Invariants of Higher-Dimensional Knots and Topological Quantum Field Theories

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Abstract

The present work can be divided into two main parts: $i)$ the first deals with the functional-integral quantization via BV formalism of BF theories in any dimension, and $ii)$ the second with (isotopy-) invariants of higher-dimensional knots in Euclidean spaces. We briefly comment on the motivations and the results of part $i)$ and $ii)$.

BF theory in 3 dimensions (with the addition of a so-called "cosmological term") is just another way of writing the Chern–Simons action functional. One sees immediately that, on the contrary to Chern–Simons theory, BF theory in 3 dimensions admits a straightforward generalizations to arbitrary dimensions. Cattaneo, Cotta-Ramusino and Longoni explicitly produced cohomology classes of the space of imbeddings of the circle S^1 into $\mathbb{R}^m$, for $m > 3$; these classes are the natural generalizations in any dimension of the perturbative knot invariants coming from the perturbative expansions in Chern–Simons theory in 3 dimensions. Therefore, BF theories in higher dimensions seem to be the natural Topological Quantum Field Theories (shortly, TQFT) to interpret such cohomology classes as perturbative expansions. However, functional-integral quantization of BF theories in arbitrary dimensions requires more care than in 3 dimensions, due to the presence of reducible symmetries; hence, we have to resort to the so-called BV formalism. At this point we may produce a BV-observable (i.e., a generalization of usual gauge-invariant functionals) for BF theories in all dimensions related to higher-dimensional knots, i.e. imbeddings of spheres of codimension 2 into $\mathbb{R}^m$. We discuss the functional-integral quantization of such an observable which is expected to yield an invariant of the imbedding. Finally, we compute explicitly the terms of order 2 and 3 of the perturbative expansion of the v.e.v. of this observable.

Part $ii)$ is directly linked to part $i)$, although we point out that the mathematical results are independent of the TQFT-framework. Here we study the properties of the functions on the space of imbeddings coming from perturbative invariants of the Vacuum Expectation Value (shortly v.e.v.) of the observable described above. The term of order 2 can be identified with the Bott invariant Θ_2 for odd spheres of codimension 2. The term of order 3 yields a function Θ_3 on the space of imbeddings of even spheres of codimension 2; a modification of this function is shown to be an isotopy-invariant for $m = 4$. A characterization of the general case is also given.

Zusammenfassung

Diese Dissertation besteht aus zwei Hauptteilen: $i)$ im ersten Teil beschäftigen wir uns mit der "functional integral quantization" von topologischen Quantenfeldtheorien (kurz, QFT) vom BF Typ in beliebiger Dimension und $ii)$ im zweiten Teil mit (Isotopie-) Invarianten von Einbettungen von Sphären der Kodimension 2 in Euklidischen Räumen. Wir beschreiben kurz die Motivationen und die Resultate von beiden Teilen.

Das wohl-bekannte Chern–Simons Funktional kann durch das Funktional für topologischen QFT vom BF Typ geschrieben werden (unter Einschluss eines so-genannten "kosmologischen Terms"). Man sieht dann sofort, dass das Funktional für topologischen QFT vom BF Typ in höheren Dimesionen definiert werden kann, ganz anders als das Chern–Simons Funktional. Cattaneo, Cotta-Ramusino und Longoni haben explizit Kohomologie-Klassen auf dem Raum der Einbettungen von S^1 in $\mathbb{R}^m$ (mit $m > 3$) konstruiert, welche die natürliche Verallgemeinerung von den störungstheoretischen Invarianten von Knoten aus der Chern–Simons Theorie bilden. Deswegen scheinen topologischen Quantenfeldtheorien vom BF Typ die natürlichen Kandidaten, um die Klassen von Cattaneo, Cotta-Ramusino und Longoni störungstheoretisch zu interpretieren. Die "functional integral quantization" von topologischen QFT vom BF Typ bringt wegen der Präsenz von reduziblen Symmetrien jedoch zusätzliche Schwierigkeiten im Vergleich zur Dimension 3, weshalb wir den so-genannten BV Formalismus anwenden. Auf diese Weise können wir BV Observablen (eine natürliche Verallgemeinerung von eichinvarianten Funktionalen) definieren, welche mit Einbettungen von Sphären der Kodimension 2 in $\mathbb{R}^m$ im im Zusammenhang stehen. Wir analysieren dann im Detail die "functional integral quantization" von dieser Observable und berechnen die Terme von Ordnung 2 und 3 in deren störungstheoretischen Entwicklung explizit.

Der zweite Teil ist zu dem ersten eng verbunden, obwohl die mathematischen Resultate vom physikalischen Hintergrund unabhängig sind. Insbesondere analysieren wir die Eigenschaften von den Termen von Ordnung 2 und 3 in der störungstheoretischen Entwicklung der oben beschriebenen Observablen. Dabei stellt sich heraus, dass der Term der Ordnung 2 mit der Invarianten von Bott für Einbettungen von Sphären der Kodimension 2 in ungeraddimensionalen Euklidischen Räumen identifiziert werden kann. Der Term von Ordnung 3 ist eine Funktion auf der Menge von Einbettungen von der Sphäre von Kodimension 2 in geraddimensionalen Euklidischen Räumen. Im einfachsten Fall $m = 4$ liefert eine Modifikation von dieser Funktion eine Invariante. Auch im allgemeinen Fall erhalten wir eine Charakterisierung.

Contents

Chapter 1

Introduction

By definition, a knot in a given smooth 3-manifold M is a smooth imbedding of the circle S^1 into M, i.e. a smooth injective map of S^1 to M with everywhere injective tangent map. One of the most fundamental concepts in the theory of knots is that of *isotopy*: two knots in M are isotopic, if, roughly speaking, they can continuously deformed into each other. More precisely, two knots are isotopic, if they can be joined by a continuous curve in the space of knots. Isotopy is an equivalence relation on the space of knots. Intimately linked to the notion of isotopy is that of *knot invariant*: a knot invariant is a function on the space of knots with values in some field k, which depends only on the isotopy class of the given knot. If the field k equals $\mathbb{R}$ or $\mathbb{C}$ and one introduces a smooth structure on the space of knots, a knot invariant can be seen as a smooth function with zero exterior differential: in fact, such a function is locally constant, i.e. it is constant on connected components of the space of knots, and therefore can depend only on the isotopy class of a given knot. This is the main approach to knot invariant we are interested to.

It is well-known that, given a G-principal bundle P over the ambient space M with a flat connection A, the holonomy $\mathrm{H}(A)|_0^1(\gamma)$ of a knot γ w.r.t. the connection A is an element of G and its conjugacy class depends only on the homotopy class of γ. Of course, the holonomy depends also on the connection A. If we take the trace of the holonomy w.r.t. a given representation (ρ, V), the resulting real, resp. complex, function on the space of knots depends only on the gauge class of A. This function is of course a knot invariant, but it is rather trivial as it does not distinguish homotopic but non isotopic knots (e.g., on $\mathbb{R}^3$ all knots are homotopic, so the invariant is completely trivial). However, this is a good starting point to define more refined knot invariants. In fact, these can be obtained if one (formally) averages the trace of the holonomy over all connections with a suitable measure. This is the point where physics comes in help.

In [54], Witten considered the Chern–Simons functional $S_{\mathrm{C.S.}}$ on a smooth, oriented 3-manifold M, depending on connections on a given G-principal trivial bundle P over M, and showed that the vacuum expectation value (shortly, v.e.v.) of the trace of the holonomy of a knot γ w.r.t. some representation (ρ, V) of G w.r.t. the Chern–Simons action is a knot invariant (for $G = SU(2)$ and the fundamental representation one gets, e.g., the Jones polynomial). In formulae, the v.e.v. of the trace of the holonomy, which physicists usually call “Wilson loop”, is given by the formal functional integral

$$\int \mathcal{D}A \ \mathrm{Tr}_\rho \left[\mathrm{H}(A)|_0^1(\gamma)\right] \ \exp \mathrm{i} S_{\text{C.S.}}, \tag{1.0.1}$$

where by $\mathcal{D}A$ we denote a formal measure over all gauge-equivalence classes of connections.

Witten analyzed the properties of these v.e.v.s by making use of arguments of conformal field theory. Motivated by this discovery, physicists and mathematicians studied explicitly the computation of (1.0.1) as a perturbative expansion around the Gaussian part. Observe that in order to perform these perturbative expansions, one has to choose a way to fix the gauge-symmetries of the Chern–Simons action, which corresponds to fix a unique representative of each gauge-class of connections. There are two particularly convenient ways to do it: the former (known as the holomorphic gauge) yields to the results of Fröhlich and King [29] and of Kontsevich [39]; the latter (known as the covariant gauge) was considered by Guadagnini, Martellini and Mintchev [34] and by Bar–Natan[6]. It is the latter approach that we will consider in the following as, at least at the moment, it is the easier to generalize to higher dimensions.

The perturbative invariants (in the covariant gauge) coming from (1.0.1) at a given knot γ take the form of integrals of products of pull-backs of the propagator of the Chern–Simons theory w.r.t. γ on the open configuration space $C^0_{s,t}\left(\mathbb{R}^3, S^1; \gamma\right)$ of s ordered distinct points in S^1 and t distinct points in $\mathbb{R}^3$, such that no image of a point in S^1 via the knot γ equals a point in $\mathbb{R}^3$. The propagator in the covariant-gauge, which is the distributional kernel of the operator belonging to the quadratic part of the gauge-fixed Chern–Simons, is a form on the cartesian product $\mathbb{R}^3 \times \mathbb{R}^3$, singular on the diagonal. The first technical problem, when considering perturbative invariants, is therefore to show that such integrals do indeed converge, when some arguments approach diagonals or escape to infinity. The second problem arises in showing that they are really isotopy invariant.

Bott and Taubes (see [12]) presented the convenient mathematical setting for perturbative knot invariants coming from Chern–Simons theory (in the covariant gauge). From now on, we will denote by $\mathrm{Imb}(S^1, \mathbb{R}^3)$ the space of knots in $\mathbb{R}^3$. First of all, Bott and Taubes introduce the tautological forms θ_{ij}, for $1 \le i < j \le n$, $n \ge 2$: they are pull-backs of the $SO(3)$-invariant, normalized top-form v on S^2 w.r.t. the maps φ_{ij} from $C^0_n\left(\mathbb{R}^3\right)$, defined by

$$\varphi_{ij}\left(x_1, \ldots, x_n\right) : = \frac{x_i - x_j}{\|x_i - x_j\|}.$$

The approach of Bott and Taubes solves the problem of convergence by noting that the tautological forms θ_{ij} on the open configuration space $C^0_n\left(\mathbb{R}^3\right)$ of $n \ge 2$ points in $\mathbb{R}^3$ extend smoothly in a natural way to the Fulton–MacPherson–Axelrod–Singer (shortly, FMcPAS) compactification $C_n\left(\mathbb{R}^3\right)$ of $C^0_n\left(\mathbb{R}^3\right)$ and $C_q\left(S^1\right)$ of $C^0_q\left(S^1\right)$ (the space of q strictly ordered points in S^1). Moreover, the spaces $C^0_{s,t}\left(\mathbb{R}^3, S^1; \gamma\right)$ can also be compactified à la FMcPAS, and they piece together to give a fibration on the space of knots with typical fiber the compactification $C_{s,t}\left(\mathbb{R}^3, S^1; \gamma\right)$, for γ given. All such compactifications yield manifolds with corners, a generalization of the notion of manifold with boundary. The evaluation map

$$\mathrm{ev}\,(\gamma;t):\ =\gamma(t),\quad \gamma\in\mathrm{Imb}(S^1,\mathbb{R}^3), t\in S^1,$$

extends to smooth maps from $C_{s,t}\left(\mathbb{R}^3,S^1\right)$ to $C_{s+t}\left(\mathbb{R}^3\right)$. If we take pull-backs of tautological forms on $C_{s+t}\left(\mathbb{R}^3\right)$ w.r.t. evaluation maps and then take the push-forward w.r.t. the projection $\pi_{s,t}$ from $C_{s,t}\left(\mathbb{R}^3,S^1\right)$ onto $\mathrm{Imb}(S^1,\mathbb{R}^3)$, such that the sum of the degrees of the product of such forms equals the dimension of $C_{s,t}\left(\mathbb{R}^3,S^1;\gamma\right)$, we get a function on $\mathrm{Imb}(S^1,\mathbb{R}^3)$. Such a function is clearly well-defined, since the fiber is compact. Moreover, the generalized Stokes Theorem for the push-forward w.r.t. the fibration $C_{s,t}\left(\mathbb{R}^3,S^1\right)\stackrel{\pi_{s,t}}{\to}\mathrm{Imb}(S^1,\mathbb{R}^3)$ gives

$$\mathrm{d}\left(\pi_{s,t*}\omega\right)=(-1)^{s+3t}\pi_{s,t*}\left(\mathrm{d}\omega\right)-(-1)^{s+3t}\pi_{s,t,\partial*}\left(\iota^*_{s,t,\partial}\omega\right),$$

where $\pi_{s,t,\partial}$, resp. $\iota_{s,t,\partial}$, denotes the projection from the codimension-1 boundary of $\partial C_{s,t}\left(\mathbb{R}^3,S^1\right)$ to $\mathrm{Imb}(S^1,\mathbb{R}^3)$, resp. the injection from the boundary of $\partial C_{s,t}\left(\mathbb{R}^3,S^1\right)$ to $\mathrm{Imb}(S^1,\mathbb{R}^3)$; ω is a form on $C_{s,t}\left(\mathbb{R}^3,S^1\right)$. If ω is a product of tautological forms on $C_{s,t}\left(\mathbb{R}^3,S^1\right)$, $\mathrm{d}\omega=0$; hence, the only contributions to the exterior derivative of $(\pi_{s,t*}\omega)$ come from the codimension-1 boundary faces; if one can show that all boundary faces of a product of tautological forms vanishes, the corresponding function is automatically a knot invariant. This is what Bott and Taubes did explicitly in [9]. In this approach, knot-invariants are regarded as elements of the 0-th de Rham cohomology of $\mathrm{Imb}(S^1,\mathbb{R}^3)$.

It is important to notice that the perturbative invariants coming from Chern–Simons theory in the covariant gauge have been proved to be *Vassiliev knot invariants*, see [7] and [1]. For the sake of completeness, let us say that Vassiliev knot invariants of order s are knot invariants which, once extended canonically to the space of immersions with a finite number of transversal double points, vanish on all immersions with more than s double points; for more details on Vassiliev invariants we refer to [50].

Motivated by these results, Cattaneo, Cotta-Ramusino and Longoni contructed by the same principles cohomology classes of $\mathrm{Imb}(S^1,\mathbb{R}^m)$, for $m>3$ (see [17]). On the other hand, Bott constructed an explicit invariant of odd spheres of codimension 2 into Euclidean spaces, using again tautological forms (actually of two different kinds) and compactified configuration spaces (see [9]). (Smooth imbeddings of spheres of codimension 2 into Euclidean spaces are the natural generalization of knots in $\mathbb{R}^3$.)

Due to the perturbative origin of the construction of Bott and Taubes, it was natural to wonder if the cohomology classes of Cattaneo, Cotta-Ramusino and Longoni and the Bott invariant have a corresponding TQFT-origin. Notice, moreover, that a TQFT origin of the Bott invariant should as well produce other possible invariants of higher-dimensional knots: namely, if the Bott invariant were really a perturbative invariant coming from the perturbative expansion of a given observable for some TQFT, explicit computations of terms of higher order of the perturbative expansion of such an observable would give us a whole series of possible invariants of higher-dimensional knots.

This is the motivation for the present work, which we now describe.

Plan of the work

The present work can be divided into two main parts. The first, which comprises Chapter 3, 4 and 5, is perhaps of more interest to people already aquainted to physics as it describes the main physical framework we use to achieve the results contained in the second part. The mathematical results of second part are explicit formulae for an invariant of imbedded spheres of codimension 2 in odd-dimensional Euclidean spaces and for an invariant of imbedded 2-spheres in 4-dimensional Euclidean space; moreover, we discuss also a quasi-invariant of imbedded spheres of codimension 2 in even-dimensional Euclidean spaces. The word "quasi" means here that we find functions on the space $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of long knots in $\mathbb{R}^m$ (which we will define in the next subsection), such that their exterior derivative vanishes up to a term which is a linear combination of a finite-dimensional basis of particular forms; if we could find a sufficiently great number of such functions, we could take convenient linear combinations of these in order to get a true invariant.

We notice that the second part can be read separately from the first part; the only requirements are contained in Section 2.4 and 2.6 of Chapter 2.

Topological quantum field theories…

We describe briefly the contents and results of Chapter 3. BF theories are the natural generalizations in any dimension of the 3-dimensional Chern–Simons theory; in fact, Chern–Simons theory depends only on a given connection on a principal bundle P over a given 3-manifold M. The action functional of BF theory in dimension m is instead defined as follows: we take a compact, closed, oriented m-dimensional manifold M and a principal bundle P over it; the BF action takes the form

$$S_{BF}(A,B)\colon = \int_M \langle\, B\,,\, F_A \,\rangle\,. \tag{1.0.2}$$

Here, A is a connection on P and F_A is its curvature, while B is an $(m-2)$-form on M with values in the associated bundle $\mathrm{ad}\,P = P\times_G \mathfrak{g}$, where G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation; $\langle\ ,\ \rangle$ represents the extension on forms on M with values in $\mathrm{ad}\,P$ of a symmetric, invariant, nondegenerate bilinear form on $\mathfrak{g}$. In 3 dimensions, the BF action plus the additional "cosmological" term $\frac{1}{6}\int_M \langle\, B\,,\,[\,B\,,\,B\,]\,\rangle$ can be written as the difference of two Chern–Simons action functionals, one computed at the connection $A+B$, the other at the connection $A-B$; the BF action without cosmological term is the derivative w.r.t. t at $t=0$ of the Chern–Simons action computed at the connection $A+tB$. The advantage of BF theory is that it is obviously well-defined in any dimension and that it contains a quadratic part, from which we can always start a perturbative expansion.

On the other hand, functional integral quantization requires some care. In fact, BF theories admit a symmetry group: it is easy to see that the BF action, irrespective of the dimension, is invariant w.r.t. the following transformations of A and B:

$$A\mapsto A^g,\quad B\mapsto B^g+\mathrm{d}_{A^g}\tau_1,$$

where g is an element of the gauge group $\mathcal{G}$ of P, and τ_1 is a form of degree $m-3$ on M with values in $\operatorname{ad} P$. It is also not difficult to see that these symmetries are reducible, if $m > 3$: namely, even for the simplest case $m = 4$, if A is flat, it is possible to find a unique representative for the class $B + \mathrm{d}_A\tau_1$ by imposing conditions on B (the so-called gauge-fixing), but the addition to τ_1 of a d_A-exact form does not modify the chosen representative.

The presence of these symmetries makes the quadratic part of the action degenerate; therefore, in order to perform functional integral quantization of BF theories, we have to fix the gauge. In 3 dimensions, it is possible to resort to the BRST procedure, as it is the case for Chern–Simons theory. Alas, the BRST procedure fails in higher dimensions, as the BRST operator does not square to 0, and the BRST procedure requires a well-defined cohomology theory to produce meaningful observables. The correct way to deal with functional integral quantization of BF theories in arbitrary dimensions is to resort to the Batalin–Vilkovisky (shortly, BV) formalism. A brief introduction to functional integrals, BRST procedure and BV formalism can be found in Section 2.8 of Chapter 2. In Section 3.2 of Chapter 3, we discuss in detail the BV formalism for BF theories in arbitrary dimensions. In order to simplify the computations, we introduce in Section 3.3 a suitable “super BV formalism”; with the help of this superformalism, we find a solution of the Quantum Master Equation for BF theories in arbitrary dimensions. We then proceed to discuss the generalization of the covariant gauge-fixing for BF theories in arbitrary dimensions in Subsection 3.4.4, and in Subsection 3.4.5 we discuss the superpropagator of BF theories in the covariant gauge-fixing along the lines of [10]. In the last Section of Chapter 3, we sketch how the BV action for BF theories in any dimension can be obtained using the prescriptions of the Alexandrov–Kontsevich–Schwarz–Zaboronsky (shortly, AKSZ) formalism, [2], which we briefly recall.

Chapter 4 contains the details about the observables for BF theories in any dimension, whose v.e.v.s yield cohomology classes on $\mathrm{Imb}(S^1, \mathbb{R}^m)$. In Section 4.1, we introduce functionals depending on the fields A and B of BF theory related to knots in $\mathbb{R}^m$ generalizing the Wilson loop in 3 dimensions; we used the formal expansion in powers of the parameter κ around 0 of the Wilson loop $\mathrm{Tr}_\rho \left[\mathrm{H}(A + \kappa B)|_0^1(\gamma)\right]$, for some knot γ, which involves in turn the notion of iterated integrals, see also [23]. We notice that the formal expansion in powers of κ of the Wilson loop can be generalized in any dimension for B of arbitrary degree $d \geq 3$, although we are interested here in the particular case $d = m - 2$, where m is the dimension of M. Further, we show that, modulo equations of motion of BF theories, such functionals are indeed observables; we make use of computations performed in Section 2.1.2 and 2.1.3 regarding the holonomy of a loop. Such observables will be called from now on “generalized Wilson loops”. In Section 4.2 and 4.3, we discuss in the framework of the super BV formalism introduced in Chapter 3 the generalized Wilson loop in any odd dimension, and we show that the super BV version of the generalized Wilson loop is a BV observable and hence its v.e.v. yields cohomology classes on $\mathrm{Imb}(S^1, \mathbb{R}^{2m+1})$; we notice that the definition of these generalized Wilson loops requires the addition to the BV action for BF theories of a super analogue of the cosmological term for BF theory in 3 dimensions, if we are interested in cohomology classes of $\mathrm{Imb}(S^1, \mathbb{R}^{2m+1})$ coming from trivalent diagrams (see [17]). We also discuss generalized Wilson loops in odd dimensions with more than trivalent interaction terms; the presence of such interaction terms requires some special assumptions on the Lie group G. In Section 4.4, we consider generalized Wilson loops in the even-dimensional case: more care is required as even at the classical level the proof that the generalized Wilson loop in even dimensions (modulo equations of motion for BF theories) is gauge-invariant and represents closed forms on $\mathrm{Imb}(S^1, \mathbb{R}^{2m})$ presents some problems, when considering the terms of even order in the parameter κ. In fact, we consider the super BV versions only of terms of odd order coming from the expansion of the Wilson loop at $A + \kappa B$, and we show that the sum of these terms is indeed a BV observable, yielding also cohomology classes of $\mathrm{Imb}(S^1, \mathbb{R}^{2m})$. The definition of the super BV version of the generalized Wilson loop in even dimensions requires the presence of trivalent or more than trivalent interaction terms, whose definition restricts the choice of the Lie group G. The results of Chapter 3 and 4 are the contents of [20] and [21].

Chapter 5 describes the explicit search for a TQFT interpretation of the Bott invariant in the framework of BF theories; preliminary approaches to invariants of higher-dimensional knots from the TQFT viewpoint may be found e.g. in [25]. As we have already seen, a *higher-dimensional knot in Euclidean space* $\mathbb{R}^m$ is an imbedding of sphere of codimension 2 into $\mathbb{R}^m$; analogously, a higher-dimensional knot in S^m is an imbedding of S^{m-2} into S^m. For computational reasons, at some point of Chapter 5 it is better to resort to the special subset $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of the space of imbeddings of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$, namely those imbeddings of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$ mapping infinity of $\mathbb{R}^{m-2}$ to infinity of $\mathbb{R}^m$ and becoming a specified linear imbedding σ of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$ outside a compact subset of $\mathbb{R}^{m-2}$. Alternatively, we can view S^{m-2}, resp. S^m, as the one-point-compactification of $\mathbb{R}^{m-2}$, resp. $\mathbb{R}^m$, e.g. via the north-pole of both spheres, which we choose to denote in both cases by ∞; we consider then imbeddings of S^{m-2} into S^m which are base-point-preserving in the sense that they send ∞ in S^{m-2} to ∞ in S^m; the tangent map at any point in S^{m-2} is injective, whence it follows that $\mathrm{T}_\infty f$ is an injective map (i.e. a linear imbedding) from $\mathrm{T}_\infty S^{m-2} \cong \mathbb{R}^{m-2}$ to $\mathrm{T}_\infty S^m \cong \mathbb{R}^m$. The action of $\mathrm{Diff}_0(S^m)$ on the space of knots in S^m sees to it that it is always possible to deform a given knot in S^m into a *long knot* of $\mathbb{R}^{m-2}$ and $\mathbb{R}^m$, i.e. an element of $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$; viceversa, the stereographic projections from S^{m-2}, resp. S^m, onto $\mathbb{R}^{m-2}$, resp. $\mathbb{R}^m$, and the prescribed behavior of a long knot in a neighbourhood of infinity enable one to recover from it a base-point preserving knot in S^m, whose tangent map at ∞ takes a prescribed form governing the behavior of the knot in a small neighbourhood of ∞. Hence, in every connected component of the space of higher-dimensional knots in $\mathbb{R}^m$ there is a long knot. Moreover, a higher-dimensional knot in $\mathbb{R}^m$ specifies a unique higher-dimensional knot in S^m (with a choice of a point at infinity in S^m).

In Subsection 5.1.1, we construct an action functional $S_{\mathcal{I}}$, which we call the "$\mathcal{I}$ action" ($\mathcal{I}$ for "imbedding"), depending on a given imbedding f of S^{m-2}, on two fields α and β on S^{m-2} or $\mathbb{R}^{m-2}$ and also on the classical fields A and B of BF theory related to imbeddings of S^{m-2} into $\mathbb{R}^m$ or long knots (with some slight modifications)

$$S_{\mathcal{I}}\left(A, B; \alpha, \beta; f\right) :\, = \int_{S^{m-2}} \langle\, \alpha \,,\, \mathrm{d}_{f^*A}\beta + f^*B \,\rangle\,,$$

where f is a higher-dimensional knot in $\mathbb{R}^m$, α, resp. β, is a 0-form on S^{m-2} with values in $\mathrm{ad}\, f^*P$, resp. an $(m-3)$-form on S^{m-2} with values in $\mathrm{ad}\, f^*P\colon = f^*P \times_G \mathfrak{g}^*$; f^*A denotes the pull-back connection on f^*P; if we consider the $\mathcal{I}$ action for long knots, we replace S^{m-2} by $\mathbb{R}^{m-2}$ and the bundle f^*P by the trivial bundle $\mathbb{R}^{m-2} \times G$. Accordingly, α is a 0-form on $\mathbb{R}^{m-2}$ with values in $\mathfrak{g}$, and β is an $(m-3)$-form on $\mathbb{R}^{m-2}$ with values in $\mathfrak{g}^*$. Finally, in both cases, $\langle\ ,\ \rangle$ denotes the canonical duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ extended to forms on $\S^{m-2}$ with values in $\mathrm{ad}\, f^*P$ or $\mathrm{ad}^*\, f^*P$, or $\mathfrak{g}$ and $\mathfrak{g}^*$. We consider the functional $\mathcal{T}_\mathcal{I}$ for BF theories related to imbeddings of S^{m-2} or long knots in $\mathbb{R}^m$ given by the partition function of the $\mathcal{I}$ action w.r.t. the fields α and β.

We discuss then in Subsection 5.1.2 and 5.1.3 the formal properties of the partition function $\mathcal{T}_\mathcal{I}$, first its gauge-invariance and then the isotopy-invariance of the v.e.v. of $\mathcal{T}_\mathcal{I}$. The main arguments in the proof of isotopy-invariance of the v.e.v. of $\mathcal{T}_\mathcal{I}$ w.r.t. BF theories is linked directly to the diffeomorphism-invariance of BF theories. Until now, all formal computations are available not only for imbeddings of S^{m-2} into $\mathbb{R}^m$, but also for elements of $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2}, \mathbb{R}^m\big)$ with some slight modifications.

In Section 5.2, we discuss how to compute explicitly the observable $\mathcal{T}_\mathcal{I}$ as a perturbative series, namely we discuss the possible symmetries of the $\mathcal{I}$ action, and we see that, if A and B satisfy the equations of motion of the BF action, a reducibility problem similar to that for BF theories arises; we have also to deal with possible 0-modes of the $\mathcal{I}$ action. We also discuss briefly the 3-dimensional case. Perturbative expansion of $\mathcal{T}_\mathcal{I}$ requires also the BV formalism, which we discuss in detail in Section 5.4; we produce a BV action for the $\mathcal{I}$ action for A and B solutions of the equations of motion of the BF action. Later, we introduce a covariant gauge-fixing for the $\mathcal{I}$ action and compute the corresponding superpropagator; we perform explicitly the perturbative expansion of $\mathcal{T}_\mathcal{I}$, when A and B are solutions of the equations of motion. We define in Section 5.5 the extension to the super BV formalism for BF theories of the functional coming from the perturbative expansion of $\mathcal{T}_\mathcal{I}$, and we prove explicitly that it is indeed a BV observable, whose v.e.v. yields possible invariants of higher-dimensional knots. Finally, we notice that it is possible to construct such a BV observable *before* performing perturbative expansion when A and B are solutions of the equations of motion of the BF action by introducing a product BV structure for the $BF + \mathcal{I}$ action, reminiscent of the product Poisson structure for the product of two common Poisson manifolds; we refer to Subsection 5.4.5.

... and invariants of higher-dimensional knots

The second part of this work contains more mathematics than the first part, and we want to point out that the mathematical results of Section 6.3 and 6.4 can be understood also without the knowledge of the physical arguments sketched in Chapter 3, 4 and 5.

In Section 6.1 we discuss explicitly the perturbative expansion of the v.e.v. of the BV observable coming from the perturbative expansion of the partition function of the $\mathcal{I}$ action introduced in the preceding chapter: we give the superpropagator for BF theories coming from the covariant gauge-fixing (which was already discussed in subsection 3.4.5 of Chapter 3) and the relevant Feynman rules. We also discuss how to remove the possible divergences coming from the perturbative expansion.

A fundamental ingredient in the discussion of Section 6.3, 6.4 and 6.5 is the FMcPAS compactification of configuration spaces: the definitions and the main properties of FMcPAS construction are discussed in detail in Section 2.4.

In Section 6.3, we compute explicitly the Feynman diagrams of order 2 of the perturbative expansion of the v.e.v. of the partition function of the $\mathcal{I}$ action in the covariant gauge-fixing; it can be written as a sum of integrals over configuration spaces:

$$\begin{aligned}\Theta_2 &= \frac{1}{8}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{1234}^2 - \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{123} = \\ &= \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23} + \frac{1}{2}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} - \int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12}.\end{aligned} \tag{1.0.3}$$

Here, the integrals over the compactified configuration spaces $C_{4,0} \equiv C_4\left(\mathbb{R}^{m-2}\right)$ and $C_{3,1} \equiv C_{3,1}\left(\mathbb{R}^m, \mathbb{R}^{m-2}\right)$ are push-forwards w.r.t. the fibrations $\mathrm{Imb}_{\infty,\sigma}\left(\mathbb{R}^{m-2},\mathbb{R}^m\right) \times C_{4,0}$ and $C_{3,1}$ over $\mathrm{Imb}_{\infty,\sigma}\left(\mathbb{R}^{m-2},\mathbb{R}^m\right)$. The forms θ_{ij} and η_{ij} are respectively the tautological forms coming from the superpropagators of the BF action, resp. $\mathcal{I}$ action, in the covariant gauge; they are defined as (the smooth extensions to compactified configuration spaces) pull-backs of the normalized, $SO(m)$-invariant top-form of S^{m-1}, resp. the normalized, $SO(m-2)$-invariant top-form of S^{m-3} by the natural maps from $\mathrm{Imb}_{\infty,\sigma}\left(\mathbb{R}^{m-2},\mathbb{R}^m\right) \times C_{4,0}$ or $C_{3,1}$ into S^{m-1} or S^{m-3}. The notations η_{123} and η_{1234} mean respectively the cyclic sums of η-forms: $\eta_{12}+\eta_{23}+\eta_{31}$ and $\eta_{12}+\eta_{23}+\eta_{34}+\eta_{41}$.

The function (1.0.3) vanishes in even dimensions; hence, from now on, we consider m odd in the discussion of the properties of (1.0.3). Moreover, by the generalized Stokes Theorem we are able to show that (1.0.3) yields indeed a locally constant function on $\mathrm{Imb}_{\infty,\sigma}\left(\mathbb{R}^{m-2},\mathbb{R}^m\right)$, i.e. an isotopy-invariant of higher-dimensional knots in odd-dimensional Euclidean spaces. Since the integrands are all closed, the proof boils down to compute the boundary contributions to the integrals appearing in (1.0.3): the sum of the contributions coming from principal faces vanish, and in Section 6.5 we display the vanishing lemmata we use to prove the vanishing of the remaining contributions.

Moreover, (1.0.3) is written with the help of cyclic sums of the η-forms; if we replace in (1.0.3) the configuration spaces $C_4\left(\mathbb{R}^{m-2}\right)$ and $C_{3,1}\left(\mathbb{R}^m, \mathbb{R}^{m-2}\right)$ by $C_4\left(S^{m-2}\right)$ and $C_{3,1}\left(\mathbb{R}^m, S^{m-2}\right)$ and the η-forms by pull-backs of natural projections from $C_{4,0}$ and $C_{3,1}$ onto $C_2\left(S^{m-2}\right)$ of a form η of degree $m-3$ on $C_2\left(S^{m-2}\right)$ satisfying

- the derivative of η is the pull-back to $C_2\left(S^{m-2}\right)$ of the Poincaré dual of the diagonal in $S^{m-2} \times S^{m-2}$;
- the restriction of η to the boundary of $C_2\left(S^{m-2}\right)$ (which is diffeomorphic to the sphere bundle $\mathrm{S}\left(S^{m-2}\right)$ of S^{m-2}) equals minus the global angular form of $\mathrm{S}\left(S^{m-2}\right)$ (for the construction of the global angular form of a sphere bundle, see Section 2.7 in Chapter 2 and
- η is odd w.r.t. the extension to $C_2\left(S^{m-2}\right)$ of the involution exchanging the arguments in $S^{m-2} \times S^{m-2}$.

Since the Poincaré dual of the diagonal in $S^{m-2} \times S^{m-2}$ is nonzero, η-forms are in this setting not closed; however, any cyclic sum of η-forms is closed. Since (1.0.3) can be written using only cyclic sums of η-forms, we can apply also the generalized Stokes Theorem to (1.0.3), seen as a function on $\mathrm{Imb}(S^{m-2}, \mathbb{R}^m)$, to prove that it is indeed a locally constant function; the two additional requirements on η are essential in the proof.

This function is not altogether new as it was already introduced by Bott in [9] as the first invariant of odd spheres of codimension 2 in Euclidean space contructed by means of configuration space integrals.

In Section 6.4, we compute and discuss the term of order 3 coming from the perturbative expansion of the v.e.v. of the partition function of the $\mathcal{I}$ action in the covariant gauge-fixing. We shall see that it is a sum of integrals as (1.0.3):

$$\begin{aligned}\Theta_3 = \frac{1}{3}\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{34}\eta_{56} + \int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{45} - \\ - \int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{34} + \frac{1}{3}\int_{C_{6,0}} \theta_{14}\theta_{25}\theta_{36}\eta_{12}\eta_{23}\eta_{31} - \\ - \int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{23} + \int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{34} - \\ - \int_{C_{4,2}} \theta_{16}\theta_{36}\theta_{56}\theta_{25}\theta_{45}\eta_{12} + \frac{1}{3}\int_{C_{3,3}} \theta_{14}\theta_{25}\theta_{36}\theta_{45}\theta_{46}\theta_{56}.\end{aligned} \tag{1.0.4}$$

The notations are as in (1.0.3).

The first result is that (1.0.4) vanishes in odd dimensions. Moreover, as for (1.0.3), the generalized Stokes Theorem implies that the sum of the principal faces vanish and all contributions to the exterior derivative of (1.0.4) coming from hidden faces vanish except where all vertices collapse together; the details of the proof are contained in Section 6.4, 6.5 and in particular in Subsection 6.5.3. A detailed discussion of the contribution of the face where all vertices collapse together is given in Subsection 6.5.4; fundamental ingredients are the description of boundary faces of compactified configuration spaces and biinvariant forms on Stiefel manifolds (see respectively Section 2.4 and 2.6 of Chapter 2).

It is shown that in 4 dimensions this degenerate contribution vanishes, after the addition to (1.0.4) of an explicit counterterm; hence, we find a perturbative invariant for imbedded 2-spheres in 4-dimensional Euclidean space.

It is also shown that this degenerate contribution is given by (pull-backs of) a finite linear combination of a basis of biinvariant forms on Stiefel manifolds $V_{m,m-2}$ for m even and strictly bigger than 4, again after the addition of an explicit counterterm; we say in this case that the corrected function (1.0.4) is a quasi-invariant.

The perturbative expansion of the v.e.v. of the partition function of the $\mathcal{I}$ action gives possible invariants at higher orders: the vanishing lemmata of Section 6.5 imply that particular combinations of configuration space integrals (in diagrammatic form, such integrals are generalizations of the diagrams associated to (1.0.3) and (1.0.4)) yield

i) invariants of imbedded spheres of codimension 2 in odd-dimensional Euclidean spaces;

ii) invariants of imbedded 2-spheres in Euclidean 4-dimensional space;

iii) quasi-invariants, in the sense explained above of imbedded spheres of codimension 2 in even-dimensional Euclidean spaces; thus, by summing up a sufficiently great number such quasi-invariants in a convenient way, it is possible to get invariants (in analogy to what Bott and Taubes did in [12] to deal with the degenerate contribution for their invariants coming from the collapse of all vertices).

Chapter 2

Preliminaries

The aim of this chapter is to introduce the mathematical objects we need in the next chapters. Some of the concepts we are going to introduce are old hat (e.g. the theory of principal bundles and the theory of connections). But other things are (as far as we know) new, e.g. the relationship between connections and bundle isomorphisms or the biinvariant cohomology of Stiefel manifolds; therefore, we are going to skip obvious details at the beginning, deferring proofs where we introduce some new concept.

2.1 Principal bundles and connections

In this section we define the main objects we need, namely principal bundles, associated bundles, connections on principal bundles and forms with values in associated bundles. We refer e.g. to [8], [32] and [38] for more details.

2.1.1 Principal bundles

From now on, M will denote either a compact, closed, connected, oriented smooth manifold of dimension m or $\mathbb{R}^m$, the m-dimensional Euclidean space. G will denote a Lie group, and $\mathfrak{g}$ the corresponding Lie algebra; the identity if G is usually denoted by e.

We begin by defining what is a principal G-bundle $P \to M$.

Definition 2.1.1. A principal bundle $P \xrightarrow{\pi} M$ is a smooth manifold P equipped with a smooth (right) action of G and a smooth G-invariant surjective map π, such that

- G acts freely on P, i.e.

$$pg = p \Rightarrow g = e, \quad \text{for general } p \in P$$

  and transitively on each fiber $P_x := \pi^{-1}(\{x\})$.

- P is locally trivial in the following sense: any point $x \in M$ has a neighbourhood U, such that there is a smooth isomorphism φ_U from $\pi^{-1}(U)$ to $U \times G$, such that

$$\varphi_U\,(p\,g) = (\varphi_U(p))\;g, \quad \forall\, y \in U, g \in G;$$
$$\mathrm{pr}_1 \circ \varphi_U = \pi|_{\pi^{-1}(U)},$$

where pr_1 denotes the projection from $U \times G$ onto the first factor. The map φ_U is called a local trivialization of P.

A smooth section of the principal bundle P over M is a smooth map σ
We need also the definition of vector bundle.

Definition 2.1.2. A real vector bundle $E \xrightarrow{\pi} M$ of rank k is a smooth manifold E endowed with a smooth surjective map π, such that E is locally trivial over M, i.e. for any $x \in M$ there is an open neighbourhood U and a diffeomorphism ϕ_U from $\pi^{-1}(U)$ to $U \times \mathbb{R}^k$, such that

$$\varphi_U\,(v + \lambda\, w) = \varphi_U(v) + \lambda\varphi_U(w), \quad \forall\, v, w \in E, \lambda \in \mathbb{R}.;$$
$$\mathrm{pr}_1 \circ \varphi_U = \pi|_{\pi^{-1}(U)},$$

where pr_1, as in Definition 2.1.1 denotes the projection onto the first component in $U \times \mathbb{R}^k$.

A complex vector bundle is simply obtained by replacing $\mathbb{R}^k$ by $\mathbb{C}^k$, and the local trivialization φ_U has to be $\mathbb{C}$-linear.

A smooth section of a vector bundle E over M is a smooth map σ from M to E, such that $\pi \circ \sigma = \mathrm{id}$.

Remark 2.1.3. Of course, there is a more general concept, namely that of fiber bundle, where the fiber can be either a Lie group (in this case, the fiber bundle is a principal bundle) or a (real or complex) vector space (in this case, the fiber bundle is a vector bundle).

A fiber bundle $F \xrightarrow{\pi} M$ is a smooth manifold, which is also locally trivial over M in the sense of Definition 2.1.1 or 2.1.2, the only difference being that now the fiber $\pi^{-1}(\{x\})$, for $x \in M$, is not constrained to be a Lie group or a vector space.

We assume now that we have a (smooth) representation of G, which we denote by (ρ, V), where V is a real (or complex) finite-dimensional vector space and ρ is a (smooth) Lie-group-homomorphism from G to $\mathrm{Aut}(V)$, the group of invertible linear transformations of V. There is a free right-action of G on the product bundle $P \times V$:

$$\mathrm{R}_g\,(p, v) :\; = \left(p\,g, \rho\left(g^{-1}\right) v\right), \quad \forall\, p \in P, v \in V, g \in G.$$

We may then take the quotient of $P \times V$; in this way, we get a smooth vector bundle $P \times_G V$ on M with typical fiber V (which is called the bundle associated to P via ρ,or shortly the associated bundle to P via ρ), as we now briefly illustrate: the canonical projection from $P \times_G V$ onto M, which we denote again by π, is given by

$$\pi\,([p, v]) :\; = \pi(p),$$

where by $[p, v]$ we denote the equivalence class of (p, v) in $P \times V$. The G-invariance of π ensures that the projection on $P \times_G V$ is well-defined.

Without giving the explicit computations, we give the explicit formulae for local trivializations for $P \times_G V$. First of all, we notice

$$\pi^{-1}(U) = \{[p, v] \in P \times_G V : \pi(p) \in U\},$$

for U open. We choose a point $x \in M$ and we denote by U, resp. φ_U, the trivializing neighbourhood of x, the trivialization at x, of P. We define then

$$\overline{\varphi}_U([p, v]) := (\pi(p), \rho(\mathrm{pr}_2 \circ \varphi_U(p)) v),$$

where pr_2 denotes the projection onto the second factor of $U \times G$. Its inverse, $\overline{\varphi}_U^{-1}$, is simply

$$\overline{\varphi}_U^{-1}(x, v) := \left[\varphi_U^{-1}(x, e), v\right].$$

This makes $P \times_G V$ into a smooth vector bundle over M. In particular, we will be interested in the vector bundles $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ and $\mathrm{ad}^* P := P \times_G \mathfrak{g}^*$.

Remark 2.1.4. We notice that we have considered only bundles associated to linear representations of G. Of course, we may consider also more general representations F, where F denotes a smooth manifold endowed with a smooth G-action. By the same procedure as for the bundle associated to a linear representation, we get a smooth fiber bundle structure with typical fiber F.

The gauge group $\mathcal{G}$ of P is the set of all G-equivariant automorphisms of P, i.e. as the set of all diffeomorphisms σ of P, such that

$$\pi \circ \sigma = \pi, \quad \sigma \circ \mathrm{R}_g = \mathrm{R}_g \circ \sigma, \quad \forall g \in G.$$

Since G acts transitively on each fiber and since σ is fiber-preserving, there is a smooth identification between the group of gauge transformations of P (it is a group w.r.t. the composition) and the group of smooth G-equivariant maps from P to G. A map σ from P to G is said to be G-equivariant, if

$$\sigma(p\, g) = g^{-1}\, \sigma(p)\, g, \quad \forall\, p \in P, g \in G.$$

One of the main facts about associated bundles (we refer to [8]) is the following

Theorem 2.1.5. *The set of smooth sections of the associated bundle $P \times_G V$ is isomorphic to the set of smooth G-equivariant maps on P with values in V.*

Therefore, the gauge group can be identified with the set $\Gamma(M, \mathrm{Ad}\, P)$ of all sections of the bundle $\mathrm{Ad}\, P := P \times_G G$, where G becomes a G-space by conjugation. For P trivial, it can be identified with the group $\Gamma(M, G)$ of maps from M to G.

We denote by $\Omega^*(N; V)$ the space of V-valued forms on a manifold N. By $\Omega^*_{\mathrm{bas}}(P; V)$ we denote the invariant, horizontal forms on P taking values in V, i.e. forms ω on P with values in V with the additional properties:

$$\mathrm{R}_g^* \omega = \rho\left(g^{-1}\right) \omega, \quad \iota_{X_\xi} \omega = 0, \quad \forall\, \xi \in \mathfrak{g};$$

in the above formulae, the representation ρ acts only on the V-part of ω, and by X_ξ, $\xi \in \mathfrak{g}$, we denote the fundamental vertical vector field on P defined by

$$X_\xi(p) := \mathrm{T}_e \mathrm{L}_p(\xi), \quad p \in P.$$

By L_p we denote the fiber inclusion $g \mapsto p\, g$, for any $g \in G$.

We borrow again from [8] the following

Theorem 2.1.6. *The following isomorphism holds*

$$\Omega^*_{bas}(P;V) \cong \Omega^*(M, P \times_G V).$$

We need the concept of connection.

Definition 2.1.7. A connection 1-form A on P is a 1-form on P with values in $\mathfrak{g}$, such that following identities hold

$$\mathrm{R}_g^* A = \mathrm{Ad}\left(g^{-1}\right) A, \quad g \in G; \quad \iota_{X_\xi} A = \xi, \quad \forall\, \xi \in \mathfrak{g}.$$

The curvature of the connection A is the 2-form on P with values in $\mathfrak{g}$ defined by

$$F_A \colon = \mathrm{d}A + \frac{1}{2}[A\,,\,A].$$

It is not difficult to check that F_A is basic, hence defines a 2-form on M with values in $\mathrm{ad}\,P$.

Finally, given a linear representation (ρ, V) of G, we can define on the space of forms on P with values in V the following operator:

$$\mathrm{d}_A\omega \colon = \mathrm{d}\omega + \widehat{\rho}(A)(\omega),$$

where $\widehat{\rho}$ denotes the tangent map at the identity of ρ; since ρ is a Lie group homomorphism from G to $\mathrm{Aut}(V)$, its tangent map at the identity provides a Lie-algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}(V)$. Clearly, $\widehat{\rho}$ acts on the $\mathfrak{g}$-part of A. E.g. if $(\rho, V) = (\mathrm{Ad}, \mathfrak{g})$, then $\widehat{\rho} = \mathrm{ad}$. It is immediate to verify that the operator d_A, which is called the covariant derivative w.r.t. A, maps basic forms on basic forms, whence it descends on forms on M with values in $P \times_G V$.

The covariant derivative w.r.t. A is clearly $\mathbb{R}$-linear and enjoys the Leibnitz rule

$$\mathrm{d}_A\left(\alpha \wedge \omega\right) = \mathrm{d}\alpha \wedge \omega + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}_A\omega, \quad \alpha \in \Omega^*\left(M\right), \omega \in \Omega^*\left(M, P \times_G V\right),$$

where the wedge product means wedge multiplication between the form parts of both factors.

An important equality satisfied by the curvature F_A is the Bianchi identity

$$\mathrm{d}_A F_A = 0.$$

We say that a connection A on P is flat, if its curvature vanishes.

It is also not difficult to see (we refer to [8]) that the space of connections on P is an affine space, modeled on $\Omega^*\left(M, \mathrm{ad}\,P\right)$, which we usually denote by $\mathcal{A}$.

