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Globalization for equivariant AKSZ theories over closed manifolds

Master's Thesis

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Abstract

In this work we study two constructions in the context of the Batalin-Vilkovisky formalism. The first one is an equivariant extension of AKSZ theories, while the second one is a globalization procedure for split AKSZ theories over closed manifolds. More specifically, we address the compatibility of the two constructions in the classical setting and we derive a modified version of the Classical Master Equation that holds for globalized equivariant AKSZ theories. Finally, we compute the globalized equivariant action in a couple of examples, namely supersymmetric Yang-Mills theory and Donaldson-Witten theory.

Contents

Introduction

In the early 1980's Igor Batalin and Grigori Vilkovisky introduced a new gauge-fixing technique built on the backbone of the BRST formalism, thus retaining the cohomological flavor of the latter. The novel element is the use of tools coming from supergeometry and, in particular, graded symplectic geometry. This was clarified by Albert Schwarz in his 1993 paper [14], where he laid the mathematical foundations of the Batalin-Vilkovisky formalism. The idea of this formalism is to make the space of fields into a graded symplectic manifold and to try and define the path integral

$$\int e^{\frac{i}{\hbar}S}$$

by integrating over Lagrangian submanifolds. In his paper, Schwarz proved that under some conditions this procedure only depends on the homology class of the Lagrangian submanifolds over which one integrates. Gauge-fixing is then understood to be a choice of a Lagrangian submanifold (or its homology class to be precise) and Schwarz's theorem, which also goes by the name of BV-Stokes' theorem, can be understood as a formulation of gauge-independence.

Although the BV formalism is very powerful, it is not easy to construct BV theories. In the late 90's, a general method for constructing such theories was proposed by Alexander, Kontsevitch, Schwarz and Zaboronsky. The resulting theories go by name of *AKSZ models*. The basic idea, which comes from the realm of topological field theories, is to construct the space of fields as the space of (graded) morphisms

$$F = \mathrm{Map}(T[1]\Sigma, N)$$

between the shifted tangent bundle to some closed oriented manifold Σ and a graded symplectic manifold N. In the celebrated AKSZ paper ([1]), the authors show that this mapping space can be used to construct a BV theory. In particular, there is a distinguished function S on F, the *BV action*, that satisfies the *Classical Master Equation* (CME)

$$\{S, S\} = 0.$$

The value of the AKSZ construction, besides its elegance, is the fact that it can be used to recover well-know theories, such as the Poisson Sigma Model, Chern-Simons theory and BF theory.

In this thesis, we will work with constructions in the context of AKSZ theories. In 2020 Bonechi, Cattaneo, Qiu and Zabzine proposed an extension of the BV formalism to the case where there is a Lie algebra $\mathfrak{g}$ acting infinitesimally on Σ ([3]). The point of the construction is that the space of functions $C^\infty(F)$ becomes a $\mathfrak{g}$-differential graded algebra and we can thus use the techniques coming from equivariant cohomology. In their paper, the authors show that it is possible to take the $\mathfrak{g}$-action into account at the cost of shrinking the space of observables (in order to retain gauge-independence). They also show that a modified version of the CME for the *equivariant BV action* S^c holds, namely

$$\frac{1}{2}\{S^c, S^c\} + \sum_a S_{\hat{L}_{v_a}} \otimes u^a = 0.$$

The Poisson bracket no longer vanishes, but terms involving the Lie derivative along fundamental vector fields coming from the $\mathfrak{g}$-action show up.

The second construction we are interested appears in [4] and [9]. In the first paper, from Bonechi, Cattaneo and Mnev, the authors apply the construction to the Poisson Sigma Model, while in the second one, by Cattaneo, Moshayedi and Wernli, the machinery is extended to arbitrary *split* AKSZ theories, i.e. theories whose target manifold is a shifted cotangent bundle. The idea behind the construction is perturb around solutions of the kinetic part of the BV action, which can be identified with points in the target manifold. To see how this procedure depends on the chosen point, the authors use methods coming from formal geometry and construct the *globalized BV action* $\tilde{S}$, which satisfies the *differential Classical Master Equation*

$$d\tilde{S} + \frac{1}{2}\{\tilde{S}, \tilde{S}\} = 0.$$

The question we address in this thesis is whether these two constructions, which are unrelated to one another, are compatible and, if so, how the CME needs to be modified. It turns out that they are compatible and that it is possible to construct a globalized action $\tilde{S}^c$ that takes the $\mathfrak{g}$-action into account. The main result is the *equivariant differential Classical Master Equation*

$$d\tilde{S}^c + \frac{1}{2}\{\tilde{S}^c, \tilde{S}^c\} + \sum_a S_{\hat{L}_{v_a}} \otimes u^a = 0.$$

Here is the structure of this work. In Chapter 1 we provide the necessary background on supermanifolds and graded geometry. The main resources we have followed are [10], [5] and [11] for the part on supermanifolds, while we have drawn from [17] for the part on graded geometry.

In Chapter 2 we discuss graded symplectic geometry following Schwarz's paper [14]. The most important result of the chapter is the BV-Stokes theorem, which we have already mentioned. We also give a complete description of odd-symplectic manifolds, which behave in a rather rigid way, compared to ordinary symplectic manifolds.

In the last two chapters, we delve into the Batalin-Vilkovisky formalism. In particular, in Chapter 3 we give a complete description of the BV formalism and of the AKSZ construction following [13]. In Chapter 4 we examine the two aforementioned constructions and we present all result in full detail. Finally, we explain how to combine them into a unique framework and we compute the equivariant globalized action for a couple of toy examples, namely supersymmetric Yang-Mills theory and Donaldson-Witten theory.

In the appendices we have gathered some background material on differential graded algebras and formal geometry, as well as some computations which we thought best to postpone in order not to clutter the presentation.

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Chapter 1

Supergeometry

In the last fifty years, supergeometry has risen to prominence as the mathematical framework of quantum field theory and supersymmetry. The idea is that we would like to add anticommuting functions to the algebra of ordinary smooth functions on a manifold. There are different ways of doing this. In the approach proposed by deWitt, called the "concrete" approach, one artificially adds "odd points". We will however take a different viewpoint, the "algebro-geometric" approach, which was pioneered by Leites. This approach consists in adding anticommuting variables locally and then gluing the local patches. One thus obtains a sheaf of generalized functions which carry a parity and commute accordingly.

In the first part of this chapter, we cover super linear algebra, i.e. the study of vector space that carry an internal $\mathbb{Z}_2$-grading. Then, we introduce supermanifolds and we describe the main properties thereof. It is worth to note that the theory of supermanifolds unfolds in a completely natural way, that is classical results (existence of partitions of unity, inverse function theorem, etc.) almost always have a super counterpart.

In the last part of the chapter, we generalize supermanifolds by allowing functions to carry a $\mathbb{Z}$-grading. The need for such objects again originated from physics, where they are for instance the backbone of BRST models.

For supergeometry, the main references we have used are [5], [16], while for $\mathbb{Z}$-graded geometry we have used the wonderful paper [17] by Vysoky.

§1.1 Super linear algebra

In this section we discuss the first steps towards the realm of supergeometry, i.e. the category of super vector spaces, and we show how linear algebra can be generalized to account for these new objects.

Definition 1.1.1. A *super vector space* over a characteristic 0 field k is a k-vector space V that has a decomposition $V = V_0 \oplus V_1$. The elements in V_0 are called *even*, the ones in V_1 *odd*.

By definition, a super vector space carries a $\mathbb{Z}_2$-grading. If V_0 has dimension p and V_1 has dimension q, we say that V has dimension $p|q$.

Let W be another k-super vector space. The space of *all* morphisms carries an induced $\mathbb{Z}_2$-grading

$$\mathbf{Hom}(V, W) = \mathbf{Hom}(V, W)_0 \oplus \mathbf{Hom}(V, W)_1,$$

given by degree-preserving (resp. reversing) morphisms. The space of morphisms from V to W is defined to be $\mathrm{Hom}(V, W) = \mathbf{Hom}(V, W)_0$, i.e. the degree-preserving ones.

Fix bases $e_1, \ldots, e_p$ and $e_{p+1}, \ldots, e_{p+q}$ for V_0 and V_1 respectively. Then, with respect to this basis, a morphism $\phi\colon V \to W$ is represented by a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where B and C (respectively A and D) vanish if ϕ is even (respectively odd).

The category of super vector spaces admits similar constructions as in classical linear algebra. If above we take $W = k$ we obtain the notion of algebraic dual. The direct sum of two spaces is defined in the obvious way. A little less obvious is the definition of the tensor product of two spaces, whose homogeneous parts are given by

$$(V \otimes W)_i = \sum_{j+k=i} V_j \otimes W_k.$$

We now want to consider algebras and the super counterparts.

Definition 1.1.2. A *superalgebra* is a super vector space endowed with a multiplication map, i.e. a linear map $A \otimes A \to A$. Note that this is equivalent to requiring that

$$p(ab) = p(a) + p(b)$$

for all $a, b \in A$ homogeneous. Here, we denote by $p(a)$ the degree of a.

This implies that the even part of A_0 of A is a commutative ring in the ordinary sense.

A superalgebra is called *(super)commutative* if $ab = (-1)^{p(a)p(b)}ba$ for all homogeneous elements a, b.

Example 1.1.3. Let V be a super vector space. Then, the space of endomorphisms

$$\mathrm{End}(V) = \mathrm{Hom}(V, V)$$

together with composition of maps is a superalgebra.

Example 1.1.4. Let V be a (purely even) k-vector space and consider its exterior algebra

$$\bigwedge V = T(V)/I,$$

where $T(V)$ denotes the tensor algebra of V and I is the ideal generated by elements of the form $x \otimes x$. The obvious multiplication map makes the exterior algebra of V into a commutative superalgebra. If V is finite-dimensional, it is isomorphic to the Grassmann algebra $k[\theta^1, \ldots, \theta^q]$, with $\theta^i\theta^j = -\theta^j\theta^i$ for all i, j.

In the last example, we considered the tensor algebra and the exterior algebra of purely even vector space. We can do this in the super category as well. Indeed, let V be a super vector space and consider the n-fold tensor product $V^{\otimes n}$. The symmetric group S_n acts on decomposable elements. For $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$ homogeneous elements, we have

$$\sigma \cdot v_1 \otimes \cdots \otimes v_n = (-1)^{p(s)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Here, $p(s)$ is the number of inversions of odd elements, i.e.

$$p(s) = |\{(i, j) \mid i < j,\ \sigma(i) > \sigma(j),\ v_i, v_j \mathrm{odd}\}|.$$

Then, the *symmetric nth-power* of V is defined to be quotient

$$S^n V = V^{\otimes n}/I,$$

where I is the ideal generated by elements of the form

$$v_1 \otimes \dots v_n - \sigma \cdot v_1 \otimes \cdots \otimes v_n, \quad \sigma \in S_n.$$

The *symmetric algebra* of V is readily defined as

$$S^\bullet V = \bigoplus_{n \in \mathbb{N}} S^n V.$$

Note that $S^\bullet V$ is a commutative superalgebra.

If in the above with replace the ideal I with the ideal J generated by elements of the form

$$v_1 \otimes \dots v_n + \sigma \cdot v_1 \otimes \cdots \otimes v_n, \quad \sigma \in S_n,$$

we obtain the *nth exterior power* of V, denoted by $\Lambda^n V$. The *exterior algebra* of V is then given by

$$\Lambda^\bullet V = \bigoplus_{n \in \mathbb{N}} \Lambda^n V.$$

The notion of Lie algebra also generalizes in a straightforward way.

Definition 1.1.5. A *super Lie algebra* is a super vector space $\mathfrak{g}$ endowed with a *super Lie bracket*, i.e. a morphism $[\cdot, \cdot] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties:

- $[a, b] = (-1)^{p(a)p(b)}[b, a]$,
- the *super Jacobi identity*

  $$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0$$

  for all homogeneous elements $a, b, c \in \mathfrak{g}$.

Example 1.1.6. Any superalgebra becomes a superalgebra when endowed with the *super-commutator*

$$[a, b] = ab - (-1)^{p(a)p(b)}ba.$$

The notion of trace of an endomorphism needs a bit of tweaking. Let $T \in \mathbf{End}(V)$ be a homogeneous endomorphism and fix a basis for V. If

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the corresponding matrix representation of T, we define $\mathrm{str}(T) = \mathrm{tr}(A) - \mathrm{tr}(D)$ to be the *supertrace* of T. It is straightforward to check that this definition does not depend on the choice of basis and that

$$\mathrm{str}(ST) = (-1)^{p(S)p(T)}\mathrm{str}(TS)$$

for all homogeneous morphisms S, T.

The notion of superalgebra allows us to speak of modules. Let A be a commutative superalgebra over the (characteristic 0) field k.

Definition 1.1.7. A module over A is a super vector space M on which A acts on the left in a way that is compatible with the respective gradings. In other words, the action

$$A \otimes M \to M$$
$$a \otimes m \mapsto am$$

is required to be a morphism of super vector spaces.

A left A-module M can be considered as a right A-module under the action

$$ma = (-1)^{p(a)p(m)} am$$

for $a \in A$ and $m \in M$.

For two A-modules M, N, the space of all morphisms $M \to N$ splits in the sum of degree-preserving and degree-reversing maps, i.e.

$$\mathbf{Hom}(M, N) = \mathbf{Hom}(M, N)_0 \oplus \mathbf{Hom}(M, N)_1.$$

and has a natural A-module structure.

Just as we did in the case of super vector spaces, we can construct some interesting modules. If we take $N = A$, we obtain the module dual to M. The tensor product $M \otimes N$ is defined to be the quotient of $M \otimes_k N$ by the relations

$$ma \otimes n = m \otimes an, \quad a \in A.$$

The following notion will be very important when we discuss super vector bundles. An A-module is called *free* if it admits a basis, i.e. even elements $e_1, \dots, e_p$ and odd elements $e_{p+1}, \dots, e_{p+q}$ such that

$$A = Ae_1 \otimes \cdots \otimes Ae_{p+q}.$$

To specify the dimension, we also write $A^{p|q}$.

Morphisms of free modules can be described in a similar way as in the case of super vector spaces. Let $\{e_1, \dots, e_{p+q}\}$ and $f_1, \dots, f_{r+s}$ be bases for $A^{p|q}$ and $A^{r|s}$ respectively and $T\colon A^{p|q} \to A^{r|s}$ a morphism. An element $x \in A^{p|q}$ is given by $x = \sum_i e_i x^i$. If we set

$$Te_j = \sum_i f_i t^i_j,$$

T is represented by the matrix (t^i_j). T is even (odd) if and only the diagonal blocks are even (odd) and the off-diagonal blocks are odd (even). Since we allow odd morphisms of modules, the definition of the supertrace is extended in the following way. For

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we define $\mathrm{str}(T) = \mathrm{tr}(A) - (-1)^{p(T)}\mathrm{tr}(D)$.

The most important step in going from linear algebra to super linear algebra is the generalization of the notion of determinant. As far as supergeometry is concerned, its use is crucial in the theory of integration over supermanifolds, as it allows for coordinate-independent definitions, just in ordinary differential geometry.

To gain some insight into the subtleties of this matter, let us consider an even square matrix, i.e. a matrix of the form

$$X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If we want the relation $\operatorname{str}(e^X) = e^{\operatorname{str}(X)}$ to be valid, we need to define the *superdeterminant* (or *Berezinian*, after Felix Berezin, who first defined it) of X to be $\operatorname{Ber}(X) = \det(A)\det(D^{-1})$. Thus, unlike in ordinary linear algebra, the superdeterminant is not defined for arbitrary matrices.

It is not difficult to prove that X is invertible if and only if both A and D are invertible over A_0 (see [16]). If X is invertible, we have the decomposition

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}.$$

If we want Ber to be multiplicative, we have to define

$$\operatorname{Ber}(X) = \det\bigl(A - BD^{-1}C\bigr)\det(D)^{-1}.$$

Then, Ber is an even map and it is multiplicative. For a proof of this statement, see for example [5].

We can also define the Berezinian $\operatorname{Ber}(A)$ of a free module A. Every basis $e_1, \dots, e_{p+q}$ for A defines a basis for $\operatorname{Ber}(A)$, which we denote by $[e_1, \dots, e_{p+q}]$, and if T is an automorphism of A, we require

$$[Te_1, \dots, Te_{p+q}] = \operatorname{Ber}(T)[e_1, \dots, e_{p+q}].$$

Note that $\operatorname{Ber}(A)$ has rank $1|0$ if q is even and $0|1$ if q is odd.

Remark 1.1.8. The definition of Berezinian of a free module that we gave is rather clumsy. There are more invariant ways of defining it, for example using cohomological tools, but we will not need such machinery. A complete account of this approach can be found in [11] and [10].

§1.2 Supermanifolds

There are two different ways of defining supermanifolds. The first one, called the "concrete approach", artificially adds "odd" points by changing the local model. We will however focus on the second approach, called "algebro-geometric". It is somewhat more abstract but it highlights a very important fact, namely that basically all the information of a (super)manifold is carried by its algebra of smooth functions.

Definition 1.2.1. Let X be a topological space. A *sheaf* of rings is an assignment

$$U \mapsto \mathcal{O}(U),$$

where U is open and $\mathcal{O}(U)$ is a ring, satisfying the following properties:

- for every pair of nested open sets $V \subseteq U$, there exists a "restriction" map

  $$r_{U,V} \colon \mathcal{O}(U) \to \mathcal{O}(V).$$

  The restriction maps are required to satisfy $r_{U,U} = id_{\mathcal{O}(U)}$ and the cocycle condition $r_{V,W} r_{U,V} = r_{U,W}$ for $W \subseteq V \subseteq U$.

- If we have an open cover $\{U_i\}_{i\in I}$ of X and $f_i \in \mathcal{O}(U_i)$ are elements (also called *sections*) such that for all $i, j \in I$,

$$r_{U_i, U_i\cap U_j}(f_i) = r_{U_j, U_i\cap U_j}(f_j),$$

  there exists a unique $f \in \mathcal{O}(X)$ such that $r_{X,U_i}(f) = f_i$ for all $i \in I$.

An assignment that only satisfies the first condition is called a *presheaf*. Of course, we can also have sheaves of groups, vector spaces, algebras, etc.

Example 1.2.2. Let M be a smooth manifold. Then, the assignment $U \mapsto C^\infty(U)$ defines a sheaf of commutative algebras over M. The second condition is satisfied due to the well-known fact that manifolds admit partitions of unity.

If $\mathcal{O}$ is a sheaf of rings over X and $x \in X$ is a point, the *stalk* of $\mathcal{O}$ at x is defined as the colimit

$$\mathcal{O}_x = \varinjlim_{x\in U} \mathcal{O}(U)$$

over all open neighborhoods of x.

Definition 1.2.3. A *locally ringed space* is a topological space together with a sheaf of rings such that the stalk at every point is a local ring, i.e. it admits a unique maximal ideal.

Here is an explicit construction of the above colimit. We declare two sections $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ to be equivalent if there exists a neighborhood W of x contained in $U \cap V$ such that

$$r_{U,W}(f) = r_{V,W}(g).$$

This is easily shown to be an equivalence relation. Then, the stalk is given (up to isomorphism) by

$$\mathcal{O}_x = \coprod_{x\in U} U \Big/ \sim$$

with the algebra structure of the $\mathcal{O}(U)$'s passing to the quotient. Going back to the above example, we see that manifolds are locally ringed spaces, since the maximal ideal in $C^\infty(U)$ are of the form $I_x = \{f \in C^\infty(U) \mid f(x) = 0\}$.

Locally ringed spaces form a category. A morphism of locally ringed spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a continuous map $\phi\colon X \to Y$ together with a *pullback* $\phi^*\colon \mathcal{O}_Y \to \mathcal{O}_X$ which is a morphism of sheaves. This means the following things:

- for any open set U, there is a morphism of rings $\phi^*_U\colon \mathcal{O}(U) \to \mathcal{O}\big(\phi^{-1}(U)\big)$;
- these morphisms are compatible with restriction maps, that is for all $V \subseteq U \subseteq Y$ the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\phi^*_U} & \mathcal{O}_X\big(\phi^{-1}(U)\big) \\ {\scriptstyle r_{U,V}}\downarrow & & \downarrow{\scriptstyle r_{\phi^{-1}(U),\phi^{-1}(V)}} \\ \mathcal{O}_Y(V) & \xrightarrow{\phi^*_V} & \mathcal{O}_X\big(\phi^{-1}(V)\big) \end{array}$$

  commutes;

- for all $x \in X$, the induced map $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is local, i.e. it preserves the maximal ideals.

Remarks 1.2.4. (i) Technically speaking, the pullback ϕ^* is a morphism from $\mathcal{O}_Y$ to $\phi_*\mathcal{O}_X$, the pushforward of $\mathcal{O}_X$, defined by

$$\phi_*\mathcal{O}_X(U) = \mathcal{O}_X\big(\phi^{-1}(U)\big).$$

We will not be too pedantic on this.

(ii) In the case of rings of functions, the pullback takes a very natural form. For $f \in \mathcal{O}_Y(U)$ and a point $x \in X$, we have

$$\phi_U^*(f)(x) = f(\phi(x)).$$

Indeed, by the above characterization of the maximal ideals of $C^\infty(U)$, if a function takes a non-zero value at some point it is invertible in some neighborhood of that point. Suppose now that there exists some function g such that $\phi^*(g)(x) \neq g(\phi(x))$. By adding a constant to g, we can assume that $\phi^*(g)(x) = 0$. Thus, g is invertible in a neighborhood of $\phi(x)$, which contradicts the fact that $\phi^*(g)(x) = 0$.

To go from geometry to supergeometry, we just replace commutative rings with supercommutative rings. The definition of local ring changes slightly, as we require the existence of a unique maximal homogeneous ideal. The most basic example of a super ringed space is a *superdomain*, i.e. an open set $U \subseteq \mathbb{R}^p$ together with the sheaf $C^{\infty\, p|q}$ defined by

$$U \supseteq V \mapsto C^\infty(V)[\theta^1, \ldots, \theta^q].$$

We will denote this superdomain by $U^{p|q}$ and we will use it as a local model to define supermanifolds.

