq, \\
&+\xi\left(\frac{4g^{7/2}}{\Omega^2}T - \eta^{3/2}\frac{g^{3/2}}{E}\right)\dot{g}^+g^+\frac{m\dot{q}}{2T}\cdot Qq - \xi(\eta^{3/2}-1)\xi^+\frac{m\dot{q}}{2T}\cdot Qq \\
=& \frac{4g^{7/2}}{\Omega^2}TEL_g g^+\xi, \qquad (55)
\end{aligned}$$

and

$$\frac{2g^{3/2}}{\Omega}g^+m\dot{q}\cdot\iota_Q\delta q = \frac{4g^{3/2}}{\Omega}Tg^+\xi. \qquad (56)$$

The resulting expression for $\mathrm{d}\iota_Q\beta^0 - \mathrm{d}f^0$ is then

$$\begin{aligned}
\mathrm{d}\iota_Q\beta^0 - \mathrm{d}f^0 &= \partial_t\left[\frac{4g^{7/2}}{\Omega^2}TEL_g g^+\xi + \frac{4g^{3/2}}{\Omega}Tg^+\xi - 2\xi g^+\left(g - \frac{2g^{3/2}}{\Omega}T\right)\right]\mathrm{d}t \\
&= \partial_t\left(2g^+g\xi + \frac{4g^{7/2}}{\Omega^2}TEL_g g^+\xi\right),
\end{aligned}$$

which is exactly the total derivative in Equation (54). Hence

$$\begin{aligned}
&\mathcal{L}_Q\iota_Q\beta^1\mathrm{d}t = \Delta L^1\mathrm{d}t + \mathrm{d}\iota_Q\beta^0 - \mathrm{d}f^0 \\
\Rightarrow&\Delta L^1\mathrm{d}t = \mathcal{L}_Q\iota_Q\beta^1\mathrm{d}t - \mathrm{d}\iota_Q\beta^0 + \mathrm{d}f^0.
\end{aligned}$$

Computation of ΔL^0: Here we want to compute

$$\Delta L^0 = \mathcal{L}_Q\iota_Q\beta^0.$$

Using Equations (55) and (56) we have

$$\mathcal{L}_Q\iota_Q\beta^0 = \mathcal{L}_Q\iota_Q\left(\xi\beta^1 + \frac{2g^{3/2}}{\Omega}g^+m\dot{q}\cdot\delta q\right) = \mathcal{L}_Q\left(\frac{4g^{7/2}}{\Omega^2}TEL_g g^+\xi + \frac{4g^{3/2}}{\Omega}Tg^+\xi\right).$$

Noting that we can write the expression in the brackets as $A\xi$ where

$$A = \frac{4g^{7/2}}{\Omega^2}TEL_g g^+ + \frac{2g^{3/2}}{\Omega}Tg^+,$$

with $t(A) = 1$, $|A| = -1$, we can compute the action of the Chevalley-Eilenberg part to be

$$\mathcal{L}_\gamma (A\xi) = \partial_t(\xi A)\xi - A\xi\dot{\xi} = \dot{\xi}A\xi - A\xi\dot{\xi} = 0.$$

The Koszul-Tate part is given by

$$\begin{aligned}
\mathcal{L}_{\delta_{KT}} \left(\frac{4g^{7/2}}{\Omega^2} TEL_g g^+ \xi + \frac{4g^{3/2}}{\Omega} Tg^+\xi \right) &= \frac{4g^{7/2}}{\Omega^2} TEL_g^2 \xi + \frac{4g^{3/2}}{\Omega} TEL_g \xi \\
&= -\left(2\sqrt{ET} - \frac{T}{\sqrt{g}} - \sqrt{g}E \right) \xi + 2 \left(\sqrt{\frac{E}{T}} - \frac{1}{\sqrt{g}} \right) T\xi \\
&= -\left(\frac{T}{g} - E \right) \sqrt{g}\xi = \Delta L^0,
\end{aligned}$$

finishing the proof.

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