The gauge group acts on $\mathcal{A}$ on the right by taking the pull-back:

$$(A; \sigma) \mapsto \sigma^* A =\colon A^\sigma.$$

If we use the G-equivariancy and the verticality of connections, explicitly the action of a gauge transformation σ is given by

$$(A;\sigma) \mapsto A^{\sigma} = \mathrm{Ad}\left(\sigma^{-1}\right) A + \sigma^{*}\left(\omega_{\mathrm{MC}}\right),$$

where on the right-hand side of the above identity we have denoted by σ the equivariant map from P to G associated to σ, and by ω_{MC} we have denoted the Maurer–Cartan form of G, i.e.

$$\left(\omega_{\mathrm{MC}}\right)_g \left(X_g\right) := \mathrm{T}_g \mathrm{L}_g^{-1}(X_g), \quad \forall\, g \in G, X_g \in \mathrm{T}_g G.$$

$\mathcal{G}$ acts also on $\Omega^{*}\left(M, P \times_G V\right)$, since the pull-back of a basic form on P with values in V by a gauge transformation is again basic. Explicitly, using the G-equivariancy and the horizontality of a general basic form, we get

$$(\omega;\sigma) \mapsto \rho\left(\sigma^{-1}\right) \omega, \quad \omega \in \Omega_{\mathrm{basic}}\left(P, V\right).$$

2.1.2 Representations of Lie groups and bundle morphisms

Later, we are going to introduce the parallel transport, as a function on the space of loops in M, denoted by $\mathrm{L}M$, via iterated integrals; for later purposes, it is better to view the parallel transport as a function on $\mathrm{L}M$ taking values in some representation of the Lie algebra $\mathfrak{g}$ of G. In the case P trivial, it is very easy to define the parallel transport with values in a given representation. In the case P nontrivial, we have to restrict ourselves to special representations of $\mathfrak{g}$, namely those coming from representation of G: in fact, since now P is nontrivial, we have to consider forms on M with values in the nontrivial associated bundle $\mathrm{ad}\, P$, and therefore we cannot simply take the image of such a form under a given Lie algebra representation. The correct method of generalizing this to the case P nontrivial is briefly sketched.

We assume (V,ρ) to be a linear representation of G, i.e. a vector space V and a homomorphism ρ from G to the group of linear automorphisms of V.

The representation ρ induces in turn a representation of G on $\mathrm{End}(V)$, which we denote by $\overline{\rho}$, by the rule:

$$\overline{\rho}\,(g)\,(\varphi) := \rho(g) \circ \varphi \circ \rho\,(g)^{-1}, \quad \forall \varphi \in \mathrm{End}(V), g \in G.$$

It is clear that $\overline{\rho}$ is a homomorphism, and that $\overline{\rho}$ restricts in an obvious way to $\mathrm{Aut}(V)$.

We consider the Lie-algebra homomorphism $\widehat{\rho}$ defined by

$$\begin{aligned} \widehat{\rho}\colon \mathfrak{g} &\longrightarrow \mathrm{End}(V) \\ X &\longmapsto \mathrm{T}_e \rho\,(X)\,. \end{aligned} \tag{2.1.1}$$

Lemma 2.1.8. *The following identity holds:*

$$\widehat{\rho} \circ \mathrm{Ad}(g) = \overline{\rho}(g) \circ \widehat{\rho}, \quad \forall g \in G. \tag{2.1.2}$$

Proof. By definition of the Adjoint action of G on its Lie algebra $\mathfrak{g}$, it holds:

$$\begin{aligned} \widehat{\rho} \circ \mathrm{Ad}(g) &= \mathrm{T}_e \rho \circ \mathrm{T}_e \mathrm{c}(g) = \mathrm{T}_e(\rho \circ \mathrm{c}(g)) = \\ &= \mathrm{T}_e(\overline{\rho}(g) \circ \rho) = \mathrm{T}_{\mathrm{id}} \overline{\rho}(g) \circ \mathrm{T}_e \rho = \\ &= \overline{\rho}(g) \circ \widehat{\rho}. \end{aligned}$$

Since ρ is a homomorphism of groups, it follows

$$(\rho \circ \mathrm{c}(g))\,(h) = \rho\,(\mathrm{c}(g)h) = \rho(g) \circ \rho(h) \circ \rho(g)^{-1} = (\overline{\rho}(g) \circ \rho)\,(h), \quad \forall g, h \in G.$$

Moreover, since $\overline{\rho}(g)$ is linear, its tangent map at the identity equals itself. $\square$

We construct, by means of $\widehat{\rho}$, a bundle morphism from $\mathrm{ad}\,P$ to the associated bundle $\mathrm{End}_P(V)\colon = P \times_G \mathrm{End}(V)$, where the right action of G on $\mathrm{End}(V)$ is given by

$$\mathrm{L}_g(\varphi)\colon = \overline{\rho}(g)(\varphi), \quad \forall g \in G, \varphi \in \mathrm{End}(V).$$

The map from $\mathrm{ad}\,P$ to $\mathrm{End}_P(V)$, which we denote by Φ_ρ, is defined as follows:

$$\Phi_\rho\,([(p, X)])\colon = [(p, \widehat{\rho}(X))]\,,$$

where by $[(p, X)]$ we have denoted an equivalence class in $\mathrm{ad}\,P$, $p \in P$ and $X \in \mathfrak{g}$. We have to show that the map Φ_ρ is well-defined, in the following sense: let $\left(\widetilde{p}, \widetilde{X}\right)$ be another representative of the class $[(p, X)]$. This is equivalent to the existence of $g \in G$, such that

$$\widetilde{p} = pg, \quad \widetilde{X} = \mathrm{Ad}\left(g^{-1}\right) X.$$

It follows

$$\begin{aligned}
\Phi_\rho\left(\left[\left(\widetilde{p}, \widetilde{X}\right)\right]\right) &= \left[\left(\widetilde{p}, \widehat{\rho}\left(\widetilde{X}\right)\right)\right] = \left[\left(pg, \widehat{\rho}\left(\mathrm{Ad}\left(g^{-1}\right) X\right)\right)\right] = \\
&= \left[\left(pg, \overline{\rho}\left(g^{-1}\right) \widehat{\rho}(X)\right)\right] = [(p, \widehat{\rho}(X))] = \\
&= \Phi_\rho\,([(p, X)])\,,
\end{aligned}$$

and the last identity follows from the definition of $\mathrm{End}_P(V)$. Hence, the map Φ_ρ is well-defined. Moreover, it is not difficult to see that it preserves the fibers.

We have shown

Theorem 2.1.9. *We assume we are given a G-principal bundle P over the smooth manifold M.*

Any linear representation (V, ρ) of G induces, via its tangent map at the identity e, denoted by $\widehat{\rho}$, a bundle morphism Φ_ρ from the adjoint bundle $\mathrm{ad}\,P$ to the associated bundle $\mathrm{End}_P(V)$:

$$\begin{aligned}
\Phi_\rho\colon P \times_{\mathrm{Ad}} \mathfrak{g} &\longrightarrow \mathrm{End}_P(V)\colon = P \times_{\overline{\rho}} \mathrm{End}(V) \\
[(p, X)] &\longmapsto [(p, \widehat{\rho}(X))]\,.
\end{aligned} \tag{2.1.3}$$

Given a principal bundle P, we already know that there is a canonical isomorphism between basic forms on P with values in some representation $(\rho; V)$ of $\mathfrak{g}$ and forms on M with values in the associated bundle $P \times_\rho V$. We want to show that the map (2.1.1) induces a morphism from basic forms on P with values in $\mathfrak{g}$ and basic forms on P with values in (ρ, V); hence, there is also a morphism between forms on M with values in $\mathrm{ad}\,P$ and in the associated bundle $\mathrm{End}_P(V)$.

We assume we are given a form ω on P of degree q with values in $\mathfrak{g}$. We then associate to it the form $\overline{\omega}$ with values in V by the rule

$$\overline{\omega}_p\left(X_1, \cdots, X_q\right):=\widehat{\rho}\left[\omega_p\left(X_1 \cdots, X_q\right)\right], \quad \forall p \in P, X_i \in \mathrm{T}_p P, 1 \leq i \leq q.$$

Clearly, $\overline{\omega}$ is a form of degree q with values in $\operatorname{End}(V)$.

We claim that, if ω is basic, so is $\overline{\omega}$. We begin by showing that $\overline{\omega}$ is G-equivariant w.r.t. the representation $\overline{\rho}$:

$$\begin{aligned}
\mathrm{R}_g^*(\overline{\omega}) &= \widehat{\rho}\left(\mathrm{R}_g^* \omega\right)= \\
&= \widehat{\rho}\left[\operatorname{Ad}\left(g^{-1}\right) \omega\right]= \\
&= \overline{\rho}\left(g^{-1}\right) \widehat{\rho}(\omega)= \\
&= \overline{\rho}\left(g^{-1}\right) \overline{\omega};
\end{aligned}$$

the third equality is a consequence of ω being G-equivariant. Next, we take $\xi \in \mathfrak{g}$. The contraction of $\overline{\omega}$ by the vertical vector field X_ξ generated by ξ equals

$$\iota_{X_\xi} \overline{\omega}=\widehat{\rho}\left(\iota_{X_\xi} \omega\right)=0,$$

since ω is horizontal.

We assume we have a connection on P at hand, represented by the 1-form A with values in $\mathfrak{g}$. The connection A specifies a covariant derivative on forms on P with values in $\operatorname{End}(V)$ by the rule

$$\mathrm{d}_A \omega:=\mathrm{d} \omega+[\,\widehat{\rho}(A)\,,\, \omega\,], \quad \omega \in \Omega^*(P, \operatorname{End}(V)), \tag{2.1.4}$$

where the Lie-bracket here denotes the usual commutator of linear endomorphisms on V. Since ρ is a group-morphism, $\widetilde{\rho}$ is a Lie-algebra morphism, and hence

$$\overline{\mathrm{d}_A \omega}=\mathrm{d}_A \overline{\omega}, \quad \forall \omega \in \Omega^*(P, \mathfrak{g}).$$

Since $\operatorname{End}(V)$ is an associative algebra, we may multiply forms on P with values in $\operatorname{End}(V)$ in an associative way by taking the usual wedge product of the form part and multiplication in $\operatorname{End}(V)$ of the $\operatorname{End}(V)$-part; this defines an associative product on $\Omega^*(P, \operatorname{End}(V))$, which we denote again by $\wedge$. Moreover, by the Leibnitz rule of the Lie-bracket in $\operatorname{End}(V)$, it follows that the operator specified by (2.1.4) satisfies a graded Leibnitz rule w.r.t. the product on $\Omega^*(P, \operatorname{End}(V))$:

$$\mathrm{d}_A(\alpha \wedge \beta)=\mathrm{d}_A \alpha \wedge \beta+(-1)^{\operatorname{deg} \alpha} \alpha \wedge \mathrm{d}_A \beta, \quad \forall \alpha, \beta \in \Omega^*(P, \operatorname{End}(V)).$$

Since the operator d_A maps clearly basic forms on basic forms, it descends to an operator on $\Omega^*\left(M, \operatorname{End}_P(V)\right)$. The product of two basic forms on P with values in $\operatorname{End}(V)$ is immediately seen to be also basic, whence it follows that it descends to an associative product on forms on M with values in $\operatorname{End}_P(V)$.

In summary, we have seen that a representation (ρ, V) of the group G gives rise to an algebra bundle over M associated to P and that it induces a morphism of graded Lie-algebras between $\Omega^*(M, \operatorname{ad} P)$ and $\Omega^*\left(M, \operatorname{End}_P(V)\right)$; moreover, a connection A on P specifies a covariant derivative on both graded Lie-algebras in a compatible way through the representation ρ. Additionally, the covariant derivative on $\Omega^*\left(M, \operatorname{End}_P(V)\right)$ satisfies the graded Leibnitz rule w.r.t. the associative product on $\Omega^*\left(M, \operatorname{End}_P(V)\right)$, which is clearly a graded associative algebra.

2.1.3 Horizontality, holonomy and parallel transport

In this Subsection we introduce notions and prove results that we will use in Section 2.3 of Chapter 2 and later in Chapter 4; we notice that the results proven here hold also for nontrivial principal bundles, although the results of Section 2.3 and of Chapter 4 are proven under the simplifying assumption that the principal bundles we are considering are trivial. The geometry of loop spaces and of principal bundles thereover with the help of the notion of holonomy and parallel transport were also studied in [19].

We assume A to be a general connection 1-form on P. We want to give another characterization of connections.

A tangent vector X to P at p is said to be vertical, if it satisfies $\mathrm{T}_p\pi(X) = 0$. It is not difficult to see that the vertical space V_pP, for any $p \in P$, is isomorphic to $\mathfrak{g}$ via

$$\xi \mapsto \mathrm{T}_e\mathrm{L}_p(\xi), \quad \xi \in \mathfrak{g}.$$

It turns out that the vertical spaces at all points $p \in P$ can be glued together to give a smooth vector bundle, the vertical bundle $\mathrm{V}P$. There is no canonical way to define of a complement of $\mathrm{V}P$ w.r.t. the Whitney sum, i.e. there is no canonical complementary bundle $\mathrm{H}P$ such that $\mathrm{V}P \oplus \mathrm{H}P = \mathrm{T}P$. Namely, the choice of a complementary bundle depends on the choice of a connection.

Definition 2.1.10. Given a connection 1-form A on P, a tangent vector X_p to P at p is said to be A-horizontal, if the following equation holds:

$$A_p\left(X_p\right) = 0.$$

In fact, a connection on P can be equivalently characterized as a smooth assignment to any $p \in P$ of a subspace $\mathrm{H}_pP \subset \mathrm{T}_pP$ such that

- $\mathrm{T}_pP = \mathrm{V}_pP \oplus \mathrm{H}_pP$;
- $\mathrm{H}_{pg}P = \mathrm{T}_p\mathrm{L}_g\left(\mathrm{H}_pP\right)$, for any $g \in G$.

Given a connection in this sense, we can define a corresponding connection 1-form by the assignment

$$A_p\left(X_p\right) := \xi_{X_p},$$

where ξ_{X_p} is the unique element of $\mathfrak{g}$, corresponding to the vertical part of X_p w.r.t. the splitting induced by the connection. Obviously, the A-horizontal space at $p \in P$ is exactly the kernel of A_p. It is not difficult to see that the 1-form A is a connection 1-form.

On the other hand, given a connection 1-form A on P, the corresponding splitting of $\mathrm{T}P$ is given by

$$X_p = \left(X_p - X_p^A\right) + X_p^A, \quad X_p^A := \mathrm{T}_e\mathrm{L}_p\left(A_p(X_p)\right), \quad p \in P.$$

The G-equivariancy of A ensures that the corresponding distribution is G-equivariant.

We take a curve γ on M; by the word "curve", we mean a piecewise smooth map from the unit interval I to M. An A-horizontal lift of γ based at $p \in P$ is a smooth curve on P lying over γ, with initial point p and such that all its tangent directions are A-horizontal.

We quote from [38] the following Theorem.

Theorem 2.1.11. *Given a connection 1-form A on P and a curve γ in M, there is a unique A-horizontal lift of γ, which we denote by $\widetilde{\gamma}_{A,p}$. The horizontal lift $\widetilde{\gamma}_{A,p}$ is (piecewise) smooth, if γ is (piecewise) smooth.*

We take a loop γ, i.e. a closed curve in M with $\gamma(0) = \gamma(1)$. We choose a point $p \in P$ lying over $\gamma(0)$. We can then find a unique A-horizontal lift $\widetilde{\gamma}_{A,p}$ of γ based at p. Since $\widetilde{\gamma}_{A,p}$ is a lift of γ, $\widetilde{\gamma}_{A,p}(1)$ also lies over $\gamma(0)$.

Definition 2.1.12. The holonomy of γ w.r.t. the connection A and a base point p over $\gamma(0)$ is the unique element of G, which we usually denote by $\mathrm{H}(A;\gamma;p)$, satisfying

$$\widetilde{\gamma}_{A,p}(1) = p\ \mathrm{H}(A;\gamma;p)\,, \tag{2.1.5}$$

where $\widetilde{\gamma}_{A,p}$ is the unique A-horizontal lift, based at $p \in P$ over $\gamma(0)$, of the loop γ. (The fact that such an element $\mathrm{H}(A;\gamma;p)$ exists and is unique follows from the fact that G operates transitively on each fiber of P.)

The group G and the gauge group $\mathcal{G}$ operate on P, resp. $\mathcal{A}$. The holonomy depends on a given loop γ, on a connection A on P and on a base point $p \in P_{\gamma(0)}$. The next Lemma shows how the holonomy behaves w.r.t. the action of G, resp. $\mathcal{G}$, on P, resp. $\mathcal{A}$.

Lemma 2.1.13. *We assume g, resp. σ, to be an element of G, resp. a gauge transformation; we denote by g_σ the G-equivariant function from P to G canonically associated to σ. The following formulae hold:*

$$\mathrm{H}(A;\gamma;pg) = \mathrm{c}\left(g^{-1}\right)\mathrm{H}(A;\gamma;p)\,; \tag{2.1.6}$$

$$\mathrm{H}(\sigma^* A;\gamma;p) = \mathrm{c}\left(g_\sigma(p)^{-1}\right)\mathrm{H}(A;\gamma;p)\,. \tag{2.1.7}$$

Proof. We begin by showing identity (2.1.6). By the very definition of the holonomy, it holds:

$$\widetilde{\gamma}_{A,pg}(1) = pg\ \mathrm{H}(A;\gamma;pg)\,,$$

where $\widetilde{\gamma}_{A,pg}$ is the unique A-horizontal lift of γ based at pg. We consider the smooth curve $\mathrm{R}_g\left(\widetilde{\gamma}_{A,p}\right)$. This curve is based at pg, and the equivariancy of A w.r.t. the action of G implies:

$$A_{\mathrm{R}_g(\widetilde{\gamma}_{A,p}(t))}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{R}_g\left(\widetilde{\gamma}_{A,p}(t)\right)\right) = A_{\mathrm{R}_g(\widetilde{\gamma}_{A,p}(t))}\left[\mathrm{T}_{\widetilde{\gamma}_{A,p}(t)}\mathrm{R}_g\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\gamma}_{A,p}(t)\right)\right] =$$
$$= \mathrm{Ad}\left(g^{-1}\right)\left[A_{\widetilde{\gamma}_{A,p}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\gamma}_{A,p}(t)\right)\right] = 0.$$

Hence, the curve $\mathrm{R}_g\left(\widetilde{\gamma}_{A,p}\right)$ is also A-horizontal, obviously lies over γ (by the G-invariance of π) and is based at pg. By the uniqueness of A-horizontal lifts, it follows $\mathrm{R}_g\left(\widetilde{\gamma}_{A,p}\right) = \widetilde{\gamma}_{A,pg}$. Therefore, it holds

$$\widetilde{\gamma}_{A,pg}(1) = \widetilde{\gamma}_{A,p}(1)g = p\ \mathrm{H}(A;\gamma;p)\,g.$$

Identity (2.1.6) holds, since the action of G on P is free.

The second identity can be proved along the same lines. In fact, by the very definition of the holonomy, it follows:

$$\widetilde{\gamma}_{\sigma^* A,p}(1) = p \ \mathrm{H}(\sigma^* A; \gamma; p) \, .$$

We claim now

$$\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}\right) = \widetilde{\gamma}_{A,\sigma(p)}. \tag{2.1.8}$$

Both curves $\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}\right)$ and $\widetilde{\gamma}_{A,\sigma(p)}$ lie over γ since $\pi \circ \sigma = \pi$, and are clearly based at $\sigma(p)$. To prove equation (2.1.8), it remains to show that both curves are A-horizontal. The claim follows then by the uniqueness of A-horizontal lifts.

We compute

$$A_{\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}(t)\right)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}(t)\right)\right) = (\sigma^* A)_{\widetilde{\gamma}_{\sigma^* A,p}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{\gamma}_{\sigma^* A,p}(t)\right) = 0,$$

where the last identity is a consequence of the $\sigma^* A$-horizontality of $\widetilde{\gamma}_{\sigma^* A,p}$. Hence, identity (2.1.8) holds true. Therefore, we obtain

$$\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}(1)\right) = \sigma(p) \ \mathrm{H}(\sigma^* A; \gamma; p) \overset{\text{By (2.1.8)}}{=} \widetilde{\gamma}_{A,\sigma(p)}(1) = \sigma(p) \ \mathrm{H}(A; \gamma; \sigma(p)) \, .$$

The fact that the action of G is free implies

$$\begin{aligned}\mathrm{H}(\sigma^* A; \gamma; p) = \mathrm{H}(A; \gamma; \sigma(p)) = \mathrm{H}(A; \gamma; p g_\sigma(p)) &\overset{\text{By (2.1.6)}}{=} \\ = \mathrm{c}\left(g_\sigma(p)^{-1}\right) \ \mathrm{H}(A; \gamma; p) \, .\end{aligned}$$

The second claim follows. ☐

The next object we want to define is the parallel transport w.r.t. A along γ.

Definition 2.1.14. We assume γ to be a general smooth curve in M, not necessarily closed. Let t be an element of the unit interval, and let $p \in P$ and $\widetilde{p} \in P$ be such that

$$\pi(p) = \gamma(0), \quad \pi\left(\widetilde{p}\right) = \gamma(t).$$

We define the parallel transport from p to $\widetilde{p}$ along γ w.r.t. A, which we denote by $\mathrm{H}(A; \gamma; t; p, \widetilde{p})$, as the unique element of G defined by the rule

$$\widetilde{\gamma}_{A,p}(t) = \widetilde{p} \ \mathrm{H}(A; \gamma; t; p, \widetilde{p}) \, . \tag{2.1.9}$$

The definition makes sense, because $\widetilde{\gamma}_{A,p}$ is a lift of p, and thus $\widetilde{\gamma}_{A,p}(t)$ lies in the same fiber as $\widetilde{p}$.

The parallel transport defined by equation (2.1.9) satisfies two identities similar to (2.1.6) and (2.1.7).

Lemma 2.1.15. *We assume we are given a connection A, a smooth curve γ in M, $p \in P_{\gamma(0)}$ and $\widetilde{p} \in P_{\gamma(t)}$, for some t in the unit interval, general elements h and k of G and a gauge transformation $\sigma \in \mathcal{G}$.*

Then the parallel transport satisfies the two following identities:

$$\mathrm{H}(A;\gamma;t;pg,\widetilde{p}h) = h^{-1}\,\mathrm{H}(A;\gamma;t;p,\widetilde{p})\,g; \tag{2.1.10}$$
$$\mathrm{H}(\sigma^* A;\gamma;t;p,\widetilde{p}) = g_\sigma\left(\widetilde{p}\right)^{-1}\mathrm{H}(A;\gamma;t;p,\widetilde{p})\,g_\sigma(p). \tag{2.1.11}$$

Proof. We begin by showing identity (2.1.10). By definition of the the parallel transport, it holds:

$$\widetilde{\gamma}_{A,p}(t) = \widetilde{p}h\ \mathrm{H}(A;\gamma;t;p,\widetilde{p}h) = \widetilde{p}\ \mathrm{H}(A;\gamma;t;p,\widetilde{p})\,.$$

Since the action of G on P is free, we get

$$h\,\mathrm{H}(A;\gamma;t;p,\widetilde{p}h) = \mathrm{H}(A;\gamma;t;p,\widetilde{p}) \Rightarrow \mathrm{H}(A;\gamma;t;p,\widetilde{p}h) = h^{-1}\,\mathrm{H}(A;\gamma;t;p,\widetilde{p})\,,\quad \forall h\in G. \tag{2.1.12}$$

On the other hand, it holds:

$$\widetilde{\gamma}_{A,pg}(t) = \widetilde{p}\ \mathrm{H}(A;\gamma;t;pg,\widetilde{p}) = \widetilde{\gamma}_{A,p}(t)g = \widetilde{p}\ \mathrm{H}(A;\gamma;t;p,\widetilde{p})\,g.$$

Again, it follows:

$$\mathrm{H}(A;\gamma;t;pg,\widetilde{p}) = \mathrm{H}(A;\gamma;t;p,\widetilde{p})\,g,\quad \forall g\in G. \tag{2.1.13}$$

Combining (2.1.12) and (2.1.13), we get (2.1.10).

As for the second identity, we make use again of (2.1.8). We get

$$\sigma\left(\widetilde{\gamma}_{\sigma^* A,p}(t)\right) = \sigma\left(\widetilde{p}\right)\ \mathrm{H}(\sigma^* A;\gamma;t;p,\widetilde{p}) = \widetilde{\gamma}_{A,\sigma(p)}(t) = \sigma\left(\widetilde{p}\right)\mathrm{H}(A;\gamma;t;\sigma(p),\sigma\left(\widetilde{p}\right))\,.$$

The previous equation yields:

$$\begin{aligned}\mathrm{H}(\sigma^* A;\gamma;t;p,\widetilde{p}) = \mathrm{H}(A;\gamma;t;\sigma(p),\sigma\left(\widetilde{p}\right)) = \mathrm{H}\big(A;\gamma;t;pg_{\sigma(p)},\widetilde{p}g_\sigma\left(\widetilde{p}\right)\big) = \\ = g_\sigma\left(\widetilde{p}\right)^{-1}\mathrm{H}(A;\gamma;t;p,\widetilde{p})\,g_\sigma(p),\end{aligned}$$

where the last identity is a consequence of (2.1.10). Hence, the claim follows. ☐

2.1.4 Isomorphisms of principal bundles

Definition 2.1.16. We assume we are given two different principal bundles $P \xrightarrow{\pi} M$, $\widetilde{P} \xrightarrow{\widetilde{\pi}} M$ over the same manifold M and with the same Lie group G; the right action of G on P, resp. $\widetilde{P}$, will be denoted by $\mathrm{R}_\bullet$, resp. $\widetilde{\mathrm{R}}_\bullet$.

An isomorphism τ from P to $\widetilde{P}$ is a diffeomorphism from P to $\widetilde{P}$ enjoying the additional properties:

$$\widetilde{\pi}\circ\tau = \pi;\quad \tau\circ\mathrm{R}_g = \widetilde{\mathrm{R}}_g\circ\tau,\quad \forall g\in G.$$

Definition 2.1.17. We assume that we are in the same situation of the previous Definition. The fibered product of P and $\widetilde{P}$, which we denote by $P\odot\widetilde{P}$, is defined as

$$P\odot\widetilde{P}\colon = \left\{(p,\widetilde{p})\in P\times\widetilde{P}\colon \pi(p) = \widetilde{\pi}\left(\widetilde{p}\right)\right\}.$$

There is a natural map $\overline{\pi}$ from the fiber product $P \odot \widetilde{P}$ to M, which is simply

$$\overline{\pi}\,(p, \widetilde{p}) :\, = \pi(p) = \widetilde{\pi}\,(\widetilde{p})\,, \quad (p, \widetilde{p}) \in P \odot \widetilde{P}.$$

Additionally, $P \odot \widetilde{P}$ receives a right $G \times G$-action:

$$(p, \widetilde{p}; (g, h)) \mapsto (pg, \widetilde{p}h)\,, \quad \forall\, (p, \widetilde{p}) \in P \odot \widetilde{P}, (g, h) \in G \times G.$$

We claim that $P \odot \widetilde{P}$ is a principal $G \times G$-bundle over M, with projection $\overline{\pi}$.

Proof. If U is an open set of M, we denote by φ_U, resp. $\widetilde{\varphi}_U$, the trivialization of P over U, resp. of $\widetilde{P}$ over U. We define a trivialization $\overline{\varphi}_U$ of $P \odot \widetilde{P}$ over U via

$$\begin{aligned} \overline{\varphi}\colon \overline{\pi}^{-1}\,(U) &\longrightarrow U \times (G \times G) \\ (p, \widetilde{p}) &\longmapsto (\pi(p); (\mathrm{pr}_2 \circ \varphi_U)\,(p), (\mathrm{pr}_2 \circ \widetilde{\varphi}_U)\,(\widetilde{p}))\,. \end{aligned}$$

These maps are invertible; in fact, their inverses are given by

$$\begin{aligned} \overline{\varphi}_U^{-1}\colon U \times (G \times G) &\longrightarrow \overline{\pi}^{-1}\,(U) \\ (x; g, h) &\longmapsto \left(\varphi_U^{-1}\,(x, g)\,, \widetilde{\varphi}_U^{-1}\,(x, h)\right). \end{aligned}$$

It is clear from their definition that the maps $\overline{\varphi}_U$ and their inverses are smooth, as they are compositions of smooth maps. Hence, we have a trivialization of the fiber product $P \odot \widetilde{P}$.

For the sake of completeness, we write down explicitly the transition maps of the $G \times G$-principal bundle $P \odot \widetilde{P}$

$$\begin{aligned} \overline{\varphi}_{U,V}\colon U \cap V &\longrightarrow \mathrm{Diff}(G \times G) \\ x &\longmapsto \mathrm{R}_{\varphi_{U,V}(x)} \times \mathrm{R}_{\widetilde{\varphi}_{U,V}(x)}, \end{aligned}$$

where $\varphi_{U,V}$, resp. $\widetilde{\varphi}_{U,V}$, are the transition maps of P, resp. $\widetilde{P}$. ☐

Remark 2.1.18. We notice that the fiber product of two principal bundles over the same base space and with the same structure group is the analogue of the Whitney sum of vector bundles for principal bundles.

We notice that there is a left action of $G \times G$ also on G, by the rule:

$$\begin{aligned} \widetilde{\mathrm{c}}\colon G \times G &\longrightarrow \mathrm{Diff}(G) \\ (g, h) &\longmapsto \widetilde{\mathrm{c}}\,(g, h)\,k :\, = hkg^{-1}. \end{aligned}$$

Definition 2.1.19. We assume the same hypotheses as in the previous definitions. The set of smooth $G \times G$-equivariant maps from $P \odot \widetilde{P}$ to G, which we denote by $C^\infty\left(P \odot \widetilde{P}, G\right)^{G \times G}$, is the subset of $C^\infty\left(P \odot \widetilde{P}, G\right)$ of maps K satisfying

$$K(pg, \widetilde{p}h) = \widetilde{\mathrm{c}}\left(g^{-1}, h^{-1}\right) K(p, \widetilde{p})\,, \quad \forall\, (p, \widetilde{p}) \in P \odot \widetilde{P}, (g, h) \in G \times G. \quad (2.1.14)$$

Theorem 2.1.20. *The set of isomorphisms from P to $\widetilde{P}$ is in one-to-one correspondence with $C^\infty\left(P \odot \widetilde{P}, G\right)^{G\times G}$.*

Proof. We consider an isomorphism τ from P to $\widetilde{P}$ and we take a point $(p, \widetilde{p}) \in P \odot \widetilde{P}$. We consider $\tau(p)$. Since $(\widetilde{\pi} \circ \tau)(p) = \pi(p) = \widetilde{\pi}(\widetilde{p})$, it holds

$$\tau(p) = \widetilde{p}\, G_\tau(p, \widetilde{p})\,,$$

for a unique element $G_\tau(p, \widetilde{p})$ of G. The map G_τ is clearly smooth, since τ is smooth.

It remains to show that it satisfies (2.1.14). Since τ is G-equivariant, it follows by the very definition of G_τ:

$$\tau(pg) = \widetilde{p}\, G_\tau(pg, \widetilde{p}) = \tau(p)g = \widetilde{p}\, G_\tau(p, \widetilde{p})\, g,$$

whence it follows

$$G_\tau(pg, \widetilde{p}) = G_\tau(p, \widetilde{p})\, g^{-1}, \quad \forall g \in G. \tag{2.1.15}$$

On the other hand, it holds

$$\tau(p) = \widetilde{p}\, G_\tau(p, \widetilde{p}) = \widetilde{p}\, g\, G_\tau(p, \widetilde{p}g)\,,$$

whence it follows

$$G_\tau(p, \widetilde{p}g) = g^{-1} G_\tau(p, \widetilde{p})\,, \quad \forall g \in G. \tag{2.1.16}$$

Combining (2.1.15) and (2.1.16), we see that G_τ is $G \times G$-equivariant.

Conversely, we assume we are given an element K of $C^\infty\left(P \odot \widetilde{P}, G\right)^{G\times G}$. We define a corresponding map τ_K from P to $\widetilde{P}$ as follows:

$$\tau_K\,(p) :\ = \widetilde{p}\, K(p, \widetilde{p})\,,$$

where $\widetilde{p}$ is a general element of $\widetilde{P}$, such that $(p, \widetilde{p})$ is in $P \odot \widetilde{P}$.

First of all, we show that the map τ_K is well-defined: assume $\widetilde{q}$ is another point in $\widetilde{P}$, such that $\widetilde{\pi}(\widetilde{p}) = \widetilde{\pi}(\widetilde{q})$. Since $\widetilde{p}$ and $\widetilde{q}$ belong to the same fiber, there exists an element g in G, such that $\widetilde{q} = \widetilde{p}g$. It follows

$$\tau_K\,(p) = \widetilde{q}\, K(p, \widetilde{q}) = \widetilde{p}\, g\, K(p, \widetilde{p}g) \overset{\text{By equivariancy}}{=} \widetilde{p}\, K(p, \widetilde{p})\,.$$

Therefore, τ_K is well-defined.

It also satisfies $\widetilde{\pi} \circ \tau_K = \pi$ by its very definition, and the G-equivariancy follows from

$$\tau_K(pg) = \widetilde{p}\, K(pg, \widetilde{p}) \overset{\text{By equivariancy}}{=} \widetilde{p}\, K(p, \widetilde{p})\; g = \tau_K(p)g, \quad \forall g \in G.$$

It remains to show that τ_K is invertible. We show this by finding an inverse. We define the map $\sigma_K : \widetilde{P} \to P$ by

$$\sigma_K(\widetilde{p}) :\ = p\, K(p, \widetilde{p})^{-1}\,,$$

where $\pi(p) = \widetilde{\pi}\left(\widetilde{\pi}\right)$, and K^{-1} denotes the inverse in G of K. As before, one can show that σ_K is well-defined, that it is G-equivariant and satisfies $\pi \circ \sigma_K = \widetilde{\pi}$. We claim that σ_K is the inverse of τ_k. In fact,

$$(\sigma_K \circ \tau_K)(p) = \sigma_K\left(\widetilde{p}K(p,\widetilde{p})\right) = \sigma_K\left(\widetilde{p}\right)K(p,\widetilde{p}) = pK(p,\widetilde{p})^{-1}(p,\widetilde{p}) = p.$$

Analogously, one can show that $\tau_K \circ \sigma_K = \mathrm{id}_{\widetilde{P}}$. $\square$

We assume furthermore that P, resp. $\widetilde{P}$, is endowed with a connection A, resp. $\widetilde{A}$. A general tangent vector to $P \odot \widetilde{P}$ at $(p,\widetilde{p})$ can be written as $(X_p, X_{\widetilde{p}})$, where X_p, resp. $X_{\widetilde{p}}$, is a tangent vector to P at p, resp. to $\widetilde{P}$ at $\widetilde{p}$. By the very definition of the projection $\overline{\pi}$ from $P \odot \widetilde{P}$ onto M, the tangent vector $(X_p, X_{\widetilde{p}})$ is vertical if and only if

$$\mathrm{T}_{(p,\widetilde{p})}\overline{\pi}\left(X_p, X_{\widetilde{p}}\right) = \mathrm{T}_p\pi\left(X_p\right) = \mathrm{T}_{\widetilde{p}}\widetilde{\pi}\left(X_{\widetilde{p}}\right) = 0.$$

Hence, both components of $(X_p, X_{\widetilde{p}})$ have to be vertical.

On P, resp. $\widetilde{P}$, the connection A, resp. $\widetilde{A}$, specifies a smooth splitting of the tangent bundle of P, resp. $\widetilde{P}$, into vertical and A-horizontal vectors, resp. $\widetilde{A}$-horizontal vectors. Hence, we can write any tangent vectors X_p and $X_{\widetilde{p}}$ to P at p and to $\widetilde{P}$ at $\widetilde{p}$ as

$$X_p = X_p^v + X_p^h, \quad X_{\widetilde{p}} = X_{\widetilde{p}}^v + X_{\widetilde{p}}^h.$$

Therefore, any tangent vector $(X_p, X_{\widetilde{p}})$ to $P \odot \widetilde{P}$ at $(p,\widetilde{p})$ has a unique splitting

$$(X_p, X_{\widetilde{p}}) = \left(X_p^v, X_{\widetilde{p}}^v\right) + \left(X_p^h, X_{\widetilde{p}}^h\right). \tag{2.1.17}$$

To the splitting (2.1.17) belong obviously the connection 1-form

$$\left(A \oplus \widetilde{A}\right)_{(p,\widetilde{p})}\left(X_p, X_{\widetilde{p}}\right) : = \left(A_p\left(X_p\right), \widetilde{A}_{\widetilde{p}}\left(X_{\widetilde{p}}\right)\right).$$

(We notice that the connection $A \oplus \widetilde{A}$ is $\mathfrak{g} \oplus \mathfrak{g}$-valued, as $P \odot \widetilde{P}$ is a $G \times G$-principal bundle.)

It is immediate to see that $A \oplus \widetilde{A}$ is flat if both A and $\widetilde{A}$ are flat.

If we are given a smooth bundle-isomorphism Φ from P to $\widetilde{P}$, we can construct a morphism $\Phi \odot \Phi^{-1}$ from $P \odot \widetilde{P}$ to $\widetilde{P} \odot P$ as follows:

$$\left(\Phi \odot \Phi^{-1}\right)(p,\widetilde{p}) : = \left(\Phi(p), \Phi^{-1}\left(\widetilde{p}\right)\right), \quad \forall\ (p,\widetilde{p}) \in P \odot \widetilde{P}.$$

Since Φ is an isomorphism, it is clear that $\Phi \odot \Phi^{-1}$ is also an isomorphism. Moreover, the G-equivariancy of Φ ensures that $\Phi \odot \Phi^{-1}$ is $G \times G$-equivariant.

There is also a natural connection 1-form $\widetilde{A} \oplus A$ on $\widetilde{P} \odot P$. Pulling back $\widetilde{A} \oplus A$ w.r.t. $\Phi \odot \Phi^{-1}$ gives a connection 1-form on $P \odot \widetilde{P}$. Clearly, by the very definition of $\Phi \odot \Phi^{-1}$, saying that the pull-back of $\widetilde{A} \oplus A$ w.r.t. $\Phi \oplus \Phi^{-1}$ equals $A \oplus \widetilde{A}$, i.e.

$$\left(\Phi \odot \Phi^{-1}\right)^*\left(\widetilde{A} \oplus A\right) = A \oplus \widetilde{A},$$

is equivalent to

$$\Phi^*\widetilde{A} = A.$$

We consider now the space of loops $\mathrm{L}M$, i.e. the set of all curves γ from the unit interval I to M satisfying $\gamma(0) = \gamma(1)$. We consider the following two smooth maps from $\mathrm{L}M \times \mathrm{I}$ to M, namely:

$$\mathrm{ev}\,(\gamma; t) : \; = \gamma(t); \tag{2.1.18}$$

$$\mathrm{ev}_0\,(\gamma; t) : \; = \gamma(0). \tag{2.1.19}$$

Via the maps (2.1.18) and (2.1.19), we may construct two principal G-bundles on $\mathrm{L}M$ via pull-back:

$$\mathrm{ev}^* P = \{(\gamma; t; p) \in \mathrm{L}M \times \mathrm{I} \times P \colon \pi(p) = \gamma(t)\}\,;$$
$$\mathrm{ev}_0^* P = \{(\gamma; p; t) \in \mathrm{L}M \times \mathrm{I} \times P \colon \pi(p) = \gamma(0)\}\,.$$

Moreover, we may construct the fiber product of $\mathrm{ev}^* P$ and $\mathrm{ev}_0^* P$:

$$\mathrm{ev}_0^* P \odot \mathrm{ev}^* P = \{(\gamma; t; p, \widetilde{p}) \in \mathrm{L}M \times \mathrm{I} \times P \times P \colon \pi(p) = \gamma(0), \quad \pi\,(\widetilde{p}) = \gamma(t)\}\,.$$

For a chosen connection A on P, Lemma 2.1.15 implies that $\mathrm{H}(A; \bullet; \bullet; \bullet, \bullet)$ (which, from now on, we denote by $\mathrm{H}(A)|_0^\bullet$, when its arguments are not specified) is an element of $C^\infty\,(\mathrm{ev}_0^* P \odot \mathrm{ev}^* P, G)^{G\times G}$. Therefore, by Theorem 2.1.20, the parallel transport $\mathrm{H}(A)|_0^\bullet$ induces an isomorphism from $\mathrm{ev}_0^* P$ to $\mathrm{ev}^* P$, which we denote by Φ_A in order to make explicit its dependence on a chosen connection A.

We want to inspect more carefully the dependence on A of the isomorphism Φ_A. First, we write down explicitly the isomorphism Φ_A:

$$\Phi_A\,(\gamma; t; p) = (\gamma; t; \widetilde{p}\ \mathrm{H}(A; \gamma; t; p, \widetilde{p}))\,,$$

where $\pi(p) = \gamma(0)$, and $\widetilde{p} \in \widetilde{P}$ is any point obeying $\widetilde{\pi}\,(\widetilde{p}) = \gamma(t)$.

We know that the gauge group $\mathcal{G}$ operates on $\mathcal{A}$. If we take an element $\sigma \in \mathcal{G}$ and we take the pull-back of the connection A w.r.t. σ, the morphism Φ_A changes as follows:

$$\begin{aligned}\Phi_{\sigma^* A}\,(\gamma; t; p) &= (\gamma; t; \widetilde{p}\ \mathrm{H}(\sigma^* A; \gamma; t; p, \widetilde{p})) = \\ &\overset{\text{By (2.1.11)}}{=} (\gamma; t; \widetilde{p}\, g_\sigma\,(\widetilde{p})^{-1}\ \mathrm{H}(A; \gamma; t; p, \widetilde{p})\ g_\sigma(p)) = \\ &= \left(\gamma; t; \sigma^{-1}\,(\widetilde{p})\ \mathrm{H}(A; \gamma; t; p\, g_\sigma(p), \widetilde{p})\right) = \\ &= \left(\gamma; t; \sigma^{-1}\,(\widetilde{p}\ \mathrm{H}(A; \gamma; t; \sigma(p), \widetilde{p}))\right) = \\ &= \sigma^{-1}\,(\Phi_A\,(\gamma; t; \sigma(p)))\,.\end{aligned} \tag{2.1.20}$$

To interpret in the correct way the previous computation, we need a digression.

We assume f to be a smooth map from the manifold L to M and P to be a G-principal bundle over M, with projection π.

Lemma 2.1.21. *Every gauge transformation of P induces a gauge transformation of the pull-back bundle $f^* P$.*

Proof. Let $\sigma \in \mathcal{G}_P$ be a general gauge transformation. By definition,

$$f^*P = \{(l, p) \in L \times P \colon f(l) = \pi(p)\}\,.$$

Let (l, p) be a general point in f^*P. We define the morphism σ_f from f^*P to itself as follows:

$$\sigma_f\,(l, p) :\, = (l, \sigma(p))\,.$$

We have to show that σ_f is well-defined, that it respects the fiber, that it is G-equivariant and that it is bijective.

Since $\pi \circ \sigma = \pi$, it follows that the image of a point in f^*P still belongs to f^*P. By its very definition, σ_f respects the fiber. The G-equivariancy follows by the definition of the right G-action on f^*P.

Finally, σ_f has an inverse, which is simply $(\sigma_f)^{-1} = \sigma_f^{-1}$. We notice that the map taking σ to σ_f is clearly a homomorphism of groups from $\mathcal{G}_P$ to $\mathcal{G}_{f^*P}$. $\square$

Given two different smooth maps f, g from L to M, we may form two different principal bundles over L, namely f^*P and g^*P. If we assume the set of G-equivariant isomorphisms from f^*P to g^*P to be nonempty (i.e. if the two bundles are isomorphic), the gauge group $\mathcal{G}_P$ operates from the left on this set as follows:

$$(\sigma, \tau) \mapsto \sigma_g \circ \tau \circ \sigma_f^{-1}, \tag{2.1.21}$$

where $\sigma \in \mathcal{G}$ and τ is a G-equivariant isomorphism from f^*P to g^*P.

With the help of the previous considerations, we may rewrite Identity (2.1.20) as follows:

$$\Phi_{\sigma^* A} = \sigma_{\mathrm{ev}}^{-1} \circ \Phi_A \circ \sigma_{\mathrm{ev}_0}, \quad \forall A \in \mathcal{A}, \sigma \in \mathcal{G}_P. \tag{2.1.22}$$

This may be restated as follows: the parallel transport $\mathrm{H}(\bullet)|_0^\bullet$ induces a $\mathcal{G}$-equivariant map from $\mathcal{A}$ to the set of G-equivariant isomorphisms from $\mathrm{ev}_0^*\,P$ to $\mathrm{ev}^*\,P$.

If we restrict the bundles $\mathrm{ev}_0^*\,P$ and $\mathrm{ev}^*\,P$ to the subset $\mathrm{L}M \times \{0\}$, we get

$$\begin{aligned}
\mathrm{ev}_0^*\,P|_{\mathrm{L}M\times\{0\}} &\cong \{(\gamma; t; p) \in \mathrm{ev}_0^*\,P \colon t = 0\} \cong \mathrm{ev}(0)^*P;\\
\mathrm{ev}^*\,P|_{\mathrm{L}M\times\{0\}} &\cong \{(\gamma; t; p) \in \mathrm{ev}^*\,P \colon t = 0\} \cong \{(\gamma; 0; p) \in \mathrm{ev}^*\,P \colon \gamma(0) = \pi(p)\} \cong\\
&\cong \mathrm{ev}(0)^*P,
\end{aligned}$$

where $\mathrm{ev}(0)(\gamma) :\, = \gamma(0)$.

Since Φ_A is a G-equivariant isomorphism from $\mathrm{ev}_0^*\,P$ to $\mathrm{ev}^*\,P$, it respects fibers. Hence, Φ_A maps the restriction to $\mathrm{L}M \times \{0\}$ of $\mathrm{ev}_0^*\,P$ to the restriction to the same set of $\mathrm{ev}^*\,P$. It is nonetheless interesting to compute explicitly the induced isomorphism of $\mathrm{ev}(0)^*P$. By the definition of Φ_A, it follows:

$$\Phi_A|_{\mathrm{L}M\times\{0\}}\,(\gamma; 0; p) = (\gamma; 0; \widetilde{p}\,\mathrm{H}(A; \gamma; 0; p, \widetilde{p})),$$

where in this case $\pi\,(\widetilde{p}) = \gamma(0) = \pi(p)$. Therefore, we may choose $\widetilde{p} = p$, and it follows

$$\widetilde{\gamma}_{A,p}(0) = p,$$

whence $\mathrm{H}(A; \gamma; 0; p, p) = e$. Hence, the restriction to $\mathrm{L}M \times \{0\}$ equals the identity on $\mathrm{ev}(0)^*P$.

On the other hand, we may restrict the bundles $\mathrm{ev}_0^* P$ and $\mathrm{ev}^* P$ to $\mathrm{L}M \times \{1\}$. In this case, we get:

$$\mathrm{ev}_0^* P|_{\mathrm{L}M\times\{1\}} = \{(\gamma; 1; p) \in \mathrm{ev}_0^* P \colon \pi(p) = \gamma(0)\} \cong \mathrm{ev}(0)^* P;$$
$$\mathrm{ev}^* P|_{\mathrm{L}M\times\{1\}} = \{(\gamma; 1; p) \in \mathrm{ev}_0^* P \colon \pi(p) = \gamma(1) = \gamma(0)\} \cong \mathrm{ev}(0)^* P.$$

So, restricting the two bundles $\mathrm{ev}_0^* P$ and $\mathrm{ev}^* P$ to the subset $\mathrm{L}M \times \{1\}$, we get the same bundle as before, namely $\mathrm{ev}(0)^* P$. But the restriction of Φ_A is not the identity, as the following computation shows:

$$\Phi_A|_{\mathrm{L}M\times\{1\}}(\gamma; 1; p) = (\gamma; 1; \widetilde{p}\,\mathrm{H}(A; \gamma; 1; p, \widetilde{p})),$$

where $\widetilde{p} \in P$ obeys $\pi(\widetilde{p}) = \gamma(1) = \gamma(0) = \pi(p)$. Hence, we may choose $\widetilde{p} = p$, and we get

$$\Phi_A|_{\mathrm{L}M\times\{1\}}(\gamma; 1; p) = (\gamma; 1; p\,\mathrm{H}(A; \gamma; 1; p, p)).$$

The following identity holds:

$$p\,\mathrm{H}(A; \gamma; 1; p, p) = \widetilde{\gamma}_{A,p}(1) = p\,\mathrm{H}(A; \gamma; p).$$

Since the action of G on each fiber is free, it follows:

$$\mathrm{H}(A; \gamma; p) = \mathrm{H}(A; \gamma; 1; p, p), \quad \forall \gamma \in \mathrm{L}M, p \in P.$$

Hence, the restriction of Φ_A to $\mathrm{L}M \times \{1\}$ equals the automorphism of $\mathrm{ev}(0)^* P$ defined by the holonomy $\mathrm{H}(A; \bullet; \bullet)$ (which we denote by $\mathrm{H}(A)|_0^1$, when we do not want to specify its arguments), which, according to Identity (2.1.6), defines a G-equivariant map from $\mathrm{ev}(0)^* P$ to itself.