Definition 1.2.5. A *supermanifold* of dimension $p|q$ is a locally super ringed space $M = (|M|, \mathcal{O})$ such that:

- $|M|$ is Hausdorff and second countable;
- $(M, \mathcal{O})$ is locally isomorphic to $U^{p|q}$.

Examples 1.2.6. (i) Any superdomain is a supermanifold, in particular $\mathbb{R}^{p|q}$.

(ii) Consider $\mathbb{R}^{p^2+q^2|2pq}$, which can be identified with the space of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, D are the even coordinates and B, C are the odd ones. Then, the open subset given by the invertible matrices, i.e. with $(\det A)(\det D) \neq 0$, is a supermanifold. We denote this space by $GL(p|q)$.

(iii) Consider a real vector bundle $E \to M$ over a smooth manifold M. Then, we can construct the *exterior bundle* $\Lambda E \to M$, that is the bundle having as fiber over $x \in M$ the exterior algebra ΛE_x of the fiber over x in E. This is a very important example of supermanifold. Indeed, a theorem of Marjorie Batchelor ([2]) shows that every supermanifold is isomorphic to the exterior bundle of some real vector bundle. This is true in the smooth category, while it is not true in the analytic one. For a full discussion, see [11]. Batchelor's theorem gives insight into the structure of supermanifolds but it must be used with care, because the isomorphism whose existence it guarantees is not canonical.

Supermanifolds differ from ordinary manifolds in that their rings of smooth functions contain nilpotent elements (coming from the anticommuting coordinates). By getting rid of these elements, we can recover a smooth manifold. Let us illustrate how to formalize this intuition.

An arbitrary function on $f \in \mathcal{O}(U) \cong C^\infty(V)[\theta^1, \ldots, \theta^q]$ can be written as

$$f = \sum_I f_I \theta^I,$$

where $f_I \in C^\infty(V)$, $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots q\}$ is an r-shuffle and $\theta^I = \theta^{i_1} \ldots \theta^{i_r}$. Let f_0 be the degree zero function in the above sum. Then, it is not hard to see that for any $x \in M$, $f(x)$ is the unique real number λ such that the function $f - \lambda$ is not invertible in any neighborhood of x.

Consider $\mathcal{J}(U) = \{s \in \mathcal{O}(U) \mid s_0 = 0 \text{ on } U\}$, which defines a subsheaf of $\mathcal{O}$. Then, the quotient $\mathcal{O}^\sim = \mathcal{O}/\mathcal{J}$ is a sheaf and we can consider f_0 a representative of a class in $\mathcal{O}^\sim(U)$.

Remark 1.2.7. The fact that $\mathcal{O}^\sim$ is a sheaf, and not a just a presheaf, is a non-trivial statement that is a consequence of the fact that supermanifolds, as in the ordinary case, admit partitions of unity. The discussion is standard, although technical, and can be found in full detail in [5].

Going back to our discussion, we see that the assignment

$$f \mapsto f_0$$

defines a morphism of sheaves $\mathcal{O} \to \mathcal{O}^\sim$.

We define the *reduced manifold* $M^\sim$ of M to be the smooth manifold $(|M|, \mathcal{O}^\sim)$. The assignment $M \mapsto M^\sim$ goes through on morphisms as well. Indeed, let

$$\phi\colon M \to N$$
$$\phi^*\colon \mathcal{O}_N \to \mathcal{O}_M$$

a morphism of supermanifolds. It is easy to see that the subsheaves $\mathcal{J}_N, \mathcal{J}_M$ are preserved by ϕ^*. This means that ϕ^* induces a morphism

$$(\phi^\sim)^*\colon \mathcal{O}_N^\sim \to \mathcal{O}_M^\sim$$

and thus we have a morphism of smooth manifolds $\phi^\sim\colon M^\sim \to N^\sim$.

The strength of the theory of supermanifold lies in the fact that, from the point of view of maps, it works the same way as for ordinary manifolds. Indeed, one can think of a morphism of supermanifold as a (local) assignment

$$(x^1, \ldots, x^m) \mapsto (y^1, \ldots, y^n),$$

where the x^i's are coordinates on the domain and the y^j's are coordinates on the codomain. Before stating the main theorem, let us discuss a lemma that we will need and that highlights a technique that is frequently used in these situations, namely polynomial approximation.

Let M be a supermanifold and $m \in |M|$ a point. Let us consider the stalk $\mathcal{O}_m$ at m and let us denote by the equivalence class of a function f by $[f]_m$. Moreover, consider the ideal

$$\mathcal{K}_m = \{[f]_m \in \mathcal{O}_m \mid f^\sim(m) = 0\}.$$

By definition, there exists a neighborhood of m that isomorphic to a superdomain $U^{p|q}$. Let x^i, θ^j be the pullbacks of the coordinates on $U^{p|q}$. Without loss of generality, we may assume that $(x^i)^\sim(m) = 0$.

Lemma 1.2.8 (Hadamard's lemma)**.** *(i) $\mathcal{K}_m$ is generated by $[x^i]_m, [\theta^j]_m$. Moreover, the ideals $\mathcal{K}_m^k$ are preserved by morphisms of supermanifolds for all $k \geq 0$.*

(ii) If $k > q$ and f is a section defined on a neighborhood of m such that $[f]_{m'} = 0$ for all m' in some neighborhood of m, then $[f]_m = 0$.

(iii) if f is a local section around m and $k \geq 0$, there is a polynomial p in the $[x^i]_m, [\theta^j]_m$ such that $f - p \in \mathcal{K}_m^k$.

Proof. See [16, Lemma 4.3.2]. □

Theorem 1.2.9 (Global Chart Theorem)**.** *Let $U^{p|q}$ be a superdomain and M a supermanifold. Then there exists a bijection between morphisms $M \to U^{p|q}$ and tuples of p even functions $f_1, \ldots, f_p$ and q odd functions $g_1, \ldots, g_q$ in $\mathcal{O}(M)$ such that*

$$\big(f_1^\sim(m), \ldots, f_p^\sim(m)\big) \in |U|$$

for all $m \in |M|$.

Proof. If we are given a morphism $\phi\colon M \to U^{p|q}$, we just define

$$f_i = \phi^*(t^i), \quad g_j = \phi^*(\theta^j). \tag{1.1}$$

The other direction is trickier. We will to prove that given such functions f_i, g_j, there exists a unique morphism ϕ that satisfies (1.1).

Uniqueness. Let ϕ_1, ϕ_2 be two such morphisms. Since the x^i's form a coordinate system on $\left|U^{p|q}\right| = U$, we have that $\phi_1^\sim = \phi_2^\sim$. Clearly, we have that $\phi_1^*(u) = \phi_2^*(u)$ for all polynomials $u \in C^\infty(U)[\theta^1, \ldots, \theta^q]$. Let now $u \in C^\infty(V)[\theta^1, \ldots, \theta^q]$, where $V \subseteq U$ is open and define $g = \phi_1^*(u) - \phi_2^*(u)$. Let $x \in M$ be a point and $y = \phi_1^\sim(x) = \phi_2^\sim(x)$. By (iii) of Lemma 1.2.8, there exists a polynomial p of degree k such that

$$[u]_y - [p]_y \in \mathcal{K}_y^k.$$

This shows that $[g]_x \in \mathcal{K}_{M,x}^k$. If we take k large enough (from the start), noting that x is arbitrary, we obtain $g = 0$ by (ii) of Lemma 1.2.8.

Existence. Since we have already established uniqueness, we can construct the morphism locally and it will automatically glue to a global one. Thus, we can assume M is covered by one chart. It is also enough to construct a morphism $C^\infty(U) \to \mathcal{O}(M)_0$ taking x^i to f_i, since the restrictions $\phi^*(\theta^j) = g_j$ uniquely determine the values of ϕ^* on polynomials in the θ^j's.

The subtlety here is the fact that even and odd coordinates can mix. Let us see how to deal with this phenomenon. Let us decompose $f_i = r_i + n_i$, where $r_i \in C^\infty(M)$ and

$$n_i = \sum_{|I|\geq 1} n_{iI}\varphi^I$$

is the nilpotent part of f_i (here, we denote by φ^j the odd coordinates on M). For $g \in C^\infty(U)$, we define

$$\phi^*(g) = \sum_\gamma \frac{1}{\gamma!}(\partial^\gamma g)(r_1, \dots, r_p)n^\gamma.$$

This defines an element of $\mathcal{O}(M)$ which is even, since the f_i were even. The only thing left to verify is that ϕ^* is morphism of rings, which we leave to the reader as an easy exercise. □

Now, we want to look at the local and infinitesimal structure of supermanifolds, which basically means defining a suitable notion of tangent space at a point. We will take the algebraic approach, i.e. that of derivations on the algebra of smooth functions. A (homogeneous) derivation of parity $p(D)$ of a k-superalgebra A is a k-linear map $D\colon A \to A$ that satisfies

$$D(ab) = D(a)b + (-1)^{p(a)p(D)}aD(b)$$

for all $a, b \in A$.

Definition 1.2.10. A *vector field* on an open set U of a supermanifold $(M, \mathcal{O})$ is a derivation of $\mathcal{O}(U)$.

As in the classical space, we expect the space of derivations to have a nice structure. Indeed, it turns out to be a (super)vector bundle, i.e. a sheaf of $\mathcal{O}$-modules on M that is locally isomorphic to $\mathcal{O}^{p|q}$, where $p|q$ is the dimension of M. This is easily seen because we can find a basis over a coordinate chart. Indeed, consider a superdomain $U^{p|q}$ and the associated coordinate system $(x^1, \dots, x^p, \theta^1, \dots, \theta^q)$. Then we have the derivations $\frac{\partial}{\partial x^i}$, defined in the usual way, and the derivations $\frac{\partial}{\partial \theta^j}$, defined on monomials of degree 1 by

$$\frac{\partial}{\partial \theta^j}(f\theta^i) = f\delta^i_j$$

and extended to arbitrary functions by the derivation property. Together, they form a basis for the module of derivations of $C^\infty(U)[\theta^1, \dots, \theta^q]$. The proof of this statement goes along the same lines of the proof of the Global Chart Theorem, i.e. one shows that any derivation can be written as a linear combination of the $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j}$ by first proving that they give the same value when applied to polynomials in the x^i, θ^j and then invoking Lemma 1.2.8. We call this sheaf of derivations *tangent sheaf* and we denote it, as usual, by TM. We can use it to recover the notion of differential forms. We define the *cotangent sheaf* to a supermanifold $(M, \mathcal{O})$ to be the sheaf

$$\Omega^1_M = \mathrm{Hom}_{\mathcal{O}}(TM, \mathcal{O}).$$

Differential forms of degree $p \geq 2$ are defined to be sections of the sheaves

$$\Omega^p_M = \underbrace{\Omega^1_M \wedge \cdots \wedge \Omega^1_M}_{p \text{ times}}.$$

One can then go on to define the de Rham complex and develop de Rham theory for supermanifolds. We will not delve into this as it is outside of the scope of this presentation. A full account can be found in [11].

Vector fields and differential forms describe a supermanifold locally and globally. The infinitesimal information is encoded in the stalks of the sheaf of smooth functions on M.

Definition 1.2.11. A *tangent vector* to M at a point m is a linear map $v\colon \mathcal{O}_m \to \mathbb{R}$ that satisfies the Leibniz identity

$$v(fg) = v(f)g(m) + -1^{p(f)p(v)} f(m)v(f)$$

for all (homogeneous) $f, g \in \mathcal{O}_m$.

We denote the tangent space at m by $T_m M$. Note that it is a super vector space and

$$\left.\frac{\partial}{\partial x^i}\right|_m, \left.\frac{\partial}{\partial \theta^j}\right|_m$$

is a basis.

Given a morphism $\phi\colon M \to N$ of supermanifolds, we can recover a notion of differential (or tangent map) exactly as in the classical case. Indeed, we have the differential at m of ϕ

$$\begin{aligned} d_m\phi\colon T_m &\to T_{\phi^\sim(m)}N \\ v &\mapsto v \circ \phi^*. \end{aligned}$$

The assignment $M \mapsto T_m M$ is functorial as a consequence of the chain rule, which keeps being valid in this context. If $\psi\colon U^{p|q} \to V^{r|s}$ is a morphism of superdomains and $(x^i, \theta^j), (y^k, \varphi^l)$ are coordinates on $U^{p|q}, V^{r|s}$ respectively, then

$$\frac{\partial \psi^*(f)}{\partial x^i} = \sum_k \frac{\partial \psi^*(y^k)}{\partial x^i} \psi^* \left(\frac{\partial f}{\partial x^k} \right)$$

for all smooth functions f on $V^{r|s}$. The proof of the chain rule is again by polynomial approximation. We leave the details to the reader.

In matrix form, the differential is represented by the Jacobian

$$J\psi = \begin{pmatrix} \frac{\partial g}{\partial x} & -\frac{\partial g}{\partial \theta} \\ \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial \theta} \end{pmatrix},$$

where $g^i = \psi^*(y^i)$ and $\lambda^j = \psi^*(\varphi^j)$.

The properties of the differential determine the local properties of a morphism. As in the classical case, we say a morphism $\phi\colon M \to N$ is an *immersion* at $m \in M$ if $d_m\phi$ is injective and a *submersion* if it is surjective. Then, we have the following theorem, which is the super counterpart to the ordinary immersion (submersion) theorem.

Examples 1.2.12. (i) Consider superdomains $U^{p|q}$ and $V^{r|s}$ with coordinates (x^i, θ^j) and (y^k, φ^l) respectively (without loss of generality, we may assume that U and V contain the origin). Then, the map

$$\begin{aligned} \phi^\sim\colon U &\to U \times V \\ m &\mapsto (m, 0) \end{aligned}$$

together with the pullback ϕ^* defined on coordinates by

$$\phi^*(x^i) = x^i, \quad \phi^*(\theta^j) = \theta^j, \quad \phi^*(y^k) = \phi^*(\varphi^l) = 0$$

is an immersion.

(ii) The morphism $\psi \colon U^{p|q} \times V^{r|s} \to U^{p|q}$ given by

$$\begin{aligned} \psi^\sim \colon U \times V &\to U \\ (u, v) &\mapsto u \end{aligned}$$

and ψ^* which acts on coordinates as

$$\psi^*(x^i) = x^i, \quad \psi^*(\theta^j) = \theta^j$$

is a submersion.

We say that the immersion and the submersion in the above example are in standard form.

Theorem 1.2.13. *Suppose $\phi \colon M \to N$ is an immersion (submersion) at $m \in M$. Then, there exist charts α at m and β at $\phi^\sim(m)$ such that $\beta\phi\alpha^{-1}$ is in standard form.*

Proof. The proof relies on the super analogue of the Inverse Function Theorem. We refer the reader to [16, Sec. 4.4] for a full account. □

This discussion allows us to introduce in a clean way the notion of submanifold.

Definition 1.2.14. Let M be a supermanifold. A *submanifold* of M is a pair (S, i), where S supermanifold and $i \colon S \to M$ is an immersion such that $i^\sim \colon S^\sim \to M^\sim$ is an embedding onto a closed or locally closed submanifold of $M^\sim$.

As in the classical case, the standard form for immersions clarifies what submanifolds look like locally.

§1.3 Integration on supermanifolds

One key aspect of supergeometry is the fact that there exists a way of making sense of integration. If we look at integration, two potential issues come to mind, namely the definition of the integral of a locally defined function and the problem of patching up local integrals in a suitable way. The second one is readily taken care of, since supermanifolds admit partitions of unity. Thus, the only thing to address is how to integrate with respect to odd coordinates. The answer to this question is the *Berezin integral*, named after the mathematician who first discovered it, Felix Berezin. In this section, we discuss the Berezin integral in full details and we also describe an invariant way of treating integrals, namely through berezinians, the super analogues of top forms.

Before getting into integration on supermanifolds, let us first consider the purely odd case, that is integration on a Grassmann algebra $R[\theta^1, \ldots, \theta^q]$. We define

$$\int \colon R[\theta^1, \ldots, \theta^q] \to R$$

to be the linear map such that

$$\int \theta^I = \begin{cases} 0, & |I| < q \\ 1, & I = (1, \dots, q) \end{cases}$$

Basically, the odd integral selects the coefficient of the highest term of an element of $R[\theta^1, \dots, \theta^q]$.

Let us now use this notion to define the Berezin integral on a superdomain $U^{p|q}$. For a compactly supported section $f \in C^\infty[\theta^1, \dots, \theta^q]$, we define

$$\int_U f$$

by first performing an odd integral to obtain a compactly supported function on U and then just integrating it in the classical sense.

The point of this definition is that it gives a well-behaved object under changes of coordinates. Consider an isomorphism of supermanifolds $\phi\colon U^{p|q} \to V^{p|q}$. Then, we have the following theorem.

Theorem 1.3.1.

$$\int_V s = \int \phi^*(s) \operatorname{Ber}(J\phi)$$

for all compactly supported sections s on V.

Proof. See [16, Thm. 4.6.1]. □

In order to integrate over ordinary oriented manifolds, one picks a top-dimensional differential form and locally uses the identification $C^\infty(U) \cong \Omega^{\mathrm{top}}(U)$. The point is that top forms behave well under change of coordinates, i.e. the Jacobian of the transformation shows up, which is then taken care of by the change of variables formula for integrals in Euclidean space. Let us now see how to generalize this to the supergeometric setting.

Let M be a supermanifold of dimension $p|q$. The basic object that can be integrated on supermanifolds is a (compactly supported) section of the *Berezinian line bundle* $\operatorname{Ber}(M)$, defined to be the Berezinian $\operatorname{Ber}(\Omega^1_M)$ of the sheaf of 1-forms on M.

Locally, on a chart with coordinates (x^i, θ^j), the Berezinian line bundle is generated by the element $dx^1 \wedge \dots dx^p \wedge d\theta^1 \wedge \dots \wedge d\theta^q$. If (y_k, φ^l) are different coordinates, the corresponding sections of the Berezinian bundle are related by

$$dy^1 \wedge \dots dy^p \wedge d\varphi^1 \wedge \dots \wedge d\varphi^q = \operatorname{Ber}(J\phi) dx^1 \wedge \dots dx^p \wedge d\theta^1 \wedge \dots \wedge d\theta^q,$$

where ϕ is the automorphism taking the coordinates (y^k, φ^l) to (x^i, θ^j).

If σ is a compactly supported section of $\operatorname{Ber}(M)$, we can define its integral by

$$\int_M \sigma = \sum_\alpha \int_{U_\alpha} \rho_\alpha \sigma,$$

where $\{U_\alpha\}$ is a cover of M by charts, $\{\rho_\alpha\}$ is a partition of unity subordinate to this cover and the integrals over the U_α's are computed by using the charts to obtain integrals over superdomains, just like in the classical case.

Up to this point we have tacitly assumed that the reduced manifold $M^\sim$ is oriented. As in ordinary differential geometry, there are more general objects, called *densities*, which can be integrated on non-orientable supermanifolds. The density bundle is constructed analogously to the Berezinian bundle. Note that if (U, ϕ) and (V, ψ) are overlapping trivializing open sets for $\mathrm{Ber}(M)$, the transition function is given by $\mathrm{Ber}\left(J(\psi\phi^{-1})\right)$. If (U_α, ϕ_α) is a trivializing open cover for $\mathrm{Ber}(M)$, the bundle of s-densities is defined to be the line bundle with same trivializing open cover and transition functions

$$\mathrm{sign}\,\mathrm{Ber}\left(J(\phi_\beta\phi_\alpha^{-1})\right)\left|\mathrm{Ber}\left(J(\phi_\beta\phi_\alpha^{-1})\right)\right|^s,$$

for $s > 0$. If $M^\sim$ is oriented, the sign factor disappears.

§1.4 $\mathbb{Z}$-graded algebra

In the previous sections we have generalized linear algebra by adding a notion of parity ($\mathbb{Z}_2$-grading) to vector spaces and we used it to define supermanifolds, which differ from ordinary manifolds in that local coordinates may anticommute.

We now want to discuss a further generalization, namely $\mathbb{Z}$-graded linear algebra. This will be the first step towards $\mathbb{Z}$-graded geometry, which will play a fundamental role later on.

Consider a sequence of abelian groups $A = (A_k)_{k\in\mathbb{Z}}$. We will call such a sequence a *graded abelian group*. A morphism $\phi\colon A \to B$ of graded abelian groups is a sequence of group morphisms $\phi_k\colon A_k \to B_k$. We define the tensor product of two graded abelian groups A, B to be the graded abelian group $A \otimes B$ whose components are given by

$$(A \otimes B)_n = \bigoplus_{k+l=n} A_k \otimes B_l.$$

Let now $R = (R_k)_{k\in\mathbb{Z}}$ be a graded abelian group. We say that R is *graded ring* if it is equipped with a morphism of graded abelian groups $\mu\colon R \otimes R \to R$ that is associative and for which there exists an element $1 \in R$ that satisfies

$$\mu(1, r) = \mu(r, 1) = r$$

for all $r \in R$.

The graded ring R is called *commutative* if

$$\mu(r, s) = (-1)^{|r||s|}\mu(s, r).$$

for all homogeneous elements $r, s \in R$. We will simplify the notation by putting $rs = \mu(r, s)$.

A morphism $\phi\colon R \to S$ is a morphism of graded abelian groups that preserves the multiplication maps and that satisfies

$$\phi(rs) = \phi(r)\phi(s).$$

The notions of ideal, local graded ring and local ring morphism are defined in the usual way.

Graded vector spaces and algebras and their morphisms are defined analogously. For the sake of this paper, we will only treat real vector spaces. We can consider morphisms that shift the degree of elements. Let V, W be graded vector spaces. A collection of linear maps $\phi_k\colon V_k \to W_{k+l}$ is called a *graded linear map of degree l*. Similarly to the $\mathbb{Z}_2$-graded case, we denote the space of maps of degree l by $\mathrm{Hom}_l(V,W)$. Then, $\big(\mathrm{Hom}_l(V,W)\big)$ carries the structure of a graded vector space. In particular, if $W_0 = \mathbb{R}$ and $W_i = 0$ for all $i \neq 0$, we obtain the dual V^* of V.

