Derived Poisson and coisotropic structures

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ABSTRACT. This thesis consists of five papers on the topic of derived Poisson geometry. Following the work [Cal+17] defining Poisson structures on (derived) Artin stacks, we define coisotropic structures in the same context. The key input is a certain algebraic result (Poisson additivity) relating n-shifted Poisson algebras to $(n-1)$-shifted Poisson algebras. We also explain relations between the classical BRST complex computing a derived version of Hamiltonian reduction to a coisotropic intersection and provide many interesting examples of n-shifted Poisson structures.

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CHAPTER 1

Preface

1. Introduction

1.1. Poisson geometry. Symplectic and Poisson manifolds appeared in abstract formulations of Hamiltonian mechanics. For instance, if $M = \mathbf{R}^n$ with coordinates $\{q_i\}$ is equipped with a potential function $V(q_1, \dots, q_n)$, the *phase space* of classical mechanics of a particle moving on M is described by the cotangent bundle $\mathrm{T}^*M = \mathbf{R}^{2n}$ with coordinates $\{p_i, q_i\}$ equipped with a Hamiltonian function

$$H(p_1, \dots, p_n, q_1, \dots, q_n) = \sum_i \frac{p_i^2}{2m} + V(q_1, \dots, q_n).$$

The time evolution of a function f on T^*M is then described by the Hamilton equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\},$$

where

$$\{H, f\} = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \tag{1}$$

is the *Poisson bracket* on the algebra of functions $C^\infty(\mathrm{T}^*M)$.

While classical mechanics describes individual particles, the fields which influence the particles are described by classical field theory. Classical field theory is specified by the space of fields $\mathcal{F}$ (which is typically an infinite-dimensional manifold of functions or sections of vector bundles over the spacetime manifold X) and the action functional $S\colon \mathcal{F} \to \mathbf{R}$. A *classical solution* is then a critical point of the action functional, i.e. a point $\phi \in \mathcal{F}$ such that $\frac{\partial S}{\partial \phi_i}(\phi) = 0$.

Note that classical mechanics can be considered as a one-dimensional classical field theory. Namely, one may consider the space of fields to be $\mathcal{F} = \mathrm{Map}(\mathbf{R}, M)$, the space of smooth functions from $\mathbf{R}$ to M, which we parametrize by a collection of functions $\{q_i(t)\}$. We take the action functional to be

$$S = \int \left(\sum_i \frac{m\dot{q}_i^2}{2} - V(q_1(t), \dots, q_n(t))\right) \mathrm{d}t.$$

The Hamilton equation for the coordinate functions $q_i(t)$ is then exactly the condition that they constitute a critical point of the action functional S.

Given an interpretation of classical mechanics in terms of Poisson manifolds, one might wonder whether there exists an abstract Poisson approach to arbitrary classical field theories. One complication is that the space of classical solutions, i.e. the subset of $\mathcal{F}$ of critical points of S, is usually not a smooth manifold if S has degenerate critical points. Such a situation often arises in physics when one considers systems with symmetries.

To deal with this problem we have to enlarge the class of spaces we consider. For instance, we may instead consider NQ-manifolds (see [Ale+97], [Roy02]). These are smooth manifolds Z equipped with a graded sheaf of commutative algebras $\mathcal{O}_Z$ and a vector field Q of degree 1 such that $[Q, Q] = 0$. Given an NQ-manifold, its underlying space $\mathrm{H}^0(Z)$ is given by considering Z equipped with the sheaf of commutative algebras $\mathrm{H}^0(\mathcal{O}_Z, Q)$ given by the zeroth cohomology of $\mathcal{O}_Z$. We may think of an NQ-manifold as a better approximation to the singular space $\mathrm{H}^0(Z)$. For instance, in the BRST formalism the quotient of a manifold M by a group action G is modeled by the NQ-manifold $[M/\mathfrak{g}]$ which is the manifold M equipped with the sheaf $\mathcal{O}_M \otimes \wedge^\bullet \mathfrak{g}^*$ where Q is the Chevalley–Eilenberg differential.

Batalin and Vilkovisky in [BV83] realized that the set of classical solutions of an action functional has a natural NQ-manifold avatar given by the *derived critical locus* $\mathrm{dCrit}(S)$. Namely, as a manifold it is given by $\mathcal{F}$ equipped with the sheaf $\wedge^\bullet \mathrm{T}_{\mathcal{F}}$ where Q is given by contraction with $\mathrm{d}_{\mathrm{dR}}S$. The underlying space $\mathrm{H}^0(\mathrm{dCrit}(S))$ is the critical locus of S, i.e. the set of classical solutions. Moreover, as observed in [BV83], the NQ-manifold $\mathrm{dCrit}(S)$ carries a natural Poisson bracket of degree 1 (a (-1)-shifted Poisson structure, i.e. a $\mathbb{P}_0$-structure) defined similarly to (1). A modern interpretation of this result is that $\mathrm{dCrit}(S)$ is the (-1)-shifted cotangent bundle $\mathrm{T}^*[-1]\mathcal{F}$ twisted by the one-form $\mathrm{d}_{\mathrm{dR}}S$.

Combining the Batalin–Vilkovisky approach with locality of field theories, Costello and Gwilliam [CG17] have advocated an approach to classical field theories in terms of factorization algebras on spacetime valued in $\mathbb{P}_0$-algebras. Moreover, there is a natural notion of quantization of such factorization algebras giving rise to factorization algebras of quantum observables. Many field theories have been mathematically rigorously analyzed in this framework. For instance, one-dimensional topological quantum mechanics turns out to be closely related to the theory of Fedosov quantization [GLL17] while the two-dimensional curved β-γ system (a holomorphic version of topological quantum mechanics) is closely related to the Witten genus [Cos10] and chiral differential operators [GGW16].

The approach of Costello and Gwilliam studies the moduli space of classical solutions only in a formal neighborhood of a given solution. A global approach (that is, the definition of a Poisson structure on a derived stack) was developed more recently: the paper [Pan+13] studies (shifted) symplectic structures on derived stacks and the paper [Cal+17] studies (shifted) Poisson structures on derived stacks.

2. Papers included

Here we list abstracts of the papers included in the thesis.

(1) *Poisson reduction as a coisotropic intersection* [Saf17a].

We give a definition of coisotropic morphisms of shifted Poisson (i.e. $\mathbb{P}_n$) algebras which is a derived version of the classical notion of coisotropic submanifolds. Using this we prove that an intersection of coisotropic morphisms of shifted Poisson algebras carries a Poisson structure of shift one less. Using an interpretation of Hamiltonian spaces as coisotropic morphisms we show that the classical BRST complex computing derived Poisson reduction coincides with the complex computing coisotropic intersection. Moreover, this picture admits a quantum version using brace algebras and their modules: the quantum BRST complex is quasi-isomorphic to the complex computing tensor product of brace modules.

(2) *Braces and Poisson additivity* [Saf18].

We relate the brace construction introduced by Calaque and Willwacher to an additivity functor. That is, we construct a functor from brace algebras associated to an operad $\mathcal{O}$ to associative algebras in the category of homotopy $\mathcal{O}$-algebras. As an example, we identify the category of $\mathbb{P}_{n+1}$-algebras with the category of associative algebras in $\mathbb{P}_n$-algebras. We also show that under this identification there is an equivalence of two definitions of derived coisotropic structures in the literature.

(3) *Derived coisotropic structures I: affine case* (with Valerio Melani) [MS16].

We define and study coisotropic structures on morphisms of commutative dg algebras in the context of shifted Poisson geometry, i.e. $\mathbb{P}_n$-algebras. Roughly speaking, a coisotropic morphism is given by a $\mathbb{P}_{n+1}$-algebra acting on a $\mathbb{P}_n$-algebra. One of our main results is an identification of the space of such coisotropic structures with the space of Maurer–Cartan elements in a certain dg Lie algebra of relative polyvector fields. To achieve this goal, we construct a cofibrant replacement of the operad controlling coisotropic morphisms by analogy with the Swiss-cheese operad which can be of independent interest. Finally, we show that morphisms of shifted Poisson algebras are identified with coisotropic structures on their graph.

(4) *Derived coisotropic structures II: stacks and quantization* (with Valerio Melani) [MS17].

We extend results about n-shifted coisotropic structures from part I of this work to the setting of derived Artin stacks. We show that an intersection of coisotropic morphisms carries a Poisson structure of shift one less. We also compare non-degenerate shifted coisotropic structures and shifted Lagrangian structures and show that there is a natural equivalence between the two spaces in agreement with the classical result. Finally, we define quantizations of n-shifted coisotropic structures and show that they exist for $n > 1$.

(5) *Poisson-Lie structures as shifted Poisson structures* [Saf17b].

Classical limits of quantum groups give rise to multiplicative Poisson structures such as Poisson-Lie and quasi-Poisson structures. We relate them to the notion of a shifted Poisson structure which gives a conceptual framework for understanding classical (dynamical) r-matrices, quasi-Poisson groupoids and so on. We also propose a notion of a symplectic realization of shifted Poisson structures and show that Manin pairs and Manin triples give examples of such.

3. Background

3.1. Derived algebraic geometry. The original approach to higher Poisson and symplectic structures was using the language of NQ-manifolds [Ale+97] [Roy02]. Recall that an NQ-manifold is a manifold whose algebra of functions is enhanced to a commutative dg algebra. We want to treat two NQ-manifold N_1 and N_2 as equivalent if there is a morphism $N_1 \to N_2$ of NQ-manifolds such that $C^\infty(N_2) \to C^\infty(N_1)$ is a quasi-isomorphism, i.e. it induces an isomorphism on cohomology. The corresponding category is a bit unwieldy, so instead we will switch to the setting of derived algebraic geometry.

Consider the category $\underline{\mathrm{CAlg}}^{\leq 0}$ of commutative dg algebras over a field k of characteristic zero concentrated in non-positive cohomological degrees (*connective cdgas*). Let $W \subset \underline{\mathrm{CAlg}}^{\leq 0}$ be the class of quasi-isomorphisms, i.e. morphisms of commutative dg algebras $A_1 \to A_2$ which induce isomorphisms on cohomology. We can then formally invert quasi-isomorphisms to obtain a new category $\underline{\mathrm{CAlg}}^{\leq 0}[W^{-1}]$ where quasi-isomorphisms become isomorphisms. It turns out to be more useful to instead consider the corresponding ∞-category that we will denote by $\mathrm{CAlg}^{\leq 0} = \mathrm{N}(\underline{\mathrm{CAlg}}^{\leq 0})[W^{-1}]$. Explicitly, we may consider the Dwyer–Kan localization [DK80] of $\underline{\mathrm{CAlg}}^{\leq 0}$ with respect to W to obtain a simplicial category which can then be turned into an ∞-category (a quasi-category, i.e. a weak Kan complex) by applying the homotopy coherent nerve (see e.g. [Lur09]).

The key property that makes the ∞-category $\mathrm{CAlg}^{\leq 0}$ manageable is that $\underline{\mathrm{CAlg}}^{\leq 0}$ admits the structure of a combinatorial simplicial model category. In particular, we may identify $\mathrm{CAlg}^{\leq 0}$ with the homotopy coherent nerve of the simplicial category $\underline{\mathrm{CAlg}}^{\leq 0, cof}$ of cofibrant connective cdgas.

The ∞-category of *derived affine schemes* is then defined to be the opposite ∞-category to that of connective cdgas: $\mathrm{dAff} = (\mathrm{CAlg}^{\leq 0})^{op}$. The ∞-category of *derived prestacks* is the ∞-category of presheaves: $\mathrm{dPSt} = \mathrm{Fun}(\mathrm{dAff}^{op}, \mathcal{S})$, where $\mathcal{S}$ is the ∞-category of spaces (∞-groupoids, i.e. Kan complexes). The ∞-category $\mathcal{S}$ has important subcategories Gpd of groupoids and Set of sets. Let us also denote by $\mathrm{Aff} = (\mathrm{CAlg}^0)^{op}$ the category of affine schemes defined to be the opposite of that of (non dg) commutative algebras. Functorial approaches to various flavors of algebraic geometry are summarized in the following diagram:

$$\begin{tikzcd}
\mathrm{Aff}^{op} \arrow[r, "\text{schemes, algebraic spaces}"] \arrow[dr, "\text{algebraic stacks}"] \arrow[dd, hook] & \mathrm{Set} \arrow[d, hook] \\
& \mathrm{Gpd} \arrow[d, hook] \\
\mathrm{dAff}^{op} \arrow[r, "\text{derived algebraic stacks}"] & \mathcal{S}
\end{tikzcd}$$

One can think about passing from affine schemes to derived prestacks as solving two distinct problems. Passing from affine schemes to derived affine schemes improves intersections, i.e. finite limits. For instance, an iterated intersection of smooth affine schemes can be non-lci and so will have an unbounded cotangent complex. However, an iterated derived intersection will always have a bounded cotangent complex. Passing from derived affine schemes to derived prestacks improves quotients, i.e. colimits. Namely, freely adding colimits to the ∞-category of derived affine schemes we obtain the ∞-category of derived prestacks.

Geometric structures on derived affine schemes which are functorial for arbitrary maps can be canonically extended to derived prestacks. Consider an ∞-category $\mathcal{C}$ and a functor $F\colon \mathrm{dAff}^{op} \to \mathcal{C}$. Then we can define $F\colon \mathrm{dPSt}^{op} \to \mathcal{C}$ as a right Kan extension along the inclusion $\mathrm{dAff}^{op} \hookrightarrow \mathrm{dPSt}^{op}$. So, given a derived prestack X we have

$$F(X) = \lim_{S \to X} F(S),$$

where the limit is over derived affine schemes mapping to X. The following are some examples of such geometric constructions:

- We have a functor $\mathcal{O}\colon \mathrm{dAff}^{op} \to \mathrm{CAlg}$ to the ∞-category of commutative dg algebras which is just the usual inclusion $\mathrm{CAlg}^{\leq 0} \to \mathrm{CAlg}$ of connective cdgas into all cdgas. We may think of $\mathcal{O}$ as the commutative dg algebra of global functions on the corresponding derived affine scheme.
- We have a functor $\mathrm{QCoh}\colon \mathrm{dAff}^{op} \to \mathrm{Cat}_\infty$ to the ∞-category of ∞-categories given by sending a connective commutative dg algebra A to Mod_A, the ∞-category of A-modules. One may think of QCoh as the ∞-category of complexes of quasi-coherent sheaves on the corresponding derived affine scheme.

Note that even though this gives a definition of the corresponding geometric structure on an arbitrary prestack, it may fail to have nice properties. For instance, the ∞-category $\mathrm{QCoh}(X)$ for a general prestack will fail to be compactly generated (in particular, dualizable as a stable presentable ∞-category) while it is so when X is a derived affine scheme.

3.2. Poisson geometry. Recall that a *Poisson algebra* is a commutative algebra A equipped with a Lie bracket $\{-,-\}$ which satisfies the Leibniz rule $\{a, bc\} = \{a, b\}c+\{a, c\}b$. Let X be a smooth affine scheme. A *Poisson structure* on X is the structure of a Poisson algebra on $\mathcal{O}(X)$, the algebra of global functions.

This can be reformulated as follows. Let us define the graded commutative algebra of polyvector fields $\mathrm{Pol}(X, 0) = \Gamma(X, \mathrm{Sym}(\mathrm{T}_X[-1]))$ whose degree n part is $\Gamma(X, \wedge^n\mathrm{T}_X)$. It carries a natural Lie algebra structure of degree -1 given by the *Schouten* bracket which extends the usual bracket of vector fields. Then a Poisson structure on X is the same as a bivector $\pi \in \Gamma(X, \wedge^2\mathrm{T}_X)$ which satisfies the Jacobi equation $[\pi, \pi] = 0$.

An important class of subschemes of Poisson schemes is given by coisotropic subschemes. Suppose $L \subset X$ is a smooth closed subscheme of a Poisson scheme X. We say that L is *coisotropic* if the Poisson bivector π of X vanishes along the composite

$$\Gamma(X, \wedge^2\mathrm{T}_X) \longrightarrow \Gamma(L, \wedge^2\mathrm{T}_X\big|_L) \longrightarrow \Gamma(L, \wedge^2\mathrm{N}_{L/X}),$$

where $\mathrm{N}_{L/X} = \mathrm{T}_X|_L/\mathrm{T}_L$ is the normal bundle to L.

Given a Poisson scheme X, we may consider its deformation quantization which we briefly recall. Let A be an associative algebra flat over $k[\![\hbar]\!]$ such that $A_0 = A/\hbar$ is commutative. Then A_0 carries a natural Poisson structure. Indeed, the map $A \to A_0$ is surjective, so given two elements $a, b \in A_0$ we can consider their arbitrary lifts $\tilde{a}, \tilde{b} \in A$. By assumption the commutator $[\tilde{a}, \tilde{b}]$ vanishes modulo $\hbar$ and by flatness we may define

$$\{a, b\} = \frac{[\tilde{a}, \tilde{b}]}{\hbar} \pmod{\hbar} \in A_0.$$

A *deformation quantization* of X is then such an associative algebra A together with an isomorphism of Poisson algebras $A_0 \cong \mathcal{O}(X)$.

The importance of coisotropic subschemes is given by the following observation. If $L \subset X$ is a subscheme, $\mathcal{O}(L)$ is naturally a module over $\mathcal{O}(X)$. Suppose that A is a deformation quantization of X together with an A-module M. Moreover, assume that we have an isomorphism of $\mathcal{O}(X)$-modules $\mathcal{O}(L) \cong M/\hbar$. Then L is necessarily a coisotropic subscheme. In fact, the coisotropic condition already appears when one considers deformation quantization modulo $\hbar^2$.

An important class of Poisson structures is given by those which are nondegenerate. Namely, a Poisson structure π on X induces a map of vector bundles $\pi^\sharp\colon \mathrm{T}^*_X \to \mathrm{T}_X$. We say π is *nondegenerate* if the map $\pi^\sharp$ is an isomorphism. In this case we can invert it to obtain an isomorphism $\mathrm{T}_X \to \mathrm{T}^*_X$ which corresponds to a two-form ω on X. Moreover, the Jacobi equation $[\pi, \pi] = 0$ becomes equivalent to the condition that ω is closed with respect to the de Rham differential d_{dR}. In other words, a nondegenerate Poisson structure on X is the same as a *symplectic structure* on X.

Given a symplectic scheme X we say that a smooth closed subscheme $L \subset X$ is *Lagrangian* if ω restricts to zero on L and L is half-dimensional. It is then easy to see that $L \subset X$ is a coisotropic subscheme.

3.3. Shifted symplectic geometry. Our next goal will be to explain how to extend the definition of Poisson structures to derived prestacks. We begin with an easier case of symplectic structures to explain the salient features in a simpler setting.

Given a connective cdga A, we have its cotangent complex $\mathbb{L}_A \in \mathrm{Mod}_A$ which is an A-module equipped with a universal derivation $\mathrm{d}_{\mathrm{dR}}\colon A \to \mathbb{L}_A$. This allows us to define the de Rham algebra $\mathrm{DR}(A) = \mathrm{Sym}(\mathbb{L}_A[-1])$. It is a cdga equipped with an additional *weight* grading where $\mathbb{L}_A$ is put in weight 1. Moreover, it comes with a de Rham differential d_{dR} which raises the weight and cohomological degree by 1; we call it a *mixed structure*. Thus, we say that the de Rham algebra $\mathrm{DR}(A)$ is a graded mixed cdga. We denote the ∞-category of graded mixed cdgas by $\mathrm{CAlg}^{gr,\epsilon}$. The de Rham complex defines a functor

$$\mathrm{DR}\colon \mathrm{dAff}^{op} \longrightarrow \mathrm{CAlg}^{gr,\epsilon},$$

so we can define its value on a derived prestack to be given by the right Kan extension.

For a derived prestack X we define a closed two-form of degree n to be an infinite series $\omega = \omega_2 + \omega_3 + \dots$, where ω_p has weight p and degree $n+2$ in $\mathrm{DR}(X)$, such that $(\mathrm{d} + \mathrm{d}_{\mathrm{dR}})\omega = 0$, where d is the cohomological differential on $\mathrm{DR}(X)$. The latter condition decomposes into a series of equations

$$\begin{aligned} \mathrm{d}\omega_2 &= 0 \\ \mathrm{d}_{\mathrm{dR}}\omega_2 + \mathrm{d}\omega_3 &= 0 \\ &\dots \end{aligned}$$

In other words, we may think of ω_2 as a differential two-form on X closed up to a coherent homotopy specified by $\omega_3, \omega_4, \dots$. Closed two-forms of degree n form an ∞-groupoid $\mathcal{A}^{2,cl}(X, n) \in \mathcal{S}$.

To discuss nondegeneracy of two-forms, let us assume that X itself has a cotangent complex $\mathbb{L}_X \in \mathrm{QCoh}(X)$ which is again uniquely specified by a universal property. For a morphism $S \to X$ from a derived affine scheme S we have a natural pullback map

$$\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])) \longrightarrow \Gamma(S, \mathrm{Sym}(\mathbb{L}_S[-1]))$$

of graded cdgas. In particular, using the universal property of the limit, we get a pullback map of graded cdgas

$$\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])) \longrightarrow \mathrm{DR}(X)$$

which is not necessarily an equivalence.

An important class of derived prestacks is given by derived Artin stacks which contain in particular quotients $[X/G]$ of derived affine schemes X by algebraic groups G. It is shown in [Pan+13] that the map $\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])) \to \mathrm{DR}(X)$ is an equivalence when X is a derived Artin stack. Thus, a closed two-form of degree n on X defines a degree n element of $\Gamma(X, \wedge^2\mathbb{L}_X)$. Let us recall that a derived prestack X is *locally of finite presentation* if for every inverse system S_i of derived affine schemes the natural map

$$\operatorname{colim} \mathrm{Hom}(S_i, X) \longrightarrow \mathrm{Hom}(\lim S_i, X)$$

is an equivalence. It follows that a derived prestack X locally of finite presentation, if it admits a cotangent complex, has a perfect, i.e. dualizable, cotangent complex. In particular, in this situation we may define the tangent complex $\mathbb{T}_X \in \mathrm{QCoh}(X)$ as the dual of $\mathbb{L}_X$.

Combining the two results above, let us assume X is a derived Artin stack locally of finite presentation equipped with a closed two-form ω of degree n. Then we get an element of $\Gamma(X, \wedge^2\mathbb{L}_X)$ of degree n and hence a map $\omega^\sharp\colon \mathbb{T}_X \to \mathbb{L}_X$. An *$n$-shifted symplectic structure* on X is a closed two-form of degree n such that $\omega^\sharp$ is an equivalence. n-shifted symplectic structures on X form an ∞-groupoid we denote by $\mathrm{Symp}(X, n) \subset \mathcal{A}^{2,cl}(X, n)$.

For a smooth scheme X, the ∞-groupoid $\mathcal{A}^{2,cl}(X, 0)$ is the set of closed two-forms on X and $\mathrm{Symp}(X, 0) \subset \mathcal{A}^{2,cl}(X, 0)$ is the subset of symplectic structures on X in the usual sense. An interesting example of shifted symplectic structures is given by considering the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ of a reductive algebraic group G. Then $\mathcal{A}^{2,cl}(\mathrm{B}G, 2)$ is the set of G-invariant symmetric bilinear pairings on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The ∞-groupoid $\mathrm{Symp}(\mathrm{B}G, 2)$ of 2-shifted symplectic structures is the set of nondegenerate G-invariant symmetric bilinear pairings on $\mathfrak{g}$.

One may also define isotropic and Lagrangian subschemes in the derived setting. There are two main differences to the classical situation:

- In the classical setting being a Lagrangian is a property of a subscheme $L \subset X$. In the derived setting it is given by extra structure on L.
- In the derived setting we may consider Lagrangian structures on arbitrary morphisms $L \to X$, not necessarily on subschemes.

The definition of an isotropic structure is similar to the classical one. Namely, suppose X is a derived prestack equipped with a closed two-form ω of degree n and $f\colon L \to X$ is a morphism of derived prestacks. Then an *n-shifted isotropic structure* on f is a nullhomotopy of $f^*\omega$ in $\mathcal{A}^{2,cl}(L, n)$.

Now suppose L and X are derived Artin stacks locally of finite presentation. Then an n-shifted isotropic structure on f provides dotted arrows in the diagram

$$\begin{tikzcd}
\mathbb{T}_L \arrow[r] \arrow[d, dashed] & f^*\mathbb{T}_X \arrow[r] \arrow[d, "\omega^\sharp"] & \mathrm{N}_{L/X} \arrow[d, dashed] \\
\mathrm{N}^*_{L/X}[n] \arrow[r] & f^*\mathbb{L}_X[n] \arrow[r] & \mathbb{L}_L[n]
\end{tikzcd}$$

An *n-shifted Lagrangian structure* on f is an n-shifted isotropic structure such that $\mathbb{T}_L \to \mathrm{N}^*_{L/X}[n]$ is an equivalence. Note that this is equivalent to the condition that $\mathrm{N}_{L/X} \to \mathbb{L}_L[n]$ is an equivalence since both rows are fiber sequences in $\mathrm{QCoh}(L)$.

An interesting construction allows one to reduce the shift of a symplectic structure as follows. Suppose X is equipped with an n-shifted symplectic structure ω and $f_1\colon L_1 \to X$ and $f_2\colon L_2 \to X$ are equipped with n-shifted Lagrangian structures. By definition it means that we have nullhomotopies of $f_1^*\omega$ and $f_2^*\omega$. Then the pullback of ω to $L_1 \times_X L_2$ has two trivializations. Comparing the two trivializations produces a closed two-form of degree $n-1$ on $L_1 \times_X L_2$. It is shown in [Pan+13] that this is in fact an $(n-1)$-shifted symplectic structure.

3.4. Shifted Poisson geometry. We begin by explaining the notion of an n-shifted Poisson structure on a derived affine scheme. Note that the definition of a Poisson algebra naturally extends to commutative dg algebras. Moreover, once we add a cohomological grading we may consider Poisson brackets $\{-,-\}$ of nontrivial cohomological degree. So, we may define a $\mathbb{P}_n$*-algebra* to be a commutative dg algebra A eqiupped with a Lie bracket $\{-,-\}$ of cohomological degree $1-n$ satisfying the Leibniz rule $\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b$. We see that a $\mathbb{P}_1$-algebra is merely a dg Poisson algebra. A $\mathbb{P}_2$-algebra is a Gerstenhaber algebra.

We can organize $\mathbb{P}_n$-structures into an ∞-groupoid in the following way. Let $\underline{\mathrm{Alg}}_{\mathbb{P}_n}$ be the category of $\mathbb{P}_n$-algebras and denote by $W \subset \mathrm{Alg}_{\mathbb{P}_n}$ the class of quasi-isomorphisms. We then have the ∞-category

$$\mathrm{Alg}_{\mathbb{P}_n} = \mathrm{N}(\underline{\mathrm{Alg}}_{\mathbb{P}_n})[W^{-1}]$$

and a forgetful functor forget: $\mathrm{Alg}_{\mathbb{P}_n} \to \mathrm{CAlg}$ to the ∞-category of commutative dg algebras.

For a commutative dg algebra A we denote by $\mathrm{Pois}(A, n-1)$, the ∞-groupoid of $(n-1)$-*shifted Poisson structures*, the fiber of forget over $A \in \mathrm{CAlg}$.

We may also describe the ∞-groupoid $\mathrm{Pois}(A, n)$ of n-shifted Poisson structures in terms of polyvectors as follows. For a commutative dg algebra A we denote by

$$\mathrm{Pol}(A, n) = \mathrm{Hom}(\mathrm{Sym}(\Omega^1_A[n+1]), A)$$

the graded commutative dg algebra of n*-shifted polyvector fields* on A. It naturally carries a graded $\mathbb{P}_{n+2}$-algebra structure where the multiplication has weight 0 and the Lie bracket (the analog of the Schouten bracket) has weight -1. Let $\mathrm{Pol}(A, n)^{\geq 2}[n+1]$ denote the completion of $\mathrm{Pol}(A, n)$ in weights at least 2 shifted so that it becomes a pro-nilpotent dg Lie algebra. A Maurer–Cartan element in $\mathrm{Pol}(A, n)^{\geq 2}[n+1]$ is a series $\pi = \pi_2 + \pi_3 + \dots$, where π_p is a weight p element of $\mathrm{Pol}(A, n)$, which satisfy a series of equations

$$\mathrm{d}\pi_2 = 0$$

$$\frac{1}{2}[\pi_2, \pi_2] + \mathrm{d}\pi_3 = 0$$

$$\dots$$

Thus, a Maurer–Cartan element in $\mathrm{Pol}(A, n)^{\geq 2}[n+1]$ is a bivector on A which satisfies the Jacobi identity up to coherent homotopy given by $\pi_3, \pi_4, \dots$. It is then shown in [Mel16] that for a cofibrant cdga A the ∞-groupoid $\mathrm{Pois}(A, n)$ is equivalent to the ∞-groupoid of Maurer–Cartan elements in $\mathrm{Pol}(A, n)^{\geq 2}[n+1]$.

Let us now explain how to extend the definition of n-shifted Poisson structures from derived affine schemes to more general derived prestacks following [Cal+17]. Note that as opposed to the de Rham algebra $\mathrm{DR}(-)$, the construction of n-shifted polyvectors $\mathrm{Pol}(-, n)$

is not functorial with respect to arbitrary maps. That is, for an arbitrary map of derived affine schemes $f\colon S_1 \to S_2$, there is no natural map $\mathrm{Pol}(S_2, n) \to \mathrm{Pol}(S_1, n)$ (such a map exists if f is formally étale). In particular, one cannot define $\mathrm{Pol}(-, n)$ on derived prestacks by a right Kan extension.

As in *loc. cit.* we will restrict our attention to derived Artin stacks X locally of finite presentation. The idea is that an n-shifted Poisson structure should be again a Maurer–Cartan element in $\mathrm{Pol}(X, n)^{\geq 2}[n+1]$ where $\mathrm{Pol}(X, n) = \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1]))$. Note that, as opposed to the affine setting, there is no natural Lie bracket on $\mathrm{Pol}(X, n)$.

Recall the de Rham stack X_{dR} defined as $\mathrm{Hom}(S, X_{dR}) = \mathrm{Hom}(\mathrm{H}^0(S)^{red}, X)$ for a derived affine scheme S. By construction X_{dR} admits the zero cotangent complex. Since the map $\mathrm{Pol}(X/X_{dR}, n) \to \mathrm{Pol}(X, n)$ is an equivalence, an n-shifted Poisson structure on X should be the same as a family of n-shifted Poisson structures along the fibers of $p\colon X \to X_{\mathrm{dR}}$. One approach would be to define an n-shifted Poisson structure on X to be a $\mathbb{P}_{n+1}$-structure on the prestack $p_*\mathcal{O}_X$ of commutative dg algebras on X_{dR}. However, one can see that the corresponding deformation complex is not equivalent to $\mathrm{Pol}(X, n)^{\geq 2}[n+1]$. Instead, [Cal+17] construct a prestack $\mathcal{B}_X$ of *graded mixed cdgas* on X_{dR}, en enhanced version of $p_*\mathcal{O}_X$. Moreover, there is a prestack $\mathbb{D}_{X_{\mathrm{dR}}}$ of graded mixed cdgas on X_{dR} which is an enhanced version of $\mathcal{O}_{X_{\mathrm{dR}}}$. Then an n-shifted Poisson structure on X is a $\mathbb{P}_{n+1}$-structure on $\mathcal{B}_X$ linear over $\mathbb{D}_{X_{\mathrm{dR}}}$. The key result of [Cal+17] is that an n-shifted Poisson structure on X is indeed equivalent to a Maurer–Cartan element in $\mathrm{Pol}(X, n)^{\geq 2}[n+1]$ with respect to a certain dg Lie structure also defined using $\mathcal{B}_X$.

4. Main results

In the thesis we extend the foundational results of [Cal+17] on shifted Poisson geometry in the following main directions:

- We define n-shifted coisotropic structures. Suppose X carries an n-shifted Poisson structure and $f\colon L \to X$ is a morphism of derived Artin stacks locally of finite presentation. Then an n-shifted coisotropic structure on f should give a nullhomotopy of $\pi_2 \in \Gamma(X, \wedge^2\mathbb{T}_X)$ along the composite
$$\Gamma(X, \wedge^2\mathbb{T}_X) \longrightarrow \Gamma(L, f^* \wedge^2 \mathbb{T}_X) \longrightarrow \Gamma(L, \wedge^2\mathrm{N}_{L/X})$$
together with higher coherences. To achieve this we define a dg Lie algebra $\mathrm{Pol}(f, n)$ of relative n-shifted polyvectors together with a map to $\mathrm{Pol}(X, n)$ which fits into a fiber sequence of graded dg Lie algebras
$$\mathrm{Pol}(L/X, n-1)[n] \longrightarrow \mathrm{Pol}(f, n)[n+1] \longrightarrow \mathrm{Pol}(X, n)[n+1].$$
- There are many interesting examples of shifted symplectic structures, see e.g. [Cal15], [Saf16], [Spa16] and [PS16]. We compute ∞-groupoids of n-shifted Poisson and coisotropic structures on certain stacks and connect them to more classical topics such as the theory of quantum groups.

4.1. Poisson reduction as a coisotropic intersection [Saf17a]. In [Pan+13] the authors have defined the notion of an n-shifted symplectic structure on a derived stack X and an n-shifted Lagrangian structure on a morphism of derived stacks $L \to X$. Moreover, they prove that given two n-shifted Lagrangian morphisms $L_1 \to X$ and $L_2 \to X$, their intersection $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted symplectic structure. In [Cal+17] the authors have extended this work by defining n-shifted Poisson structures on a derived stack X so that an n-shifted Poisson structure on a derived affine scheme $\operatorname{Spec} A$ is the same as a $\mathbb{P}_{n+1}$-algebra structure on A. The question is then left to define coisotropic structures in this context.

In [Saf17a] we propose a definition of a coisotropic structure on a morphism of commutative dg algebras $A \to B$ where A is a $\mathbb{P}_{n+1}$-algebra and B is a $\mathbb{P}_n$-algebra. Namely, we may consider the algebra of $(n-1)$-shifted polyvectors on B given by $\mathrm{Pol}(B, n-1) = \mathrm{Sym}(\mathrm{T}_B[-n])$. It carries a natural Poisson bracket (the Schouten bracket of polyvector fields) making it into a $\mathbb{P}_{n+1}$-algebra. Moreover, the Poisson structure on B gives a differential $[\pi_B, -]$ on $\mathrm{Pol}(B, n-1)$. Denote by $\mathrm{Pol}_\pi(B, n-1)$ the $\mathbb{P}_{n+1}$-algebra with the differential twisted by $[\pi_B, -]$. Then an n-shifted coisotropic structure on $A \to B$ is a lift of this morphism to a morphism of $\mathbb{P}_{n+1}$-algebras $A \to \mathrm{Pol}_\pi(B, n-1)$. We moreover show that if $A \to B_1$ and $A \to B_2$ carry n-shifted coisotropic structures, then the derived intersection $B_1 \otimes^{\mathbb{L}}_A B_2$ carries a natural $(n-1)$-shifted Poisson structure mimicking the Lagrangian intersection theorem of [Pan+13].

Let us recall from [Cal15] that a Hamiltonian G-manifold M can be considered as a 1-shifted Lagrangian morphism $[M/G] \to [\mathfrak{g}^*/G]$. Moreover, the reduced space is the Lagrangian intersection $[M/G] \times_{[\mathfrak{g}^*/G]} [\mathrm{pt}/G]$. As a litmus test for our definitions of shifted coisotropic structures, we show that a Poisson algebra B equipped with an action of a Lie algebra $\mathfrak{g}$ and a moment map $\mu\colon \mathfrak{g} \to B$ gives rise to a 1-shifted coisotropic morphism $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \to \mathrm{C}^\bullet(\mathfrak{g}, B)$, where the Chevalley–Eilenberg complex $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$ is a $\mathbb{P}_2$ (i.e. a Gerstenhaber) algebra with respect to the so-called big bracket [Kos92]. Moreover, the classical BRST complex [KS87] can be computed as a coisotropic intersection $\mathrm{C}^\bullet(\mathfrak{g}, B) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))} \mathrm{C}^\bullet(\mathfrak{g}, k)$ giving a Poisson version of the Lagrangian intersection $[M/G] \times_{[\mathfrak{g}^*/G]} [\mathrm{pt}/G]$.

This algebraic approach to coisotropic structures admits a straightforward quantization. Just like associative algebras are quantum versions of Poisson (i.e. $\mathbb{P}_1$) algebras, brace (i.e. $\mathbb{E}_2$) algebras are quantum versions of Gerstenhaber (i.e. $\mathbb{P}_2$) algebras. In the paper we also study quantum moment maps in terms of quantization of the coisotropic morphism $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \to \mathrm{C}^\bullet(\mathfrak{g}, B)$. Namely, on the quantum level the moment map gives rise to a brace algebra $\mathrm{HH}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g})$ of Hochschild cochains and its brace module $\mathrm{HH}^\bullet(\mathrm{U}\mathfrak{g}, B)$, the notion we define.

4.2. Braces and Poisson additivity [Saf18]. A completely abstract definition of shifted coisotropic structures was proposed in [Cal+17] which rests on a certain algebraic result about shifted Poisson algebras called Poisson additivity. A result by Dunn and Lurie [Lur17, Theorem 5.1.2.2] asserts that there is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_{n+1}} \cong \mathrm{Alg}(\mathrm{Alg}_{\mathbb{E}_n})$$

between the ∞-category of $\mathbb{E}_{n+1}$-algebras and the ∞-category of algebra objects in the ∞-category of $\mathbb{E}_n$-algebras. In particular, this allows us to think of $\mathbb{E}_n$-algebras as "algebras with n coherent multiplications". As $\mathbb{P}_n$-algebras are classical versions of $\mathbb{E}_n$-algebras, one expects a similar equivalence for $\mathbb{P}_n$-algebras. Namely, there should be an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Alg}_{\mathbb{P}_{n+1}} \cong \mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_n})$$

which we call the Poisson additivity theorem. Such a result was announced by Rozenblyum, but the proof has not appeared in the literature yet. Assuming this result, the authors of [Cal+17] proposed the following concise definition of n-shifted coisotropic morphisms: these are given by a pair of an algebra and a module in the ∞-category of $\mathbb{P}_n$-algebras. By Poisson additivity an algebra in $\mathbb{P}_n$-algebras is a $\mathbb{P}_{n+1}$-algebra, so this definition looks very close to the definition given in [Saf17a] except for a different model of an action of a $\mathbb{P}_{n+1}$-algebra on a $\mathbb{P}_n$-algebra.

In the present paper we prove the Poisson additivity theorem whose proof is strongly influenced by the papers [Tam00] of Tamarkin and [CW15] of Calaque and Willwacher. Namely, given a Hopf dg cooperad $\mathcal{C}$ (i.e. a cooperad in commutative dg algebras) the authors of [CW15] construct a new dg operad $\mathrm{Br}_{\mathcal{C}}$ of braces. The first main result is that under a mild technical assumption on $\mathcal{C}$ there is a functor of ∞-categories

$$\mathrm{Alg}_{\mathrm{Br}_{\mathcal{C}}} \longrightarrow \mathrm{Alg}(\mathrm{Alg}_{\Omega\mathcal{C}}),$$

where $\Omega\mathcal{C}$ is the dg operad given by the cobar construction on $\mathcal{C}$. In the case $\mathcal{C} = \mathrm{co}\mathbb{P}_n = (\mathbb{P}_n)^*$, the cooperad of $\mathbb{P}_n$-coalgebras, we have $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n} \cong \mathbb{P}_{n+1}$ and $\Omega\mathrm{co}\mathbb{P}_n \cong \mathbb{P}_n$. Thus, we get a functor of ∞-categories

$$\mathrm{Alg}_{\mathbb{P}_{n+1}} \longrightarrow \mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_n}).$$

The second main result is that this functor is compatible with symmetric monoidal structures and is an equivalence thus proving the Poisson additivity theorem.

Having provided the proof of Poisson additivity, we now have two definitions of shifted coisotropic structures: the algebraic one from [Saf17a] (elaborated on in [MS16]) and the abstract one from [Cal+17] which uses Poisson additivity. In fact, both definitions naturally give rise to spaces (i.e. ∞-groupoids) of shifted coisotropic structures. Given a morphism $f\colon A \to B$ of commutative dg algebras we have the algebraically-defined space $\mathrm{Cois}^{MS}(f, n)$ and the space $\mathrm{Cois}^{CPTVV}(f, n)$ whose definition uses Poisson additivity. The third main result of the paper is that there is an equivalence of spaces $\mathrm{Cois}^{MS}(f, n) \cong \mathrm{Cois}^{CPTVV}(f, n)$ which shows that the two definitions of shifted coisotropic structures coincide.

4.3. Derived coisotropic structures I: affine case [MS16], with Valerio Melani. Following the definition of shifted coisotropic structures given in [Saf17a] and [Saf18], we study the homotopy theory of shifted coisotropic structures. Namely, the definition given in [Saf17a] can be reformulated as a construction of a two-colored operad $\mathbb{P}_{[n+1,n]}$ whose algebras are triples (A, B, f) of a $\mathbb{P}_{n+1}$-algebra A, a $\mathbb{P}_n$-algebra B and a morphism $f\colon A \to \mathrm{Pol}_\pi(B, n-1)$ of $\mathbb{P}_{n+1}$-algebras. Our first main result is a construction of a cofibrant replacement

$$\widetilde{\mathbb{P}}_{[n+1,n]} \to \mathbb{P}_{[n+1,n]}$$

of colored operads. This allows one to explicitly describe the space of n-shifted coisotropic structures $\mathrm{Cois}^{MS}(f, n)$ as $\widetilde{\mathbb{P}}_{[n+1,n]}$ is semi-free.

The construction of $\widetilde{\mathbb{P}}_{[n+1,n]}$ is given by a relative version of the brace construction of [CW15]. Namely, given a dg cooperad $\mathcal{C}_1$ and a Hopf dg cooperad $\mathcal{C}_2$ together with a morphism $\Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}$ we construct a colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ whose algebras are triples (A, B, f) of a $\Omega\mathcal{C}_1$-algebra A, a $\Omega\mathcal{C}_2$-algebra B and an ∞-morphism of $\Omega\mathcal{C}_1$-algebras $f\colon A \to \mathbf{Z}(B)$, where $\mathbf{Z}(B)$ is the operadic convolution algebra of B shown to be a $\mathrm{Br}_{\mathcal{C}_2}$-algebra in [CW15].

It was shown in [Mel16] that the space $\mathrm{Pois}(A, n)$ of n-shifted Poisson structures on A can be computed as a Maurer–Cartan space of the dg Lie algebra

$$\mathrm{Pol}^{\geq 2}(A, n)[n+1] = \mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathbb{L}_A[n+1]), A)[n+1]$$

of n-shifted polyvector fields. The second main result of the paper is an identification of the space $\mathrm{Cois}^{MS}(f, n)$ of n-shifted coisotropic structures with a Maurer–Cartan space of a certain dg Lie algebra $\mathrm{Pol}^{\geq 2}(f, n)[n+1]$ of relative n-shifted polyvector fields. In particular, it implies that given a smooth closed subscheme $f\colon L \hookrightarrow X$ of a smooth scheme, the space $\mathrm{Cois}^{MS}(f, 0)$ of 0-shifted coisotropic structures on f is the *set* of ordinary Poisson structures on X for which L is coisotropic in the usual sense. Thus, the derived notion of coisotropic morphisms subsumes the classical notion of coisotropic subschemes.

4.4. Derived coisotropic structures II: derived stacks and quantization [MS17], with Valerio Melani. Following up on our previous work [MS16], in [MS17] we extend the results from commutative dg algebras to derived stacks. Namely, we give a definition of an n-shifted coisotropic structure on a morphism of derived stacks $f\colon L \to X$ and compute the space of such n-shifted coisotropic structures in terms of a dg Lie algebra $\mathrm{Pol}^{\geq 2}(f, n)[n+1]$ of relative n-shifted polyvector fields. Moreover, we show that if a derived stack X carries an n-shifted Poisson structure and morphisms of derived stacks $L_1 \to X$ and $L_2 \to X$ carry n-shifted coisotropic structures, the derived intersection $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted Poisson structure.

One of the results of [Cal+17] and [Pri17b] is a comparison between the space of nondegenerate n-shifted Poisson structures on a derived stack X and the space of n-shifted symplectic structures on X. Our second main result identifies nondegenerate n-shifted coisotropic structures on a morphism of derived stacks $f\colon L \to X$ with n-shifted Lagrangian structures on f (generalizing the case of 0-shifted Lagrangians from [Pri16]). In particular, this implies that if $L \to X$ carries an n-shifted symplectic structure, there is an induced $(n-1)$-shifted Poisson structure on L, a result that has been found very useful (see [Saf17b], [HP17], [Toë18] for examples of applications).

Finally, we discuss the notion of quantization of n-shifted coisotropic morphisms. While the quantization of 0-shifted Poisson structures on derived stacks is difficult (see, however, [Pri17a]), it was shown in [Cal+17] that there are canonical quantizations of n-shifted Poisson stacks for $n \geq 1$ obtained from formality. Generalizing this result, we show that there is a canonical quantization of n-shifted coisotropic morphisms of derived stacks for $n \geq 2$.

4.5. Poisson-Lie structures as shifted Poisson structures [Saf17b]. A relationship between multiplicative Poisson structures and higher symplectic geometry has a long history. A prototypical instance is given by the following sequence of results. Suppose $\mathcal{G} \rightrightarrows X$ is a smooth algebraic groupoid over a smooth scheme X and let $\mathcal{L}$ be its Lie algebroid over X. A multiplicative Poisson structure on $\mathcal{G}$ (i.e. the structure of a Poisson groupoid on $\mathcal{G}$) gives rise to a Lie bialgebroid structure on $\mathcal{L}$ [MX94]. In turn, we can regard a Lie bialgebroid $\mathcal{L}$ as a Dirac structure in its double $\mathcal{L} \oplus \mathcal{L}^*$ which is a Courant algebroid [LWX97]. Dirac structures in Courant algebroids can be viewed as 2-shifted Lagrangian morphisms

[Roy02]. By one of the results of [MS17], this gives rise to a 1-shifted Poisson structure. Thus, Poisson groupoids are naturally associated with 1-shifted Poisson structures. In the present paper we elucidate this relationship.

Suppose G is an affine algebraic group and let $\mathrm{B}G = [\mathrm{pt}/G]$ be its classifying stack. One of our results is a computation of the space $\mathrm{Pois}(\mathrm{B}G, n)$ of n-shifted Poisson structures for $n \geq 1$. First, we show that it is trivial for $n \geq 3$. The space $\mathrm{Pois}(\mathrm{B}G, 2)$ is a set of Casimir elements, i.e. elements of $\mathrm{Sym}^2(\mathfrak{g})^G$. The space $\mathrm{Pois}(\mathrm{B}G, 1)$ is the groupoid of quasi-Poisson structures on G with morphisms given by twists. We also relate classical (dynamical) r-matrices to 1-shifted Poisson morphisms.

Recall that in [MS17] we have shown that if $L \to X$ is an n-shifted Lagrangian morphism, there is a natural $(n-1)$-shifted Poisson structure on L. Thus, we may define a *symplectic realization* of an $(n-1)$-shifted Poisson stack L to be such an n-shifted Lagrangian morphism. We show that for $n = 1$ we recover the usual notion of symplectic realizations of Poisson structures in terms of symplectic groupoids [Wei83]. Suppose $G \subset D$ is a subgroup such that on the level of Lie algebras $\mathfrak{g} \subset \mathfrak{d}$ is a Manin pair, i.e. $\mathfrak{d}$ carries a nondegenerate pairing and $\mathfrak{g}$ is a Lagrangian subalgebra. In particular, in such a situation there is a natural quasi-Poisson structure on G [AK00]. We show that the 2-shifted Lagrangian morphism $\mathrm{B}G \to \mathrm{B}D$ is a symplectic realization of the corresponding 1-shifted Poisson structure on $\mathrm{B}G$. Moreover, we show that Manin triples (G, G^*, D) give rise to symplectic realizations of the 1-shifted coisotropic morphism $\mathrm{pt} \to \mathrm{B}G$.

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CHAPTER 2

Collected papers

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Poisson reduction as a coisotropic intersection

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Abstract

We give a definition of coisotropic morphisms of shifted Poisson (i.e. $\mathbb{P}_n$) algebras which is a derived version of the classical notion of coisotropic submanifolds. Using this we prove that an intersection of coisotropic morphisms of shifted Poisson algebras carries a Poisson structure of shift one less. Using an interpretation of Hamiltonian spaces as coisotropic morphisms we show that the classical BRST complex computing derived Poisson reduction coincides with the complex computing coisotropic intersection. Moreover, this picture admits a quantum version using brace algebras and their modules: the quantum BRST complex is quasi-isomorphic to the complex computing tensor product of brace modules.

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Introduction

The goal of the present paper is to introduce the notion of a coisotropic structure on shifted Poisson algebras on the level of 1-categories and show that it satisfies some expected properties such as:

- Moment maps provide examples of coisotropic structures,
- A derived intersection $B_1 \otimes^{\mathbb{L}}_A B_2$ of coisotropic maps $A \to B_1$ and $A \to B_2$, where A is an n-shifted Poisson algebra, carries an $(n-1)$-shifted Poisson structure up to homotopy.

The homotopy theory of such coisotropic structures is further studied in [MS16] and [MS17].

Coisotropic intersections Motivated by Lagrangian Floer theory and Donaldson–Thomas theory, Behrend and Fantechi [BF10] showed that the cohomology of the algebra of functions on a derived intersection of two holomorphic Lagrangian submanifolds of a complex symplectic manifold carries a (-1)-shifted Poisson ($\mathbb{P}_0$) structure.

Pantev, Toën, Vaquié and Vezzosi [PTVV] gave a derived-geometric interpretation of this result. Namely, it was shown that a derived intersection of two algebraic Lagrangians carries a

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(-1)-shifted symplectic structure. More generally, they have shown that a derived intersection of two Lagrangians in an n-shifted symplectic stack is $(n-1)$-shifted symplectic.

Baranovsky and Ginzburg [BG09] generalized the Behrend–Fantechi result in a different direction. Namely, they have shown that the cohomology of the algebra of functions on the derived intersection of two coisotropic subvarieties of a Poisson variety carries a $\mathbb{P}_0$-structure. It is thus natural to ask whether one can lift the Baranovsky–Ginzburg construction to the chain level.

Calaque, Pantev, Toën, Vaquié and Vezzosi [CPTVV] introduced n-shifted Poisson structures on derived stacks and derived coisotropic structures on morphisms of stacks. Let us recall their definitions in the affine setting. Let A be a commutative dg algebra. By a theorem of Melani [Mel14], an n-shifted Poisson structure on A is the same as a $\mathbb{P}_{n+1}$-structure on A, i.e. a Poisson bracket of cohomological degree $-n$. For B another commutative dg algebra, CPTVV define a coisotropic structure on a morphism $A \to B$ to be the same as a $\mathbb{P}_n$-structure on B together with the data of an associative action of A on B in the category of $\mathbb{P}_n$-algebras. To define such a notion, they use a result announced by Rozenblyum (Poisson additivity) which identifies $\mathbb{P}_{n+1}$-algebras with associative algebra objects in the ∞-category of $\mathbb{P}_n$-algebras. This definition is expected to give rather easily a $\mathbb{P}_n$-structure on a coisotropic intersection. However, Poisson additivity is not given by explicit formulas, so the explicit Poisson structure on the coisotropic intersection would be difficult to write down.

In this paper we develop coisotropic structures in the affine setting, i.e. for arbitrary commutative differential graded algebras. We model an action of the $\mathbb{P}_{n+1}$-algebra A on a $\mathbb{P}_n$-algebra B by a $\mathbb{P}_{n+1}$-morphism $A \to \mathrm{Z}(B)$ (Definition 1.8). Here

$$\mathrm{Z}(B) = \mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_B[n]), B)$$

is the complex of $(n-1)$-shifted polyvector fields with the differential twisted by the Poisson structure on B which is a derived version of the Poisson center of B.

Note that the $\mathbb{P}_{n+1}$-structure on $\mathrm{Z}(B)$ is very explicit: it is given by the Schouten bracket (i.e. by the commutator of multiderivations). Using this definition we prove the following theorem (Theorem 1.18).

Theorem. *Let A be a $\mathbb{P}_{n+1}$-algebra and $A \to B_1,\ A \to B_2$ two coisotropic morphisms. Then the derived intersection $B_1 \otimes^{\mathbb{L}}_A B_2$ carries a homotopy $\mathbb{P}_n$-structure. Moreover, the natural projection $B_1^{\mathrm{op}} \otimes B_2 \to B_1 \otimes^{\mathbb{L}}_A B_2$ is a $\mathbb{P}_n$-morphism where B_1^{op} denotes the same commutative dg algebra with the opposite Poisson bracket.*

The proof of this theorem uses ideas from Koszul duality. Since one can identify by Poisson additivity a $\mathbb{P}_{n+1}$-algebra with an associative algebra object in $\mathbb{P}_n$-algebras, one expects the Koszul dual coalgebra of a $\mathbb{P}_{n+1}$-algebra to carry a compatible $\mathbb{P}_n$-structure; indeed, it is given by explicit formulas using the bar complex (Proposition 1.14). Similarly, we show that the Koszul dual to the A-module B_i carries a homotopy $\mathbb{P}_n$-structure given by the coisotropic structure. Finally, the derived tensor product $B_1 \otimes^{\mathbb{L}}_A B_2$ can be written as an underived cotensor product on the Koszul dual side.

After the present paper was posted on the ArXiv, the proof of Poisson additivity was written down in [Saf16]. In the same paper it was shown that our definition of coisotropic morphisms is equivalent to the one of [CPTVV].

Moment maps We give an application of derived coisotropic intersection to Hamiltonian reduction.

Let us recall that given a symplectic manifold X with a G-action preserving the symplectic form, a moment map is a G-equivariant morphism $\mu\colon X \to \mathfrak{g}^*$ which is a Hamiltonian for the G-action. Hamiltonian reduction is defined to be the quotient

$$X//G = \mu^{-1}(0)/G.$$

If 0 is a regular value for μ and the G-action on $\mu^{-1}(0)$ is free and proper, the quotient is a symplectic manifold as shown by Marsden and Weinstein [MW74]. If one of these conditions fails, the quotient is only a stratified symplectic manifold which hints that it is a shadow of a derived symplectic structure.

Indeed, passing to the setting of derived algebraic geometry we can rewrite

$$X//G \cong \mathrm{pt}/G \times_{\mathfrak{g}^*/G} X/G.$$

Moreover, as shown in [Cal13] and [Saf13], Hamiltonian G-spaces are the same as Lagrangians in the 1-shifted symplectic stack $\mathfrak{g}^*/G$. Therefore, $X//G$ is a Lagrangian intersection and so carries a derived symplectic structure.

In this paper we show similar statements in the affine Poisson setting. Namely, if B is a $\mathbb{P}_1$-algebra (a dg Poisson algebra) with $\mu\colon \operatorname{Sym}\mathfrak{g} \to B$ a moment map for a $\mathfrak{g}$-action on B we show that the induced morphism

$$\mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}\mathfrak{g}) \to \mathrm{C}^\bullet(\mathfrak{g}, B)$$

is coisotropic. Here $\mathrm{C}^\bullet(\mathfrak{g}, -)$ is the Chevalley–Eilenberg cochain complex and $\mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}\mathfrak{g})$ is the $\mathbb{P}_2$-algebra (i.e. Gerstenhaber algebra) of functions on the quotient $\mathfrak{g}^*/G$ with G formal.

The coisotropic intersection

$$\mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}\mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B)$$

is thus a derived Poisson reduction which we show to be quasi-isomorphic (as a commutative dg algebra) to the classical BRST complex as defined by Kostant and Sternberg [KS87].

Let us note that this perspective on Poisson reduction is somewhat orthogonal to the one obtained by computing coisotropic reduction of $\mu^{-1}(0) \subset X$ using the BFV complex (see e.g. [Sta96]). Indeed, in that approach one considers a coisotropic *reduction* of the 0-shifted coisotropic morphism $\mu^{-1}(0) \to X$. On the other hand, in our approach we consider a coisotropic *intersection* of the 1-shifted coisotropic morphism $X/G \to \mathfrak{g}^*/G$. The precise relationship between the two approaches is not clear to the author.

Quantization We also develop quantum versions of our results in the sense of deformation quantization. Namely, while deformation quantizations of $\mathbb{P}_1$-algebras are dg algebras, deformation quantizations of $\mathbb{P}_2$-algebras are $\mathbb{E}_2$-algebras, i.e. algebras over the operad of little disks, which we model by brace algebras following [MS99]. We introduce a notion of a brace module M over a brace algebra A which provides deformation quantization of the notion of a coisotropic morphism from a $\mathbb{P}_2$-algebra A to a $\mathbb{P}_1$-algebra M. One way to think of it is as follows: the pair (brace algebra, brace module) is conjectured to be the same as an algebra over the Swiss-cheese operad introduced by Voronov [Vor98]. We prove the following quantum version of the coisotropic intersection theorem (Theorem 3.10).

Theorem. *Let A be a brace algebra, B_1 a left brace module and B_2 a right brace module over A. Then the derived tensor product $B_1 \otimes^{\mathbb{L}}_A B_2$ carries a natural dg algebra structure such that the projection $B_1^{\mathrm{op}} \otimes B_2 \to B_1 \otimes^{\mathbb{L}}_A B_2$ is an algebra morphism, where B_1^{op} is the algebra with the opposite multiplication.*

We apply this result to quantum moment maps. Recall that a quantum moment map is given by a morphism of associative algebras $\mathrm{U}\mathfrak{g} \to B$, where B is an associative algebra. These are to be thought of as deformation quantizations of Poisson maps $\mathrm{Sym}\,\mathfrak{g} \to B$ (classical moment map), where B is a Poisson algebra.

A quantization of the $\mathbb{P}_2$-algebra $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})$ is the brace algebra $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g})$, the Hochschild cochain complex of the universal enveloping algebra $\mathrm{U}\mathfrak{g}$. We show that a quantum moment map $\mathrm{U}\mathfrak{g} \to B$ makes $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$ into a brace module over $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g})$. The tensor product

$$\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$$

computing derived quantum Hamiltonian reduction is therefore a dg algebra which is shown to be quasi-isomorphic to the quantum BRST complex [KS87].

This point of view on quantum Hamiltonian reduction allows one to generalize ordinary (i.e. $\mathbb{E}_1$) Hamiltonian reduction to $\mathbb{E}_n$-algebras (algebras over the operad of little n-disks) which we sketch in Section 4.5.

Both classical and quantum constructions can be put on the same footing if one starts with a deformation quantization for which we use the language of Beilinson–Drinfeld algebras [CG16, Section 2.4]. We end the paper with some theorems that interpolate between classical coisotropic intersections and tensor products of brace modules.

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Notation We work over a field k of characteristic zero. We adopt the cohomological grading convention. By a dga we mean a differential graded algebra over k not necessarily non-positively graded. For A a dga and M and N two modules we denote by $M \otimes^{\mathbb{L}}_A N$ the resolution given by the two-sided bar complex.

An (n, m)-shuffle $\sigma \in S_{n,m}$ is a permutation $\sigma \in S_{n+m}$ such that $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+m)$.

1. Shifted Poisson algebras

1.1 Polyvector fields Let A be a cdga. We denote by $\mathrm{T}_A = \mathrm{Der}(A, A)$ the A-module of derivations which is a dg Lie algebra over k. We define the complex of $(n-1)$-shifted polyvector fields to be

$$\mathrm{Pol}(A, n-1) = \mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n]), A).$$

$\mathrm{Pol}(A, n-1)$ has a natural *weight* grading under which Ω^1_A has weight -1 and we can decompose

$$\mathrm{Pol}(A, n-1) = \bigoplus_k \mathrm{Pol}(A, n-1)^k = \bigoplus_k \mathrm{Hom}_A(\mathrm{Sym}^k_A(\Omega^1_A[n]), A).$$

We denote by $\lrcorner$ the natural duality pairing between $\mathrm{Pol}(A, n-1)$ and $\mathrm{Sym}_A(\Omega^1[n])$. Given a polyvector $v \in \mathrm{Pol}(A, n-1)^k$ we define

$$v(a_1, \dots, a_k) = v \lrcorner (\mathrm{d_{dR}} \otimes \dots \otimes \mathrm{d_{dR}})(a_1 \otimes \dots \otimes a_k), \tag{1}$$

where the formal symbol $\mathrm{d_{dR}}$ is put in degree $-n$ to fix the signs and $a_i \in A$. The symmetry of v implies that

$$v(a_1, a_2, \dots, a_k) = (-1)^{|a_1||a_1|+n} v(a_2, a_1, \dots, a_k).$$

We define the Schouten bracket of $v \in \mathrm{Pol}(A, n-1)^k$ and $w \in \mathrm{Pol}(A, n-1)^l$ to be

$$\begin{aligned}[v, w](a_1, \dots, a_{k+l-1}) = &\sum_{\sigma \in S_{l,k-1}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_1} v(w(a_{\sigma(1)}, \dots, a_{\sigma(l)}), a_{\sigma(l+1)}, \dots, a_{\sigma(k+l-1)}) \\ &- \sum_{\sigma \in S_{k,l-1}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_2} w(v(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(k+l-1)}),\end{aligned}$$

where $(-1)^\epsilon$ denotes the sign coming from the Koszul sign rule applied to the permutation σ of a_i and the signs ϵ_i are

$$\begin{aligned}\epsilon_1 &= (|w| + l)(k+1)n + |v|n \\ \epsilon_2 &= (|v| - kn)(|w| - ln) + n(k+1)(|w| + 1) + |v|n.\end{aligned}$$

The product of polyvector fields is defined to be

$$(v \cdot w)(a_1, \dots, a_{k+l}) = \sum_{\sigma \in S_{k.l}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_1} v(a_{\sigma(1)}, \dots, a_{\sigma(k)}) w(a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}),$$

where the sign is

$$\epsilon_1 = |w|kn + \sum_{i=1}^{k} |a_{\sigma(i)}|(nl + |w|).$$

1.2 Algebras Let us begin with the basic object in this section which is a weak (and shifted) version of Poisson algebras.

Definition 1.1. A $\widehat{\mathbb{P}}_n$*-algebra* is a cdga A together with an L_∞-algebra structure of degree $1-n$ such that the L_∞ operations l_k are polyderivations with respect to the multiplication. More explicitly, l_k are multilinear operations of degree $1-(k-1)n$ satisfying the following equations:

- (Symmetry).

$$l_k(a_1, \dots, a_i, a_{i+1}, \dots, a_k) = (-1)^{|a_i||a_{i+1}|+n} l_k(a_1, \dots, a_{i+1}, a_i, \dots, a_k).$$

- (Leibniz rule).

$$l_k(a_1, \dots, a_k a_{k+1}) = l_k(a_1, \dots, a_k) a_{k+1} + (-1)^{|a_k||a_{k+1}|} l_k(a_1, \dots, a_{k+1}) a_k.$$

- (Jacobi identity).

$$0 = \sum_{k=1}^{m} (-1)^{nk(m-k)} \sum_{\sigma \in S_{k,m-k}} \mathrm{sgn}(\sigma)^n (-1)^\epsilon l_{m-k+1}(l_k(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(m)}),$$

  where ϵ is the sign coming from the Koszul sign rule.

Given a $\widehat{\mathbb{P}}_n$-algebra A, the opposite algebra A^{op} is defined to be the same cdga together with operations $l_k^{\mathrm{op}} = (-1)^{k+1} l_k$.

There is also a strict version of Poisson algebras as follows.

Definition 1.2. A $\mathbb{P}_n$*-algebra* is a $\widehat{\mathbb{P}}_n$-algebra such that the operations l_k vanish for $k > 2$. In this case we denote the operation l_2 by $\{a, b\}$.

Definition 1.3. A *morphism of* $\widehat{\mathbb{P}}_n$*-algebras* $f\colon A \to B$ is a chain map of complexes $f\colon A \to B$ strictly preserving the multiplication and the L_∞ operations l_k.

Here is an important example of a $\mathbb{P}_{n+1}$-algebra. Observe that the Schouten bracket on $\mathrm{Pol}(A, n-1)$ has cohomological degree $-n$.

Proposition 1.4. *Let A be a cdga. The product and Schouten bracket define a $\mathbb{P}_{n+1}$-structure on the complex of $(n-1)$-shifted polyvector fields $\mathrm{Pol}(A, n-1)$.*

A $\mathbb{P}_n$-structure on a cdga A is given by a bivector $\pi_A \in \mathrm{Pol}(A, n-1)$ of degree $n+1$, so that

$$\{a, b\} := \pi_A(a, b).$$

The Jacobi identity for the bracket then becomes

$$[\pi_A, \pi_A] = 0.$$

Given a $\mathbb{P}_n$-algebra A, we can naturally produce a $\mathbb{P}_{n+1}$-algebra $\mathrm{Z}(A)$ as follows.

Definition 1.5. Let A be a $\mathbb{P}_n$-algebra. Its *Poisson center* is the $\mathbb{P}_{n+1}$-algebra given by the completion

$$\mathrm{Z}(A) = \widehat{\mathrm{Pol}}(A, n-1)$$

of the algebra of $(n-1)$-shifted polyvector fields with respect to the weight grading. The Lie bracket is given by the Schouten bracket. The differential has two components: the differential on the module of Kähler differentials and $[\pi_A, -]$.

Remark 1.6. Suppose A is a non-dg Poisson algebra. Then $\mathrm{Z}(A)$ coincides with the Lichnerowicz–Poisson complex $C^\bullet_{LP}(A, A)$, see [Fre06, Section 1.4.8], whose zeroth cohomology is the space of Casimir functions. See also [CW13, Theorem 2] for a relation between $\mathrm{Z}(A)$ and a Poisson analog of the Hochschild complex.

Remark 1.7. We believe that if A is cofibrant as a commutative dg algebra, $\mathrm{Z}(A)$ is a model of the center of $A \in \mathrm{Alg}_{\mathbb{P}_n}$ in the sense of [Lu, Definition 5.3.1.6]. We will return to this comparison in a future work.

We have a morphism

$$\mathrm{Z}(A) \to A$$

of commutative dg algebras given by projecting to the weight zero part of polyvector fields.

1.3 Modules Let A be a $\mathbb{P}_{n+1}$-algebra and M a cdga.

Definition 1.8. A *coisotropic structure* on a morphism of commutative dg algebras $f\colon A \to M$ is a $\mathbb{P}_n$-algebra structure on M and a lift

$$\begin{tikzcd} A \arrow[r, dashed, "\tilde{f}"] \arrow[dr, "f"'] & \mathrm{Z}(M) \arrow[d] \\ & M, \end{tikzcd}$$

where $\tilde{f}\colon A \to \mathrm{Z}(M)$ is a morphism of $\mathbb{P}_{n+1}$-algebras.

Here is a way to unpack this definition. A coisotropic structure consists of maps

$$f_k\colon A \to \mathrm{Hom}_M(\mathrm{Sym}^k(\Omega^1_M[n]), M)$$

for $k \geq 0$, where $f_0 = f$ is the original morphism. We define the maps

$$f_k\colon A \otimes M^{\otimes k} \to M[-nk]$$

by

$$f_k(a; m_1, \dots, m_k) := f_k(a)(m_1, \dots, m_k).$$

They satisfy the following equations:

- (Symmetry).

$$f_k(a; m_1, \dots, m_i, m_{i+1}, \dots, m_k) = (-1)^{|m_i||m_{i+1}|+n} f_k(a; m_1, \dots, m_{i+1}, m_i, \dots, m_k) \tag{2}$$

  for every $a \in A$ and $m_i \in M$.
- (Derivation).

$$f_k(a; m_1, \dots, m_k m_{k+1}) = f_k(a; m_1, \dots, m_k) m_{k+1} + (-1)^{|m_k||m_{k+1}|} f_k(a; m_1, \dots, m_{k+1}) m_k \tag{3}$$

  for every $a \in A$ and $m_i \in M$.
- (Compatibility with the differential).

$$\begin{aligned} &\mathrm{d} f_k(a; m_1, \dots, m_k) = \\ &\quad f_k(\mathrm{d}a; m_1, \dots, m_k) + \sum_{i=1}^{k} (-1)^{|a| + \sum_{j=1}^{i-1} |m_j| + nk} f_k(a; m_1, \dots, \mathrm{d}m_i, \dots, m_k) \\ &- \sum_{i=1}^{k} (-1)^{n(|a|+i-1) + |m_i| \sum_{j=i+1}^{k} |m_j|} \{f_{k-1}(a; m_1, \dots, \hat{m}_i, \dots, m_k), m_i\} \\ &+ \sum_{i<j} (-1)^{|m_i| \sum_{l=1}^{i-1} |m_l| + |m_j| \sum_{l=1, l\neq i}^{j-1} |m_l| + n(i+j) + |a|} f_{k-1}(a; \{m_i, m_j\}, m_1, \dots, \hat{m}_i, \dots, \hat{m}_j, \dots m_k) \end{aligned} \tag{4}$$

  for every $a \in A$ and $m_i \in M$.

- (Compatibility with the brackets).
  For every $a_1, a_2 \in A$ and $m_i \in M$ we have

$$
\begin{aligned}
&f_k(\{a_1, a_2\}; m_1, \dots, m_k) \\
&\quad = \sum_{i+j=k+1} \sum_{\sigma \in S_{j,i-1}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_1} f_i(a_1; f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) \\
&\quad - \sum_{i+j=k+1} \sum_{\sigma \in S_{j,i-1}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_2} f_i(a_2; f_j(a_1; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}),
\end{aligned}
\tag{5}
$$

  where the signs are

$$
\begin{aligned}
\epsilon_1 &= (|a_2| + j)(i+1)n + |a_1|n \\
\epsilon_2 &= (|a_1| - jn)(|a_2| - in) + n(j+1)(|a_2| + 1) + |a_1|n.
\end{aligned}
$$

- (Compatibility with the product).
  For every $a_1, a_2 \in A$ and $m_i \in M$ we have

$$
\begin{aligned}
&f_k(a_1 a_2; m_1, \dots, m_k) \\
&\quad = \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \mathrm{sgn}(\sigma)^n (-1)^{\epsilon+\epsilon_1} f_i(a_1; m_{\sigma(1)}, \dots, m_{\sigma(i)}) f_j(a_2; m_{\sigma(i+1)}, \dots, m_{\sigma(k)}),
\end{aligned}
\tag{6}
$$

  where the sign is $\epsilon_1 = |a_2|ni + \sum_{l=1}^{i} |m_{\sigma(l)}|(nj + |a_2|)$.

Remark 1.9. Equation (5) for $k = 0$ reads as

$$f_0(\{a_1, a_2\}) = (-1)^{|a_1|n} f_1(a_1; f_0(a_2)) - (-1)^{n(|a_2|+1)+|a_1||a_2|} f_1(a_2; f_0(a_1)).$$

In particular, the kernel of f_0 is closed under the Poisson bracket and so $\mathrm{Spec}\, M \to \mathrm{Spec}\, A$ is a coisotropic subscheme in the usual sense.

Example 1.10. Several examples of coisotropic structures as above are constructed in [JS15, Examples 3.20 and 3.21] from shifted Lagrangian structures.

Remark 1.11. The above definition can be made into a two-colored operad $\mathbb{P}_{[n+1,n]}$ so that a $\mathbb{P}_{[n+1,n]}$-algebra is given by a triple of a $\mathbb{P}_{n+1}$-algebra A, a $\mathbb{P}_n$-algebra B and a morphism of $\mathbb{P}_{n+1}$-algebras $A \to \mathrm{Z}(B)$. This allows one to define an ∞-groupoid of coisotropic structures which is studied in [MS16]. In particular, in [MS17, Section 2.3] Melani and the author show that arbitrary smooth coisotropic subschemes possess a coisotropic structure in this sense up to homotopy.

1.4 Koszul duality For a complex A we denote by $\mathrm{T}_\bullet(A[1])$ the tensor coalgebra. As a complex,

$$\mathrm{T}_\bullet(A[1]) \cong \bigoplus_{k=0}^{\infty} A^{\otimes k}[k].$$

We denote an element of $A^{\otimes k}$ by $[a_1|\dots|a_k]$ for $a_i \in A$. The canonical element in $A^{\otimes 0}$ is denoted by $[]$.

The coproduct is given by deconcatenation, i.e.

$$\Delta[a_1|\dots|a_k] = \sum_{i=0}^{k} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_k].$$

Let us denote by $\wedge$ the concatenation product:

$$[a_1|\dots|a_i]\wedge[a_{i+1}|\dots|a_k]=[a_1|\dots|a_k].$$

Note that the deconcatenation coproduct and concatenation product do not form a bialgebra structure.

If A is a cdga, we can introduce the bar differential on $\mathrm{T}_\bullet(A[1])$ and a commutative multiplication given by shuffles. That is,

$$\begin{aligned}\mathrm{d}[a_1|\dots|a_k]=&\sum_{i=1}^k(-1)^{\sum_{q=1}^{i-1}|a_q|+i-1}[a_1|\dots|\mathrm{d}a_i|\dots|a_k]\\&+\sum_{i=1}^{k-1}(-1)^{\sum_{q=1}^{i}|a_q|+i}[a_1|\dots|a_ia_{i+1}|\dots|a_k]\end{aligned}$$

and

$$[a_1|\dots|a_k]\cdot[a_{k+1}|\dots|a_{k+m}]=\sum_{\sigma\in S_{k,m}}(-1)^\epsilon[a_{\sigma(1)}|\dots|a_{\sigma(k+m)}],$$

where the sign ϵ is determined by assigning degrees $|a_i|-1$ to a_i. The element $[]\in\mathrm{T}_\bullet(A[1])$ is the unit for the shuffle product. We refer the reader to [GJ90, Section 1] for a detailed explanations of all signs involved.

Now let A be a $\mathbb{P}_{n+1}$-algebra. Then we can define a Lie bracket on $\mathrm{T}_\bullet(A[1])$ by

$$\begin{aligned}&\{[a_1|\dots|a_k],[b_1|\dots|b_m]\}\\=&\sum_{i,j}(-1)^{\epsilon+|a_i|+n+1}([a_1|\dots|a_{i-1}]\cdot[b_1|\dots|b_{j-1}])\wedge[\{a_i,b_j\}]\wedge([a_{i+1}|\dots|a_k]\cdot[b_{j+1}|\dots|b_m]).\end{aligned}\tag{7}$$

The sign ϵ is determined by the following rule: an element b moving past $\{a.-\}$ produces a sign $(-1)^{(|b|+1)(|a|+n)}$. For instance,

$$\{[a],[b|c]\}=(-1)^{|a|+n+1}[\{a,b\}|c]+(-1)^{|b|(|a|+n)+1}[b|\{a,c\}].$$

Remark 1.12. The same Poisson bracket was previously introduced by Fresse [Fre06, Section 3] under the name "shuffle Poisson bracket".

Definition 1.13. A $\mathbb{P}_n$-*bialgebra* is a $\mathbb{P}_n$-algebra $\tilde{A}$ together with a coassociative comultiplication $\tilde{A}\to\tilde{A}\otimes\tilde{A}$ which is a morphism of $\mathbb{P}_n$-algebras.

Proposition 1.14. *The differential, multiplication, comultiplication and bracket defined above endow* $\mathrm{T}_\bullet(A[1])$ *with a* $\mathbb{P}_n$*-bialgebra structure.*

Proof. See [GJ90, Proposition 4.1] for the proof that $\mathrm{T}_\bullet(A[1])$ is a commutative dg bialgebra. We just need to show that the bracket is compatible with the other operations.

Let us first show that the Lie bracket is compatible with the coproduct. We will omit some obvious signs arising from a permutation of a and b.

$$
\begin{aligned}
&\{\Delta[a_1|\dots|a_k], \Delta[b_1|\dots|b_m]\} \\
&\qquad = \sum_{i,j} \{[a_1|\dots|a_i]\otimes[a_{i+1}|\dots|a_k], [b_1|\dots|b_j]\otimes[b_{j+1}|\dots|b_m]\} \\
&\qquad = \sum_{i,j} (-1)^\epsilon \{[a_1|\dots|a_i], [b_1|\dots|b_j]\}\otimes([a_{i+1}|\dots|a_k]\cdot[b_{j+1}|\dots|b_m]) \\
&\qquad + \sum_{i,j} (-1)^\epsilon ([a_1|\dots|a_i]\cdot[b_1|\dots|b_j])\otimes\{[a_{i+1}|\dots|a_k], [b_{j+1}|\dots|b_m]\} \\
&\qquad = \sum_{i,j,p,q} (-1)^\epsilon ([a_1|\dots|a_{p-1}]\cdot[b_1|\dots|b_{q-1}])\wedge[\{a_p, b_q\}]\wedge([a_{p+1}|\dots|a_i]\cdot[b_{q+1}|\dots|b_j]) \\
&\qquad\quad \otimes([a_{i+1}|\dots|a_k]\cdot[b_{j+1}|\dots|b_m]) \\
&\qquad + \sum_{i,j,p,q} (-1)^\epsilon ([a_1|\dots|a_i]\cdot[b_1|\dots|b_j]) \\
&\qquad\quad \otimes([a_{i+1}|\dots|a_{p-1}]\cdot[b_{j+1}|\dots|b_{q-1}])\wedge[\{a_p, b_q\}]\wedge([a_{p+1}|\dots|a_k]\cdot[b_{q+1}|\dots|b_m]) \\
&\qquad = \Delta\{[a_1|\dots|a_k], [b_1|\dots|b_m]\}.
\end{aligned}
$$

In the last equality we have used that the tensor coalgebra with a shuffle product is a bialgebra.

The fact that the Lie bracket is symmetric is obvious from the graded commutativity of the shuffle product.

The Jacobi identity and the Leibniz rule are morphisms $f\colon \mathrm{T}_\bullet(A[1])^{\otimes 3}\to \mathrm{T}_\bullet(A[1])$ satisfying

$$\Delta_{\mathrm{T}_\bullet(A[1])}\circ f = (f\otimes m + m\otimes f)\circ\Delta_{\mathrm{T}_\bullet(A[1])^{\otimes 3}},$$

where $m\colon \mathrm{T}_\bullet(A[1])^{\otimes 3}\to \mathrm{T}_\bullet(A[1])$ is the multiplication map.

These are uniquely determined by the projections

$$\mathrm{T}_\bullet(A[1])^{\otimes 3}\to \mathrm{T}_\bullet(A[1])\to A[1]$$

to cogenerators. Therefore, to check the relevant identities, we just need to see that the components landing in A are all zero.

- (Jacobi identity). The Lie bracket has a component in A only if both arguments are in A. Therefore, the Jacobi identity in $\mathrm{T}_\bullet(A[1])$ reduces to the Jacobi identity in A itself.
- (Leibniz rule). The Leibniz rule

  $$\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b, \quad a, b, c\in \mathrm{T}_\bullet(A[1])$$

  has components in A only if either a or b are 1. In that case the Leibniz rule is tautologically true.
- (Compatibility with the differential). The compatibility relation

  $$\mathrm{d}\{a, b\} = (-1)^{n+1}\{\mathrm{d}a, b\} + (-1)^{|a|+n+1}\{a, \mathrm{d}b\}$$

  has components in A if either both a and b are in A or one of them is in A and the other one is in $A^{\otimes 2}$. In the first case the compatibility of the bracket on $\mathrm{T}_\bullet(A[1])$ with the differential reduces to the compatibility of the bracket on A with the differential. In the second case the A component of the equation is

  $$(-1)^{|b_1|}\{a, b_1\}b_2 + (-1)^{|b_1|(|a|+n+1)}b_1\{a, b_2\} = (-1)^{|b_1|}\{a, b_1 b_2\}.$$

  After multiplying through by $(-1)^{|b_1|}$ we get the Leibniz rule for the bracket on A.

□

Remark 1.15. For A a $\mathbb{P}_{n+1}$-algebra the coalgebra $\mathrm{T}_\bullet(A[1])^{\mathrm{cop}}$ with the opposite coproduct is isomorphic to $\mathrm{T}_\bullet(A^{\mathrm{op}}[1])$ as a $\mathbb{P}_n$-bialgebra via

$$[a_1|\dots|a_k] \mapsto (-1)^{k+\sum_{i<j}(|a_i|+1)(|a_j|+1)}[a_k|\dots|a_1]. \tag{8}$$

1.5 Coisotropic intersection Let us now describe a relative version of the previous statement. Let A be a $\mathbb{P}_{n+1}$-algebra and $f\colon A \to M$ a coisotropic morphism. We are going to define a $\mathbb{P}_n$-algebra structure on $\mathrm{T}_\bullet(A[1]) \otimes M$, the one-sided bar complex of M. As before, we denote elements of $\mathrm{T}_\bullet(A[1]) \otimes M$ by $[a_1|\dots|a_k|m]$.

Recall that the bar differential is given by

$$\begin{aligned}
\mathrm{d}[a_1|\dots|a_k|m] = &\sum_{i=1}^{k}(-1)^{\sum_{q=1}^{i-1}|a_q|+i-1}[a_1|\dots|\mathrm{d}a_i|\dots|a_k|m] \\
&+(-1)^{\sum_{q=1}^{k}|a_q|+k}[a_1|\dots|a_k|\mathrm{d}m] \\
&+\sum_{i=1}^{k-1}(-1)^{\sum_{q=1}^{i}|a_q|+i}[a_1|\dots|a_ia_{i+1}|\dots|a_k|m] \\
&+(-1)^{\sum_{q=1}^{k}|a_q|+k}[a_1|\dots|a_km].
\end{aligned}$$

One has an obvious coaction map making $\mathrm{T}_\bullet(A[1]) \otimes M$ into a left dg $\mathrm{T}_\bullet(A[1])$-comodule. As a graded $\mathrm{T}_\bullet(A[1])$-comodule, $\mathrm{T}_\bullet(A[1]) \otimes M$ is cofree.

Introduce a commutative multiplication on $\mathrm{T}_\bullet(A[1])\otimes M$ where the multiplication on $\mathrm{T}_\bullet(A[1])$ is given by shuffles as before and the multiplication on M is coming from its cdga structure. The L_∞ operations we are about to introduce are multiderivations, so by the relation

$$[a_1|\dots|a_k|m] = [a_1|\dots|a_k|1]\cdot[m]$$

it is enough to specify them when the arguments are either in $\mathrm{T}_\bullet(A[1])$ or in M. If all arguments are in $\mathrm{T}_\bullet(A[1])$, we define the brackets as before. We let

$$l_{k+1}([a_1|\dots|a_p|1],[m_1],\dots,[m_k]) = (-1)^{(\sum_{q=1}^{p}|a_q|+p)(1-nk)}[a_1|\dots|a_{p-1}|f_k(a_p;m_1,\dots,m_k)] \tag{9}$$

and

$$l_2([m_1],[m_2]) = [\{m_1,m_2\}], \tag{10}$$

where the Poisson bracket on the right is the bracket in M. All the other brackets are defined to be zero.

Definition 1.16. A *left* $\widehat{\mathbb{P}}_n$*-comodule* $\tilde{M}$ over a $\mathbb{P}_n$-bialgebra $\tilde{A}$ is a $\widehat{\mathbb{P}}_n$-algebra $\tilde{M}$ together with a coassociative left coaction map $\tilde{M} \to \tilde{A} \otimes \tilde{M}$ which is a morphism of $\widehat{\mathbb{P}}_n$-algebras.

Proposition 1.17. *The differential, coaction, multiplication and L_∞ operations defined above make* $\mathrm{T}_\bullet(A[1]) \otimes M$ *into a left* $\widehat{\mathbb{P}}_n$*-comodule over* $\mathrm{T}_\bullet(A[1])$.

Proof. To prove compatibility of the L_∞ operations with the coaction, it is enough to assume each argument is either in M or in $\mathrm{T}_\bullet(A[1])$. If all arguments are in $\mathrm{T}_\bullet(A[1])$, the compatibility with the coaction was checked in Proposition 1.14. If all arguments are in M and $k = 2$ we have

$$\Delta l_2([m_1],[m_2]) = [] \otimes [\{m_1,m_2\}]$$

and

$$l_2(\Delta([m_1]),\Delta([m_2])) = l_2([]\otimes[m_1],[]\otimes[m_2]) = []\otimes[\{m_1,m_2\}].$$

If all but one arguments are in M and k is arbitrary we have

$$\begin{aligned}
&l_k(\Delta[a_1|\dots|a_p|1],[]\otimes[m_1],\dots,[]\otimes[m_{k-1}])\\
&\quad=\sum_{i=0}^p l_k([a_1|\dots|a_i]\otimes[a_{i+1}|\dots|a_p|1],[]\otimes[m_1],\dots,[]\otimes[m_{k-1}])\\
&\quad=\sum_{i=0}^p(-1)^{\sum_{q=1}^i|a_q|(1-(k-1)n)}[a_1|\dots|a_i]\otimes l_k([a_{i+1}|\dots|a_p|1],[m_1],\dots,[m_{k-1}])\\
&\quad=\sum_{i=0}^p(-1)^{\sum_{q=1}^p|a_q|(1-(k-1)n)}[a_1|\dots|a_i]\otimes[a_{i+1}|\dots|f_{k-1}(a_p;m_1,\dots,m_{k-1})]
\end{aligned}$$

and

$$\begin{aligned}
&\Delta l_k([a_1|\dots|a_p|1],[m_1],\dots,[m_{k-1}])\\
&\quad=(-1)^{\sum_{q=1}^p|a_q|(1-(k-1)n)}\Delta[a_1|\dots|a_{p-1}|f_{k-1}(a_p;m_1,\dots,m_{k-1})]\\
&\quad=(-1)^{\sum_{q=1}^p|a_q|(1-(k-1)n)}\sum_{i=0}^p[a_1|\dots|a_i]\otimes[a_{i+1}|\dots|f_{k-1}(a_p;m_1,\dots,m_{k-1})].
\end{aligned}$$

Therefore, as before it is enough to check symmetry, the Leibniz rule and Jacobi identity only after projecting to M. The operation l_k has a component in M if either all but one arguments are in M and one argument is in A or $k=2$ and both arguments are in M.

- (Symmetry). Symmetry is clear for $l_2(m_1,m_2)$. For $l_k(a,m_1,\dots,m_{k-1})$ symmetry in the m_i variables follows from the symmetry property (2) of f_{k-1}.
- (Leibniz rule). If $k=2$ we need to check that

  $$l_2([m_1],[m_2m_3]) = l_2([m_1],[m_2])[m_3] + (-1)^{|m_2||m_3|}l_2([m_1],[m_3])[m_2].$$

  This is just an expression for the Leibniz rule in M. For any k we also need to check that

  $$\begin{aligned}l_k([a|1],[m_1],\dots,[m_{k-1}m_k]) &= l_k([a|1],[m_1],\dots,[m_{k-1}])[m_k]\\ &\quad+(-1)^{|m_{k-1}||m_k|}l_k([a|1],[m_1],\dots,[m_k])[m_{k-1}].\end{aligned}$$

  This immediately follows from the derivation property (3) of f_{k-1}
- (Jacobi identity).

  The Jacobi identity has a component in M in the following four cases:
  1. All arguments are in M. In this case we get the Jacobi identity for the bracket in M.
  2. One argument is in A, the rest are in M.

     The Jacobi identity is

     $$\begin{aligned}
     0 =&(-1)^{nk}l_{k+1}([\mathrm{d}a|1],[m_1],\dots,[m_k])\\
     &+\mathrm{d}l_{k+1}([a|1],[m_1],\dots,[m_k])\\
     &+\sum_i(-1)^{|a|+\sum_{j=1}^{i-1}|m_j|+nk+1}l_{k+1}([a|1],[m_1],\dots,[\mathrm{d}m_i],\dots,[m_k])\\
     &+\sum_{i<j}(-1)^{\epsilon'}l_k([a|1],\{[m_i],[m_j]\},\dots)\\
     &+\sum_i(-1)^{|m_i|\sum_{j=i+1}^k|m_j|+in}\{l_k([a|1],[m_1],\dots,\widehat{[m_i]},\dots,[m_k]),[m_i]\},
     \end{aligned}$$

where the sign is

$$\epsilon' = |m_i| \sum_{p=1}^{i-1} |m_p| + |m_j| \sum_{p=1, p\neq i}^{j-1} |m_p| + n(i+j) + (|a|+1)(1-n).$$

Substituting l_k in terms of f_{k-1} from equation (9) we obtain

$$\begin{aligned}
0 =& (-1)^{nk}(-1)^{|a|(1-nk)} f_k(\mathrm{d}a; m_1, \dots, m_k) \\
&+ (-1)^{(|a|+1)(1-nk)} \mathrm{d} f_k(a; m_1, \dots, m_k) \\
&+ \sum_i (-1)^{|a| + \sum_{j=1}^{i-1} |m_j| + nk + (|a|+1)(1-nk)} f_k(a; m_1, \dots, \mathrm{d}m_i, \dots, m_k) \\
&+ \sum_{i<j} (-1)^{|m_i| \sum_{p=1}^{i-1} |m_p| + |m_j| \sum_{p=1, p\neq i}^{j-1} |m_p| + n(i+j) + (|a|+1)nk} f_{k-1}(a; \{m_i, m_j\}, \dots) \\
&+ \sum_i (-1)^{|m_i| \sum_{j=i+1}^{k} |m_j| + in + (|a|+1)(1-n(k-1))} \{ f_{k-1}(a; m_1, \dots, \hat{m}_i, \dots, m_k), m_i \}.
\end{aligned}$$

After clearing out the signs, the equation coincides with (4).

3. Two arguments are in A, the rest are in M.
   The Jacobi identity is

$$\begin{aligned}
0 =& (-1)^{|a_1|+n+1} l_{k+1}([\{a_1, a_2\}|1], [m_1], \dots, [m_k]) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \operatorname{sgn}(\sigma)^n (-1)^{\epsilon_m} (-1)^{nj + (|a_1|+1)(1+nj)} \times \\
&\quad l_{i+1}([a_1|1], l_{j+1}([a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \operatorname{sgn}(\sigma)^n (-1)^{\epsilon_m} (-1)^{n(j+1) + (|a_2|+1)(nj+|a_1|)} \times \\
&\quad l_{i+1}([a_2|1], l_{j+1}([a_1|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]).
\end{aligned}$$

Substituting l_k in terms of f_{k-1} from equation (9) we obtain

$$\begin{aligned}
0 =& (-1)^{|a_1|+n+1} (-1)^{(|a_1|+|a_2|-n+1)(1-nk)} f_k(\{a_1, a_2\}; m_1, \dots, m_k) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \operatorname{sgn}(\sigma)^n (-1)^{\epsilon_m + (|a_1|+1)n(k+1) + (|a_2|+1)(1+nj) + nj} \times \\
&\quad f_i(a_1; f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \operatorname{sgn}(\sigma)^n (-1) \times \\
&\quad f_i(a_2; f_j(a_1; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}),
\end{aligned}$$

where the last sign is

$$\epsilon' = \epsilon_m + (|a_2|+1)(1 - n(k+1) + |a_1|) + (|a_1|+1)(1-nj) + n(j+1).$$

After rearranging the signs, we get (5).

4. One argument is in $(A[1])^{\otimes 2}$, the rest are in M.

The Jacobi identity is

$$
\begin{aligned}
0 &= (-1)^{nk} l_{k+1}(\mathrm{d}[a_1|a_2|1], [m_1], \dots, [m_k]) + \mathrm{d}l_{k+1}([a_1|a_2|1], [m_1], \dots, [m_k]) \\
&\quad + \sum_{\substack{i+j=k \\ i,j>0}} \sum_{\sigma \in S_{j,i}} \mathrm{sgn}(\sigma)^n (-1)^{nk(j+1)+\epsilon} \times \\
&\qquad l_{i+1}(l_{j+1}([a_1|a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]).
\end{aligned}
$$

The projection of each term to M is

$$
\begin{aligned}
&l_{k+1}(\mathrm{d}[a_1|a_2|1], [m_1], \dots, [m_k]) \\
&\quad = (-1)^{|a_1|+1} l_{k+1}([a_1 a_2|1], [m_1], \dots, [m_k]) \\
&\quad + (-1)^{|a_1|+|a_2|} l_{k+1}([a_1|f_0(a_2)], [m_1], \dots, [m_k]) \\
&\quad = (-1)^{|a_1|+1+(|a_1|+|a_2|+1)(1-nk)} f_k(a_1 a_2; m_1, \dots, m_k) \\
&\quad + (-1)^{|a_1|+|a_2|(1+\sum_{i=1}^k |m_i|)+(|a_1|+1)(1-nk)} f_k(a_1; m_1, \dots, m_k) f_0(a_2), \\
&\mathrm{d}l_{k+1}([a_1|a_2|1], [m_1], \dots, [m_k]) = (-1)^{|a_1|+1+(|a_1|+|a_2|)(1-nk)} f_0(a_1) f_k(a_2; m_1, \dots, m_k), \\
&l_{i+1}(l_{j+1}([a_1|a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
&\quad = (-1)^{(|a_1|+|a_2|)(1-nj)} l_{i+1}([a_1|f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)})], [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
&\quad = (-1)^{(|a_1|+|a_2|)(1-nj)+(|a_2|+\sum_{i=1}^j |m_{\sigma(i)}|+nj)\sum_{p=1}^{k-j}|m_{\sigma(j+p)}|+(|a_1|+1)(1-ni)} \times \\
&\qquad f_i(a_1; m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)})
\end{aligned}
$$

Let us denote by $\overline{\sigma} \in S_{i,j}$ the shuffle obtained from σ by swapping the blocks $\sigma(1), \dots, \sigma(j)$ and $\sigma(j+1), \dots, \sigma(k)$. That is, $\overline{\sigma}(p) = \sigma(j+p)$ for $1 \leq p \leq i$ and $\overline{\sigma}(p) = \sigma(p-i)$ for $i < p \leq k$. Denote by $\overline{\epsilon}$ the Koszul sign corresponding to the shuffle $\overline{\sigma}$. We have

$$\mathrm{sgn}(\overline{\sigma}) = \mathrm{sgn}(\sigma)(-1)^{j(k-j)}$$

and

$$(-1)^{\overline{\epsilon}} = (-1)^{\epsilon}(-1)^{\sum_{i=1}^j |m_{\sigma(i)}| \sum_{p=1}^{k-j} |m_{\sigma(j+p)}|}.$$

The Jacobi identity becomes

$$
\begin{aligned}
0 &= (-1)^{|a_1|+(|a_1|+|a_2|)(1-nk)} f_k(a_1 a_2; m_1, \dots, m_k) \\
&\quad - (-1)^{|a_1|+|a_2|(1+\sum_{i=1}^k |m_i|)+|a_1|(1-nk)} f_k(a_1; m_1, \dots, m_k) f_0(a_2) \\
&\quad - (-1)^{|a_1|+(|a_1|+|a_2|)(1-nk)} f_0(a_1) f_k(a_2; m_1, \dots, m_k) \\
&\quad + \sum_{i+j=k; i,j>0} \sum_{\overline{\sigma} \in S_{i,j}} \mathrm{sgn}(\overline{\sigma})^n (-1)^{\epsilon'} f_i(a_1; m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}),
\end{aligned}
$$

where the sign is

$$\epsilon' = (|a_1|+|a_2|)(1-nj) + (|a_2|+nj)\sum_{p=1}^{k-j} |m_{\sigma(j+p)}| + (|a_1|+1)(1-ni) + n(j+k).$$

Rearranging the signs, we obtain (6).

□

In the same way we can make $M \otimes \mathrm{T}_\bullet(A[1])$ into a $\widehat{\mathbb{P}}_n$-algebra compatibly with the right coaction of $\mathrm{T}_\bullet(A[1])$. The bar differential on $M \otimes \mathrm{T}_\bullet(A[1])$ is given by

$$\begin{aligned}\mathrm{d}[m|a_1|\ldots|a_n] =& [\mathrm{d}m|a_1|\ldots|a_n] \\ &+\sum_{i=1}^{n}(-1)^{\sum_{q=1}^{i-1}|a_q|+i-1+|m|}[m|a_1|\ldots|\mathrm{d}a_i|\ldots|a_n] \\ &+(-1)^{|m|+|a_1|+1}[ma_1|\ldots|a_n] \\ &+\sum_{i=1}^{n-1}(-1)^{\sum_{q=1}^{i}|a_q|+i+|m|}[m|a_1|\ldots|a_ia_{i+1}|\ldots|a_n].\end{aligned}$$

Moreover, $M \otimes \mathrm{T}_\bullet(A[1])$ is isomorphic to $\mathrm{T}_\bullet(A^{\mathrm{op}}[1])^{\mathrm{cop}} \otimes M^{\mathrm{op}}$ as right $\mathrm{T}_\bullet(A[1])$-comodules using the isomorphism (8). Here M^{op} represents the same cdga with the opposite bracket and the coisotropic structure given by $f_k^{\mathrm{op}} = (-1)^k f_k$. Using the previous theorem, we can make $M \otimes \mathrm{T}_\bullet(A[1])$ into a right $\widehat{\mathbb{P}}_n$-comodule over $\mathrm{T}_\bullet(A[1])$.

Let us now combine left and right comodules.

Theorem 1.18. *Let A be a $\mathbb{P}_{n+1}$-algebra and $A \to M$ and $A \to N$ two coisotropic morphisms. Then the two-sided bar complex $N \otimes_A^{\mathbb{L}} M$ has a natural structure of a $\widehat{\mathbb{P}}_n$-algebra such that the natural projection $N^{\mathrm{op}} \otimes M \to N \otimes_A^{\mathbb{L}} M$ is morphism of $\widehat{\mathbb{P}}_n$-algebras.*

Proof. Let $\tilde{A} = \mathrm{T}_\bullet(A[1])$, $\tilde{N} = N \otimes \tilde{A}$ and $\tilde{M} = \tilde{A} \otimes M$. Then $\tilde{A}$ is a $\mathbb{P}_n$-bialgebra, $\tilde{N}$ a right $\widehat{\mathbb{P}}_n$-comodule and $\tilde{M}$ a left $\widehat{\mathbb{P}}_n$-comodule over $\tilde{A}$.

We will first show that the cotensor product $\tilde{N} \otimes^{\tilde{A}} \tilde{M}$ is closed under the $\widehat{\mathbb{P}}_n$-structures coming from $\tilde{N} \otimes \tilde{M}$.

Recall that

$$\tilde{N} \otimes^{\tilde{A}} \tilde{M} := \mathrm{eq}(\tilde{N} \otimes \tilde{M} \rightrightarrows \tilde{N} \otimes \tilde{A} \otimes \tilde{M}),$$

where the two maps are coactions on $\tilde{M}$ and $\tilde{N}$ and the equalizer is the strict equalizer in the category of complexes. By definition the coaction

$$\tilde{M} \xrightarrow{\Delta_M} \tilde{A} \otimes \tilde{M}$$

is a morphism of $\widehat{\mathbb{P}}_n$-algebras, so

$$\tilde{N} \otimes \tilde{M} \xrightarrow{\mathrm{id}_{\tilde{N}} \otimes \Delta_M} \tilde{N} \otimes \tilde{A} \otimes \tilde{M}$$

is also a morphism of $\widehat{\mathbb{P}}_n$-algebras, but the forgetful functor from $\widehat{\mathbb{P}}_n$-algebras to complexes creates limits, so the equalizer is also a $\widehat{\mathbb{P}}_n$-algebra.

To conclude the proof of the theorem, we are going to construct an isomorphism

$$\tilde{N} \otimes^{A} \tilde{M} \cong N \otimes_A^{\mathbb{L}} M.$$

The coproduct $\Delta\colon \tilde{A} \to \tilde{A} \otimes \tilde{A}$ induces an isomorphism

$$\Delta\colon \tilde{A} \to \mathrm{eq}(\tilde{A} \otimes \tilde{A} \rightrightarrows \tilde{A} \otimes \tilde{A} \otimes \tilde{A}),$$

where the two maps are $\Delta \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta$. Therefore,

$$N \otimes_A^{\mathbb{L}} M = N \otimes \mathrm{T}_\bullet(A[1]) \otimes M \xrightarrow{\mathrm{id}_N \otimes \Delta \otimes \mathrm{id}_M} N \otimes \mathrm{T}_\bullet(A[1]) \otimes \mathrm{T}_\bullet(A[1]) \otimes M$$

induces an isomorphism $N \otimes_A^{\mathbb{L}} M \xrightarrow{\sim} \tilde{N} \otimes^{A} \tilde{M}$. $\square$

Remark 1.19. Suppose $A \to M$ and $A \to N$ are two coisotropic morphisms as in the previous Theorem. Then any model of their derived intersection is quasi-isomorphic to the two-sided bar construction $N \otimes^{\mathbb{L}}_A M$ and hence by the homotopy transfer theorem [LV12, Section 10.3] we get an induced homotopy $\mathbb{P}_n$-structure on the given model.

2. Classical Hamiltonian reduction

Let $\mathfrak{g}$ be a finite-dimensional dg Lie algebra over k concentrated in non-positive degrees. In this section we apply results of the previous section to the $\mathbb{P}_2$-algebra $A = \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})$. The results of this section generalize in a straightforward way to n-shifted Hamiltonian reduction in which case we replace A by the $\mathbb{P}_{n+2}$-algebra $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-n]))$.

2.1 Chevalley-Eilenberg complex Let V be a $\mathfrak{g}$-representation. The Chevalley–Eilenberg complex $\mathrm{C}^\bullet(\mathfrak{g}, V)$ is defined to be

$$\mathrm{C}^\bullet(\mathfrak{g}, V) = \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), V)$$

with the differential

$$\begin{aligned}(\mathrm{d}f)(x_1, \dots, x_n) &= \mathrm{d}f(x_1, \dots, x_n) \\ &+ \sum_{i=1}^n (-1)^{\sum_{p=1}^{i-1}|x_p| + |f| + n + 1} f(x_1, \dots, \mathrm{d}x_i, \dots, x_n) \\ &+ \sum_{i<j} (-1)^{|x_i|\sum_{p=1}^{i-1}|x_p| + |x_j|\sum_{p=1, p\neq i}^{j-1}|x_p| + i + j + |f|} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) \\ &+ \sum_i (-1)^{|x_i|(\sum_{p=1}^{i-1}|x_p| + |f| + n + 1) + |f| + i + 1} x_i f(x_1, \dots, \widehat{x}_i, \dots, x_n). \qquad (11)\end{aligned}$$

Here $|f|$ is the degree of f in $\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), V)$ and we have used the décalage isomorphism as in (1) to identify $\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), -)$ with antisymmetric functions on $\mathfrak{g}$.

The product

$$\smile \colon \mathrm{C}^\bullet(\mathfrak{g}, A) \otimes \mathrm{C}^\bullet(\mathfrak{g}, B) \to \mathrm{C}^\bullet(\mathfrak{g}, A \otimes B) \qquad (12)$$

is defined to be

$$(v \smile w)(x_1, \dots, x_{k+l}) = \sum_{\sigma \in S_{k.l}} \mathrm{sgn}(\sigma)(-1)^{\epsilon + \epsilon_1} v(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \otimes w(x_{\sigma(k+1)}, \dots, x_{\sigma(k+l)}),$$

where the sign is

$$\epsilon_1 = |w|k + \sum_{i=1}^k |x_{\sigma(i)}|(l + |w|).$$

Remark 2.1. Due to our finiteness assumptions on $\mathfrak{g}$, we have an isomorphism

$$\mathrm{C}^\bullet(\mathfrak{g}, V) \cong \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes V.$$

In particular, if V is a semi-free commutative algebra, so is $\mathrm{C}^\bullet(\mathfrak{g}, V)$.

The algebra $\mathrm{Sym}\,\mathfrak{g}$ has the Kirillov–Kostant Poisson structure given on the generators by $\pi(x_1, x_2) = [x_1, x_2]$ for $x_i \in \mathfrak{g}$. The center of this $\mathbb{P}_1$-algebra can be computed to be

$$\mathrm{Z}(\mathrm{Sym}\,\mathfrak{g}) \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})$$

with the bracket

$$[v,w](x_1,\dots,x_{k+l-1}) = \sum_{\sigma\in S_{l,k-1}} \mathrm{sgn}(\sigma)(-1)^{\epsilon+\epsilon_1} v(w(x_{\sigma(1)},\dots,x_{\sigma(l)}),x_{\sigma(l+1)},\dots,x_{\sigma(k+l-1)})$$
$$- \sum_{\sigma\in S_{k,l-1}} \mathrm{sgn}(\sigma)(-1)^{\epsilon+\epsilon_2} w(v(x_{\sigma(1)},\dots,x_{\sigma(k)}),x_{\sigma(k+1)},\dots,x_{\sigma(k+l-1)}),$$

where $(-1)^\epsilon$ denotes the sign coming from the Koszul sign rule applied to the permutation σ of x_i and the signs ϵ_i are

$$\epsilon_1 = (|w|+l)(k+1)+|v|$$
$$\epsilon_2 = (|v|-k)(|w|-l)+(k+1)(|w|+1)+|v|.$$

2.2 Hamiltonian reduction Let B be a $\mathbb{P}_1$-algebra with a $\mathfrak{g}$-action preserving the Poisson bracket. We denote by $a\colon \mathfrak{g}\to \mathrm{Der}(B)$ the action map.