2.1.5 Some consequences of flatness

We assume we have chosen a flat connection A on P. A well-known fact about flat connections states that the holonomy $\mathrm{H}(A; \gamma; p)$ of a loop γ, based at $p \in P$ over $\gamma(0)$, depends only on the homotopy class of the loop γ, if the connection A is flat. We refer to Section 7 of Chapter 2 of [38] for more details on the relationship between flat connections and holonomy. Since the holonomy w.r.t. A depends on a pair $(\gamma; p)$, with $\gamma \in \mathrm{L}M$ and on a point $p \in P$ lying over $\gamma(0)$, we may view the holonomy w.r.t. A as a map from the pull-back bundle

$$\mathrm{ev}(0)^* P \colon = \{(\gamma; p) \in \mathrm{L}M \times P \colon \pi(p) = \gamma(0) = \mathrm{ev}(0)(\gamma)\}$$

to G. The bundle $\mathrm{ev}(0)^* P$ receives an obvious right action of G.

Moreover, Identity (2.1.6) implies that $\mathrm{H}(A)|_0^1$ is G-equivariant w.r.t. the conjugation on G. This is equivalent to the fact that the holonomy w.r.t. A induces a gauge-transformation on $\mathrm{ev}(0)^* P$, which we denote by Φ_A.

There is a natural map $\widetilde{\mathrm{ev}(0)}$ from $\mathrm{ev}(0)^* P$ to P, defined as follows $\widetilde{\mathrm{ev}(0)}(\gamma; p) :=$ p. Clearly, $\widetilde{\mathrm{ev}(0)}$ is G-equivariant; hence, the pull-back of A w.r.t. $\widetilde{\mathrm{ev}(0)}$ is again a connection on $\mathrm{ev}(0)^* P$, which we denote as $\mathrm{ev}(0)^* A$.

Theorem 2.1.22. *If A is flat, the gauge transformation of $\mathrm{ev}(0)^* P$ induced by the holonomy w.r.t. A stabilizes the connection $\mathrm{ev}(0)^* A$, i.e. $\Phi_A^* \mathrm{ev}(0)^* A = \mathrm{ev}(0)^* A$.*

Before proving Theorem 2.1.22, we need some preliminary facts.

A general tangent vector at a point $(\gamma; p) \in \mathrm{ev}(0)^* P$ can be written as a couple of tangent vectors $(X_\gamma; X_p)$, where $X_\gamma \in \mathrm{T}_\gamma \mathrm{L}M$ and $X_p \in \mathrm{T}_p P$. The condition for the tangent vector $(X_\gamma; X_p)$ at $(\gamma; p) \in \mathrm{ev}(0)^* P$ to be vertical is

$$X_\gamma = 0, \quad \text{implying in turn} \quad \mathrm{T}_p\pi(X_p) = \mathrm{T}_\gamma \mathrm{ev}(0)(X_\gamma) = 0,$$

since $(X_\gamma; X_p)$ is tangent to $\mathrm{ev}(0)^* P$. Therefore, a general vertical vector at $(\gamma; p) \in \mathrm{ev}(0)^* P$ can be uniquely written as $(0; X_p)$, where X_p is vertical at p.

On the other hand, the tangent vector $(X_\gamma; X_p)$ at $(\gamma; p) \in \mathrm{ev}(0)^* P$ is $\mathrm{ev}(0)^* A$-horizontal if

$$\left(\mathrm{ev}(0)^* A\right)_{(\gamma;p)} (X_\gamma; X_p) = A_p\left(X_p\right) = 0.$$

Hence, a tangent vector $(X_\gamma; X_p)$ at $(\gamma; p)$ is $\mathrm{ev}(0)^* A$-horizontal if and only if its P-component is A-horizontal.

Since the connection A specifies a splitting of the tangent bundle of P into the vertical bundle and the A-horizontal bundle, we can split any tangent vector $(X_\gamma; X_p)$ at $(\gamma; p) \in \mathrm{ev}(0)^* P$ into a unique sum

$$(X_\gamma; X_p) = \left(0; X_p^v\right) + \left(X_\gamma; X_p^h\right), \tag{2.1.23}$$

where X_p^v, resp. X_p^h, denotes the vertical, resp. A-horizontal, part of the vector X_p. The splitting (2.1.23) plays a pivotal rôle in the proof of Lemma 2.1.22.

We need also the following technical

Lemma 2.1.23. *We assume we have two curves γ_1 and γ_2, such that $\gamma_1(1) = \gamma_2(0)$; additionally, we take three points p, p_1 and p_2 in P, such that*

$$\gamma_1(0) = \pi(p), \quad \gamma_1(1) = \gamma_2(0) = \pi(p_1), \quad \gamma_2(1) = \pi(p_2).$$

We define the curve $\gamma_2 \circ \gamma_1$ as

$$\gamma_2 \circ \gamma_1\,(t) : = \begin{cases} \gamma_1(2t), & t \in \left[0; \frac{1}{2}\right] \\ \gamma_2\left(2t - 1\right), & t \in \left[\frac{1}{2}; 1\right] \end{cases}.$$

Clearly, the composition $\gamma_2 \circ \gamma_1$ is piecewise smooth.

Then, the following identity holds

$$\mathrm{H}(A; \gamma_2 \circ \gamma_1; 1; p, p_2) = \mathrm{H}(A; \gamma_2; 1; p_1, p_2)\ \mathrm{H}(A; \gamma_1; 1; p, p_1)\,. \tag{2.1.24}$$

Lemma 2.1.24. *We assume we have a curve γ on M, and two points p, $\widetilde{p}$ in P, such that*

$$\gamma(0) = \pi(p), \quad \gamma(1) = \pi(\widetilde{p}).$$

We define the inverse curve γ^{-1} of γ as

$$\gamma^{-1}\,(t) : = \gamma\,(1 - t)\,.$$

Then, the following identity holds:

$$\mathrm{H}\left(A; \gamma^{-1}; 1; \widetilde{p}, p\right) = \mathrm{H}(A; \gamma; 1; p, \widetilde{p})^{-1}\,. \tag{2.1.25}$$

Proof of Lemma 2.1.23. By the very definition of the parallel transport, we get

$$\widetilde{\gamma_2 \circ \gamma_1}_{A,p}(1) = p_2 \; \mathrm{H}(A; \gamma_2 \circ \gamma_1; 1; p, p_2)\,,$$

where $\widetilde{\gamma_2 \circ \gamma_1}_{A,p}$ is the unique A-horizontal lift of $\gamma_2 \circ \gamma_1$ based at p.

On the other hand, we consider the composition $\widetilde{\gamma_2}_{A,\widetilde{\gamma_1}_{A,p}(1)} \circ \widetilde{\gamma_1}_{A,p}$

$$\left(\widetilde{\gamma_2}_{A,\widetilde{\gamma_1}_{A,p}(1)} \circ \widetilde{\gamma_1}_{A,p}\right)(t) :\;= \begin{cases} \widetilde{\gamma_1}_{A,p}(2t), & t \in \left[0, \frac{1}{2}\right] \\ \widetilde{\gamma_2}_{A,\widetilde{\gamma}_{A,p}(1)}(2t-1), & t \in \left[\frac{1}{2}; 1\right]. \end{cases}$$

By its very definition, it is clear that $\widetilde{\gamma_2}_{A,\widetilde{\gamma_1}_{A,p}(1)} \circ \widetilde{\gamma_1}_{A,p}$ lies over $\gamma_2 \circ \gamma_1$ and that it is based at p; since it is the composition of two A-horizontal curves, it is also A-horizontal, whence it follows that

$$\widetilde{\gamma_2}_{A,\widetilde{\gamma_1}_{A,p}(1)} \circ \widetilde{\gamma_1}_{A,p} = \widetilde{\gamma_2 \circ \gamma_1}_{A,p}.$$

Therefore, we get

$$\begin{aligned} \widetilde{\gamma_2}_{A,\widetilde{\gamma_1}_{A,p}(1)} \circ \widetilde{\gamma_1}_{A,p} = \widetilde{\gamma_2}_{A,\widetilde{\gamma}_{A,p}(1)}(1) = \widetilde{\gamma_2}_{A,p_1}(1) \; \mathrm{H}(A; \gamma_1; 1; p, p_1) = \\ = p_2 \; \mathrm{H}(A; \gamma_2; 1; p_1, p_2) \; \mathrm{H}(A; \gamma_1; 1; p, p_1)\,, \end{aligned}$$

and the claim is an immediate consequence. ☐

Proof of Lemma 2.1.24. We consider the curve

$$\widetilde{\gamma}^{-1}_{A,p \;\; \mathrm{H}(A;\gamma;1;p,\widetilde{p})^{-1}}(t) :\; = \widetilde{\gamma}_{A,p}(1-t) \; \mathrm{H}(A; \gamma; 1; p, \widetilde{p})^{-1}.$$

It is immediate to see that this curve lies over γ^{-1}; moreover, it is based on $\widetilde{p}$, since

$$\begin{aligned} \widetilde{\gamma}_{A,p}(1-t) \; \mathrm{H}(A; \gamma; 1; p, \widetilde{p})^{-1} = \widetilde{p} \; \mathrm{H}(A; \gamma; 1; p, \widetilde{p}) \; \mathrm{H}(A; \gamma; 1; p, \widetilde{p})^{-1} = \\ = \widetilde{p}. \end{aligned}$$

Since the A-horizontal bundle is defines a G-invariant distribution, $\widetilde{\gamma}^{-1}_{A,p \;\; \mathrm{H}(A;\gamma;1;p,\widetilde{p})^{-1}}$ is A-horizontal, and by the uniqueness of A-horizontal lifts of smooth curves, we get

$$\widetilde{\gamma}^{-1}_{A,p \;\; \mathrm{H}(A;\gamma;1;p,\widetilde{p})^{-1}} = \widetilde{\gamma^{-1}}_{A,\widetilde{p}},$$

whence the claim follows. ☐

Proof of Theorem 2.1.22. We consider a general tangent vector $(X_\gamma; X_p)$ at some point $(\gamma; p)$ in $\mathrm{ev}(0)^*P$, and we assume X_p to be A-horizontal at p. We may assume that $(X_\gamma; X_p)$ is the tangent vector at $(\gamma; p)$ of a smooth curve $(\gamma_s; p_s)$ ($s \in \mathrm{I}$), where γ_s is a smooth curve in $\mathrm{L}M$ (a curve of loops) and p_s is an A-horizontal curve in P. Moreover, since $(\gamma_s; p_s)$ is in $\mathrm{ev}(0)^*P$, it follows that p_s lies over $\gamma_s(0)$, for any $s \in \mathrm{I}$.

We define a smooth curve $\overline{\gamma}_s$, for any $s \in \mathrm{I}$, by the rule

$$\overline{\gamma}_s(t) :\; = \gamma_{st}(0). \tag{2.1.26}$$

We then construct the map Γ on the unit square $\mathrm{I}\times\mathrm{I}$ by means of the curves (2.1.26) as follows:

$$\Gamma(t,s) :\, = \begin{cases} \overline{\gamma}_s\,(3t)\,, & t\in\left[0;\frac{1}{3}\right] \\ \gamma_s\,(3t-1)\,, & t\in\left[\frac{1}{3};\frac{2}{3}\right] \\ \overline{\gamma}_s\,(3-3t)\,, & t\in\left[\frac{2}{3},1\right]. \end{cases}$$

First of all, for any $s\in\mathrm{I}$, $\Gamma(\bullet,s)$ is a closed curve. Namely,

$$\Gamma(0,s)=\overline{\gamma}_s(0)=\gamma_0(0)=\gamma(0),\quad \Gamma(1,s)=\overline{\gamma}_s(0)=\gamma(0).$$

It follows also that $\Gamma(\bullet,s)$ is based in $\gamma(0)$, for any s in the unit interval. Finally, it is clear that Γ is a homotopy of γ. We notice that $\Gamma(\bullet,s)$ is piecewise smooth w.r.t. $t\in\mathrm{I}$.

Since any $\Gamma(\bullet,s)=\Gamma_s$ is homotopic to γ and since the connection A is flat, we get

$$\mathrm{H}(A;\Gamma_s;p)=\mathrm{H}(A;\gamma;p)\,.$$

On the other hand, the curve Γ_s may be written as the composition $\overline{\gamma}_s^{-1}\circ\gamma_s\circ\overline{\gamma}_s$, and each piece of Γ_s is smooth. It follows by Lemma 2.1.23 and 2.1.24

$$\mathrm{H}(A;\gamma;p)=\mathrm{H}(A;\Gamma_s;p)=\mathrm{H}(A;\overline{\gamma}_s;1;p,p_s)^{-1}\ \mathrm{H}(A;\gamma_s;p_s)\ \mathrm{H}(A;\overline{\gamma}_s;1;p,p_s)\,.$$

We define, for any $s\in\mathrm{I}$, the curve $\overline{p}_s(t)$ by the rule

$$\overline{p}_s\,(t) :\, = p_{ts},$$

where t is also in the unit interval. The curve $\overline{p}_s$ is clearly A-horizontal, as it is a reparametrization of an A-horizontal curve. Moreover, it lies over $\overline{\gamma}_s$:

$$\pi\left(\overline{p}_s(t)\right)=\pi\left(p_{st}\right)=\gamma_{st}(0)=\overline{\gamma}_s(t),\quad \forall t\in\mathrm{I},$$

and is based at p, since $\overline{p}_s(0)=p_0=p$. It follows immediately that $\mathrm{H}(A;\overline{\gamma}_s;1;p,p_s)=\mathrm{id}$, for all $s\in\mathrm{I}$, whence it follows

$$\mathrm{H}(A;\gamma;p)=\mathrm{H}(A;\gamma_s;p_s)\,,\quad \forall s\in\mathrm{I},$$

which is equivalent to the constancy of the holonomy along the curve $(\gamma_s;p_s)$, and this yields in turn

$$\mathrm{T}_{(\gamma;p)}\mathrm{H}(A)|_0^1\,(X_\gamma;X_p)=0,\quad \text{if } X_p \text{ is } A\text{-horizontal.}$$

An explicit computations gives

$$\begin{aligned}\mathrm{T}_{(\gamma;p)}\Phi_A\,(X_\gamma;X_p)=\big(X_\gamma;\mathrm{T}_p\mathrm{R}_{\mathrm{H}(A;\gamma;p)}\,(X_p)\,+\\ +\,\mathrm{T}_{\mathrm{H}(A;\gamma;p)}\mathrm{L}_p\left[\mathrm{T}_{(\gamma;p)}\mathrm{H}(A)|_0^1\,(X_\gamma;X_p)\right]\big)\,,\end{aligned}$$

for any tangent vector $(X_\gamma;X_p)$ on $\mathrm{ev}(0)^*P$ at $(\gamma;p)$.

If X_p is A-horizontal, the above equation simplifies to

$$\mathrm{T}_{(\gamma;p)}\Phi_A\,(X_\gamma;X_p)=\big(X_\gamma;\mathrm{T}_p\mathrm{R}_{\mathrm{H}(A;\gamma;p)}\,(X_p)\big)\,.$$

Finally, we get

$$\begin{aligned}\Phi_A^*\left(\mathrm{ev}(0)^*A\right)_{(\gamma;p)}\left(X_\gamma;X_p\right) &= \left(\mathrm{ev}(0)^*A\right)_{(\gamma;p\ \mathrm{H}(A;\gamma;p))}\left[\left(X_\gamma;\mathrm{T}_p\mathrm{R}_{\mathrm{H}(A;\gamma;p)}\left(X_p\right)\right)\right]=\\ &= A_{p\ \mathrm{H}(A;\gamma;p)}\left[\mathrm{T}_p\mathrm{R}_{\mathrm{H}(A;\gamma;p)}\left(X_p\right)\right]=\\ &= \mathrm{Ad}\left(\mathrm{H}(A;\gamma;p)^{-1}\right)\left[A_p\left(X_p\right)\right]=\\ &= 0,\end{aligned}$$

if $(X_\gamma;X_p)$ is $\mathrm{ev}(0)^*A$-horizontal and by G-equivariancy of A. Hence, any $\mathrm{ev}(0)^*A$-horizontal vector at $(\gamma;p)$ is also $\Phi_A^*\left(\mathrm{ev}(0)^*A\right)$-horizontal.

It follows immediately that $\Phi_A^*\left(\mathrm{ev}(0)^*A\right)=\mathrm{ev}(0)^*A$, since at any point $(\gamma;p)\in\mathrm{ev}(0)^*P$ a tangent vector $(X_\gamma;X_p)$ can be decomposed in a unique way into a vertical piece and an $\mathrm{ev}(0)^*A$-horizontal piece; by the above computations, $\Phi_A^*\left(\mathrm{ev}(0)^*A\right)$ and $\mathrm{ev}(0)^*A$ agree on any tangent vector, and hence they are equal. □

The identity $\Phi_A^*\left(\mathrm{ev}(0)^*A\right)=\mathrm{ev}(0)^*A$ can be written alternatively as

$$\mathrm{d}_{\mathrm{ev}(0)^*A}\,\mathrm{H}(A)|_0^1=0.$$

For a function σ on $\mathrm{ev}(0)^*P$ with values in G, we define

$$\begin{aligned}\left(\mathrm{d}_{\mathrm{ev}(0)^*A}\sigma\right)_{(\gamma;p)}\left(X_\gamma;X_p\right):=&\left(\mathrm{T}_{(\gamma;p)}\sigma\right)\left(X_\gamma;X_p\right)+\mathrm{T}_e\mathrm{R}_{\sigma(\gamma;p)}\left[A_p\left(X_p\right)\right]-\\ &-\mathrm{T}_e\mathrm{L}_{\sigma(\gamma;p)}\left[A_p\left(X_p\right)\right].\end{aligned}$$

We return to the principal bundles $\mathrm{ev}^*\,P$ and $\mathrm{ev}_0^*\,P$. We denote by $\widetilde{\mathrm{ev}}$, resp. $\widetilde{\mathrm{ev}_0}$, the natural map from $\mathrm{ev}^*\,P$, resp. $\mathrm{ev}_0^*\,P$, to P, given by

$$\widetilde{\mathrm{ev}}\left(\gamma;t;p\right):=p,\quad\text{resp.}\quad\widetilde{\mathrm{ev}_0}\left(\gamma;t;\widetilde{p}\right):=\widetilde{p}.$$

We denote then by $\mathrm{ev}^*\,A$, resp. $\mathrm{ev}_0^*\,P$, the pull-back of A w.r.t. $\widetilde{\mathrm{ev}}$, resp. $\widetilde{\mathrm{ev}_0}$; $\mathrm{ev}^*\,A$ and $\mathrm{ev}_0^*\,A$ are clearly connections on $\mathrm{ev}^*\,P$ and $\mathrm{ev}_0^*\,P$.

We consider the fiber product of $\mathrm{ev}_0^*\,P$ and $\mathrm{ev}^*\,P$, and a point $(\gamma;t;p,\widetilde{p})$ in it. We denote by $(X_\gamma;X_t;X_p,X_{\widetilde{p}})$ a general tangent vector to $\mathrm{ev}_0^*\,P\odot\mathrm{ev}^*\,P$ at $(\gamma;t;p,\widetilde{p})$, where X_γ is tangent to $\mathrm{L}M$ at the loop γ, X_t is tangent to $[0;1]$ at t, and X_p, resp. $X_{\widetilde{p}}$, is tangent to P at p, resp. $\widetilde{p}$.

The condition on $(X_\gamma;X_t;X_p,X_{\widetilde{p}})$ to be vertical may be translated into

$$X_\gamma=0,\quad X_t=0\rightarrow\begin{cases}\mathrm{T}_p\pi(X_p) & =\mathrm{T}_{(\gamma,t)}\mathrm{ev}_0\left(X_\gamma;X_t\right)=0,\\ \mathrm{T}_{\widetilde{p}}\pi\left(X_{\widetilde{p}}\right) & =\mathrm{T}_{(\gamma,t)}\mathrm{ev}\left(X_\gamma;X_t\right)=0,\end{cases}$$

where the first identities follow from the definition of the bundle projection from $\mathrm{ev}_0^*\,P\odot\mathrm{ev}^*\,P$ onto $\mathrm{L}M\times[0;1]$, and the second one is a consequence of $(X_\gamma;X_t;X_p,X_{\widetilde{p}})$ to be tangent to $\mathrm{ev}_0^*\,P\odot\mathrm{ev}^*\,P$. Hence, a tangent vector $(X_\gamma;X_t;X_p,X_{\widetilde{p}})$ to the fiber product $\mathrm{ev}_0^*\,P\odot\mathrm{ev}^*\,P$ at $(\gamma;t;p,\widetilde{p})$ is vertical if and only if its P-pieces are vertical and its $\mathrm{L}M\times[0;1]$-piece vanishes.

As we already know, the connection A specifies a splitting of the tangent bundle of P. Therefore, any tangent vector X_p to P at p can be written uniquely as $X_p=$

$X_p^v + X_p^h$, where X_p^v, resp. X_p^h, denotes the vertical part, resp. the horizontal part, of X_p.

Therefore,

$$(X_\gamma; X_t; X_p, X_{\widetilde{p}}) = \left(0; 0; X_p^v, X_{\widetilde{p}}^v\right) + \left(X_\gamma; X_t; X_p^h, X_{\widetilde{p}}^h\right), \tag{2.1.27}$$

for any tangent vector $(X_\gamma; X_t; X_p, X_{\widetilde{p}})$ to $\mathrm{ev}_0^* P \odot \mathrm{ev}^* P$ at $(\gamma; t; p, \widetilde{p})$.

The splitting (2.1.27) specifies in an obvious way a G-invariant distribution on $\mathrm{ev}_0^* P \odot \mathrm{ev}^* P$; the corresponding connection 1-form is given explicitly by

$$\left(\mathrm{ev}_0^* A \oplus \mathrm{ev}^* A\right)_{(\gamma;t;p,\widetilde{p})} (X_\gamma; X_t; X_p, X_{\widetilde{p}}) :\, = \left(A_p\left(X_p\right), A_{\widetilde{p}}\left(X_{\widetilde{p}}\right)\right).$$

It is immediate to verify that the connection $\mathrm{ev}_0^* A \oplus \mathrm{ev}^* A$ is flat, if A is flat.

Analogous definitions and results hold for $\mathrm{ev}^* P \odot \mathrm{ev}_0^* P$; we denote the connection constructed from A on the fiber product $\mathrm{ev}^* P \odot \mathrm{ev}_0^* P$ by $\mathrm{ev}^* A \oplus \mathrm{ev}_0^* A$.

We define the following $G \times G$-equivariant isomorphism from $\mathrm{ev}_0^* P \odot \mathrm{ev}^* P$ to $\mathrm{ev}^* P \odot \mathrm{ev}_0^* P$

$$\Phi_A \odot \Phi_A^{-1}(\gamma; t; p, \widetilde{p}) :\, = \Big(\gamma; t; \widetilde{p}\ \mathrm{H}(A; \gamma; t; p, \widetilde{p}), p\ \mathrm{H}(A; \gamma; t; p, \widetilde{p})^{-1}\Big).$$

We have at our disposal all elements needed to state the fundamental

Theorem 2.1.25. *If A flat, then the following identity holds:*

$$\left(\Phi_A \odot \Phi_A^{-1}\right)^* \left(\mathrm{ev}^* A \oplus \mathrm{ev}_0^* A\right) = \mathrm{ev}_0^* A \oplus \mathrm{ev}^* A. \tag{2.1.28}$$

Proof. It suffices to show that $\mathrm{ev}_0^* A \oplus \mathrm{ev}^* A$-horizontal vectors in $\mathrm{ev}_0^* P \odot \mathrm{ev}^* P$ are also horizontal w.r.t. its pull-back by $\Phi_A \odot \Phi_A^{-1}$.

In order to show this claim, we view a general horizontal vector $(X_\gamma; X_t; X_p; X_{\widetilde{p}})$, where the subscripts are related to the base points of the respective tangent vectors, as the initial tangent direction of an $\mathrm{ev}_0^* A \oplus \mathrm{ev}^* A$-horizontal curve, which we denote by $\left(\gamma_s; \overline{t}_s; p_s, \widetilde{p}_s\right)$; we notice that p_s and $\widetilde{p}_s$ are A-horizontal curves. Here, $\overline{t}_s$ is a curve in $[0;1]$ starting at $\overline{t}_0 = t$.

Further, by means of γ_s and $\overline{t}_s$, we define the piecewise smooth family of curves Γ as

$$\Gamma(t,s) :\, = \begin{cases} \gamma_{ts}(0), & t \in \left[0; \frac{1}{4}\right] \\ \gamma_s\left((4t-1)\overline{t}_s\right), & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \gamma_{(3-4t)s}\left((3-4t)\overline{t}_s\right), & t \in \left[\frac{1}{2}; \frac{3}{4}\right] \\ \gamma\left((4-4t)\overline{t}_0\right), & t \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

It is immediate to see that Γ is a piecewise smooth homotopy of the piecewise smooth loop $\gamma_{\overline{t}_0}^{-1} \circ \gamma_{\overline{t}_0}$, where $\gamma_{\overline{t}_0}(s) :\, = \gamma(s\overline{t}_0)$.

By Lemma 2.1.23 and 2.1.24, it follows immediately

$$\mathrm{H}(A; \gamma_s; t_s; p_s, \widetilde{p}_s) = \mathrm{H}(A; \gamma; t; p, \widetilde{p}),$$

using that the horizontal curves p_{ts} and $\widetilde{p}_{ts}$ lie over $\gamma_{ts}(0)$ and $\gamma_{ts}(tt_s)$ and are based in p and $\widetilde{p}$ respectively.

The claim follows by arguments very similar to those used in the final steps of the proof of Theorem 2.1.22. $\square$

We notice that Identity (2.1.28) may be also rewritten as

$$\begin{aligned}\mathrm{T}_{(\gamma;t;p,\widetilde{p})}\mathrm{H}(A)|_0^\bullet\,(X_\gamma;X_t;X_p,X_{\widetilde{p}}) = &-\mathrm{T}_e\mathrm{R}_{\mathrm{H}(A;\gamma;t;p,\widetilde{p})}\left[A_{\widetilde{p}}(X_{\widetilde{p}})\right]+\\ &+\mathrm{T}_e\mathrm{L}_{\mathrm{H}(A;\gamma;t;p,\widetilde{p})}\left[A_p(X_p)\right]:\iff\\ \mathrm{d}\,\mathrm{H}(A)|_0^\bullet = -\,\mathrm{ev}^*A\ \mathrm{H}(A)|_0^\bullet &+\mathrm{H}(A)|_0^\bullet\ \mathrm{ev}_0^*A.\end{aligned}$$

We are interested in the following more general situation: we consider the space $\mathrm{L}M\times\triangle_n$, where $\triangle_n$ denotes the standard n-dimensional simplex. We consider the following natural maps from $\mathrm{L}M\times\triangle_n$ into M:

$$\mathrm{ev}_{i,n}:=\mathrm{ev}\circ\pi_{i,n},\quad 1\leq i\leq n,\quad \mathrm{ev}(0)_n:=\mathrm{ev}(0)\circ\pi_n,$$

where $\pi_{i,n}(\gamma;t_1,\dots,t_n):=(\gamma;t_i)$ and $\pi(\gamma;t_1,\dots,t_n):=\gamma$. Accordingly to the previous computations, we define the bundles $\mathrm{ev}_{i,n}^*P$ and $\mathrm{ev}(0)_n^*P$ over $\mathrm{L}M\times\triangle_n$ and their fiber product $\mathrm{ev}(0)_n^*P\odot\mathrm{ev}_{i,n}^*P$.

Given a connection A on P, a natural connection on the fiber product $\mathrm{ev}(0)_n^*P\odot\mathrm{ev}_{i,n}^*P$ is given by

$$\mathrm{ev}(0)_n^*A\oplus\mathrm{ev}_{i,n}^*A:=\left(\mathrm{ev}(0)_n^*A,\mathrm{ev}_{i,n}^*A\right).$$

(We recall that the fiber product is a $G\times G$-principal bundle over $\mathrm{L}M\times\triangle_n$.) We denote by $\mathrm{ev}_{i,n}^*A\oplus\mathrm{ev}(0)_n^*A$ the corresponding natural connection on $\mathrm{ev}_{i,n}^*P\odot\mathrm{ev}(0)_n^*P$.

For any $1\leq i\leq n$, the bundle $\mathrm{ev}(0)_n^*P$ is isomorphic to $\mathrm{ev}_{i,n}^*P$ via the map

$$\Phi_{A,i,n}(\gamma;t_1,\dots,t_n;p):=\left(\gamma;t_1,\dots,t_n;\widetilde{p}\ \mathrm{H}(A;\gamma;t_i;p,\widetilde{p})\right),$$

where $\pi(p)=\gamma(0)$, and $\widetilde{p}$ is any point in P over $\gamma(t_i)$.

We consider the isomorphism $\Phi_{A,i,n}\odot\Phi_{A,i,n}^{-1}$ from $\mathrm{ev}(0)_n^*P\odot\mathrm{ev}_{i,n}^*P$ to $\mathrm{ev}_{i,n}^*P\odot\mathrm{ev}(0)_n^*P$

$$\Phi_{A,i,n}\odot\Phi_{A,i,n}^{-1}(\gamma;t_1,\dots;p,\widetilde{p}):=\left(\gamma;t_1,\dots;\widetilde{p}\ \mathrm{H}(A;\gamma;t_i;p,\widetilde{p}),p\ \mathrm{H}(A;\gamma;t_i;p,\widetilde{p})^{-1}\right).$$

An obvious corollary of Theorem 2.1.25 is given by

Theorem 2.1.26. *If A is flat, the following identity holds*

$$\left(\Phi_{A,i,n}\odot\Phi_{A,i,n}^{-1}\right)^*\left(\mathrm{ev}_{i,n}^*A\oplus\mathrm{ev}(0)_n^*A\right)=\mathrm{ev}(0)_n^*A\oplus\mathrm{ev}_{i,n}^*A.$$

From Theorem 2.1.26 we easily recover the following identity

$$\Phi_{A,i,n}^*\left(\mathrm{ev}_{i,n}^*A\right)=\mathrm{ev}(0)_n^*A,$$

if A is flat.

The principal bundles $\mathrm{ev}_{i,n}^*P$ and $\mathrm{ev}(0)_n^*P$ are pull-backs of P w.r.t. $\mathrm{ev}_{i,n}$, resp. $\mathrm{ev}(0)_n$. As we have already seen, a connection A on P determines natural connections $\mathrm{ev}_{i,n}^*A$, resp. $\mathrm{ev}(0)_n^*A$, on $\mathrm{ev}_{i,n}^*P$, resp. $\mathrm{ev}(0)_n^*P$; moreover, Theorem 2.1.26 shows immediately that the isomorphism $\Phi_{A,i,n}$ intertwines the connections $\mathrm{ev}_{i,n}^*A$ and $\mathrm{ev}(0)_n^*A$.

We consider a general form ω on M with values in some associated bundle $P\times_G V$ of P, for some representation (ρ, V) of G; by the same label we denote the unique basic form on P with values in V corresponding to ω. We may then take the pull-back of ω w.r.t. $\widetilde{\mathrm{ev}}_{i,n}$, hence getting a basic form on $\mathrm{ev}_{i,n}^* P$ with values in the representation V, which we denote by $\mathrm{ev}_{i,n}^*\,\omega$. If we take the pull-back of $\mathrm{ev}_{i,n}^*\,\omega$ w.r.t. $\Phi_{A,i,n}$, we get a basic form on $\mathrm{ev}(0)_n^* P$ with values in V, descending to a form on $\mathrm{L}M\times\triangle_n$ with values in the associated bundle $\mathrm{ev}(0)_n^* P\times_G V$. We denote the result of all these operations by $\widehat{\omega}_{i,n}$, for $1\leq i\leq n$. In other words,

$$\widehat{\omega}_{A,i,n}:=\left(\widetilde{\mathrm{ev}}_{i,n}\circ\Phi_{A,i,n}\right)^*\omega,$$

where $\widetilde{\mathrm{ev}}_{i,n}$ is the natural map from $\mathrm{ev}_{i,n}^* P$ to P.

The gauge group $\mathcal{G}$ of P operates from the right on $\Omega^*\left(M, P\times_G V\right)$ by the rule

$$\omega^\sigma:=\sigma^*\left(\omega\right),\quad \sigma\in\mathcal{G},$$

where by ω we have denoted the unique basic form on P with values in V associated to a form ω on M with values in $P\times_G V$. A slight modification of (2.1.20) implies

$$\begin{aligned}
\widehat{\omega^\sigma}_{A^\sigma,i,n}&=\left(\sigma\circ\widetilde{\mathrm{ev}}_{i,n}\circ\Phi_{A^\sigma,i,n}\right)^*\omega=\\
&=\left(\widetilde{\mathrm{ev}}_{i,n}\circ\sigma\circ\Phi_{A^\sigma,i,n}\right)^*\omega=\\
&=\left(\widetilde{\mathrm{ev}}_{i,n}\circ\Phi_{A,i,n}\circ\sigma\right)^*\omega=\\
&=\widehat{\omega}_{A,i,n}^\sigma,
\end{aligned}$$

where the σ in the last identity is the gauge-transformation of $\mathrm{ev}(0)_n^* P$ induced by $\sigma\in\mathcal{G}$.

We consider the covariant derivative w.r.t. the connection $\mathrm{ev}(0)_n^* A$ of $\widehat{\omega}_{i,n}$:

$$\begin{aligned}
\mathrm{d}_{\mathrm{ev}(0)_n^* A}\widehat{\omega}_{i,n}&=\mathrm{d}\widehat{\omega}_{i,n}+\widehat{\rho}\left(\mathrm{ev}(0)_n^* A\right)\widehat{\omega}_{i,n}=\\
&=\widehat{\mathrm{d}\omega}_{i,n}+\widehat{\rho}\left(\mathrm{ev}(0)_n^* A\right)\left(\Phi_{A,i,n}\circ\widetilde{\mathrm{ev}}_{i,n}\right)^*\omega=\\
&=\widehat{\mathrm{d}\omega}_{i,n}+\left(\Phi_{A,i,n}\circ\widetilde{\mathrm{ev}}_{i,n}\right)\left[\widehat{\rho}(A)\omega\right]=\\
&=\widehat{\mathrm{d}_A\omega}_{i,n};
\end{aligned}$$

the third equality is a direct consequence of Theorem 2.1.26, if A is flat.

We notice also that, since ω is a basic form on P, the form $\widehat{\omega}_{i,n}$ can be also written as

$$\widehat{\omega}_{i,n}=\mathrm{Ad}\left(\mathrm{H}(A)|_0^\bullet\right)^{-1}\mathrm{ev}_{i,n}^*\,\omega.$$

We end with some comments about the isomorphisms $\Phi_{A,i,n}$. If we denote by $\iota_{\alpha,n}$, for $0\leq\alpha\leq n$, the inclusion of $\mathrm{L}M\times\triangle_{n-1}$ into $\mathrm{L}M\times\triangle_n$ given by

$$\iota_{\alpha,n}\left(\gamma;t_1,\dots,t_{n-1}\right):=\begin{cases}\left(\gamma;0,t_1,\dots,t_{n-1}\right), & \alpha=0\\ \left(\gamma;t_1,\dots,t_\alpha,t_\alpha,\dots,t_{n-1}\right), & 1\leq\alpha\leq n-1\\ \left(\gamma;t_1,\dots,t_{n-1},1\right), & \alpha=n,\end{cases}$$

we also denote by $\widetilde{\iota}_{\alpha,n}$ the natural map from $\mathrm{ev}_{i,n-1}^* P$ into $\mathrm{ev}_{i,n}^* P$.

Given two manifolds M and N, G-principal bundle P over N and a smooth map f from M to N, it is possible to pull-back any form on N with values in an associated bundle $P \times_G V$ of P, for some representation (ρ, V); the result is a form on M with values in the pull-back bundle $f^* (P \times_G V) \cong f^* P \times_G V$. If we denote by $\widetilde{f}$ the natural map from $f^* P$ to P and by ω a general form on N with values in $P \times_G V$, it is well-known that the basic form on $f^* P$ with values in V corresponding to $f^* \omega$ is the pull-back w.r.t. $\widetilde{f}$ of the basic form on P with values in V corresponding to ω.

Therefore, in order to compute the result of the restriction by $\iota_{\alpha,n}$ of any form $\widehat{\omega}_{i,n}$ as a form on $\mathrm{L}M \times \triangle_n$ with values in $\mathrm{ev}(0)_n^* P \times_G V$, it suffices to compute the result of the following compositions of bundle maps:

$$\widetilde{\mathrm{ev}}_{i,n} \circ \Phi_{A,i,n} \circ \widetilde{\iota}_{0,n} = \begin{cases} \widetilde{\mathrm{ev}(0)}_n, & i = 1 \\ \widetilde{\mathrm{ev}}_{i-1,n-1} \circ \Phi_{A,i-1,n-1}, & 2 \leq i \leq n \end{cases},$$

$$\widetilde{\mathrm{ev}}_{i,n} \circ \Phi_{A,i,n} \circ \widetilde{\iota}_{\alpha,n} = \begin{cases} \widetilde{\mathrm{ev}}_{i,n-1} \circ \Phi_{A,i,n-1}, & 1 \leq i \leq \alpha \\ \widetilde{\mathrm{ev}}_{\alpha,n-1} \circ \Phi_{A,\alpha,n-1}, & i = \alpha + 1 \\ \widetilde{\mathrm{ev}}_{i,n-1} \circ \Phi_{A,i,n-1}, & \alpha + 2 \leq i \leq n \end{cases},$$

$$\widetilde{\mathrm{ev}}_{i,n} \circ \Phi_{A,i,n} \circ \widetilde{\iota}_{n,n} = \begin{cases} \widetilde{\mathrm{ev}}_{i,n-1} \circ \Phi_{A,i,n-1}, & 1 \leq i \leq n-1 \\ \widetilde{\mathrm{ev}(0)} \circ \Phi_A \circ \widetilde{\pi}_n, & i = 1. \end{cases}$$

2.2 Definition and main properties of the push-forward

Let M be a manifold and $\mathcal{E} \xrightarrow{\pi} M$ a smooth fiber bundle with typical fiber F, with F an oriented compact manifold, possibly with boundaries and corners. It is possible to pick also a noncompact fiber, but in this case we have to restrict to forms on $\mathcal{E}$ with compact support on each fiber. Let m, resp. e, resp. f, denote the dimensions of M, resp. $\mathcal{E}$, resp. F (so $e = f + m$).

We pick a form ω in $\Omega^p(\mathcal{E})$, where $p \geq f$; we then define the push-forward $\pi_* \omega$ of the form ω w.r.t. π as the form in $\Omega^{p-f}(M)$ satisfying the following identity:

$$\int_M \pi_* \omega \wedge \eta = \int_{\mathcal{E}} \omega \wedge \pi^* \eta \quad , \forall\, \eta \in \Omega^{m+f-p}(M). \tag{2.2.1}$$

In the case $p < f$ we define $\pi_* \omega = 0$. We now list without proof the main properties of the push-forward:

$$\begin{aligned} &\pi_*(\pi^* \alpha \wedge \beta) = (-1)^{f \deg \alpha} \alpha \wedge \pi_* \beta, && \forall\, \alpha \in \Omega^*(M),\ \forall\, \beta \in \Omega^*(\mathcal{E}), \\ &\pi_*(\alpha \wedge \pi^* \beta) = \pi_* \alpha \wedge \beta, && \forall\, \alpha \in \Omega^*(\mathcal{E}),\ \forall\, \beta \in \Omega^*(M), \\ &\mathrm{d}\pi_* \alpha = (-1)^f \pi_* \mathrm{d}\alpha - (-1)^f \pi_{\partial *} \iota^* \alpha, && \forall\, \alpha \in \Omega^*(\mathcal{E}), \end{aligned} \tag{2.2.2}$$

where $\iota : \mathcal{E}_\partial \to \mathcal{E}$ is the canonical injection of the fiber bundle with typical fiber ∂F, and $\pi_\partial : \iota(\mathcal{E}_\partial) \to M$ is the corresponding projection. For more details we refer to [8] and [32].

Another important property we use throughout the paper is given by the following Lemma; for a proof, see [32]. We consider two manifolds M and N and suppose that $\mathcal{E} \xrightarrow{\pi} M$, resp. $\mathcal{F} \xrightarrow{\tilde{\pi}} N$, is a fiber bundle over the manifold M, resp. N. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a bundle morphism with base map $\psi : M \to N$. We cast all these maps in the following commutative square:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\varphi} & \mathcal{F} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow{\psi} & N \end{array}$$

We additionally assume that φ induces orientation preserving diffeomorphisms of the fibers.

Lemma 2.2.1. *Under the above assumptions, the following identity holds:*

$$(\pi_* \circ \varphi^*)\alpha = (\psi^* \circ \tilde{\pi}_*)\alpha, \quad \forall\, \alpha \in \Omega^p(\mathcal{F}). \tag{2.2.3}$$

We assume we have a fiber bundle $\mathcal{E} \to \mathcal{F}$ and we assume additionally that $\mathcal{F} \to M$ is also a fiber bundle; we denote the projections by π_1, resp. π_2.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\pi_1} & \mathcal{F} \\ & & \big\downarrow{\scriptstyle\pi_2} \\ & & M \end{array}$$

If we compose the two projections we obtain a fiber bundle $\mathcal{E} \to M$, with projection $\pi = \pi_2 \circ \pi_1$, whose orientation will be determined by the orientation of the resulting fiber, the product manifold of the fibers of the two bundles. Then we obtain the following

Lemma 2.2.2. *With the above hypotheses, the following identity holds*

$$\pi_*\alpha = \pi_{2*}(\pi_{1*}\alpha), \quad \forall \alpha \in \Omega^p(\mathcal{E}). \tag{2.2.4}$$

This is just Fubini's Theorem for repeated integration, and the definition of the push-forward is consistent with the orientation choices.

We list two important properties of the push-forward, which we will use later in We assume $\mathcal{E}$ and M to be both G-spaces, for some Lie group G, and we assume the G-action on M and $\mathcal{E}$ to be compatible in the following sense: if we denote by φ_g^M, resp. $\varphi_g^{\mathcal{E}}$, the action of an element g of G on M, resp. $\mathcal{E}$, the projection π enjoys

$$\pi \circ \varphi_g^{\mathcal{E}} = \varphi_g^M \circ \pi, \quad \forall\, g \in G.$$

Lemma 2.2.3. *The push-forward w.r.t. π enjoys the following property*

$$\varphi_g^{M*}\left(\pi_*\omega\right) = (-1)^{\mathrm{or}\,\varphi_g^{\mathcal{E}}|_F}\pi_*\left(\varphi_g^{\mathcal{E}*}\omega\right), \quad \forall\, g \in G, \omega \in \Omega^*\left(\mathcal{E}\right).$$

We have denoted by $(-1)^{\mathrm{or}\,\varphi_g^{\mathcal{E}}|_F}$ the orientation sign of $\varphi_g^{\mathcal{E}}$ on each fiber (we recall once again that the orientation of a fiber bundle is specified by an orientation of the base manifold and of the fiber).

Furthermore, we take ξ in the Lie algebra of G, and we denote by X_ξ^M, resp. $X_\xi^{\mathcal{E}}$, the corresponding fundamental vector field on M, resp. $\mathcal{E}$. In other words, if we denote by L_x^M, resp. $\mathrm{L}_p^{\mathcal{E}}$, for $x \in M$, resp. $p \in \mathcal{E}$, the map

$$\mathrm{L}_x^M(g) := \varphi_g^M(x), \quad \text{resp.} \quad \mathrm{L}_p^{\mathcal{E}}(g) := \varphi_g^{\mathcal{E}}(p),$$

then X_ξ^M, resp. $X_\xi^{\mathcal{E}}$, the vector field

$$X_\xi^M(x) := \mathrm{T}_e\mathrm{L}_x^M(\xi), \quad \text{resp.} \quad X_\xi^{\mathcal{E}}(p) := \mathrm{T}_e\mathrm{L}_p^{\mathcal{E}}(\xi).$$

We finally notice that the action of G on M and on $\mathcal{E}$ is not necessarily free, hence X_ξ^M or $X_\xi^{\mathcal{E}}$ can be also zero.

Lemma 2.2.4. *The push-forward w.r.t. π enjoys the following property*

$$\iota_{X_\xi^M}(\pi_*\omega) = \pi_*\left(\iota_{X_\xi^{\mathcal{E}}}\omega\right), \quad \forall\, \xi \in \mathfrak{g}, \omega \in \Omega^*(\mathcal{E}).$$

The proof of Lemma 2.2.3 and 2.2.4 is an easy consequence of equation (2.2.1).

We end this section by defining the push-forward of forms on $\mathcal{E} \to M$ with values in some finite dimensional vector space W. This is simply given by

$$\pi_*(\alpha \otimes \boldsymbol{v}) := \pi_*\alpha \otimes \boldsymbol{v}$$

on generators and extended by linearity.

Finally, it is well-known (we refer e.g. to [32] and [33]) that, given a smooth map $f\colon N_1 \to N_2$ between two smooth manifolds N_1 and N_2 and a bundle $\mathcal{N}_2 \to N_2$, forms on N_1 taking values in the pull-back bundle $f^*\mathcal{N}_2 \to N_1$ are generated by pull-backs of sections of $\mathcal{N}_2$ as an $\Omega^*(N_1)$-module. We use this fact to extend in an obvious way the notion of push-forward on forms on a fibration $\mathcal{E} \xrightarrow{\pi} M$ over M with values in the pull-back bundle $\pi^*\mathcal{N}$, for some bundle $\mathcal{N}$ over M. This serves when we want to define the parallel transport via iterated integrals, which we introduce and discuss in detail in the next section, in the case of a nontrivial principal bundle P.

2.3 Parallel transport as a function on $\mathrm{L}M$

We consider a trivial principal bundle $P \to M$. We also pick a connection A on M; by d_A we denote the corresponding covariant derivative. We pick an element a in $\Omega^1(M, \operatorname{ad} P)$, and we may define another connection starting from A, namely $A + a$. For the sake of simplicity, we assume A to be flat. We apply to $A + a$ the canonical injection ι from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$, so as to obtain a 1-form on M with values in $\mathcal{U}(\mathfrak{g})$; we omit to write ι before $A + a$. We then define

$$\widehat{a} := \mathrm{H}(A)|_0^\bullet\, \mathrm{ev}^*(a)\, \left(\mathrm{H}(A)|_0^\bullet\right)^{-1},$$

where by ev we have denoted the evaluation map $\mathrm{ev}(\gamma; t) := \gamma(t)$ from $\mathrm{L}M \times [0,1]$ to M, and by $\mathrm{H}(A)|_0^\bullet$ we denote here the inverse of the parallel transport w.r.t. the connection A, viewed as a function on $\mathrm{L}M \times [0,1]$ with values in G; the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ receives in an obvious way a representation of G, and hence we may view $\mathrm{H}(A)|_0^\bullet$ as a map from $\mathrm{L}M \times [0;1]$ with values in $\mathrm{Aut}\,(\mathcal{U}(\mathfrak{g}))$.