An operation that is quite common when dealing with graded vector spaces is that of shifting the degree. Let V be a graded vector space. For $l \in \mathbb{Z}$ we define the graded vector space $V[l]$ by

$$(V[l])_k = V_{k+l}.$$

Thus, a graded linear map $\phi\colon V \to W$ of degree l is a graded linear map of degree 0 from V to $W[l]$.

We define the dimension of a graded vector space V to be the sum of the dimensions of its components:

$$\dim(V) = \sum_{k\in\mathbb{Z}} \dim(V_k).$$

If V is finite-dimensional, a *basis* of V is an ordered collection $\big(\{\xi_\mu\}_{\mu\in I_k}\big)_{k\in\mathbb{Z}}$ of bases for the components of V.

Example 1.4.1. This example will be crucial when we discuss $\mathbb{Z}$-graded manifolds. Let $(n_k)_{k\in\mathbb{Z}}$ be a sequence of non-negative numbers. We define $\mathbb{R}^{(n_k)}$ to be graded vector space having components $(\mathbb{R}^{(n_k)})_j = \mathbb{R}^{n_j}$.

A *graded algebra* is a graded vector space A together with a map $\mu\colon A\times A \to A$ that makes A into a graded ring. If we have a collection $(A_\alpha)_{\alpha\in\mathcal{A}}$ of graded algebras, we can construct their direct product $\prod_{\alpha\in\mathcal{A}} A_\alpha$ by

$$\left(\prod_{\alpha\in\mathcal{A}} A_\alpha\right)_k = \prod_{\alpha\in\mathcal{A}} (A_\alpha)_k$$

with component-wise multiplication.

Let us now look at some important examples, some of which we have already encountered in the $\mathbb{Z}_2$ graded setting.

Example 1.4.2. (i) Given a graded vector space V, denote by $T^p(V)$ the graded vector space given by

$$T^p(V)_k = \bigoplus_{k_1+\cdots+k_p=k} V_{k_1}\otimes\cdots\otimes V_{k_p}.$$

the tensor algebra is then

$$T(V) = \bigoplus_{p\in\mathbb{Z}} T^p(V),$$

where the direct sum of graded vector spaces is defined in the obvious way. The multiplication map is given by

$$\begin{aligned} \mu\colon\, & T(V)\otimes T(V) \to T(V) \\ & (v_1\otimes\cdots\otimes v_p)\otimes(w_1\otimes\cdots\otimes w_q) \mapsto (v_1\otimes v_p\otimes w_1\cdots\otimes w_q). \end{aligned}$$

As in the ordinary case, the tensor algebra can be more elegantly defined through a universal property (see [17] for more details).

(ii) Again, let V be a graded vector space and consider the ideal $I \subseteq T(V)$ generated by the graded set having as kth component

$$\{v\otimes w - (-1)^{|v||w|} w\otimes v \mid |v| + |w| = k\}.$$

The quotient $S(V) = T(V)/I$ is called the symmetric algebra of V and it is a commutative graded algebra when endowed with the product coming from $T(V)$.

(iii) If instead of the ideal I we consider the ideal J generated by the graded set having kth component

$$\{v \otimes w + (-1)^{|v||w|} w \otimes v \mid |v| + |w| = k\},$$

we get the exterior algebra $\Lambda V = T(V)/J$.

(iv) Let A be a graded algebra and V be a graded vector space. We define the *extended symmetric algebra of V with coefficients in A*

$$\bar{S}(V, A) = \prod_{p=0}^{\infty} A \otimes S^p(V).$$

The product is defined using the Koszul sign convention:

$$(a \otimes \sigma)(b \otimes \tau) = (-1)^{|\sigma||b|}(ab) \otimes (\sigma\tau).$$

In practice, we will have $A_k = 0$ for $k < 0$ and $V_k = 0$ for $k \leq 0$. In this case, the extended symmetric algebra coincides with a simpler object, namely

$$S(V, A) = \bigotimes_{p=0}^{\infty} A \otimes S^p(V).$$

Let us take a closer at look at the extended symmetric algebra of a graded vector space V with $\dim(V) = n$. To this end, let $\{\xi_\mu\}$ be a total basis for V and define

$$\mathbb{N}_k^n = \left\{ I = (i_1, \ldots, i_n) \in (\mathbb{N} \cup \{0\})^n \;\middle|\; \sum_{\mu=1}^{n} i_\mu |\xi_\mu| = k,\; i_\mu \in \{0, 1\} \text{ if } |\xi_\mu| \text{ is odd} \right\}.$$

Note that this set might be infinite. However, it can be written as a disjoint union

$$\mathbb{N}_k^n = \coprod_{p=0}^{\infty} N_k^n(p),$$

where

$$\mathbb{N}_k^n(p) = \left\{ I \in \mathbb{N}_k^n \;\middle|\; \sum_{\mu=1}^{n} i_\mu = p \right\},$$

which is a finite set for all $p \geq 0$.

For $I \in (\mathbb{N} \cup \{0\})^n$, let us write

$$\xi^I = \xi^{i_1} \ldots \xi^{i_n},$$

where the product is in the symmetric algebra of V. Then, it is easy to see that the collection $\{\xi^I\}_{I \in \mathbb{N}_k^n(p)}$ is a basis of $S^p(V)_k$, which in particular shows that $S^p(V)$ is finite-dimensional (note that $\coprod_{p\in\mathbb{Z}} \mathbb{N}_k^n(p)$ is still a finite set).

If we further make the simplification that $A_k = 0$ for $k \neq 0$, it is straightforward to see that every element in $\hat{S}(V, A)$ is a formal power series in the ξ_μ's with coefficients in A. This will be the case in our applications to $\mathbb{Z}$-graded geometry.

Let us briefly discuss graded modules and derivations. Given a graded algebra A, a graded A-module is a graded vector space together with a morphism

$$\lambda\colon A \otimes V \to V$$

that satisfies

$$\lambda\big((ab), x\big) = \lambda\big(a, \lambda(b, x)\big)$$
$$\lambda(1, x) = x$$

for all $x \in V$ and $a, b \in A$. We will drop this notation and just write $\lambda(a, x) = ax$.

Let V, W be graded A-modules. A map $\phi\colon V \to W$ is a morphism of A-modules of degree l if it shifts the degree of homogeneous elements by l and

$$\phi(ax) = (-1)^{|a|l} a\phi(x)$$

for all $x \in V$ and homogeneous $a \in A$.

Then, a *derivation of degree* l with values in V is a morphism $D\colon A \to V$ of degree l that additionally satisfies the graded Leibniz identity:

$$D(ab) = D(a)b + (-1)^{|a|l} aD(b)$$

for all homogeneous $a, b \in A$.

Clearly, derivations form an A-module with the obvious A-action. Given two derivations $D_1, D_2\colon A \to V$, their composition is not necessarily a derivation, but their *graded commutator*

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$$

is.

Example 1.4.3. This example will be crucial in our discussion of graded vector bundles. Consider a topological space X and a sheaf $\mathcal{A}$ over X of commutative graded algebras. Additionally, let V be a finite-dimensional graded vector space. Then, the assignment

$$U \mapsto \mathcal{A}[V](U) = \mathcal{A}(U) \otimes V$$

defines a sheaf of graded vector spaces. To verify the sheaf property one picks a total basis $\{\theta_\mu\}$ for V. Then, every local section $s \in \mathcal{A}[V](U)$ is a linear combination

$$s = \sum_\mu a_\mu \otimes \theta_\mu,$$

where $a_\mu \in \mathcal{A}(U)$. The assertion then easily follows from the fact that $\mathcal{A}$ is a sheaf.

Notice that $\mathcal{A}[V](U)$ has a natural $\mathcal{A}(U)$-module structure that makes $\mathcal{A}[V]$ into a sheaf of graded $\mathcal{A}$-modules.

We say that a sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules is *freely and finitely generated* if it is isomorphic to $\mathcal{A}[V]$ for some finite-dimensional graded vector space V.

It would be tempting to define the rank of $\mathcal{F} \cong mathcal A[V]$ to be the dimension of V, but it turns out that this is not a good definition as it is not uniquely determined. If we add the condition that there exists an open set U and an ideal $J \subseteq \mathcal{A}(U)$ such that $\mathcal{A}(U)/J \cong \mathbb{R}$, then $\dim(V)$ is uniquely determined and we can speak of the rank of $\mathcal{F}$. We refer the reader to [17, Sec. 2.4] for a detailed discussion of this subtle matter. In what follows, we will always be able to speak of the rank of the sheaves of modules we consider.

§1.5 $\mathbb{Z}$-graded manifolds

In this section we use the tools we have introduced to define $\mathbb{Z}$-graded manifolds. The intuitive picture we should keep in mind is similar to the case of supermanifolds, in that we want to define manifolds where coordinates have a degree in $\mathbb{Z}$ and commute accordingly. However, this generalization turns out to be a bit more subtle than in the $\mathbb{Z}_2$-graded case. Once one gets past this initial hurdle, the theory unfolds naturally.

In order to define graded manifolds, the first step is identifying the local model. The naive definition would be to say that a graded domain is an open set U in some Euclidean space together with the algebra of functions $C^\infty(U) \otimes S(V)$ for some finite-dimensional graded vector space V. However, this definition turns out to be bad. Indeed, the assignment

$$U \mapsto C^\infty(U) \otimes S(V)$$

is only a presheaf, not a sheaf. To see this, consider the following example (see also [17]). Let V be the graded vector space with $V_{-2} = V_2 = \mathbb{R}$ and $V_k = 0$ for $k \neq -2, 2$. Also, let $U = \coprod_{i \in \mathbb{Z}} (i, i+1)$ and consider a total basis $\{\eta, \xi\}$ for V, where $|\eta| = -2, |\xi| = 2$. Then, we can define local sections $s_i \in C^\infty(i, i+1) \otimes S(V)$ of degree 0 by

$$s_i = \eta^i \xi^i.$$

The s_i's agree on the overlaps but they do not glue to a polynomial in $C^\infty(U) \otimes S(V)$.

Let us see how to work around this issue. Let $(n_j)_{j \in \mathbb{Z}}$ be a sequence of non-negative integers such that $\sum_{j \in \mathbb{Z}} n_j < \infty$. Instead of considering the graded vector space $\mathbb{R}^{(n_j)}$, which we defined earlier, we will work with $\mathbb{R}_*^{(n_j)}$, which is the same graded vector space except in degree 0, where we set $\big(\mathbb{R}_*^{(n_j)}\big)_0 = 0$.

Definition 1.5.1. A *graded domain* is an open set $U \subseteq \mathbb{R}^{n_0}$ together with the algebra of functions

$$C^\infty_{(n_j)}(U) = \bar{S}\big(\mathbb{R}_*^{(n_j)}, C^\infty_{n_0}(U)\big),$$

i.e. the extended symmetric algebra of $\mathbb{R}_*^{(n_j)}$ with coefficients in the algebra of ordinary smooth functions on U. We will denote this graded domain by $U^{(n_j)}$.

It is easy to verify that this does indeed define a sheaf that makes $(U, C^\infty_{(n_j)}(U))$ into a graded locally ringed space. The maximal ideal in the stalk $C^\infty(U)_x$ at $x \in U$ takes the usual form

$$J_x = \{[f]_x \in C^\infty_{(n_j)}(U)_x \mid f(x) = 0\},$$

where $f(x)$ denotes the evaluation of f at x, defined exactly as in the $\mathbb{Z}_2$-graded case (modulo the obvious adjustments).

Recall the description of elements of $C^\infty_{(n_j)}(U)$. Any homogeneous section s can be written as a formal sum

$$s = \sum_{I \in \mathbb{N}^{n_*}_{|s|}} s_I \xi^I,$$

where the s_I's are smooth functions on U and $\{\xi_\mu\}_{\mu=1}^{n_*}$ is a total basis for $\mathbb{R}_*^{(n_j)}$ (here $n_* = \sum_{j \neq 0} n_j$). There exists a canonical sheaf morphism $\beta \colon C^\infty \to C^\infty_{n_0}$ defined on homogeneous sections as follows:

$$\beta_V(s) = \begin{cases} s_0, & |s| = 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $s \in C^\infty(V)$ and $V \subseteq U$ open. The kernel of β defines the sheaf of purely graded sections

$$\mathcal{J}^{\mathrm{pg}}_{(n_j)} = \ker(\beta).$$

With these notions, one can formulate the Hadamard Lemma as for supermanifolds. The reader can find a detailed discussion in [17].

Let $U^{(n_j)}, V^{(m_k)}$ be graded domains. A *morphism of graded domains* $U^{(n_j)} \to V^{m_k}$ is a pair

$$\begin{aligned} &\phi\colon U \to V \\ &\phi^*\colon C^\infty_{(m_k)}(V) \to C^\infty_{(n_j)}(U), \end{aligned}$$

where the first is just an ordinary smooth map and the second is a local morphism of graded rings. In order to avoid cumbersome notation, we will denote a morphism (ϕ, ϕ^*) just by ϕ.

Remark 1.5.2. Again, ϕ^* would take values in the pushforward sheaf $\phi_* C^\infty_{(n_j)}(U)$, but we will not be pedantic. It is clear what we mean.

One can then proceed to study the structure of morphism of graded domain. It turns out that the characterization we gave for superdomains (supermanifolds) holds in the $\mathbb{Z}$-graded case as well and the proof is very similar.

Theorem 1.5.3. *Let $U^{(n_j)}, V^{(m_k)}$ be graded domains. Let $(y^1, \dots, y^{m_0})$ be coordinates on V and $\theta^1, \dots, \theta^{m_*}$ be a total basis for $\mathbb{R}^{(m_k)}_*$ (here, we set $m_* = \sum_{k \neq 0} m_k$). Then, there is a bijection between morphisms $\phi\colon U^{(n_j)} \to V^{(m_k)}$ and tuples $(\bar{x}^1, \dots, \bar{x}^{m_0}, \bar{\theta}^1, \dots, \bar{\theta}^{m_*})$, where $\bar{x}^i \in \mathcal{J}^{\mathrm{pg}}_{(n_j)}(U)_0$ and $\bar{\theta}^\nu \in C^\infty_{(n_j)}(U)_{|\theta^\nu|}$ for all $1 \leq i \leq m_0, 1 \leq j \leq m_*$. More explicitly, the relation between these functions and ϕ is given by*

$$\bar{x}^i = \phi^*(y^i) - y^i \phi, \quad \bar{\theta}^\nu = \phi^*(\theta^\nu).$$

Proof. See the discussion in [17, Sec. 3.2]. □

What this theorem basically says is that to specify a morphism of graded domains, one only needs to declare what its pullback does on the coordinates of the codomain.

Now that we have our local model, we can define graded manifolds.

Definition 1.5.4. A *graded manifold* is a graded locally ringed space (M, C^∞) such that

(i) M is a second countable Hausdorff topological space,

(ii) for every $x \in M$, there exists an open set $U \ni x$ and an isomorphism of graded locally ringed spaces $C^\infty|_U \cong \hat{U}^{(n_j)}$, where (n_j) is a (fixed) sequence of non-negative integers with $\sum_{j \in \mathbb{N}} n_j < \infty$ and $\hat{U} \subseteq \mathbb{N}^{n_0}$ is open.

(n_j) is called the *dimension* of (M, C^∞). We will often not bother to write the sheaf of functions C^∞ and just denote (M, C^∞) by M.

Morphisms of graded manifolds are defined in the usual way.

We saw that we can think of supermanifolds as ordinary manifolds surrounded by a "cloud" of odd stuff. That is because we saw that to every supermanifold we can canonically associate a reduced manifold by quotienting out the nilpotent elements of its algebra of functions. The same intuition holds in the $\mathbb{Z}$-graded case, as the sheaf morphism β we defined above glues to a well defined morphism i_M with pullback

$$i_M^*\colon C^\infty \to C_M^\infty,$$

where C_M^∞ is the sheaf of ordinary smooth functions on M (see [17, Prop. 3.21]). Given a function $f \in C^\infty(U)$ and a point $x \in U$, we can define the evaluation of f at x by

$$f(x) = i_M^*(f)(x).$$

At the infinitesimal level, a graded manifold is described by its tangent spaces. The tangent space to M at $x \in M$ is the graded vector space

$$T_x M = \operatorname{Der}(C_x^\infty, \mathbb{R})$$

of real graded derivations of the stalk of C^∞ at x.

Remark 1.5.5. There is some ambiguity in our notation, as $T_x M$ could denote both the tangent space to the graded manifold and to the underlying smooth manifold. The context will clarify what we mean on a case-by-case basis.

Given a morphism

$$\phi\colon (M, C_M^\infty) \to (N, C_N^\infty)$$

of graded manifolds, we get an induced map

$$\phi_x\colon C_{N,\phi(x)}^\infty \to C_{M,x}^\infty$$

of the stalks. Thus, we can define the differential of ϕ at x to be the morphism

$$\begin{aligned} d_x\phi\colon T_x M &\to T_{\phi(x)} N \\ v &\mapsto v\phi_x. \end{aligned}$$

As with supermanifolds, if (x^i, ξ^μ) are local coordinates on M, the tangent vectors

$$\left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial \xi^\mu}\right|_x$$

form a basis for $T_x M$.

The stalks of C^∞ carry more information than just what happens pointwise. Indeed, they carry local information. In differential geometry, one way to define tangent vectors is as derivations of the algebra of functions defined in a neighborhood of the base point. We can do the same for graded manifolds. Indeed, if $x \in M$ and $U \ni x$ is a neighborhood, there exists a canonical isomorphism of graded vector spaces

$$T_m M = \operatorname{Der}(C_x^\infty, \mathbb{R}) \cong \operatorname{Der}\big(C^\infty(U), \mathbb{R}\big).$$

Here, $\mathbb{R}$ is viewed as a $C^\infty(U)$ module by setting

$$f \cdot \lambda = f(x)\lambda, \quad f \in C^\infty(U), \lambda \in \mathbb{R}.$$

Explicitly, the above isomorphism is given by precomposing with the projection

$$\pi_{U,x}\colon C^\infty \to C_x^\infty.$$

We leave the details to the reader (see also [17, Prop. 4.8]).

Let us take a closer look at the sheaf of vector fields

$$U \mapsto \mathfrak{X}_M(U) = \mathrm{Der}\big(C^\infty(U), \mathbb{R}\big).$$

The proof of the fact that it is indeed a sheaf is standard. It uses the existence of partitions of unity on graded manifolds. We refer the reader to [17, Sec. 4.2] for the technical details. It is also routine to prove that it is locally freely and finitely generated. Indeed, a basis over a coordinate chart (U, x^i, ξ^μ) is given by

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^\mu}.$$

Example 1.5.6. On graded manifolds there is a canonical vector field E, the *Euler vector field*, that has no classical counterpart. It is defined on homogeneous functions $f \in C^\infty(M)$ by

$$Ef = |f|f.$$

It is manifestly a derivation of degree 0. In local coordinates (x^i, ξ^μ) it looks like

$$E = \sum_\mu |\xi^\mu| \xi^\mu \frac{\partial}{\partial \xi^\mu}.$$

As usual, the space of vector fields is not closed under composition, but it is closed under the *graded commutator*

$$[X, Y] = XY - (-1)^{|X||Y|} YX,$$

which makes $\mathfrak{X}_M$ into a sheaf of *graded Lie algebras* (whose definition is analogous to that of Lie superalgebras).

Definition 1.5.7. A *graded vector bundle* over a graded manifold M is a locally freely and finitely generated sheaf of C^∞-modules.

Examples 1.5.8. (i) We already seen that the sheaf of vector fields $\mathfrak{X}_M$ over a graded manifold M is graded vector bundle.

(ii) Given a graded vector bundle E, its dual E^* is also freely and finitely generated. Hence, E^* is a graded vector bundle.

(iii) The sheaf Ω_M of *differential forms* on M is given by

$$U \mapsto \Omega_M(U) = \mathrm{Hom}_{C^\infty(U)}\big(\mathfrak{X}_M(U), C^\infty(U)\big),$$

Note that it is canonically isomorphic to $\mathfrak{X}_M^*$.

Differential forms of higher order are defined as usual (see [17, Sec. 6.1] for a different viewpoint). The de Rham differential is defined over a coordinate chart (U, x^i, ξ^μ) as

$$d = \sum_i dx^i \frac{\partial}{\partial x^i} + \sum_\mu d\xi^\mu \frac{\partial}{\partial \xi^\mu}$$

and it glues to a well defined global differential over M.

If $X = \sum_i X^i \frac{\partial}{\partial x^i} + \sum_\mu X^\mu \frac{\partial}{\partial \xi^\mu}$ is a vector field over M, we can define the *interior product* or *contraction* with X locally by

$$i_X = \sum_i X^i \frac{\partial}{\partial(dx^i)} + \sum_i X^\mu \frac{\partial}{\partial(d\xi^\mu)},$$

which again glues to a well-defined global derivation of degree -1 on the space of differential forms.

Lastly, we can introduce the Lie derivative with respect to X, which is given by Cartan's magic formula

$$L_X = [i_X, d] = i_X d + d i_X,$$

the graded commutator of i_X and d.

These three objects satisfy the usual rules of Cartan calculus, namely

$$[L_X, L_Y] = L_{[X,Y]}, \quad [L_X, d] = 0, \quad [L_X, i_Y] = i_{[X,Y]}.$$

We can also consider the Lie derivative of a Berezinian (here, we view M as a supermanifold with $\mathbb{Z}_2$-grading coming from the reduction modulo 2 of the $\mathbb{Z}$-grading). The definition of the Lie derivative on Berezinians goes along the same lines as the ordinary one. We refer the reader to [6] for a detailed discussion. Let μ be a Berezinian and X a vector field. Since the Berezinian bundle has rank $1|0$ (or $0|1$), there exists as function f such that

$$L_X \mu = f\mu.$$

We call f the *divergence of* X with respect to μ and we denote it by $\mathrm{div}_\mu(X)$. In terms of integrals, the definition of divergence of a vector field looks like

$$\int_M X(f)\mu = -\int_M f \mathrm{div}_\mu(X)$$

for all $f \in C_c^\infty(M)$.