Definition 2.2. A $\mathfrak{g}$-equivariant morphism of complexes $\mu\colon \mathfrak{g}\to B$ is a *moment map* for the $\mathfrak{g}$-action on B if the equation

$$\{\mu(x),b\} = a(x).b$$

is satisfied for all $x\in\mathfrak{g}$ and $b\in B$. In this case we say that the $\mathfrak{g}$-action is *Hamiltonian.*

Remark 2.3. One can replace $\mathfrak{g}$-equivariance in the definition of the moment map with the condition that the induced map $\mathrm{Sym}\,\mathfrak{g}\to B$ is a morphism of $\mathbb{P}_1$-algebras.

Definition 2.4. Suppose B is a $\mathbb{P}_1$-algebra equipped with a $\mathfrak{g}$-action and a moment map $\mu\colon \mathfrak{g}\to B$. Its *Hamiltonian reduction* is

$$B/\!/\,\mathrm{Sym}\,\mathfrak{g} := \mathrm{C}^\bullet(\mathfrak{g},k)\otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g},\mathrm{Sym}\,\mathfrak{g})}\mathrm{C}^\bullet(\mathfrak{g},B).$$

We will introduce a $\widehat{\mathbb{P}}_1$-structure on this complex later in Corollary 2.7. Let us just mention a different complex used in derived Hamiltonian reduction called the classical BRST complex [KS87]

$$\mathrm{C}^\bullet(\mathfrak{g},\mathrm{Sym}(\mathfrak{g}[1])\otimes B).$$

Here the differential on $\mathrm{Sym}(\mathfrak{g}[1])\otimes B$ is the Koszul differential: given

$$x_1\wedge\cdots\wedge x_n\otimes b\in\mathrm{Sym}(\mathfrak{g}[1])\otimes B$$

we let

$$\mathrm{d}(x_1\wedge\cdots\wedge x_n\otimes b) = \sum_{i=1}^n (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1}|x_q|+i-1)}\mathrm{d}x_i\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_n\otimes b$$
$$-\sum_{i=1}^n(-1)^{|x_i|\sum_{q=i+1}^n(|x_q|+1)+\sum_{q=1}^{i-1}(|x_q|+1)+|x_i|}x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_n\otimes\mu(x_i)b$$
$$+(-1)^{\sum_{q=1}^n|x_q|+n}x_1\wedge\cdots\wedge x_n\otimes\mathrm{d}b.$$

One can introduce a Poisson bracket on the classical BRST complex as follows. As a graded commutative algebra, the classical BRST complex is generated by $\mathfrak{g}^*[-1]$, $\mathfrak{g}[1]$ and B. We keep the bracket on B and let the bracket between an element $\phi\in\mathfrak{g}^*[-1]$ and an element $x\in\mathfrak{g}[1]$ be the natural pairing: $\{\phi,x\} := \phi(x)$. Then d is a derivation of the bracket precisely due to the moment map equation. In this way the classical BRST complex becomes a $\mathbb{P}_1$-algebra.

2.3 Hamiltonian reduction as a coisotropic intersection As a plain graded commutative algebra, $\mathrm{C}^\bullet(\mathfrak{g}, B) \cong B \otimes \mathrm{Sym}(\mathfrak{g}^*[-1])$, so its module of derivations is isomorphic to

$$\mathrm{T}_B \otimes \mathrm{Sym}(\mathfrak{g}^*[-1]) \oplus B \otimes \mathfrak{g}[1] \otimes \mathrm{Sym}(\mathfrak{g}^*[-1])$$

with the differential given by the sum of internal differentials on each term and the action map $\mathfrak{g} \to \mathrm{T}_B$. Therefore, the Poisson center of $\mathrm{C}^\bullet(\mathfrak{g}, B)$ is

$$\mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{g}, B)) \cong \mathrm{C}^\bullet(\mathfrak{g}, \widehat{\mathrm{Sym}}(\mathrm{T}_B[-1]) \otimes \widehat{\mathrm{Sym}}(\mathfrak{g})).$$

Given a Hamiltonian $\mathfrak{g}$-action on B, let us define the morphism

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}) \to \mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{g}, B))$$

as follows. The cdga $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g})$ is generated by $\mathrm{C}^\bullet(\mathfrak{g}, k)$ and $\mathfrak{g} \subset \mathrm{Sym}\, \mathfrak{g}$. We let

$$\mathrm{C}^\bullet(\mathfrak{g}, k) \hookrightarrow \mathrm{C}^\bullet(\mathfrak{g}, \widehat{\mathrm{Sym}}(\mathrm{T}_B[-1]) \otimes \widehat{\mathrm{Sym}}(\mathfrak{g}))$$

be the natural embedding. The map

$$\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{g}, \widehat{\mathrm{Sym}}(\mathrm{T}_B[-1]) \otimes \widehat{\mathrm{Sym}}(\mathfrak{g}))$$

is given by $x \mapsto \mu(x) - x$ for $v \in \mathfrak{g}$.

Proposition 2.5. *Let B be a $\mathbb{P}_1$-algebra with a Hamiltonian $\mathfrak{g}$-action. Then the morphism*

$$\mathrm{C}^\bullet(\mathfrak{g}, \mu)\colon \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}) \to \mathrm{C}^\bullet(\mathfrak{g}, B)$$

is coisotropic.

Proof. It is enough to check that the morphism we have defined on generators commutes with the differential and the brackets.

Indeed, it is clear that the embedding $\mathrm{C}^\bullet(\mathfrak{g}, k) \hookrightarrow \mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{g}, B))$ commutes with differentials. For $x \in \mathfrak{g}$

$$\mathrm{d}\mu(x) + [\pi, \mu(x)] - \mathrm{d}x - (-1)^{|x|}a(x) = \mathrm{d}\mu(x) - \mathrm{d}x = \mu(\mathrm{d}x) - \mathrm{d}x,$$

where in the first equality we have used the moment map equation

$$[\pi, \mu(x)](b) = (-1)^{|x|}\{\mu(x), b\} = (-1)^{|x|}a(x).b.$$

It is also clear that the morphism commutes with brackets as B Poisson-commutes with $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \hookrightarrow \mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{g}, B))$. □

Example 2.6. Let $B = k$ with the trivial $\mathfrak{g}$-action and $\mu = 0$.

The morphism $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}) \to \mathrm{C}^\bullet(\mathfrak{g}, k)$ given by the counit $\mathrm{Sym}\, \mathfrak{g} \to k$ possesses a coisotropic structure given by the composite of the antipode $S\colon \mathrm{Sym}\, \mathfrak{g} \to \mathrm{Sym}\, \mathfrak{g}$ with the completion map

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}) \xrightarrow{S} \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g}) \to \mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{g}, k)) \cong \mathrm{C}^\bullet(\mathfrak{g}, \widehat{\mathrm{Sym}}(\mathfrak{g})).$$

Corollary 2.7. *The Poisson reduction*

$$B/\!/ \mathrm{Sym}\, \mathfrak{g} = \mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\, \mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B)$$

carries a natural $\widehat{\mathbb{P}}_1$-structure. Moreover, there is a zig-zag of quasi-isomorphisms of cdgas between $B/\!/ \mathrm{Sym}\, \mathfrak{g}$ and the classical BRST complex.

Proof. Combining Proposition 2.5 with Theorem 1.18, we see that

$$\mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B)$$

carries a $\widehat{\mathbb{P}}_1$-structure.

The two-sided bar complex $k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B$ is the geometric realization of the simplicial complex $V_\bullet$ where

$$V_n = k \otimes (\mathrm{Sym}\,\mathfrak{g})^{\otimes n} \otimes B.$$

We also denote by $W^1_\bullet$ the simplicial complex whose geometric realization is

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes^{\mathbb{L}}_{\mathrm{Sym}(\mathfrak{g}^*[-1])} \mathrm{Sym}(\mathfrak{g}^*[-1])$$

and by $W^2_\bullet$ the constant simplicial complex with $W^2_0 = \mathrm{Sym}(\mathfrak{g}^*[-1])$.

The two-sided bar complex $\mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B)$ is computed as the geometric realization of the simplicial complex $V_\bullet \otimes W^1_\bullet$ with the Chevalley-Eilenberg differential (11). The multiplication map gives a weak equivalence of simplicial complexes $W^1_\bullet \to W^2_\bullet$ which extends to a weak equivalence of simplicial complexes $V_\bullet \otimes W^1_\bullet \to V_\bullet \otimes W^2_\bullet$ which acts as the identity on $V_\bullet$. This implies that the multiplication map gives a quasi-isomorphism of cdgas

$$\mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B) \to \mathrm{C}^\bullet(\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B).$$

We have a quasi-isomorphism of $\mathfrak{g}$-representations

$$\mathrm{Sym}(\mathfrak{g}[1]) \otimes B \to k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B$$

given by the symmetrization

$$x_1 \wedge \cdots \wedge x_n \otimes b \mapsto \sum_{\sigma \in S_n} (-1)^\epsilon [x_{\sigma(1)}| \dots |x_{\sigma(n)}|b].$$

This gives a quasi-isomorphism of cdgas

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \to \mathrm{C}^\bullet(\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B).$$

Combining these two quasi-isomorphisms we obtain a quasi-isomorphism

$$B/\!/\,\mathrm{Sym}\,\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B)$$

to the classical BRST complex. ☐

Remark 2.8. We do not know whether the classical BRST complex is quasi-isomorphic to $B/\!/\,\mathrm{Sym}\,\mathfrak{g}$ as a $\hat{\mathbb{P}}_1$-algebra for general $\mathfrak{g}$. However, let's restrict to the case $\mathfrak{g}$ is an abelian Lie algebra.

We have a splitting of the multiplication map

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}(\mathfrak{g}^*[-1])^{\otimes n} \otimes \mathrm{Sym}(\mathfrak{g}^*[-1]) \to \mathrm{Sym}(\mathfrak{g}^*[-1])$$

given by sending $x \mapsto x \otimes 1^{\otimes n} \otimes 1$. This gives a splitting

$$\mathrm{C}^\bullet(\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B) \to \mathrm{C}^\bullet(\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g})} \mathrm{C}^\bullet(\mathfrak{g}, B) = B/\!/\,\mathrm{Sym}\,\mathfrak{g}.$$

It is easy to check that the composite map

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \to \mathrm{C}^\bullet(\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{Sym}\,\mathfrak{g}} B) \to B/\!/\,\mathrm{Sym}\,\mathfrak{g}$$

is compatible with the Poisson structures.

3. Brace algebras

In this section we introduce quantum versions of $\mathbb{P}_2$-algebras called brace algebras introduced by Gerstenhaber and Voronov, see [GV95] and [GV94]. By a theorem of McClure and Smith [MS99] the brace operad controlling brace algebras is a model of the chain operad of little disks $\mathbb{E}_2$.

3.1 Algebras

Definition 3.1. A *brace algebra* A is a dga together with brace operations $A\otimes A^{\otimes n}\to A[-n]$ for $n>0$ denoted by $x\{y_1,\dots,y_n\}$ satisfying the following equations:

- (Associativity).

  $$x\{y_1,\dots,y_n\}\{z_1,\dots,z_m\}=\sum(-1)^\epsilon x\{z_1,\dots,z_{i_1},y_1\{z_{i_1+1},\dots\},\dots,y_n\{z_{i_n+1},\dots\},\dots,z_m\},$$

  where the sum goes over the locations of the y_i insertions and the length of each y_i brace. The sign is

  $$\epsilon=\sum_{p=1}^n(|y_p|+1)\sum_{q=1}^{i_p}(|z_q|+1).$$

- (Higher homotopies).

  $$\begin{aligned}\mathrm{d}(x\{y_1,\dots,y_n\})&=(\mathrm{d}x)\{y_1,\dots,y_n\}\\&+\sum_i(-1)^{|x|+\sum_{q=1}^{i-1}|y_q|+i}x\{y_1,\dots,\mathrm{d}y_i,\dots,y_n\}\\&+\sum_i(-1)^{|x|+\sum_{q=1}^{i}|y_q|+i+1}x\{y_1,\dots,y_iy_{i+1},\dots,y_n\}\\&-(-1)^{(|y_1|+1)|x|}y_1\cdot x\{y_2,\dots,y_n\}\\&-(-1)^{|x|+\sum_{q=1}^{n-1}|y_q|+n}x\{y_1,\dots,y_{n-1}\}\cdot y_n.\end{aligned}$$

- (Distributivity).

  $$\sum_{k=0}^n(-1)^{|x_2|(\sum_{q=1}^k|y_q|+k)}x_1\{y_1,\dots,y_k\}x_2\{y_{k+1},\dots,y_n\}=(x_1\cdot x_2)\{y_1,\dots,y_n\}.$$

In the axioms we use a shorthand notation $x\{\}\equiv x$.

Remark 3.2. These axioms coincide with the ones in [GV95] if one flips the sign of the differential.

For instance, the second axiom for $n=1$ is equivalent to

$$xy-(-1)^{|x||y|}yx=(-1)^{|x|}\mathrm{d}(x\{y\})-(-1)^{|x|}(\mathrm{d}x)\{y\}+x\{\mathrm{d}y\}.$$

In other words, the multiplication is commutative up to homotopy.

One has the *opposite* brace algebra A^{op} defined as follows. The product on A^{op} is opposite to that of A:

$$a\cdot^{\mathrm{op}}b:=(-1)^{|a||b|}b\cdot a$$

while the braces on A^{op} are defined by

$$x\{y_1,\dots,y_n\}^{\mathrm{op}}=(-1)^{\sum_{i<j}(|y_i|+1)(|y_j|+1)+n}x\{y_n,\dots,y_1\}.$$

3.2 Modules Let A be a brace algebra. We are now going to define modules over such algebras.

Definition 3.3. A *left brace A-module* is a dga M together with a left A-module structure and brace operations $M \otimes A^{\otimes n} \to M[-n]$ denoted by $m\{x_1, \dots, x_n\}$ satisfying the following equations:

- (Compatibility). For any $x, y_i \in A$ one has
$$(x \cdot 1)\{y_1, \dots, y_n\} = x\{y_1, \dots, y_n\} \cdot 1.$$
- (Associativity). For any $m \in M$ and $x_i, y_i \in A$ one has
$$m\{x_1, \dots, x_n\}\{y_1, \dots, y_m\} = \sum (-1)^{\epsilon} \times \\ m\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots\}, \dots, x_n\{y_{i_n+1}, \dots\}, \dots, y_m\},$$
where the sign is
$$\epsilon = \sum_{p=1}^{n} (|x_p| + 1) \sum_{q=1}^{i_p} (|y_q| + 1).$$
- (Higher homotopies). For any $m \in M$ and $x_i \in A$ one has
$$\begin{aligned} \mathrm{d}(m\{x_1, \dots, x_n\}) = {} & (\mathrm{d}m)\{x_1, \dots, x_n\} \\ & + \sum (-1)^{|m| + \sum_{q=1}^{i-1} |x_q| + i} m\{x_1, \dots, \mathrm{d}x_i, \dots, x_n\} \\ & + \sum (-1)^{|m| + \sum_{q=1}^{i} |x_q| + i + 1} m\{x_1, \dots, x_i x_{i+1}, \dots, x_n\} \\ & - (-1)^{|m|(|x_1|+1)} x_1 \cdot m\{x_2, \dots, x_n\} \\ & - (-1)^{|m| + \sum_{q=1}^{n-1} |x_q| + n} m\{x_1, \dots, x_{n-1}\} \cdot x_n. \end{aligned}$$
- (Distributivity). For any $m, n \in M$ and $x_i \in A$ one has
$$(mn)\{x_1, \dots, x_p\} = \sum_{k=0}^{p} (-1)^{|n|(\sum_{q=1}^{k} |x_q| + k)} m\{x_1, \dots, x_k\} n\{x_{k+1}, \dots, x_p\}.$$

Example 3.4. If A is a brace algebra, then it is a left brace A-module using the brace operations on A itself.

Remark 3.5. Note that left brace modules are unrelated to the general notion of modules over an algebra over an operad, [LV12, Section 12.3.1]. Our definition is analogous to the notion of a left module over an associative algebra while an operadic module over an associative algebra is a bimodule.

We define *right brace A-modules* to be left brace A^{op}-modules. If M is a left brace A-module, then M^{op} is naturally a right brace A-module with the brace operations mirror reversed.

3.3 Koszul duality Let A be a brace algebra. Recall from Section 1.4 the bar complex $\mathrm{T}_\bullet(A[1])$ which is a dg coalgebra. Since A is not commutative, the shuffle product is not compatible with the differential, so we introduce a slightly different product.

A product
$$\mathrm{T}_\bullet(A[1]) \otimes \mathrm{T}_\bullet(A[1]) \to \mathrm{T}_\bullet(A[1])$$

is uniquely specified by the projection to the cogenerators

$$A^{\otimes n} \otimes A^{\otimes m} \to A[1-n-m].$$

We let the maps with $n = 1$ be given by the brace operations and the maps with $n \neq 1$ be zero. Our sign conventions are such that

$$[x] \cdot [y_1|\dots|y_n] = [x\{y_1, \dots, y_n\}] + \dots,$$

i.e. the leading term carries no extra sign.

Extending the product to the whole tensor coalgebra we obtain

$$[x_1|\dots|x_n] \cdot [y_1|\dots|y_m] = \sum_{\{i_p, l_p\}_{p=1}^n} (-1)^\epsilon \times \\ [y_1|\dots|y_{i_1}|x_1\{y_{i_1+1}, \dots, y_{i_1+l_1}\}|\dots|x_n\{y_{i_n+1}, \dots, y_{i_n+l_n}\}|\dots|y_n],$$

where the sign is

$$\epsilon = \sum_{p=1}^n (|x_p| + 1) \sum_{q=1}^{i_p} (|y_q| + 1).$$

Example 3.6. Let A be a commutative algebra considered as a brace algebra with vanishing brace operations. Then the product defined above coincides with the shuffle product.

The following statement is shown in [GV94, Lemma 9].

Proposition 3.7. *Let A be a brace algebra. The multiplication on $\mathrm{T}_\bullet(A[1])$ defined above makes it into a dg bialgebra.*

Proof. By definition the product is compatible with the comultiplication and we only have to check associativity and the Leibniz rule for d.

It is enough to check the components of the identities landing in $A[1]$.

- (Associativity). The equation

$$([x] \cdot [y_1|\dots|y_n]) \cdot [z_1|\dots|z_m] = [x] \cdot ([y_1|\dots|y_n] \cdot [z_1|\dots|z_m])$$

  has the following A component:

$$x\{y_1, \dots, y_n\}\{z_1, \dots, z_m\} \\ = \sum_{\{i_p, l_p\}_{p=1}^n} (-1)^\epsilon x\{z_1, \dots, z_{i_1}, y_1\{z_{i_1+1}, \dots, z_{i_1+l_1}\}, \dots, y_n\{z_{i_n+1}, \dots, z_{i_n+l_n}\}, \dots, z_n\}.$$

  This exactly coincides with the associativity property for brace algebras.
  If we replace $[x]$ by $[x_1|\dots|x_m]$ for $m > 1$, the associativity equation will have a trivial A component.
- (Derivation). The equation

$$\mathrm{d}([x] \cdot [y_1|\dots|y_n]) = [\mathrm{d}x] \cdot [y_1|\dots|y_n] + (-1)^{|x|+1}[x] \cdot \mathrm{d}[y_1|\dots|y_n]$$

has the following A component:

$$\begin{aligned}\mathrm{d}(x\{y_1,\dots,y_n\}) + (-1)^{|x|+\sum_{q=1}^{n-1}|y_q|+n} x\{y_1,\dots,y_{n-1}\}\cdot y_n = \\ -(-1)^{(|y_1|+1)|x|} y_1\cdot x\{y_2,\dots,y_n\} + (\mathrm{d}x)\{y_1,\dots,y_n\} \\ + \sum_{i=1}^n (-1)^{\sum_{q=1}^{i-1}|y_q|+|x|+i} x\{y_1,\dots \mathrm{d}y_i,\dots,y_n\} \\ + \sum_{i=1}^{n-1} (-1)^{\sum_{q=1}^{i}|y_q|+|x|+i+1} x\{y_1,\dots,y_i y_{i+1},\dots,y_n\}.\end{aligned}$$

This follows from the higher homotopy identities for brace algebras.

The equation

$$\mathrm{d}([x_1|x_2]\cdot[y_1|\dots|y_n]) = \mathrm{d}[x_1|x_2]\cdot[y_1|\dots|y_n] + (-1)^{|x_1|+|x_2|}[x_1|x_2]\cdot\mathrm{d}[y_1|\dots|y_n]$$

has the following A component:

$$\begin{aligned}\sum_{m=0}^n (-1)^{(|x_2|+1)(\sum_{q=1}^m |y_q|+m)} (-1)^{|x_1|+\sum_{q=1}^m|y_q|+m+1} x_1\{y_1,\dots,y_m\} x_2\{y_{m+1},\dots,y_n\} \\ = (-1)^{|x_1|+1}(x_1x_2)\{y_1,\dots,y_n\}.\end{aligned}$$

This follows from the distributivity property for brace algebras.

If we instead have $[x_1|\dots|x_m]$ for $m>2$, this equation will have a trivial A component.

□

Remark 3.8. It is not difficult to see that $\mathrm{T}_\bullet(A[1])^{\mathrm{cop}} \cong \mathrm{T}_\bullet(A^{\mathrm{op}}[1])$ under the isomorphism (8). Here $(\dots)^{\mathrm{cop}}$ refers to the same dg algebra with the opposite coproduct and A^{op} is the opposite brace algebra.

Let us move on to a relative version of this statement. Let A be a brace algebra as before and M a left brace A-module. Recall the differential on the bar complex $\mathrm{T}_\bullet(A[1])\otimes M$. We are going to define a dg algebra structure on $\mathrm{T}_\bullet(A[1])\otimes M$ compatibly with the left coaction of $\mathrm{T}_\bullet(A[1])$ such that M and $\mathrm{T}_\bullet(A[1])$ are subalgebras. Thus, we just need to define a braiding morphism

$$M\otimes \mathrm{T}_\bullet(A[1]) \to \mathrm{T}_\bullet(A[1])\otimes M.$$

Compatibility with the $\mathrm{T}_\bullet(A[1])$-comodule structure allows one to uniquely reconstruct this map from the composite

$$M\otimes \mathrm{T}_\bullet(A[1]) \to \mathrm{T}_\bullet(A[1])\otimes M \to M.$$

We define it using the brace A-module structure on M. That is, the product is given by

$$[m]\cdot[x_1|\dots|x_n|1] = \sum_{i=0}^n (-1)^{|m|(\sum_{q=1}^i |x_q|+i)}[x_1|\dots|x_i|m\{x_{i+1},\dots,x_n\}].$$

Proposition 3.9. *Let M be a left brace A-module. The previous formula defines a dga structure on $\mathrm{T}_\bullet(A[1])\otimes M$ compatibly with the left $\mathrm{T}_\bullet(A[1])$-comodule structure.*

Proof. By construction the product on $\mathrm{T}_\bullet(A[1])\otimes M$ is compatible with the $\mathrm{T}_\bullet(A[1])$-coaction, so we just need to check the associativity of the product and the derivation property of d. Due to the compatibility with the $\mathrm{T}_\bullet(A[1])$-coaction, it is enough to check the properties after projection to M.

- (Associativity). The equation

$$[mn]\cdot[x_1|\dots|x_p|1]=[m]\cdot([n]\cdot[x_1|\dots|x_p|1])$$

has the M component identified with the distributivity property of left brace modules. Similarly, the equation

$$[m]\cdot([x_1|\dots|x_n|1]\cdot[y_1|\dots|y_m|1])=([m]\cdot[x_1|\dots|x_n|1])\cdot[y_1|\dots|y_m|1]$$

has the M-component identified with the associativity property of left brace modules.
- (Derivation). The equation

$$\mathrm{d}([m]\cdot[x_1|\dots|x_n|1])=[\mathrm{d}m]\cdot[x_1|\dots|x_n|1]+(-1)^{|m|}[m]\cdot\mathrm{d}[x_1|\dots|x_n|1]$$

has the M component identified with the higher homotopy identities of left brace modules.

□

We have the same statement for right brace A-modules. Indeed, one can replace A by A^{op} in the previous proposition and observe that the bar complexes $\mathrm{T}_\bullet(A[1])^{\mathrm{cop}}\otimes M$ and $M\otimes\mathrm{T}_\bullet(A[1])$ are isomorphic.

We can combine left and right modules as follows.

Theorem 3.10. *Let A be a brace algebra, M a left brace A-module and N a right brace A-module. Then the intersection $N\otimes^{\mathbb{L}}_A M$ carries a natural dga structure so that the projection $N^{\mathrm{op}}\otimes M\to N\otimes^{\mathbb{L}}_A M$ is a morphism of dg algebras.*

Proof. By Proposition 3.7 the bar complex $\mathrm{T}_\bullet(A[1])$ is a dg bialgebra.

Now let $\tilde{M}=\mathrm{T}_\bullet(A[1])\otimes M$ and $\tilde{N}=N\otimes\mathrm{T}_\bullet(A[1])$. By the previous proposition $\tilde{M}$ is a left $\mathrm{T}_\bullet(A[1])$-comodule while $\tilde{N}$ is a right $\mathrm{T}_\bullet(A[1])$-comodule.

The two-sided bar complex $N\otimes^{\mathbb{L}}_A M$ is isomorphic to the cotensor product $\tilde{N}\otimes^{\mathrm{T}_\bullet(A[1])}\tilde{M}$. As both $\tilde{N}$ and $\tilde{M}$ are dg algebras which are compatible with the coaction of $\mathrm{T}_\bullet(A[1])$, their cotensor product is also a dga. □

Remark 3.11. Given a model for the derived tensor product of the right A-module N and a left A-module M, it is quasi-isomorphic to the two-sided bar complex $N\otimes^{\mathbb{L}}_A M$, so by homotopy transfer one can induce a homotopy associative structure on the given model.

4. Quantum Hamiltonian reduction

4.1 Hochschild cohomology Let A be a dga and B an A-bimodule. We define the Hochschild cochain complex $\mathrm{CC}^\bullet(A,B)$ to be the graded vector space

$$\mathrm{CC}^\bullet(A,B)=\bigoplus_{n=0}^{\infty}\mathrm{Hom}(A^{\otimes n},B)[-n]$$

with the differential

$$\begin{aligned}(\mathrm{d}f)(x_1,\dots,x_n) &= \mathrm{d}f(x_1,\dots,x_n)\\ &\quad+\sum_{i=1}^{n}(-1)^{|f|+\sum_{q=1}^{i-1}|x_q|+i+1}f(x_1,\dots,\mathrm{d}x_i,\dots,x_n)\\ &\quad+\sum_{i=1}^{n-1}(-1)^{|f|+\sum_{q=1}^{i}|x_q|+i}f(x_1,\dots,x_ix_{i+1},\dots,x_n)\\ &\quad+(-1)^{|f|(|x_1|+1)}x_1f(x_2,\dots,x_n)+(-1)^{\sum_{q=1}^{n-1}|x_q|+|f|+n}f(x_1,\dots,x_{n-1})x_n.\end{aligned}$$

Given two A-bimodules B_1 and B_2 we have a cup product map

$$\mathrm{CC}^\bullet(A,B_1)\otimes\mathrm{CC}^\bullet(A,B_2)\to\mathrm{CC}^\bullet(A,B_1\otimes B_2)$$

given by

$$(f_1\smile f_2)(x_1,\dots,x_n)=\sum_{i=0}^{n}(-1)^{|f_2|(\sum_{q=1}^{i}|x_q|+i)}f_1(x_1,\dots,x_i)\otimes f_2(x_{i+1},\dots,x_n).$$

A relation between Hochschild and Chevalley–Eilenberg cohomology is given by the following construction. Let V be a $\mathrm{U}\mathfrak{g}$-bimodule. Then V^{ad} is a $\mathfrak{g}$-representation with the action given by

$$x.v := xv-(-1)^{|x||v|}vx,\quad x\in\mathfrak{g},\quad v\in V.$$

Consider $f\in\mathrm{Hom}((\mathrm{U}\mathfrak{g})^{\otimes n},V)[-n]\subset\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},V)$. We get an element $\tilde{f}\in\mathrm{C}^\bullet(\mathfrak{g},V)$ by the following formula:

$$\tilde{f}(x_1,\dots,x_n)=(-1)^{\sum_{q=1}^{n-1}(n-q)|x_q|}\sum_{\sigma\in S_n}(-1)^\epsilon f(x_{\sigma(1)},\dots,x_{\sigma(n)}),$$

where ϵ is given by the Koszul sign rule with x_i in degree $|x_i|+1$.

The following theorem can be found in [CR11, Theorem 2.5].

Proposition 4.1. *Let $A=\mathrm{U}\mathfrak{g}$ and V be a $\mathrm{U}\mathfrak{g}$-bimodule. Then the morphism*

$$\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},V)\to\mathrm{C}^\bullet(\mathfrak{g},V)$$

we have defined is a quasi-isomorphism. Moreover, it is compatible with cup products.

4.2 Hochschild cohomology and braces Gerstenhaber and Voronov [GV95] observed that the Hochschild cochain complex $\mathrm{CC}^\bullet(A,A)$ is a brace algebra which was the motivating example. We define the brace operations as follows:

$$\begin{aligned}&x\{x_1,\dots,x_n\}(a_1,\dots,a_m)\\ &\quad=\sum(-1)^\epsilon x(a_1,\dots,a_{i_1},x_1(a_{i_1+1},\dots,a_{i_1+l_1}),\dots,x_n(a_{i_n+1},\dots,a_{i_n+l_n}),\dots,a_m),\end{aligned}\tag{13}$$

where the sign is determined by the following rule: x_i moving past a_j produces the sign $(|x_i|+1)(|a_j|+1)$.

A multiplication on A determines a degree 2 element m of $\mathrm{CC}^\bullet(A,A)$ via

$$m(x,y)=(-1)^{|x|+1}xy.$$

The differential on $\mathrm{CC}^\bullet(A,A)$ is the sum of the natural differential on $\oplus_n \mathrm{Hom}((A[1])^{\otimes n}, A)$ and the differential $m\{f\} + (-1)^{|f|} f\{m\}$. The cup product on $\mathrm{CC}^\bullet(A,A)$ is given by the formula $f_1 \smile f_2 = (-1)^{|f_1|+1} m\{f_1, f_2\}$.

We will also need a variation of this example. Let B be a dga and $\mu\colon A \to B$ a morphism. Using the brace operations as above, one can turn $\mathrm{CC}^\bullet(A,B)$ into a left brace $\mathrm{CC}^\bullet(A,A)$-module.

We can also use the Hochschild cochain complex to give an interpretation of brace modules similar to Definition 1.8.

Proposition 4.2. *Let A be a brace algebra and M a left brace A-module with the module structure given by a morphism of algebras $f_0\colon A \to M$. Then we have a lift*

$$\begin{tikzcd} \mathrm{T}_\bullet(A[1]) \arrow[r, "f_0"] \arrow[dr, dashed, "f"'] & \mathrm{T}_\bullet(M[1]) \\ & \mathrm{T}_\bullet(\mathrm{CC}^\bullet(M,M)[1]), \arrow[u] \end{tikzcd}$$

where f is a morphism of dg bialgebras.

Proof. A morphism of coalgebras $f\colon \mathrm{T}_\bullet(A[1]) \to \mathrm{T}_\bullet(\mathrm{CC}^\bullet(M,M)[1])$ is uniquely specified by the composite $\mathrm{T}_\bullet(A[1]) \to \mathrm{T}_\bullet(\mathrm{CC}^\bullet(M,M)[1]) \to \mathrm{CC}^\bullet(M,M)[1]$ which consists of morphisms

$$f_{m,n}\colon M^{\otimes m} \otimes A^{\otimes n} \to M[1-n-m].$$

We define $f_{m,n} = 0$ for $m > 1$. The operations $f_{1,n}$ are given by

$$f_{1,n}(m, x_1, \ldots, x_n) = m\{x_1, \ldots, x_n\}.$$

A straightforward computation shows that the first two axioms in Definition 3.3 are equivalent to the compatibility of f with the multiplications and the last two axioms are equivalent to the compatibility of f with the differentials. $\square$

Remark 4.3. A triple (A, M, f) of a brace algebra A, a dga M and a morphism of brace algebras $f\colon A \to \mathrm{CC}^\bullet(M,M)$ is expected (see [Kon99, Section 2.5]) to be the same as an algebra over chains on the two-dimensional Swiss-cheese operad. The corresponding statement in the topological setting has been proved in [Th10]. Partial progress has been made in [DTT09] where the authors show that the pair $(\mathrm{CC}^\bullet(M,M), M)$ is indeed an algebra over the Swiss-cheese operad.

4.3 Hamiltonian reduction Let B be a dg algebra with a $\mathfrak{g}$-action. We denote by

$$a\colon \mathfrak{g} \to \mathrm{Der}(B)$$

the action morphism. Under deformation quantization the notion of a moment map for Poisson algebras (Definition 2.2) is deformed as follows.

Definition 4.4. A $\mathfrak{g}$-equivariant morphism $\mu\colon \mathfrak{g} \to B$ is a *quantum moment map* if the equation

$$[\mu(x), b] = a(x).b$$

is satisfied for all $x \in \mathfrak{g}$ and $b \in B$.

We refer to [Et07] for details on quantum moment maps.

Remark 4.5. As in the case of classical moment maps, one can replace $\mathfrak{g}$-equivariance by the condition that μ extends to a morphism of dg algebras $\mathrm{U}\mathfrak{g} \to B$.

Definition 4.6. Suppose B is a dga equipped with a $\mathfrak{g}$-action and a quantum moment map $\mu\colon \mathrm{U}\mathfrak{g} \to B$. Its *quantum Hamiltonian reduction* is

$$B//\mathrm{U}\mathfrak{g} = \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B).$$

In this bar complex we use the left $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})$-module structure on $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$ coming from the moment map $\mathrm{U}\mathfrak{g} \to B$ and the right $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})$-module structure on $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k)$ coming from the counit. We put a dga structure on $B//\mathrm{U}\mathfrak{g}$ in Corollary 4.7.

There is a quantum version of the BRST complex introduced in [KS87]. As a complex, it has the following description. We will assume that the Lie algebra $\mathfrak{g}$ is unimodular, i.e. the representation $\det(\mathfrak{g})$ is trivial.

Recall the Koszul complex $\mathrm{Sym}(\mathfrak{g}[1]) \otimes B$ that we have defined in Section 2.2. We are going to deform it to the Chevalley–Eilenberg differential as follows. Given

$$x_1 \wedge \cdots \wedge x_n \otimes b \in \mathrm{Sym}(\mathfrak{g}[1]) \otimes B$$

we let

$$\begin{aligned}
\mathrm{d}(x_1 \wedge \cdots \wedge x_n \otimes b) = &\sum_{i=1}^n (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1}|x_q|+i-1)} \mathrm{d}x_i \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes b \\
&- \sum_{i=1}^n (-1)^{|x_i|\sum_{q=i+1}^n(|x_q|+1)+\sum_{q=1}^{i-1}(|x_q|+1)+|x_i|} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes \mu(x_i) b \\
&+ (-1)^{\sum_{q=1}^n |x_q|+n} x_1 \wedge \cdots \wedge x_n \otimes \mathrm{d}b \\
&+ \sum_{i<j} (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1}|x_q|+i)+(|x_j|+1)(\sum_{q=1,q\neq i}^{j-1}|x_q|+j-1)} \times \\
&\qquad \times [x_i, x_j] \wedge x_1 \wedge \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n \otimes b.
\end{aligned} \tag{14}$$

The quantum BRST complex is then

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B).$$

We refer the reader to [KS87, Section 6] for a detailed description of the quantum BRST complex together with a dga structure.

4.4 Hamiltonian reduction as an intersection Let B be a dga with a Hamiltonian action of $\mathfrak{g}$. Recall that $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$ is then a left brace module over $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})$. Similarly, $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k)$ is a left brace module using the counit map $\mathrm{U}\mathfrak{g} \to k$ and hence $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k)^{\mathrm{op}}$ is a right brace module. Using Theorem 3.10 we therefore have a natural multiplication on the tensor product of $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k)$ and $\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$.

Corollary 4.7. *The quantum Hamiltonian reduction*

$$B//\mathrm{U}\mathfrak{g} = \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$$

carries a natural dga structure. Moreover, it is quasi-isomorphic to the quantum BRST complex.

Proof. The zig-zag of quasi-isomorphisms mentioned in the statement of the theorem is as follows:

$$\begin{tikzcd} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B) \arrow[d] \\ \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{U}\mathfrak{g}} B) \\ \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \arrow[u] \arrow[d] \\ \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B). \end{tikzcd}$$

- The morphism
$$\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes^{\mathbb{L}}_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g},\mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B) \to \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{U}\mathfrak{g}} B)$$
is given by the cup product. The fact that it is a quasi-isomorphism is proved as in Corollary 2.7.
- The morphism
$$\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \hookrightarrow \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k \otimes^{\mathbb{L}}_{\mathrm{U}\mathfrak{g}} B)$$
is given by including the Chevalley–Eilenberg chain complex into the bar complex.
- The morphism
$$\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \to \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B)$$
is the restriction morphism which is a quasi-isomorphism by Proposition 4.1.

□

Remark 4.8. As for the classical BRST complex, we do not know if the quasi-isomorphism above is compatible with the multiplication.

4.5 $\mathbb{E}_n$ Hamiltonian reduction The interpretation of quantum Hamiltonian reduction as a tensor product of brace modules allows one to formulate an $\mathbb{E}_n$ version of quantum Hamiltonian reduction. In this section we sketch what such a notion looks like in the ∞-categorical setting. We refer to [Gin13] for some basics of $\mathbb{E}_n$-algebras that we will use.

Let $\mathbb{E}_n$ be the chain operad of little n-cubes. For instance, the operad $\mathbb{E}_1$ is quasi-isomorphic to the associative operad and $\mathbb{E}_2$ is quasi-isomorphic to the brace operad. Given a morphism of $\mathbb{E}_n$-algebras $f\colon A \to B$ one has the $\mathbb{E}_n$-*centralizer* $\mathrm{Z}(f)$ which is an $\mathbb{E}_n$-algebra satisfying a certain universal property [Gin13, Definition 24]. For $f = \mathrm{id}\colon A \to A$ we denote $\mathrm{Z}(\mathrm{id}) = \mathrm{Z}(A)$, the center of A, which is an associative algebra object in $\mathbb{E}_n$-algebras, i.e. an $\mathbb{E}_{n+1}$-algebra by Dunn–Lurie additivity [Lu, Theorem 5.1.2.2]. Note that in the case of associative algebras (i.e. $n = 1$), $\mathrm{Z}(A)$ coincides with the Hochschild complex and its zeroth cohomology is the center of A in the usual sense.

One has a forgetful functor from $\mathbb{E}_n$-algebras to Lie algebras which on the level of underlying complexes is $A \mapsto A[n-1]$. The left adjoint to this forgetful functor is called the universal enveloping $\mathbb{E}_n$-algebra functor and is denoted by $\mathrm{U}_{\mathbb{E}_n}$, see [Gin13, Section 7.5].

Let B be an $\mathbb{E}_n$-algebra with an action of the Lie algebra $\mathfrak{g}$, i.e. we have a morphism of Lie algebras $a\colon \mathfrak{g} \to \mathbb{T}_B$ to the tangent complex of B.

Definition 4.9. A *quantum moment map* for the $\mathfrak{g}$-action on B is a morphism of Lie algebras $\mathfrak{g} \to B[n-1]$ fitting into the diagram

$$\begin{tikzcd} B[n-1] \arrow[r] & \mathbb{T}_B \\ & \mathfrak{g} \arrow[u, "a"'] \arrow[ul, dashed] \end{tikzcd}$$

of Lie algebras.

By adjunction the morphism of Lie algebras $\mathfrak{g} \to B[n-1]$ gives rise to a morphism of $\mathbb{E}_n$-algebras $\mu\colon \mathrm{U}_{\mathbb{E}_n}(\mathfrak{g}) \to B$. By the defining property of centralizers we see that $\mathrm{Z}(\mu)$ is a left module over the $\mathbb{E}_{n+1}$-algebra $\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}(\mathfrak{g}))$ in the ∞-category of $\mathbb{E}_n$-algebras. Alternatively, we can view the pair $(\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}(\mathfrak{g})), \mathrm{Z}(\mu))$ as an $\mathbb{E}_n$-algebra in the ∞-category LMod of pairs of an associative algebra and a module.

Remark 4.10. Note that using [Gin13, Theorem 14] one can identify $\mathrm{Z}(\mu) \cong \mathrm{C}^\bullet(\mathfrak{g}, B)$.

Let $\epsilon\colon \mathrm{U}_{\mathbb{E}_n}\mathfrak{g} \to k$ be the counit map. If $n > 0$ one can choose an isomorphism

$$\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}\mathfrak{g}) \cong \mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}\mathfrak{g})^{\mathrm{op}}$$

making $\mathrm{Z}(\epsilon)$ into a right module over $\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}\mathfrak{g})$. Thus, if we denote by BiMod the ∞-category of triples (A, M, N) of an associative algebra A, a left A-module M and a right A-module N, then we see that the triple $(\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}\mathfrak{g}), \mathrm{Z}(\mu), \mathrm{Z}(\epsilon))$ becomes an $\mathbb{E}_n$-algebra in BiMod. In particular, for any $\mathbb{E}_n$-algebra (A, M, N) in BiMod, the bar construction $N \otimes_A M$ is still an $\mathbb{E}_n$-algebra.

Definition 4.11. Let B be an $\mathbb{E}_n$-algebra with a $\mathfrak{g}$-action and a moment map $\mu\colon \mathrm{U}_{\mathbb{E}_n}(\mathfrak{g}) \to B$. Its $\mathbb{E}_n$ *Hamiltonian reduction* is the $\mathbb{E}_n$-algebra

$$B//\mathrm{U}_{\mathbb{E}_n}(\mathfrak{g}) = \mathrm{Z}(\epsilon) \otimes_{\mathrm{Z}(\mathrm{U}_{\mathbb{E}_n}\mathfrak{g})} \mathrm{Z}(\mu).$$

For instance, consider the case $n = 1$. Then $\mathrm{U}_{\mathbb{E}_1}(\mathfrak{g})$ coincides with the usual enveloping algebra. Indetifying centralizers with the Hochschild complex, we get the formula

$$B//\mathrm{U}(\mathfrak{g}) = \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, k) \otimes_{\mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, \mathrm{U}\mathfrak{g})} \mathrm{CC}^\bullet(\mathrm{U}\mathfrak{g}, B)$$

recovering quantum Hamiltonian reduction given in Definition 4.6.

5. Classical limits

In this section we relate some constructions in Section 1 to those in Section 3. Namely, we formulate precisely in which sense constructions in Section 3 are quantizations. Along the way we also relate Baranovsky and Ginzburg's construction [BG09] of the Poisson structure on a coisotropic intersection to our formulas.

5.1 Beilinson–Drinfeld algebras A precise sense in which associative algebras are quantizations of Poisson algebras is given by Beilinson–Drinfeld ($\mathbb{BD}$) algebras [CG16, Section 2.4]. Let us recall the definition.

Definition 5.1. A $\mathbb{BD}_1$-*algebra* is a dgla A over $k[\![\hbar]\!]$ together with an associative $k[\![\hbar]\!]$-linear multiplication satisfying the relations

- $\hbar\{x, y\} = xy - (-1)^{|x||y|} yx$,
- $\{x, yz\} = \{x, y\}z + (-1)^{|x||y|} y\{x, z\}$.

To understand this definition, recall that dg algebras are naturally Lie algebras with the bracket given by the commutator. The notion of a $\mathbb{BD}_1$-algebra then captures the fact that the Lie bracket vanishes to the first order at $\hbar = 0$. In the classical limit we have an isomorphism of operads

$$\mathbb{BD}_1/\hbar \cong \mathbb{P}_1$$

while in the quantum case $\hbar \neq 0$ we have

$$\mathbb{BD}_1[\hbar^{-1}] \cong \mathrm{Ass} \otimes_k k(\!(\hbar)\!)$$

since the bracket is then uniquely determined from the multiplication. In other words, the operad $\mathbb{BD}_1$ interpolates between the Poisson operad $\mathbb{P}_1$ and the associative operad Ass.

Remark 5.2. One can show that $\mathbb{BD}_1(n)$ is free as a $k[\![\hbar]\!]$-module.

Let us also mention that there is a canonical isomorphism of operads

$$\mathbb{P}_1 \otimes k[\hbar]/\hbar^2 \xrightarrow{\sim} \mathbb{BD}_1/\hbar^2$$

given by sending the multiplication to $\frac{ab+(-1)^{|a||b|}ba}{2}$.

Given a $\mathbb{BD}_1$-algebra A, we let A^{op} be the opposite algebra with the operations

$$a \cdot^{\mathrm{op}} b = (-1)^{|a||b|} b \cdot a$$
$$\{a, b\}^{\mathrm{op}} = -\{a, b\}.$$

There are also lower-dimensional and higher-dimensional versions of the $\mathbb{BD}_n$ operad.

Definition 5.3. A $\mathbb{BD}_0$-*algebra* is a complex A over $k[\![\hbar]\!]$ together with a degree 1 Lie bracket and a unital commutative multiplication satisfying the relations

- $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\mathrm{d}(b) + \hbar\{a, b\}$,
- $\{x, yz\} = \{x, y\}z + (-1)^{|y||z|}\{x, z\}y$.

In the classical limit we have an isomorphism

$$\mathbb{BD}_0/\hbar \cong \mathbb{P}_0$$

since then the multiplication is compatible with the differential. In the quantum case $\hbar \neq 0$ we have

$$\mathbb{BD}_0[\hbar^{-1}] \cong \widehat{\mathbb{E}}_0 \otimes k(\!(\hbar)\!),$$

where the operad $\widehat{\mathbb{E}}_0$ is contractible, i.e. quasi-isomorphic to the operad $\mathbb{E}_0$ controlling complexes with a distinguished vector.

Let $\mathrm{co}\mathbb{BD}_1$ be the cooperad obtained as the $k[\![\hbar]\!]$-linear dual to the operad $\mathbb{BD}_1$. It has a natural Hopf structure, so following Calaque and Willwacher [CW13] we can consider its brace construction

$$\mathbb{BD}_2 = \mathrm{Br}_{\mathrm{co}\mathbb{BD}_1}.$$

By construction we have that

$$\mathbb{BD}_2[\hbar^{-1}] = \mathrm{Br}_{\mathrm{co}\mathbb{BD}_1[\hbar^{-1}]} \cong \mathrm{Br}_{\mathrm{coAss}} \otimes k(\!(\hbar)\!),$$

where $\mathrm{Br}_{\mathrm{coAss}}$ is a dg operad quasi-isomorphic to the operad Br controlling brace algebras, but where the product is merely A_∞.

Moreover,

$$\mathbb{BD}_2/\hbar \cong \mathrm{Br}_{\mathrm{co}\mathbb{P}_1} \cong \mathbb{P}_2,$$

where the last quasi-isomorphism is given by [CW13, Theorem 4] (we remind the reader that $e_n \cong \mathbb{P}_n$ for $n \geq 2$).

5.2 Modules Let us now describe modules over $\mathbb{BD}_1$-algebras.

Recall that a coisotropic morphism $A \to B$ for A a $\mathbb{P}_1$-algebra is the data of a $\mathbb{P}_0$-algebra on B and a morphism of $\mathbb{P}_1$-algebras

$$A \to \mathrm{Z}(B) \cong \widehat{\mathrm{Sym}}(\mathrm{T}_B),$$

where for simplicity we have assumed that Ω^1_B is dualizable as a dg module over B.

For B a commutative graded algebra we denote by $\widehat{\mathrm{D}}_\hbar(B)$ the completed algebra of $\hbar$-differential operators. That is, it is an algebra over $k[[\hbar]]$ generated by elements of B and T_B with the relations

$$vw - (-1)^{|v||w|}wv = \hbar[v, w], \quad v, w \in \mathrm{T}_B$$
$$vb - (-1)^{|v||b|}bv = \hbar v.b, \quad v \in \mathrm{T}_B,\ w \in B$$

completed with respect to the increasing filtration given by the order of differential operators.

If B is a $\mathbb{BD}_0$-algebra, the data of the differential on B determines a Maurer–Cartan element in $\widehat{\mathrm{D}}_\hbar(B)$ and we denote by $\mathrm{Z}(B)$, the $\mathbb{BD}_0$-*center* of B, the algebra $\widehat{\mathrm{D}}_\hbar(B)$ with the differential twisted by that Maurer–Cartan element. It is clear that $\mathrm{Z}(B)$ is a $\mathbb{BD}_1$-algebra.

More generally, if B is a commutative graded algebra, the data of a Maurer–Cartan element in $\widehat{\mathrm{D}}_\hbar(B)$ will be called a $\widehat{\mathbb{BD}}_0$-algebra structure on B. Note that $\mathbb{BD}_0$-structures correspond to those Maurer–Cartan elements which have order at most 2.

Remark 5.4. Suppose B_0 is a cofibrant commutative dg algebra over k. We can trivially extend it to a $\mathbb{BD}_0$-algebra $B = B_0 \otimes k[[\hbar]]$ with the bracket defined to be zero. Then we expect that $\widehat{\mathrm{D}}_\hbar(B)$ coincides with the center of $B \in \mathrm{Alg}_{\mathbb{BD}_0}$ in the sense of [Lu, Definition 5.3.1.6]. Note that given any $\mathbb{BD}_0$-algebra B, its $\mathbb{BD}_0$-center at $\hbar = 0$ becomes $(\widehat{\mathrm{Sym}}(\mathrm{T}_{B_0}), [\pi_{B_0}, -])$, the $\mathbb{P}_0$-center of $B_0 = B/\hbar$.

Let A be another $\mathbb{BD}_1$-algebra.

Definition 5.5. A *left* $\mathbb{BD}_1$*-module* over A is a $\mathbb{BD}_0$-algebra B together with a morphism $A \to \mathrm{Z}(B)$ of $\mathbb{BD}_1$-algebras.

A *right* $\mathbb{BD}_1$*-module* over A is the same as a left $\mathbb{BD}_1$-module over A^{op}.

It is clear that the definition at $\hbar = 0$ reduces to the definition of a coisotropic morphism. Thus, one can talk about *quantizations* of a given coisotropic morphism $A_0 \to B_0$: these are $\mathbb{BD}_1$-algebras A and $\mathbb{BD}_0$-algebras B reducing to the given algebras A_0, B_0 at $\hbar = 0$ together with a left $\mathbb{BD}_1$-module structure on B.

One can similarly define $\mathbb{BD}_2$-modules as follows. Given a complex B, we denote by $\mathrm{co}\mathbb{BD}_1(B)$ the cofree conilpotent $\mathbb{BD}_1$-coalgebra on B. Given a $\mathbb{BD}_1$-algebra B, we define its *center* to be the complex

$$\mathrm{Z}(B) = \mathrm{Hom}(\mathrm{co}\mathbb{BD}_1(B), B)$$

twisted by the differential given by the $\mathbb{BD}_1$-structure on B. By the results of [CW13], $\mathrm{Z}(B)$ is a $\mathbb{BD}_2 = \mathrm{Br}_{\mathrm{co}\mathbb{BD}_1}$-algebra, so we can give the following definition.

Let A be a $\mathbb{BD}_2$-algebra.

Definition 5.6. A *left* $\mathbb{BD}_2$*-module* over A is a $\mathbb{BD}_1$-algebra B together with a morphism $A \to \mathrm{Z}(B)$ of $\mathbb{BD}_2$-algebras.

Remark 5.7. Just like for Poisson algebras, we expect that the $\mathbb{BD}_1$-center of a $\mathbb{BD}_1$-algebra $B \in \mathrm{Alg}_{\mathbb{BD}_1}$ satisfies the universal property of [Lu, Definition 5.3.1.6]. See also Remark 1.7.

5.3 From $\mathbb{BD}_1$ to $\mathbb{BD}_0$ We are now going to prove a $\mathbb{BD}_1$-version of Theorem 1.18.

Let A be a $\mathbb{BD}_1$-algebra. In particular, it is a dga and so we have a dg coalgebra $\mathrm{T}_\bullet(A[1])$. Introduce a commutative multiplication on $\mathrm{T}_\bullet(A[1])$ given by the shuffle product and the Lie bracket given by (7).

Since A is not necessarily commutative, the differential is not compatible with the shuffle product. But its failure is exactly captured by the bracket.

Proposition 5.8. *Let A be a $\mathbb{BD}_1$-algebra. The differential, multiplication and the bracket make $\mathrm{T}_\bullet(A[1])$ into a $\mathbb{BD}_0$-algebra compatibly with the coalgebra structure.*

Proof. To prove the claim we just need to show that the relation between the differential on $\mathrm{T}_\bullet(A[1])$ and the product is exactly the one that appears in the definition of $\mathbb{BD}_0$-algebras.

Due to the compatibility of the operations with the coproduct on $\mathrm{T}_\bullet(A[1])$, we just need to check the corresponding relation after projection to $A[1]$.

For $a, b \in A$ we have

$$\begin{aligned}\mathrm{d}([a]\cdot[b]) &= \mathrm{d}([a|b] + (-1)^{(|a|+1)(|b|+1)}[b|a]) \\ &= [\mathrm{d}a|b] + (-1)^{|a|+1}[a|\mathrm{d}b] + (-1)^{(|a|+1)(|b|+1)}[\mathrm{d}b|a] + (-1)^{|a|(|b|+1)}[b|\mathrm{d}a] \\ &\quad + (-1)^{|a|+1}[ab] + (-1)^{|a|(|b|+1)}[ba].\end{aligned}$$

Similarly, we have

$$[\mathrm{d}a]\cdot[b] + (-1)^{|a|+1}[a]\cdot[\mathrm{d}b] = [\mathrm{d}a|b] + (-1)^{|a|(|b|+1)}[b|\mathrm{d}a] + (-1)^{|a|+1}[a|\mathrm{d}b] + (-1)^{(|a|+1)(|b|+1)}[\mathrm{d}b|a].$$

Their difference is given by

$$\begin{aligned}(-1)^{|a|+1}[ab] + (-1)^{|a|(|b|+1)}[ba] &= \hbar(-1)^{|a|+1}[\{a,b\}] \\ &= \hbar\{[a],[b]\}.\end{aligned}$$

□

We can also add modules in the picture. Let M be a left $\mathbb{BD}_1$-module. Then $\mathrm{T}_\bullet(A[1]) \otimes M$ carries a differential and L_∞ brackets given by equations (9) and (10). Moreover, $\mathrm{T}_\bullet(A[1]) \otimes M$ carries a natural multiplication.

Proposition 5.9. *Let A be a $\mathbb{BD}_1$-algebra and M a left $\mathbb{BD}_1$-module. Then $\mathrm{T}_\bullet(A[1]) \otimes M$ carries a natural structure of a left $\widehat{\mathbb{BD}}_0$-comodule over $\mathrm{T}_\bullet(A[1])$.*

Proof. By construction the differential and the brackets on $\mathrm{T}_\bullet(A[1]) \otimes M$ are compatible with the $\mathrm{T}_\bullet(A[1])$-comodule structure, so we just have to check that the projection of the differential on M has symbol given by the brackets.

The differential lands in M in the following two cases:

1. The symbol of $\mathrm{d}\colon M \to M$ is given by the Poisson bracket on M since it is a $\mathbb{BD}_0$-algebra.
2. The symbol of the action map $A \otimes M \to M$ is given by the $\hbar = 0$ limit of the action map $A \to \mathrm{D}_\hbar(M)$ which coincides with the Poisson brackets on $\mathrm{T}_\bullet(A[1]) \otimes M$ given by formula (9).

$\square$

Finally, suppose M and N are a left and right $\mathbb{BD}_1$-modules over A respectively. Then on the two-sided bar complex

$$N \otimes \mathrm{T}_\bullet(A[1]) \otimes M$$

we can introduce the usual bar differential and the shuffle product.

Theorem 5.10. *Let A be a $\mathbb{BD}_1$-algebra, M a left $\mathbb{BD}_1$-module and N a right $\mathbb{BD}_1$-module over A. Then the two-sided bar complex $N \otimes \mathrm{T}_\bullet(A[1]) \otimes M$ has a $\widehat{\mathbb{BD}}_0$-structure.*

At $\hbar = 0$ this construction recovers the $\widehat{\mathbb{P}}_0$-structure of Theorem 1.18.

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Braces and Poisson additivity

Pavel Safronov

Abstract

We relate the brace construction introduced by Calaque and Willwacher to an additivity functor. That is, we construct a functor from brace algebras associated to an operad $\mathcal{O}$ to associative algebras in the category of homotopy $\mathcal{O}$-algebras. As an example, we identify the category of $\mathbb{P}_{n+1}$-algebras with the category of associative algebras in $\mathbb{P}_n$-algebras. We also show that under this identification there is an equivalence of two definitions of derived coisotropic structures in the literature.

Introduction

This paper is devoted to a proof of an equivalence of symmetric monoidal ∞-categories

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \cong \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

between the ∞-category of $\mathbb{P}_{n+1}$-algebras and the ∞-category of associative algebras in the ∞-category of $\mathbb{P}_n$-algebras. Here $\mathbb{P}_{n+1}$ is the operad which controls dg commutative algebras together with a Poisson bracket of degree $-n$.

Braces

Let $\mathcal{O}$ be a dg operad and $\mathcal{C}$ its Koszul dual cooperad which is assumed to be Hopf. Following Tamarkin's work [Tam00] on the deformation complex of a $\mathbb{P}_n$-algebra, Calaque and Willwacher [CW15] introduced an operad $\mathrm{Br}_{\mathcal{C}}$ of brace algebras which acts on the deformation complex of any homotopy $\mathcal{O}$-algebra. Moreover, they have remarked that the brace construction is an analog of the Boardman–Vogt tensor product $\mathbb{E}_1 \otimes \Omega\mathcal{C}$ of operads. This is suggested by the following examples:

- if $\mathbf{1}$ is the trivial cooperad, $\mathrm{Br}_{\mathbf{1}} \cong \mathbb{E}_1$;
- if coAss is the cooperad of coassociative coalgebras, $\mathrm{Br}_{\mathrm{coAss}}\{1\} \cong \mathbb{E}_2$;
- if coComm is the cooperad of cocommutative coalgebras, $\mathrm{Br}_{\mathrm{coComm}} \cong \mathrm{Lie}$;
- if $\mathrm{co}\mathbb{P}_n$ is the cooperad of $\mathbb{P}_n$-coalgebras, $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\} \cong \mathbb{P}_{n+1}$.

In this paper we explain to what extent this is true. Namely, suppose $\mathcal{C}$ is a Hopf cooperad satisfying a minor technical assumption. We construct a functor of ∞-categories

$$\mathcal{A}\mathrm{lg}_{\mathrm{Br}_{\mathcal{C}}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathcal{O}}) \tag{1}$$

from the ∞-category of $\mathrm{Br}_{\mathcal{C}}$-algebras to the ∞-category of associative algebras in the ∞-category of $\mathcal{O}$-algebras. Let us note that we did not assume that $\mathcal{O}$ is a Hopf operad and the symmetric monoidal structure on $\mathcal{A}\mathrm{lg}_{\mathcal{O}}$ comes from the Koszul dual side.

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Unfortunately, we do not know if (1) is an equivalence in general, but we do show that it is an equivalence in two examples of interest: namely, Lie algebras and Poisson algebras.

Suppose $\mathcal{C} = \mathrm{coComm}$. As we have mentioned, $\mathrm{Br}_{\mathrm{coComm}} \cong \mathrm{Lie}$ and so we get a functor

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}).$$

We show that it is an equivalence and in fact coincides with the functor which sends a Lie algebra $\mathfrak{g}$ to the associative algebra object in the category of Lie algebras $0 \times_{\mathfrak{g}} 0$ (see Proposition 2.13). We also show that the same functor can be constructed as follows. Given a Lie algebra $\mathfrak{g}$, the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative bialgebra, i.e. an associative algebra object in cocommutative coalgebras. Identifying cocommutative coalgebras with Lie algebras using Koszul duality we obtain the same functor (see Proposition 2.11). Let us mention that the underlying Lie algebra structure on $\mathrm{add}(\mathfrak{g})$ is canonically trivial by Proposition 2.15.

Note that $\mathrm{Br}_{\mathrm{coComm}}$ is an important operad in itself and appears for instance in the description of the Atiyah bracket of vector fields, see § 2.1.1.

Poisson additivity

The additivity functor is more interesting in the case of $\mathbb{P}_{n+1}$-algebras. So, take $\mathcal{C} = \mathrm{co}\mathbb{P}_n$. Since $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\} \cong \mathbb{P}_{n+1}$, we obtain a functor

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}).$$

The following statement combines Propositions 2.19–2.21 and Theorem 2.22.

Theorem. *The additivity functor*

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

is an equivalence of symmetric monoidal ∞-categories.

Moreover, the diagrams

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r] \arrow[d] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d] \\ \mathcal{A}\mathrm{lg}_{\mathrm{Comm}} & \mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r] \arrow[d, shift left] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d, shift left] \\ \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r] \arrow[u, shift left, "\mathrm{Sym}"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \arrow[u, shift left, "\mathrm{Sym}"] \end{tikzcd}$$

commute.

Rozenblyum has given an independent proof of this result in the language of factorization algebras. This statement is a Poisson version of the additivity theorem [Lur17, Theorem 5.1.2.2] for $\mathbb{E}_n$-algebras proved by Dunn and Lurie: one has an equivalence

$$\mathcal{A}\mathrm{lg}_{\mathbb{E}_{n+1}} \cong \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{E}_n})$$

of symmetric monoidal ∞-categories, where $\mathbb{E}_n$ is the operad of little n-disks.

One has the following explicit description of the additivity functor for Poisson algebras which uses some ideas of Tamarkin (see [Tam00] and [Tam07]). For simplicity, we describe the construction in the case of non-unital $\mathbb{P}_{n+1}$-algebras. In the case of unital $\mathbb{P}_{n+1}$-algebras one has to take care of the natural curving appearing on the Koszul dual side, but otherwise the construction is identical (see § 2.5). If A is a commutative algebra, we can consider its Harrison complex $\mathrm{coLie}(A[1])$ which is a Lie coalgebra. If A is moreover a $\mathbb{P}_{n+1}$-algebra, then the Harrison complex $\mathrm{coLie}(A[1])[n-1]$ has a natural structure of an $(n-1)$-shifted Lie bialgebra (Definition 2.16) which defines a functor

$$\mathrm{Alg}_{\mathbb{P}_{n+1}} \longrightarrow \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}}.$$

Given a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is a cocommutative bialgebra. If $\mathfrak{g}$ is moreover an $(n-1)$-shifted Lie bialgebra, then $\mathrm{U}(\mathfrak{g})$ acquires a natural cobracket making it into an associative algebra object in $\mathbb{P}_n$-coalgebras. Thus, we get a functor

$$\mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}).$$

Applying Koszul duality we identify $\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}$ with the category of $\mathbb{P}_n$-algebras thus giving the required additivity functor.

An important point we have neglected in this discussion is that at the very end one has to pass from the localization of the category of associative algebras in $\mathbb{P}_n$-coalgebras to the ∞-category of (homotopy) associative algebras in the localization of the category of $\mathbb{P}_n$-coalgebras. That is, we have a natural functor

$$\mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n})[W_{\mathrm{Kos}}^{-1}] \longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}[W_{\mathrm{Kos}}^{-1}]),$$

where W_{Kos} is a certain natural class of weak equivalences we define in the paper, Alg is the 1-category of associative algebras and $\mathcal{A}\mathrm{lg}$ is the ∞-category of (homotopy) associative algebras. The fact that this functor is an equivalence is not automatic: the corresponding rectification statement was proved in [Lur17, Theorem 4.1.8.4] under the assumption that the model category in question is a monoidal model category while the monoidal structure on $\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}$ does not even preserve colimits. Furthermore, if we do not pass to the Koszul dual side, the localization functor (where W_{qis} is the class of quasi-isomorphisms)

$$\mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_n})[W_{\mathrm{qis}}^{-1}] \longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{Alg}_{\mathbb{P}_n}[W_{\mathrm{qis}}^{-1}])$$

is *not* an equivalence. Indeed, $\mathrm{Alg}(\mathrm{Alg}_{\mathbb{P}_n})$ is equivalent to the category of commutative algebras while we prove that $\mathcal{A}\mathrm{lg}(\mathrm{Alg}_{\mathbb{P}_n}[W_{\mathrm{qis}}^{-1}])$ is equivalent to the ∞-category of $\mathbb{P}_{n+1}$-algebras.

Let us note that the underlying commutative structure on $\mathrm{add}(A)$ for a $\mathbb{P}_{n+1}$-algebra A coincides with the commutative structure on A and the underlying Lie structure on $\mathrm{add}(A)$ is trivial. However, the underlying $\mathbb{P}_n$-structure on $\mathrm{add}(A)$ is not necessarily commutative.

One motivation for developing Poisson additivity is the recent work of Costello and Gwilliam [CG16] that formalizes algebras of observables in quantum field theories. In that work a topological quantum field theory is described by a locally constant factorization algebra on the spacetime manifold valued in $\mathbb{E}_0$-algebras. Since locally constant factorization algebras on $\mathbb{R}^n$ are the same as $\mathbb{E}_n$-algebras, we see that observables in an n-dimensional topological quantum field theory are described by $\mathbb{E}_n \otimes \mathbb{E}_0 = \mathbb{E}_n$-algebras. Similarly, classical topological field theories are described by locally constant factorization algebras valued in $\mathbb{P}_0$-algebras, which in the case of $\mathbb{R}^n$ are the same as $\mathbb{E}_n$-algebras in $\mathbb{P}_0$-algebras. Our result thus shows that observables in an n-dimensional classical topological field theory are described by a $\mathbb{P}_n$-algebra (a natural result one expects by extrapolating from the case of topological quantum mechanics which is $n = 1$).

Coisotropic structures

Another motivation is given by the theory of shifted Poisson geometry developed by Calaque–Pantev–Toën–Vaquié–Vezzosi and, more precisely, derived coisotropic structures. Recall that an n-shifted Poisson structure on an affine scheme $\operatorname{Spec} A$ for A a commutative dg algebra is described by a $\mathbb{P}_{n+1}$-algebra structure on A. Now suppose $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of affine schemes. In [CPTVV17] the following notion of derived coisotropic structures was introduced. Assume the statement of Poisson additivity. Then one can realize A as an associative algebra in $\mathbb{P}_n$-algebras and a coisotropic structure on f is a lift of the natural action of A on B in commutative algebras to $\mathbb{P}_n$-algebras. Let us denote by $\mathrm{Cois}^{\mathrm{CPTVV}}(f,n)$ the space of such coisotropic structures.

A more explicit definition of derived coisotropic structures was given in [Saf17] and [MS16] which does not rely on Poisson additivity. An action of a $\mathbb{P}_{n+1}$-algebra A on a $\mathbb{P}_n$-algebra B was modeled by a certain colored operad $\mathbb{P}_{[n+1,n]}$ and a derived coisotropic structure was defined to be the lift of the natural action of A on B in commutative algebras to an algebra over the operad $\mathbb{P}_{[n+1,n]}$. Let us denote by $\mathrm{Cois}^{\mathrm{MS}}(f,n)$ the space of such coisotropic structures.

In this paper we show that these two notions coincide. The following statement is Corollary 3.9.

THEOREM. *Suppose $f\colon A \to B$ is a morphism of commutative dg algebras. One has a natural equivalence*

$$\mathrm{Cois}^{\mathrm{MS}}(f,n) \cong \mathrm{Cois}^{\mathrm{CPTVV}}(f,n)$$

of spaces of n-shifted coisotropic structures.

This statement is proved by developing a relative analog of the Poisson additivity functor. Namely, Theorem 3.8 asserts that the ∞-category of $\mathbb{P}_{[n+1,n]}$-algebras is equivalent to the ∞-category of pairs (A, M), where A is an associative algebra and M is an A-module in the ∞-category of $\mathbb{P}_n$-algebras.

Notations

- Given a relative category $(\mathcal{C}, W)$ we denote by $\mathcal{C}[W^{-1}]$ the underlying ∞-category.
- We work over a field k of characteristic zero; Ch denotes the category of chain complexes of k-modules and $\mathcal{C}h$ the underlying ∞-category.
- Given a topological operad $\mathcal{O}$, we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the category of $\mathcal{O}$-algebras in a symmetric monoidal category $\mathcal{C}$ and by $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$ the ∞-category of $\mathcal{O}$-algebras in a symmetric monoidal ∞-category $\mathcal{C}$. If $\mathcal{O}$ is a dg operad, the category of $\mathcal{O}$-algebras in complexes is simply denoted by $\mathrm{Alg}_{\mathcal{O}}$.
- All operads are non-unital unless specified otherwise. We denote by $\mathcal{O}^{\mathrm{un}}$ the operad of unital $\mathcal{O}$-algebras.
- All non-counital coalgebras are conilpotent.

1. Operads

1.1 Relative categories

In the paper we will extensively use relations between relative categories and ∞-categories, so let us recall the necessary facts.

DEFINITION 1.1. A *relative category* $(\mathcal{C}, W)$ consists of a category $\mathcal{C}$ and a subcategory $W \subset \mathcal{C}$ which has the same objects as $\mathcal{C}$ and contains all isomorphisms in $\mathcal{C}$.

We will call morphisms belonging to W *weak equivalences*. A functor of relative categories $(\mathcal{C}, W_{\mathcal{C}}) \to (\mathcal{D}, W_{\mathcal{D}})$ is a functor that preserves weak equivalences.

Recall that given a category $\mathcal{C}$ its nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category. Similarly, if $\mathcal{C}$ is a relative category, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category equipped with a system of morphisms W and we introduce the notation

$$\mathcal{C}[W^{-1}] = \mathrm{N}(\mathcal{C})[W^{-1}],$$

where the localization functor on the right is defined in [Lur17, Proposition 4.1.7.2]. In particular, $\mathcal{C}[W^{-1}]$ is an ∞-category which we call the *underlying* ∞*-category* of the relative category $(\mathcal{C}, W)$.

We will also need a construction of symmetric monoidal ∞-categories from ordinary symmetric monoidal categories. We say that $\mathcal{C}$ is a *relative symmetric monoidal category* if the functor $x \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves weak equivalences for every object $x \in \mathcal{C}$.

Proposition 1.2. *Let $\mathcal{C}$ be a relative symmetric monoidal category. Then the localization $\mathcal{C}[W^{-1}]$ admits a natural structure of a symmetric monoidal ∞-category.*

If $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a (lax) symmetric monoidal functor of relative symmetric monoidal categories, then its localization induces a (lax) symmetric monoidal functor of ∞-categories

$$F\colon \mathcal{C}_1[W^{-1}] \longrightarrow \mathcal{C}_2[W^{-1}].$$

Proof. Given a symmetric monoidal category $\mathcal{C}$ we can construct the symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ as in [Lur17, Construction 2.0.0.1]. The class of weak equivalences W defines a system in the underlying ∞-category of $\mathcal{C}^{\otimes}$ which is compatible with the tensor product and hence by [Lur17, Proposition 4.1.7.4] we can construct a symmetric monoidal ∞-category $(\mathcal{C}')^{\otimes}[W^{-1}]$ whose underlying ∞-category is equivalent to $\mathcal{C}[W^{-1}]$.

A (lax) symmetric monoidal functor $\mathcal{C}_1 \to \mathcal{C}_2$ gives rise to a (lax) symmetric monoidal functor $\mathcal{C}_1^{\otimes} \to \mathcal{C}_2^{\otimes}$ of ∞-categories. Consider the composite

$$\mathcal{C}_1^{\otimes} \longrightarrow \mathcal{C}_2^{\otimes} \longrightarrow \mathcal{C}_2^{\otimes}[W^{-1}].$$

Since $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ preserves weak equivalences, by the universal property of the localization we obtain a (lax) symmetric monoidal functor

$$F\colon \mathcal{C}_1^{\otimes}[W^{-1}] \longrightarrow \mathcal{C}_2^{\otimes}[W^{-1}]. \qquad \square$$

For instance, let Ch be the symmetric monoidal category of chain complexes of k-vector spaces. Let $W_{\mathrm{qis}} \subset \mathrm{Ch}$ be the class of quasi-isomorphisms. Since $M \otimes -$ preserves quasi-isomorphisms for any $M \in \mathrm{Ch}$, we obtain a natural symmetric monoidal structure on the ∞-category

$$\mathcal{C}\mathrm{h} = \mathrm{Ch}[W_{\mathrm{qis}}^{-1}]$$

of chain complexes.

We will repeatedly use the following method to prove that a functor between ∞-categories is an equivalence. Consider a commutative diagram of ∞-categories.

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "F"] \arrow[dr, "G_1"'] & & \mathcal{C}_2 \arrow[dl, "G_2"] \\ & \mathcal{D} & \end{tikzcd}$$

Assuming G_1 and G_2 have left adjoints G_1^L and G_2^L respectively, we obtain a natural transformation $G_2^L \to FG_1^L$ of functors $\mathcal{D} \to \mathcal{C}_2$. We say that the original diagram satisfies the *left Beck–Chevalley condition* if this natural transformation is an equivalence. The following is a corollary of the ∞-categorical version of the Barr–Beck theorem proved in [Lur17, Corollary 4.7.3.16].

PROPOSITION 1.3. *Suppose*

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr] \arrow[dr, "G_1"'] & & \mathcal{C}_2 \arrow[dl, "G_2"] \\ & \mathcal{D} & \end{tikzcd}$$

is a commutative diagram of ∞-categories such that:

(i) *the functors G_1 and G_2 admit left adjoints;*

(ii) *the diagram satisfies the left Beck–Chevalley condition;*

(iii) *the ∞-categories $\mathcal{C}_i$ admit and G_i preserve geometric realizations of simplicial objects;*

(iv) *the functors G_i are conservative.*

Then the functor $\mathcal{C}_1 \to \mathcal{C}_2$ is an equivalence.

1.2 Operads

Our definitions and notations for operads follows those of Loday and Vallette [LV12]. Unless specified otherwise, by an operad we mean an operad in chain complexes.

Given a symmetric sequence V, we define the shift $V[n]$ to be the symmetric sequence with

$$(V[n])(m) = V(m)[n].$$

Let sgn_m be the one-dimensional sign representation of S_m. We will also use the notation $V\{n\}$ to denote the symmetric sequence with

$$(V\{n\})(m) = V(m) \otimes \mathrm{sgn}_m^{\otimes n}[n(m-1)].$$

If V is an operad or a cooperad, so is $V\{n\}$.

Let $\mathbf{1}$ be the trivial operad. Recall that an *augmentation* on an operad $\mathcal{O}$ is a morphism of operads

$$\mathcal{O} \longrightarrow \mathbf{1}.$$

In particular, one obtains a splitting of symmetric sequences

$$\mathcal{O} \cong \overline{\mathcal{O}} \oplus \mathbf{1}.$$

Similarly, one has a notion of a *coaugmentation* on a cooperad $\mathcal{C}$.

Given an augmented operad $\mathcal{O}$, its bar construction $\mathrm{B}\mathcal{O}$ is defined to be the cofree cooperad on $\overline{\mathcal{O}}[1]$ equipped with the bar differential which consists of two terms: one coming from the differential on $\mathcal{O}$ and one coming from the product on $\mathcal{O}$. Similarly, given a coaugmented cooperad $\mathcal{C}$ we have its cobar construction $\Omega\mathcal{C}$. We refer to [LV12, § 6.5] for details. In particular, one has a quasi-isomorphism of operads

$$\Omega\mathrm{B}\mathcal{O} \xrightarrow{\sim} \mathcal{O}.$$

Most of the operads of interest that control non-unital algebras satisfy $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) \cong k$ and hence possess a unique augmentation. However, operads controlling unital algebras tend not to have an augmentation, so, following Hirsh and Millès [HM12], we relax the condition a bit.

Definition 1.4. A *semi-augmentation* on an operad $\mathcal{O}$ is a morphism of the underlying graded symmetric sequences $\epsilon\colon \mathcal{O} \to \mathbf{1}$ which is not necessarily compatible with the differential and the product such that the composite

$$\mathbf{1} \longrightarrow \mathcal{O} \xrightarrow{\epsilon} \mathbf{1}$$

is the identity.

Given a semi-augmented operad $\mathcal{O}$, one can still consider the bar construction $\mathrm{B}\mathcal{O}$, but the corresponding differential no longer squares to zero. Instead, we obtain a curved cooperad equipped with a curving $\theta\colon \mathcal{C}(1) \to k[2]$ (see [HM12, Definition 3.2.1] for a complete definition).

We refer to [HM12, § 3.3] for explicit formulas for the differential and the curving on the bar construction of a semi-augmented operad. Moreover, it is also shown there that the cobar construction $\Omega\mathcal{C}$ on a coaugmented curved cooperad $\mathcal{C}$ is a dg operad equipped with a natural semi-augmentation.

Finally, we refer to [LV12, § 7] for Koszul duality for augmented operads and to [HM12, § 4] for Koszul duality for semi-augmented operads. The important point that we will use in the paper is that the Koszul dual cooperad $\mathcal{C}$ of $\mathcal{O}$ is naturally equipped with a quasi-isomorphism

$$\Omega\mathcal{C} \xrightarrow{\sim} \mathcal{O}$$

which gives a semi-free resolution of the operad $\mathcal{O}$. Such a quasi-isomorphism is equivalently given by a degree 1 (curved) Koszul twisting morphism $\mathcal{C} \to \mathcal{O}$.

1.3 Operadic algebras

Given an operad $\mathcal{O}$ we denote by $\mathrm{Alg}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras in chain complexes. Similarly, for a cooperad $\mathcal{C}$ we denote by $\mathrm{CoAlg}_{\mathcal{C}}$ the category of conilpotent $\mathcal{C}$-coalgebras. To simplify the notation, we let

$$\mathrm{Alg} = \mathrm{Alg}_{\mathrm{Ass}^{\mathrm{un}}}$$

be the category of unital associative algebras.

If $\mathcal{C}$ is a curved cooperad, we denote by $\mathrm{CoAlg}_{\mathcal{C}}$ the category of curved conilpotent $\mathcal{C}$-coalgebras (see [HM12, Definition 5.2.1]) which are cofibrant. Note that morphisms strictly preserve the differential.

Remark 1.5. Positselski in [Pos11, § 9] considers a closely related category of curved coassociative coalgebras $k-\mathrm{coalg}_{\mathrm{cdg}}$ whose morphisms do not strictly preserve the differential.

If A is an $\mathcal{O}$-algebra, then $A[-n]$ is an $\mathcal{O}\{n\}$-algebra; similarly, if C is a $\mathcal{C}$-coalgebra, then $C[-n]$ is a $\mathcal{C}\{n\}$-coalgebra.

Now consider a (curved) cooperad $\mathcal{C}$ equipped with a (curved) Koszul twisting morphism $\mathcal{C} \to \mathcal{O}$. Given an $\mathcal{O}$-algebra A we define its bar construction to be

$$\mathrm{B}(A) = \mathcal{C}(A) = \bigoplus_{n=0}^{\infty} (\mathcal{C}(n) \otimes A^{\otimes n})^{S_n}$$

equipped with the bar differential (see [HM12, § 5.2.3]). Given a curved $\mathcal{C}$-coalgebra C we define its cobar construction to be

$$\Omega(C) = \mathcal{O}(C) = \bigoplus_{n=0}^{\infty} (\mathcal{O}(n) \otimes C^{\otimes n})_{S_n}$$

equipped with the cobar differential (see [HM12, § 5.2.5]). Note that the cobar differential squares to zero. In particular, we get a bar-cobar adjunction

$$\Omega\colon \mathrm{CoAlg}_{\mathcal{C}} \rightleftarrows \mathrm{Alg}_{\mathcal{O}} \colon \mathrm{B}$$

such that for any $\mathcal{O}$-algebra A the natural projection

$$\Omega \mathrm{B} A \longrightarrow A$$

is a quasi-isomorphism.

Let us denote by $W_{\mathrm{qis}} \subset \mathrm{Alg}_{\mathcal{O}}$ the class of morphisms of $\mathcal{O}$-algebras which are quasi-isomorphisms of the underlying complexes. Let us also denote by $W_{\mathrm{Kos}} \subset \mathrm{CoAlg}_{\mathcal{C}}$ the class of morphisms of (curved) $\mathcal{C}$-coalgebras which become quasi-isomorphisms after applying the cobar functor Ω. The class of weak equivalences W_{Kos} is independent of the choice of the operad $\mathcal{O}$ as shown by the following statement. Let us denote by $\Omega_{\mathcal{O}}\colon \mathrm{CoAlg}_{\mathcal{C}} \to \mathrm{Alg}_{\mathcal{O}}$ the cobar construction associated to the operad $\mathcal{O}$.

Proposition 1.6. *Suppose $\mathcal{C} \to \mathcal{O}_1$ is a (curved) Koszul twisting morphism and $\mathcal{O}_1 \to \mathcal{O}_2$ a quasi-isomorphism of operads. Consider a morphism of (curved) $\mathcal{C}$-coalgebras $C_1 \to C_2$. Then*

$$\Omega_{\mathcal{O}_1}(C_1) \longrightarrow \Omega_{\mathcal{O}_1}(C_2)$$

is a quasi-isomorphism if and only if

$$\Omega_{\mathcal{O}_2}(C_1) \longrightarrow \Omega_{\mathcal{O}_2}(C_2)$$

is a quasi-isomorphism.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
\Omega_{\mathcal{O}_1}(C_1) \arrow[r] \arrow[d] & \Omega_{\mathcal{O}_1}(C_2) \arrow[d] \\
\Omega_{\mathcal{O}_2}(C_1) \arrow[r] & \Omega_{\mathcal{O}_2}(C_2)
\end{tikzcd}$$

The morphisms

$$\Omega_{\mathcal{O}_1}(C_i) \longrightarrow \Omega_{\mathcal{O}_2}(C_i)$$

are filtered quasi-isomorphisms where the filtration is defined as in [Val14, Proposition 2.3]. The filtration is also complete and bounded below and hence the morphisms $\Omega_{\mathcal{O}_1}(C_i) \to \Omega_{\mathcal{O}_2}(C_i)$ are quasi-isomorphisms.

Therefore, the top morphism is a quasi-isomorphism if and only if the bottom morphism is a quasi-isomorphism. □

Proposition 1.7. *Suppose $\mathcal{C} \to \mathcal{O}$ is a (curved) Koszul twisting morphism. Then the adjunction*

$$\Omega\colon \mathrm{CoAlg}_{\mathcal{C}} \rightleftarrows \mathrm{Alg}_{\mathcal{O}} \colon \mathrm{B}$$

descends to an adjoint equivalence of ∞-categories

$$\Omega\colon \mathrm{CoAlg}_{\mathcal{C}}[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathcal{O}}[W_{\mathrm{qis}}^{-1}] \colon \mathrm{B}.$$

Proof. First of all, we have to show that B and Ω preserve weak equivalences. Indeed, by definition Ω creates weak equivalences. Now suppose $A_1 \to A_2$ is a quasi-isomorphism and consider the morphism $\mathrm{B}A_1 \to \mathrm{B}A_2$. We have a commutative diagram

$$\begin{tikzcd} \Omega\mathrm{B}A_1 \arrow[r] \arrow[d] & \Omega\mathrm{B}A_2 \arrow[d] \\ A_1 \arrow[r] & A_2 \end{tikzcd}$$

where the two vertical morphisms are quasi-isomorphisms by [HM12, Proposition 5.2.8] and the bottom morphism is a quasi-isomorphism by assumption. Therefore, $\Omega\mathrm{B}A_1 \to \Omega\mathrm{B}A_2$ is also a quasi-isomorphism and hence by definition $\mathrm{B}A_1 \to \mathrm{B}A_2$ is a weak equivalence.

Therefore, we get an adjunction $\Omega \dashv \mathrm{B}$ of the underlying ∞-categories. To show that it is an adjoint equivalence we have to show that the unit and the counit of the adjunction are weak equivalences. Indeed, again by [HM12, Proposition 5.2.8] the counit of the adjunction is a weak equivalence. Next, suppose C is a (curved) $\mathcal{C}$-coalgebra and consider the unit of the adjunction $C \to \mathrm{B}\Omega C$. To show that it is a weak equivalence, consider the morphisms

$$\Omega C \longrightarrow \Omega\mathrm{B}\Omega C \longrightarrow \Omega C.$$

By construction the composite morphism is the identity; the second morphism is the counit of the adjunction and hence is a quasi-isomorphism. Therefore, the first morphism is a quasi-isomorphism and hence $C \to \mathrm{B}\Omega C$ is a weak equivalence. ◻

We introduce the notation

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}} = \mathrm{Alg}_{\mathcal{O}}[W_{\mathrm{qis}}^{-1}]$$

for the ∞-category of $\mathcal{O}$-algebras and by the previous proposition it can also be modeled by the relative category $(\mathrm{CoAlg}_{\mathcal{C}}, W_{\mathrm{Kos}})$.

Suppose $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of operads which is a quasi-isomorphism in each arity. The forgetful functor

$$\mathrm{Alg}_{\mathcal{O}_2} \longrightarrow \mathrm{Alg}_{\mathcal{O}_1}$$

automatically preserves quasi-isomorphisms and hence induces a functor on the level of ∞-categories.

Proposition 1.8. *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a quasi-isomorphism of operads. Then the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}_2} \longrightarrow \mathrm{Alg}_{\mathcal{O}_1}$$

induces an equivalence of ∞-categories

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}_2} \longrightarrow \mathcal{A}\mathrm{lg}_{\mathcal{O}_1}.$$

Proof. Indeed, by [BM07, Theorem 4.1] and [Hin97, Theorem 4.7.4] the induction/restriction functors provide a Quillen equivalence between $\mathrm{Alg}_{\mathcal{O}_1}$ and $\mathrm{Alg}_{\mathcal{O}_2}$ and hence induce an equivalence of the underlying ∞-categories. ◻

Finally, recall that the forgetful functor $\mathrm{Alg}_{\mathcal{O}} \to \mathrm{Ch}$ creates sifted colimits since the category $\mathrm{Alg}_{\mathcal{O}}$ can be written as the category of algebras over a monad which preserves sifted colimits. The same statement is true on the level of ∞-categories.

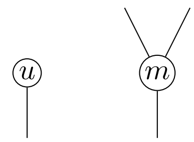


Figure 1. Generating operations of $\mathrm{Ass}^{\mathrm{un}}$.

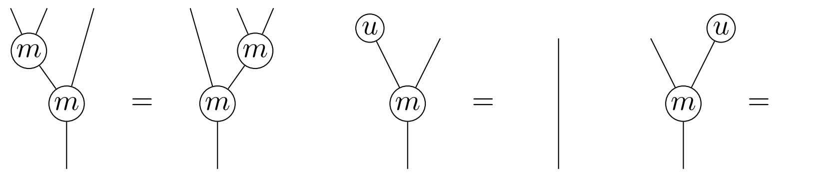


Figure 2. Relations in $\mathrm{Ass}^{\mathrm{un}}$.

Proposition 1.9. *Let I be a set and $\mathcal{O}$ an I-colored dg operad. The forgetful functor*

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}} \longrightarrow \mathrm{Fun}(I, \mathcal{C}\mathrm{h})$$

creates sifted colimits.

Proof. By [Lur17, Proposition 1.3.4.24] we just need to show that the forgetful functor $\mathrm{Alg}_{\mathcal{O}} \to \mathrm{Fun}(I, \mathrm{Ch})$ creates homotopy sifted colimits which follows by [PS14, Proposition 7.8]. ☐

1.4 Examples

Let us show how the bar-cobar duality works for unital algebras over the associative and Poisson operads to compare it to the classical bar-cobar duality.

We begin with the case of the associative operad considered in [HM12, § 6]. Let $\mathcal{O} = \mathrm{Ass}^{\mathrm{un}}$ be the operad governing unital associative algebras. The operad $\mathrm{Ass}^{\mathrm{un}}$ is quadratic and is generated by the symmetric sequence V with $V(0) = k$ and $V(2) = k[S_2]$ whose elements are shown on Figure 1.

The relations in $\mathrm{Ass}^{\mathrm{un}}$ have the form shown on Figure 2. This gives an inhomogeneous quadratic-linear-constant presentation of the operad $\mathrm{Ass}^{\mathrm{un}}$. Given such an operad $\mathcal{O}$, we denote by $q\mathcal{O}$ the operad with the same generators and where we only keep the quadratic part of the relations. Recall that the underlying graded cooperad of the Koszul dual is defined from the quadratic part of the relations, the differential uses the linear part and the curving comes from the constant part. In the relations we have there are no linear terms, so the Koszul dual cooperad coincides with the Koszul dual cooperad of the quadratic operad $q\mathrm{Ass}^{\mathrm{un}}$ equipped with a curving. From the relations we see that

$$q\mathrm{Ass}^{\mathrm{un}} \cong \mathrm{Ass} \oplus \mathbb{E}_0,$$

where $\mathbb{E}_0$ is the operad governing complexes together with a distinguished vector and where $\oplus$ refers to the product in the category of operads.

By [HM12, Proposition 6.1.4] the Koszul dual of $q\mathrm{Ass}^{\mathrm{un}}$ is given by

$$(q\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}} = \mathrm{Ass}^{\mathrm{i}} \oplus \mathrm{co}\mathbb{E}_0\{-1\},$$

where $\mathrm{co}\mathbb{E}_0 \cong \mathbb{E}_0$ is the cooperad of complexes together with a functional and $\oplus$ now denotes the product in the category of conilpotent cooperads, i.e. the conilpotent cooperad cogenerated by $\mathrm{Ass}^{\mathrm{i}}$ and $\mathrm{co}\mathbb{E}_0$. We define

$$(\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}} = (q\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}}$$

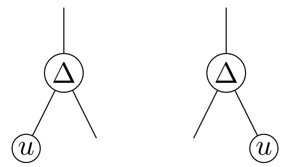


Figure 3. Curving on $(\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}}$.

as graded cooperads. The degree -2 part of $(\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}}(1)$ is the two-dimensional vector space spanned by trees shown in Figure 3 and we define the curving $\theta\colon (\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}}(1) \to k[2]$ to take value -1 on both of these. Let us denote

$$\mathrm{coAss}^{\theta}\{1\} = (\mathrm{Ass}^{\mathrm{un}})^{\mathrm{i}}.$$

The cooperad coAss^{θ} governs coassociative coalgebras C together with a coderivation $d\colon C \to C$ of degree 1 and a curving $\theta\colon C \to k[2]$ satisfying the equations

$$\begin{aligned} d^2 x &= \theta(x_{(1)})x_{(2)} - x_{(1)}\theta(x_{(2)}) \\ \theta(dx) &= 0, \end{aligned}$$

where

$$\Delta(x) = x_{(1)} \otimes x_{(2)}.$$

Given a unital associative dg algebra A, its bar complex is

$$\mathrm{B}(A) = \overline{\mathrm{T}}_{\bullet}(A[1] \oplus k[2])$$

equipped with the following differential. Let us denote elements of the bar complex by $[x_1|\cdots|x_n]$ with elements of $k[2]$ denoted by $*$. Then

$$\begin{aligned} d[x_1|\cdots|x_n] = &\sum_{i=1}^{n}(-1)^{\sum_{q=1}^{i-1}(|x_q|+1)}[x_1|\cdots|dx_i|\cdots|x_n] \\ &+ \sum_{i=1}^{n-1}(-1)^{\sum q=1^i(|x_q|+1)}[x_1|\cdots|x_i x_{i+1}|\cdots|x_n] \end{aligned}$$

with $d* = 1 \in A$ and $* \cdot x = x \cdot * = 0$. The curving is given by $\theta([*]) = 1$.

Remark 1.10. Given a unital associative dg algebra A equipped with a semi-augmentation, Positselski in [Pos11, § 6.1] considers the bar construction to be $\mathrm{T}_{\bullet}(\overline{A}[1])$ equipped with a natural curving and the standard bar differential.

Similarly, given a curved coalgebra C, its cobar complex is

$$\Omega(C) = \mathrm{T}^{\bullet}(C[-1])$$

whose elements we denote by $[x^1|\cdots|x^n]$ for $x^i \in C$ such that the differential is

$$\begin{aligned} d[x^1|\cdots|x^n] &= \sum_{i=1}^{n}(-1)^{\sum_{q=1}^{i-1}(|x^q|+1)}[x^1|\cdots|dx^i|\cdots|x^n] \\ &\quad + \sum_{i=1}^{n}(-1)^{\sum_{q=1}^{i-1}(|x^q|+1)}[x^1|\cdots|\theta(x^i)x^{i+1}|\cdots|x^n] \\ &\quad + \sum_{i=1}^{n}(-1)^{\sum_{q=1}^{i-1}(|x^q|+1)+|x^i_{(1)}|+1}[x^1|\cdots|x^i_{(1)}|x^i_{(2)}|\cdots|x^n]. \end{aligned}$$

Let us similarly work out the Koszul dual of the operad of unital $\mathbb{P}_n$-algebras. Recall that a $\mathbb{P}_n$-algebra is a dg Poisson algebra whose Poisson bracket has degree $1-n$. We denote by $\mathcal{O} = \mathbb{P}_n^{\mathrm{un}}$ the operad controlling such algebras. It is generated by the symmetric sequence V with $V(0) = k\cdot u$ and $V(2) = k\cdot m \oplus \mathrm{sgn}_2^{\otimes n}\cdot\{\}$ with the following relations:

$$\begin{aligned} a(bc) &= (ab)c \\ \{\{a,b\},c\} &= (-1)^{n+|b||c|+1}\{\{a,c\},b\} + (-1)^{|a|(|b|+|c|)+1}\{\{b,c\},a\} \\ \{a,bc\} &= \{a,b\}c + (-1)^{|b||c|}\{a,c\}b \\ 1\cdot a &= a \\ \{1,a\} &= 0. \end{aligned}$$

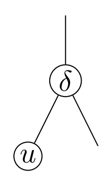
In particular, we see again that

$$q\mathbb{P}_n^{\mathrm{un}} \cong \mathbb{P}_n \oplus \mathbb{E}_0.$$

Note that the Koszul dual of the operad of non-unital $\mathbb{P}_n$-algebras is $\mathbb{P}_n^{\text{!`}} \cong \mathrm{co}\mathbb{P}_n\{n\}$. Therefore, by [HM12, Proposition 6.1.4] the Koszul dual of $\mathbb{P}_n^{\mathrm{un}}$ is

$$(\mathbb{P}_n^{\mathrm{un}})^{\text{!`}} \cong \mathrm{co}\mathbb{P}_n\{n\} \oplus \mathrm{co}\mathbb{E}_0\{-1\}$$

equipped with the curving $\theta\colon (\mathbb{P}_n^{\mathrm{un}})^{\text{!`}}(1) \to k[2]$ which sends the tree



to -1, where δ is the cobracket in $\mathrm{co}\mathbb{P}_n\{n\}$. We denote

$$\mathrm{co}\mathbb{P}_n^{\theta}\{n\} = (\mathbb{P}_n^{\mathrm{un}})^{\text{!`}}.$$

A curved coalgebra C over the cooperad $\mathrm{co}\mathbb{P}_n^{\theta}$ is given by the following data:

- a cocommutative comultiplication on C;
- a cobracket $\delta\colon C \to C\otimes C[1-n]$ for which we use Sweedler's notation
$$\delta(x) = x^{\delta}_{(1)} \otimes x^{\delta}_{(2)},$$
which satisfies the coalgebraic version of the Jacobi identity;
- a coderivation $d\colon C\to C$ of degree 1;
- a curving $\theta\colon C \to k[1+n]$.

Together these satisfy the relations

$$\begin{aligned} d^2x &= \theta(x^{\delta}_{(1)})x^{\delta}_{(2)} \\ \theta(dx) &= 0. \end{aligned}$$

1.5 Hopf operads

Recall that a *Hopf operad* is an operad in counital cocommutative coalgebras. Dually, a *Hopf cooperad* is a cooperad in unital commutative algebras. Alternatively, recall that the category of symmetric sequences has two tensor structures: the composition product which is merely monoidal and which we use to define operads and cooperads and the Hadamard product which is symmetric monoidal. The Hadamard tensor product defines a symmetric monoidal structure on the category of cooperads and one can define a Hopf cooperad to be a unital commutative algebra in the category of cooperads.

One can similarly define a notion of a curved Hopf cooperad to be a unital commutative algebra in the category of curved cooperads.

Given a Hopf operad $\mathcal{O}$ and two $\mathcal{O}$-algebras A_1, A_2 the tensor product of the underlying complexes is also an $\mathcal{O}$-algebra using

$$\mathcal{O}(n)\otimes(A\otimes B)^{\otimes n}\to\mathcal{O}(n)\otimes A^{\otimes n}\otimes\mathcal{O}(n)\otimes B^{\otimes n}\to A\otimes B.$$

Dually, one defines the tensor product of two (curved or dg) $\mathcal{C}$-coalgebras for a Hopf cooperad $\mathcal{C}$ to be the tensor product of the underlying chain complexes.

The operads $\mathrm{Ass}^{\mathrm{un}}$ and $\mathbb{P}_n^{\mathrm{un}}$ we are interested in are Hopf operads. For instance, for $\mathbb{P}_n^{\mathrm{un}}$ we have

$$\Delta(m)=m\otimes m,\quad \Delta(\{\})=\{\}\otimes m+m\otimes\{\}.$$

By duality we get Hopf cooperad structures on $\mathrm{coAss}^{\mathrm{cu}}=(\mathrm{Ass}^{\mathrm{un}})^*$ and $\mathrm{co}\mathbb{P}_n^{\mathrm{cu}}=(\mathbb{P}_n^{\mathrm{un}})^*$, the cooperads of counital coassociative coalgebras and counital $\mathbb{P}_n$-coalgebras respectively.

Note, however, that the curved cooperad coAss^{θ} admits no Hopf structure. Indeed, the degree zero part of $\mathrm{coAss}^{\theta}(0)$ is trivial and hence one cannot define a unit. To remedy this problem, we introduce the following modification. Given an operad $\mathcal{O}$ we denote by $\mathcal{O}^{\mathrm{un}}=\mathcal{O}\oplus k$ the symmetric sequence which coincides with $\mathcal{O}$ in arities at least 1 and which is $\mathcal{O}(0)\oplus k$ in arity zero. Similarly, we define the symmetric sequence $\mathcal{C}^{\mathrm{cu}}=\mathcal{C}\oplus k$ for a cooperad $\mathcal{C}$.

Definition 1.11. A *Hopf unital structure* on an operad $\mathcal{O}$ is the structure of a Hopf operad on $\mathcal{O}^{\mathrm{un}}$ such that the natural inclusion $\mathcal{O}\to\mathcal{O}^{\mathrm{un}}$ is a morphism of operads and such that the counit on $\mathcal{O}^{\mathrm{un}}(0)=\mathcal{O}(0)\oplus k$ is given by the projection on the second factor.

Definition 1.12. A *Hopf counital structure* on a (curved) cooperad $\mathcal{C}$ is the structure of a (curved) Hopf cooperad on $\mathcal{C}^{\mathrm{cu}}$ such that the natural projection $\mathcal{C}^{\mathrm{cu}}\to\mathcal{C}$ is a morphism of (curved) cooperads and such that the unit on $\mathcal{C}^{\mathrm{cu}}(0)=\mathcal{C}(0)\oplus k$ is given by inclusion into the second factor.

For instance, Ass and $\mathbb{P}_n$ have Hopf unital structures given by the Hopf operads $\mathrm{Ass}^{\mathrm{un}}$ and $\mathbb{P}_n^{\mathrm{un}}$. Similarly, the curved cooperads coAss^{θ} and $\mathrm{co}\mathbb{P}_n^{\theta}$ have Hopf counital structures given by the cooperads $\mathrm{coAss}^{\theta,\mathrm{cu}}$ and $\mathrm{co}\mathbb{P}_n^{\theta,\mathrm{cu}}$ respectively, where, for instance, $\mathrm{coAss}^{\theta,\mathrm{cu}}$ governs curved counital coassociative coalgebras.

Given a Hopf operad $\mathcal{O}^{\mathrm{un}}$, the counits assemble to give a map of operads $\mathcal{O}^{\mathrm{un}}\to\mathrm{Comm}^{\mathrm{un}}$. We can equivalently define a Hopf unital structure on an operad $\mathcal{O}$ to be a Hopf operad $\mathcal{O}^{\mathrm{un}}$ together with an isomorphism of operads

$$\mathcal{O}\cong\mathcal{O}^{\mathrm{un}}\times_{\mathrm{Comm}^{\mathrm{un}}}\mathrm{Comm}.$$

Using the morphism $\mathcal{O}^{\mathrm{un}}\to\mathrm{Comm}^{\mathrm{un}}$ we see that the unital commutative algebra k admits a natural $\mathcal{O}^{\mathrm{un}}$-algebra structure.

Definition 1.13. Let $\mathcal{O}^{\mathrm{un}}$ be a Hopf operad. An *augmented* $\mathcal{O}^{\mathrm{un}}$*-algebra* is an $\mathcal{O}^{\mathrm{un}}$-algebra A together with a morphism of $\mathcal{O}^{\mathrm{un}}$-algebras $A \to k$.

Dually, for a Hopf cooperad $\mathcal{C}^{\mathrm{cu}}$ we define the notion of a coaugmented $\mathcal{C}^{\mathrm{cu}}$-coalgebra. We denote by $\mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}^{\mathrm{un}}}$ the category of augmented $\mathcal{O}^{\mathrm{un}}$-algebras and by $\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}$ the category of coaugmented $\mathcal{C}^{\mathrm{cu}}$-coalgebras. Note that a coaugmented $\mathcal{C}^{\mathrm{cu}}$-coalgebra $C \to k$ is automatically assumed to be conilpotent in the sense that the non-counital coalgebra $\overline{C}$ is conilpotent.

Lemma 1.14. *Let $\mathcal{O}$ be an operad with a Hopf unital structure. Then we have an equivalence of categories*

$$\mathrm{Alg}_{\mathcal{O}} \cong \mathrm{Alg}^{\mathrm{aug}}_{\mathcal{O}^{\mathrm{un}}}.$$

Dually, if $\mathcal{C}$ is a dg cooperad with a Hopf counital structure, then we have an equivalence of categories

$$\mathrm{CoAlg}_{\mathcal{C}} \cong \mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}.$$

Proof. Given an $\mathcal{O}^{\mathrm{un}}$-algebra A, we have the unit

$$k \longrightarrow A$$

coming from the inclusion $k \to \mathcal{O}^{\mathrm{un}}(0) \cong \mathcal{O}(0) \oplus k$ into the second factor. The coaugmentation $A \to k$ splits the unit and hence one has an isomorphism of complexes

$$A \cong \overline{A} \oplus k.$$

The $\mathcal{O}^{\mathrm{un}}$-algebra structure on A gives rise to the operations

$$\mathcal{O}(0) \longrightarrow A, \quad \mathcal{O}(n) \otimes \overline{A}^{\otimes m} \longrightarrow A,$$

where $m \leqslant n$.

The morphism $\mathcal{O}(0) \to A \to k$ is zero by the definition of the counit on $\mathcal{O}^{\mathrm{un}}(0)$. The morphisms $\mathcal{O}(n) \otimes \overline{A}^{\otimes m} \to A$ of $m < n$ are uniquely determined from the ones for $m = n$. But since $A \to k$ is a morphism of $\mathcal{O}^{\mathrm{un}}$-algebras, the diagram

$$\begin{tikzcd} \mathcal{O}(n) \otimes \overline{A}^{\otimes n} \arrow[r] \arrow[d, "0"] & A \arrow[d] \\ \mathcal{O}(n) \arrow[r] & k \end{tikzcd}$$

is commutative which implies that the composite

$$\mathcal{O}(n) \otimes \overline{A}^{\otimes m} \to A \to k$$

factors through $\overline{A}$. Therefore, the augmented $\mathcal{O}^{\mathrm{un}}$-algebra structure on A is uniquely determined by the $\mathcal{O}$-algebra structure on $\overline{A}$.