Remark 2.3.1. We notice that the parallel transport $\mathrm{H}(A)|_0^\bullet$ is, in the more general case of a nontrivial bundle P, a section of the bundle $\mathrm{Ad}\,\mathrm{ev}^* P$ over $\mathrm{L}M \times [0;1]$. In this case, the form $\widehat{a}$ is defined as the pull-back of a w.r.t. the composition of Φ_A with the (lift to $\mathrm{ev}^* P$ of the) evaluation map ev (see Subsection 2.1.4 and 2.1.5); hence, $\widehat{a}$ is a form on $\mathrm{L}M \times [0;1]$ with values in $\mathrm{ad}\,\mathrm{ev}_0^* P$.

Since A is flat, $\mathrm{H}(A)|_0^\bullet$ enjoys by Theorem 2.1.25 the following property:

$$\mathrm{d}\,\mathrm{H}(A)|_0^\bullet = -\,\mathrm{ev}_0^* A\ \mathrm{H}(A)|_0^\bullet + \mathrm{H}(A)|_0^\bullet\ \mathrm{ev}^* A, \tag{2.3.1}$$

where $\mathrm{ev}_0 : \mathrm{L}M \times [0;1] \to M$ is defined by $\mathrm{ev}_0(\gamma; t) := \gamma(0)$; we define further, for $n \in \mathbb{N}$, the maps $\pi_n : \mathrm{L}M \times \triangle_n \to \mathrm{L}M$ by $\pi_n(\gamma; t_1, \dots, t_n) := \gamma$, and by $\triangle_n$ we denote the n-simplex

$$\triangle_n := \Big\{(t_1, \dots, t_n) \in [0,1]^n : 0 \le t_1 \le \dots \le t_n \le 1\Big\}, \tag{2.3.2}$$

with orientation form specified by $\mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n$. By $\mathrm{H}(A)|_0^1$ we denote the inverse of the holonomy along the loop γ, considered as a function on $\mathrm{L}M$ taking values in G; it is immediate to see that the holonomy $\mathrm{H}(A)|_0^1$ is simply the restriction to the subspace $\mathrm{L}M \times \{1\}$ of the parallel transport $\mathrm{H}(A)|_0^\bullet$. In the case P nontrivial, $\mathrm{H}(A)|_0^1$ is a section of the bundle $\mathrm{Ad}\,\mathrm{ev}(0)^* P$ over $\mathrm{L}M$.

It follows from its definition that $\widehat{a}$ is a 1-form on $\mathrm{L}M \times [0,1]$ with values in $\mathcal{U}(\mathfrak{g})$. We can now define the *parallel transport* w.r.t. $A + a$ from 1 to 0 as the formal series in $\mathcal{U}(\mathfrak{g})$

$$\mathrm{H}(A+a)|_0^1 := \mathrm{H}(A)|_0^1 + \sum_{n \ge 1} \pi_{n*}\left(\widehat{a}_{1,n} \wedge \dots \wedge \widehat{a}_{n,n}\right) \mathrm{H}(A)|_0^1, \tag{2.3.3}$$

where $\widehat{a}_{i,n} := \pi_{i,n}^* \widehat{a}$ and $\pi_{i,n}(\gamma; t_1, \dots, t_n) := (\gamma; t_i)$. It follows from its very definition that the parallel transport is an element of $\Omega^0(\mathrm{L}M; \mathcal{U}(\mathfrak{g}))$.

Remark 2.3.2. We can define the parallel transport with free final point w.r.t. the connection $A + a$ by

$$\mathrm{H}(A+a)|_0^\bullet := 1 + \sum_{n \ge 1} \pi_{n+1,n+1*}\left(\widehat{a}_{1,n+1} \wedge \dots \wedge \widehat{a}_{n,n+1}\right),$$

with the same notations as above; it follows from its very definition that this particular parallel transport is a map $\mathrm{L}M \times [0,1] \to \mathcal{U}(\mathfrak{g})$. The parallel transport as a function on $\mathrm{L}M \times [0,1]$ with free initial point is defined analogously by the formula

$$\mathrm{H}(A+a)|_\bullet^1 := 1 + \sum_{n \ge 1} \pi_{1,n+1*}\left(\widehat{a}_{2,n+1} \wedge \dots \wedge \widehat{a}_{n+1,n+1}\right).$$

Further, we can define the parallel transport with free end-points as a function on $\mathrm{L}M \times \triangle_2$:

$$\mathrm{H}(A+a)|_\bullet^\bullet := 1 + \sum_{n \ge 1} \pi_{1,\widehat{n},n+2*}\left(\widehat{a}_{2,n+2} \wedge \dots \wedge \widehat{a}_{n+1,n+2}\right),$$

where $\pi_{1,\widehat{n},n+2}(\gamma; s_1, s_2, \dots, s_{n+1}, s_{n+2}) := (\gamma; s_1, s_{n+2})$.

Theorem 2.3.3. *If we denote by* $\mathrm{d}_{\mathrm{ev}(0)^*A}$ *the covariant derivative w.r.t. the connection* $\mathrm{ev}(0)^*A$ *on forms on* $\mathrm{L}M$ *with values in* $\mathcal{U}(\mathfrak{g})$*, then the following identity holds, for any flat connection* A *on* P *and any* $a \in \Omega^1(M, \mathrm{ad}\, P)$*:*

$$\mathrm{d}_{\mathrm{ev}(0)^*A}\, \mathrm{H}(A+a)|_0^1 = -\,\mathrm{ev}(0)^*a \wedge \mathrm{H}(A+a)|_0^1 + \mathrm{H}(A+a)|_0^1\, \mathrm{ev}(0)^*a + \\ - \left[\int\limits_{0\leq s\leq 1} \mathrm{H}(A+a)|_0^s \wedge \widehat{F_{A+a}}_s \wedge \mathrm{H}(A+a)|_s^1\right] \mathrm{H}(A)|_0^1.$$

Remark 2.3.4. We have written

$$\int\limits_{0\leq s\leq 1} \mathrm{H}(A+a)|_0^s \wedge \widehat{F_{A+a}}_s \wedge \mathrm{H}(A+a)|_s^1 := \pi_{1*}\left[\mathrm{H}(A+a)|_0^\bullet \wedge \widehat{F_{A+a}} \wedge \mathrm{H}(A+a)|_\bullet^1\right],$$

where we have used again the notations in Remark 2.3.2.

Proof. We shall apply the generalized Stokes Theorem to the push-forward w.r.t. the maps π_n; we note that the n-simplex $\triangle_n$ has a boundary, and that this boundary can be written as

$$\partial\triangle_n = \bigcup_{\alpha=0}^n (\partial\triangle_n)_\alpha,$$

where each $(\partial\triangle_n)_\alpha \cong \triangle_{n-1}$. With our choice of orientation of the simplices—see after (2.3.2)—the first face of the boundary (which corresponds to the equation $t_1 = 0$) comes equipped with opposite orientation -1, while the second has orientation 1, the third has opposite orientation -1 again, and so forth. In other words,

$$\mathrm{or}((\partial\triangle_n)_\alpha) = (-1)^{\alpha+1}\,\mathrm{or}(\triangle_{n-1}). \tag{2.3.4}$$

We apply the covariant derivative w.r.t A_0 to the n-th term of the series, and we obtain:

$$\mathrm{d}_{\mathrm{ev}(0)^*A}\pi_{n*}\left[\widehat{a}_{1,n} \wedge \cdots\right] \mathrm{H}(A)|_0^1 = (-1)^n \pi_{n*}\mathrm{d}_{\pi_n^*\,\mathrm{ev}(0)^*A}\left[\widehat{a}_{1,n} \wedge \cdots\right] \mathrm{H}(A)|_0^1 + \\ -(-1)^n \pi_{\partial_n *}\left[\iota_{\partial_n}^* \widehat{a}_{1,n} \wedge \cdots \wedge \widehat{a}_{n,n}\right] \mathrm{H}(A)|_0^1, \tag{2.3.5}$$

where $\pi_{\partial_n} : \mathrm{L}M \times \partial\triangle_n \to \mathrm{L}M$ denotes the projection onto the first factor, while $\iota_{\partial_n} : \mathrm{L}M \times \partial\triangle_n \to \mathrm{L}M \times \triangle_n$ is the canonical injection of the boundary of the simplex into the simplex itself. We have used the identity

$$\mathrm{d}_{\mathrm{ev}(0)^*A}\, \mathrm{H}(A)|_0^1 = 0,$$

which follows from Theorem 2.1.22, since A is flat.

We now consider the two terms on the right hand side of (2.3.5) separately, and we begin with the second term, which we call "the n-th boundary term" from now on. Since $\partial\triangle_n = \bigcup_{\alpha=0}^n \partial\triangle_{n,\alpha}$, we can write

$$\iota_{\partial_n}^*\left[\widehat{a}_{1,n} \wedge \cdots \wedge \widehat{a}_{n,n}\right] = \sum_{\alpha=0}^n \iota_{\partial_{n,\alpha}}^*\left[\widehat{a}_{1,n} \wedge \cdots \wedge \widehat{a}_{n,n}\right],$$

and $\iota_{\partial_{n,\alpha}} : \mathrm{L}M\times(\partial\triangle_n)_\alpha \to \mathrm{L}M\times\triangle_n$ is the canonical injection of the α-th face of the boundary. Considering the orientations of the faces, we obtain for the n-th boundary term the following expression:

$$\sum_{\alpha=0}^{n}(-1)^{\alpha+1}\pi_{n-1*}\iota^*_{\partial_{n,\alpha}}\Big[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\Big].$$

We begin with the first face $\alpha = 0$; it is not difficult to prove the following identities

$$\pi_{j,n}\circ\iota_{\partial_{n,0}} = \begin{cases}\iota(0)\circ\pi_{n-1} & j=1,\\ \pi_{j-1,n-1} & j\neq 1;\end{cases} \tag{2.3.6}$$

similarly, one shows for $\alpha = n$

$$\pi_{j,n}\circ\iota_{\partial_{n,n}} = \begin{cases}\pi_{j,n-1} & j\neq n,\\ \iota(1)\circ\pi_{n-1} & j=n.\end{cases} \tag{2.3.7}$$

We have denoted by $\iota(0)$, resp. $\iota(1)$, the injection of $\mathrm{L}M$ into $\mathrm{L}M\times[0,1]$ given by $\iota(0)(\gamma) := (\gamma;0)$, resp. $\iota(1)(\gamma) := (\gamma;1)$. for $\alpha\neq 0,n$, it holds

$$\pi_{j,n}\circ\iota_{\partial_{n,\alpha}} = \begin{cases}\pi_{j,n-1} & j<\alpha,\\ \pi_{\alpha,n-1} & j=\alpha,\alpha+1,\\ \pi_{j-1,n-1} & j>\alpha+1.\end{cases} \tag{2.3.8}$$

It is therefore an easy consequence of (2.3.6) (2.3.8) and (2.3.7) that (we recall that $\iota(0)\,\mathrm{H}(A)|_0^\bullet = 1$ by its very definition)

$$\begin{aligned}
\iota^*_{\partial_{n,0}}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right] &= \pi^*_{n-1}\,\mathrm{ev}(0)^*a\wedge\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1};\\
\iota^*_{\partial_{n,\alpha}}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right] &= \widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{(a\wedge a)}_{\alpha,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1};\\
\iota^*_{\partial_{n,n}}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right] &= \widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\wedge\pi^*_{n-1}\,\mathrm{ev}(0)^*a.
\end{aligned}$$

We consider now the first term under the action of the push-forward w.r.t. π_{n-1}:

$$\begin{aligned}
\pi_{n-1*}\iota^*_{\partial_{n,0}}\Big[\widehat{a}_{1,n}\wedge\cdots\Big]\,\mathrm{H}(A)|_0^1 &= \pi_{n-1*}\Big[\pi^*_{n-1}\,\mathrm{ev}(0)^*a\wedge\widehat{a}_{1,n-1}\wedge\cdots\Big]\,\mathrm{H}(A)|_0^1\\
&= (-1)^{n-1}\,\mathrm{ev}(0)^*a\wedge\pi_{n-1*}\Big[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\Big]\,\mathrm{H}(A)|_0^1.
\end{aligned}$$

Similarly, we obtain for $\alpha = n$

$$\pi_{n-1*}\iota^*_{\partial_{n,n}}\Big[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\Big]\,\mathrm{H}(A)|_0^1 = \pi_{n-1*}\Big[\widehat{a}_{1,n-1}\wedge\cdots\Big]\,\mathrm{H}(A)|_0^1\wedge\mathrm{ev}(0)^*a,$$

and for $\alpha\neq 0,n$ we obtain

$$\pi_{n-1*}\iota^*_{\partial_{n,\alpha}}\Big[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\Big]\,\mathrm{H}(A)|_0^1 = \pi_{n-1*}\Big[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{(a\wedge a)}_{\alpha,n-1}\wedge\cdots\Big]\,\mathrm{H}(A)|_0^1.$$

Finally, we obtain the following expression for the n-th boundary term of (2.3.5):

$$\sum_{\alpha=1}^{n-1}(-1)^{\alpha+1}\pi_{n-1*}\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{(a\wedge a)}_{\alpha,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1-$$
$$-(-1)^{n-1}\,\mathrm{ev}(0)^*a\wedge\pi_{n-1*}\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1+$$
$$+(-1)^{n+1}\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1\wedge\mathrm{ev}(0)^*a.$$

We then consider the first term of (2.3.5), and by the Leibnitz rule we obtain

$$\pi_{n*}\mathrm{d}_{\pi_n^*\,\mathrm{ev}(0)^*A}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right]=\sum_{i=1}^{n}(-1)^{i+1}\pi_{n*}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{\mathrm{d}_Aa}_{i,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right];$$

here we have used

$$\mathrm{d}_{\pi_1^*\,\mathrm{ev}(0)^*A}\widehat{a}=\widehat{\mathrm{d}_Aa},$$

which is a consequence of Theorem 2.1.25, being A flat.

Summing up both contributions to (2.3.5) with the correct signs, we obtain for the left hand side of (2.3.5)

$$\sum_{i=1}^{n}(-1)^{n+i+1}\pi_{n*}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{\mathrm{d}_Aa}_{i,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right]\mathrm{H}(A)|_0^1+$$
$$+\sum_{\alpha=1}^{n-1}(-1)^{n+\alpha}\pi_{n-1*}\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{(a\wedge a)}_{\alpha,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1-$$
$$-\,\mathrm{ev}(0)^*a\wedge\pi_{n-1*}\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1+$$
$$+\left[\widehat{a}_{1,n-1}\wedge\cdots\wedge\widehat{a}_{n-1,n-1}\right]\mathrm{H}(A)|_0^1\wedge\mathrm{ev}(0)^*a.$$

We begin by summing up all terms which contain before them $\mathrm{ev}(0)^*a$, and we obtain $-\,\mathrm{ev}(0)^*a\wedge\mathrm{H}(A+a)|_0^1$; similarly, by summing up all terms with $\mathrm{ev}(0)^*a$ on the right, we obtain $\mathrm{H}(A+a)|_0^1\wedge\mathrm{ev}(0)^*a$. By recalling the definition of the curvature of the connection $A+a$, the sum of the remaining terms gives

$$\sum_{n\geq 1}\sum_{i=1}^{n}(-1)^{n+i+1}\pi_{n*}\left[\widehat{a}_{1,n}\wedge\cdots\wedge\widehat{F_{A+a}}_{i,n}\wedge\cdots\wedge\widehat{a}_{n,n}\right]\mathrm{H}(A)|_0^1. \tag{2.3.9}$$

For $1\leq i\leq n, n\geq 1$, we write the projection π_n as the composition of three projections as follows $\pi_n=\pi_1\circ\pi_{1,n-i+1}\circ\pi_{i,i,n}$, where

$$\begin{aligned}\pi_{i,i,n}:\ \mathrm{L}M\times\triangle_n&\to\mathrm{L}M\times\triangle_{n-i+1};\\ (\gamma;s_1,\dots,s_n)&\mapsto(\gamma,s_i,\dots,s_n);\\ \pi_{1,n-i+1}:\ \mathrm{L}M\times\triangle_{n-i+1}&\to\mathrm{L}M\times[0;1];\\ (\gamma;s_1,\dots,s_{n-i+1})&\mapsto(\gamma,s_1);\\ \pi_1:\ \mathrm{L}M\times[0;1]&\to\mathrm{L}M;\\ (\gamma;s)&\mapsto\gamma.\end{aligned}$$

We notice the useful identities:

$$\pi_{j,n} = \pi_{j-i+1,n+i-1} \circ \pi_{i,i,n}, \quad i \le j \quad \text{and} \quad \pi_{j,n} = \pi_{k,i} \circ \pi_{1,i,n}, \quad j \le i-1,$$

where $\pi_{1,i,n}(\gamma; t_1, \ldots, t_m) : \ = (\gamma; t_1, \ldots, t_i)$. With the help of these identities, we may rewrite the n-th term in (2.3.9) as follows:

$$\pi_{n*}\left[\widehat{a}_{1,n} \wedge \cdots \wedge \widehat{F_{A+a}}_{i,n} \wedge \cdots \wedge \widehat{a}_{n,n}\right] =$$
$$= \pi_{n*}\left[\pi^*_{1,i,n}\left(\widehat{a}_{1,i} \wedge \cdots \wedge \widehat{a}_{i-1,i}\right) \wedge \pi^*_{i,n,n}\left(\widehat{F_{A+a}}_{1,n-i+1} \wedge \widehat{a}_{2,n-i+1} \wedge \cdots\right)\right].$$

We use the following identity

$$\pi_{n*} = (-1)^{n-i} \pi_{1*} \circ \pi_{1,n-i+1*} \circ \pi_{i,n,n*}. \tag{2.3.10}$$

We notice the appearance of the sign $(-1)^{n-i}$; this follows from our earlier conventions on push-forwards and from the choice of the standard orientation $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$.

Therefore, using the first identity in (2.2.2), we may write

$$\pi_{n*}\left[\widehat{a}_{1,n} \wedge \cdots \wedge \widehat{F_{A+a}}_{i,n} \wedge \cdots \wedge \widehat{a}_{n,n}\right] =$$
$$= (-1)^{n-i} \pi_{1*} \circ \pi_{1,n-i+1*} \left\{ \pi_{i,n,n*}\left[\pi^*_{1,i,n}\left(\widehat{a}_{1,i} \wedge \cdots \wedge \widehat{a}_{i-1,i}\right)\right] \wedge \right.$$
$$\left. \wedge \left(\widehat{F_{A+a}}_{1,n-i+1} \wedge \widehat{a}_{2,n-i+1} \wedge \cdots\right) \right\}.$$

At this point, it is useful to notice the following commutative diagram

$$\begin{array}{ccc}
\mathrm{L}M \times \triangle_n & \xrightarrow{\pi_{1,i,n}} & \mathrm{L}M \times \triangle_i \\
{\scriptstyle \pi_{i,n,n}} \big\downarrow & & \big\downarrow {\scriptstyle \pi_{i,i}} \\
\mathrm{L}M \times \triangle_{n-i+1} & \xrightarrow{\pi_{1,n-i+1}} & \mathrm{L}M \times [0,1].
\end{array}$$

This means that $\pi_{1,n-i+1} \circ \pi_{i,n,n} = \pi_{i,i} \circ \pi_{1,i,n}$. This permits also to apply Lemma 2.2.1 to $\pi_{i,n,n*} \circ \pi^*_{1,i,n}$, transforming it into $\pi^*_{1,n-i+1} \circ \pi_{i,i*}$. Hence, we get

$$(-1)^{n-i} \pi_{1*} \circ \pi_{1,n-i+1*} \left\{ \pi_{i,n,n*}\left[\pi^*_{1,i,n}\left(\widehat{a}_{1,i} \wedge \cdots \wedge \widehat{a}_{i-1,i}\right)\right] \wedge \right.$$
$$\left. \wedge \left(\widehat{F_{A+a}}_{1,n-i+1} \wedge \widehat{a}_{2,n-i+1} \wedge \cdots\right) \right\} =$$
$$= (-1)^{n-i} \pi_{1*} \circ \pi_{1,n-i+1*} \left\{ \pi^*_{1,n-i+1}\left[\pi_{i,i*}\left(\widehat{a}_{1,i} \wedge \cdots \widehat{a}_{i-1,i}\right) \wedge \widehat{F_{A+a}}\right] \wedge \right.$$
$$\left. \wedge \left(\widehat{a}_{2,n-i+1} \wedge \cdots \wedge \widehat{a}_{n-i+1,n-i+1}\right) \right\} =$$

$$= (-1)^{n-i}\pi_{1*}\Bigg[\pi_{i,i*}\left(\widehat{a}_{1,i}\wedge\cdots\widehat{a}_{i-1,i}\right)\wedge\widehat{F_{A+a}}\wedge$$

$$\wedge\,\pi_{1,n-i+1*}\left(\widehat{a}_{2,n-i+1}\wedge\cdots\wedge\widehat{a}_{n-i+1,n-i+1}\right)\Bigg].$$

The last equality is again a consequence of the first identity in (2.2.2).

We may finally write (2.3.9) as

$$-\sum_{n\geq 1}\sum_{i=1}^{n}\pi_{1*}\Bigg[\pi_{i,i*}\left(\widehat{a}_{1,i}\wedge\cdots\widehat{a}_{i-1,i}\right)\wedge\widehat{F_{A+a}}\wedge$$

$$\wedge\,\pi_{1,n-i+1*}\left(\widehat{a}_{2,n-i+1}\wedge\cdots\wedge\widehat{a}_{n-i+1,n-i+1}\right)\Bigg]\,\mathrm{H}(A)|_0^1,$$

and this gives clearly the claim. ☐

Remark 2.3.5. Similar identities can be proved for the two other cases in which we consider parallel transports as functions on $\mathrm{L}M\times[0,1]$, resp. on $\mathrm{L}M\times\triangle_2$. We obtain for the first case the result

$$\mathrm{d}_{\pi^*\,\mathrm{ev}(0)^*A}\,\mathrm{H}(A+a)|_0^\bullet = -\pi_1^*\,\mathrm{ev}(0)^*a\wedge\mathrm{H}(A+a)|_0^\bullet+\mathrm{H}(A+a)|_0^\bullet\wedge\widehat{a}-$$
$$-\,\pi_{2,2*}\left[\pi_{1,2}^*\left(\mathrm{H}(A+a)|_0^\bullet\wedge\widehat{F_{A+a}}\right)\wedge\mathrm{H}(A+a)|_\bullet^\bullet\right].$$

An analogous identity holds for the holonomy as a function depending on the final point:

$$\mathrm{d}_{\pi^*\,\mathrm{ev}(0)^*A}\,\mathrm{H}(A+a)|_\bullet^1 = -\widehat{a}\wedge\mathrm{H}(A+a)|_\bullet^1+\mathrm{H}(A+a)|_\bullet^1\wedge\pi_1^*\left[\mathrm{H}(A)|_0^1\;\mathrm{ev}(0)^*a\;\left(\mathrm{H}(A)|_0^1\right)^{-1}\right]-$$
$$-\,\pi_{1,2*}\left[\mathrm{H}(A+a)|_\bullet^\bullet\wedge\pi_{2,2}^*\left(\widehat{F_{A+a}}\wedge\mathrm{H}(A+a)|_\bullet^1\right)\right].$$

For the second case, we get

$$\mathrm{d}_{\pi_2^*\,\mathrm{ev}(0)^*A}\,\mathrm{H}(A+a)|_\bullet^\bullet = -\pi_{1,2}^*\widehat{a}\wedge\mathrm{H}(A+a)|_\bullet^\bullet+\mathrm{H}(A+a)|_\bullet^\bullet\wedge\pi_{2,2}^*\widehat{a}-$$
$$-\,\pi_{\widehat{2},3*}\left[\pi_{\widehat{3},3}^*\,\mathrm{H}(A+a)|_\bullet^\bullet\wedge\pi_{2,3}^*\widehat{F_{A+a}}\wedge\pi_{\widehat{1},3}^*\,\mathrm{H}(A+a)|_\bullet^\bullet\right],$$

where $\pi_{\widehat{j},3}:\mathrm{L}M\times\triangle_3\to\mathrm{L}M\times\triangle_2$ forgets the j-th point of the 3-simplex. We have preferred to adopt the notation

$$\int_{t_1\leq s\leq t_2}\mathrm{H}(A+a)|_{t_1}^s\wedge\widehat{F_{A+a}}_s\wedge\mathrm{H}(A+a)|_s^{t_2}$$

for the third term in the three above expressions, where $t_1\leq t_2$ can be fixed or can be understood as variables, given the case in the specific context.

2.3.1 Some remarks on the n-simplex $\triangle_n$

The fundamental ingredient in the construction of $\mathrm{H}(A+a)|_0^1$ is the n-simplex $\triangle_n$, which is defined as the set of n-tuples $(t_1,\dots,t_n)$ with $0 \le t_1 \le \dots \le t_n \le 1$.

In order to compute the push-forward w.r.t. the forgetful projection π_n (and also w.r.t. $\pi_{n+1,n+1}$ or $\pi_{1,n+1}$), we need first specify an orientation of the simplex. This can be done by choosing an orientation form, which is the restriction to $\triangle_n$ of the standard orientation form on $\mathbb{R}^n$, i.e.

$$\mathrm{dvol}_{\triangle_n} :\, = \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n. \tag{2.3.11}$$

Thus, we may then integrate n-forms on $\triangle_n$ by the rule: if $\omega = f\mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n$ is a form of highest degree on $\triangle_n$, where f is a smooth function on $\triangle_n$, its integral is given by

$$\int_{\triangle_n} \omega :\, = \int_0^1 \cdots \int_0^{t_2} f(t_1,\dots,t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

Clearly, by Fubini's theorem, we may also write e.g.

$$\int_{\triangle_n} \omega = \int_0^1 \int_{t_i}^1 \cdots \int_{t_i}^{t_{i+2}} \int_0^{t_i} \cdots \int_0^{t_2} f(t_1,\dots,t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_{i-1} \mathrm{d}t_{i+1} \cdots \mathrm{d}t_n \mathrm{d}t_i,$$

for any $1 \le i \le n$.

Now that we have specified a particular orientation of $\triangle_n$, we may define the push-forward w.r.t. the projection π_n from $\mathrm{L}M \times \triangle_n$ onto $\mathrm{L}M$ as

$$\pi_{n*}\left[f(\gamma; t_1,\dots)\, \mathrm{d}t_{i_1} \wedge \dots \wedge \mathrm{d}t_{i_p} \wedge \pi_n^*(\omega)\right] :\, = \left[\int_{\triangle_n} f(\gamma; t_1,\dots)\, \mathrm{dvol}_{\triangle_n}\right] \wedge \omega, \tag{2.3.12}$$

if $\{i_1,\dots,i_p\} = \{1,\dots,n\}$ or if it is a permutation thereof (clearly, the sign of the permutation must be included); otherwise, we set the push-forward to be 0. In equation (2.3.12), f is a smooth function on $\mathrm{L} \times \triangle_n$, while ω is a smooth form on $\mathrm{L}M$. One can verify directly that equation (2.3.12) is a special case of our definition of push-forward.

Similarly, the orientation form (2.3.11) specifies in an obvious way an orientation form on the n-simplex bounded from below by $1 \le s \le 1$ and from above by $0 \le s \le t \le 1$ (we denote such a simplex by $\triangle_{n,s,t}$)

$$\int_{\triangle_{n,s,t}} f(t_1,\dots)\, \mathrm{dvol}_{n,s,t} :\, = \int_s^t \cdots \int_s^{t_2} f(t_1,\dots,t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

Fubini's theorem may be applied to this situation accordingly.

Such an orientation choice, along with the first two identities in (2.2.2), gives obvious formulae for the push-forwards w.r.t. $\pi_{n+1,n+1}(\gamma; t_1,\dots,t_{n+1}) = (\gamma; t_{n+1})$, w.r.t. $\pi_{1,n+1}(\gamma; t_1,\dots,t_{n+1}) = (\gamma, t_1)$, w.r.t. $\pi_{i,n,n}(\gamma; t_1,\dots,t_n) = (\gamma; t_i,\dots,t_n)$ and w.r.t. $\pi_{1,i,n}(\gamma; t_1,\dots,t_n) = (\gamma; t_1,\dots,t_i)$, for $1 \le n$ and $1 \le i \le n$.

With all conventions in mind and using the first and the second property in (2.2.2), we may explicitly prove equation (2.3.10). In fact, by the definition of the push-forward

π_{n*} and $\pi_{1*} \circ \pi_{1,n-i+1*} \circ \pi_{i,n,n*}$, we need only consider a form on $\mathrm{L}M \times \triangle_n$ of the shape

$$f(\gamma; t_1, \dots)\, \mathrm{dvol}_n \wedge \pi_n^*(\omega),$$

where the notations are as in equation (2.3.12). In other cases, both push-forwards are 0 by definition, and equation (2.3.10) is a trivial consequence. We notice the identity $\pi_n = \pi_1 \circ \pi_{1,n-i+1} \circ \pi_{i,n,n}$; hence we may rewrite the above form as

$$f(\gamma; t_1, \dots)\, \mathrm{dvol}_n \wedge \pi_n^*(\omega) = f(\gamma; t_1, \dots)\, \mathrm{dvol}_n \wedge \pi_{i,n,n}^* \left\{\pi_{1,n-i+1}^* \left[\pi_1^*(\omega)\right]\right\}.$$

Therefore, we get by Fubini's Theorem

$$\begin{aligned}
&\pi_{1*} \circ \pi_{1,n-i+1*} \circ \pi_{i,n,n*} \left[f(\gamma; t_1, \dots)\, \mathrm{dvol}_n \wedge \pi_n^*(\omega)\right] = \\
&= \pi_{1*} \circ \pi_{1,n-i+1*} \circ \pi_{i,n,n*} \left[f(\gamma; t_1, \dots)\, \mathrm{dvol}_n \wedge \pi_{i,n,n}^* \left\{\pi_{1,n-i+1}^* \left[\pi_1^*(\omega)\right]\right\}\right] = \\
&= (-1)^{n-i} \pi_{1*} \circ \pi_{1,n-i+1*} \circ \pi_{i,n,n*} \left[f(\gamma; t_1, \dots)\, \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_{i-1} \wedge \right. \\
&\qquad\qquad \mathrm{d}t_{i+1} \wedge \cdots \wedge \mathrm{d}t_n \wedge \mathrm{d}t_i \wedge \qquad \wedge \pi \\
&= (-1)^{n-i} \left[\int_{\triangle_n} f(\gamma; t_1, \dots)\, \mathrm{dvol}_{\triangle_n}\right] \wedge \omega.
\end{aligned}$$

Hence, equation (2.3.10) follows directly from these computations.

A last word about the orientation of the boundary faces of the n-simplex $\triangle_n$ and Formula (2.3.4).

The boundary $\partial\triangle_n$ of the simplex can be decomposed into a union of $n+1$ different sets as follows:

$$\partial\triangle_n = \bigcup_{\alpha=0}^{n} \left(\partial\triangle_n\right)_\alpha,$$

where

$$\begin{aligned}
\left(\partial\triangle_n\right)_0 &:= \left\{(t_1, \dots, t_n) \in \triangle_n : t_1 = 0\right\}; \\
\left(\partial\triangle_n\right)_\alpha &:= \left\{(t_1, \dots, t_n) \in \triangle_n : t_\alpha = t_{\alpha+1}\right\}, \quad 1 \le \alpha \le n-1; \\
\left(\partial\triangle_n\right)_n &:= \left\{(t_1, \dots, t_n) \in \triangle_n : t_n = 1\right\}.
\end{aligned}$$

It is clear that any $\left(\partial\triangle_n\right)_\alpha$ is isomorphic to the $n-1$-simplex $\triangle_{n-1}$; the orientation form on $\triangle_{n-1}$ is dvol_{n-1}.

Clearly, the 0-th boundary face $\left(\partial\triangle_n\right)_0$ has $-\frac{\partial}{\partial t_1}$ as outward-directed (normalized) vector field (since $t_i \ge 0$, for all $1 \le i \le n$). Since the orientation form of the boundary ∂M of a smooth, oriented manifold M is induced from the orientation form of M by the rule

$$\mathrm{dvol}_{\partial M} := \iota_v \mathrm{dvol}_M,$$

where ι_v denotes the contraction of the orientation form of M by a given vector-field v, which has to be outward-directed from ∂M, it follows immediately that the 0-th boundary face of $\triangle_n$ has orientation form $-\mathrm{dvol}_{n-1}$, hence we say that it has orientation -1. In a similar way, it can be proved that any face $\left(\partial\triangle_n\right)_\alpha$ has orientation form $(-1)^{\alpha+1}\mathrm{dvol}_{n-1}$, hence the orientation of the $n+1$ faces of $\triangle_n$ is given by the rule

$$\mathrm{or}\left(\partial\triangle_n\right)_\alpha = (-1)^{\alpha+1}.$$

2.4 Compactification of configuration spaces

In this section we discuss configuration spaces and their compactification á la Fulton–MacPherson–Axelrod–Singer (FMcPAS for short). Standard references for this subject are [30], [3] and [12]. Other (equivalent) compactifications are also proposed in [41] and [47].

2.4.1 The FMcPAS-compactification of the configuration space of n points in a compact manifold M

First of all, we need to recall the notion of configuration space of n points in a compact, m-dimensional manifold, which we will denote by M.

Definition 2.4.1. The *open configuration space of n points in M*, which is usually denoted by $C_n^0(M)$, is defined as

$$C_n^0(M) \colon = \{(x_1, \ldots, x_n) \in M^n \colon i \neq j \Rightarrow x_i \neq x_j, \ \forall i, j\}. \tag{2.4.1}$$

In order to define a good compactification of $C_n^0(M)$, we have to recall briefly the (differential-geometric) definition of the blow-up of a manifold M along a given submanifold N: the *geometric blow-up of M along N*, which we denote by $\mathrm{Bl}\,(M, N)$, is obtained, roughly speaking, by replacing the submanifold $N \subset M$ by its unit normal bundle in $\mathrm{T}M$. The construction of the normal bundle involves usually the choice of a Riemannian metric on M. Alternatively, if we do not want to introduce an explicit Riemannian metric on M, we may view the unit normal bundle of N as the quotient of the tangent bundle of M by the tangent bundle of N (viewed as a subbundle of the restriction of $\mathrm{T}M$ on N) and by the group of dilations $\mathbb{R}^+$, which acts on the left simply by multiplication. We prefer to view the normal bundle in the second setting.

If we consider an ordered subset $S = \left\{i_1, \ldots, i_{|S|}\right\}$ of $\{1, \ldots, n\}$, with cardinality $|S|$ bigger than 1, we denote by Δ_S the principal diagonal in $M^{|S|}$, defined explicitly as follows

$$\Delta_S \colon = \left\{\left(x_{i_1}, \ldots, x_{i_{|S|}}\right) \in M^{|S|} \colon x_i = x_j, \quad \forall i, j \in S\right\}.$$

Clearly, there is a smooth imbedding of the open configuration space $C_n^0(M)$ into M^n. Recalling the above definition of blow-up, there is a smooth imbedding of $C_n^0(M)$ into $\mathrm{Bl}\left(M^{|S|}, \Delta_S\right)$, for any subset of $\{1, \ldots, n\}$ of cardinality bigger than 1, simply given by projecting from $C_n^0(M)$ onto $M^{|S|}$. Hence, we may consider the smooth imbedding

$$C_n^0(M) \hookrightarrow M^n \times \prod_{\substack{S \subseteq \{1, \ldots, n\} \\ |S| \geq 2}} \mathrm{Bl}\left(M^{|S|}, \Delta_S\right). \tag{2.4.2}$$

Definition 2.4.2. The *compactified configuration space of n points*, denoted by $C_n(M)$, is defined as the closure of $C_n^0(M)$ in $M^n \times \prod_{\substack{S \subset \{1, \ldots, n\} \\ |S| \geq 2}} \mathrm{Bl}\left(M^{|S|}, \Delta_S\right)$.

The presence of the blow-ups makes $C_n(M)$ into a manifold with corners (locally modeled on $\mathbb{R}^m \times \mathbb{R}_+^n$, for some m and n). An important feature of $C_n(M)$ is the combinatoric of the boundaries. The presence of the blow-ups makes also $C_n(M)$ into a stratified space: namely, if we consider an ordered subset $S = \left\{i_1, \ldots, i_{|S|}\right\} \subset \{1, \ldots, n\}$ labeling the points in M collapsing together, we have to consider classes of $2 \leq k \leq |S|$-dimensional vectors modulo global translations and scalings, with at least two distinct components, corresponding to elements in the unit normal bundle of the diagonal labeled by S. When all components of the equivalence class of the $|S|$-dimensional vector (a normal vector to Δ_S) over the diagonal Δ_S are all distinct, all directions of collapse to the diagonal Δ_S can be distinguished. When two or more components of a unit normal vector over the diagonal Δ_S are equal, we have to "magnify" in order to distinguish the directions of collapse of such points; the magnification process is taken into account by the presence of other equivalence classes of vectors with $2 \leq k < |S|$-components (i.e. unit normal vectors on the diagonals of the fibers of the unit normal bundle of the diagonal Δ_S), with at least two of them distinct. The components of such equivalence classes are labeled by the subset of S, corresponding to the nondistinct components of the given equivalence classes. If these equivalence classes also have nondistinct components, we have to resort to other equivalence classes with less components, until the magnification process comes to an end, i.e. when we arrive to a situation when we can distinguish all directions of collapse.

The stratification of $C_n(M)$ can be visualized as follows: a boundary face is indicated by a parenthesization of $\{1, \ldots, n\}$, such that the parenthesization labels only nested subsets, i.e. any two subsets are either disjoint or one is contained in the other, and any parenthesis contains at least two elements. E.g. if $n = 5$, a possible boundary face is

$$\mathcal{B} = \{1\,\{2\,\{3, 4\}\}\,5\}\,.$$

We explain its significance: the first internal parentheses indicate that the points labeled by 2, 3 and 4 collapse together, and the innermost parentheses indicate that we have to "magnify" on the coordinates 3 and 4, in order to distinguish the directions of their collapse. More generally, each internal parentheses indicate that the point therein collapse together; parentheses within parentheses mean that we have to magnify on the points in the innermost parentheses. The codimension of a stratum, indicated by a chosen parenthesization, is equal to the number of parentheses (e.g., if we take again the above example, the codimension of the corresponding stratum is 2). For our purposes, we need only consider the boundary faces of codimension 1

We quote now some important facts concerning the properties $C_n(M)$.

First of all, there are natural projections

$$\begin{aligned} \pi_{S,n}\colon C_n^0(M) &\to C_S^0(M), \\ (x_1, \ldots, x_n) &\mapsto (x_{i_1}, \ldots, x_{i_{|S|}}), \end{aligned} \tag{2.4.3}$$

where $S = \left\{i_1, \ldots, i_{|S|}\right\} \subset \{1, \ldots, n\}$.

Theorem 2.4.3. *For any ordered subset $S = \left\{i_1, \ldots, i_{|S|}\right\} \subset \{1, \ldots, n\}$, the map $\pi_{S,n}$ defined in (2.4.3) lifts to a smooth map between $C_n(M)$ and $C_S(M)$.*

Another important feature of $C_n(M)$ concerns imbeddings. If we consider a smooth imbedding f between two compact manifolds M and N, it extends obviously to a smooth map, denoted by $C_p^0(f)$, between the corresponding open configuration spaces $C_p^0(M)$ and $C_p^0(N)$. The injectivity of the tangential map of f at any point of M implies the following

Theorem 2.4.4. *The map $C_p^0(f)$ lifts to a smooth map between $C_p(M)$ and $C_p(N)$.*

2.4.2 The compactification of the configuration space $C_n^0(\mathbb{R}^m)$

We discuss here the compactification of configuration spaces of $M = \mathbb{R}^m$. As $\mathbb{R}^m$ is not compact, the above construction does not work. We recall but that $\mathbb{R}^m$ may be compactified by the addition of exactly one point (the point "at infinity"). The one-point-compactification of $\mathbb{R}^m$ is diffeomorphic to the m-sphere S^m, which is compact. We then consider a base point p on S^m, e.g. the north pole; so, $S^m \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^m$ (sometimes, the point p will be denoted by ∞).

Definition 2.4.5. The compactification of $C_n^0(\mathbb{R}^m)$, which we denote by $C_n(\mathbb{R}^m)$, is defined by means of the commutative diagram

$$\begin{array}{ccc} C_n(\mathbb{R}^m) := \iota_p^*\left(C_{n+1}(S^m)\right) & \xrightarrow{\overline{\iota}_p} & C_{n+1}(S^m) \\ {\scriptstyle \pi_p}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{n+1,n+1}} \\ \{p\} & \xrightarrow{\iota_p} & S^m, \end{array} \tag{2.4.4}$$

where ι_p is the inclusion of $\{p\}$ into S^m.

The interior of $C_n(\mathbb{R}^m)$ consists, by construction, of the $n+1$-tuples in S^m, with distinct entries, where the last entry has to be p; the remaining n entries must be also different from p. Hence, we get $\overset{\circ}{C}_n(\mathbb{R}^m) = C_n^0(S^m \setminus \{p\}) \cong C_n^0(\mathbb{R}^m)$. Alternatively, the compactification $C_n(\mathbb{R}^m)$ is obtained by compactifying the open configuration space $C_n^0(S^m)$ along the same lines of Definition 2.4.2, but where we replace $a)$ all diagonals (not containing p) by their respective blow-ups and $b)$ all k-tuples of components all equal to p, labeled by subsets S of $\{1, \ldots, n\}$ of cardinality at least 1, also by their respective blow-ups. We notice that in the latter case, the blow up consists simply of $|S|$-tuples in $\mathrm{T}_p S^m$ such that the sum of their squares w.r.t. to a given norm on S^m equals 1. The combinatoric of the boundaries of $C_n(\mathbb{R}^m)$ is more complicated, since we have to take into account not only the boundary faces corresponding to the collapse of two or more points, but also the boundary faces "at infinity", where one or more points go to infinity.

The manifold $C_n(\mathbb{R}^m)$ enjoys many of the properties enjoyed by $C_n(M)$, with M compact: namely, forgetful projections, e.g. the maps $\pi_{S,n}$ from (2.4.3) from $C_n^0(\mathbb{R}^m)$ into $C_n^0(\mathbb{R}^{|S|})$, lift to smooth maps from $C_n(\mathbb{R}^m)$ into $C_n(\mathbb{R}^{|S|})$, and similarly elements of $\mathrm{Imb}_\infty(\mathbb{R}^{m-2}, \mathbb{R}^m)$ lift to smooth maps from $C_n(R^{m-2})$ into $C_n(\mathbb{R}^m)$.

We consider the open configuration space $C_2^0(\mathbb{R}^m)$. There is a smooth map φ_{12} from $C_2^0(\mathbb{R}^m)$ to S^{m-1}

$$\begin{aligned} \varphi_{12} \colon C_2^0(\mathbb{R}^m) &\to S^{m-1}, \\ (x_1, x_2) &\mapsto \frac{x_1 - x_2}{\|x_1 - x_2\|}, \end{aligned} \tag{2.4.5}$$

where in this case $\| \ \|$ denotes Euclidean norm in $\mathbb{R}^m$. The important relationship between $C_2(\mathbb{R}^m)$ and the map (2.4.5) is encoded (we refer to [12] for an explicit proof) in the following

Theorem 2.4.6. *The map φ_{12} lifts to a smooth map from $C_2(\mathbb{R}^m)$ into S^{m-1}.*

2.4.3 Including submanifolds

In this subsection, we want to discuss another interesting feature of FMcPAS compactification of configuration spaces, namely the possibility of including submanifolds. In fact, given a compact submanifold $S \overset{f}{\hookrightarrow} M$ of the compact manifold M, we may define the (open) configuration space of s points in S and t points in M as

$$C^0_{s,t}(M,S;f) \colon = \left\{ (x_1,\ldots,x_s;y_1,\ldots,y_t) \in S^s \times M^t \colon \begin{array}{ll} x_i \neq x_j, & i \neq j; \\ y_k \neq y_l, & k \neq l; \\ f(x_m) \neq y_n, & \forall m,n \end{array} \right\}. \tag{2.4.6}$$

This definition may be extended to a family of imbeddings $S \hookrightarrow M$, viewing such an extension as a fibration over $\mathrm{Imb}(S,M)$, with typical fiber at $f \in \mathrm{Imb}(S,M)$ the space $C_{s,t}(M,S;f)$.

Our aim is to define a suitable compactification of this space, denoted by $C_{s,t}(M,S)$, which has to be also a fibration over the space of imbeddings from S into M.

Definition 2.4.7. The space $C_{s,t}(M,S)$ is defined by means of the commutative diagram

$$\begin{array}{ccc} C_{s,t}(M,S) \colon = \mathrm{ev}_s^*\,(C_{s+t}(M)) & \xrightarrow{\overline{\mathrm{ev}}_s} & C_{s+t}(M) \\ {\scriptstyle \overline{\pi}_s}\big\downarrow & & \big\downarrow{\scriptstyle \pi_s} \\ \mathrm{Imb}(S,M) \times C_s(S) & \xrightarrow{\mathrm{ev}_s} & C_s(M), \end{array} \tag{2.4.7}$$

where the map π_s is the (smooth lift to $C_{s+t}(M)$ of the) projection from $C^0_{s+t}(M)$ onto the first s factors, while the map ev_s is the (smooth lift to $\mathrm{Imb}(S,M) \times C_s(S)$ of the) usual evaluation map, extended to open configuration spaces.

It turns out that this construction does the job: namely, the space $C_{s,t}(M,S)$ is a smooth manifold with corners, and the maps $\overline{\mathrm{ev}}_s$ and $\overline{\pi}_s$ are smooth (we refer to [12] for more details).

We have considered only the case of imbeddings of a compact submanifold in a compact manifold; we will consider smooth imbeddings of a compact manifold in $\mathbb{R}^m$, for some m. The compact configuration space $C_{s,t}(\mathbb{R}^m,S)$ is defined by means of $C_{s+t}(\mathbb{R}^m)$ and the commutative square given in (2.4.7), replacing $\mathrm{Imb}\,SM$ by $\mathrm{Imb}(S,\mathbb{R}^m)$.

2.4.4 Explicit realizations of boundary faces of compactified configuration spaces

The group $\mathfrak{S}_n$ of permutations of n factors operates freely on the compactification $C_n(M)$. If we consider the boundary face of codimension 1 describing the collapse of $2 \leq |S| \leq n$ points, we may always put ourselves in the case when the collapsing points are labeled by the indices $\{1, \ldots, |S|\}$, by permuting the factors. We therefore need only describe this special boundary face; for simplicity, we write $k = |S|$.

We consider the action of the group G, the semidirect product of the group of dilations $\mathbb{R}^+$ and $\mathbb{R}^m$ (viewed as the abelian group of global translations), viewed as a representation of $\mathbb{R}^+$, on the open configuration space $C_k^0(\mathbb{R}^m)$:

$$((x_1, \ldots, x_k) \, ; (\lambda, v)) \mapsto (\lambda x_1 + v, \ldots, \lambda x_k + v) \, , \; x_i \neq x_j, \quad i \neq j, \lambda \in \mathbb{R}^+, v \in \mathbb{R}^m.$$

(we notice that this G-action is free).

We briefly comment on the sense of the action of translations and dilations. The precise definition of the differential-geometric blow-up of a manifold M along a given submanifold $S \subset M$ involves the unit normal bundle of $S \subset M$. In the special case at hand of the diagonal $\Delta_k \subset M^k$, at a given point $x \in M$, seen as a point in the diagonal Δ_k, the unit normal bundle of $\Delta_k \subset M^k$ at x takes the explicit form, having chosen a Riemannian metric on M, which we denote by $\langle \ , \ \rangle$ (we denote by the same symbol its extension to M^k and its restriction to S):

$$S\left(\mathrm{N}_x \Delta_k\right) = \left\{ (x_1, \ldots, x_k) \in \bigoplus^k \mathrm{T}_x M \colon \sum_{i=1}^k x_i = 0, \quad \sum_{i=1}^k \|x_i\|^2 = 1 \right\}, \tag{2.4.8}$$

where $\| \ \|$ denotes the norm w.r.t. the Riemannian metric $\langle \ , \ \rangle$.

We see immediately that both conditions in (2.4.8) rule out the possibility that all vectors x_i are equal. Moreover, the second condition implies that $S\left(\mathrm{N}_x \Delta_k\right)$ may be seen as a compact submanifold of $\left(\mathbb{R}^m\right)^k$.