So far, we have not discussed any examples of graded vector bundles. Let us briefly describe a few.

Examples 1.5.9. (i) Let M be a smooth manifold. Then, we can consider its tangent bundle shifted by 1, denoted by $T[1]M$. What this means it that we take TM and declare the fibers to have degree 1. If (U, x^i) is a local trivialization, we have a local frame for TM given by the $\frac{\partial}{\partial x^i}$'s. Then, we can interpret them as coordinates of degree 1. Note that under this interpretation, there is a canonical identification

$$C^\infty(T[1]M) \cong \Omega^\bullet(M).$$

(ii) We can do the same thing with the cotangent bundle T^*M, but this time we shift it by -1. Locally, we have a frame given by the dx^i's, which we think of as coordinates of degree -1. Then, there is a canonical identification

$$C^\infty(T^*[-1]M) \cong \mathfrak{X}^\bullet(M),$$

of the space of functions on $T^*[-1]M$ with the space of multivector fields on M.

(iii) Let us recall the definition of the Chevalley-Eilenberg complex. Given a Lie algebra $\mathfrak{g}$, we define the real Chevalley-Eilenberg cochains to be

$$\Omega^\bullet_{CE}(\mathfrak{g}) = \mathrm{Hom}(\Lambda^\bullet, \mathbb{R}).$$

The Lie bracket $[\cdot, \cdot] \colon \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ induces by transposition a map

$$d_{CE} \colon \mathfrak{g}^* \to (\mathfrak{g} \wedge \mathfrak{g})^* \cong \mathfrak{g}^* \wedge \mathfrak{g}^*,$$

which readily extends to a map

$$d_{CE} \colon \Lambda^\bullet \mathfrak{g}^* \to \Lambda^{\bullet+1} \mathfrak{g}^*$$

by imposing the Leibniz rule. It is straightforward to check that d_{CE} squares to zero as a consequence of the Jacobi identity for $[\cdot, \cdot]$.

Consider now the graded manifold $M = \mathfrak{g}[1]$. It is easy to see that smooth functions on M can be identified with the Chevalley-Eilenberg complex of $\mathfrak{g}$. Thus, M is equipped with a *cohomological vector field*, i.e. a degree 1 vector field that squares to 0.

Chapter 2

Graded symplectic geometry

In this chapter we introduce symplectic structures on graded symplectic manifolds following the seminal work of Schwarz in [14]. The theory differs from ordinary symplectic geometry in that graded symplectic manifolds are more rigid than ordinary ones. Indeed, we will prove that basically the only example of such an object is the shifted cotangent bundle to a smooth manifold endowed with standard symplectic structure. We also discuss Lagrangian submanifolds and we set up a natural way of performing integration on them. The most important theorem we prove is the BV-Stokes' theorem, which roughly asserts that under some mild conditions on the integrand the operation of integrating over (closed oriented) Lagrangian submanifolds only depends on the homology class of the Lagrangian. This theorem, first proved by Batalin and Vilkovisky in the early 1980's, is the starting point of the Batalin-Vilkovisky formalism, which is a powerful gauge-fixing technique in the context of perturbative quantum field theory. We will discuss this at length in the next chapter.

Apart from Schwarz's paper [14], we have also drawn material from Pavel Mnev's book [13].

§2.1 Basic definitions

In this section we define graded symplectic manifolds and we discuss how to generalize a few notions from ordinary symplectic geometry. Although our definitions are general, in our applications to the BV formalism we will only focus on odd-symplectic structures, i.e. symplectic structure that have odd internal degree.

In what follows, all graded manifolds are assumed to be finite-dimensional. We will also assume that the associated reduced manifold is compact.

Definition 2.1.1. Let M be a graded manifold. A *symplectic structure* (sometimes also called P-structure) on M is a closed non-degenerate 2-form $\omega \in \Omega^2(M)$. Here, non-degeneracy means that the induced map

$$\begin{aligned} \omega^\sharp \colon TM &\to T^*M \\ v &\mapsto \omega(v, \cdot) \end{aligned}$$

is a bundle isomorphism. We say that ω has degree n if

$$L_E\omega = n\omega,$$

where E is the Euler vector field on M.

Example 2.1.2. Let N be a smooth manifold. Then, the shifted cotangent bundle $T^*[1]N$ carries a natural symplectic structure, i.e. the one coming from the Liouville 1-form on N. Note that this yields a degree 1 symplectic structure on $T^*[1]N$. In local coordinates (x^i, ξ_μ), it looks like

$$\omega_{std} = \sum_i dx^i \wedge d\xi_i.$$

In ordinary symplectic geometry, Darboux's theorem ensures that all symplectic manifolds locally look like $T^*\mathbb{R}^n$ with the standard symplectic structure. Graded symplectic manifolds are more rigid, as this holds globally.

Remark 2.1.3. The de Rham cohomology of a graded supermanifold is particularly simple. Indeed, let τ be a closed k-form with internal degree n. Then, $L_E\tau = n\tau$. By Cartan's magic formula, we have

$$n\tau = L_E\tau = i_E d\tau + d i_E \tau = d i_E \tau,$$

which means that $\tau = \frac{1}{n} d i_E \tau$ if $n \neq 0$. Thus, symplectic structures of degree bigger than 0 are automatically exact.

For the sake of clarity, let us forget about the $\mathbb{Z}$-grading and let us only focus on its parity. In other words, let us consider graded symplectic manifolds as supermanifolds with the $\mathbb{Z}_2$-grading coming from reduction modulo 2. Then, we have the following theorem.

Theorem 2.1.4. *Let (M, ω) be an odd-symplectic manifold. Then, there exists a smooth manifold N and a symplectomorphism*

$$\psi\colon (M, \omega) \to (T^*[1]N, \omega_{std}).$$

Moreover, one take $N = M^\sim$, i.e. the reduction of M.

Proof. By Batchelor's theorem (see Chapter 1), we can assume that M is an odd vector bundle $\pi\colon M \to N$. Let us consider the restriction $\omega|_N$ of ω to N, viewed as the zero section of M. In local coordinates (x^i, ξ^μ) on M, where the x^i's are coordinates on the base, we have

$$\omega|_N = \sum_{i,\mu} \omega_{i\mu}(x) dx^i \wedge d\xi^\mu.$$

Thus, we get a pairing between the fibers of M and the fibers of $T^*[1]N$ and hence a bundle morphism

$$\begin{aligned} \phi\colon M &\to T^*[1]N \\ (x, v) &\mapsto \omega|_N(x)(\cdot, v). \end{aligned}$$

By non-degeneracy of ω, this is a bundle isomorphism. Here, we are making the obvious identification between the fibers of M and the tangent space to the fibers.

The only thing left to check is that the symplectic structures ω and $\omega_0 = \phi^*\omega_{std}$ are equivalent. Note that we can connect ω_0 and ω via the path

$$\omega_t = (1-t)\omega_0 + t\omega.$$

Note that ω_t is a symplectic form, since they coincide on $N \subseteq M$, which together with the fact that ω, ω_0 are non-degenerate on N gives non-degeneracy on $T^*[1]N$.

The proof of the equivalence of ω_0 and ω goes along the same lines as Moser's trick (see [15]). We want to find a (time-dependent) vector field X such that $\frac{d}{dt}\omega_t = L_{X_t}\omega_t$.

Such a vector field can then be integrated to an isotopy ρ_t such that $\rho_t^*\omega_t = \omega_0$, thus proving that ω_0 and ω are equivalent. Note that

$$L_{X_t}\omega_t = di_{X_t}\omega_t + i_{X_t}d\omega_t = di_{X_t}\omega_t.$$

Since $\frac{d}{dt}\omega_t$ is closed and odd, it is exact (see the above remark). Thus, we can find a smooth family of odd 1-forms μ_t such that $d\mu_t = di_{X_t}\omega_t$ (see [12] for a proof of smoothness). To solve this equation, it is sufficient to solve

$$\mu_t = i_{X_t}\omega_t,$$

which can be done uniquely thanks to the non-degeneracy of ω_t. □

§2.2 Lagrangian submanifolds and the BV theorem

In this section we introduce Lagrangian submanifolds, which play a fundamental role in our application of graded symplectic geometry to gauge theories. We then define a suitable notion of integration over Lagrangian submanifolds and we prove the BV theorem, which roughly says that under some conditions on the integrand, the integral only depends on the homology class of the Lagrangian. This is the starting point of the BV formalism, which we will develop in the next chapter.

Definition 2.2.1. Let (M, ω) be an odd-symplectic manifold of dimension $n|n$. An embedded submanifold $L \subseteq M$ is called *Lagrangian* if

(i) L is isotropic, i.e. $i^*\omega = 0$, where $i\colon L \to M$ is the inclusion map,

(ii) L is maximal, i.e. it is not a proper submanifold of any other isotopic submanifold.

Remark 2.2.2. Note that if M has dimension $n|n$ and the reduced manifold $L^\sim \subseteq M^\sim$ has dimension k, then L has dimension $n|n-k$.

Example 2.2.3. Let $(T^*[1]N, \omega_{std})$ be the shifted cotangent bundle to the smooth manifold N endowed with the standard symplectic structure. Let $K \subseteq N$ be a k-dimensional embedded submanifold. Then, we can consider the conormal bundle

$$N^*K = \{(x, l) \in T^*N \mid x \in K, l|_{T_xK} = 0\},$$

which is a rank $n-k$ subbundle of T^*N. Thus, the shifted conormal bundle $N^*[1]K$ is a $(k|n-k)$-dimensional submanifold of $T^*[1]N$. This is the most basic example of a Lagrangian submanifold of $T^*[1]N$.

The previous example is of fundamental importance because of the next theorem. Indeed, it is basically the only type of Lagrangian submanifold.

Theorem 2.2.4 (Schwarz)**.** *Let (M, ω) be a graded symplectic manifold and let $N = M^\sim$ be the associated reduced manifold. If $L \subseteq M$ is a Lagrangian submanifold with reduced manifold $K = L^\sim$, there exists a symplectomorphism*

$$\psi\colon (M, \omega) \to (T^*[1]N, \omega_{std})$$

taking L to $N^*[1]K$.

Proof. See [14, Thm. 4] □

This theorem is particularly important because Lagrangian submanifolds are a key ingredient of the Batalin-Vilkovisky formalism, which is going to be the subject of the next chapters. Indeed, we are going to discuss how to perform integration over Lagrangian submanifolds and being able to treat them as odd conormal bundles greatly simplifies matters. In order to do so, we need to analyze volume elements (i.e. Berezinians) on odd symplectic manifolds.

Let us consider an odd symplectic manifold (M, ω) and an arbitrary Berezinian μ on M. We can define an operator on $C^\infty(M)$, denoted by Δ_μ, by

$$\Delta_\mu f = \frac{1}{2}\mathrm{div}_\mu X_f,$$

where X_f is the Hamiltonian vector field of $f \in C^\infty(M)$. If in local coordinates (x^i, ξ^μ) the volume element looks like

$$\mu = \rho dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \cdots \wedge d\xi^\mu,$$

where ρ is a locally defined function, Δ_μ takes the form

$$\Delta_\mu = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi^i} + \frac{1}{2}\left\{\log \rho, \cdot\right\}.$$

We refer the reader to [13] for a full account.

The Berezinian μ is said to be *compatible* with ω if around any point there exist coordinates (x^i, ξ^μ) in which μ takes the form

$$\mu = dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \wedge \cdots \wedge d\xi^n.$$

In this case, the above operator takes the simpler form

$$\Delta_\mu = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi^i}$$

and it squares to zero. In the next chapter we are going to see the importance of this last property. odd symplectic manifold endowed with a compatible Berezinian are sometimes referred to as *manifolds*. This terminology is due to Schwarz ([14]).

Now, let L be a Lagrangian submanifold of M and suppose that the SP-structure is locally given by a function $\rho > 0$ that only depends on the x coordinates. Without loss of generality, we can assume that $M = T^*[1]N$ and $L = N^*[1]K$, for $K \subseteq N$ submanifold. Let us assume that K is oriented and that it can be singled out locally by the equations $x^{k+1} = \cdots = x^n = 0$. Then, the SP-structure on M induced a volume element

$$\mathrm{d}\lambda = \sqrt{\rho} dx^1 \wedge \cdots \wedge dx^k \wedge d\xi^{k+1} \wedge \cdots \wedge d\xi^n.$$

One can either prove this directly or use the elegant argument due to Schwarz in [14, Lemma 4]. We have now all the tools to prove the central theorem of this chapter. It goes by different names, like BV-Stokes' and Schwarz's theorem.

Theorem 2.2.5. *Let (M, ω) be an odd symplectic manifold endowed with a compatible Berezinian μ. Also, let $f \in C^\infty(M)$. Then, the following hold:*

(i)

$$\int_L \Delta_\mu f \,\mathrm{d}\lambda = 0$$

for all $L \subseteq M$ closed oriented Lagrangian submanifold.

(ii) if $\Delta_\mu f = 0$,

$$\int_{L_0} f \,\mathrm{d}\lambda_0 = \int_{L_1} f \,\mathrm{d}\lambda_1$$

for all closed oriented Lagrangian submanifolds $L_0, L_1 \subseteq M$ such that $L_0^\sim$ and $L_1^\sim$ are homologous, i.e. they define the same element of $H_n(M^\sim;\mathbb{R})$.

Proof. We have seen that we can assume that $(M,\omega) = (T^*[1]N, \omega_{std})$, where $N = M^\sim$. Let us consider local coordinates $x^1,\dots,x^n$ on N. Then, we get induced coordinates (x^i,η^μ) on TN and (x^i,ξ_μ) on T^*N. We define the *odd Fourier transform* to be the map

$$\mathcal{F}\colon C^\infty(T[1]N) \to C^\infty(T^*[1]N)$$

given by

$$(\mathcal{F}\omega)(x,\xi) = \int e^{\sum_\mu \xi_\mu \eta^\mu} \omega(x,\eta)\rho(x)^{-\frac{1}{2}} \,\mathrm{d}\eta,$$

where we assume that the Berezinian locally looks like $\mu = \rho dx^1 \wedge \dots dx^n \wedge d\eta^1 \wedge \dots \wedge d\eta^n$ with ρ only depending on the x coordinates. The above integral is a purely odd Berezin integral. Note that the odd Fourier transform has an inverse given by

$$(\mathcal{F}^{-1}\sigma)(x,\eta) = \int e^{\sum_\mu \xi_\mu \eta^\mu} \sigma(x,\eta)\sqrt{\rho(x)} \,\mathrm{d}\xi.$$

We claim that the odd Fourier transform intertwines Δ_μ and the de Rham operator on N, that is the diagram

$$\begin{array}{ccc} C^\infty(T[1]N) & \xrightarrow{\mathcal{F}} & C^\infty(T^*[1]N) \\ \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle \Delta_\mu} \\ C^\infty(T[1]N) & \xrightarrow{\mathcal{F}} & C^\infty(T^*[1]N) \end{array}$$

commutes. Indeed, this follows from the obvious identities

$$\frac{\partial(\mathcal{F}\omega)}{\partial \xi_i} = \mathcal{F}(\eta^i\omega), \quad \frac{\partial(\mathcal{F}\omega)}{\partial x^i} = \mathcal{F}\left(\frac{\partial\omega}{\partial x^i}\right) - \alpha^{-1}\frac{\partial\alpha}{\partial x^i}\mathcal{F}\omega,$$

where we have set $\alpha = \sqrt{\rho}$ for simplicity. Under the identification $C^\infty(T[1]N) \cong \Omega^\bullet(N)$, the de Rham operator looks like $d = \sum_i \eta^i \frac{\partial}{\partial x^i}$. Thus, we get

$$\begin{aligned} \mathcal{F}(d\omega) &= \mathcal{F}\left(\sum_i \eta^i \frac{\partial\omega}{\partial x^i}\right) \\ &= \sum_i \mathcal{F}\left(\eta^i \frac{\partial\omega}{\partial x^i}\right) \\ &= \sum_i \frac{\partial}{\partial \xi_i}\mathcal{F}\left(\frac{\partial\omega}{\partial x^i}\right) \\ &= \sum_i \left(\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial x^i}\mathcal{F}\omega + \alpha^{-1}\frac{\partial\alpha}{\partial x^i}\mathcal{F}\omega\right) \\ &= \Delta_\mu(\mathcal{F}\omega). \end{aligned}$$

Let us now consider a closed oriented Lagrangian submanifold L. Without loss of generality, we can assume it is of the form $N^*[1]K$, for $K \subseteq N$ closed and oriented. Then, if $\omega \in C^\infty(T[1]N) \cong \Omega^\bullet(N)$, we have the identity

$$\int_K \omega = \int_L \mathcal{F}\omega \, \mathrm{d}\lambda. \tag{2.1}$$

Indeed, let us consider a coordinate chart (x^i) on N such that K is defined by the equations $x^{k+1} = \cdots = x^n = 0$. By linearity, we can also assume ω is a monomial in the η^μ's. By the description of the volume element on L we gave above, the only monomials that give a non-zero contribution in the right hand side of (2.1) are of the form $\omega = \gamma\eta^1 \dots \eta^k$, for $\gamma \in C^\infty(N)$. Indeed, for such a monomial the odd Fourier transform gives

$$\begin{aligned}\mathcal{F}(\gamma\eta^1 \dots \eta^k) &= \frac{\partial}{\partial\xi_1}\mathcal{F}(\gamma\eta^2 \dots \eta^k)\\ &= \cdots = \frac{\partial}{\partial\xi_1} \dots \frac{\partial}{\partial\xi_k}\mathcal{F}(\gamma)\\ &= \frac{\partial}{\partial\xi_1} \dots \frac{\partial}{\partial\xi_k}(\alpha^{-1}\gamma\xi_1 \dots \xi^n)\\ &= \alpha^{-1}\gamma\xi_{k+1} \dots \xi_n.\end{aligned}$$

Upon integration against $\mathrm{d}\lambda = \alpha dx^1 \wedge \cdots \wedge dx^k \wedge d\xi_{k+1} \wedge \cdots \wedge d\xi_n$, we get precisely

$$\int \gamma dx^1 \wedge \dots dx^k = \int \omega,$$

which is what we wanted prove. The above integrals are over local coordinate open sets. The global statement is obtained by using a partition of unity as usual.

We are finally ready to prove our theorem. Let us start with (i). Consider a function $f \in C^\infty(T^*[1]N)$. By surjectivity of the odd Fourier transform, there exists a function $g \in C^\infty(T[1]N)$ such that $f = \mathcal{F}g$. Then, we have

$$\begin{aligned}\int_L \Delta_\mu f \, \mathrm{d}\lambda &= \int_L \Delta_\mu(\mathcal{F}g) \, \mathrm{d}\lambda\\ &= \int_L \mathcal{F}(dg) \, \mathrm{d}\lambda\\ &= \int_K dg = 0,\end{aligned}$$

where the last equality follows from Stokes' theorem together with the fact that K was assumed to be closed. This proves (i). The other assertion is proved similarly. Let us assume $\Delta_\mu f = 0$. Then, we have

$$\begin{aligned}\int_{L_0} f \, \mathrm{d}\lambda_0 - \int_{L_1} f \, \mathrm{d}\lambda_1 &= \int_{L_0} \mathcal{F}g \, \mathrm{d}\lambda_0 - \int_{L_1} \mathcal{F}g \, \mathrm{d}\lambda_1\\ &= \int_{K_0} g - \int_{K_1} g.\end{aligned}$$

The assumption $\Delta_\mu f = 0$ implies that $dg = 0$, i.e. that g is closed. moreover, by hypothesis K_0 and K_1 (viewed as singular chains) are homologous. Thus, we can write $K_0 - K_1 = \partial C$ and apply Stokes' theorem (in the context of de Rham theory) to get

$$\int_{K_0} g - \int_{K_1} g = \int_{\partial C} g = \int_C dg = 0.$$

This concludes the proof of (ii). □

Chapter 3

The Batalin-Vilkovisky formalism

The BV formalism was introduced by Igor Batalin and Grigori Vilkovisky in the 1980's in the context of gauge theories. It is combines features that appear in the BRST formalism, which brings cohomological methods to the forefront, and graded symplectic geometry. It is the most general and powerful gauge-fixing method available.

In this chapter, we first brush up on gauge theories from a qualitative point of view and we describe the BRST formalism as a warm-up. Then, discuss the BV formalism in detail and one of the most powerful methods of producing BV theories, that is the AKSZ construction, named after Alexander, Kontsevitch, Schwarz and Zaboronsky. Finally, we discuss the Poisson Sigma Model and Chern-Simons theory as examples of BV theories that arise from the AKSZ construction.

§3.1 Motivation

In this first section we give an overview of the reasons why the BV formalism was conceived. In particular, we address the path integral formulation of quantum field theory and the problem of gauge-fixing. We also take a quick look at the BRST formalism, which preceded the BV formalism and, one might argue, inspired it.

Let M be a smooth manifold and $\mathcal{F}$ a sheaf over M. A *classical field theory* is given by:

(i) the space of *fields* $F = \Gamma(M, \mathcal{F})$, i.e. the space of sections over M of $\mathcal{F}$;

(ii) an *action functional* $S\colon F \to \mathbb{R}$ given by

$$S(\phi) = \int_M L(\phi, \partial\phi, \partial^2\phi, \dots),$$

where the integrand is a function of the field ϕ and its derivatives up to a fixed finite order.

Examples of sheaves over M that we might encounter are spaces of functions and sections of principal bundles over M. Another example, which we will be working with quite a bit in what follows, are mapping spaces. Given M, N smooth manifold, we define the *mapping space*

$$\mathrm{Map}(M, N) = \{\phi\colon M \to N \text{ smooth}\}.$$

Examples 3.1.1. (i) (Scalar field theory) Assume M is a Riemannian manifold with metric tensor $g = \langle \cdot, \cdot \rangle$. We choose $\mathcal{F} = C^\infty(M)$ and the Lagrangian density given by $L = T - V$, where

$$T = \frac{1}{2}\langle \operatorname{grad}\phi, \operatorname{grad}\phi \rangle$$

is the kinetic term and V is the potential, which usually the form

$$V = \frac{1}{2}\phi^2 + \sum_{n=3}^{\infty} \frac{1}{n!} g_n \phi^n.$$

This is a generalization of a point mass moving in a potential.