The statement for coalgebras is proved similarly. □

The same construction works for a curved cooperad $\mathcal{C}$. Note that since $\mathcal{C}^{\mathrm{cu}}$ is a Hopf cooperad, the category $\mathrm{CoAlg}_{\mathcal{C}}$ inherits a natural symmetric monoidal structure. Explicitly, given two $\mathcal{C}$-coalgebras C_1, C_2, the underlying graded vector space of their tensor product is defined to be $C_1 \otimes C_2 \oplus C_1 \oplus C_2$.

Remark 1.15. The cooperads coAss and $\mathrm{co}\mathbb{P}_n$ are already Hopf cooperads and this gives a *different* symmetric monoidal structure on the category $\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}$ which we will not consider in the paper.

We will need a certain compatibility between the Hopf structure on $\mathcal{C}^{\mathrm{cu}}$ and weak equivalences.

Definition 1.16. A Hopf unital structure on a (curved) cooperad $\mathcal{C}$ is *admissible* if the tensor product functor

$$C\otimes -\colon \mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}\longrightarrow \mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}$$

preserves weak equivalences for any $C\in\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}$.

We are now going to show that some standard examples of Hopf cooperads are admissible.

Proposition 1.17. *One has the following quasi-isomorphisms of $\mathcal{O}$-algebras for any pair C_1, C_2 of (curved) conilpotent $\mathcal{C}$-coalgebras:*

(i) *for* $\mathcal{O}=\mathrm{Ass}\{-1\}$ *and* $\mathcal{C}=\mathrm{coAss}$,

$$\Omega(C_1\oplus C_2\oplus C_1\otimes C_2)\longrightarrow \Omega(C_1)\oplus\Omega(C_2)\oplus\Omega(C_1)\otimes\Omega(C_2)[-1];$$

(ii) *for* $\mathcal{O}=\mathrm{Lie}\{-1\}$ *and* $\mathcal{C}=\mathrm{coComm}$,

$$\Omega(C_1\oplus C_2\oplus C_1\otimes C_2)\longrightarrow \Omega(C_1)\oplus\Omega(C_2);$$

(iii) *for* $\mathcal{O}=\mathbb{P}_n\{-n\}$ *and* $\mathcal{C}=\mathrm{co}\mathbb{P}_n$,

$$\Omega(C_1\oplus C_2\oplus C_1\otimes C_2)\longrightarrow \Omega(C_1)\oplus\Omega(C_2)\oplus\Omega(C_1)\otimes\Omega(C_2)[-n];$$

(iv) *for* $\mathcal{O}=\mathrm{Ass}^{\mathrm{un}}\{-1\}$ *and* $\mathcal{C}=\mathrm{coAss}^{\theta}$,

$$\Omega(C_1\oplus C_2\oplus C_1\otimes C_2)\longrightarrow \Omega(C_1)\otimes\Omega(C_2)[-1];$$

(v) *for* $\mathcal{O}=\mathbb{P}_n^{\mathrm{un}}\{-n\}$ *and* $\mathcal{C}=\mathrm{co}\mathbb{P}_n^{\theta}$,

$$\Omega(C_1\oplus C_2\oplus C_1\otimes C_2)\longrightarrow \Omega(C_1)\otimes\Omega(C_2)[-n].$$

Proof. For simplicity in cases (1) and (3) we add units and augmentations using Lemma 1.14.

In the case $\mathcal{C}=\mathrm{coAss}$ a morphism of associative algebras

$$p\colon A=\mathrm{T}^\bullet(C_1[-1]\oplus C_2[-1]\oplus C_1\otimes C_2[-1])\longrightarrow \mathrm{T}^\bullet(C_1[-1])\otimes\mathrm{T}^\bullet(C_2[-1])$$

is uniquely specified on the generators and we define it to be zero on $C_1\otimes C_2[-1]$ and the obvious inclusions on the first two summands. It is clear that p is compatible with the cobar differentials.

Elements of A are given by words

$$[x_1|\cdots|y_k|\cdots|(x_i,y_i)|\cdots],$$

where $x_n\in C_1$, $y_n\in C_2$ and $(x_n,y_n)\in C_1\otimes C_2$. The coproduct on $C_1\otimes C_2$ is given by

$$\begin{aligned}\Delta(x,y)&=x\otimes y+(-1)^{|x||y|}y\otimes x+(-1)^{|x_{(2)}||y_{(1)}|}(x_{(1)},y_{(1)})\otimes(x_{(2)},y_{(2)})+(x,y_{(1)})\otimes y_{(2)}\\&\quad+(-1)^{|x||y_{(1)}|}y_{(1)}\otimes(x,y_{(2)})+x_{(1)}\otimes(x_{(2)},y)+(-1)^{|y||x_{(2)}|}(x_{(1)},y)\otimes x_{(2)}.\end{aligned}$$

We define a splitting

$$i\colon \mathrm{T}^\bullet(C_1[-1])\otimes \mathrm{T}^\bullet(C_2[-1]) \longrightarrow A$$

to be the multiplication, e.g.

$$i([x_1|x_2]\otimes[y_1|y_2|y_3]) = [x_1|x_2|y_1|y_2|y_3].$$

It is again clear that i is compatible with the cobar differentials and that $p\circ i = \mathrm{id}$. Note, however, that i is merely a morphism of chain complexes and is not compatible with the multiplication.

Let $\{F_nC_i\}_{n\geqslant 1}$ be the coradical filtrations on C_i which satisfy

$$\Delta(F_nC_i)\subset \bigoplus_{a+b=n} F_aC_i\otimes F_bC_i.$$

We introduce a filtration on the coalgebra $C_1\oplus C_2\oplus C_1\otimes C_2$ by declaring that

$$F_nC_1\otimes F_mC_2\subset F_{n+m-1}(C_1\otimes C_2).$$

It induces a filtration on the algebra A by declaring that $[x_1|\cdots|x_n]$ with $x_i\in F_{n_i}$ lies in $F_{\sum n_i - n}A$ and similarly for $\mathrm{T}^\bullet(C_1[-1])\otimes \mathrm{T}^\bullet(C_2[-1])$. The filtrations are bounded below (by zero) and complete. Moreover, the morphism p is clearly compatible with the filtrations, so it is enough to prove that p is a quasi-isomorphism after passing to the associated graded algebras with respect to the filtration. This allows us to assume that the coproducts on C_i are zero.

We are now going to construct a homotopy $\mathrm{id}\overset{h}{\sim} i\circ p$ such that

$$a - i(p(a)) = dh(a) + h(da). \tag{2}$$

The homotopy h on monomials $a\in A$ is constructed by the following algorithm akin to bubble sort.

– Suppose a has no factors in $C_1\otimes C_2$. If a has no elements in the wrong order (i.e. elements of C_2 followed by an element of C_1), then $h(a)=0$. Otherwise, write $a = b\cdot[y|x]\cdot c$, where c has no elements in the wrong order and define inductively

$$h(a) = (-1)^{|x||y|+|y|+1+|b|}b\cdot[(x,y)]\cdot c + (-1)^{(|x|+1)(|y|+1)}h(b\cdot[x|y]\cdot c). \tag{3}$$

– If a has factors in $C_1\otimes C_2$, we define it recursively by the formula

$$h(a) = (-1)^{|b|+|x|+|y|+1}b\cdot[(x,y)]\cdot h(c), \tag{4}$$

where $a = b\cdot[(x,y)]\cdot c$ such that b has no factors in $C_1\otimes C_2$.

Clearly, the second step reduces the number of elements in $C_1\otimes C_2$, so in finite time we arrive at an expression without factors in $C_1\otimes C_2$. Given a monomial a without factors in $C_1\otimes C_2$ we define the number of *inversions* to be the number of elements of C_2 left of an element of C_1. For instance, the expression $[y_1|x_1|x_2|y_2]$ with $x_i\in C_1$ and $y_i\in C_2$ has two inversions. It is immediate that the first step of the algorithm reduces the number of inversions by 1 and hence it also terminates in finite time.

Let us make a preliminary observation that the equation

$$\begin{aligned} b\cdot[(x,y)]ip(c) &= (-1)^{|y|}b\cdot[x|y]\cdot h(c)+(-1)^{|x|(|y|+1)}b\cdot[y|x]\cdot h(c)\\ &\quad+(-1)^{|b|+|x|+1}h(b\cdot[x|y]\cdot c)+(-1)^{|b|+|x||y|+|y|+1}h(b\cdot[y|x]\cdot c) \end{aligned} \tag{5}$$

holds if b does not contain elements of $C_1\otimes C_2$. Indeed, if c contains elements of $C_1\otimes C_2$, both sides are zero by (4). Otherwise, it is enough to assume that c is ordered. In that case

$$(-1)^{|b|+|x||y|+|y|+1}h(b\cdot[y|x]\cdot c)=b\cdot[(x,y)]\cdot c+(-1)^{|x|+|b|}h(b\cdot[x|y]\cdot c)$$

by (3) and hence (5) holds in this case. Let us now show that thus constructed homotopy h satisfies (2).

An element a with 0 inversions is completely ordered and by definition h annihilates it. da also has 0 inversions, so $h(da)=0$, but we also have $a=i(p(a))$. Next, suppose a has no factors in $C_1\otimes C_2$. Suppose we have checked (2) for all monomials with at most k inversions and consider a monomial $a=b\cdot[y|x]\cdot c$ with $k+1$ inversions. We have

$$\begin{aligned} dh(a) &= (-1)^{|x||y|+|y|+1+|b|}db\cdot[(x,y)]\cdot c+(-1)^{|x||y|+|y|+1}b\cdot[(dx,y)]\cdot c\\ &\quad+(-1)^{(|x|+1)(|y|+1)}b\cdot[(x,dy)]\cdot c+(-1)^{|x||y|+|x|+|y|}b\cdot[x|y]\cdot c\\ &\quad+b\cdot[y|x]\cdot c+(-1)^{|x|(|y|+1)}b\cdot[(x,y)]\cdot dc+(-1)^{(|x|+1)(|y|+1)}dh(b\cdot[x|y]\cdot c),\\ da &= db\cdot[y|x]\cdot c+(-1)^{|b|}b\cdot[dy|x]\cdot c+(-1)^{|b|+|y|+1}b\cdot[y|dx]\cdot c\\ &\quad+(-1)^{|b|+|x|+|y|}b\cdot[y|x]\cdot dc \end{aligned}$$

and

$$\begin{aligned} h(da) &= (-1)^{|x||y|+|y|+|b|}db\cdot[(x,y)]\cdot c+(-1)^{(|x|+1)(|y|+1)}h(db\cdot[x|y]\cdot c)\\ &\quad+(-1)^{|x||y|+|x|+|y|}b\cdot[(x,dy)]\cdot c+(-1)^{(|x|+1)|y|+|b|}h(b\cdot[x|dy]\cdot c)\\ &\quad+(-1)^{(|x|+1)|y|}b\cdot[(dx,y)]\cdot c+(-1)^{(|x|+1)(|y|+1)+|b|}h(b\cdot[dx|y]\cdot c)\\ &\quad+(-1)^{|x|(|y|+1)+1}b\cdot[(x,y)]\cdot dc+(-1)^{|x||y|+|b|}h(b\cdot[x|y]\cdot dc) \end{aligned}$$

We also have

$$ip(b\cdot[x|y]c)=(-1)^{(|x|+1)(|y|+1)}ip(a)$$

and by inductive assumption

$$h(d(b\cdot[x|y]\cdot c))+dh(b\cdot[x|y]\cdot c)=b\cdot[x|y]\cdot c-(-1)^{(|x|+1)(|y|+1)}ip(a).$$

Combining these equations we get

$$\begin{aligned} dh(a)+h(da) &= (-1)^{|x||y|+|x|+|y|}b\cdot[x|y]\cdot c+a+(-1)^{(|x|+1)(|y|+1)}b\cdot[x|y]\cdot c-ip(a)\\ &= a-ip(a). \end{aligned}$$

So far we have proved (2) for monomials with no factors in $C_1\otimes C_2$. Now suppose we know the formula holds for monomials with at most k factors in $C_1\otimes C_2$ and consider a monomial a with $k+1$ factors in $C_1\otimes C_2$. Then

$$\begin{aligned} dh(a) &= (-1)^{|b|+|x|+|y|+1}db\cdot[(x,y)]\cdot h(c)+(-1)^{|x|+|y|+1}b\cdot[(dx,y)]\cdot h(c)\\ &\quad+(-1)^{|y|+1}b\cdot[(x,dy)]\cdot h(c)+b\cdot[(x,y)]\cdot dh(c)+(-1)^{|y|}b\cdot[x|y]\cdot h(c)\\ &\quad+(-1)^{|x|(|y|+1)}b\cdot[y|x]\cdot h(c), \end{aligned}$$

$$\begin{aligned}da = db\cdot[(x,y)]\cdot c &+ (-1)^{|b|}b\cdot[(dx,y)]\cdot c + (-1)^{|b|+|x|}b\cdot[(x,dy)]\cdot c\\ &+ (-1)^{|b|+|x|+1}b\cdot[x|y]\cdot c + (-1)^{|b|+|x||y|+|y|+1}b\cdot[y|x]\cdot c\\ &+ (-1)^{|b|+|x|+|y|+1}b\cdot[(x,y)]\cdot dc\end{aligned}$$

and

$$\begin{aligned}h(da) = (-1)^{|b|+|x|+|y|}db\cdot[(x,y)]\cdot h(c) &+ (-1)^{|x|+|y|}b\cdot[(dx,y)]\cdot h(c) + (-1)^{|y|}b\cdot[(x,dy)]\cdot h(c)\\ &+ (-1)^{|b|+|x|+1}h(b\cdot[x|y]\cdot c) + (-1)^{|b|+|x||y|+|y|+1}h(b\cdot[y|x]\cdot c) + b\cdot[(x,y)]\cdot h(dc).\end{aligned}$$

Therefore,

$$h(da) + dh(a) = a$$

by using (5). This finishes the proof for $\mathcal{C} = \mathrm{coAss}$.

Observe now that the morphism p is compatible with the shuffle coproducts on both sides by looking at the generators. If C_1 and C_2 are both cocommutative, then the shuffle coproduct is compatible with the differentials. Therefore, passing to the primitives we obtain the statement for $\mathcal{C} = \mathrm{coComm}$.

In the case $\mathcal{C} = \mathrm{co}\mathbb{P}_n$ we can assume that the cobar differential involving the cobrackets are absent exactly as in the case of $\mathcal{C} = \mathrm{coAss}$. But then the statement is obtained by applying the symmetric algebra to the statement for $\mathcal{C} = \mathrm{coComm}$.

Finally, the curved cases $\mathcal{C} = \mathrm{coAss}^\theta$ and $\mathcal{C} = \mathrm{co}\mathbb{P}_n^\theta$ are reduced to the uncurved cases $\mathcal{C} = \mathrm{coAss}$ and $\mathcal{C} = \mathrm{co}\mathbb{P}_n$ after passing to the associated gradeds. ☐

The statement has the following important corollaries.

Corollary 1.18. *The natural Hopf counital structures on the following cooperads are admissible:*

- $\mathcal{C} = \mathrm{coAss}$;
- $\mathcal{C} = \mathrm{coAss}^\theta$;
- $\mathcal{C} = \mathrm{coComm}$;
- $\mathcal{C} = \mathrm{co}\mathbb{P}_n$;
- $\mathcal{C} = \mathrm{co}\mathbb{P}_n^\theta$.

Proof. Suppose C, D, E are three (curved) conilpotent $\mathcal{C}$-coalgebras with a morphism $D \to E$ such that $\Omega D \to \Omega E$ is a quasi-isomorphism. We have to show that

$$\Omega(C\otimes D) \longrightarrow \Omega(C\otimes E)$$

is a quasi-isomorphism as well.

If $\mathcal{C} = \mathrm{coComm}$, the statement follows from the commutative diagram

$$\begin{tikzcd}\Omega(C\otimes D) \arrow[r] \arrow[d, "\sim"] & \Omega(C\otimes E) \arrow[d, "\sim"] \\ \Omega(C)\oplus\Omega(D) \arrow[r, "\sim"] & \Omega(C)\oplus\Omega(E)\end{tikzcd}$$

with vertical weak equivalences provided by Proposition 1.17.

If $\mathcal{C} = \mathrm{coAss}$ or $\mathcal{C} = \mathrm{co}\mathbb{P}_n$, the statement follows from the commutative diagram

$$\begin{tikzcd} \Omega(C \otimes D) \arrow[r] \arrow[d, "\sim"] & \Omega(C \otimes E) \arrow[d, "\sim"] \\ \Omega(C) \oplus \Omega(D) \oplus \Omega(C) \otimes \Omega(D)[-1] \arrow[r, "\sim"] & \Omega(C) \oplus \Omega(E) \oplus \Omega(C) \otimes \Omega(E)[-1] \end{tikzcd}$$

with vertical weak equivalence given by the same proposition.

Finally, if $\mathcal{C} = \mathrm{coAss}^\theta$ or $\mathcal{C} = \mathrm{co}\mathbb{P}_n^\theta$, the statement follows from the commutative diagram below.

$$\begin{tikzcd} \Omega(C \otimes D) \arrow[r] \arrow[d, "\sim"] & \Omega(C \otimes E) \arrow[d, "\sim"] \\ \Omega(C) \oplus \Omega(D) \arrow[r, "\sim"] & \Omega(C) \oplus \Omega(E) \end{tikzcd}$$

In all these cases we are using the fact that the tensor product of complexes preserves quasi-isomorphisms. $\square$

Consider the symmetric monoidal structure on the category of non-unital algebras $\mathrm{Alg}_{\mathrm{Ass}}$ where the tensor product of A_1 and A_2 is

$$A_1 \otimes A_2 \oplus A_1 \oplus A_2.$$

One also has a symmetric monoidal structure on the category of non-counital coalgebras $\mathrm{CoAlg}_{\mathrm{coAss}}$ where the tensor product of C_1 and C_2 is

$$C_1 \otimes C_2 \oplus C_1 \oplus C_2.$$

Similarly, one introduces the symmetric monoidal structures on the categories $\mathrm{Alg}_{\mathbb{P}_n}$ and $\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}$. Finally, consider the Cartesian symmetric monoidal structure on the category of Lie algebras $\mathrm{Alg}_{\mathrm{Lie}}$.

Corollary 1.19. *The adjoint equivalences*

$$\mathrm{CoAlg}_{\mathrm{coAss}}[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathrm{Ass}}[W_{\mathrm{qis}}^{-1}],$$
$$\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathrm{Lie}}[W_{\mathrm{qis}}^{-1}]$$

and

$$\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathbb{P}_n}[W_{\mathrm{qis}}^{-1}]$$

are symmetric monoidal.

Proof. By Corollary 1.18 the cooperads coAss, coComm and $\mathrm{co}\mathbb{P}_n$ have Hopf admissible structures. Therefore, by Proposition 1.2 we obtain a lax symmetric monoidal equivalence

$$\mathrm{CoAlg}_{\mathrm{coAss}}[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathrm{Ass}}[W_{\mathrm{qis}}^{-1}]$$

and similarly for the other cooperads. But by Proposition 1.17 these are in fact symmetric monoidal functors. $\square$

Therefore, we can use the relative symmetric monoidal category $(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}, W_{\mathrm{Kos}})$ as a model for the symmetric monoidal ∞-category $\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}$.

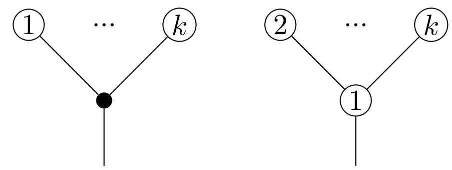


FIGURE 4. Generating operations of $\mathrm{Br}_{\mathcal{C}}$.

2. Brace bar duality

2.1 Brace construction

Suppose $\mathcal{C}$ is a (curved) cooperad equipped with a Hopf counital structure (see Definition 1.12). Let us briefly recall from [CW15] the definition of the associated operad of braces $\mathrm{Br}_{\mathcal{C}}$. Its operations are parametrized by rooted trees with 'external' vertices that are colored white in our pictures and 'internal' vertices that are colored black. An external vertex with r children is labeled by elements of $\mathcal{C}^{\mathrm{cu}}(r)$ while an internal vertex with r children is labeled by elements of $\overline{\mathcal{C}}(r)[-1]$.

The coproduct on $\mathcal{C}^{\mathrm{cu}}$ defines a morphism

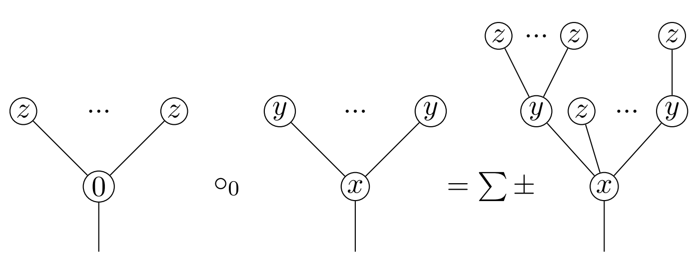
$$\mathcal{C}^{\mathrm{cu}}(r) \longrightarrow \mathcal{C}^{\mathrm{cu}}(r+r') \otimes \mathcal{C}^{\mathrm{cu}}(1)^{\otimes r} \otimes \mathcal{C}^{\mathrm{cu}}(0)^{\otimes r'} \longrightarrow \mathcal{C}^{\mathrm{cu}}(r+r'),$$

where in the second morphism we use $\mathcal{C}^{\mathrm{cu}}(1) \to k$ given by the counit on the cooperad $\mathcal{C}^{\mathrm{cu}}$ and $\mathcal{C}^{\mathrm{cu}}(0) \cong \mathcal{C}(0) \oplus k \to k$ given by projection on the second summand. The operadic composition is given by grafting rooted trees with labels obtained by applying the coproduct on $\mathcal{C}^{cu}$ and combining different labels using the Hopf structure. The differential on internal vertices coincides with the cobar differential (in particular, if $\mathcal{C}$ is curved, it contains an extra curving term); the differential of an external vertex splits it into an internal and an external vertex. We refer to [DW15, Formula (8.14)] for an explicit description of the differential.

The operad $\mathrm{Br}_{\mathcal{C}}$ is generated by trees shown in Figure 4, where the leaves are labeled by the unit element of $\mathcal{C}^{\mathrm{cu}}(0)$. For a $\mathrm{Br}_{\mathcal{C}}$-algebra A we denote the operation given by the corolla with an internal vertex by $m(c|x_1, \dots, x_r)$ where $c \in \overline{\mathcal{C}}(r)[-1]$ and $x_i \in A$. The operation given by the corolla with an external vertex is denoted by $x\{c|y_1, \dots, y_r\}$, where $c \in \mathcal{C}^{\mathrm{cu}}(r)$. We denote by $\Omega\mathcal{C} \to \mathrm{Br}_{\mathcal{C}}$ the natural morphism which sends a generator in $\overline{\mathcal{C}}[-1]$ to the corresponding corolla with an internal vertex.

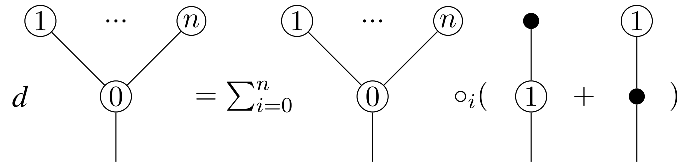
The generating operations satisfy the following three relations.

(i) (Associativity).



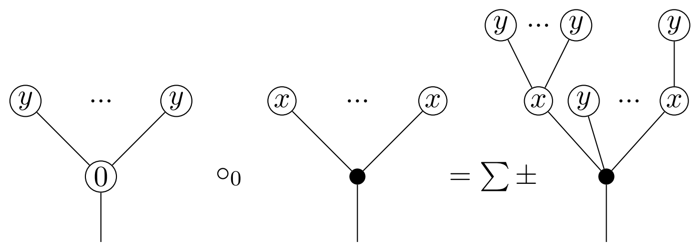
(6)

(ii) (Higher homotopies).



(7)

(iii) (Distributivity).



(8)

Let us give some examples of the brace construction that we will use. Note that the second corolla in Figure 4 with $k = 2$ gives rise to a pre-Lie structure on any $\mathrm{Br}_{\mathcal{C}}$-algebra. However, in general the pre-Lie operation is not compatible with the differential.

The simplest example is the case $\mathcal{C} = \mathbf{1}$, the trivial cooperad. In this case $\overline{\mathcal{C}} = 0$ and hence the only operations are given by braces. However, since $\mathcal{C}(n) = 0$ for $n \geqslant 2$, we can only have vertices of valence 1 and 0, i.e. the operations of $\mathrm{Br}_{\mathbf{1}}$ are parametrized by linear chains and hence $\mathrm{Br}_{\mathbf{1}}(n) \cong k[S_n]$. It is immediate to see that the pre-Lie operation in this case gives rise to an associative multiplication.

Proposition 2.1. *One has an isomorphism of operads*

$$\mathrm{Ass} \cong \mathrm{Br}_{\mathbf{1}}.$$

For $\mathcal{C} = \mathrm{coAss}$ we obtain an A_∞ structure of degree 1 together with degree 0 brace operations $x\{y_1, \dots, y_n\}$. Recall the brace operad Br introduced by Gerstenhaber and Voronov [GV95] and let A_∞ be the operad controlling A_∞ algebras.

Proposition 2.2. *We have a pushout of operads.*

$$\begin{tikzcd} A_\infty \arrow[r] \arrow[d] & \mathrm{Br}_{\mathrm{coAss}}\{1\} \arrow[d] \\ \mathrm{Ass} \arrow[r] & \mathrm{Br} \end{tikzcd}$$

Now consider $\mathcal{C} = \mathrm{co}\mathbb{P}_n$. Calaque and Willwacher [CW15] following some ideas of Tamarkin [Tam00] introduced a morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\}) \longrightarrow \mathrm{Br}_{\mathrm{co}\mathbb{P}_n}, \tag{9}$$

on generators by the following rule.

– The generators

$$\underline{x_1 \cdots x_k} \in \mathrm{coLie}\{1-n\}(k) \subset \mathrm{co}\mathbb{P}_{n+1}\{1\}(k)$$

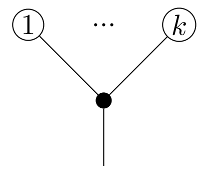


Figure 5. Image of $\underline{x_1 \cdots x_k}$.

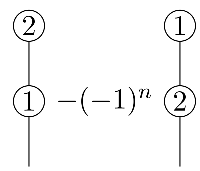


Figure 6. Image of $x_1 \wedge x_2$.

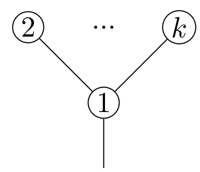


Figure 7. Image of $x_1 \wedge \underline{x_2 \cdots x_k}$.

are sent to the tree drawn in Figure 5 with the root labeled by the element

$$\underline{x_1 \cdots x_k} \in \mathrm{coLie}\{1-n\}(k) \subset \mathrm{co}\mathbb{P}_n(k).$$

Here $\underline{x_1 \cdots x_k}$ is the image of the k-ary comultiplication under the projection

$$\mathrm{coAss}\{1-n\} \to \mathrm{coLie}\{1-n\}.$$

– The generator

$$x_1 \wedge x_2 \in \mathrm{coComm}\{1\}(2) \subset \mathrm{co}\mathbb{P}_{n+1}\{1\}(2)$$

is sent to the linear combination of trees shown in Figure 6.

– The generators

$$x_1 \wedge \underline{x_2 \cdots x_k} \in \mathrm{co}\mathbb{P}_{n+1}\{1\}(k)$$

for $k > 2$ are sent to the tree shown in Figure 7 with the root labeled by the element

$$\underline{x_2 \cdots x_k} \in \mathrm{coLie}\{1-n\}(k-1) \subset \mathrm{co}\mathbb{P}_n(k-1).$$

– The rest of the generators are sent to zero.

Proposition 2.3 (Calaque–Wilwacher). *The morphism of operads*

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\}) \longrightarrow \mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$$

is a quasi-isomorphism.

Note that we have a quasi-isomorphism of operads

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}) \longrightarrow \mathbb{P}_{n+1}$$

and hence we get a zig–zag of quasi-isomorphisms between the operads $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\}$ and $\mathbb{P}_{n+1}$.

One can similarly define a morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}) \to \mathrm{Br}_{\mathrm{co}\mathbb{P}^{\theta}_n}$$

in the following way.

- The generators $y \in \mathrm{coLie}^{\theta}\{1-n\} \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}$ are sent to the tree drawn in Figure 5 with the root labeled by $x \in \mathrm{coLie}^{\theta}\{1-n\} \subset \mathrm{co}\mathbb{P}^{\theta}_n$.
- The generator
$$x_1 \wedge x_2 \in \mathrm{coComm}\{1\}(2) \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(2)$$
is sent to the linear combination of trees shown in Figure 6.
- Suppose $y \in \mathrm{coLie}^{\theta}\{1-n\}(k-1) \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k-1)$ for $k > 2$ and let $x \wedge y$ be the image of y under $\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k-1) \to \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k)$.
The generators $x \wedge y \in \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k)$ are sent to the tree shown in Figure 7 with the root labeled by the element $y \in \mathrm{coLie}^{\theta}\{1-n\}(k-1) \subset \mathrm{co}\mathbb{P}^{\theta}_n(k-1)$.
- The rest of the generators are sent to zero.

The following statement is a version of Proposition 2.3.

Proposition 2.4. *The morphism of operads*

$$\Omega(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}) \to \mathrm{Br}_{\mathrm{co}\mathbb{P}^{\theta}_n}$$

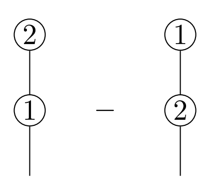
is a quasi-isomorphism.

This proposition implies that we have a zig–zag of weak equivalences between the operads $\mathrm{Br}_{\mathrm{co}\mathbb{P}^{\theta}_n}\{n\}$ and $\mathbb{P}^{\mathrm{un}}_{n+1}$.

Finally, consider the case $\mathcal{C} = \mathrm{coComm}$. By construction we have a morphism of operads

$$\mathrm{Lie} \to \mathrm{Br}_{\mathcal{C}}$$

given by sending the Lie bracket to the following combination.



Proposition 2.5. *The morphism of operads* $\mathrm{Lie} \to \mathrm{Br}_{\mathrm{coComm}}$ *is a quasi-isomorphism.*

Proof. Introduce a grading on $\mathrm{co}\mathbb{P}_n$ by setting the cobracket to be of weight 1 and the comultiplication of weight 0. In this way $\mathrm{co}\mathbb{P}^{\mathrm{cu}}_n$ becomes a graded Hopf cooperad, i.e. a cooperad in graded commutative dg algebras. This induces a grading on the brace construction $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$, where the weight of a tree is given by the sum of the weights of the labels. The morphism (9) is compatible with the gradings if we introduce the grading on $\mathrm{co}\mathbb{P}_{n+1}$ where again the cobracket has weight 1 and the comultiplication has weight 0.

Let $L_\infty = \Omega(\mathrm{coComm}\{1\})$ be the operad controlling L_∞ algebras. Passing to weight 0 components and using Proposition 2.3, we obtain a quasi-isomorphism

$$L_\infty \longrightarrow \mathrm{Br}_{\mathrm{coComm}}$$

which by construction factors as

$$L_\infty \xrightarrow{\sim} \mathrm{Lie} \to \mathrm{Br}_{\mathrm{coComm}}$$

and the claim follows. □

2.1.1 *Aside: the operad* $\mathrm{Br}_{\mathrm{coComm}}$. Let us explain the role the operad $\mathrm{Br}_{\mathrm{coComm}}$ plays in Lie theory (see also Proposition 2.11).

Suppose that $\mathfrak{g}$ is a Lie algebra with a lift of the structure to a $\mathrm{Br}_{\mathrm{coComm}}$-algebra. That is, $\mathfrak{g}$ has a pre-Lie structure $x \circ y$ and degree L_∞ brackets $\{x_1, \dots, x_n\}$ such that

$$\begin{aligned}
[x, y] &= x \circ y - (-1)^{|x||y|} y \circ x \\
\{x, y\} &= d(x \circ y) - (dx) \circ y - (-1)^{|x|} x \circ dy,
\end{aligned}$$

where $[x, y]$ is the original degree 0 Lie bracket. Note that the degree 1 L_∞ brackets are uniquely determined from the pre-Lie operation. Then the $\mathrm{Br}_{\mathrm{coComm}}$-algebra structure allows one to replace the morphism of Lie algebras $0 \to \mathfrak{g}$ by a fibration in the following way.

Consider the complex

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}[-1] \tag{10}$$

with the identity differential. We define an L_∞ algebra structure on $\widetilde{\mathfrak{g}}$ as follows:

- the bracket on the first term is the original bracket $[-, -]$;
- the L_∞ structure on the second term is given by the operations $\{-, \dots, -\}$;
- the L_∞ brackets $[x, \mathrm{s}y_1, \dots, \mathrm{s}y_n]$ where $x \in \mathfrak{g}$ and $\mathrm{s}y_i \in \mathfrak{g}[-1]$ land in $\mathfrak{g}[-1]$ and are given by the symmetric braces $\mathfrak{g} \otimes \mathrm{Sym}(\mathfrak{g}) \to \mathfrak{g}$.

It is immediate that $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a fibration of L_∞ algebras (i.e. it is a degreewise surjective morphism) and, moreover, that the morphism $0 \to \widetilde{\mathfrak{g}}$ is a quasi-isomorphism.

In particular, we see that the L_∞ algebra structure on

$$\Omega\mathfrak{g} = 0 \oplus_{\mathfrak{g}} 0 \cong \mathfrak{g}[-1]$$

is given by the degree 1 L_∞ brackets $\{-, \dots, -\}$.

Example 2.6. Let A be a commutative dg algebra over a field k and denote by $\mathrm{T}_A = \mathrm{Der}_k(A, A)$ the complex of derivations. Suppose ∇ is a flat torsion-free connection on the underlying graded algebra, i.e. it defines a morphism of graded vector spaces

$$\nabla \colon \mathrm{T}_A \otimes_k \mathrm{T}_A \to \mathrm{T}_A$$

such that

$$[v, w] = \nabla_v w - (-1)^{|v||w|} \nabla_w v, \tag{11}$$
$$\nabla_{[v,w]} = \nabla_v \nabla_w - (-1)^{|v||w|} \nabla_w \nabla_v \tag{12}$$

for two vector fields $v, w \in \mathrm{T}_A$.

Then the Lie algebra structure on T_A given by the commutator of derivations is lifted to a $\mathrm{Br}_{\mathrm{coComm}}$-algebra structure, where the pre-Lie structure is given by the connection:

$$v \circ w = \nabla_v w.$$

Indeed, (11) implies that $\nabla_v w$ lifts the Lie bracket of vector fields and (12) implies that it is indeed a pre-Lie bracket.

Therefore, we see that the L_∞ structure on

$$\Omega \mathrm{T}_A = \mathrm{T}_A[-1]$$

is given by

$$\{v, w\} = d(\nabla_v w) - \nabla_{dv} w - (-1)^{|v|} \nabla_v (dw).$$

In this way we discover exactly the Atiyah bracket of vector fields as defined by Kapranov, see [Kap99, § 2.5].

2.2 Additivity for brace algebras

Recall that we have a morphism of operads $\Omega\mathcal{C} \to \mathrm{Br}_{\mathcal{C}}$. In particular, a brace algebra has a bar complex which is a $\mathcal{C}$-coalgebra. Now we are going to introduce an associative multiplication on the bar complex making the diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Br}_{\mathcal{C}}} \arrow[r] \arrow[d, dashed, "\mathrm{B}"] & \mathrm{Alg}_{\Omega\mathcal{C}} \arrow[d, "\mathrm{B}"] \\ \mathrm{Alg}(\mathrm{CoAlg}_{\mathcal{C}}) \arrow[r] & \mathrm{CoAlg}_{\mathcal{C}} \end{tikzcd}$$

commute.

Remark 2.7. Consider a bialgebra $C \in \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}})$. The unit of the symmetric monoidal structure on $\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}$ is given by k with the identity coaugmentation. Therefore, the unit morphism for C is a morphism of coaugmented $\mathcal{C}^{\mathrm{cu}}$-coalgebras

$$k \to C.$$

Compatibility with the coaugmentation on C implies that the unit morphism $k \to C$ coincides with the coaugmentation $k \to C$.

Let A be a $\mathrm{Br}_{\mathcal{C}}$-algebra and consider $\mathcal{C}^{\mathrm{cu}}(A)$, the cofree conilpotent $\mathcal{C}^{\mathrm{cu}}$-coalgebra, equipped with the bar differential. It is naturally coaugmented using the decomposition

$$\mathcal{C}^{\mathrm{cu}}(0) \cong \mathcal{C}(0) \oplus k.$$

We introduce the unit on $\mathcal{C}^{\mathrm{cu}}(A)$ to be given by the coaugmentation $k \to \mathcal{C}^{\mathrm{cu}}(A)$. The multiplication

$$\mathcal{C}^{\mathrm{cu}}(A) \otimes \mathcal{C}^{\mathrm{cu}}(A) \to \mathcal{C}^{\mathrm{cu}}(A)$$

is uniquely specified on the cogenerators by a morphism

$$\mathcal{C}^{\mathrm{cu}}(A) \otimes \mathcal{C}^{\mathrm{cu}}(A) \to A.$$

In turn, it is defined via the composite

$$\mathcal{C}^{\mathrm{cu}}(A) \otimes \mathcal{C}^{\mathrm{cu}}(A) \to A \otimes \mathcal{C}^{\mathrm{cu}}(A) \to A,$$

where the first morphism is induced by the counit on $\mathcal{C}^{\mathrm{cu}}$ and the second morphism is given by braces, i.e. the morphism

$$\mathcal{C}^{\mathrm{cu}}(n) \otimes A \otimes A^{\otimes n} \to A$$

is given by applying the second corolla in Figure 4 with the root labeled by the element of $\mathcal{C}^{\mathrm{cu}}(n)$ and the leaves labeled by the unit $k \to \mathcal{C}^{\mathrm{cu}}(0)$. The following statement is shown in [MS16, Proposition 3.4].

PROPOSITION 2.8. *This defines a unital dg associative multiplication on $\mathcal{C}^{\mathrm{cu}}(A)$ compatible with the $\mathcal{C}^{\mathrm{cu}}$-coalgebra structure.*

For a $\mathrm{Br}_{\mathcal{C}}$-algebra A we denote by $\mathrm{B}A = \mathcal{C}^{\mathrm{cu}}(A)$ the bar complex equipped with the bar differential and the above associative multiplication. This defines a functor

$$\mathrm{B}\colon \mathrm{Alg}_{\mathrm{Br}_{\mathcal{C}}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}).$$

Now suppose the Hopf unital structure on $\mathcal{C}$ is admissible. Then the symmetric monoidal structure on $\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}$ preserves weak equivalences and hence $\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathcal{C}^{\mathrm{cu}}}[W^{-1}_{\mathrm{Kos}}]$ is a symmetric monoidal ∞-category.

Since the bar functor preserves weak equivalences, the composite functor

$$\mathrm{Alg}_{\mathrm{Br}_{\mathcal{C}}}[W^{-1}_{\mathrm{qis}}] \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathcal{C}})[W^{-1}_{\mathrm{Kos}}] \longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathcal{C}}[W^{-1}_{\mathrm{Kos}}])$$

gives rise to the additivity functor

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Br}_{\mathcal{C}}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\Omega\mathcal{C}}). \tag{13}$$

We do not know if the functor (13) is an equivalence in general, but we prove that it is an equivalence for Lie and Poisson algebras. Note that in the case $\mathcal{C} = \mathrm{coAss}$ we obtain a functor

$$\mathcal{A}\mathrm{lg}_{\mathbb{E}_2} \cong \mathcal{A}\mathrm{lg}_{\mathrm{Br}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg})$$

which we expect coincides with the Dunn–Lurie equivalence [Lur17, Theorem 5.1.2.2].

2.3 Additivity for Lie algebras

In this section we work out how the additivity functor (13) looks like for Lie algebras and prove that it is an equivalence. We consider the cooperad $\mathcal{C} = \mathrm{coComm}$.

We have a functor

$$\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{coComm}^{\mathrm{cu}}})$$

which sends a Lie algebra to its universal enveloping algebra. The following is proved e.g. in [Car07, Theorem 3.8.1].

THEOREM 2.9 (Cartier–Milnor–Moore). *The universal enveloping algebra*

$$\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{coComm}^{\mathrm{cu}}})$$

is an equivalence of categories.

LEMMA 2.10. *The equivalence of categories*

$$\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{\sim} \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{coComm}^{\mathrm{cu}}})$$

preserves weak equivalences.

Proof. Recall that weak equivalences in $\mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{coComm}^{\mathrm{cu}}})$ are created by the forgetful functor to $\mathrm{CoAlg}_{\mathrm{coComm}}$. Given a dg Lie algebra $\mathfrak{g}$, the Poincaré–Birkhoff–Witt (PBW) theorem gives an identification of cocommutative coalgebras $\mathrm{U}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g})$, hence for a morphism of dg Lie algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$ we have a commutative diagram

$$\begin{tikzcd} \Omega\mathrm{U}(\mathfrak{g}_1) \arrow[r] \arrow[d, "\sim"] & \Omega\mathrm{U}(\mathfrak{g}_2) \arrow[d, "\sim"] \\ \mathfrak{g}_1[-1] \arrow[r] & \mathfrak{g}_2[-1] \end{tikzcd}$$

where the vertical morphisms are quasi-isomorphisms. Therefore, the bottom morphism is a quasi-isomorphism iff the top morphism is a quasi-isomorphism. $\square$

We get two functors

$$\mathrm{Alg}_{\mathrm{Br}_{\mathrm{coComm}}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}),$$

where one is given by the brace bar construction B and the other one is given by the composite

$$\mathrm{Alg}_{\mathrm{Br}_{\mathrm{coComm}}} \xrightarrow{\text{forget}} \mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{\mathrm{U}} \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}),$$

where the forgetful functor is given by Proposition 2.5. In fact, these are equivalent.

Proposition 2.11. *There is a natural weak equivalence*

$$\mathrm{U} \circ \mathrm{forget} \xrightarrow{\sim} \mathrm{B}$$

of functors

$$\mathrm{Alg}_{\mathrm{Br}_{\mathrm{coComm}}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}).$$

Proof. Suppose $\mathfrak{g}$ is a $\mathrm{Br}_{\mathrm{coComm}}$-algebra. In particular, $\mathfrak{g}$ is a pre-Lie algebra and we have to produce a natural isomorphism of dg cocommutative bialgebras

$$\mathrm{U}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{g}),$$

where $\mathrm{Sym}(\mathfrak{g})$ is equipped with the associative product using the pre-Lie structure and the differential uses the shifted L_∞ brackets on $\mathfrak{g}$.

The morphism $\mathrm{U}(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})$ is uniquely determined by a map

$$\mathfrak{g} \to \mathrm{Sym}(\mathfrak{g})$$

on generators which we define to be the obvious inclusion. We refer to [OG08, Theorem 2.12] for the claim that it extends to an isomorphism of cocommutative bialgebras $\mathrm{U}(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})$. The compatibility with the differential is obvious as the differential on $\mathfrak{g} \cong \mathrm{coComm}(1) \otimes \mathfrak{g}$ is simply given by the differential on $\mathfrak{g}$. $\square$

The ∞-category of Lie algebras is pointed, i.e. the initial and final objects coincide. Therefore, we can consider the loop functor

$$\Omega\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$$

given by sending a Lie algebra $\mathfrak{g}$ to its loop object $0 \times_{\mathfrak{g}} 0$. More explicitly, since the monoidal structure on $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ is Cartesian, by [Lur17, Proposition 4.1.2.10] we can identify $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$ with the ∞-category of Segal monoids, i.e. simplicial objects $M_\bullet$ of $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ such that M_0 is contractible and the natural maps $M_n \to M_1 \times_{M_0} \times \cdots \times_{M_0} M_1$ are equivalences. Under this identification the loop object of $\mathfrak{g}$ is defined to be the simplicial object underlying the Cech nerve of $0 \to \mathfrak{g}$:

$$0 \leftleftarrows 0 \times_{\mathfrak{g}} 0 \; \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \; 0 \times_{\mathfrak{g}} 0 \times_{\mathfrak{g}} 0 \qquad \cdots$$

Its left adjoint is the classifying space functor

$$\mathrm{B}\colon \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \longrightarrow \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$$

which sends a Segal monoid in Lie algebras to its geometric realization. The following is proved in [Toë13, Lemma 5.3] and [GH16, Corollary 2.7.2].

Proposition 2.12. *The adjunction*

$$\mathrm{B}\colon \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} : \Omega$$

is an equivalence of symmetric monoidal ∞-categories.

Since the operads $\mathrm{Br}_{\mathrm{coComm}}$ and Lie are quasi-isomorphic, the additivity functor (13) becomes

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}).$$

Observe that now we have constructed two functors $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$: the loop functor Ω and the additivity functor add.

Proposition 2.13. *The additivity functor* $\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \to \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$ *is equivalent to the loop functor Ω.*

Proof. Since the classifying space functor B is an inverse to the loop functor Ω by Proposition 2.12, we have to prove that there is an equivalence $\mathrm{B} \circ \mathrm{add} \cong \mathrm{id}$ of functors $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \to \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$. By Proposition 2.11 we can write add as the composite

$$\begin{aligned} \mathrm{Alg}_{\mathrm{Lie}}[W_{\mathrm{qis}}^{-1}] &\xrightarrow{\mathrm{U}} \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \\ &\longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}]) \\ &\cong \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}), \end{aligned}$$

where the last equivalence is given by the Chevalley–Eilenberg complex which realizes the bar construction for Lie algebras.

Therefore, the statement will follow once we show that the composite

$$\begin{aligned} \mathrm{Alg}_{\mathrm{Lie}}[W_{\mathrm{qis}}^{-1}] &\xrightarrow{\mathrm{U}} \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \\ &\longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}]) \\ &\xrightarrow{\mathrm{B}} \mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}] \end{aligned} \tag{14}$$

is equivalent to the Chevalley–Eilenberg complex functor

$$\mathrm{C}_\bullet\colon \mathrm{Alg}_{\mathrm{Lie}}[W_{\mathrm{qis}}^{-1}] \longrightarrow \mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}].$$

For an algebra A in a symmetric monoidal ∞-category $\mathcal{C}$, a right A-module M and a left A-module N let us denote by $\mathrm{Bar}_\bullet(M, A, N)$ the simplicial object of $\mathcal{C}$ underlying the two-sided bar construction whose n-simplices are given by $M \otimes A^{\otimes n} \otimes N$. The composite (14) applied to a Lie algebra $\mathfrak{g}$ is then by definition $|\mathrm{Bar}_\bullet(k, \mathrm{U}\mathfrak{g}, k)| \cong k \otimes^{\mathbb{L}}_{\mathrm{U}\mathfrak{g}} k$ and is a cocommutative coalgebra since k and $\mathrm{U}\mathfrak{g}$ are.

For a dg Lie algebra $\mathfrak{g}$ we define following [Lur11, § 2.2] the Lie algebra $\mathrm{Cn}(\mathfrak{g})$ which as a graded vector space is $\mathrm{Cn}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}[1]$ equipped with the following dg Lie algebra structure.

– The differential is given by the identity differential from the second term to the first term and the internal differentials on the two summands of $\mathfrak{g}$.
– The Lie bracket on the first term is the original Lie bracket on $\mathfrak{g}$.
– The Lie bracket between $\mathbf{s}^{-1}x \in \mathfrak{g}[1]$ and $y \in \mathfrak{g}$ lands in $\mathfrak{g}[1]$ and is given by

$$[\mathbf{s}^{-1}x, y] = \mathbf{s}^{-1}[x, y].$$

– The Lie bracket on the last term is zero.

We have a quasi-isomorphism of $\mathfrak{g}$-modules $0 \to \mathrm{Cn}(\mathfrak{g})$. Therefore, after taking the universal enveloping algebra we obtain a weak equivalence of left $\mathrm{U}\mathfrak{g}$-modules in cocommutative coalgebras

$$k \longrightarrow \mathrm{U}(\mathrm{Cn}(\mathfrak{g})) \cong \mathrm{U}\mathfrak{g} \otimes \mathrm{Sym}(\mathfrak{g}[1]),$$

where on the right we have used the PBW isomorphism.

Therefore, we have a weak equivalence of Segal monoids

$$\mathrm{Bar}_\bullet(k, \mathrm{U}\mathfrak{g}, k) \to \mathrm{Bar}_\bullet(k, \mathrm{U}\mathfrak{g}, \mathrm{U}(\mathrm{Cn}(\mathfrak{g}))).$$

The homotopy colimit of $\mathrm{Bar}_\bullet(k, \mathrm{U}\mathfrak{g}, \mathrm{U}(\mathrm{Cn}(\mathfrak{g})))$ is a strict colimit since $\mathrm{U}(\mathrm{Cn}(\mathfrak{g}))$ is a semi-free left $\mathrm{U}\mathfrak{g}$-module. But its strict colimit is

$$k \otimes_{\mathrm{U}\mathfrak{g}} \mathrm{U}(\mathrm{Cn}(\mathfrak{g})) \cong \mathrm{C}_\bullet(\mathfrak{g})$$

as dg cocommutative coalgebras. Therefore, we obtain a natural equivalence $|\mathrm{Bar}_\bullet(k, \mathrm{U}\mathfrak{g}, k)| \cong \mathrm{C}_\bullet(\mathfrak{g})$ and the claim follows. □

Combining the previous proposition with Proposition 2.12 we obtain the following corollary.

Corollary 2.14. *The additivity functor*

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$$

is an equivalence of symmetric monoidal ∞-categories.

Note that the underlying Lie algebra of $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$ is canonically trivial. Indeed, let $\mathcal{C}\mathrm{h} \longrightarrow \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ be the functor which sends a complex $\mathfrak{g}$ to the trivial Lie algebra $\mathfrak{g}[-1]$.

Proposition 2.15. *The diagram*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r, "\mathrm{add}"] \arrow[d] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \arrow[d] \\ \mathcal{C}\mathrm{h} \arrow[r] & \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \end{tikzcd}$$

is commutative.

Proof. Indeed, the PBW theorem implies that we have a commutative diagram of categories

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Lie}} \arrow[r, "\mathrm{U}"] \arrow[d] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \arrow[d] \\ \mathrm{Ch} \arrow[r, "\mathrm{Sym}"] & \mathrm{CoAlg}_{\mathrm{coComm}} \end{tikzcd}$$

i.e. the underlying cocommutative coalgebra of $\mathrm{U}(\mathfrak{g})$ is isomorphic to $\mathrm{Sym}(\mathfrak{g})$. But $\mathrm{Sym}(\mathfrak{g})$ coincides with the bar construction of a trivial Lie algebra $\mathfrak{g}[-1]$ and the claim follows. □

2.4 Additivity for Poisson algebras

This section is devoted to an explicit description of the additivity functor (13) in the case of $\mathbb{P}_n$-algebras and to showing that it is a symmetric monoidal equivalence of ∞-categories.

Consider the cooperad $\mathcal{C} = \mathrm{co}\mathbb{P}_n$. Since we have a zig–zag of quasi-isomorphisms between the operads $\mathbb{P}_{n+1}$ and $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\}$, the additivity functor (13) becomes

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}). \tag{15}$$

Now we are going to give a different perspective on this functor closer to Tamarkin's papers [Tam00] and [Tam07]. This new perspective will elucidate the formulas for the morphism (9) and allow us to show that the additivity functor is a symmetric monoidal equivalence.

We are going to introduce yet another version of the bar-cobar duality for Poisson algebras, this time the dual object will be a Lie bialgebra.

DEFINITION 2.16. An *n-shifted Lie bialgebra* is a dg Lie algebra $\mathfrak{g}$ together with a degree $-n$ Lie coalgebra structure $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}[-n]$ satisfying the cocycle relation

$$\delta([x, y]) = (\mathrm{ad}_x \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_x)\delta(y) - (-1)^{n+|x||y|}(\mathrm{ad}_y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_y)\delta(x). \tag{16}$$

We will say an n-shifted Lie bialgebra is *conilpotent* if the underlying Lie coalgebra is so and we denote the category of n-shifted conilpotent Lie bialgebras by $\mathrm{BiAlg}_{\mathrm{Lie}_n}$. Weak equivalences in $\mathrm{BiAlg}_{\mathrm{Lie}_n}$ are created by the forgetful functor

$$\mathrm{BiAlg}_{\mathrm{Lie}_n} \longrightarrow \mathrm{CoAlg}_{\mathrm{coLie}}.$$

The commutative bar-cobar adjunction

$$\Omega\colon \mathrm{CoAlg}_{\mathrm{coLie}} \rightleftarrows \mathrm{Alg}_{\mathrm{Comm}} \colon \mathrm{B}$$

extends to a bar-cobar adjunction

$$\Omega\colon \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \rightleftarrows \mathrm{Alg}_{\mathbb{P}_{n+1}} \colon \mathrm{B} \tag{17}$$

so that the diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{P}_{n+1}} \arrow[r, shift left] \arrow[d] & \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \arrow[l, shift left] \arrow[d] \\ \mathrm{Alg}_{\mathrm{Comm}} \arrow[r, shift left] & \mathrm{CoAlg}_{\mathrm{coLie}} \arrow[l, shift left] \end{tikzcd}$$

commutes where the vertical functors are the forgetful functors.

Explicitly, if A is a $\mathbb{P}_{n+1}$-algebra, consider $\mathfrak{g} = \mathrm{coLie}(A[1])[n-1]$, the cofree conilpotent shifted Lie coalgebra equipped with the bar (i.e. Harrison) differential. By the cocycle equation a Lie bracket on $\mathfrak{g}$ is uniquely determined by its projection to cogenerators and the map

$$\mathrm{coLie}(A[1])[n-1] \otimes \mathrm{coLie}(A[1])[n-1] \longrightarrow \mathrm{coLie}(A[1])[n-1] \longrightarrow A[n]$$

is defined to be the Lie bracket

$$A[n] \otimes A[n] \longrightarrow A[n].$$

The Jacobi identity is obvious. Compatibility of the bracket on $\mathfrak{g}$ with the bar differential follows from the Leibniz rule for A.

Conversely, if $\mathfrak{g}$ is an $(n-1)$-shifted Lie bialgebra, consider $A = \mathrm{Sym}(\mathfrak{g}[-n])$ equipped with the cobar differential using the Lie coalgebra structure on $\mathfrak{g}$. The Lie bracket on A by the Leibniz rule is defined on generators to be the Lie bracket on $\mathfrak{g}$. The compatibility of the cobar differential on A with the Lie structure can be checked on generators where it coincides with the cocycle (16).

By the definition of weak equivalences it is clear that the adjunction (17) induces an adjoint equivalence on ∞-categories since the unit and counit of the adjunction are weak equivalences after forgetting down to commutative algebras and Lie coalgebras.

By the Cartier–Milnor–Moore theorem (Theorem 2.9) the universal enveloping algebra functor induces an equivalence of categories

$$\mathrm{U}\colon \mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{\sim} \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}).$$

If $\mathfrak{g}$ is an $(n-1)$-shifted Lie bialgebra, we can define a $\mathbb{P}_n$-coalgebra structure on $\mathrm{U}(\mathfrak{g})$ as follows. By construction $\mathrm{U}(\mathfrak{g})$ is a cocommutative bialgebra and we define the cobracket on the generators to be the cobracket on $\mathfrak{g}$. The coproduct on $\mathrm{U}(\mathfrak{g})$ is conilpotent and the cobracket on $\mathrm{U}(\mathfrak{g})$ is conilpotent if the cobracket on $\mathfrak{g}$ is so. In this way we construct the following commutative diagram.

$$\begin{tikzcd} \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \arrow[r, "\sim"] \arrow[d] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \arrow[d] \\ \mathrm{Alg}_{\mathrm{Lie}} \arrow[r, "\sim"] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \end{tikzcd}$$

Note that if $A \in \mathrm{Alg}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{co}\mathbb{P}^{\mathrm{cu}}_n})$ has an associative multiplication and a compatible $\mathbb{P}_n$-coalgebra structure, then the space of primitive elements is closed under the cobracket by the Leibniz rule for the $\mathbb{P}_n$-coalgebra structure. Thus, the inverse functor in both cases is simply given by the functor of primitive elements.

Now we are going to show that the brace bar construction for $\mathcal{C} = \mathrm{co}\mathbb{P}_n$ is compatible with the bar-cobar duality between $\mathbb{P}_{n+1}$-algebras and $(n-1)$-shifted Lie bialgebras in the following sense.

Proposition 2.17. *The composite functor*

$$\begin{aligned} \mathrm{Alg}_{\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}} &\longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \\ &\longrightarrow \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \\ &\longrightarrow \mathrm{Alg}_{\mathbb{P}_{n+1}} \\ &\xrightarrow{\text{forget}} \mathrm{Alg}_{\Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\})} \end{aligned}$$

is weakly equivalent to the forgetful functor

$$\mathrm{Alg}_{\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}} \longrightarrow \mathrm{Alg}_{\Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\})}$$

given by (9).

Proof. Let A be a $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$-algebra. The functor

$$\mathrm{Alg}_{\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n})$$

sends A to $\mathrm{co}\mathbb{P}_n(A)$ equipped with the bar differential using the homotopy $\mathbb{P}_n$-algebra structure on A.

We can identify $\mathrm{co}\mathbb{P}_n \cong \mathrm{coComm} \circ \mathrm{coLie}\{1-n\}$ as symmetric sequences and hence after passing to primitives in $\mathrm{co}\mathbb{P}_n(A)$ we obtain $\mathfrak{g} = \mathrm{coLie}(A[1-n])[n-1]$ equipped with the bar differential using the homotopy commutative algebra structure on A. The Lie bracket on $\mathfrak{g}$ is obtained by antisymmetrizing the associative multiplication on $\mathrm{co}\mathbb{P}_n(A)$ and from the explicit description of the multiplication on $\mathrm{co}\mathbb{P}_n(A)$ given in § 2.2 we see that the projection of the bracket on the cogenerators

$$\mathrm{coLie}(A[1-n])[n-1] \otimes \mathrm{coLie}(A[1-n])[n-1] \longrightarrow A$$

is defined by the morphism

$$A \otimes \mathrm{coLie}(A[1-n])[n-1] \longrightarrow A$$

given by the brace operations $x\{c|y_1, \ldots, y_m\}$ where $x, y_i \in A$ and $c \in \mathrm{coLie}\{1-n\}(m)$.

We conclude that $\mathrm{coLie}(A[1-n])[n-1]$ is the Koszul dual of a homotopy $\mathbb{P}_{n+1}$-algebra A whose homotopy commutative multiplication is encoded in the differential on $\mathfrak{g}$ which comes from the homotopy commutative multiplication in the $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$-algebra structure. The rest of the homotopy $\mathbb{P}_{n+1}$ structure on A coincides with the one given by the morphism (9) by inspection. □

By the previous proposition the two functors of ∞-categories

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

given either by localizing the brace bar functor for $\mathcal{C} = \mathrm{co}\mathbb{P}_n$ or by localizing the bar-cobar duality between $\mathbb{P}_{n+1}$-algebras and shifted Lie bialgebras coincide.

Proposition 2.18. *The functor*

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

has a natural symmetric monoidal structure.

Proof. The functor $\Omega\colon \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \to \mathrm{Alg}_{\mathbb{P}_{n+1}}$ is symmetric monoidal, so its right adjoint $\mathrm{B}\colon \mathrm{Alg}_{\mathbb{P}_{n+1}} \to \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}}$ has a lax symmetric monoidal structure. Moreover, by Proposition 1.17, B becomes strictly symmetric monoidal after localization. Finally, the universal enveloping algebra functor

$$\mathrm{U}\colon \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n})$$

has an obvious symmetric monoidal structure and the claim follows. □

As a corollary, we get a sequence of functors

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+2}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}}) \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})).$$

But $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}(\mathcal{C})) \cong \mathcal{A}\mathrm{lg}_{\mathbb{E}_2}(\mathcal{C})$ for any symmetric monoidal ∞-category $\mathcal{C}$ by the Dunn–Lurie additivity theorem [Lur17, Theorem 5.1.2.2]. Iterating this construction, we get a symmetric monoidal functor

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+m}} \longrightarrow \mathcal{A}\mathrm{lg}_{\mathbb{E}_m}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}).$$

The additivity functor (15) interacts in the obvious way with the commutative and the Lie structures on a $\mathbb{P}_n$-algebra as shown by the next three propositions.

PROPOSITION 2.19. *The diagram*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r] \arrow[d] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d] \\ \mathcal{A}\mathrm{lg}_{\mathrm{Comm}} & \mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \arrow[l] \end{tikzcd}$$

is commutative.

Proof. The claim immediately follows from the below commutative diagram of operads.

$$\begin{tikzcd} \Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\}) \arrow[r] & \mathrm{Br}_{\mathrm{co}\mathbb{P}_n} \\ \Omega(\mathrm{coLie}\{1-n\}) \arrow[r] \arrow[u] & \Omega(\mathrm{co}\mathbb{P}_n) \arrow[u] \end{tikzcd}$$

□

Let us denote by

$$\overline{\mathrm{Sym}}\colon \mathrm{Alg}_{\mathrm{Lie}} \longrightarrow \mathrm{Alg}_{\mathbb{P}_n}$$

the functor which sends a Lie algebra $\mathfrak{g}$ to the non-unital $\mathbb{P}_n$-algebra $\overline{\mathrm{Sym}}(\mathfrak{g}[1-n])$, the reduced symmetric algebra on $\mathfrak{g}[1-n]$ with the Poisson bracket induced by the Leibniz rule from the bracket on $\mathfrak{g}$. Recall that the symmetric monoidal structure on $\mathrm{Alg}_{\mathbb{P}_n}$ we consider is the one transferred under the equivalence $\mathrm{Alg}_{\mathbb{P}_n} \cong \mathrm{Alg}^{\mathrm{aug}}_{\mathbb{P}_n^{\mathrm{un}}}$ from the usual tensor product of augmented $\mathbb{P}_n$-algebras. In particular, under this equivalence the functor $\overline{\mathrm{Sym}}$ corresponds to $\mathfrak{g} \mapsto \mathrm{Sym}(\mathfrak{g}[1-n])$ which is symmetric monoidal. The functor $\overline{\mathrm{Sym}}$ preserves weak equivalences, so we obtain a symmetric monoidal functor of ∞-categories

$$\overline{\mathrm{Sym}}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \longrightarrow \mathcal{A}\mathrm{lg}_{\mathbb{P}_n}.$$

PROPOSITION 2.20. *The diagram*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r, "\sim"] \arrow[d, "\overline{\mathrm{Sym}}"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \arrow[d, "\overline{\mathrm{Sym}}"] \\ \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \end{tikzcd}$$

is commutative.

Proof. Let us denote by

$$\Omega_{\mathrm{Lie}}\colon \mathrm{CoAlg}_{\mathrm{coComm}} \longrightarrow \mathrm{Alg}_{\mathrm{Lie}}$$

the cobar complex for Lie algebras and by

$$\Omega_{\mathbb{P}_n}\colon \mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n} \longrightarrow \mathrm{Alg}_{\mathbb{P}_n}$$

the cobar complex for $\mathbb{P}_n$-algebras which we can factor as

$$\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n} \xrightarrow{\Omega_{\mathrm{Lie}}} \mathrm{BiAlg}_{\mathrm{Lie}_{n-2}} \longrightarrow \mathrm{Alg}_{\mathbb{P}_n}.$$

Consider the symmetric monoidal functors

$$\mathrm{triv}\colon \mathrm{CoAlg}_{\mathrm{coComm}} \longrightarrow \mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}$$

and

$$\mathrm{triv}\colon \mathrm{Alg}_{\mathrm{Lie}} \longrightarrow \mathrm{BiAlg}_{\mathrm{Lie}_{n-2}}$$

which assign trivial cobrackets. We have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{CoAlg}_{\mathrm{coComm}} \arrow[r, "\Omega_{\mathrm{Lie}}"] \arrow[d, "\mathrm{triv}"] & \mathrm{Alg}_{\mathrm{Lie}} \arrow[d, "\mathrm{triv}"] \arrow[dr, "\overline{\mathrm{Sym}}"] & \\
\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n} \arrow[r, "\Omega_{\mathrm{Lie}}"] & \mathrm{BiAlg}_{\mathrm{Lie}_{n-2}} \arrow[r, "\Omega"] & \mathrm{Alg}_{\mathbb{P}_n}
\end{tikzcd}$$

Therefore, the claim will follow once we show that the diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Lie}} \arrow[r, equal] \arrow[d, "\overline{\mathrm{Sym}}"] & \mathrm{Alg}_{\mathrm{Lie}} \arrow[r, "\mathrm{U}"] \arrow[d, "\mathrm{triv}"] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \arrow[d, "\mathrm{triv}"] \\
\mathrm{Alg}_{\mathbb{P}_{n+1}} \arrow[r] & \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \arrow[r, "\mathrm{U}"] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n})
\end{tikzcd}$$

commutes up to a weak equivalence. It is clear that the square on the right commutes strictly and we are reduced to showing commutativity of the square on the left.

We can factor the functor

$$\overline{\mathrm{Sym}}\colon \mathrm{Alg}_{\mathrm{Lie}} \longrightarrow \mathrm{Alg}_{\mathbb{P}_{n+1}}$$

as

$$\mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{\mathrm{triv}} \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{P}_{n+1}}.$$

Therefore, the composite

$$\mathrm{Alg}_{\mathrm{Lie}} \xrightarrow{\mathrm{triv}} \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \xrightarrow{\Omega} \mathrm{Alg}_{\mathbb{P}_{n+1}} \xrightarrow{\mathrm{B}} \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}}$$

is weakly equivalent to $\mathrm{triv}\colon \mathrm{Alg}_{\mathrm{Lie}} \to \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}}$ and hence the remaining square commutes up to a weak equivalence. $\square$

The functor $\overline{\mathrm{Sym}}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \to \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}}$ has a right adjoint $\mathrm{forget}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \to \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ given by forgetting the commutative algebra structure. Since $\overline{\mathrm{Sym}}$ is symmetric monoidal, the right adjoint forget is lax symmetric monoidal and hence sends associative algebras to associative algebras. Therefore, the commutativity data of Proposition 2.20 gives rise to a diagram of right adjoints

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r, "\sim"] \arrow[dr, Rightarrow] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \\
\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[u, "\mathrm{forget}"] \arrow[r] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[u, "\mathrm{forget}"']
\end{tikzcd}$$

which commutes up to a natural transformation.

Proposition 2.21. *The diagram of right adjoints*

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r, "\sim"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \\
\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[u, "\mathrm{forget}"] \arrow[r] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[u, "\mathrm{forget}"]
\end{tikzcd}$$

commutes.

Proof. Since the functor forget: $\mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \to \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ is lax monoidal, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \arrow[r] & \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \\
\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[r] \arrow[u, "\mathrm{forget}"] & \mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \arrow[u, "\mathrm{forget}"']
\end{tikzcd}$$

where the horizontal functors are given by forgetting the associative algebra structure.

Forgetting the algebra structure is conservative, so it will be enough to prove that the outer square in

$$\begin{tikzcd}
\mathrm{Alg}_{\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathrm{Lie}} \arrow[d] \\
\mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \arrow[r] \arrow[d] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \arrow[d] \\
\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n} \arrow[r] \arrow[d] & \mathrm{CoAlg}_{\mathrm{coComm}} \arrow[d] \\
\mathrm{Alg}_{\mathbb{P}_n} \arrow[r, "\mathrm{forget}"] & \mathrm{Alg}_{\mathrm{Lie}}
\end{tikzcd}$$

commutes up to a weak equivalence. That is, suppose A is a $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$-algebra. Then we have to show that the natural morphism of dg Lie algebras

$$\Omega_{\mathrm{Lie}}\mathrm{B}_{\mathrm{Lie}}A \longrightarrow \Omega_{\mathbb{P}_n}\mathrm{B}_{\mathbb{P}_n}A$$

is a quasi-isomorphism, where $\Omega_{\dots}$ and $\mathrm{B}_{\dots}$ are the respective cobar and bar constructions. Let us note that $\mathrm{B}_{\mathrm{Lie}}A$ is the bar complex with respect to the L_∞ structure on A underlying the homotopy $\mathbb{P}_n$-structure on A.

But this is a quasi-isomorphism since the diagram

$$\begin{tikzcd}
\Omega_{\mathrm{Lie}}\mathrm{B}_{\mathrm{Lie}}A \arrow[rr] \arrow[dr, "\sim"'] & & \Omega_{\mathbb{P}_n}\mathrm{B}_{\mathbb{P}_n}A \arrow[dl, "\sim"] \\
& A &
\end{tikzcd}$$

commutes. □

We will end this section by proving that the additivity functor for $\mathbb{P}_n$-algebras is an equivalence.

Theorem 2.22. *The additivity functor*

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. By Proposition 2.18 the additivity functor is symmetric monoidal, so we just have to show that it is an equivalence of ∞-categories.

Consider the following diagram.

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r] \arrow[d, "\text{forget}"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d, "\text{forget}"] \\ \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[r] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \end{tikzcd}$$

By Corollary 2.14 the bottom functor is an equivalence. The forgetful functor

$$\text{forget}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$$

is conservative and preserves sifted colimits since they are created by the forgetful functor to chain complexes by Proposition 1.9. Similarly, the forgetful functor

$$\text{forget}\colon \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$$

is conservative. Let $\mathcal{O} = \mathbb{P}_n$ or Lie. Sifted colimits in $\mathcal{A}\mathrm{lg}_{\mathcal{O}}$ are created by the forgetful functor to chain complexes and since sifted colimits in $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathcal{O}})$ are created by the forgetful functor to $\mathcal{A}\mathrm{lg}_{\mathcal{O}}$, we conclude that sifted colimits in $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathcal{O}})$ are created by the forgetful functor to chain complexes.

By Proposition 2.21 the diagram commutes and by Proposition 2.20 the diagram of left adjoints commutes. Therefore, Proposition 1.3 applies and the claim follows. □

A natural question is whether the Dunn–Lurie additivity functor (see [Lur17, Theorem 5.1.2.2])

$$\mathcal{A}\mathrm{lg}_{\mathbb{E}_{n+1}} \xrightarrow{\sim} \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{E}_n})$$

is compatible with the Poisson additivity functor in the following sense. Suppose that $n \geqslant 2$. Then we have an equivalence of Hopf operads $\mathbb{P}_n \cong \mathbb{E}_n$ provided by the formality of the operad $\mathbb{E}_n$ which gives an equivalence of symmetric monoidal ∞-categories $\mathcal{A}\mathrm{lg}_{\mathbb{E}_n} \cong \mathcal{A}\mathrm{lg}_{\mathbb{P}_n}$.

Conjecture 2.23. Suppose $n \geqslant 2$. Then the diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d, "\sim"] \\ \mathcal{A}\mathrm{lg}_{\mathbb{E}_{n+1}} \arrow[r, "\sim"] & \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{E}_n}) \end{tikzcd}$$

is commutative.

2.5 Additivity for unital Poisson algebras

Let us explain the necessary modifications one performs to construct the additivity functor for unital Poisson algebras. Recall from § 1.4 the bar-cobar adjunction for unital Poisson algebras

$$\Omega\colon \mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n^\theta} \rightleftarrows \mathrm{Alg}_{\mathbb{P}_n^{\mathrm{un}}} : \mathrm{B},$$

where $\mathrm{co}\mathbb{P}_n^\theta$ is the cooperad of curved $\mathbb{P}_n$-coalgebras. One similarly has a bar-cobar adjunction

$$\Omega\colon \mathrm{CoAlg}_{\mathrm{coLie}^\theta} \rightleftarrows \mathrm{Alg}_{\mathrm{Comm}^{\mathrm{un}}} : \mathrm{B},$$

where coLie^θ is the cooperad of curved Lie coalgebras, that is, graded Lie coalgebras $\mathfrak{g}$ together with a coderivation d of degree 1 satisfying

$$\begin{aligned} d^2(x) &= \theta(x^\delta_{(1)})x^\delta_{(2)} \\ \theta(dx) &= 0, \end{aligned}$$

where $\delta(x) = x^\delta_{(1)} \otimes x^\delta_{(2)}$.

The cobar construction is given by the same formula as in the non-curved case except that we add the curving to the differential. The bar construction of a unital commutative algebra A is given by

$$\mathrm{coLie}(A[1] \oplus k[2])$$

with a bar differential and the curving which is given by projecting to $k[2]$.

DEFINITION 2.24. An *n-shifted curved Lie bialgebra* is a curved Lie coalgebra $\mathfrak{g}$ of degree $-n$ together with a degree 0 Lie bracket satisfying the cocycle relation (16).

We denote the category of n-shifted conilpotent curved Lie bialgebras by $\mathrm{BiAlg}^\theta_{\mathrm{Lie}_n}$ with weak equivalences created by the forgetful functor

$$\mathrm{BiAlg}^\theta_{\mathrm{Lie}_n} \longrightarrow \mathrm{CoAlg}_{\mathrm{coLie}^\theta}.$$

The commutative bar-cobar adjunction

$$\Omega\colon \mathrm{CoAlg}_{\mathrm{coLie}^\theta} \rightleftarrows \mathrm{Alg}_{\mathrm{Comm}^{\mathrm{un}}} : \mathrm{B},$$

extends to a bar-cobar adjunction

$$\Omega\colon \mathrm{BiAlg}^\theta_{\mathrm{Lie}_{n-1}} \rightleftarrows \mathrm{Alg}_{\mathbb{P}^{\mathrm{un}}_{n+1}} : \mathrm{B}.$$

The only modification from the case of non-unital Poisson algebras is the formula for the Lie bracket. Suppose A is a unital $\mathbb{P}_{n+1}$-algebra. Then as a graded vector space

$$\mathrm{B}(A) = \mathrm{coLie}(A[1] \oplus k[2])[n-1].$$

By the cocycle (16), the Lie bracket on $\mathrm{coLie}(A[1] \oplus k[2])$ is uniquely determined after projection to cogenerators and the morphism

$$\begin{aligned} \mathrm{coLie}(A[1] \oplus k[2])[n-1] \otimes \mathrm{coLie}(A[1] \oplus k[2])[n-1] &\longrightarrow \mathrm{coLie}(A[1] \oplus k[2])[n-1] \\ &\longrightarrow A[n] \oplus k[n+1] \end{aligned}$$

has the zero component in $k[n+1]$ and its $A[n]$ component is defined to be the bracket

$$A[n] \otimes A[n] \longrightarrow A[n].$$

The universal enveloping algebra construction gives a functor

$$\mathrm{U}\colon \mathrm{BiAlg}^\theta_{\mathrm{Lie}_{n-1}} \longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}^\theta_n}).$$

Therefore, we can define the additivity functor

$$\mathrm{add}_{\mathbb{P}^{\mathrm{un}}_n} : \mathcal{A}\mathrm{lg}_{\mathbb{P}^{\mathrm{un}}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}^{\mathrm{un}}_n})$$

to be given by the composite

$$\begin{aligned}\mathrm{Alg}_{\mathbb{P}^{\mathrm{un}}_{n+1}}[W_{\mathrm{qis}}^{-1}] &\longrightarrow \mathrm{BiAlg}^{\theta}_{\mathrm{Lie}_{n_1}}[W_{\mathrm{Kos}}^{-1}] \\ &\longrightarrow \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}^{\theta}_n})[W_{\mathrm{Kos}}^{-1}] \\ &\longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}^{\theta}_n}[W_{\mathrm{Kos}}^{-1}]).\end{aligned}$$

The following statement is proved as for non-unital Poisson algebras by analyzing the forgetful functor to Lie algebras.

Theorem 2.25. *The additivity functor*

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}^{\mathrm{un}}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}^{\mathrm{un}}_{n}})$$

is an equivalence of symmetric monoidal ∞-categories.

3. Coisotropic structures

In this section we show that two definitions of derived coisotropic structures given in [CPTVV17] and [MS16] are equivalent. Both definitions are given first in the affine setting and then extended in the same way to derived stacks, so it will be enough to prove equivalence on the affine level.

3.1 Two definitions

Let us introduce the following notations. Given a category $\mathcal{C}$ we denote by $\mathrm{Arr}(\mathcal{C})$ the category of morphisms in $\mathcal{C}$, i.e. the functor category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Given a symmetric monoidal category $\mathcal{C}$ we denote by $\mathrm{LMod}(\mathcal{C})$ the category of pairs (A, M), where A is a unital associative algebra in $\mathcal{C}$ and M is a left A-module. Let us denote by $\mathcal{L}\mathrm{Mod}(\mathcal{C})$ the same construction for a symmetric monoidal ∞-category $\mathcal{C}$.

Introduce the notation

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{(n+1,n)}} = \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}).$$

We have a forgetful functor $\mathcal{A}\mathrm{lg}_{\mathbb{P}_{(n+1,n)}} \to \mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Comm}})$ which sends a pair (A, B) with the action map $A\otimes B \to B$ to the morphism of commutative algebras $A \to B$ given by the composite $A \stackrel{\mathrm{id}\otimes 1}{\to} A \otimes B \to B$.

The following definition of coisotropic structures is due to Calaque *et al.* [CPTVV17, Definition 3.4.3].

Definition 3.1. Let $f\colon A \to B$ be a morphism of commutative dg algebras. The *space of n-shifted coisotropic structures* $\mathrm{Cois}^{\mathrm{CPTVV}}(f, n)$ is defined to be the fiber of

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{(n+1,n)}} \longrightarrow \mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Comm}})$$

at the given morphism f.

To relate this space of n-shifted coisotropic structures to the space of n-shifted Poisson structures on A, one has to use the Poisson additivity functor

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}).$$

A more explicit definition of the space of coisotropic structures was given in [MS16] and [Saf17] as follows. Let B be a $\mathbb{P}_n$-algebra. We define its *strict center* to be the $\mathbb{P}_{n+1}$-algebra

$$\mathrm{Z}^{\mathrm{str}}(B) = \mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_B[n]), B)$$

with the differential twisted by $[\pi_B, -]$. We refer to [Saf17, § 1.1] for explicit formulas for the $\mathbb{P}_{n+1}$-structure. One can also consider its center

$$\mathrm{Z}(B) = \mathrm{Hom}_k(\mathrm{co}\mathbb{P}_n^{\mathrm{cu}}(B), B)[-n]$$

with the differential twisted by the $\mathbb{P}_n$-structure on B. By results of [CW15], $\mathrm{Z}(B)$ is a $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\}$-algebra and hence in particular a homotopy $\mathbb{P}_{n+1}$-algebra. Moreover, the natural inclusion

$$\mathrm{Z}^{\mathrm{str}}(B) \to \mathrm{Z}(B)$$

is compatible with the homotopy $\mathbb{P}_{n+1}$ structures on both sides and is a quasi-isomorphism if B is cofibrant as a commutative algebra.

Let $\mathbb{P}_{[n+1,n]}$ be the colored operad whose algebras consist of a pair (A, B, f) where A is a $\mathbb{P}_{n+1}$-algebra, B is a $\mathbb{P}_n$-algebra and $f\colon A \to \mathrm{Z}^{\mathrm{str}}(B)$ is a $\mathbb{P}_{n+1}$-morphism. The projection $\mathrm{Z}^{\mathrm{str}}(B) \to B$ is a morphism of commutative algebras, so we get a natural forgetful functor

$$\mathrm{Alg}_{\mathbb{P}_{[n+1,n]}} \to \mathrm{Arr}(\mathrm{Alg}_{\mathrm{Comm}}).$$

Definition 3.2. Let $f\colon A \to B$ be a morphism of commutative dg algebras. The *space of n-shifted coisotropic structures* $\mathrm{Cois}^{\mathrm{MS}}(f, n)$ is defined to be the fiber of

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Comm}})$$

at the given morphism f.

Our goal will be to construct an equivalence of ∞-categories $\mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \to \mathcal{A}\mathrm{lg}_{\mathbb{P}_{(n+1,n)}}$ which is compatible with the forgetful functor to $\mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Comm}})$ which will show that the spaces $\mathrm{Cois}^{\mathrm{MS}}(f, n)$ and $\mathrm{Cois}^{\mathrm{CPTVV}}(f, n)$ are equivalent. We will construct the equivalence as a relative version of the additivity functor (13).

3.2 Relative additivity for Lie algebras

We begin with the relative analog of the additivity functor for Lie algebras.

Let us introduce a Swiss-cheese operad of Lie algebras. Given a dg Lie algebra $\mathfrak{h}$ the Chevalley–Eilenberg complex $\mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$ carries a convolution Lie bracket. We let $\mathrm{Lie}_{[1,0]}$ be the colored operad whose algebras consist of a pair of dg Lie algebras $(\mathfrak{g}, \mathfrak{h})$ together with a map of dg Lie algebras $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$.

Remark 3.3. Following [MS16, § 3.3], given the morphism of operads

$$\Omega\,\mathrm{coComm}\{1\} \to \mathrm{Lie} \to \mathrm{Br}_{\mathrm{coComm}},$$

one can consider the colored operad $\mathrm{SC}(\mathrm{coComm}, \mathrm{coComm})$ whose algebras are given by a pair $(\mathfrak{g}, \mathfrak{h})$ of L_∞ algebras together with an ∞-morphism $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$. The colored operad $\mathrm{Lie}_{[1,0]}$ is given by a quotient where we declare both $\mathfrak{g}$ and $\mathfrak{h}$ to be strict Lie algebras and the morphism $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$ to be a strict morphism.

Consider the morphism $\mathfrak{h} \to \mathrm{Der}(\mathfrak{h})$ given by the adjoint action and denote by $\widetilde{\mathrm{Der}}(\mathfrak{h})$ its cone. Note that we have an obvious inclusion $\widetilde{\mathrm{Der}}(\mathfrak{h}) \subset \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$. We denote by $\mathrm{Lie}^{\mathrm{str}}_{[1,0]}$ the quotient operad of $\mathrm{Lie}_{[1,0]}$ where we set all components of the map $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$ having $\mathfrak{h}$-arity at least 2 to be zero. Thus, a $\mathrm{Lie}^{\mathrm{str}}_{[1,0]}$-algebra is a pair $(\mathfrak{g},\mathfrak{h})$ of dg Lie algebras together with a map of Lie algebras $\mathfrak{g}\to\widetilde{\mathrm{Der}}(\mathfrak{h})$.

Given a pair $(\mathfrak{g},\mathfrak{h})\in\mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}$ we can construct a dg Lie algebra $\mathfrak{k}$ as follows. As a graded vector space we define $\mathfrak{k}=\mathfrak{g}\oplus\mathfrak{h}$. The differential on $\mathfrak{k}$ is the sum of the differentials on $\mathfrak{g}$ and $\mathfrak{h}$ and the differential $\mathfrak{g}\to\mathfrak{h}$ coming from the composite

$$\mathfrak{g}\longrightarrow \mathrm{C}^\bullet(\mathfrak{h},\mathfrak{h})[1]\longrightarrow\mathfrak{h}[1].$$

The Lie bracket on $\mathfrak{k}$ is the sum of Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$ and the action map of $\mathfrak{g}$ on $\mathfrak{h}$ given by the map $\mathfrak{g}\to\mathrm{Der}(\mathfrak{h})$. In this way we see that a pair $(\mathfrak{g},\mathfrak{h})\in\mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}$ is the same as a dg Lie algebra extension

$$0\longrightarrow\mathfrak{h}\longrightarrow\mathfrak{k}\longrightarrow\mathfrak{g}\longrightarrow 0.$$

One can similarly construct an L_∞ extension from the data of a general object of $\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$. Since every L_∞ algebra is quasi-isomorphic to a dg Lie algebra, the following statement should be obvious.

Proposition 3.4. *The forgetful functor*

$$\mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}\longrightarrow\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$$

induces an equivalence on the underlying ∞-categories.

Proof. By Proposition 1.8 it is enough to show that the projection $\mathrm{Lie}_{[1,0]}\to\mathrm{Lie}^{\mathrm{str}}_{[1,0]}$ is a quasi-isomorphism.

Let $\mathrm{coComm}^{\mathrm{cu}}$ be the cooperad of counital cocommutative coalgebras. One can identify the underlying graded colored symmetric sequence $\mathrm{Lie}_{[1,0]}$ with $\mathrm{Lie}\circ(\mathrm{coComm}^{\mathrm{cu}}\circ\mathrm{Lie}\oplus\mathbf{1})$ so that $\mathrm{Lie}_{[1,0]}(\mathcal{A}^{\otimes n},\mathcal{B}^{\otimes m})$ has n operations in $\mathrm{coComm}^{\mathrm{cu}}$ and m operations in Lie or $\mathbf{1}$. The projection

$$\mathrm{Lie}_{[1,0]}(\mathcal{A}^{\otimes 0},\mathcal{B}^{\otimes m})\to\mathrm{Lie}^{\mathrm{str}}(\mathcal{A}^{\otimes 0},\mathcal{B}^{\otimes m})$$

is an isomorphism. To simplify the notation, we prove that the projection

$$\mathrm{Lie}_{[1,0]}(\mathcal{A}^{\otimes 1},\mathcal{B}^{\otimes m})\to\mathrm{Lie}^{\mathrm{str}}(\mathcal{A}^{\otimes 1},\mathcal{B}^{\otimes m})\tag{18}$$

is a quasi-isomorphism since the case of higher n is handled similarly. Thus, we can identify the underlying graded symmetric sequence $\mathrm{Lie}_{[1,0]}(\mathcal{A}^{\otimes 1},\mathcal{B}^{\otimes -})$ wth $\mathrm{Lie}\circ_{(1)}(\mathrm{coComm}^{\mathrm{cu}}\circ\mathrm{Lie})$, where $\circ_{(1)}$ is the infinitesimal composite (see [LV12, § 6.1]).

Let V be a complex and consider $\mathrm{Lie}(V)$, the free Lie algebra on V, and its homology $\mathrm{C}_\bullet(\mathrm{Lie}(V),\mathrm{Lie}(V))$. Let $\mathrm{C}_{\leqslant 1}(\mathrm{Lie}(V),\mathrm{Lie}(V))$ be the quotient of $\mathrm{C}_\bullet(\mathrm{Lie}(V),\mathrm{Lie}(V))$ where we consider only components of weight $\leqslant 1$ and mod out by the image of the Chevalley–Eilenberg differential from weight 2 to weight 1. The projection

$$\mathrm{C}_\bullet(\mathrm{Lie}(V),\mathrm{Lie}(V))\to\mathrm{C}_{\leqslant 1}(\mathrm{Lie}(V),\mathrm{Lie}(V))$$

is a quasi-isomorphism which can be seen as follows. The left-hand side by definition computes the derived tensor product $k\otimes^{\mathbb{L}}_{\mathrm{T}(V)}\mathrm{Lie}(V)$. But k has a two-term free resolution

$$(\mathrm{T}(V)\otimes V\to\mathrm{T}(V))\xrightarrow{\sim}k$$

as a $\mathrm{T}(V)$-module and computing the derived tensor product using this resolution one exactly obtains $\mathrm{C}_{\leqslant 1}(\mathrm{Lie}(V),\mathrm{Lie}(V))$.

To conclude the proof, observe that the coefficient of $V^{\otimes m}$ in $\mathrm{C}_\bullet(\mathrm{Lie}(V),\mathrm{Lie}(V))$ is isomorphic as a complex to $\mathrm{Lie}_{[1,0]}(\mathcal{A}^{\otimes 1},\mathcal{B}^{\otimes m})$ while its coefficient in $\mathrm{C}_{\leqslant 1}(\mathrm{Lie}(V),\mathrm{Lie}(V))$ is isomorphic to $\mathrm{Lie}^{\mathrm{str}}(\mathcal{A}^{\otimes 1},\mathcal{B}^{\otimes m})$. □

Now we are going to introduce the bar construction

$$\mathrm{B}\colon \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}).$$

Consider a pair $(\mathfrak{g}, \mathfrak{h}) \in \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$. We send it to the pair $(\mathrm{U}(\mathfrak{g}), \mathrm{C}_\bullet(\mathfrak{h}))$ of a cocommutative bialgebra and a cocommutative coalgebra. The action map

$$\mathrm{U}\mathfrak{g} \otimes \mathrm{C}_\bullet(\mathfrak{h}) \longrightarrow \mathrm{C}_\bullet(\mathfrak{h})$$

is constructed as follows. Since $\mathrm{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$, it is enough to specify the action $\mathfrak{g} \otimes \mathrm{C}_\bullet(\mathfrak{h}) \to \mathrm{C}_\bullet(\mathfrak{h})$ that we denote by $x.c$ for $x \in \mathfrak{g}$ and $c \in \mathrm{C}_\bullet(\mathfrak{h})$ satisfying the equations

$$(x.c)_{(1)} \otimes (x.c)_{(2)} = (x.c_{(1)}) \otimes c_{(2)} + (-1)^{|x||c_{(1)}|} c_{(1)} \otimes (x.c_{(2)}), \quad x \in \mathfrak{g}, c \in \mathrm{C}_\bullet(\mathfrak{h})$$
$$[x, y].c = x.(y.c) - (-1)^{|x||y|} y.(x.c), \quad x, y \in \mathfrak{g}, c \in \mathrm{C}_\bullet(\mathfrak{h}).$$

Since $\mathrm{C}_\bullet(\mathfrak{h})$ is cofree, by the first equation it is enough to specify the map $\mathfrak{g} \otimes \mathrm{C}_\bullet(\mathfrak{h}) \to \mathfrak{h}[1]$ which we define to be adjoint to the given map $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$. It is then easy to see that since the map $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$ is compatible with Lie brackets, the second equation is satisfied and since it is compatible with the differentials, so is the map $\mathfrak{g} \otimes \mathrm{C}_\bullet(\mathfrak{h}) \to \mathrm{C}_\bullet(\mathfrak{h})$. This defines the functor

$$\mathrm{B}\colon \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}).$$

We can also introduce the cobar construction

$$\Omega\colon \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \longrightarrow \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}.$$

Consider an element $(A, C) \in \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}})$ where A is an algebra and C is an A-module. By Theorem 2.9 we can identify $A \cong \mathrm{U}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ of primitive elements. We send $(\mathrm{U}(\mathfrak{g}), C)$ to the pair $(\mathfrak{g}, \mathfrak{h} = \Omega C)$, where ΩC is the Harrison complex $\mathrm{Lie}(\overline{C}[-1])$. Let us denote the action map $\mathrm{U}\mathfrak{g} \otimes C \to C$ by $x.c$ for $x \in \mathfrak{g}$ and $c \in C$. The morphism $\mathfrak{g} \longrightarrow \mathfrak{h}[1]$ is defined by the composite

$$\mathfrak{g} \longrightarrow \overline{C}[-1] \longrightarrow \mathrm{Lie}(\overline{C}[-1]) = \mathfrak{h},$$

where the first map is given by the action map $x.1$. Since $\mathfrak{h}$ is semi-free, the morphism $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is uniquely determined by the map

$$\mathfrak{g} \otimes \overline{C}[-1] \longrightarrow \overline{C}[-1] \longrightarrow \mathrm{Lie}(\overline{C}[-1]),$$

where the first map is the action of $\mathfrak{g}$ on C. The fact that the thus constructed morphism $\mathfrak{g} \to \widetilde{\mathrm{Der}}(\mathfrak{h})$ is a morphism of Lie algebras follows from the associativity of the action map $\mathrm{U}\mathfrak{g} \otimes C \to C$. The compatibility of the morphism $\mathfrak{g} \to \widetilde{\mathrm{Der}}(\mathfrak{h})$ with the differential follows from the compatibility of the action map $\mathrm{U}(\mathfrak{g}) \otimes C \to C$ with coproducts. This defines the functor

$$\Omega\colon \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \longrightarrow \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}.$$

Note that in this way we obtain an adjunction

$$\Omega\colon \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \rightleftarrows \mathrm{Alg}^{\mathrm{str}}_{\mathrm{Lie}_{[1,0]}} : \mathrm{B}.$$

Indeed, the counit and unit morphisms

$$\Omega\mathrm{B}(\mathfrak{g}, \mathfrak{h}) \longrightarrow (\mathfrak{g}, \mathfrak{h}), \quad (A, C) \to \mathrm{B}\Omega(A, C)$$

are defined to be the identities in the first slot and the counit and unit of the usual bar-cobar adjunction in the second slot.

PROPOSITION 3.5. *The adjunction*

$$\Omega\colon \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \rightleftarrows \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}} : \mathrm{B}$$

induces an adjoint equivalence

$$\mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \rightleftarrows \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}}[W_{\mathrm{qis}}^{-1}]$$

on the underlying ∞-categories.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \arrow[r, shift left] \arrow[d] & \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{str}}_{[1,0]}} \arrow[l, shift left] \arrow[d] \\ \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \times \mathrm{CoAlg}_{\mathrm{coComm}} \arrow[r, shift left] & \mathrm{Alg}_{\mathrm{Lie}} \times \mathrm{Alg}_{\mathrm{Lie}} \arrow[l, shift left] \end{tikzcd}$$

where the vertical functors are the obvious forgetful functors. Since they reflect weak equivalences, it is enough to show the bottom adjunction induces an equivalence after localization. Indeed,

$$\mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}}) \underset{\mathrm{U}}{\rightleftarrows} \mathrm{Alg}_{\mathrm{Lie}}$$

is an equivalence by Theorem 2.9 and

$$\mathrm{CoAlg}_{\mathrm{coComm}} \rightleftarrows \mathrm{Alg}_{\mathrm{Lie}}$$

induces an ∞-categorical equivalence by Proposition 1.7. □

The composite functor

$$\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}[W_{\mathrm{qis}}^{-1}] \xrightarrow{\mathrm{B}} \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \longrightarrow \mathcal{L}\mathrm{Mod}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}])$$

defines a relative additivity functor for Lie algebras:

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}).$$

THEOREM 3.6. *The additivity functor*

$$\mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$$

is an equivalence of ∞-categories.

Proof. Consider a commutative diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \arrow[d, "\mathrm{B}"] & \\ \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \arrow[r, "G_1"] \arrow[d, "L_1"] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \times \mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}] \arrow[d, "L_2"] \\ \mathcal{L}\mathrm{Mod}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}]) \arrow[r, "G_2"] & \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}]) \times \mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}] \end{tikzcd}$$

where the functors G_1, G_2 are the obvious forgetful functors.

The functor

$$\mathrm{B}\colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}]$$

is an equivalence by Proposition 3.5, so we just need to show that the functor L_1 is an equivalence. The localization functor

$$\mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{coComm}})[W_{\mathrm{Kos}}^{-1}] \longrightarrow \mathcal{A}\mathrm{lg}(\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}])$$

is an equivalence by results of § 2.3, so the functor L_2 is an equivalence.

The functor G_1 is conservative. Sifted colimits in $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}}$ are created by the forgetful functor to $\mathcal{C}\mathrm{h} \times \mathcal{C}\mathrm{h}$ by Proposition 1.9, so the functor G_1 preserves sifted colimits. The functor G_2 is conservative; it preserves sifted colimits by [Lur17, Corollay 4.2.3.7].

The left adjoint to G_1 post-composed with the forgetful functor to $\mathrm{CoAlg}_{\mathrm{coComm}}[W_{\mathrm{Kos}}^{-1}]$ is given by the functor $(A, V) \mapsto (A, A \otimes V)$. By [Lur17, Corollary 4.2.4.4] the functor G_2 also admits a left adjoint given by the same formula, so the diagram of left adjoints commutes. Therefore, Proposition 1.3 applies and thus L_1 is an equivalence. □

Remark 3.7. One can also construct the relative additivity functor $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \cong \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$ as follows. The colored operad $\mathrm{Lie}_{[1,0]}^{\mathrm{str}}$ is quadratic whose Koszul dual is the cooperad of a pair of cocommutative coalgebras C_1, C_2 together with a morphism $C_1 \to C_2$. Finally, a relative version of Proposition 2.12 gives an equivalence $\mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \cong \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$.

3.3 Relative additivity for Poisson algebras

We now proceed to the construction of the additivity functor

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathcal{L}\mathrm{Mod}(\mathrm{Alg}_{\mathbb{P}_n}).$$

Consider a pair $(A, B) \in \mathrm{Alg}_{\mathbb{P}_{[n+1,n]}}$. Let $\mathfrak{g}$ be the Koszul dual $(n-1)$-shifted Lie bialgebra to A constructed in § 2.4. As a graded vector space, we can identify $\mathfrak{g} \cong \mathrm{coLie}(A[1])[n-1]$. Let us also denote by $\mathrm{B}B$ the Koszul dual coaugmented $\mathbb{P}_n$-coalgebra; as a graded vector space, we can identify $\mathrm{B}B \cong \mathrm{co}\mathbb{P}_n^{\mathrm{cu}}(B[n])$. Recall also that $\mathrm{U}(\mathfrak{g})$ is an associative algebra in $\mathbb{P}_n$-coalgebras. Now we want to construct the action map

$$a\colon \mathrm{U}(\mathfrak{g}) \otimes \mathrm{B}B \longrightarrow \mathrm{B}B$$

of $\mathbb{P}_n$-coalgebras. Such a map by associativity is uniquely determined by the map

$$\mathfrak{g} \otimes \mathrm{B}B \longrightarrow \mathrm{B}B$$

and since $\mathrm{B}B$ is cofree as a graded $\mathbb{P}_n$-coalgebra, this map is uniquely determined by projection to the cogenerators

$$\mathfrak{g} \otimes \mathrm{B}B \longrightarrow B[n].$$

We define this map to be adjoint to the map

$$\mathrm{coLie}(A[1])[n-1] \longrightarrow A[n] \longrightarrow \mathrm{Z}(B)[n].$$

Recall the description of the Koszul dual Lie bialgebra to a $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$-algebra such as $\mathrm{Z}(B)$ from the proof of Proposition 2.17. Using this description, we see that the associativity of the action map a follows from the compatibility of the morphism $A \longrightarrow \mathrm{Z}(B)$ with Lie brackets and the compatibility of a with the differential follows from the compatibility of $A \longrightarrow \mathrm{Z}(B)$ with the C_∞ structure. Compatibility with the $\mathbb{P}_n$-coalgebra structures is obvious by construction.

In this way we obtain a functor

$$\mathrm{Alg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathbb{P}_n})$$

and we let the additivity functor

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathcal{L}\mathrm{Mod}(\mathrm{CoAlg}_{\mathbb{P}_n}[W_{\mathrm{Kos}}^{-1}])$$

be the composite

$$\mathrm{Alg}_{\mathbb{P}_{[n+1,n]}}[W_{\mathrm{qis}}^{-1}] \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathbb{P}_n})[W_{\mathrm{Kos}}^{-1}] \longrightarrow \mathcal{L}\mathrm{Mod}(\mathrm{CoAlg}_{\mathbb{P}_n}[W_{\mathrm{Kos}}^{-1}]).$$

Theorem 3.8. *The additivity functor*

$$\mathrm{add}\colon \mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n})$$

is an equivalence of ∞-categories.

Proof. Suppose B is a $\mathbb{P}_n$-algebra. In particular, $B[n-1]$ is a Lie algebra and we have a morphism

$$\mathrm{Z}(B)[n] \longrightarrow \mathrm{C}^\bullet(B[n-1], B[n])$$

of Lie algebras induced by the morphism of cooperads $\mathrm{coComm} \to \mathrm{co}\mathbb{P}_n$. Compatibility with the Lie algebra structures on both sides is clear as both are given by convolution brackets.

This gives a forgetful functor

$$G_1\colon \mathrm{Alg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$$

which sends a pair (A, B) with a morphism of $\mathbb{P}_{n+1}$-algebras $A \longrightarrow \mathrm{Z}^{\mathrm{str}}(B)$ to the pair of Lie algebras $(A[n], B[n-1])$ with a morphism of Lie algebras

$$A[n] \longrightarrow \mathrm{Z}^{\mathrm{str}}(B)[n] \longrightarrow \mathrm{C}^\bullet(B[n-1], B[n-1])[1].$$

The forgetful functor G_1 has a left adjoint

$$F_1\colon \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{Alg}_{\mathbb{P}^{\mathrm{str}}_{[n+1,n]}}$$

which is constructed as follows. Consider a pair $(\mathfrak{g}, \mathfrak{h}) \in \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$ equipped with a morphism $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$. Then $A = \overline{\mathrm{Sym}}(\mathfrak{g}[-n])$ is a $\mathbb{P}_{n+1}$-algebra and $B = \overline{\mathrm{Sym}}(\mathfrak{h}[1-n])$ is a $\mathbb{P}_n$-algebra. We can identify

$$\mathrm{Z}^{\mathrm{str}}(B) \cong \mathrm{C}^\bullet(\mathfrak{h}, B)$$

as Lie algebras. Using this identification we obtain a morphism of $\mathbb{P}_n$-algebras

$$A \longrightarrow \mathrm{Z}^{\mathrm{str}}(B)$$

defined to be the Lie map $\mathfrak{g} \to \mathrm{C}^\bullet(\mathfrak{h}, \mathfrak{h})[1]$ on the generators of A. This concludes the construction of the functor F_1.

Now consider the following diagram.

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbb{P}^{\mathrm{str}}_{[n+1,n]}} \arrow[r] & \mathrm{LMod}(\mathrm{CoAlg}_{\mathbb{P}_n}) \\
\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \arrow[u, "F_1"] \arrow[r, "\mathrm{B}"] & \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \arrow[u, "\mathrm{triv}"']
\end{tikzcd} \tag{19}$$

Consider an object $(\mathfrak{g}, \mathfrak{h}) \in \mathrm{Alg}_{\mathrm{Lie}_{[1,0]}}$. Under the composite

$$\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{Alg}_{\mathbb{P}^{\mathrm{str}}_{[n+1,n]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathbb{P}_n})$$

the underlying associative algebra can be identified with $\mathrm{U}\mathfrak{g}$ with the trivial cobracket by Proposition 2.20. Similarly, the underlying module can be identified with $\mathrm{C}_\bullet(\mathfrak{h})$ with the trivial cobracket using the weak equivalence

$$\mathrm{C}_\bullet(\mathfrak{h}) \xrightarrow{\sim} \mathrm{B}_{\mathbb{P}_n}(\overline{\mathrm{Sym}}(\mathfrak{h}[1-n])).$$

It is easy to see that the action of $\mathrm{U}\mathfrak{g}$ on $\mathrm{C}_\bullet(\mathfrak{h})$ under this equivalence coincides with the action given by the composite

$$\mathrm{Alg}_{\mathrm{Lie}_{[1,0]}} \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{coComm}}) \longrightarrow \mathrm{LMod}(\mathrm{CoAlg}_{\mathbb{P}_n})$$

and hence the diagram (19) commutes up to a weak equivalence.

Denote by

$$F_2 \colon \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathbb{P}_n} : G_2$$

the free-forgetful adjunction and consider the following diagram of ∞-categories.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \arrow[r, "\mathrm{add}_{\mathbb{P}_n}"] \arrow[d, shift left, "G_1"] & \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[d, shift left, "G_2"] \\
\mathcal{A}\mathrm{lg}_{\mathrm{Lie}_{[1,0]}} \arrow[u, shift left, "F_1"] \arrow[r, "\mathrm{add}_{\mathrm{Lie}}"] & \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \arrow[u, shift left, "F_2"]
\end{tikzcd} \tag{20}$$

By Theorem 3.6 the functor $\mathrm{add}_{\mathrm{Lie}}$ is an equivalence. Moreover, the commutativity of diagram (19) implies that

$$\mathrm{add}_{\mathbb{P}_n} \circ F_1 \cong F_2 \circ \mathrm{add}_{\mathrm{Lie}}.$$

To check that the natural morphism

$$\mathrm{add}_{\mathrm{Lie}} \circ G_1 \to G_2 \circ \mathrm{add}_{\mathbb{P}_n}$$

is an equivalence, it is enough to check it in $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \times \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ since the forgetful functor

$$\mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \longrightarrow \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}) \times \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$$

is conservative. By Proposition 2.21 the corresponding morphism in $\mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathrm{Lie}})$ is an equivalence. It is also obvious that the corresponding morphism in $\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}$ is an equivalence since the diagram (20) can be forgotten to the following commutative diagram.