On the other hand, taking the complement of the principal diagonal in $\oplus^k \mathrm{T}_x M$, we may define on this space the sane action of $\mathbb{R}^+ \ltimes \mathrm{T}_x M$ as on $C_k^0(\mathbb{R}^m)$; this gives a metric-free characterization of the unit normal bundle of the principal diagonal in M^k. Clearly, in any equivalence class of such k-tuples there is one and only one representative, such that the sum of its components vanishes and the sum of the squares of the norms of its components equals 1.

It is also clear that if two or more vectors in the unit normal bundle at x of the diagonal Δ_k are equal, so are also the corresponding components in the quotient of the complement of the principal diagonal in $\oplus^k \mathrm{T}_x M$ modulo dilations and global translations.

We want to describe, for later purposes, which compactification of $C_k^0(\mathbb{R}^m) / G$ corresponds to the boundary face where k points in M collapse together. We imbed the subset $\widehat{C}_k^0(\mathbb{R}^m) : \ = C_k^0(\mathbb{R}^m) / G$ in the product

$$C_k^0(\mathbb{R}^m)/G \hookrightarrow \prod_{\substack{R \subseteq \{1, \ldots, k\} \\ |R| \geq 2}} S^{|R|m-1} \cap \mathcal{H}_R. \tag{2.4.9}$$

The imbedding is explicitly realized as follows: for any subset $R \subseteq \{1, \ldots, k\}$ of cardinality strictly bigger than 1, take the projection from $\widehat{C}_k^0(\mathbb{R}^m)$ onto $\widehat{C}_R^0(\mathbb{R}^m)$, given by projecting onto the components labeled by elements of R, and then map this

$|S|$-tuple (modulo global translations and scaling) to its unique representative such that

$$\sum_{i\in R} x_i = 0, \quad \sum_{i\in R} \|x_i\|^2 = 1,$$

where $\| \ \|$ denotes Euclidean norm in $\mathbb{R}^m$. This is the motivation of our notation: by $S^{|R|m-1} \cap \mathcal{H}_R$ we have denoted the set of $|R|$-tuples of vectors in $\mathbb{R}^m$, indexed by elements in R, such that the sum of the squares of their components in the Euclidean norm is 1 (whence we may view such an $|R|$-tuple as an element of the sphere $S^{|R|m-1}$) and such that the sum of their components vanishes (whence we may view such a $|R|$-tuple as belonging to the intersection of m hyperplanes in $\mathbb{R}^{|}R|$). (We notice that in the compactification of $\widehat{C}^0_R(\mathbb{R}^m)$, there is no need to compactify along the principal diagonal, because such a configuration is automatically ruled out by the isomorphism between $\widehat{C}^0_k(\mathbb{R}^m)$ and $S^{km-1} \cap \mathcal{H}_R \cap C^0_k(\mathbb{R}^m)$.)

We then define $\widehat{C}_k(\mathbb{R}^m)$ as the closure of $C^0_k(\mathbb{R}^m)\,/G$ imbedded as in (2.4.9). Because of all the blow-ups, $\widehat{C}_k(\mathbb{R}^m)$ is also a manifold with corners. E.g. , $\widehat{C}^0_2(\mathbb{R}^m) = S^{m-1}$, since any equivalence class in $\widehat{C}^0_2(\mathbb{R}^m)$ has a unique representative with vanishing second entry and normalized first entry. It follows that $\widehat{C}_2(\mathbb{R}^m) = S^{m-1}$.

For a given compact manifold M, the bundle $\widehat{C}_k(\mathrm{T}M)$ is defined fiberwise by $\widehat{C}_k(\mathrm{T}M)_x : = \widehat{C}_k(\mathrm{T}_x M) \cong \widehat{C}_k(\mathbb{R}^m)$. In the simplest case $k = 2$, we get $C_2(M) = \mathrm{Bl}\left(M^2, \Delta\right)$, and the boundary is exactly the sphere bundle of M. More generally, the boundary face corresponding to the collapse of the first k points in $C_n(M)$ is the pulled-back bundle over $C_{n-k+1}(M)$ of $\widehat{C}_k(\mathrm{T}M)$ w.r.t. to the projection from $C_{n-k+1}(M)$ onto the first point.

For the case $M = \mathbb{R}^m$, there are additional codimension-1 faces to be considered, namely those faces, where the first l points escape to infinity. The description of such a face involves again the spaces $\widehat{C}_l(\mathbb{R}^m)$: in fact, it is the trivial fibration $C_{n-l}(\mathbb{R}^m) \times \widehat{C}_{l+1}(\mathbb{R}^m)$.

We consider $C_{s,t}(\mathbb{R}^m, S)$, where S is a compact submanifold of $\mathbb{R}^m$. By the explicit construction of $C_{s,t}(\mathbb{R}^m, S)$, it follows that the codimension 1 boundary faces are of four distinct types:

a) $2 \le p \le s$ points in S collapse together;

b) $2 \le r \le t$ points in $\mathbb{R}^m$ collapse together;

c) $1 \le p \le s$ points in S and $1 \le r \le t$ points in $\mathbb{R}^m$ collapse together in S;

d) $1 \le q \le s$ points escape together to infinity.

We consider the action of $\mathfrak{S}_s \times \mathfrak{S}_t$ on $C_{s,t}(\mathbb{R}^m, S)$, where an element of $\mathfrak{S}_s \times \mathfrak{S}_t$ is a couple of permutations, the first one in $\mathfrak{S}_s$ and the second one in $\mathfrak{S}_t$, with componentwise multiplication. By the action of $\mathfrak{S}_s \times \mathfrak{S}_t$ we may put ourselves in the situation where $a)$ the first p points in S collapse together, $b)$ the first r points in $\mathbb{R}^m$ collapse together $c)$ the first p points in S and the first r points in $\mathbb{R}^m$ collapse together in S and $d)$ the first q points in $\mathbb{R}^m$ escape to infinity.

The description of the first, second and fourth special boundary face is easy: the face $a)$ is a fibration over $C_{s-p+1,t}(\mathbb{R}^m, S)$, obtained by pulling back the bundle $\widehat{C}_p(\mathrm{T}S)$ w.r.t. the projection onto the first coordinate in S.

The face $b)$ is the trivial fibration $C_{s,t-r+1}(\mathbb{R}^m, S) \times \widehat{C}_r(\mathbb{R}^m)$.

The face $d)$ is also a trivial fibration, namely $C_{s,t-q}(\mathbb{R}^m, S) \times \widehat{C}_{q+1}(\mathbb{R}^m)$. To understand this, we have to recall the characterization of $C_n(\mathbb{R}^m)$: we replace any $|S|$-tuple consisting of components all equal to p ($p \in S^m$ corresponds to the point at infinity of $\mathbb{R}^m$), where S is a subset of $\{1, \dots, n\}$ of cardinality at least 1, by its blow-up, which is in this case simply the sphere in $\oplus_S \mathrm{T}_p S^m$, in the sense of FMcPAS. Hence, the interior of the fiber of the face at infinity, where points labeled by S escape to infinity, is given by (assuming for simplicity that $S = \{1, \dots, q\}$, $q = |S|$)

$$(w_1, \dots, w_q) \in \oplus^q \mathrm{T}_p S^m \cong (\mathbb{R}^m)^q \quad \text{such that} \quad \sum_{i=1}^{q} \|w_i\|^2 = 1,$$

and the w_i are all nonzero and distinct. It is immediate to verify that in any class of distinct vectors of $q+1$ vectors in $\mathrm{T}_p S^m$ modulo global translations and dilations (i.e. an element of $\widehat{C}^0_{q+1}(\mathbb{R}^m)$) there is one and only one q-tuple of nonzero and distinct vectors in $\mathrm{T}_p S^m$ such that the sum of their squares equals 1 (e.g. by setting the $q+1$-th component to 0). The compactification of the faces of infinity is then clearly realized, by means of the preceding identification, via the same machinery leading to the compactification of the faces of type $b)$ and it is given by $\widehat{C}_{q+1}(\mathbb{R}^m)$; we only notice that, in this case, the possibility that all components w_i coincide, $i = 1, \dots, q$, is not ruled out automatically.

The face $c)$ is more complicated; we describe first its interior, and later we give a glimpse of its compactification in the spirit of the FMcPAS-compactification. We introduce the space $\mathcal{I}(\mathbb{R}^d, \mathbb{R}^m)$ of linear, injective maps from $\mathbb{R}^d$ to $\mathbb{R}^m$; here, d is the dimension of the imbedded submanifold $S \subset M$.

We then define the set

$$\mathcal{I}(\mathbb{R}^d, \mathbb{R}^m) \ltimes C^0_{p,r} := \left\{ (\alpha; x_1, \dots, x_p; y_1, \dots, y_r) \in \mathcal{I}(\mathbb{R}^d, \mathbb{R}^m) \times \mathbb{R}^d \times \mathbb{R}^m : \begin{array}{ll} x_i \neq x_j, & i \neq j \\ y_k \neq y_l, & k \neq l \\ \alpha(x_i) \neq y_j, & \forall i, j \end{array} \right\} \tag{2.4.10}$$

We define on the above set the following equivalence relation:

$$(\alpha; x_1, \dots; y_1, \dots) \simeq (\widetilde{\alpha}; \widetilde{x}_1, \dots; \widetilde{y}_1, \dots) \Longleftrightarrow$$
$$\Longleftrightarrow \begin{cases} \widetilde{\alpha} = \alpha, \\ \exists\, (\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^d : \begin{array}{l} \widetilde{x}_i = \lambda x_i + \xi, \quad 1 \leq i \leq p \\ \widetilde{y}_i = \lambda y_i + \alpha(\xi), \quad 1 \leq i \leq r. \end{array} \end{cases} \tag{2.4.11}$$

We denote by $\mathcal{I}(\mathbb{R}^d, \mathbb{R}^m) \ltimes \widehat{C}^0_{p,r}$ the quotient of $\mathcal{I}(\mathbb{R}^d, \mathbb{R}^m) \ltimes C^0_{p,r}$ w.r.t. the previous equivalence relation. For later purposes, we need to introduce the *Stiefel manifold* $V_{m,d}$, which is viewed in this context as the space of linear, *isometric* maps from $\mathbb{R}^d$ into $\mathbb{R}^m$, hence as a submanifold of $\mathcal{I}(\mathbb{R}^d, \mathbb{R}^m)$.

We want to explain briefly why we have considered the manifold $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$ and the previous equivalence relation. Given an imbedding f from S into $\mathbb{R}^m$, $C_{s,t}(M,S,f)$ is also realized by a sequence of blow-ups in the spirit of (2.4.2), where we have to consider blow-ups along diagonals in S, diagonals in M and submanifolds of the form $\Delta_{p,r}(M,S,f)$ of $S^p \times M^r$, for $1 \leq p \leq s$ and $1 \leq r \leq t$, defined by

$$\Delta_{p,r}(M,S,f)\colon = \big\{(x_1,\dots,x_p;y_1,\dots,y_r) \in \Delta_p^S \times \Delta_r^M \colon f(x_1) = y_1\big\}\,.$$

(We notice that the submanifold $\Delta_{p,r}(M,S,f)$ is diffeomorphic to S.)

For $x \in S$, the unit normal bundle of $\Delta_{p,r}(M,S,f)$ is realized explicitly as follows, upon picking two Riemannian metrics $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_S$ on M, resp. S:

$$S\left(\mathrm{N}_x\Delta_{p,r}(M,S,f)\right) = \left\{\begin{array}{l} (x_1,\dots;y_1,\dots) \in \bigoplus^p \mathrm{T}_xS \oplus \bigoplus^r \mathrm{T}_{f(x)}M\colon \\ \sum_{i=1}^p \langle\, x_i\,,\,v\,\rangle_S + \sum_{j=1}^r \langle\, y_j\,,\,\mathrm{T}_xf(v)\,\rangle_M = 0, \quad \forall v \in \mathrm{T}_xS, \\ \sum_{i=1}^p \|x_i\|_S^2 + \sum_{j=1}^r \|y_j\|_M^2 = 1 \end{array}\right\}. \tag{2.4.12}$$

Since we are considering an imbedding f, its tangent map at any point $x \in S$ defines an injective map from T_xS into $\mathrm{T}_{f(x)}M$. As for the sense of the equivalence relation w.r.t. global translations and scalings, it is not difficult to see that in any equivalence class in $C_{p,r}(M,S,f)_x$ w.r.t. the group action (2.4.11) there is exactly one representative belonging to $S\left(\mathrm{N}_x\Delta_{p,r}(M,S,f)\right)$, with all distinct internal and external components and such that the image through the tangent map T_xf of any internal component is distinct from any external component; hence, it gives a metric-free representation of the unit normal bundle of $\Delta_{p,r}(M,S,f)$. It is also clear that, at any point $x \in S$, the unit normal bundle of $\Delta_{p,r}(M,S,f)$ is a compact submanifold of $\left(\mathbb{R}^d\right)^p \times (\mathbb{R}^m)^r$; moreover, the possibility that all vectors in $\mathbb{R}^d \cong \mathrm{T}_xS$ are equal and that their image w.r.t. T_xf equal the vectors in $\mathbb{R}^m \cong \mathrm{T}_{f(x)}M$ is automatically ruled out.

There is a right action of the group $SO(d)$ on $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big) \ltimes C^0_{p,r}$:

$$(g;(\alpha;x_1,\dots;y_1,\dots)) \longmapsto \left(\alpha\circ g^{-1};g(x_1),\dots;y_1,\dots\right).$$

Clearly, the equivalence relation (2.4.11) is equivariant w.r.t. the action of $SO(d)$; therefore, the action of $SO(d)$ on $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)\ltimes C^0_{p,r}$ descends to an action on $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)\ltimes \widehat{C}^0_{p,r}$. S is an oriented Riemannian manifold with metric $\langle\ ,\ \rangle_S$, whence we may construct the principal bundle $\mathrm{SO}(S)$ of oriented orthonormal frames of $\mathrm{T}S$; it is an $SO(d)$-principal bundle over S. Finally, we consider the associated bundle $\mathrm{SO}(S)\times_{SO(d)} \left(\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big) \ltimes \widehat{C}^0_{p,r}\right)$ over S.

We construct a map Φ_α from $\mathrm{Imb}(S,\mathbb{R}^m) \times U_\alpha$ to $\left(\mathrm{SO}(S) \times \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)\right)|_{U_\alpha}$ in the following way:

$$\Phi_\alpha((x,f))\colon = \left(\phi^{-1}_{\alpha,x};\mathrm{T}_xf\circ\phi^{-1}_{\alpha,x}\right), \quad f \in \mathrm{Imb}(S,\mathbb{R}^m)\,, \quad x \in U_\alpha, \tag{2.4.13}$$

where U_α belongs to a trivializing covering of $\mathrm{T}S$, with trivialization ϕ_α constructed as follows:

$$\phi_\alpha(v) :\ = \Big(\pi(v); \Big(\langle\, v\,,\, e_1^\alpha(\pi(v))\,\rangle_{S,\pi(v)}\,, \ldots, \langle\, v\,,\, e_d^\alpha(\pi(v))\,\rangle_{S,\pi(v)}\Big)\Big)\,, \quad v \in \mathrm{T}S|_{U_\alpha}.$$

By $\{e_1^\alpha, \ldots, e_d^\alpha\}$ we have denoted an oriented orthonormal frame on U_α w.r.t. the metric $\langle\ \ ,\ \ \rangle_S$. Finally, $\phi_{\alpha,x}^{-1}$, for $x \in S$, is simply

$$\phi_{\alpha,x}^{-1}\,(\lambda_1, \ldots, \lambda_d) :\ = \phi_\alpha^{-1}\,(x, (\lambda_1, \ldots, \lambda_d))\,.$$

Clearly, $\mathrm{T}_x f \circ \phi_{\alpha,x}^{-1}$ defines an element of $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$.

Since the maps ϕ_α define a local trivialization of $\mathrm{T}S$, it holds

$$\phi_{\alpha,x}^{-1} = \phi_{\beta,x}^{-1} \circ g_{\beta\alpha}(x),$$

if $U_\alpha \cap U_\beta \neq$, and $g_{\beta\alpha}(x)$ denotes the transition function from the trivialization over U_β to that over U_α; $g_{\beta\alpha}(x)$ clearly belongs to $SO(d)$. Unless S is parallelizable, the map Φ_α is only locally defined. By the previous computations, it follows on a nonempty overlapping $U_\alpha \cap U_\beta$:

$$\Phi_\alpha(x, f) = \Phi_\beta(x, f) g_{\beta\alpha}(x),$$

where $SO(d)$ acts on the right on $\mathrm{SO}(S)$, and furthermore it acts on $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$ via

$$(g; \alpha) \mapsto \alpha \circ g^{-1}, \quad g \in SO(d), \quad \alpha \in \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)\,.$$

Hence, it is easy to see that the maps (2.4.13) glue together to a well-defined map Φ from $S \times \mathrm{Imb}(S,\mathbb{R}^m)$ to $\mathrm{SO}(S) \times_{SO(d)} \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$. If $S = \mathbb{R}^m$ (whence it follows that S is parallelizable), $\mathrm{SO}(S)$ is trivial, and we get $\mathrm{SO}(S) \times_{SO(m-2)} \mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) = S \times \mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$.

There is a natural fibration over $\mathrm{SO}(S) \times_{SO(d)} \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$, given by the obvious forgetful projection from $\mathrm{SO}(S) \times_{SO(d)} \Big(\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big) \ltimes \widehat{C}_{p,r}^0\Big)$ onto $\mathrm{SO}(S) \times_{SO(d)} \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$. We consider also the projection $\pi_{p,r}$ from $C_{s-p+1,t-r}(\mathbb{R}^n, S)$ onto $S \times \mathrm{Imb}(S,\mathbb{R}^m)$, onto the first point.

The interior of the face $c)$ of $C_{s,t}(\mathbb{R}^m, S)$, which we denote by $\widehat{C}_{p,r}^0$, is the fibration over $C_{s-p+1,t-r}(\mathbb{R}^m, S)$ defined by means of the commutative square:

$$\begin{array}{ccc} \widehat{C}_{p,s}^0 & \xrightarrow{\overline{\Phi\circ\pi_{p,r}}} & \mathrm{SO}(S) \times_{SO(d)} \Big(\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big) \ltimes \widehat{C}_{p,r}^0\Big) \\ {\scriptstyle \overline{\mathrm{pr}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ C_{s-p+1,t-r}(\mathbb{R}^m, S) & \xrightarrow{\Phi\circ\pi_{p,r}} & \mathrm{SO}(S) \times_{SO(d)} \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)\,. \end{array} \tag{2.4.14}$$

For later purposes, we briefly discuss which compactification of the space $V_{m,d} \ltimes \widehat{C}_{p,r}^0$ corresponds to the actual boundary of type $c)$. For a given $\alpha \in \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$, we define the space

$$\widehat{C}_{p,r}^0(\alpha) :\ = \pi^{-1}\left(\{\alpha\}\right), \tag{2.4.15}$$

where π denotes here the surjective projection from $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m) \ltimes \widehat{C}^0_{p,r}$ onto $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$.

The space $\widehat{C}^0_{p,r}(\alpha)$ can be imbedded in a product spheres as follows:

$$\begin{aligned}\widehat{C}^0_{p,r}(\alpha) \hookrightarrow & \prod_{\substack{S_1\subseteq\{1,\dots,p\}\\ 2\leq|S_1|\leq p}} \left(S^{|S_1|d-1}\cap\mathcal{H}_{S_1}\right)\times\\ &\times\prod_{\substack{S_2\subseteq\{1,\dots,r\}\\ 2\leq|S_2|\leq r}} \left(S^{|S_2|m-1}\cap\mathcal{H}_{S_2}\right)\times\\ &\times\prod_{\substack{S_1\cup S_2\subset\{1,\dots,p\}\cup\{1,\dots,r\}\\ 2\leq|S_1|+|S_2|\leq p+r}} S^{|S_1|d+|S_2|m-1}\cap\mathcal{H}_{S_1\cup S_2};\end{aligned}\tag{2.4.16}$$

where the imbedddings for $S=S_1$ or $S=S_2$ are as in (2.4.9). The case $S=S_1\cup S_2$ needs some explanations: first of all, we take the projection from $\widehat{C}^0_{p,r}$ onto $\widehat{C}^0_{S_1,S_2}$, simply given by projecting onto the components labeled by $S_1\cup S_2$, and then we identify such a $|S_1|+|S_2|$-tuple in $\left(\mathbb{R}^d\right)^{|S_1|}\times(\mathbb{R}^m)^{|S_2|}$ (modulo global translations and scaling) with its unique representative enjoying

$$\begin{aligned}&\sum_{i\in S_1}\|x_i\|^2+\sum_{j\in S_2}\|y_j\|^2=1,\\ &\sum_{i\in S_1}\langle\, x_i\,,\,v\,\rangle+\sum_{j\in S_2}\langle\, y_j\,,\,\alpha(v)\,\rangle=0,\quad \forall v\in\mathbb{R}^d.\end{aligned}$$

The former equation defines a sphere $S^{|S_1|d+|S_2|m-1}$ and the latter defines the intersection $\mathcal{H}_{S_1\cup S_2}$ of d hyperplanes in $\mathbb{R}^{|S_1|+|S_2|}$, and $\langle\ ,\ \rangle$, resp. $\|\ \|$, denotes Euclidean scalar product, resp. norm, in both $\mathbb{R}^d$ and $\mathbb{R}^m$.

The compactification of $\widehat{C}^0_{p,r}(\alpha)$, which we denote by $\widehat{C}_{p,r}(\alpha)$, is simply the closure of $\widehat{C}^0_{p,r}(\alpha)$, embedded as in (2.4.16). Therefore, the compactification of $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)\ltimes\widehat{C}^0_{p,r}$ is simply given by replacing $\widehat{C}^0_{p,r}(\alpha)$ at any $\alpha\in\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$ by $\widehat{C}_{p,r}(\alpha)$. It follows easily that to get the compactified face corresponding to the collapse of the first $1\leq p\leq s$ points in S and of the first $1\leq r\leq t$ points in $\mathbb{R}^m$ to S, we have to replace $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)\ltimes\widehat{C}^0_{p,r}$ in the commutative diagram (2.4.14) by $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)\ltimes\widehat{C}_{p,r}$.

2.4.5 Long knots in $\mathbb{R}^m$ and configuration spaces

For later purposes, it is convenient at this point to introduce a special class of imbeddings from $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$, the so-called *long knots in* $\mathbb{R}^m$. As we have already seen, the k-dimensional Euclidean space $\mathbb{R}^k$ may be compactified by the addition of one point, the so-called "point at infinity". We consider the space $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of long knots, namely a subspace of the space of imbeddings of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$ with the property of being base-point preserving (in the case at hand, "base-point preserving" means that, viewing $\mathbb{R}^{m-2}$ as S^{m-2} minus one point p and $\mathbb{R}^m$ as S^m minus one point q, any element of $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ sends p to q, or, in other words, sends the point at infinity in $\mathbb{R}^{m-2}$ into the point at infinity in $\mathbb{R}^m$) We also specify additionally a linear imbedding of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$, which we denote by σ.

Definition 2.4.8. The set $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ is defined as

$$\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) := \big\{f \in \mathrm{Imb}_\infty(\mathbb{R}^{m-2},\mathbb{R}^m) \colon \exists \text{ a compact subset } \Omega \text{ of } \mathbb{R}^{m-2} \\ f|_{\mathbb{R}^{m-2}\setminus\Omega} = \sigma\big\}\,, \tag{2.4.17}$$

$\mathrm{Imb}_\infty(\mathbb{R}^{m-2},\mathbb{R}^m)$ denotes the space of base-point preserving imbeddings from S^{m-2} into S^m, where we specify two base-points in S^{m-2} and S^m, both denoted by ∞. The action of the group of diffeomorphisms of S^m (which contains $SO(m+1)$) sees to it that any knot in S^m (i.e. an imbedding of S^{m-2} into S^m) can be deformed so as to give a long knot.

With the help of $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, we may define the configuration spaces $C_{s,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2}\big)$

Definition 2.4.9. The open configuration space $C^0_{s,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2},f\big)$ is defined analogously as (2.4.6), but now f has to belong to $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$. Therefore, $C^0_{s,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2}\big)$ is now seen as a fibration over $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$.

The compactification thereof, which is denoted by $C_{s,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2}\big)$, is again a fibration over $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, specified by the commutative diagram

$$\begin{array}{ccc} C_{s,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2}\big) := \mathrm{ev}_s^*\left(C_{s+t}(\mathbb{R}^m)\right) & \xrightarrow{\overline{\mathrm{ev}}_s} & C_{s+t}(\mathbb{R}^m) \\ {\scriptstyle \overline{\pi}_{1,s}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{1,s}} \\ \mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times C_s\big(\mathbb{R}^{m-2}\big) & \xrightarrow{\mathrm{ev}_s} & C_s(\mathbb{R}^m)\,, \end{array} \tag{2.4.18}$$

where the notations are as in the commutative diagram (2.4.4).

This space is also a manifold with corners; in this case, the boundary faces are of six types: $a)$ $2 \le p \le s$ points in $\mathbb{R}^{m-2}$ collapse together, $b)$ $2 \le q \le t$ points in $\mathbb{R}^m$ collapse together, $c)$ $1 \le p \le s$ points in $\mathbb{R}^{m-2}$ and $1 \le q \le t$ points in $\mathbb{R}^m$ collapse together in $\mathbb{R}^{m-2}$, $d)$ $1 \le q \le s$ points in $\mathbb{R}^{m-2}$ escape to infinity in $\mathbb{R}^{m-2}$, $e)$ $1 \le r \le t$ points in $\mathbb{R}^m$ tend to infinity in $\mathbb{R}^m$ and finally $f)$ $1 \le q \le s$ points in $\mathbb{R}^{m-2}$ and $1 \le r \le t$ points in $\mathbb{R}^m$ escape to infinity. The boundary faces of type $a)$, $b)$, $c)$ and $e)$ need not be discussed, because they are completely analogous to the boundary faces appearing in $C_{s,t}(\mathbb{R}^m,S)$. A more careful discussion is needed for the remaining boundary faces, namely those of type $d)$ and of type $f)$. When the first p points in $\mathbb{R}^{m-2}$ escape to infinity, we get the trivial fibration $C_{s-p,t}\big(\mathbb{R}^m,\mathbb{R}^{m-2}\big) \times \widehat{C}_{p+1}\big(\mathbb{R}^{m-2}\big)$ over $C_{s-p,t}(\mathbb{R}^m,\mathbb{R}^{m-2})$; similarly, any face where p points escape to infinity in $\mathbb{R}^{m-2}$ is realized starting from this special face via permutations on the fibration.

We describe now the codimension-1 boundary face where the first q points in $\mathbb{R}^{m-2}$ and the first r points in $\mathbb{R}^m$ escape to infinity. It is better to first view explicitly a given face at infinity, when e.g. the first q points in $\mathbb{R}^{m-2}$ escape to infinity. The interior of this face can be characterized by the inclusion (see also [12])

$$\begin{aligned} &(x_1,\dots,x_{s-q};y_1,\dots,y_t;z_1,\dots,z_q) \mapsto \\ &\mapsto (Rz_1,Rz_2,\dots,Rz_q,x_1,\dots,x_{s-q};y_1,\dots,y_t)\,, \end{aligned} \tag{2.4.19}$$

where $R > 0$ tends to infinity and the z_i's are all nonzero and distinct vectors in $\mathbb{R}^{m-2}$, and such that the sum of their squares equals 1.

Analogously, when the first q points in $\mathbb{R}^m$ tend to infinity in $\mathbb{R}^m$, we may view the interior of the corresponding face via the inclusion

$$\begin{aligned}&(x_1, \ldots, x_s; y_1, \ldots, y_{t-q}; w_1, \ldots, w_q) \mapsto \\ &\mapsto (x_1, \ldots, x_s; Rw_1, \ldots, Rw_q, y_1, \ldots, y_{t-q}),\end{aligned}$$

where R again tends to infinity, and the w_i's are all nonzero and distinct vectors in $\mathbb{R}^m$ such that the sum of their squares equals 1.

The special class of imbeddings we are considering permits a good characterization of the codimension-1 boundary face where the first p points in $\mathbb{R}^{m-2}$ and the first q points in $\mathbb{R}^m$ tend to infinity, since any element of $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ maps ∞ in $\mathbb{R}^{m-2}$ to ∞ in $\mathbb{R}^m$. If σ is the linear imbedding outside a compact set in $\mathbb{R}^{m-2}$, this face is then realized by the inclusion

$$\begin{aligned}&(x_1, \ldots, x_{s-p}; y_1, \ldots, y_{t-q}; z_1, \ldots, z_p; w_1, \ldots, w_q) \mapsto \\ &\mapsto (Rz_1, \ldots, Rz_p, x_1, \ldots, x_{s-p}; Rw_1, \ldots, Rw_q, y_1, \ldots, y_{t-q}).\end{aligned}$$

where $z_i \in \mathbb{R}^{m-2}$ and $w_j \in \mathbb{R}^m$, $1 \le i \le p$ and $1 \le j \le q$, are all nonzero and distinct in their respective spaces, such that additionally no w_j equals the image through σ of any z_i and such that the sum of their squares (w.r.t. the Euclidean norms in $\mathbb{R}^{m-2}$ and $\mathbb{R}^m$ for example) equals 1.

2.4.6 Orientation conventions for principal faces

Definition 2.4.10. We call a boundary face of codimension 1 of $C_{s,t}(\mathbb{R}^m, S)$ *principal*, if one of the following situations occur:

- the face describes the collapse of exactly two points in S or in $\mathbb{R}^m$;
- the face describes the collapse in S of exactly one point in S and one point in $\mathbb{R}^m$;
- the face describes the situation, when exactly one point in $\mathbb{R}^m$ escapes to infinity.

All other faces are called *hidden*.

In subsequent definitions and computations, we need to compute push-forwards along boundary faces of codimension 1 of configuration spaces $C_{s,t}(\mathbb{R}^m, S)$. The central object of most computations is the generalized Stokes Theorem; in order to compute correctly the boundary contributions of integration along fibers, we need to know how to orient the boundary, which decomposes into principal and hidden faces. We are not interested in the orientations of other types of faces, since we will later prove that, in the cases under consideration, there are no contributions coming from hidden faces and from principal faces at infinity.

We notice that an orientation for $C_{s,t}(\mathbb{R}^m, S)$ is fixed by the restriction to $C^0_{s,t}(\mathbb{R}^m, S)$ of an orientation of the product manifold $S^s \times (\mathbb{R}^m)^t$. We fix such an orientation: namely, we choose as orientation form

$$d\mathrm{vol}_{S^s\times(\mathbb{R}^m)^t} := \pi_1^*\,(d\mathrm{vol}_S) \wedge \cdots \wedge \pi_s^*\,(d\mathrm{vol}_S) \wedge \pi_{s+1}^*\,(d\mathrm{vol}_{\mathbb{R}^m}) \wedge \cdots \pi_{s+t}^*\,(d\mathrm{vol}_{\mathbb{R}^m})\,, \tag{2.4.20}$$

where π_i denotes the projection from $S^s \times (\mathbb{R}^m)^t$ onto the i-th factor, and $d\mathrm{vol}_S$, resp. $d\mathrm{vol}_{\mathbb{R}^m}$, denotes the orientation form of S, resp. $\mathbb{R}^m$.

As we have already seen, the cartesian product of the permutation groups $\mathfrak{S}_s$ and $\mathfrak{S}_t$ (which is also a group with its product structure) act freely and transitively on $C_{s,t}(\mathbb{R}^m, S)$. This means that, once we have fixed background principal faces of $C_{s,t}(\mathbb{R}^m, S)$, one for each type, we can obtain any other principal face of the same type via permutations. Permutations may be orientation-preserving or -reversing, depending on two factors: the signs of the permutations involved and the dimensions d of S and m of $\mathbb{R}^m$. Hence, in order to determine the orientation of a given principal face, we need only the orientation of three chosen background face. First, we consider the principal faces of the first type where the first and the second point in S, resp. in $\mathbb{R}^m$, collapse together. The orientation choice (2.4.20) implies immediately that the orientation of these two special faces is 1, resp. $(-1)^{sd}$. We may explicitly characterize these faces via the natural inclusion maps (considering only the interior of $C_{n-1}(\mathbb{R}^m)$)

$$\begin{aligned} \partial_{1,2,\mathrm{int}}(f; x_1, x_2, \ldots, x_{s-1}; y_1, \ldots, y_t; v) &:= (f; x_1, x_1 + \varepsilon v, x_2, \ldots, x_{s-1};\\ &\qquad y_1, \ldots, y_t)\,,\\ \partial_{1,2,\mathrm{ext}}(f; x_1, x_2, \ldots, x_s; y_1, \ldots, y_{t-1}, w) &:= (f; x_1, \ldots, x_s; y_1, y_1 + \varepsilon w, \ldots, y_{t-1}), \end{aligned} \tag{2.4.21}$$

where v, resp. w is a unit vector of $\mathbb{R}^d$, resp. of $\mathbb{R}^m$, and $\varepsilon > 0$ is small.

We consider the special face of the second type, where the first point of $\mathbb{R}^m$ collapse to the first point of S. We may also view this face via the inclusion map:

$$\partial_{1,s+1}\,(f; x_1, \ldots, x_s; y_1, \ldots, y_{t-1}; w) := (f; x_1, \ldots, x_s; f(x_1) + \varepsilon w, y_1, \ldots, y_{t-1})\,, \tag{2.4.22}$$

where w is a normalized vector in $\mathbb{R}^m$, and $\varepsilon > 0$ is also small. To this face, we assign the canonical orientation $(-1)^{(s-1)d}$, which is a consequence of the choice of orientation (2.4.20). By the transitive action of $\mathfrak{S}_s \times \mathfrak{S}_t$ on $C_{s,t-1}(\mathbb{R}^m, S)$ and of $\mathfrak{S}_s \times \mathfrak{S}_{t-1}$ on $C_{s,t-1}(\mathbb{R}^m, S)$, the orientation of any other principal face of the second type may be obtained from the chosen orientation of the background face in (2.4.22).

If we consider the configuration space $C_{s,t}(\mathbb{R}^m, S)$, there are four cases to discuss: when d and m are both even, when d is even and m is odd, when d is odd and m is even, and when d and m are both odd.

If d and m are both even, all principal faces have orientation 1, since permutations of s, resp. t, elements induce always orientation-preserving maps.

If d is even and m is odd, any permutation of s elements induces an orientation-preserving diffeomorphism on corresponding configuration spaces. On the other hand, since m is odd, a permutation of t elements induce an orientation-preserving or -reversing diffeomorphism on corresponding configuration spaces, if and only if is even, resp. odd. Analogously, when d is odd and m is even, the rôles are interchanged, i.e. permutations of s elements induce orientation-preserving or -reversing diffeomorphisms, if their signs are even, resp. odd, while permutations of t elements are always orientation preserving.

If we consider the case d and m both odd, then any element σ of $\mathfrak{S}_s \times \mathfrak{S}_t$ induces an orientation-preserving, resp. -reversing, diffeomorphism on $C_{s,t}(\mathbb{R}^m, S)$, for any $s \geq 1$, $t \geq 1$, if and only if, writing $\sigma = \sigma_s \times \sigma_t$, the sign of σ_s and the sign of σ_t are equal, resp. different.

Any principal face of $C_{s,t}(\mathbb{R}^m, S)$ of the first type for S, resp. $\mathbb{R}^m$, is characterized by a map of the form

$$\sigma_s \circ \partial_{1,2,\mathrm{int}} \circ \sigma_{s-1}, \quad \text{resp.} \quad \tau_t \circ \partial_{1,2,\mathrm{ext}} \circ \tau_{t-1},$$

where $\sigma_j \in \mathfrak{S}_j$, for $j = s, s-1$, and $\tau_k \in \mathfrak{S}_k$, $k = t, t-1$. Similarly, a principal face of the second type may be characterized by a map of the form

$$(\sigma_s \times \tau_t) \circ \partial_{1,s+1} \circ \widetilde{\sigma}_s,$$

where σ_s and $\widetilde{\sigma}_s$ are in $\mathfrak{S}_s$ and $\tau_t \in \mathfrak{S}_t$. Hence, we get the following formula for the orientation of principal faces:

$$\begin{aligned} \mathrm{or}\,(\sigma_s \circ \partial_{1,2,\mathrm{int}} \circ \sigma_{s-1}) &= (-1)^{(\sigma_s + \sigma_{s-1})d}; \\ \mathrm{or}\,(\tau_t \circ \partial_{1,2,\mathrm{ext}} \circ \tau_{t-1}) &= (-1)^{(\tau_t + \tau_{t-1})m + sd}; \\ \mathrm{or}\,((\sigma_s \times \tau_t) \circ \partial_{1,s+1} \circ \widetilde{\sigma}_s) &= (-1)^{(\sigma_s + \widetilde{\sigma}_s + s - 1)d}(-1)^{\tau_t m}. \end{aligned} \tag{2.4.23}$$

2.5 Some comments on $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$

As we have already seen, an important element in the characterization of boundary faces of compactified relative configuration spaces $C_{s,t}(\mathbb{R}^m, S)$, when points in S and points from $\mathbb{R}^m$ collapse together in S is the space $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$.

First of all, we construct a map $\Lambda_{m,d}$ from $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$ into the Stiefel manifold $V_{m,d}$, which we prefer to view as the space of orthonormal systems of d vectors in $\mathbb{R}^m$; in fact, picking the standard orthonormal bases of $\mathbb{R}^d$ and $\mathbb{R}^m$, any element of $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$ can be seen as a matrix with d columns and m rows, and the column vectors are linearly independent. For this purpose, we use the Gram–Schmidt procedure: if we are given d linearly independent vectors $\{a_1, \dots, a_d\}$ in $\mathbb{R}^m$, one can produce out of it an orthonormal system of d vectors, denoted by $\{e_1^a, \dots, e_d^a\}$,

$$\begin{aligned} e_1^a &:= \frac{a_1}{\|a_1\|}; \\ \widetilde{e}_i^a &:= a_i - \sum_{j=1}^{i-1} \left\langle\, a_i \,,\, e_j^a \,\right\rangle e_j^a, \quad e_i^a := \frac{\widetilde{e}_i^a}{\|\widetilde{e}_i^a\|}, \quad i = 1, \dots, d. \end{aligned}$$

We denote by $\{e_1, \dots, e_d\}$ the standard orthonormal basis w.r.t. the Euclidean scalar product on $\mathbb{R}^d$. Taking an element $\alpha \in \mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$, we consider the system of d linearly independent vectors in $\mathbb{R}^m$ given by $\{\alpha(e_1), \dots, \alpha(e_d)\}$ (i.e., the column vectors of α)

we denote by $\{e_1^\alpha, \ldots, e_d^\alpha\}$ the system of d orthonormal vectors constructed by means of the Gram–Schmidt procedure starting from $\{\alpha(e_1), \ldots, \alpha(e_d)\}$. We define the map $\Lambda_{m,d}$ from $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$ to $V_{m,d}$ as follows

$$\Phi_{m,d}(\alpha) :\ = \{e_1^\alpha, \ldots, e_d^\alpha\}, \quad \forall \alpha \in \mathcal{I}(\mathbb{R}^d,\mathbb{R}^m). \tag{2.5.1}$$

We notice that there is also a map $\Psi_{m,d}$ from $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$ into $\mathrm{Gr}_{m,d}$, which is simply given by

$$\Psi_{m,d}(\alpha) :\ = \langle \alpha(e_1), \ldots, \alpha(e_d) \rangle, \tag{2.5.2}$$

where $\langle \alpha(e_1), \ldots, \alpha(e_d) \rangle$ denotes the subspace spanned by the column vectors of α.

The map $\Lambda_{m,d}$ is clearly surjective, since linear isometries of $\mathbb{R}^d$ into $\mathbb{R}^m$ are a subset of $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$; similarly, $\Psi_{m,d}$ is also surjective.

We notice that the element $\Lambda_{m,d}(\alpha)$ can be written also as

$$\Phi_{m,d}(\alpha) = \alpha \circ \mathsf{M}(\alpha),$$

where $\mathsf{M}(\alpha)$ is an upper triangular $d \times d$ matrix with positive eigenvalues, depending *non-linearly* on the entries of α (this matrix arises in the Gram–Schmidt procedure). Any such matrix can be smoothly connected to the identity in the space of upper triangular matrices with positive eigenvalues:

$$\mathsf{M}_t(\alpha) :\ = (1-t)\,\mathrm{id} + t\mathsf{M}(\alpha).$$

It is also clear that, if α lies already in $V_{m,d}$, then $\mathsf{M}(\alpha)$ equals the identity: the Gram-Schmidt procedure maps the orthogonal system to itself.

Remark 2.5.1. In fact, denoting by $\iota_{m,d}$ the natural inclusion of $V_{m,d}$ into $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$, we get $\Phi_{m,d} \circ \iota_{m,d} = \mathrm{id}$. On the other hand, we can define a map Φ from $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m) \times [0,1]$ to $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$ as follows:

$$\widehat{\Lambda}_{m,d}(\alpha; t) :\ = \alpha \circ \mathsf{M}_t(\alpha).$$

The map $\widehat{\Lambda}_{m,d}$ obviously satisfies

$$\widehat{\Lambda}_{m,d}(\alpha; 0) = \alpha, \quad \widehat{\Lambda}_{m,d}(\alpha; 1) = \alpha \circ \mathsf{M}(\alpha) = \iota_{m,d} \circ \Phi_{m,d}(\alpha),$$

which is equivalent to the fact that $\widehat{\Lambda}_{m,d}$ is a *deformation retraction* of $\mathcal{I}(\mathbb{R}^d,\mathbb{R}^m)$ to $V_{m,d}$.

2.6 Stiefel and Grassmann manifolds

The Stiefel manifold $V_{m,d}$ is the set of orthonormal systems of d vectors in $\mathbb{R}^m$, while the Grassmann manifold $\mathrm{Gr}_{m,d}$ is defined as the set of d-dimensional subspaces of $\mathbb{R}^m$; alternatively, $V_{m,d}$ is the set of linear isometries of $\mathbb{R}^d$ into $\mathbb{R}^m$. There is another characterization of Stiefel and Grassmann manifolds as *homogeneous spaces*; for details about the equivalence of the two settings, we refer to [33].

Definition 2.6.1. The Stiefel manifold $V_{m,d}$, for any two positive integers $d \leq m$, is the homogeneous space

$$V_{n,m} :\ = SO(n)/SO(n-m),$$

where here $SO(m-d)$ is embedded as a subgroup of $SO(n)$ as follows:

$$\mathsf{B} \in SO(m-d) \hookrightarrow \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathsf{B} \end{bmatrix}. \tag{2.6.1}$$

In Formula (2.6.1), id is the identity isomorphism of the Euclidean m-dimensional space, the 0 on the upper right side is a $(n-m,m)$-matrix with zero entries and the 0 on the lower left side is its transpose.

We consider additionally the group $SO(d)$, which can be imbedded in $SO(m)$ as follows:

$$\mathsf{A} \in SO(d) \hookrightarrow \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathrm{id} \end{bmatrix}; \tag{2.6.2}$$

the notations are the same as in Formula (2.6.1) and id denotes the identity in the Euclidean $m-n$-dimensional space. Additionally, we consider the subgroup $SO(d) \oplus SO(m-d)$ of $SO(m)$, consisting of matrices of the form

$$\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{bmatrix}, \quad \mathsf{A} \in SO(d), \quad \mathsf{B} \in SO(m-d).$$

Definition 2.6.2. The Grassmann manifold $\mathrm{Gr}_{m,d}$, for m and d as above, is the homogeneous space

$$\mathrm{Gr}_{m,d} \colon = SO(m)/SO(d) \oplus SO(m-d).$$

2.6.1 Invariant forms on homogeneous spaces

We recall the definition of a homogeneous space.

Definition 2.6.3. We assume we are given a Lie-group G and a Lie-subgroup H of G.

The homogeneous space G/H is defined as the space of left cosets

$$G/H \colon = \{gH : \ g \in G\}$$

We define the projection π_H by $\pi_H(g) : \ = gH$; this map is clearly surjective. We endow G/H with the usual quotient topology.

It is customary to denote a class in G/H with representative g by $\overline{g}$, for $g \in G$.

The main fact about homogeneous spaces is encoded in the following

Theorem 2.6.4. *There is a unique smooth structure on G/H, such that π_H is smooth and G/H is a smooth manifold.*

It is possible to endow G/H with the structure of a left G-space, i.e. there is a smooth left action of G on G/H, defined as follows

$$G \times G/H \to G/H, \quad (g, \overline{x}) \mapsto \overline{gx}.$$

This action is well-defined, but it is in general not free. We will denote from now on the left action of an element $g \in G$ by L_g.

The Lie group G has also the structure of a left G-space, since left multiplication induces a smooth, free, transitive action of G on itself, which, for any $g \in G$, is also denoted by L_g.

We say that a form $\omega \in \Omega^*(G)$ is *invariant* (w.r.t. the left action of G on itself), if it satisfies $\mathrm{L}_g^* \omega = \omega$, for all $g \in G$. There is an obvious algebra homomorphism σ_G from $\Omega_G^*(G)$ to $\bigwedge^* \mathfrak{g}^*$, defined by the equation

$$\sigma(\omega) = \omega_e,$$

with inverse

$$\sigma_G^{-1}(\alpha)_g = \alpha\left(\mathrm{T}_g \mathrm{L}_g^{-1} \bullet, \dots\right).$$

details on these computations can be found in [33]. The usual exterior derivative of forms maps obviously invariant forms to invariant forms. Twisting the exterior derivative by the isomorphism σ_G we get a differential on $\bigwedge^* \mathfrak{g}^*$, which is usually denoted by $\delta_{\mathfrak{g}}$. In other words, the differential $\delta_{\mathfrak{g}}$ is given explicitly by $\delta_{\mathfrak{g}} = \sigma_G \circ \mathrm{d} \circ \sigma_G^{-1}$. The definition of σ_G and σ_G^{-1}, along with the invariance, yield the following explicit formula for $\delta_{\mathfrak{g}}$:

$$(\delta_{\mathfrak{g}}\alpha)(X_0, \dots, X_p) := \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha\Big([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p\Big), \tag{2.6.3}$$

for a given element α of $\bigwedge^p \mathfrak{g}^*$ and elements X_i of $\mathfrak{g}$; the hat on an argument means that it is omitted. The differential $\delta_{\mathfrak{g}}$ is then immediately seen to be equal to the differential in *Lie algebra cohomology with values in the trivial* $\mathfrak{g}$*-module*. Hence, we may consider the invariant cohomology of G: namely, a closed, invariant differential form on G is exact in invariant cohomology, if there exists an invariant form, whose differential equals the given closed form. The invariant cohomology of G is then in one-to-one correspondence with the Lie-algebra cohomology of $\mathfrak{g}$ with values in the trivial $\mathfrak{g}$-module.

Motivated by these results, we give the following

Definition 2.6.5. A differential form ω on the homogeneous space G/H is said to be G-invariant, if it satisfies

$$\mathrm{L}_g^* \omega = \omega, \quad \forall g \in G.$$

It is clear that the set of invariant forms on G/H is a differential algebra (the exterior derivative commutes with the action of G); we denote this algebra by $\Omega_G^*(G/H)$.

We want to characterize more precisely invariant forms on homogeneous spaces, and therefore we need some background. Again, we refer to [33] for a detailed discussion of what follows.

Given a Lie-group G, we denote by c the conjugation of G, by Ad, resp. Ad^*, the adjoint, resp. coadjoint, action of G on $\mathfrak{g}$, resp. on $\mathfrak{g}^*$. Finally, by ad, resp. ad^*, we denote the adjoint action of $\mathfrak{g}$ on itself, resp. on $\mathfrak{g}^*$. Since H is a Lie-subgroup, $\mathfrak{h}$ (the Lie-algebra of H) is a Lie-subalgebra of $\mathfrak{g}$; moreover, it is clearly stable under the adjoint action of H. We denote by $\mathfrak{h}^\perp$ the *annihilator of* $\mathfrak{h}$:

$$\mathfrak{h}^\perp := \{\alpha \in \mathfrak{g}^* : \alpha(X) = 0, \quad \forall X \in \mathfrak{h}\}.$$

Since for any $h \in H$, $\mathrm{Ad}(h)$ maps $\mathfrak{h}$ into itself, $\mathfrak{h}^{\perp}$ is stable under the coadjoint action of H. Hence, there is a representation of H on $\mathfrak{h}^{\perp}$. This action can be extended in an obvious way to the exterior algebra $\bigwedge^* \mathfrak{h}^{\perp}$.