(ii) (Chern-Simons theory) Let M be a closed connected oriented 3-manifold and consider the trivial principal bundle $P = M \times SU(2) \to M$. The space of fields is given by

$$F = \Omega^1(M, \mathfrak{su}(2)),$$

i.e. the space of $SU(2)$-connections on P. The action is given by

$$S(A) = \int_M \operatorname{tr} \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A$$

We will see more examples along the way in a different guise.

The idea of path integrals is to define the *partition function*

$$Z(\hbar) = \int_F e^{\frac{i}{\hbar}S(\phi)} \mathcal{D}\phi.$$

However, this definition poses serious problems. First of all, the space of fields might be some very nasty space over which integrating does not make sense. Secondly, it is not clear what the measure $\mathcal{D}\phi$ should be.

In *perturbative quantum field theory*, one defines the above integral as an asymptotic series for $\hbar \to 0$. The reason behind this idea is that there exists a asymptotic expansion of the above integral in a very specific case. Let M be an oriented smooth n-manifold, μ a compactly supported top form and $f \in C^\infty(M)$. Let us put

$$\operatorname{Crit}(f) = \{x \in M \mid x \text{ is critical for } f\}.$$

Then, we have the asymptotics

$$\int_M e^{\frac{i}{\hbar}f(x)} \sim \sum_{x_0 \in \operatorname{Crit}(f)} e^{\frac{i}{\hbar}f(x_0)} \sqrt{\frac{(2\pi\hbar)^n}{|\det Hf(x_0)|}} e^{\frac{\pi i}{4} \operatorname{sgn} H(f)} \mu_{x_0} + O(\hbar^{\frac{n}{2}+1}), \quad \hbar \to 0,$$

where μ_{x_0} is the density given by μ at x_0. This is called the *stationary phase formula.* We refer the reader to [13] for a complete discussion.

This formula is only valid under the above assumptions. In more general contexts, one defines the path integral above to be equal to this expansion. However, this brings up more questions. Indeed, a quick look at the stationary phase formula suffices to observe that a necessary condition for it to even make sense is that f be a Morse function, i.e. its critical points be non-degenerate. Most of the time, this is not the case. In fact, we will be focusing on *gauge theories*, i.e. theories that carry some local symmetry. In this setting, isolated points come in orbits, which means that they are not isolated, hence degenerate. The cure for this problem goes under the name of *gauge-fixing*, a procedure that reduces the degrees of freedom of the theory, roughly speaking, and eliminates degeneracies. The important part is then showing that the theory is independent of the choice of gauge.

There are a number of gauge-fixing procedures, such as the Faddeev-Popov technique, the BRST formalism and the Batalin-Vilkovisky formalism. The latter is the most modern and powerful technique. Since it shares some key elements with it, let us first take a look at the BRST model.

Definition 3.1.2. A *classical BRST theory* is given by the following data:

(i) a $\mathbb{Z}$-graded manifold F, which is called the *space of fields*;

(ii) a vector field $Q \in \mathfrak{X}(F)_1$ satisfying $Q^2 = 0$, i.e. a cohomological vector field (as defined at the end of Chapter 1);

(iii) a function $S \in C^\infty(F)_0$, called the *BRST action*, satisfying $Q(S) = 0$.

The cohomological vector field Q carries information about the gauge symmetry of the theory and the fact that the action is a Q-cocycle can be interpreted as gauge-invariance.

Example 3.1.3. Let M be a smooth manifold together with a smooth action of a compact Lie group G. Furthermore, suppose we have a smooth invariant function $f \in C^\infty(M)^G$. We define the space of fields to be

$$F = X \times \mathfrak{g}[1].$$

Then, the smooth functions on the space of fields are given by

$$C^\infty(F) = C^\infty(M) \otimes \Lambda^\bullet \mathfrak{g}^* = \Omega^\bullet(\mathfrak{g}, C^\infty(M)),$$

that is the Chevalley-Eilenberg complex with coefficients in $C^\infty(M)$. The cohomological vector field Q is then given by by the Chevalley-Eilenberg differential d twisted by the module C^∞. In other words, at the point $(x, v) \in M$,

$$Q_{(x,v)} = id \otimes d + \alpha(v) \otimes id,$$

where $\alpha\colon \mathfrak{g} \to \mathfrak{X}(M)$ is the infinitesimal $\mathfrak{g}$-action induced by the G-action on M.

Definition 3.1.4. A *quantum BRST theory* is a classical BRST theory (F, Q, S) together with a Berezinian μ on F such that

$$\mathrm{div}_\mu Q = 0.$$

Note that

$$\int_F Q(f)\mu = -\int_F \mathrm{div}_\mu(F)\mu = 0.$$

for all $f \in C_c^\infty(F)$. We then get a well defined real map

$$\int_F \mu\colon H_Q\big(C_c^\infty(M)\big) \to \mathbb{R},$$

called the *BRST integral*.

Let us consider the power series

$$e^{\frac{i}{\hbar}S} = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n S^n \in C^\infty(F)[[\hbar]].$$

The action of Q on it is defined term-wise. It is a straightforward to prove using induction that

$$Q(e^{\frac{i}{\hbar}S}) = Q(S)e^{\frac{i}{\hbar}S},$$

which means that if S is a Q-cocycle, so is $e^{\frac{i}{\hbar}S}$. Moreover, it easy to check that if we change S by a Q-exact term, the exponential changes by a Q-exact term. Indeed,

$$e^{\frac{i}{\hbar}(S+Q\psi)} - e^{\frac{i}{\hbar}S} = Q\left(e^{\frac{i}{\hbar}S}Q\left(\frac{i}{\hbar}\psi\right)\sum_{n=0}^{\infty} Q\left(\frac{i}{\hbar}\psi\right)^n\right)$$

for $\psi \in C_c^\infty(F)_{-1}$. This is the key observation that leads to the idea of gauge-fixing in BRST models. If the partition function

$$Z(\hbar) = \int_F e^{\frac{i}{\hbar}S}\mu$$

is ill-defined due to degeneracy of the critical point of S, we can (hopefully) modify S using a suitable ψ in order to remove said degeneracies. The above discussion then shows that BRST theories have gauge-invariant built in by definition. The function ψ goes under the name of *gauge-fixing fermion*.

§3.2 BV formalism

We are finally ready to introduce the Batalin-Vilkovisky formalism. Its main features are the cohomological nature, which it shares with BRST theories, and the use of graded symplectic geometry.

Definition 3.2.1. A *Hamiltonian differential graded manifold* of degree k (most of the time we will just write *dg*) is a $\mathbb{Z}$-graded manifold M endowed with:

(i) a symplectic structure $\omega \in \Omega^2(M)_k$ of (internal) degree k;

(ii) a function (the *Hamiltonian*) $H \in C^\infty(M)_{k+1}$ that satisfies

$$\{H, H\} = 0,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket of degree $-k$ induced by ω.

Note that $Q = \{H, \cdot\} \in \mathfrak{X}(M)_1$ is a cohomological vector field.

A *classical BV theory* is a Hamiltonian dg manifold (M, ω, H) of degree -1. M is called the *space of BV fields* and H is called the *BV action*. It is customary to denote the action by S.

Remark 3.2.2. Hamiltonian dg manifold of degree different from -1 also show up. For instance, the case $k = 0$ corresponds to the BFV formalism, which is useful in extending the BV formalism to manifolds with boundary. As we will only be working with closed manifolds, we will not delve into this.

Other degrees show up when constructing AKSZ sigma models. We will discuss this later on.

Example 3.2.3. Let us see how the BV formalism generalizes the BRST one. Let (F, Q, S) be a classical BRST theory. We define the space of BV fields to be $F_{BV} = T^*[-1]F$. Note that F_{BV} carries the canonical symplectic structure ω coming from the Liouville 1-form. Observe also that ω has degree -1 since we shifted the cotangent bundle by -1.

Let $p\colon F_{BV} \to F$ be the projection to the base manifold F. The BV action on F_{BV} is given by

$$S_{BV} = p^*S + \tilde{Q},$$

where the first term is the pullback of S by p and $\tilde{Q}$ is the function on F_{BV} corresponding to Q under the identification $C^\infty(T^*[-1]F) \cong \mathfrak{X}^\bullet(F)$ (we saw this in the case of smooth manifold and it holds for graded manifolds as well. Indeed, one can even define multivector fields on graded manifolds by imposing this identification. We refer the reader to [17] for this point of view on multivector fields and differential forms).

The cohomological vector field is accordingly given by

$$Q_{BV} = X_{p^*S} + Q^\sharp,$$

where the first term is the Hamiltonian vector field of p^*S and the second one is the cotangent lift of Q.

Let us move to the quantum setting.

Definition 3.2.4. A *quantum BV theory* is a graded manifold F endowed with:

(i) a degree -1 symplectic form $\omega \in \Omega^2(F)_{-1}$;

(ii) a Berezinian μ compatible with ω;

(iii) a function $S \in C^\infty(F)_0$, the *master action*, satisfying the *quantum master equation (QME)*

$$\frac{1}{2}\{S, S\} - i\hbar\Delta_\mu S = 0,$$

where Δ_μ is the *BV Laplacian*, i.e. the operator on $C^\infty(F)$ defined by

$$\Delta_\mu f = \frac{1}{2}\mathrm{div}_\mu X_f,$$

the divergence with respect to μ of the Hamiltonian vector field of $f \in C^\infty(F)$.

The idea of gauge-fixing in BV theories comes from the BV theorem, which we discussed in Chapter 2. Indeed, we can define the partition function

$$Z(\hbar) = \int_L e^{\frac{i}{\hbar}S},$$

where $L \subseteq F$ is a Lagrangian submanifold. Then, if S has some degenerate critical points on L, we can deform L to obtain a different Lagrangian submanifold L' on which (hopefully) the critical points of S are non-degenerate, so that we can apply the stationary phase formula. The BV theorem ensures that the integral does not change when performing this procedure.

Before introducing observables, let us spend a few words on the algebraic aspects of the BV formalism. A particularly important role is played by the BV Laplacian Δ_μ.

Definition 3.2.5. A *BV algebra* is a $\mathbb{Z}$-graded commutative real algebra A together with

(i) a degree 1 bracket $\{\cdot,\cdot\} : A_k \otimes A_l \to A_{k+l+1}$ satisfying

- skew symmetry: $\{x,y\} = -(-1)^{(|x|+1)(|y|+1)}\{y,x\}$,
- the Leibniz identity:

$$\{x, yz\} = \{x,y\}\, z + (-1)^{(|x|+1)|y|} y \{x,z\}, \tag{3.1}$$

- the Jacobi identity:

$$\{x, \{y,z\}\} = \{\{x,y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x,z\}\}.$$

(ii) an $\mathbb{R}$-linear map (the *BV Laplacian*) $\Delta\colon A_j \to A_{j+1}$ such that

- $\Delta^2 = 0$,
- $\Delta(1) = 0$,
- $\Delta\{x,y\} = \{\Delta x, y\} + (-1)^{|x|+1}\{x, \Delta y\}$,
- $\{\cdot,\cdot\}$ measures the failure of the Leibniz identity for Δ:

$$\Delta(xy) = (\Delta x)y + (-1)^{|x|} x\Delta y + (-1)^{|x|}\{x,y\}.$$

It is easy to see that for a quantum BV theory (F,ω,S,μ) the BV Laplacian Δ_μ satisfies the above requirements. Hence, $\left(C^\infty(F), \{\cdot,\cdot\}, \Delta_\mu\right)$ is BV algebra. In what follows, we simplify the notation by just writing Δ instead of Δ_μ.

An element $S \in A_0$ is said to satisfy the *Classical Master Equation (CME)* if $\{S,S\} = 0$. S satisfies the *Quantum Master Equation (QME)* if

$$\frac{1}{2}\{S,S\} - i\hbar\Delta S = 0.$$

It is easy, although a bit tedious, to prove using induction and the properties of BV algebras that S satisfies the QME if and only if $\Delta e^{\frac{i}{\hbar}S} = 0$.

Definition 3.2.6. A *BV observable* is an element $O \in C^\infty(F)[[\hbar]]$ that satisfies

$$\Delta O + \frac{i}{\hbar}\{S,O\} = 0.$$

Let us unravel the reason behind this definition. The end goal is to be able to define the expectation value of O as

$$\langle O\rangle = \frac{1}{Z(\hbar)}\int_L e^{\frac{i}{\hbar}S} O.$$

To do so, we need the integrand to be Δ_μ-closed (in order to have gauge-invariance). We compute

$$\Delta(e^{\frac{i}{\hbar}S}O) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \{S^n, O\}.$$

A straightforward inductive argument shows that $\{S^n, O\} = nS^{n-1}\{S, O\}$, which gives

$$\begin{aligned}\Delta(e^{\frac{i}{\hbar}S}O) &= e^{\frac{i}{\hbar}S}\Delta O + \left\{e^{\frac{i}{\hbar}S}, O\right\} \\ &= e^{\frac{i}{\hbar}S}\Delta O + \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{i}{\hbar}\right)^n S^{n-1}\{S, O\} \\ &= e^{\frac{i}{\hbar}S}\left(\Delta O + \frac{i}{\hbar}\{S, O\}\right) \\ &= e^{\frac{i}{\hbar}S}\left(\Delta O + \frac{i}{\hbar}QO\right)\end{aligned}$$

where we have used the QME in the first equality. Thus, $\Delta(e^{\frac{i}{\hbar}S}O) = 0$ if and only if

$$\Delta_S O = \left(\Delta O + \frac{i}{\hbar}Q\right)O = 0.$$

§3.3 AKSZ sigma models

BV theories are quite abstract and difficult to construct. In this section we discuss a systematic way of producing BV theories due to Alexander, Kontsevitch, Schwarz and Zaboronsky ([1]). Their idea originally was to produce topological quantum field theories, but it turns out that their technique fits nicely in the BV framework.

Definition 3.3.1. An *AKSZ theory* consists of the following data:

(i) a closed oriented smooth n-manifold, called the *source*,

(ii) a Hamiltonian dg manifold (N, ω_N, θ_N) of degree $n-1$ where the symplectic form is required to be exact:

$$\omega_N = d_N\alpha_N, \quad \alpha \in \Omega^1(N)_{n-1}.$$

Here, d_N denotes the de Rham operator on N.

Given an AKSZ theory, we define the space of fields to be

$$F = \mathrm{Map}(T[1]M, N),$$

the space of graded morphisms from $T[1]M$ to N.

Remark 3.3.2. Note that F is generally an infinite-dimensional graded manifold. We will ignore this fact in what follows in order not clutter the discussion.

We would like to make F into a graded symplectic manifold. To do so, we will define a map that allows us to lift differential forms on the target to differential forms on the space of fields. Consider the evaluation map

$$ev\colon T[1]M \times F \to N.$$

If $\sigma \in \Omega^p(N)$, we can pull it back by ev to obtain

$$ev^*\sigma \in \Omega^p(T[1]M \times F) \cong \bigoplus_{i=0}^{p}\Omega^i(T[1]M) \otimes \Omega^{p-i}(F)$$

and we single out the component τ of $ev^*\sigma$ in $\Omega^0(T[1]M) \otimes \Omega^p(F)$. Recall that

$$\Omega^0(T[1]M) = C^\infty(T[1]M) \cong \Omega^\bullet(M)$$

so we can interpret τ as an $\Omega^p(F)$-valued non-homogeneous form on M. In order to get a p-form on F, we integrate τ over M, disregarding terms of (form) degree less than n. This construction gives a map

$$\mathbb{T}\colon \Omega^p(N)_j \to \Omega^p(F)_{j-n}.$$

The internal degree drops by n because we are integrating n-forms on M (identified with elements of $C^\infty(T[1]M)_n$) to numbers. We call $\mathbb{T}$ the *transgression map*.

Using the transgression map, we can define the BV symplectic form on the space of fields F as follows:

$$\omega = (-1)^n \mathbb{T}\omega_N \in \Omega^2(F)_{-1}.$$

If $\omega_N = \sum_{i,j} \omega_{ij} d_N x^i \wedge d_N x^j$ in local coordinates (x^i) on N, then

$$\omega = (-1)^n \int_M \sum_{i,j} (ev^*\omega_{ij}) \delta X^i \wedge \delta X^j,$$

where $X^i = ev^* x^i$.

In order to define the BV action, we first observe that we can lift certain operators on the source and target to F. This relies on the fact that the space of fields is defined to be a mapping space and hence its tangent spaces have a simple description (see [7]). Let us consider a mapping space of graded manifolds $\mathrm{Map}(X, Y)$. Then, the tangent space at $f \in \mathrm{Map}(X, Y)$ is given by $\Gamma(X, f^*TY)$. Thus, if D is a vector field on X, we can lift it to a vector field $\hat{D}$ on $\mathrm{Map}(X, Y)$ by defining

$$\hat{D}_f(x) = d_x f(D_x), \quad f \in \mathrm{Map}(X, Y),\ x \in X.$$

We can do a similar thing for a vector field Q on Y. Indeed, we can define a vector field $\hat{Q}$ on $\mathrm{Map}(X, Y)$ by

$$\hat{Q}_f(x) = Q(f(x)), \quad f \in \mathrm{Map}(X, Y),\ x \in X.$$

Going back to AKSZ models, we have two degree 1 cohomological vector fields, namely the de Rham differential on M, viewed as derivation of $C^\infty(T[1]M)$, and the vector field Q_N on N coming from the Hamiltonian dg manifold structure. Thus, we get degree 1 vector fields $\hat{d}_M, \hat{Q}_N$ on F which commute. This means that the vector field $Q = \hat{d}_M + \hat{Q}_N$ is a cohomological vector field on F.

Finally, we define the BV action on F to be

$$S = i_{\hat{d}_M} \mathbb{T}\alpha_N + \mathbb{T}\theta_N.$$

Note that the properties of $\mathbb{T}$ ensure that S has degree 0. Locally, if $\alpha = \sum_i \alpha_i d_N x^i$, we get

$$S = \int_M \sum_i (ev^*\alpha_i) d_M X^i + ev^*\theta_N.$$

Theorem 3.3.3. *(F, ω, S) is a classical BV theory. Moreover,* *(i)* $\mathbb{T}$ *is a chain map.*

(ii) ω *is exact.*

Proof. We will first prove that $\mathbb{T}$ is a chain map, meaning that $\mathbb{T}d_N\phi = (-1)^n\delta\mathbb{T}\phi$ for all forms ϕ on N. It is clear that ev^* is a chain map. Thus,

$$ev^*(d_N\phi) = \delta(ev^*\phi)$$

Recall that

$$\Omega^p(T[1]M \times F) \cong \bigoplus_{i=0}^{p} \Omega^i(T[1]M) \otimes \Omega^{p-i}(F)$$

and that when transgressing, we only keep the component of $ev^*\phi$ in $\Omega^n(M) \otimes \Omega^p(F)$. The de Rham differential on $T[1]M \times F$ is equal to $d_{T[1]M} \otimes id + id \otimes \delta$, which on the relevant component of $ev^*\phi$ reads $d_M \otimes id + id \otimes \delta$. Observe that

$$(id \otimes \delta)(a \otimes b) = (-1)^n a \otimes \delta b$$

for all decomposable elements $a \otimes b \in \Omega^n(M) \otimes \Omega^p(F)$. Also,

$$\int_M (d_M \otimes id)(ev^*\phi) = 0$$

because of Stokes' theorem and the assumption that M is closed.

Thus,

$$\begin{aligned}\int_M ev^*(d_N\phi) &= \int_M (id \otimes \delta)ev^*\phi \\ &= (-1)^n\delta \int_M ev^*\phi \\ &= (-1)^n\delta\mathbb{T}\phi.\end{aligned}$$

Since $\mathbb{T}$ is chain map, we have

$$\begin{aligned}\delta\omega &= (-1)^n\delta\mathbb{T}\omega_N \\ &= \mathbb{T}d_N\omega_N = 0,\end{aligned}$$

which shows that ω is closed.

We have already shown that ω has degree -1. The only things left to check are the fact that Q is cohomological and that it is the Hamiltonian vector field of S. By inspection of the definition of $\hat{d}_M$ and $\hat{Q}_N$, we see that they both square to zero. Thus,

$$\begin{aligned}Q^2 &= (\hat{d}_M + \hat{Q})_N^2 \\ &= \hat{d}_M^2 + \hat{d}_M\hat{Q}_N + \hat{Q}_N\hat{d}_M + \hat{Q}_N^2 = 0.\end{aligned}$$

The second claim follows from the following computation:

$$\begin{aligned}i_Q\omega &= (-1)^n(i_{\hat{d}_M}\mathbb{T}\omega_N + i_{\hat{Q}}\mathbb{T}\omega_N) \\ &= i_{\hat{d}_M}\delta\mathbb{T}\alpha_N + (-1)^n\mathbb{T}i_{Q_N}\omega_N \\ &= L_{\hat{d}_M}\mathbb{T}\alpha_N + \delta i_{\hat{d}_M}\mathbb{T}\alpha_N + (-1)^n\mathbb{T}d_N\theta_N \\ &= \delta i_{\hat{d}_M}\mathbb{T}\alpha_N + \delta\mathbb{T}\theta_N = \delta S,\end{aligned}$$

where we have used the identities

$$\begin{aligned}&i_{\hat{Q}_N}\mathbb{T}\omega_N = (-1)^n\mathbb{T}(i_{Q_N}\omega_N) \\ &L_{\hat{d}_M}\mathbb{T}\alpha_N = 0.\end{aligned}$$

Since $Q^2 = 0$, S satisfies the CME. This concludes the proof. □

§3.4 Examples

In this section we discuss some examples of BV theories arising from the AKSZ construction.