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \arrow[r, "\mathrm{id}"] \arrow[d, shift left, "\mathrm{forget}"] & \mathcal{A}\mathrm{lg}_{\mathbb{P}_n} \arrow[d, shift left, "\mathrm{forget}"] \\
\mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[u, shift left, "\overline{\mathrm{Sym}}"] \arrow[r, "\mathrm{id}"] & \mathcal{A}\mathrm{lg}_{\mathrm{Lie}} \arrow[u, shift left, "\overline{\mathrm{Sym}}"]
\end{tikzcd}$$

The forgetful functor G_1 is conservative and it preserves sifted colimits since they are created by the forgetful functor to $\mathcal{C}\mathrm{h} \times \mathcal{C}\mathrm{h}$ by Proposition 1.9. Similarly, the forgetful functor G_2 is conservative and preserves sifted colimits since sifted colimits in $\mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathcal{O}})$ are created by the forgetful functor to $\mathcal{A}\mathrm{lg}_{\mathcal{O}} \times \mathcal{A}\mathrm{lg}_{\mathcal{O}}$ and hence by the forgetful functor to $\mathcal{C}\mathrm{h} \times \mathcal{C}\mathrm{h}$. Therefore, by Proposition 1.3 the functor $\mathrm{add}_{\mathbb{P}_n}$ is an equivalence. □

Corollary 3.9. *Given a morphism $f\colon A \to B$ of dg commutative algebras, there is a canonical equivalence of spaces of n-shifted coisotropic structures*

$$\mathrm{Cois}^{\mathrm{MS}}(f, n) \xrightarrow{\sim} \mathrm{Cois}^{\mathrm{CPTVV}}(f, n).$$

Proof. To prove the claim we have to show that we have a commutative diagram of ∞-categories.

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{\mathbb{P}_{[n+1,n]}} \arrow[rr, "\mathrm{add}_{\mathbb{P}_n}"] \arrow[dr] & & \mathcal{L}\mathrm{Mod}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \arrow[dl] \\ & \mathrm{Arr}(\mathcal{A}\mathrm{lg}_{\mathrm{Comm}}) & \end{tikzcd}$$

The forgetful functor $\mathrm{Alg}_{\mathbb{P}_n} \to \mathrm{Alg}_{\mathrm{Comm}}$ under Koszul duality corresponds to the functor $\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n} \to \mathrm{CoAlg}_{\mathrm{coLie}}$ which is given by taking primitive elements. We are going to show that the diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{P}_{[n+1,n]}} \arrow[r] \arrow[dd] & \mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \arrow[d] \\ & \mathrm{Arr}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \arrow[d] \\ \mathrm{Arr}(\mathrm{Alg}_{\mathrm{Comm}}) \arrow[r] & \mathrm{Arr}(\mathrm{CoAlg}_{\mathrm{coLie}}) \end{tikzcd}$$

strictly commutes which will prove the claim.

Consider an object $(A, B) \in \mathrm{Alg}_{\mathbb{P}_{[n+1,n]}}$. Let $\mathfrak{g}$ be the $(n-1)$-shifted Lie bialgebra Koszul dual to A and $\mathrm{B}_{\mathbb{P}_n} B$ the coaugmented $\mathbb{P}_n$-coalgebra Koszul dual to B. The functor

$$\mathrm{LMod}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n}) \longrightarrow \mathrm{Arr}(\mathrm{CoAlg}^{\mathrm{coaug}}_{\mathrm{co}\mathbb{P}^{\mathrm{cu}}_n})$$

sends the action map $\mathrm{U}\mathfrak{g} \otimes \mathrm{B}_{\mathbb{P}_n} B \to \mathrm{B}_{\mathbb{P}_n} B$ to the morphism $\mathrm{U}\mathfrak{g} \to \mathrm{B}_{\mathbb{P}_n} B$ which is given by the image of $1 \in B$. After passing to primitives we obtain a morphism

$$\mathfrak{g} \longrightarrow \mathrm{B}_{\mathrm{Comm}} B \tag{21}$$

of Lie coalgebras. But as a Lie coalgebra we can identify $\mathfrak{g} \cong \mathrm{B}_{\mathrm{Comm}} A$ and the morphism (21) is the image of $A \to B$ under the commutative bar construction. □

Remark 3.10. In [MS16] we construct a forgetful map from n-shifted Poisson structures to $(n-1)$-shifted Poisson structures on a commutative algebra A using the natural correspondence

$$\mathrm{Pois}(A, n-1) \longleftarrow \mathrm{Cois}^{\mathrm{MS}}(\mathrm{id}, n) \xrightarrow{\sim} \mathrm{Pois}(A, n).$$

We can relate it to the additivity functor as follows. Let A be a $\mathbb{P}_{n+1}$-algebra and $\mathfrak{g}$ the Koszul dual $(n-1)$-shifted Lie bialgebra. Then $\mathrm{U}(\mathfrak{g}) \in \mathrm{Alg}(\mathrm{CoAlg}_{\mathrm{co}\mathbb{P}_n})$ is naturally a module over itself which by Corollary 3.9 gives a coisotropic structure on the identity $A \to A$, i.e. an element of $\mathrm{Cois}^{\mathrm{MS}}(\mathrm{id}, n)$. The underlying $\mathbb{P}_n$-algebra structure in $\mathrm{Pois}(A, n-1)$ is then the Koszul dual $\mathbb{P}_n$-algebra to the $\mathbb{P}_n$-coalgebra $\mathrm{U}(\mathfrak{g})$. But this exactly coincides with the forgetful functor

$$\mathcal{A}\mathrm{lg}_{\mathbb{P}_{n+1}} \xrightarrow{\sim} \mathcal{A}\mathrm{lg}(\mathcal{A}\mathrm{lg}_{\mathbb{P}_n}) \longrightarrow \mathcal{A}\mathrm{lg}_{\mathbb{P}_n}.$$

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Derived coisotropic structures I: affine case

Valerio Melani[1] · **Pavel Safronov**[2,3]

Abstract We define and study coisotropic structures on morphisms of commutative dg algebras in the context of shifted Poisson geometry, i.e. $\mathbb{P}_n$-algebras. Roughly speaking, a coisotropic morphism is given by a $\mathbb{P}_{n+1}$-algebra acting on a $\mathbb{P}_n$-algebra. One of our main results is an identification of the space of such coisotropic structures with the space of Maurer–Cartan elements in a certain dg Lie algebra of relative polyvector fields. To achieve this goal, we construct a cofibrant replacement of the operad controlling coisotropic morphisms by analogy with the Swiss-cheese operad which can be of independent interest. Finally, we show that morphisms of shifted Poisson algebras are identified with coisotropic structures on their graph.

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Contents

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Introduction

This paper is a continuation of the work of Calaque et al. [3] on shifted Poisson structures on derived stacks. In this paper we introduce shifted coisotropic structures on morphisms of commutative dg algebras and show that they possess certain expected properties:

- Suppose A is a $\mathbb{P}_{n+1}$-algebra, i.e. a commutative dg algebra with a Poisson bracket of degree $-n$. Then the identity morphism $A \to A$ carries a unique n-shifted coisotropic structure.
- A morphism of commutative algebras $A \to B$ is compatible with n-shifted Poisson structures iff its graph $A \otimes B \to B$ has an n-shifted coisotropic structure.

We generalize these results to general derived Artin stacks in [21].

Classical setting

Let us recall two ways of defining Poisson structures and coisotropic embeddings in the classical setting. Suppose X is a smooth scheme over a characteristic zero field k. Then one has the following equivalent definitions of a Poisson structure on X:

(1) The structure sheaf $\mathcal{O}_X$ of X is a sheaf of k-linear Poisson algebras where the multiplication coincides with the original commutative multiplication on $\mathcal{O}_X$.
(2) X carries a bivector $\pi_X \in \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$ such that $[\pi_X, \pi_X] = 0$.

The equivalence of the two definitions is clear: a bivector π_X is the same as an antisymmetric biderivation $\mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{O}_X$; an easy computation shows that the equation $[\pi_X, \pi_X] = 0$ is then equivalent to the Jacobi identity for the corresponding biderivation.

Now suppose X is a scheme carrying a Poisson structure in one of the senses above and consider a smooth closed subscheme $i\colon L \hookrightarrow X$. Then one has the following equivalent definitions of L being coisotropic:

(1) The ideal sheaf $\mathcal{I}_L$ defining L is a coisotropic ideal, i.e. it is closed under the Poisson bracket on $\mathcal{O}_X$.

(2) Let $\mathrm{N}_{L/X}$ be the normal bundle of L. The composite

$$\mathrm{N}^*_{L/X} \longrightarrow i^*\mathrm{T}^*_X \xrightarrow{\pi_X} i^*\mathrm{T}_X \longrightarrow \mathrm{N}_{L/X}$$

is zero.

The equivalence of the two definitions is well-known and follows from the identification $\mathrm{N}^*_{L/X} \cong \mathcal{I}_L/\mathcal{I}_L^2$.

Shifted Poisson algebras

Now suppose A is a commutative dg algebra. To generalize the first definition of a Poisson structure we simply replace the Poisson bracket of degree 0 by one of degree $-n$.

To generalize the second definition, consider

$$\mathbf{Pol}(A, n) = \mathrm{Hom}_A(\mathrm{Sym}_A(\mathbb{L}_A[n+1]), A),$$

the algebra of n-shifted polyvector fields, where $\mathbb{L}_A$ is the cotangent complex of A. It is equipped with a commutative multiplication, a Poisson bracket of degree $-n-1$ generalizing the Schouten bracket and an additional grading such that $\mathbb{L}_A$ has weight -1, i.e. $\mathbf{Pol}(A, n)$ is a graded $\mathbb{P}_{n+2}$-algebra. A bivector of the correct degree is a morphism of graded complexes $k(2)[-1] \to \mathbf{Pol}(A, n)[n+1]$ and the Jacobi identity can be succinctly summarized by assuming that the morphism $k(2)[-1] \to \mathbf{Pol}(A, n)[n+1]$ is a map of graded dg Lie algebras.

Therefore, we can give the following two definitions of an n-shifted Poisson structure on a cdga A:

(1) A carries a $\mathbb{P}_{n+1}$-algebra structure compatible with the original commutative structure on A.
(2) One has a morphism of graded dg Lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}(A, n)[n+1].$$

In fact, both definitions give a whole *space* $\mathrm{Pois}(A, n)$ of n-shifted Poisson structures. An equivalence of the two spaces is the main theorem of [20] (see also Theorem 4.5).

Shifted coisotropic structures

Now suppose $f\colon A \to B$ is a morphism of commutative algebras. One can generalize the classical definitions of coisotropic submanifolds to n-shifted coisotropic structures as follows.

To generalize the first definition, let us consider a certain colored dg operad $\mathbb{P}_{[n+1,n]}$ introduced in [24]. Its algebras are triples (A, B, F) consisting of a $\mathbb{P}_{n+1}$-algebra A, a $\mathbb{P}_n$-algebra B and a morphism of $\mathbb{P}_{n+1}$-algebras $F\colon A \to \mathrm{Z}(B)$. Here $\mathrm{Z}(B)$ is the so-called Poisson center and is given by twisting the differenital of $\mathbf{Pol}(B, n-1)$ by $[\pi_B, -]$. In particular, we obtain a morphism of commutative algebras $A \to \mathrm{Z}(B) \to B$. To establish a relation between this definition and the classical definition of coisotropic ideals, we show in Sect. 3.5 that the homotopy fiber $\mathrm{U}(A, B)$ of the morphism $A \to B$ carries a canonical structure of non-unital $\mathbb{P}_{n+1}$-algebra such that the projection $\mathrm{U}(A, B) \to A$ is a morphism of $\mathbb{P}_{n+1}$-algebras.

To generalize the second definition, consider the algebra of n-shifted relative polyvectors

$$\mathbf{Pol}(B/A, n-1) = \mathrm{Hom}_B(\mathrm{Sym}_B(\mathbb{L}_{B/A}[n-1]), B).$$

It is equipped with a graded $\mathbb{P}_{n+1}$-algebra structure as before. There is a natural morphism of commutative algebras

$$\mathbf{Pol}(A, n) \longrightarrow \mathbf{Pol}(B/A, n-1)$$

induced from the morphism $\mathbb{L}_{B/A} \to f^*\mathbb{L}_A[1]$ of cotangent complexes. In Sect. 4.2 we show that the pair $(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$ is naturally a graded $\mathbb{P}_{[n+2,n+1]}$-algebra and thus we obtain a graded non-unital $\mathbb{P}_{n+2}$-algebra $\mathbf{Pol}(f, n) = \mathrm{U}(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$. Thus, we can consider morphisms of graded dg Lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}(f, n)[n+1].$$

To see that this indeed generalizes the classical definition, consider the case when f is surjective which corresponds to a closed embedding of a subscheme. Then the morphism $\mathbf{Pol}(A, n) \to \mathbf{Pol}(B/A, n-1)$ is surjective as well and by Proposition 4.12 we can identify $\mathbf{Pol}(f, n)$ with its strict kernel. Thus, bivectors in $\mathbf{Pol}(f, n)$ are bivectors in $\mathbf{Pol}(A, n)$ which vanish when restricted to the normal bundle of B.

Therefore, we can make the following two definitions of an n-shifted coisotropic structure on a cdga morphism $f\colon A \to B$:

(1) One has a $\mathbb{P}_{[n+1,n]}$-algebra structure on (A, B) such that the induced morphism $A \to \mathrm{Z}(B) \to B$ of commutative algebras is homotopic to f.
(2) One has a morphism of graded dg Lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}(f, n)[n+1].$$

Let us denote the space of n-shifted coisotropic structures in the first sense by $\mathrm{Cois}(f, n)$. The following theorem is the first main result of the paper (Theorem 4.16):

Theorem *Let $f\colon A \to B$ be a morphism of commutative dg algebras. Then one has an equivalence of spaces*

$$\mathrm{Cois}(f, n) \cong \mathrm{Map}_{\mathrm{Alg}^{gr}_{Lie}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]).$$

Let us mention that our first definition was based on [3, Definition 3.4.4], which is given as follows. By a theorem proved independently by Rozenblyum and the second author (see [25, Theorem 2.22]) there is an equivalence of ∞-categories

$$\mathbf{Alg}_{\mathbb{P}_{n+1}} \xrightarrow{\sim} \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}) \tag{1}$$

beween $\mathbb{P}_{n+1}$-algebras and associative algebra objects in $\mathbb{P}_n$-algebras. Thus, one can define an n-shifted coisotropic structure in terms of an associative action of the $\mathbb{P}_{n+1}$-algebra A on the $\mathbb{P}_n$-algebra B. An equivalence of the two definitions is shown in [25, Corollary 3.8].

Braces and the Swiss-cheese operad

Let us briefly explain some intermediate constructions that go into the proof of Theorem 4.16 which can be of independent interest.

To prove the claim, we have to construct a cofibrant replacement of the colored operad $\mathbb{P}_{[n+1,n]}$, which is done as follows. First of all, we have to replace the center $\mathrm{Z}(B)$ by a homotopy center $\mathbf{Z}(B)$ which is defined as an operadic convolution algebra associated to the cooperad $\mathrm{co}\mathbb{P}_n$ Koszul dual to $\mathbb{P}_n$. A priori the homotopy center $\mathbf{Z}(B)$ associated to some cooperad $\mathcal{C}$ is merely a Lie algebra, but as shown by Calaque and Willwacher [4], it carries a canonical action of the so-called brace operad $\mathrm{Br}_{\mathcal{C}}$ if $\mathcal{C}$ is a Hopf cooperad, i.e. a cooperad in commutative algebras, which is the case for $\mathcal{C} = \mathrm{co}\mathbb{P}_n$. Moreover, they have constructed a morphism from a resolution of $\mathbb{P}_{n+1}$ to $\mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$. Thus, if B is a $\mathbb{P}_n$-algebra, $\mathbf{Z}(B)$ becomes a homotopy $\mathbb{P}_{n+1}$-algebra.

We can now generalize the problem of construction of a cofibrant replacement for $\mathbb{P}_{[n+1,n]}$ as follows. Let $\mathcal{C}_1$ be a dg cooperad and $\mathcal{C}_2$ a Hopf dg cooperad together with a morphism $\Omega(\mathcal{C}_1) \to \mathrm{Br}_{\mathcal{C}_2}$. In Sect. 3.3 we construct a certain semi-free colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ whose algebras are given by triples (A, B, F), where A is an $\Omega(\mathcal{C}_1)$-algebra, B is an $\Omega(\mathcal{C}_2)$-algebra and $F\colon A \to \mathbf{Z}(B)$ is an ∞-morphism of $\Omega(\mathcal{C}_1)$-algebras. In the case of Poisson algebras we get a cofibrant colored operad $\widetilde{\mathbb{P}}_{[n+1,n]}$ whose algebras are triples (A, B, F) consisting of a homotopy $\mathbb{P}_{n+1}$-algebra A, homotopy $\mathbb{P}_n$-algebra B and an ∞-morphism of homotopy $\mathbb{P}_{n+1}$-algebras $F\colon A \to \mathbf{Z}(B)$. Moreover, by Proposition 3.19 the natural morphism $\widetilde{\mathbb{P}}_{[n+1,n]} \to \mathbb{P}_{[n+1,n]}$ is a weak equivalence.

Let us mention why we call $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ the Swiss-cheese construction. Consider the colored operad $\mathrm{SC} = \mathrm{SC}(\mathrm{co}\mathbb{E}_2, \mathrm{coAss})$. Its algebras are triples (A, B, F) consisting of an $\mathbb{E}_2$-algebra A, an $\mathbb{E}_1$-algebra B and a morphism of $\mathbb{E}_2$-algebras $A \to \mathrm{HH}^\bullet(B, B)$, where $\mathrm{HH}^\bullet(B, B)$ is the Hochschild cochain complex endowed with its natural $\mathbb{E}_2$-structure. It is thus expected that SC is a model for the chain operad on the two-dimensional Swiss-cheese operad of Voronov (we refer to [26] for a similar statement in the topological setting).

We also generalize the construction of Calaque–Willwacher of the brace operad to a relative setting of a morphism of $\Omega\mathcal{C}$-algebras $A \to B$. In this case one can consider a triple of a convolution algebra of A, a convolution algebra of B and a relative convolution algebra. For instance, one can interpret $\mathbf{Pol}(B/A, n-1)$ as a relative convolution algebra of the morphism of graded $\mathbb{P}_{n+1}$-algebras $A \to B \to \mathbf{Pol}(B, n-1)$, where both A and B are equipped with the zero Poisson bracket. Therefore, the construction of the graded $\mathbb{P}_{[n+2,n+1]}$-algebra on the pair $(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$ follows from the action of the convolution algebra of A on the relative convolution algebra constructed in Proposition 3.9.

Poisson morphisms

Let us finish the introduction by stating the last main result of the paper. Recall that a morphism of Poisson manifolds $Y \to X$ is Poisson iff its graph $Y \to \overline{Y} \times X$ is coisotropic, where $\overline{Y}$ is the same manifold with the opposite Poisson structure. We show that a similar statement holds in the derived context as well. Namely, we have the following statement (see Theorem 4.21).

Theorem *Let A, B be $\mathbb{P}_{n+1}$-algebras, and let $f\colon A \to B$ be a morphism of commutative algebras. Then the space* $\mathrm{Pois}(f, n)$ *of lifts of f to a morphism of $\mathbb{P}_{n+1}$-algebras is equivalent to the space of coisotropic structures on $A \otimes \overline{B} \to B$, where $\overline{B}$ is the same algebra B with the opposite $\mathbb{P}_{n+1}$-structure.*

Here by a decomposable n-shifted coisotropic structure on $A \otimes B$ we mean one whose underlying n-shifted Poisson structure on $A \otimes B$ is obtained by a sum of n-shifted Poisson structures on A and B separately.

The main idea of the proof is to use the Poisson additivity functor (1) to reduce the statement to the following basic algebraic fact (see Lemma 4.20). Let A and B be associative algebras and consider the right B-module B_B. Then lifts of B_B to an (A, B)-bimodule are equivalent to specifying a morphism of associative algebras $A \to B$ since an (A, B)-bimodule ${}_AB_B$ gives rise to a morphism of associative algebras $A \to \mathrm{End}_{\mathbf{RMod}_B}(B) \cong B$.

1 Basic definitions

1.1 Model categories

Let k be a field of characteristic zero and let dg_k be the symmetric monoidal category of unbounded chain complexes of k-vector spaces. Moreover, it carries a projective model structure, whose weak equivalences are the quasi-isomorphisms and whose fibrations are degree-wise surjections. Together these structures are compatible in the sense that dg_k forms a symmetric monoidal model category.

Throughout the paper we are going to work in the setting of a general model category M following [3, Sect. 1.1], which the reader can safely assume to be the category of chain complexes dg_k or the category of diagrams in it. In this section we will list the necessary assumptions on the model category M.

Consider M, a closed symmetric monoidal combinatorial model category. In addiction to this, suppose M is enriched over dg_k, or, equivalently, M is a symmetric monoidal dg_k-model algebra in the sense of Hovey (see [14, Definition 4.2.20]). It is shown in [3, Appendix A.1.1] that such an M is a stable model category. Furthermore, we will make the following assumptions on M:

(1) The unit $\mathbf{1}_M$ of M is cofibrant.
(2) Suppose $f: A \to B$ is a cofibration and C an object of M. Then for any morphism $A \otimes C \to D$ the strict pushout of the diagram

$$\begin{tikzcd} A \otimes C \arrow[r] \arrow[d] & D \\ B \otimes C & \end{tikzcd}$$

is also a model for the homotopy pushout.
(3) If A is a cofibrant object, then the functor $A \otimes -: M \to M$ preserves weak equivalences.
(4) M is a tractable model category.
(5) Finite products and filtered colimits preserve weak equivalences.

We denote by $\mathcal{M}$ the localization of the model category M which is a symmetric monoidal dg category.

1.2 Graded mixed objects

Most of the results in this section can be found in [3, Sect. 1.1], so we will be brief; the reader is invited to consult the reference for details.

Let V be a complex and ϵ a square-zero endomorphism of V of degree 1 such that $[\mathrm{d}, \epsilon] = 0$. We can then twist the differential on V by ϵ, i.e. consider the same underlying graded vector space equipped with the differential $\mathrm{d} + \epsilon$. This construction does not preserve quasi-isomorphisms and so does not make sense in the underlying ∞-category. We use the realization functors of graded mixed objects as replacements for this construction: these preserve weak equivalences and make sense for any $\mathcal{M}$.

Let us recall the symmetric monoidal model categories M^{gr} and $M^{gr,\epsilon}$ of graded and graded mixed objects of M respectively. We denote by $\mathcal{M}^{gr}$ and $\mathcal{M}^{gr,\epsilon}$ the corresponding ∞-categories.

Definition 1.1 The category of *graded objects* in M

$$M^{gr} = \mathrm{CoMod}_{\mathcal{O}(\mathbf{G}_m)}(M)$$

is the category of comodules over $\mathcal{O}(\mathbf{G}_m) \cong k[t, t^{-1}]$ with $\deg(t) = 0$.

Explicitly, an object of M^{gr} consists of a collection $\{A(n)\}_{n \in \mathbb{Z}}$ of objects of M with tensor product defined by

$$(A \otimes B)(n) = \bigoplus_{n_1+n_2=n} A(n_1) \otimes B(n_2).$$

Note that the braiding isomorphism does not involve Koszul signs with respect to this grading. We will refer to this grading as the *weight* grading.

Now consider the commutative bialgebra $B = k[x, t, t^{-1}]$ with $\deg(x) = -1$, $\deg(t) = 0$, where $\Delta(t) = t \otimes t$ and $\Delta(x) = x \otimes 1 + t \otimes x$.

Definition 1.2 The category of *graded mixed objects* in M

$$M^{gr,\epsilon} = \mathrm{CoMod}_B(M)$$

is the category of comodules over B.

Explicitly, objects of $M^{gr,\epsilon}$ are graded objects $\{A(n)\}_{n\in\mathbb{Z}}$ together with operations

$$\epsilon\colon A(n) \to A(n+1)[1]$$

such that $\epsilon^2 = 0$. Equivalently, $M^{gr,\epsilon}$ is the category of comodules over $k[x]$ in M^{gr}, where the weight and degree of x are both -1. Since $k[x]$ is dualizable, $M^{gr,\epsilon}$ is equivalently the category of modules over $k[\epsilon]$ in M^{gr}, where the weight and degree of ϵ are both 1.

The category $M^{gr,\epsilon}$ has a projective model structure whose weak equivalences and fibrations are defined componentwise.

Consider the functor $\mathrm{triv}\colon \mathcal{M} \to \mathcal{M}^{gr,\epsilon}$ which associates to any object of $\mathcal{M}$ the same object with the trivial graded mixed structure. Let $\mathbf{1}_M(0)$ be the trivial graded mixed object concentrated in weight 0. Then $\mathrm{triv}(V) = V \otimes \mathbf{1}_M(0)$. It is naturally a symmetric monoidal functor.

Definition 1.3 The *realization functor*

$$|-|\colon \mathcal{M}^{gr,\epsilon} \longrightarrow \mathcal{M}$$

is the right adjoint to the trivial functor $\mathrm{triv}\colon \mathcal{M} \to \mathcal{M}^{gr,\epsilon}$.

Explicitly, $|A| = \underline{\mathrm{Map}}_{\mathcal{M}^{gr,\epsilon}}(\mathbf{1}_M(0), A)$, where $\underline{\mathrm{Map}}_{\mathcal{M}^{gr,\epsilon}}(-,-)$ is the $\mathcal{M}$-enriched Hom. The realization functor has the following strict model. $\mathbf{1}_M(0)$ has a cofibrant replacement in the projective model structure given by $\tilde{k} \otimes \mathbf{1}_M$ for $\tilde{k} = k[z, w]$, where $\deg(z) = 0$, $\deg(w) = 1$ and the weights of both z and w are 1. We define $\mathrm{d}z = w$ and $\epsilon z = wz$. The natural morphism $\tilde{k} \to k$ given by setting $z = w = 0$ is a weak equivalence. Assume $A \in M^{gr,\epsilon}$ is fibrant in the projective model structure. Then

$$|A| = \underline{\mathrm{Map}}_{M^{gr,\epsilon}}(\mathbf{1}_M(0), A) \cong \underline{\mathrm{Hom}}_{M^{gr,\epsilon}}(1_M \otimes \tilde{k}, A) \in M.$$

Post-composing the realization functor with the forgetful functor $M \to \mathrm{dg}_k$ we get the following description:

$$|A| \cong \prod_{n\geq 0} A(n)$$

with the differential given by twisting the original differential by ϵ. Since $|-|$ is a right adjoint to a symmetric monoidal functor, it naturally has a structure of a lax monoidal functor.

Definition 1.4 The *left realization* functor

$$|-|^l\colon \mathcal{M}^{gr,\epsilon} \to \mathcal{M}$$

is the left adjoint to the trivial functor $\mathrm{triv}\colon \mathcal{M} \to \mathcal{M}^{gr,\epsilon}$.

Explicitly, it is given by $A \mapsto (A \otimes_{k[\epsilon]} k)^{\mathbf{G}_m}$. We have a strict model of $|-|^l$ which is a functor $M^{gr,\epsilon} \to M$ given by $A \mapsto (A \otimes_{k[\epsilon]} \tilde{k})^{\mathbf{G}_m}$. In the case $M = \mathrm{dg}_k$ we have explicitly

$$|A|^l \cong \bigoplus_{n \leq 0} A(n)$$

with the differential given by twisting the original differential by ϵ.

We have the following statements about our strict models of the realization functors.

Proposition 1.5 *The realization functor $|-|\colon M^{gr,\epsilon} \to M$ preserves weak equivalences between fibrant objects.*

Proof The internal Hom $\underline{\mathrm{Hom}}_{M^{gr,\epsilon}}(-,-)$ is a Quillen bifunctor in the projective model structure. Therefore, $\underline{\mathrm{Hom}}_{M^{gr,\epsilon}}(\mathbf{1}_M \otimes \tilde{k}, -)$ preserves weak equivalences between fibrant objects. □

Proposition 1.6 *The left realization functor $|-|^l\colon M^{gr,\epsilon} \to M$ preserves weak equivalences.*

Proof By definition the functor on $A \in M^{gr,\epsilon}$ is given by

$$|A|^l = (A \otimes_{\mathbf{1}_M \otimes k[\epsilon]} (\mathbf{1}_M \otimes \tilde{k}))^{\mathbf{G}_m},$$

The functor of $\mathbf{G}_m$-invariants clearly preserves weak equivalences, so we just need to show that the functor

$$\mathrm{Mod}_{k[\epsilon]}(M^{gr}) \to M^{gr}$$

given by

$$A \mapsto A \otimes_{k[\epsilon]} \tilde{k}$$

preserves weak equivalences.

But $\tilde{k}$ is cofibrant as a $k[\epsilon]$-module and hence $\mathbf{1}_M \otimes \tilde{k}$ is flat over $\mathbf{1}_M \otimes k[\epsilon]$ by our assumptions on the model category. □

An important feature of the realization functors are the natural filtrations that they carry. For a graded mixed object $A \in M^{gr,\epsilon}$ we define $|A|^{\geq n}$ to be the realization of $A \otimes k(-n)$. Similarly, we define $|A|^{l,\leq n}$ to be the left realization of $A \otimes k(-n)$

Proposition 1.7 *Suppose $A \in M^{gr,\epsilon}$ is a graded mixed object. Then*

$$|A|^{\geq(n+1)} \to |A|^{\geq n} \to A(n)$$

is a cofiber sequence.

Similarly,

$$A(n) \to |A|^{l,\leq n} \to |A|^{l,\leq(n-1)}$$

is a fiber sequence.

Since $|-|^l$ is left adjoint to a symmetric monoidal functor, it is naturally an oplax symmetric monoidal functor.

Proposition 1.8 *Suppose $A_1, A_2 \in M^{gr,\epsilon}$ are two objects concentrated in non-positive degrees. Then the natural morphism*

$$|A_1 \otimes A_2|^l \to |A_1|^l \otimes |A_2|^l$$

is an isomorphism.

We denote by $M^{\leq 0} \subset M$ the full subcategory of objects concentrated in non-positive weights and similarly for $M^{\leq 0,\epsilon} \subset M^{gr,\epsilon}$. The previous Proposition implies that we have a well-defined functor

$$|-|^l \colon \mathrm{CAlg}(M^{\leq 0,\epsilon}) \to \mathrm{CAlg}(M).$$

We also have a functor of Tate realization which combines both left and ordinary realizations.

Definition 1.9 The *Tate realization functor*

$$|-|^t \colon \mathcal{M}^{gr,\epsilon} \longrightarrow \mathcal{M}$$

is defined to be

$$|A|^t = \operatorname*{colim}_{i\geq 0} |A \otimes k(-i)|.$$

Finally, we will need a certain weakening of the notion of a graded mixed object. By definition the data of a mixed structure on a graded object $A \in M^{gr}$ boils down to a morphism of graded Lie algebras

$$\epsilon_A \colon k(2)[-1] \longrightarrow \mathrm{Hom}_{M^{gr}}(A, A).$$

Replacing strict morphisms by ∞-morphisms we arrive at the following definition:

Definition 1.10 A *weak graded mixed object* is a graded object $A \in M^{gr}$ equipped with endomorphisms $\epsilon_1, \epsilon_2, \dots$ of A where ϵ_i has degree 1 and weight i such that

$$(\mathrm{d} + \epsilon_1 + \epsilon_2 + \cdots)^2 = 0.$$

Note that graded mixed objects correspond to the case where $\epsilon_i = 0$ for $i > 1$. Similarly, we can define ∞-morphisms of weak graded mixed objects:

Definition 1.11 An ∞-*morphism* of weak graded mixed objects $f : A \to B$ is given by a collection of maps $f_0, f_1, \dots : A \to B$, where f_i has degree 0 and weight i such that for $f = f_0 + f_1 + \cdots$ we have

$$f \circ (\mathrm{d} + \epsilon_1^A + \cdots) = (\mathrm{d} + \epsilon_1^B + \cdots) \circ f.$$

It is obvious that weak graded mixed objects form a category whose localization is equivalent to $\mathcal{M}^{gr,\epsilon}$, see [3, Sect. 3.3.4].

1.3 Operads

Our conventions about operads follow those of [6] and [17]. All operads we consider are operads in chain complexes.

Recall that a *symmetric sequence* V is a sequence of chain complexes $V(n) \in \mathrm{dg}_k$ together with an action of S_n on $V(n)$. The category of symmetric sequences is monoidal with respect to the composition product and an operad is an algebra in the category of symmetric sequences. We denote the category of operads as Op_k.

We denote by $\mathrm{Tree}_m(n)$ the groupoid of planar trees with labeled n incoming edges and m vertices. The morphisms are not necessarily planar isomorphisms between trees. For instance, the groupoid $\mathrm{Tree}_2(n)$ has components parametrized by $(p, n-p)$-shuffles σ for any p, where a shuffle σ corresponds to the tree $\mathbf{t}_\sigma$ as shown in Fig. 1.

We will also be interested in the set $\mathrm{Isom}_{\pitchfork}(n, r)$ of *pitchforks* with n incoming edges and $r+1$ vertices, see Fig. 2 for an example and [8, Sect. 2] for more details. The groupoid $\mathrm{Tree}_3(n)$ has trees of two kinds: pitchforks in $\mathrm{Isom}_{\pitchfork}(n, 2)$ and the complement $\mathrm{Tree}_3^0(n)$.

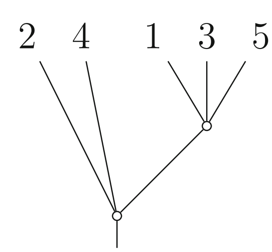
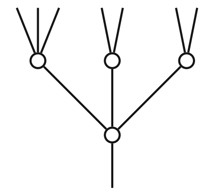


Fig. 1 The tree $\mathbf{t}_\sigma$ corresponding to a (2, 3)-shuffle σ

Fig. 2 A pitchfork in $\mathrm{Isom}_{\pitchfork}(7,3)$



Given a tree $\mathbf{t} \in \mathrm{Tree}_m(n)$ and a symmetric sequence $\mathcal{O}$ we define $\mathcal{O}(\mathbf{t})$ to be the tensor product

$$\mathcal{O}(\mathbf{t}) = \bigotimes_i \mathcal{O}(n_i)$$

where the tensor product is over the vertices of $\mathbf{t}$ and n_i is the number of incoming edges at vertex i.

Given an operad $\mathcal{O}$, a tree $\mathbf{t} \in \mathrm{Tree}_m(n)$ defines a multiplication map

$$m_{\mathbf{t}} \colon \mathcal{O}(\mathbf{t}) \to \mathcal{O}(n).$$

Similarly, for a cooperad $\mathcal{C}$ we have a comultiplication map

$$\Delta_{\mathbf{t}} \colon \mathcal{C}(n) \to \mathcal{C}(\mathbf{t}).$$

All cooperads are assumed to be coaugmented and conilpotent and we denote by $\mathcal{C} \cong \overline{\mathcal{C}} \oplus \mathbf{1}$ the natural splitting.

Given a symmetric sequence $\mathcal{P}$, the free operad $\mathrm{Free}(\mathcal{P})$ has operations parametrized by trees $\mathbf{t}$ whose vertices are labeled by operations in $\mathcal{P}$. Given a cooperad $\mathcal{C}$ we define its cobar complex $\Omega\mathcal{C}$ to be the free operad on $\overline{\mathcal{C}}[-1]$. The differential on the generators $X \in \overline{\mathcal{C}}(n)[-1]$ is given by

$$\mathrm{d}X = -\mathbf{s}\mathrm{d}_1(\mathbf{s}^{-1}X) - \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(\mathbf{s}^{-1}X)) \tag{2}$$

where d_1 is the differential on the symmetric sequence $\mathcal{C}$.

The following lemma is standard, and its proof can be found for example in [17, Proposition 6.5.6].

Lemma 1.12 *The cobar differential* d *on* $\Omega\mathcal{C}$ *squares to zero.*

We will also need a slight generalization of the above construction.

Definition 1.13 A *curved cooperad* is a dg cooperad $\mathcal{C}$ together with a morphism of symmetric sequences $\theta \colon \mathcal{C} \to \mathbf{1}[2]$ such that

$$(\theta \otimes \mathrm{id}_{\mathcal{C}} - \mathrm{id}_{\mathcal{C}} \otimes \theta)\Delta(x) = 0$$

for any $x \in \mathcal{C}$.

Note that this definition is slightly more restrictive than the corresponding notion in [13], but it will suffice for our purposes. Given a curved cooperad $\mathcal{C}$ we can also consider its cobar complex $\Omega\mathcal{C}$, we refer to [13, Sect. 3.3.6] for explicit formulas for the differentials on $\Omega\mathcal{C}$.

Given an operad $\mathcal{O}$ and a complex A, we define the free $\mathcal{O}$-algebra on A to be

$$\mathcal{O}(A) = \bigoplus_n (\mathcal{O}(n) \otimes A^{\otimes n})_{S_n}.$$

Similarly, for a cooperad $\mathcal{C}$ and a complex A, we define the cofree conilpotent $\mathcal{C}$-coalgebra on A to be

$$\mathcal{C}(A) = \bigoplus_n (\mathcal{C}(n) \otimes A^{\otimes n})^{S_n}.$$

We will also be interested in colored symmetric sequences and colored operads. Let $\mathcal{V}$ be a set. A $\mathcal{V}$-colored symmetric sequence is a collection of complexes $\mathcal{V}(v_1^{\otimes n_1} \otimes \cdots \otimes v_m^{\otimes n_m}, v_0)$ for every collection of elements $v_0, v_1, \ldots, v_m \in \mathcal{V}$ together with an action of $S_{n_1} \times \cdots \times S_{n_m}$. As before, the category of $\mathcal{V}$-colored symmetric sequences has a composition product and a $\mathcal{V}$-colored operad is defined to be an algebra object in the category of $\mathcal{V}$-colored symmetric sequences. We denote the category of colored operads by $\mathcal{V}\mathrm{Op}_k$; in particular, if the set of colors has two elements, we denote it by $2\mathrm{Op}_k$.

The following theorem is due to Hinich [12] in the uncolored case; the colored case is treated in [1,2] and [22].

Theorem 1.14 *The category of (colored) dg operads $\mathcal{V}\mathrm{Op}_k$ has a model structure which is transferred from the model structure on $\mathcal{V}$-colored symmetric sequences by the free-forgetful adjunction.*

Considering the coradical filtration on a conilpotent cooperad $\mathcal{C}$, one has the following:

Proposition 1.15 *Let $\mathcal{C}$ be a conilpotent (colored) cooperad. Then $\Omega\mathcal{C}$ is cofibrant.*

Here are our main examples of operads and cooperads:

- If A is an object of M, End_A is a dg operad with $\mathrm{End}_A(n) = \mathrm{Hom}_M(A^{\otimes n}, A)$. Similarly, if we have a pair of objects $A, B \in M$, then $\mathrm{End}_{A,B}$ is a $\{\mathcal{A}, \mathcal{B}\}$-colored dg operad with

$$\mathrm{End}_{A,B}(\mathcal{A}^{\otimes n} \otimes \mathcal{B}^{\otimes m}, \mathcal{A}) = \mathrm{Hom}(A^{\otimes n} \otimes B^{\otimes m}, A)$$

  and similarly for $\mathrm{End}_{A,B}(-, \mathcal{B})$.
- Comm is the operad of unital commutative algebras, Lie is the operad of Lie algebras. $\mathbb{P}_n$ is the operad of unital shifted Poisson algebras with the commutative multiplication of degree 0 and the Poisson bracket of degree $1-n$. We denote by $\mathrm{Comm}^{\mathrm{nu}}$ and $\mathbb{P}_n^{\mathrm{nu}}$ the non-unital versions of the operads Comm and $\mathbb{P}_n$.

- The operads Lie and $\mathbb{P}_n$ can also be upgraded to operads in graded complexes where we set the weight of the bracket to be -1 and the weight of the multiplication to be zero.
- coComm is the cooperad of non-counital cocommutative coalgebras, coLie is the cooperad of Lie coalgebras. $\mathrm{co}\mathbb{P}_n$ is the cooperad of non-counital shifted Poisson coalgebras with the cocommutative comultiplication of degree 0 and the Poisson cobracket of degree $1-n$. We denote by $\mathrm{coComm}^{\mathrm{cu}}$ and $\mathrm{co}\mathbb{P}_n^{\mathrm{cu}}$ the counital versions of the cooperads coComm and $\mathrm{co}\mathbb{P}_n$.
- If $\mathcal{O}$ is a symmetric sequence, we denote by $\mathcal{O}\{n\}$ the symmetric sequence defined by

$$\mathcal{O}\{n\}(m) = \mathcal{O}(m) \otimes \mathrm{sgn}_m^{\otimes n}[n(m-1)],$$

where sgn_m is the sign representation of S_m. If $\mathcal{O}$ is a (co)operad, then so is $\mathcal{O}\{n\}$. For instance, $\mathrm{Lie}\{n\}$ is the operad of Lie algebras with bracket of degree $-n$.

Given a dg operad $\mathcal{O}$, we can consider $\mathcal{O}$-algebras in M. We denote the corresponding category by $\mathrm{Alg}_{\mathcal{O}}(M)$ and its localization by $\mathbf{Alg}_{\mathcal{O}}(\mathcal{M})$. We also introduce shorthands

$$\mathrm{Alg}_{\mathrm{Lie}}^{gr} = \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{dg}_k^{gr}), \qquad \mathrm{Alg}(M) = \mathrm{Alg}_{\mathrm{Ass}}(M).$$

1.4 Lie algebras

Given a nilpotent L_∞ algebra $\mathfrak{g}$, the set of Maurer–Cartan elements is defined to be the set of degree 1 elements $x \in \mathfrak{g}$ such that

$$\mathrm{d}x + \sum_{n\geq 2} \frac{1}{n!}[x, \ldots, x]_n = 0.$$

When there is no confusion, we will omit the subscript on the bracket. Similarly, if $\mathfrak{g}$ is a curved nilpotent L_∞ algebra with curvature θ, the Maurer–Cartan equation is

$$\theta + \mathrm{d}x + \sum_{n\geq 2} \frac{1}{n!}[x, \ldots, x]_n = 0.$$

Let $\Omega_\bullet$ be the cosimplicial commutative algebra of polynomial differential forms on simplices. For instance, $\Omega_0 = k$ and $\Omega_1 = k[x, y]$ with $\deg(x) = 0$, $\deg(y) = 1$ and $\mathrm{d}x = y$.

Definition 1.16 The *space of Maurer–Cartan elements* $\underline{\mathrm{MC}}(\mathfrak{g})$ is the simplicial set of Maurer–Cartan elements in $\mathfrak{g} \otimes \Omega_\bullet$.

Now suppose that $\mathfrak{g}$ is a (curved) L_∞ algebra equipped with a decreasing filtration $\mathfrak{g} = \mathfrak{g}^a \supset \mathfrak{g}^{a+1} \supset \cdots$ such that $[\mathfrak{g}^{a_1}, \ldots, \mathfrak{g}^{a_n}]_n \subset \mathfrak{g}^{\sum a_i + 1 - n}$ and such that the quotients $\mathfrak{g}/\mathfrak{g}^a$ are nilpotent. We call such filtrations *admissible*. For an L_∞ algebra $\mathfrak{g}$ equipped with such an admissible filtration, we define

$$\underline{\mathrm{MC}}(\mathfrak{g}) = \lim_a \underline{\mathrm{MC}}(\mathfrak{g}/\mathfrak{g}^a).$$

Definition 1.17 Let $\mathfrak{g}$ be a (curved) L_∞ algebra and $x \in \mathfrak{g}$ a Maurer–Cartan element. The *L_∞ algebra $\mathfrak{g}$ twisted by x* has the same underlying graded vector space; the brackets are defined by

$$[x_1, \dots, x_n]_n = \sum_{k\geq 0} \frac{1}{k!}[x, \dots, x, x_1, \dots, x_n]_{n+k}.$$

Lemma 1.18 *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two admissible filtered L_∞ algebras with a pair of morphisms $p\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $i\colon \mathfrak{g}_2 \to \mathfrak{g}_1$ such that $p \circ i = \mathrm{id}_{\mathfrak{g}_2}$. Then the homotopy fiber of*

$$\underline{\mathrm{MC}}(\mathfrak{g}_1) \to \underline{\mathrm{MC}}(\mathfrak{g}_2)$$

at a Maurer–Cartan element $x \in \mathfrak{g}_2$ is equivalent to the space of Maurer–Cartan elements in the L_∞ algebra $\ker p$ *twisted by the element $i(x)$.*

Proof By assumption the morphism $p\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is surjective. By [27, Theorem 3.2] the induced morphism $\underline{\mathrm{MC}}(\mathfrak{g}_1) \to \underline{\mathrm{MC}}(\mathfrak{g}_2)$ is a fibration of simplicial sets. Hence the homotopy fiber is equivalent to the strict fiber, i.e. the inverse limit of fibers of $\underline{\mathrm{MC}}(\mathfrak{g}_1/\mathfrak{g}_1^n) \to \underline{\mathrm{MC}}(\mathfrak{g}_2/\mathfrak{g}_2^n)$.

The set of m-simplices of the strict fiber of $\underline{\mathrm{MC}}(\mathfrak{g}_1/\mathfrak{g}_1^n) \to \underline{\mathrm{MC}}(\mathfrak{g}_2/\mathfrak{g}_2^n)$ at $x \in \mathfrak{g}_2$ is isomorphic to the set of Maurer-Cartan elements $y \in \mathfrak{g}_1 \otimes \Omega_m$ such that $p(y) = x$. Using i we can identify

$$\mathfrak{g}_1 \cong \mathfrak{g}_2 \oplus \ker p$$

as filtered L_∞ algebras.

Therefore, the set of m-simplices of the strict fiber consists of elements $y_0 \in \ker p$ satisfying the equation

$$\theta + \mathrm{d}(y_0 + i(x)) + \sum_{n\geq 2} \frac{1}{n!}[y_0 + i(x), \dots, y_0 + i(x)]_n.$$

Let us now expand this equation. The term not involving y_0 is

$$\theta + \mathrm{d}i(x) + \sum_{n\geq 2} \frac{1}{n!}[i(x), \dots, i(x)]_n = 0$$

by the Maurer–Cartan equation for $i(x)$. Therefore, we get the equation

$$\mathrm{d}y_0 + \sum_{n\geq 2} \frac{1}{(n-1)!}[i(x), \ldots, i(x), y_0]_n$$
$$+ \sum_{n\geq 2} \frac{1}{n!} \sum_{k\geq 0} \frac{n!}{k!(n-k)!}[i(x), \ldots, i(x), y_0, \ldots, y_0]_n,$$

i.e. the Maurer–Cartan equation in the L_∞ algebra $\ker p$ twisted by $i(x)$. □

Recall that the operad Lie is an operad in graded complexes where the weight of the bracket is -1. The operad of non-curved L_∞-algebras can also be enhanced to an operad in graded complexes by assigning weight $1-n$ to the bracket $[-, \ldots, -]_n$. Alternatively, one can consider the coComm{1} as a graded cooperad with coproduct of weight 1 and define the L_∞ operad as $\Omega(\mathrm{coComm}\{1\})$.

Given a graded L_∞ algebra $\mathfrak{g}$ we denote by

$$\mathfrak{g}^{\geq m} = \prod_{n\geq m} \mathfrak{g}(n)$$

its completion in weights $\geq m$. Then $\mathfrak{g}^{\geq 2}$ carries an admissible filtration

$$\mathfrak{g}^{\geq 2} \supset \mathfrak{g}^{\geq 3} \supset \cdots$$

A version of the following statement was proved by the first author in [20, Sect. 4]. Let $k(2)[-1]$ be the trivial graded L_∞ algebra in degree 1 and weight 2.

Proposition 1.19 *Let $\mathfrak{g}$ be a graded L_∞ algebra in dg_k. There is an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{L}_\infty}}(k(2)[-1], \mathfrak{g}) \cong \underline{\mathrm{MC}}(\mathfrak{g}^{\geq 2}).$$

Similarly, if $\mathfrak{g}$ is a graded dg Lie algebra, then there is an equivalence of spaces

$$\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathfrak{g}) \cong \underline{\mathrm{MC}}(\mathfrak{g}^{\geq 2}).$$

Proof Recall that in the model category of L_∞ algebras every object is fibrant, so we just need to find a cofibrant replacement L for $k(2)[-1]$.

By [17, Lemma 6.5.14] the symmetric sequence $L_\infty \circ_\alpha \mathrm{coComm}\{1\}$ equipped with the Koszul differential α is quasi-isomorphic to the unit symmetric sequence, so the natural morphism

$$L_\infty(\overline{\mathrm{Sym}}_\bullet(V[1])[-1]) \to V$$

is a quasi-isomorphism for any complex V. Here $\overline{\mathrm{Sym}}_\bullet$ is the reduced symmetric algebra and the free L_∞-algebra $L_\infty(\overline{\mathrm{Sym}}_\bullet(V[1])[-1])$ is equipped with the Koszul differential α.

Let $V = k(2)[-1]$, then $\overline{\mathrm{Sym}}_{\bullet}(V[1])[-1] = \mathrm{span}\{p_2, p_3, \dots\}$, where p_n has weight n and degree 1. Let $p = \sum_{i=2}^{\infty} p_i$. Then the Koszul differential gives

$$\mathrm{d}p + \sum_{n=2}^{\infty}[p, \dots, p] = 0.$$

Therefore, if we define L_0 to be the free graded L_∞ algebra on the generators $p_2, p_3, \dots$ equipped with the differential as above, then the natural morphism $L_0 \to k(2)[-1]$ given by projecting on p_2 is a quasi-isomorphism. Moreover, by construction L_0 is cofibrant.

Therefore, one has equivalences of spaces

$$\begin{aligned}\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{L}_\infty}}(k(2)[-1], \mathfrak{g}) &\cong \mathrm{Hom}_{\mathrm{Alg}^{gr}_{\mathrm{L}_\infty}, \bullet}(L_0, \mathfrak{g}) \\ &\cong \mathrm{Hom}_{\mathrm{Alg}^{gr}_{\mathrm{L}_\infty}}(L_0, \mathfrak{g} \otimes \Omega_\bullet).\end{aligned}$$

The latter Hom is easy to compute and it exactly gives the set of Maurer–Cartan elements in $\mathfrak{g}^{\geq 2} \otimes \Omega_\bullet$. □

2 Operadic resolutions

In this section we collect some useful results which describe spaces of $\mathcal{O}$-algebra structures for an operad $\mathcal{O}$ and spaces of $\mathcal{O}$-algebra morphisms.

2.1 Deformations of algebras

Let $\mathcal{C}$ be a coaugmented dg cooperad so that we can split $\mathcal{C} \cong \overline{\mathcal{C}} \oplus \mathbf{1}$. For an operad $\mathcal{P}$ we introduce the convolution algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$, a dg Lie algebra, as follows. As a complex it is defined to be

$$\mathrm{Conv}(\mathcal{C}, \mathcal{P}) = \prod_n \mathrm{Hom}_{S_n}(\overline{\mathcal{C}}(n), \mathcal{P}(n)).$$

For brevity we denote

$$\mathrm{Conv}(\mathcal{C}; A) = \mathrm{Conv}(\mathcal{C}, \mathrm{End}_A).$$

We introduce a pre-Lie structure on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ by

$$(f \bullet g)(X) = \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} \mu_{\mathbf{t}}((f \otimes g)\Delta_{\mathbf{t}}(X))$$

for any $f, g \in \mathrm{Conv}(\mathcal{C}, \mathcal{P})$ and $X \in \mathcal{C}(n)$. The Lie bracket is defined to be

$$[f, g] = f \bullet g - (-1)^{|f||g|} g \bullet f.$$

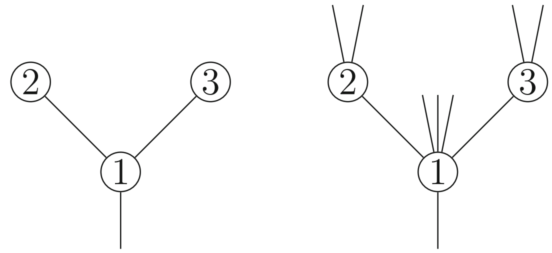


Fig. 3 A rooted tree **t** and an element of $\mathrm{Tree}_3(\mathbf{t}, 7)$

Here is a more explicit description of the pre-Lie structure. Recall that operations of arity m in the pre-Lie operad are parametrized by rooted trees with m vertices [5]. Given a rooted tree $\mathbf{t}$ we denote by $\mathrm{Tree}_m(\mathbf{t}, n)$ the groupoid of trees obtained by attaching incoming edges at each vertex of the tree $\mathbf{t}$ such that the total number of incoming edges is n (Fig. 3). The action of the rooted tree $\mathbf{t}$ on the elements $f_1, \ldots, f_m \in \mathrm{Conv}(\mathcal{C}; A)$ is given by the sum over the trees $\mathbf{t}' \in \pi_0(\mathrm{Tree}_m(\mathbf{t}, n))$ where each term in the sum is given by the composition associated to the tree $\mathbf{t}'$ with vertices labeled by the elements f_i.

For example, consider the tree $\mathbf{t}$ defining the pre-Lie bracket. Then $\mathrm{Tree}_2(\mathbf{t}, n) \cong \mathrm{Tree}_2(n)$ whose connected components are parametrized by trees $\mathbf{t}_\sigma$ associated to the shuffles $\sigma \in S_{p,n-p}$. Therefore,

$$(f \bullet g)(X; a_1, \ldots, a_n) = \sum_{p=0}^{n} \sum_{\sigma \in S_{p,n-p}} \pm f(X^{\mathbf{t}_\sigma}_{(1)}; \\ g(X^{\mathbf{t}_\sigma}_{(2)}; a_{\sigma(1)}, \ldots, a_{\sigma(p)}), a_{\sigma(p+1)}, \ldots, a_{\sigma(n)}), \tag{3}$$

where the sign arises from the permutation of $\{f, g, X^{\mathbf{t}_\sigma}_{(1)}, X^{\mathbf{t}_\sigma}_{(2)}, a_1, \ldots, a_n\}$ and the tree $\mathbf{t}_\sigma$ is the tree corresponding to the shuffle σ.

If $\mathcal{P}$ is an operad in M, we can enhance $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ to a Lie algebra in M by considering the internal Hom in M and similarly for $\mathrm{Conv}(\mathcal{C}; A)$ in the case A is an object of M.

If $\mathcal{C}$ is a graded cooperad, we introduce a graded Lie algebra structure on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ by considering the internal grading on $\mathcal{C}$ and putting $\mathcal{P}$ in weight 1. In this way $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ acquires a Lie structure of weight -1. Note that graded morphisms $\Omega\mathcal{C} \to \mathcal{P}$ give rise to elements of $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ which are pure of weight 1.

Finally, if $\mathcal{C}$ is a curved cooperad, we obtain a curved Lie algebra structure on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ as follows. The curving on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ is the weight 1, degree 2 element obtained as a composite

$$\mathcal{C}(1) \xrightarrow{\theta} k[2] \to \mathcal{P}(1)[2],$$

where the second morphism is the unit morphism in $\mathcal{P}$.

The following statement is proved by considering the coradical filtration on $\mathcal{C}$:

Lemma 2.1 *Suppose $\mathcal{C}$ is a conilpotent (curved) cooperad. Then* $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ *is pronilpotent.*

Using this Lemma we have the following statement.

Proposition 2.2 *Assume $\mathcal{O}$ is an operad with a weak equivalence $\Omega\mathcal{C} \xrightarrow{\sim} \mathcal{O}$, where $\mathcal{C}$ is a conilpotent (curved) cooperad. Then we have an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{Op}_k}(\mathcal{O}, \mathcal{P}) \cong \underline{\mathrm{MC}}(\mathrm{Conv}(\mathcal{C}, \mathcal{P})).$$

Proof Note that since $\mathcal{C}$ is conilpotent, $\Omega\mathcal{C}$ is cofibrant. We have a sequence of equivalences of spaces

$$\mathrm{Map}_{\mathrm{Op}_k}(\mathcal{O}, \mathcal{P}) \cong \mathrm{Map}_{\mathrm{Op}_k}(\Omega\mathcal{C}, \mathcal{P}) \cong \underline{\mathrm{Hom}}_{\mathrm{Op}_k,\bullet}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\mathrm{Op}_k}(\Omega\mathcal{C}, \mathcal{P} \otimes \Omega_\bullet).$$

An operad morphism $f\colon \Omega\mathcal{C} \to \mathcal{P}$ is uniquely specified by a degree 0 map of symmetric sequences $f_0\colon \overline{\mathcal{C}}[-1] \to \mathcal{P}$ satisfying the equation

$$\begin{aligned}
\mathrm{d}(f_0(\mathbf{s}X)) &= f(\mathrm{d}\mathbf{s}X)\\
&= f\left(-\mathbf{s}\mathrm{d}X - \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (\mathbf{s}\otimes\mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(X))\right)\\
&= -f_0(\mathbf{s}\mathrm{d}X) - f\left(\sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (\mathbf{s}\otimes\mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(X))\right)
\end{aligned}$$

for any $\mathbf{s}X \in \mathcal{C}_\circ(n)[-1]$. Since f is a morphism of operads, the last term can also be written in terms of f_0, so we obtain

$$\mathrm{d}(f_0(\mathbf{s}X)) + f_0(\mathbf{s}\mathrm{d}X) + \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} \mu_{\mathbf{t}}((f_0\mathbf{s}\otimes f_0\mathbf{s})\Delta_{\mathbf{t}}(X)) = 0.$$

Finally, identifying degree 0 maps $f_0\colon \overline{\mathcal{C}}[-1] \to \mathcal{P}$ with degree 1 maps $f_0\mathbf{s}\colon \overline{\mathcal{C}} \to \mathcal{P}$ we get exactly the Maurer–Cartan equation in $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$. Since the simplicial set of Maurer–Cartan elements in a dg Lie algebra $\mathfrak{g}$ is defined to be the set of Maurer–Cartan elements in $\mathfrak{g}\otimes\Omega_\bullet$ and $\mathrm{Conv}(\mathcal{C}, \mathcal{P}\otimes\Omega_\bullet) \cong \mathrm{Conv}(\mathcal{C}, \mathcal{P})\otimes\Omega_\bullet$, we obtain an equivalence of spaces

$$\mathrm{Map}_{\mathrm{Op}_k}(\mathcal{O}, \mathcal{P}) \cong \underline{\mathrm{MC}}(\mathrm{Conv}(\mathcal{C}, \mathcal{P})).$$

□

We will also need a variant of the convolution algebra

$$\mathrm{Conv}^0(\mathcal{C}, \mathcal{P}) = \prod_n \mathrm{Hom}_{S_n}(\mathcal{C}(n), \mathcal{P}(n))$$

which does not use a coaugmentation on $\mathcal{C}$.

The Lie bracket on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ extends to one on $\mathrm{Conv}^0(\mathcal{C}, \mathcal{P})$. Note that in contrast to $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$, the Lie algebra $\mathrm{Conv}^0(\mathcal{C}, \mathcal{P})$ is not pronilpotent.

2.2 Harrison complex

Suppose A is a commutative algebra in M. We will begin by constructing a canonical resolution of A, i.e. a graded mixed commutative algebra A^ϵ, free as a graded commutative algebra, together with a weak equivalence $|A^\epsilon|^l \xrightarrow{\sim} A$. For this we will use the canonical cobar-bar resolution, the reader is referred to [17, Sect. 11.2] for details.

Let coLie^θ be the cooperad of curved Lie coalgebras. Then as in [13, Sect. 6.1] one can construct a weak equivalence of operads

$$\Omega(\mathrm{coLie}^\theta\{1\}) \to \mathrm{Comm}.$$

The cooperad coLie^θ admits a weight grading where we set the weights of comultiplication and curving to be -1. Under this grading coLie^θ is concentrated in non-positive weights. Similarly, Comm has a filtration by the number of generating operations. Moreover, the morphism $\Omega(\mathrm{coLie}^\theta\{1\}) \to \mathrm{Comm}$ is compatible with the filtrations.

We define $A^\epsilon = \mathrm{Sym}(\mathrm{coLie}^\theta\{1\}(A)) = \mathrm{Comm} \circ \mathrm{coLie}^\theta\{1\} \circ A$ as a commutative algebra, where $\circ$ is the composition product of symmetric sequences with A considered as a symmetric sequence in arity 0. We define the grading on A^ϵ using the grading on $\mathrm{coLie}^\theta\{1\}$. The mixed structure on A^ϵ consists of two terms and coincides with the differential on the cobar-bar resolution for which we refer to [13, Sect. 5.2].

The counit morphism $\mathrm{coLie}^\theta\{1\} \to \mathbf{1}$ induces a morphism

$$\mathrm{coLie}^\theta\{1\}(A) \to A$$

which defines a morphism of graded commutative algebras

$$\mathrm{Sym}(\mathrm{coLie}^\theta\{1\}(A)) \to A.$$

It is easy to see that it is compatible with the mixed structure, hence one obtains a morphism

$$f\colon |A^\epsilon|^l \to A.$$

The following statement is [17, Theorem 11.3.6] and [16, Theorem 15]:

Proposition 2.3 *Let A be a commutative algebra in M cofibrant as an object of M. Then the morphism $f\colon |A^\epsilon|^l \to A$ is a weak equivalence.*

Lemma 2.4 *Let A be a commutative algebra in M cofibrant as an object of M. Then $|A^\epsilon|^l$ is a cofibrant commutative algebra.*

Proof Consider a filtration on the resolution A^ϵ as follows. Let A^ϵ_n be the symmetric algebra on $\mathrm{coLie}^\theta\{1\}(A)$ in weights at least $-n$; the mixed structure on A^ϵ restricts to one on A^ϵ_n. In particular, $A^\epsilon_0 = \mathrm{Sym}(A)$ with the trivial mixed structure.

Since A is cofibrant as an object of M, the object

$$\bigoplus_{0\leq m\leq n} (\mathrm{coLie}^{\theta}\{1\}(m) \otimes A^{\otimes m})^{S_m}$$

is cofibrant as well. $|A^{\epsilon}|^{l}$ is given as the colimit of the direct system

$$|A^{\epsilon}_0|^{l} \to |A^{\epsilon}_1|^{l} \to \cdots,$$

but each arrow is a cofibration of commutative algebras and hence $|A^{\epsilon}|^{l} \in \mathrm{CAlg}(M)$ is cofibrant. □

Lemma 2.5 *Let* $B \in \mathrm{CAlg}(M^{\leq 0,\epsilon})$ *be a graded mixed commutative algebra in* M *concentrated in non-positive weights. We have an isomorphism*

$$|\Omega^1_B|^{l} \cong \Omega^1_{|B|^{l}}$$

of $|B|^{l}$*-modules, where on the left we use the functor*

$$|-|^{l}\colon \mathrm{Mod}_B(M^{gr,\epsilon}) \to \mathrm{Mod}_{|B|^{l}}(M).$$

Proof Recall the following explicit construction of the module of Kähler differentials. Let A be a commutative algebra in M.

Denote by $i\colon \mathrm{Sym}_2(A) \to A \otimes A$ the space of S_2-invariants. We denote $m_s\colon \mathrm{Sym}_2(A) \to A$ the multiplication map given by $m_s = \frac{1}{2} m \circ i$. The module of Kähler differentials $\Omega^1_A \in \mathrm{Mod}_A(M)$ is defined to be the coequalizer of A-modules

$$\Omega^1_A = \mathrm{coeq}\left(A \otimes \mathrm{Sym}_2(A) \underset{\mathrm{id}\otimes m_s}{\overset{(m\otimes \mathrm{id})\circ(\mathrm{id}\otimes i)}{\rightrightarrows}} A \otimes A \right),$$

where $A \otimes \mathrm{Sym}_2(A)$ and $A \otimes A$ are the free A-modules on $\mathrm{Sym}_2(A)$ and A respectively.

Since $|-|^{l}$ is a left adjoint, it preserves colimits and therefore the claim follows from the explicit description of the module of Kähler differentials given above. □

We are now going to define the Harrison chain and cochain complexes for a commutative algebra $A \in \mathrm{CAlg}(M)$ which will be certain graded mixed A-modules whose realizations represent the cotangent and tangent complex respectively.

Define

$$\mathrm{Harr}_{\bullet}(A, A) = A \otimes \mathrm{coLie}\{1\}(A) \cong \Omega^1_{A^{\epsilon}} \otimes_{A^{\epsilon}} A \in \mathrm{Mod}_A(M^{gr,\epsilon})$$

with the mixed structure coming from the one on $\Omega^1_{A^{\epsilon}}$. By construction we have a morphism of graded mixed A-modules

$$\mathrm{Harr}_{\bullet}(A, A) \to \Omega^1_A$$

given by

$$f \otimes g \mapsto f \mathrm{d}_{\mathrm{dR}} g$$

in weight 0, where we consider the trivial graded mixed structure on Ω^1_A.

Proposition 2.6 *Suppose $A \in \mathrm{CAlg}(M)$ is a cofibrant commutative algebra in M. Then the morphism*

$$\mathrm{Harr}_\bullet(A, A) \to \Omega^1_A$$

induces a weak equivalence

$$|\mathrm{Harr}_\bullet(A, A)|^l \to \Omega^1_A$$

of A-modules.

Proof By [3, Proposition A.1.4] the forgetful functor $\mathrm{CAlg}(M) \to M$ preserves cofibrant objects, so A is cofibrant as an object of M.

By Proposition 2.3 and Lemma 2.4 the morphism of A-modules

$$\Omega^1_{|A^\epsilon|^l} \otimes_{|A^\epsilon|^l} A \to \Omega^1_A$$

is a weak equivalence.

By Lemma 2.5 we get an isomorphism

$$\Omega^1_{|A^\epsilon|^l} \otimes_{|A^\epsilon|^l} A \cong |\Omega^1_{A^\epsilon}|^l \otimes_{|A^\epsilon|^l} A.$$

But since $|-|^l$ preserves colimits and is monoidal,

$$|\Omega^1_{A^\epsilon}|^l \otimes_{|A^\epsilon|^l} A \cong |\Omega^1_{A^\epsilon} \otimes_{A^\epsilon} A|^l = \mathrm{Harr}_\bullet(A, A).$$

□

Let us now introduce the Harrison cochain complex. Let

$$\mathrm{Harr}^\bullet(A, A) = \underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M^{gr,\epsilon})}(\mathrm{Harr}_\bullet(A, A), A) \cong \underline{\mathrm{Hom}}_M(\mathrm{coLie}^\theta\{1\}(A), A).$$

As graded objects we can identify

$$\mathrm{Harr}^\bullet(A, A) \cong \mathrm{Conv}(\mathrm{coLie}^\theta\{1\}; A)(-1).$$

The multiplication and unit on A defines an element $m \in \mathrm{Conv}(\mathrm{coLie}^\theta\{1\}; A)$ satisfying the curved Maurer–Cartan equation. It is not difficult to check that the mixed structure on $\mathrm{Harr}^\bullet(A, A)$ is given by $[m, -]$.

Let $\mathrm{Der}^{int}(A, A) \in \mathrm{Alg}_{\mathrm{Lie}}(M)$ be the Lie algebra of derivations of A. Consider the morphism of graded objects

$$\mathrm{Der}^{int}(A, A) \to \mathrm{Harr}^{\bullet}(A, A)$$

induced from the morphism $\mathrm{Der}^{int}(A, A) \to \underline{\mathrm{Hom}}_M(A, A)$.

Proposition 2.7 *Suppose $A \in \mathrm{CAlg}(M)$ is a fibrant and cofibrant commutative algebra in M. Then the morphism*

$$\mathrm{Der}^{int}(A, A) \to \mathrm{Harr}^{\bullet}(A, A)$$

induces a weak equivalence of Lie algebras in M

$$\mathrm{Der}^{int}(A, A) \to |\mathrm{Harr}^{\bullet}(A, A)|.$$

Proof The compatibility with the Lie brackets is obvious since the weight 0 part of $\mathrm{Harr}^{\bullet}(A, A)$ is $\underline{\mathrm{Hom}}_M(A, A)$ and the pre-Lie structure (3) restricts to the composition of endomorphisms.

The morphism $\mathrm{Harr}_{\bullet}(A, A) \to \Omega^1_A$ induces the morphism

$$\mathrm{Der}^{int}(A, A) \to \mathrm{Harr}^{\bullet}(A, A)$$

after applying $\underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M)}(-, A)$. Since A is fibrant, the functor $\underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M)}(-, A)$ preserves weak equivalences between cofibrant objects. By construction $|\mathrm{Harr}_{\bullet}(A, A)|^l$ is a semi-free A-module, hence it is cofibrant. Since A is cofibrant, Ω^1_A is also a cofibrant A-module. Therefore,

$$\mathrm{Der}^{int}(A, A) \to \underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M)}(|\mathrm{Harr}_{\bullet}(A, A)|^l, A)$$

is a weak equivalence by Proposition 2.6. The statement follows by observing that the natural morphism

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M)}(|\mathrm{Harr}_{\bullet}(A, A)|^l, A) &\to |\underline{\mathrm{Hom}}_{\mathrm{Mod}_A(M^{gr,\epsilon})}(\mathrm{Harr}_{\bullet}(A, A), A)| \\ &= |\mathrm{Harr}^{\bullet}(A, A)|\end{aligned}$$

is an isomorphism. $\square$

3 Brace construction

3.1 Braces

Let $\mathcal{C}$ be a cooperad. In Sect. 2.1 we have shown how to make the convolution algebras $\mathrm{Conv}(\mathcal{C}; A)$ and $\mathrm{Conv}^0(\mathcal{C}; A)$ into pre-Lie algebras for any complex A. In this section we introduce two important generalizations of this construction.

Recall from [25, Definition 1.12] the following notion. Given a symmetric sequence $\mathcal{C}$ we define $\mathcal{C}^{\mathrm{cu}}$ to coincide with $\mathcal{C}$ in arities at least 1 and $\mathcal{C}^{\mathrm{cu}}(0) = \mathcal{C}(0) \oplus k$ in arity 0.

Definition 3.1 A *Hopf counital structure* on a (curved) cooperad $\mathcal{C}$ is the structure of a (curved) Hopf cooperad on $\mathcal{C}^{\mathrm{cu}}$ such that the natural projection $\mathcal{C}^{\mathrm{cu}} \to \mathcal{C}$ is a morphism of (curved) cooperads and such that the unit of the Hopf structure on $\mathcal{C}^{\mathrm{cu}}(0) = \mathcal{C}(0) \oplus k$ is given by inclusion into the second factor.

Remark 3.2 If $\mathcal{C}$ is a coaugmented cooperad, $\mathcal{C}^{\mathrm{cu}}$ will not in general inherit the coaugmentation.

Calaque and Willwacher extend the action of the pre-Lie operad preLie on $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A)$ to an action of the operad $\mathrm{preLie}_{\mathcal{C}}$ whose definition we will now sketch. We refer to [4, Sect. 3.1] for details.

Recall that the operations in the pre-Lie operad preLie are parametrized by rooted trees with numbered vertices. The operadic composition $\mathbf{t}_1 \circ_m \mathbf{t}_2$ is given by grafting the root of the tree $\mathbf{t}_2$ into the m-th vertex of $\mathbf{t}_1$. Now suppose $\mathcal{C}$ is a cooperad with a Hopf counital structure. The operad $\mathrm{preLie}_{\mathcal{C}}$ has operations parametrized by rooted trees where each vertex is labeled by an operation of $\mathcal{C}^{\mathrm{cu}}$ whose arity is equal to the number of incoming edges at the given vertex. The operadic composition is again given by grafting trees and it uses the Hopf cooperad structure on $\mathcal{C}^{\mathrm{cu}}$. The action of $\mathrm{preLie}_{\mathcal{C}}$ on $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A)$ is essentially the same as the pre-Lie structure on $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A)$ and uses the Hopf structure on $\mathcal{C}^{\mathrm{cu}}$ to multiply the label on the rooted tree by the label in the convolution algebra.

Given a Maurer–Cartan element f in a dg Lie algebra, the same Lie algebra with the differential $\mathrm{d}+[f, -]$ is still a dg Lie algebra. This is no longer true for pre-Lie algebras and one has to twist the operad. Using the general notion of twisting introduced in [7] one can construct the brace operad $\mathrm{Br}_{\mathcal{C}}$ which has operations parametrized by rooted trees where "external" vertices are labeled by elements of $\mathcal{C}^{\mathrm{cu}}$ and the rest of the vertices, "internal" vertices, are labeled by elements of $\overline{\mathcal{C}}[-1]$. In the pictures we will draw, external vertices are colored white and internal vertices are colored black. The generating operations of $\mathrm{Br}_{\mathcal{C}}$ are shown in Fig. 4, where the root is labeled by an element of $\mathcal{C}$. We refer to [7, Sect. 9] for an explicit description of the differential on $\mathrm{Br}_{\mathcal{C}}$, but roughly it is obtained by the sum over all vertices of the following terms:

- If the vertex is external, we replace it by the first expression shown in Fig. 5 and apply the composition in the pre-Lie operad.
- If the vertex is internal, we replace it by the second expression shown in Fig. 5 and apply the composition in the pre-Lie operad.
- We discard all trees which have internal vertices with fewer than 2 children.

Consider a morphism of operads

$$\Omega\mathcal{C} \to \mathrm{Br}_{\mathcal{C}} \tag{4}$$

defined in the following way. The operad $\Omega\mathcal{C}$ is freely generated by $\overline{\mathcal{C}}[-1]$ and we send a generator $\mathbf{s}x \in \overline{\mathcal{C}}[-1]$ to the first corolla shown in Fig. 4 with the internal vertex labeled by x.

Fig. 4 Generating operations of $\mathrm{Br}_{\mathcal{C}}$

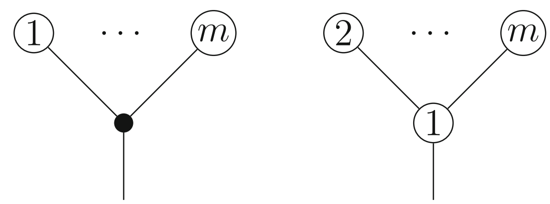
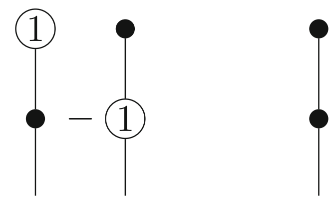
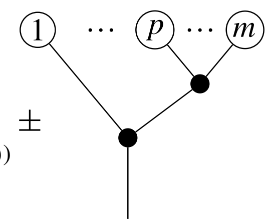


Fig. 5 Differential on $\mathrm{Br}_{\mathcal{C}}$



To see that the morphism (4) is compatible with the differential, note that the differential applied to the first tree in Fig. 4 is equal to the sum of trees of the form

$$\sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(m))} \pm$$



where the labels of the two internal vertices are obtained by applying the comultiplication in the cooperad $\mathcal{C}$ to the original label which is exactly the image of the cobar differential on $\Omega\mathcal{C}$.

Suppose now A is a $\mathrm{Br}_{\mathcal{C}}$-algebra. Applying the forgetful morphism (4) we get an $\Omega\mathcal{C}$-algebra structure on A and hence a differential on the cofree $\mathcal{C}^{\mathrm{cu}}$-coalgebra $\mathcal{C}^{\mathrm{cu}}(A)$. Moreover, we get an associative product on $\mathcal{C}^{\mathrm{cu}}(A)$ defined in the following way. A multiplication

$$\mathcal{C}^{\mathrm{cu}}(A)\otimes\mathcal{C}^{\mathrm{cu}}(A)\to\mathcal{C}^{\mathrm{cu}}(A)$$

of cofree conilpotent $\mathcal{C}$-coalgebras is uniquely determined by the projection to the cogenerators

$$\mathcal{C}^{\mathrm{cu}}(A)\otimes\mathcal{C}^{\mathrm{cu}}(A)\to A,$$

i.e. by morphisms

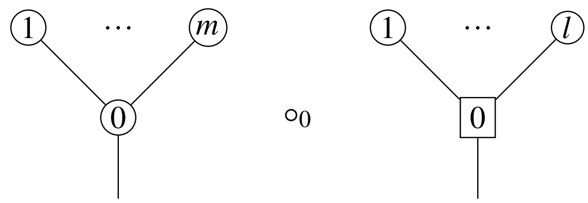
$$\mathcal{C}^{\mathrm{cu}}(l)\otimes A^{\otimes l}\otimes\mathcal{C}^{\mathrm{cu}}(m)\otimes A^{\otimes m}\to A.$$

We let the morphisms with $l=1$ be given by the second corollas in Fig. 4 and those with $l\neq 1$ are defined to be zero. The unit is defined to be the inclusion of k into the second summand of $\mathcal{C}^{\mathrm{cu}}(A)\cong\mathcal{C}(A)\oplus k$.

Remark 3.3 The multiplication

$$\mathcal{C}^{\mathrm{cu}}(l) \otimes A^{\otimes l} \otimes \mathcal{C}^{\mathrm{cu}}(m) \otimes A^{\otimes m} \to \mathcal{C}^{\mathrm{cu}}(A)$$

is given by the composition



In the composition the $\mathcal{C}^{\mathrm{cu}}$-label of the square vertex is the label of the output in $\mathcal{C}^{\mathrm{cu}}(A)$ and the number of incoming edges at the square vertex is the arity of the operation in $\mathcal{C}^{\mathrm{cu}}(A)$.

Proposition 3.4 *Let A be a $\mathrm{Br}_{\mathcal{C}}$-algebra. Thus defined multiplication defines on $\mathcal{C}^{\mathrm{cu}}(A)$ a structure of a dg associative algebra which is compatible with the $\mathcal{C}^{\mathrm{cu}}$-coalgebra structure.*

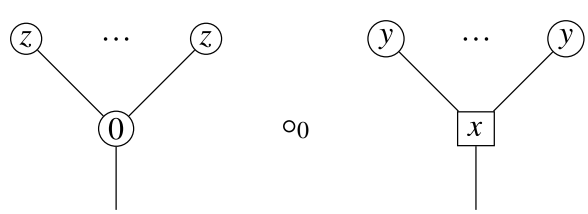
Proof We refer to [24, Proposition 3.3] for a detailed proof in the case $\mathcal{C} = \mathrm{coAss}$.

By construction the product is compatible with the $\mathcal{C}^{\mathrm{cu}}$-coalgebra structure, so we just need to check the associativity of the product and the fact that the differential d on $\mathcal{C}^{\mathrm{cu}}(A)$ is a derivation of the product. To check these axioms, it is enough to check that these hold after projections to the cogenerators A.

- (Associativity). The only nontrivial equation is expressed by the commutative diagram

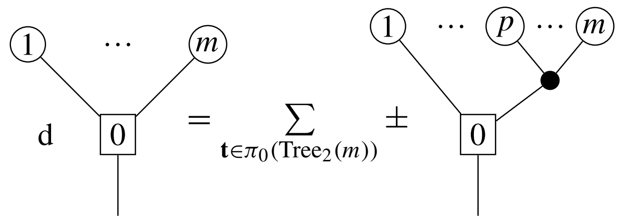
$$\begin{tikzcd}
(A \otimes \mathcal{C}^{\mathrm{cu}}(l) \otimes A^{\otimes l}) \otimes \mathcal{C}^{\mathrm{cu}}(m) \otimes A^{\otimes m} \arrow[r, "\sim"] \arrow[dd] & A \otimes (\mathcal{C}^{\mathrm{cu}}(l) \otimes A^{\otimes l} \otimes \mathcal{C}^{\mathrm{cu}}(m) \otimes A^{\otimes m}) \arrow[d] \\
 & A \otimes \mathcal{C}^{\mathrm{cu}}(A) \arrow[d] \\
A \otimes \mathcal{C}^{\mathrm{cu}}(m) \otimes A^{\otimes m} \arrow[r] & A
\end{tikzcd}$$

  We denote by x the element of the first A factor, by y elements of $A^{\otimes l}$ and by z elements of $A^{\otimes m}$. Then the composition along the bottom-left corner is given by



which coincides with the composition along the top-right corner following Remark 3.3.

- (Derivation). The differential on $\mathcal{C}^{cu}(A)$ is given by the sum



The compatibility of the differential and the multiplication then immediately follows from the compatibility of the differential and the composition in the operad $\mathrm{Br}_{\mathcal{C}}$ since the multiplication on $\mathcal{C}^{cu}(A)$ is defined in terms of the composition.

□

This proposition motivates the following definition.

Definition 3.5 An *∞-morphism of* $\mathrm{Br}_{\mathcal{C}}$*-algebras* $A \to B$ is a morphism of dg associative algebras $\mathcal{C}^{cu}(A) \to \mathcal{C}^{cu}(B)$ compatible with the $\mathcal{C}^{cu}$-coalgebra structures.

Suppose $\mathcal{C}^{cu}(A) \to \mathcal{C}^{cu}(B)$ is an ∞-morphism of $\mathrm{Br}_{\mathcal{C}}$-algebras. Using the canonical unit morphism $\mathrm{coComm} \to \mathcal{C}^{cu}$ we obtain a morphism $\mathrm{Sym}(A) \to \mathrm{Sym}(B)$ which preserves the cocommutative comultiplication and the multiplication. In particular, it induces a *strict* morphism of Lie algebras $A \to B$ after passing to primitive elements.

Let us now consider the case when $\mathcal{C}^{cu}$ is a graded Hopf cooperad, i.e. a cooperad in graded commutative dg algebras. The operad $\mathrm{Br}_{\mathcal{C}}$ inherits the grading given by the sum of all weights of labels in $\mathcal{C}^{cu}$. Now recall that $\mathrm{Br}_{\mathcal{C}}$ acts on $\mathrm{Conv}^0(\mathcal{C}^{cu}\{n\}, A)$ for any n. If $\mathcal{C}^{cu}\{n\}$ has a structure of a graded module over $\mathcal{C}^{cu}$, then the natural grading on the convolution algebra $\mathrm{Conv}^0(\mathcal{C}^{cu}\{n\}, A)$ makes it into a graded algebra over the graded operad $\mathrm{Br}_{\mathcal{C}}$.

Remark 3.6 Note that we do not assume that the grading on $\mathcal{C}^{cu}\{n\}$ coincides with the grading on $\mathcal{C}^{cu}$. In fact, in our main example of polyvector fields the two gradings are distinct.

Fix the number n.

Definition 3.7 Let A be an $\Omega(\mathcal{C}\{n\})$-algebra whose structure is determined by a Maurer–Cartan element $f \in \mathrm{Conv}(\mathcal{C}\{n\}; A)$. The *center* of A is the shifted convolution algebra

$$\mathbf{Z}(A) = \mathrm{Conv}^0(\mathcal{C}^{cu}\{n\}; A)[-n]$$

twisted by the Maurer–Cartan element f.

Note that by construction the center $\mathbf{Z}(A)$ is naturally an algebra over $\mathrm{Br}_{\mathcal{C}}\{n\}$.

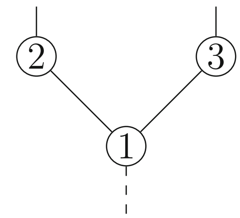


Fig. 6 An example of an operation in $\mathrm{preLie}^{\rightarrow}$ of arity $((\mathcal{A} \to \mathcal{A})^{\otimes 2} \otimes (\mathcal{A} \to \mathcal{B}), (\mathcal{A} \to \mathcal{B}))$

3.2 Relative brace construction

Let us now describe a relative version of these constructions. Consider a pair of complexes A, B. Then we can define a relative convolution algebra to be the complex

$$\mathrm{Conv}^0(\mathcal{C}; A, B) = \mathrm{Hom}_k(\mathcal{C}(A), B).$$

We are now going to introduce a certain algebraic structure on the triple

$$(\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A), \mathrm{Conv}^0(\mathcal{C}; A, B), \mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B))$$

which generalizes the pre-Lie structure on the convolution algebra.

Consider the set of colors $\mathcal{V} = \{\mathcal{A} \to \mathcal{A}, \mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{B}\}$. We introduce a $\mathcal{V}$-colored version of the pre-Lie operad denoted by $\mathrm{preLie}^{\rightarrow}$ in the following way. Operations of $\mathrm{preLie}^{\rightarrow}$ are parametrized by rooted trees with edges of two types: of type $\mathcal{A}$ that we denote by solid lines and of type $\mathcal{B}$ that we denote by dashed lines. We disallow any vertices which have incoming edges of different types or those that have incoming edges of type $\mathcal{B}$ but an outgoing edge of type $\mathcal{A}$. A color of the vertex is determined by the type of inputs and outputs and we denote it as e.g. $\mathcal{A} \to \mathcal{B}$. To resolve ambiguities, we draw incoming edges to leaves (recall that in the case of the ordinary pre-Lie operad we de not draw incoming edges to leaves following [5]). One can read off the arity of the operation parametrized by a rooted tree in the following way. Each vertex has a color determined by incoming and outgoing edges and so does the whole graph and this determines the arity. See Fig. 6 for an example. The operadic composition is given by grafting trees exactly in the same way as in the case of the pre-Lie operad.

The colored operad $\mathrm{preLie}^{\rightarrow}$ acts on the triple

$$(\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A), \mathrm{Conv}^0(\mathcal{C}; A, B), \mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B))$$

exactly in the same way as in the case of the usual pre-Lie operad. Namely, given a rooted tree we substitute elements of the convolution algebras into vertices based on colors:

- if a vertex has color $\mathcal{A} \to \mathcal{A}$, we substitute an element of $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A)$,
- if a vertex has color $\mathcal{A} \to \mathcal{B}$, we substitute an element of $\mathrm{Conv}^0(\mathcal{C}; A, B)$,
- if a vertex has color $\mathcal{B} \to \mathcal{B}$, we substitute an element of $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B)$.

After such a substitution one reads off the result by composing the morphisms using the pattern given by the rooted tree.

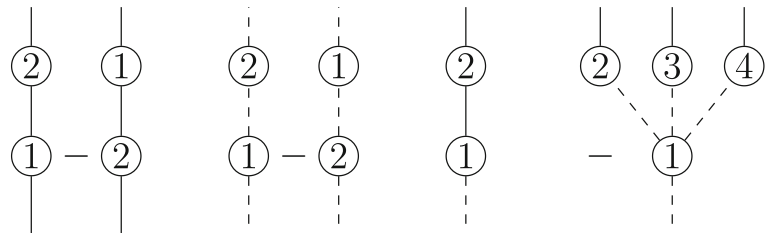


Fig. 7 L_∞ brackets on a preLie$^{\rightarrow}$ algebra

Given a preLie$^{\rightarrow}$-algebra (C_1, C_2, C_3) one has a natural L_∞ structure on

$$C_1 \oplus C_2[-1] \oplus C_3$$

given by expressions in Fig. 7. Here the first three trees give rise to ordinary Lie brackets and the last tree gives rise to an L_∞ operation.

Since $\mathcal{C}^{\mathrm{cu}}$ is a Hopf cooperad, one can similarly define a $\mathcal{V}$-colored operad $\mathrm{preLie}^{\rightarrow}_{\mathcal{C}}$ whose operations are parametrized by rooted trees with edges of two types as in the case of preLie$^{\rightarrow}$ and whose vertices are parametrized by elements of $\mathcal{C}$. As before, the triple $(\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A), \mathrm{Conv}^0(\mathcal{C}; A, B), \mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B))$ is an algebra over the colored operad $\mathrm{preLie}^{\rightarrow}_{\mathcal{C}}$.

Using the forgetful map from $\mathrm{preLie}^{\rightarrow}_{\mathcal{C}}$-algebras to L_∞-algebras, one can apply the general formalism of twistings of [7] to construct the colored operad $\mathrm{Br}^{\rightarrow}_{\mathcal{C}}$ whose operations are parametrized by rooted trees with dashed and solid edges and external and internal vertices as before.

Suppose

$$f = (f_1, f_2, f_3) \in (\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A), \mathrm{Conv}^0(\mathcal{C}; A, B), \mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B))$$

is a Maurer–Cartan element in the underlying L_∞-algebra. We can twist the differential on $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; A)$ using f_1, we can twist the differential on $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B)$ using f_3 and we can twist the differential on $\mathrm{Conv}^0(\mathcal{C}; A, B)$ using all three elements. Then as before the triple

$$(\mathrm{Conv}^0_{f_1}(\mathcal{C}^{\mathrm{cu}}; A), \mathrm{Conv}^0_f(\mathcal{C}; A, B), \mathrm{Conv}^0_{f_3}(\mathcal{C}^{\mathrm{cu}}; B))$$

becomes an algebra over the colored operad $\mathrm{Br}^{\rightarrow}_{\mathcal{C}}$ if we assign f to internal vertices.

Suppose now (C_1, C_2, C_3) is any $\mathrm{Br}^{\rightarrow}_{\mathcal{C}}$-algebra. In particular, C_1 and C_3 are $\mathrm{Br}_{\mathcal{C}}$-algebras. One naturally has an $\Omega\mathcal{C}$-algebra structure on C_2 given by sending an element $\mathbf{s}x \in \overline{\mathcal{C}}[-1]$ to the first corolla shown in Fig. 8 where the internal vertex is labeled by the element x. In particular, $\mathrm{Conv}(\mathcal{C}; C_2)$ has a Maurer–Cartan element that we denote by f. Note that this should not be confused with the previous occurrence of Maurer–Cartan elements in $\mathrm{Conv}^0(\mathcal{C}^{\mathrm{cu}}; B)$.

We have a morphism

$$C_3 \to \mathrm{Conv}^0_f(\mathcal{C}^{\mathrm{cu}}; C_2) \tag{5}$$

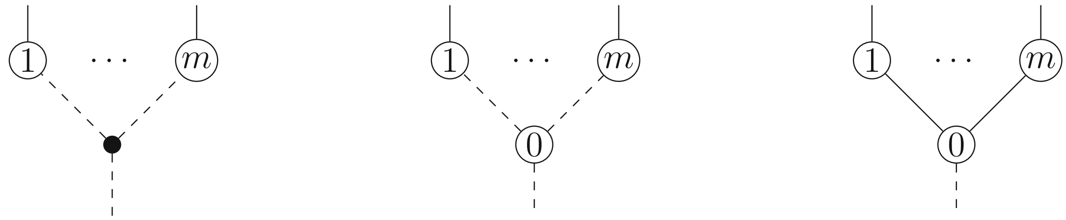


Fig. 8 An $\Omega\mathcal{C}$-structure on C_2, the morphism $C_3 \to \mathrm{Conv}_f(\mathcal{C}; C_2)$ and the ∞-morphism $C_1 \to \mathrm{Conv}_f(\mathcal{C}; C_2)$ respectively

defined in the following way. The second corolla shown in Fig. 8 defines a morphism

$$\mathcal{C}^{\mathrm{cu}}(m) \otimes C_1 \otimes C_2^{\otimes m} \to C_2,$$

where the element of $\mathcal{C}^{\mathrm{cu}}(m)$ labels vertex 0. By adjunction this gives the required morphism.

Similarly, we define the morphism

$$\mathcal{C}^{\mathrm{cu}}(C_1) \longrightarrow \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2) \tag{6}$$

in the following way. The third corolla shown in Fig. 8 defines a morphism

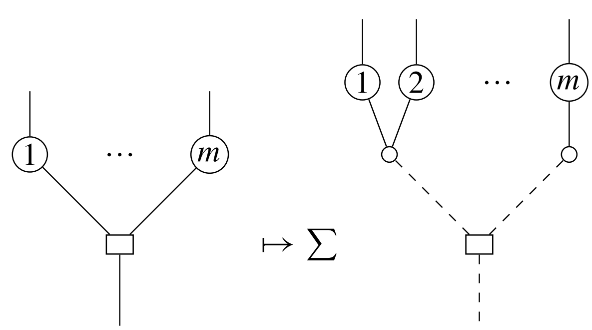
$$\mathcal{C}^{\mathrm{cu}}(m) \otimes C_1^{\otimes m} \otimes C_2 \longrightarrow C_2$$

which by adjunction gives the morphism

$$\mathcal{C}^{\mathrm{cu}}(C_1) \longrightarrow \mathrm{Hom}(C_2, C_2) \to \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2).$$

Remark 3.8 One can give the following pictorial representation of the morphism

$$\mathcal{C}^{\mathrm{cu}}(m) \otimes C_1^{\otimes m} \longrightarrow \mathcal{C}^{\mathrm{cu}}(\mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2)) :$$



Here the label of the rectangle on the left is the element $c \in \mathcal{C}^{\mathrm{cu}}(m)$. The labels of the unmarked vertices on the right are given by applying the coproduct in the cooperad $\mathcal{C}^{\mathrm{cu}}$ to c with respect to the corresponding picthfork.

Proposition 3.9 *Let* (C_1, C_2, C_3) *be a* $\mathrm{Br}_{\mathcal{C}}^{\rightarrow}$*-algebra. Then the morphism* (5)

$$C_3 \to \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2)$$

is a morphism of $\mathrm{Br}_{\mathcal{C}}$*-algebras.*

Similarly, the morphism (6)

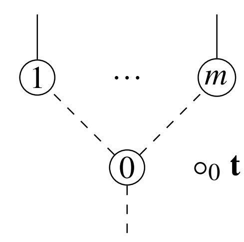
$$C_1 \to \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2)$$

is an ∞*-morphism of* $\mathrm{Br}_{\mathcal{C}}$*-algebras.*

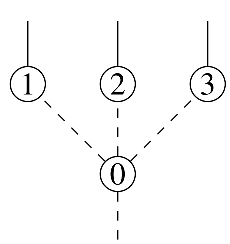
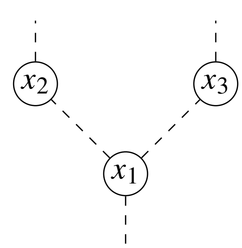
Proof For the first statement we have to show that the diagram

$$\begin{tikzcd}
\mathrm{Br}_{\mathcal{C}}(m) \otimes C_3^{\otimes m} \arrow[r] \arrow[d] & \mathrm{Br}_{\mathcal{C}}(m) \otimes \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2)^{\otimes m} \arrow[d] \\
C_3 \arrow[r] & \mathrm{Conv}_f^0(\mathcal{C}^{\mathrm{cu}}; C_2)
\end{tikzcd}$$

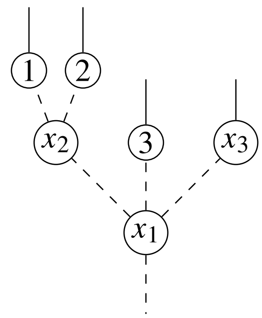
commutes. Pick $\mathbf{t} \in \mathrm{Br}_{\mathcal{C}}(m)$ and $x_1, \ldots, x_m \in C_3$. The composition along the bottom-left corner is given by applying composition using the pattern given by the trees

 $\circ_0$ $\mathbf{t}$

where the vertices of $\mathbf{t}$ are labeled by the elements x_i. Similarly, the composition along the top-right corner is given by the sum over numbers $n_1, \ldots, n_m$ of trees given by attaching n_i incoming edges at vertex i of $\mathbf{t}$. The two expressions obviously coincide. For instance, in the composition

 $\circ_0$ 

the term with $n_1 = 1$, $n_2 = 2$ and $n_3 = 0$ is



For the second statement we have to check that

$$\mathcal{C}^{\mathrm{cu}}(C_1) \to \mathcal{C}^{\mathrm{cu}}(\mathrm{Conv}^0_f(\mathcal{C}^{\mathrm{cu}}; C_2))$$

is a morphism compatible with the differentials and multiplications. The computation is similar to the proof of the first statement and uses the description of the morphism given in Remark 3.8. □

Remark 3.10 If C is a $\mathrm{Br}_{\mathcal{C}}$-algebra and D is an $\Omega\mathcal{C}$-algebra, then an ∞-morphism of $\mathrm{Br}_{\mathcal{C}}$-algebras $C \to D$ is essentially the same as the notion of a brace module from [24, Definition 3.2] when $\mathcal{C} = \mathrm{coAss}$. In this case an analog of the second statement in the previous proposition is [24, Proposition 4.2].

3.3 Swiss-cheese construction

Recall from [26] that a Swiss-cheese algebra consists of an $\mathbb{E}_2$-algebra A, an $\mathbb{E}_1$-algebra B and an $\mathbb{E}_2$-morphism $A \to \mathrm{HH}^\bullet(B)$ to the Hochschild cohomology of B. A model of the $\mathbb{E}_2$ operad is given by the brace operad which can be obtained in our notation as $\mathrm{Br}_{\mathrm{coAss}}\{1\}$, the brace construction on the Hopf cooperad of coassociative coalgebras. In this section we construct a colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ generalizing the Swiss-cheese operad using the brace construction. An algebra over $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ will be an $\Omega\mathcal{C}_1$-algebra A, an $\Omega(\mathcal{C}_2\{n\})$-algebra B and an ∞-morphism of $\Omega\mathcal{C}_1$-algebras $A \to \mathbf{Z}(B)$. The construction of the colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ will be modeled after the resolution of the operad controlling morphisms of $\mathcal{O}$-algebras constructed in [19, Sect. 2].

Fix a number n. Let $\mathcal{C}_1$ be a cooperad and $\mathcal{C}_2$ a cooperad with a Hopf counital structure together with an operad morphism

$$F\colon \Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}\{n\}.$$

From this data we define a semi-free colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ in the following way.

The set of colors of $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ is $\{\mathcal{A}, \mathcal{B}\}$. The operad is semi-free on the colored symmetric sequence $P(\mathcal{C}_1, \mathcal{C}_2)$ whose nonzero elements are

$$P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{A}^{\otimes m}, \mathcal{A}) = \overline{\mathcal{C}_1}(m)$$
$$P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{B}^{\otimes l}, \mathcal{B}) = \overline{\mathcal{C}_2}\{n\}(l)$$
$$P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{A}^{\otimes m} \otimes \mathcal{B}^{\otimes l}, \mathcal{B}) = \mathcal{C}_1(m) \otimes \mathcal{C}_2^{\mathrm{cu}}\{n\}(l)[n+1].$$

The colored operad $\mathrm{Free}(P(\mathcal{C}_1, \mathcal{C}_2)[-1])$ has operations parametrized by trees with edges of two types: those of color $\mathcal{A}$ that we denote by solid lines and those of color $\mathcal{B}$ that we denote by dashed lines. The vertices of the trees are labeled by generating operations in $P(\mathcal{C}_1, \mathcal{C}_2)$. We define a differential on $\mathrm{Free}(P(\mathcal{C}_1, \mathcal{C}_2)[-1])$ in the following way. The differentials in arities $(\mathcal{A}^{\otimes -}, \mathcal{A})$ and $(\mathcal{B}^{\otimes -}, \mathcal{B})$ are the usual cobar differentials (2). The differential on an element $\mathbf{s}^{-n} X \otimes Y$ for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2^{\mathrm{cu}}\{n\}(l)$ has four components:

(1)

$$\mathrm{d}_1(\mathbf{s}^{-n} X \otimes Y) = (-1)^n \mathbf{s}^{-n} \mathrm{d}_1 X \otimes Y + (-1)^{n+|X|} \mathbf{s}^{-n} X \otimes \mathrm{d}_1 Y$$

where d_1 are the internal differentials on the complexes $\mathcal{C}_1(m)$ and $\mathcal{C}_2^{\mathrm{cu}}\{n\}(l)$.

(2)

$$\mathrm{d}_2(\mathbf{s}^{-n} X \otimes Y) = (-1)^n (\mathbf{s}^{-n-1} \otimes 1) \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(m))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(X)) \circ_0 Y,$$

where we use the following notation. Let $\Delta_{\mathbf{t}}(X) = X_{(0)} \otimes X_{(1)}$ with $X_{(0)}$ the label of the root. We denote by $(\mathbf{t}, \Delta_{\mathbf{t}}(X)) \circ_0 Y$ the tree $\mathbf{t}$ with additional l dashed incoming edges at the root which is labeled by $X_{(0)} \otimes Y$. The right-hand side consists of a composition of an operation in $P(\mathcal{A}^{\otimes -}, \mathcal{A})$ and $P(\mathcal{A}^{\otimes -} \otimes \mathcal{B}^{\otimes -}, \mathcal{B})$. See Fig. 9 for an example.

(3)

$$\mathrm{d}_3(\mathbf{s}^{-n} X \otimes Y) = (-1)^n (\mathbf{s}^{-n-1} \otimes 1) X \circ_0 \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(l))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(Y)) + (-1)^n (1 \otimes \mathbf{s}^{-n-1}) X \circ_1 \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(l))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(Y)),$$

where we use the following notation. Let $\Delta_{\mathbf{t}}(Y) = Y_{(0)} \otimes Y_{(1)}$ with $Y_{(0)}$ the label of the root. We denote by $X \circ_0 (\mathbf{t}, \Delta_{\mathbf{t}}(Y))$ the tree $\mathbf{t}$ with additional m solid incoming edges at the root which is labeled by $X \otimes Y_{(0)}$. Similarly, $X \circ_1 (\mathbf{t}, \Delta_{\mathbf{t}}(Y))$ is the tree $\mathbf{t}$ with additional m solid incoming edges at the other node which is labeled by $X \otimes Y_{(1)}$. See Fig. 10 for an example.

(4)

$$\mathrm{d}_4(\mathbf{s}^{-n} X \otimes Y) = \sum_r \sum_{\mathbf{t} \in \mathrm{Isom}_{\pitchfork}(m,r)} F(\mathbf{t}, \Delta_{\mathbf{t}}(X), Y).$$

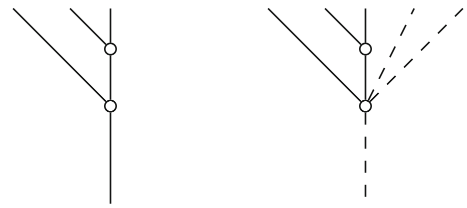


Fig. 9 A tree $\mathbf{t}$ and $\mathbf{t} \circ_0 Y$ with $l = 2$

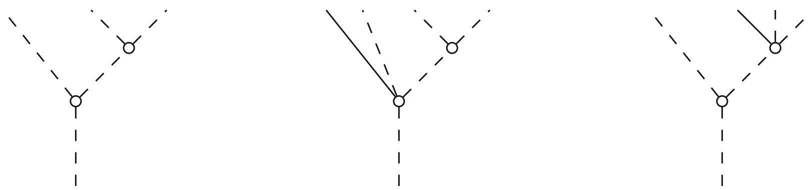


Fig. 10 A tree $\mathbf{t}$, $X \circ_0 \mathbf{t}$ and $X \circ_1 \mathbf{t}$ with $m = 1$

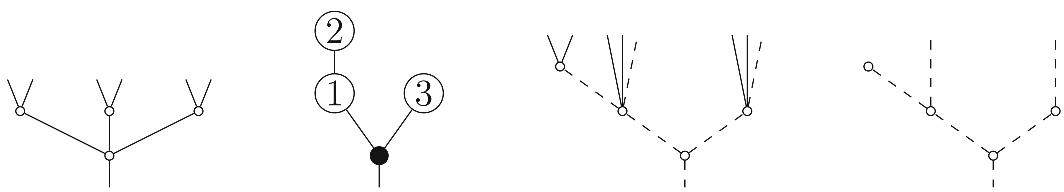


Fig. 11 A pitchfork $\mathbf{t}$, a rooted tree $F(\mathbf{s}X_{(0)})$, an example of $\tilde{\mathbf{t}}$ and $\tilde{\mathbf{t}}_{dashed}$

Here $F(\mathbf{t}, \Delta_{\mathbf{t}}(X), Y)$ is defined in the following way. Denote by $X_{(i)} \in \mathcal{C}_1$ the labels of the vertices in $\Delta_{\mathbf{t}}(X)$ with $X_{(0)}$ the label of the root. The image of $\mathbf{s}X_{(0)}$ under $F\colon \Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}\{n\}$ is a rooted tree $F(\mathbf{s}X_{(0)})$ labeled by r elements $Z_{(i)} \in \mathcal{C}_2^{\mathrm{cu}}$. Consider the composition $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$. We consider the following set of trees $\tilde{\mathbf{t}}$: a tree $\tilde{\mathbf{t}}$ is obtained from $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$ by adding an arbitrary number of incoming dashed edges to vertices so that the total number of incoming dashed edges is l. Let us denote by $\tilde{\mathbf{t}}_{dashed}$ the tree obtained from $\tilde{t}$ by erasing all solid edges. We let $\Delta_{\mathbf{t}_{dashed}}(Y) = Y_{(1)} \otimes \cdots \otimes Y_{(r)}$. The labelings of vertices of $\tilde{\mathbf{t}}$ are of two kinds: external vertices are labeled by the tensor product $X_{(i)} \otimes Y_{(i)} Z_{(i)}$ where $Y_{(i)} Z_{(i)}$ is the product in the Hopf cooperad $\mathcal{C}_2^{\mathrm{cu}}$ and they belong to the operations in $P(\mathcal{A}^{\otimes -} \otimes \mathcal{B}^{\otimes -}, \mathcal{B})$; the internal vertices are simply labeled by elements of $\mathcal{C}_2$ and they belong to the operations in $P(\mathcal{B}^{\otimes -}, \mathcal{B})$. We refer to Fig. 11 for an example. We define $F(\mathbf{t}, \Delta_{\mathbf{t}}(X), Y)$ to be the sum over all such trees $\tilde{\mathbf{t}}$.