We consider now the projection π_H; it can be shown that the tangent map of π_H at the identity induces a linear isomorphism between $\mathfrak{g}/\mathfrak{h}$ and the tangent space of G/H at the class of the identity (we refer again to [33] for more details); we denote this map by $\mathrm{T}_e\pi_H$. Hence, its dual map induces an isomorphism between the dual space of $\mathrm{T}_{\overline{e}}G/H$ and the space $\mathfrak{h}^{\perp}$. Consequently, its extension to exterior algebras induces again an isomorphism between $\bigwedge^* (\mathrm{T}_{\overline{e}}G/H)^*$ and $\bigwedge^* \mathfrak{h}^{\perp}$.

By definition of G/H and equivariancy of π_H, $\pi_H\left(c(h)g\right) = \mathrm{L}_h\, \pi_H(g)$, for all g in G and h in H; infinitesimally at the identity, we get

$$\mathrm{T}_e\pi_H \circ \mathrm{Ad}(h) = \mathrm{T}_{\overline{e}}\mathrm{L}_h \circ \mathrm{T}_e\pi_H, \quad \forall h \in H. \tag{2.6.4}$$

The equivariancy of π_H w.r.t. the action of G implies that the pull-back of any invariant form on G/H gives a G-invariant form on G.

We denote by $\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H$ the subalgebra of $\bigwedge^* \mathfrak{h}^{\perp}$ of H-invariant elements. If we take a differential form $\omega \in \Omega^*_G(G/H)$, then the alternating functional $\tau_G\left(\omega\right)$ defined by

$$\tau_G(\omega) :\, = \bigwedge\nolimits^* \left(\mathrm{T}_e\pi_H\right)^* \omega_{\overline{e}} \tag{2.6.5}$$

belongs to $\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H$; this a consequence of being ω invariant and of equation (2.6.4).

Moreover, it holds

Theorem 2.6.6. *The assignment τ_G defined by equation (2.6.5) is an isomorphism, which makes the following square commutative:*

$$\begin{array}{ccc} \Omega^*_G(G/H) & \xrightarrow{\tau_G} & \left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H \\ {\scriptstyle \pi_H^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ \Omega^*_G(G) & \xrightarrow{\sigma_G} & \bigwedge^* \mathfrak{g}^*, \end{array} \tag{2.6.6}$$

where ι denotes the natural inclusion. Furthermore, the graded algebra $\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H$ is stable w.r.t. the action of $\delta_{\mathfrak{g}}$; therefore, we may view $\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H$ endowed with the differential $\delta_{\mathfrak{g}}$ also as a differential graded algebra.

2.6.2 Homogeneous spaces with right actions from special subgroups and biinvariant forms

First of all, we give particular assumptions on the group G, which fit in for later purposes.

Condition 2.6.7. We assume G to be a Lie-group, with two Lie-subgroups H and K, satisfying the requirements:

- K and H commute, i.e. for any two elements $h \in H$ and $k \in K$, the relation $hk = kh$ holds;

- $H \cap K = \{e\}$, where e is the identity of G.

A triple (G, H, K), where both hypotheses are satisfied, is given e.g. by $G = SO(m)$, $H = SO(m-d)$ and $K = SO(d)$, for m and d positive integers obeying $d \leq m$, where $SO(m-d)$ is imbedded in $SO(m)$ via (2.6.1) and $SO(d)$ via (2.6.2).

Remark 2.6.8. The second condition is not crucial for the next computations. It is useful only in order to get a right *free* K-action on the homogeneous space G/H.

We return back to the homogeneous space G/H; under the above assumptions, G/H receives a right K-action as follows:

$$\begin{aligned} (G/H) \times K &\longrightarrow G/H, \\ (\overline{g}, k) &\longmapsto \overline{gk}. \end{aligned} \tag{2.6.7}$$

First of all, this action is well-defined. In fact, if we take two distinct representatives g, $\widetilde{g}$ of the class $x \in G/H$, then

$$g, \widetilde{g} \in x \Longleftrightarrow \exists\, h \in H \colon \widetilde{g} = gh.$$

It follows

$$\widetilde{g}k = ghk = gkh \Longrightarrow \widetilde{g}k \sim_H gk, \quad \forall k \in K,$$

where the second equality follows by the commutativity of H and K. Hence, left G-cosets are mapped by right multiplication by K into left G-cosets. Moreover, by the second assumption, the action of K on G/H is obviously free: taking a class $\overline{g}$ in G/H, if, for some k in K $\overline{g}k = \overline{g}$, it follows

$$\overline{g}k = \overline{gk} = \overline{g} \Longrightarrow gk = gh, \quad h \in H \Longrightarrow k = h.$$

But then k would also belong to H, contradicting $H \cap K = \{e\}$. Hence, given the above assumptions on H and K, the homogeneous space G/H inherits a free right K-action. The subgroup K operates also by right multiplication on G. The projection π_H is equivariant w.r.t. the right action of K on both G and G/H:

$$\pi_H\left(\mathrm{R}_k(g)\right) = \pi_H(gk) = \overline{gk} = \overline{g}k = \left(\mathrm{R}_k \circ \pi_H\right)(g),$$

where R_k denotes right multiplication by $k \in K$.

Definition 2.6.9. Under the above hypotheses on G, H and K, a form ω on G/H is said to be *right K-invariant*, if it satisfies

$$\mathrm{R}_k^*\, \omega = \omega, \quad \forall k \in K. \tag{2.6.8}$$

The algebra of K-invariant differential forms on G/H is denoted by $\Omega^*(G/H)_K$. Since both G- and K-actions are compatible, it makes sense to consider the algebra of left G-invariant, right K-invariant forms (shortly, *biinvariant forms* on G/H, denoted by $\Omega^*_G(G/H)_K$; it is clearly a subalgebra of $\Omega^*_G(G/H)$.

We consider the Lie-algebra $\mathfrak{k}$ of K and the adjoint action of K on $\mathfrak{h}$; since H and K commute, it follows

$$\mathrm{Ad}(k)X = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} c(k)\exp tX = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp tX = X, \quad X \in \mathfrak{h}, \quad k \in K.$$

Hence, the adjoint action of K on $\mathfrak{h}$ is trivial. It follows that $\mathfrak{h}^{\perp}$ is invariant w.r.t. the coadjoint action of K. Hence, we get a representation of K on $\mathfrak{h}^{\perp}$; moreover, the representation of H on $\mathfrak{h}^{\perp}$ obviously commutes with the representation of K.

We consider the conjugation on G by elements of K; we easily get

$$\pi_H \circ \mathrm{c}(k) = \mathrm{L}_k \circ \mathrm{R}_k^{-1} \circ \pi_H.$$

Taking the tangent map on both sides at the identity, we get

$$\mathrm{T}_e\pi_H \circ \mathrm{Ad}(k) = \mathrm{T}_{\overline{k^{-1}}}\mathrm{L}_k \circ \mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H.$$

We take an element of $\omega \in \Omega_G^p(G/H)_K$. For an arbitrary $k \in K$, we consider

$$\begin{aligned}
\left(\bigwedge\nolimits^p \mathrm{Ad}^*(k)\right)\tau_G(\omega) &= \tau_G(\omega)\left(\mathrm{Ad}\left(k^{-1}\right)\bullet,\dots\right) = \omega_{\overline{e}}\left(\mathrm{T}_e\pi_H \circ \mathrm{Ad}\left(k^{-1}\right)\bullet,\dots,\right) = \\
&= \omega_{\overline{e}}\left(\left(\mathrm{T}_{\overline{k^{-1}}}\mathrm{L}_k \circ \mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \omega_{\mathrm{L}_{k^{-1}}(\overline{k})}\left(\left(\mathrm{T}_{\overline{k^{-1}}}\mathrm{L}_k \circ \mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \omega_{\overline{k}}\left(\left(\mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \omega_{\mathrm{R}_k(\overline{e})}\left(\left(\mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \omega_{\overline{e}}\left(\mathrm{T}_e\pi_H\bullet,\dots\right) = \\
&= \tau_G\left(\omega\right),
\end{aligned}$$

where we have used explicitly the biinvariance of ω. Therefore, τ_G maps the algebra of G-invariant, K-invariant forms on G/H into $\left[\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H\right]_K$, the algebra of H-invariant, K-invariant elements of $\bigwedge^* \mathfrak{h}^{\perp}$. On the other hand, since τ_G is an isomorphism from $\Omega_G^*(G/H)_K$ in $\left(\bigwedge^* \mathfrak{h}^{\perp}\right)_H$, given an element $\alpha \in \left[\left(\bigwedge^p \mathfrak{h}^{\perp}\right)_H\right]_K$, there exists exactly one element of $\omega \in \Omega_G^p(G/H)$, such that

$$\alpha = \bigwedge\nolimits^p \left(\mathrm{T}_e\pi_H\right)^* \omega_{\overline{e}}.$$

We will show that ω is also K-invariant: since α is in $\left[\left(\bigwedge^p \mathfrak{h}^{\perp}\right)_H\right]_K$, it follows

$$\begin{aligned}
\omega_{\overline{e}}\left(\mathrm{T}_e\pi_H\bullet,\dots\right) &= \alpha\left(\bullet,\dots\right) = \\
&= \bigwedge\nolimits^p \mathrm{Ad}^*(k)\alpha\left(\bullet,\dots\right) = \alpha\left(\mathrm{Ad}\left(k^{-1}\right)\bullet,\dots\right) = \\
&= \omega_{\overline{e}}\left(\mathrm{T}_e\pi_H \circ \mathrm{Ad}\left(k^{-1}\right)\bullet,\dots\right) = \\
&= \omega_{\overline{e}}\left(\left(\mathrm{T}_{\overline{k^{-1}}}\mathrm{L}_k \circ \mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \omega_{\overline{k}}\left(\left(\mathrm{T}_{\overline{e}}\mathrm{R}_k^{-1} \circ \mathrm{T}_e\pi_H\right)\bullet,\dots\right) = \\
&= \left(\mathrm{R}_k^*\,\omega\right)_{\overline{e}}\left(\mathrm{T}_e\pi_H\bullet,\dots\right);
\end{aligned}$$

in the fifth equation, we made use of the left-invariance of ω. Since the dual map of $\mathrm{T}_{\overline{e}}\pi_H$ is an isomorphism between $\mathfrak{h}^\perp$ and the dual of $\mathrm{T}_{\overline{e}}G/H$, it follows

$$\omega_{\overline{e}} = (\mathrm{R}_k^*\,\omega)_{\overline{e}}\,, \quad \forall k \in K. \tag{2.6.9}$$

Since the right action of K and the left action of G commute, along with the G-invariance of ω, imply directly that the form ω is also K-invariant. Namely

$$\begin{aligned}
(\mathrm{R}_k^*\,\omega)_{\overline{g}}(\bullet,\cdots) &= \omega_{\mathrm{L}_g\,\overline{k}}\left(\left(\mathrm{T}_{\overline{k}}\mathrm{L}_g\right)\circ\left(\mathrm{T}_{\overline{gk}}\mathrm{L}_g^{-1}\right)\circ\left(\mathrm{T}_{\overline{g}}\mathrm{R}_k\right)\bullet,\cdots\right) = \\
&= \omega_{\overline{k}}\left(\left(\mathrm{T}_{\overline{e}}\mathrm{R}_k\right)\circ\left(\mathrm{T}_g\mathrm{L}_g^{-1}\right)\bullet,\cdots\right) = \\
&= \omega_{\mathrm{R}_k\,\overline{e}}\left(\left(\mathrm{T}_{\overline{e}}\mathrm{R}_k\right)\circ\left(\mathrm{T}_g\mathrm{L}_g^{-1}\right)\bullet,\cdots\right) = \omega_{\overline{e}}\left(\left(\mathrm{T}_{\overline{g}}\mathrm{L}_g^{-1}\right)\bullet,\cdots\right) = \\
&= \omega_{\overline{g}}(\bullet,\cdots)\,, \quad \forall g \in G, \quad \forall k \in K.
\end{aligned}$$

Hence, ω is also K-invariant. The map τ_G restricts therefore to an isomorphism between $\Omega_G^*(G/H)_K$ and $\left[\left(\bigwedge^*\mathfrak{h}^\perp\right)_H\right]_K$. We may state the results of all these computations in the following

Theorem 2.6.10. *Given a Lie-group G, which has two Lie-subgroups H and K satisfying the two hypotheses in Condition (2.6.7), the isomorphism τ_G of Theorem 2.6.6 restricts to an isomorphism (denoted again by τ_G)*

$$\Omega_G^*(G/H)_K \xrightarrow{\tau_G} \left[\left(\bigwedge^*\mathfrak{h}^\perp\right)_H\right]_K, \tag{2.6.10}$$

making the following square obviously commutative

$$\begin{array}{ccc}
\Omega_G^*(G/H)_K & \xrightarrow{\tau_G} & \left[\left(\bigwedge^*\mathfrak{h}^\perp\right)_H\right]_K \\
{\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\
\Omega_G^*(G/H) & \xrightarrow{\tau_G} & \left(\bigwedge^*\mathfrak{g}^\perp\right)_H.
\end{array} \tag{2.6.11}$$

Moreover, the graded algebra $\left[\left(\bigwedge^*\mathfrak{h}^\perp\right)_H\right]_K$ is stable under the differential $\delta_\mathfrak{g}$; we will denote the (restriction of the) differential $\delta_\mathfrak{g}$ on $\left[\left(\bigwedge^*\mathfrak{h}^\perp\right)_H\right]_K$ by the same symbol $\delta_\mathfrak{g}$.

2.6.3 The Lie-algebra of $SO(m)$

The Lie-algebra of $SO(m)$, which is usually denoted by $\mathfrak{so}(m)$, is given by the set

$$\mathfrak{so}(m)\colon = \left\{\mathsf{X} \in \mathfrak{gl}(m)\colon \mathsf{X}^t = -\mathsf{X}\right\}. \tag{2.6.12}$$

Here, we have denoted by $\mathfrak{gl}(m)$ the associative algebra of $m\times m$-matrices with real entries. The Lie-algebra $\mathfrak{gl}(m)$ is generated by the matrices E_{ij}, where $i, j \in \{1,\ldots,m\}$:

$$(\mathsf{E}_{ij})_{kl}\,:\, = \delta_{ik}\delta_{jl},$$

i.e., the matrices E_{ij} have all zero entries but the entry (i,j), which is 1. It is clear that such matrices are a basis of $\mathfrak{gl}(m)$. A basis of $\mathfrak{so}(m)$ may be then displayed via the basis E_{ij}: for $1 \leq i < j \leq m$, we define

$$\overline{\mathsf{E}}_{ij}\colon = \mathsf{E}_{ij} - \mathsf{E}_{ji}.$$

Hence,

$$\mathfrak{so}(m) = \left\langle \overline{\mathsf{E}}_{ij} \colon 1 \leq i < j \leq m \right\rangle . \tag{2.6.13}$$

We need explicit formulae for the structure constants of $\mathfrak{so}(m)$ w.r.t. the basis $\overline{\mathsf{E}}_{ij}$. It suffices to compute the structure constants of $\mathfrak{gl}(m)$ w.r.t. the basis E_{ij}: for this purpose, we have to compute the products $\mathsf{E}_{ij}\mathsf{E}_{kl}$, for any four indices i, j, k, l:

$$(\mathsf{E}_{ij})_{mn} (\mathsf{E}_{kl})_{np} = \delta_{im}\delta_{jn}\delta_{kn}\delta_{lp} = \delta_{im}\delta_{jk}\delta_{lp} = \delta_{jk} (\mathsf{E}_{il})_{mp} .$$

Hence, it follows

$$[\, \mathsf{E}_{ij} \, , \, \mathsf{E}_{kl} \,] = \delta_{jk}\mathsf{E}_{il} - \delta_{il}\mathsf{E}_{kj}.$$

A simple computation ensures that

$$[\, \overline{\mathsf{E}}_{ij} \, , \, \overline{\mathsf{E}}_{kl} \,] = \delta_{jk}\overline{\mathsf{E}}_{il} - \delta_{jl}\overline{\mathsf{E}}_{ik} - \sigma_{jl}\delta_{ik}\overline{\mathsf{E}}_{jl} + \sigma_{jk}\delta_{il}\overline{\mathsf{E}}_{jk}, \tag{2.6.14}$$

where

$$\sigma_{ij} = \begin{cases} 1 & , \quad i < j \\ -1 & , \quad i > j. \end{cases}$$

The Lie-algebra $\mathfrak{gl}(m)$ possesses as *Killing form*

$$\langle \mathsf{X} \, , \, \mathsf{Y} \rangle : \, = \operatorname{Tr} \operatorname{ad}(\mathsf{X}) \operatorname{ad}(\mathsf{Y}), \tag{2.6.15}$$

where Tr denotes the trace of a linear endomorphism on $\mathfrak{gl}(m)$, and ad denotes the adjoint action of $\mathfrak{so}(m)$ on itself; the Killing form can be also written as

$$\langle \mathsf{X} \, , \, \mathsf{Y} \rangle = 2m \operatorname{Tr}(\mathsf{XY}) - 2 \operatorname{Tr} \mathsf{X} \operatorname{Tr} \mathsf{Y}, \quad \forall \mathsf{X}, \mathsf{Y} \in \mathfrak{gl}(m),$$

where now the trace is the usual trace of matrices. If we now restrict the Killing form to $\mathfrak{so}(m)$, we get also a $SO(m)$-invariant bilinear form on $\mathfrak{so}(m)$, which reduces to

$$\langle \mathsf{X} \, , \, \mathsf{Y} \rangle = 2m \operatorname{Tr} \mathsf{XY} = -4m \sum_{1 \leq i < j \leq m} \mathsf{X}_{ij}\mathsf{Y}_{ij}, \quad \forall \mathsf{X}, \mathsf{Y} \in \mathfrak{so}(m),$$

where $\mathsf{X} = \sum_{1 \leq i < j \leq m} \mathsf{X}_{ij}\overline{\mathsf{E}}_{ij}$, and analogously for Y. Using the explicit definition of the generators $\overline{\mathsf{E}}_{ij}$, it can be proven that

$$\left\langle \, \overline{\mathsf{E}}_{ij} \, , \, \overline{\mathsf{E}}_{kl} \, \right\rangle = -4m\delta_{ik}\delta_{jl},$$

i.e. the Killing form is nondegenerate on $\mathfrak{so}(m)$, and that the basis $\overline{\mathsf{E}}_{ij}$ is orthogonal. We may further normalize the Killing form, multiplying it by $-4m$, whence the basis $\overline{\mathsf{E}}_{ij}$ becomes orthonormal w.r.t. the corrected Killing form

$$\langle \mathsf{X} \, , \, \mathsf{Y} \rangle = -\frac{1}{2} \operatorname{Tr}(\mathsf{XY}) . \tag{2.6.16}$$

(We will call the last bilinear form "Killing form"). Since (2.6.16) is $SO(m)$-invariant, the Riesz map Φ from $\mathfrak{so}(m)$ to $\mathfrak{so}(m)^*$ w.r.t. the Killing form is $SO(m)$-equivariant: $\Phi \circ \operatorname{Ad}(\mathsf{A}) = \operatorname{Ad}^*(\mathsf{A}) \circ \Phi$, for all $\mathsf{A} \in SO(m)$.

The abelian Lie-group $SO(2)$ is imbedded in $SO(m)$ via the map (2.6.1); therefore, the Lie-subalgebra $\mathfrak{so}(2)$ of $\mathfrak{so}(m)$ is given by a copy of $\mathfrak{so}(2)$ imbedded in $\mathfrak{so}(m)$ as follows:

$$\mathfrak{so}(2)\colon = \left\{ \mathsf{Y} \in \mathfrak{so}(m) \colon \mathsf{Y} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{X} \end{bmatrix}, \quad \mathsf{X} \in \mathfrak{so}(2) \right\}. \tag{2.6.17}$$

In the previous formula, the 0 on the upper left side is the $(m-2) \times (m-2)$-matrix with all zero entries, while the 0 on the upper right side is a $(m-2) \times 2$-matrix with all zero entries, and the remaining 0 is its transpose. In other words,

$$\mathfrak{so}(2) = \left\langle \overline{\mathsf{E}}_{m-1,m} \right\rangle.$$

Therefore,

$$\mathfrak{so}(2)^{\perp} = \left\{ \mathsf{Z} \in \mathfrak{so}(m) \colon \mathsf{Z} = \begin{bmatrix} \mathsf{X} & \mathsf{Y} \\ -\mathsf{Y}^t & 0 \end{bmatrix}, \mathsf{X} \in \mathfrak{so}(m-2), \mathsf{Y} \in M_{m-2,2} \right\}$$

In the last expression, $M_{m-2,2}$ denotes the space of real $(m-2) \times 2$-matrices. Hence, the annihilator $\mathfrak{so}(2)^{\perp}$ of $\mathfrak{so}(m)^*$ corresponds to the orthogonal complement of $\mathfrak{so}(2)$ under Φ. Sometimes, in order to simplify the notations, we will write an element of $\mathfrak{so}(2)^{\perp}$ as a 2-tuple $(\mathsf{X}, \mathsf{Y}) \in \mathfrak{so}(m-2) \times M_{m-2,2}$.

Finally, we want to give an explicit expression for the adjoint action of $SO(m-2) \oplus SO(2)$ on $\mathfrak{so}(2)^{\perp}$; here, any element of $SO(m-2) \oplus SO(2)$ has the form

$$\begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{bmatrix}, \quad \mathsf{A} \in SO(m-2), \quad \mathsf{B} \in SO(2).$$

By a direct computation we get

$$\mathrm{Ad}\,(\mathsf{A} \oplus \mathsf{B}) \begin{bmatrix} \mathsf{X} & \mathsf{Y} \\ -\mathsf{Y}^t & 0 \end{bmatrix} = \begin{bmatrix} \mathsf{AXA}^t & \mathsf{AYB}^t \\ -\mathsf{BY}^t\mathsf{A}^t & 0, \end{bmatrix}, \quad \mathsf{A} \in SO(m-2), \quad \mathsf{B} \in SO(2). \tag{2.6.18}$$

Since we have identified the annihilator $\mathfrak{so}(2)^{\perp}$ with the orthogonal complement of $\mathfrak{so}(2)$ w.r.t. the Killing form via the map Φ, we may view the exterior algebra $\bigwedge^* \mathfrak{so}(2)^{\perp}$ of the annihilator as the exterior algebra of the orthogonal complement of $\mathfrak{so}(2)$. Hence, the invariant subalgebra $\left(\bigwedge^* \mathfrak{so}(2)^{\perp}\right)_{SO(2)}$ (in the dual sense) may be identified with the invariant subalgebra w.r.t. the action (2.6.18) of the exterior algebra of the orthogonal complement of $\mathfrak{so}(2)$.

2.6.4 Biinvariant forms on $V_{4,2}$

In this special case, the computations simplify considerably, and we are able to produce all biinvariant forms on $V_{4,2}$.

The Lie-algebra $\mathfrak{so}(4)$ is spanned by the following six linearly independent vectors

$$\mathfrak{so}(4) = \left\langle \overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}, \overline{\mathsf{E}}_{34} \right\rangle.$$

We consider two copies of $SO(2)$: the first one is imbedded in $SO(4)$ via the map (2.6.2), the second one is imbedded via the map (2.6.1). If we consider the Lie algebra of the second copy of $SO(2)$, we find that its generator is $\overline{\mathsf{E}}_{34}$. Therefore, we may write

$$\mathfrak{so}(2)^{\perp} = \left\{ \begin{bmatrix} \mathsf{X} & \mathsf{Y} \\ -\mathsf{Y}^t & 0 \end{bmatrix} : \mathsf{X} \in \mathfrak{so}(2), \quad \mathsf{Y} \in \mathfrak{gl}(2) \right\}.$$

The adjoint action of $SO(2) \oplus SO(2)$ on $\mathfrak{so}(2)^{\perp}$ is given by the following formula, which is a direct consequence of Formula (2.6.18):

$$\mathrm{Ad}(\mathsf{A} \oplus \mathsf{B}) \begin{bmatrix} \mathsf{X} & \mathsf{Y} \\ -\mathsf{Y}^t & 0 \end{bmatrix} = \begin{bmatrix} \mathsf{X} & \mathsf{A}\mathsf{Y}\mathsf{B}^t \\ -\mathsf{B}\mathsf{Y}^t\mathsf{A}^t & 0 \end{bmatrix}, \quad \mathsf{A}, \mathsf{B} \in SO(2); \tag{2.6.19}$$

the formula does not contain the adjoint action of the first copy of $SO(2)$ on its Lie algebra, as $SO(2)$ is abelian. Therefore, the generator $\overline{\mathsf{E}}_{12}$ of the first copy of $\mathfrak{so}(2)$ is obviously $SO(2) \oplus SO(2)$-invariant. This implies that, for $m = 4$, there exists an $SO(2) \oplus SO(2)$-invariant vector; equivalently, there exists a biinvariant 1-form on $V_{4,2}$. Moreover, this is the unique biinvariant 1-form on $V_{4,2}$. Namely, since $-\,\mathrm{id} \in SO(2)$, any $SO(2) \oplus SO(2)$-invariant functional $\alpha \in \mathfrak{so}(2)^{\perp}$ is a multiple of $\overline{\mathsf{E}}^*_{12}$, the dual of $\overline{\mathsf{E}}_{12}$, as the following argument shows:

$$\alpha\left(\overline{\mathsf{E}}_{13}\right) = \alpha\left(-\overline{\mathsf{E}}_{13}\right) = 0,$$

by the $\mathrm{SO}(2) \oplus SO(2)$ invariance of α, taking $\mathsf{A} = \mathrm{id}$ and $\mathsf{B} = -\,\mathrm{id}$. Similar computations yield

$$\alpha\left(\overline{\mathsf{E}}_{14}\right) = \alpha\left(\overline{\mathsf{E}}_{23}\right) = \alpha\left(\overline{\mathsf{E}}_{24}\right) = 0.$$

Therefore, if α is $SO(2) \oplus SO(2)$-invariant, the only surviving component of α is the $\overline{\mathsf{E}}^*_{12}$-component, and this yields

$$\alpha = \lambda\, \overline{\mathsf{E}}^*_{12}, \quad \lambda \in \mathbb{R}.$$

For the sake of simplicity, we will denote by α the invariant functional $\overline{\mathsf{E}}^*_{12}$, and also the corresponding unique biinvariant form on $V_{4,2}$.

Next, we consider alternating $SO(2) \oplus SO(2)$-invariant functionals of degree 2 in $\mathfrak{so}(2)^{\perp}$. We take a general element η of $\bigwedge^2 \mathfrak{so}(2)^{\perp}$, invariant under the action (2.6.19). First of all, choosing as before $\mathsf{A} = \mathrm{id}$ and $\mathsf{B} = -\,\mathrm{id}$, we get

$$\begin{aligned} \eta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}\right) = 0, \quad \eta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{14}\right) = 0, \\ \eta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{23}\right) = 0, \quad \eta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{24}\right) = 0. \end{aligned}$$

In fact, the $\mathrm{id} \oplus SO(2)$-invariance implies e.g.

$$\eta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}\right) = \eta\left(\overline{\mathsf{E}}_{12}, -\overline{\mathsf{E}}_{13}\right).$$

All other cases follow by similar computations.

We now show that $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$ and $\overline{\mathsf{E}}^*_{13} \overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$ are $SO(2) \oplus SO(2)$-invariant.

Any element of $SO(2)$ can be uniquely written in the form $\mathsf{A}\colon = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$, where $c\colon = \cos\theta$ and $s\colon = \sin\theta$, with θ varying in $[0, 2\pi)$.

We compute explicitly the action of $\mathsf{A}^t \oplus \mathrm{id}$, resp. $\mathrm{id} \oplus \mathsf{A}^t$, on the generators $\left\{\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\overline{\mathsf{E}}_{24}\right\}$:

$$\begin{aligned}
\mathrm{Ad}(\mathsf{A}^t \oplus \mathrm{id})\overline{\mathsf{E}}_{13} &= c\,\overline{\mathsf{E}}_{13} + s\,\overline{\mathsf{E}}_{23}, & \mathrm{Ad}(\mathrm{id} \oplus \mathsf{A}^t)\overline{\mathsf{E}}_{13} &= c\,\overline{\mathsf{E}}_{13} + s\,\overline{\mathsf{E}}_{14}, \\
\mathrm{Ad}(\mathsf{A}^t \oplus \mathrm{id})\overline{\mathsf{E}}_{14} &= c\,\overline{\mathsf{E}}_{13} + s\,\overline{\mathsf{E}}_{24}, & \mathrm{Ad}(\mathrm{id} \oplus \mathsf{A}^t)\overline{\mathsf{E}}_{14} &= -s\,\overline{\mathsf{E}}_{13} + c\,\overline{\mathsf{E}}_{14}, \\
\mathrm{Ad}(\mathsf{A}^t \oplus \mathrm{id})\overline{\mathsf{E}}_{23} &= -s\,\overline{\mathsf{E}}_{13} + c\,\overline{\mathsf{E}}_{23}, & \mathrm{Ad}(\mathrm{id} \oplus \mathsf{A}^t)\overline{\mathsf{E}}_{23} &= c\,\overline{\mathsf{E}}_{23} + s\,\overline{\mathsf{E}}_{24}, \\
\mathrm{Ad}(\mathsf{A}^t \oplus \mathrm{id})\overline{\mathsf{E}}_{24} &= -s\,\overline{\mathsf{E}}_{14} + c\,\overline{\mathsf{E}}_{24}, & \mathrm{Ad}(\mathrm{id} \oplus \mathsf{A}^t)\overline{\mathsf{E}}_{24} &= -s\,\overline{\mathsf{E}}_{23} + c\,\overline{\mathsf{E}}_{24}.
\end{aligned} \tag{2.6.20}$$

If we choose $\theta = \frac{\pi}{2}$, the $SO(2) \oplus SO(2)$-invariance implies:

$$\begin{aligned}
&\eta\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}\right) = \eta\left(\overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right), \\
&\eta\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{23}\right) = \eta\left(\overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{24}\right), \\
&\eta\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{24}\right) = -\eta\left(\overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = \eta\left(\overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = 0.
\end{aligned}$$

Therefore, η may be written as follows:

$$\eta = \lambda\,\Theta + \mu\,\Omega,$$

where Θ denotes $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$ and Ω denotes $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} + \overline{\mathsf{E}}^*_{23} \wedge \overline{\mathsf{E}}^*_{24}$, and λ and μ are real numbers. For the sake of simplicity, we denote by Θ and Ω the corresponding biinvariant forms.

Returning to the general case, the identity $c^2 + s^2 = 1$ implies immediately that $\overline{\mathsf{E}}_{13} \wedge \overline{\mathsf{E}}_{14} + \overline{\mathsf{E}}_{23} \wedge \overline{\mathsf{E}}_{24}$ and $\overline{\mathsf{E}}_{13} \wedge \overline{\mathsf{E}}_{23} + \overline{\mathsf{E}}_{14} \wedge \overline{\mathsf{E}}_{24}$ are $SO(2) \oplus SO(2)$-invariant. Since $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$ corresponds to $\overline{\mathsf{E}}_{13} \wedge \overline{\mathsf{E}}_{23} + \overline{\mathsf{E}}_{14} \wedge \overline{\mathsf{E}}_{24}$ under the $SO(4)$-equivariant isomorphism Φ between $\mathfrak{so}(4)$ and $\mathfrak{so}(4)^*$ induced by the Killing-form, $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$ is also $SO(2) \oplus SO(2)$-invariant.

Hence, any $SO(2) \oplus SO(2)$-invariant element of $\bigwedge^2 \mathfrak{so}(2)^\perp$ is a linear combination of $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} + \overline{\mathsf{E}}^*_{23} \wedge \overline{\mathsf{E}}^*_{24}$ and $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{24}$.

We consider now a general element ξ of $\left(\bigwedge^3 \mathfrak{so}(2)^\perp\right)_{SO(2)\oplus SO(2)}$. The $SO(2) \oplus SO(2)$-invariance implies

$$\xi\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = \xi\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{24}\right) = \xi\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right) = \xi\left(\overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right) = 0.$$

Namely, if we consider e.g. the first expression, taking $\mathsf{B} = -\,\mathrm{id} \in SO(2)$, and using the $SO(2) \oplus SO(2)$-invariance, we get

$$\xi\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = \xi\left(-\overline{\mathsf{E}}_{13}, -\overline{\mathsf{E}}_{14}, -\overline{\mathsf{E}}_{23}\right) = -\xi\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right).$$

Therefore, ξ must be of the form

$$\xi = \alpha \wedge \eta,$$

where η has to be an alternating functional in $\bigwedge^2 \mathfrak{so}(2)^\perp$, depending only on the generators $\left\{\overline{\mathsf{E}}^*_{13}, \overline{\mathsf{E}}^*_{14}, \overline{\mathsf{E}}^*_{23}, \overline{\mathsf{E}}^*_{24}\right\}$. Since ξ and α are both invariant, η has to be invariant.

By what we have proved above, η is a linear combination of Ω and Θ. Hence, ξ has the general form

$$\xi = \alpha \wedge [\lambda \Omega + \mu\, \Theta]\,,$$

with λ and μ real numbers.

We take a general element θ of $\bigwedge^4 \mathfrak{so}(2)^\perp$, invariant under the action of $SO \oplus SO(2)$. As in the previous cases, the $SO(2) \oplus SO(2)$-invariance implies

$$\theta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = \theta\left(\overline{\mathsf{E}}_{12}, -\overline{\mathsf{E}}_{13}, -\overline{\mathsf{E}}_{14}, -\overline{\mathsf{E}}_{23}\right) = -\theta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}\right) = 0.$$

Similar computations yield:

$$\begin{aligned}\theta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{24}\right) &= 0,\\ \theta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right) &= 0,\\ \theta\left(\overline{\mathsf{E}}_{12}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right) &= 0.\end{aligned}$$

Therefore, θ takes the form

$$\theta = \lambda\, \overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{23} \wedge \overline{\mathsf{E}}^*_{24},$$

with λ a real number. Since $\overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{23} \wedge \overline{\mathsf{E}}^*_{24} = \frac{1}{2}\Omega^2$, it follows immediately that θ is $SO(2) \oplus SO(2)$-invariant.

Finally, we consider an element γ of $\bigwedge^5 \mathfrak{so}(2)^\perp$, invariant under the action (2.6.19). Such an element must be a real multiple of $\overline{\mathsf{E}}^*_{12} \wedge \overline{\mathsf{E}}^*_{13} \wedge \overline{\mathsf{E}}^*_{14} \wedge \overline{\mathsf{E}}^*_{23} \wedge \overline{\mathsf{E}}^*_{24}$, which may alternatively written as

$$\gamma = \frac{\lambda}{2}\, \alpha \wedge \Omega^2,$$

and we see immediately that γ is also $\mathrm{id} \oplus SO(2)$-invariant.

We may summarize all the results so far in the following

Theorem 2.6.11. *We consider the Stiefel manifold $V_{4,2}$. Then, the following isomorphisms hold:*

$$\begin{aligned}&\Omega^0_{SO(4)}(V_{4,2}) \cong \mathbb{R}; && \Omega^1_{SO(4)}(V_{4,2}) \cong \mathbb{R};\\ &\Omega^2_{SO(4)}(V_{4,2}) \cong \mathbb{R}^2; && \Omega^3_{SO(4)}(V_{4,2}) \cong \mathbb{R}^2;\\ &\Omega^4_{SO(4)}(V_{4,2}) \cong \mathbb{R}; && \Omega^5_{SO(4)}(V_{4,2}) \cong \mathbb{R}.\end{aligned}$$

Proof. The isomorphisms are provided by Theorem 2.6.10, since Ω_1, Ω_2, Θ and H are linearly independent, as well as $\alpha \wedge \Omega_i$, $i = 1, 2$, $\alpha \wedge \Theta$ and $\alpha \wedge \mathrm{H}$.

The preceding computations show that

$$\begin{aligned}&\left(\mathfrak{so}(2)^\perp\right)_{SO(2)} = \langle \alpha \rangle\,; && \left(\bigwedge^2 \mathfrak{so}(2)^\perp\right)_{SO(2)} = \langle \Omega, \Theta \rangle\,;\\ &\left(\bigwedge^3 \mathfrak{so}(2)^\perp\right)_{SO(2)} = \langle \alpha \wedge \Omega, \alpha \wedge \Theta \rangle\,; && \left(\bigwedge^4 \mathfrak{so}(2)^\perp\right)_{SO(2)} = \left\langle \Omega^2 \right\rangle\,;\\ &\left(\bigwedge^5 \mathfrak{so}(2)^\perp\right)_{SO(2)} = \left\langle \alpha \wedge \Omega^2 \right\rangle\,.\end{aligned}$$

$\square$

2.6.5 Biinvariant cohomology of $V_{4,2}$

We begin this subsubsection with the explicit computation of the differentials of the basis vectors of $so(4)^*$:

$$\begin{aligned}
&\delta\left(\overline{\mathsf{E}}^*_{12}\right) = \overline{\mathsf{E}}^*_{13}\wedge\overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14}\wedge\overline{\mathsf{E}}^*_{24}, && \delta\left(\overline{\mathsf{E}}^*_{13}\right) = -\overline{\mathsf{E}}^*_{12}\wedge\overline{\mathsf{E}}^*_{23} + \overline{\mathsf{E}}^*_{14}\wedge\overline{\mathsf{E}}^*_{34},\\
&\delta\left(\overline{\mathsf{E}}^*_{14}\right) = -\overline{\mathsf{E}}^*_{12}\wedge\overline{\mathsf{E}}^*_{24} - \overline{\mathsf{E}}^*_{13}\wedge\overline{\mathsf{E}}^*_{34}, && \delta\left(\overline{\mathsf{E}}^*_{23}\right) = \overline{\mathsf{E}}^*_{12}\wedge\overline{\mathsf{E}}^*_{13} + \overline{\mathsf{E}}^*_{24}\wedge\overline{\mathsf{E}}^*_{34},\\
&\delta\left(\overline{\mathsf{E}}^*_{24}\right) = \overline{\mathsf{E}}^*_{12}\wedge\overline{\mathsf{E}}^*_{14} - \overline{\mathsf{E}}^*_{23}\wedge\overline{\mathsf{E}}^*_{34}, && \delta\left(\overline{\mathsf{E}}^*_{34}\right) = \overline{\mathsf{E}}^*_{13}\wedge\overline{\mathsf{E}}^*_{14} + \overline{\mathsf{E}}^*_{23}\wedge\overline{\mathsf{E}}^*_{24}.
\end{aligned} \tag{2.6.21}$$

These equations are easy consequences of the definition of the differential δ and of equation (2.6.14).

The first equation in (2.6.21) shows immediately that Θ is exact in invariant cohomology; in fact, $\Theta = \mathrm{d}\alpha$.

On the other hand, since $V_{4,2} = SO(4)/SO(2)$, the pull-back by the canonical projection from $SO(4)$ onto $V_{4,2}$ of Ω is the exterior derivative of the $SO(4)$-invariant form on $SO(4)$, associated to $\overline{\mathsf{E}}^*_{34}$. Hence, Ω is closed, but it is not exact in invariant cohomology.

A general biinvariant 2-form on $V_{4,2}$ takes the form

$$\eta = \lambda\,\Omega + \mu\,\Theta, \quad \lambda,\mu\in\mathbb{R}.$$

Since both components are biinvariant and closed, every biinvariant form on $V_{4,2}$ is automatically closed.

We compute the differential of $\alpha\wedge\Omega$ and $\alpha\wedge\Theta$. We begin by computing the differential of $\alpha\wedge\Omega$:

$$\mathrm{d}\left(\alpha\wedge\Omega\right) = \delta\alpha\wedge\Omega = \Theta\wedge\Omega = 0,$$

by the definition of Θ and Ω.

The differential of $\alpha\wedge\Theta$ equals

$$\mathrm{d}\left(\alpha\wedge\Theta\right) = \Theta\wedge\Theta = -2\,\overline{\mathsf{E}}^*_{13}\wedge\overline{\mathsf{E}}^*_{14}\wedge\overline{\mathsf{E}}^*_{23}\wedge\overline{\mathsf{E}}^*_{24} \neq 0.$$

At degree 3, we know that a general biinvariant form on $V_{4,2}$ may be written as

$$\xi = \lambda\,\alpha\wedge\Omega + \mu\,\alpha\wedge\Theta, \quad \lambda,\mu\in\mathbb{R}.$$

Therefore, the differential of ξ equals

$$\mathrm{d}\xi = -\frac{\mu}{2}\,\Theta^2 \overset{!}{=} 0 \Longleftrightarrow \mu = 0.$$

Hence, any closed biinvariant 3-form on $V_{4,2}$ is multiple of $\alpha\wedge\Omega$. On the other hand, such a form cannot be exact in biinvariant cohomology, since any biinvariant 2 form is closed.

At degree 4, we know that there is exactly a biinvariant form, namely $\Theta \wedge \Theta$; since Θ is exact in invariant cohomology, it follows immediately that $\Theta \wedge \Theta$ is also exact.

Finally, the unique biinvariant 5-form $\alpha \wedge \Theta \wedge \Theta$ is by dimensional reasons closed, and it cannot be exact in invariant cohomology, since the only biinvariant 4-form is closed.

These results may be summarized in the following

Theorem 2.6.12. *The biinvariant cohomology of $V_{4,2}$ is given by*

$$\begin{aligned} &\mathrm{H}^0_{SO(4)}\left(V_{4,2}\right)_{SO(2)} \cong \mathbb{R}; && \mathrm{H}^1_{SO(4)}\left(V_{4,2}\right)_{SO(2)} = 0; \\ &\mathrm{H}^2_{SO(4)}\left(V_{4,2}\right)_{SO(2)} \cong \mathbb{R}; && \mathrm{H}^3_{SO(4)}\left(V_{4,2}\right)_{SO(2)} \cong \mathbb{R}; \\ &\mathrm{H}^4_{SO(4)}\left(V_{4,2}\right)_{SO(2)} = 0; && \mathrm{H}^5_{SO(4)}\left(V_{4,2}\right)_{SO(2)} \cong \mathbb{R}. \end{aligned}$$

Proof. The unique biinvariant 1-form on $V_{4,2}$ is α, which is clearly not closed. Hence, the only closed invariant form of degree 1 is 0.

At degree 2, any biinvariant form is closed; but Ω is not exact, while Θ is exact. Hence, any invariant cohomology class is represented by a multiple of the class of Ω.

Any closed biinvariant 3-form on $V_{4,2}$ is a multiple of $\alpha \wedge \Omega$, which cannot be exact in biinvariant cohomology.

Any closed biinvariant 4-form on $V_{4,2}$ is a multiple of Θ^2, which is exact in invariant cohomology.

Finally, any closed biinvariant cohomology class on $V_{4,2}$ of degree 5 is a multiple of $\alpha \wedge \Omega^2$, which is closed but not exact in invariant cohomology. $\square$

Last, we notice that, by a tedious but straightforward computation, it is possible to show that the pull-back w.r.t. the projection from $SO(4)$ onto $V_{4,2}$ the closed form $\alpha \wedge \Omega$ is not exact in $SO(4)$.

2.6.6 Invariant forms on the Grassmann manifold $\mathrm{Gr}_{m,m-2}$

In this subsection, we want to investigate the existence of left $SO(m)$-invariant forms on $\mathrm{Gr}_{m,m-2}$. The Grassmann manifold $\mathrm{Gr}_{m,m-2}$ is a homogeneous space of the form G/H, where $G = SO(m)$, and $H = SO(m-2) \oplus SO(2)$; alternatively, it may be described as the space of $m-$-dimensional subspaces of $\mathbb{R}^m$.

To characterize explicitly left $SO(m)$-invariant forms (from now on, shortly, invariant forms) on $\mathrm{Gr}_{m,m-2}$, we use Theorem 2.6.6, hence we need only investigate the existence of $SO(m-2) \oplus SO(2)$-invariant elements of the exterior product of the orthogonal complement of the Lie-algebra of $SO(m-2) \oplus SO(2)$ w.r.t. the Killing form on $\mathfrak{so}(m)$. The Lie-algebra of $SO(m-2) \oplus SO(2)$ is clearly $\mathfrak{so}(m-2) \oplus \mathfrak{so}(2)$:

$$\mathfrak{so}(m-2) \oplus \mathfrak{so}(2) = \left\{ \begin{bmatrix} \mathsf{X} & 0 \\ 0 & \mathsf{Y} \end{bmatrix} \in \mathfrak{gl}(m) \colon \mathsf{X} \in \mathfrak{so}(m-2), \quad \mathsf{Y} \in \mathfrak{so}(2) \right\}.$$

Therefore, the annihilator of $\mathfrak{so}(m-2) \oplus \mathfrak{so}(2)$ can be identified with

$$\begin{aligned} \left(\mathfrak{so}(m-2) \oplus \mathfrak{so}(2)\right)^{\perp} &= \left\{ \begin{bmatrix} 0 & \mathsf{X} \\ -\mathsf{X}^t & 0 \end{bmatrix} \in \mathfrak{gl}(m) \colon \mathsf{Y} \in M_{m-2,2} \right\} = \\ &= \left\langle \overline{\mathsf{E}}_{i,m-1}, \quad \overline{\mathsf{E}}_{i,m} \colon i = 1, \dots, m-2 \right\rangle \cong M_{m,m-2}. \end{aligned}$$

The adjoint action of $SO(m-2) \oplus SO(2)$ is easily computed:

$$\mathrm{Ad}\,(\mathsf{A} \oplus \mathsf{B}) \begin{bmatrix} 0 & \mathsf{X} \\ -\mathsf{X}^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{AXB}^t \\ -\mathsf{BX}^t\mathsf{A}^t & 0 \end{bmatrix}. \tag{2.6.22}$$

Sometimes we represent an element of $(\mathfrak{so}(m-2) \oplus \mathfrak{so}(2))^\perp$ by $\mathsf{X} \in M_{m-2,2}$. An invariant form on $\mathrm{Gr}_{m,m-2}$ is uniquely represented by a linear combination of exterior monomials of $\overline{\mathsf{E}}^*_{i,m-1}$ and $\overline{\mathsf{E}}^*_{i,m}$, $i = 1, \dots, m-2$, invariant under the action (2.6.22).

First of all, invariant forms on $\mathrm{Gr}_{m,m-2}$ of odd degree cannot exist. An invariant form η on $\mathrm{Gr}_{m,m-2}$ of odd degree is represented by an element of $\left[\bigwedge^* \left((\mathfrak{so}(m-2) \oplus \mathfrak{so}(2))^\perp\right)\right]$, invariant w.r.t. the action of $SO(m-2) \oplus SO(2)$, which we denote by the same symbol. If we let $\mathrm{id} \oplus -\mathrm{id}$ act on the orthogonal complement of $\mathfrak{so}(m-2) \oplus \mathfrak{so}(2)$, we get

$$\mathrm{Ad}(\mathrm{id} \oplus -\mathrm{id})\mathsf{X} = -\mathsf{X}, \quad \forall \mathsf{X} \in M_{m-2,2}.$$

Hence, if the form η has odd degree, the action of $\mathrm{id} \oplus -\mathrm{id}$ on η is simply multiplication by -1. If, moreover, η is $SO(m-2) \oplus SO(2)$-invariant, then

$$\left[\bigwedge^{\deg \eta} \mathrm{Ad}\,(\mathrm{id} \oplus -\mathrm{id})\,\eta\right] = \eta = -\eta = 0.$$

Therefore, there are no invariant forms of odd degree on $\mathrm{Gr}_{m,m-2}$.

Since the exterior derivative of an invariant form on $\mathrm{Gr}_{m,m-2}$ is also invariant, the any invariant form on $\mathrm{Gr}_{m,m-2}$ is closed in invariant cohomology. In fact, a general nonzero invariant form ω on $\mathrm{Gr}_{m,m-2}$ has to be of even degree. Its exterior derivative has then odd degree and is also invariant, hence it has to vanish. Moreover, it cannot be exact in invariant cohomology: if it were exact, then it would exist a nonzero invariant form of odd degree, whose exterior differential equals ω.