The Poisson Sigma Model. Let Σ a closed oriented surface and (M, π) a Poisson manifold. Consider a bundle morphism $(X, \eta)\colon T\Sigma \to T^*N$, i.e. a pair of smooth maps $X\colon \Sigma \to M$ and $\eta\colon T\Sigma \to T^*M$ such that the diagram

$$\begin{array}{ccc} T\Sigma & \xrightarrow{\eta} & T^*M \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & M \end{array}$$

commutes and η is linear on fibers. Note that η can be viewed as a section

$$\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*M).$$

The action of the Poisson Sigma Model is given by

$$S(X, \eta) = \int_\Sigma \langle \eta \overset{\wedge}{,} dX \rangle + \frac{1}{2}\langle \pi(X), \eta \wedge \eta \rangle.$$

Let us unravel this definition. In the first term, $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between T^*N and TN. Note that $dX \in \Gamma(\Sigma, T^*\Sigma \otimes X^*TM)$, which we pair with η to get a 2-form on Σ. In the second term, we pair $\pi(X) \in \Gamma(\Sigma, \Lambda^2 X^*TM)$ with $\eta \wedge \eta \in \Omega^2(\Sigma, \Lambda^2 X^*T^*M)$.

If $\pi = \sum_{i,j} \pi^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$ and $\eta = \sum_k \eta_k dx^k$ in local coordinates on M, we have

$$S(X, \eta) = \int_\Sigma \sum_k \eta_k dX^k + \sum_{i,j} \pi^{ij}(X) \eta_i \wedge \eta_j$$

Let us now take a look at the Poisson Sigma Model through the lens of the AKSZ construction.

Σ will be our source manifold. As target manifold, we take

$$N = T[1]^*M.$$

Note that N is a graded symplectic manifold when endowed with the canonical symplectic structure ω_N. Note that ω_N is exact, since it is the differential of the Liouville 1-form on N, and that it has internal degree 1, as we shifted T^*M by 1. We will make N into a Hamiltonian dg manifold of degree 1. We can use the identification

$$C^\infty(T^*[1]M) \cong \mathfrak{X}^\bullet(M) \tag{3.2}$$

to view the Poisson structure $\pi \in \mathfrak{X}^2(M)$ as a degree 2 function θ_N on $T^*[1]M$. In local coordinates (x^i, p_j) on $T^*[1]M$,

$$\theta_N = \frac{1}{2} \sum_{i,j} \pi^{ij} p_i p_j.$$

Its Hamiltonian vector field is given by $Q_N = [\pi, \cdot]_{NS}$ under the identification (3.2). Here, $[\cdot,\cdot]_{NS}$ is the Nijenhuis-Schouten bracket on multivector fields, i.e. the extension of the usual Lie bracket on vector fields by imposing the Leibniz rule. Locally, we have the expression

$$Q_N = -\sum_{i,j} \pi^{ij} p_i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j,k} \frac{\partial \pi^{ij}}{\partial x^k} p_i p_j \frac{\partial}{\partial p_k}.$$

The requirement that $[\pi,\pi]_{NS} = 0$ (π is a Poisson structure) implies that Q_N is cohomological.

According to the AKSZ prescription, the space of fields is given by

$$F = \mathrm{Map}(T[1]\Sigma, T^*[1]M).$$

Let $\tilde{X}^i = ev^* x^i$ and $\tilde{\eta}_j = ev^* p_j$ be the pullback of local coordinates (x^i, p_j) on $T^*[1]M$. Note that since ev^* is a morphism of graded algebras, and hence it preserves degrees, $\tilde{X}^i = X^i + \eta_i^+ + \beta_i^+$, where

- X^i is a 0-form, i.e. a function on M, valued in functions on F o degree 0,
- η_i^+ is a 1-form on M valued in functions on F of degree -1,
- β_i^+ is a 2-form on M valued in functions on F of degree -2.

Similarly, $\tilde{\eta}_j = \beta_j + \eta_j + X^{+j}$, where

- β_j is a 0-form on M valued in functions on F of degree 1,
- η_j is a 1-form on M valued in functions on F of degree 0,
- X^{+j} is a 2-form on M valued in functions on F of degree -1.

Remark 3.4.1. A couple of remarks are in order. First of all, we denoted the first component of $\tilde{X}$ by X. This is not by accident, as X recovers the base map $X\colon \Sigma \to M$ we had earlier. The same reasoning applies to the second component of $\tilde{\eta}$, which we denoted by η.

The second remark regards the notation

$$\tilde{X} = X + \eta^+ + \beta^+, \quad \tilde{\eta} = \beta + \eta + X^+,$$

which comes from physics. We say that $\tilde{x}$ and $\tilde{\eta}$ are *superfields*. The components with positive internal degree are called *fields* and the ones with negative internal degree are called *antifields*. To denote the latter, we used a + sign.

The canonical symplectic form on $T^*[1]M$ is given in local coordinates by

$$\omega_N = \sum_i d_N p_i \wedge d_N x^i$$

Using this, we can explicitly write down ω as

$$\begin{aligned}\omega &= \int_\Sigma \sum_i \delta\tilde{\eta}_i \wedge \delta\tilde{X}^i \\ &= \int_\Sigma \delta X^i \wedge \delta\tilde{X}^{+i} + \delta\eta_i \wedge \delta\eta_i^+ + \delta\beta_i \wedge \beta_i^+\end{aligned}$$

and S as

$$S = \int_\Sigma \sum_i \tilde{\eta}_i \wedge \delta \tilde{X}^i + \frac{1}{2} \sum_{i,j} (ev^* \pi^{ij}) \tilde{\eta}_i \wedge \tilde{\eta}_j$$

This expression is similar in structure to the action of the Poisson Sigma Model we gave above, but X, η are replaced by $\tilde{X}, \tilde{\eta}$. If we expand in the homogeneous field components, we get the above action and a number of auxiliary terms that roughly speaking come from gauge transformations (see [13, Ex. 4.9.3] for a more detailed discussion on this last point).

Remark 3.4.2. The Poisson Sigma Model is an example of a *split AKSZ theory*, i.e. an AKSZ theory where the space of fields is she shifted cotangent bundle to some smooth manifold, together with the canonical symplectic structure. More precisely, an AKSZ theory with a d-dimensional closed manifold M as source space is called *split* if the target has the form $T^*[d-1]N$, for some smooth manifold N. These theories will play an important role in the next chapter.

Let us now take a look at a non-split AKSZ theory.

Chern-Simons theory. Let M be a closed oriented 3-manifold and $\mathfrak{g}$ a Lie algebra endowed with the Killing form, that is the non-degenerate ad-invariant pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (X, Y) &\mapsto \operatorname{tr}\big(\operatorname{ad}(X)\operatorname{ad}(Y)\big). \end{aligned}$$

Let us fix an orthonormal basis $\{e_a\}$ for $\mathfrak{g}$ and consider the graded manifold $N = \mathfrak{g}[1]$. If $x \in \mathfrak{g}$, we have

$$x = \sum_a \psi^a(x) e_a$$

for uniquely determined real numbers $\psi^a(x)$. Thus, we get linear maps $\psi^a \colon \mathfrak{g}[1] \to \mathbb{R}$, which we can conveniently use to define a $\mathfrak{g}$-valued coordinate on N

$$\psi = \sum_a \psi^a e_a \in C^\infty(N) \otimes \mathfrak{g}.$$

Note that $\psi^a \in C^\infty(N) \cong \Lambda^\bullet \mathfrak{g}^*$ have degree 1. We define a symplectic form of degree 2 on N by

$$\omega_N = \frac{1}{2} \langle \delta\psi, \delta\psi \rangle = \frac{1}{2} \sum_a \delta\psi^a \wedge \delta\psi^a,$$

which is exact, being the differential of

$$\alpha_N = \frac{1}{2} \langle \psi, \delta\psi \rangle = \frac{1}{2} \sum_a \psi^a \wedge \delta\psi^a.$$

The cohomological vector field, as we saw at the beginning of the chapter, is given by the Chevalley-Eilenberg differential. It can be compactly written as

$$Q_N = \frac{1}{2} \langle [\psi, \psi], \frac{\partial}{\partial \psi} \rangle = \frac{1}{2} \sum_{a,b,c} f_{ab}^c \psi^a \psi^b \frac{\partial}{\partial \psi^c},$$

where the f_{ab}^c are the structure constants of $\mathfrak{g}$ with respect to $\{e_a\}$. The corresponding Hamiltonian is given by

$$\theta_N = \frac{1}{6} \langle \psi, [\psi, \psi] \rangle = \frac{1}{6} \sum_{a,b,c} f_{ab}^c \psi^a \psi^b \psi^c.$$

The AKSZ space of fields is given by

$$F = \mathrm{Map}(T[1]M, \mathfrak{g}[1]) \cong \Omega^\bullet(M) \otimes \mathfrak{g}[1]$$

Let us put $\tilde{A} = ev^*\psi$. Then $\tilde{A} = c + A + A^+ + c^+$, where

- $c \in \Omega^0(M, \mathfrak{g})$ and has internal degree 1,
- $A \in \Omega^1(M, \mathfrak{g})$ and has internal degree 0,
- $A^+ \in \Omega^2(M, \mathfrak{g})$ and has internal degree -1,
- $c^+ \in \Omega^2(M, \mathfrak{g})$ and has internal degree -2.

Upon transgression, we get the BV symplectic form on F

$$\omega = -\frac{1}{2}\int_M \langle \delta\tilde{A}, \delta\tilde{A}\rangle = -\int_M \langle \delta A, \delta A^+\rangle + \langle \delta c, \delta c^+\rangle.$$

Similarly, the BV action on F obtained by transgression is given by

$$S = \int_M \frac{1}{2}\langle \tilde{A}, d_M\tilde{A}\rangle + \frac{1}{6}\langle \tilde{A}, [\tilde{A}, \tilde{A}]\rangle,$$

which upon expansion looks like

$$S = \int_M \frac{1}{2}\langle A, d_M A\rangle + \frac{1}{6}\langle A, [A, A]\rangle + \int_M \langle A^+, (d_M c + [A, c])\rangle + \int_M \frac{1}{2}\langle c^+, [c, c]\rangle$$

The first term is the action of Chern-Simons theory we discussed at the beginning of the chapter. We refer the reader to [13, Ex. 4.9.2].

Chapter 4

Globalization for equivariant AKSZ theories

This is the main chapter of this work. Here, we present two unrelated constructions due to Bonechi, Cattaneo, Qiu, Zabzine ([3]) and Cattaneo, Moshayedi, Wernli ([9]) respectively. The first one is a generalization of the BV formalism to the case when there is an infinitesimal Lie algebra action on the source of an AKSZ theory. The way to deal with it is to modify the algebra of functions on the space of fields and use tools from equivariant cohomology. The drawback of this approach is that we have to shrink the space of observables in order to retain gauge-invariance. The second construction, which also shows up in [4] in the case of the Poisson Sigma Model, is a way of studying solutions of the CME locally using methods coming from formal geometry.

It is natural to wonder whether these constructions are compatible. It turns out they are. In the last part of the chapter, we address this question and we show how the Classical Master Equation needs to be modified. Finally, we take a look at a couple of examples which were already considered in [3].

§4.1 Equivariant AKSZ theories

In this section we review [3] in full detail, with particular emphasis on the treatment of equivariant AKSZ theories, following . When trying to incorporate a Lie algebra action on the source space into the BV framework a difficulty arises, namely the fact that the CME ceases to hold. Thus, it is necessary to modify the BV formalism in order to retain gauge-invariance.

Before introducing equivariance, let us see how to modify the BV formalism when the QME

$$\frac{1}{2}\{S,S\} - i\hbar\Delta S = 0$$

for the BV action S fails. Recall that if O is an observable if and only if it satisfies $\Delta_S O = \Delta O + \frac{i}{\hbar} QO = 0$, where $Q = \{S, \cdot\}$ is the Hamiltonian vector field of S. This condition makes it possible to apply the BV theorem to the integral

$$\int_L e^{\frac{i}{\hbar}S} O.$$

In order words, under this condition the expectation value of O is gauge-invariant. The twisted Laplacian Δ_S pops up in the equivalent formulation of the QME

$$e^{-\frac{i}{\hbar}S}\Delta(e^{\frac{i}{\hbar}S}O).$$

Let now T be the failure of the QME, i.e.

$$T = \frac{1}{2}\{S, S\} - i\hbar\Delta S.$$

The same computation gives

$$e^{-\frac{i}{\hbar}S}\Delta(e^{\frac{i}{\hbar}S}O) = \Delta O + \frac{i}{\hbar}QO + \left(\frac{i}{\hbar}\right)^2 TO = \Delta_{S,T}O. \tag{4.1}$$

The BV theorem says that $\int_L e^{\frac{i}{\hbar}S}$ is invariant under Lagrangian homotopies if we only allows Lagrangians submanifolds on which T vanishes. From now on, we will only consider such submanifolds. Then, the BV theorem ensures that

(i) $\int_L e^{\frac{i}{\hbar}S} = 0$ if O is proportional to T,

(ii) $\int_L e^{\frac{i}{\hbar}S}$ is invariant under Lagrangian homotopies of L if $\Delta_{S,T}O$ is proportional to T.

Thus, it is tempting to work modulo the ideal $\mathcal{I}_T$ generated by T. Note that $\Delta_{S,T}$ squares to zero, since

$$\Delta_{S,T}^2 O = e^{-\frac{i}{\hbar}S}\Delta^2(e^{\frac{i}{\hbar}S}).$$

Also, (4.1) shows that T is proportional to $\Delta_{S,T}1$, which implies that $\Delta_{S,T}T = 0$. Thus,

$$\Delta T + \frac{i}{\hbar}QT = \Delta_{S,T}T = 0,$$

where we have used the fact that T has degree 1 and hence $T^2 = 0$.

The problem with working modulo $\mathcal{I}_T$ is that we cannot take the quotient, since it is not a Poisson ideal:

$$\begin{aligned}\Delta_{S,T}(TO) &= (\Delta T)O - T\Delta O - \{T, O\} + \frac{i}{\hbar}Q(T)O - \frac{i}{\hbar}TQ(O) + \left(\frac{i}{\hbar}\right)^2 T(TO)\\ &= -T\Delta O - \{T, O\} - \frac{i}{\hbar}TQ(O)\\ &= -T\Delta_{S,T}O - \{T, O\}.\end{aligned}$$

From this computation, we see that if $\{T, O\} \in \mathcal{I}_T$, then $\Delta_{S,T}$ preserves $\mathcal{I}_T$. Thus, we restrict our attention to

$$\mathcal{N}_T = \{O \in A \,|\, \{T, O\} \in \mathcal{I}_T\},$$

where A denotes the BV algebra of functions. Note that $\mathcal{N}_T$ is a subalgebra of A and that

$$A_T = \mathcal{N}_T/\mathcal{I}_T$$

is a Poisson algebra with the induced Poisson structure. Moreover, by our discussion above, $\Delta_{S,T}$ descends to A_T. We define the space of *quantum observables* as

$$\{[O] \in A_T \,|\, \Delta_{S,T}[O] = 0\}.$$

The BV theorem in the context of quantum observables ensures that the definition

$$\int_L e^{\frac{i}{\hbar}S}[O] = \int_L e^{\frac{i}{\hbar}S}O, \tag{4.2}$$

for any representative O of $[O]$, is good and that the integral does not change under Lagrangian homotopies of L (we remind the reader that we only allow Lagrangian submanifolds where $T = 0$).

We are now ready to discuss equivariant AKSZ theories.

Let Σ be a closed oriented d-manifold and (M, ω_M, θ_M) a Hamiltonian dg manifold of degree $d-1$. The AKSZ space of fields is

$$F = \mathrm{Map}(T[1]\Sigma, M).$$

Suppose now that we have a Lie algebra $\mathfrak{g}$ acting infinitesimally on Σ, i.e. a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(\Sigma)$. For $X \in \mathfrak{g}$, we denote its image under the action by $v_X \in \mathfrak{X}(\Sigma)$. We have two distinguished derivations of $C^\infty(T[1]\Sigma) \cong \Omega^\bullet(\Sigma)$ namely the Lie derivative L_{v_X} along v_X, which has degree 0, and the contraction i_{v_X}, which has degree -1. In other words, they are graded vector fields on $T[1]\Sigma$ and can be consequently lifted to vector fields $\hat{L}_{v_X}, \hat{i}_{v_X}$ on F (see our discussion of AKSZ sigma models in Chapter 3). Thus, the BV algebra of functions $C^\infty(F)$ carries three operators, i.e. the BV vector field Q, the Lie derivative $\hat{L}_{v_X}$ and the contraction $\hat{i}_{v_X}$, which make it into a $\mathfrak{g}$-differential graded algebra (see Appendix A for more details). To prove this assertion, we need to verify that these operators satisfy the rules of Cartan calculus. These are straightforward computations. For instance, as Q is the sum of $\hat{d}_\Sigma$ and of the lift $\hat{Q}_M$ of the cohomological vector field Q_M on M, we get

$$[\hat{L}_{v_X}, Q] = [\hat{L}_{v_X}, \hat{d}_\Sigma + \hat{Q}_M] = [\hat{L}_{v_X}, \hat{d}_\Sigma].$$

The rightmost term is the lift of $[L_{v_X}, d_\Sigma]$, which is 0. This proves that $[\hat{L}_{v_X}, Q] = 0$. The other identities of Cartan calculus are left to the reader as an easy exercise.

Notation. If D is a vector field on F, we denote its Hamiltonian by S_D.

In order to take the $\mathfrak{g}$-action into account, the idea is to replace the algebra $A = C^\infty(F)$ with the algebra

$$A[u] = C^\infty(F) \otimes S\mathfrak{g}^*,$$

where $S\mathfrak{g}^*$ denotes the symmetric algebra of $\mathfrak{g}^*$. Let $\{e_a\}$ be a basis for $\mathfrak{g}$ and let $\{u^a\}$ be the dual basis. We define the equivariant extension of the BV action as

$$S^c = S - \sum_a S_{\hat{i}_{v_a}} \otimes u^a,$$

where for simplicity's sake we have put $v_a = v_{e_a}$.

The $\mathfrak{g}$ action on $A[u]$ is given by $\bar{L}_X = \hat{L}_{v_X} \otimes id + id \otimes L_X$, here L_X denotes the coadjoint action of $\mathfrak{g}$ on $S\mathfrak{g}^*$ (we have collected the details in Appendix A). For a vector $X = \sum_a X^a e_a$, we have

$$L_X = -\sum_{a,b,c} X^a f^c_{ab} u^b \frac{\partial}{\partial u^c}.$$

We claim that

$$\bar{L}_X S^c = 0$$

for all $X \in \mathfrak{g}$. Indeed,

$$\bar{L}_{v_X} S^c = (\hat{L}_{v_X} \otimes id) S^c + (id \otimes L_X) S^c.$$

The first term is given by

$$\begin{aligned}
(\hat{L}_{v_X} \otimes id) S^c &= (\hat{L}_{v_X} \otimes id)\Big(S - \sum_a S_{\hat{i}_{v_a}} \otimes u^a\Big) \\
&= (\hat{L}_{v_X} \otimes id)\Big(S_{\hat{d}_\Sigma} + S_{\hat{Q}_M} - \sum_a S_{\hat{i}_{v_a}} \otimes u^a\Big) \\
&= \{S_{\hat{L}_{v_X}}, S_{\hat{d}_\Sigma}\} + \{S_{\hat{L}_{v_X}}, S_{\hat{Q}_M}\} - \sum_a \{S_{\hat{L}_{v_X}}, S_{\hat{i}_{v_a}}\} \otimes u^a \\
&= S_{[\hat{L}_{v_X}, \hat{d}_\Sigma]} + S_{[\hat{L}_{v_X}, \hat{Q}_M]} - \sum_a \{S_{[\hat{L}_{v_X}, \hat{i}_{v_a}]} \otimes u^a \\
&= \sum_a S_{\hat{i}_{[v_a, v_X]}} \otimes u^a \\
&= \sum_{a,b,c} X^b f^c_{ab} S_{\hat{i}_{v_c}} \otimes u^a.
\end{aligned}$$

The second term is similarly computed:

$$\begin{aligned}
(id \otimes L_X) S^c &= (id \otimes L_X)\Big(S - \sum_a S_{\hat{i}_{v_a}} \otimes u^a\Big) \\
&= -\sum_a S_{\hat{i}_{v_a}} \otimes L_X(u^a) \\
&= \sum_{a,b,c,d} X^d f^c_{db} S_{\hat{i}_{v_a}} \otimes u^b \delta^a_c \\
&= \sum_{a,b,d} X^d f^a_{db} S_{\hat{i}_{v_a}} \otimes u^b.
\end{aligned}$$

Since $f^c_{ab} = -f^c_{ba}$, the two terms cancel out.