Remark 3.11 Let us informally explain the differentials. The first differential d_1 is simply the sum of the internal differentials on $\mathcal{C}_1$ and $\mathcal{C}_2$. The differentials d_2 and d_3 are analogous to the cobar differentials (2) on $\Omega\mathcal{C}_1$ and $\Omega\mathcal{C}_2$ respectively. Finally, the differential d_4 expresses the fact that $A \to \mathbf{Z}(B)$ is an ∞-morphism of $\Omega\mathcal{C}_1$-algebras and the complicated structure comes from the fact that $\mathbf{Z}(B)$ is an $\Omega\mathcal{C}_1$-algebra via the morphism $F\colon \Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}\{n\}$.

Lemma 3.12 *The total differential* d *on* $\mathrm{Free}(\mathcal{P}(\mathcal{C}_1, \mathcal{C}_2)[-1])$ *squares to zero.*

Proof The claim in arities $(\mathcal{A}^{\otimes m}, \mathcal{A})$ and $(\mathcal{B}^{\otimes l}, \mathcal{B})$ follows from Lemma 1.12.

Let us split the differentials on the generators in arities $(\mathcal{A}^{\otimes -}, \mathcal{A})$ and $(\mathcal{B}^{\otimes -}, \mathcal{B})$ as $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_{\mathcal{A}}$ and $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_{\mathcal{B}}$ respectively.

Given an element $\mathbf{s}^{-n}X \otimes Y$ for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2^{\mathrm{cu}}(l)$ the expression $\mathrm{d}^2(\mathbf{s}^{-n}X \otimes Y)$ splits into the following combinations:

(1) $\mathrm{d}_1^2(\mathbf{s}^{-n}X \otimes Y)$,
(2) $(\mathrm{d}_1\mathrm{d}_2 + \mathrm{d}_2\mathrm{d}_1)(\mathbf{s}^{-n}X \otimes Y)$,
(3) $(\mathrm{d}_1\mathrm{d}_3 + \mathrm{d}_3\mathrm{d}_1)(\mathbf{s}^{-n}X \otimes Y)$,
(4) $(\mathrm{d}_1\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_1)(\mathbf{s}^{-n}X \otimes Y)$,
(5) $(\mathrm{d}_2^2 + \mathrm{d}_{\mathcal{A}}\mathrm{d}_2)(\mathbf{s}^{-n}X \otimes Y)$,
(6) $(\mathrm{d}_3^2 + \mathrm{d}_{\mathcal{B}}\mathrm{d}_3)(\mathbf{s}^{-n}X \otimes Y)$,
(7) $(\mathrm{d}_2\mathrm{d}_3 + \mathrm{d}_3\mathrm{d}_2)(\mathbf{s}^{-n}X \otimes Y)$,
(8) $(\mathrm{d}_2\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_2)(\mathbf{s}^{-n}X \otimes Y)$,
(9) $(\mathrm{d}_3\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_3)(\mathbf{s}^{-n}X \otimes Y)$,
(10) $(\mathrm{d}_4^2 + \mathrm{d}_{\mathcal{B}}\mathrm{d}_4)(\mathbf{s}^{-n}X \otimes Y)$.

We claim that each of these is zero. It is obvious for terms of type (1). Terms of type (2) and (3) vanish due to compatibility of the cooperad structure on $\mathcal{C}_1$ and $\mathcal{C}_2^{\mathrm{cu}}$ respectively with the differentials. The vanishing of terms of type (5) and (6) follows as in Lemma 1.12. The vanishing of the terms of type (7), (8), (9) is obvious as the corresponding modifications of the trees are independent.

Differentials on both $\Omega\mathcal{C}_1$ and $\mathrm{Br}_{\mathcal{C}_2}\{n\}$ have a linear and a quadratic component. Therefore, the compatibility of the morphism $F\colon \Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}\{n\}$ with differentials has two implications. First, the compatibility of the linear parts of the differentials implies the vanishing of terms of type (4). Second, the compatibility of the quadratic parts of the differentials implies the vanishing of terms of type (10). □

Definition 3.13 The *Swiss-cheese operad* $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ is the colored operad $\mathrm{Free}(\mathcal{P}(\mathcal{C}_1, \mathcal{C}_2)[-1])$ equipped with the above differential.

We define the L_∞ algebra $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B)$ as follows. As a complex,

$$\begin{aligned}\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B) = \mathrm{Conv}(\mathcal{C}_1; A) \oplus \mathrm{Conv}(\mathcal{C}_2\{n\}; B)\\ \oplus \mathrm{Hom}(\mathcal{C}_1(A) \otimes \mathcal{C}_2^{\mathrm{cu}}\{n\}(B), B)[-n-1].\end{aligned}$$

The L_∞ operations are given by the following rule:

- The first two terms have the standard convolution algebra brackets.
- The first two terms act on the third term by precomposition, i.e. by using the pre-Lie structure on the convolution algebras.
- The second term acts on the third term by post-composition.
- Given $R_1, \dots, R_q \in \mathrm{Conv}(\mathcal{C}_2\{n\}; B)$ and $T_1, \dots, T_r \in \mathrm{Hom}(\mathcal{C}_1(A) \otimes \mathcal{C}_2^{cu}\{n\}(B), B)$, their bracket is

$$[R_1, \dots, R_q.T_1, \dots, T_r](X \otimes Y; a_1, \dots, a_m, b_1, \dots, b_l)$$

for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2^{\mathrm{cu}}\{n\}(l)$ is given by the sum over pitchforks $\mathbf{t} \in \mathrm{Isom}_{\pitchfork}(m, r)$ where each term is given as follows. Let $\Delta_{\mathbf{t}}(X) = X_{(0)} \otimes \cdots$ where $X_{(0)}$ is assigned to the root and recall the tree $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$. The value of the bracket is given by the sum over all ways of assigning $T_1, \dots, T_r$ to the white external vertices and $R_1, \dots, R_q$ to the black internal vertices of $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$ and then reading off the composition using the pattern given by the tree.

Proposition 3.14 *The space of morphisms* $\mathrm{Map}_{2\mathrm{Op}_k}(\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2), \mathrm{End}_{A,B})$ *is equivalent to the space of Maurer–Cartan elements in the* L_∞ *algebra* $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B)$.

The proof of this Proposition is similar to the proof of Proposition 2.2, so we omit it.

Let B be a $\Omega(\mathcal{C}_2\{n\})$-algebra and consider its center $\mathbf{Z}(B)$ (see Definition 3.7), which is a $\mathrm{Br}_{\mathcal{C}_2}\{n\}$-algebra. Using the morphism

$$F\colon \Omega\mathcal{C}_1 \to \mathrm{Br}_{\mathcal{C}_2}\{n\}$$

one defines an $\Omega\mathcal{C}_1$-algebra structure on $\mathbf{Z}(B)$. From Proposition 3.14 and the formulas for the L_∞ brackets we get the following description of $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$-algebras.

Corollary 3.15 *An algebra over the colored operad* $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ *is an* $\Omega\mathcal{C}_1$*-algebra* A*, an* $\Omega(\mathcal{C}_2\{n\})$*-algebra* B *and an* ∞*-morphism of* $\Omega\mathcal{C}_1$*-algebras* $A \to \mathbf{Z}(B)$.

Let us now define a small model of the colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$. Suppose $\Omega\mathcal{C}_1 \to \mathcal{O}_1$ and $\Omega(\mathcal{C}_2\{n\}) \to \mathcal{O}_2$ are quasi-isomorphisms of operads. An algebra over $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ is a triple consisting of a homotopy $\mathcal{O}_1$-algebra A, a homotopy $\mathcal{O}_2$-algebra B and an ∞-morphism of homotopy $\mathcal{O}_1$-algebras $A \to \mathbf{Z}(B)$. Define the colored operad $\mathrm{SC}(\mathcal{O}_1, \mathcal{O}_2)$ to be the quotient of $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ whose algebras are triples consisting of a *strict* $\mathcal{O}_1$-algebra A, a *strict* $\mathcal{O}_2$-algebra B and a *strict* morphism of homotopy $\mathcal{O}_1$-algebras $A \to \mathbf{Z}(B)$.

For the following statement we assume that $\mathcal{C}_i$ and $\mathcal{O}_i$ admit an increasing exhaustive filtration $\{F_n\}_{n\geq 0}$ satisfying the following properties:

(1) $F_0\mathcal{C}_i = \mathbf{1}$ and $F_0\mathcal{O}_i = \mathbf{1}$.
(2) The morphisms $\Omega\mathcal{C}_i \to \mathcal{O}_i$ are compatible with filtrations.

Proposition 3.16 *The projection* $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2) \to \mathrm{SC}(\mathcal{O}_1, \mathcal{O}_2)$ *is a quasi-isomorphism.*

Proof Consider an intermediate operad $\mathrm{SC}'(\mathcal{O}_1, \mathcal{O}_2)$ whose algebras are triples consisting of a strict $\mathcal{O}_1$-algebra A, a strict $\mathcal{O}_2$-algebra B and an ∞-*morphism* of homotopy $\mathcal{O}_1$-algebras $A \to \mathbf{Z}(B)$. Thus, we have a sequence of projections

$$\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2) \longrightarrow \mathrm{SC}'(\mathcal{O}_1, \mathcal{O}_2) \longrightarrow \mathrm{SC}(\mathcal{O}_1, \mathcal{O}_2).$$

We will prove that each morphism is a quasi-isomorphism.

Define the colored operad $(\mathcal{O}_1)_{\mathcal{A}} \oplus (\mathcal{O}_2)_{\mathcal{B}}$ whose set of colors is $\{\mathcal{A}, \mathcal{B}\}$ and whose algebras are given by pairs of an $\mathcal{O}_1$-algebra and an $\mathcal{O}_2$-algebra and similarly for $(\Omega\mathcal{C}_1)_{\mathcal{A}} \oplus (\Omega(\mathcal{C}_2\{n\}))_{\mathcal{B}}$. By construction we have a pushout of operads.

$$\begin{tikzcd}
(\Omega\mathcal{C}_1)_{\mathcal{A}} \oplus (\Omega(\mathcal{C}_2\{n\}))_{\mathcal{B}} \arrow[r, "\sim"] \arrow[d] & (\mathcal{O}_1)_{\mathcal{A}} \oplus (\mathcal{O}_2)_{\mathcal{B}} \arrow[d] \\
\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2) \arrow[r] & \mathrm{SC}'(\mathcal{O}_1, \mathcal{O}_2)
\end{tikzcd}$$

But the left vertical morphism is a cofibration and the top morphism is a quasi-isomorphism, hence the bottom morphism $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2) \to \mathrm{SC}'(\mathcal{O}_1, \mathcal{O}_2)$ is a quasi-isomorphism.

We can identify the projection $\mathrm{SC}'(\mathcal{O}_1, \mathcal{O}_2)$ in arities $(\mathcal{A}^{\otimes -} \otimes \mathcal{B}^0, \mathcal{B})$ with the morphism of symmetric sequences $\mathcal{C}_1 \circ_{\mathrm{d}} \mathcal{O}_1 \to \mathbf{1}$, where the differential d comes from the morphism $\Omega\mathcal{C}_1 \to \mathcal{O}_1$. But since the latter morphism is a quasi-isomorphism, by [17, Theorem 6.6.2] the former morphism is a quasi-isomorphism as well. The claim in arities $(\mathcal{A}^{\otimes -} \otimes \mathcal{B}^l, \mathcal{B})$ is proved similarly. □

3.4 Relative Poisson algebras

In this section we define the main operad to be used in this paper.

Definition 3.17 Let B be a $\mathbb{P}_n$-algebra. Its *strict Poisson center* is defined to be the $\mathbb{P}_{n+1}$-algebra

$$\mathrm{Z}(B) = \mathrm{Hom}_{\mathrm{Mod}_B}(\mathrm{Sym}_B(\Omega^1_B[n]), B).$$

The Lie bracket on $\mathrm{Z}(B)$ is given by the Schouten bracket of polyvector fields, we refer to [24, Sect. 1.1] for explicit formulas.

Definition 3.18 A unital $\mathbb{P}_{[n+1,n]}$-algebra consists of the following triple:

- A unital $\mathbb{P}_{n+1}$-algebra A,
- A unital $\mathbb{P}_n$-algebra B,
- A morphism of unital $\mathbb{P}_{n+1}$-algebras $f\colon A \to \mathrm{Z}(B)$.

We denote by $\mathbb{P}_{[n+1,n]}$ the colored operad controlling such algebras, see [24, Sect. 1.3] for explicit relations in the operad. Similarly, we denote by $\mathbb{P}^{\mathrm{nu}}_{[n+1,n]}$ the non-unital version of this operad.

Given a $\mathbb{P}_{[n+1,n]}$-algebra (A, B, f), the morphism

$$A \longrightarrow \mathrm{Z}(B) \longrightarrow B$$

is strictly compatible with the multiplications, so it defines a forgetful functor

$$\mathrm{Alg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{Arr}(\mathrm{Alg}_{\mathrm{Comm}}).$$

Our goal now is to define a cofibrant resolution of the operad $\mathbb{P}_{[n+1,n]}$. The cooperad $\mathrm{co}\mathbb{P}^{\theta}_n$ of curved non-unital $\mathbb{P}_n$-coalgebras has a Hopf counital structure $\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_n$ given by the cooperad of curved counital $\mathbb{P}_n$-coalgebras. Calaque and Willwacher [4] define a morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}) \to \mathrm{Br}_{\mathrm{co}\mathbb{P}^{\theta}_n} \tag{7}$$

on the generators by the following rule:

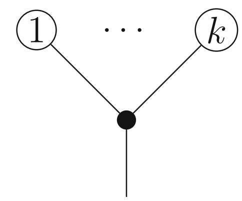


Fig. 12 Image of $\underline{x_1 \dots x_k}$

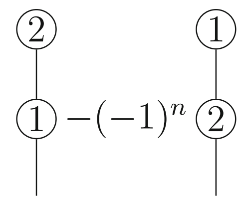


Fig. 13 Image of $x_1 \wedge x_2$

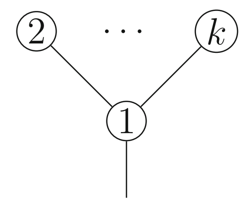


Fig. 14 Image of $x_1 \wedge \underline{x_2 \dots x_k}$

- The generators

$$x \in \mathrm{coLie}^{\theta}\{1-n\}(k) \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k)$$

  are sent to the tree drawn in Fig. 12 with the root labeled by the element

$$x \in \mathrm{coLie}^{\theta}\{1-n\}(k) \subset \mathrm{co}\mathbb{P}^{\theta}_{n}(k).$$

- The generator

$$x_1 \wedge x_2 \in \mathrm{coComm}\{1\}(2) \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(2)$$

  is sent to the linear combination of trees shown in Fig. 13.
- Given $y \in \mathrm{coLie}^{\theta}\{1-n\}(k-1) \subset \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k-1)$ for $k > 2$, we denote by $x \wedge y$ its image under $\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k-1) \to \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k)$.
  The generators

$$x \wedge y \in \mathrm{co}\mathbb{P}^{\theta}_{n+1}\{1\}(k)$$

  are sent to the tree shown in Fig. 14 with the root labeled by the element

$$y \in \mathrm{coLie}^{\theta}\{1-n\}(k-1) \subset \mathrm{co}\mathbb{P}^{\theta}_{n}(k-1).$$

- The rest of the generators are sent to zero.

Note that the composite

$$\Omega(\mathrm{coComm}\{n+1\}) \subset \Omega(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}) \to \mathrm{Br}_{\mathrm{co}\mathbb{P}_n}\{n\}$$

gives a strict Lie structure and it coincides with the morphism Lie $\to$ preLie as easily seen from Fig. 13. We define

$$\widetilde{\mathbb{P}}_{[n+1,n]} = \mathrm{SC}(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}^{\theta}_{n}),$$

so by Corollary 3.15 a $\widetilde{\mathbb{P}}_{[n+1,n]}$-algebra is a homotopy unital $\mathbb{P}_{n+1}$-algebra A, a homotopy unital $\mathbb{P}_n$-algebra B and an ∞-morphism of homotopy unital $\mathbb{P}_{n+1}$-algebras $A \to \mathbf{Z}(B)$.

Similarly, one has a morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}\{1\}) \to \mathrm{Br}_{\mathrm{co}\mathbb{P}_n}$$

and thus we can define the non-unital version of the operad $\widetilde{\mathbb{P}}^{\mathrm{nu}}_{[n+1,n]}$:

$$\widetilde{\mathbb{P}}^{\mathrm{nu}}_{[n+1,n]} = \mathrm{SC}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_{n}).$$

Proposition 3.19 *The natural morphism of colored operads*

$$\widetilde{\mathbb{P}}^{\mathrm{nu}}_{[n+1,n]} \longrightarrow \mathbb{P}^{\mathrm{nu}}_{[n+1,n]}$$

is a quasi-isomorphism.

Proof Let $\mathbb{P}^{w}_{[n+1,n]}$ be the colored operad obtained as a quotient of $\widetilde{\mathbb{P}}^{\mathrm{nu}}_{[n+1,n]}$ whose algebras are strict non-unital $\mathbb{P}_{n+1}$-algebras A, strict non-unital $\mathbb{P}_n$-algebras B and a strict morphism of homotopy $\mathbb{P}_{n+1}$-algebras $A \to \mathbf{Z}(B)$. We have morphisms of colored operads

$$\widetilde{\mathbb{P}}^{\mathrm{nu}}_{[n+1,n]} \longrightarrow \mathbb{P}^{w}_{[n+1,n]} \longrightarrow \mathbb{P}^{\mathrm{nu}}_{[n+1,n]},$$

where the first morphism is a quasi-isomorphism by Proposition 3.16 where we note that $\mathbb{P}^{w}_{[n+1,n]} = \mathrm{SC}(\mathbb{P}^{\mathrm{nu}}_{n+1}, \mathbb{P}^{\mathrm{nu}}_{n})$. Therefore, it is enough to show that the second morphism is a quasi-isomorphism. For this it is enough to show that the morphism in arities $(\mathcal{A}^{\otimes m} \otimes \mathcal{B}^{\otimes -}, \mathcal{B})$ is a quasi-isomorphism. The following argument is similar to the proof of [25, Proposition 3.4].

The relevant morphism is an isomorphism if $m = 0$. We will give the proof for $m = 1$ since the case of higher m is similar.

If B is a $\mathbb{P}_n$-algebra and M a $\mathbb{P}_n$-module over B, we can consider its Poisson homology $\mathrm{C}^{\mathbb{P}_n}_{\bullet}(B, M)$ which as a graded vector space is isomorphic to

$$\mathrm{co}\mathbb{P}_n(M[n]) \otimes_k B \cong \mathrm{Sym}_{\geq 1}(\mathrm{coLie}(M[1])[n-1]) \otimes_k B.$$

We also have the *canonical* chain complex $\mathrm{C}^{can}_\bullet(B, M)$ which as a graded vector space is isomorphic to $\mathrm{Sym}_{\geq 1}(\Omega^1_M[n]) \otimes_k B$. We refer to [10, Sect. 1.3, Section 1.4.2] for explicit formulas for the differentials on these complexes.

Consider an arbitrary complex V. We can identify the colored operad $\mathbb{P}^w_{[n+1,n]}$ in arity $(\mathcal{A}^{\otimes 1} \otimes \mathcal{B}^{\otimes l}, \mathcal{B})$ with the coefficient of $V^{\otimes l}$ in $\mathrm{C}^{\mathbb{P}_n}_\bullet(\mathbb{P}_n(V), \mathbb{P}_n(V))$. Similarly, we can identify the colored operad $\mathbb{P}^{\mathrm{nu}}_{[n+1,n]}$ in the same arity with the coefficient of $V^{\otimes l}$ in $\mathrm{C}^{can}_\bullet(\mathbb{P}_n(V), \mathbb{P}_n(V))$. But since $\mathbb{P}_n(V)$ is free, the natural projection

$$\mathrm{C}^{\mathbb{P}_n}_\bullet(\mathbb{P}_n(V), \mathbb{P}_n(V)) \to \mathrm{C}^{can}_\bullet(\mathbb{P}_n(V), \mathbb{P}_n(V))$$

is a quasi-isomorphism which proves the claim. □

One proves similarly that the morphism of colored operads $\widetilde{\mathbb{P}}_{[n+1,n]} \to \mathbb{P}_{[n+1,n]}$ is a quasi-isomorphism.

Let us now explain how to construct graded versions of these operads. Recall that throughout the paper we consider the grading on $\mathbb{P}_n$ such that the bracket has weight -1 and the multiplication has weight 0. In this section we consider a different convention where the bracket has weight 0, multiplication weight 1 and the unit weight -1. In particular, A is a graded $\mathbb{P}_n$-algebra with respect to the current convention iff $A \otimes k(-1)$ is a graded $\mathbb{P}_n$-algebra in the original convention.

Define the grading on the Hopf cooperad $\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_n$ to be such that the comultiplication has weight 0 and the cobracket has weight 1. It is compatible with the Hopf structure making it into a graded Hopf cooperad. Also observe that the morphisms

$$\Omega(\mathrm{co}\mathbb{P}^{\theta}_n\{n\}) \longrightarrow \mathbb{P}_n$$

and (7) are compatible with gradings. Thus, if A is a graded $\mathbb{P}_n$-algebra, its Poisson center $\mathbf{Z}(A)$ becomes a graded homotopy $\mathbb{P}_{n+1}$-algebra.

3.5 From relative Poisson algebras to Poisson algebras

Suppose that one has a $\mathbb{P}_{[n+1,n]}$-algebra (A, B, f) in M. In this section we show how to produce a $\mathbb{P}_{n+1}$-structure on the homotopy fiber of the underlying map of commutative algebras $A \to B$.

Recall that the strict Poisson center of a $\mathbb{P}_n$-algebra B is defined to be the algebra of polyvectors

$$\mathrm{Z}(B) = \mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_B[n]), B)$$

with the differential twisted by the Maurer–Cartan element $[\pi_B, -]$ defining the Poisson bracket.

Definition 3.20 Let B be a $\mathbb{P}_n$-algebra in M. Its *strict deformation complex* is defined to be the algebra of polyvectors

$$\mathrm{Def}(B) = \mathrm{Hom}_B(\mathrm{Sym}^{\geq 1}_B(\Omega^1_B[n]), B)$$

with the differential twisted by $[\pi_B, -]$.

As for the center, the deformation complex $\mathrm{Def}(B)[-n]$ is a $\mathbb{P}_{n+1}$-algebra, albeit non-unital.

We have a fiber sequence

$$\mathrm{Def}(B)[-n] \to \mathrm{Z}(B) \to B$$

in M. Rotating it, we obtain a homotopy fiber sequence

$$B[-1] \to \mathrm{Def}(B)[-n] \to \mathrm{Z}(B), \tag{8}$$

where the morphism $B[n-1] \to \mathrm{Def}(B)$ is given by $b \mapsto [\pi_B, b]$. Note that $B[n-1]$ is a Lie algebra with respect to the Poisson bracket.

Proposition 3.21 *Let B be a $\mathbb{P}_n$-algebra in M. Then the morphism*

$$B[n-1] \to \mathrm{Def}(B)$$

given by $b \mapsto [\pi_B, b]$ is a morphism of Lie algebras.

Proof Indeed, compatibility with the brackets is equivalent to the equation

$$\{\{b_1, b_2\}, b_3\} = \{b_1, \{b_2, b_3\}\} - (-1)^{|b_1||b_2|}\{b_2, \{b_1, b_3\}\}$$

for $b_i \in B[n-1]$ which is exactly the Jacobi identity in the Lie algebra $B[n-1]$. □

Since the morphism $\mathrm{Def}(B)[-n] \to \mathrm{Z}(B)$ is a morphism of non-unital $\mathbb{P}_{n+1}$-algebras, the sequence (8) can be upgraded to a fiber sequence in $\mathbf{Alg}_{\mathbb{P}^{\mathrm{nu}}_{n+1}}(\mathcal{M})$. In particular, $B[-1]$ carries a homotopy non-unital $\mathbb{P}_{n+1}$-structure such that the underlying Lie structure by Proposition 3.21 coincides with the one on B.

Remark 3.22 The fiber sequence of non-unital $\mathbb{P}_{n+1}$-algebras (8) is a $\mathbb{P}_n$-analog of the sequence

$$B[-1] \to \mathrm{Def}(B)[-n] \to \mathrm{HH}^\bullet_{\mathbb{E}_n}(B)$$

of non-unital $\mathbb{E}_{n+1}$-algebras constructed by Francis in [9, Theorem 4.25] (see also [18, Sect. 5.3.2]) when B is an $\mathbb{E}_n$-algebra.

Next, suppose (A, B, f) is a $\mathbb{P}_{[n+1,n]}$-algebra in M. Then we can construct a commutative diagram of non-unital $\mathbb{P}_{n+1}$-algebras

$$\begin{tikzcd}
B[-1] \arrow[r] \arrow[d, equal] & \mathrm{U}(A, B) \arrow[r] \arrow[d] & A \arrow[d] \\
B[-1] \arrow[r] & \mathrm{Def}(B)[-n] \arrow[r] & \mathrm{Z}(B)
\end{tikzcd}$$

where both rows are homotopy fiber sequences and the square on the right is Cartesian. This defines $\mathrm{U}(A, B)$, a non-unital $\mathbb{P}_{n+1}$-algebra, which as a Lie algebra fits into a fiber sequence

$$B[n-1] \to \mathrm{U}(A, B)[n] \to A[n].$$

Proceeding as in [3, Definition 1.4.15] it defines a forgetful functor

$$\mathrm{U}\colon \mathbf{Alg}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M}) \longrightarrow \mathbf{Alg}_{\mathbb{P}^{\mathrm{nu}}_{n+1}}(\mathcal{M}).$$

Let us now give explicit formulas for the Lie brackets on $\mathrm{U}(A, B)$. From the $\mathbb{P}_{n+1}$-morphism $A \to \mathrm{Z}(B)$ we obtain a morphism of Lie algebras

$$A[n] \to \mathrm{Z}(B)[n] \to \mathrm{Hom}(\mathrm{Sym}(B[n]), B[n]),$$

where the object on the right is equipped with the convolution Lie bracket. Therefore, by results of [15, Sect. 3] we obtain an L_∞ structure on $A[n] \oplus B[n-1]$. The brackets are as follows:

- The bracket on $A[n]$ is the Poisson bracket on A,
- The bracket on $B[n-1]$ is the Poisson bracket on B,
- The mixed brackets $A[n] \otimes (B[n-1])^{\otimes k} \to B[n-k]$ are given by the components $f_k\colon A \to \mathrm{Hom}(\mathrm{Sym}^k(B[n]), B)$ of the morphism above.

3.6 Mixed structures on relative Poisson algebras

Let A be a graded $\mathbb{P}_{n+1}$-algebra in $\mathcal{M}$. Moreover, suppose we have a morphism

$$a\colon k(2)[-1] \longrightarrow A[n]$$

of graded Lie algebras. The bracket defines a morphism $A[n] \to \mathrm{Der}(A, A) \otimes k(1)$ of graded Lie algebras and hence we obtain a morphism

$$k(2)[-1] \longrightarrow \mathrm{Der}(A, A) \otimes k(1)$$

of graded Lie algebras, i.e. the graded commutative algebra A is enhanced to a graded mixed algebra. Let A_0 be the weight 0 component of A. By the universal property of the de Rham algebra (see [3, Proposition 1.3.8]) we obtain a morphism

$$\mathbf{DR}^{int}(A_0) \longrightarrow A$$

of graded mixed commutative algebras.

More explicitly, given an element $a \in A$ we construct a graded mixed structure on A given by $\epsilon_A = [a, -]$. The morphism $\mathbf{DR}^{int}(A_0) \to A$ is given by

$$x \mathrm{d}_{\mathrm{dR}} y_1 \mathrm{d}_{\mathrm{dR}} y_2 \dots \mathrm{d}_{\mathrm{dR}} y_n \mapsto x[a, y_1][a, y_2] \dots [a, y_n].$$

We proceed to a relative version of this construction. Fix a graded $\mathbb{P}_{[n+1,n]}$-algebra (A, B, f) in M and suppose

$$(a, b)\colon k(2)[-1] \longrightarrow \mathrm{U}(A, B)[n]$$

is a ∞-morphism of graded L_∞-algebras. By Proposition 1.19 it is given by a collection of elements $a = a_2 + \cdots$ and $b = b_2 + \cdots$ satisfying the Maurer–Cartan equation, where the subscript denotes the weight. Twisting the L_∞ brackets on $\mathrm{U}(A, B)[n]$ by (a, b) as in Definition 1.17, we obtain mixed structures on A and B. Using the explicit description of the L_∞ brackets on $\mathrm{U}(A, B)[n]$ given in Sect. 3.5, we see that the mixed structure ϵ_A on A is given by $\epsilon_A = [a, -]$ while the mixed structure ϵ_B on B is given by

$$\epsilon_B = [b, -] + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} f_k(a; b, \ldots, b, -).$$

It is clear from these formulas that the mixed structures ϵ_A and ϵ_B are derivations of the commutative multiplication, thus both A and B become weak graded mixed commutative algebras. Moreover, the morphism $f_0\colon A \to B$ extends to an ∞-morphism of weak graded mixed commutative algebras whose components are given by $\frac{1}{k!} f_k(-; b, \ldots, b)$. The corresponding relation between the mixed structures on A and B simply follows by observing that twisting the differential on an L_∞-algebra produces a mixed structure.

In this way we construct an ∞-morphism of L_∞-algebras

$$\mathrm{U}(A, B)[n] \to \mathrm{Der}(A \to B, A \to B) \otimes k(1),$$

where we consider $A \to B$ as a commutative algebra in $\mathrm{Arr}(M)$. Moreover, this can be enhanced to a natural transformation of functors

$$\mathbf{Alg}^{gr}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M})^{\sim} \to \mathbf{Alg}^{gr}_{\mathrm{Lie}}.$$

Considering again $A_0 \to B_0$ as a commutative algebra in $\mathrm{Arr}(\mathcal{M})$ we can identify

$$\mathbf{DR}^{int}(A_0 \to B_0) \cong (\mathbf{DR}^{int}(A_0) \to \mathbf{DR}^{int}(B_0)).$$

Therefore, given a graded $\mathbb{P}_{[n+1,n]}$-algebra (A, B, f) we obtain a diagram

$$\begin{tikzcd}
\mathbf{DR}^{int}(A_0) \arrow[r] \arrow[d] & \mathbf{DR}^{int}(B_0) \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

of graded mixed commutative algebras in $\mathcal{M}$, where A and B are endowed with the graded mixed structure constructed above.

4 Coisotropic structures on affine derived schemes

In Sects. 2 and 3 we have established the necessary facts about convolution and relative convolution algebras. In this section we use those notions to define polyvector and relative polyvector fields. We also define the notion of Poisson and coisotropic structures and compute these spaces explicitly in terms of Maurer–Cartan elements in the Lie algebras of polyvectors and relative polyvectors respectively.

4.1 Poisson structures and polyvectors

Let A be a commutative algebra in M which we view as a graded $\mathbb{P}_{n+1}$-algebra with the trivial bracket. Recall from Definition 3.7 its center $\mathbf{Z}(A)$ which is a graded $\mathrm{Br}_{\mathrm{co}\mathbb{P}^{\theta}_{n+1}}$-algebra and hence a graded homotopy $\mathbb{P}_{n+2}$-algebra.

Definition 4.1 Let A be a commutative algebra in M. Its algebra of *n-shifted polyvectors* $\mathbf{Pol}(A, n)$ is the graded homotopy $\mathbb{P}_{n+2}$-algebra $\mathbf{Z}(A)$.

By considering the internal Hom in M, we can upgrade $\mathbf{Pol}(A, n)$ to a graded homotopy $\mathbb{P}_{n+2}$-algebra in M that we denote by $\mathbf{Pol}^{int}(A, n)$ and call the algebra of *internal n-shifted polyvectors*.

Let $\mathrm{CAlg}^{fet}(M) \subset \mathrm{CAlg}(M)$ be the wide subcategory of commutative algebras in M where we only consider morphisms which are formally étale, i.e. morphisms $A \to B$ such that the pullback morphism $\Omega^1_A \otimes_A B \to \Omega^1_B$ is a quasi-isomorphism. Denote by $\mathbf{CAlg}^{fet}(\mathcal{M})$ its localization. Then as in [3, Definition 1.4.15] one can upgrade $\mathbf{Pol}^{int}(-, n)$ to a functor of ∞-categories

$$\mathbf{Pol}^{int}(-, n)\colon \mathbf{CAlg}^{fet}(\mathcal{M}) \longrightarrow \mathbf{Alg}_{\mathbb{P}_{n+2}}(\mathcal{M}^{gr}).$$

The complex

$$\mathrm{Pol}(A, n) = \mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]), A)$$

carries a natural graded $\mathbb{P}_{n+2}$-algebra structure where Ω^1_A has weight -1 and where the Lie bracket is given by the Schouten bracket (see [24, Sect. 1.1] for explicit formulas). The following statement explains the term "polyvectors":

Proposition 4.2 *Let A be a bifibrant commutative algebra in M. Then one has an equivalence of graded $\mathbb{P}_{n+2}$-algebras*

$$\mathbf{Pol}(A, n) \cong \mathrm{Pol}(A, n).$$

Proof By definition we have

$$\mathbf{Pol}(A, n) \cong \mathrm{Hom}_A(\mathrm{Sym}_A(\mathrm{Harr}_\bullet(A, A)[n+1])A).$$

The morphism $\mathrm{Harr}_\bullet(A, A) \to \Omega^1_A$ induces a morphism

$$\mathrm{Pol}(A, n) \longrightarrow \mathbf{Pol}(A, n)$$

which is strictly compatible with the homotopy $\mathbb{P}_{n+2}$-structures by [4, Theorem 1].

Since A is a cofibrant commutative algebra, it is also cofibrant in M, so by Proposition 2.6 the morphism $\mathrm{Harr}_\bullet(A, A) \to \Omega^1_A$ is a weak equivalence between cofibrant A-modules and hence so is

$$\mathrm{Sym}_A(\mathrm{Harr}_\bullet(A, A)[n+1]) \longrightarrow \mathrm{Sym}_A(\Omega^1_A[n+1]).$$

Since A is fibrant, the functor $\mathrm{Hom}_A(-, A)$ preserves weak equivalences. $\square$

Corollary 4.3 *For any commutative algebra A in $\mathcal{M}$ we have an equivalence of graded objects*

$$\mathbf{Pol}(A, n) \cong \mathrm{Hom}_A(\mathrm{Sym}_A(\mathbb{L}_A[n+1]), A).$$

In particular, if $\mathbb{L}_A$ is perfect, we get an equivalence

$$\mathbf{Pol}(A, n) \cong \mathrm{Sym}_A(\mathbb{T}_A[-n-1]).$$

Definition 4.4 Let A be a commutative algebra in $\mathcal{M}$, The *space of n-shifted Poisson structures* $\mathrm{Pois}(A, n)$ is given by the fiber of the forgetful functor

$$\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M})^{\sim} \to \mathbf{CAlg}(\mathcal{M})^{\sim}$$

at $A \in \mathbf{CAlg}(\mathcal{M})$.

One has an explicit way to compute the space of shifted Poisson structures in terms of the algebra of polyvectors. The following theorem is a version of [3, Theorem 1.4.9] and [20, Theorem 3.2]:

Theorem 4.5 *Let A be a commutative algebra in $\mathcal{M}$. One has an equivalence of spaces*

$$\mathrm{Pois}(A, n) \cong \mathrm{Map}_{\mathbf{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(A, n)[n+1]).$$

Proof Assume A is a bifibrant commutative algebra in M. By [23] (see also [28]) we can identify the space $\mathrm{Pois}(A, n)$ with the homotopy fiber of

$$\mathrm{Map}_{\mathrm{Op}_k}(\mathbb{P}_{n+1}, \mathrm{End}_A) \to \mathrm{Map}_{\mathrm{Op}_k}(\mathrm{Comm}, \mathrm{End}_A)$$

at the given commutative structure on A.

We have a commutative diagram of spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Op}_k}(\mathbb{P}_{n+1}, \mathrm{End}_A) \arrow[r] \arrow[d, "\sim"] & \mathrm{Map}_{\mathrm{Op}_k}(\mathrm{Comm}, \mathrm{End}_A) \arrow[d, "\sim"] \\ \underline{\mathrm{MC}}(\mathrm{Conv}(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{n+1\}; A)) \arrow[r] & \underline{\mathrm{MC}}(\mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A)) \end{tikzcd}$$

with the vertical weak equivalences given by Proposition 2.2.

By Lemma 1.18 we can identify the fiber of the bottom map with the space of Maurer–Cartan elements in the dg Lie algebra

$$\mathrm{Hom}_k(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}^{\theta}(A[1])[n]), A) \cong \mathrm{Hom}_A(\mathrm{Sym}^{\geq 2}_A(\mathrm{Harr}_{\bullet}(A, A)[n+1]), A)$$

which can be identified with the completion of $\mathbf{Pol}(A, n)$ in weights 2 and above. The claim therefore follows from Proposition 1.19. $\square$

Remark 4.6 The main difference with the formulation we give and the one in [20] and [3] is that we consider the non-strict version of polyvectors. To obtain the same result, one should couple Theorem 4.5 with Proposition 4.2.

If A is a $\mathbb{P}_{n+1}$-algebra, we define the opposite algebra to have the same multiplication and the bracket $\{a, b\}^{\mathrm{opp}} = -\{a, b\}$. This defines a morphism of spaces

$$\mathrm{opp}\colon \mathrm{Pois}(A, n) \longrightarrow \mathrm{Pois}(A, n).$$

It is easy to see that under the equivalence given by Theorem 4.5 it corresponds to the involution on $\mathbf{Pol}(A, n)$ given by multiplication by $(-1)^{k+1}$ in weight k.

4.2 Relative polyvectors

Suppose $f\colon A \to B$ is a morphism of commutative algebras in M. We regard A as a graded $\mathbb{P}_{n+1}$-algebra with the trivial bracket and B as a graded $\mathbb{P}_n$-algebra with the trivial bracket. In particular, the composite

$$A \xrightarrow{f} B \longrightarrow \mathbf{Pol}^{int}(B, n-1)$$

is strictly compatible with the $\mathbb{P}_{n+1}$-structures and we denote it by $\tilde{f}$.

Recall from Sect. 3.2 the complex

$$\mathrm{Conv}^0(\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_{n+1}\{n+1\}; A, \mathbf{Pol}^{int}(B, n-1)).$$

The morphism $\tilde{f}$ defines a Maurer–Cartan element in the L_∞ algebra

$$\mathcal{L}^0(\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_{n+1}\{n+1\}; A, \mathbf{Pol}^{int}(B, n-1))$$

and hence we can consider the twisted convolution algebra

$$\mathrm{Conv}^0_{\tilde{f}}(\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_{n+1}\{n+1\};\, A, \mathbf{Pol}^{int}(B, n-1)).$$

As shown in the same section, it carries a natural structure of a graded homotopy $\mathbb{P}_{n+1}$-algebra which we denote by $\mathbf{Pol}(B/A, n-1)$. Similarly, we can define its internal version $\mathbf{Pol}^{int}(B/A, n-1)$ which is a graded homotopy $\mathbb{P}_{n+1}$-algebra in M.

Proposition 4.7 *Let $f\colon A \to B$ be a cofibrant diagram of commutative algebras in M, where B is also fibrant. Then one has an equivalence of graded objects*

$$\mathbf{Pol}^{int}(B/A, n-1) \cong \mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_{B/A}[n]), B).$$

Proof Since B is bifibrant as a commutative algebra, it is enough to prove that the projection

$$\mathrm{Sym}(\mathrm{coLie}(A[1])[n]) \otimes \mathrm{Sym}_B(\Omega^1_B[n]) \to \mathrm{Sym}(\Omega^1_{B/A}[n])$$

induced by the morphism $\Omega^1_B \to \Omega^1_{B/A}$ induces a weak equivalence after passing to left realizations. This will follow once we prove that

$$\mathrm{Harr}_\bullet(A, A) \otimes_A B \oplus \Omega^1_B \to \Omega^1_{B/A}$$

induces a weak equivalence of B-modules after passing to left realizations. Here the grading is inherited from the Harrison complex on the left-hand side and given by putting the Kähler differentials in weight 0. The mixed structure on the right-hand side is trivial. The mixed structure on the left-hand side is a sum of two terms:

(1) The first term is the usual Harrison differential.
(2) The second term is given by the composite

$$A \otimes B \xrightarrow{\mathrm{d_{dR}}\otimes \mathrm{id}} \Omega^1_A \otimes_A B \to \Omega^1_B,$$

where $A \otimes B$ is the weight -1 part of $\mathrm{Harr}_\bullet(A, A) \otimes_A B$.

Since A is cofibrant, by Proposition 2.6 it is enough to prove that

$$\Omega^1_A \otimes_A B[1](-1) \oplus \Omega^1_B \to \Omega^1_{B/A},$$

where the left-hand side is equipped with the mixed structure given by the pullback of differential forms $\Omega^1_A \otimes_A B \to \Omega^1_B$ induces a weak equivalence after passing to left realizations. But since $A \to B$ is a cofibrant diagram, the natural sequence of B-modules

$$\Omega^1_A \otimes_A B \to \Omega^1_B \to \Omega^1_{B/A}$$

is exact and by Proposition 1.7 this finishes the proof. □

Using the morphism $f: A \to B$ one can regard B as a commutative algebra in $\mathrm{Mod}_A(M)$. In particular, we can compute internal strict polyvectors of B in $\mathrm{Mod}_A(M)$ which we denote by $\mathrm{Pol}_A^{int}(B, n-1)$. It is easy to see that the composite

$$\mathrm{Pol}_A^{int}(B, n-1) \to \mathbf{Pol}^{int}(B, n-1) \to \mathbf{Pol}^{int}(B/A, n-1)$$

is strictly compatible with the homotopy $\mathbb{P}_{n+1}$-structures.

Remark 4.8 Note that the map $\mathbf{Pol}^{int}(B, n-1) \to \mathbf{Pol}^{int}(B/A, n-1)$ is *not* compatible with differentials and is not model-independent. However, it is easy to see that the composite map $\mathrm{Pol}_A^{int}(B, n-1) \to \mathbf{Pol}^{int}(B/A, n-1)$ is compatible with the differentials.

Corollary 4.9 *Let $A \to B$ be a cofibrant diagram of commutative algebras in M, where B is also fibrant. Then the morphism*

$$\mathrm{Pol}_A^{int}(B, n-1) \longrightarrow \mathbf{Pol}^{int}(B/A, n-1)$$

is a weak equivalence of homotopy $\mathbb{P}_{n+1}$-algebras in M.

Proof It is enough to prove that the morphism is a weak equivalence of graded objects in M which follows from Propositions 4.7 and 4.2. □

By Proposition 3.9 we have an ∞-morphism of graded homotopy $\mathbb{P}_{n+1}$-algebras

$$\mathbf{Pol}^{int}(A, n) \to \mathbf{Z}(\mathbf{Pol}^{int}(B/A, n-1))$$

and hence $(\mathbf{Pol}^{int}(A, n), \mathbf{Pol}^{int}(B/A, n-1))$ forms a graded homotopy $\mathbb{P}_{[n+2,n+1]}$-algebra in M. Similarly, $(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$ is a graded homotopy $\mathbb{P}_{[n+2,n+1]}$-algebra in complexes.

Definition 4.10 Let $f: A \to B$ be a morphism of commutative algebras in M. The *algebra of relative n-shifted polyvector fields* is the graded homotopy non-unital $\mathbb{P}_{n+2}$-algebra

$$\mathbf{Pol}(f, n) = \mathrm{U}(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1)).$$

As before, we can upgrade it to a functor

$$\mathbf{Pol}(-, n)\colon \mathrm{Arr}(\mathbf{CAlg}^{fet}(\mathcal{M})) \to \mathbf{Alg}_{\mathbb{P}_{[n+2,n+1]}}(\mathcal{M}^{gr}).$$

Note that as a graded vector space we can identify

$$\mathbf{Pol}(f, n)[n+1] \cong \mathbf{Pol}(A, n)[n+1] \oplus \mathbf{Pol}(B/A, n-1)[n].$$

In particular, we get a morphism of graded vector spaces

$$\mathbf{Pol}(f, n)[n+1] \to \mathbf{Pol}(B/A, n-1)[n] \to \mathbf{Pol}(B, n-1)[n].$$

Restricting to weight 0 we get the morphism $A[n+1] \oplus B[n] \to B[n]$ which is *not* compatible with the differentials. Nevertheless, in highest weight this phenomenon does not occur.

Proposition 4.11 *The morphism* $\mathbf{Pol}^{\geq 1}(f, n)[n+1] \to \mathbf{Pol}^{\geq 1}(B, n-1)[n]$ *is a morphism of filtered* L_∞ *algebras.*

Proof Recall from Sect. 3.5 that the L_∞ brackets on $\mathbf{Pol}(A, n)[n+1] \oplus \mathbf{Pol}(B/A, n-1)[n]$ are given by the original brackets on each summand and the mixed L_∞ brackets between $\mathbf{Pol}(A, n)$ and $\mathbf{Pol}(B/A, n-1)$ which land in $\mathbf{Pol}(B/A, n-1)$.

The map $\mathbf{Pol}(B/A, n-1)[n] \to \mathbf{Pol}(B, n-1)[n]$ is compatible with the Lie structures, so we need to show that the image of the mixed brackets vanishes. By definition they come from the action of the convolution algebra

$$\mathbf{Pol}(A, n) = \mathrm{Conv}^0(\mathrm{co}\mathbb{P}_{n+1}^{\theta,\mathrm{cu}}\{n+1\}; A)$$

on the relative convolution algebra

$$\mathbf{Pol}(B/A, n-1) = \mathrm{Conv}^0(\mathrm{co}\mathbb{P}_{n+1}^{\theta,\mathrm{cu}}\{n+1\}; A, \mathbf{Pol}(B, n-1))$$

given by the last tree in Fig. 8. But elements in $\mathbf{Pol}^{\geq 1}(A, n)$ have at least one A input, so the result in $\mathrm{Conv}^0(\mathrm{co}\mathbb{P}_{n+1}^{\theta,\mathrm{cu}}\{n+1\}; A, \mathbf{Pol}(B, n-1))$ will have at least one A input. However, the projection map

$$\mathrm{Conv}^0(\mathrm{co}\mathbb{P}_{n+1}^{\theta,\mathrm{cu}}\{n+1\}; A, \mathbf{Pol}(B, n-1)) \to \mathbf{Pol}(B, n-1)$$

annihilates all such elements which proves the claim. $\square$

We get a diagram of Lie algebras

$$\begin{tikzcd} & \mathbf{Pol}^{\geq 2}(f, n)[n+1] \arrow[dl] \arrow[dr] & \\ \mathbf{Pol}^{\geq 2}(B, n-1)[n] & & \mathbf{Pol}^{\geq 2}(A, n)[n+1] \end{tikzcd}$$

The Lie algebra $\mathbf{Pol}(f, n)[n+1]$ is quite complicated, so in practice we will use the following strict model. Define the graded non-unital $\mathbb{P}_{n+2}$-algebra

$$\mathrm{Pol}(f, n) = \ker(\mathrm{Pol}(A, n) \longrightarrow \mathrm{Pol}_A(B, n-1)),$$

where the graded non-unital $\mathbb{P}_{n+2}$-structure on $\mathrm{Pol}(f, n)$ is induced from the one on $\mathrm{Pol}(A, n)$.

Proposition 4.12 *Suppose* $f: A \to B$ *is a bifibrant object of* $\mathrm{Arr}(\mathrm{CAlg}(M))$. *Moreover, assume that*

$$\mathrm{Pol}(A, n) \longrightarrow \mathrm{Pol}_A(B, n-1)$$

is surjective. Then we have a quasi-isomorphism of L_∞ algebras

$$\mathrm{Pol}(f,n)[n+1] \cong \mathbf{Pol}(f,n)[n+1].$$

Proof The Lie algebra $\mathbf{Pol}(f,n)[n+1]$ is given by the complex

$$\mathbf{Pol}(A,n)[n+1] \oplus \mathbf{Pol}(B/A,n-1)[n]$$

with the differential from the first term to the second term given by the morphism

$$\mathbf{Pol}(A,n) \longrightarrow \mathbf{Pol}(B/A,n-1)$$

and the following brackets:

- The convolution bracket on $\mathbf{Pol}(A,n)$.
- The convolution bracket on $\mathbf{Pol}(B/A,n-1)$.
- The action of $\mathbf{Pol}(A,n)$ on $\mathbf{Pol}(B/A,n-1)$.

By Proposition 4.2 the morphism $\mathrm{Pol}(A,n) \to \mathbf{Pol}(A,n)$ is a quasi-isomorphism, so we can replace the above dg Lie algebra with

$$\mathrm{Pol}(A,n)[n+1] \oplus \mathbf{Pol}(B/A,n-1)[n].$$

Moreover, the inclusion $\mathrm{Pol}_A(B,n-1) \to \mathbf{Pol}(B/A,n-1)$ is a quasi-isomorphism. Extending it to a deformation retract and applying the homotopy transfer theorem [17, Theorem 10.3.9] we obtain a quasi-isomorphic L_∞ structure on the complex

$$\widetilde{\mathrm{Pol}}(f,n)[n+1] = \mathrm{Pol}(A,n)[n+1] \oplus \mathrm{Pol}_A(B,n-1)[n]$$

with the differential twisted by the morphism

$$\mathrm{Pol}(A,n) \longrightarrow \mathrm{Pol}_A(B,n-1).$$

Using the explicit formulas for the transferred structure, we see that all the brackets

$$[-,\dots,-]_m\colon (\mathrm{Pol}(A,n)[n+1])^{\otimes m} \to \mathrm{Pol}_A(B,n-1)[2+n-m]$$

vanish for $m > 1$. Therefore, the morphism

$$\mathrm{Pol}(f,n)[n+1] \to \widetilde{\mathrm{Pol}}(f,n)[n+1]$$

is strictly compatible with the L_∞ brackets and is a quasi-isomorphism by the surjectivity assumption. □

4.3 Coisotropic structures

Definition 4.13 Let $f\colon A \to B$ be a morphism of commutative algebras in $\mathcal{M}$. The *space of n-shifted coisotropic structures* $\mathrm{Cois}(f, n)$ is defined to be the fiber of the forgetful functor

$$\mathbf{Alg}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M})^{\sim} \longrightarrow \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M}))^{\sim}$$

at $f \in \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M}))$.

One has the following alternative point of view on coisotropic structures following [3, Sect. 3.4]. First of all, one has the following additivity statement for Poisson structures shown independently by Rozenblyum and the second author, see [25, Theorem 2.22]:

Theorem 4.14 *One has an equivalence of ∞-categories*

$$\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M}) \cong \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})).$$

In other words, a $\mathbb{P}_{n+1}$-algebra is equivalent to an associative algebra object in $\mathbb{P}_n$-algebras. Let us now denote by $\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))$ the ∞-category of pairs (A, B) of an associative algebra object A in $\mathbb{P}_n$-algebras and a left A-module B.

The following is [25, Corollary 3.8]:

Theorem 4.15 *One has a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathbf{Alg}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M}) \arrow[rr, "\sim"] \arrow[dr] & & \mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})) \arrow[dl] \\
& \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M})) &
\end{tikzcd}$$

Therefore, a coisotropic structure on the morphism $f\colon A \to B$ consists of a $\mathbb{P}_{n+1}$-algebra structure on A, a $\mathbb{P}_n$-algebra structure on B and an associative action of A on B, all compatible with the original morphism f.

Returning to our definition of coisotropic structures, we have forgetful maps

$$\begin{tikzcd}
& \mathrm{Cois}(f, n) \arrow[dl] \arrow[dr] & \\
\mathrm{Pois}(B, n-1) & & \mathrm{Pois}(A, n).
\end{tikzcd}$$

One has the following explicit way to compute this diagram of spaces.

Theorem 4.16 *Let $f\colon A \to B$ be a morphism of commutative algebras in $\mathcal{M}$. Then one has an equivalence of spaces*

$$\mathrm{Cois}(f, n) \cong \mathrm{Map}_{\mathbf{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]) \cong \underline{\mathrm{MC}}(\mathbf{Pol}^{\geq 2}(f, n)[n+1])$$

compatible with the diagram of spaces

$$\begin{tikzcd} & \mathrm{Cois}(f, n) \arrow[dl] \arrow[dr] & \\ \mathrm{Pois}(B, n-1) & & \mathrm{Pois}(A, n) \end{tikzcd}$$

on the left and the diagram of Lie algebras

$$\begin{tikzcd} & \mathbf{Pol}^{\geq 2}(f, n)[n+1] \arrow[dl] \arrow[dr] & \\ \mathbf{Pol}^{\geq 2}(B, n-1)[n] & & \mathbf{Pol}^{\geq 2}(A, n)[n+1] \end{tikzcd}$$

on the right.

We will give a proof of this theorem in the next section. Here is one application of the above theorem. Recall that classically the identity morphism from a Poisson manifold to itself is coisotropic. Similarly, we show that in the derived context there is a *unique* coisotropic structure on the identity.

Proposition 4.17 *Let A be a commutative algebra in $\mathcal{M}$ and* $\mathrm{id}\colon A \to A$ *the identity morphism. Then the forgetful map*

$$\mathrm{Cois}(\mathrm{id}, n) \to \mathrm{Pois}(A, n)$$

is a weak equivalence.

Proof We have a fiber sequence of graded Lie algebras

$$\mathbf{Pol}(A/A, n-1)[n] \longrightarrow \mathbf{Pol}(\mathrm{id}, n)[n+1] \longrightarrow \mathbf{Pol}(A, n)[n+1].$$

But $\mathbf{Pol}(A/A, n-1) \cong k$, hence the morphism $\mathbf{Pol}(\mathrm{id}, n)[n+1] \to \mathbf{Pol}(A, n)[n+1]$ becomes an equivalence after applying $\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], -)$. □

Definition 4.18 The forgetful map

$$\mathrm{Pois}(A, n) \longrightarrow \mathrm{Pois}(A, n-1)$$

is the composite

$$\mathrm{Pois}(A, n) \xleftarrow{\sim} \mathrm{Cois}(\mathrm{id}, n) \longrightarrow \mathrm{Pois}(A, n-1).$$

This forgetful map is the classical analog of the forgetful functor $\mathrm{Alg}_{\mathbb{E}_{n+1}} \to \mathrm{Alg}_{\mathbb{E}_n}$.

4.4 Proof of Theorem 4.16

Assume $f\colon A \to B$ is a bifibrant object in $\mathrm{Arr}(M)$ with respect to the projective model structure. Then $\mathrm{Cois}(f, n)$ is equivalent to the homotopy fiber of

$$\mathrm{Map}_{2\mathrm{Op}_k}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \to \mathrm{Map}_{\mathrm{Op}_k}(\mathrm{Comm}, \mathrm{End}_{A\to B})$$

at $f\colon A \to B$.

By Proposition 2.2 we can identify

$$\mathrm{Map}_{\mathrm{Op}_k}(\mathrm{Comm}, \mathrm{End}_{A\to B}) \cong \mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A \to B),$$

where we can identify $\mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A \to B)$ as a complex with

$$\ker(\mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A) \oplus \mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; B) \to \mathrm{Hom}(\mathrm{coLie}^{\theta}\{1\}(A), B)).$$

Recall from [8, Sect. 3.2] (a related construction also appears in [11]) the cylinder L_∞-algebra

$$\mathcal{L}(\mathrm{coLie}^{\theta}\{1\}; A, B) = \mathrm{Cyl}(\mathrm{coLie}^{\theta}\{1\}; A, B)^f$$

which fits into a *homotopy* fiber sequence of complexes

$$\begin{aligned}\mathcal{L}(\mathrm{coLie}^{\theta}\{1\}; A, B) &\longrightarrow \mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A) \oplus \mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; B)\\ &\longrightarrow \mathrm{Hom}(\mathrm{coLie}^{\theta}\{1\}(A), B).\end{aligned}$$

Since $A \to B$ is a cofibration in M and B is fibrant, the obvious morphism

$$\mathrm{Conv}(\mathrm{coLie}^{\theta}\{1\}; A \to B) \to \mathcal{L}(\mathrm{coLie}^{\theta}\{1\}; A, B)$$

is a quasi-isomorphism. Moreover, it is strictly compatible with the L_∞ brackets.

Combining Propositions 3.19 and 3.14 we obtain a commutative diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{2\mathrm{Op}_k}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Map}_{2\mathrm{Op}_k}(\mathrm{Comm}, \mathrm{End}_{A\to B}) \arrow[d, "\sim"] \\
\underline{\mathrm{MC}}(\mathcal{L}(\mathrm{co}\mathbb{P}^{\theta}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}^{\theta}_{n}; A, B)) \arrow[r] & \underline{\mathrm{MC}}(\mathcal{L}(\mathrm{coLie}^{\theta}\{1\}; A, B))
\end{tikzcd}$$

By Lemma 1.18 we can identify the fiber of the bottom map with the space of Maurer–Cartan elements in the L_∞ algebra

$$\begin{aligned}
&\mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}^\theta(A[1])[n]), A)[n+1]\\
&\oplus\mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}^\theta(B[1])[n-1]), B)[n]\\
&\oplus\mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}^\theta(A[1])[n])\otimes\mathrm{Sym}(\mathrm{coLie}^\theta(B[1])[n-1]), B)[n]\\
&\oplus\mathrm{Hom}(\mathrm{Sym}^{\geq 1}(\mathrm{coLie}^\theta(A[1])[n])\otimes\mathrm{Sym}^{\geq 1}(\mathrm{coLie}^\theta(B[1])[n-1]), B)[n].
\end{aligned}$$

We can identify the first term with the completion of $\mathbf{Pol}(A, n)[n+1]$ in weights 2 and above. The rest of the terms are identified with the completion of $\mathbf{Pol}(B/A, n-1)[n]$ in weights 2 and above. It is easy to see that this identification is compatible with the Lie bracket on $\mathbf{Pol}(A, n)[n+1]$ and the L_∞ brackets on $\mathbf{Pol}(B/A, n-1)[n]$.

The mixed brackets given by the aciton of $\mathbf{Pol}(A, n)[n+1]$ on $\mathbf{Pol}(B/A, n-1)[n]$ can be identified as follows. Recall from Proposition 3.9 that the ∞-morphism of $\mathrm{Br}_{\mathrm{co}\mathbb{P}^\theta_{n+1}}$-algebras

$$\mathbf{Pol}(A, n)[n+1] \longrightarrow \mathrm{Conv}^0_f(\mathrm{co}\mathbb{P}^{\theta,\mathrm{cu}}_{n+1}, \mathbf{Pol}(B/A, n-1)[n])$$

has an underlying strict Lie morphism which factors through

$$\mathbf{Pol}(A, n)[n+1] \longrightarrow \mathrm{End}(\mathbf{Pol}(B/A, n-1)[n])$$

and which is given by the precomposition map of the convolution algebra of A on the relative convolution algebra of B/A. But this exactly coincides with the mixed bracket in $\mathcal{L}(\mathrm{co}\mathbb{P}^\theta_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}^\theta_n; A, B)$ and hence we have identified $\mathrm{Cois}(f, n)$ with the space of Maurer–Cartan elements in $\mathbf{Pol}(f, n)^{\geq 2}$.

The claim therefore follows from Proposition 1.19.

4.5 Poisson morphisms

Recall that classically a Poisson morphism $f\colon X \to Y$ between two Poisson manifolds can be characterized by the property that the graph $X \to X \times \overline{Y}$ is coisotropic, where $\overline{Y}$ is the same manifold with the opposite Poisson structure. In this section we define a derived notion of Poisson morphisms and show that an analogue of this result holds in the derived setting as well.

Definition 4.19 Suppose A and B are commutative algebras in $\mathcal{M}$ and $f\colon A \to B$ is a morphism of commutative algebras. We define the *space* $\mathrm{Pois}(f, n)$ *of n-shifted Poisson structures* on the morphism f to be the fiber of the forgetful functor

$$\mathrm{Arr}(\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M}))^\sim \longrightarrow \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M}))^\sim$$

at $f\colon A \to B$.

We can also consider the morphism $f: A \to B$ as a commutative algebra in $\mathrm{Arr}(\mathcal{M})$. In particular, Theorem 4.5 gives an efficient method of computing the space of Poisson structures on f.

We begin with the following general paradigm. Suppose $\mathcal{C}$ is a monoidal ∞-category which admits sifted colimits and which are preserved by the tensor product. Denote by $\mathbf{BMod}(\mathcal{C})$ the ∞-category of triples (A, B, M), where $A, B \in \mathbf{Alg}(\mathcal{C})$ and M is an (A, B)-bimodule. Similarly, let $\mathbf{RMod}(\mathcal{C})$ be the ∞-category of pairs (B, M) where $B \in \mathbf{Alg}(\mathcal{C})$ and M is a right B-module. We have a functor

$$\mathbf{Alg}(\mathcal{C}) \to \mathbf{RMod}(\mathcal{C})$$

given by sending an associative algebra B to the canonical right B-module B.

Lemma 4.20 *We have a Cartesian square of ∞-categories*

$$\begin{tikzcd} \mathrm{Arr}(\mathbf{Alg}(\mathcal{C})) \arrow[r] \arrow[d, "\text{target}"] & \mathbf{BMod}(\mathcal{C}) \arrow[d] \\ \mathbf{Alg}(\mathcal{C}) \arrow[r] & \mathbf{RMod}(\mathcal{C}). \end{tikzcd} \tag{9}$$

Proof This is a slight variant of [18, Corollary 4.8.5.6].

Let $\mathbf{LMod}_{\mathcal{C}}$ be the ∞-category of $\mathcal{C}$-module categories and $\mathbf{LMod}^*_{\mathcal{C}}$ the ∞-category of pointed $\mathcal{C}$-module categories, i.e.

$$\mathbf{LMod}^*_{\mathcal{C}} = (\mathbf{LMod}_{\mathcal{C}})_{\mathcal{C}/}.$$

By [18, Theorem 4.8.4.1] we have a Cartesian diagram of ∞-categories

$$\begin{tikzcd} \mathbf{BMod}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Arr}(\mathbf{LMod}_{\mathcal{C}}) \arrow[d] \\ \mathbf{Alg}(\mathcal{C}) \times \mathbf{Alg}(\mathcal{C}) \arrow[r] & \mathbf{LMod}_{\mathcal{C}} \times \mathbf{LMod}_{\mathcal{C}} \end{tikzcd}$$

where $\mathbf{BMod}(\mathcal{C}) \to \mathrm{Arr}(\mathbf{LMod}_{\mathcal{C}})$ sends a triple (A, B, M) to the functor

$$- \otimes_A M : \mathbf{RMod}_A(\mathcal{C}) \longrightarrow \mathbf{RMod}_B(\mathcal{C})$$

determined by M. Therefore, the pullback P of the diagram (9) fits into a Cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Arr}(\mathbf{LMod}^*_{\mathcal{C}}) \arrow[d] \\ \mathbf{Alg}(\mathcal{C}) \times \mathbf{Alg}(\mathcal{C}) \arrow[r] & \mathbf{LMod}^*_{\mathcal{C}} \times \mathbf{LMod}^*_{\mathcal{C}} \end{tikzcd}$$

By [18, Theorem 4.8.5.5] the functor $\mathbf{Alg}(\mathcal{C}) \to \mathbf{LMod}^*_{\mathcal{C}}$ given by $A \mapsto \mathbf{RMod}_A(\mathcal{C})$ is fully faithful. Therefore, we have a Cartesian diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Arr}(\mathbf{Alg}(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Arr}(\mathbf{LMod}^*_{\mathcal{C}}) \arrow[d] \\ \mathbf{Alg}(\mathcal{C}) \times \mathbf{Alg}(\mathcal{C}) \arrow[r] & \mathbf{LMod}^*_{\mathcal{C}} \times \mathbf{LMod}^*_{\mathcal{C}} \end{tikzcd}$$

and hence $P \cong \mathrm{Arr}(\mathbf{Alg}(\mathcal{C}))$. □

To prove the following theorem we are going to use the above lemma in the cases when $\mathcal{C} = \mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})$ and $\mathcal{C} = \mathbf{CAlg}(\mathcal{M})$.

Theorem 4.21 *Let $f\colon A \to B$ be a morphism of commutative algebras in $\mathcal{M}$ and $g\colon A \otimes B \to B$ its graph, i.e. $g(x \otimes y) = f(x)y$. Then we have a Cartesian square of spaces*

$$\begin{tikzcd} \mathrm{Pois}(f, n) \arrow[r] \arrow[d] & \mathrm{Pois}(A, n) \times \mathrm{Pois}(B, n) \arrow[d, "\mathrm{id} \times \mathrm{opp}"] \\ \mathrm{Cois}(g, n) \arrow[r] & \mathrm{Pois}(A \otimes B, n). \end{tikzcd}$$

Proof In the proof we will repeatedly use the statements of Poisson additivity given by Theorems 4.14 and 4.15.

If $\mathcal{C}$ is a symmetric monoidal ∞-category, we have a Cartesian square of ∞-categories

$$\begin{tikzcd} \mathbf{BMod}(\mathcal{C}) \arrow[r] \arrow[d] & \mathbf{Alg}(\mathcal{C}) \times \mathbf{Alg}(\mathcal{C}) \arrow[d] \\ \mathbf{LMod}(\mathcal{C}) \arrow[r] & \mathbf{Alg}(\mathcal{C}), \end{tikzcd} \tag{10}$$

where the functor $\mathbf{Alg}(\mathcal{C}) \times \mathbf{Alg}(\mathcal{C}) \to \mathbf{Alg}(\mathcal{C})$ is given by sending $(A, B) \mapsto A \otimes B^{op}$.

Let F be the fiber of

$$\mathbf{BMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))^{\sim} \to \mathbf{BMod}(\mathbf{CAlg}(\mathcal{M}))^{\sim}$$

at (A, B, B). Taking the fiber of (10) applied to the forgetful functor

$$(\mathcal{C}_1 = \mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})) \longrightarrow (\mathcal{C}_2 = \mathbf{CAlg}(\mathcal{M}))$$

we get a Cartesian square of spaces

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Pois}(A, n) \times \mathrm{Pois}(B, n) \arrow[d, "\mathrm{id} \times \mathrm{opp}"] \\ \mathrm{Cois}(g, n) \arrow[r] & \mathrm{Pois}(A \otimes B, n), \end{tikzcd}$$

so we have to show that $F \cong \mathrm{Pois}(f, n)$.

Now consider the Cartesian square

$$\begin{tikzcd} \mathrm{Arr}(\mathbf{Alg}(\mathcal{C})) \arrow[r] \arrow[d, "\text{target}"] & \mathbf{BMod}(\mathcal{C}) \arrow[d] \\ \mathbf{Alg}(\mathcal{C}) \arrow[r] & \mathbf{RMod}(\mathcal{C}). \end{tikzcd} \tag{11}$$

given by Lemma 4.20. Taking the fiber of (11) applied to the forgetful functor

$$(\mathcal{C}_1 = \mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})) \longrightarrow (\mathcal{C}_2 = \mathbf{CAlg}(\mathcal{M}))$$

we get a Cartesian square of spaces

$$\begin{tikzcd} \mathrm{Pois}(f, n) \arrow[r] \arrow[d] & F \arrow[d] \\ \mathrm{Pois}(B, n) \arrow[r] & \mathrm{Cois}(\mathrm{id}, n) \end{tikzcd}$$

The bottom morphism is induced by the functor

$$\mathbf{Alg}(\mathcal{C}) \to \mathbf{LMod}(\mathcal{C})$$

splitting the obvious forgetful functor

$$\mathbf{LMod}(\mathcal{C}) \to \mathbf{Alg}(\mathcal{C})$$

and hence $\mathrm{Pois}(B, n) \to \mathrm{Cois}(\mathrm{id}, n)$ is a weak equivalence by Proposition 4.17. Therefore, $\mathrm{Pois}(f, n) \to F$ is also a weak equivalence which proves the claim. □

In particular, the fibers of the horizontal morphisms of the Cartesian diagram of Theorem 4.21 are thus equivalent. It follows that given two Poisson structures π_A and π_B on A and B respectively, lifting a map of algebras $f : A \to B$ to a Poisson map is equivalent to giving a coisotropic structure on the graph $g : A \otimes B \to B$, where $A \otimes B$ is endowed with the $\mathbb{P}_{n+1}$-structure $(\pi_A; -\pi_B)$.

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Derived coisotropic structures II: stacks and quantization

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Abstract We extend results about n-shifted coisotropic structures from part I of this work to the setting of derived Artin stacks. We show that an intersection of coisotropic morphisms carries a Poisson structure of shift one less. We also compare non-degenerate shifted coisotropic structures and shifted Lagrangian structures and show that there is a natural equivalence between the two spaces in agreement with the classical result. Finally, we define quantizations of n-shifted coisotropic structures and show that they exist for $n > 1$.

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Contents

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Introduction

This paper is a continuation of [15] where we have defined a notion of an n-shifted coisotropic structure on a morphism of commutative dg algebras. In this paper we extend this definition to derived Artin stacks. Some of our results are as follows:

- An intersection of n-shifted coisotropic morphisms carries an $(n-1)$-shifted Poisson structure.
- A non-degenerate n-shifted coisotropic structure is the same as an n-shifted Lagrangian structure.
- Let $f\colon L \to X$ be a morphism of derived stacks equipped with an n-shifted coisotropic structure. If $n > 1$, then f admits a canonical deformation quantization.

Shifted Poisson algebras

Recall the operad $\mathbb{P}_n$ which controls commutative dg algebras equipped with a Poisson bracket of degree $1-n$. An important feature of the operad $\mathbb{P}_n$ is the *additivity property*, which is the equivalence

$$\mathbf{Alg}_{\mathbb{P}_{n+1}} \cong \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n})$$

of symmetric monoidal ∞-categories. Such an equivalence has been constructed by Rozenblyum and, independently, by the second author in [23]. In other words, one can think of a $\mathbb{P}_{n+1}$-algebra as an associative algebra object in $\mathbb{P}_n$-algebras. Therefore, we can define an action of a $\mathbb{P}_{n+1}$-algebra A on a $\mathbb{P}_n$-algebra B to be simply the data of a structure of a left A-module on B. Moreover, in [15] we have constructed an explicit two-colored operad $\mathbb{P}_{[n+1,n]}$ which models such actions, i.e. there is an equivalence of ∞-categories

$$\mathbf{Alg}_{\mathbb{P}_{[n+1,n]}} \cong \mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}),$$

where **LMod** is the ∞-category of pairs of an associative algebra and a left module.

Given a commutative algebra A, one defines the space $\mathrm{Pois}(A, n)$ of n-shifted Poisson structures on A to be the space of lifts of A to a $\mathbb{P}_{n+1}$-algebra. This space has the following alternative description. There is a graded $\mathbb{P}_{n+2}$-algebra $\mathbf{Pol}(A, n)$ of n-shifted polyvectors on A, and it was shown in [14] that there is an equivalence of spaces

$$\mathrm{Pois}(A, n) \cong \mathrm{Map}_{\mathbf{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(A, n)[n+1]),$$

where $k(2)[-1]$ is the trivial graded Lie algebra concentrated in weight 2 and cohomological degree 1.

A similar definition was given for n-shifted coisotropic structures in [15]. Suppose $f\colon A \to B$ is a morphism of commutative dg algebras. We can consider f as an object of $\mathbf{LMod}(\mathbf{CAlg})$, where $\mathbf{CAlg}$ is the ∞-category of commutative dg algebras. In other words, f endows B with an action of A. Then the space $\mathrm{Cois}(f, n)$ of n-shifted coisotropic structures on $f\colon A \to B$ is the space of lifts of f along the forgetful functor

$$\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}) \longrightarrow \mathbf{LMod}(\mathbf{CAlg}).$$

One can construct a graded $\mathbb{P}_{[n+2,n+1]}$-algebra $\mathbf{Pol}(f, n) = (\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$ and it was shown in [15, Theorem 4.15] that there is an equivalence of spaces

$$\mathrm{Cois}(f, n) \cong \mathrm{Map}_{\mathbf{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]).$$

Note that we can identify

$$\begin{aligned} \mathbf{Pol}(A, n) &\cong \mathrm{Hom}_A(\mathrm{Sym}_A(\mathbb{L}_A[n+1]), A) \\ \mathbf{Pol}(B/A, n-1) &\cong \mathrm{Hom}_B(\mathrm{Sym}_B(\mathbb{L}_{B/A}[n]), B) \end{aligned}$$

as graded commutative algebras and the action on the level of commutative algebras is induced from the morphism $\mathbb{L}_{B/A} \to \mathbb{L}_A \otimes_A B[1]$.

Shifted Poisson structures on stacks

Recall that a Poisson structure on a smooth scheme X is defined to be the structure of a k-linear Poisson algebra on the structure sheaf $\mathcal{O}_X$. If X is a derived Artin stack, we no longer have the structure sheaf $\mathcal{O}_X$ as an object of a category of sheaves of k-modules, so it is not clear how to extend the above definition of an n-shifted Poisson structure to stacks.

To a derived Artin stack X we can associate its de Rham stack X_{DR} together with a projection $q\colon X \to X_{DR}$. Since the cotangent complex of the de Rham stack X_{DR} is trivial, it is reasonable to expect that an n-shifted Poisson structure on X is the same as a relative n-shifted Poisson structure on $q\colon X \to X_{DR}$. This simplifies the problem as now the fibers of q are affine formal derived stacks, in the sense of [5, Section 2.2].

Even though they are not affine schemes, it is shown in [5] that they are controlled by a certain *graded mixed* commutative dg algebra.

More precisely, the general theory of *formal localization* developed in [5, Section 2] produces a graded mixed commutative algebra $\mathbb{D}_{X_{DR}}$ enhancing the structure sheaf $\mathcal{O}_{X_{DR}}$ and a graded mixed commutative algebra $\mathcal{B}_X$ enhancing the pushforward $q_*\mathcal{O}_X$. Then an n-shifted Poisson structure on a derived Artin stack X is a lift of $\mathcal{B}_X$ to a $\mathbb{D}_{X_{DR}}$-linear $\mathbb{P}_{n+1}$-algebra. Note that one has to introduce certain twists $\mathcal{B}_X(\infty)$ and $\mathbb{D}_{X_{DR}}(\infty)$ to fully capture all polyvectors, but we ignore this technical difference in the introduction.

The same procedure works almost verbatim in the relative setting and we can define the space $\mathrm{Cois}(f, n)$ of n-shifted coisotropic structures on a morphism $f\colon L \to X$ of derived Artin stacks (see Definition 2.1). Note that for our purposes it is useful to include the data of an n-shifted Poisson structure on X in the definition of the space $\mathrm{Cois}(f, n)$ and not fix it in advance.

One can similarly define the notion of relative polyvectors $\mathbf{Pol}(f, n)$ which is again a graded $\mathbb{P}_{[n+2,n+1]}$-algebra associated to a morphism $f\colon L \to X$ of derived Artin stacks. Extending [15, Theorem 4.15] to the setting of derived stacks, we obtain the following result (see Theorem 2.7):

Theorem *Let $f\colon L \to X$ be a morphism of derived Artin stacks. Then we have an equivalence of spaces*

$$\mathrm{Cois}(f, n) \cong \mathrm{Map}_{\mathbf{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]).$$

Here are some examples of coisotropic structures described in Sect. 2.3.

- (Classical case). Suppose $f\colon L \hookrightarrow X$ is a smooth closed subscheme of a smooth scheme X. Then we show that the *space* of 0-shifted Poisson structures on X is equivalent to the *set* of ordinary Poisson structures on X and the *space* of 0-shifted coisotropic structures on $f\colon L \hookrightarrow X$ is equivalent to the *subset* of the set of Poisson structures on X for which L is coisotropic in the classical sense.
- (Identity). We show that the space of n-shifted coisotropic structures on the identity morphism $\mathrm{id}\colon X \to X$ is equivalent to the space of n-shifted Poisson structures on the target. In other words, identity has a unique coisotropic structure. This has an interesting consequence: the forgetful morphisms

  $$\mathrm{Pois}(X, n-1) \longleftarrow \mathrm{Cois}(\mathrm{id}, n) \xrightarrow{\sim} \mathrm{Pois}(X, n)$$

  between spaces of shifted coisotropic and shifted Poisson structures assemble to give a forgetul map $\mathrm{Pois}(X, n) \to \mathrm{Pois}(X, n-1)$ from n-shifted Poisson structures on X to $(n-1)$-shifted Poisson structures on X. This map is nontrivial in general even though the underlying bivector of the corresponding $(n-1)$-shifted Poisson structure can be shown to be zero.
- (Graph). Suppose X and Y are derived Artin stacks equipped with n-shifted Poisson structures. Moreover, suppose $f\colon X \to Y$ is a morphism compatible with the Poisson structures. Then we show that the graph $X \to \overline{X} \times Y$ carries a canonical n-shifted coisotropic structure, where $\overline{X}$ is the same stack as X but equipped with the

opposite n-shfited Poisson structure. In fact, this gives a complete characterization of n-shifted coisotropic structures on the graph (see Theorem 2.9).

In [21] it was shown that given two Lagrangians $L_1, L_2 \to X$ in an n-shifted symplectic stack X, their derived intersection $L_1 \times_X L_2$ carries a canonical $(n-1)$-shifted symplectic structure. We extend this result to coisotropic structures in the following statement (see Theorem 3.6).

Theorem *Suppose X is an n-shifted Poisson stack and $L_1, L_2 \to X$ are two morphisms equipped with compatible n-shifted coisotropic structures. Then the intersection $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted Poisson structure such that the natural projection*

$$L_1 \times_X L_2 \longrightarrow \overline{L}_1 \times L_2$$

is a morphism of $(n-1)$-shifted Poisson stacks.

We remark that this statement gives a nice conceptual explanation of the main result of [1], generalizing it to a much broader context.

Non-degenerate coisotropic structures

One may ask more generally how the theory of shifted symplectic and shifted Lagrangian structures of [21] relates to the theory of shifted Poisson and shifted coisotropic structures. Classically, a Poisson structure whose bivector induces an isomorphism $\mathrm{T}^*_X \xrightarrow{\sim} \mathrm{T}_X$ is the same as a symplectic structure. It was shown in [5,17] that the subspace $\mathrm{Pois}^{nd}(f, n) \subset \mathrm{Pois}(f, n)$ of non-degenerate n-shifted Poisson structures, i.e. those that induce an equivalence $\mathbb{L}_X \xrightarrow{\sim} \mathbb{T}_X[-n]$, is equivalent to the space $\mathrm{Symp}(X, n)$ of n-shifted symplectic structures.

Suppose that $f\colon L \to X$ is equipped with an n-shifted coisotropic structure. Then we obtain a natural morphism of fiber sequences

$$\begin{tikzcd}
\mathbb{L}_{L/X}[-1] \arrow[r] \arrow[d] & f^*\mathbb{L}_X \arrow[r] \arrow[d] & \mathbb{L}_L \arrow[d] \\
\mathbb{T}_L[-n] \arrow[r] & f^*\mathbb{T}_X[-n] \arrow[r] & \mathbb{T}_{L/X}[1-n]
\end{tikzcd}$$

It is thus natural to define the subspace $\mathrm{Cois}^{nd}(f, n) \subset \mathrm{Cois}(f, n)$ of non-degenerate n-shifted coisotropic structures to be those that induce equivalences $\mathbb{L}_X \to \mathbb{T}_X[-n]$ and $\mathbb{L}_{L/X} \to \mathbb{T}_L[1-n]$ (which automatically implies that $\mathbb{L}_L \to \mathbb{T}_{L/X}[1-n]$ is an equivalence as well). We prove the following result (see Theorem 4.22).

Theorem *Suppose $f\colon L \to X$ is a morphism of derived Artin stacks. Then we have an equivalence*

$$\mathrm{Cois}^{nd}(f, n) \cong \mathrm{Lagr}(f, n)$$

of spaces of non-degenerate n-shifted coisotropic structures and n-shifted Lagrangian structures on f.

Let us mention that the proof for $n = 0$ was previously given by Pridham in [20] using a slightly different notion of coisotropic structures. We closely follow his proof to obtain the result for all n for our definition of n-shifted coisotropic structures. The main idea is to prove a stronger result by showing that there is an equivalence of spaces equipped with a natural (co)filtration: $\mathrm{Lagr}(f, n)$ is filtered by the maximal weight of the form and $\mathrm{Cois}^{nd}(f, n)$ is filtered by the maximal weight of the polyvector. The proof then proceeds by induction by developing obstruction theory where the inductive step is a simple problem in linear algebra.

We believe an alternative proof can be given along the lines of the proof of [5] by using the Darboux lemma for shifted Lagrangians from [9].

Quantization

We conclude this paper with a description of deformation quantization of shifted coisotropic structures. Recall that a deformation quantization of a $\mathbb{P}_n$-algebra A can be formulated in terms of lifts of A to a $\mathbb{BD}_n$-algebra (the Beilinson–Drinfeld operad $\mathbb{BD}_n$ is reviewed in Sect. 5.1). Since the notion of an n-shifted Poisson structure on a stack is reduced to a $\mathbb{P}_{n+1}$-algebra structure on $\mathcal{B}_X$, one can similarly define the notion of a deformation quantization of an n-shifted Poisson stack. One has the following results on quantizations of n-shifted Poisson structures on derived Artin stacks:

- If $n \geq 1$, it is shown in [5] that every n-shifted Poisson stack admits a deformation quantization by using the formality of the $\mathbb{E}_n$ operad.
- If $n = 0$, it is shown in [19] that non-degenerate 0-shifted Poisson structures (i.e. 0-shifted symplectic structures) admit a *curved* deformation quantization. In fact, the paper gives a complete characterization of such deformation quantizations and shows that they are unobstructed beyond the first order in $\hbar$.
- If $n = -1$, it is shown in [18] that non-degenerate (-1)-shifted Poisson structures (i.e. (-1)-shifted symplectic structures) admit deformation quantizations in a twisted sense if X is Gorenstein with a choice of the square root $\omega_X^{1/2}$ of the dualizing sheaf.

We formulate the notion of deformation quantization for n-shifted coisotropic structures and show that using the formality of the $\mathbb{E}_n$ operad one can prove the following result (see Theorem 5.15).

Theorem *Suppose $n \geq 2$. Then any n-shifted coisotropic structure on a morphism $f : L \to X$ of derived Artin stacks admits a deformation quantization.*

Let us mention that [20] shows that non-degenerate 0-shifted coisotropic structures (i.e. 0-shifted Lagrangians) admit curved deformation quantizations if we again include a certain twist necessary to deal with the first-order obstruction (see also [2]). Moreover, we expect that the results of [20] can be extended to similarly provide quantizations of 1-shifted Lagrangians thus treating the only remaining case.

One might wonder how the above theorem relates to the statement of the non-formality of the Swiss-cheese operad SC_n shown by Livernet [11]. Recall that the operad $\mathbb{E}_n$ has two filtrations: the cohomological one and the one related to the Poisson operad. These coincide for $n \geq 2$ but differ for $n = 1$ and $n = 0$. For instance, the cohomological filtration for $\mathbb{E}_1 \cong \mathrm{Ass}$ is trivial while the Poisson filtration is nontrivial (it is the so-called PBW filtration). Livernet shows that the cohomological filtration on $\mathrm{C}_\bullet(\mathrm{SC}_n)$ is nontrivial for $n \geq 2$. We expect that there is similarly a Poisson filtration on $\mathrm{C}_\bullet(\mathrm{SC}_n)$ whose associated graded is $\mathbb{P}_{[n,n-1]}$. It is obvious that the two filtrations are different for $n \leq 2$ by looking at the associated gradeds; we expect that the two filtrations moreover differ for all n. That is, the Poisson filtration on $\mathrm{C}_\bullet(\mathrm{SC}_n)$ is trivial for $n \geq 3$ while the cohomological filtration is nontrivial. Let us also note a related drastic difference between the operads $\mathbb{P}_n$ and $\mathbb{P}_{[n+1,n]}$: while $\mathbb{P}_n$ admits a minimal model with a quadratic differential, the differential on the minimal model $\tilde{\mathbb{P}}_{[n+1,n]}$ for $\mathbb{P}_{[n+1,n]}$ constructed in [15, Section 3.4] has higher terms.

1 Recollections from part I

In this section we recall some results from [5, 15] that will be used in the paper.

1.1 Notations

We will use the following notations used in [15]:

- k denotes a field of characteristic zero. dg_k is the category of cochain complexes of k-vector spaces considered with its projective model structure. The standard tensor product on dg_k makes it a symmetric monoidal model category. The associated symmetric monoidal ∞-category will be denoted by $\mathbf{dg}_k$.
- We denote by M a symmetric monoidal model dg category as in [15, Section 1.1] and by $\mathcal{M}$ its underlying ∞-category.
- M^{gr} and $M^{gr,\epsilon}$ denote the model categories of (weight) graded objects in M and of graded mixed objects in M respectively. We refer to [15, Section 1.2] and [5, Section 1.1] for more details. The associated ∞-categories will be denoted by $\mathcal{M}^{gr}$ and $\mathcal{M}^{gr,\epsilon}$ respectively.
- If $\mathcal{C}$ is any ∞-category, then $\mathcal{C}^\sim$ denotes its ∞-groupoid of equivalences. We will also refer to $\mathcal{C}^\sim$ as the underlying space of $\mathcal{C}$. The ∞-category of morphisms in $\mathcal{C}$ will be denoted by $\mathrm{Arr}(\mathcal{C}) = \mathrm{Fun}(\Delta^1, \mathcal{C})$.
- Given a dg operad $\mathcal{P}$, the category of $\mathcal{P}$-algebras in M is denoted by $\mathrm{Alg}_{\mathcal{P}}(M)$. The ∞-category of $\mathcal{P}$-algebras in $\mathcal{M}$ is denoted by $\mathbf{Alg}_{\mathcal{P}}(\mathcal{M})$. In the case of $\mathcal{P}$ being the operad Ass of associative algebras, we will simply use the notation $\mathrm{Alg}(M)$ and $\mathbf{Alg}(\mathcal{M})$. Similarly, for the operad Comm of commutative algebras we will use $\mathrm{CAlg}(M)$ and $\mathbf{CAlg}(\mathcal{M})$.
- When considering Lie and Poisson algebras in M^{gr} or $\mathcal{M}^{gr}$ we will always assume that the bracket is of weight -1. In the case $M = \mathrm{dg}_k$, we will use the simpler notations $\mathrm{Alg}^{gr}_{\mathrm{Lie}}$ and $\mathbf{Alg}^{gr}_{\mathrm{Lie}}$ instead of the full $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{dg}^{gr}_k)$ and $\mathbf{Alg}_{\mathrm{Lie}}(\mathbf{dg}^{gr}_k)$.

- We denote by $\mathbf{BMod}(\mathcal{M})$ the ∞-category of triples (A, B, L), where $A, B \in \mathbf{Alg}(\mathcal{M})$ and L is an (A, B)-bimodule. In a similar way, we denote by $\mathbf{RMod}(\mathcal{M})$ (resp. $\mathbf{LMod}(\mathcal{M})$) the ∞-category of pairs (A, L) where $A \in \mathbf{Alg}(\mathcal{M})$ and L is a right (resp. left) A-module.
- By a derived Artin stack we mean a derived Artin stack locally of finite presentation over k.

1.2 Formal localization and Poisson structures on derived stacks

Let X be a derived Artin stack. Recall that the de Rham stack X_{DR} is defined to be

$$X_{DR}(A) = X(\mathrm{H}^0(A)^{red})$$

for any cdga A concentrated in non-positive cohomological degrees. It comes equipped with a projection $q\colon X \to X_{DR}$, whose fibers are formal completions of X.

Moreover, there are two naturally defined prestacks of graded mixed commutative algebras on X_{DR}, denoted by $\mathbb{D}_{X_{DR}}$ and $\mathcal{B}_X$; they are to be thought as derived versions of the crystalline structure sheaf and of the sheaf of principal parts respectively. More precisely, we have the following equivalences of prestacks of commutative algebras on X_{DR}:

$$|\mathbb{D}_{X_{DR}}| \cong \mathcal{O}_{X_{DR}}, \qquad |\mathcal{B}_X| \cong q_*\mathcal{O}_X.$$

Just as in the classical case, we have a morphism $\mathbb{D}_{X_{DR}} \to \mathcal{B}_X$ that we think of as a $\mathbb{D}_{X_{DR}}$-linear structure on $\mathcal{B}_X$.

As functors to the category of graded mixed modules, both $\mathbb{D}_{X_{DR}}$ and $\mathcal{B}_X$ admit natural twistings $\mathbb{D}_{X_{DR}}(\infty)$ and $\mathcal{B}_X(\infty)$, that are now prestacks of commutative algebras in Ind-objects in the category of graded mixed modules. For details on these constructions, see [5, Sections 1.5 and 2.4.2]. Notice that in particular $\mathcal{B}_X(\infty)$ is a commutative algebra in the category of $\mathbb{D}_{X_{DR}}(\infty)$-modules.

With these notations, we can now give the definition of shifted Poisson structures, see [5, Section 3.1].

Definition 1.1 Let X be a derived Artin stack. The *space* $\mathrm{Pois}(X, n)$ *of n-shifted Poisson structures* on X is the space of lifts of the commutative algebra $\mathcal{B}_X(\infty)$ in the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules to a $\mathbb{P}_{n+1}$-algebra.

Note that if $X = \operatorname{Spec} A$ is an affine derived scheme, this definition recovers [15, Definition 4.4]. Just as in the affine case, we can give an alternative definition of shifted Poisson structures. First, define the graded $\mathbb{P}_{n+2}$-algebra of n-shifted polyvectors on X to be

$$\mathbf{Pol}(X, n) := \Gamma(X_{DR}, \mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n))$$

where $\mathbf{Pol}^t(\mathcal{B}_X/\mathbb{D}_{X_{DR}}, n)$ is the Tate realization of the algebra of shifted $\mathbb{D}_{X_{DR}}$-linear multiderivations of $\mathcal{B}_X$. Again, we refer to [5] for more details on this construction.

Using the main theorem of [14] and its extended version [15, Theorem 4.5] one immediately obtains the following result (see also [5, Theorem 3.1.2]).

Theorem 1.2 *With notations as above, there is a canonical equivalence of spaces*

$$\mathrm{Pois}(X, n) \cong \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(X, n)[n+1]).$$

We remark that the above theorem is somewhat reassuring, since it gives an expected alternative description of Poisson structures on derived stacks in terms of bivectors.

Example 1.3 Suppose X is a smooth scheme and $n = 0$. Then

$$\mathbf{Pol}(X, 0) \cong \Gamma(X, \mathrm{Sym}(\mathrm{T}_X[-1]))$$

where T_X is the tangent bundle of X, and the $\mathbb{P}_2$-structure on the right is given by the Schouten bracket. The completion $\mathbf{Pol}(X, 0)^{\geq 2}$ in weights at least 2 is concentrated in degree at least 2. Using [15, Proposition 1.19], we have an equivalence of spaces

$$\mathrm{Pois}(X, 0) \cong \underline{\mathrm{MC}}(\mathbf{Pol}(X, 0)^{\geq 2}[1]).$$

Since the Lie algebra $\mathbf{Pol}(X, 0)^{\geq 2}[1]$ is concentrated in degree at least 1, the space of Maurer–Cartan elements is discrete. Its elements correspond to bivectors $\pi_X \in \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$ satisfying $[\pi_X, \pi_X] = 0$, i.e. we recover the usual notion of a Poisson structure.

The same argument shows that there are no nontrivial n-shifted Poisson structures on smooth schemes for $n > 0$.

1.3 Coisotropic structures on algebras

Recall from [15, Section 3.4] the colored operad $\mathbb{P}_{[n+1,n]}$ whose algebras in $\mathcal{M}$ are pairs of objects (A, B) in $\mathcal{M}$ together with the following additional structure:

- a $\mathbb{P}_{n+1}$-structure on A;
- a $\mathbb{P}_n$-structure on B;
- a morphism of $\mathbb{P}_{n+1}$-algebras $A \to \mathbf{Z}(B)$, where $\mathbf{Z}(B)$ is the Poisson center of B, considered with its natural structure of a $\mathbb{P}_{n+1}$-algebra in $\mathcal{M}$.

Recall also that there is a canonical morphism of commutative algebras $\mathbf{Z}(B) \to B$. The fiber of this map is denoted by $\mathbf{Def}(B)[-n]$, and we have a fiber sequence

$$B[-1] \longrightarrow \mathbf{Def}(B)[-n] \longrightarrow \mathbf{Z}(B)$$

of non-unital $\mathbb{P}_{n+1}$-algebras (see [15, Section 3.5]). In particular, it follows that if (A, B) is a $\mathbb{P}_{[n+1,n]}$-algebra, then the fiber $\mathbf{U}(A, B)$ of the underlying morphism of commutative algebras $A \to B$ fits in the Cartesian square

$$\begin{tikzcd} \mathbf{U}(A, B) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbf{Def}(B)[-n] \arrow[r] & \mathbf{Z}(B) \end{tikzcd}$$

of non-unital $\mathbb{P}_{n+1}$-algebras in $\mathcal{M}$. In particular, we obtain a forgetful functor

$$\mathbf{Alg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathbf{Alg}_{\mathbb{P}^{\mathrm{nu}}_{n+1}}.$$

Using the morphism of commutative algebras $\mathbf{Z}(B) \to B$, we also get a natural forgetful map

$$\phi\colon\ \mathbf{Alg}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M})^{\sim} \to \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M}))^{\sim}$$

to the ∞-groupoid of morphisms of commutative algebras, sending a $\mathbb{P}_{[n+1,n]}$-algebra (A, B) to the underlying map $A \to B$. This forgetful functor was used in [15, Section 4.3] to define coisotropic structures on a morphism $f: A \to B$ in $\mathbf{CAlg}(\mathcal{M})$. The following is [15, Definition 4.12].

Definition 1.4 Let $f: A \to B$ be a map of commutative algebra objects in the ∞-category $\mathcal{M}$. The *space* $\mathrm{Cois}(f, n)$ *of n-shifted coisotropic structures on f* is the fiber of the forgetful functor ϕ, taken at f.

We also have an alternative operadic description of the space $\mathrm{Cois}(f, n)$. The following is [23, Theorem 2.22].

Theorem 1.5 *Let $\mathcal{M}$ be a symmetric monoidal dg category satisfying our starting assumption. Then there is an equivalence of ∞-categories*

$$\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M}) \simeq \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})).$$

Thanks to the above result one can think of a $\mathbb{P}_{n+1}$-algebra as an associative algebra in the ∞-category of $\mathbb{P}_n$-algebras. In particular, this allows us to obtain the following important theorem, which is [23, Corollary 3.8] (see also [15, Theorem 4.14]), which gives an alternative characterization of coisotropic structures.

Theorem 1.6 *There is a commutative diagram of ∞-categories*

$$\begin{tikzcd} \mathbf{Alg}_{\mathbb{P}_{[n+1,n]}}(\mathcal{M}) \arrow[rr, "\sim"] \arrow[dr] & & \mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})) \arrow[dl] \\ & \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M})) & \end{tikzcd}$$

We immediately get the following corollary about coisotropic structures.

Corollary 1.7 *Let $f: A \to B$ be a morphism in* $\mathbf{CAlg}(\mathcal{M})$. *Then the space* $\mathrm{Cois}(f, n)$ *is equivalent to the fiber of*

$$\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))^{\sim} \longrightarrow \mathrm{Arr}(\mathbf{CAlg}(\mathcal{M}))^{\sim}$$

taken at f.

In other words, consider $f: A \to B$ in $\mathbf{CAlg}(\mathcal{M})$. Then a n-shifted coisotropic structure on f amounts to a $\mathbb{P}_{n+1}$-structure on A, a $\mathbb{P}_n$-structure on B and an action of A on B as $\mathbb{P}_n$-algebras, all compatible with the given commutative morphism f.

We end this section by recalling a third important characterization of the space of coisotropic structures on algebras. Let again $f: A \to B$ be a morphism in $\mathbf{CAlg}(\mathcal{M})$. It induces a natural morphism of graded commutative algebras

$$\mathbf{Pol}(A, n) \longrightarrow \mathbf{Pol}(B/A, n-1)$$

between algebras of shifted polyvectors. In [15, Section 4.2] we showed that the pair $(\mathbf{Pol}(A, n), \mathbf{Pol}(B/A, n-1))$ becomes a $\mathbb{P}_{[n+2,n+1]}$-algebra and we denote the underlying non-unital $\mathbb{P}_{n+2}$-algebra by $\mathbf{Pol}(f, n)$. We also have its internal version, i.e. there is an $\mathbb{P}_{[n+2,n+1]}$-algebra $(\mathbf{Pol}^{int}(A, n), \mathbf{Pol}^{int}(B/A, n-1))$ in $\mathcal{M}$ and we denote its underlying non-unital $\mathbb{P}_{n+2}$-algebra in $\mathcal{M}$ by $\mathbf{Pol}^{int}(f, n)$. The following is [15, Theorem 4.15].

Theorem 1.8 *With notations as above there is an equivalence of spaces*

$$\mathrm{Cois}(f, n) \simeq \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]).$$

2 Coisotropic structures on derived stacks

In this section we generalize definitions of coisotropic structures from the affine case to the case of general stacks.

2.1 Definitions

Let $f: L \to X$ be a map of derived Artin stacks. The map f descends to a map between the de Rham stacks $f_{DR}: L_{DR} \to X_{DR}$, which in turn induces a pullback functor (simply denoted by f^*, with a slight abuse of notation) from prestacks on X_{DR} to prestacks on L_{DR}. The functor f^* is just the precomposition of prestacks with f.

By definition of $\mathbb{D}_{X_{DR}}$, one immediately gets an equivalence $\mathbb{D}_{L_{DR}} \cong f^*\mathbb{D}_{X_{DR}}$. As for the sheaves of principal parts, f induces a natural algebra map

$$f^*_{\mathcal{B}}: f^*\mathcal{B}_X \to \mathcal{B}_L$$

preserving the $\mathbb{D}_{L_{DR}}$-linear structures. It follows that there exists an induced morphism

$$f^*_{\mathcal{B}}(\infty): \ f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$$

of $\mathbb{D}_{L_{DR}}(\infty)$-algebras.

Let us denote by $\mathcal{C}_X$ the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules in the ∞-category of functors

$$(\mathrm{dAff}/X_{DR})^{op} \longrightarrow \mathrm{Ind}(\mathbf{dg}_k^{gr,\epsilon}).$$

Similarly, we let $\mathcal{C}_L$ be the ∞-category of $\mathbb{D}_{L_{DR}}(\infty)$-modules in the category of functors

$$(\mathrm{dAff}/L_{DR})^{op} \longrightarrow \mathrm{Ind}(\mathbf{dg}_k^{gr,\epsilon}).$$

By the above discussion, the map $f^*_{\mathcal{B}}(\infty)$ is naturally a morphism of commutative algebras in $\mathcal{C}_L$.

Moreover, f induces a symmetric monoidal pullback ∞-functor $\mathcal{C}_X \to \mathcal{C}_L$, so that in particular there is a well defined functor

$$\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{C}_X) \longrightarrow \mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{C}_L).$$

For example, suppose that X is endowed with a n-shifted Poisson structure. This corresponds to a $\mathbb{P}_{n+1}$-structure on $\mathcal{B}_X(\infty)$ in the ∞-category $\mathcal{C}_X$, so that $f^*\mathcal{B}_X(\infty)$ becomes a $\mathbb{P}_{n+1}$-algebra in $\mathcal{C}_L$. This means that there is an induced map of spaces

$$\mathrm{Pois}(X, n) \longrightarrow \mathrm{Pois}(f^*\mathcal{B}_X(\infty), n),$$

where $\mathrm{Pois}(f^*\mathcal{B}_X(\infty), n)$ is the space of compatible $\mathbb{P}_{n+1}$-structures on $f^*\mathcal{B}_X(\infty)$, viewed as an element in $\mathbf{CAlg}(\mathcal{C}_L)$.

Definition 2.1 Let $f: L \to X$ be a map of derived Artin stacks. The *space* $\mathrm{Cois}(f, n)$ *of n-shifted coisotropic structures on f* is the fiber product

$$\begin{array}{ccc} \mathrm{Cois}(f, n) & \longrightarrow & \mathrm{Pois}(X, n) \\ \downarrow & & \downarrow \\ \mathrm{Cois}(f^*_{\mathcal{B}}(\infty), n) & \longrightarrow & \mathrm{Pois}(f^*\mathcal{B}_X(\infty), n) \end{array}$$

where $\mathrm{Cois}(f^*_{\mathcal{B}}(\infty), n)$ is the space of coisotropic structures on $f^*_{\mathcal{B}}(\infty)$ in the sense of Definition 1.4.

Remark 2.2 Our definition is equivalent to the one given in [5, Section 3.4] if one identifies $\mathbb{P}_{[n+1,n]}$-algebras with $\mathbb{P}_{(n+1,n)}$-algebras following [23, Theorem 3.7].

Recall from [15, Section 4.3] that if $\mathcal{M}$ is a nice enough symmetric monoidal ∞-category, then for every $f: A \to B$ in $\mathbf{CAlg}(\mathcal{M})$ the space $\mathrm{Cois}(f, n)$ comes equipped with two natural forgetful maps

$$\mathrm{Pois}(B, n-1) \longleftarrow \mathrm{Cois}(f, n) \longrightarrow \mathrm{Pois}(A, n).$$

In particular, it follows that for every map $f: L \to X$ of derived Artin stacks we have a similar correspondence of spaces

$$\begin{tikzcd} & \mathrm{Cois}(f, n) \arrow[dl] \arrow[dr] & \\ \mathrm{Pois}(L, n-1) & & \mathrm{Pois}(X, n). \end{tikzcd}$$

2.2 Relative polyvectors for derived stacks

In this subsection we give an alternative definition of coisotropic structures on a morphism of derived Artin stacks using the notion of relative polyvectors. The goal is to prove an analogue of Theorem 1.8 in the more general case of derived stacks.

For every map of derived stacks $f: L \to X$ the morphism

$$f_{\mathcal{B}}^*: f^*\mathcal{B}_X \to \mathcal{B}_L$$

we introduced in the previous subsection is a map of commutative algebras in the ∞-category of $\mathbb{D}_{L_{DR}}$-modules. Similarly, after twisting we get a morphism of commutative algebras

$$f_{\mathcal{B}}^*(\infty): f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$$

in the ∞-category $\mathcal{C}_L$ of $\mathbb{D}_{L_{DR}}(\infty)$-modules. In particular, we can consider its algebra of n-shifted relative polyvectors. That is, we have a graded $\mathbb{P}_{[n+2,n+1]}$-algebra

$$(\mathbf{Pol}^{int}(f^*\mathcal{B}_X(\infty), n), \mathbf{Pol}^{int}(\mathcal{B}_L(\infty)/f^*\mathcal{B}_X(\infty), n-1))$$

in $\mathcal{C}_L$.

Remark 2.3 The fact that the ∞-category $\mathcal{C}_L$ can be presented as a model category (and thus that we can use the constructions and the results of [15]) is a consequence of [27, Section 2.3.2]. Alternatively, one can observe that in this particular case polyvectors are in fact functorial, and hence can be defined objectwise (see [5, Remark 2.4.8] for more details).

We have a graded commutative algebra

$$\mathbf{Pol}(L/X, n) = \Gamma(L, \mathrm{Sym}_{\mathcal{O}_L}(\mathbb{T}_{L/X}[-n-1]))$$

as in [5, Definition 2.3.7]. The following lemma is a straightforward consequence of formal localization, as studied in [5, Section 2].

Lemma 2.4 *There is an equivalence of graded commutative cdgas*

$$\mathbf{Pol}(L/X, n) \cong \Gamma(L_{DR}, \mathbf{Pol}^{int}(\mathcal{B}_L(\infty)/f^*\mathcal{B}_X(\infty), n-1)).$$

The morphism of $\mathbb{P}_{n+2}$-algebras

$$\mathbf{Pol}(X, n) \cong \Gamma(X_{DR}, \mathbf{Pol}^{int}(\mathcal{B}_X(\infty), n)) \longrightarrow \Gamma(L_{DR}, \mathbf{Pol}^{int}(f^*\mathcal{B}_X(\infty), n))$$

moreover induces a graded $\mathbb{P}_{[n+2,n+1]}$-algebra structure on the pair

$$(\mathbf{Pol}(X, n), \mathbf{Pol}(L/X, n-1)).$$

Definition 2.5 Let $f : L \to X$ be a morphism of derived Artin stacks. The *algebra of relative n-shifted polyvectors* is the graded non-unital $\mathbb{P}_{n+2}$-algebra

$$\mathbf{Pol}(f, n) = \mathbf{U}(\mathbf{Pol}(X, n), \mathbf{Pol}(L/X, n-1)).$$

Proposition 2.6 *For a morphism $f : L \to X$ of derived Artin stacks there is a fiber sequence of graded non-unital $\mathbb{P}_{n+2}$-algebras*

$$\mathbf{Pol}(L/X, n-1)[-1] \longrightarrow \mathbf{Pol}(f, n) \longrightarrow \mathbf{Pol}(X, n).$$

The connecting homomorphism $\mathbf{Pol}(X, n) \to \mathbf{Pol}(L/X, n-1)$ *is induced from the morphism* $\mathbb{L}_{L/X} \to f^*\mathbb{L}_X[1]$.