All these results may be summarized in the following

Theorem 2.6.13. *For the Grassmann manifold* $\mathrm{Gr}_{m,m-2}$*, the following equalities hold:*

$$\Omega^{2k+1}_{SO(m)}\left(\mathrm{Gr}_{m,m-2}\right) = 0\,;$$
$$\Omega^{2k}_{SO(m)}\left(\mathrm{Gr}_{m,m-2}\right) = \mathrm{H}^{2k}_{SO(m)(\mathrm{Gr}_{m,m-2})}\,.$$

Finally, we show that, in any dimension, a nonzero invariant form of degree 2 on $\mathrm{Gr}_{m,m-2}$ always exist, whence the invariant cohomology of $\mathrm{Gr}_{m,m-2}$ does not vanish.

Proposition 2.6.14. *The element* $\Omega \in \bigwedge^2 [\mathfrak{so}(m-2) \oplus \mathfrak{so}(2)]^\perp$*, defined by*

$$\Omega \colon = \sum_{i=1}^{m-2} \overline{\mathsf{E}}_{i,m-1} \wedge \overline{\mathsf{E}}_{i,m},$$

is invariant w.r.t. the action (2.6.18).

Proof. Since the two actions commute, it suffices to show that Ω is invariant separately w.r.t. the adjoint action of $SO(m-2)$ and of $SO(2)$.

A simple computation shows:

$$\mathrm{Ad}(\mathrm{id}\oplus\mathsf{B})\overline{\mathsf{E}}_{i,m-1} = b_{11}\overline{\mathsf{E}}_{i,m-1}+b_{21}\overline{\mathsf{E}}_{i,m}, \quad \mathrm{Ad}(\mathrm{id}\oplus\mathsf{B})\overline{\mathsf{E}}_{i,m} = b_{12}\overline{\mathsf{E}}_{i,m-1}+b_{22}\overline{\mathsf{E}}_{i,m},$$

where $\mathsf{B}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in SO(2)$. Recalling the definition of the adjoint action on the exterior product $\bigwedge^2 \mathfrak{so}(2)^{\perp}$, for any $i = 1, \ldots, m-2$, we obtain

$$\begin{aligned} \bigwedge^2 \mathrm{Ad}(\mathrm{id}\oplus\mathsf{B})\left(\overline{\mathsf{E}}_{i,m-1}\wedge\overline{\mathsf{E}}_{i,m}\right) &= \left(\mathrm{Ad}(\mathrm{id}\oplus\mathsf{B})\overline{\mathsf{E}}_{i,m-1}\right)\wedge\left(\mathrm{Ad}(\mathrm{id}\oplus\mathsf{B})\overline{\mathsf{E}}_{i,m}\right) = \\ &= \left(b_{11}\overline{\mathsf{E}}_{i,m-1}+b_{21}\overline{\mathsf{E}}_{i,m}\right)\wedge\left(b_{12}\overline{\mathsf{E}}_{i,m-1}+b_{22}\overline{\mathsf{E}}_{i,m}\right) = \\ &= \det\mathsf{B}\ \left(\overline{\mathsf{E}}_{i,m-1}\wedge\overline{\mathsf{E}}_{i,m}\right) = \\ &= \overline{\mathsf{E}}_{i,m-1}\wedge\overline{\mathsf{E}}_{i,m}, \end{aligned}$$

since B has determinant 1. Since every summand of Ω is $SO(2)$-invariant, their sum is also invariant.

For what concerns the $SO(m-2)$-invariance, a simple computation yields

$$\mathrm{Ad}(\mathsf{A}\oplus\mathrm{id})\overline{\mathsf{E}}_{i,m-1} = \sum_{j=1}^{m-2} a_{ji}\overline{\mathsf{E}}_{j,m-1}, \quad \mathrm{Ad}(\mathsf{A}\oplus\mathrm{id})\overline{\mathsf{E}}_{i,m} = \sum_{j=1}^{m-2} a_{ji}\overline{\mathsf{E}}_{j,m},$$

where

$$\mathsf{A} \colon = \begin{bmatrix} a_{11} & \cdots & a_{1,m-2} \\ \vdots & \ddots & \vdots \\ a_{m-2,1} & \cdots & a_{m-2,m-2} \end{bmatrix}$$

is a general element of $SO(m-2)$. We therefore get

$$\begin{aligned} \bigwedge^2 \mathrm{Ad}(\mathsf{A}\oplus\mathrm{id})\left(\overline{\mathsf{E}}_{i,m-1}\wedge\overline{\mathsf{E}}_{i,m}\right) &= \left(\mathrm{Ad}(\mathsf{A}\oplus\mathrm{id})\overline{\mathsf{E}}_{i,m-1}\right)\wedge\left(\mathrm{Ad}(\mathsf{A}\oplus\mathrm{id})\overline{\mathsf{E}}_{i,m}\right) = \\ &= \sum_{j,k=1}^{m-2} a_{ji}a_{ki}\left(\overline{\mathsf{E}}_{j,m-1}\wedge\overline{\mathsf{E}}_{k,m}\right) = \\ &= \sum_{j\neq k} a_{ji}a_{ki}\left(\overline{\mathsf{E}}_{j,m-1}\wedge\overline{\mathsf{E}}_{k,m}\right) + \sum_{j=1}^{m-2} a_{ji}^2\left(\overline{\mathsf{E}}_{j,m-1}\wedge\overline{\mathsf{E}}_{j,m}\right). \end{aligned}$$

If we now sum over $i = 1, \dots, m-2$ the last expression, we get

$$\begin{aligned}
\bigwedge^2 \mathrm{Ad}(\mathsf{A} \oplus \mathrm{id})\Omega &= \sum_{i=1}^{m-2} \left[\bigwedge^2 \mathrm{Ad}(\mathsf{A} \oplus \mathrm{id}) \left(\overline{\mathsf{E}}_{i,m-1} \wedge \overline{\mathsf{E}}_{i,m} \right) \right] = \\
&= \sum_{i=1}^{m-2} \left[\sum_{j \neq k} a_{ji} a_{ki} \left(\overline{\mathsf{E}}_{j,m-1} \wedge \overline{\mathsf{E}}_{k,m} \right) + \sum_{j=1}^{m-2} a_{ji}^2 \left(\overline{\mathsf{E}}_{j,m-1} \wedge \overline{\mathsf{E}}_{j,m} \right) \right] = \\
&= \sum_{j \neq k} \left(\sum_{i=1}^{m-2} a_{ji} a_{ki} \right) \left(\overline{\mathsf{E}}_{j,m-1} \wedge \overline{\mathsf{E}}_{k,m} \right) + \sum_{j=1}^{m-2} \left(\sum_{i=1}^{m-2} a_{ji}^2 \right) \left(\overline{\mathsf{E}}_{j,m-1} \wedge \overline{\mathsf{E}}_{j,m} \right) = \\
&= \sum_{i=1}^{m-2} \overline{\mathsf{E}}_{i,m-1} \wedge \overline{\mathsf{E}}_{i,m} = \\
&= \Omega.
\end{aligned}$$

In the last equations, we have used the fact that the row- and column-vectors of A are orthonormal vectors in the Euclidean $m-2$-dimensional space.

Hence, Ω is also $SO(m-2)$-invariant. □

We denote again by Ω the corresponding invariant 2-form; it is clearly nonzero.

Remark 2.6.15. We notice that Ω descends from a closed biinvariant form on $V_{m,m-2}$. Moreover, it is also possible to find a nontrivial biinvariant 3-form Ξ on $V_{m,m-2}$; Ξ is obtained from the canonical invariant 3-form on $SO(m)$ associated to

$$\xi\left(\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3\right) := \langle \mathsf{X}_1 \,,\, [\, \mathsf{X}_2 \,,\, \mathsf{X}_3 \,] \rangle , \quad \mathsf{X}_i \in \mathfrak{so}(m), i = 1,2,3,$$

where $\langle \ , \ \rangle$ denotes the normalized Killing form on $\mathfrak{so}(m)$. This shows also that biinvariant forms on $V_{m,m-2}$ exist at any degree (except at degree 1, if $m > 4$).

$SO(4)$-invariant forms on $\mathrm{Gr}_{4,2}$

In this case, similarly to the study of $SO(4)$-invariant forms on $V_{4,2}$, the computations simplify, and we may display the whole invariant coholomogy of $\mathrm{Gr}_{4,2}$.

The generators of $\left(\mathfrak{so}(2) \oplus so(2)\right)^{\perp}$ are

$$\left(\mathfrak{so}(2) \oplus so(2)\right)^{\perp} = \left\langle \overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}, \overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24} \right\rangle .$$

We will write a general element of $SO(2) \oplus SO(2)$ as $\mathsf{A} \oplus \mathsf{B} = \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{bmatrix}$, where A and B are in $SO(2)$.

First of all, by Theorem 2.6.13 there are no invariant forms at degree 1 and 3.

We then consider a general element η of $\left[\bigwedge^2 \left(\mathfrak{so}(2) \oplus so(2)\right)^{\perp} \right]_{SO(2) \oplus SO(2)}$.

Using the same arguments as in the discussion on biinvariant forms on $V_{4,2}$, any invariant form η on $\mathrm{Gr}_{4,2}$ is a linear combination of Ω and Θ, where Ω and Θ are the two biinvariant 2-forms on $V_{4,2}$:

$$\eta\left(\overline{\mathsf{E}}_{13}, \overline{\mathsf{E}}_{14}\right) = \eta\left(\overline{\mathsf{E}}_{23}, \overline{\mathsf{E}}_{24}\right) .$$

Since $(\mathfrak{so}(2) \oplus so(2))^\perp$ has dimension 4, any element of $\left[\bigwedge^4 (\mathfrak{so}(2) \oplus so(2))^\perp\right]_{SO(2)\oplus SO(2)}$ is a real multiple of $\Omega \wedge \Omega$.

Summarizing all these results, we get the following

Theorem 2.6.16. *The invariant cohomology of* $\mathrm{Gr}_{4,2}$ *is given by*

$$\begin{aligned} &\mathrm{H}^0_{SO(4)}(\mathrm{Gr}_{4,2}) \cong \mathbb{R}; && \mathrm{H}^1_{SO(4)}(\mathrm{Gr}_{4,2}) = 0; \\ &\mathrm{H}^2_{SO(4)}(\mathrm{Gr}_{4,2}) \cong \mathbb{R}^2; && \mathrm{H}^3_{SO(4)}(\mathrm{Gr}_{4,2}) = 0; \\ &\mathrm{H}^4_{SO(4)}(\mathrm{Gr}_{4,2}) \cong \mathbb{R}. \end{aligned}$$

2.7 The universal global angular form

In this section, we construct the universal global angular form by using a fermionic integral representation. This is analogous to the construction of the Mathai–Quillen representative [42] of the Thom class (see [8] and [24] and references therein). we recall that a global angular form on an oriented sphere bundle $\mathcal{S} \xrightarrow{p} M$ is a form ϑ on $\mathcal{S}$ satisfying $p_*\vartheta = 1$ and $\mathrm{d}\vartheta = -p^*e$, where e is a representative of the Euler class of the bundle; we refer also to [13], where a global angular form is constructed by means of cohomological arguments.

We notice also, referring again to [13], that the Thom-class of an oriented vector bundle $E \xrightarrow{p} M$ can be constructed by means of the global angular form of the corresponding sphere bundle $\mathrm{S}(E) \xrightarrow{p} M$ (we assume E to possess a bundle metric).

Let $Q \to M$ be an $SO(n)$-principal bundle (not necessarily $SO(M)$). Let E the associated vector bundle $Q \times_{SO(n)} E_n$ with E_n the n-dimensional Euclidean vector space. We denote by $\langle\ ,\ \rangle$ the corresponding scalar product. We consider the associated unit sphere bundle $\mathcal{S} = Q \times_{SO(n)} S^{n-1}$ as the base manifold of $\widehat{\mathcal{S}} = Q \times S^{n-1}$. We summarize all these bundles and the respective projections in the following commutative square:

$$\begin{array}{ccc} Q & \xleftarrow{\ \widehat{p}\ } & Q \times S^{n-1} = \widehat{\mathcal{S}} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\widehat{\pi}}\downarrow \\ M & \xleftarrow{\ p\ } & Q \times_{SO(n)} S^{n-1} = \mathcal{S}. \end{array}$$

We denote by θ a connection 1-form on Q. By abuse of notation, we denote again by the same symbol its pull-back w.r.t. $\widehat{p}$ (which is again a connection on $\widehat{\mathcal{S}}$), and by F its curvature. Finally, we denote by x the canonical euclidean coordinates on $\mathbb{R}^n$ (with S^{n-1} defined as the locus of $\langle\, x\,,\, x\,\rangle = 1$). We may consider x as an equivariant function on $\widehat{S}$ with values in $\mathbb{R}^n$ (which inherits the canonical representation of $SO(n)$), and by ∇x its corresponding covariant derivative, yielding a basic 1-form on $\widehat{\mathcal{S}}$ with values in $\mathbb{R}^n$. (Here, the right action of $SO(n)$ on $\widehat{\mathcal{S}}$ is defined by $(q,x)O := (qO, O^{-1}x)$.)

From now, by basic we will mean every form on $\widehat{\mathcal{S}}$, which is horizontal and invariant w.r.t. the action of $SO(n)$.

Our aim is to construct a global angular form on the trivial sphere bundle $\widehat{\mathcal{S}}$ in terms of the monomials

$$\epsilon[x; F, k; \nabla x, l] \doteq \epsilon_{a_1 \dots a_{2k+l+1}}\, x^{a_1}\, F^{a_2 a_3} \dots F^{a_{2k} a_{2k+1}} \\ (\nabla x)^{a_{2k+2}} \dots (\nabla x)^{a_{2k+l+1}}, \quad (2.7.1)$$

where $2k+l+1 = n$, $\epsilon_{ij\dots n}$ is the totally antisymmetric tensor and sums over repeated indices are understood. Observe that these monomials are basic in $\widehat{\mathcal{S}} \to \mathcal{S}$ since x, ∇x and F are horizontal and equivariant.

Our first task is to write a generating function for these monomials. To do so, we consider $\Pi T\mathbb{R}^n$. We go on denoting by x the (even) coordinates on the base and denote by ρ_i, collectively ρ, the n Grassmann coordinates on the fiber. We introduce then Berezin integration $\int [D\rho]$ by the rules:

- $\int [D\rho] P(\rho) = 0$, for any polynomial P in the odd variables ρ_i of degree strictly less than n;
- $\int [D\rho] \rho_1 \cdots \rho_n = 1$.

These two rules determine a unique Berezin integral on any polynomial in the Grassmann variables ρ (any smooth function in the variables ρ has the form of a polynomial of maximal degree n).

The generating function we are looking for reads

$$\Psi = \int [D\rho]\ \langle\, \rho\, ,\, x \,\rangle\ \exp S, \quad (2.7.2)$$

where

$$S\colon = \langle\, \rho\, ,\, \nabla x \,\rangle + \frac{\lambda}{2} \langle\, \rho\, ,\, F\rho \,\rangle\, , \quad (2.7.3)$$

and λ is a parameter. For the next discussion we need to introduce also the following generating function of basic n-forms:

$$\Phi = \int [D\rho]\ \exp S. \quad (2.7.4)$$

To prove that the forms generated by Φ and Ψ are actually basic just observe that the action of $SO(n)$ on x, ∇x and F can be compensated for by a change of variables corresponding to the fundamental representation of $SO(n)$ on the vector space generated by $\{\rho^i\}$.

Remark 2.7.1. The Thom class on $P \times_{SO(n)} \mathbb{R}^n$ can be written as a basic form on $P \times \mathbb{R}^n$ as

$$U = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{(2\pi t)^{n/2}} \int [D\rho]\ \exp\left(-\frac{1}{2t} \langle\, x\, ,\, x \,\rangle + \langle\, \rho\, ,\, \nabla x \,\rangle - \frac{t}{2} \langle\, \rho\, ,\, F\rho \,\rangle \right),$$

for any $t > 0$ [8].

So, apart form a multiplicative constant, Φ is the restriction of $U|_{t=-\lambda}$ to $P \times S^{n-1}$, while Ψ is the restriction of the form obtained contracting $U|_{t=-\lambda}$ with the radial vector field $r\, \frac{\partial}{\partial r}$.

Now we have the following

Lemma 2.7.2. Φ *and* Ψ *obey the equation:*

$$d\Psi = (-1)^{n+1}\left(n - 2\,\lambda\,\frac{\partial}{\partial\lambda} + \frac{1}{\lambda}\right)\Phi. \tag{2.7.5}$$

Proof. When differentiating a form given as in (2.7.4) or (2.7.2), we apply the following rules:

1. ρ is odd with respect to exterior derivative;
2. ρ behaves "as if" it were covariantly closed.

To justify the second rule, we first notice that, given any $n\times n$ matrix X, integration by parts shows that

$$\int [D\rho]\,\left\langle X\,\rho\,,\,\frac{\partial}{\partial\rho}\right\rangle f = \operatorname{Tr}X\int [D\rho]\,f. \tag{2.7.6}$$

(With commuting variables we would have the same relation with a minus sign on the r.h.s.)

As a consequence,

$$\int [D\rho]\;\delta f = 0,$$

with

$$\delta f = \left\langle -\theta\rho\,,\,\frac{\partial}{\partial\rho}\right\rangle f, \tag{2.7.7}$$

because θ takes values in $\mathfrak{so}(n)$. Therefore,

$$\mathrm{d}\int [D\rho]\,f = (-1)^n\int [D\rho]\,\tilde{\mathrm{d}}f,$$

where the new exterior derivative $\tilde{\mathrm{d}}$ is defined by $\mathrm{d}\pm\delta$. Introducing the covariant derivative

$$\tilde{\tilde{\nabla}} = \tilde{\mathrm{d}} + \theta\cdot,$$

we get from (2.7.7) that $\tilde{\tilde{\nabla}}\rho = 0$, that is, rule 2. In particular, we have

$$\begin{aligned}
\tilde{\mathrm{d}}\,\langle\,\rho\,,\,x\,\rangle &= -\left\langle\,\rho\,,\,\tilde{\tilde{\nabla}}x\,\right\rangle = -\,\langle\,\rho\,,\,\nabla x\,\rangle\,,\\
\tilde{\mathrm{d}}\,\langle\,\rho\,,\,\nabla x\,\rangle &= -\left\langle\,\rho\,,\,\tilde{\tilde{\nabla}}\nabla x\,\right\rangle = -\,\langle\,\rho\,,\,Fx\,\rangle\,,
\end{aligned}$$

since on x-variables $\tilde{\tilde{\nabla}} = \nabla$, and

$$\tilde{\mathrm{d}}\,\langle\,\rho\,,\,F\rho\,\rangle = 0,$$

by the Bianchi identity. Therefore,

$$\mathrm{d}\Psi = (-1)^{n+1}\,A + (-1)^n\,B,$$

with

$$A = \int [D\rho]\ \langle\, \rho\,,\, \nabla x \,\rangle\ \exp S,$$
$$B = \int [D\rho]\ \langle\, \rho\,,\, x \,\rangle\ \langle\, \rho\,,\, Fx \,\rangle\ \exp S,$$

and S defined in (2.7.3). Now, simple manipulations and the use of (2.7.6) show that

$$\begin{aligned} A &= \int [D\rho]\ \left\langle\, \rho\,,\, \frac{\partial}{\partial\rho} \,\right\rangle \exp S - \lambda \int [D\rho]\ \langle\, \rho\,,\, F\rho \,\rangle\ \exp S = \\ &= n\,\Phi - 2\,\lambda\,\frac{\partial}{\partial\lambda}\Phi. \end{aligned}$$

Similarly, we get

$$\begin{aligned} B &= -\frac{1}{\lambda} \int [D\rho]\ \langle\, \rho\,,\, x \,\rangle \left\langle\, x\,,\, \frac{\partial}{\partial\rho} \,\right\rangle \exp S + \\ &\quad + \frac{1}{\lambda} \int [D\rho]\ \langle\, \rho\,,\, x \,\rangle\ \langle\, x\,,\, \nabla x \,\rangle\ \exp S = \\ &= -\frac{1}{\lambda} \int [D\rho]\ \left(\left\langle\, x\,,\, \frac{\partial}{\partial\rho} \,\right\rangle \langle\, \rho\,,\, x \,\rangle \right) \exp S = \\ &= -\frac{1}{\lambda}\,\Phi, \end{aligned}$$

where we have used the constraint $\langle\, x\,,\, x \,\rangle = 1$ and the ensuing identity

$$0 = \mathrm{d}\,\langle\, x\,,\, x \,\rangle = 2\,\langle\, x\,,\, \nabla x \,\rangle\,.$$

□

To exploit (2.7.5), it is convenient to expand Φ and Ψ in powers of λ:

$$\Phi = \sum_{k=0}^{\infty} \lambda^k\, \Phi_k,$$
$$\Psi = \sum_{k=0}^{\infty} \lambda^k\, \Psi_k.$$

Notice that these are actually finite sums. By performing the integrations we get

$$\Phi_k = \frac{(-1)^{k+\lfloor \frac{n}{2} \rfloor}}{2^k\, k!\,(n-2k)!}\,\epsilon[F, k; \nabla x, n-2k], \tag{2.7.8}$$

for $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$, and

$$\Psi_k = \frac{(-1)^{k+\lceil \frac{n-2}{2} \rceil}}{2^k\, k!\,(n-2k-1)!}\,\epsilon[x; F, k; \nabla x, n-2k-1], \tag{2.7.9}$$

for $k = 0, 1, \dots, \lceil \frac{n-2}{2} \rceil$. Applying (2.7.5) to the power expansions, we get

$$\Phi_0 = 0, \tag{2.7.10}$$

$$\mathrm{d}\Psi_k = (-1)^{n+1} \left[(n-2k)\, \Phi_k + \Phi_{k+1} \right]. \tag{2.7.11}$$

Then we have the following

Lemma 2.7.3. *The form*

$$\overline{\vartheta} \doteq \sum_{k=0}^{s} C_k\, \Psi_k \in \Omega^{n-1}_{\text{basic}}(Q \times S^{n-1}), \tag{2.7.12}$$

with $s = \lceil \frac{n-2}{2} \rceil$, induces a global angular form ϑ on S if and only if the coefficients C_k are defined by

$$C_k = \begin{cases} (-1)^{k+s}\, \frac{(s-k)!}{2^{k+1}\, \pi^{s+1}} & \textit{for } n = 2s+2 \\ (-1)^{k+s}\, \frac{(2s-2k)!}{2^{s-k+1}\, (2\pi)^s\, (s-k)!} & \textit{for } n = 2s+1 \end{cases} \tag{2.7.13}$$

Proof. The forms $\overline{\vartheta}$ and ϑ are related by the formula

$$\overline{\vartheta} = \widehat{\pi}^* \vartheta. \tag{2.7.14}$$

The first property a global angular form has to satisfy is $p_* \vartheta = 1$. By the surjectivity of π and by (2.7.14), it suffices to show that $\widehat{p}_* \overline{\vartheta} = 1$. Since $\widehat{p}_*$ selects the θ-independent part in

$$\Psi_0 = \frac{(-1)^s}{(n-1)!}\, \epsilon_{i_1 \dots i_n}\, x^{i_1}\, (\nabla x)^{i_2} \dots (\nabla x)^{i_n},$$

this property is satisfied if and only if we set the correct normalization:

$$C_0 = \frac{(-1)^s}{\Omega_{n-1}}, \tag{2.7.15}$$

where Ω_{n-1} is the volume of the unit $(n-1)$-sphere; that is,

$$\Omega_{2s+1} = \frac{2\, \pi^{s+1}}{s!},$$

$$\Omega_{2s} = \frac{2\, (2\pi)^s}{(2s-1)!!}.$$

Next we use (2.7.10) and (2.7.11) to get

$$\mathrm{d}\overline{\vartheta} = (-1)^{n+1} \sum_{k=0}^{s} \left[(n-2k)\, C_k + C_{k-1} \right]\, \Phi_k + (-1)^{n+1}\, C_s\, \Phi_{s+1}.$$

Now recall that the differential of a global angular form must be basic on $\widehat{\mathcal{S}} \to Q$ (in particular it has to be the pullback w.r.t. p of a representative of the Euler class). By (2.7.14), together with the surjectivity of $\widehat{\pi}$, it is sufficient to show the identity

$\mathrm{d}\overline{\vartheta} = -\widehat{p}^* \pi^* e$, where e is a representative of the Euler class. All Φ_k with $k \leq s$ contain a form on S^{n-1}, so they cannot be $\widehat{p}$-basic (i.e., S^{n-1}-independent). Therefore, we must choose the coefficients C_k so that the terms in square brackets vanish. This yields a recursion rule that, once the initial condition is fixed by (2.7.15), has the unique solution (2.7.13).

Now observe that the last term Φ_{s+1} vanishes when n is odd. Therefore, ϑ is closed in this case, and this is enough to prove that it is a global angular form. If n is even, however,

$$\Phi_{s+1} = \int [D\rho] \,\exp\left(\frac{1}{2}\langle\, \rho\,,\, F\rho\,\rangle\right) = \operatorname{Pfaff} F,$$

with Pfaff denoting the Pfaffian, and the recursion fixes

$$C_s = \frac{1}{(2\pi)^{s+1}}.$$

As a consequence, in this case we get

$$\mathrm{d}\vartheta = \frac{-1}{(2\pi)^{n/2}} \operatorname{Pfaff} F.$$

Since the r.h.s. is minus (a representative of the pullback to $Q \times S^{n-1}$ of) the Euler class, the lemma is proved. $\square$

We can rewrite the results of the Lemma and (2.7.9) as follows. In the odd-dimensional case, $n = 2s+1$—cf. [28]—one has

$$\bar{\eta} = \frac{1}{2\,(4\pi)^s} \sum_{k=0}^{s} \frac{1}{k!\,(s-k)!}\, \epsilon[x; F, k; \nabla x, 2s-2k]. \tag{2.7.16}$$

In the even-dimensional case, $n = 2s+2$, we get instead

$$\bar{\eta} = \frac{1}{2\,\pi^{s+1}} \sum_{k=0}^{s} \frac{1}{4^k} \frac{(s-k)!}{k!\,(2s-2k+1)!}\, \epsilon[x; F, k; \nabla x, 2s-2k+1]. \tag{2.7.17}$$

Also observe that if one denotes by T the antipodal map on the fiber crossed with identity on the base, one has

$$T^* \vartheta = (-1)^n\, \vartheta.$$

Remark 2.7.4. From (2.7.16), we see that, *in the odd-dimensional case*, $\overline{\vartheta}$ can also be given the following expression:

$$\overline{\vartheta} = \frac{1}{2} \frac{1}{s!\,(4\pi)^s} \int [D\rho]\, \langle\, \rho\,,\, x\,\rangle\, \widetilde{S}^s = \frac{1}{2} \int [D\rho]\, \langle\, \rho\,,\, x\,\rangle \exp\left(\frac{1}{4\pi}\widetilde{S}\right),$$

with

$$\widetilde{S} = \langle\, \rho\,,\, F\,\rho\,\rangle - \langle\, \rho\,,\, \nabla x\,\rangle^2 = \langle\, \rho\,,\, (F + \nabla x\, \nabla x)\,\rho\,\rangle\,.$$

This is in accordance with the interpretation given in [10] of ϑ as one half of the Euler class of the tangent bundle along the fiber $T_{S^{n-1}}\mathcal{S}$.

2.8 Functional integrals and perturbative expansion

This section serves as a brief introduction to functional integrals and to their perturbative expansion of such quantities; we refer for more details to [56] and [46].

Roughly speaking, a functional integral is an integral over an infinite-dimensional space X of a given functional on X; usually, such a space is the space of functions in one or several variables or of sections of given vector bundles. We refer to arguments of a functional integrals as to "fields", in spite of the physical origin of functional integrals. Of course, the definition implies the notion of measure on an infinite-dimensional, which lacks in general at the moment of a rigorous definition.

If $\{\phi\}$ denotes collectively all fields, we denote the functional integral of a general functional $F(\phi)$ as

$$\int \mathcal{D}\phi \, F(\phi)\,;$$

the notation $\mathcal{D}\phi$ refers to a "formal" measure on X, which usually does not really exist.

We consider the following "Gaussian integral"

$$\mathcal{Z}(J) :\, = \int \mathcal{D}\phi \,\exp\left[-\frac{1}{2}\langle\, \phi\,,\, \mathsf{K}\phi\,\rangle + \langle\, J\,,\, \phi\,\rangle\right], \tag{2.8.1}$$

where $\langle\ ,\ \rangle$ denotes an inner product of the space of fields and K denotes a self-adjoint operator on the space of fields, such that the quadratic bilinear form $\langle\ ,\mathsf{K}\ \rangle$ is symmetric and positive-definite; the "source" J belongs to the same space of fields as the ϕ's. (If we replace $-\frac{1}{2}$ by $\frac{1}{2}$, we assume the quadratic form $\langle\ ,\ \rangle$ to be negative definite.)

Hence, the operator K possesses an inverse, which we denote by K^{-1}, and which is usually referred to as to the *propagator of* K. In most cases, the inner product $\langle\ ,\ \rangle$ is a variant of the L_2-product of functions or of sections on some vector bundle (e.g. differential forms with values in a vector bundle), and the operator K is an elliptic self-adjoint differential operator (e.g. exterior derivative, covariant derivative, Hodge Laplacian, etc. . .). Its propagator is given by the convolution with a distributional form, which we also call the propagator of K, denoted by Δ, depending on two arguments, usually on the cartesian product of the manifold M where the field are defined.

Formally, the integral (2.8.1) can be computed as to give

$$\mathcal{Z}(J) = \exp\left[\frac{1}{2}\left\langle\, J\,,\, \mathsf{K}^{-1}J\,\right\rangle\right], \tag{2.8.2}$$

where we have implicitly divided by the formal determinant of K. Of course, it is also possible to define rigorously, at least for elliptic differential operators, what is exactly understood as the determinant of K; we refer to [46] for more details on elliptic differential operators and their determinants.

Remark 2.8.1. Although for simplicity we have assumed K to be positive-definite, Formula (2.8.2) makes sense assuming only the quadratic form associated to K to be nondegenerate.

From equation (2.8.2), we derive the following useful expression for the propagator K^{-1}:

$$\Delta(x,y)=\frac{\partial\mathcal{Z}\left(J\right)}{\partial J(x)\partial J(y)}|_{J=0},$$

where the "functional derivatives" $\frac{\partial}{\partial J(x)}$ are the components of the gradient w.r.t. the inner product $\langle\ ,\ \rangle$. We notice that the propagator $\Delta(x,y)$ is in most cases a singular quantity: in fact, it presents singularities on the diagonal $\{x=y\}$. We skip for a moment the problem of the singularity of the propagator.

We consider a more general functional in the form of the exponential of

$$S\left(\phi\right)=-\frac{1}{2}\left\langle\,\phi\,,\,\mathsf{K}\phi\,\right\rangle+V_I\left(\phi\right), \tag{2.8.3}$$

where K is as before a positive-definite, self-adjoint operator on the space of fields, and $V_I\left(\phi\right)$ is a polynomial or a formal power series in the fields ϕ.

It is not difficult to check that formally the *partition function $\mathcal{Z}_S$ w.r.t. the action S* takes the form

$$\mathcal{Z}_S:\ =\int\mathcal{D}\phi\ \exp\left[-\frac{1}{2}\left\langle\,\phi\,,\,\mathsf{K}\phi\,\right\rangle+V_I\left(\phi\right)\right]=\exp\left[V_I\left(\frac{\partial}{\partial J(x)}\right)\right]\mathcal{Z}\left(J\right)|_{J=0}.$$

Since $V_I\left(J\right)$ is a polynomial or a formal power series in the fields, the expression $V_I\left(\frac{\partial}{\partial J(x)}\right)$ means simply that we replace all fields by the functional derivative w.r.t. $J(x)$, hence getting a polydifferential operator on the space of fields.

We assume ϕ to be functions on $\mathbb{R}^m$ for simplicity, and we assume $\langle\ ,\ \rangle$ to be the usual L_2-product on functions on $\mathbb{R}^m$. We then define the n-point correlation function as

$$\left\langle\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right\rangle:\ =\frac{1}{\mathcal{Z}(0)}\left[\frac{\partial}{\partial J(x_1)}\cdots\frac{\partial}{\partial J(x_n)}\mathcal{Z}\left(J\right)\right]|_{J=0}. \tag{2.8.4}$$

The usual Wick theorem (see [56] e.g.) can then be applied to the computation of (2.8.4) to give

$$\left\langle\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right\rangle=\begin{cases}0, & \text{if } n \text{ is odd}\\ \sum_\sigma\Delta\big(x_{\sigma(1)},x_{\sigma(2)}\big)\cdots\Delta\big(x_{\sigma(2m-1)},x_{\sigma((2m)}\big), & \text{if } n=2m,\end{cases} \tag{2.8.5}$$

where σ runs over all possible ways to pair the elements of $\{1,\dots,2m\}$. The Wick theorem, as exposed in equation (2.8.5), is the groundstone for the evaluation of the partition function $\mathcal{Z}_S$. Namely, if $V_I\left(\phi\right)$ has the form of a product of integrals of powers of ϕ, like e.g.

$$V_I\left(\phi\right)=\int\phi(x_1)^{d_1}\mathrm{d}x_1\cdots\int\phi\left(x_k\right)^{d_k}\mathrm{d}x_k,$$

we get

$$\begin{aligned}\mathcal{Z}_S=\sum_{n\geq0}\frac{1}{n!}\int\mathrm{d}x_1^1\cdots\int\mathrm{d}x_k^1\cdots\int\mathrm{d}x_1^n\cdots\int\mathrm{d}x_k^n\\ \left\langle\phi\left(x_1^1\right)^{d_1^1}\cdots\phi\left(x_k^1\right)^{d_k^1}\cdots\phi\left(x_1^n\right)^{d_1^n}\cdots\phi\left(x_k^n\right)^{d_k^n}\right\rangle;\end{aligned} \tag{2.8.6}$$

the correlation function $\left\langle \phi\left(x_1^1\right)^{d_1^1} \cdots \phi\left(x_k^1\right)^{d_k^1} \cdots \phi\left(x_1^n\right)^{d_1^n} \cdots \phi\left(x_k^n\right)^{d_k^n} \right\rangle$ can be then explicitly computed in terms of the propagator Δ with the help of Wick theorem.

Remark 2.8.2. Correlations functions (2.8.4) are usually ill-defined as they are given in terms of products of propagators. As we have already noticed, the propagator has singularities when both arguments collapse. Therefore, in order to give sense to correlation functions, we have to "renormalize" the divergences coming from singularities of propagators. Luckily, for our later purposes, singularities when arguments of propagators collapse can be renormalized simply by noting that the propagators extend to suitable compactifications of configuration spaces.

For computational purposes, it is sometimes better to resort to a pictorial language in order to write down explicitly expressions like (2.8.6): namely, the propagator $\Delta(x, y)$ can be represented by a segment joining the point x to the point y, and moreover any point appearing at the end of more than one segment denotes an argument which has to be integrated over.

The diagrams constructed from correlation functions in the way indicated above are called Feynman diagrams, in honor of the physicist who developed the technique of functional integrals and perturbative expansion.

2.8.1 The Faddeev–Popov procedure

We come now to some peculiarities of functional integrals. The main assumption made in order to evaluate correlation functions is the invertibility of the operator K in the quadratic part of the integrand in the functional S. In most cases, like BF theories, which we are going to analyze in the next chapter, there is a large group of symmetries $\mathcal{G}$ of the action S, which extend symmetries of the quadratic part of S. This causes the operator K to be degenerate, hence noninvertible. Apparently, in such frameworks, perturbative expansion is not possible.

However, we can apply, under particular assumptions, the Faddeev–Popov trick in order to make the quadratic part of the action nondegenerate. We refer to chapter 25 of [46] for a brief, yet illuminating, discussion of the Faddeev–Popov trick, noting here only the main features.

The main idea is to reduce the functional integral $\mathcal{Z}_S$ to an integral over a subset Σ of the space of fields X, where the quadratic part of S is nondegenerate. Assuming that the action of $\mathcal{G}$ on P is free (although in most cases it is not so, which requires more care), the subset of X we are interested is a local section of the principal bundle X with group $\mathcal{G}$ (which intersects each orbit locally exactly once), given by the zero set of a function F on X with values in a (possibly infinite-dimensional) vector space, which is frequently chosen to be the Lie algebra of $\mathcal{G}$; the function F is called the *gauge-fixing condition*. The gauge-fixing condition is usually expressed as the kernel of some self-adjoint elliptic differential operators, and it is a consequence of a generalization of the implicit function theorem in the infinite-dimensional setting and of standard results in the theory of elliptic differential operators that the zero set of such a function gives a local section of X.

The Faddeev–Popov trick consists in localizing the functional integral over X on the local section Σ by multiplying the integrand by the product of a formal delta function in $F(\phi)$ and the ratio of the volume of the intersection of the orbit with the section Σ. Typically, the ratio of the volume of the intersection of the orbit with the section Σ is expressed as the determinant of a self-adjoint elliptic differential operator M acting on the Lie algebra $\mathcal{G}$, given by the tangential map of F on the fundamental vector field associated to an element of the Lie algebra of $\mathcal{G}$; M is also self-adjoint and elliptic.

Mimicking the finite-dimensional case, the formal delta function can be rewritten also as a functional integral, at the cost of introducing additional fields with values in the Lie algebra of $\mathcal{G}$:

$$\delta(F(\phi)) = \int \mathcal{D}\lambda \, \exp \mathrm{i} \langle \lambda, F(\phi) \rangle,$$

where we denote also by $\langle \ , \ \rangle$ a $\mathcal{G}$-invariant symmetric bilinear form on the Lie algebra of $\mathcal{G}$; typically, $\langle \ , \ \rangle$ coincides with the inner product on the space of fields ϕ. Additionally, the ratio of the volume of the intersection of the orbit at a given ϕ with the section Σ, as a determinant of an elliptic differential operator M on the Lie algebra of $\mathcal{G}$, can be also written as an integral in the following way:

$$\det \mathsf{M} = \int \mathcal{D}c\mathcal{D}\overline{c} \, \exp \langle c, \mathsf{M}\overline{c} \rangle,$$

where c and $\overline{c}$ are *anticommuting fields* on the same manifold, where the fields ϕ are defined, with values in the Lie algebra of $\mathcal{G}$. The sense of introducing anticommuting fields is that we want to avoid to invert the operator M, which depends on the fields ϕ. In fact, the inverse of the operator M is also usually given in terms of a distributional kernel; such a term has not the form of an integral of a given quadratic form (we say that such a term is *nonlocal*) and the functional integration of a nonlocal term is much more difficult to perform, since we have also keep track also of the renormalization of the divergences coming from the singularities of the distributional kernel of M. Using anticommuting fields, the determinant of M can be clearly written as a local functional.

In summary, we can write the partition function of S as

$$\mathcal{Z}_S = \int \mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c} \, \exp\left[S(\phi) + \mathrm{i}\langle \lambda, F(\phi) \rangle + \langle c, \mathsf{M}\overline{c} \rangle\right]. \tag{2.8.7}$$

The gauge-fixed action

$$S_{\text{g.f.}}(\phi; \lambda; c; \overline{c}) := S(\phi) + \mathrm{i}\langle \lambda, F(\phi) \rangle + \langle c, \mathsf{M}\overline{c} \rangle$$

has now an invertible quadratic part; namely, the quadratic part of the action S is nondegenerate, since we have fixed its symmetries, and the operator M is also invertible (this is a consequence of the infinite-dimensional version of the implicit function theorem). Hence, we can start a perturbative expansion in the same spirit of (2.8.6), with the caveat that we have to introduce new propagators and new interaction terms.

2.8.2 BRST procedure

We see immediately from equation (2.8.7) that the gauge-fixed partition function of the action S depends explicitly on a choice of gauge-fixing condition. Since we want the partition function to be independent of the gauge-fixing condition, at least when we slightly vary the gauge-fixing condition, which, in mathematical terms, corresponds to an "infinitesimal" homotopy of the corresponding section Σ (i.e. a smooth curve of sections Σ_t, for small t, starting at the section Σ), it suffices to show that the derivative of the gauge-fixed partition function w.r.t. Σ_t at $t = 0$ vanishes. We want therefore to develop an algebraic setting in order to verify independence of the partition on the gauge-fixing condition.

This can be done by introducing the following operator, denoted by δ_{BRST} on the graded algebra generated by the fields $\{\phi; \lambda; c; \overline{c}\}$:

$$\delta_{\mathrm{BRST}}\phi = X_c\,(\phi)\,, \quad \delta_{\mathrm{BRST}}c = -\frac{1}{2}[\,c\,,\,c\,], \quad \delta_{\mathrm{BRST}}\overline{c} = \mathrm{i}\,\lambda, \quad \delta_{\mathrm{BRST}}\lambda = 0, \tag{2.8.8}$$

and requiring that δ_{BRST} is a derivation. The field c is called Faddeev–Popov, $\overline{c}$ the antifield to the Faddeev–Popov and λ the Lagrange multiplier. To the fields $\{\phi; \lambda; c; \overline{c}\}$ we assign a new gradation, the *ghost number*, by the rules

$$\mathrm{gh}\,\phi = 0, \quad \mathrm{gh}\,\lambda = 0, \quad \mathrm{gh}\,c = 1, \quad \mathrm{gh}\,\overline{c} = -1.$$

Clearly, the gauge-fixed action has ghost number 0, the sum of the ghost numbers of its pieces. $X_c\,(\phi)$ denotes the fundamental vector field on X associated to c, seen as an anticommuting element of the Lie algebra of $\mathcal{G}$, and $\langle\ \ ,\ \ \rangle$ denotes the Lie bracket in the Lie algebra of $\mathcal{G}$; we notice that the bracket of c with itself is nonzero, since c is anticommuting. It is not difficult to verify by equation (2.8.8) that δ_{BRST} squares to 0 by the graded Jacobi identity and by the Lie algebra isomorphism between the Lie algebra of $\mathcal{G}$ and the fundamental vertical vector fields on the space of fields ϕ, hence defining a differential on the fields of ghost number 1, the BRST differential. It is not difficult to see that δ_{BRST} can be seen as an anticommuting vector field of ghost number 1 on the space of fields $\{\phi; \lambda; c; \overline{c}\}$, which we denote by the same symbol.

The gauge-fixed action can be written as

$$S_{\text{g.f.}}\,(\phi; \lambda; c; \overline{c}) = S\,(\phi) + \delta_{\mathrm{BRST}}\Psi\,(\phi; \overline{c})\,, \tag{2.8.9}$$

where the functional $\Psi\,(\phi; \overline{c}) : \ = \langle\,\overline{c}\,,\,F\,(\phi)\,\rangle$ has ghost number -1; it is customary to call Ψ the gauge-fixing fermion associated to F.

Since δ_{BRST} is a differential and since S is $\mathcal{G}$-invariant by assumption, it follows immediately that the gauge-fixed action is δ_{BRST}-closed:

$$\delta_{\mathrm{BRST}}S_{\text{g.f.}} = 0.$$

We assume that we have a smooth family F_t of gauge-fixing conditions starting at $F_0 = F$, which translates into a smooth family of gauge-fixing fermions Ψ_t starting at Ψ; we denote by $\widetilde{\Psi}$ the derivative at 0 of Ψ_t. If we denote by $\mathcal{Z}_S(t)$ the gauge-fixed partition function w.r.t. the gauge-fixing condition Ψ_t, the derivative w.r.t. t of $\mathcal{Z}_S(t)$ at $t = 0$ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{Z}_S(t) = \int \mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c}\;\delta_{\mathrm{BRST}}\widetilde{\Psi}\,(\phi; \overline{c})\;\exp S_{\text{g.f.}}\,(\phi; \lambda; c; \overline{c})\,.$$

Since $S_{\text{g.f.}}(\phi;\lambda;c;\overline{c})$ is δ_{BRST}-closed, we can rewrite the above equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{Z}_S(t)=\int\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c}\;\delta_{\mathrm{BRST}}\left[\widetilde{\Psi}\left(\phi;\overline{c}\right)\,\exp S_{\text{g.f.}}\left(\phi;\lambda;c;\overline{c}\right)\right].$$

If the vector field δ_{BRST} is divergence-free w.r.t. the formal measure $\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c}$, it is not difficult to see that the infinitesimal variation of $\mathcal{Z}_S(t)$ vanishes, hence proving that the partition function of S is independent of the gauge-fixing condition.

More generally, we consider *vacuum expectation values of functionals $\mathcal{O}(\phi)$ w.r.t the action S* (shortly, v.e.v.'s), i.e.

$$\langle\mathcal{O}\rangle:=\int\mathcal{D}\phi\;\mathcal{O}\left(\phi\right)\,\exp S\left(\phi\right),\tag{2.8.10}$$

where $\mathcal{O}(\phi)$ is a functional on the space of fields X, which takes usually the form of a polynomial or of a formal power series in the fields ϕ. The action S is as before.

In order to compute the functional integral (2.8.10), we need to fix the symmetries. Using the Faddeev–Popov procedure, we write the gauge fixed v.e.v. of $\mathcal{O}$ w.r.t. the action S

$$\langle\mathcal{O}\rangle_{\Psi}=\int\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c}\;\mathcal{O}\left(\phi\right)\,\exp S_{\text{g.f.}}\left(\phi;\lambda;c;\overline{c}\right),$$

where the gauge-fixing condition is encoded in the gauge-fixing fermion Ψ.

We want to find conditions on the functional $\mathcal{O}$, such that the v.e.v. of $\mathcal{O}$ is also invariant on the gauge-fixing condition. Since the gauge-fixed action $S_{\text{g.f.}}(\phi;\lambda;c;\overline{c})$ is δ_{BRST}-closed and δ_{BRST} is a differential, it turns out that a sufficient condition on $\mathcal{O}$ is that

$$\delta_{\mathrm{BRST}}\mathcal{O}=0.\tag{2.8.11}$$

A functional $\mathcal{O}=\mathcal{O}(\phi)$ satisfying equation (2.8.11) is called an *observable*. If we assume $\mathcal{O}$ to be $\mathcal{G}$-invariant, equation (2.8.11) is an easy consequence of the $\mathcal{G}$-invariance, since the BRST differential δ_{BRST} on ϕ encodes the infinitesimal action of $\mathcal{G}$ on ϕ.

Summarizing the results so far, we get the following scheme for the BRST procedure: given a set of fields ϕ, a $\mathcal{G}$-invariant functional $S=S(\phi)$, which is called the action, with $\mathcal{G}$ is an infinite-dimensional Lie group, a gauge-fixing fermion $\Psi=\Psi(\phi;\overline{c})$ encoding a gauge-fixing condition making the quadratic part of the action nondegenerate, with $\overline{c}$ anticommuting fields taking values in Lie algebra of $\mathcal{G}$, and a functional $\mathcal{O}=\mathcal{O}(\phi)$, we define the gauge-fixed action corresponding to S by equation (2.8.9), where the BRST differential δ_{BRST} is defined as in equation (2.8.8) and the corresponding vector field is assumed to be divergence-free w.r.t. the formal measure $\mathcal{D}\phi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\overline{c}$.

Then the gauge-fixed v.e.v. of $\mathcal{O}$ w.r.t. the action S, denoted by $\langle\mathcal{O}\rangle_{\Psi}$ enjoys the properties:

- if $\mathcal{O}$ is BRST-closed, $\langle\mathcal{O}\rangle_{\Psi}$ does not depend on Ψ (at least in a small neighbourhood of the local section defined by Ψ);
- if $\mathcal{O}$ is δ_{BRST}-exact, $\langle\mathcal{O}\rangle_{\Psi}$ vanishes for any choice of the gauge-fixing fermion Ψ.