Thus, the equivariant action S^c sits inside the Cartan model for equivariant cohomology, namely

$$A[u]^{\mathfrak{g}} = \{O \in A[u] \,|\, \forall X \in \mathfrak{g} \colon \bar{L}_X O = 0\}.$$

It is immediate to see that the Hamiltonian vector field of S^c is

$$Q^c = Q - \sum_a \hat{i}_{v_a} \otimes u^a.$$

Note that it is not cohomological. In other words, the CME does not hold. Let us see now how to modify it. We start by computing

$$\begin{aligned}
\{S^c, S^c\} &= \Big\{S - \sum_a S_{\hat{i}_{v_a}} \otimes u^a, S - \sum_b S_{\hat{i}_{v_b}} \otimes u^b\Big\} \\
&= \{S, S\} - \sum_a \{S_{\hat{i}_{v_a}}, S\} \otimes u^a - \sum_a \{S, S_{\hat{i}_{v_b}}\} \otimes u^b + \sum_{a,b} \{S_{\hat{i}_{v_a}}, S_{\hat{i}_{v_b}}\} \otimes u^a u^b \\
&= -2 \sum_a \{S_{\hat{i}_{v_a}}, S\} \otimes u^a + \sum_{a,b} \{S_{\hat{i}_{v_a}}, S_{\hat{i}_{v_b}}\} \otimes u^a u^b,
\end{aligned}$$

where we have used the rules of BV algebras and the fact that S has degree 0 and $S_{\hat{i}_{v_a}}$ has degree -2. The second term in the above computation vanishes, since $u^a u^b = u^b u^a$ and

$$\{S_{\hat{i}_{v_a}}, S_{\hat{i}_{v_b}}\} = -\{S_{\hat{i}_{v_b}}, S_{\hat{i}_{v_a}}\}.$$

Moreover,

$$\begin{aligned}
\{S_{\hat{i}_{v_a}}, S\} &= \{S_{\hat{i}_{v_a}}, S_{\hat{d}_\Sigma} + S_{\hat{Q}_M}\} \\
&= \{S_{\hat{i}_{v_a}}, S_{\hat{d}_\Sigma}\} \\
&= S_{[\hat{i}_{v_a}, \hat{d}_\Sigma]} \\
&= S_{\hat{L}_{v_a}}.
\end{aligned}$$

Thus, we get the *equivariant Classical Master Equation*

$$\frac{1}{2}\{S^c, S^c\} + \sum_a S_{\hat{L}_{v_a}} \otimes u^a = 0.$$

Let us now take a look at observables using our discussion at the beginning of the chapter. The failure of the QME is measured by

$$\begin{aligned}
T &= \frac{1}{2}\{S^c, S^c\} - i\hbar\Delta S^c \\
&= -\sum_a S_{\hat{L}_{v_a}} \otimes u^a - i\hbar\Delta S^c \\
&= -\sum_a (S_{\hat{L}_{v_a}} + i\hbar\Delta S_{\hat{i}_{v_a}}) \otimes u^a - i\hbar\Delta S
\end{aligned}$$

We assume that the vector fields $\{\Delta S_{\hat{d}_\Sigma}, \cdot\}, \{\Delta S_{\hat{i}_{v_X}}, \cdot\}$ vanish for all $X \in \mathfrak{g}$. This has several consequences:

- $\Delta S_{\hat{L}_{v_X}} = \Delta\{S_{\hat{i}_{v_X}}, S_{\hat{d}_\Sigma}\} = 0.$
- $\Delta T = 0$. This is now immediate. Since

  $$\Delta T + \frac{i}{\hbar} Q T = 0,$$

  we get $Q^c T = 0$, which implies that S^c is an element of

  $$\mathcal{N}_T = \{O \in A[u] \,|\, [T, O] \in \mathcal{I}_T\},$$

  i.e. the algebra of functions that Poisson-commute with T up to a multiple of T.
- $[\Delta, \hat{L}_{v_X}] = \Delta\{S_{\hat{L}_{v_X}}, \cdot\} - \{S_{\hat{L}_{v_X}}, \Delta(\cdot)\} = \{\Delta S_{\hat{L}_{v_X}}, \cdot\} = 0.$
- $\Delta S_{[\hat{i}_{v_a}, \hat{i}_{v_b}]} = \Delta\{S_{\hat{i}_{v_a}}, S_{\hat{L}_{v_b}}\} = 0$, since Δ is a derivation of the Poisson bracket. Thus, we obtain the relations $\sum_c f^c_{ab} \Delta S_{\hat{i}_{v_c}} = 0$.
- Let $T' = -\sum_a S_{\hat{L}_{v_a}} \otimes u^a - i\hbar\Delta S_{\hat{Q}_M}$. Then,

  $$\{T, O\} = \{T', O\} \tag{4.3}$$

  for all $O \in A[u]$. This follows immediately from our assumptions on $\Delta S_{\hat{i}_{v_X}}, \Delta S_{\hat{d}_\Sigma}$.

The problem we want to address is the fact that Q^c does not square to zero. The idea to fix this is to consider a smaller algebra of functions on which it does. Let us define

$$\mathcal{N}'_T = \{O \in A[u] \,|\, \forall X \in \mathfrak{g}\colon \bar{L}_X O = \{\Delta S_{\hat{Q}_M}, O\} = 0\}.$$

Note that (4.3) together with the fact that $\sum_a u^a L_{e_a} = 0$ (since the structure constants of $\mathfrak{g}$ are skew-symmetric) implies that $\mathcal{N}'_T \subseteq \mathcal{N}_T$. The following proposition ([3, Prop. 3.1]) explains the importance of $\mathcal{N}'_T$.

Proposition 4.1.1. *$\mathcal{N}'_T$ is a Poisson subalgebra of $\mathcal{N}_T$. Moreover, it is invariant under $Q^c, \Delta_{S,T}$ and it contains T.*

Proof. The proof is standard but a bit tedious. In order not to clutter our discussion, we have moved it to Appendix C. □

Note that Q^c squares to zero when restricted to $\mathcal{N}'_T$, since

$$(Q^c)^2 = -\sum_a \hat{L}_{v_a} \otimes u^a$$

and $\sum_a u^a L_{e_a} = 0$, so that

$$(Q^c)^2 O = -\sum_a \bar{L}_{e_a} O = 0$$

for all $O \in \mathcal{N}'_T$.

The algebra $(\mathcal{N}'_T, Q^c)$ is called the algebra of *classical equivariant BV preobservables.* An *observable* is a preobservable that is closed under Q^c.

Let us now define the ideal $\mathcal{I}'_T = \mathcal{I}_T \cap \mathcal{N}'_T$. Then, we have the following lemma ([3, Lemma 3.4]).

Lemma 4.1.2. *$\mathcal{I}'_T$ is a $\Delta_{S,T}$-invariant Poisson ideal in $\mathcal{N}'_T$.*

Proof. Let $U \in \mathcal{N}'_T$ and let us pick an element in $\mathcal{I}'_T$, which can be written in the form OT, for some $O \in \mathcal{N}'_T$. Then, we have

$$\{U, OT\} = \{U, O\}\, T \pm O\, \{U, T\} = \{U, O\}\, T,$$

since $\{U, T\} = \{U, T'\} = 0$ (see the definition of T' above). Moreover, $\mathcal{N}'_T$ is a Poisson subalgebra of $\mathcal{N}_T$, which means that $\{U, O\} \in \mathcal{N}'_T$. This shows that $\mathcal{I}'_T$ is a Poisson ideal.

To prove $\Delta_{S,T}$-invariance, we compute

$$\Delta_{S,T}(OT) = (\Delta_{S,T} O)T \pm \{O, T\} = (\Delta_{S,T} O)T.$$

Then, Lemma C.0.1 and Lemma C.0.2 imply that $\Delta_{S,T} O \in \mathcal{N}'_T$. This concludes the proof. □

As a consequence of this lemma, we are allowed to quotient by $\mathcal{I}'_T$ and still get a Poisson algebra $\mathcal{N}'_T / \mathcal{I}'_T$, called the algebra of *quantum equivariant preobservables.* As we saw at the beginning of the chapter, $\Delta_{S,T}$ descends to the quotient. A *quantum observable* is a $\Delta_{S,T}$-closed preobservable.

§4.2 Globalization

We now move on from equivariant AKSZ theories to globalization of split AKSZ theories. The idea is to replace the target space, i.e. a cotangent bundle T^*M to some smooth manifold M with a simpler space, that is T^*T_xM, for $x \in M$. Again, this is done using a formal exponential map. In this section we discuss in detail how this is done following [9].

Let us consider a split AKSZ theory with space of fields

$$F = \mathrm{Map}(T[1]\Sigma, T^*[d-1]M),$$

where Σ is a closed oriented d-manifold and M is a smooth manifold. The target is assumed to be a dg Hamiltonian manifold with the canonical symplectic structure $\omega = d_M\alpha$, Hamiltonian θ_M and corresponding cohomological vector field Q_M. The BV action is given by transgression:

$$S = S_{\hat{d}_\Sigma} + S_{\hat{Q}_M} = i_{\hat{d}_\Sigma}\mathbb{T}\alpha + \mathbb{T}\theta$$

Let $\varphi\colon TM \to M$ be a formal exponential map on M and $x \in M$ a point. Let us define the *cotangent lift of* φ

$$\begin{aligned} \varphi_x^\sharp\colon T^*T_xM &\to T^*M \\ (y,\xi) &\mapsto \big(\varphi_x(y), (d_y\varphi_x)^{-1,*}\xi\big) \end{aligned}$$

Remark 4.2.1. In the definition of $\varphi_x^\sharp$ we have abused notation for the sake of clarity. Indeed, $\varphi_x^\sharp$ is not defined for all $y \in T_xM$. However, since $d_0\varphi_x = id_{T_xM}$, there is a neighborhood of $0 \in T_xM$ on which $d_y\varphi_x$ is invertible.

Using $\varphi_x^\sharp$, we can "linearize" the target space. The goal is to define an AKSZ theory with space of fields

$$F_x = \mathrm{Map}(T[1]\Sigma, T^*[d-1]T_xM).$$

The target is a symplectic manifold when endowed with the canonical symplectic structure $\omega_x = d_{T_xM}\alpha_x$. As (formal) Hamiltonian, we take

$$\theta_x = T(\varphi_x^\sharp)^*\theta_M.$$

Thus, we get the linearized action on F_x

$$S_x = S_{\hat{d}_\Sigma} + S_{\hat{Q}_x},$$

where here $S_{\hat{d}_\Sigma} = i_{\hat{d}_\Sigma}\mathbb{T}\alpha_x$, α_x being the Liouville 1-form on T_xM, and Q_x is the Hamiltonian vector field of θ_x. Occasionally, we will a slightly different notation, namely for a function f on the target, we will write $S_f = S_{\hat{X}_f}$, where X_f is the Hamiltonian vector field of f. For instance, we will write $S_{\theta_x} = S_{\hat{Q}_x}$.

This construction gives an AKSZ theory in a (formal) neighborhood of a point in M. We now want to investigate more closely the dependence on the point and thus globalize over M. More specifically, if we define $\hat{S}(x) = S_x$, we want to compute $d_x\hat{S}$. Since we have a formal exponential map, we also have a Grothendieck connection, which comes with a 1-form $R \in \Omega^1(M, TM \otimes \hat{S}T^*M)$ (see Appendix B). R takes values in formal vertical vector fields, i.e. (formal) functions on the target, so we can consider the function S_R on the space of (linearized fields). To make this more precise, the evaluation R_x of R at $x \in M$ is a 1-form that takes values in formal functions on F_x.

Claim 4.2.2. $d_x\hat{S} = -\{S_R, \hat{S}\}$.

Proof.

$$\begin{aligned} d_x\hat{S} &= d_x S_{\hat{d}_\Sigma} + d_x S_{T\varphi^{\sharp,*}\theta} \\ &= S_{d_x T\varphi^{\sharp,*}\theta} \\ &= -S_{L_R T\varphi^{\sharp,*}\theta} \\ &= -S_{\left[R, T\varphi^{\sharp,*}\theta\right]} = -\{S_R, S_{T\varphi^{\sharp,*}\theta}\}, \end{aligned}$$

where we have used the fact that $d_x S_{\hat{d}_\Sigma} = 0$ and the equality, discussed in Appendix B,

$$d_x T\varphi^{\sharp,*}\theta + L_R T\varphi^{\sharp,*}\theta = 0.$$

On the other hand, $\{S_R, S_{\hat{d}_\Sigma}\} = S_{\left[\hat{X}_R, \hat{d}_\Sigma\right]} = 0$, as $[\hat{X}_R, \hat{d}_\Sigma] = 0$. In this calculation, X_R is the Hamiltonian vector field corresponding to R. Thus,

$$\{S_R, S_{T\varphi^{\sharp,*}\theta}\} = \{S_R, \hat{S}\}.$$

This proves the claim. □

If we let $\tilde{S} = \hat{S} + S_R$, we get

$$\begin{aligned} d_x\tilde{S} &= d_x\hat{S} + d_x S_R \\ &= -\{S_R, \hat{S}\} + d_x S_R \\ &= -\{S_R, \hat{S}\} + S_{d_x R} \\ &= -\{S_R, \hat{S}\} - \frac{1}{2} S_{[R,R]} \\ &= -\{S_R, \hat{S}\} - \frac{1}{2}\{S_R, S_R\}. \end{aligned}$$

In the fourth equality we have used flatness of the Grothendieck connection, phrased in terms of the Maurer-Cartan equation (see Appendix B)

$$d_x R + \frac{1}{2}[R, R] = 0.$$

On the other hand,

$$\begin{aligned} \{\tilde{S}, \tilde{S}\} &= \{\hat{S}, \hat{S}\} + \{\hat{S}, S_R\} + \{S_R, \hat{S}\} + \{S_R, S_R\} \\ &= 2\{S_R, \hat{S}\} + \{S_R, S_R\}. \end{aligned}$$

Putting everything together, we get the *differential Classical Master Equation*

$$d_x\tilde{S} + \frac{1}{2}\{\tilde{S}, \tilde{S}\} = 0.$$

§4.3 Globalization of equivariant AKSZ models

We have seen how to deal with AKSZ models that have an additional Lie algebra action on the source manifold. In a completely unrelated fashion, we have seen how to study split AKSZ models in a formal neighborhood of a point. It is natural to wonder whether the two constructions are compatible and, if so, what the CME looks like. It is not difficult to show that it is indeed the case. In this section, we discuss this matter in detail and we take a look at a couple of examples.

Let us consider a split AKSZ theory with space of fields

$$F = \mathrm{Map}(T[1]\Sigma, T^*[d-1]M).$$

Suppose we have a Lie algebra $\mathfrak{g}$ acting infinitesimally on Σ. We will use the notation we used at the beginning of the chapter. Let $\{e_a\}$ be a basis for $\mathfrak{g}$ and $\{u^a\}$ the dual basis. The equivariant BV action on $F_x = \mathrm{Map}(T[1]\Sigma, T^*[d-1]T_xM)$ is given by

$$S_x^c = S_x - \sum_a S_{\hat{i}_{v_a}} \otimes u^a,$$

where S is the BV action on F.

To study the dependence on the basepoint $x \in M$, we consider the map $x \mapsto S_x^c$, which we denote by $\hat{S}^c$. Adding S_R, we obtain $\tilde{S}^c = \tilde{S} - \sum_a S_{\hat{i}_{v_a}} \otimes u^a$. As we did before, we want to compute $d_x\tilde{S}^c$. Note that

$$d_x S_{\hat{i}_{v_a}} = d_x(i_{\hat{i}_{v_a}} \mathbb{T}\omega_x) = 0,$$

where ω_x is the canonical symplectic structure on $T^*[d-1]T_xM$. Hence, $d_x\tilde{S}^c = d_x\tilde{S}$.

We have already seen that the commutativity of the u^a's and the skew-symmetry of the bracket $\{S_{\hat{i}_{v_a}}, S_{\hat{i}_{v_b}}\}$ imply that

$$\left\{ \sum_a u^a S_{\hat{i}_{v_a}}, \sum_b u^b S_{\hat{i}_{v_b}} \right\} = 0.$$

Thus,

$$\begin{aligned}
\{\tilde{S}^c, \tilde{S}^c\} &= \{\tilde{S}, \tilde{S}\} - \left\{ \sum_a S_{\hat{i}_{v_a}} \otimes u^a, \tilde{S} \right\} - \left\{ \tilde{S}, \sum_a S_{\hat{i}_{v_a}} \otimes u^a \right\} \\
&= \{\tilde{S}, \tilde{S}\} + \sum_a \{S_{\hat{i}_{v_a}}, \tilde{S}\} \otimes u^a + \sum_a \{\tilde{S}, S_{\hat{i}_{v_a}}\} \otimes u^a \\
&= \{\tilde{S}, \tilde{S}\} + \sum_a \{S_{\hat{i}_{v_a}}, S_{\hat{d}_\Sigma}\} \otimes u^a + \sum_a \{S_{\hat{d}_\Sigma}, S_{\hat{i}_{v_a}}\} \otimes u^a \\
&= \{\tilde{S}, \tilde{S}\} + 2\sum_a \{S_{\hat{i}_{v_a}}, S_{\hat{d}_\Sigma}\} \otimes u^a \\
&= \{\tilde{S}, \tilde{S}\} + 2\sum_a S_{\hat{L}_{v_a}} \otimes u^a.
\end{aligned}$$

In the third equality, we have used the relations

$$\{S_{\hat{i}_{v_a}}, S_{\hat{Q}_x}\} = \{S_{\hat{i}_{v_a}}, S_R\} = 0,$$

while in the third we have used the fact that $\left|S_{\hat{i}_{v_a}}\right| = -2$ and $\left|S_{\hat{d}_\Sigma}\right| = 0$, which gives by graded skew-symmetry of the Poisson bracket

$$\{S_{\hat{i}_{v_a}}, S_{\hat{d}_\Sigma}\} = \{S_{\hat{d}_\Sigma}, S_{\hat{i}_{v_a}}\} = S_{\hat{L}_{v_a}}.$$

Putting all of this together and using the differential CME, we get that

$$d_x\tilde{S}^c + \frac{1}{2}\{\tilde{S}^c, \tilde{S}^c\} + \sum_a S_{\hat{L}_{v_a}} \otimes u^a = 0,$$

which we call the *equivariant differential Classical Master Equation.*

To end the chapter, let us take a look at a couple of examples.

Two-dimensional SUSY Yang-Mills theory. Let us recall the AKSZ construction of supersymmetric Yang-Mills theory (see also [3]). Let Σ be a compact oriented surface and consider a Lie algebra $\mathfrak{l}$. We want to construct an AKSZ theory with target

$$T^*[1](\mathfrak{l}[1] \times \mathfrak{l}[2]),$$

endowed with the canonical symplectic structure. Let $\{T_\alpha\}$ be a basis for $\mathfrak{l}$ and consider coordinates (with respect to this basis) c, ϕ. This means that the map

$$\begin{aligned} &\mathfrak{l} \to \mathbb{R}^n \\ &X \mapsto \big(c_1(X), \dots, c_n(X)\big) \end{aligned}$$

gives the coordinate representation of the vectors in $\mathfrak{l}$ (and similarly for ϕ). Let ξ, η be the corresponding momenta of degree $0, -1$ respectively, i.e. the cotangent fiber coordinates (note that the fiber is $(\mathfrak{l}[1] \times \mathfrak{l}[2])^*[1] \cong \mathfrak{l}^* \times \mathfrak{l}^*[-1]$).

The Hamiltonian on the target is defined as

$$\theta = \sum_\alpha \left(\frac{1}{2}\xi_\alpha\, [c, c]^\alpha + \eta_\alpha\, [c, \phi]^\alpha + \eta_\alpha \phi^\alpha\right)$$

Suppose now that there is a Lie algebra $\mathfrak{g}$ acting infinitesimally on Σ. We want to perform globalization on the equivariant BV action. To this end, we need to pick a formal exponential map on the target. In this case there is an easy choice, namely the map

$$\begin{aligned} \varphi\colon T(\mathfrak{l}[1] \times \mathfrak{l}[2]) &\to \mathfrak{l}[1] \times \mathfrak{l}[2] \\ (x, v) &\mapsto x + v. \end{aligned}$$

Fix the first entry to obtain the map

$$\begin{aligned} \varphi_x\colon \mathfrak{l}[1] \times \mathfrak{l}[2] &\to \mathfrak{l}[1] \times \mathfrak{l}[2] \\ v &\mapsto x + v. \end{aligned}$$

Its cotangent lift is given by

$$\begin{aligned} \varphi_x^\sharp\colon T^*(\mathfrak{l}[1] \times \mathfrak{l}[2]) &\to T^*(\mathfrak{l}[1] \times \mathfrak{l}[2]) \\ (v, \sigma) &\mapsto (x + v, \sigma). \end{aligned}$$

We compute the (formal) Hamiltonian $\theta_x = T(\varphi_x^\sharp)^*\theta$, where the T denotes Taylor expansion around $v = 0$. For the sake of clarity, we drop the summation signs and use the Einstein convention. We have

$$\begin{aligned} \big((\varphi_x^\sharp)^*\theta\big)(v, \sigma) &= \theta(x + v, \sigma) \\ &= \frac{1}{2}\xi_\alpha(\sigma)\, [c(x + v), c(x + v)]^\alpha \\ &\quad + \eta_\alpha(\sigma)\, [c(x + v), \phi(x + v)]^\alpha + \xi_\alpha(\sigma)\phi^\alpha(x + v). \end{aligned}$$

Using linearity of c, ϕ, ξ, η, it is straightforward to compute the Taylor expansion

$$\begin{aligned}\theta_x(v,\sigma) = \xi_\alpha(\sigma)\,[c(x), c(v)]^\alpha + \frac{1}{2}\xi_\alpha(\sigma)\,[c(v), c(v)]^\alpha \\ + \eta_\alpha(\sigma)\,[c(x), \phi(v)]^\alpha + \eta_\alpha(\sigma)\,[c(v), \phi(x)]^\alpha \\ + \eta_\alpha(\sigma)\,[c(v), \phi(v)]^\alpha + \xi_\alpha(\sigma)\phi^\alpha(v),\end{aligned}$$

that is

$$\begin{aligned}\theta_x = \xi_\alpha\,[c(x), c]^\alpha + \frac{1}{2}\xi_\alpha\,[c, c]^\alpha \\ + \eta_\alpha\,[c(x), \phi]^\alpha + \eta_\alpha\,[c, \phi(x)]^\alpha \\ + \eta_\alpha\,[c, \phi]^\alpha + \xi_\alpha\phi^\alpha.\end{aligned}$$

The Grothendieck connection for the formal exponential map φ is given by

$$R = \sum_\alpha \left(\delta c^\alpha \frac{\partial}{\partial c^\alpha} + \delta\phi^\alpha \frac{\partial}{\partial \phi^\alpha}\right),$$

where we used δ to denote the de Rham differential on $\mathfrak{l}[1] \times \mathfrak{l}[2]$.