Proof The first claim follows from definitions since we have a fiber sequence of graded non-unital $\mathbb{P}_{n+2}$-algebras

$$\mathbf{Pol}(L/X, n-1)[-1] \longrightarrow \mathbf{U}(\mathbf{Pol}(X, n), \mathbf{Pol}(L/X, n-1)) \longrightarrow \mathbf{Pol}(X, n).$$

Moreover, we have a fiber sequence of graded non-unital commutative algebras

$$\mathbf{U}(\mathbf{Pol}(X, n), \mathbf{Pol}(L/X, n-1)) \longrightarrow \mathbf{Pol}(X, n) \longrightarrow \mathbf{Pol}(L/X, n-1)$$

and the second claim follows from the fact that by the general machinery of formal localization of [5, Section 2], the map

$$\mathbb{L}^{int}_{\mathcal{B}_L(\infty)/f^*\mathcal{B}_X(\infty)} \to \mathbb{L}^{int}_{f^*\mathcal{B}_X(\infty)}[1] \otimes_{f^*\mathcal{B}_X(\infty)} \mathcal{B}_L(\infty).$$

corresponds exactly to the morphism $\mathbb{L}_{L/X} \to f^*\mathbb{L}_X[1]$. □

We are now ready to prove our first main result relating the space of n-shifted coisotropic structures of Definition 1.4 to the algebra of relative polyvectors introduced above. The following theorem is an extension of Theorem 1.8 to derived Artin stacks.

Theorem 2.7 *Let $f : L \to X$ be a map of derived Artin stacks. Then we have an equivalence of spaces*

$$\mathrm{Cois}(f, n) \simeq \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f, n)[n+1]).$$

Proof Let again $\mathcal{C}_L$ be the ∞-category of $\mathbb{D}_{L_{DR}}(\infty)$-modules. As above, there is a morphism of commutative algebras in $\mathcal{C}_L$

$$f^*_{\mathcal{B}}(\infty)\colon\ f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$$

whose algebra of relative n-shifted polyvectors fits into a fiber sequence

$$\begin{aligned}&\mathbf{Pol}(\mathcal{B}_L(\infty)/f^*\mathcal{B}_X(\infty), n-1)[n] \longrightarrow \mathbf{Pol}(f^*_{\mathcal{B}}(\infty), n)[n+1]\\ &\quad\longrightarrow \mathbf{Pol}(f^*\mathcal{B}_X(\infty), n)[n+1]\end{aligned}$$

of graded Lie algebras.

By definition, the graded $\mathbb{P}_{n+2}$-algebra $\mathbf{Pol}(f, n)$ fits into a Cartesian square

$$\begin{tikzcd}
\mathbf{Pol}(f, n) \arrow[r] \arrow[d] & \mathbf{Pol}(X, n) \arrow[d] \\
\mathbf{Pol}(f^*_{\mathcal{B}}(\infty), n) \arrow[r] & \mathbf{Pol}(f^*\mathcal{B}_X(\infty), n)
\end{tikzcd}$$

in the category of graded non-unital $\mathbb{P}_{n+2}$-algebras

Moreover, it follows from Theorem 1.8 applied in $\mathcal{C}_L$ that the space of n-shifted coisotropic structures on $f^*_{\mathcal{B}}(\infty)$ has an explicit description in terms of $\mathbf{Pol}(f^*_{\mathcal{B}}(\infty), n)$; namely, one has an equivalence

$$\mathrm{Cois}(f^*_{\mathcal{B}}(\infty), n) \cong \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f^*_{\mathcal{B}}(\infty), n)[n+1]).$$

On the other hand, Theorem 1.2 tells us that

$$\mathrm{Pois}(X, n) \simeq \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(X, n)[n+1]),$$

and similarly

$$\mathrm{Pois}(f^*\mathcal{B}_X(\infty), n) \simeq \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}(f^*\mathcal{B}_X(\infty), n)[n+1]),$$

so that we immediately get the desired equivalence. □

The alternative characterization of coisotropic structures given by Theorem 2.7 is of a more geometric nature than Definition 2.1. This demonstrates why this definition is a sensible generalization of the classical notion, as explained in the following examples.

2.3 Examples

(1) **Smooth schemes.** Let L be a smooth subscheme of a smooth scheme X, and let $f: L \to X$ be the corresponding immersion. Suppose X is endowed with a

classical Poisson structure π_X, i.e. $\pi_X \in \mathrm{Pois}(X, 0)$. The graded $\mathbb{P}_2$-algebra of 0-shifted polyvectors on X is

$$\mathbf{Pol}(X, 0) \cong \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathrm{T}_X[-1])),$$

where the weight grading is given by putting the tangent bundle T_X in weight 1 and the bracket is the usual Schouten bracket. Denote by $\mathrm{N}_{L/X} \cong \mathbb{T}_{L/X}[1]$ the normal bundle to the subscheme L.
By Proposition 2.6 we get a graded L_∞ structure on the graded complex

$$\mathbf{Pol}(f, 0)[1] \cong \Gamma(X, \mathrm{Sym}(\mathrm{T}_X[-1]))[1] \oplus \Gamma(L, \mathrm{Sym}(\mathrm{N}_{L/X}[-1]))$$

with the differential twisted by the morphism f^*.

A Maurer–Cartan element in $\mathbf{Pol}(f, 0)^{\geq 2}[1]$ is an element $\pi_X \in \mathrm{H}^0(X, \wedge^2 \mathrm{T}_X)$, so let us analyze the possible brackets of such an element. The bracket $[\pi_X, ..., \pi_X]_n$ has degree 2 and weight $n+1$. Therefore, $[\pi_X, ..., \pi_X]_n = 0$ for $n > 2$. The projection $\mathbf{Pol}(f, 0)[1] \to \mathbf{Pol}(X, 0)[1]$ has a structure of a graded L_∞ morphism, hence $[\pi_X, \pi_X]$ in $\mathbf{Pol}(f, 0)^{\geq 2}[1]$ is the standard Schouten bracket. Let us denote by $f^*\pi_X$ the image of π_X in $\Gamma(L, \wedge^2(\mathrm{N}_{L/X}))$. Then the Maurer–Cartan equation for π_X in $\mathbf{Pol}(f, 0)^{\geq 2}[1]$ splits into two:

$$[\pi_X, \pi_X] = 0 \quad f^*\pi_X = 0.$$

The first equation is the integrability equation for the Poisson structure on X and the second equation is equivalent to $L \to X$ being coisotropic with respect to the Poisson structure π_X.

By degree reasons the space of Maurer–Cartan elements in $\mathbf{Pol}(f, 0)^{\geq 2}[1]$ is discrete and hence $\mathrm{Cois}(f, 0)$ is a subset of $\mathrm{Pois}(X, 0)$ of Poisson structures for which the subscheme L is coisotropic in the usual sense. In other words, in the classical context the morphism $L \to X$ has a coisotropic structure iff $L \to X$ is a coisotropic submanifold in the usual sense.

Here is an alternative way to obtain the same conclusion. Assume first L and X are affine. In this case the morphism

$$f^* \colon \Gamma(X, \mathrm{Sym}(\mathrm{T}_X[-1])) \longrightarrow \Gamma(L, \mathrm{Sym}(\mathrm{N}_{L/X}[-1]))$$

is surjective and hence by [15, Proposition 4.11] $\mathbf{Pol}(f, 0)[1]$ is equivalent to the algebra of polyvectors $\mathrm{Pol}(f, 0)[1]$ on X which vanish when pulled back to $\mathrm{N}_{L/X}$. This gives the result in the affine case and the general case follows by descent.

(2) **Identity.** Let X be a derived Artin stack and consider the identity morphism $\mathrm{id} \colon X \to X$. The homotopy fiber sequence of graded dg Lie algebras

$$\mathbf{Pol}(X/X, n-1)[n] \to \mathbf{Pol}(\mathrm{id}, n)[n+1] \to \mathbf{Pol}(X, n)[n+1]$$

implies that the projection $\mathbf{Pol}(\mathrm{id}, n) \to \mathbf{Pol}(X, n)$ is a quasi-isomorphism in weights ≥ 1 since $\mathbb{T}_{X/X} = 0$. Therefore, the natural projection

$$\mathrm{Cois}(\mathrm{id}, n) \longrightarrow \mathrm{Pois}(X, n)$$

is a weak equivalence, i.e. the identity morphism has a unique coisotropic structure for any n-shifted Poisson structure on X.
An interesting consequence of this statement is that we obtain a forgetful map $\mathrm{Pois}(X, n) \to \mathrm{Pois}(X, n-1)$ given as the composite

$$\mathrm{Pois}(X, n) \cong \mathrm{Cois}(\mathrm{id}, n) \longrightarrow \mathrm{Pois}(X, n-1).$$

(3) **Point.** Let X be a derived Artin stack and consider the projection $p\colon X \to \mathrm{pt}$. The homotopy fiber sequence of graded dg Lie algebras

$$\mathbf{Pol}(X/\mathrm{pt}, n-1)[n] \to \mathbf{Pol}(p, n)[n+1] \to \mathbf{Pol}(\mathrm{pt}, n)[n+1]$$

implies that the morphism $\mathbf{Pol}(X, n-1)[n] \to \mathbf{Pol}(p, n)[n+1]$ is a quasi-isomorphism in weights ≥ 1. Therefore, the natural morphism

$$\mathrm{Cois}(p, n) \to \mathrm{Pois}(X, n-1)$$

is a weak equivalence.

Note that this is a shifted Poisson analogue of a well-known statement for shifted symplectic structures: a Lagrangian structure on $X \to \mathrm{pt}$ where the point is equipped with its unique n-shifted symplectic structure is the same as an $(n-1)$-shifted symplectic structure on X.

Let us now give a more general procedure to construct coisotropic structures. Let X, Y be derived Artin stacks together with a morphism $f\colon X \to Y$. In analogy with [15, Section 4.5], we give the following definition.

Definition 2.8 Let $f\colon X \to Y$ be a morphism of derived Artin stacks, and let

$$f^*_{\mathcal{B}}(\infty)\colon f^*\mathcal{B}_Y(\infty) \to \mathcal{B}_X(\infty)$$

be the induced map of $\mathbb{D}_{X_{DR}}(\infty)$-algebras. We define *the space* $\mathrm{Pois}(f, n)$ *of n-Poisson structures on* f to the the fiber product

$$\begin{tikzcd}
\mathrm{Pois}(f, n) \arrow[r] \arrow[d] & \mathrm{Pois}(X, n) \times \mathrm{Pois}(Y, n) \arrow[d] \\
\mathrm{Pois}(f^*_{\mathcal{B}}(\infty), n) \arrow[r] & \mathrm{Pois}(\mathcal{B}_X(\infty), n) \times \mathrm{Pois}(f^*\mathcal{B}_Y(\infty), n)
\end{tikzcd}$$

where $\mathrm{Pois}(f^*_{\mathcal{B}}(\infty), n)$ is defined as in [15, Definition 4.18].

In the special case where $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are derived affine schemes, one sees that $\mathrm{Pois}(f, n)$ is indeed equivalent to the fiber of

$$\mathrm{Arr}(\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathbf{dg}_k))^{\sim} \longrightarrow \mathrm{Arr}(\mathbf{CAlg}(\mathbf{dg}_k))^{\sim}$$

taken at the given map $f : A \to B$. This means Definition 2.8 is a generalization of [15, Definition 4.18].

Denote by $g : X \to X \times Y$ the graph of f, that is to say the morphism given by $\mathrm{id} \times f$. The following result is an extension of [15, Theorem 4.20] to the case of general derived stacks.

Theorem 2.9 *With notations as above, there is a cartesian square of spaces*

$$\begin{tikzcd}
\mathrm{Pois}(f, n) \arrow[r] \arrow[d] & \mathrm{Pois}(X, n) \times \mathrm{Pois}(Y, n) \arrow[d] \\
\mathrm{Cois}(g, n) \arrow[r] & \mathrm{Pois}(X \times Y, n)
\end{tikzcd}$$

where the map on the right sends two Poisson structures π_X and π_Y to the Poisson structure given by $(\pi_X; -\pi_Y)$.

Proof Consider the map of $\mathbb{D}_{X_{DR}}(\infty)$-algebras

$$f^*_{\mathcal{B}}(\infty) \colon f^*\mathcal{B}_Y(\infty) \to \mathcal{B}_X(\infty).$$

By [15, Theorem 4.20], we know that there is a fiber square of spaces

$$\begin{tikzcd}
\mathrm{Pois}(f^*_{\mathcal{B}}(\infty), n) \arrow[r] \arrow[d] & \mathrm{Pois}(\mathcal{B}_X(\infty), n) \times \mathrm{Pois}(f^*\mathcal{B}_Y(\infty), n) \arrow[d] \\
\mathrm{Cois}(g^*_{\mathcal{B}}(\infty), n) \arrow[r] & \mathrm{Pois}(f^*\mathcal{B}_Y(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathcal{B}_X(\infty), n)
\end{tikzcd}$$

where $g^*_{\mathcal{B}}(\infty)$ is the induced map

$$g^*_{\mathcal{B}}(\infty) \colon f^*\mathcal{B}_Y(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathcal{B}_X(\infty) \to \mathcal{B}_X(\infty).$$

It follows that in order to prove the proposition, it will suffice to show that there is an equivalence

$$f^*\mathcal{B}_Y(\infty)\otimes_{\mathbb{D}_{X_{DR}}(\infty)}\mathcal{B}_X(\infty)\cong g^*\mathcal{B}_{X\times Y}(\infty)$$

of $\mathbb{D}_{X_{DR}}(\infty)$-modules. This can be checked directly: for every affine A, given an A-point of X_{DR}, the value of $g^*\mathcal{B}_{X\times Y}(\infty)$ on A is by definition $\mathbb{D}((X\times Y)_A)(\infty)$, where $(X\times Y)_A$ is the fiber product

$$\begin{tikzcd}
(X\times Y)_A \arrow[r] \arrow[d] & X \arrow[r] & X\times Y \arrow[d] \\
\operatorname{Spec} A \arrow[r] & X_{DR} \arrow[r] & X_{DR}\times Y_{DR}
\end{tikzcd}$$

But $(X\times Y)_A$ is naturally equivalent to $X_A\times_A Y_A$, so that

$$\mathbb{D}((X\times Y)_A)(\infty)\cong\mathbb{D}(X_A)(\infty)\otimes_{\mathbb{D}_{X_{DR}}(\infty)}\mathbb{D}(Y_A)(\infty)$$

which concludes the proof. $\square$

Notice that Theorem 2.9 gives further examples of coisotropic structures: for every n-shifted Poisson derived Artin stack X, the map to $\mathrm{pt}=\operatorname{Spec} k$ is naturally a Poisson map, where $\operatorname{Spec} k$ is considered with its trivial n-Poisson structure. The graph of this map is the identity map on X, which therefore admits a canonical coisotropic structure, already constructed in Example 2 above. Notice also that the space of Poisson maps $X\to\mathrm{pt}$ is equivalent to the space $\mathrm{Pois}(X,n)$ of n-shifted Poisson structures on X. We therefore get an equivalence $\mathrm{Pois}(X,n)\cong\mathrm{Cois}(\mathrm{id},n)$ exactly as in Example 2.

The identity morphism $X\to X$ is also a Poisson morphism. Its graph is the diagonal $X\to X\times X$, which then admits a canonical coisotropic structure.

3 Coisotropic intersections

In this section we state and prove our second main result which extends the Lagrangian intersection theorem (see [21, Theorem 2.9]) in the context of shifted Poisson structures.

3.1 Affine case

We begin with an affine version of the coisotropic intersection theorem. We consider the symmetric monoidal dg category $\mathcal{M}$ as in [15, Section 1.1].

Proposition 3.1 *Consider a diagram*

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B_1 & & B_2
\end{tikzcd}$$

in the ∞*-category* $\mathbf{CAlg}(\mathcal{M})$ *of commutative algebras in* $\mathcal{M}$*. Suppose we are given coisotropic structures in* $\mathrm{Cois}(f, n)$ *and* $\mathrm{Cois}(g, n)$*, such that the* $\mathbb{P}_{n+1}$*-structures on* A *coincide. Then the tensor product* $B_1 \otimes_A B_2$ *carries a natural* $\mathbb{P}_n$*-structure such that the map*

$$B_1^{op} \otimes B_2 \longrightarrow B_1 \otimes_A B_2$$

is a Poisson morphism, where B_1^{op} *is the algebra* B_1 *taken with the opposite Poisson structure.*

The proposition above is an easy consequence of Proposition 3.3 below, which is a slightly more general result.

Remark 3.2 Note that [22, Theorem 1.9] gives an alternative way to construct coisotropic intersections. Unfortunately, we do not know if the induced $\mathbb{P}_n$ structures on $B_1 \otimes_A B_2$ are equivalent.

Consider two objects $A, B \in \mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M})$. We say that a diagram in $\mathbf{CAlg}(\mathcal{M})$ of the form

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & L & \end{tikzcd}$$

is an *affine* n*-shifted coisotropic correspondence from* A *to* B if the induced map $A \otimes B^{op} \to L$ is endowed with an n-shifted coisotropic structure, relative to the given n-shifted Poisson structure on $A \otimes B^{op}$. Given such an affine n-shifted coisotropic correspondence we obtain a $\mathbb{P}_n$-algebra structure on L.

Proposition 3.3 *Let* A*,* B *and* C *be objects of* $\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M})$*, and suppose we are given affine* n*-shifted coisotropic correspondences* $A \to L_1 \leftarrow B$ *from* A *and* B *and* $B \to L_2 \leftarrow C$ *from* B *to* C*. Let* $L_{12} = L_1 \otimes_B L_2$*. Then* $A \to L_{12} \leftarrow C$ *is an affine* n*-shifted coisotropic correspondence from* A *to* C *such that the projection* $L_1 \otimes L_2 \to L_{12}$ *is a morphism of* $\mathbb{P}_n$*-algebras.*

Proof Interpreting the given coisotropic structures as in Corollary 1.7, we can interpret

$$A, B, C \in \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))$$

and

$$L_1 \in {}_A\mathbf{BMod}_B(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})), \qquad L_2 \in {}_B\mathbf{BMod}_C(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})).$$

Therefore, using composition of bimodules we see that $L_{12} \in {}_A\mathbf{BMod}_C(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))$, i.e. L_{12} is an affine n-shifted coisotropic correspondence from A to C. The last statement follows from the existence of a natural projection $L_1 \otimes L_2 \to L_1 \otimes_B L_2$ in the ∞-category $\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})$. $\square$

Remark 3.4 The above Proposition 3.3 gives a way to *compose* affine coisotropic correspondences. Using the Poisson additivity proven in [23] and the construction of the Morita (∞, m)-category of $\mathbb{E}_m$-algebras given in [8] and [25, Section 3] it is indeed possible to construct an (∞, m)-category whose objects are $\mathbb{P}_n$-algebras in $\mathcal{M}$, and whose i-morphisms are i-fold coisotropic correspondences. We will come back to these questions in a future work.

3.2 Intersections of derived stacks

Notice that Proposition 3.1 recovers in particular the constructions in [1] for affine schemes. More generally, derived algebraic geometry provides a suitable general context to interpret the results of Baranovsky and Ginzburg: we now extend Proposition 3.1 to general derived stacks, giving a general conceptual explanation for the Gerstenhaber algebra structure constructed in [1].

First, we need a lemma.

Lemma 3.5 *Consider a diagram of derived Artin stacks*

$$\begin{tikzcd} K \arrow[r, "\phi"] & X \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[d] \\ & Z \arrow[r] & W \end{tikzcd}$$

where the square on the right is Cartesian. Then the diagram of quasi-coherent sheaves on K

$$\begin{tikzcd} \mathbb{T}_{K/X} \arrow[r] \arrow[d] & \mathbb{T}_{K/Y} \arrow[d] \\ \mathbb{T}_{K/Z} \arrow[r] & \mathbb{T}_{K/W} \end{tikzcd}$$

is Cartesian.

Proof From the diagram of stacks, one immediately gets two fiber sequences of quasi-coherent sheaves on K

$$\mathbb{T}_{K/Y} \longrightarrow \mathbb{T}_{K/W} \longrightarrow \phi^* i^* \mathbb{T}_{Y/W}$$
$$\mathbb{T}_{K/Z} \longrightarrow \mathbb{T}_{K/W} \longrightarrow \phi^* j^* \mathbb{T}_{Z/W}$$

and therefore the pullback of

$$\begin{tikzcd} & \mathbb{T}_{K/Y} \arrow[d] \\ \mathbb{T}_{K/Z} \arrow[r] & \mathbb{T}_{K/W} \end{tikzcd}$$

is equivalent to the fiber of the morphism $\mathbb{T}_{K/W} \to \phi^* i^* \mathbb{T}_{Y/W} \oplus \phi^* j^* \mathbb{T}_{Z/W}$. But by general properties of Cartesian squares, $\mathbb{T}_{X/W} \cong i^* \mathbb{T}_{Y/W} \oplus j^* \mathbb{T}_{Z/W}$, and hence we get that $\phi^* \mathbb{T}_{X/W} \cong \phi^* i^* \mathbb{T}_{Y/W} \oplus \phi^* j^* \mathbb{T}_{Z/W}$. We conclude by observing that the fiber of the map

$$\mathbb{T}_{K/W} \to \phi^* \mathbb{T}_{X/W}$$

is equivalent to $\mathbb{T}_{K/X}$. □

We have the following analogue of Proposition 3.1 for general derived stacks.

Theorem 3.6 *Consider a diagram*

$$\begin{tikzcd} L_1 \arrow[dr, "f"] & & L_2 \arrow[dl, "g"'] \\ & X & \end{tikzcd}$$

of derived Artin stacks. Suppose we are given n-shifted coisotropic structures $\gamma_1 \in \mathrm{Cois}(f, n)$ *and* $\gamma_2 \in \mathrm{Cois}(g, n)$ *on the morphisms* f *and* g*, such that the n-shifted Poisson structures on* X *coincide. Then the derived intersection* $Y := L_1 \times_X L_2$ *carries a natural* $(n-1)$*-shifted Poisson structure, such that the map*

$$Y \longrightarrow L_1 \times L_2$$

is a morphism of $(n-1)$*-shifted Poisson derived stacks, where* L_1 *is equipped with the opposite Poisson structure.*

Proof The Cartesian diagram of stacks

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "i"] & L_1 \arrow[d, "f"] \\ L_2 \arrow[r, "g"] & X \end{tikzcd}$$

induces a commutative square of $\mathbb{D}_{Y_{DR}}(\infty)$-algebras

$$\begin{tikzcd} j^* f^* \mathcal{B}_X(\infty) \cong i^* g^* \mathcal{B}_X(\infty) \arrow[r] \arrow[d] & j^* \mathcal{B}_{L_1}(\infty) \arrow[d] \\ i^* \mathcal{B}_{L_2}(\infty) \arrow[r] & \mathcal{B}_Y(\infty) \end{tikzcd}$$

By definition the two coisotropic structures γ_1 and γ_2 produce two $\mathbb{P}_{[n+1,n]}$-structures on the maps

$$j^* f^* \mathcal{B}_X(\infty) \to j^* \mathcal{B}_{L_1}(\infty) \quad \text{and} \quad i^* g^* \mathcal{B}_X(\infty) \to i^* \mathcal{B}_{L_2}(\infty)$$

so that by Proposition 3.1 we obtain a natural $\mathbb{P}_n$-structure on the coproduct

$$j^*\mathcal{B}_{L_1}(\infty) \otimes_{i^*g^*\mathcal{B}_X(\infty)} i^*\mathcal{B}_{L_2}(\infty) \ .$$

Our goal is now to show that this coproduct is actually equivalent to $\mathcal{B}_Y(\infty)$, which would immediately conclude the proof. Notice that the twist by $k(\infty)$ commutes with colimits, so that is enough to show that

$$j^*\mathcal{B}_{L_1} \otimes_{i^*g^*\mathcal{B}_X} i^*\mathcal{B}_{L_2} \cong \mathcal{B}_Y$$

as $\mathbb{D}_{Y_{DR}}$-algebras.

Let Spec $A \to Y_{DR}$ be an A-point of Y_{DR}. We want to prove that $j^*\mathcal{B}_{L_1} \otimes_{i^*g^*\mathcal{B}_X} i^*\mathcal{B}_{L_2}$ and $\mathcal{B}_Y$ coincide on the point Spec $A \to Y_{DR}$. By definition, the value of $\mathcal{B}_Y$ on this point is $\mathbb{D}(Y_A)$, where Y_A is the perfect formal derived stack over Spec A constructed as the fiber product

$$\begin{tikzcd} Y_A \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} A \arrow[r] & Y_{DR} \end{tikzcd}$$

Since the $(-)_{DR}$ construction is defined as a right adjoint, it automatically commutes with limits, so that $Y_{DR} \cong L_{1,DR} \times_{X_{DR}} L_{2,DR}$. In particular, any A-point of Y_{DR} has corresponding A-points of $L_{1,DR}$, $L_{2,DR}$ and X_{DR}, for which one can define fibers $L_{1,A}$, $L_{2,A}$ and X_A. Therefore, we need to show that

$$\mathbb{D}(Y_A) \cong \mathbb{D}(L_{1,A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2,A})$$

as graded mixed dg algebras.

We start by remarking that the fiber square

$$\begin{tikzcd} Y_A \arrow[r] \arrow[d] & L_{1,A} \arrow[d] \\ L_{2,A} \arrow[r] & X_A \end{tikzcd}$$

induces a map of graded mixed cdgas

$$\mathbb{D}(L_{1,A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2,A}) \to \mathbb{D}(Y_A)$$

by the universal property of the coproduct. In order to prove that this map is an equivalence, it is enough to check it at the level of commutative algebras, forgetting the graded mixed structures. But the forgetful functor

$$\mathbf{CAlg}(\mathbf{dg}_k^{gr,\epsilon}) \longrightarrow \mathbf{CAlg}(\mathbf{dg}_k)$$

comes by definition from the forgetful functor

$$\mathrm{CoMod}_B(\mathbf{dg}_k) \longrightarrow \mathbf{dg}_k$$

where B is the bialgebra $B = k[t, t^{-1}] \otimes_k k[x]$ and $\mathrm{CoMod}_B(\mathbf{dg}_k)$ is the category of B-comodules in $\mathbf{dg}_k$, as explained in [15, Section 1.2]. In particular, this means that forgetting the graded mixed structure preserves colimits, so that the underlying commutative algebra of the pushout of

$$\begin{tikzcd}
\mathbb{D}(X_A) \arrow[r] \arrow[d] & \mathbb{D}(L_{1,A}) \\
\mathbb{D}(L_{2,A}) &
\end{tikzcd}$$

is exactly the tensor product of commutative algebras $\mathbb{D}(L_{1,A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2,A})$. Since the stacks X_A, $L_{1,A}$, $L_{2,A}$ are all *algebraisable* (in the sense of Section 2.2 of [5]), by [5, Theorem 2.2.2] we have equivalences of commutative algebras

$$\begin{aligned}
&\mathbb{D}(L_{1,A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2,A}) \cong \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{1,A}}[-1]) \\
&\quad \otimes_{\mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/X_A}[-1])} \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{2,A}}[-1]) \\
&\mathbb{D}(Y_A) \cong \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/Y_A}[-1])
\end{aligned}$$

We can now just apply Lemma 3.5 to the diagram of algebraisable stacks

$$\begin{tikzcd}
\mathrm{Spec}(A^{red}) \arrow[r] & Y_A \arrow[r] \arrow[d] & L_{1,A} \arrow[d] \\
& L_{2,A} \arrow[r] & X_A
\end{tikzcd}$$

and get a Cartesian square of A^{red}-modules

$$\begin{tikzcd}
\mathbb{T}_{A^{red}/Y_A} \arrow[r] \arrow[d] & \mathbb{T}_{A^{red}/L_{1,A}} \arrow[d] \\
\mathbb{T}_{A^{red}/L_{2,A}} \arrow[r] & \mathbb{T}_{A^{red}/X_A}
\end{tikzcd}$$

From this we deduce a pushout diagram of A^{red}-algebras

$$\begin{tikzcd}
\mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/X_A}[-1]) \arrow[r] \arrow[d] & \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{1,A}}[-1]) \arrow[d] \\
\mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{2,A}}[-1]) \arrow[r] & \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/Y_A}[-1])
\end{tikzcd}$$

which is exactly what we wanted. □

Example 3.7 Let G be an affine algebraic group. Assume that $\mathrm{B}G$ carries a 1-shifted Poisson structure and the basepoint $\mathrm{pt} \to \mathrm{B}G$ carries a coisotropic structure. By Theorem 3.6 we obtain an ordinary Poisson structure on $G \cong \mathrm{pt} \times_{\mathrm{B}G} \mathrm{pt}$ which is easily seen to be multiplicative, i.e. G carries a Poisson–Lie structure. It is shown in [24, Corollary 2.11] that in fact the space of 1-shifted coisotropic structures on $\mathrm{pt} \to \mathrm{B}G$ is equivalent to the set of Poisson–Lie structures on G.

Remark 3.8 Following Remark 3.4, one could generalize Proposition 3.3 to possibly non-affine coisotropic correspondences: this will likely lead to a construction of the full (∞, m)-category of coisotropic correspondences, even in the non-affine case. Namely, there exists an (∞, m)-category whose objects are n-shifted Poisson stacks, and whose i-morphisms are i-fold coisotropic correspondences. This has to be considered as a derived incarnation of the so-called *Poisson category* studied by Weinstein [28]. We plan to give a detailed construction of this category in a future paper.

4 Non-degenerate coisotropic structures

The purpose of this section is to introduce the notion of non-degeneracy of a coisotropic structure. This is a relative version of non-degenerate Poisson structures, as treated in [5] or [17]. Our main result is a proof of [5, Conjecture 3.4.5], stating that the space of non-degenerate coisotropic structures is equivalent to the space of Lagrangian structures in the sense of [21].

4.1 Definition of non-degeneracy

We begin by first looking at the affine case. Recall the following notion from [5, Corollary 1.4.24].

Definition 4.1 Let A be a commutative algebra in $\mathcal{M}$ such that $\mathbb{L}_A^{int}$ is a dualizable A-module. Suppose moreover that A is equipped with an n-shifted Poisson structure. We say that it is *non-degenerate* if the induced morphism

$$\pi_A^\sharp \colon \mathbb{L}_A \longrightarrow \mathbb{T}_A[-n]$$

is an equivalence.

Suppose we have a cofibrant algebra $A \in \mathrm{CAlg}_M$. Then in this case we adopt the convention that $\pi_A^\sharp(f \mathrm{d}_{\mathrm{dR}} g) = \pm f[\pi_2, g]$, where π_2 is the underlying bivector of π, and $\pm$ is the Koszul sign.

Equivalently, as in [5, Definition 1.4.18], a $\mathbb{P}_{n+1}$-algebra A is non-degenerate if the morphism

$$\mathbf{DR}^{int}(A) \longrightarrow \mathbf{Pol}^{int}(A, n)$$

induced by the Poisson bracket is an equivalence in $\mathcal{M}^{gr}$.

We now deal with the case of relative Poisson algebras. Let (A, B) be a $\mathbb{P}_{[n+1,n]}$-algebra in $\mathcal{M}$, and let $f : A \to B$ be the underlying morphism in $\mathbf{CAlg}_{\mathcal{M}}$. Using the description of $\mathbb{P}_{[n+1,n]}$-structures in terms of relative polyvectors we see that the induced map

$$\mathbb{L}_{B/A}[-1] \longrightarrow f^*\mathbb{L}_A \xrightarrow{f^*\pi_A^\sharp} f^*\mathbb{T}_A[-n] \longrightarrow \mathbb{T}_{B/A}[-n+1]$$

is null-homotopic. Therefore, we get a morphism of fiber sequences of B-modules:

$$\begin{tikzcd}
\mathbb{L}_{B/A}[-1] \arrow[r] \arrow[d, dotted] & f^*\mathbb{L}_A \arrow[r] \arrow[d, "f^*\pi_A^\sharp"] & \mathbb{L}_B \arrow[d, dotted] \\
\mathbb{T}_B[-n] \arrow[r] & f^*\mathbb{T}_A[-n] \arrow[r] & \mathbb{T}_{B/A}[-n+1].
\end{tikzcd} \tag{1}$$

Note that if $\pi_A^\sharp$ and one of the dotted maps are equivalences, so is the other dotted map.

Definition 4.2 Let $f : A \to B$ be a morphism of commutative algebras in $\mathcal{M}$ equipped with an n-shifted coisotropic structure.

- We say that the coisotropic structure is *non-degenerate* if the n-shifted Poisson structure on A is non-degenerate and one of the dotted maps in diagram (1) is an equivalence.
- The space $\mathrm{Cois}^{nd}(f, n)$ of *non-degenerate n-shifted coisotropic structures* on f is the subspace of $\mathrm{Cois}(f, n)$ consisting of non-degenerate points.

We will now generalize the notion of non-degeneracy to stacks. Let $f : L \to X$ be a morphism of derived Artin stacks. Suppose we are given an n-shifted coisotropic structure on f in the sense of Definition 2.1. This means that, in particular, we have a map of graded dg Lie algebras

$$k(2)[-1] \longrightarrow \mathbf{Pol}(X, n)[n+1]$$

such that the induced map

$$k(2)[-1] \longrightarrow \mathbf{Pol}(L/X, n-1)[n+1]$$

is homotopic to zero. Looking at weight 2 components, the shifted Poisson structure on X induces by adjunction a morphism of perfect complexes on X

$$\pi_X^\sharp : \mathbb{L}_X \to \mathbb{T}_X[-n],$$

and the coisotropic condition implies that the induced map $\mathbb{L}_{L/X} \to \mathbb{T}_{L/X}[-n+2]$ is homotopic to zero. This in turn implies the existence of dotted arrows in the diagram

$$\begin{tikzcd} \mathbb{L}_{L/X}[-1] \arrow[r] \arrow[d, dotted] & f^*\mathbb{L}_X \arrow[r] \arrow[d, "\pi_X^\sharp"] & \mathbb{L}_L \arrow[d, dotted] \\ \mathbb{T}_L[-n] \arrow[r] & f^*\mathbb{T}_X[-n] \arrow[r] & \mathbb{T}_{L/X}[-n+1] \end{tikzcd} \tag{2}$$

where both horizontal rows are fiber sequences of perfect complexes on L.

Definition 4.3 Let $f\colon L \to X$ be a morphism of derived Artin stacks equipped with an n-shifted coisotropic structure.

- We say that the coisotropic structure is *non-degenerate* if the n-shifted Poisson structure on X is non-degenerate and one of the dotted maps in diagram (2) is an equivalence.
- We denote the subspace of non-degenerate points by $\mathrm{Cois}^{nd}(f,n) \subset \mathrm{Cois}(f,n)$.

Example 4.4 Suppose $i\colon L \hookrightarrow X$ is a smooth closed coisotropic subscheme of a smooth scheme X carrying a (0-shifted) Poisson structure. Then the diagram (2) becomes

$$\begin{tikzcd} \mathrm{N}^*_{L/X} \arrow[r] \arrow[d, dotted] & i^*\mathrm{T}^*_X \arrow[r] \arrow[d, "\pi_X^\sharp"] & \mathrm{T}^*_L \arrow[d, dotted] \\ \mathrm{T}^*_L \arrow[r] & i^*\mathrm{T}_X \arrow[r] & \mathrm{N}_{L/X} \end{tikzcd}$$

The bivector π_X is non-degenerate iff it underlines a symplectic structure. The coisotropic structure on i is non-degenerate iff $\mathrm{T}^*_L \to \mathrm{N}_{L/X}$ is an isomorphism which is the case precisely when $L \hookrightarrow X$ is Lagrangian.

By Theorem 2.7, the datum of a coisotropic structure on $f\colon L \to X$ is equivalent to the datum of a $\mathbb{P}_{n+1}$-structure on $\mathcal{B}_X(\infty)$ and a compatible n-shifted coisotropic structure on the map $f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$ in the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

Corollary 4.5 *Let $f\colon L \to X$ be a morphism of derived Artin stacks. An n-shifted coisotropic structure on f is non-degenerate in the sense of Definition 4.3 if and only if the corresponding n-shifted Poisson structure on $\mathcal{B}_X(\infty)$ and the n-shifted coisotropic structure on $f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$ are both non-degenerate in the sense on Definitions 4.1 and 4.2.*

This is an immediate consequence of the general correspondence between geometric differential calculus on derived stacks and algebraic differential calculus on the associated prestacks of Tate principal parts as described in [5].

Alternatively, one can avoid using twists by $k(\infty)$ as follows. Consider the prestack of graded mixed commutative algebras $f^*\mathcal{B}_X$ and define the space of *Tate n-shifted*

Poisson structures on $f^*\mathcal{B}_X$ to be

$$\mathrm{Pois}^t(f^*\mathcal{B}_X, n) := \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}^t(f^*\mathcal{B}_X, n)[n+1]),$$

where $\mathbf{Pol}^t(f^*\mathcal{B}_X, n)$ is the Tate realization of the prestack of bigraded mixed $\mathbb{P}_{n+2}$-algebras $\mathbf{Pol}^{int}(f^*\mathcal{B}_X, n)$. We have an equivalence of graded $\mathbb{P}_{n+2}$-algebras

$$\mathbf{Pol}^t(f^*\mathcal{B}_X, n) \cong \mathbf{Pol}(f^*\mathcal{B}_X(\infty), n)$$

and hence an equivalence of spaces

$$\mathrm{Pois}^t(f^*\mathcal{B}_X, n) \cong \mathrm{Pois}(f^*\mathcal{B}_X(\infty), n).$$

Similarly, consider the map of prestacks of graded mixed algebras $f^*_{\mathcal{B}} \colon f^*\mathcal{B}_X \to \mathcal{B}_L$, and define the space of *Tate n-shifted coisotropic structures* on $f^*_{\mathcal{B}}$ to be

$$\mathrm{Cois}^t(f^*_{\mathcal{B}}, n) := \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}^t(f^*_{\mathcal{B}}, n)[n+1]).$$

Again, we have an equivalence

$$\mathrm{Cois}^t(f^*_{\mathcal{B}}, n) \simeq \mathrm{Cois}(f^*_{\mathcal{B}}(\infty), n).$$

We also have obviously defined subspaces of non-degenerate structures $\mathrm{Pois}^{t,nd}(f^*\mathcal{B}_X, n)$ and $\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}, n)$. Notice that by definition we have a Cartesian square

$$\begin{tikzcd}
\mathrm{Cois}^{nd}(f, n) \arrow[r] \arrow[d] & \mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}, n) \arrow[d] \\
\mathrm{Pois}^{t,nd}(\mathcal{B}_X, n) \arrow[r] & \mathrm{Pois}^{t,nd}(f^*\mathcal{B}_X, n).
\end{tikzcd}$$

4.2 Symplectic and Lagrangian structures

We recall the notions of shifted symplectic and shifted Lagrangian structures, defined and studied in [5,21].

Definition 4.6 Let $A \in \mathbf{CAlg}_{\mathcal{M}}$. The *space of closed 2-forms of degree n on A* is

$$\mathcal{A}^{2,cl}(A, n) := \mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-1], \mathbf{DR}(A)[n+1]).$$

In particular, every closed 2-form ω of degree n has an underlying 2-form $\omega_2 \in \mathrm{Sym}^2_A(\mathbb{L}_A[-1])$. If the A-module $\mathbb{L}_A$ is dualizable, this in turn gives rise to a morphism

$$\omega^\sharp \colon \mathbb{T}_A \to \mathbb{L}_A[n].$$

Explicitly, suppose A is a cofibrant commutative algebra in M, and suppose the underlying 2-form is written as $\omega_2 = \sum_{i,j} \omega_{ij} \mathrm{d}_{\mathrm{dR}} a_i \mathrm{d}_{\mathrm{dR}} a_j$. Then our convention is that

$$\omega^\sharp(v) = \pm \sum_{i,j} 2\omega_{ij}[v, a_i]\mathrm{d}_{\mathrm{dR}} a_j,$$

where $v \in \mathbb{T}_A$ and $\pm$ is the Koszul sign.

Definition 4.7 Let again $A \in \mathbf{CAlg}_{\mathcal{M}}$, and suppose moreover that the cotangent complex $\mathbb{L}_A$ is dualizable.

- We say that a point $\omega \in \mathcal{A}^{2,cl}(A, n)$ is *non-degenerate* if the above map $\omega^\sharp$ is an equivalence.
- The space $\mathrm{Symp}(A, n)$ of *n-shifted symplectic structures* on A is the subspace of $\mathcal{A}^{2,cl}(A, n)$ of non-degenerate forms.

Suppose now $f\colon A \to B$ is a morphism in $\mathbf{CAlg}_{\mathcal{M}}$. There is an induced map $\mathbf{DR}(A) \to \mathbf{DR}(B)$ of graded mixed cdgas and denote by $\mathbf{DR}(f)$ the fiber of this map.

Definition 4.8 With notations as above, the space $\mathrm{Isot}(f, n)$ of *n-shifted isotropic structures* is

$$\mathrm{Isot}(f, n) := \mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-1], \mathbf{DR}(f)[n+1]).$$

Informally, elements of $\mathrm{Isot}(f, n)$ are closed 2-forms of degree n on A, whose restriction to B is homotopic to zero. In other words, there is a fiber sequence of spaces

$$\mathrm{Isot}(f, n) \to \mathcal{A}^{2,cl}(A, n) \to \mathcal{A}^{2,cl}(B, n).$$

Let $\lambda \in \mathrm{Isot}(f, n)$, and suppose that $\mathbb{L}_A$ and $\mathbb{L}_B$ are both dualizable. The point λ produces a map $\mathbb{T}_A \to \mathbb{L}_A[n]$ of A-modules, such that the composite

$$\mathbb{T}_B \longrightarrow f^*\mathbb{T}_A \xrightarrow{f^*\omega_A^\sharp} f^*\mathbb{L}_A[n] \longrightarrow \mathbb{L}_B[n]$$

is null-homotopic. This yields a diagram of B-modules

$$\begin{tikzcd} \mathbb{T}_B \arrow[r] \arrow[d, dotted] & f^*\mathbb{T}_A \arrow[r] \arrow[d, "f^*\omega_A^\sharp"] & \mathbb{T}_{B/A}[1] \arrow[d, dotted] \\ \mathbb{L}_{B/A}[n-1] \arrow[r] & f^*\mathbb{L}_A[n] \arrow[r] & \mathbb{L}_B[n], \end{tikzcd} \tag{3}$$

where both rows are fiber sequences. As before, if $\omega_A^\sharp$ and one of the dotted maps are equivalences, so is the other dotted map.

Definition 4.9 Let $f\colon A \to B$ be a morphism in $\mathbf{CAlg}_{\mathcal{M}}$ and suppose both $\mathbb{L}_A$ and $\mathbb{L}_B$ are dualizable.

- We say that a point $\lambda \in \mathrm{Isot}(f, n)$ is *non-degenerate* if $\omega_A^\sharp$ and of the dotted maps in diagram (3) is an equivalence.
- The space $\mathrm{Lagr}(f, n)$ of *n-shifted Lagrangian structures* is the subspace of $\mathrm{Isot}(f, n)$ consisting of non-degenerate points.

These algebraic notions can be used to introduce the concepts of symplectic and Lagrangian structures for general derived stacks.

Definition 4.10 Let X be a derived Artin stack. The space $\mathrm{Symp}(X, n)$ of n-shifted symplectic structures on X is

$$\mathrm{Symp}(X, n) := \mathrm{Symp}(\mathcal{B}_X, n),$$

where we regard $\mathcal{B}_X$ as a commutative algebra in the category of $\mathbb{D}_{X_{DR}}$-modules.

Remark 4.11 By [5, Proposition 2.4.15] this notion recovers the original global definition of an n-shifted symplectic structure given by [21, Definition 1.18].

The following is [5, Theorem 3.2.4] and [17, Theorem 3.33].

Theorem 4.12 *Let X be a derived Artin stack. There is an equivalence of spaces*

$$\mathrm{Pois}^{nd}(X) \simeq \mathrm{Symp}(X, n).$$

In the relative case the definitions are analogous. Recall that given a map $f\colon L \to X$ of derived stacks we have an induced map

$$f_{\mathcal{B}}^*\colon f^*\mathcal{B}_X \to \mathcal{B}_L$$

of commutative algebras in the category of $\mathbb{D}_{L_{DR}}$-modules. Notice that any shifted symplectic structure on X gives in particular a shifted symplectic structure on $f^*\mathcal{B}_X$; in other words, there is a natural map of spaces

$$\mathrm{Symp}(X, n) \simeq \mathrm{Symp}(\mathcal{B}_X, n) \longrightarrow \mathrm{Symp}(f^*\mathcal{B}_X, n).$$

Definition 4.13 Let $f\colon L \to X$ be a map of derived Artin stacks. The space $\mathrm{Lagr}(f, n)$ of *n-shifted Lagrangian structures* on f is given by the pullback

$$\begin{tikzcd}
\mathrm{Lagr}(f, n) \arrow[r] \arrow[d] & \mathrm{Lagr}(f_{\mathcal{B}}^*, n) \arrow[d] \\
\mathrm{Symp}(X, n) \arrow[r] & \mathrm{Symp}(f^*\mathcal{B}_X, n).
\end{tikzcd}$$

4.3 Compatible pairs

The remainder of the section is devoted to proving a derived analogue of Example 4.4. In particular, we would like to compare the spaces $\mathrm{Cois}^{nd}(f,n)$ and $\mathrm{Lagr}(f,n)$. With this goal in mind, we begin by developing a general formalism of compatibility for pairs (γ,λ) formed by a coisotropic and a Lagrangian structure in a general symmetric monoidal ∞-category, following the approach of [20]. The notion of compatibility of an n-shifted Poisson and an n-shifted symplectic structure has previously appeared in [5, Definition 1.4.20] and [17, Definition 1.20].

Consider a map $f\colon A\to B$ in CAlg_M, such that both A and B are cofibrant, and consider the graded homotopy $\mathbb{P}_{[n+2,n+1]}$-algebra

$$(\mathbf{Pol}(A,n),\mathbf{Pol}(B/A,n-1))$$

constructed in [15]. We pick a *strict* graded $\mathbb{P}_{[n+2,n+1]}$ algebra, which is equivalent to the latter, and we denote it by

$$(\mathrm{Pol}(A,n),\mathrm{Pol}(B/A,n-1)).$$

We denote by $\mathrm{Pol}(f,n)$ the homotopy fiber of the underlying morphism of graded commutative algebras. Consider an element $\gamma\in\mathrm{Pol}(f,n)^{\geq 2}[n+2]$. By the results of [15, Section 3.6], we know that γ induces a pair of compatible k-linear derivations $\phi_{\gamma,A}$ and $\phi_{\gamma,B}$ on $\mathrm{Pol}(A,n)$ and $\mathrm{Pol}(B/A,n-1)$ respectively. Restricted to weight zero they fit in the diagram

$$\begin{tikzcd}
\Omega^1_A[-1] \arrow[r] \arrow[d, "\phi_{\gamma,A}"] & \Omega^1_B[-1] \arrow[d, "\phi_{\gamma,B}"] \\
\mathrm{Pol}(A,n) \arrow[r] & \mathrm{Pol}(B/A,n-1).
\end{tikzcd}$$

Using the universal property of the symmetric algebra we obtain a diagram of commutative algebras

$$\begin{tikzcd}
\mathrm{DR}(A) \arrow[r] \arrow[d, "\mu(-,\gamma)_A"] & \mathrm{DR}(B) \arrow[d, "\mu(-,\gamma)_B"] \\
\mathrm{Pol}(A,n) \arrow[r] & \mathrm{Pol}(B/A,n-1).
\end{tikzcd}$$

Remark 4.14 The construction above also applies to the general case of coisotropic structure on a morphism of derived Artin stacks. For example, let $L\hookrightarrow X$ be a smooth coisotropic submanifold of a Poisson manifold. Then Oh–Park [16] and Cattaneo–Felder [3] construct a certain homotopy $\mathbb{P}_1$-algebra which as a graded commutative algebra coincides with $\Gamma(L,\mathrm{Sym}(\mathrm{N}_{L/X}[-1]))$. We expect that it coincides with $\mathbf{Pol}(L/X,-1)$ with the differential twisted by $\phi_{\gamma,L}$.

The vertical arrows are the identity on weight 0 while their value on weight 1 generators is given by

$$\mu(a\mathrm{d}_{\mathrm{dR}}x, \gamma)_A = a\phi_{\gamma,A}(x), \quad \mu(b\mathrm{d}_{\mathrm{dR}}y, \gamma)_B = b\phi_{\gamma,B}(y),$$

where $a, x \in A$ and $b, y \in B$. Moreover, if γ satisfies the Maurer–Cartan equation, i.e. it corresponds to an ∞-morphism $k(2)[-1] \to \mathrm{Pol}(f, n)[n+1]$ of graded Lie algebras, the above diagram is a diagram of weak graded mixed algebras.

In particular, observe that every $\gamma \in \mathrm{Pol}(f, n)[n+2]$ induces a map

$$\mu(-, \gamma)\colon \mathrm{DR}(f) \to \mathrm{Pol}(f, n)$$

of commutative algebras, where $\mathrm{DR}(f)$ is the homotopy fiber of $\mathrm{DR}(A) \to \mathrm{DR}(B)$. Moreover, if γ defines a coisotropic structure then $\mu(-, \gamma)$ is a map of graded mixed commutative algebras.

Consider now the commutative dg algebra $k[\epsilon]$, where ϵ is of degree 0 and satisfies $\epsilon^2 = 0$. If $\mathfrak{g}$ is a graded dg Lie algebra, then $\mathfrak{g} \otimes k[\epsilon]$ is still a graded dg Lie algebra, and there is a natural projection of graded Lie algebras $\mathfrak{g} \otimes k[\epsilon] \to \mathfrak{g}$ sending ϵ to zero. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the operator given by $p - 1$ in degree p. Then $\mathrm{id} + \epsilon\sigma$ gives a section $\mathfrak{g} \to \mathfrak{g} \otimes k[\epsilon]$.

The projection $\mathfrak{g} \otimes k[\epsilon] \to \mathfrak{g}$ induces a morphism of spaces

$$\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathfrak{g} \otimes k[\epsilon]) \to \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathfrak{g}),$$

whose fibers have the following nice characterization.

Lemma 4.15 *The fiber of the map*

$$\mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathfrak{g} \otimes k[\epsilon]) \to \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathfrak{g}),$$

at a point corresponding to a Maurer–Cartan element $x \in \mathfrak{g}^{\geq 2}$ *is given by* $\mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-1], \mathfrak{g}_x)$*, where* $\mathfrak{g}_x$ *is the graded module* L *equipped with the mixed structure* $[x, -]$.

The proof of this lemma is a straightforward computation, and we omit it. Following [17], it is convenient to introduce an auxiliary space.

Definition 4.16 Let again $f\colon A \to B$ be a morphism of commutative algebras in M. The *tangent space of* n*-shifted coisotropic structures on* f is

$$\mathrm{TCois}(f, n) = \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathrm{Pol}(f, n)[n+1] \otimes k[\epsilon]).$$

In particular, we get a natural map

$$\mathrm{id} + \epsilon\sigma\colon \mathrm{Cois}(f, n) \longrightarrow \mathrm{TCois}(f, n).$$

By Lemma 4.15 a point of $\mathrm{TCois}(f, n)$ is given by an n-shifted coisotropic structure γ on f together with a morphism of graded mixed complexes $k(2)[-1] \to \mathrm{Pol}_\gamma(f, n)[n+1]$, where $\mathrm{Pol}_\gamma(f, n)[n+1]$ is $\mathrm{Pol}(f, n)[n+1]$ equipped with the mixed structure $[\gamma, -]$.

Consider now the natural projection

$$\mathrm{TCois}(f, n) \times \mathrm{Lagr}(f, n) \to \mathrm{Cois}(f, n) \times \mathrm{Lagr}(f, n),$$

which simply forgets the ϵ-component on $\mathrm{TCois}(f, n)$. The above constructions provide a section of this map.

Lemma 4.17 *The section*

$$\begin{aligned} \Phi\colon \mathrm{Cois}(f, n) \times \mathrm{Lagr}(f, n) &\longrightarrow \mathrm{TCois}(f, n) \times \mathrm{Lagr}(f, n) \\ (\gamma, \lambda) &\longmapsto (\gamma + \epsilon(\sigma(\gamma) - \mu(\lambda, \gamma)), \lambda). \end{aligned}$$

is well-defined.

Proof We have to check that if γ is a Maurer–Cartan element and λ is a closed element, then $\gamma + \epsilon(\sigma(\gamma) - \mu(\gamma, \lambda))$ is also a Maurer–Cartan element.

For this it is enough to show that $a_1 = \gamma + \epsilon\sigma(\gamma)$ and $a_2 = \gamma - \epsilon\mu(\lambda, \gamma)$ separately satisfy the Maurer–Cartan equations. Indeed, $\mathrm{id} + \epsilon\sigma$ is a morphism of Lie algebras and hence sends Maurer–Cartan elements to Maurer–Cartan elements. For the second expression we can compute

$$[a_2, a_2] = [\gamma - \epsilon\mu(\lambda, \gamma), \gamma - \epsilon\mu(\lambda, \gamma)] = [\gamma, \gamma] - 2\epsilon[\gamma, \mu(\lambda, \gamma)].$$

Since $\mu(-, \gamma)$ is a morphism of weak graded mixed commutative algebras (see [15, Definition 1.11]), we have $[\gamma, \mu(\lambda, \gamma)] = \mu(\mathrm{d}_{\mathrm{dR}}\lambda, \gamma)$. Therefore,

$$\mathrm{d}a_2 + \frac{1}{2}[a_2, a_2] = \mathrm{d}\gamma + \frac{1}{2}[\gamma, \gamma] - \epsilon\mu((\mathrm{d} + \mathrm{d}_{\mathrm{dR}})\lambda, \gamma) = 0.$$

□

We will repeatedly consider the following construction. Suppose $p\colon T \to S$ is a morphism of simplicial sets equipped with two sections $s\colon S \to T$ and $0\colon S \to T$. The section 0 will be implicit and we encode the rest of the data in the following diagram:

$$\begin{tikzcd} T \arrow[r, "p"'] & S \arrow[l, bend right, "s"'] \end{tikzcd}$$

The following is introduced in [17, Definition 1.23].

Definition 4.18 The *vanishing locus* of the diagram

$$\begin{tikzcd} T \arrow[r, "p"'] & S \arrow[l, bend right, "s"'] \end{tikzcd}$$

is defined to be the homotopy limit of

$$\begin{tikzcd} S \arrow[r, shift left, "s"] \arrow[r, shift right, "0"'] & T \arrow[r, "p"] & S \end{tikzcd}$$

In other words, the vanishing locus of such a diagram parametrizes points $x \in S$ together with a homotopy $s(x) \sim 0(x)$ in $p^{-1}(x)$. Given this definition, we can define compatibility between coisotropic and Lagrangian structures.

Definition 4.19 We define the space $\mathrm{Comp}(f, n)$ of *compatible pairs* to be the vanishing locus of

$$\begin{tikzcd} \mathrm{TCois}(f, n) \times \mathrm{Lagr}(f, n) \arrow[r] & \mathrm{Cois}(f, n) \times \mathrm{Lagr}(f, n) \arrow[l, bend right, "\Phi"'] \end{tikzcd}$$

In other words, elements of $\mathrm{Comp}(f, n)$ are given by pairs (γ, λ) of a coisotropic and a Lagrangian structure on f together with a homotopy from $\mu(\lambda, \gamma)$ to $\sigma(\gamma)$ in $\mathrm{Pol}_\gamma(f, n)$.

Definition 4.20 The space of *non-degenerate compatible pairs* $\mathrm{Comp}^{nd}(f, n)$ is defined to be the homotopy fiber product

$$\begin{tikzcd} \mathrm{Comp}^{nd}(f, n) \arrow[r] \arrow[d] & \mathrm{Comp}(f, n) \arrow[d] \\ \mathrm{Cois}^{nd}(f, n) \arrow[r] & \mathrm{Cois}(f, n). \end{tikzcd}$$

In particular, the space $\mathrm{Comp}^{nd}(f, n)$ comes equipped with two projections to $\mathrm{Cois}^{nd}(f, n)$ and $\mathrm{Lagr}(f, n)$, giving a correspondence

$$\begin{tikzcd} & \mathrm{Comp}^{nd}(f, n) \arrow[dl] \arrow[dr] & \\ \mathrm{Cois}^{nd}(f, n) & & \mathrm{Lagr}(f, n). \end{tikzcd}$$

Proposition 4.21 *The map*

$$\mathrm{Comp}^{nd}(f,n) \to \mathrm{Cois}^{nd}(f,n)$$

is an equivalence.

Proof If γ is a non-degenerate coisotropic structure, then $\mu(-,\gamma)\colon \mathrm{DR}(f) \to \mathrm{Pol}_\gamma(f,n)$ is an equivalence. In particular, the space of two-forms $\lambda \in \mathrm{DR}(f)$ such that $\mu(\lambda,\gamma) \sim \sigma(\gamma)$ is contractible. □

It follows that given any morphism $f\colon A \to B$ in CAlg_M, we get a map of spaces

$$\mathrm{Cois}^{nd}(f,n) \to \mathrm{Lagr}(f,n).$$

In particular, consider a map $f\colon L \to X$ of derived stacks. We have an induced map $f^*_{\mathcal{B}}\colon f^*\mathcal{B}_X \to \mathcal{B}_L$ of $\mathbb{D}_{L_{DR}}$-algebras; twisting by $k(\infty)$, we get $f^*_{\mathcal{B}}(\infty)$: $f^*\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$ of commutative algebras in the category of graded mixed $\mathbb{D}_{L_{DR}}(\infty)$-modules. From the above discussion, we know that there is a morphism

$$\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}},n) \simeq \mathrm{Cois}^{nd}(f^*_{\mathcal{B}}(\infty),n) \to \mathrm{Lagr}(f^*_{\mathcal{B}}(\infty),n) \simeq \mathrm{Lagr}(f^*_{\mathcal{B}},n).$$

Together with Theorem 4.12, this produces a map

$$\mathrm{Cois}^{nd}(f,n) \to \mathrm{Lagr}(f,n).$$

The following is the main result of this section, which has been stated as a conjecture in [5, Conjecture 3.4.5].

Theorem 4.22 *Let $f\colon L \to X$ be a map of derived Artin stacks. Then the above map*

$$\mathrm{Cois}^{nd}(f,n) \to \mathrm{Lagr}(f,n)$$

is an equivalence.

Note that a treatment of this result in the case $n = 0$ was given in [20] and our proof is a slight generalization of that.

4.4 Reduction to graded mixed algebras

Our first step in the proof of Theorem 4.22 is to notice that we can reduce the problem to an algebraic question: in fact, by Theorem 4.12, we only need to show that the map

$$\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}},n) \to \mathrm{Lagr}(f^*_{\mathcal{B}},n)$$

is an equivalence.

We claim that both $\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}, n)$ and $\mathrm{Lagr}(f^*_{\mathcal{B}}, n)$ can obtained as global sections of prestacks. Consider the internal relative Tate polyvectors

$$(\mathbf{Pol}^{int,t}(f^*\mathcal{B}_X, n), \mathbf{Pol}^{int,t}(\mathcal{B}_L/f^*\mathcal{B}_X, n-1)).$$

This is a graded $\mathbb{P}_{[n+2,n+1]}$-algebra in the ∞-category $\mathrm{Fun}((\mathrm{dAff}/L_{DR})^{op}, \mathrm{dg}_k)$, or equivalently a diagram of graded $\mathbb{P}_{[n+2,n+1]}$-algebras in cochain complexes. Let $\mathbf{Pol}^{int,t}(f^*_{\mathcal{B}}, n)$ be the fiber of the underlying map of commutative algebras. Then there is an equivalence

$$\mathrm{Cois}^t(f^*_{\mathcal{B}}, n) \simeq \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathrm{Pol}^t(f^*_{\mathcal{B}}, n)),$$

where $\mathrm{Pol}^t(f^*_{\mathcal{B}}, n)$ is the realization (i.e. the limit) of the diagram $\mathbf{Pol}^{int,t}(f^*_{\mathcal{B}}, n)$.

Now define the prestack

$$\begin{array}{rccc} \underline{\mathrm{Cois}}^t(f^*_{\mathcal{B}}, n): & (\mathrm{dAff}/L_{DR})^{op} & \longrightarrow & \mathrm{sSet} \\ & (\mathrm{Spec}\, A \to L_{DR}) & \longmapsto & \mathrm{Map}_{\mathrm{Alg}^{gr}_{\mathrm{Lie}}}(k(2)[-1], \mathbf{Pol}^{int,t}(f^*_{\mathcal{B}}, n)(A)). \end{array}$$

Consider the sub-prestack of $\underline{\mathrm{Cois}}^t(f^*_{\mathcal{B}}, n)$ of non-degenerate coisotropic structures, and denote it by $\underline{\mathrm{Cois}}^{t,nd}(f^*_{\mathcal{B}}, n)$. By definition, by taking global sections (that is to say, taking the realization) of the prestack $\underline{\mathrm{Cois}}^{t,nd}(f^*_{\mathcal{B}})$ we get the space $\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}, n)$.

Similarly, we can define the prestack

$$\begin{array}{rccc} \underline{\mathrm{Isot}}: & (\mathrm{dAff}/L_{DR})^{op} & \to & \mathrm{sSet} \\ & (\mathrm{Spec}\, A \to L_{DR}) & \mapsto & \mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-n-2], \mathbf{DR}^{int}(f^*_{\mathcal{B}})(A)) \end{array}$$

of n-shifted isotropic structures on $f^*_{\mathcal{B}}$, and define $\underline{\mathrm{Lagr}}(f^*_{\mathcal{B}}, n)$ to be the sub-prestack of $\underline{\mathrm{Isot}}(f^*_{\mathcal{B}}, n)$ of Lagrangian structures. Once again, global sections of $\underline{\mathrm{Lagr}}(f^*_{\mathcal{B}}, n)$ are identified with the space $\mathrm{Lagr}(f^*_{\mathcal{B}}, n)$.

The construction of Sect. 4.3 produces a map of prestacks

$$\underline{\mathrm{Cois}}^{t,nd}(f^*_{\mathcal{B}}, n) \longrightarrow \underline{\mathrm{Lagr}}(f^*_{\mathcal{B}}, n). \tag{4}$$

Taking global sections on both sides we get back $\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}, n) \to \mathrm{Lagr}(f^*_{\mathcal{B}}, n)$.

It follows that in order to prove Theorem 4.22, it is enough to show that the above map (4) is an equivalence. This can be checked object-wise; in particular, we need to show that for every $\mathrm{Spec}\, A \to L_{DR}$, the induced map

$$\mathrm{Cois}^{t,nd}(f^*_{\mathcal{B}}(A), n) \to \mathrm{Lagr}(f^*_{\mathcal{B}}(A), n)$$

is an equivalence. Notice that in this case, $f^*_{\mathcal{B}}(A)$ is a map between graded mixed cdgas. In particular, Theorem 4.22 will follow from the following result.

Theorem 4.23 *Let M be the model category of graded mixed complexes, and let $f: A \to B$ be a morphism in* CAlg_M. *Then the map*

$$\mathrm{Cois}^{t,nd}(f, n) \to \mathrm{Lagr}(f, n)$$

is an equivalence.

Remark 4.24 By the correspondence between Tate realizations and twists by $k(\infty)$, the above theorem says that the map

$$\mathrm{Cois}^{nd}(f(\infty), n) \to \mathrm{Lagr}(f(\infty), n)$$

is an equivalence, where $f(\infty): A(\infty) \to B(\infty)$ is the twist of f by $k(\infty)$.

The proof of Theorem 4.23 occupies the rest of this section.

4.5 Filtrations

In order to make notations a bit simpler, we will omit the underlying map f and the shift n in what follows. Moreover, we will also remove the reference to Tate realization. In particular, the space of Tate n-coisotropic structures $\mathrm{Cois}^t(f, n)$ will be denoted Cois, and similarly for Cois^{nd}, TCois, TCois^{nd}, Lagr.

Notice that the weight grading of $\mathrm{Pol}^t(f, n)$ automatically defines a filtration on it: we denote by $\mathrm{Pol}^t(f, n)^{\leq p}$ the graded Poisson algebra

$$\mathrm{Pol}^t(f, n)^{\leq p} = \bigoplus_{i \leq p} \mathrm{Pol}^t(f, n)^i$$

where $\mathrm{Pol}^t(f, n)^i$ is the weight i part of $\mathrm{Pol}^t(f, n)$. Note that $\mathrm{Pol}^t(f, n)^{\leq p}$ is naturally a quotient of $\mathrm{Pol}^t(f, n)$. This in turn induces a filtration on the space Cois, and we set

$$\mathrm{Cois}^{\leq p} := \mathrm{Map}_{\mathrm{Lie}^{gr}}(k(2)[-1], \mathrm{Pol}^t(f, n)[n+1]^{\leq p}),$$

and similarly $\mathrm{Cois}^{nd, \leq p}$ is the subspace of $\mathrm{Cois}^{\leq p}$ of non-degenerate coisotropic structures. Note that non-degeneracy is merely a condition on the underlying bivector. We define in a similar way the spaces $\mathrm{TCois}^{\leq p}$ and $\mathrm{TCois}^{nd, \leq p}$.

The same construction also applies to the space Lagr. In fact, Lagr is by definition the subspace of

$$\mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-n-2], \mathrm{DR}(f))$$

given by non-degenerate maps. But using the weight grading on $\mathrm{DR}(f)$, one can define graded mixed modules $\mathrm{DR}(f)^{\leq p}$, a quotient of $\mathrm{DR}(f)$, and set $\mathrm{Lagr}^{\leq p}$ to be the subspace of

$$\mathrm{Map}_{\mathrm{dg}^{gr,\epsilon}}(k(2)[-n-2], \mathrm{DR}(f)^{\leq p})$$

given by non-degenerate maps.

Notice that all our previous constructions of μ and σ in Sect. 4.3 are compatible with these filtrations and

$$\mathrm{TCois}^{\leq p} \times \mathrm{Lagr}^{\leq p} \to \mathrm{Cois}^{\leq p} \times \mathrm{Lagr}^{\leq p}$$

admits a natural section Φ_p. We define $\mathrm{Comp}^{\leq p}$ to be the vanishing locus of the section Φ_p.

In particular, for every p we have a commutative square

$$\begin{tikzcd} \mathrm{Comp}^{\leq p+1} \arrow[r] \arrow[d] & \mathrm{Lagr}^{\leq p+1} \arrow[d] \\ \mathrm{Comp}^{\leq p} \arrow[r] & \mathrm{Lagr}^{\leq p}. \end{tikzcd}$$

In order to prove that the map $\mathrm{Comp}^{nd} \to \mathrm{Lagr}$ is an equivalence, it is enough to show that $\mathrm{Comp}^{nd,\leq p} \to \mathrm{Lagr}^{\leq p}$ is an equivalence for every $p \geq 2$. We will thus proceed by induction on p.

First, let us unpack the compatibility between coisotropic and Lagrangian structures.

Lemma 4.25 *Suppose γ is an n-shifted coisotropic structure on f and λ an n-shifted Lagrangian structure on f inducing morphisms*

$$\gamma_A^\sharp \colon \mathbb{L}_A \to \mathbb{T}_A[-n], \qquad \gamma_B^\sharp \colon \mathbb{L}_B \to \mathbb{T}_{B/A}[1-n]$$

and

$$\lambda_A^\sharp \colon \mathbb{T}_A \to \mathbb{L}_A[n], \qquad \lambda_B^\sharp \colon \mathbb{T}_{B/A} \to \mathbb{L}_B[n-1].$$

The compatibility between γ and λ in weight 2 is equivalent to the relations

$$\gamma_A^\sharp \circ \lambda_A^\sharp \circ \gamma_A^\sharp \cong \gamma_A^\sharp, \qquad \gamma_B^\sharp \circ \lambda_B^\sharp \circ \gamma_B^\sharp \cong \gamma_B^\sharp.$$

Proof Suppose

$$\lambda_A = \sum_{i,j} \omega_{ij} \mathrm{d_{dR}} a_i \mathrm{d_{dR}} a_j$$

is the two-form on A underlying λ and γ_A is the bivector on A underlying γ. By definition

$$\mu(\lambda, \gamma)_A = \pm \sum_{i,j} \omega_{ij}[\gamma_A, a_i][\gamma_A, a_j]$$

and the compatibility relation is that this bivector is homotopic to γ_A itself. Since $\mathbb{L}_A$ is dualizable, we can identify bivectors with antisymmetric maps $\mathbb{L}_A \to \mathbb{T}_A$. Let us now compute the induced maps. Suppose $a' \in A$. Then

$$\begin{aligned}[\mu(\lambda_A, \gamma_A), a'] &= \pm \sum_{i,j} \omega_{ij}[[\gamma_A, a_i][\gamma_A, a_j], a'] \\ &= \pm \sum_{i,j} 2\omega_{ij}[\gamma_A, a_i][[\gamma_A, a_j], a'] \\ &= \pm \sum_{i,j} 2\omega_{ij}[\gamma_A, a_i][[\gamma_A, a'], a_j].\end{aligned}$$

But we have

$$(\lambda_A^\sharp \circ \gamma_A^\sharp)(a') = \pm \sum_{i,j} 2\omega_{ij}[[\gamma_A, a'], a_i]\mathrm{d}_{\mathrm{dR}} a_j$$

and thus the compatibility relation boils down to a homotopy

$$\gamma_A^\sharp \circ \lambda_A^\sharp \circ \gamma_A^\sharp \cong \gamma_A^\sharp.$$

For the second statement observe that we have a commutative diagram

$$\begin{tikzcd} \mathrm{DR}(f)^{\geq 2}[n+2] \arrow[r] \arrow[d, "\mu(-,\gamma)"] & \mathbb{L}_{B/A} \otimes \mathbb{L}_B[n-1] \arrow[d, "\widetilde{\gamma}_B^\sharp \otimes \gamma_B^\sharp"] \\ \mathrm{Pol}^t(f, n)^{\geq 2}[n+2] \arrow[r] & \mathbb{T}_B \otimes \mathbb{T}_{B/A}[1-n] \end{tikzcd}$$

where $\widetilde{\gamma}_B^\sharp \colon \mathbb{L}_{B/A} \to \mathbb{T}_B[1-n]$ is the dual of $\gamma_B^\sharp$. Proceeding as before we see that

$$\left(\widetilde{\gamma}_B^\sharp \otimes \gamma_B^\sharp\right)\left(\lambda_B^\sharp\right) \cong \gamma_B^\sharp \circ \lambda_B^\sharp \circ \gamma_B^\sharp$$

where we have identified $\mathbb{L}_{B/A} \otimes \mathbb{L}_B \cong \mathrm{Hom}_B(\mathbb{T}_{B/A}, \mathbb{L}_B)$ and $\mathbb{T}_B \otimes \mathbb{T}_{B/A} \cong \mathrm{Hom}_B(\mathbb{L}_B, \mathbb{T}_{B/A})$. Therefore, we obtain that the compatibility of γ_B and λ_B is equivalent to the relation

$$\gamma_B^\sharp \circ \lambda_B^\sharp \circ \gamma_B^\sharp \cong \gamma_B^\sharp.$$

□

The base of the induction is given by the following statement.

Proposition 4.26 *The projection*

$$\mathrm{Comp}^{nd,\leq 2} \to \mathrm{Lagr}^{\leq 2}$$

is an equivalence.

Proof The points of the space $\mathrm{Comp}^{nd,\leq 2}$ are in particular pairs (γ, λ), where γ is a weight 2 cocycle in $\mathrm{Pol}^t(f, n)[n+2]$ and λ is a weight 2 cocycle in $\mathrm{DR}(f)[n+2]$. Moreover, γ is non-degenerate, in the sense that the induced vertical morphisms

$$\begin{tikzcd} \mathbb{L}_A \arrow[r] \arrow[d, "\gamma_A^\sharp"] & \mathbb{L}_B \arrow[d, "\gamma_B^\sharp"] \\ |\mathbb{T}_A^{int}[-n]|^t \arrow[r] & |\mathbb{T}_{B/A}^{int}[-n+1]|^t \end{tikzcd}$$

are equivalences. Similarly, λ defines a Lagrangian structure, so that we also have vertical morphisms

$$\begin{tikzcd} |\mathbb{T}_A^{int}[-n]|^t \arrow[r] \arrow[d, "\lambda_A^\sharp"] & |\mathbb{T}_{B/A}^{int}[-n+1]|^t \arrow[d, "\lambda_B^\sharp"] \\ \mathbb{L}_A \arrow[r] & \mathbb{L}_B \end{tikzcd}$$

which are again equivalences.

By Lemma 4.25 we have

$$\gamma_A^\sharp \circ \lambda_A^\sharp \circ \gamma_A^\sharp \cong \gamma_A^\sharp, \qquad \gamma_B^\sharp \circ \lambda_B^\sharp \circ \gamma_B^\sharp \cong \gamma_B^\sharp$$

and since both $\gamma_A^\sharp$ and $\gamma_B^\sharp$ are equivalences, we see that $\lambda_A^\sharp$ and $\lambda_B^\sharp$ are uniquely determined to be the inverses of $\gamma_A^\sharp$ and $\gamma_B^\sharp$. □

4.6 Obstructions

For the inductive step, suppose we are given a non-degenerate compatible pair $(\gamma, \lambda) \in \mathrm{Comp}^{nd,\leq p}$. We start by studying the obstruction to extend it to something in $\mathrm{Comp}^{nd,\leq p+1}$. For this purpose let us define the obstruction spaces

$$\begin{aligned} \mathrm{Obs}(p+1, \mathrm{Lagr}) &= \mathrm{DR}(f)^{p+1}[n+3] \\ \mathrm{Obs}(p+1, \mathrm{Cois}) &= \mathrm{Pol}^t(f, n)^{p+1}[n+3] \\ \mathrm{Obs}(p+1, \mathrm{TCois}) &= \mathrm{Pol}^t(f, n)^{p+1}[n+3] \otimes k[\epsilon], \end{aligned}$$

where we apply the Dold–Kan correspondence to the complexes on the right to turn a complex into a simplicial set.

We denote by $\underline{\mathrm{Obs}}(p+1,-)$ the corresponding trivial bundles; for example,

$$\underline{\mathrm{Obs}}(p+1, \mathrm{Lagr}) = \mathrm{Obs}(p+1, \mathrm{Lagr}) \times \mathrm{Lagr}^{\leq p}.$$

This obstruction bundle has a natural non-trivial section. Suppose $\lambda \in \mathrm{DR}(f)^{\leq p}$ defines a Lagrangian structure in $\mathrm{Lagr}^{\leq p}$. Then we can also consider λ as living in the whole $\mathrm{DR}(f)$, and take $\mathrm{d_{dR}}\lambda$. We define the map

$$\mathrm{Lagr}^{\leq p} \to \mathrm{Obs}(p+1, \mathrm{Lagr})$$

by sending λ to $P_{p+1}\mathrm{d_{dR}}\lambda$, where P_{p+1} is the projection to the weight $(p+1)$-component of $\mathrm{DR}(f)$ and $\mathrm{d_{dR}}$ is the mixed structure on $\mathrm{DR}(f)$. More explicitly, if the weight component of λ are $\lambda_2 + \cdots + \lambda_p$, then λ is sent to $\mathrm{d_{dR}}\lambda_p$. In particular, this gives a section $s\colon \mathrm{Lagr}^{\leq p} \to \underline{\mathrm{Obs}}(p+1, \mathrm{Lagr})$.

The following lemma explains why we think of $\mathrm{Obs}(\mathrm{Lagr}, p+1)$ as obstruction spaces.

Lemma 4.27 *There is a natural equivalence between* $\mathrm{Lagr}^{\leq(p+1)}$ *and the vanishing locus of*

$$\begin{tikzcd} \underline{\mathrm{Obs}}(p+1, \mathrm{Lagr}) \arrow[r] & \mathrm{Lagr}^{\leq p} \arrow[l, bend right, "s"'] \end{tikzcd}.$$

Proof Let $\mathrm{DR}(f)^{[2,p]}$ be the sub-complex of $\mathrm{DR}(f)$ obtained by considering only weights in the interval $[2, p]$. In other words, $\mathrm{DR}(f)^{[2,p]}$ is the complex

$$\mathrm{DR}(f)^{[2,p]} = \bigoplus_{i=2}^{p} \mathrm{DR}(f)^i$$

equipped with the differential $d + \mathrm{d_{dR}}$.

By definition, m-simplices of $\mathrm{Lagr}^{\leq p+1}$ are identified with closed elements in the complex

$$\Omega^\bullet(\Delta^m) \otimes \mathrm{DR}(f)^{[2,p+1]}[n+2].$$

On the other hand, the vanishing locus of the section s has m-simplices given by closed elements $\lambda_2 + \cdots + \lambda_p$ in

$$\Omega^\bullet(\Delta^m) \otimes \mathrm{DR}(f)^{[2,p]}[n+2].$$

together with an element λ_{p+1} in $\Omega^\bullet(\Delta^m) \otimes \mathrm{DR}(f)(p+1)[n+2]$ such that $\mathrm{d_{dR}}\lambda_p + \mathrm{d}\lambda_{p+1} = 0$. Therefore, they are identified with elements $\lambda_2 + \cdots + \lambda_{p+1}$ of Lagr^{p+1}. $\square$

We can also obtain a similar interpretation of the obstruction spaces for Cois and TCois. More specifically, define a section

$$\mathrm{Cois}^{\leq p} \to \mathrm{Obs}(\mathrm{Cois}, p+1)$$

of the obstruction bundle for Cois by sending an element $\gamma \in \mathrm{Cois}^{\leq p}$ to $\frac{1}{2} P_{p+1}[\gamma, \gamma]$, where once again P_{p+1} is the projection on weight $(p+1)$, and the bracket is the one on $\mathrm{Pol}^t(f, n)$. The same formula defines a section

$$\mathrm{TCois}^{\leq p} \to \mathrm{Obs}(\mathrm{TCois}, p+1)$$

of the obstruction bundle $\underline{\mathrm{Obs}}(\mathrm{TCois}, p+1)$.

The proof of the following lemma is completely analogous to the proof of Lemma 4.27.

Lemma 4.28 *There are natural equivalences between the spaces* $\mathrm{Cois}^{\leq p+1}$, $\mathrm{TCois}^{\leq p+1}$ *and the vanishing loci of the sections*

$$\mathrm{Cois}^{\leq p} \to \underline{\mathrm{Obs}}(\mathrm{Cois}, p+1)$$

and

$$\mathrm{TCois}^{\leq p} \to \underline{\mathrm{Obs}}(\mathrm{TCois}, p+1)$$

defined above.

Putting all of them together, we obtain a diagram

$$\begin{tikzcd}
\underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr}) \arrow[r] \arrow[d] & \underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}) \arrow[d] \\
\mathrm{TCois}(A, n)^{\leq p} \times \mathrm{Lagr}(A, n)^{\leq p} \arrow[r] \arrow[u, bend right] & \mathrm{Cois}(A, n)^{\leq p} \times \mathrm{Lagr}(A, n)^{\leq p}. \arrow[u, bend right] \arrow[l, bend right, "\Phi_p"']
\end{tikzcd}$$

Moreover, the lemmas above show that taking the vanishing loci vertically we obtain the obvious projection

$$\mathrm{TCois}^{\leq p+1} \times \mathrm{Lagr}^{\leq p+1} \longrightarrow \mathrm{Cois}^{\leq p+1} \times \mathrm{Lagr}^{\leq p+1}.$$

On the other hand, the vanishing locus of the bottom section is by definition $\mathrm{Comp}^{\leq p}$. Our next goal is to show that the top map in the diagram also admits a canonical section.

Suppose we are given two elements γ and δ of $\mathrm{Pol}^t(f, n)$. By the results of [15, Section 3.6], γ can be used to induce a commutative diagram of algebras

$$\begin{tikzcd}
\mathrm{DR}(A) \arrow[r] \arrow[d, "\mu(-,\gamma)_A"] & \mathrm{DR}(B) \arrow[d, "\mu(-,\gamma)_B"] \\
\mathrm{Pol}^t(A, n) \arrow[r] & \mathrm{Pol}^t(B/A, n-1).
\end{tikzcd}$$

Similarly, δ defines derivations $\psi_{\delta,A}$ and $\psi_{\delta,B}$ on $\mathrm{Pol}^t(A, n)$ and $\mathrm{Pol}^t(B/A, n-1)$, compatible with the $\mathbb{P}_{[n+2,n+1]}$-structure. In particular, their restriction to weight zero define a commutative diagram

$$\begin{tikzcd}
\Omega^1_A \arrow[r] \arrow[d] & \Omega^1_B \arrow[d] \\
\mathrm{Pol}^t(A, n) \arrow[r] & \mathrm{Pol}^t(B/A, n-1).
\end{tikzcd}$$

Putting all together, we can now construct two derivations which fit vertically in the diagram

$$\begin{tikzcd}
\mathrm{DR}(A) \arrow[r] \arrow[d, "\nu(-,\gamma,\delta)_A"] & \mathrm{DR}(B) \arrow[d, "\nu(-,\gamma,\delta)_B"] \\
\mathrm{Pol}^t(A, n) \arrow[r] & \mathrm{Pol}^t(B/A, n-1).
\end{tikzcd}$$

More explicitly, $\nu(-, \gamma, \delta)_A$ is an A-linear derivation $\mathrm{DR}(A) \to \mathrm{Pol}^t(A, n)$ relative to $\mu(-, \gamma)_A$ and similarly for $\nu(-, \gamma, \delta)_B$. On weight 1 generators they are given by

$$\nu(a\mathrm{d}_{\mathrm{dR}}x, \gamma, \delta)_A = \mu(a, \gamma)_A \psi_{\delta,A}(x) = a\psi_{\delta,A}(x)$$

and

$$\nu(b\mathrm{d}_{\mathrm{dR}}y, \gamma, \delta)_B = \mu(b, \gamma)_B \psi_{\delta,B}(y) = b\psi_{\delta,B}(y).$$

Passing to homotopy fibers, we obtain a map

$$\nu(-, \gamma, \delta)\colon \mathrm{DR}(f) \to \mathrm{Pol}^t(f, n).$$

Define the morphism

$$\underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}) \to \underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr})$$

to be given by

$$
\begin{aligned}
(\gamma, \lambda, \delta_\gamma, \delta_\lambda) \mapsto (&\gamma + \epsilon\sigma(\gamma) - \epsilon\mu(\lambda, \gamma), \lambda, \delta_\gamma + \epsilon\sigma(\delta_\gamma) - \epsilon\nu(\lambda, \gamma, \delta_\gamma) \\
&- \epsilon\mu(\delta_\lambda, \gamma), \delta_\lambda),
\end{aligned}
$$

where we denoted by δ_γ and δ_λ the elements of the obstruction spaces. Notice that the above formula clearly gives a section of the projection

$$\underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr}) \to \underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}).$$

In particular, we obtain a diagram

$$
\begin{tikzcd}
\underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr}) \arrow[r] \arrow[d, shift right] & \underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}) \arrow[l, bend right] \arrow[d, shift right] \\
\mathrm{TCois}(A, n)^{\leq p} \times \mathrm{Lagr}(A, n)^{\leq p} \arrow[r] \arrow[u, bend right] & \mathrm{Cois}(A, n)^{\leq p} \times \mathrm{Lagr}(A, n)^{\leq p} \arrow[l, bend right, "\Phi_p"'] \arrow[u, bend right]
\end{tikzcd}
$$

We are now going to show that the top section is compatible with other sections. First, we need a preliminary Lemma.

Lemma 4.29 *Let* $\gamma \in \mathrm{Pol}^t(f, n)^{\leq p}$ *and* $\lambda \in \mathrm{DR}(f)^{\leq p}$*, and suppose* γ *is of degree* $n+2$*. Then we have*

$$\frac{1}{2}\nu(\lambda, \gamma, [\gamma, \gamma]) + \mu(\mathrm{d_{dR}}\lambda, \gamma) = [\gamma, \mu(\lambda, \gamma)].$$

Proof For fixed γ, the three terms are induced by pairs of maps $\mathrm{DR}(A) \to \mathrm{Pol}^t(A, n)$ and $\mathrm{DR}(B) \to \mathrm{Pol}^t(B/A, n-1)$, which are all derivations determined by their values in weight 0 and 1. It follows that it is enough to prove the lemma for λ in weight 0 or 1.

- Suppose $\lambda \in \mathrm{DR}(f)$ is of weight 0. By definition $\nu(-, \gamma, [\gamma, \gamma])$ is induced by the pair of derivations $\psi_{[\gamma,\gamma],A}$ and $\psi_{[\gamma,\gamma],B}$, which are respectively A-linear and B-linear. In particular, they are both zero in weight 0, so that also $\nu(\lambda, \gamma, [\gamma, \gamma])$ is zero if λ is of weight 0.
  We are thus left with proving that if λ has weight 0, then
  $$\mu(\mathrm{d_{dR}}\lambda, \gamma) = [\gamma, \mu(\lambda, \gamma)] = [\gamma, \lambda]$$
  which follows from the definition of μ.
- Suppose now $\lambda \in \mathrm{DR}(f)$ is a generator of weight 1 of the form $\lambda = \mathrm{d_{dR}}g$. Then $\mu(\mathrm{d_{dR}}\lambda, \gamma) = 0$, and we need to prove that
  $$\frac{1}{2}\nu(\mathrm{d_{dR}}g, \gamma, [\gamma, \gamma]) = [\gamma, \mu(\mathrm{d_{dR}}g, \gamma)].$$

Using the definition of ν and μ this reduces to

$$\frac{1}{2}[[\gamma,\gamma],g]=[\gamma,[\gamma,g]]$$

which follows from the Jacobi identity.

□

Proposition 4.30 *The section* $\underline{\mathrm{Obs}}(p+1,\mathrm{Cois}\times\mathrm{Lagr})\to\underline{\mathrm{Obs}}(p+1,\mathrm{TCois}\times\mathrm{Lagr})$ *constructed above commutes with the obstruction maps. Moreover, the induced map on vanishing loci of the vertical sections is equivalent to*

$$\Phi_{p+1}\colon \mathrm{Cois}(f,n)^{\leq(p+1)}\times\mathrm{Lagr}(f,n)^{\leq(p+1)}\longrightarrow\mathrm{TCois}(f,n)^{\leq(p+1)}\times\mathrm{Lagr}(f,n)^{\leq(p+1)}.$$

Proof We begin by analyzing the commutativity of

$$\begin{tikzcd}
\mathrm{Cois}^{\leq p}\times\mathrm{Lagr}^{\leq p} \arrow[r] \arrow[d, "\Phi_p"] & \underline{\mathrm{Obs}}(p+1,\mathrm{Cois}\times\mathrm{Lagr}) \arrow[d] \\
\mathrm{TCois}^{\leq p}\times\mathrm{Lagr}^{\leq p} \arrow[r] & \underline{\mathrm{Obs}}(p+1,\mathrm{TCois}\times\mathrm{Lagr}).
\end{tikzcd}$$

Consider an element $(\gamma,\lambda)\in\mathrm{Cois}^{\leq p}\times\mathrm{Lagr}^{\leq p}$. Its image in $\underline{\mathrm{Obs}}(p+1,\mathrm{Cois}\times\mathrm{Lagr})$ is

$$\left(\gamma,\lambda,\frac{1}{2}P_{p+1}[\gamma,\gamma],\mathrm{d}_{\mathrm{dR}}\lambda_p\right).$$

If we send it to $\underline{\mathrm{Obs}}(p+1,\mathrm{TCois}\times\mathrm{Lagr})$ via the top section, we get

$$\begin{aligned}&(\gamma+\epsilon\sigma(\gamma)-\epsilon\mu(\lambda,\gamma),\lambda,\\&\frac{1}{2}P_{p+1}[\gamma,\gamma]+\epsilon\frac{p}{2}P_{p+1}[\gamma,\gamma]-\epsilon\nu\left(\lambda,\gamma,\frac{1}{2}P_{p+1}[\gamma,\gamma]\right)-\epsilon\mu(\mathrm{d}_{\mathrm{dR}}\lambda_p,\gamma),\mathrm{d}_{\mathrm{dR}}\lambda_p).\end{aligned}$$

Alternatively, we can apply Φ_p to the pair (γ,λ), getting

$$\Phi_p(\gamma,\lambda)=(\gamma+\epsilon\sigma(\gamma)-\epsilon\mu(\lambda,\gamma),\lambda).$$

The image of $\Phi_p(\gamma,\lambda)$ in $\underline{\mathrm{Obs}}(p+1,\mathrm{TCois}\times\mathrm{Lagr})$ is

$$\left(\gamma+\epsilon\sigma(\gamma)-\epsilon\mu(\lambda,\gamma),\lambda,\frac{1}{2}P_{p+1}[\gamma,\gamma]+\epsilon P_{p+1}[\gamma,\sigma(\gamma)-\mu(\lambda,\gamma)],\mathrm{d}_{\mathrm{dR}}\lambda_p\right).$$

This means that we are left with proving that

$$\frac{p}{2}P_{p+1}[\gamma,\gamma]-\nu\left(\lambda,\gamma,\frac{1}{2}P_{p+1}[\gamma,\gamma]\right)-\mu(\mathrm{d}_{\mathrm{dR}}\lambda_p,\gamma)=P_{p+1}[\gamma,\sigma(\gamma)-\mu(\lambda,\gamma)]$$

in $\mathrm{Pol}^t(f, n)^{p+1}$. A straightforward computation shows that $[\gamma, \sigma(\gamma)] = \frac{1}{2}\sigma([\gamma, \gamma])$, so that $\frac{p}{2}[\gamma, \gamma] = P_{p+1}[\gamma, \sigma(\gamma)]$ in $\mathrm{Pol}^t(f, n)^{p+1}$. The rest of the terms are dealt with using Lemma 4.29.

This proves the first part of the proposition. For the second statement, let (γ', λ') be an element in $\mathrm{Cois}^{\leq p+1} \times \mathrm{Lagr}^{\leq p+1}$, and write

$$\gamma' = \gamma + \gamma_{p+1} \quad \text{and} \quad \lambda' = \lambda + \lambda_{p+1},$$

where γ and λ have no component of weight $p+1$, while γ_{p+1} and λ_{p+1} are concentrated in weight $p+1$. Then the image of (γ', λ') under Φ_{p+1} in $\mathrm{TCois}^{\leq(p+1)} \times \mathrm{Lagr}^{\leq(p+1)}$ is

$$(\gamma + \gamma_{p+1} + \epsilon\sigma(\gamma + \gamma_{p+1}) - \epsilon\mu(\lambda + \lambda_{p+1}, \gamma + \gamma_{p+1}), \lambda + \lambda_{p+1}).$$

By weight reasons

$$\mu(\lambda + \lambda_{p+1}, \gamma + \gamma_{p+1}) = \mu(\lambda, \gamma) + \mu(\lambda_{p+1}, \gamma) + \nu(\lambda, \gamma, \gamma_{p+1})$$

and hence the induced map on vanishing loci coincides with Φ_{p+1}, which concludes the proof. $\square$

Thanks to the proposition, we end up with a diagram

$$\begin{tikzcd}
\underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr}) \arrow[r] \arrow[d, shift right] & \underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}) \arrow[l, bend right] \arrow[d, shift right] \\
\mathrm{TCois}^{\leq p} \times \mathrm{Lagr}^{\leq p} \arrow[r] \arrow[u, bend right] & \mathrm{Cois}^{\leq p} \times \mathrm{Lagr}^{\leq p} \arrow[l, bend right, "\Phi_p"'] \arrow[u, bend right] \\
\mathrm{TCois}^{\leq(p+1)} \times \mathrm{Lagr}^{\leq(p+1)} \arrow[r] \arrow[u] & \mathrm{Cois}^{\leq(p+1)} \times \mathrm{Lagr}^{\leq(p+1)} \arrow[l, bend right, "\Phi_{p+1}"'] \arrow[u]
\end{tikzcd}$$

By definition, the vanishing loci of the horizontal sections Φ_p and Φ_{p+1} are $\mathrm{Comp}^{\leq p}$ and $\mathrm{Comp}^{\leq p+1}$ respectively. Let us denote by $\underline{\mathrm{Obs}}(\mathrm{Comp}, p+1)$ the horizontal vanishing locus of the top section. Therefore, we obtain a diagram

$$\begin{tikzcd}
\underline{\mathrm{Obs}}(p+1, \mathrm{TCois} \times \mathrm{Lagr}) \arrow[r] \arrow[d, shift right] & \underline{\mathrm{Obs}}(p+1, \mathrm{Cois} \times \mathrm{Lagr}) \arrow[l, bend right] \arrow[d, shift right] & \underline{\mathrm{Obs}}(p+1, \mathrm{Comp}) \arrow[l] \arrow[d, shift right] \\
\mathrm{TCois}^{\leq p} \times \mathrm{Lagr}^{\leq p} \arrow[u, bend right] & \mathrm{Cois}^{\leq p} \times \mathrm{Lagr}^{\leq p} \arrow[l, bend right, "\Phi_p"'] \arrow[u, bend right] & \mathrm{Comp}^{\leq p} \arrow[l] \arrow[u, bend right] \\
\mathrm{TCois}^{\leq(p+1)} \times \mathrm{Lagr}^{\leq(p+1)} \arrow[u] & \mathrm{Cois}^{\leq(p+1)} \times \mathrm{Lagr}^{\leq(p+1)} \arrow[l, bend right, "\Phi_{p+1}"'] \arrow[u] & \mathrm{Comp}^{\leq(p+1)} \arrow[l] \arrow[u]
\end{tikzcd}$$

By the commutation of limits, we see that $\mathrm{Comp}^{\leq(p+1)} \to \mathrm{Comp}^{\leq p}$ realizes $\mathrm{Comp}^{\leq(p+1)}$ as a vanishing locus of

$$\begin{tikzcd} \mathrm{Comp}^{\leq p} & \underline{\mathrm{Obs}}(\mathrm{Comp},\, p+1). \arrow[l] \arrow[from=1-1, to=1-2, bend left] \end{tikzcd}$$

By definition, the obstruction bundle $\underline{\mathrm{Obs}}(\mathrm{Comp},\, p+1)$ has a quite explicit description: its fiber over a compatible pair $(\gamma, \lambda) \in \mathrm{Comp}^{\leq p}$ is given by the homotopy fiber of

$$\begin{array}{ccc} \mathrm{Pol}^t(f, n)^{p+1} \oplus \mathrm{DR}(f)^{p+1} & \longrightarrow & \mathrm{Pol}^t(f, n)^{p+1} \\ (\gamma_{p+1}, \lambda_{p+1}) & \longmapsto & \sigma(\gamma_{p+1}) - \nu(\lambda_2, \gamma_2, \gamma_{p+1}) - \mu(\lambda_{p+1}, \gamma_2). \end{array}$$

Note that by weight reasons $\nu(\lambda, \gamma, \gamma_{p+1}) = \nu(\lambda_2, \gamma_2, \gamma_{p+1})$ and $\mu(\lambda_{p+1}, \gamma) = \mu(\lambda_{p+1}, \gamma_2)$ in $\mathrm{Pol}^t(f, n)^{p+1}$. Comparing the obstruction bundles for Comp and Lagr, we get a diagram

$$\begin{tikzcd} \underline{\mathrm{Obs}}(p+1, \mathrm{Comp}) \arrow[r] \arrow[d, shift right] & \underline{\mathrm{Obs}}(p+1, \mathrm{Lagr}) \arrow[d, shift right] \\ \mathrm{Comp}^{\leq p} \arrow[r] \arrow[u, bend right, shift right] & \mathrm{Lagr}^{\leq p} \arrow[u, bend right, shift right] \\ \mathrm{Comp}^{\leq(p+1)} \arrow[r] \arrow[u] & \mathrm{Lagr}^{\leq(p+1)} \arrow[u] \end{tikzcd}$$

where the horizontal arrows are given by the natural projections.

We denote by $\underline{\mathrm{Obs}}(p+1, \mathrm{Comp}^{nd})$ the restriction of the obstruction bundle $\underline{\mathrm{Obs}}(p+1, \mathrm{Comp})$ to the subspace $\mathrm{Comp}^{nd} \subset \mathrm{Comp}$.

Proposition 4.31 *The projection*

$$\underline{\mathrm{Obs}}\left(p+1, \mathrm{Comp}^{nd}\right) \to \underline{\mathrm{Obs}}(p+1, \mathrm{Lagr})$$

is an equivalence.

Proof Let (γ, λ) be a non-degenerate compatible pair in weight $\leq p$. The proposition is equivalent to showing that the natural projection

$$\mathrm{Pol}^t(f, n)^{p+1} \oplus \mathrm{DR}(f)^{p+1} \to \mathrm{DR}(f)^{p+1}$$

is an equivalence. It is then enough to show that the map

$$\begin{array}{ccc} \mathrm{Pol}^t(f, n)^{p+1} & \to & \mathrm{Pol}^t(f, n)^{p+1} \\ x & \mapsto & \sigma(x) - \nu(\lambda_2, \gamma_2, x) \end{array}$$

is an equivalence.

We claim that $\nu(\lambda_2, \gamma_2, x)$ is equivalent to $(p+1)x$ if x has weight $p+1$. To show this, it is enough to show the claim for its components $\nu(\lambda_2, \gamma_2, x)_A$ and $\nu(\lambda_2, \gamma_2, x)_B$.

By construction, if λ has weight 1, $\nu(\lambda, \gamma_2, x)_A$ is a derivation in $x \in \mathrm{Pol}^t(A, n)$. But $\nu(\lambda, \gamma_2, x)_A$ is a derivation in λ relative to $\mu(-, \gamma_2)_A$. Since $\mathrm{DR}(A)$ is generated in weight 1, we conclude that $\nu(\lambda, \gamma_2, x)_A$ is a derivation in x for λ of any weight. Therefore, it is enough to prove that $\nu(\lambda_2, \gamma_2, x)_A$ is homotopic to px for x of weight $p = 0$ and $p = 1$. The case $p = 0$ is clear since then $\nu(\lambda_2, \gamma_2, x)_A = 0$. Now suppose x has weight 1. Let

$$\lambda_A = \sum_{i,j} a_{ij} \mathrm{d}_{\mathrm{dR}} f_i \mathrm{d}_{\mathrm{dR}} f_j$$

be the two-form on A underlying λ_2 and γ_A the bivector on A underlying γ_2. Then

$$\nu(\lambda_2, \gamma_2, x)_A = \pm \sum_{i,j} 2a_{ij} [\gamma_A, f_i][x, f_j].$$

But $\sum_{i,j} 2a_{ij}[\gamma_2, f_i]\mathrm{d}_{\mathrm{dR}} f_j \in T_A \otimes \Omega^1_A \cong \mathrm{Hom}(T_A, T_A)$ coincides with $\gamma_A^\sharp \circ \lambda_A^\sharp$. Since γ_2 and λ_2 are compatible, $\gamma_A^\sharp \circ \lambda_A^\sharp \cong \mathrm{id}$ and hence $\nu(\lambda_2, \gamma_2, x)_A \cong x$. The claim for $\nu(\lambda_2, \gamma_2, x)_B$ is proved similarly.

We obtain that

$$\begin{array}{ccc} \mathrm{Pol}^t(f, n)^{p+1} & \to & \mathrm{Pol}^t(f, n)^{p+1} \\ x & \mapsto & \sigma(x) - \nu(\lambda_2, \gamma_2, x) \end{array}$$

is homotopic to $(p - (p+1))\mathrm{id} = -\mathrm{id}$, and it is thus an equivalence. □

Finally, we obtain the required statement.

Proposition 4.32 *The projection*

$$\mathrm{Comp}^{nd}(f, n) \to \mathrm{Lagr}(f, n)$$

is an equivalence.

Proof By Proposition 4.26 the map $\mathrm{Comp}^{nd}(f, n) \to \mathrm{Lagr}(f, n)^{\leq 2}$ is an equivalence. Suppose that we have already proved that

$$\mathrm{Comp}^{nd}(f, n)^{\leq p} \to \mathrm{Lagr}(f, n)^{\leq p}$$

is an equivalence for some p. We have a diagram

$$\begin{tikzcd}
\underline{\mathrm{Obs}}(p+1, \mathrm{Comp}^{nd}) \arrow[r, "\sim"] \arrow[d] & \underline{\mathrm{Obs}}(p+1, \mathrm{Lagr}) \arrow[d] \\
\mathrm{Comp}^{nd, \leq p}(f, n) \arrow[r, "\sim"] \arrow[u, bend right] & \mathrm{Lagr}^{\leq p}(f, n) \arrow[u, bend right] \\
\mathrm{Comp}^{nd, \leq (p+1)}(f, n) \arrow[r] \arrow[u] & \mathrm{Lagr}^{\leq (p+1)}(f, n) \arrow[u]
\end{tikzcd}$$

where the bottom row is obtained as a vanishing locus of the vertical maps and the top map is an equivalence by Proposition 4.31. Therefore, $\mathrm{Comp}^{nd, \leq (p+1)}(f, n) \to \mathrm{Lagr}^{\leq (p+1)}(f, n)$ is also an equivalence.

We have

$$\mathrm{Comp}^{nd}(f, n) = \lim_p \mathrm{Comp}^{nd, \leq p}(f, n), \qquad \mathrm{Lagr}(f, n) = \lim_p \mathrm{Lagr}^{\leq p}(f, n)$$

and therefore $\mathrm{Comp}^{nd}(f, n) \to \mathrm{Lagr}(f, n)$ is also an equivalence. □

These results, together with Proposition 4.21, conclude the proof of Theorem 4.23, which in turn finally implies Theorem 4.22.

5 Quantization

In this section we describe quantizations of n-shifted Poisson stacks and n-shifted coisotropic structures and show that any n-shifted coisotropic structure admits a formal quantization for $n > 1$.

5.1 Beilinson–Drinfeld operads

Recall that one has a family of dg operads $\mathbb{P}_n$ for $n \in \mathbb{Z}$ which, as we recall, are Hopf. Therefore, the ∞-category $\mathbf{Alg}_{\mathbb{P}_n}$ is endowed with a symmetric monoidal structure. Similarly, one has a family of dg Hopf operads $\mathbb{E}_n$ for $n \geq 0$ which are defined to be

$$\mathbb{E}_n = \mathrm{C}_\bullet(\mathrm{E}_n; k),$$

where E_n is the topological operad of little n-disks.

Remark 5.1 We consider unital versions of operads E_n. For instance,

$$\mathbb{E}_0(0) = k, \qquad \mathbb{E}_0(1) = k, \qquad \mathbb{E}_0(n) = 0, \ n > 1.$$

Therefore, the operad $\mathbb{E}_0$ controls complexes with a distinguished element.

The Beilinson–Drinfeld operads $\mathbb{BD}_n$ are operads providing an interpolation between the operads $\mathbb{P}_n$ and $\mathbb{E}_n$. That is, they are graded Hopf dg operads over $k[\![\hbar]\!]$ with $\hbar$ of weight 1 together with equivalences

$$\mathbb{BD}_n/\hbar \cong \mathbb{P}_n, \qquad \mathbb{BD}_n[\hbar^{-1}] \cong \mathbb{E}_n(\!(\hbar)\!).$$

The known definition of the operads $\mathbb{BD}_n$ is non-uniform in n and they are defined separately for $\mathbb{BD}_0$, $\mathbb{BD}_1$ and $\mathbb{BD}_n$ for $n \geq 2$. The following is [4, Definition 2.4.0.1].

Definition 5.2 A $\mathbb{BD}_0$-*algebra* is a dg $k[\![\hbar]\!]$-module together with a degree 1 Lie bracket $\{-,-\}$ and a unital commutative multiplication satisfying the relations

- $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|}a\mathrm{d}(b) + \hbar\{a, b\}$,
- $\{x, yz\} = \{x, y\}z + (-1)^{|y||z|}\{x, z\}y$.

These relations define a dg operad $\mathbb{BD}_0$. We introduce a weight grading by assigning weight -1 to $\{-,-\}$ and weight 0 to the multiplication. Note that we have an isomorphism $\mathbb{BD}_0 \cong \mathbb{P}_0[\![\hbar]\!]$ of bigraded operads; we transfer the Hopf structure from $\mathbb{P}_0$ to one on $\mathbb{BD}_0$ using this isomorphism.