We use tildes to denote the fields obtained by pulling back the coordinates c, ϕ, ξ, η by the evaluation map. The globalized action then reads

$$\begin{aligned}\tilde{S} &= S_{\hat{d}_\Sigma} + S_{\theta_x} + S_R \\ &= \int_\Sigma \tilde{\xi}_\alpha d_\Sigma \tilde{c}^\alpha + \tilde{\eta}_\alpha d_\Sigma \tilde{\phi}^\alpha \\ &+ \int_\Sigma \tilde{\xi}_\alpha\left([c(x), \tilde{c}]^\alpha + \frac{1}{2}[\tilde{c}, \tilde{c}]^\alpha + \tilde{\phi}^\alpha\right) + \tilde{\eta}_\alpha\left([c(x), \tilde{\phi}]^\alpha + [\tilde{c}, \phi(x)]^\alpha + [\tilde{c}, \tilde{\phi}]\right) \\ &+ \int_\Sigma \tilde{\xi}_\alpha \delta\tilde{c}^\alpha + \tilde{\eta}_\alpha \delta\tilde{\phi}^\alpha\end{aligned}$$

(we have omitted the wedge symbol $\wedge$ in the above formula). To get the globalized *equivariant* action, one just replaces the de Rham differential d_Σ with the Cartan differential $d_G = d_\Sigma - \sum_a \hat{i}_{v_a} \otimes u^a$.

Equivariant Donaldson-Witten theory. Let us recall how to get DW theory from the AKSZ construction (see also [3]). Let Σ be a four-dimensional closed oriented manifold and let $\mathfrak{l}$ be a Lie algebra endowed with a non-degenerate symmetric pairing $\langle\cdot,\cdot\rangle$. Then, the target space is defined to be $\mathfrak{l}[1] \oplus \mathfrak{l}[2]$ together with the symplectic structure

$$\omega = \langle \delta c, \delta\phi\rangle,$$

where c, ϕ are the coordinates of degree 1 and 2 respectively. Let $\{e_i, T_\alpha\}$ be a basis for $\mathfrak{l}[1] \oplus \mathfrak{l}[2]$. Then, using the obvious identifications

$$\frac{\partial}{\partial c^i} = e_i, \quad \frac{\partial}{\partial \phi^\alpha} = T_\alpha,$$

ω can be written as

$$\omega = \sum_{i,\alpha} \langle e_i, T_\alpha\rangle \delta c^i \wedge \delta\phi^\alpha \tag{4.4}$$

The Hamiltonian on the target is

$$\theta = \frac{1}{2}\langle \phi, \phi \rangle + \frac{1}{2}\langle \phi, [c, c] \rangle.$$

We want to view this as a *split* AKSZ theory. To do this, note that we have a canonical isomorphism

$$\begin{aligned} \Lambda \colon \mathfrak{l} &\to \mathfrak{l}^* \\ x &\mapsto \langle x, \cdot \rangle, \end{aligned}$$

which we can use to endow $\mathfrak{l}^*$ with a Lie bracket $[\cdot, \cdot]_*$ by pulling back the one on $\mathfrak{l}$. Explicitly, for $l, l' \in \mathfrak{l}$,

$$[l, l']_* = [\Lambda^{-1} l, \Lambda^{-1} l'].$$

Thus, we can identify

$$\mathfrak{l}[1] \oplus \mathfrak{l}[2] \cong \mathfrak{l}[1] \oplus \mathfrak{l}^*[2] = T^*[3](\mathfrak{l}[1]). \tag{4.5}$$

Define a pairing

$$\begin{aligned} \mathfrak{l} \times \mathfrak{l}^* &\to \mathbb{R} \\ (x, v) &\mapsto \langle x, \Lambda^{-1} v \rangle. \end{aligned}$$

We will use the notation $\langle x \,|\, v \rangle$ to denote the pairing.

Let now γ be the coordinate on $\mathfrak{l}^*$ obtained from c using Λ, i.e. $\gamma = c\Lambda^{-1}$. The symplectic form on $T^*[3](\mathfrak{l}[1])$ corresponding to ω under the above isomorphism (4.5) is then given by

$$\omega' = \langle \delta\phi \,|\, \delta\gamma \rangle$$

and the Hamiltonian is given by

$$\theta' = \frac{1}{2}\big\langle \phi, \phi \big\rangle + \frac{1}{2}\big\langle \phi \,|\, [\gamma, \gamma]_* \big\rangle.$$

We now choose the formal exponential map

$$\begin{aligned} \varphi \colon T\mathfrak{l}[1] &\to \mathfrak{l}[1] \\ (v, w) &\mapsto v + w. \end{aligned}$$

Fix $v \in \mathfrak{l}[1]$. The cotangent lift of $\varphi_v \colon w \mapsto \varphi(v, w)$ is given by

$$\begin{aligned} \varphi_v^\sharp \colon T^*\mathfrak{l}[1] &\to T^*\mathfrak{l}[1] \\ (w, \xi) &\mapsto (v + w, \xi). \end{aligned}$$

We have

$$(\varphi_v^\sharp)^* \theta'_v(w, \xi) = \frac{1}{2}\big\langle \phi(\xi), \phi(\xi) \big\rangle + \frac{1}{2}\big\langle \phi(\xi) \,\big|\, [\gamma(v + w), \gamma(v + w)]_* \big\rangle$$

Thus, the Taylor expansion $\theta'_v = T(\varphi_v^\sharp)^* \theta'$ reads

$$\theta'_v(w, \xi) = \big\langle \phi(\xi) \,\big|\, [\gamma(v), \gamma(w)]_* \big\rangle + \frac{1}{2}\big\langle \phi(\xi) \,\big|\, [\gamma(w), \gamma(w)]_* \big\rangle,$$

i.e.

$$\theta'_v = \big\langle \phi \,\big|\, [\gamma(v), \gamma]_* \big\rangle + \frac{1}{2}\big\langle \phi \,\big|\, [\gamma, \gamma]_* \big\rangle.$$

The Grothendieck connection is given by

$$R = \sum_{\alpha} \delta\gamma^{\alpha} \frac{\partial}{\partial \gamma^{\alpha}}.$$

Thus, the globalized action is given by

$$\begin{aligned}
\tilde{S}^c &= S_{\hat{d}_\Sigma} + S_{\theta'_v} + S_R \\
&= \int_\Sigma \langle \tilde{\phi} \,|\, d_\Sigma \tilde{\gamma} \rangle \\
&+ \int_\Sigma \langle \tilde{\phi} \,|[\tilde{\gamma}(v), \tilde{\gamma}]_* \rangle \\
&+ \int_\Sigma \delta\tilde{\gamma}^{\alpha} \tilde{\phi}^{\alpha}.
\end{aligned}$$

Again, we have used tildes to denote the pullback of the coordinates ϕ, γ by the evaluation map (see the AKSZ construction in Chapter 3). Again, to get the globalized equivariant action, one just replaces the de Rham differential d_Σ with the Cartan differential.

Appendix A

Differential graded algebras

In this appendix we give a brief of the theory of graded differential algebras and its relation to equivariant cohomology. The reader can find a complete account in [**tu2020**].

In what follows, $\mathfrak{g}$ will be a finite-dimensional Lie algebra.

Definition A.0.1. A *graded $\mathfrak{g}$-differential algebra* is a graded commutative algebra

$$A = \bigoplus_{n=0}^{\infty} A^n$$

together with graded derivations d, i_X, L_X of degree $1, -1, 0$ respectively for all $X \in \mathfrak{g}$. i_X, L_X are assumed to depend linearly on $X \in \mathfrak{g}$ and to satisfy the rules of Cartan calculus, i.e.

(i) $d^2 = 0$,

(ii) $[L_X, d] = 0$,

(iii) $[i_X, d] = L_X$,

(iv) $[i_X, i_Y] = 0$,

(v) $[L_X, L_Y] = L_{[X,Y]}$,

(vi) $[L_X, i_Y] = i_{[X,Y]}$

for all $X, Y \in \mathfrak{g}$.

Example A.0.2. Let G be a Lie group and M be a G-manifold, i.e. a manifold together with a smooth G-action. Then, there is also a $\mathfrak{g}$-action on M, namely the Lie algebra morphism $a\colon \mathfrak{g} \to \mathfrak{X}(M)$ given by

$$a(X)_p = \left.\frac{d}{dt}\right|_0 e^{-tX} p.$$

The graded commutative algebra $\Omega^\bullet(M)$ of differential forms on M is a $\mathfrak{g}$-differential algebra. Indeed, we have the de Rham operator d, the contractions i_X and the Lie derivative L_X for all $X \in \mathfrak{g}$, where by i_X and L_X we mean the contraction and the Lie derivative with respect to the vector fields $a(X)$.

If A is a $\mathfrak{g}$-differential algebra, we define the *horizontal subalgebra* to be $A_h = \bigcap_{X\in\mathfrak{g}} \ker i_X$ and the *invariant subalgebra* to be $A^{\mathfrak{g}} = \bigcap_{X\in\mathfrak{g}} \ker L_X$. Furthermore, we define the *basic subalgebra* to be $A_b = A_h \cap A^{\mathfrak{g}}$. Note that A_b is a differential subcomplex of A.

At this point, we would like to define a cohomology theory for $\mathfrak{g}$-differential algebras that captures the influence of $\mathfrak{g}$. In order to do so, we define the *Weil algebra* to be $\mathfrak{g}$-differential algebra

$$W(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \bigoplus_{2i+j=n} S^i\mathfrak{g}^* \otimes \Lambda^j\mathfrak{g}^*.$$

The g-da graded derivations are defined as follows. Let $\{t_a\}$ be a basis for $\mathfrak{g}$, $u^a \in S^1\mathfrak{g}^*$ and $\theta^a \in \Lambda^1\mathfrak{g}^*$ be the corresponding generators of the symmetric and exterior algebras (respectively). The *Weil differential* is the defined by

$$\begin{aligned} d_W u^a &= [\theta, u]^a, \\ d_W \theta^a &= u^a + \frac{1}{2}[\theta, \theta]^a. \end{aligned}$$

The contraction is given by $i_{t_a} = \frac{\partial}{\partial\theta^a}$ and $L_{t_a} = [i_{t_a}, d_W]$. The reason behind this definition has to do with a certain universal property that the Weil algebra satisfies. We refer the reader to [**tu2020**] for more details.

Given a $\mathfrak{g}$-differential algebra A, we define its *equivariant cohomology* to be given by

$$H_{\mathfrak{g}}(A) = H\big(A_{\mathfrak{g}}\big),$$

where $A_{\mathfrak{g}} = (W\mathfrak{g} \otimes A)_b$.

We can also consider a different $\mathfrak{g}$-da algebra, namely the *Cartan algebra*

$$A[u] = A \otimes S(\mathfrak{g}^*)$$

endowed with the differential $d_C = d - \sum_a i_{t_a} \otimes u^a$. It turns out that the Cartan model gives the same cohomology of the Weil model.

To end this section, let us remark that the relevance of this machinery is related to G-manifolds. Indeed, one can define a notion of "equivariant cohomology" for G-manifolds and it turns out that the equivariant cohomology of the algebra of differential forms is isomorphic to it. This provides a nice framework to work with G-manifolds, as the definition of equivariant cohomology of a G-manifold is rather abstract, while it is much more feasible to compute the equivariant cohomology of a graded differential algebra. We refer the reader to [**tu2020**] for a full account.

Appendix B

Basics of formal geometry

In this appendix we give a swift introduction to formal coordinates following [4]. We also discuss the notion of Grothendieck connection associated to a formal exponential map, which plays an important role in globalization techniques for perturbative quantum field theory.

Let M be a smooth manifold.

Definition B.0.1. A *formal exponential map* is a smooth map $\varphi\colon TM \supseteq U \to M$, where U is a neighborhood of the zero section of TM, such that

(i) $\varphi(x, 0) = x$ for all $x \in M$.

(ii) $d_0\varphi_x = id_{T_xM}$ for all $x \in M$, where

$$\begin{aligned}\varphi_x\colon T_xM &\to M\\ y &\mapsto \varphi(x,y),\end{aligned}$$

Let $f \in C^\infty(M)$ and consider the pullback $\varphi^* f = f\varphi$. If $(x, y) \in U$, we have $T_{(x,y)TM} \cong T_xM \times T_y(T_xM)$. We can then separate the base and the fiber part of the differential $d(\varphi^* f) = df d\varphi\colon T(TM) \to TM$. We will denote them by $d_x(\varphi^* f) = df d_x\varphi$ and $d_y(\varphi^* f) = df d_y\varphi$ respectively. By (ii), we can assume up to shrinking U that $d_y\varphi$ is invertible. Hence, we get the relation

$$d_x(\varphi^* f) = d_y(\varphi^* f)(d_y\varphi)^{-1}d_x\varphi \tag{B.1}$$

The basic object we want to consider is the Taylor of $\varphi^* f$ in the fiber coordinates around the origin, which we denote by $T\varphi^* f$. This is a section of $\hat{S}T^*M$, the extended symmetric algebra of the cotangent bundle T^*M. If we put $\sigma = T\varphi^* f$, the Taylor expansion of (B.1) gives

$$d_x\sigma = d_y\sigma(d_y\varphi)^{-1}d_x\varphi.$$

We can do this more generally. If σ is a section of $\hat{S}T^*M$, we define

$$R(\sigma) \in \Gamma(T^*M \otimes \hat{S}T^*M)$$

to be the Taylor expansion of $-d_y\sigma(d_y\varphi)^{-1}d_x\varphi$. The assignment

$$\begin{aligned}\mathfrak{X}(M)\times\Gamma(\hat{S}T^*M) &\to \Gamma(\hat{S}T^*M)\\ (X,\sigma) &\mapsto i_X R(\sigma).\end{aligned}$$

defines a connection on $\hat{S}T^*M$, called *Grothendieck connection.* Writing it down in coordinates, one can check that it is flat, i.e. is satisfies the Maurer-Cartan equation

$$d_x R + \frac{1}{2}[R, R] = 0.$$

The bracket $[\cdot,\cdot]$ is the Lie bracket of vector fields. Indeed, it is easy to check that $i_X R$ is a derivation of $\hat{S}T^*M$, that is a *formal vertical vector fields.* Note that

$$\mathrm{Der}(\hat{S}T^*M) = \Gamma(TM \otimes \hat{S}T^*M).$$

Our discussion can be rephrased in terms of R, meaning that if $\sigma = T\varphi^* f$ for $f \in C^\infty(M)$,

$$d_x\sigma + R(\sigma) = 0.$$

The converse is also true. We refer the reader to [8] for a more detailed discussion on the properties of R.

Similar constructions can be carried out with formal vertical multivector fields and differential forms. Let us denote them respectively by

$$\hat{\mathfrak{X}}(TM) = \Gamma(\Lambda TM \otimes \hat{S}T^*M), \quad \hat{\Omega}(TM) = \Gamma(\Lambda T^*M \otimes \hat{S}T^*M).$$

Given multivector fields or differential forms, we can use φ to produce formal vertical multivector fields or differential forms by Taylor expanding. Thus, we get maps

$$T\varphi^* \colon \mathfrak{X}(M) \to \hat{\mathfrak{X}}(TM), \quad T\varphi^* \colon \Omega(M) \to \hat{\Omega}(TM).$$

Here, the pullback φ^* of multivector fields is defined to be the pushforward φ_*^{-1}, which is well-defined because of how we defined formal exponential maps.

Since R takes values in (formal vertical) vector fields, we can take the Lie derivative of sections along R. If $\sigma \in \hat{\Omega}(TM)$, say, $L_R\sigma$ is a 1-form on M that takes values in formal vertical differential forms. As above, the equation

$$d_x\sigma + L_R\sigma = 0$$

is satisfied if and only if σ lies in the image of $T\varphi^*$.

Appendix C

Computations

In this appendix, we report the proof of Proposition 4.1.1. The argument is taken from [3].

The proof uses a couple of lemmata.

Lemma C.0.1. *For all $X \in \mathfrak{g}$:*

(i) $[\bar{L}_X, \Delta] = 0$,

(ii) $[\bar{L}_X, Q^c] = 0$,

(iii) $\bar{L}_X T = 0$,

(iv) $[\bar{L}_X, \Delta_{S,T}] = 0$.

Proof. We saw that $[\hat{L}_{v_X}, \Delta] = 0$. Moreover, Δ manifestly commutes with L_X. This proves the first claim.

As far the second one is concerned, we have

$$\begin{aligned}[\bar{L}_X, Q^c] &= [\bar{L}_X, \hat{d}_\Sigma] + [\bar{L}_X, \hat{Q}] - [\bar{L}_X, \sum_a \hat{i}_{v_a} \otimes u^a] \\ &= [\hat{L}_{v_X}, \hat{d}_\Sigma] \otimes id - \sum_{a,b} [\hat{L}_{v_a}, \hat{i}_{v_b}] \otimes u^b - \sum_{a,b} \hat{i}_{v_a} \otimes [L_{e_b}, u^a] = 0.\end{aligned}$$

Let us move to the third claim. Recall that

$$T = -\sum_a (S_{\hat{L}_{v_a}} + i\hbar \Delta S_{\hat{i}_{v_a}}) \otimes u^a - i\hbar \Delta S.$$

We have

$$\begin{aligned}(\hat{L}_{v_b} \otimes id)\left(\sum_a S_{\hat{L}_{v_a}} \otimes u^a\right) &= \sum_a \hat{L}_{v_b} S_{\hat{L}_{v_a}} \otimes u^a \\ &= \sum_a S_{\hat{L}_{[v_b, v_a]}} \otimes u^a \\ &= \sum_{a,c} f^c_{ba} S_{\hat{L}_{v_c}} \otimes u^a \\ &= -(id \otimes L_{e_b})\left(\sum_a S_{\hat{L}_{v_a}} \otimes u^a\right).\end{aligned}$$

Recall the relation $\sum_c f^c_{ab} S_{\hat{i}_{v_c}} = 0$ we proved in Chapter 3. Since we assumed that $\{\Delta S_{\hat{i}_{v_a}}, \cdot\} = 0$, we get

$$\begin{aligned}\sum_b \bar{L}_{e_a}(\Delta S_{\hat{i}_{v_b}} \otimes u^b) &= \sum_b (id \otimes L_{e_a})(\Delta S_{\hat{i}_{v_b}} \otimes u^b) \\ &= \sum_{b,c} -f^c_{ab} \Delta S_{\hat{i}_{v_c}} \otimes u^b = 0.\end{aligned}$$

Finally, since $[\hat{L}_X, \Delta] = 0$, we have

$$\hat{L}_{v_a} \Delta S = \Delta \hat{L}_{v_a} S = 0$$

and $L_{e_a} \Delta S = 0$. Putting everything together, we get the third claim.

The last claim follows from the previous ones and the definition of $\Delta_{S,T}$. □

For a function $f \in C^\infty(F)$, let us denote its Hamiltonian vector field by X_f.

Lemma C.0.2. *In the context of Lemma C.0.1, the following hold:*

(i) $[X_{\Delta S_{\hat{Q}_M}}, \Delta] = 0,$

(ii) $[X_{\Delta S_{\hat{Q}_M}}, Q^c] = 0,$

(iii) $X_{\Delta S_{\hat{Q}_M}}(T) = 0,$

(iv) $[X_{\Delta S_{\hat{Q}_M}}, \Delta_{S,T}] = 0.$

Proof. Let $f \in C^\infty(F)$. Then, we have

$$\begin{aligned}[X_{\Delta S_{\hat{Q}_M}}, \Delta] f &= X_{\Delta S_{\hat{Q}_M}} \Delta f - \Delta X_{\Delta S_{\hat{Q}_M}} f \\ &= \{\Delta S_{\hat{Q}_M}, \Delta f\} - \Delta \{\Delta S_{\hat{Q}_M}, f\} \\ &= \{\Delta S_{\hat{Q}_M}, \Delta f\} - \{\Delta S_{\hat{Q}_M}, \Delta f\} = 0,\end{aligned}$$

where we have used the fact that Δ is a derivation of the Poisson bracket. This shows the first claim.

As far as the second claim goes, it is sufficient to prove that $\{\Delta S_{\hat{Q}_M}, S^c\} = 0$. We compute

$$\begin{aligned}\Delta S_{\hat{Q}_M}, S^c\} &= \{\Delta S^c, S^c\} \\ &= \frac{1}{2} \{S^c, S^c\} \\ &= -\sum_a \Delta S_{\hat{L}_{v_a}} \otimes u^a = 0.\end{aligned}$$

Here, the first equality follows from the assumptions

$$\{\Delta S_{\hat{d}_\Sigma}, \cdot\} = \{\Delta S_{\hat{i}_{v_X}}, \cdot\} = 0.$$

The second equality follows from the fact that Δ is a derivation of $\{\cdot, \cdot\}$ and the third equality follows from the equivariant CME.

Let us prove the third claim. For degree reasons, $\{T, T\} = 0$. Thus,

$$\begin{aligned} 0 = \{T, T\} &= \{T', T\} \\ &= \left\{ \sum_a S_{\hat{L}_{v_a}} \otimes u^a + i\hbar \Delta S_{\hat{Q}_M}, T \right\} \\ &= \sum_a \bar{L}_{e_a}(T) \otimes u^a + i\hbar \{\Delta S_{\hat{Q}_M}, T\} \\ &= i\hbar \{\Delta S_{\hat{Q}_M}, T\} \end{aligned}$$

In the third equality, we have used (iii) from Lemma C.0.1. The last claim follows immediately from the first three. □

We can finally prove Proposition 4.1.1.

Proof. Lemma C.0.1 shows that $[\bar{L}_X, Q^c] = 0$ for all $X \in \mathfrak{g}$. We are going to show that $[X_{\Delta S_{\hat{Q}_M}}, Q^c] = 0$, i.e. $\{\Delta S_{\hat{Q}_M}, S^c\} = 0$. Indeed,

$$\begin{aligned} \{\Delta S_{\hat{Q}_M}, S^c\} &= \{\Delta S^c, S^c\} \\ &= \frac{i}{\hbar} \{T + \sum_a S_{\hat{L}_{v_a}} \otimes u^a, S^c\} \\ &= \frac{i}{\hbar} (Q^c T + \sum_a \bar{L}_{e_a} S^c) = 0. \end{aligned}$$

This proves that $\mathcal{N}'_T$ is invariant under Q^c.

Invariance under $\Delta_{S,T}$ is an immediate consequence of the last item in both lemmas, while the fact that T belongs to $\mathcal{N}'_T$ follows from the third item thereof. □

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