The following definition is given in [4, Section 2.4.2].

Definition 5.3 A $\mathbb{BD}_1$-algebra is a dg Lie algebra $(A, \{-,-\})$ over $k[\![\hbar]\!]$ equipped with an associative multiplication satisfying the relations

- $\hbar\{x, y\} = xy - (-1)^{|x||y|}yx$,
- $\{x, yz\} = \{x, y\}z + (-1)^{|y||z|}\{x, z\}y$.

We introduce an additional weight grading such that $\{-,-\}$ has weight -1 and that the multiplication has weight 0, thus making $\mathbb{BD}_1$ into a graded dg operad. The Hopf structure on $\mathbb{BD}_1$ is defined so that

$$\Delta(m) = m \otimes m, \qquad \Delta(\{-,-\}) = \{-,-\} \otimes m + m \otimes \{-,-\},$$

where $m \in \mathbb{BD}_1(2)$ is the product and $\{-,-\} \in \mathbb{BD}_1(2)$ is the Lie bracket.

Finally, E_n has a Postnikov tower which gives a Hopf filtration of $\mathbb{E}_n$ and we define $\mathbb{BD}_n$ for $n \geq 2$ to be the graded operad obtained as the Rees construction with respect to this filtration.

Remark 5.4 Note that with respect to this filtration $\mathrm{gr}\,\mathbb{E}_1 \cong \mathbb{E}_1$ while $\mathbb{P}_1 \not\cong \mathbb{E}_1$, i.e. the filtration on $\mathbb{E}_1$ induced by the operad $\mathbb{BD}_1$ is different from the Postnikov filtration. The same remark applies to the case $n = 0$.

Let us now state two important results about the operads $\mathbb{BD}_n$.

Theorem 5.5 (Formality of the operad of little n-disks) *Suppose $n \geq 2$. Then one has an equivalence of graded Hopf dg operads*

$$\mathbb{BD}_n \cong \mathbb{P}_n[\![\hbar]\!]$$

compatible with the equivalence $\mathbb{BD}_n/\hbar \cong \mathbb{P}_n$.

The statement for $n = 2$ was proved by Tamarkin [26] using the existence of rational Drinfeld associators, by Kontsevich [10] and Lambrechts–Volić [12] for all $n \geq 2$ and $k \supset \mathbb{R}$ and finally by Fresse–Willwacher [7] for all $n \geq 2$ and all fields k of characteristic zero.

Remark 5.6 The space of formality isomorphisms $\mathbb{E}_n \cong \mathbb{P}_n$ as dg Hopf operads is nontrivial and is described in [6, Corollary 5]. For instance, the space of formality isomorphisms $\mathbb{E}_2 \cong \mathbb{P}_2$ has connected components parametrized by the set of Drinfeld associators.

The following result has been announced by Rozenblyum:

Theorem 5.7 *Suppose $n \geq 0$. Then one has an equivalence of $k[\![\hbar]\!]$-linear symmetric monoidal ∞-categories*

$$\mathbf{Alg}_{\mathbb{BD}_{n+1}}(\mathcal{M}) \cong \mathbf{Alg}(\mathbf{Alg}_{\mathbb{BD}_n}(\mathcal{M})).$$

Inverting $\hbar$ in this equivalence, we obtain an equivalence of symmetric monoidal ∞-categories

$$\mathbf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{M}) \cong \mathbf{Alg}(\mathbf{Alg}_{\mathbb{E}_n}(\mathcal{M}))$$

constructed by Dunn and Lurie (see [13, Theorem 5.1.2.2]). In the other extreme, setting $\hbar = 0$ we obtain an equivalence of symmetric monoidal ∞-categories

$$\mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M}) \cong \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}))$$

constructed in [23].

Remark 5.8 It is expected that for $n \geq 2$ the equivalence of Theorem 5.7 is compatible with Theorem 5.5 in that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Alg}_{\mathbb{BD}_{n+1}}(\mathcal{M}) \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathbf{Alg}_{\mathbb{P}_{n+1}}(\mathcal{M}[\![\hbar]\!]) \arrow[d, "\sim"] \\
\mathbf{Alg}(\mathbf{Alg}_{\mathbb{BD}_n}(\mathcal{M})) \arrow[r, "\sim"] & \mathbf{Alg}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}[\![\hbar]\!]))
\end{tikzcd}$$

where the vertical functor on the right is given by Poisson additivity, i.e. the statement of Theorem 5.7 for $\hbar = 0$.

5.2 Deformation quantization for Poisson structures

Once we have the definition of the $\mathbb{BD}_n$ operads we can define the notion of deformation quantization.

Definition 5.9 Let A be a $\mathbb{P}_{n+1}$-algebra in $\mathcal{M}$. A *deformation quantization* of A is a $\mathbb{BD}_{n+1}$-algebra $A_\hbar$ together with an equivalence of $\mathbb{P}_{n+1}$-algebras $A_\hbar/\hbar \cong A$.

Remark 5.10 Suppose $n = 0$ and fix a (non-dg) Poisson algebra A. A deformation quantization of A in the classical sense is given by an associative algebra $A_\hbar$ flat over $k[\![\hbar]\!]$ whose multiplication is commutative at $\hbar = 0$ together with an isomorphism $A_\hbar/\hbar \cong A$ of Poisson algebras, where $A_\hbar/\hbar$ is equipped with the induced Lie bracket $(ab - ba)/\hbar$. It is easy to see that one can therefore lift $A_\hbar$ to a $\mathbb{BD}_1$-algebra and the flatness condition is necessary to ensure that the derived tensor product $A_\hbar \otimes_{k[\![\hbar]\!]} k$ coincides with the ordinary tensor product.

One can similarly give a definition for general stacks, see [5, Section 3.5.1].

Definition 5.11 Let X be a derived Artin stack equipped with an n-shifted Poisson structure. A *deformation quantization* of X is a lift of the $\mathbb{P}_{n+1}$-algebra $\mathcal{B}_X(\infty)$ in the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules to a $\mathbb{BD}_{n+1}$-algebra $\mathcal{B}_{X,\hbar}(\infty)$.

The following is [5, Theorem 3.5.4] and immediately follows from Theorem 5.5.

Theorem 5.12 *Let X be a derived Artin stack equipped with an n-shifted Poisson structure for $n > 0$. Then deformation quantizations exist.*

Given a derived Artin stack X we have a symmetric monoidal ∞-category $\mathrm{Perf}(X)$ of perfect complexes which by [5, Corollary 2.4.12] can be identified with

$$\mathrm{Perf}(X) \cong \Gamma(X_{DR}, \mathbf{Mod}^{perf}_{\mathcal{B}_X(\infty)}),$$

where $\mathbf{Mod}^{perf}_{\mathcal{B}_X(\infty)}$ is the prestack on X_{DR} of symmetric monoidal ∞-categories of perfect $\mathcal{B}_X(\infty)$-modules.

Now suppose X has an n-shifted Poisson structure which admits a deformation quantization. Then $\mathcal{B}_X(\infty)$ has a deformation over $k[\![\hbar]\!]$ as an $\mathbb{E}_{n+1}$-algebra. Therefore, using the equivalence $\mathbf{Alg}_{\mathrm{E}_{n+1}} \cong \mathbf{Alg}_{\mathrm{E}_n}(\mathbf{Alg})$ we obtain that $\mathbf{Mod}^{perf}_{\mathcal{B}_X(\infty)}$ has a deformation $\mathbf{Mod}_{\mathcal{B}_{X,\hbar}(\infty)}$ as a prestack of E_n-monoidal ∞-categories. Hence we conclude that $\mathrm{Perf}(X)$ itself inherits a deformation over $k[\![\hbar]\!]$ as an E_n-monoidal ∞-category.

5.3 Deformation quantization for coisotropic structures

Now we define deformation quantization in the relative setting. Recall that by [23, Theorem 3.7] one can identify $\mathbb{P}_{[n+1,n]}$-algebras with a pair of an associative algebra A and a left A-module B in the ∞-category of $\mathbb{P}_n$-algebras.

Definition 5.13 Let (A, B) be a $\mathbb{P}_{[n+1,n]}$-algebra. A *deformation quantization* of (A, B) is a pair $(A_\hbar, B_\hbar) \in \mathbf{LMod}(\mathbf{Alg}_{\mathbb{BD}_n})$ together with an equivalence $(A_\hbar, B_\hbar)/\hbar \cong (A, B)$ of objects of $\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n})$.

Note that a deformation quantization of a $\mathbb{P}_{[n+1,n]}$-algebra (A, B) in particular gives a deformation quantization of the $\mathbb{P}_{n+1}$-algebra A by Theorem 5.7.

Definition 5.14 Let $f: L \to X$ be a morphism of derived Artin stacks equipped with an n-shifted coisotropic structure. A *deformation quantization* of f is given by a deformation quantization $\mathcal{B}_{X,\hbar}(\infty)$ of X and a deformation quantization $(f^*\mathcal{B}_{X,\hbar}(\infty), \mathcal{B}_{L,\hbar}(\infty))$ of the $\mathbb{P}_{[n+1,n]}$-algebra $(f^*\mathcal{B}_X(\infty), \mathcal{B}_L(\infty))$ in the ∞-category of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

As in the case of shifted Poisson structures, we have the following obvious result:

Theorem 5.15 *Let $f: L \to X$ be a morphism of derived Artin stacks equipped with an n-shifted coisotropic structure for $n > 1$. Then deformation quantizations of f exist.*

Proof Indeed, Theorem 5.5 gives an equivalence of symmetric monoidal ∞-categories

$$\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}[\![\hbar]\!])) \cong \mathbf{LMod}(\mathbf{Alg}_{\mathbb{BD}_n}(\mathcal{M}))$$

and we have a functor $\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}) \to \mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M}[\![\hbar]\!])$ sending a $\mathbb{P}_n$-algebra A to $A \otimes k[\![\hbar]\!]$.

Combining these two functors, we obtain a functor

$$\mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_n}(\mathcal{M})) \longrightarrow \mathbf{LMod}(\mathbf{Alg}_{\mathbb{BD}_n}(\mathcal{M}))$$

which gives a deformation quantization of any $\mathbb{P}_{[n+1,n]}$-algebra. The result follows by applying the functor to the $\mathbb{P}_{[n+1,n]}$-algebra $(f^*\mathcal{B}_X(\infty), \mathcal{B}_L(\infty))$ and noticing that the $\mathbb{BD}_{n+1}$-algebra obtained as the quantization of $f^*\mathcal{B}_X(\infty)$ is canonically equivalent to the pullback under f of the quantization of $\mathcal{B}_X(\infty)$. □

Given a morphism of derived Artin stacks $f: L \to X$, the symmetric monoidal functor $f^*: \mathrm{Perf}(X) \to \mathrm{Perf}(L)$ can be realized as the composite

$$\begin{aligned}\mathrm{Perf}(X) &\cong \Gamma(X_{DR}, \mathbf{Mod}^{perf}_{\mathcal{B}_X(\infty)}) \\ &\to \Gamma(L_{DR}, \mathbf{Mod}^{perf}_{f^*\mathcal{B}_X(\infty)}) \\ &\to \Gamma(L_{DR}, \mathbf{Mod}^{perf}_{\mathcal{B}_L(\infty)}) \\ &\cong \mathrm{Perf}(L).\end{aligned}$$

Note that $f^*: \mathrm{Perf}(X) \to \mathrm{Perf}(L)$ promotes the pair $(\mathrm{Perf}(X), \mathrm{Perf}(L))$ to an object of $\mathbf{LMod}(\mathbf{Alg}_{\mathrm{E}_{n-1}}(\mathrm{St}))$, where St is the ∞-category of small stable dg categories. Therefore, as before we see that given an n-shifted coisotropic structure we get a deformation over $k[\![\hbar]\!]$ of the pair $(\mathrm{Perf}(X), \mathrm{Perf}(L))$ as an object of $\mathbf{LMod}(\mathbf{Alg}_{\mathrm{E}_{n-1}}(\mathrm{St}))$. A bit more explicit description of this ∞-category is given by the following statement.

Conjecture 5.16 *Let SC_n be the n-dimensional topological Swiss–cheese operad. Then one has equivalences of symmetric monoidal ∞-categories*

$$\mathbf{Alg}_{\mathrm{SC}_{n+m}} \cong \mathbf{Alg}_{\mathrm{E}_n}(\mathbf{Alg}_{\mathrm{SC}_m}) \cong \mathbf{Alg}_{\mathrm{SC}_m}(\mathbf{Alg}_{\mathrm{E}_n}).$$

This conjecture is a slight generalization of the Dunn–Lurie additivity statement to stratified factorization algebras. Given this statement we see that a quantization of an n-shifted coisotropic structure gives rise to a deformation of the pair $(\mathrm{Perf}(X), \mathrm{Perf}(L))$ as an algebra over SC_n.

Example 5.17 Let $G \subset D$ be an inclusion of affine algebraic groups such that $\mathfrak{d} = \mathrm{Lie}(D)$ carries an element $c \in \mathrm{Sym}^2(\mathfrak{d})^D$ and the inclusion $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{d}$ is coisotropic with respect to c. Then it is shown in [24, Proposition 2.9] that $\mathrm{B}D$ carries a 2-shifted Poisson structure and $\mathrm{B}G \to \mathrm{B}D$ a coisotropic structure. In particular, we may apply Theorem 5.15 to this coisotropic.

Fix a Drinfeld associator which provides an equivalence of symmetric monoidal ∞-categories

$$\mathbf{Alg}_{\mathbb{E}_2} \cong \mathbf{Alg}_{\mathbb{P}_2}.$$

Therefore, we obtain an equivalence of ∞-categories

$$\mathbf{LMod}(\mathbf{Alg}_{\mathbb{E}_2}) \cong \mathbf{LMod}(\mathbf{Alg}_{\mathbb{P}_2}).$$

Thus, we obtain a quantization $\widetilde{\mathrm{Rep}}(D)$ of $\mathrm{B}D$ which is a braided monoidal category and a quantization $\widetilde{\mathrm{Rep}}(G)$ of $\mathrm{B}G$ which is a monoidal category. Moreover, $\widetilde{\mathrm{Rep}}(G)$ becomes a monoidal module category over $\widetilde{\mathrm{Rep}}(D)$.

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POISSON-LIE STRUCTURES AS SHIFTED POISSON STRUCTURES

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Abstract. Classical limits of quantum groups give rise to multiplicative Poisson structures such as Poisson-Lie and quasi-Poisson structures. We relate them to the notion of a shifted Poisson structure which gives a conceptual framework for understanding classical (dynamical) r-matrices, quasi-Poisson groupoids and so on. We also propose a notion of a symplectic realization of shifted Poisson structures and show that Manin pairs and Manin triples give examples of such.

Contents

Introduction

Shifted Poisson geometry. Shifted symplectic and Poisson structures introduced in [Pan+13] and [Cal+17] respectively give a way to define higher symplectic and Poisson structures on algebraic stacks. That is, if X is an algebraic stack or a derived scheme, its cotangent bundle T^*_X can be naturally enhanced to a cotangent complex $\mathbb{L}_X$. In this way we can talk about differential forms and polyvectors of a nontrivial cohomological degree.

Another way to view the "shift" in a shifted Poisson structure is through the prism of higher deformation quantization. Given an ordinary Poisson structure on a smooth variety X one can consider its deformation quantization which gives rise to a deformation of the algebra of global functions $\mathcal{O}(X)$ as an associative algebra or the category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ as a plain category (i.e. the deformation no longer has a monoidal structure). Similarly, given a 1-shifted Poisson structure on a derived algebraic stack X, its deformation quantization (see [Cal+17, Section 3.5]) gives rise to a deformation of $\mathrm{QCoh}(X)$ as a monoidal category. More generally, a deformation quantization of an n-shifted Poisson stack X gives rise to a deformation of $\mathrm{QCoh}(X)$ as an $\mathbb{E}_n$-monoidal category, where $\mathbb{E}_n$ is the operad of little n-disks.

One can also talk about Lagrangian (coisotropic) morphisms $f\colon L \to X$ if X has an n-shifted symplectic (Poisson) structure. In contrast to the classical setting, f is not required to be a closed immersion and being a Lagrangian is no longer a condition, but an extra structure. Similar to the classical setting, deformation quantization of shifted coisotropic structures (see [MS17, Section 5]) gives rise to a deformation of modules. For instance, given a morphism $f\colon L \to X$, the category $\mathrm{QCoh}(L)$ becomes a module category over $\mathrm{QCoh}(X)$. A deformation quantization of a 1-shifted coisotropic structure on f gives rise to a deformation of $(\mathrm{QCoh}(X), \mathrm{QCoh}(L))$ as a pair (monoidal category, module category).

Many examples of shifted symplectic and Lagrangian structures are known. One of the goals of the present paper is to provide interesting and nontrivial examples of shifted Poisson and coisotropic structures.

Poisson-Lie structures. Recall that a Poisson-Lie structure on a group G is given by a multiplicative Poisson bivector $\pi \in \wedge^2 \mathrm{T}_G$. A more general notion is that of a quasi-Poisson group where one is given in addition a trivector $\phi \in \wedge^3(\mathfrak{g})$ which measures the failure of the Jacobi identity for π. In addition, given an element $\lambda \in \wedge^2(\mathfrak{g})$ we can twist quasi-Poisson structures, so they form a groupoid.

It is well-known that multiplicative objects on a group G give rise to an object on the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$. For instance, a multiplicative line bundle on G (i.e. a central extension of G) gives rise to a gerbe on $\mathrm{B}G$. Thus, it is natural to ask whether a multiplicative Poisson structure on G gives rise to some structure on $\mathrm{B}G$.

Let $\mathrm{Pois}(X, n)$ be the space (i.e. an ∞-groupoid) of n-shifted Poisson structures on a derived algebraic stack X. The following result is given by Proposition 2.6 and Theorem 2.9.

Theorem. *Let G be an algebraic group. We have the following classification results:*

- *The space $\mathrm{Pois}(\mathrm{B}G, n)$ for $n > 2$ is trivial. That is, every n-shifted Poisson structure on $\mathrm{B}G$ for $n > 2$ is canonically zero.*
- *The space $\mathrm{Pois}(\mathrm{B}G, 2)$ is equivalent to the set $\mathrm{Sym}^2(\mathfrak{g})^G$.*
- *The space $\mathrm{Pois}(\mathrm{B}G, 1)$ is equivalent to the groupoid of quasi-Poisson structures on G.*

These computations rely on the following trick introduced in [Cal+17, Section 3.6.2]. Let $\widehat{G}$ be the formal completion of G at the unit and $G_{\mathrm{dR}} = G/\widehat{G}$. Then we can identify $\mathrm{B}G \cong (\mathrm{B}\widehat{G})/G_{\mathrm{dR}}$. Since the cotangent complex to G_{dR} is trivial (i.e. $\mathrm{B}\widehat{G} \to \mathrm{B}G$ is formally étale), we can compute polyvectors on $\mathrm{B}G$ as G_{dR}-invariant polyvectors on $\mathrm{B}\widehat{G}$. The latter stack is an example of a formal affine stack and its polyvectors are straightforward to compute.

This classification reflects the well-known expectation that to consider a deformation quantization of $\operatorname{Rep} G \cong \mathrm{QCoh}(\mathrm{B}G)$ as a monoidal category one has to endow G with a quasi-Poisson structure and to deformation quantize it as a braided monoidal category this structure has to be quasi-triangular, i.e. it should come from a Casimir element $c \in \mathrm{Sym}^2(\mathfrak{g})^G$.

We also have a relative analog of the previous statement. Given a morphism of derived algebraic stacks $f\colon L \to X$ we denote by $\mathrm{Cois}(f, n)$ the space of pairs of an n-shifted Poisson structure on X and an n-shifted coisotropic structure on $L \to X$. The following is Proposition 2.10.

Theorem. *Let $H \subset G$ be a closed subgroup. We have the following classification results:*

- *The space* $\mathrm{Cois}(\mathrm{B}H \to \mathrm{B}G, 2)$ *is equivalent to the set* $\ker\left(\mathrm{Sym}^2(\mathfrak{g})^G \to \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{h})^H\right)$.
- *The space* $\mathrm{Cois}(\mathrm{B}H \to \mathrm{B}G, 1)$ *is equivalent to the groupoid of quasi-Poisson structures on G for which H is coisotropic.*

Given a morphism $f\colon X_1 \to X_2$ of algebraic stacks we denote by $\mathrm{Pois}(X_1 \to X_2, n)$ the space of triples of n-shifted Poisson structures on X_i and a compatibility between them making f into a Poisson morphism. Then the previous statement for $H = \mathrm{pt}$ implies (see Corollary 2.11) that $\mathrm{Pois}(\mathrm{pt} \to \mathrm{B}G, 1)$ is equivalent to the set of Poisson-Lie structures on G. Similar statements arise if one replaces $\mathrm{B}G$ by its formal completion $\mathrm{B}\widehat{G}$ and one replaces quasi-Poisson structures on G by quasi-Lie bialgebra structures on $\mathfrak{g}$.

Given an n-shifted Poisson structure on X there is a natural way to extract an $(n-1)$-shifted Poisson structure on X. This is a classical shadow of the natural forgetful functor from $\mathbb{E}_n$-monoidal categories to $\mathbb{E}_{n-1}$-monoidal categories. Thus, one can ask how the above statements behave under this forgetful map.

Let us recall that if G is a Poisson-Lie group, there is a dual Poisson-Lie group G^* whose completion at the unit is the formal group associated with Lie algebra $\mathfrak{g}^*$. We say it is formally linearizable if there is a Poisson isomorphism between the formal completion of $\mathfrak{g}^*$ at the origin with the Kirillov–Kostant–Souriau Poisson structure and the formal completion of G^* at the unit with its Poisson-Lie structure.

The following statement combines Proposition 2.16 and Proposition 2.18.

Theorem.

- *Suppose $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ defines a 2-shifted Poisson structure on $\mathrm{B}G$. Its image under $\mathrm{Pois}(\mathrm{B}G, 2) \to \mathrm{Pois}(\mathrm{B}G, 1)$ is given by the quasi-Poisson structure $(\pi = 0, \phi)$ on G, where*
$$\phi = -\frac{1}{6}[c_{12}, c_{23}].$$
- *Suppose G carries a Poisson-Lie structure defining a 1-shifted Poisson morphism $\mathrm{pt} \to \mathrm{B}\widehat{G}$. Its image under $\mathrm{Pois}(\mathrm{pt} \to \mathrm{B}\widehat{G}, 1) \to \mathrm{Pois}(\mathrm{pt} \to \mathrm{B}\widehat{G}, 0)$ is trivial iff the dual Poisson-Lie group G^* is formally linearizable.*

We also give an interpretation of classical (dynamical) r-matrices as follows. For a group H the stack $[\mathfrak{h}^*/H] \cong \mathrm{T}^*[1](\mathrm{B}H)$ has a natural 1-shifted Poisson structure. The following is Proposition 5.6.

Theorem. *Let $H \subset G$ be a closed subgroup and $U \subset \mathfrak{h}^*$ an H-invariant open subscheme. Then the space of pairs of*

- *A 2-shifted Poisson structure on $\mathrm{B}G$,*
- *A 1-shifted Poisson structure on the composite $[U/H] \to \mathrm{B}H \to \mathrm{B}G$ compatible with the given 2-shifted Poisson structure on $\mathrm{B}G$ and the 1-shifted Poisson structure on $[U/H] \subset [\mathfrak{h}^*/H]$*

is equivalent to the set of quasi-triangular classical dynamical r-matrices with base U.

For instance, let us consider the case $H = \mathrm{pt}$ in which case we recover ordinary classical quasi-triangular r-matrices. Then the above statement recovers the well-known prescription that a quantization of a quasi-triangular classical r-matrix gives rise to a braided monoidal deformation of $\mathrm{Rep}\, G$ together with a monoidal deformation of the forgetful functor $\mathrm{Rep}\, G \to \mathrm{Vect}$.

To explain the general case, let us first observe that $\mathrm{QCoh}([\mathfrak{h}^*/H]) \cong \mathrm{Mod}_{\mathrm{Sym}(\mathfrak{h})}(\mathrm{Rep}\, H)$. Its monoidal deformation quantization is given by the monoidal category

$$\mathcal{H}_H = \mathrm{Mod}_{\mathrm{U}(\mathfrak{h})}(\mathrm{Rep}\, H)$$

equivalent to the so-called category of Harish-Chandra bimodules, i.e. $\mathrm{U}(\mathfrak{h})$-bimodules whose diagonal $\mathfrak{h}$-action is H-integrable. Then a deformation quantization of a classical dynamical r-matrix with base $\mathfrak{h}^*$ is given by a braided monoidal deformation of $\mathrm{Rep}\, G$ together with a monoidal functor $\mathrm{Rep}\, G \to \mathcal{H}_H$. If the dynamical r-matrix has poles (i.e. $U \neq \mathfrak{h}^*$), then this will be a lax monoidal functor which is only generically (i.e. over U) monoidal.

This theorem explains the seemingly different geometric interpretations of dynamical r-matrices in terms of quasi-Poisson G-spaces (see [EE03, Section 2.2]) and dynamical Poisson groupoids (see [EV98]). Indeed, consider the G-space $Y = U \times G$. Then $[Y/G] \to \mathrm{B}G$ is equivalent to $U \to \mathrm{B}G$. We may also consider the groupoid $\mathcal{G} = U \times G \times U \rightrightarrows U$. Then $U \to [U/\mathcal{G}]$ is also equivalent to $U \to \mathrm{B}G$.

We show how some standard (dynamical) r-matrices can be constructed from Lagrangian correspondences in section 5.4.

Quasi-Poisson groupoids. The description of quasi-Poisson structures on a group in terms of 1-shifted Poisson structures on its classifying stack can be generalized to groupoids. Namely, let us consider a source-connected smooth affine groupoid $\mathcal{G} \rightrightarrows X$ over a smooth affine scheme. The following is Theorem 3.29.

Theorem. *The space of 1-shifted Poisson structures on $[X/\mathcal{G}]$ is equivalent to the groupoid of quasi-Poisson structures on $\mathcal{G}$ with morphisms given by twists.*

Let us note that in [Bon+18] quasi-Poisson groupoids are also described in terms of Maurer–Cartan elements. However, the two approaches are opposite. In [Bon+18] the authors show that the notion of a quasi-Poisson groupoid is Morita-invariant and so it defines a geometric structure on the stack $[X/\mathcal{G}]$ independent of the presentation. In this paper we start with a manifestly Morita-invariant notion of a 1-shifted Poisson structure on $[X/\mathcal{G}]$

and show that it is equivalent to a quasi-Poisson structure on $\mathcal{G}$ thus showing that the latter is Morita-invariant.

A much easier computation can be performed for 1-shifted symplectic structures (see Proposition 3.31).

Theorem. *The space of 1-shifted symplectic structures on $[X/\mathcal{G}]$ is equivalent to the groupoid of quasi-symplectic structures on $\mathcal{G}$ with morphisms given by twists.*

One may identify 1-shifted symplectic structures on a derived stack with the space of *nondegenerate* 1-shifted Poisson structures (see [Cal+17, Theorem 3.2.4] and [Pri17, Theorem 3.33]). Thus, the groupoid of quasi-symplectic structures on $\mathcal{G}$ is identified with the groupoid of nondegenerate quasi-Poisson structures on $\mathcal{G}$. Explicitly, it means the following. A quasi-Poisson structure on $\mathcal{G}$ gives rise to a quasi-Lie bialgebroid structure on its underlying Lie algebroid $\mathcal{L}$. In turn, a quasi-Lie bialgebroid $\mathcal{L}$ gives rise to a Courant algebroid $\mathcal{L} \oplus \mathcal{L}^*$ (see [Roy02]). The 1-shifted Poisson structure on $[X/\mathcal{G}]$ is nondegenerate iff the Courant algebroid $\mathcal{L} \oplus \mathcal{L}^*$ is exact.

Symplectic realizations. Given a Poisson manifold X the cotangent bundle T^*_X becomes a Lie algebroid with respect to the so-called Koszul bracket; moreover, this Lie algebroid has a compatible symplectic structure. Thus, one might ask if it integrates to a symplectic groupoid $\mathcal{G} \rightrightarrows X$ which is a groupoid equipped with a multiplicative symplectic structure on $\mathcal{G}$. Conversely, given such a symplectic groupoid we get an induced Poisson structure on X. Symplectic groupoids give symplectic realizations [Wei83] of Poisson manifolds, thus one might ask for a similar notion in the setting of shifted symplectic structures. We warn the reader that what we only restrict to symplectic realizations which are symplectic groupoids (see Remark 4.10).

Given a symplectic groupoid $\mathcal{G} \rightrightarrows X$ we have an induced 1-shifted symplectic structure on $[X/\mathcal{G}]$ together with a Lagrangian structure on the projection $X \to [X/\mathcal{G}]$ (see Proposition 3.32). Now, given any n-shifted Lagrangian $L \to Y$ by the results of [MS17, Section 4] we get an induced $(n-1)$-shifted Poisson structure on L. In the case of the 1-shifted Lagrangian $X \to [X/\mathcal{G}]$ we get an unshifted Poisson structure on X coming from the symplectic groupoid.

Thus, we define symplectic realizations of n-shifted Poisson stacks X to be lifts of those to $(n+1)$-shifted Lagrangians $X \to Y$. It is expected that any n-shifted Poisson stack X has a unique formal symplectic realization, i.e. a symplectic realization $X \to Y$ which is an equivalence of reduced stacks (a nil-isomorphism). We refer to [Cal16, Section 3] for a discussion of this. The work [Spa16] in fact uses this as a definition of n-shifted Poisson structures.

We illustrate symplectic realizations by showing that the Feigin–Odesskii [FO98] Poisson structure on $\mathrm{Bun}_P(E)$, the moduli space of P-bundles on an elliptic curve E for P a parabolic subgroup of a simple group G, admits a symplectic realization given by the 1-shifted Lagrangian $\mathrm{Bun}_P(E) \to \mathrm{Bun}_M(E) \times \mathrm{Bun}_G(E)$, where M is the Levi factor of P. In particular, by taking the Čech nerve of this map we recover a symplectic groupoid integrating the Feigin–Odesskii Poisson structure.

Recall that a Manin pair is a pair $\mathfrak{g} \subset \mathfrak{d}$ of Lie algebras where $\mathfrak{d}$ is equipped with an invariant nondegenerate pairing and $\mathfrak{g} \subset \mathfrak{d}$ is Lagrangian. Suppose that the Manin pair $\mathfrak{g} \subset \mathfrak{d}$

integrates to a group pair $G \subset D$. It is known that it induces a quasi-Poisson structure on G. Given a quasi-Poisson group G we get an induced 1-shifted Poisson structure on $\mathrm{B}G$ and we show that a Manin pair gives its symplectic realization which is a 2-shifted Lagrangian $\mathrm{B}G \to \mathrm{B}D$.

Similarly, given a shifted coisotropic morphism $C \to X$ we propose a notion of symplectic realizations for those which are given by $(n+1)$-shifted Lagrangian correspondences

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

where Y carries an $(n+1)$-shifted symplectic structure, $X \to Y$ and $\tilde{X} \to Y$ are Lagrangian and so is $C \to \tilde{X} \times_Y X$. Assuming a certain conjecture (Conjecture 4.5) on a compatibility between Lagrangian and coisotropic intersections, we see that $C \to X$ inherits an n-shifted coisotropic structure.

Recall that a Manin triple is a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ where $\mathfrak{g} \subset \mathfrak{d}$ and $\mathfrak{g}^* \subset \mathfrak{d}$ are Manin pairs and $\mathfrak{g}$ and $\mathfrak{g}^*$ intersect transversely. Again suppose the Manin triple integrates to a triple of groups (D, G, G^*). It is known that it induces a Poisson-Lie structure on G.

Given a Manin triple (D, G, G^*) we obtain a 2-shifted Lagrangian correspondence

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl] \arrow[dr] & \\ \mathrm{B}G^* \arrow[dr] & & \mathrm{B}G \arrow[dl] \\ & \mathrm{B}D & \end{tikzcd}$$

In particular, $\mathrm{pt} \to \mathrm{B}G$ and $\mathrm{pt} \to \mathrm{B}G^*$ carry a 1-shifted coisotropic structure and hence G and G^* become Poisson-Lie groups, thus Manin triples give symplectic realizations of Poisson-Lie structures.

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1. Poisson and coisotropic structures on stacks

In this section we remind the reader the necessary basics of shifted Poisson and shifted coisotropic structures on derived stacks as defined in [Cal+17], [MS16] and [MS17].

1.1. **Formal geometry.** Recall the notion of a graded mixed cdga from [Cal+17, Section 1.1, Section 1.5] and the ind-object $k(\infty)$. Given a graded object A we denote $A(\infty) = A \otimes k(\infty)$. Explicitly, a graded mixed cdga is a cdga A together with an extra ***weight*** grading and a square-zero derivation ϵ of degree 1 and weight 1. Given a graded mixed cdga A its realization is

$$|A| \cong \prod_{n\geq 0} A(n)$$

with the differential $\mathrm{d}_A + \epsilon$ and its Tate realization is

$$|A|^t \cong \operatorname*{colim}_{m\to\infty} \prod_{n\geq -m} A(n)$$

with the same differential. We can also identify

$$|A|^t \cong |A \otimes k(\infty)|.$$

Here are two important examples that we will use in this paper:

- Let X be a derived Artin stack. Then one has the graded mixed cdga $\Omega^\epsilon(X)$ of differential forms on X (see [Pan+13] where it is denoted by $\mathbf{DR}(X)$). As a graded cdga it can be identified with

$$\Omega^\epsilon(X) \cong \Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1]))$$

  where the weight of $\mathbb{L}_X$ is 1. One can think of the mixed structure as the de Rham differential d_{dR}. We denote by $\Omega^\bullet(X)$ its realization.
- Let $\mathfrak{g}$ be a Lie algebra and A a commutative algebra with a $\mathfrak{g}$-action. Then one can define

$$\mathrm{C}^\epsilon(\mathfrak{g}, A) \cong \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), A)$$

  as a graded cdga with the mixed structure given by the Chevalley–Eilenberg differential d_{CE}. We denote by $\mathrm{C}^\bullet(\mathfrak{g}, A)$ its realization.

Let $\mathfrak{g}$ be a Lie algebra and $\widehat{G}$ the corresponding formal group. Throughout the paper we will be interested in the stack

$$\mathrm{B}\mathfrak{g} = \mathrm{B}\widehat{G}.$$

It is easy to see that $\mathcal{O}(\mathrm{B}\mathfrak{g})$ coincides with the Lie algebra cohomology of $\mathfrak{g}$ which follows for instance from [Lur11, Theorem 2.4.1]. Even though $\mathrm{B}\mathfrak{g}$ is not affine, many of its properties are essentially determined by its algebra of functions $\mathcal{O}(\mathrm{B}\mathfrak{g})$, more precisely by its variant $\mathbb{D}(\mathrm{B}\mathfrak{g})$ that we will construct shortly.

For an affine scheme S we define the graded mixed cdga $\mathbb{D}(S)$ to be

$$\mathbb{D}(S) = \Omega^\epsilon(S_{\mathrm{red}}/S).$$

For a general stack X we define

$$\mathbb{D}(X) = \lim_{S\to X} \mathbb{D}(S),$$

where the limit is over affine schemes S mapping to X. We can identify

$$|\mathbb{D}(X)| \cong \mathcal{O}(X).$$

Given a morphism $X \to Y$ of stacks we define the relative de Rham space to be

$$(X/Y)_{\mathrm{dR}} = X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y.$$

Proposition 1.1. *Let $X \to Y$ be a morphism of affine schemes where X is reduced. Then we have an equivalence of graded mixed cdgas*

$$\mathbb{D}((X/Y)_{\mathrm{dR}}) \cong \Omega^{\epsilon}(X/Y).$$

Proof. Let us begin by constructing a morphism $\Omega^{\epsilon}(X/Y) \to \mathbb{D}((X/Y)_{\mathrm{dR}})$ of graded mixed cdgas. Given $\operatorname{Spec} A \to (X/Y)_{\mathrm{dR}}$ we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_{\mathrm{red}} \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] & \\ X \arrow[r] & (X/Y)_{\mathrm{dR}} \arrow[dr] & \\ & & Y \end{tikzcd}$$

Moreover, we have pullbacks

$$\Omega^{\epsilon}(X/Y) \longrightarrow \Omega^{\epsilon}(A_{\mathrm{red}}/Y) \longrightarrow \Omega^{\epsilon}(A_{\mathrm{red}}/A) = \mathbb{D}(A).$$

This is compatible with pullbacks along A and hence we obtain a morphism of graded mixed cdgas

$$\Omega^{\epsilon}(X/Y) \longrightarrow \mathbb{D}((X/Y)_{\mathrm{dR}}) = \lim_{\operatorname{Spec} A \to (X/Y)_{\mathrm{dR}}} \mathbb{D}(A).$$

Since $((X/Y)_{\mathrm{dR}})_{\mathrm{red}} \cong X$, the underlying morphism of graded cdgas is an equivalence by [Cal+17, Proposition 2.2.7]. $\square$

Given a stack X we have its de Rham space X_{dR} which has the functor of points

$$X_{\mathrm{dR}}(R) = X(\mathrm{H}^0(R)_{red}).$$

For a smooth scheme X, its de Rham space can be constructed as the quotient of X by its infinitesimal groupoid. Now let X be a derived Artin stack and consider $p\colon X \to X_{\mathrm{dR}}$. Then $p_*\mathcal{O}_X$ is a commutative $\mathcal{O}_{X_{\mathrm{dR}}}$-algebra. Following [Cal+17, Definition 2.4.11], one can enhance the pair $(p_*\mathcal{O}_X, \mathcal{O}_{X_{\mathrm{dR}}})$ to prestacks of graded mixed cdgas over X_{dR} denoted by $\mathcal{B}_X = \mathbb{D}_{X/X_{\mathrm{dR}}}$ and $\mathbb{D}_{X_{\mathrm{dR}}}$ respectively so that $\mathcal{B}_X$ is a $\mathbb{D}_{X_{\mathrm{dR}}}$-cdga. Moreover, we have equivalences of prestacks

$$|\mathcal{B}_X| \cong p_*\mathcal{O}_X, \qquad |\mathbb{D}_{X_{\mathrm{dR}}}| \cong \mathcal{O}_{X_{\mathrm{dR}}}.$$

1.2. **Maurer–Cartan spaces.** Let us recall the necessary results about Maurer–Cartan spaces. Let $\mathfrak{g}$ be a nilpotent dg Lie algebra and $\Omega_\bullet = \Omega^\bullet(\Delta^n)$ the simplicial algebra of polynomial differential forms on simplices. We define $\underline{\mathrm{MC}}(\mathfrak{g})$ to be the simplicial set of Maurer–Cartan elements in $\mathfrak{g} \otimes \Omega_\bullet$. More generally, if $\mathfrak{g}$ is a pro-nilpotent dg Lie algebra, we define $\underline{\mathrm{MC}}(\mathfrak{g})$ to be the inverse limit of Maurer–Cartan spaces of the filtration.

We will use the following useful way to compute Maurer–Cartan spaces [Hin97, Proposition 2.2.3]:

Proposition 1.2. *Suppose $\mathfrak{g}$ is a nilpotent dg Lie algebra concentrated in non-negative degrees. Then $\underline{\mathrm{MC}}(\mathfrak{g})$ is equivalent to the nerve of the following* **Deligne groupoid***:*

- *Its objects are Maurer–Cartan elements in $\mathfrak{g}$.*

- *Its morphisms from x to y are given by elements $\lambda \in \mathfrak{g}^0$ and a Maurer–Cartan element $\alpha(t) \in \mathfrak{g} \otimes k[t]$ satisfying the following equations:*
$$\begin{aligned} \frac{d\alpha(t)}{dt} + \mathrm{d}\lambda + [\alpha(t), \lambda] &= 0 \\ \alpha(0) &= x \\ \alpha(1) &= y. \end{aligned}$$

If $\mathfrak{g}$ is a graded dg Lie algebra with a bracket of weight -1, its completion in weights ≥ 2 denoted by $\mathfrak{g}^{\geq 2}$ is a pro-nilpotent dg Lie algebra. Moreover, suppose $\mathfrak{g}_\bullet$ is a cosimplicial graded dg Lie algebra. It follows from [MS16, Proposition 1.17] that the functor $\underline{\mathrm{MC}}((-)^{\geq 2})\colon \mathrm{Alg}^{gr}_{\mathrm{Lie}} \to \mathrm{SSet}$ preserves homotopy limits, so we obtain the following statement. See also [Hin97, Theorem 4.1] and [Ban17, Theorem 3.11] for a closely related statement.

Proposition 1.3. *We have an equivalence of spaces*
$$\underline{\mathrm{MC}}(\mathrm{Tot}(\mathfrak{g}_\bullet)^{\geq 2}) \cong \mathrm{Tot}(\underline{\mathrm{MC}}(\mathfrak{g}_\bullet^{\geq 2})).$$

We will also use the following lemma to compute totalizations of cosimplicial groupoids (see [Hol08a, Corollary 2.11]):

Lemma 1.4. *Let $\mathcal{G}^\bullet$ be a cosimplicial groupoid*
$$\mathcal{G}^0 \rightrightarrows \mathcal{G}^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$
Its totalization is equivalent to the following groupoid $\mathcal{G}$:

- *Objects of $\mathcal{G}$ are objects a of $\mathcal{G}^0$ together with an isomorphism $\alpha\colon \mathrm{d}^1(x) \to \mathrm{d}^0(x)$ in $\mathcal{G}^1$ satisfying $s^0(\alpha) = \mathrm{id}_a$ and $\mathrm{d}^0(\alpha) \circ \mathrm{d}^2(\alpha) = \mathrm{d}^1(\alpha)$.*
- *Morphisms in $\mathcal{G}$ from (a, α) to (a', α') are given by isomorphisms $\beta\colon a \to a'$ in $\mathcal{G}^0$ such that*
$$\begin{tikzcd} \mathrm{d}^1(a) \arrow[r, "\mathrm{d}^1(\beta)"] \arrow[d, "\alpha"] & \mathrm{d}^1(a') \arrow[d, "\alpha'"] \\ \mathrm{d}^0(a) \arrow[r, "\mathrm{d}^0(\beta)"] & \mathrm{d}^0(a') \end{tikzcd}$$
commutes.

1.3. **Poisson and coisotropic structures.** Recall that the operad $\mathbb{P}_n$ is a dg operad controlling commutative dg algebras with a degree $1-n$ Poisson bracket. Given an ∞-category $\mathcal{C}$ we denote by $\mathcal{C}^\sim$ the underlying ∞-groupoid of objects.

Definition 1.5. Let A be a commutative dg algebra. The ***space of n-shifted Poisson structures*** $\mathrm{Pois}(A, n)$ is defined to be the homotopy fiber of the forgetful map
$$\mathbf{Alg}^\sim_{\mathbb{P}_{n+1}} \longrightarrow \mathbf{Alg}^\sim_{\mathrm{Comm}}$$
at $A \in \mathbf{Alg}_{\mathrm{Comm}}$.

For instance, if A is a smooth commutative algebra, the space $\mathrm{Pois}(A, 0)$ is discrete and given by the set of Poisson structures on $\mathrm{Spec}\, A$.

Definition 1.6. Let X be a derived Artin stack. The ***space of** n**-shifted Poisson structures*** $\mathrm{Pois}(X, n)$ is defined to be the space of n-shifted Poisson structures on $\mathcal{B}_X(\infty)$ as a $\mathbb{D}_{X_{\mathrm{dR}}}(\infty)$-algebra.

Let us briefly explain how to think about the twist $k(\infty)$ in the definition. Suppose A is a graded mixed commutative algebra. Then a compatible strict $\mathbb{P}_{n+1}$-algebra structure on A is given by a Poisson bracket of weight 0 such that the mixed structure is a biderivation. We can also weaken the compatibility with the mixed structure so that a weak $\mathbb{P}_{n+1}$-algebra structure on A would be given by a sequence of Poisson brackets $\{[-,-]_n\}_{n\geq 0}$ of weight n which are biderivations with respect to the total differential $(\mathrm{d}+\epsilon)$.

Similarly, a weak $\mathbb{P}_{n+1}$-algebra structure on $A(\infty)$ is given by a sequence of Poisson brackets $\{[-,-]_n\}_{n\in\mathbb{Z}}$ such that for any fixed elements $x, y \in A$ the expressions $[x, y]_n$ are zero for negative enough n.

We define coisotropic structures as follows. Consider the colored operad $\mathbb{P}_{[n+1,n]}$ whose algebras are triples (A, B, F), where A is a $\mathbb{P}_{n+1}$-algebra, B is a $\mathbb{P}_n$-algebra and $F\colon A \to \mathrm{Z}(B)$ is a $\mathbb{P}_{n+1}$-morphism, where

$$\mathrm{Z}(B) = \mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_B[n]), B)$$

is the ***Poisson center*** with the differential twisted by $[\pi_B, -]$. Given such a $\mathbb{P}_{[n+1,n]}$-algebra, the composite $A \to \mathrm{Z}(B) \to B$ is a morphism of commutative algebras which gives a forgetful functor

$$\mathbf{Alg}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{Arr}(\mathbf{Alg}_{\mathrm{Comm}}).$$

Definition 1.7. Let $f\colon A \to B$ be a morphism of commutative dg algebras. The ***space of** n**-shifted coisotropic structures*** $\mathrm{Cois}(f, n)$ is defined to be the homotopy fiber of the forgetful map

$$\mathbf{Alg}^{\sim}_{\mathbb{P}_{[n+1,n]}} \longrightarrow \mathrm{Arr}(\mathbf{Alg}_{\mathrm{Comm}})^{\sim}$$

at $f \in \mathrm{Arr}(\mathbf{Alg}_{\mathrm{Comm}})$.

Suppose (A, B, F) is a $\mathbb{P}_{[n+1,n]}$-algebra and let $f\colon A \to B$ be the induced morphism of commutative algebras. If we denote the homotopy fiber of f by $\mathrm{U}(A, B)$, it is shown in [MS16, Section 3.5] that $\mathrm{U}(A, B)[n]$ has a natural dg Lie algebra structure such that

$$B[n-1] \longrightarrow \mathrm{U}(A, B)[n] \longrightarrow A[n]$$

becomes a fiber sequence of Lie algebras. Moreover, if $A \to B$ is surjective we can identify $\mathrm{U}(A, B)$ with the strict kernel of $A \to B$ with the Lie bracket induced from A.

Suppose $f\colon L \to X$ is a morphism of derived Artin stacks. We denote the induced morphism on de Rham spaces by

$$f_{\mathrm{dR}}\colon L_{\mathrm{dR}} \longrightarrow X_{\mathrm{dR}}.$$

Moreover, we get a pullback morphism $f^*_{\mathrm{dR}}\mathcal{B}_X \to \mathcal{B}_L$ of $\mathbb{D}_{L_{\mathrm{dR}}}$-algebras. Now, if X in addition has an n-shifted Poisson structure, we obtain a natural n-shifted Poisson structure on $f^*_{\mathrm{dR}}\mathcal{B}_X(\infty)$.

Definition 1.8. Let $f\colon L \to X$ a morphism of derived Artin stacks. The ***space of** n**-shifted coisotropic structures*** $\mathrm{Cois}(f, n)$ is defined to be the space of pairs (γ_L, π_X) of an n-shifted coisotropic structure γ_L on $f^*_{\mathrm{dR}}\mathcal{B}_X(\infty) \to \mathcal{B}_L(\infty)$ as $\mathbb{D}_{L_{\mathrm{dR}}}(\infty)$-algebras and an

n-shifted Poisson structure π_X on X such that the induced n-shifted Poisson structures on $f^*_{\mathrm{dR}}\mathcal{B}_X(\infty)$ coincide.

Note that the space $\mathrm{Cois}(f,n)$ in particular contains the information of an n-shifted Poisson structure on X so that we get a diagram of spaces

$$\begin{tikzcd} & \mathrm{Cois}(f,n) \arrow[dl] \arrow[dr] & \\ \mathrm{Pois}(L,n-1) & & \mathrm{Pois}(X,n) \end{tikzcd}$$

For instance, consider the identity morphism $\mathrm{id}\colon X\to X$. By [MS16, Proposition 4.16] the morphism $\mathrm{Cois}(\mathrm{id},n)\to\mathrm{Pois}(X,n)$ is an equivalence, so we obtain a forgetful map

$$\mathrm{Pois}(X,n)\cong\mathrm{Cois}(\mathrm{id},n)\to\mathrm{Pois}(X,n-1)$$

allowing us to reduce the shift. We will analyze this map in some examples in section 2.4.

If $f\colon Y\to X$ is a morphism of derived Artin stacks, one can define the ***space of n-shifted Poisson morphisms*** $\mathrm{Pois}(f,n)$ as the space of compatible pairs of n-shifted Poisson structures on X and Y, we refer to [MS17, Definition 2.8] for a precise definition. The following statement ([MS17, Theorem 2.8]) allows us to reduce the computation of the space of Poisson morphisms to the space of coisotropic morphisms:

Theorem 1.9. *Let $f\colon Y\to X$ be a morphism of derived Artin stacks and denote by $g\colon Y\to Y\times X$ its graph. One has a Cartesian diagram of spaces*

$$\begin{tikzcd} \mathrm{Pois}(f,n) \arrow[r] \arrow[d] & \mathrm{Pois}(Y,n)\times\mathrm{Pois}(X,n) \arrow[d] \\ \mathrm{Cois}(g,n) \arrow[r] & \mathrm{Pois}(Y\times X,n) \end{tikzcd}$$

Note that in the diagram the morphism $\mathrm{Pois}(Y,n)\times\mathrm{Pois}(X,n)\to\mathrm{Pois}(Y\times X,n)$ is given by sending $(\pi_Y,\pi_X)\mapsto\pi_Y-\pi_X$.

This is a generalization of the classical statement that a morphism of Poisson manifolds $Y\to X$ is Poisson iff its graph is coisotropic.

1.4. Polyvectors.

Let us briefly sketch a computationally-efficient way of describing Poisson and coisotropic structures in terms of Maurer–Cartan spaces.

Definition 1.10. Let A be a commutative dg algebra. We define the ***algebra of n-shifted polyvector fields*** $\mathrm{Pol}(A,n)$ to be the graded $\mathbb{P}_{n+2}$-algebra

$$\mathrm{Pol}(A,n)=\mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]),A)$$

with the Poisson bracket given by the Schouten bracket of polyvector fields.

If A is a graded mixed cdga, one can consider two variants. First, let $\mathrm{Pol}^{int}(A,n)$ be the bigraded mixed $\mathbb{P}_{n+2}$-algebra defined as above. Then we define

$$\widetilde{\mathrm{Pol}}(A,n)=|\mathrm{Pol}^{int}(A,n)|$$
$$\mathrm{Pol}(A,n)=|\mathrm{Pol}^{int}(A,n)|^t.$$

We will only consider the latter variant of polyvector fields in this paper, see Remark 1.17 for the difference.

If X is a derived stack, $\mathcal{B}_X$ is a prestack of $\mathbb{D}_{X_{\mathrm{dR}}}$-linear graded mixed cdgas on X_{dR} and hence $\mathrm{Pol}(\mathcal{B}_X, n)$ is a prestack of graded $\mathbb{P}_{n+2}$-algebras on X_{dR}. We define

$$\mathrm{Pol}(X, n) = \Gamma(X_{\mathrm{dR}}, \mathrm{Pol}(\mathcal{B}_X, n)).$$

Let us also denote by $\mathrm{Pol}(X, n)^{\geq 2}$ the completion of this graded dg Lie algebra in weights ≥ 2. The following is [Cal+17, Theorem 3.1.2].

Theorem 1.11. *Let X be a derived Artin stack. Then one has an equivalence of spaces*

$$\mathrm{Pois}(X, n) \cong \underline{\mathrm{MC}}(\mathrm{Pol}(X, n)^{\geq 2}[n+1]).$$

Similarly, if $f\colon L \to X$ is a morphism of derived Artin stacks, one can define the relative algebra of polyvectors $\mathrm{Pol}(L/X, n-1)$ which is a graded $\mathbb{P}_{n+1}$-algebra. The following statement shows that we have control over $\mathrm{Pol}(X, n)$ as a graded cdga (but not as a Lie algebra).

Proposition 1.12. *One has equivalences of graded cdgas*

$$\mathrm{Pol}(X, n) \cong \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1]))$$

and

$$\mathrm{Pol}(L/X, n-1) \cong \Gamma(L, \mathrm{Sym}(\mathbb{T}_{L/X}[-n])).$$

Observe that we have a morphism of graded cdgas

$$\mathrm{Pol}(X, n) \longrightarrow \mathrm{Pol}(L/X, n-1)$$

induced by the natural morphism $f^*\mathbb{T}_X \to \mathbb{T}_{L/X}[1]$. It is shown in [MS17, Section 2.2] that we can upgrade the pair $(\mathrm{Pol}(X, n), \mathrm{Pol}(L/X, n-1))$ to a graded $\mathbb{P}_{[n+2,n+1]}$-algebra; denote the ***algebra of relative n-shifted polyvectors*** by

$$\mathrm{Pol}(f, n) = \mathrm{U}(\mathrm{Pol}(X, n), \mathrm{Pol}(L/X, n-1)).$$

Theorem 1.13. *Let $f\colon L \to X$ be a morphism of derived Artin stacks. Then one has an equivalence of spaces*

$$\mathrm{Cois}(f, n) \cong \underline{\mathrm{MC}}(\mathrm{Pol}(f, n)^{\geq 2}[n+1]).$$

1.5. **The case of a classifying stack.** Let G be an affine algebraic group and denote by $\widehat{G}$ its formal completion at the unit. $\widehat{G} \subset G$ is a normal subgroup and hence $\mathrm{B}\widehat{G}$ carries an action of $[G/\widehat{G}] \cong G_{\mathrm{dR}}$.

Identifying $\mathfrak{g}$ with right-invariant vector fields we get an isomorphism of graded cdgas

$$\Omega^\epsilon(G) \cong \mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G)).$$

Lemma 1.14. *The isomorphism*

$$\Omega^\epsilon(G) \cong \mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G))$$

is compatible with the mixed structures, where the $\mathfrak{g}$-action on $\mathcal{O}(G)$ is given by infinitesimal left translations.

The graded mixed cdga $\mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G))$ has a bialgebra structure transferred from $\Omega^\epsilon(G)$ whose coproduct is uniquely determined by the following properties:

- The diagram
$$\begin{tikzcd} \mathcal{O}(G) \arrow[r, "\Delta"] \arrow[d] & \mathcal{O}(G) \otimes \mathcal{O}(G) \arrow[d] \\ \Omega^\epsilon(G) \arrow[r, "\Delta"] & \Omega^\epsilon(G) \otimes \Omega^\epsilon(G) \end{tikzcd}$$
is commutative.
- For $\alpha \in \mathfrak{g}^* \subset \mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G))$ we have
$$\Delta(\alpha) = \alpha \otimes 1 + \mathrm{Ad}_{g_{(1)}}(1 \otimes \alpha),$$
where $g_{(1)}$ is the coordinate on the first factor of $\mathcal{O}(G)$ in $\Omega^\epsilon(G) \otimes \Omega^\epsilon(G)$.

Suppose V is a representation of G. Then $V^{\widehat{G}} = \mathrm{C}^\epsilon(\mathfrak{g}, V)$ is a representation of G_{dR}, i.e. it carries a coaction of $\mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G))$ with the formulas generalizing the ones above.

The following statement was proved in [Cal+17, Proposition 3.6.3] by a different method.

Theorem 1.15. *One has an equivalence of graded mixed cdgas*
$$\mathbb{D}(\mathrm{B}\mathfrak{g}) \cong \mathrm{C}^\epsilon(\mathfrak{g}, k)$$
compatible with the actions of G_{dR} on both sides.

Proof. Since
$$\mathrm{B}\mathfrak{g} \cong [G_{\mathrm{dR}}/G],$$
we have
$$\mathbb{D}(\mathrm{B}\mathfrak{g}) \cong \mathbb{D}(G_{\mathrm{dR}})^G,$$
where we regard $\mathbb{D}(G_{\mathrm{dR}})$ as a $\mathbb{D}(G)$-comodule and take the corresponding derived invariants. Since G is reduced, $\mathbb{D}(G) \cong \mathcal{O}(G)$. Moreover, by Proposition 1.1 we get $\mathbb{D}(G_{\mathrm{dR}}) \cong \Omega^\epsilon(G)$. Therefore,
$$\mathbb{D}(\mathrm{B}\mathfrak{g}) \cong \Omega^\bullet(G)^G.$$
By Lemma 1.14 we identify
$$\Omega^\bullet(G) \cong \mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{O}(G))$$
as graded mixed cdgas and hence
$$\mathbb{D}(\mathrm{B}\mathfrak{g}) \cong \mathrm{C}^\epsilon(\mathfrak{g}, k).$$
□

Recall that $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-n]))$ is naturally a $\mathbb{P}_{n+2}$-algebra with the so-called *big bracket* [KS92] given by contracting $\mathfrak{g}^*$ and $\mathfrak{g}$. Moreover, it is naturally graded if we assign weight 1 to $\mathfrak{g}[-n]$.

Proposition 1.16. *One has an equivalence of graded $\mathbb{P}_{n+2}$-algebras*
$$\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n) \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-n])).$$

Proof. We have $\mathbb{T}_{\mathbb{D}(\mathrm{B}\mathfrak{g})} \cong \mathrm{C}^{\epsilon}(\mathfrak{g}, \mathfrak{g}[1])$, where $\mathfrak{g}[1]$ is in internal weight -1. Therefore,
$$\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n) \cong |\operatorname{Sym}_{\mathbb{D}(\mathrm{B}\mathfrak{g})}(\mathbb{T}_{\mathbb{D}(\mathrm{B}\mathfrak{g})}[-n-1])|^t \cong \mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}(\mathfrak{g}[-n])).$$
□

Remark 1.17. Note that from the proof we see that the answer is different if we do not take the Tate realization. In that case the weight p part of $\widetilde{\mathrm{Pol}}(\mathrm{B}\mathfrak{g}, n)$ is
$$\mathrm{C}^{\geq p}(\mathfrak{g}, \operatorname{Sym}^p(\mathfrak{g}[-n]))$$
while the weight p part of $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n)$ is
$$\mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}^p(\mathfrak{g}[-n])).$$

$\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n)$ has a natural action of $\Omega^\bullet(G) \cong \mathcal{O}(G_{\mathrm{dR}})$ given by Lemma 1.14 and we have
$$\mathrm{Pol}(\mathrm{B}G, n) \cong \mathrm{Pol}(\mathrm{B}\mathfrak{g}, n)^{G_{\mathrm{dR}}}.$$

Now suppose $H \subset G$ is a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Let $f\colon \mathrm{B}H \to \mathrm{B}G$ be the induced map on classifying stacks and $\hat{f}\colon \mathrm{B}\mathfrak{h} \to \mathrm{B}\mathfrak{g}$ be its completion. Since $\mathbb{T}_{\mathrm{B}\mathfrak{h}/\mathrm{B}\mathfrak{g}} \cong \mathfrak{g}/\mathfrak{h}$ as $\mathfrak{h}$-representations, we have a quasi-isomorphism of graded mixed *cdgas*
$$\mathrm{Pol}(\mathrm{B}\mathfrak{h}/\mathrm{B}\mathfrak{g}, n-1) \cong \mathrm{C}^\bullet(\mathfrak{h}, \operatorname{Sym}(\mathfrak{g}/\mathfrak{h}[-n])).$$
Since the projection $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n) \to \mathrm{Pol}(\mathrm{B}\mathfrak{h}/\mathrm{B}\mathfrak{g}, n-1)$ is surjective, by [MS16, Proposition 4.10] we can identify
$$\mathrm{Pol}(\hat{f}, n) \cong \ker(\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n) \longrightarrow \mathrm{Pol}(\mathrm{B}\mathfrak{h}/\mathrm{B}\mathfrak{g}, n-1)),$$
where the Lie bracket is induced from the one on $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, n)$.

Finally, as before we identify
$$\mathrm{Pol}(f, n) \cong \mathrm{Pol}(\hat{f}, n)^{H_{\mathrm{dR}}}.$$

2. Poisson-Lie groups

In this section we show that many classical notions from Poisson-Lie theory have a natural interpretation in terms of shifted Poisson structures.

2.1. **Classical notions.**

We begin by recalling some standard notions from Poisson-Lie theory. Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. Recall from section 1.5 that $\mathrm{C}^\bullet(\mathfrak{g}, \operatorname{Sym}(\mathfrak{g}[-1]))$ is naturally a $\mathbb{P}_3$-algebra.

Definition 2.1. A ***quasi-Lie bialgebra structure*** on $\mathfrak{g}$ is the data of $\delta\colon \mathfrak{g} \to \wedge^2(\mathfrak{g})$ and $\phi \in \wedge^3(\mathfrak{g})$ satisfying the following equations:

$$\mathrm{d}_{\mathrm{CE}}\delta = 0 \tag{1}$$

$$\frac{1}{2}[\delta, \delta] + \mathrm{d}_{\mathrm{CE}}\phi = 0 \tag{2}$$

$$[\delta, \phi] = 0. \tag{3}$$

A ***Lie bialgebra structure*** on $\mathfrak{g}$ is a quasi-Lie bialgebra structure with $\phi = 0$.

Denote the natural $\mathbb{P}_2$ bracket on $\mathrm{Sym}(\mathfrak{g}[-1])$ by $[\![-,-]\!]$. The relation between this bracket and the big $\mathbb{P}_3$ bracket on $\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-1]))$ is given by

$$[\![x,x]\!] = [x, \mathrm{d}x], \qquad x \in \mathrm{Sym}(\mathfrak{g}[-1]).$$

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra. The ***groupoid*** $\mathrm{QLieBialg}(\mathfrak{g})$ ***of quasi-Lie bialgebra structures on*** $\mathfrak{g}$ is defined as follows:

- Objects of $\mathrm{QLieBialg}(\mathfrak{g})$ are quasi-Lie bialgebra structures on $\mathfrak{g}$.
- Morphisms of $\mathrm{QLieBialg}(\mathfrak{g})$ from (δ_1, ϕ_1) to (δ_2, ϕ_2) are given by $\lambda \in \wedge^2(\mathfrak{g})$ satisfying

$$\delta_2 = \delta_1 + \mathrm{d}_{\mathrm{CE}}\lambda$$
$$\phi_2 = \phi_1 + [\delta_1, \lambda] - \frac{1}{2}[\![\lambda, \lambda]\!].$$

Similarly, one has the notion of a ***quasi-Poisson group*** G where one replaces the cobracket δ by a bivector $\pi \in \Gamma(G, \wedge^2(\mathrm{T}_G))$, see e.g. [KM04, Definition 2.1]. We denote by $\mathrm{QPois}(G)$ the groupoid of quasi-Poisson structures on G with morphisms given by twists. Quasi-Poisson groups with $\phi = 0$ are called ***Poisson groups***.

Remark 2.3. We have a natural "differentiation functor" $\mathrm{QPois}(G) \to \mathrm{QLieBialg}(\mathfrak{g})$ which is fully faithful by definition.

Given a quasi-Poisson group (G, π, ϕ), the linear part of π at the unit defines a quasi-Lie bialgebra structure on $\mathfrak{g}$. Conversely, if $\mathfrak{g}$ is a quasi-Lie bialgebra, the group G has at most one compatible quasi-Poisson structure.

Recall that a G-scheme X is simply a smooth scheme X together with a G-action $G \times X \to X$. We denote its components by $R_x \colon G \to X$ and $L_g \colon X \to X$ for $x \in X$ and $g \in G$. Let $a \colon \mathfrak{g} \to \Gamma(X, \mathrm{T}_X)$ be the infinitesimal action map.

Definition 2.4. Let (G, π_G, ϕ) be a quasi-Poisson group. A ***quasi-Poisson*** G***-scheme*** is a G-scheme X together with a bivector $\pi_X \in \Gamma(X, \wedge^2 \mathrm{T}_X)$ satisfying

$$\pi_X(gx) = L_{g,*}\pi_X(x) + R_{x,*}\pi_G(g) \tag{4}$$

$$\frac{1}{2}[\pi_X, \pi_X] + a(\phi) = 0. \tag{5}$$

Definition 2.5. Let G be a group and X a G-scheme. Let $\mathrm{QPois}(G, X)$ be the following groupoid:

- Objects of $\mathrm{QPois}(G, X)$ are quasi-Poisson group structures on G and bivectors π_X on X making X into a quasi-Poisson G-scheme.
- Morphisms of $\mathrm{QPois}(G, X)$ from (π_G, ϕ, π_X) to (π'_G, ϕ', π'_X) are given by elements $\lambda \in \wedge^2(\mathfrak{g})$ defining a morphism $(\pi_G, \phi) \to (\pi'_G, \phi')$ in $\mathrm{QPois}(G)$ and satisfying

$$\pi'_X = \pi_X + a(\lambda).$$

2.2. **Shifted Poisson structures on** $\mathrm{B}G$. In this section we classify shifted Poisson structures on the classifying stack $\mathrm{B}G$. The following statement also appears in [Cal+17, Section 3.6.2]:

Proposition 2.6. *The space of* n*-shifted Poisson structures on* $\mathrm{B}G$ *is trivial if* $n > 2$*. Moreover, we can identify the space* $\mathrm{Pois}(\mathrm{B}G, 2)$ *with the set* $\mathrm{Sym}^2(\mathfrak{g})^G$.

Proof. The algebra of n-shifted polyvectors is

$$\mathrm{Pol}(\mathrm{B}G, n) \cong \mathrm{C}^\bullet(G, \mathrm{Sym}(\mathfrak{g}[-n]))$$

as a graded complex. In particular, we obtain a graded homotopy $\mathbb{P}_{n+2}$ structure on $\mathrm{C}^\bullet(G, \mathrm{Sym}(\mathfrak{g}[-n]))$. Elements of weight at least 2 have cohomological degree at least $2n$. But Maurer–Cartan elements in a $\mathbb{P}_{n+2}$-algebra are in degree $n+2$. Therefore, for $n > 2$ the space of Maurer–Cartan elements in $\mathrm{Pol}(\mathrm{B}G, n)^{\geq 2}$ is contractible.

For $n = 2$ Maurer–Cartan elements are degree 4 elements in $\mathrm{Pol}(\mathrm{B}G, 2)^{\geq 2}$, i.e. elements $c \in \mathrm{Sym}^2(\mathfrak{g})$, satisfying

$$\mathrm{d}c = 0, \qquad [c, c] = 0, \qquad [c, c, c] = 0, \qquad \dots$$

The bracket $[-, \dots, -]_m$ in a graded homotopy $\mathbb{P}_4$-algebra has weight $1-m$ and degree $5-4m$. Therefore, $[c, \dots, c]_m$ is an element of weight $m+1$ and degree 5 which is automatically zero if $m > 1$. Therefore, the Maurer–Cartan equation reduces to $\mathrm{d}c = 0$, i.e. $c \in \mathrm{Sym}^2(\mathfrak{g})^G$. □

For example, identifying nondegenerate n-shifted Poisson structures with n-shifted symplectic structures by [Cal+17, Theorem 3.2.4] we get the following statement which was proved in [Pan+13] for reductive groups.

Proposition 2.7. *Let G be an algebraic group. The space $\mathrm{Symp}(\mathrm{B}G, 2)$ of 2-shifted symplectic structures on $\mathrm{B}G$ is equivalent to the subset of $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ consisting of non-degenerate quadratic forms.*

The space of 1-shifted Poisson structures is more interesting. We begin with the case of Lie algebras.

Theorem 2.8. *Let $\mathfrak{g}$ be a Lie algebra of an algebraic group G. Then the space of 1-shifted Poisson structures $\mathrm{Pois}(\mathrm{B}\mathfrak{g}, 1)$ is equivalent to the groupoid $\mathrm{QLieBialg}(\mathfrak{g})$ of quasi-Lie bialgebra structures on $\mathfrak{g}$.*

Proof. We can identify

$$\mathrm{Pois}(\mathrm{B}\mathfrak{g}, 1) \cong \underline{\mathrm{MC}}(\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]).$$

Since $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2} \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}^{\geq 2}(\mathfrak{g}[-1]))$, the dg Lie algebra $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]$ is concentrated in non-negative degrees and hence by Proposition 1.2 we can identify its ∞-groupoid of Maurer–Cartan elements with the Deligne groupoid.

A Maurer–Cartan element α in $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]$ is given by a pair of elements $\delta\colon \mathfrak{g} \to \wedge^2(\mathfrak{g})$ in weight 2 and $\phi \in \wedge^3(\mathfrak{g})$ in weight 3 which satisfy equations (1) - (3) defining the quasi-Lie bialgebra structure on $\mathfrak{g}$.

A 1-morphism in the Deligne groupoid of $\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]$ is given by a t-parameter family of quasi-Lie bialgebra structures $\delta(t)$ and $\phi(t)$ and an element $\lambda \in \wedge^2(\mathfrak{g})$ which satisfy the equations

$$\frac{d\delta(t)}{dt} - \mathrm{d}_{\mathrm{CE}}\lambda = 0$$
$$\frac{d\phi(t)}{dt} - [\delta(t), \lambda] = 0.$$

The first equation implies that

$$\delta(t) = \delta_0 + t\mathrm{d}_{\mathrm{CE}}\lambda.$$

Substituting it in the second equation and integrating, we obtain

$$\phi(t) = \phi_0 + t[\delta_0, \lambda] + \frac{t^2}{2}[\mathrm{d}_{\mathrm{CE}}\lambda, \lambda] = 0.$$

In this way we see that the Deligne groupoid is isomorphic to the groupoid $\mathrm{QLieBialg}(\mathfrak{g})$ of quasi-Lie bialgebra structures on $\mathfrak{g}$. □

We also have a global version of the previous statement.

Theorem 2.9. *Let G be an algebraic group. Then the space of 1-shifted Poisson structures* $\mathrm{Pois}(\mathrm{B}G, 1)$ *is equivalent to the groupoid* $\mathrm{QPois}(G)$ *of quasi-Poisson structures on G.*

Proof. By Theorem 1.11

$$\mathrm{Pois}(\mathrm{B}G, 1) \cong \underline{\mathrm{MC}}(\mathrm{Pol}(\mathrm{B}G, 1)^{\geq 2}[2]).$$

Moreover, we can identify the latter space with

$$\underline{\mathrm{MC}}((\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2})^{G_{\mathrm{dR}}}[2]).$$

We can present G_{dR}-invariants on a complex V as a totalization of the cosimplicial object

$$V \rightrightarrows V \otimes \Omega^\bullet(G) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \dots.$$

Therefore, by Proposition 1.3 we can identify $\underline{\mathrm{MC}}(\mathrm{Pol}(\mathrm{B}G, 1)^{\geq 2}[2])$ with the totalization of

$$\underline{\mathrm{MC}}(\mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]) \rightrightarrows \underline{\mathrm{MC}}(\Omega^\bullet(G) \otimes \mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \dots$$

The latter is a cosimplicial diagram of groupoids and we can use Lemma 1.4 to compute its totalization $\mathcal{G}$. Let us begin by describing the groupoid $\mathcal{G}^1 = \underline{\mathrm{MC}}(\Omega^\bullet(G) \otimes \mathrm{Pol}(\mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2])$. Degree 1 elements in the corresponding dg Lie algebra are given by elements

$$\begin{aligned}
\delta &\in \mathcal{O}(G) \otimes \mathfrak{g}^* \otimes \wedge^2(\mathfrak{g}) \\
\phi &\in \mathcal{O}(G) \otimes \wedge^3(\mathfrak{g}) \\
P &\in \Omega^1(G) \otimes \wedge^2(\mathfrak{g}).
\end{aligned}$$

The Maurer–Cartan equation for $\delta + \phi + P$ boils down to the equations expressing the fact that the pair $\delta(g), \phi(g)$ defines a Lie bialgebra structure on $\mathfrak{g}$ for any $g \in G$ and the following equations:

$$\begin{aligned}
\mathrm{d}_{\mathrm{dR}}P &= 0 \\
\mathrm{d}_{\mathrm{dR}}\delta + \mathrm{d}_{\mathrm{CE}}P &= 0 \\
\mathrm{d}_{\mathrm{dR}}\phi + [P, \delta] &= 0.
\end{aligned}$$

Morphisms from (δ, ϕ, P) to (δ', ϕ', P') in $\mathcal{G}^1$ are given by elements $\pi \in \mathcal{O}(G) \otimes \wedge^2(\mathfrak{g})$ satisfying the equations

$$\begin{aligned}\delta' &= \delta + \mathrm{d}_{\mathrm{CE}}\pi \\ \phi' &= \phi + [\delta, \pi] - \frac{1}{2}[\![\pi, \pi]\!] \\ P' &= P + \mathrm{d}_{\mathrm{dR}}\pi.\end{aligned}$$

Thus, by Lemma 1.4 an object of $\mathcal{G}$ consists of a quasi-Lie bialgebra structure (δ, ϕ) on $\mathfrak{g}$ together with a bivector $\pi \in \mathcal{O}(G) \otimes \wedge^2(\mathfrak{g})$ satisfying the equations

$$\pi(e) = 0 \tag{6}$$

$$\pi(xy) = \pi(x) + \mathrm{Ad}_x\pi(y) \tag{7}$$

$$\mathrm{Ad}_g(\delta) = \delta + \mathrm{d}_{\mathrm{CE}}\pi \tag{8}$$

$$\mathrm{Ad}_g(\phi) = \phi + [\delta, \pi] - \frac{1}{2}[\![\pi, \pi]\!] \tag{9}$$

$$\mathrm{Ad}_g(\delta) = \mathrm{d}_{\mathrm{dR}}\pi. \tag{10}$$

Equation (6) follows from (7) which expresses the fact that $\pi \in \Gamma(G, \wedge^2\mathrm{T}_G)$ is multiplicative. Denote by $\widetilde{\delta}\colon \mathfrak{g} \to \wedge^2(\mathfrak{g})$ the linear part of π at the unit. Then the infinitesimal version of (7) is

$$\mathrm{d}_{\mathrm{dR}}\pi = \widetilde{\delta} + \mathrm{d}_{\mathrm{CE}}\pi.$$

Combining this with equations (8) and (10) we see that $\delta = \widetilde{\delta}$. Therefore, equation (9) becomes

$$\mathrm{Ad}_g(\phi) - \phi = [\mathrm{d}_{\mathrm{dR}}\pi, \pi] + \frac{1}{2}[\![\pi, \pi]\!].$$

If we denote by $[\pi, \pi]_{\mathrm{Sch}}$ the Schouten bracket of the bivector of π, then we can rewrite the above equation as

$$\mathrm{Ad}_g(\phi) - \phi = -\frac{1}{2}[\pi, \pi]_{\mathrm{Sch}}.$$

This shows that an object of $\mathcal{G}$ is a quasi-Poisson structure (π, ϕ) on G and (δ, ϕ) is the induced Lie bialgebra structure on $\mathfrak{g}$.

Morphisms $(\delta, \phi, \pi) \to (\delta', \phi', \pi')$ in $\mathcal{G}$ are given by morphisms in $\mathcal{G}^0$, i.e. by twists $\lambda \in \wedge^2(\mathfrak{g})$ of quasi-Lie bialgebra structures $(\delta, \phi) \to (\delta', \phi')$. The compatibility of the twist λ with the bivectors is the equation

$$\pi + \lambda = \mathrm{Ad}_g(\lambda) + \pi'$$

which shows that λ is also a twist of the quasi-Poisson structure (π, ϕ) into (π', ϕ'). In this way we have identified objects of $\mathcal{G}$ with quasi-Poisson structures on $\mathcal{G}$ and morphisms with twists of those, i.e. $\mathcal{G} \cong \mathrm{QPois}(G)$. □

2.3. **Coisotropic structures.** In this section we study coisotropic structures in $\mathrm{B}G$. Denote by $\mathrm{QPois}_H(G) \subset \mathrm{QPois}(G)$ the following subgroupoid:

- Objects of $\mathrm{QPois}_H(G)$ are objects (π, ϕ) of $\mathrm{QPois}(G)$ such that the image of π under
$$\mathcal{O}(G) \otimes \wedge^2(\mathfrak{g}) \to \mathcal{O}(H) \otimes \wedge^2(\mathfrak{g}/\mathfrak{h})$$
is zero and ϕ is in the kernel of $\wedge^3(\mathfrak{g}) \to \wedge^3(\mathfrak{g}/\mathfrak{h})$.

- Morphisms in $\mathrm{QPois}_H(G)$ are morphisms $(\pi, \phi) \to (\pi', \phi')$ in $\mathrm{QPois}(G)$ given by λ in the kernel of $\wedge^2(\mathfrak{g}) \to \wedge^2(\mathfrak{g}/\mathfrak{h})$.

Note that $(\pi, \phi = 0)$ is an object of $\mathrm{QPois}_H(G)$ iff π makes G into a Poisson-Lie group and $H \subset G$ into a coisotropic subgroup.

Proposition 2.10. *Let $H \subset G$ be a closed subgroup and $f\colon \mathrm{B}H \to \mathrm{B}G$ the induced map on classifying stacks.*

The space of 2-shifted coisotropic structures $\mathrm{Cois}(f, 2)$ *is equivalent to the set*
$$\ker(\mathrm{Sym}^2(\mathfrak{g})^G \to \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{h})^H).$$
The space of 1-shifted coisotropic structures $\mathrm{Cois}(f, 1)$ *is equivalent to the groupoid* $\mathrm{QPois}_H(G)$.

Proof. Both statements immediately follow from the description of relative polyvectors $\mathrm{Pol}(f, n)$ as a kernel of
$$\mathrm{C}^\bullet(G, \mathrm{Sym}(\mathfrak{g}[-n])) \to \mathrm{C}^\bullet(H, \mathrm{Sym}(\mathfrak{g}/\mathfrak{h}[-n])).$$
□

Corollary 2.11. *Let $f\colon \mathrm{pt} \to \mathrm{B}G$ be the inclusion of the basepoint. Then the space of 1-shifted Poisson structures* $\mathrm{Pois}(f, 1)$ *is equivalent to the set of Poisson-Lie structures on G.*

Proof. By Theorem 1.9 we have an equivalence of spaces $\mathrm{Pois}(f, 1) \cong \mathrm{Cois}(f, 1)$ and by Proposition 2.10 we have an equivalence of spaces $\mathrm{Cois}(f, 1) \cong \mathrm{QPois}_{\{e\}}(G)$. But it is immediate that the latter groupoid is in fact a set whose objects have $\phi = 0$. □

Recall that we have a diagram of spaces
$$\begin{tikzcd} & \mathrm{Cois}(f,2) \arrow[dl] \arrow[dr] & \\ \mathrm{Pois}(\mathrm{B}H, 1) & & \mathrm{Pois}(\mathrm{B}G, 2). \end{tikzcd}$$
Thus, given a Casimir element $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ vanishing on $\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{h})^H$ we obtain a quasi-Poisson structure on H, let us now describe it explicitly. For simplicity we will describe the induced quasi-Lie bialgebra structure on $\mathfrak{h}$.

Pick a splitting $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ as vector spaces. Let $\{e_i\}$ be a basis of $\mathfrak{h}$ and $\{\tilde{e}_i\}$ a basis of $\mathfrak{g}/\mathfrak{h}$. Let $\{e^i\}$ and $\{\tilde{e}^i\}$ be the dual bases. We denote the structure constants as follows:
$$\begin{aligned} [e_i, e_j] &= f_{ij}^k e_k \\ [e_i, \tilde{e}_j] &= A_{ij}^k e_k + B_{ij}^k \tilde{e}_k \\ [\tilde{e}_i, \tilde{e}_j] &= C_{ij}^k e_k + D_{ij}^k \tilde{e}_k \end{aligned}$$
Let us also split $c = P + Q$, where $P \in \mathrm{Sym}^2(\mathfrak{h})$ and $Q \in \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h}$. We denote its components by P^{ij} and Q^{ij}.

The cobracket $\delta \in \mathfrak{h}^* \otimes \wedge^2(\mathfrak{h})$ has components
$$\delta_k^{ij} = \frac{1}{2}(A_{ka}^j Q^{ia} - A_{ka}^i Q^{ja}) \tag{11}$$

and the associator $\phi \in \wedge^3(\mathfrak{h})$ has components

$$\phi^{ijk} = \frac{1}{8} f^i_{ab} P^{aj} P^{bk} + \frac{1}{4} Q^{ia}(\gamma^k_{ab} Q^{jb} - \gamma^j_{ab} Q^{kb}) + \frac{1}{8} P^{ia}(\alpha^k_{ab} Q^{jb} - \alpha^j_{ab} Q^{kb}). \tag{12}$$

A tedious computation shows that the above formulas indeed define a quasi-Lie bialgebra structure on $\mathfrak{h}$. One can also see that if c is nondegenerate and both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are Lagrangian in $\mathfrak{g}$, then the formulas reduce to those of [Dri89, Section 2] and [AKS00, Section 2.1].

Remark 2.12. Consider an exact sequence of $\mathfrak{h}$-representations

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0.$$

The connecting homomorphism gives rise to a morphism $\mathfrak{g}/\mathfrak{h} \to \mathfrak{h}[1]$ in the derived category of $\mathfrak{h}$-representations. Since $\mathfrak{h} \subset \mathfrak{g}$ is coisotropic, the Casimir element induces a morphism of $\mathfrak{h}$-representations $\mathfrak{h}^* \to \mathfrak{g}/\mathfrak{h}$. Combining the two, we get a morphism $\mathfrak{h}^* \to \mathfrak{h}[1]$, i.e. an element of $\mathrm{H}^1(\mathfrak{h}, \mathfrak{h}\otimes\mathfrak{h})$. The formula (11) gives an explicit representative for this cohomology class given a splitting $\mathfrak{g}/\mathfrak{h} \subset \mathfrak{g}$.

Proposition 2.13. *Let $\hat{f}\colon \mathrm{B}\mathfrak{h} \to \mathrm{B}\mathfrak{g}$. Under the forgetful map*

$$\mathrm{Cois}(\hat{f}, 2) \to \mathrm{Pois}(\mathrm{B}\mathfrak{h}, 1)$$

the induced quasi-Lie bialgebra structure on $\mathfrak{h}$ is given by formulas (11) and (12).

Proof. Pick $c \in \ker(\mathrm{Sym}^2(\mathfrak{g})^G \to \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{h})^H)$ as above. We are going to construct a $\mathbb{P}_{[3,2]}$-algebra structure on the pair $(\mathrm{C}^\bullet(\mathfrak{g}, k), \mathrm{C}^\bullet(\mathfrak{h}, k))$ such that the $\mathbb{P}_2$-structure on $\mathrm{C}^\bullet(\mathfrak{h}, k)$ is induced by the above quasi-Lie bialgebra structure on $\mathfrak{h}$. That is, we have to define a morphism of $\mathbb{P}_3$-algebras

$$F\colon \mathrm{C}^\bullet(\mathfrak{g}, k) \longrightarrow \mathrm{Z}(\mathrm{C}^\bullet(\mathfrak{h}, k)) \cong \mathrm{C}^\bullet(\mathfrak{h}, \mathrm{Sym}(\mathfrak{h}[-1])).$$

The Poisson bracket on $\mathrm{C}^\bullet(\mathfrak{g}, k)$ is given by

$$\{e^i, e^j\} = P^{ij}, \qquad \{e^i, \tilde{e}^j\} = Q^{ij}$$

and the Poisson bracket on $\mathrm{C}^\bullet(\mathfrak{h}, \mathrm{Sym}(\mathfrak{h}[-1]))$ is given by the canonical pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$.

The differentials on $\mathrm{C}^\bullet(\mathfrak{g}, k)$ are

$$\mathrm{d}e^i = A^i_{jk} e^j \tilde{e}^k + \frac{1}{2} C^i_{jk} \tilde{e}^j \tilde{e}^k + \frac{1}{2} f^i_{jk} e^j e^k$$

and

$$\mathrm{d}\tilde{e}^i = B^i_{jk} e^j \tilde{e}^k + \frac{1}{2} D^i_{jk} \tilde{e}^j \tilde{e}^k.$$

The differentials on $\mathrm{C}^\bullet(\mathfrak{h}, \mathrm{Sym}(\mathfrak{h}[-1]))$ are

$$\mathrm{d}e^i = \frac{1}{2} f^i_{jk} e^j e^k + \phi^{ijk} e_j e_k - \delta^{ij}_k e^k e_j$$

and

$$\mathrm{d}e_i = f^k_{ij} e^j e_k + \frac{1}{2} \delta^{jk}_i e_j e_k$$

G-invariance of the Casimir c implies the following equations:

$$0 = A^i_{jk}P^{ja} + C^i_{jk}Q^{aj} + A^a_{jk}P^{ji} + C^a_{jk}Q^{ij} \tag{13}$$

$$0 = A^i_{jk}Q^{ak} - f^i_{kj}P^{ka} + A^a_{jk}Q^{ik} - f^a_{kj}P^{ki} \tag{14}$$

$$0 = -A^i_{jk}Q^{ja} - B^a_{jk}P^{ij} - D^a_{jk}Q^{ij} \tag{15}$$

$$0 = -f^i_{kj}Q^{ka} + B^a_{jk}Q^{ik} \tag{16}$$

$$0 = B^i_{jk}Q^{ja} + B^a_{jk}Q^{ji}. \tag{17}$$

Since $\mathrm{C}^\bullet(\mathfrak{g}, k)$ is generated as a commutative algebra by $\mathfrak{g}^*$, it is enough to define F on the generators: we define

$$F(e^i) = e^i + \frac{1}{2}P^{ij}e_j, \qquad F(\tilde{e}^i) = Q^{ji}e_j.$$

It is easy to see that thus defined F is compatible with the Poisson brackets. We are now going to show that it is also compatible with differentials.

Consider $\tilde{e}^i \in (\mathfrak{g}/\mathfrak{h})^*$. Then

$$\mathrm{d}\tilde{e}^i = B^i_{jk}e^j\tilde{e}^k + \frac{1}{2}D^i_{jk}\tilde{e}^j\tilde{e}^k$$

and hence

$$F(\mathrm{d}\tilde{e}^i) = B^i_{jk}\left(e^j + \frac{1}{2}P^{ja}e_a\right)Q^{bk}e_b + \frac{1}{2}D^i_{jk}Q^{aj}Q^{bk}e_ae_b.$$

On the other hand,

$$\mathrm{d}F(\tilde{e}^a) = Q^{ia}f^k_{ij}e^je_k + \frac{1}{2}Q^{ia}\delta^{jk}_i e_je_k.$$

Therefore,

$$Q^{ci}\delta^{ab}_c = \frac{1}{2}(B^i_{jk}P^{ja}Q^{bk} + D^i_{jk}Q^{aj}Q^{bk}) - (a \leftrightarrow b).$$

which holds by (15).

Now consider $e^i \in \mathfrak{h}^*$. Then

$$\mathrm{d}e^i = A^i_{jk}e^j\tilde{e}^k + \frac{1}{2}C^i_{jk}\tilde{e}^j\tilde{e}^k + \frac{1}{2}f^i_{jk}e^je^k$$

and hence

$$F(\mathrm{d}e^i) = A^i_{jk}\left(e^j + \frac{1}{2}P^{ja}e_a\right)Q^{bk}e_b + \frac{1}{2}C^i_{jk}Q^{aj}Q^{bk}e_ae_b + \frac{1}{2}f^i_{jk}\left(e^j + \frac{1}{2}P^{ja}e_a\right)\left(e^k + \frac{1}{2}P^{kb}e_b\right).$$

On the other hand,

$$\mathrm{d}F(e^i) = \frac{1}{2}f^i_{jk}e^je^k + \phi^{iab}e_ae_b - \delta^{ib}_j e^je_b + \frac{1}{2}P^{ib}\left(f^k_{bj}e^je_k + \frac{1}{2}\delta^{jk}_b e_je_k\right).$$

Therefore,

$$\begin{aligned} P^{ik}f^b_{kj} - 2\delta^{ib}_j &= 2A^i_{jk}Q^{bk} + f^i_{jk}P^{kb} \\ \phi^{iab} + \frac{1}{4}P^{ic}\delta^{ab}_c &= \frac{1}{4}A^i_{jk}P^{ja}Q^{bk} - \frac{1}{4}A^i_{jk}P^{jb}Q^{ak} + \frac{1}{2}C^i_{jk}Q^{aj}Q^{bk} + \frac{1}{8}f^i_{jk}P^{ja}P^{kb}. \end{aligned}$$

These can be checked using (13) and (14). ◻

Let us recall that one can identify G-spaces with spaces over $\mathrm{B}G$: given a G-space X we get $Y = [X/G] \to \mathrm{B}G$; conversely, given $Y \to \mathrm{B}G$ we form the G-space $X = \mathrm{pt} \times_{\mathrm{B}G} Y$. The following statements extend this analogy to the Poisson setting.

Proposition 2.14. *Let X be a G-scheme and $f\colon [X/G] \to \mathrm{B}G$ the projection map. Then we have an equivalence of groupoids*

$$\mathrm{Cois}(f, 1) \cong \mathrm{QPois}(G, X).$$

Proof. We have a fiber sequence of Lie algebras

$$\mathrm{Pol}([X/G]/\mathrm{B}G, 0)[1] \longrightarrow \mathrm{Pol}(f, 1)[2] \longrightarrow \mathrm{Pol}(\mathrm{B}G, 1)[2]$$

We have quasi-isomorphisms

$$\mathrm{Pol}(\mathrm{B}G, 1) \cong \mathrm{C}^\bullet(G, \mathrm{Sym}(\mathfrak{g}[-1])), \qquad \mathrm{Pol}([X/G]/\mathrm{B}G, 0) \cong \mathrm{C}^\bullet(G, \mathrm{Pol}(X, 0)).$$

The connecting homomorphism

$$a\colon \mathrm{Pol}(\mathrm{B}G, 1) \longrightarrow \mathrm{Pol}([X/G]/\mathrm{B}G, 0)$$

is then induced by the action map $\mathfrak{g} \to \Gamma(X, \mathrm{T}_X)$.

The Lie algebra $\mathrm{Pol}(f, 1)^{\geq 2}[2]$ is concentrated in non-negative degrees, so by Proposition 1.2 we can identify its Maurer–Cartan space with the Deligne groupoid. The degree 3 elements in $\mathrm{Pol}(f, 1)^{\geq 2}$ are elements (π_G, ϕ) in $\mathrm{Pol}(\mathrm{B}G, 1)$ and $\pi_X \in \Gamma(X, \wedge^2 \mathrm{T}_X)$. The Maurer–Cartan equation reduces to the Maurer–Cartan equations in $\mathrm{Pol}(\mathrm{B}G, 1)$ which imply by Theorem 2.9 that (π_G, ϕ) define a quasi-Poisson structure on G and the equations

$$\mathrm{d}\pi_X + a(\pi_G) = 0$$
$$\frac{1}{2}[\pi_X, \pi_X] + [\pi_X, \pi_G] + a(\phi) = 0.$$

The other brackets vanish due to weight reasons. But the morphism

$$\mathrm{Pol}(f, 1)[2] \longrightarrow \mathrm{Pol}([X/G], 0)[1]$$

is a morphism of Lie algebras which shows that $[\pi_X, \pi_G] = 0$. Therefore, the second equation is equivalent to

$$\frac{1}{2}[\pi_X, \pi_X] + a(\phi) = 0,$$

and therefore we recover equations (4) and (5) in the definition of quasi-Poisson G-spaces.

Morphisms $(\pi_X, \pi_G, \phi) \to (\pi'_X, \pi'_G, \phi')$ are given by degree 2 elements in $\mathrm{Pol}(f, 1)^{\geq 2}$ which are elements $\lambda \in \wedge^2(\mathfrak{g})$. These elements, firstly, define a morphism $(\pi_G, \phi) \to (\pi'_G, \phi')$ in $\mathrm{QPois}(G)$ by Theorem 2.9 and, secondly, satisfy the equation

$$\pi'_G - \pi_G = a(\lambda).$$

In this way we have identified the Maurer–Cartan space of $\mathrm{Pol}(f, 1)^{\geq 2}[2]$ with $\mathrm{QPois}(G, X)$. □

Combining Proposition 2.10 and Proposition 2.14 we obtain the following statement:

Corollary 2.15. *We have an equivalence of groupoids*

$$\mathrm{QPois}(G, G/H) \cong \mathrm{QPois}_H(G).$$

In other words, given a coisotropic subgroup $H \subset G$ of a quasi-Poisson group G we get a natural quasi-Poisson structure on the homogeneous G-scheme $[G/H]$.

2.4. **Forgetting the shift.** In this section we relate n-shifted Poisson structures on classifying spaces to $(n-1)$-shifted Poisson structures. We begin with the case $n = 2$. If $c \in \mathrm{Sym}^2(\mathfrak{g})^G$, recall the standard notation

$$c_{12} = c \otimes 1 \in \mathrm{U}(\mathfrak{g})^{\otimes 3}$$
$$c_{23} = 1 \otimes c \in \mathrm{U}(\mathfrak{g})^{\otimes 3}.$$

Proposition 2.16. *The forgetful map*

$$\mathrm{Pois}(\mathrm{B}G, 2) \longrightarrow \mathrm{Pois}(\mathrm{B}G, 1)$$

sends a Casimir element $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ to the quasi-Poisson structure with $\pi = 0$ and

$$\phi = -\frac{1}{6}[c_{12}, c_{23}].$$

In particular, if G is semisimple, the forgetful map

$$\mathrm{Pois}(\mathrm{B}G, 2) \longrightarrow \mathrm{Pois}(\mathrm{B}G, 1)$$

is an equivalence.

Proof. The first claim follows from Proposition 2.13 by explicitly computing (12) in this case.

Since G is semisimple, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Pois}(\mathrm{B}G, 2) \arrow[r] \arrow[d, "\sim"] & \mathrm{Pois}(\mathrm{B}G, 1) \arrow[d, "\sim"] \\ \mathrm{Sym}^2(\mathfrak{g})^G \arrow[r] & \wedge^3(\mathfrak{g})^G \end{tikzcd}$$

But it is well-known that the map $\mathrm{Sym}^2(\mathfrak{g})^G \to \wedge^3(\mathfrak{g})^G$ is an isomorphism, see [Kos50, Théorème 11.1]. ☐

If $\mathfrak{g}$ is a Lie bialgebra, the space $\mathfrak{g}$ carries the Kirillov–Kostant–Souriau Poisson structure induced from the Lie cobracket. Recall the following notion (see e.g. [EEM05]):

Definition 2.17. A Poisson-Lie group G is ***formally linearizable*** at the unit if we have an isomorphism of formal Poisson schemes

$$\widehat{\mathfrak{g}}_{\{0\}} \to \widehat{G}_{\{e\}}$$

inducing identity on tangent spaces at the origin.

Let G^* be an algebraic group whose Lie algebra is $\mathfrak{g}^*$.

Proposition 2.18. *Let G be a Poisson-Lie group. The image of the Poisson structure under*

$$\mathrm{Pois}(\mathrm{pt} \to \mathrm{B}\mathfrak{g}, 1) \longrightarrow \mathrm{Pois}(\mathrm{pt} \to \mathrm{B}\mathfrak{g}, 0)$$

is zero iff G^* is formally linearizable.

Proof. The forgetful map on augmented shifted Poisson algebras can be obtained as a composite

$$\mathbf{Alg}^{\mathrm{aug}}_{\mathbb{P}_{n+1}} \xrightarrow{\sim} \mathbf{Alg}(\mathbf{Alg}^{\mathrm{aug}}_{\mathbb{P}_n}) \longrightarrow \mathbf{Alg}^{\mathrm{aug}}_{\mathbb{P}_n}.$$

Recall the construction of the first functor from [Saf16]. Using Koszul duality we can identify

$$\mathbf{Alg}^{\mathrm{aug}}_{\mathbb{P}_n} \cong \mathrm{Alg}_{\mathrm{co}\mathbb{P}_n}[W^{-1}_{\mathrm{Kos}}],$$

where W_{Kos} is a certain class of weak equivalences of coaugmented $\mathbb{P}_n$-coalgebras. Then the image of the augmented $\mathbb{P}_2$-algebra $\mathrm{C}^\bullet(\mathfrak{g}, k)$ is given by the associative algebra object in Poisson coalgebras $\mathrm{U}(\mathfrak{g}^*)$, whose Poisson cobracket is induced from the Lie algebra structure on $\mathfrak{g}^*$.

We have a commutative diagram

$$\begin{tikzcd}
\mathbf{Alg}^{\mathrm{aug}}_{\mathbb{P}_n} \arrow[r, "\sim"] \arrow[d] & \mathrm{Alg}_{\mathrm{co}\mathbb{P}_n}[W^{-1}_{\mathrm{Kos}}] \arrow[d] \\
\mathbf{Alg}^{\mathrm{aug}}_{\mathrm{Comm}} \arrow[r, "\sim"] & \mathrm{Alg}_{\mathrm{coLie}}[W^{-1}_{\mathrm{Kos}}]
\end{tikzcd}$$

where the horizontal equivalences are given by Koszul duality, the vertical functor on the left is the obvious forgetful functor and the vertical functor on the right is given by taking the primitive elements in a coaugmented $\mathbb{P}_n$-coalgebra.

The Poisson coalgebra $\mathrm{U}(\mathfrak{g}^*)$ has primitives given by the Lie coalgebra $\mathfrak{g}^*$. Thus, the Koszul dual of the coaugmented Poisson coalgebra $\mathrm{U}(\mathfrak{g}^*)$ is a certain augmented homotopy $\mathbb{P}_1$-algebra A lifting the standard augmented cdga structure on $\mathrm{C}^\bullet(\mathfrak{g}, k)$.

Now consider $\mathrm{C}^\bullet(\mathfrak{g}, k)$ with the zero Poisson bracket. Trivialization of the 0-shifted Poisson structure on $\mathrm{B}G$ is a weak equivalence of augmented $\mathbb{P}_1$-algebras

$$A \cong \mathrm{C}^\bullet(\mathfrak{g}, k)$$

being the identity on the underlying augmented commutative algebras.

Passing to the Koszul dual side, it corresponds to a weak equivalence of coaugmented $\mathbb{P}_1$-coalgebras

$$\mathrm{U}(\mathfrak{g}^*) \cong \mathrm{Sym}(\mathfrak{g}^*)$$

inducing the identity on the primitives. Since both $\mathbb{P}_1$-coalgebras are concentrated in degree zero, this boils down to an *isomorphism* of coaugmented $\mathbb{P}_1$-coalgebras which is the identity on $\mathfrak{g}^*$. But $\mathrm{U}(\mathfrak{g}^*)$ is the Poisson coalgebra of distributions on $\widehat{G^*}_{\{e\}}$ and $\mathrm{Sym}(\mathfrak{g}^*)$ is the Poisson coalgebra of distributions on $\widehat{\mathfrak{g}^*}_{\{0\}}$. □

3. Poisson groupoids

Let $\mathcal{G} \rightrightarrows X$ be a groupoid with associated Lie algebroid $\mathcal{L}$ on X. Recall that $\mathcal{G}$ is called a ***Poisson groupoid*** if we are given a multiplicative Poisson structure π on $\mathcal{G}$. Similarly, it is called a ***quasi-Poisson groupoid*** if we are given a multiplicative bivector π on $\mathcal{G}$ and a trivector $\phi \in \Gamma(X, \wedge^3\mathcal{L})$ satisfying some equations generalizing those of quasi-Poisson groups. Infinitesimal versions of quasi-Poisson groupoids are known as ***quasi-Lie bialgebroids***. The goal of this section is to relate these notions to shifted Poisson structures.

3.1. **Quasi-Lie bialgebroids.** Suppose X is a smooth affine scheme and $\mathcal{G} \rightrightarrows X$ is a groupoid with associated Lie algebroid $\mathcal{L}$. Our assumption in this section will be that $\mathcal{G}$ is also an affine scheme with the source map being smooth. In particular, $\mathcal{G}$ is smooth itself. Let us recall the definition of quasi-Lie bialgebroids from [IPLGX12].

Definition 3.1. An ***almost p-differential on*** $\mathcal{L}$ is a pair of k-linear maps

$$\delta\colon \mathcal{O}(X) \to \Gamma(X, \wedge^{p-1}\mathcal{L}), \qquad \delta\colon \Gamma(X, \mathcal{L}) \to \Gamma(X, \wedge^p\mathcal{L})$$

satisfying

$$\begin{aligned} \delta(fg) &= \delta(f)g + f\delta(g) && f, g \in \mathcal{O}(X) \\ \delta(fv) &= \delta(f) \wedge v + f\delta(v) && f \in \mathcal{O}(X), v \in \Gamma(X, \mathcal{L}). \end{aligned}$$

The algebra $\Gamma(X, \mathrm{Sym}(\mathcal{L}[-1]))$ has a natural Gerstenhaber bracket that we denote by $[\![-,-]\!]$.

Definition 3.2. A ***p-differential on*** $\mathcal{L}$ is an almost p-differential satisfying

$$\begin{aligned} \delta[\![v, f]\!] &= [\![\delta v, f]\!] + [\![v, \delta f]\!] && f \in \mathcal{O}(X), v \in \Gamma(X, \mathcal{L}) \\ \delta[\![v, w]\!] &= [\![\delta v, w]\!] + [\![v, \delta w]\!] && v, w \in \Gamma(X, \mathcal{L}) \end{aligned}$$

Given $\lambda \in \Gamma(X, \wedge^p\mathcal{L})$ we have a p-differential $\mathrm{ad}(\lambda) = [\![\lambda, -]\!]$.

Definition 3.3. The ***groupoid*** $\mathrm{MultBivec}(\mathcal{L})$ ***of multiplicative bivector fields on*** $\mathcal{L}$ is defined as follows:

- Its objects are 2-differentials δ on $\mathcal{L}$.
- Morphisms from δ to δ' are given by elements $\lambda \in \Gamma(X, \wedge^2\mathcal{L})$ such that

$$\delta' = \delta + \mathrm{ad}(\lambda).$$

Definition 3.4. The ***groupoid*** $\mathrm{QLieBialgd}(\mathcal{L})$ ***of quasi-Lie bialgebroid structures on*** $\mathcal{L}$ is defined as follows:

- Its objects are pairs (δ, Ω), where δ is a 2-differential on $\mathcal{L}$ and $\Omega \in \Gamma(X, \wedge^3\mathcal{L})$ which satisfy the following equations:

$$\frac{1}{2}[\delta, \delta] = \mathrm{ad}(\Omega)$$
$$\delta\Omega = 0.$$

- Morphisms are given by elements $\lambda \in \Gamma(X, \wedge^2\mathcal{L})$ which identify two quasi-Lie bialgebroid structures by a twist (see [IPLGX12, Section 4.4]).

By construction there is an obvious forgetful map $\mathrm{QLieBialgd}(\mathcal{L}) \to \mathrm{MultBivec}(\mathcal{L})$.

Remark 3.5. The definition of quasi-Lie bialgebroids we use is given in [IPLGX12, Definition 4.6]. They were originally introduced by Roytenberg in [Roy02], but note that what Roytenberg calls a quasi-Lie bialgebroid is dual to that of [IPLGX12].

Given a groupoid $\mathcal{G}$ over X we have a notion of a left $\mathcal{G}$-space which is a space $Y \to X$ equipped with an associative action map $\mathcal{G} \times_X Y \to Y$. Note that since the morphism $\mathcal{G} \to X$ is smooth, we may consider the underived fiber product. For example, the groupoid $\mathcal{G}$ itself is a left (resp. right) $\mathcal{G}$-space if we consider $\mathcal{G} \to X$ to be the source (resp. target) map.

We consider the following associated spaces:

- The groupoid $\widehat{\mathcal{G}} \rightrightarrows X$ is the formal completion of $\mathcal{G}$ along the unit section.
- We denote $[X/\mathcal{L}] = [X/\widehat{\mathcal{G}}]$.
- Observe that $\mathcal{G}$ equipped with the target map is a right $\widehat{\mathcal{G}}$-space. Let $[\mathcal{G}/\mathcal{L}] = [\mathcal{G}/\widehat{\mathcal{G}}]$. The source map defines a projection $[\mathcal{G}/\mathcal{L}] \to X$ and the target map defines a projection $[\mathcal{G}/\mathcal{L}] \to [X/\mathcal{L}]$. The space $[\mathcal{G}/\mathcal{L}] \to X$ is a left $\mathcal{G}$-space. Moreover, we have an equivalence
$$[\mathcal{G}\backslash[\mathcal{G}/\mathcal{L}]] \cong [X/\mathcal{L}].$$

Let $\mathcal{M}$ be an $\mathcal{L}$-module, i.e. an $\mathcal{O}(X)$-module equipped with a compatible action of $\Gamma(X, \mathcal{L})$. We have the Chevalley–Eilenberg graded mixed cdga
$$\mathrm{C}^\epsilon(\mathcal{L}, \mathcal{M}) = \mathrm{Sym}_{\mathcal{O}(X)}(\mathcal{L}^*[-1]) \otimes_{\mathcal{O}(X)} \mathcal{M}$$
with the zero differential and the mixed structure generalizing the mixed structure on the Chevalley–Eilenberg complex $\mathrm{C}^\epsilon(\mathfrak{g}, \mathcal{M})$ of a Lie algebra. Let
$$\mathrm{C}^\epsilon(\mathcal{L}) = \mathrm{C}^\epsilon(\mathcal{L}, \mathcal{O}_X).$$

Denote by $\Omega^\epsilon(\mathcal{G}/t)$ the de Rham complex of forms on $\mathcal{G}$ relative to the target map $t\colon \mathcal{G} \to X$. Identifying $\mathcal{L}$ with right-invariant vector fields we get an isomorphism of graded cdgas
$$\Omega^\epsilon(\mathcal{G}/t) \cong \mathrm{C}^\epsilon(\mathcal{L}, t_*\mathcal{O}(\mathcal{G})).$$

Lemma 3.6. *The isomorphism of graded cdgas*
$$\Omega^\epsilon(\mathcal{G}/t) \cong \mathrm{C}^\epsilon(\mathcal{L}, t_*\mathcal{O}(\mathcal{G}))$$
is compatible with the mixed structures.

Using this Lemma we can now describe $\mathbb{D}([X/\mathcal{L}])$.

Theorem 3.7. *One has an equivalence of graded mixed cdgas*
$$\mathbb{D}([X/\mathcal{L}]) \cong \mathrm{C}^\epsilon(\mathcal{L})$$
compatible with the actions of $[\mathcal{L}\backslash\mathcal{G}/\mathcal{L}]$ on both sides.

Proof. The proof is similar to the proof of Theorem 1.15.

Observe that $\mathbb{D}(\mathcal{G}) \cong \mathcal{O}(\mathcal{G})$ is commutative Hopf algebroid over $\mathcal{O}(X)$ in the sense of [Rav86, Definition A1.1.1]. The fact that $[\mathcal{G}/\mathcal{L}]$ is a left $\mathcal{G}$-space translates on the level of functions to a coaction map
$$\mathbb{D}([\mathcal{G}/\mathcal{L}]) \longrightarrow \mathcal{O}(\mathcal{G}) \otimes_{\mathcal{O}(X)} \mathbb{D}([\mathcal{G}/\mathcal{L}])$$
where we use that both $\mathcal{G}$ and X are reduced, so $\mathbb{D}(\mathcal{G}) \cong \mathcal{O}(\mathcal{G})$.

Using the isomorphism
$$[X/\mathcal{L}] \cong [\mathcal{G}\backslash[\mathcal{G}/\mathcal{L}]]$$
we can identify $\mathbb{D}([X/\mathcal{L}]) \cong \mathbb{D}([\mathcal{G}/\mathcal{L}])^{\mathcal{G}}$. By Proposition 1.1 we can identify $\mathbb{D}([\mathcal{G}/\mathcal{L}]) \cong \Omega^\epsilon(\mathcal{G}/t)$ and by Lemma 3.6 we identify $\Omega^\epsilon(\mathcal{G}/t) \cong \mathrm{C}^\epsilon(\mathcal{L}, t_*\mathcal{O}(\mathcal{G}))$. We can identify $\mathcal{G}$-invariants on $t_*\mathcal{O}(\mathcal{G})$ with $\mathcal{O}([\mathcal{G}\backslash\mathcal{G}]) \cong \mathcal{O}(X)$, so we finally get
$$\mathbb{D}([X/\mathcal{L}]) \cong \mathrm{C}^\epsilon(\mathcal{L}, \mathcal{O}(X)).$$
∎

The following notion was introduced in [AC12].

Definition 3.8. A ***representation of $\mathcal{L}$ up to homotopy*** is a $\mathrm{C}^\epsilon(\mathcal{L})$-module $\mathcal{E}$ which is equivalent to $\mathrm{C}^\epsilon(\mathcal{L}) \otimes_{\mathcal{O}(X)} \mathcal{M}$ as a graded $\mathrm{C}^\epsilon(\mathcal{L})$-module for some $\mathcal{O}(X)$-module $\mathcal{M}$. A representation up to homotopy is ***perfect*** if $\mathcal{M}$ is perfect.

Note that the underlying complex $\mathcal{M}$ is uniquely determined to be the weight 0 part of the $\mathrm{C}^\epsilon(\mathcal{L})$-module $\mathcal{E}$. Given a representation up to homotopy $\mathcal{E}$ with underlying complex $\mathcal{M}$ we denote

$$\mathrm{C}^\bullet(\mathcal{L}, \mathcal{M}) = |\mathcal{E}|.$$

Representations of $\mathcal{L}$ up to homotopy form a dg category $\mathrm{Rep}\,\mathcal{L}$ defined as the full subcategory of $\mathrm{Mod}^{gr,\epsilon}_{\mathrm{C}^\epsilon(\mathcal{L})}$. We denote by $\mathrm{Rep}_{perf}\,\mathcal{L} \subset \mathrm{Rep}\,\mathcal{L}$ the full subcategory consisting of perfect complexes.

Remark 3.9. There is a fully faithful functor from the derived category of $\mathcal{L}$-representations to $\mathrm{Rep}\,\mathcal{L}$. We expect it to be an equivalence, but we will not use this statement in the paper.

Proposition 3.10. *We have an equivalence of dg categories*

$$\mathrm{Perf}([X/\mathcal{L}]) \cong \mathrm{Rep}_{perf}\,\mathcal{L}.$$

Under this equivalence we have the following:

- *The pullback functor* $\mathrm{Perf}([X/\mathcal{L}]) \to \mathrm{Perf}(X)$ *is the forgetful functor* $\mathrm{Rep}_{perf}\,\mathcal{L} \to \mathrm{Perf}(X)$.
- *The pushforward functor* $\mathrm{Perf}([X/\mathcal{L}]) \to \mathrm{Vect}$ *is given by* $\mathcal{E} \mapsto \mathrm{C}^\bullet(\mathcal{L}, \mathcal{M})$.

Proof. The stack $[X/\mathcal{L}]$ is an algebraisable affine formal derived stack, so by [Cal+17, Theorem 2.2.2] we can identify $\mathrm{Perf}([X/\mathcal{L}])$ with the full subcategory $\mathrm{Mod}^{gr,\epsilon,perf}_{\mathrm{C}^\epsilon(\mathcal{L})}$ of graded mixed $\mathrm{C}^\epsilon(\mathcal{L})$-modules $\mathcal{E}$ which are equivalent to $\mathrm{C}^\epsilon(\mathcal{L}) \otimes_{\mathcal{O}(X)} \mathcal{M}$ as graded $\mathrm{C}^\epsilon(\mathcal{L})$-modules for some perfect complex $\mathcal{M}$ on X. This proves the first claim.

The pullback $\mathrm{Perf}([X/\mathcal{L}]) \to \mathrm{Perf}(X)$ sends the $\mathrm{C}^\epsilon(\mathcal{L})$-module $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathrm{C}^\epsilon(\mathcal{L})} \mathcal{O}(X)$, i.e. it corresponds to taking the weight 0 part of $\mathcal{E}$.

The pushforward $\mathrm{Perf}([X/\mathcal{L}]) \to \mathrm{Vect}$ is given by $\mathcal{E} \mapsto \mathrm{Hom}(\mathcal{O}_X, \mathcal{E})$. The latter complex is equivalent to $|\mathcal{E}| \cong \mathrm{C}^\bullet(\mathcal{L}, \mathcal{M})$. ☐

We are now going to describe 1-shifted Poisson structures on $[X/\mathcal{L}]$. As a first step, we will need a description of $\mathrm{Pol}([X/\mathcal{L}], 1)$.

Proposition 3.11. *The graded dg Lie algebra of polyvectors* $\mathrm{Pol}([X/\mathcal{L}], 1)$ *has the following description.*

1. *The complex* $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ *is concentrated in degrees* $\geq p$.
2. *An element of* $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ *of degree* p *is an element of* $\Gamma(X, \wedge^p \mathcal{L})$.
3. *An element of* $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ *of degree* $p+1$ *is an almost* p*-differential on* $\mathcal{L}$.
4. *The differential on* $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ *from degree* p *to degree* $p+1$ *is given by* $\lambda \mapsto \mathrm{ad}(\lambda)$.
5. *The closed elements of* $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ *of degree* $p+1$ *are* p*-differentials on* $\mathcal{L}$.

Proof. Recall that $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ is the complex of p-multiderivations of $\mathbb{D}([X/\mathcal{L}]) \cong \mathrm{C}^\epsilon(\mathcal{L})$. Since $\mathcal{O}(X)$ is smooth and $\mathrm{C}^\epsilon(\mathcal{L})$ is freely generated as an $\mathcal{O}(X)$-algebra, we may consider strict multiderivations. As a graded commutative algebra, $\mathrm{C}^\epsilon(\mathcal{L})$ is generated by $\mathcal{O}(X)$ in degree 0 and $\Gamma(X, \mathcal{L}^*)$ in degree 1. Therefore, p-multiderivations are uniquely determined by their values on these spaces. In other words, an element of $\mathrm{Pol}^p([X/\mathcal{L}], 1)$ of degree $p+n$ is given by a collection of maps

$$\alpha^{(l)}\colon \mathrm{Sym}^l(\mathcal{O}(X)) \otimes_k \wedge^{p-l}\Gamma(X, \mathcal{L}^*) \to \Gamma(X, \wedge^{n-l}\mathcal{L}^*)$$

which together satisfy the Leibniz rule. Here $0 \leq l \leq p$ and $l \leq n$. In particular, these are nonzero only for $n \geq 0$ which proves the first statement.

For $n = 0$ only $\alpha^{(0)}$ is nonzero which gives an $\mathcal{O}(X)$-linear map $\Gamma(X, \wedge^p\mathcal{L}^*) \to \mathcal{O}(X)$, i.e. $\alpha^{(0)} \in \Gamma(X, \wedge^p\mathcal{L})$.

For $n = 1$ the nontrivial maps are

$$\begin{aligned}\alpha^{(0)}&\colon \wedge^p\,\Gamma(X, \mathcal{L}^*) \to \Gamma(X, \mathcal{L}^*)\\ \alpha^{(1)}&\colon \mathcal{O}(X) \otimes \wedge^{p-1}\Gamma(X, \mathcal{L}^*) \to \mathcal{O}(X).\end{aligned}$$

The Leibniz rule implies that $\alpha^{(1)} \in \Gamma(X, \wedge^{p-1}\mathcal{L} \otimes \mathrm{T}_X)$ and

$$\alpha^{(0)}(l_1, \dots, f l_p) = f\alpha^{(0)}(l_1, \dots, f l_p) + \iota_{\mathrm{d_{dR}}f}\alpha^{(1)}(l_1, \dots, l_{p-1}) \wedge l_p \tag{18}$$

for $f \in \mathcal{O}(X)$ and $l_1, \dots, l_p \in \Gamma(X, \mathcal{L}^*)$.

Using these maps we define a p-differential on $\mathcal{L}$ as follows. For $f \in \mathcal{O}(X)$ let

$$\delta(f) = \iota_{\mathrm{d_{dR}}f}\alpha^{(1)}.$$

For $s \in \Gamma(X, \mathcal{L})$ and $l_1, \dots, l_p \in \Gamma(X, \mathcal{L}^*)$ let

$$\langle l_1 \wedge \dots \wedge l_p, \delta(s)\rangle = \langle \alpha^{(0)}(l_1, \dots, l_p), s\rangle - \sum_{a=1}^{p}(-1)^{p-a}\iota_{\mathrm{d_{dR}}l_a(s)}\alpha^{(1)}(l_1, \dots, \hat{l}_a, \dots, l_p).$$

From (18) it is easy to see that the right-hand side is $\mathcal{O}(X)$-linear in $l_1, \dots, l_p$, so defines a map $\delta\colon \Gamma(X, \mathcal{L}) \to \Gamma(X, \wedge^p\mathcal{L})$. This establishes an isomorphism between pairs $\{\alpha^{(0)}, \alpha^{(1)}\}$ and almost p-differentials.

The differential on $\mathrm{Pol}([X/\mathcal{L}], 1)$ comes from the mixed structure on $\mathrm{C}^\epsilon(\mathcal{L})$, i.e. the Chevalley–Eilenberg differential. Then the last two statements are obtained by a straightforward calculation. □

Remark 3.12. We will not use a description of elements of $\mathrm{Pol}([X/\mathcal{L}], 1)$ of higher degrees, but we refer to [CM08] for the full description of the weight 1 part of $\mathrm{Pol}([X/\mathcal{L}], 1)$.

Remark 3.13. In [Bon+18, Section 2.1] the authors define a graded Lie 2-algebra $\mathcal{V}(\mathcal{G})$ of polyvector fields on a groupoid $\mathcal{G} \rightrightarrows X$. One may similarly define the graded Lie 2-algebra $\mathcal{V}(\mathcal{L})$ of polyvector fields on a Lie algebroid $\mathcal{L}$. Let $\tau\mathrm{Pol}([X/\mathcal{L}], 1)$ be the truncation where we truncate the weight p part in degrees $\leq (p+1)$. Identifying graded Lie 2-algebras with graded dg Lie algebras whose weight p part is concentrated in degree $p-2$ and $p-1$, we get an isomorphism of graded dg Lie algebras $\mathcal{V}(\mathcal{L}) \to (\tau\mathrm{Pol}([X/\mathcal{L}], 1))[2]$.

Corollary 3.14. *The space* $\underline{\mathrm{MC}}(\mathrm{Pol}^2([X/\mathcal{L}], 1)[2])$ *of Maurer–Cartan elements in the abelian dg Lie algebra* $\mathrm{Pol}^2([X/\mathcal{L}], 1)[2]$ *is equivalent to the groupoid* $\mathrm{MultBivec}(\mathcal{L})$ *of multiplicative bivectors on* $\mathcal{L}$.

Theorem 3.15. *Suppose* $\mathcal{L}$ *is the Lie algebroid of the groupoid* $\mathcal{G} \rightrightarrows X$. *Then the space* $\mathrm{Pois}([X/\mathcal{L}], 1)$ *of* 1*-shifted Poisson structures on* $[X/\mathcal{L}]$ *is equivalent to the groupoid* $\mathrm{QLieBialgd}(\mathcal{L})$ *of quasi-Lie bialgebroid structures on* $\mathcal{L}$.

Proof. The proof of this statement is completely analogous to the proof of Theorem 2.8, so we omit it. ☐

Corollary 3.16. *With notations as before, the space* $\mathrm{Cois}(X \to [X/\mathcal{L}], 1)$ *of* 1*-shifted coisotropic structures on the projection* $X \to [X/\mathcal{L}]$ *is equivalent to the set of 2-differentials* δ *on* $\mathcal{L}$ *endowing it with the structure of a Lie bialgebroid.*

Proof. As for Proposition 2.10, the claim follows from the description of $\mathrm{Pol}(X \to [X/\mathcal{L}], 1)$ as the kernel of

$$\mathrm{Pol}([X/\mathcal{L}], 1) \longrightarrow \mathrm{Pol}(X, 1).$$

Indeed, Maurer–Cartan elements in the kernel have $\Omega = 0$ and homotopies have $\lambda = 0$, i.e. all of them are given by the identity. ☐

Remark 3.17. The statement of Corollary 3.16 extends verbatim to non-affine schemes X by using Zariski descent for shifted Poisson structures. One can similarly apply descent to compute $\mathrm{Pois}([X/\mathcal{L}], 1)$ in the case of a non-affine X, but then one discovers that such "non-affine quasi-Lie bialgebroids" have a twisting class in $\mathrm{H}^1(X, \wedge^2\mathcal{L})$ which obstructs the existence of a global cobracket.

3.2. **Quasi-Poisson groupoids.** Fix a smooth affine algebraic groupoid $\mathcal{G} \rightrightarrows X$ as before with associated Lie algebroid $\mathcal{L}$. We have an embedding $\mathcal{G} \times_X \mathcal{G} \hookrightarrow \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ where the first two maps are the projections are the third map is the composition.

Definition 3.18. A bivector $\Pi \in \Gamma(\mathcal{G}, \wedge^2\mathrm{T}_{\mathcal{G}})$ is ***multiplicative*** if the embedding $\mathcal{G} \times_X \mathcal{G} \hookrightarrow \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is coisotropic with respect to $\Pi \oplus \Pi \oplus (-\Pi)$.

Given $\lambda \in \Gamma(X, \wedge^2\mathcal{L})$, we can extend it to left- and right-invariant bivectors $\lambda^L, \lambda^R \in \Gamma(\mathcal{G}, \wedge^2\mathrm{T}_{\mathcal{G}})$. The difference $\lambda^R - \lambda^L$ is then a multiplicative bivector on $\mathcal{G}$.

Definition 3.19. The ***groupoid*** $\mathrm{MultBivec}(\mathcal{G})$ ***of multiplicative bivector fields on*** $\mathcal{G}$ is defined as follows:

- Its objects are multiplicative bivector fields Π on $\mathcal{G}$.
- Morphisms from Π to Π' are given by $\lambda \in \Gamma(X, \wedge^2\mathcal{L})$ such that
$$\Pi' = \Pi + \lambda^R - \lambda^L.$$

Definition 3.20. The ***groupoid*** $\mathrm{QPoisGpd}(\mathcal{G})$ ***of quasi-Poisson groupoid structures on*** $\mathcal{G}$ is defined as follows:

- Its objects are pairs (Π, Ω), where Π is a multiplicative bivector field on $\mathcal{G}$ and $\Omega \in \Gamma(X, \wedge^3\mathcal{L})$ which satisfy the following equations:
$$\frac{1}{2}[\Pi, \Pi] = \Omega^R - \Omega^L$$
$$[\Pi, \Omega^R] = 0$$
- Morphisms are given by elements $\lambda \in \Gamma(X, \wedge^2\mathcal{L})$ which identify two quasi-Poisson groupoids by a twist (see [IPLGX12, Section 4.4]).

By construction there is an obvious forgetful map $\mathrm{QPoisGpd}(\mathcal{G}) \to \mathrm{MultBivec}(\mathcal{G})$.

In section 1.5 we have used the isomorphisms

$$[G\backslash G_{\mathrm{dR}}] \cong \mathrm{B}\mathfrak{g}, \qquad [\mathrm{B}\mathfrak{g}/G_{\mathrm{dR}}] \cong \mathrm{B}G$$

to compute $\mathrm{Pol}(\mathrm{B}G, n)$. In the groupoid setting these are generalized to isomorphisms

$$[\mathcal{G}\backslash[\mathcal{G}/\mathcal{L}]] \cong [X/\mathcal{L}], \qquad \left[{}^{[X/\mathcal{L}]}\big/_{[\mathcal{L}\backslash\mathcal{G}/\mathcal{L}]}\right] \cong [X/\mathcal{G}]$$

which can be used to compute $\mathrm{Pol}([X/\mathcal{G}], n)$. We will now present a different way to compute $\mathrm{Pol}([X/\mathcal{G}], 1)$.

Lemma 3.21. *Let $f\colon X \to Y$ be an epimorphism of derived stacks. Suppose $\mathcal{E}_X$ and $\mathcal{E}_Y$ are perfect complexes on X and Y respectively together with a map $f^*\mathcal{E}_Y \to \mathcal{E}_X$ such that the homotopy fiber of the dual map $\mathcal{E}_X^* \to f^*\mathcal{E}_Y^*$ is connective. Then the map*

$$\mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}_X) \longrightarrow \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{E}_Y)$$

is an epimorphism.

Proof. The map $f\colon X \to Y$ being an epimorphism means that for any derived affine scheme S and a map $g\colon S \to Y$ there is an étale cover $p\colon \tilde{S} \to S$ and a commutative diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{g}"] \arrow[d, "p"] & X \arrow[d, "f"] \\ S \arrow[r, "g"] & Y \end{tikzcd}$$

We have to show that the same claim is true for

$$E_X^* = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}_X) \longrightarrow E_Y^* = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{E}_Y).$$

Let S be a derived affine scheme. A morphism $S \to E_Y^*$ is the same as a pair (g, s) of a morphism $g\colon S \to Y$ and an element $s \in \mathrm{Map}_{\mathrm{QCoh}(S)}(g^*\mathcal{E}_Y, \mathcal{O}_S)$. Since $f\colon X \to Y$ is an epimorphism, we can lift g to $\tilde{g}\colon \tilde{S} \to X$ as above. Let K be the homotopy fiber of $\mathcal{E}_X^* \to f^*\mathcal{E}_Y^*$. Since it is connective, so is $\Gamma(\tilde{S}, \tilde{g}^*K)$. From the fiber sequence

$$\Gamma(\tilde{S}, \tilde{g}^*K) \longrightarrow \Gamma(\tilde{S}, \tilde{g}^*\mathcal{E}_X^*) \longrightarrow \Gamma(\tilde{S}, \tilde{g}^*f^*\mathcal{E}_Y^*)$$

we deduce that the projection

$$\mathrm{Map}_{\mathrm{QCoh}(\tilde{S})}(\tilde{g}^*\mathcal{E}_X, \mathcal{O}_{\tilde{S}}) \longrightarrow \mathrm{Map}_{\mathrm{QCoh}(\tilde{S})}(\tilde{g}^*f^*\mathcal{E}_Y, \mathcal{O}_{\tilde{S}}) \cong \mathrm{Map}_{\mathrm{QCoh}(\tilde{S})}(p^*g^*\mathcal{E}_Y, \mathcal{O}_{\tilde{S}})$$

is essentially surjective and hence p^*s admits a lift. □

Corollary 3.22. *Let $n \geq 1$. Let $L^*[n-1] = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}[1-n]) \to X$ be the total space of the perfect complex $\mathcal{L}^*[n-1]$. The cotangent stack $\mathrm{T}^*[n]([X/\mathcal{G}])$ is equivalent to the quotient of the smooth groupoid*

$$\mathrm{T}^*[n-1](\mathcal{G}) \rightrightarrows L^*[n-1].$$

Proof. Consider the map $X \to [X/\mathcal{G}]$. Its n-shifted conormal bundle gives a morphism

$$L^*[n-1] \to \mathrm{T}^*[n]([X/\mathcal{G}]).$$

Let $Y = [X/\mathcal{G}]$, $\mathcal{E}_X = \mathcal{L}[n-1]$ and $\mathcal{E}_Y = \mathbb{T}_{[X/\mathcal{G}]}[-n]$. The relative tangent complex of $X \to [X/\mathcal{G}]$ fits into a fiber sequence

$$\mathcal{L} \longrightarrow \mathbb{T}_X \longrightarrow f^*\mathbb{T}_{[X/\mathcal{G}]}$$

which gives the required morphism $f^*\mathbb{T}_{[X/\mathcal{G}]} \to \mathcal{L}[1]$. Taking the duals and rotating, we obtain a fiber sequence

$$\mathbb{L}_X[n-1] \longrightarrow \mathcal{L}^*[n-1] \longrightarrow f^*\mathbb{L}_{[X/\mathcal{G}]}[n],$$

so the fiber of $\mathcal{E}_X^* \to f^*\mathcal{E}_Y^*$ is equivalent to $\mathbb{L}_X[n-1]$ which is connective since $n \geq 1$. Therefore, by Lemma 3.21 the morphism

$$L^*[n-1] \longrightarrow \mathrm{T}^*[n]([X/\mathcal{G}])$$

is an epimorphism and hence is equivalent to the geometric realization of its nerve.

For any map $X_1 \to X_2$ of derived stacks, the self-intersection of the conormal bundle $\mathrm{N}^*(X_1/X_2) \to \mathrm{T}^*X_2$ is equivalent to $\mathrm{T}^*[-1](X_1 \times_{X_2} X_1)$. In our case we get

$$L^*[n-1] \times_{\mathrm{T}^*[n]([X/\mathcal{G}])} L^*[n-1] \cong \mathrm{T}^*[n-1](\mathcal{G})$$

which gives the required groupoid. $\square$

Proposition 3.23. *The space* $\underline{\mathrm{MC}}(\mathrm{Pol}^2([X/\mathcal{G}],1)[2])$ *of Maurer–Cartan elements in the abelian dg Lie algebra* $\mathrm{Pol}^2([X/\mathcal{G}],1)[2]$ *is equivalent to the groupoid* $\mathrm{MultBivec}(\mathcal{G})$ *of multiplicative bivector fields on* $\mathcal{G}$.

Proof. The complex $\mathrm{Pol}([X/\mathcal{G}],1)$ is given by the algebra of functions on $\mathrm{T}^*[2][X/\mathcal{G}]$. By Corollary 3.22 we can compute it as a totalization of

$$\mathcal{O}(L^*[1]) \rightrightarrows \mathcal{O}(\mathrm{T}^*[1]\mathcal{G}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \dots$$

Thus, $\underline{\mathrm{MC}}(\mathrm{Pol}^2([X/\mathcal{G}],1)[2])$ is the totalization of

$$\underline{\mathrm{MC}}(\Gamma(X,\wedge^2\mathcal{L})) \rightrightarrows \underline{\mathrm{MC}}(\Gamma(\mathcal{G},\wedge^2\mathrm{T}_{\mathcal{G}})) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \dots$$

Here the complexes in each term are concentrated in cohomological degree 0, so by Proposition 1.2 we get that $\underline{\mathrm{MC}}(\Gamma(X,\wedge^2\mathcal{L})) = */\Gamma(X,\wedge^2\mathcal{L})$, i.e. the one-object groupoid with $\Gamma(X,\wedge^2\mathcal{L})$ as the automorphism group of the object, and similarly for $\underline{\mathrm{MC}}(\Gamma(\mathcal{G},\wedge^2\mathrm{T}_{\mathcal{G}}))$.

By [IPLGX12, Proposition 2.7] the multiplicativity condition for a bivector on $\mathcal{G}$ is equivalent to the condition that the corresponding linear function on the groupoid $\mathrm{T}^*\mathcal{G} \times_{\mathcal{G}} \mathrm{T}^*\mathcal{G} \rightrightarrows L^* \times_X L^*$ is a 1-cocycle. Therefore, using Lemma 1.4 we conclude that the totalization has objects given by multiplicative bivectors on $\mathcal{G}$ and morphisms by elements $\lambda \in \Gamma(X,\wedge^2\mathcal{L})$. $\square$

Observe that Theorems 2.8 and 2.9 imply that, if G is connected, we have a Cartesian diagram of spaces

$$\begin{tikzcd} \mathrm{Pois}(\mathrm{B}G,1) \arrow[r] \arrow[d] & \mathrm{Pois}(\mathrm{B}\mathfrak{g},1) \arrow[d] \\ \underline{\mathrm{MC}}(\mathrm{C}^\bullet(G,\wedge^2\mathfrak{g})) \arrow[r] & \underline{\mathrm{MC}}(\mathrm{C}^\bullet(\mathfrak{g},\wedge^2\mathfrak{g})) \end{tikzcd}$$

where at the bottom we consider Maurer–Cartan spaces of abelian dg Lie algebras. In other words, a 1-shifted Poisson structure on $\mathrm{B}G$ is the same as a 1-shifted Poisson structure on $\mathrm{B}\mathfrak{g}$ whose bivector is integrable. We will now show that the same statement is true in the groupoid setting. Let us first state some preliminary lemmas.

Given a graded dg Lie algebra $\mathfrak{g}$ we denote by $\mathfrak{g}(n)$ its weight n component.

Lemma 3.24. *Suppose $\mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of graded dg Lie algebras such that for any $n \geq 3$ the fiber of $\mathfrak{g}_1(n) \to \mathfrak{g}_2(n)$ is concentrated in cohomological degrees ≥ 3.*

Then the diagram of spaces

$$\begin{tikzcd} \underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}) \arrow[r] \arrow[d] & \underline{\mathrm{MC}}(\mathfrak{g}_2^{\geq 2}) \arrow[d] \\ \underline{\mathrm{MC}}(\mathfrak{g}_1(2)) \arrow[r] & \underline{\mathrm{MC}}(\mathfrak{g}_2(2)). \end{tikzcd}$$

is Cartesian.

Proof. We will use the obstruction-theoretic argument given in [Pri17, Section 1.4]. By definition

$$\underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}) = \lim_n \underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}/\mathfrak{g}_1^{\geq n})$$

and the result follows once we prove that

$$\begin{tikzcd} \underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}/\mathfrak{g}_1^{\geq n}) \arrow[r] \arrow[d] & \underline{\mathrm{MC}}(\mathfrak{g}_2^{\geq 2}/\mathfrak{g}_2^{\geq n}) \arrow[d] \\ \underline{\mathrm{MC}}(\mathfrak{g}_1(2)) \arrow[r] & \underline{\mathrm{MC}}(\mathfrak{g}_2(2)) \end{tikzcd}$$

is Cartesian for any $n \geq 3$. We prove the claim by induction. Indeed, the claim is tautologically true for $n = 3$.

We prove the inductive step as follows. We have a sequence of simplicial sets

$$\underline{\mathrm{MC}}(\mathfrak{g}_i^{\geq 2}/\mathfrak{g}_i^{\geq (n+1)}) \longrightarrow \underline{\mathrm{MC}}(\mathfrak{g}_i^{\geq 2}/\mathfrak{g}_i^{\geq n}) \longrightarrow \underline{\mathrm{MC}}(\mathfrak{g}_i(n)[-1]),$$

where we treat $\mathfrak{g}_i(n)[-1]$ as an abelian dg Lie algebra.

By [Pri17, Proposition 1.29] the homotopy fiber of $\underline{\mathrm{MC}}(\mathfrak{g}_i^{\geq 2}/\mathfrak{g}_i^{\geq n}) \to \underline{\mathrm{MC}}(\mathfrak{g}_i(n)[-1])$ at $0 \in \underline{\mathrm{MC}}(\mathfrak{g}_i(n)[-1])$ is equivalent to $\underline{\mathrm{MC}}(\mathfrak{g}_i^{\geq 2}/\mathfrak{g}_i^{\geq (n+1)})$. By assumptions the map

$$\underline{\mathrm{MC}}(\mathfrak{g}_1(n)[-1]) \to \underline{\mathrm{MC}}(\mathfrak{g}_2(n)[-1])$$

is an equivalence of connected components of the trivial Maurer–Cartan element for $n \geq 3$. Therefore, the square

$$\begin{tikzcd} \underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}/\mathfrak{g}_1^{\geq (n+1)}) \arrow[r] \arrow[d] & \underline{\mathrm{MC}}(\mathfrak{g}_2^{\geq 2}/\mathfrak{g}_2^{\geq (n+1)}) \arrow[d] \\ \underline{\mathrm{MC}}(\mathfrak{g}_1^{\geq 2}/\mathfrak{g}_1^{\geq n}) \arrow[r] & \underline{\mathrm{MC}}(\mathfrak{g}_2^{\geq 2}/\mathfrak{g}_2^{\geq n}) \end{tikzcd}$$

is Cartesian which proves the inductive step. □

We are going to apply the previous statement to the map $\mathrm{Pol}([X/\mathcal{G}], 1) \to \mathrm{Pol}([X/\mathcal{L}], 1)$. First, recall the following basic fact, see e.g. [Cra03, Theorem 4].

Lemma 3.25. *Let $\mathcal{G}$ be a source-connected groupoid and V a $\mathcal{G}$-representation.*

(1) *The restriction map*

$$\mathrm{H}^0(\mathcal{G}, V) \longrightarrow \mathrm{H}^0(\mathcal{L}, V)$$

is an isomorphism.

(2) *The restriction map*
$$\mathrm{H}^1(\mathcal{G}, V) \longrightarrow \mathrm{H}^1(\mathcal{L}, V)$$
is injective.

The same statement is true for hypercohomology.

Corollary 3.26. *Let $\mathcal{G}$ be a source-connected groupoid and $V^\bullet$ a complex of $\mathcal{G}$-representations concentrated in non-negative degrees.*

(1) *The restriction map*
$$\mathrm{H}^0(\mathcal{G}, V^\bullet) \longrightarrow \mathrm{H}^0(\mathcal{L}, V^\bullet)$$
is an isomorphism.

(2) *The restriction map*
$$\mathrm{H}^1(\mathcal{G}, V^\bullet) \longrightarrow \mathrm{H}^1(\mathcal{L}, V^\bullet)$$
is injective.

Proof. In degree 0 we have
$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{G}, V^\bullet) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^0(\mathcal{G}, \mathrm{H}^0(V^\bullet)) \arrow[d, "\sim"] \\
\mathrm{H}^0(\mathcal{L}, V^\bullet) \arrow[r, "\sim"] & \mathrm{H}^0(\mathcal{L}, \mathrm{H}^0(V^\bullet))
\end{tikzcd}$$
so the map $\mathrm{H}^0(\mathcal{G}, V^\bullet) \to \mathrm{H}^0(\mathcal{L}, V^\bullet)$ is an isomorphism.

The hypercohomology spectral sequence gives long exact sequences
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\mathcal{G}, \mathrm{H}^0(V^\bullet)) \arrow[r] \arrow[d, hook] & \mathrm{H}^1(\mathcal{G}, V^\bullet) \arrow[r] \arrow[d] & \mathrm{H}^0(\mathcal{G}, \mathrm{H}^1(V^\bullet)) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^2(\mathcal{G}, \mathrm{H}^0(V^\bullet)) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(\mathcal{L}, \mathrm{H}^0(V^\bullet)) \arrow[r] & \mathrm{H}^1(\mathcal{L}, V^\bullet) \arrow[r] & \mathrm{H}^0(\mathcal{L}, \mathrm{H}^1(V^\bullet)) \arrow[r] & \mathrm{H}^2(\mathcal{L}, \mathrm{H}^0(V^\bullet)).
\end{tikzcd}$$
and the injectivity of $\mathrm{H}^1(\mathcal{G}, V^\bullet) \to \mathrm{H}^1(\mathcal{L}, V^\bullet)$ follows from the five lemma. ☐

Lemma 3.27. *The dg category* $\mathrm{Perf}([X/\mathcal{G}])$ *is equivalent to full subcategory of the derived category $\mathcal{G}$-representations consisting of perfect complexes.*

Proof. The dg category $\mathrm{QCoh}([X/\mathcal{G}])$ has a natural t-structure and let us denote by
$$\mathrm{QCoh}^+([X/\mathcal{G}]) \subset \mathrm{QCoh}([X/\mathcal{G}])$$
the full subcategory of bounded below complexes. By [GR17, Chapter I.3, Proposition 2.4.3] we can identify $\mathrm{QCoh}^+([X/\mathcal{G}])$ with the bounded below derived category of $\mathcal{G}$-representations. The claim then follows by passing to the subcategory of perfect complexes. ☐

Lemma 3.28. *Suppose $\mathcal{G}$ is source-connected. Then the natural diagram of groupoids*
$$\begin{tikzcd}
\mathrm{QPoisGpd}(\mathcal{G}) \arrow[r] \arrow[d] & \mathrm{QLieBialgd}(\mathcal{L}) \arrow[d] \\
\mathrm{MultBivec}(\mathcal{G}) \arrow[r] & \mathrm{MultBivec}(\mathcal{L})
\end{tikzcd}$$
is Cartesian.

Proof. The map $\mathrm{QLieBialgd}(\mathcal{L}) \to \mathrm{MultBivec}(\mathcal{L})$ is a fibration of groupoids (i.e. an isofibration), so it is enough to show that the map $\mathrm{QPoisGpd}(\mathcal{G}) \to \mathrm{QLieBialgd}(\mathcal{L}) \times_{\mathrm{MultBivec}(\mathcal{L})} \mathrm{MultBivec}(\mathcal{G})$ is an equivalence where we consider the strict fiber product. This map is clearly fully faithful.

Objects of both groupoids are pairs (Π, Ω) where $\Pi \in \Gamma(\mathcal{G}, \wedge^2 \mathrm{T}_{\mathcal{G}})$ and $\Omega \in \Gamma(X, \wedge^3 \mathcal{L})$ satisfying certain equations. For $\mathrm{QPoisGpd}(\mathcal{G})$ we have the following three equations:

- The bivector Π on $\mathcal{G}$ is multiplicative.
- We have an equality of multiplicative trivectors $\frac{1}{2}[\Pi, \Pi] = \Omega^R - \Omega^L$ on $\mathcal{G}$.
- We have an equality of fourvectors $[\Pi, \Omega^R] = 0$ on $\mathcal{G}$.

For $\mathrm{QLieBialgd}(\mathcal{L}) \times_{\mathrm{MultBivec}(\mathcal{L})} \mathrm{MultBivec}(\mathcal{G})$ we have the following three equations:

- The bivector Π on $\mathcal{G}$ is multiplicative.
- Let δ be the 2-differential on $\mathcal{L}$ corresponding to Π. Then we have an equality of 3-differentials $\frac{1}{2}[\delta, \delta] = \mathrm{ad}(\Omega)$.
- We have $\delta(\Omega) = 0$.

There is a natural map from the graded Lie algebra of multiplicative multivectors on $\mathcal{G}$ to the graded Lie algebra of differentials on $\mathcal{G}$. Moreover, if $\mathcal{G}$ is source-connected, it is shown in [IPLGX12, Proposition 2.35] that it is injective. Thus, the equation $\frac{1}{2}[\Pi, \Pi] = \Omega^R - \Omega^L$ holds iff $\frac{1}{2}[\delta, \delta] = \mathrm{ad}(\Omega)$.

Moreover, [IPLGX12, Equation (18)] implies that

$$[\Pi, \Omega^R] = (\delta\Omega)^R,$$

so $[\Pi, \Omega^R] = 0$ iff $\delta\Omega = 0$.

Thus, the map $\mathrm{QPoisGpd}(\mathcal{G}) \to \mathrm{QLieBialgd}(\mathcal{L}) \times_{\mathrm{MultBivec}(\mathcal{L})} \mathrm{MultBivec}(\mathcal{G})$ is an isomorphism on objects. □

Theorem 3.29. *Suppose $\mathcal{G} \rightrightarrows X$ is a smooth algebraic groupoid which is source-connected. The space $\mathrm{Pois}([X/\mathcal{G}], 1)$ of 1-shifted Poisson structures on $[X/\mathcal{G}]$ is equivalent to the groupoid $\mathrm{QPoisGpd}(\mathcal{G})$ of quasi-Poisson structures on $\mathcal{G}$.*

Proof. The map $f\colon [X/\mathcal{L}] \to [X/\mathcal{G}]$ is formally étale. Therefore, $f^*\mathbb{T}_{[X/\mathcal{G}]} \cong \mathbb{T}_{[X/\mathcal{L}]}$. By Lemma 3.27 we may identify $\mathrm{Perf}([X/\mathcal{G}]) \cong \mathrm{Rep}\,\mathcal{G}$ and let us denote by $\mathcal{E}$ the complex of $\mathcal{G}$-representations corresponding to $\mathbb{T}_{[X/\mathcal{G}]}$. Under the functor $\mathrm{Rep}\,\mathcal{G} \to \mathrm{Rep}\,\mathcal{L}$ it corresponds to a complex of $\mathcal{L}$-representations which we also denote by $\mathcal{E}$.

Let $p\colon X \to [X/\mathcal{G}]$ be the projection. Then $p^*\mathbb{T}_{[X/\mathcal{G}]}$ fits into a fiber sequence

$$p^*\mathbb{T}_{[X/\mathcal{G}]} \longrightarrow \mathbb{T}_{X/[X/\mathcal{G}]} \cong \mathcal{L} \longrightarrow \mathbb{T}_X.$$

In particular, the $\mathcal{G}$-representation $\mathcal{E}$ is concentrated in degrees ≥ -1.

We can identify

$$\mathrm{Pol}([X/\mathcal{G}], 1) \cong \mathrm{C}^\bullet(\mathcal{G}, \mathrm{Sym}(\mathcal{E}[-2])), \qquad \mathrm{Pol}([X/\mathcal{L}], 1) \cong \mathrm{C}^\bullet(\mathcal{L}, \mathrm{Sym}(\mathcal{E}[-2])).$$

Considering the weight p part, we see that $\mathrm{Sym}(\mathcal{E}[-2])$ is concentrated in degrees $\geq p$. Thus, by Corollary 3.26 the homotopy fiber of $\mathrm{Pol}^p([X/\mathcal{G}],1) \to \mathrm{Pol}^p([X/\mathcal{L}],1)$ is concentrated in degrees $\geq (p+2)$. Therefore, from Lemma 3.24 we get that the diagram of spaces

$$\begin{tikzcd} \underline{\mathrm{MC}}(\mathrm{Pol}([X/\mathcal{G}],1)^{\geq 2}[2]) \arrow[r] \arrow[d] & \underline{\mathrm{MC}}(\mathrm{Pol}([X/\mathcal{L}],1)^{\geq 2}[2]) \arrow[d] \\ \underline{\mathrm{MC}}(\mathrm{Pol}^2([X/\mathcal{G}],1)[2]) \arrow[r] & \underline{\mathrm{MC}}(\mathrm{Pol}^2([X/\mathcal{L}],1)[2]) \end{tikzcd}$$

is Cartesian. Using Theorem 3.15, Corollary 3.14, and Proposition 3.23 we obtain a Cartesian diagram

$$\begin{tikzcd} \mathrm{Pois}([X/\mathcal{G}],1) \arrow[r] \arrow[d] & \mathrm{Pois}([X/\mathcal{L}],1) \arrow[d] \\ \mathrm{MultBivec}(\mathcal{G}) \arrow[r] & \mathrm{MultBivec}(\mathcal{L}) \end{tikzcd}$$

Therefore, from Lemma 3.28 we obtain an equivalence $\mathrm{Pois}([X/\mathcal{G}],1) \cong \mathrm{QPoisGpd}(\mathcal{G})$. □

Corollary 3.30. *With notations as before, the space $\mathrm{Cois}(X \to [X/\mathcal{G}],1)$ of 1-shifted coisotropic structures on the projection $X \to [X/\mathcal{G}]$ is equivalent to the set of multiplicative Poisson bivectors Π on $\mathcal{G}$.*

Proof. The proof is identical to the proof of Corollary 3.16 since $\mathrm{Pol}(X \to [X/\mathcal{G}],1)$ is the kernel of

$$\mathrm{Pol}([X/\mathcal{G}],1) \to \mathrm{Pol}(X,1).$$

□

3.3. **Quasi-symplectic groupoids.** One also has a symplectic analog of quasi-Poisson groupoids which are quasi-symplectic groupoids, i.e. groupoids $\mathcal{G} \rightrightarrows X$ equipped with a multiplicative two-form ω on $\mathcal{G}$ and a three-form H on X satisfying some closure and non-degeneracy equations, see [Xu04, Definition 2.5]. The following statement is well-known.

Proposition 3.31. *Fix a groupoid $\mathcal{G} \rightrightarrows X$. The space of 1-shifted symplectic structures on $[X/\mathcal{G}]$ is equivalent to the following groupoid:*

- *Its objects are pairs (ω, H) endowing $\mathcal{G}$ with a structure of a quasi-symplectic groupoid.*
- *Morphisms $(\omega_1, H_1) \to (\omega_2, H_2)$ are given by two-forms $B \in \Omega^2(X)$ satisfying*

$$\omega_2 - \omega_1 = \mathrm{d}B$$
$$H_2 - H_1 = \mathrm{d}_{\mathrm{dR}} B.$$

Proof. The space of 1-shifted presymplectic structures on $[X/\mathcal{G}]$ is equivalent to the space $\mathcal{A}^{2,cl}([X/\mathcal{G}],1)$ of closed two-forms on $[X/\mathcal{G}]$ of degree 1. The latter space is equivalent to the totalization of the cosimplicial space

$$\mathcal{A}^{2,cl}(X,1) \rightrightarrows \mathcal{A}^{2,cl}(\mathcal{G},1) \;\substack{\to\\[-1ex]\to\\[-1ex]\to}\; \dots$$

If Y is a smooth affine scheme, $\mathcal{A}^{2,cl}(Y,1)$ is equivalent to the following groupoid:

- Its objects are closed three-forms H on Y,

- Its morphisms from H_1 to H_2 are given by two-forms B satisfying

$$\mathrm{d}_{\mathrm{dR}} B = H_2 - H_1.$$

Applying Lemma 1.4 we deduce that the space of 1-shifted presymplectic structures on $[X/\mathcal{G}]$ is equivalent to the following groupoid:

- Its objects are pairs (ω, H), where ω is a two-form on $\mathcal{G}$ and H is a closed three-form on X satisfying

$$\mathrm{d}\omega = 0$$
$$\mathrm{d}H + \mathrm{d}_{\mathrm{dR}}\omega = 0,$$

  where d is the Čech differential.
- Its morphisms from (ω_1, H_1) to (ω_2, H_2) are given by two-forms B on X satisfying

$$\omega_2 - \omega_1 = \mathrm{d}B$$
$$H_2 - H_1 = \mathrm{d}_{\mathrm{dR}} B.$$

The space of 1-shifted symplectic structures on $[X/\mathcal{G}]$ is the subspace of the above space for which the two-form ω is non-degenerate. For $p\colon X \to [X/\mathcal{G}]$ the natural projection, the pullback functor $p^*\colon \mathrm{QCoh}([X/\mathcal{G}]) \to \mathrm{QCoh}(X)$ is conservative, so we are looking for forms ω which induce a quasi-isomorphism

$$\omega^\sharp\colon p^*\mathbb{T}_{[X/\mathcal{G}]} \to p^*\mathbb{L}_{[X/\mathcal{G}]}[1]$$

i.e. a quasi-isomorphism

$$\begin{tikzcd} (\mathcal{L} \arrow[r, "a"] \arrow[d] & \mathrm{T}_X) \arrow[d] \\ (\mathrm{T}^*_X \arrow[r, "a^*"] & \mathcal{L}^*) \end{tikzcd}$$

of complexes of quasi-coherent sheaves on X. By passing to the total complex, this condition is equivalent to the condition that

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathrm{T}_X \oplus \mathrm{T}^*_X \longrightarrow \mathcal{L}^* \longrightarrow 0$$

is an exact sequence, i.e. $\mathcal{L} \to \mathrm{T}_X \oplus \mathrm{T}^*_X$ is a Lagrangian embedding. But the latter condition is known to be equivalent to the non-degeneracy of the quasi-symplectic groupoid, see [Bur+04, Theorem 2.2] where quasi-symplectic groupoids are called H-twisted presymplectic groupoids. □

We have a similar characterization of symplectic groupoids.

Proposition 3.32. *The space of pairs of a 1-shifted symplectic structure on $[X/\mathcal{G}]$ and a Lagrangian structure on $X \to [X/\mathcal{G}]$ is equivalent to the set of closed two-forms ω on $\mathcal{G}$ endowing it with a structure of a symplectic groupoid.*

Proof. Let us present $\Omega^\epsilon([X/\mathcal{G}])$ as the totalization of the cosimplicial graded mixed cdga

$$\Omega^\epsilon(X) \rightrightarrows \Omega^\epsilon(\mathcal{G}) \overset{\rightarrow}{\rightrightarrows} \dots$$

The pullback $\Omega^\epsilon([X/\mathcal{G}]) \to \Omega^\epsilon(X)$ is surjective, so its homotopy fiber coincides with the strict fiber. Therefore, repeating the argument of Proposition 3.31 we see that the space of pairs of a 1-shifted presymplectic structure on $[X/\mathcal{G}]$ and an isotropic structure on $X \to [X/\mathcal{G}]$ is equivalent to the set of closed multiplicative two-forms ω on $\mathcal{G}$. Note that the two-form ω can be explicitly constructed by considering the isotropic self-intersection $X \times_{[X/\mathcal{G}]} X \cong \mathcal{G}$ which carries a presymplectic structure of shift 0.

The nondegeneracy condition on $f\colon X \to [X/\mathcal{G}]$ implies that the 1-shifted presymplectic structure on $[X/\mathcal{G}]$ is 1-shifted symplectic since then the two vertical morphisms in

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d] & \mathrm{T}_X \arrow[d] \\ \mathrm{T}^*_X \arrow[r] & \mathcal{L}^* \end{tikzcd}$$

are isomorphisms which implies that the total morphism $f^*\mathbb{T}_{[X/\mathcal{G}]} \to f^*\mathbb{L}_{[X/\mathcal{G}]}[-1]$ is a quasi-isomorphism. Thus, it is enough to work out the nondegeneracy condition on the isotropic structure on f.

The nondegeneracy condition on the Lagrangian is equivalent to the fact that the natural morphism

$$\mathbb{T}_{X/[X/\mathcal{G}]} \longrightarrow \mathbb{L}_X$$

is an equivalence. We can identify $\mathbb{T}_{X/[X/\mathcal{G}]} \cong \mathrm{N}_{X/\mathcal{G}} \cong \mathcal{L}$, where $\mathcal{L}$ is the Lie algebroid of $\mathcal{G}$ and $\mathrm{N}_{X/\mathcal{G}}$ is the normal bundle to the unit section. The composite $\mathrm{N}_{X/\mathcal{G}} \to \mathbb{L}_X$ is identified with the natural morphism induced by the fact that the unit section $X \to \mathcal{G}$ is isotropic. Thus, the isotropic structure on $X \to [X/\mathcal{G}]$ is nondegenerate iff the unit section is in fact Lagrangian which is equivalent to the nondegeneracy condition of symplectic groupoids. ☐

4. Symplectic realizations

4.1. **Definitions.** Let X be a derived Artin stack. We denote by $\mathrm{Symp}(X, n)$ the space of n-shifted symplectic structures on X as defined in [Pan+13]. Similarly, given a morphism $f\colon L \to X$ of such stacks we denote by $\mathrm{Lagr}(f, n)$ the space of n-shifted Lagrangian structures, i.e. pairs of an n-shifted symplectic structure on X and a Lagrangian structure on f.

An n-shifted Poisson structure π_X on X induces a morphism $\pi_X^\sharp\colon \mathbb{L}_X \to \mathbb{T}_X[-n]$ given by contraction with the bivector. Similarly, an n-shifted coisotropic structure on $f\colon L \to X$ induces vertical morphisms of fiber sequences

$$\begin{tikzcd} \mathbb{L}_{L/X}[-1] \arrow[r] \arrow[d, "\pi_L^\sharp"] & f^*\mathbb{L}_X \arrow[r] \arrow[d, "\pi_X^\sharp"] & \mathbb{L}_L \arrow[d] \\ \mathbb{T}_L[-n] \arrow[r] & f^*\mathbb{T}_X[-n] \arrow[r] & \mathbb{T}_{L/X}[-n+1] \end{tikzcd}$$

Definition 4.1.

- An n-shifted Poisson structure on X is ***nondegenerate*** if the induced morphism $\pi_X^\sharp\colon \mathbb{L}_X \to \mathbb{T}_X[-n]$ is a quasi-isomorphism. Denote by $\mathrm{Pois}^{nd}(X, n) \subset \mathrm{Pois}(X, n)$ the subspace of such nondegenerate structures.
- An n-shifted coisotropic structure on $f\colon L \to X$ is ***nondegenerate*** if the n-shifted Poisson structure on X is so and the induced morphism $\pi_L^\sharp\colon \mathbb{L}_{L/X} \to \mathbb{T}_L[1-n]$ is

a quasi-isomorphism. Denote by $\mathrm{Cois}^{nd}(f,n)\subset \mathrm{Cois}(f,n)$ the space of subspace of such nondegenerate structures.

The following statement was proved in [Cal+17, Theorem 3.2.4] and [Pri17, Theorem 3.33].

Theorem 4.2. *One has an equivalence of spaces*

$$\mathrm{Pois}^{nd}(X,n)\cong \mathrm{Symp}(X,n).$$

Under this equivalence the two-form $\omega^\sharp\colon \mathbb{T}_X\xrightarrow{\sim}\mathbb{L}_X[n]$ is inverse to the bivector $\pi_X^\sharp\colon \mathbb{L}_X\xrightarrow{\sim}\mathbb{T}_X[-n]$.

An n-shifted Lagrangian structure on $f\colon L\to X$ induces vertical isomorphisms of fiber sequences

$$\begin{tikzcd}
\mathbb{T}_L \arrow[r] \arrow[d, "\omega_L^\sharp", "\sim"'] & f^*\mathbb{T}_X \arrow[r] \arrow[d, "\omega_X^\sharp", "\sim"'] & \mathbb{T}_{L/X}[1] \arrow[d, "\sim"'] \\
\mathbb{L}_{L/X}[n-1] \arrow[r] & f^*\mathbb{L}_X[n] \arrow[r] & \mathbb{L}_L[n]
\end{tikzcd}$$

The following statement was proved in [Pri16] for $n=0$ and [MS17, Theorem 4.22] for all n.

Theorem 4.3. *One has an equivalence of spaces*

$$\mathrm{Cois}^{nd}(f,n)\cong \mathrm{Lagr}(f,n).$$

Under this equivalence the quasi-isomorphism $\omega_L^\sharp\colon \mathbb{T}_L\xrightarrow{\sim}\mathbb{L}_{L/X}[n-1]$ induced by the Lagrangian structure is inverse to the quasi-isomorphism $\pi_L^\sharp\colon \mathbb{L}_{L/X}\xrightarrow{\sim}\mathbb{T}_L[1-n]$ induced by the coisotropic structure.

In particular, by Theorem 4.3 we get a forgetful map

$$\mathrm{Lagr}(f,n)\cong \mathrm{Cois}^{nd}(f,n)\longrightarrow \mathrm{Pois}(L,n-1).$$

Definition 4.4. Suppose L is a derived Artin stack equipped with an $(n-1)$-shifted Poisson structure. Its ***symplectic realization*** is the data of an n-shifted Lagrangian $L\to X$ whose induced $(n-1)$-shifted Poisson structure on L coincides with the original one.

Next, we are going to discuss symplectic realizations of coisotropic morphisms. Suppose $f_1\colon L_1\to X$ and $f_2\colon L_2\to X$ are two morphisms. In [Pan+13, Theorem 2.9] the authors construct a map of spaces

$$\mathrm{Lagr}(f_1,n)\times_{\mathrm{Symp}(X,n)}\mathrm{Lagr}(f_2,n)\longrightarrow \mathrm{Symp}(L_1\times_X L_2,n-1),$$

i.e. an intersection of n-shifted Lagrangians carries an $(n-1)$-shifted symplectic structure.
Similarly, in [MS17, Theorem 3.6] the authors construct a map of spaces

$$\mathrm{Cois}(f_1,n)\times_{\mathrm{Pois}(X,n)}\mathrm{Cois}(f_2,n)\longrightarrow \mathrm{Pois}(L_1\times_X L_2,n-1),$$

i.e. an intersection of n-shifted coisotropics carries an $(n-1)$-shifted Poisson structure.

Conjecture 4.5. *Coisotropic intersections have the following properties:*

(1) *Suppose $L_1\to X$ and $L_2\to X$ carry a* nondegenerate *n-shifted coisotropic structure. Then the $(n-1)$-shifted Poisson structure on $L_1\times_X L_2$ is nondegenerate.*

(2) *The diagram of spaces*

$$\begin{tikzcd} \mathrm{Cois}^{nd}(f_1,n)\times_{\mathrm{Pois}^{nd}(X,n)}\mathrm{Cois}^{nd}(f_2,n) \arrow[r] \arrow[d, "\sim"] & \mathrm{Pois}^{nd}(L_1\times_X L_2, n-1) \arrow[d, "\sim"] \\ \mathrm{Lagr}(f_1,n)\times_{\mathrm{Symp}(X,n)}\mathrm{Lagr}(f_2,n) \arrow[r] & \mathrm{Symp}(L_1\times_X L_2, n-1) \end{tikzcd}$$

is commutative.

Definition 4.6. A diagram of derived Artin stacks

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

is an n***-shifted Lagrangian correspondence*** provided we have the following data:

- An n-shifted symplectic structure on Y,
- Lagrangian structures on $X \to Y$ and $\tilde{X} \to Y$,
- A Lagrangian structure on $C \to \tilde{X}\times_Y X$, where $\tilde{X}\times_Y X$ is equipped with an $(n-1)$-shifted symplectic structure as a Lagrangian intersection.

Suppose we have an n-shifted Lagrangian correspondence as above. In particular, X carries an $(n-1)$-shifted Poisson structure and $X \to Y$ is its symplectic realization. Assuming Conjecture 4.5, we get an $(n-1)$-shifted Poisson structure on the projection $\tilde{X}\times_Y X \to X$. Since $C \to \tilde{X}\times_Y X$ carries an $(n-1)$-shifted coisotropic structure and $\tilde{X}\times_Y X \to X$ is an $(n-1)$-shifted Poisson morphism, the composite

$$C \longrightarrow \tilde{X}\times_Y X \longrightarrow X$$

also carries an $(n-1)$-shifted coisotropic structure.

Definition 4.7. Suppose $C \to X$ is a morphism of derived Artin stacks equipped with an $(n-1)$-shifted coisotropic structure. Its ***symplectic realization*** is the data of an n-shifted Lagrangian correspondence

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

whose underlying $(n-1)$-shifted coisotropic structure on $C \to X$ coincides with the original one.

Remark 4.8. Note that the picture of an n-shifted Lagrangian correspondence is completely symmetric and we also obtain an $(n-1)$-shifted coisotropic structure on $C \to \tilde{X}$. In a sense, the two $(n-1)$-shifted coisotropic structures $C \to X$ and $C \to \tilde{X}$ are dual. We refer to Remark 4.19 for an example of this.

Given a 2-shifted Lagrangian correspondence as above, using [ABB17, Corollary 2.15] we obtain a 2-shifted Lagrangian correspondence

$$\begin{tikzcd} & C \times_X C \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[dr] & & \tilde{X} \arrow[dl] \\ & Y & \end{tikzcd}$$

Therefore, the $(n-1)$-shifted Lagrangian $C \times_X C \to \tilde{X} \times_Y \tilde{X}$ gives a symplectic realization of the $(n-2)$-shifted Poisson structure on $C \times_X C$.

4.2. **Shift 0.** Our first goal is to relate our notion of symplectic realization to more classical notions in the case of ordinary (i.e. unshifted) Poisson structures.

Suppose $\mathcal{G} \rightrightarrows X$ is a smooth symplectic groupoid. Since the unit section $X \to \mathcal{G}$ is Lagrangian, we get an identification $\mathrm{N}_{X/\mathcal{G}} \cong \mathrm{T}^*_X$. But the normal bundle to the unit section is canonically identified with the Lie algebroid $\mathcal{L}$ of $\mathcal{G}$. Thus, we obtain an isomorphism of vector bundles $\mathcal{L} \cong \mathrm{T}^*_X$. By [CDW87, Proposition II.2.1], the induced Poisson structure has bivector given by the composite

$$\mathrm{T}^*_X \cong \mathcal{L} \xrightarrow{a} \mathrm{T}_X.$$

In Proposition 3.32 we have shown that the data of a symplectic groupoid is equivalent to the data of a 1-shifted Lagrangian $X \to [X/\mathcal{G}]$ which gives an a priori different 0-shifted Poisson structure on X.

Proposition 4.9. *Suppose $\mathcal{G} \rightrightarrows X$ is a smooth symplectic groupoid. The underlying Poisson structure on X coincides with the 0-shifted Poisson structure underlying the 1-shifted Lagrangian $X \to [X/\mathcal{G}]$.*

Proof. By Theorem 4.3 the underlying nondegenerate 1-shifted coisotropic structure on $X \to [X/\mathcal{G}]$ has the morphism $\pi^\sharp \colon \mathrm{T}^*_X \to \mathbb{T}_{X/[X/\mathcal{G}]}$ inverse to the quasi-isomorphism $\omega^\sharp \colon \mathbb{T}_{X/[X/\mathcal{G}]} \xrightarrow{\sim} \mathrm{T}^*_X$ induced by the Lagrangian structure. The latter morphism is given by the natural isomorphism $\mathcal{L} \cong \mathrm{T}^*_X$ for the Lie algebroid $\mathcal{L}$ underlying the symplectic groupoid $\mathcal{G}$.

The 0-shifted Poisson structure on X underlying the 1-shifted coisotropic structure on $X \to [X/\mathcal{G}]$ has its bivector given by the composite

$$\mathrm{T}^*_X \to \mathbb{T}_{X/[X/\mathcal{G}]} \to \mathrm{T}_X.$$

But it is easy to see that $\mathcal{L} \cong \mathbb{T}_{X/[X/\mathcal{G}]} \to \mathrm{T}_X$ coincides with the anchor map which gives the result. ∎

Remark 4.10. Note that the notion of a symplectic realization of a Poisson manifold introduced in [Wei83] is slightly more general. Namely, if M is a Poisson manifold, its symplectic realization was defined to be a symplectic manifold S with a submersive Poisson map

$S \to M$. In particular, S was not required to be a groupoid. However, symplectic groupoids integrating M provide particularly nice examples of symplectic realizations.

Example 4.11. Let G be a semisimple algebraic group, $P \subset G$ a parabolic subgroup and M its Levi factor. Let $(-,-)$ be a nondegenerate symmetric bilinear G-invariant pairing on $\mathfrak{g} = \mathrm{Lie}(G)$. By restriction it gives rise to a nondegenerate pairing on $\mathfrak{m} = \mathrm{Lie}(M)$. Let us also denote $\mathfrak{p} = \mathrm{Lie}(P)$. These pairings give rise to 2-shifted symplectic structures on $\mathrm{B}G$ and $\mathrm{B}M$ and it is shown in [Saf17] that the correspondence

$$\begin{tikzcd} & \mathrm{B}P \arrow[dl] \arrow[dr] & \\ \mathrm{B}M & & \mathrm{B}G \end{tikzcd}$$

has a 2-shifted Lagrangian structure coming from the exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{g} \oplus \mathfrak{m} \to \mathfrak{p}^* \to 0 \tag{19}$$

of P-representations.

Now suppose E is an elliptic curve. Denote by $\mathrm{Bun}_-(E) = \mathrm{Map}(E, \mathrm{B}(-))$ the moduli space of principal bundles on E. Then applying $\mathrm{Map}(E,-)$ to the above 2-shifted Lagrangian correspondence we obtain a 1-shifted Lagrangian correspondence

$$\begin{tikzcd} & \mathrm{Bun}_P(E) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_M(E) & & \mathrm{Bun}_G(E) \end{tikzcd}$$

using the AKSZ theorem of [Pan+13]. In this way we obtain a 0-shifted Poisson structure on $\mathrm{Bun}_P(E)$. Let us work it out explicitly. Denote $L = \mathrm{Bun}_P(E)$ and $X = \mathrm{Bun}_M(E) \times \mathrm{Bun}_G(E)$ for brevity.

Fix a principal P-bundle $F \to E$. We can identify the tangent complex to L at F with

$$\mathbb{T}_{L,F} \cong \Gamma(E, \mathrm{ad}F)[1].$$

Similarly, we can identify

$$\mathbb{T}_{L/X,F} \cong \mathrm{fib}\left(\Gamma(E, F \times_P \mathfrak{p}) \to \Gamma(E, F \times_P (\mathfrak{m} \oplus \mathfrak{g}))\right)[1].$$

Using the exact sequence (19) we can simplify it to

$$\mathbb{T}_{L/X,F} \cong \Gamma(E, F \times_P \mathfrak{p}^*).$$

Moreover, $\mathbb{L}_{L,F} \cong \Gamma(E, F \times_P \mathfrak{p}^*)$ and the equivalence $\mathbb{T}_{L/X} \cong \mathbb{L}_L$ induced by the 1-shifted Lagrangian structure on $L \to X$ is the identity.

The bivector underlying the 0-shifted Poisson structure on L is given by the composite

$$\mathbb{L}_L \xrightarrow{\sim} \mathbb{T}_{L/X} \to \mathbb{T}_L$$

which is easily seen to be given by the morphism

$$\Gamma(E, F \times_P \mathfrak{p}^*) \to \Gamma(E, F \times_P \mathfrak{p})[1]$$

induced by the morphism $\mathfrak{p}^* \to \mathfrak{p}[1]$ in the derived category of P-representations obtained as the connecting homomorphism in the sequence (19). In this way we recover the Feigin–Odesskii Poisson structure on $\mathrm{Bun}_P(E)$ constructed in [FO98].

Note that in this way we also obtain a symplectic groupoid

$$\mathrm{Bun}_P(E) \times_{\mathrm{Bun}_M(E)\times \mathrm{Bun}_G(E)} \mathrm{Bun}_P(E) \rightrightarrows \mathrm{Bun}_P(E)$$

integrating the Feigin–Odesskii Poisson structure.

4.3. **Shift 1.** In this section we discuss symplectic realizations of some 1-shifted Poisson structures.

Definition 4.12. A ***Manin pair*** is a pair of finite-dimensional Lie algebras $\mathfrak{d}$ and $\mathfrak{g}$, where $\mathfrak{d}$ is equipped with a nondegenerate invariant pairing and $\mathfrak{g}\subset\mathfrak{d}$ is a Lagrangian subalgebra.

Given a Manin pair $(\mathfrak{d},\mathfrak{g})$ choose a Lagrangian splitting $\mathfrak{d}\cong\mathfrak{g}\oplus\mathfrak{d}/\mathfrak{g}$ where we do not assume that $\mathfrak{d}/\mathfrak{g}\subset\mathfrak{d}$ is closed under the bracket. Formulas (11) and (12) then define a quasi-Lie bialgebra structure on $\mathfrak{g}$.

Remark 4.13. Changing the Lagrangian complement to $\mathfrak{g}\subset\mathfrak{d}$ corresponds to twists of the quasi-Lie bialgebra structure on $\mathfrak{g}$.

Definition 4.14. A ***Manin triple*** is a triple $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^*)$ of finite-dimensional Lie algebras, where $\mathfrak{d}$ is equipped with a nondegenerate invariant pairing and $\mathfrak{g}\subset\mathfrak{d}$ and $\mathfrak{g}^*\subset\mathfrak{d}$ are complementary Lagrangian subalgebras.

Remark 4.15. If $\mathfrak{g}\subset\mathfrak{d}$ is a Manin pair, we obtain an exact sequence of $\mathfrak{g}$-representations

$$0\longrightarrow\mathfrak{g}\longrightarrow\mathfrak{d}\longrightarrow\mathfrak{g}^*\longrightarrow 0$$

and the extra data in a Manin triple is given by a splitting of this exact sequence compatibly with the Lie brackets and the pairings. In particular, in a Manin triple $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^*)$ $\mathfrak{g}^*$ is indeed a linear dual of $\mathfrak{g}$.

Given a Manin triple $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^*)$, the Lie bracket on $\mathfrak{g}^*$ gives rise to a Lie cobracket on $\mathfrak{g}$ which is easily seen to endow $\mathfrak{g}$ with a structure of a Lie bialgebra.

Now suppose D is an algebraic group whose Lie algebra $\mathfrak{d}$ is equipped with a nondegenerate D-invariant pairing. Also suppose $G\subset D$ is a closed subgroup whose Lie algebra $\mathfrak{g}\subset\mathfrak{d}$ is Lagrangian. Such pairs are called ***group pairs*** in [AKS00]. By the results of [AKS00, Section 3.3] one obtains a quasi-Poisson structure on G such that the induced quasi-Lie bialgebra structure on $\mathfrak{g}$ coincides with the one above. In particular, we get a 1-shifted Poisson structure on $\mathrm{B}G$ by Theorem 2.9.

Proposition 4.16. *Suppose (D,G) is a group pair as above. Then the 2-shifted Lagrangian*

$$\mathrm{B}G\to\mathrm{B}D$$

is a symplectic realization of the 1-shifted Poisson structure on $\mathrm{B}G$ *corresponding to the quasi-Poisson structure on G.*

Proof. We have computed the space $\mathrm{Cois}(\mathrm{B}G\to\mathrm{B}D,2)$ in Proposition 2.10. In particular, nondegenerate coisotropic structures are given by nondegenerate pairings on D for which $\mathfrak{g}$ is Lagrangian which provides the identification $\mathrm{Cois}^{nd}(\mathrm{B}G\to\mathrm{B}D,2)\cong\mathrm{Lagr}(\mathrm{B}G\to\mathrm{B}D,2)$.

The forgetful map $\mathrm{Cois}^{nd}(\mathrm{B}G \to \mathrm{B}D, 2) \to \mathrm{Pois}(\mathrm{B}G, 1)$ was computed in Proposition 2.13 where it was shown that the underlying quasi-Lie bialgebra structure on $\mathfrak{g}$ coincides with the one given by formulas (11) and (12). $\square$

Similarly, we may consider ***group triples*** (D, G, G^*) which are triples of algebraic groups whose Lie algebras form a Manin triple. From such a data we get a Poisson-Lie structure on G and hence a 1-shifted coisotropic structure on $\mathrm{pt} \to \mathrm{B}G$ by Corollary 2.11. The two morphisms $\mathrm{B}G \to \mathrm{B}D$ and $\mathrm{B}G^* \to \mathrm{B}D$ carry an obvious 2-shifted Lagrangian structure. Moreover, their intersection $\mathrm{B}G^* \times_{\mathrm{B}D} \mathrm{B}G \cong G^*\backslash D/G$ has the zero tangent complex at the unit of D since $\mathfrak{g}^*, \mathfrak{g} \subset \mathfrak{d}$ are transverse. Therefore, the inclusion of the unit $\mathrm{pt} \to G^*\backslash D/G$ has a unique 1-shifted Lagrangian structure. In other words, we obtain a 2-shifted Lagrangian correspondence

(20)
$$\begin{tikzcd} & \mathrm{pt} \arrow[dl] \arrow[dr] & \\ \mathrm{B}G^* \arrow[dr] & & \mathrm{B}G \arrow[dl] \\ & \mathrm{B}D & \end{tikzcd}$$

Proposition 4.17. *Suppose (D, G, G^*) is a group triple as above. Then the 2-shifted Lagrangian correspondence (20) is a symplectic realization of the 1-shifted coisotropic structure on $\mathrm{pt} \to \mathrm{B}G$ corresponding to the Poisson-Lie structure on G.*

Proof. To prove the claim, we have to compare the Poisson-Lie structure on G obtained from the Manin triple with the one obtained from the 2-shifted Lagrangian correspondence. For this it is enough to compare the induced Lie cobrackets on $\mathfrak{g}$.

As observed in section 4.1, the symplectic realization of the Poisson structure on G is given by the 1-shifted Lagrangian $f\colon G \to \mathrm{B}G^* \times_{\mathrm{B}D} \mathrm{B}G^* \cong [G^*\backslash D/G^*]$. To compute the induced Poisson structure on G, let us unpack the underlying two-forms on G and $[G^*\backslash D/G^*]$.

Trivialize the tangent bundle to D and G using left translations. Given an element $d \in D$, the tangent complex to $[G^*\backslash D/G^*]$ at d is

$$\mathbb{T}_{[G^*\backslash D/G^*],d} \cong (\mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathfrak{d})$$

in degrees -1 and 0, where one of the maps is the identity and the other map is given by the composite

$$\mathfrak{g}^* \longrightarrow \mathfrak{d} \xrightarrow{\mathrm{Ad}_d} \mathfrak{d}.$$

Using the exact sequence $0 \to \mathfrak{g}^* \to \mathfrak{d} \to \mathfrak{g} \to 0$ we get a quasi-isomorphic model for the tangent complex as

$$\mathbb{T}_{[G^*\backslash D/G^*],d} \cong (\mathfrak{g}^* \xrightarrow{\mathrm{Ad}_d} \mathfrak{g}).$$

The 1-shifted symplectic structure on $[G^*\backslash D/G^*]$ induces an obvious isomorphism

$$\mathbb{T}_{[G^*\backslash D/G^*],d} \cong \mathbb{L}_{[G^*\backslash D/G^*],d}[1].$$

The Lagrangian structure on $G \to [G^*\backslash D/G^*]$ gives rise to a fiber sequence of complexes

$$\mathbb{T}_{G,g} \longrightarrow \mathbb{T}_{[G^*\backslash D/G^*],g} \longrightarrow \mathbb{L}_{G,g}[1]$$

for $g\in G$. Explicitly, it is given by the vertical fiber sequence of complexes

$$\begin{tikzcd} (0 \arrow[r] \arrow[d] & \mathfrak{g}) \arrow[d, equal] \\ (\mathfrak{g}^* \arrow[r, "\mathrm{Ad}_g"] \arrow[d, equal] & \mathfrak{g}) \arrow[d] \\ (\mathfrak{g}^* \arrow[r] & 0) \end{tikzcd}$$

Therefore, the connecting homomorphism $\mathbb{L}_{G,g}\to \mathbb{T}_{G,g}$ is given by the morphism $\mathrm{Ad}_g\colon \mathfrak{g}^*\to \mathfrak{g}$ which is the underlying bivector on G. The Lie cobracket on $\mathfrak{g}$ is given by linearizing this bivector at $g=e$, so we see that the cobracket is given by $[x,-]\colon \mathfrak{g}^*\to\mathfrak{g}$ for $x\in\mathfrak{g}$, i.e. the map $\mathfrak{g}\otimes\mathfrak{g}^*\to\mathfrak{g}$ given by the coadjoint action of $\mathfrak{g}^*$ which recovers the Lie cobracket on $\mathfrak{g}$ coming from the Manin triple. ☐

Remark 4.18. Note that in the course of the proof we have constructed a symplectic realization of the underlying Poisson-Lie structure on G as $G\to[G^*\backslash D/G^*]$. In particular, we obtain the symplectic groupoid

$$G\times_{[G^*\backslash D/G^*]} G\cong (G\times G^*)\times_D (G\times G^*)\rightrightarrows G$$

known as the Lu–Weinstein groupoid [Lu90, Section 4.2]. A similar interpretation of the Lu–Weinstein groupoid was previously given in [Sev12].

Remark 4.19. The symmetry of the Lagrangian correspondence (20) reflects Poisson-Lie duality.

5. Classical r-matrices

In this section we relate shifted Poisson structures to the notion of a classical r-matrix.

5.1. **Classical notions.** Let $r\in\mathfrak{g}\otimes\mathfrak{g}$. The classical Yang–Baxter equation is an equation in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ given by

$$\mathrm{CYBE}(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{21}$$

where, for instance, $r_{12}=r\otimes 1\in(\mathrm{U}\mathfrak{g})^{\otimes 3}$.

Now consider an element $c\in\mathrm{Sym}^2(\mathfrak{g})^G$ and the induced trivector

$$\phi = -\frac{1}{6}[c_{12}, c_{23}].$$

The data of twist from the quasi-Poisson group $(G,\pi=0,\phi)$ to $(G,\pi,0)$ is given by $\lambda\in\wedge^2(\mathfrak{g})$ satisfying

$$\frac{1}{2}[\![\lambda,\lambda]\!]=\phi,\qquad \pi=\lambda^L-\lambda^R.$$

In turn, the first equation is equivalent to the classical Yang–Baxter equation (21) for $r=2\lambda+c$.

Definition 5.1. A ***quasi-triangular classical*** r***-matrix*** is the data of $r\in\mathfrak{g}\otimes\mathfrak{g}$ such that

(1) $\mathrm{CYBE}(r) = 0$,
(2) The symmetric part of r is G-invariant.

One also has the following generalization. Let $H \subset G$ be a subgroup and $U \subset \mathfrak{h}^*$ an open dense subset. Consider a function $r\colon U \to \mathfrak{g} \otimes \mathfrak{g}$. Its differential is $\mathrm{d}_{\mathrm{dR}} r\colon U \to \mathfrak{h} \otimes \mathfrak{g} \otimes \mathfrak{g}$. We denote by $\mathrm{Alt}(\mathrm{d}_{\mathrm{dR}} r)$ the function $U \to \wedge^3(\mathfrak{g})$ obtained as its antisymmetrization.

Definition 5.2. A ***quasi-triangular classical dynamical r-matrix*** is the data of a function $r\colon U \to \mathfrak{g} \otimes \mathfrak{g}$ such that

(1) r is an $\mathfrak{h}$-equivariant function,
(2) $\mathrm{CYBE}(r) = 0$,
(3) The symmetric part of r is a constant element of $\mathrm{Sym}^2(\mathfrak{g})^G$.

Note that the data of a quasi-triangular classical dynamical r-matrix is equivalent to the data of $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ and an $\mathfrak{h}$-equivariant function $\lambda \in U \to \wedge^2(\mathfrak{g})$ such that

$$\frac{1}{2}[\![\lambda, \lambda]\!] + \mathrm{Alt}(\mathrm{d}_{\mathrm{dR}}\lambda) = \phi.$$

5.2. **Non-dynamical case.**

Proposition 5.3. *Let G be an algebraic group. The space parametrizing the pairs of*

- *A 2-shifted Poisson structure π on $\mathrm{B}G$,*
- *A 1-shifted Poisson morphism $\mathrm{pt} \to \mathrm{B}G$ with the 1-shifted Poisson structure on $\mathrm{B}G$ obtained from π*

is equivalent to the set of quasi-triangular Poisson structures on G, i.e. the set of quasi-triangular classical r-matrices.

The quasi-triangular Poisson structure is factorizable iff the underlying 2-shifted Poisson structure on $\mathrm{B}G$ is 2-shifted symplectic.

Proof. Indeed, the space of 2-shifted Poisson structures on $\mathrm{B}G$ is equivalent to the set of Casimir elements $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ by Proposition 2.6. The induced 1-shifted Poisson structure on $\mathrm{B}G$ corresponds to the quasi-Poisson structure $(\pi = 0, \phi)$, where

$$\phi = -\frac{1}{6}[c_{12}, c_{23}].$$

A 1-shifted Poisson morphism $\mathrm{pt} \to \mathrm{B}G$ is equivalent to the data of a Poisson structure π on G by Corollary 2.11. The compatibility between the two is given by an element $\lambda \in \wedge^2(\mathfrak{g})$ twisting $(0, \phi)$ into $(\pi, 0)$, i.e. we get an equation

$$\frac{1}{2}[\![\lambda, \lambda]\!] = -\frac{1}{6}[c_{12}, c_{23}]$$

and hence $r = 2\lambda + c$ is a quasi-triangular classical r-matrix. ∎

5.3. **Dynamical case.** Before we give a description of dynamical r-matrices similar to Proposition 5.3 we need to construct a canonical 1-shifted Poisson structure on $[\mathfrak{g}^*/G]$. We can identify $[\mathfrak{g}^*/G] \cong \mathrm{T}^*[1](\mathrm{B}G)$, so by [Cal16] it carries a 1-shifted symplectic structure. Replacing G by the formal completion, it has the following description. We can identify

$$\Omega^\epsilon([\mathfrak{g}^*/\widehat{G}]) \cong \mathrm{C}^\bullet(\mathfrak{g}, \Omega^\epsilon(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*[-2])) \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}^*[-2]),$$

where $\mathfrak{g}^*[-2]$ and $\mathfrak{g}[-1]$ are concentrated in weight 1. The internal differential on $\Omega^\epsilon([\mathfrak{g}^*/\widehat{G}])$ has the following two components:

- The Chevalley–Eilenberg differential,
- The Cartan differential $\alpha \in \Omega^\epsilon(\mathfrak{g}^*) \mapsto \iota_{a(-)}\alpha \in \Omega^\epsilon(\mathfrak{g}^*) \otimes \mathfrak{g}^*$, where $a\colon \mathfrak{g} \to \mathrm{T}_{\mathfrak{g}^*}$ is the action map.

We have a canonical Maurer–Cartan $\mathfrak{g}^*$-valued one-form $\theta \in \Omega^1(\mathfrak{g}^*; \mathfrak{g}^*)$ on $\mathfrak{g}^*$ which gives rise to a two-form on $[\mathfrak{g}^*/\widehat{G}]$ of degree 1. It is closed under the internal differential due to $\mathfrak{g}$-equivariance. It is also closed under the de Rham differential since θ is constant on $\mathfrak{g}^*$.

Similarly, we can identify

$$\mathrm{Pol}([\mathfrak{g}^*/\widehat{G}], 1) \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}^*[-2]),$$

where $\mathfrak{g}[-1]$ and $\mathfrak{g}^*[-2]$ are again concentrated in weight 1. The same element θ gives a degree 3 element $\pi \in \mathrm{Pol}([\mathfrak{g}^*/\widehat{G}], 1)$. The Schouten bracket pairs $\mathfrak{g}^*[-1]$ with $\mathfrak{g}[-1]$ and $\mathfrak{g}$ with $\mathfrak{g}^*[-2]$, so $[\pi, \pi] = 0$. Therefore, π defines a 1-shifted Poisson structure on $[\mathfrak{g}^*/\widehat{G}]$. The following statement is clear from the explicit formula for this 1-shifted Poisson structure.

Lemma 5.4. *The projection $[\mathfrak{g}^*/G] \to \mathrm{B}G$ is a 1-shifted Poisson morphism where $\mathrm{B}G$ has the zero 1-shifted Poisson structure.*

Both ω and π are clearly G_{dR}-invariant, so they descend to a 1-shifted symplectic and 1-shifted Poisson structure on $[\mathfrak{g}^*/G]$.

Lemma 5.5. *The 1-shifted Poisson structure π on $[\mathfrak{g}^*/G]$ is nondegenerate and is compatible with the 1-shifted symplectic structure ω.*

Proof. π induces a morphism $\mathbb{L}_{[\mathfrak{g}^*/G]} \to \mathbb{T}_{[\mathfrak{g}^*/G]}[-1]$ of complexes of quasi-coherent sheaves on $[\mathfrak{g}^*/G]$ and to check that it is an equivalence it is enough to check it after pulling back to $\mathfrak{g}^*$. But then π induces a morphism

$$\begin{tikzcd} (\mathfrak{g} \otimes \mathcal{O}_{\mathfrak{g}^*} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathfrak{g}^* \otimes \mathcal{O}_{\mathfrak{g}^*}) \arrow[d, "\mathrm{id}"] \\ (\mathfrak{g} \otimes \mathcal{O}_{\mathfrak{g}^*} \arrow[r] & \mathfrak{g}^* \otimes \mathcal{O}_{\mathfrak{g}^*}) \end{tikzcd}$$

of complexes of sheaves on $\mathfrak{g}^*$ which is indeed an equivalence.

Being a 1-shifted Poisson structure, π endows $\mathrm{Pol}([\mathfrak{g}^*/\widehat{G}], 1)$ with a graded mixed structure and by the universal property of the de Rham algebra, we obtain a morphism

$$\Omega^\epsilon([\mathfrak{g}^*/\widehat{G}], 1) \longrightarrow \mathrm{Pol}([\mathfrak{g}^*/\widehat{G}], 1)$$

which after unpacking corresponds to the identity

$$\mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}^*[-2])) \longrightarrow \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}^*[-2])).$$

To show that π is inverse to ω, we have to produce a compatibility data between ω and π in the sense of [Cal+17, Definition 1.4.20]. But since both elements are concentrated purely in weight 2, this reduces to showing that $\omega^\sharp\colon \mathbb{T}_{[\mathfrak{g}^*/\widehat{G}]} \to \mathbb{L}_{[\mathfrak{g}^*/\widehat{G}]}[1]$ is inverse to $\pi^\sharp\colon \mathbb{L}_{[\mathfrak{g}^*/\widehat{G}]} \to \mathbb{T}_{[\mathfrak{g}^*/\widehat{G}]}[-1]$ (see e.g. [Pri17, Example 1.21]) which is obvious. $\square$

From now on we will simply refer to π as the canonical 1-shifted Poisson structure on $[\mathfrak{g}^*/G]$.

Proposition 5.6. *Let G be an algebraic group, $H \subset G$ a subgroup and $U \subset \mathfrak{h}^*$ an open H-invariant subscheme. The space parametrizing the pairs of*

- *A 2-shifted Poisson structure π on $\mathrm{B}G$,*
- *A 1-shifted Poisson morphism $f\colon [U/H] \to \mathrm{B}G$ compatible with the canonical 1-shifted Poisson structure on $[U/H] \subset [\mathfrak{h}^*/H] \cong \mathrm{T}^*[1](\mathrm{B}H)$ and the 1-shifted Poisson structure on $\mathrm{B}G$ induced from π*

is equivalent to the set of quasi-triangular classical dynamical r-matrices with base U.

Proof. It is enough to prove the corresponding statement where we replace groups by their formal completion. Let $g\colon [U/\mathfrak{h}] \to [U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}$ be the graph. By Theorem 1.9 we have to compute the pullback

$$\begin{tikzcd}
\mathrm{Pois}([U/\mathfrak{h}] \to \mathrm{B}\mathfrak{g}, 1) \arrow[r] \arrow[d] & \mathrm{Pois}(\mathrm{B}\mathfrak{g}, 2) \arrow[d] \\
\mathrm{Cois}(g, 1) \arrow[r] & \mathrm{Pois}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1).
\end{tikzcd}$$

Here the map $\mathrm{Pois}(\mathrm{B}\mathfrak{g}, 2) \to \mathrm{Pois}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)$ is given by sending a 2-shifted Poisson structure c on $\mathrm{B}\mathfrak{g}$ to the sum of the canonical 1-shifted Poisson structure on $[U/\mathfrak{h}] \subset \mathrm{T}^*[1](\mathrm{B}\mathfrak{h})$ and the 1-shifted Poisson structure on $\mathrm{B}\mathfrak{g}$ given by Proposition 2.16.

We can identify

$$\mathrm{Pol}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1) \cong \mathrm{C}^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-1])) \otimes \mathrm{C}^\bullet(\mathfrak{h}, \mathrm{Sym}(\mathfrak{h}[-1] \oplus \mathfrak{h}^*[-2]) \otimes \mathcal{O}(U))$$

and

$$\mathrm{Pol}([U/\mathfrak{h}]/([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}), 0) \cong \mathrm{C}^\bullet(\mathfrak{h}, \mathrm{Sym}(\mathfrak{g}[-1]) \otimes \mathcal{O}(U)).$$

Moreover,

$$\mathrm{Pol}(g, 1) \cong \ker\left(\mathrm{Pol}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1) \longrightarrow \mathrm{Pol}([U/\mathfrak{h}]/([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}), 0)\right).$$

Let us begin by computing the space $\mathrm{Pois}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)$. Since $\mathrm{Pol}[U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)^{\geq 2}[2]$ is concentrated in non-negative degrees, by Proposition 1.2 the space $\mathrm{Pois}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)$ is equivalent to the corresponding Deligne groupoid that we now proceed to compute.

Elements of $\mathrm{Pol}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)$ of degree 2 have the following type:

- $\lambda^1 \in \wedge^2(\mathfrak{g}) \otimes \mathcal{O}(U)$,
- $\lambda^2 \in \mathfrak{g} \otimes \mathfrak{h} \otimes \mathcal{O}(U)$,
- $\lambda^3 \in \wedge^2(\mathfrak{h}) \otimes \mathcal{O}(U)$.

Elements of $\mathrm{Pol}([U/\mathfrak{h}] \times \mathrm{B}\mathfrak{g}, 1)$ of degree 3 have the following type:

- $\alpha^1 \in \wedge^2(\mathfrak{g}) \otimes \mathfrak{g}^* \otimes \mathcal{O}(U)$,
- $\alpha^2 \in \wedge^2(\mathfrak{g}) \otimes \mathfrak{h}^* \otimes \mathcal{O}(U)$,
- $\alpha^3 \in \mathfrak{g} \otimes \mathfrak{h} \otimes \mathfrak{g}^* \otimes \mathcal{O}(U)$,
- $\alpha^4 \in \mathfrak{g} \otimes \mathfrak{h} \otimes \mathfrak{h}^* \otimes \mathcal{O}(U)$,
- $\alpha^5 \in \wedge^2(\mathfrak{h}) \otimes \mathfrak{g}^* \otimes \mathcal{O}(U)$,
- $\alpha^6 \in \wedge^2(\mathfrak{h}) \otimes \mathfrak{h}^* \otimes \mathcal{O}(U)$,
- $\alpha^7 \in \mathfrak{g} \otimes \mathfrak{h}^* \otimes \mathcal{O}(U)$,

- $\alpha^8 \in \mathfrak{h}\otimes\mathfrak{h}^*\otimes\mathcal{O}(U)$,
- $\beta^1 \in \wedge^3(\mathfrak{g})\otimes\mathcal{O}(U)$,
- $\beta^2 \in \wedge^2(\mathfrak{g})\otimes\mathfrak{h}\otimes\mathcal{O}(U)$,
- $\beta^3 \in \mathfrak{g}\otimes\wedge^2(\mathfrak{h})\otimes\mathcal{O}(U)$,
- $\beta^4 \in \wedge^3(\mathfrak{h})\otimes\mathcal{O}(U)$.

Let $\rho\colon \mathfrak{h}\to\mathfrak{h}^*\otimes\mathcal{O}(U)$ be the coadjoint action of $\mathfrak{h}$ on U. Then we get the differential equations

$$\begin{aligned}
0 &= \frac{d\alpha^1}{dt} + \mathrm{d}^{\mathfrak{g}}_{\mathrm{CE}}\lambda^1, & 0 &= \frac{d\alpha^2}{dt} + \mathrm{d}^{\mathfrak{h}}_{\mathrm{CE}}\lambda^1\\
0 &= \frac{d\alpha^3}{dt} + \mathrm{d}^{\mathfrak{g}}_{\mathrm{CE}}\lambda^2, & 0 &= \frac{d\alpha^4}{dt} + \mathrm{d}^{\mathfrak{h}}_{\mathrm{CE}}\lambda^2\\
0 &= \frac{d\alpha^5}{dt}, & 0 &= \frac{d\alpha^6}{dt} + \mathrm{d}^{\mathfrak{h}}_{\mathrm{CE}}\lambda^3\\
0 &= \frac{d\beta^1}{dt} + \rho(\lambda^2), & 0 &= \frac{d\beta^2}{dt} + \rho(\lambda^3)
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
0 &= \frac{d\gamma^1}{dt} + [\alpha^1,\lambda^1] + [\beta^1,\lambda^1] + [\alpha^2,\lambda^2]\\
0 &= \frac{d\gamma^2}{dt} + [\alpha^3,\lambda^1] + [\beta^2,\lambda^1] + [\alpha^1,\lambda^2] + [\alpha^4,\lambda^2] + [\beta^1,\lambda^2] + [\alpha^2,\lambda^3]\\
0 &= \frac{d\gamma^3}{dt} + [\alpha^5,\lambda^1] + [\alpha^3,\lambda^2] + [\alpha^6,\lambda^2] + [\beta^2,\lambda^2] + [\alpha^4,\lambda^3] + [\beta^1,\lambda^3]\\
0 &= \frac{d\gamma^4}{dt} + [\alpha^5,\lambda^2] + [\alpha^6,\lambda^3] + [\beta^2,\lambda^3].
\end{aligned} \tag{23}$$

Objects of $\mathrm{Pois}([U/\mathfrak{h}]\times \mathrm{B}\mathfrak{g}, 1)$ are given by Maurer–Cartan elements $\alpha^1, ..., \alpha^6, \beta^1, \beta^2, \gamma^1, ..., \gamma^4$. Morphisms are given by elements $\lambda^1, ..., \lambda^3$ and solutions of the differential equations (22) and (23).

The groupoid $\mathrm{Cois}(g, 1)$ is the subgroupoid of $\mathrm{Pois}([U/\mathfrak{h}]\times \mathrm{B}\mathfrak{g}, 1)$ whose objects are Maurer–Cartan elements as before satisfying

$$\sum_{i=1}^6 \alpha^i = 0, \qquad \sum_{i=1}^4 \gamma^i = 0$$

and whose morphisms are elements $\lambda_1, ..., \lambda^3$ satisfying

$$\sum_{i=1}^3 \lambda_i = 0.$$

The space $\mathrm{Pois}([U/\mathfrak{h}]\to \mathrm{B}\mathfrak{g}, 1)$ is given by the homotopy fiber product of groupoids

$$\mathrm{Pois}([U/\mathfrak{h}]\to \mathrm{B}\mathfrak{g}, 1) = \mathrm{Cois}(g, 1)\times_{\mathrm{Pois}([U/\mathfrak{h}]\times\mathrm{B}\mathfrak{g},1)} \mathrm{Pois}(\mathrm{B}\mathfrak{g}, 2)$$

for which we will use the standard model, see e.g. [Hol08b, Lemma 2.2]. Let $\mathcal{G}\subset \mathrm{Pois}([U/\mathfrak{h}]\to \mathrm{B}\mathfrak{g}, 1)$ be the full subgroupoid where we assume that $\lambda^2 = 0$ and $\lambda^3 = 0$ on objects. Then it's clear that objects in $\mathcal{G}$ have no automorphisms and it is merely a set. Moreover, the inclusion is essentially surjective, hence is an equivalence. $\mathcal{G}$ is therefore the set parametrizing elements $c \in \mathrm{Sym}^2(\mathfrak{g})^G$ and $\lambda^1 \in \wedge^2(\mathfrak{g}) \otimes \mathcal{O}(U)$ satisfying some equations that we will now describe.

Let $\phi = -\frac{1}{6}[c_{12}, c_{23}]$, $\lambda^2 = 0$, $\lambda^3 = 0$. We consider solutions of (22) and (23) with the initial conditions $\beta^2(0) = \mathrm{id}_h \in \mathfrak{h}^* \otimes \mathfrak{h}$, $\gamma^1(0) = \phi$ and the rest of $\alpha^i(0), \beta^i(0), \gamma^i(0)$ zero. The nonzero equations are

$$\begin{aligned}
\alpha^1(t) &= -t\mathrm{d}^{\mathfrak{g}}_{\mathrm{CE}}\lambda^1 \\
\alpha^2(t) &= -t\mathrm{d}^{\mathfrak{h}}_{\mathrm{CE}}\lambda^1 \\
\beta^2(t) &= \mathrm{id}_{\mathfrak{h}} \\
\gamma^1(t) &= \phi + \frac{t^2}{2}[\mathrm{d}^{\mathfrak{g}}_{\mathrm{CE}}\lambda^1, \lambda^1] \\
\gamma^2(t) &= -t[\mathrm{id}_{\mathfrak{h}}, \lambda^1].
\end{aligned}$$

Note that $[\mathrm{id}_{\mathfrak{h}}, \lambda^1] = \mathrm{d}_{\mathrm{dR}}\lambda^1 \in \mathfrak{h} \otimes \wedge^2(\mathfrak{g}) \otimes \mathcal{O}(U)$. Since the value at $t = 1$ is assumed to be a coisotropic structure, we get that $\alpha^1 + \alpha^2 = 0$ and $\gamma^1 + \gamma^2 = 0$. The first equation is equivalent to $\lambda^1 \colon U \to \wedge^2(\mathfrak{g})$ being $\mathfrak{h}$-invariant while the second equation is equivalent to

$$\phi - \frac{1}{2}[\![\lambda^1, \lambda^1]\!] - \mathrm{Alt}(\mathrm{d}_{\mathrm{dR}}\lambda^1) = 0,$$

i.e. $r = 2\lambda^1 + c$ is a quasi-triangular classical dynamical r-matrix. □

Remark 5.7. Strictly speaking, since we consider algebraic functions on $U \subset \mathfrak{h}^*$, we only capture dynamical r-matrices with a rational dependence on the dynamical parameter. One can also consider trigonometric dependence on the dynamical parameter by considering open subschemes of H^*, where H is a Poisson subgroup of G (see [FM02] for a related formalism).

Let us now draw some consequences from this point of view on classical dynamical r-matrices. Fix a classical dynamical r-matrix giving rise to a 1-shifted Poisson morphism $[U/H] \to \mathrm{B}G$. The natural projection $U \to [U/H]$ has an obvious coisotropic (even Lagrangian) structure, hence the composite

$$U \longrightarrow [U/H] \longrightarrow \mathrm{B}G$$

has a 1-shifted coisotropic structure. In particular, by Proposition 2.14 we get that $U \times G$ is quasi-Poisson G-space. It is easy to see that this quasi-Poisson structure is equivalent to the one given in [EE03, Section 2.2].

Similarly, interpreting the 1-shifted coisotropic structure on $U \to \mathrm{B}G$ in terms of a Poisson groupoid over U following Corollary 3.30 we recover dynamical Poisson groupoids introduced in [EV98].

In a different direction, suppose the classical dynamical r-matrix is regular at the origin $0 \in \mathfrak{h}^*$. In other words, $0 \in U$. Then the inclusion of the origin gives a coisotropic morphism $\mathrm{B}H \to [U/H]$. Therefore, the composite

$$\mathrm{B}H \longrightarrow [U/H] \longrightarrow \mathrm{B}G$$

has a 1-shifted coisotropic structure. Again applying Proposition 2.14 we recover a structure of a quasi-Poisson G-space on G/H studied in [Lu00] and [Kar+05].

5.4. **Construction of classical r-matrices.** In this section we will explain how some manipulations with Lagrangian correspondences give rise to classical (dynamical) r-matrices.

We begin with the non-dynamical case. Suppose G is a split simple algebraic group, $B_+ \subset G$ and $B_- \subset G$ are a pair of opposite Borel subgroups sharing a common maximal torus H. The Killing form defines a nondegenerate G-invariant pairing on $\mathfrak{g}$ and hence a 2-shifted symplectic structure on $\mathrm{B}G$. The pairing restricts to a nondegenerate pairing on $\mathfrak{h}$ and hence gives rise to a 2-shifted symplectic structure on $\mathrm{B}H$. By [Saf17, Lemma 3.4] we have Lagrangian correspondences

$$\begin{tikzcd} & \mathrm{B}B_\pm \arrow[dl] \arrow[dr] & \\ \mathrm{B}H & & \mathrm{B}G \end{tikzcd}$$

Here we set $B_+ \to H$ to be the natural projection and $B_- \to H$ to be the natural projection post-composed with the inverse on H.

Composing Lagrangian correspondences for B_+ and B_- we get a Lagrangian correspondence

$$\begin{tikzcd} & \mathrm{B}B_+ \times_{\mathrm{B}H} \mathrm{B}B_- \arrow[dl] \arrow[dr] & \\ \mathrm{B}G & & \mathrm{B}G \end{tikzcd}$$

If we denote $G^* = B_+ \times_H B_-$, then $\mathrm{B}G^* \cong \mathrm{B}B_+ \times_{\mathrm{B}H} \mathrm{B}B_-$. Therefore, we get a pair of Lagrangians

$$\begin{tikzcd} \mathrm{B}G^* \arrow[dr] & & \mathrm{B}G \arrow[dl] \\ & \mathrm{B}D & \end{tikzcd}$$

where $D = G \times G$ and its Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ is equipped with the difference of the pairings on the two summands.

By construction the two subspaces $\mathfrak{g}^*, \mathfrak{g} \subset \mathfrak{d}$ are transverse. Indeed, let us split $\mathfrak{b}_+ \cong \mathfrak{n}_+ \oplus \mathfrak{h}$ and $\mathfrak{b}_- \cong \mathfrak{n}_- \oplus \mathfrak{h}$, where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the corresponding nilpotent subalgebras. Then $\mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}$ is given by pairs (x_+, x_-) of elements of $\mathfrak{g}$ such that $x_\pm \in \mathfrak{b}_\pm$ and their $\mathfrak{h}$-components add up to zero. Then the intersection $\Delta(\mathfrak{g}) \cap \mathfrak{g}^*$ is clearly zero. But since they are Lagrangian, the intersection is also transverse.

Therefore, the natural basepoint $\mathrm{pt} \to \mathrm{B}G^* \times_{\mathrm{B}D} \mathrm{B}G$ has a unique 1-shifted Lagrangian structure and hence we obtain a 2-shifted Lagrangian correspondence

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl] \arrow[dr] & \\ \mathrm{B}G^* \arrow[dr] & & \mathrm{B}G \arrow[dl] \\ & \mathrm{B}D & \end{tikzcd}$$

In particular, we obtain a 1-shifted coisotropic structure on $\mathrm{pt} \to \mathrm{B}G$. We are left to show that the underlying 1-shifted Poisson structure on $\mathrm{B}G$ can be lifted to the natural 2-shifted symplectic structure on $\mathrm{B}G$ obtained from the pairing on $\mathfrak{g}$.

From explicit formulas it is easy to see (e.g. see [AKS00, Example 2.1.5]) that the quasi-Lie bialgebra structure on $\mathfrak{g}$ coming from the Manin pair $\mathfrak{g} \overset{\Delta}{\subset} \mathfrak{g} \oplus \mathfrak{g}$ is equivalent to the one given in Proposition 2.16. It also follows from the following easy argument on the level of shifted Poisson structures.

Lemma 5.8. *Consider the* 2*-shifted Lagrangian* $\Delta\colon \mathrm{B}G \to \overline{\mathrm{B}G} \times \mathrm{B}G$ *given by the diagonal, where* $\overline{\mathrm{B}G}$ *has the opposite* 2*-shifted symplectic structure. The underlying* 1*-shifted Poisson structure on* $\mathrm{B}G$ *is equivalent to one obtained from the* 2*-shifted coisotropic structure* $\mathrm{id}\colon \mathrm{B}G \to \mathrm{B}G$.

Proof. Consider the induced 2-shifted coisotropic structure on $\Delta\colon \mathrm{B}G \to \overline{\mathrm{B}G} \times \mathrm{B}G$. The natural projection $p_2\colon \overline{\mathrm{B}G} \times \mathrm{B}G \to \mathrm{B}G$ of 2-shifted Poisson stacks is compatible with the Poisson structure, so the composite

$$\mathrm{B}G \xrightarrow{\Delta} \overline{\mathrm{B}G} \times \mathrm{B}G \xrightarrow{p_2} \mathrm{B}G$$

acquires a 2-shifted coisotropic structure whose underlying 1-shifted Poisson structure on $\mathrm{B}G$ is equivalent to the one obtained from the diagonal Lagrangian. But the space of 2-shifted coisotropic structures on the identity $\mathrm{id}\colon \mathrm{B}G \to \mathrm{B}G$ compatible with the given 2-shifted Poisson structure on $\mathrm{B}G$ is contractible by [MS16, Proposition 4.16] which gives the result. $\square$

To summarize, we have constructed a 1-shifted coisotropic structure on $\mathrm{pt} \to \mathrm{B}G$ whose underlying 1-shifted Poisson structure on $\mathrm{B}G$ lifts to a 2-shifted Poisson structure on $\mathrm{B}G$ given by the Killing form. Therefore, by Proposition 5.3 we obtain a quasi-triangular classical r-matrix on $\mathfrak{g}$. This is the so-called standard r-matrix on a simple Lie algebra.

Remark 5.9. Belavin and Drinfeld [BD84] classified all factorizable classical r-matrices on a simple Lie algebra in terms of the so-called Belavin–Drinfeld triples $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \Gamma$ are subsets of simple roots and $\tau\colon \Gamma_1 \xrightarrow{\sim} \Gamma_2$ is an isomorphism preserving the pairing satisfying a nilpotency condition. In particular, they have constructed the corresponding Manin triples and the above construction can be repeated with those Manin triples.

One can similarly give a construction of dynamical r-matrices. With notations as before, we have a 1-shifted Lagrangian correspondence [Saf17, Theorem 3.2]

$$\begin{tikzcd} & {[\mathfrak{b}_+/B_+]} \arrow[dl] \arrow[dr] & \\ {[\mathfrak{h}/H]} & & {[\mathfrak{g}/G]} \end{tikzcd}$$

Let $\mathfrak{h}^{rss} \subset \mathfrak{h}$ be the open subscheme of elements which are regular semisimple as elements of $\mathfrak{g}$. The projection $[\mathfrak{b}_+/B_+] \to [\mathfrak{h}/H]$ becomes an isomorphism over $[\mathfrak{h}^{rss}/H]$, so we obtain a 1-shifted Lagrangian correspondence

$$\begin{tikzcd} & {[\mathfrak{h}^{rss}/H]} \arrow[dl] \arrow[dr] & \\ {[\mathfrak{h}^{rss}/H]} & & {[\mathfrak{g}/G]} \end{tikzcd}$$

which is a graph of a morphism $[\mathfrak{h}^{rss}/H] \to [\mathfrak{g}/G]$. In particular, it carries a 1-shifted coisotropic structure. By Theorem 1.9 we therefore see that the morphism $[\mathfrak{h}^{rss}/H] \to [\mathfrak{g}/G]$ is 1-shifted Poisson. Combining it with Lemma 5.4, the composite

$$[\mathfrak{h}^{rss}/H] \longrightarrow [\mathfrak{g}/G] \longrightarrow \mathrm{B}G$$

is also a 1-shifted Poisson morphism and hence by Proposition 5.6 we obtain a classical dynamical r-matrix. This is known as the basic rational dynamical r-matrix, see [ES01, Section 3.1].

Remark 5.10. We may replace adjoint quotients of Lie algebras $[\mathfrak{g}/G]$ by adjoint quotients of groups $[G/G]$ in the above construction. Then we recover the basic trigonometric dynamical r-matrix.

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