2.8.3 BV procedure

The BV formalism is obtained by the BRST procedure by recalling that the gauge-fixed action $S_{\text{g.f.}}$ is BRST-closed and that the differential can be also seen as a vector field on the space of fields $\{\phi; \lambda; c; \overline{c}\}$. The main idea behind the BV formalism is to encode the BRST-closedness of the gauge-fixed action and the fact that δ_{BRST} is divergence-free into a single condition. We refer to [52] for a discussion of the BV formalism starting from the BRST procedure; here we list only the main features for later purposes. This can be done if we resort to a "shifted" version of Poisson geometry in the following sense: we associate to any field $\{\phi; \lambda; c; \overline{c}\}$ a canonical antifield (playing the rôle of the associated momentum) with opposite ghost number shifted by -1; the antifields are denoted by $\left\{\phi^+; \lambda^+; c^+; \overline{c}^+\right\}$. The fields and antifields have all a parity, given by the ghost number; they can therefore be commuting or anticommuting. When we take functional derivatives w.r.t. a field or antifield, we have to take care of the parity of the field w.r.t. which we take the functional derivative; we have to introduce for this purpose left- and right-derivatives, meaning that left- ,resp. right-derivatives, act from the left to the right, resp. from the right to the left, and to such derivatives is also associated the same parity of the corresponding field; left- and right-derivatives are also related one to another by sign factor.

Further, we define the BV antibracket $(\ ,\ )$ of two functional F and G depending on fields and antifields, which we denote collectively by $\{\phi^\alpha\}$ and $\{\phi^+_\alpha\}$, by

$$(F\,,\,G):\;=\left\langle \frac{F\overleftarrow{\partial}}{\partial\phi^\alpha}\,,\,\frac{\overrightarrow{\partial}G}{\partial\phi^+_\alpha}\right\rangle - \left\langle \frac{F\overleftarrow{\partial}}{\partial\phi^+_\alpha}\,,\,\frac{\overrightarrow{\partial}G}{\partial\phi^\alpha}\right\rangle. \tag{2.8.12}$$

The arrows on the functional derivatives label their "directions".

To any monomial on both fields and antifields we can associate a ghost number by summing up all ghost numbers of its arguments. A homogeneous functional is a sum or formal series of monomials of the same ghost number. It is not difficult to verify that the BV antibracket is a "shifted" Poisson bracket, i.e. it enjoys

- $(F\,,\,G) = -(-1)^{\mathrm{gh}\,F+1}\mathrm{gh}\,G+1\ \ (G\,,\,F)$, if the F and G are supercommuting homogeneous functionals;
- $(F\,,\,G\,H) = (F\,,\,G)\ H + (-1)^{(\mathrm{gh}\,F+1)\,\mathrm{gh}\,G}G\ \ (F\,,\,H)$, for any three homogeneous supercommuting functionals;
- $(-1)^{(\mathrm{gh}\,F+1)(\mathrm{gh}\,H+1)}\,(F\,,\,(G\,,\,H)) +$ cyclic permutations $= 0$, for any three homogeneous supercommuting functionals.

The first identity is a shifted version of the antisymmetry, the second one is a shifted version of the Leibnitz rule and the third one is a shifted version of the Jacobi identity of the usual Poisson bracket.

We define the BV action S_{BV} associated to the action S by the formula

$$S_{\mathrm{BV}}\left(\phi^\alpha;\phi^+_\alpha\right):\;=S(\phi)-\frac{1}{2}\left\langle c^+\,,\,[c\,,\,c]\right\rangle+\left\langle\phi^+\,,\,X_c(\phi)\right\rangle+\left\langle\overline{c}^+\,,\,\lambda\right\rangle, \tag{2.8.13}$$

which has clearly again ghost number 0. The BV antibracket of S_{BV} with itself vanishes: namely, a simple computation shows that it is given by the sum of the BRST variation of S and terms containing the square of δ_{BRST}. It is also an easy consequence of the definition of the BV antibracket and of the BV action that the BRST differential is the "Hamiltonian" vector field $(\, S_{\mathrm{BV}}\, ,\quad)$ generated by S_{BV} and the BV antibracket. The graded Jacobi identity and the graded Leibniz rule imply together that $\delta_{BV}\colon = (\, S_{\mathrm{BV}}\, ,\quad)$ is a differential of ghost number 1. Moreover, if we set all antifields to 0 in S_{BV}, we recover immediately the original action. We take a gauge-fixing fermion $\Psi\,(\phi;\overline{c})$; if we set

$$\phi^+_\alpha = \frac{\overrightarrow{\partial}\,\Psi}{\partial\phi^\alpha},$$

we recover the gauge-fixed action $S_{\text{g.f.}}$ w.r.t. the gauge-fixing condition F.

Finally, the divergence of the vector field associated to the BV differential δ_{BV} defines the following Laplacian of the Hamiltonian S_{BV} of δ_{BV}:

$$\Delta_{BV} S\colon = \mathrm{div}\,(\, S_{\mathrm{BV}}\, ,\quad)\,. \tag{2.8.14}$$

The BV Laplacian Δ_{BV} is easily seen to square to 0 and to be a differential w.r.t. the BV antibracket

$$\Delta_{BV}\,(\, F\, ,\, G\,) = (\, \Delta_{BV} F\, ,\, G\,) + (-1)^{\mathrm{gh}\, F}\,(\, F\, ,\, \Delta_{BV} G\,)\, ,$$

for any two homogeneous supercommuting functionals. It is not a differential w.r.t. the usual product of functionals but

$$\Delta_{BV}\,(F\, G) = \Delta_{BV} F\, G + (-1)^{\mathrm{gh}\, F}\,(\, F\, ,\, G\,) + (-1)^{\mathrm{gh}\, F} F\, \Delta_{BV} G,$$

with F and G as above.

We generalize now all these computations. We assume we have an action functional $S = S\,(\phi)$, invariant w.r.t. the action of an infinite-dimensional Lie group $\mathcal{G}$; in this general case, the Lie group is not assumed to act freely on the space of fields; moreover, it is also not necessary to assume that all fields have isomorphic stabilizators. In these more general cases, the BRST operator does not square to 0; hence, we do not have a well-defined cohomology allowing us to construct meaningful physical observables. We consider the set of all fields ϕ of S plus the Faddeev–Popov ghosts for all fields, associated to the infinitesimal symmetries of the corresponding fields, associated Faddeev–Popov antifields and corresponding Lagrange multipliers, and eventually a family of ghosts for ghosts, keeping track of the reducibility of the symmetries, if the stabilizers are in general not trivia; we associate to any such field a corresponding antifield of the same type, with opposite ghost number shifted by -1. We construct the BV antibracket $(\quad ,\quad)$ generalizing in an obvious way equation (2.8.12) and a formal integration measure $\mathcal{D}\phi^\alpha\mathcal{D}\phi^+_\alpha$ w.r.t. all fields and antifields; via the formal $\mathcal{D}\phi^\alpha\mathcal{D}\phi^+_\alpha$ measure and the BV antibracket, it is possible to construct the BV Laplacian Δ_{BV}.

A functional S_{BV} of ghost number 0 in all fields and antifields is called a BV action for the action S, if it satisfies

- $\frac{1}{2}\,(\, S_{\mathrm{BV}}\, ,\, S_{\mathrm{BV}}\,) + \Delta_{BV} S_{\mathrm{BV}} = 0$; this equation is called the Quantum Master Equation;
- If we set all antifields to 0, we recover the original action S;

- If we set all antifields to 0, the Hamiltonian vector field $(\, S_{\mathrm{BV}}\,,\quad)$ reduces to the usual BRST operator δ_{BRST} (which does not necessarily square to 0).

Given a BV action for S, we construct the operator Ω_{BV}

$$\Omega_{BV} : = (\, S_{\mathrm{BV}}\,,\quad) - \Delta_{BV}. \tag{2.8.15}$$

Since S obeys the Quantum Master Equation, Ω_{BV} squares to 0, although it is not a derivation, since the BV antibracket measures the failure of the BV Laplacian to be a derivation. A gauge-fixing condition F in the BV formalism is encoded in a gauge-fixing fermion Ψ, a functional on fields of ghost number -1, such that

$$S_{\mathrm{BV}}\left[\phi^{\alpha}; \phi^{+}_{\alpha} = \frac{\overrightarrow{\partial}\,\Psi}{\partial \phi^{\alpha}}\right]$$

has an invertible quadratic part.

For an action S possessing a BV action, for a functional $\mathcal{O} = \mathcal{O}\,(\phi)$, extending to a functional depending on fields and antifields, which we still denote by the same symbol, and for a given gauge-fixing fermion Ψ, we define the v.e.v. of $\mathcal{O}$ w.r.t. S and the gauge-fixing fermion Ψ as

$$\langle \mathcal{O} \rangle_{\Psi} : = \int_{\phi^{+}_{\alpha} = \frac{\overrightarrow{\partial}\,\Psi}{\partial \phi^{\alpha}}} \mathcal{D}\phi^{\alpha} \mathcal{D}\phi^{+}_{\alpha}\; \mathcal{O}\left(\phi^{\alpha}; \phi^{+}_{\alpha}\right)\; \exp S_{\mathrm{BV}}\left(\phi^{\alpha}; \phi^{+}_{\alpha}\right). \tag{2.8.16}$$

Finally, the sense of the nilpotent operator is that it generalizes the BRST cohomology in the following way:

- if $\mathcal{O}$ is Ω_{BV}-closed, the v.e.v. (2.8.16) is independent of the gauge-fixing fermion Ψ;

- if $\mathcal{O}$ is Ω_{BV}-exact, the v.e.v. (2.8.16) vanishes, for any choice of gauge-fixing Ψ.

A functional $\mathcal{O}$ obeying $\Omega_{BV}\mathcal{O} = 0$ is called a BV observable. It is not difficult to check that the BRST procedure for a given action, if it works, is simply a special case of the BV procedure. In the cases that we are going to discuss in the following chapters, we will see that the BRST procedure fails; however, the BV procedure still works and this permits us to perform perturbative expansion.

Chapter 3

BF theories in any dimension and BV formalism

In this chapter we discuss BF theories in any dimension. We begin with the definition of BF theories. Then we analyze in detail the classical symmetries of BF theories under particular assumptions; we briefly recall the BRST quantization procedure, which however fails for BF theories in dimensions greater than 3. We then discuss the BV quantization procedure for BF theories, displaying the BV action and introducing all the necessary ingredients for the next chapters, where we define observables for BF theories. After that, we discuss covariant gauge fixing for BF theories and then the superpropagator, which plays a pivotal rôle in the perturbative expansion. Finally, we give a brief account of the AKSZ method for finding BV actions in some situations, of which BF theories are particular cases.

3.1 BF theories

3.1.1 Some preliminary assumptions for BF theories

We begin the discussion of BF theories by stating three particular assumptions, which simplify remarkably the later computations.

Assumption 1. The manifold M is compact, and there is a flat connection A_0 on P, such that all cohomology groups $H^*_{\mathrm{d}_{A_0}}(M, \mathrm{ad}\, P)$ are trivial.

If $M = \mathbb{R}^m$, which is clearly not compact, we can take the trivial connection $A_0 = 0$ (in the particular trivialization of P $x \mapsto (x, e)$, where e is the identity of G; we then consider the corresponding covariant derivative on the space of forms with rapid decrease on $\mathbb{R}^m$ with values in $\mathfrak{g}$.

Assumption 2. The principal bundle P is trivial.

Assumption 3. The Lie algebra $\mathfrak{g}$ is endowed with a symmetric, Ad-invariant, nondegenerate bilinear form $\langle\ ,\ \rangle$ (e.g., if $\mathfrak{g}$ is semisimple, we may take the Killing form). In the following, we will extend this form to $\Omega^*(M, \mathrm{ad}\, P)$ in the usual way.

3.1.2 A brief discussion of Assumption 1

Assumption 1 is very strong; we want to discuss it briefly in view of future applications (definitions of loop observables). We assume for a moment that we consider a Lie group G, whose Lie algebra $\mathfrak{g}$ satisfies the third assumption, and a compact, oriented manifold M of dimension m. In particular, the 0-th and the m-th de Rham cohomology groups are nontrivial. We assume additionally that the Lie algebra $\mathfrak{g}$ has nontrivial invariant elements w.r.t. the adjoint action of G; this is equivalent to the nonvanishing of the 0-th cohomology group of G with values in G-module $\mathfrak{g}$ via the adjoint action. We denote a nontrivial element of the 0-th cohomology of G with values in $\mathfrak{g}$ by ξ.

It can be shown that the first of the above assumptions cannot hold true. More generally, if the 0-th cohomology group of G with values in $\mathfrak{g}$ is nontrivial, and if the manifold M is compact and oriented, then there exists no flat connection A_0 on P such that $H^*_{\mathrm{d}_{A_0}}(M, \mathrm{ad}\, P)$ is trivial.

Namely, we take a general connection A on P. The nontrivial invariant element ξ induces a nonzero section of $\mathrm{ad}\, P$. In fact, a section of $\mathrm{ad}\, P$ is well-known to be in one-to-one correspondence with G-equivariant functions on P with values in $\mathfrak{g}$; we then define the following function from P to $\mathfrak{g}$:

$$\sigma_\xi(p) :\, = \xi, \quad \forall\, p \in P. \tag{3.1.1}$$

Since ξ is invariant, it follows immediately that σ_ξ is G-equivariant, thus inducing a nonzero section of $\mathrm{ad}\, P$. Moreover, the infinitesimal version of the invariance of ξ w.r.t. the adjoint action of G on $\mathfrak{g}$ takes the form

$$[\, X \,,\, \xi\,] = 0, \quad \forall\, X \in \mathfrak{g}. \tag{3.1.2}$$

If we take the covariant derivative of the section induced by σ_ξ w.r.t. any connection A on P, we get

$$\mathrm{d}_A \sigma_\xi = \mathrm{d}\sigma_\xi + [\, A \,,\, \sigma_\xi\,] = \mathrm{d}\xi + [\, A \,,\, \xi\,] = 0,$$

which is a consequence of ξ being constant, (3.1.1) and (3.1.2).

This implies that Assumption 1 is not compatible with the case of a compact, oriented manifold M and the Lie algebra $\mathfrak{g} = \mathfrak{gl}(N)$, since multiples of the identity matrix are nontrivial invariant elements of $\mathfrak{gl}(N)$. More generally, this forces us to exclude principal bundles P, whose structure group G possesses nontrivial 0-th cohomology group with coefficients in $\mathfrak{g}$.

However, we may assume that Assumption 1 holds true for odd-dimensional compact, oriented manifolds; this is in analogy with the assumption made by Axelrod and Singer in the introduction of [3].

For the even dimensional case, we may consider special even-dimensional manifolds arising as products of two odd-dimensional manifolds M_1 and M_2, one of which (say M_1) is the base space of a principal bundle P with Lie group G, satisfying Assumption 1, endowed with a flat acyclic connection A_0. We consider on M_1 the complex $(\Lambda^* T^* M_1 \otimes \operatorname{ad} P, \mathrm{d}_{A_0})$ and on M_2 the complex $(\Lambda^* T^* M_2, \mathrm{d})$; both are elliptic complexes, and the first one is acyclic by assumption. We take the exterior tensor product of the two complexes defined on $M_1 \times M_2$, with the induced differential; this is clearly again an elliptic complex, and by the Kuenneth Theorem and the acyclicity of the complex on M_1 it is acyclic. So, the existence of odd-dimensional manifolds, for which Assumptions 1 holds true implies the existence of even-dimensional manifolds, for which Assumption 1 is valid.

In summary, we have found algebraic-topological obstructions to the existence of odd-dimensional compact, oriented manifolds for which Assumption 1 is valid, but we are still not able to produce a definitive criterion for the existence of such manifolds. We work therefore under the hope that there are Lie groups G and odd-dimensional compact, oriented manifolds, for which Assumption 1 holds.

In the case $G = GL(N)$, as we have seen before, there are no such manifolds. So, in this case, i.e. in section 8, we choose implicitly $M = \mathbb{R}^n$ with the flat trivial connection, whose corresponding covariant derivative (the exterior differential) acts on forms on $\mathbb{R}^m$ with rapid decrease.

3.1.3 The BF action functional

The fundamental ingredients that we need are a connection 1-form A on P and an $(m-2)$-form of the adjoint type B. We then construct the curvature F_A of the connection and define the classical BF action as the functional

$$S^{\mathrm{cl}} = \int_M \langle\, B \,,\, F_A \,\rangle \,. \tag{3.1.3}$$

Remark 3.1.1. A more natural setting would be to consider B as a form of the coadjoint type. In this case, one would not have to introduce an invariant bilinear form on $\mathfrak{g}$, and Assumption 3 could be discarded. Instead one would use the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. We will call these theories *canonical BF theories* and will comment more on them in subsection 3.4.3. However, for the main purposes of this paper (namely, to define loop observables), one needs anyway to consider B of the adjoint type (or to introduce an isomorphism between $\mathfrak{g}$ and its dual). So we will stick for most of the paper to the setting described in this section.

Let us first compute the Euler–Lagrange equations of motion for the BF action; they are given by the couple of equations

$$F_A = 0, \quad \mathrm{d}_A B = 0. \tag{3.1.4}$$

In the following, by "on shell" we will refer to the space of solutions with the extra condition that the connection 1-form is as in Assumption 1. Next we turn to the symmetries of this action:

$$A \mapsto A^g, \qquad B \mapsto \operatorname{Ad}(g^{-1})B + \mathrm{d}_{A^g}\tau_1,$$

where by A^g we denote the right action of the gauge group element g on the connection A, and τ_1 is an element of $\Omega^{m-3}(M, \operatorname{ad} P)$. The symmetries under which the BF action is invariant can be interpreted as the action of the semidirect product $\mathcal{G} \rtimes_{\mathrm{Ad}} \Omega^{m-3}(M, \operatorname{ad} P)$ on $\mathcal{A} \times \Omega^{m-2}(M, \operatorname{ad} P)$, where $\mathcal{A}$ denotes the space of connections on P. In infinitesimal form we obtain

$$A \mapsto A + \epsilon\, \mathrm{d}_A c, \qquad B \mapsto B + \epsilon([B, c] + \mathrm{d}_A \tau_1), \tag{3.1.5}$$

where c is in $\Omega^0(M, \operatorname{ad} P)$ (the Lie algebra of $\mathcal{G}$).

These symmetries are reducible on shell, i.e. each solution $(A_0; B_0)$ with A_0 as in Assumption 1 has as isotropy group the semidirect product $\{e\} \rtimes_{\mathrm{Ad}} \{\tau_1 \in \Omega^{m-3}(M, \operatorname{ad} P) : \mathrm{d}_{A_0} \tau_1 = 0\}$. This isotropy group is isomorphic to $\Omega^{m-4}(M, \operatorname{ad} P)/\mathrm{d}_{A_0}\Omega^{m-5}(M, \operatorname{ad} P)$, because of Assumption 1; there are in this quotient nontrivial isotropy groups isomorphic to $\Omega^{m-5}(M, \operatorname{ad} P)/\mathrm{d}_{A_0}\Omega^{m-6}(M, \operatorname{ad} P)$, and so on until we arrive at $\Omega^0(M, \operatorname{ad} P)$ which acts freely on $\Omega^1(M, \operatorname{ad} P)$. Therefore, we have to adopt the extended BRST procedure to consistently fix all the symmetries, by introducing a hierarchy of ghosts for ghosts. Unfortunately the isotropy groups off shell are different from the above groups; so we have to resort to the BV formalism which generalizes BRST and works also in this case; see the next subsections for more details on both procedure.

3.1.4 The BRST procedure

For the sake of simplicity, we restrict ourselves for the moment to the special case $m = 4$.

We first promote the 0-form c and the 1-form τ_1 appearing in the infinitesimal gauge transformations (3.1.5) to anticommuting fields of ghost number 1; A (and every variation of A which is a 1-form) and B will be given ghost number 0. We then define the BRST operator δ_{BRST} for the 4-dimensional BF theory by the rules

$$\delta_{\mathrm{BRST}} A = \mathrm{d}_A c, \quad \delta_{\mathrm{BRST}} B = [B, c] + \mathrm{d}_A \tau_1, \quad \delta_{\mathrm{BRST}} c = -\frac{1}{2}[c, c], \tag{3.1.6}$$

and

$$\delta_{\mathrm{BRST}} \tau_1 = -[\tau_1, c] + \mathrm{d}_A \tau_2, \quad \delta_{\mathrm{BRST}} \tau_2 = [\tau_2, c],$$

where τ_2 is a form in $\Omega^0(M, \operatorname{ad} P)$ to which we assign ghost number two. Then δ_{BRST} is an odd operator of ghost number 1 and a differential for the Lie bracket. By the graded Leibnitz w.r.t. the ghost number, it follows that

$$\delta^2_{\mathrm{BRST}} B = [F_A, \tau_2] \neq 0,$$

while for the other fields, $\delta^2_{\mathrm{BRST}} = 0$. We notice that a sufficient condition for δ_{BRST} to be a differential is $F_A = 0$; this is exactly the first equation in (3.1.4). Otherwise the BRST quantization procedure fails, but the BRST operator closes on shell; we can therefore apply to this situation another formalism to quantize the BF theory, namely the BV quantization procedure which works well for such a theory. A similar problem arises for any $m \geq 4$.

In general, however, because of the on-shell reducibility discussed in the last subsection, we have to introduce more ghosts for ghosts τ_k with values in $\Omega^{m-2-k}(M, \operatorname{ad} P)$,

$k = 1, \ldots, m-2$, and ghost number k. The BRST operator is defined by (3.1.6) and by the rules:

$$\begin{aligned} \delta_{\mathrm{BRST}} \tau_k &= (-1)^k [\, \tau_k \,,\, c\,] + \mathrm{d}_A \tau_{k+1}, \\ \delta_{\mathrm{BRST}} \tau_{m-2} &= (-1)^m [\, \tau_{m-2} \,,\, c\,]. \end{aligned} \tag{3.1.7}$$

It is then easy to see that $\delta^2_{\mathrm{BRST}} = 0 \bmod F_A$.

The case $m = 2$ and $m = 3$ are the only ones in which the BRST formalism works, but one can apply the BV formalism there as well obtaining equivalent results.

3.2 The Batalin–Vilkovisky quantization procedure for BF theories

We now review briefly the BV formalism [4], though in a form already adapted to BF theories. For a general account on the formalism, we refer to e.g. [48] and references therein.

We consider all the fields of the theory, i.e. the connection one-form A (which we write as $A_0 + a$, where A_0 is a flat background connection on P, and a is an element of $\Omega^1(M, \operatorname{ad} P)$), the tensorial $(m-2)$-form B of adjoint type, the ghost c with values in $\Omega^0(M, \operatorname{ad} P)$ and the ghosts τ_j, $j \in \{1, \ldots, m-2\}$, for which holds: τ_j takes values in $\Omega^j(M, \operatorname{ad} P)$ and has ghost number $\operatorname{gh} \tau_j = j$.

We then associate to each field ϕ^α a canonical "antifield," denoted by ϕ^+_α, as follows: if we assume the field ϕ^α to have degree $\deg \phi^\alpha$ and ghost number $\operatorname{gh} \phi^\alpha$, the antifield ϕ^+_α is a form on M with values in $\operatorname{ad} P$, whose degree is set to be $m - \deg \phi^\alpha$ and its ghost number is set to be $-1 - \operatorname{gh} \phi^\alpha$.

The fundamental ingredients of the BV antibracket are the left and right partial derivatives of a functional F, which we are going to define precisely in the following subsection.

To simplify the notations from now on we will denote all the fields and antifields collectively as "fields" and will use the symbols φ_α, where α runs from 1 to $(2m+2)$; $\mathcal{M} := \{\varphi_\alpha\}_\alpha$.

3.2.1 Functional derivatives

We pick a commutative algebra $\mathfrak{A}$ (usually, we take $\mathfrak{A} = \mathbb{R}$ or $\mathfrak{A} = \mathbb{C}$, but see Remark 3.2.1). We are going to consider (formal) power series of local functionals in the fields taking values in $\mathfrak{A}$. We introduce a grading, which is defined on monomials in the fields and antifields as the sum of the ghost numbers of all the fields appearing in a given monomial; it is then extended by linearity. We finally consider the graded commutative algebra $\mathcal{S}(\mathfrak{A})$ generated by such objects. We then define the left and right functional derivatives of an element F in $\mathcal{S}(\mathfrak{A})$ w.r.t. the field φ_α by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\varphi_\alpha + t\rho_\alpha) = \int_M \left\langle \rho_\alpha \,,\, \frac{\overrightarrow{\partial} F}{\partial \varphi_\alpha} \right\rangle = \int_M \left\langle \frac{F \overleftarrow{\partial}}{\partial \varphi_\alpha} \,,\, \rho_\alpha \right\rangle.$$

It follows from these definitions that the functional derivatives are in general distributional forms. For convenience of notations, however, we will denote the space of distributional forms with the same symbol Ω^* used for smooth forms since this causes no harm. So the functional derivatives of F in $\mathcal{S}(\mathfrak{A})$ w.r.t. φ_α are elements of $\Omega^{p_\alpha}(M, \operatorname{ad} P \otimes \mathfrak{A})$, with the property that

$$p_\alpha := \deg \frac{\overrightarrow{\partial} F}{\partial \varphi_\alpha} = \deg \frac{F \overleftarrow{\partial}}{\partial \varphi_\alpha} = m - \deg \varphi_\alpha. \tag{3.2.1}$$

As for the ghost number one has

$$g_\alpha := \operatorname{gh} \frac{\overrightarrow{\partial} F}{\partial \varphi_\alpha} = \operatorname{gh} \frac{F \overleftarrow{\partial}}{\partial \varphi_\alpha} = \operatorname{gh} F - \operatorname{gh} \varphi_\alpha. \tag{3.2.2}$$

From the definitions and the above introduced notations, we also obtain the following useful identities:

$$\frac{\overrightarrow{\partial} F}{\partial \varphi_\alpha} = (-1)^{p_\alpha \deg \phi_\alpha + g_\alpha \operatorname{gh} \phi_\alpha} \frac{F \overleftarrow{\partial}}{\partial \varphi_\alpha}. \tag{3.2.3}$$

Beside the manifold M, we want to consider another (possibly infinite-dimensional) manifold N (e.g., $\mathrm{L}M$). In the following we will also consider (formal) power series of functionals taking values in $\Omega^*(N; E)$, for an associative algebra E (e.g, $\mathbb{R}, \mathbb{C}, \mathcal{U}(\mathfrak{g})$ or $\operatorname{End}(V)$, for a $\mathfrak{g}$-module V). On this space we introduce two gradings: the first one is the ghost number which is defined as in the case of $\mathcal{S}(\mathfrak{A})$; the second is simply the form degree on N. We denote by $\mathcal{S}^*(N; E)$ the bigraded superalgebra generated by such functionals (this superalgebra is supercommutative iff E is). Let us notice, at last, that for E an $\mathfrak{A}$-module, $\mathcal{S}(\mathfrak{A})$ is a subalgebra of $\mathcal{S}^*(N; E)$.

For the left (resp. right) derivative of a functional in $\mathcal{S}^*(N; E)$, we use the canonical identification of $\Omega^p(M, \operatorname{ad} P \otimes \Omega^q(N; E))$ with $\Omega^{p,q}(M \times N, \operatorname{ad} P \boxtimes E)$ (respectively with $\Omega^{q,p}(N \times M, E \boxtimes \operatorname{ad} P)$). We next introduce the following notations:

$$\begin{aligned} \pi_1 &: N \times M \longrightarrow N, & (\tilde{x}, x) &\longmapsto \tilde{x}, \\ \pi_2 &: N \times M \longrightarrow M, & (\tilde{x}, x) &\longmapsto x, \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}_1 &: M \times N \longrightarrow M, & (x; \tilde{x}) &\longmapsto x, \\ \tilde{\pi}_2 &: M \times N \longrightarrow N, & (x; \tilde{x}) &\longmapsto \tilde{x}. \end{aligned}$$

We have used the following useful notation: Let $\mathcal{E} \to M$ and $\mathcal{F} \to N$ vector bundles over M, resp. N. Then we define

$$\begin{aligned} \mathcal{E} \boxtimes \mathcal{F} &:= \tilde{\pi}_1^*(\mathcal{E}) \otimes \tilde{\pi}_2^*(\mathcal{F}), \quad \text{resp.} \\ \mathcal{F} \boxtimes \mathcal{E} &:= \pi_1^*(\mathcal{F}) \otimes \pi_2^*(\mathcal{E}); \end{aligned}$$

it follows that they are vector bundles over $M \times N$, resp. $N \times M$.

With these notations we can finally define the functional derivatives of F in $\mathcal{S}^*(N;E)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\varphi_\alpha + t\rho_\alpha) = \tilde{\pi}_{2*}\left\langle \tilde{\pi}_1^*\rho_\alpha \, , \, \frac{\overrightarrow{\partial} F}{\partial\varphi_\alpha} \right\rangle = \pi_{1*}\left\langle \frac{F\overleftarrow{\partial}}{\partial\varphi_\alpha} \, , \, \pi_2^*\rho_\alpha \right\rangle .$$

The functional derivatives have now two different form degrees: one is the form degree w.r.t. M and is still given by (3.2.1); the other is the form degree w.r.t. N and remains equal to $\deg F$. The ghost number is given by (3.2.2) as before.

3.2.2 The BV antibracket

We define the BV antibracket for two elements F, G in $\mathcal{S}(\mathfrak{A})$ as the functional:

$$(\, F \, , \, G \,) := \int_M \left\langle \frac{F\overleftarrow{\partial}}{\partial\phi^\alpha} \, , \, \frac{\overrightarrow{\partial} G}{\partial\phi_\alpha^+} \right\rangle - (-1)^{(m+1)\deg\phi^\alpha} \left\langle \frac{F\overleftarrow{\partial}}{\partial\phi_\alpha^+} \, , \, \frac{\overrightarrow{\partial} G}{\partial\phi^\alpha} \right\rangle .$$

We note that this functional is again in $\mathcal{S}(\mathfrak{A})$, since we integrate over M and since functional derivatives of an element of $\mathcal{S}(\mathfrak{A})$ are again clearly power series; it is not difficult to see that the ghost number of the BV antibracket of two homogeneous elements F and G in $\mathcal{S}(\mathfrak{A})$, with ghost numbers $\operatorname{gh} F$ and $\operatorname{gh} G$, is given by $\operatorname{gh} F + \operatorname{gh} G + 1$.

Next, we define the BV antibracket for two functionals F and G in $\mathcal{S}^*(N;E)$ by the formula:

$$(\, F \, , \, G \,) := \pi_{13*}\left\langle \pi_{12}^*\frac{F\overleftarrow{\partial}}{\partial\phi^\alpha} \, , \, \pi_{23}^*\frac{\overrightarrow{\partial} G}{\partial\phi_\alpha^+} \right\rangle - (-1)^{\deg\phi^\alpha(m+1)}\pi_{13*}\left\langle \pi_{12}^*\frac{F\overleftarrow{\partial}}{\partial\phi_\alpha^+} \, , \, \pi_{23}^*\frac{\overrightarrow{\partial} G}{\partial\phi^\alpha} \right\rangle , \tag{3.2.4}$$

where we use the projections

$$\begin{aligned} \pi_{12} &: N \times M \times N \to N \times M, \quad (n_1; m; n_2) \mapsto (n_1; m); \\ \pi_{23} &: N \times M \times N \to M \times N, \quad (n_1; m; n_2) \mapsto (m; n_2); \\ \pi_{13} &: N \times M \times N \to N \times N, \quad (n_1; m; n_2) \mapsto (n_1; n_2). \end{aligned}$$

This formula needs some explanations. We assume that F and G are homogeneous as elements of $\Omega^*(N;E)$, with degrees $\deg F$, resp. $\deg G$. We consider the case of a trivial algebra bundle $\mathcal{E} = N \times E$ over N; in this case, left functional derivatives are elements of $\Omega^{\deg F, p}(N \times M, E \boxtimes \operatorname{ad} P)$, while the right ones are elements of $\Omega^{q, \deg F}(M \times N, \operatorname{ad} P \boxtimes E)$. The product that we write in this case denotes two operations: the first one is the usual wedge multiplication of the form parts, while the second one is multiplication in E of the algebra part. (We refer to the beginning of Appendix 8.1 for further details.) Therefore, in this special case, the BV antibracket of two homogeneous functionals F, G, in $\mathcal{S}^*(N;E)$ gives as a result a homogeneous element of $\mathcal{S}^*(N;E)$, with degree in N equal to $\deg F + \deg G$ and ghost number $\operatorname{gh} F + \operatorname{gh} G + 1$.

We define last the BV antibracket for two special functionals, for we will often consider this case in the following: namely, we pick a functional F in $\mathcal{S}(\mathfrak{A})$ and a functional G in $\mathcal{S}^*(N;E)$, where E is an $\mathfrak{A}$-module:

$$(F \, , \, G) := \tilde{\pi}_{2*} \left\langle \tilde{\pi}_1^* \frac{F\overleftarrow{\partial}}{\partial \phi^\alpha} \, , \, \frac{\overrightarrow{\partial} G}{\partial \phi_\alpha^+} \right\rangle - (-1)^{\deg \phi^\alpha (m+1)} \tilde{\pi}_{2*} \left\langle \tilde{\pi}_1^* \frac{F\overleftarrow{\partial}}{\partial \phi_\alpha^+} \, , \, \frac{\overrightarrow{\partial} G}{\partial \phi^\alpha} \right\rangle .$$

It is clear that in this case the BV antibracket of F and G is an element of $\mathcal{S}^*(N;E)$. For homogeneous elements, the degree of the antibracket is equal to the degree of G, while $\operatorname{gh} (F \, , \, G) = \operatorname{gh} F + \operatorname{gh} G + 1$.

3.2.3 Properties of the BV antibracket

We recall first, in a unified way, the ghost and degree properties of the antibracket. We denote by $\mathcal{S}$ the algebra of functionals (which according to the case may be $\mathcal{S}(\mathfrak{A})$ or $\mathcal{S}^*(N;E)$) and by $\mathcal{S}^{p,g}$ the subspace of homogeneous functionals of form degree p and ghost number g by (in the case of $\mathcal{S}(\mathfrak{A})$, p is necessarily zero). Then the antibracket is a bilinear operator

$$(\ , \ ) : \mathcal{S}^{p,g} \otimes \mathcal{S}^{p',g'} \to \mathcal{S}^{p+p',g+g'+1}.$$

We list (without proofs) some useful identities for the BV antibracket.

We begin with the graded commutativity

$$(F \, , \, G) = -(-1)^{\deg F \deg G + (\operatorname{gh} F + 1)(\operatorname{gh} G + 1)} (G, F),$$

which holds whenever one of the two functionals is central.

The next property is the graded Jacobi identity

$$(-1)^{\deg F \deg H + (\operatorname{gh} F + 1)(\operatorname{gh} H + 1)} \, (F \, , \, (G \, , \, H)) + \text{cyclic permutations} = 0.$$

which holds whenever two of the three functionals are central.

The last property is the graded Leibniz rule

$$(F \, , \, GH) = (F \, , \, G) \, H + (-1)^{\deg F \deg G + (\operatorname{gh} F + 1) \operatorname{gh} G} G \, (F \, , \, H) \, ,$$

which holds whenever F or G or both are central. In particular, this holds in the following important cases: i) all functionals are in $\mathcal{S}(\mathfrak{A})$; ii) all functionals are in $\mathcal{S}^*(N;E)$ with E a commutative algebra; iii) F or G or both are in $\mathcal{S}(\mathfrak{A})$ and the remaining functional(s) are in $\mathcal{S}^*(N;E)$ for E an $\mathfrak{A}$-module.

Remark 3.2.1. If we restrict ourselves to $\mathcal{S}(\mathfrak{A})$, then the above properties hold on the whole algebra. An algebra with a bracket satisfying the above properties is known as a Gerstenhaber algebra [31].

The Leibniz rule will play a key-rôle in the following section, where we define the BV operator via the BV antibracket; the functional F will be there the BV action for the BF theory. Let us in fact suppose that we have a homogeneous local functional S in $\mathcal{S}(\mathfrak{A})$ with even ghost number (usually, $\mathfrak{A} = \mathbb{R}$ and $\operatorname{gh} S = 0$). We can then define the following operator on the superalgebra $\mathcal{S}^*(N;E)$, with E an $\mathfrak{A}$-module:

$$\delta_S F := (S \, , \, F) \, .$$

It follows easily from the $\mathfrak{A}$-linearity of the BV antibracket that δ_S is a $\mathfrak{A}$-linear operator on the algebra $\mathcal{S}^*(N; E)$. The most important property of such an operator is an immediate consequence of the Leibniz rule written above; namely,

$$\delta_S(FG) = (\delta_S F)G + (-1)^{\mathrm{gh}\, F} F(\delta_S G);$$

i.e. the operator δ_S is a graded $(0, \mathrm{gh}\, S+1)$-derivation on $\mathcal{S}$. (If moreover $(\, S\, ,\, S\,) = 0$, then the Jacobi identity implies $\delta_S^2 = 0$.)

We now list some other useful properties of the derivation δ_S.

Lemma 3.2.2. *We assume that the functional F lies in $\mathcal{S}^*(N; E)$, and that we have a map $h : H \to N$ from a manifold H to the manifold N, then the following identity holds:*

$$\delta_S[h^*(F)] = h^*(\delta_S F).$$

Lemma 3.2.3. *We assume we have a functional F in $\mathcal{S}^*(H; E)$, where H is the total space of a fiber bundle over N with typical fiber a manifold B (possibly with boundaries or corners) and projection π. The integration along the fiber of the functional F yields a functional in $\mathcal{S}^*(N; E)$ with degree $\deg \pi_*(F) = \deg F - \dim B$, if we suppose additionally that F is homogeneous in the degree. Then we obtain the following identity:*

$$\delta_S[\pi_*(F)] = \pi_*(\delta_S F)$$

Lemma 3.2.4. *We assume we have a functional F in $\mathcal{S}^*(N; E)$, for a manifold N and an algebra E. Let us denote by d the exterior derivative on N. Then the following identity holds:*

$$\delta_S(\mathrm{d}F) = \mathrm{d}(\delta_S F).$$

Lemma 3.2.5. *We assume that the functional F belong to the superalgebra $\mathcal{S}^*(N; \mathfrak{g})$; let us suppose additionally that we have a $\mathfrak{g}$-module (V, ρ). The application of Tr_ρ to F gives an element of $\mathcal{S}^*(N; \mathbb{R})$. Then we obtain the following identity:*

$$\delta_S[\mathrm{Tr}_\rho(F)] = \mathrm{Tr}_\rho(\delta_S F).$$

We will only sketch a few ideas of the proofs of the above lemmata.

For Lemma 3.2.2, we have to write down explicitly the expressions for the two BV antibrackets, which in this special case involve the push-forward w.r.t the projection $\bar{\pi}_1 : H \times M \to H$, resp. $\pi_1 : N \times M \to M$, and the pull-backs w.r.t. the maps $\bar{\pi}_2 : M \times H \to H$, resp. $\pi_2 : M \times N \to M$; these maps do appear in the definition of the partial functional derivatives of F. Then we have to consider the following commutative diagram:

$$\begin{array}{ccc} M \times H & \xrightarrow{\mathrm{id}\times h} & M \times N \\ {\scriptstyle \bar{\pi}_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ H & \xrightarrow{h} & N \end{array}$$

It is easy to see that $\mathrm{id} \times h$ induces an orientation preserving map (namely, the identity map) between the fibers $(\bar{\pi}_2)^{-1}(\{e\})$ ($\cong \{e\} \times M$) and $(\pi_2)^{-1}(\pi(e))$ ($\cong \{\pi(e)\} \times M$), for $e \in H$. From Lemma 2.2.1, the claim follows.

For Lemma 3.2.3, we have to write down again explicitly the BV antibrackets on the two sides of the identity. In this case, we use the following commutative diagram:

$$\begin{array}{ccc} M\times H & \xrightarrow{\mathrm{id}\times\pi} & M\times N \\ {\scriptstyle\tilde\pi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_2} \\ H & \xrightarrow{\pi} & N \end{array}$$

The commutativity of this diagram implies that the composite bundles $\mathcal{H}_{B\times M} = (M \times H; \pi\circ\tilde\pi_2; N; B\times M)$ and $\mathcal{H}_{M\times B} = (M\times H; \pi_2\circ(\mathrm{id}\times\pi); N; M\times B)$ possess the same total space and the same base space, but have different fibers; in fact, the fiber of the first is isomorphic to $B\times M$, while the fiber of the second one is $M\times B$. We can go from a bundle to the other via a bundle isomorphism which is the identity on the total and on the base space, but which reverses the orientation of the fibers. It is well-known that the orientation of a fiber bundle is induced by the orientations both of the base space and of the fiber; this implies the following identity

$$\pi_*\circ\tilde\pi_{2*} = (-1)^{m\dim B}\,\pi_{2*}\circ(\mathrm{id}\times\pi)_*,$$

and the coefficient $(-1)^{m\dim B}$ comes from the orientation reversal of the fibers of the two bundles (for the property of the push-forward, see Lemma 2.2.2). This identity will imply the claim.

For Lemma 3.2.4 we simply apply the generalized Stokes theorem for the push-forward w.r.t. $\tilde\pi_2 : M\times N \to N$; notice that in this case the fiber, i.e. M, has no boundary. We then have to remember that the exterior derivative $\mathrm{d}_{M\times N}$ on $M\times N$ splits as $\mathrm{d}_N + \sigma\mathrm{d}_M$, where the sign σ is given by $\sigma = (-1)^{\deg_N(\omega)}$, for a form ω on $M\times N$ with degree over N equal to $\deg_N(\omega)$. We have to remember also that, in the defining equation for the right functional derivative, the test form is independent of N, therefore the exterior derivative on N applied to (the pullback w.r.t. $\tilde\pi_1$ of) the test form gives 0 as result; next, we know that the integrand has maximal degree w.r.t. M, so that the exterior derivative w.r.t. M of the integrand gives 0. The result is a consequence of all the above considerations.

Lemma 3.2.5 follows easily from the definition of the partial functional derivatives and from the fact the trace Tr_ρ acts only on the $\mathrm{End}\,V$-part of the tensor product (we recall that the functionals we are considering take their values in $\mathrm{End}(V)$ for a $\mathfrak{g}$-module V).

3.2.4 The BV Laplacian

We introduce temporarily a Riemannian metric on M and we denote by $\star$ the induced Hodge star operator. We also pick a field ϕ^α; we denote by ϕ^*_α the field (called sometimes the *Hodge dual antifield* of ϕ^α) defined by the formula

$$\phi^*_\alpha := \star\phi^+_\alpha,$$

where ϕ^+_α is the antifield of ϕ^α. It follows easily from the definition that the degree of ϕ^*_α is given by the degree of the field it is associated to, while its ghost number is given by $-1-\mathrm{gh}\,\phi^\alpha$.

If α, β are two elements in $\Omega^p(M, \mathrm{ad}\, P)$, we define

$$\langle\, \alpha\,,\, \beta\,\rangle_{\mathrm{Hodge}} := \int_M \langle\, \alpha\,,\, \star\beta\,\rangle\,.$$

Now, we define a new type of functional derivatives. We begin with functionals in the space $\mathcal{S}(\mathfrak{A})$. We denote collectively by φ_α the fields ϕ^α and their newly defined antifields ϕ^*_α.

We assume ρ_α to be a form with the same degree and ghost number as φ_α and F to be an element in $\mathcal{S}(\mathfrak{A})$; we then define the Hodge functional derivatives of F by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\varphi_\alpha + t\rho_\alpha) = \left\langle\, \rho_\alpha\,,\, \frac{\overrightarrow{\delta}\, F}{\delta\varphi_\alpha}\,\right\rangle_{\mathrm{Hodge}} = \left\langle\, \frac{F\,\overleftarrow{\delta}}{\delta\varphi_\alpha}\,,\, \rho_\alpha\,\right\rangle_{\mathrm{Hodge}}.$$

It follows from the definition that, for a homogeneous functional F, the Hodge functional derivatives w.r.t. φ_α lie in $\Omega^{\deg\varphi_\alpha}(M, \mathrm{ad}\, P)$ and possess ghost number equal to $\mathrm{gh}\, F - \mathrm{gh}\,\varphi_\alpha$. We have now at our disposal the essential elements to construct the BV Laplacian. We start defining the BV Laplacian of an element of $\mathcal{S}(\mathfrak{A})$ by the formula

$$\Delta_{BV} F := \sum_\alpha (-1)^{\mathrm{gh}\,\phi^\alpha} \left\langle\, \frac{\overrightarrow{\delta}}{\delta\phi^\alpha}\,,\, \frac{\overrightarrow{\delta}\, F}{\delta\phi^*_\alpha}\,\right\rangle_{\mathrm{Hodge}}. \tag{3.2.5}$$

The result is again a functional in $\mathcal{S}(\mathfrak{A})$, and, if F is homogeneous, then $\Delta_{BV} F$ is homogeneous of ghost number $\mathrm{gh}\, F + 1$.

Remark 3.2.6. This definition can also be extended to functionals in the space $\mathcal{S}^*(N; E)$ in analogy with the construction presented in the preceding subsection. For a homogeneous functional G in $\mathcal{S}^*(N; E)$, $\Delta_{BV} G$ is again a functional in $\mathcal{S}^*(N; E)$, whose ghost number is given by $\mathrm{gh}\, G + 1$ and whose degree is unchanged.

Remark 3.2.7. Turning to a unified notation $\mathcal{S}$, we have in general

$$\Delta_{BV} : \mathcal{S}^{p,g} \to \mathcal{S}^{p,g+1}.$$

We notice however that Δ_{BV} is in general ill-defined for all functionals in $\mathcal{S}$. It is particularly ill-defined on local functionals. The correct definition would involve some regularization. *We assume however that, independently of the regularization, $\Delta_{BV} F = 0$ whenever F depends only on one element in each pair field–antifield, as the formal definition of Δ_{BV} suggests.*

The properties of the BV Laplacian Δ_{BV} are:

- the BV Laplacian is a coboundary operator, i.e.

$$\Delta^2_{BV} = 0;$$

- the BV antibracket measures the failure of the BV Laplacian to be a derivation, i.e.

$$\Delta_{BV}(FG) = (\Delta_{BV} F)G + (-1)^{\mathrm{gh}\, F}\,(\, F\,,\, G\,) + (-1)^{\mathrm{gh}\, F} F(\Delta_{BV} G), \tag{3.2.6}$$

  where one of the functionals must be central.

The latter property in particular implies that the BV Laplacian is well-defined on the subalgebra generated by those local functionals which are killed by Δ_{BV} (e.g., those described in the previous remark).

Remark 3.2.8. If we restrict ourselves to $\mathcal{S}(\mathfrak{A})$, then the above properties hold on the whole algebra. A Gerstenhaber algebra with an operator Δ satisfying the above properties is known as a BV algebra.

Remark 3.2.9. We notice that we can define (independently of the dimension) the BV antibracket by

$$(F \, , \, G) := \left\langle \frac{F \overleftarrow{\delta}}{\delta \phi^{\alpha}} \, , \, \frac{\overrightarrow{\delta} G}{\delta \phi^{*}_{\alpha}} \right\rangle_{\text{Hodge}} - \left\langle \frac{F \overleftarrow{\delta}}{\delta \phi^{*}_{\alpha}} \, , \, \frac{\overrightarrow{\delta} G}{\delta \phi^{\alpha}} \right\rangle_{\text{Hodge}} .$$

This is the definition of the BV antibracket in its original setting [4]. This expression depends in general on the Riemannian metric on M, but in the case of BF theories the antibracket is actually independent thereof, since it is equal to the one defined in subsection 3.2.2.

3.2.5 BV cohomology and observables

We have introduced the BV Laplacian in order to deal with the quantum version of the BV formalism, which is needed when considering functional integrals with weight $\exp(\mathrm{i}/\hbar)S$, where S should be a solution of the *quantum master equation*

$$(S \, , \, S) - 2\mathrm{i}\hbar \Delta_{BV} S = 0.$$

The main consequence of the quantum master equation is that the operator

$$\Omega_{BV} := \delta_{BV} - \mathrm{i}\hbar\Delta_{BV} \tag{3.2.7}$$

is a coboundary operator of ghost number 1; it is *not* a differential, because of (3.2.6). This operator is fundamental in the BV formalism; namely, all meaningful observables in the BV formalism lie in the 0-ghost number cohomology of Ω_{BV}. This means (at least formally) that the vacuum expectation values of Ω_{BV}-cohomology classes, weighted by $\exp(\mathrm{i}/\hbar)S$, are independent of the choice of gauge fixing. In turn, the v.e.v.s of trivial Ω_{BV}-cohomology classes or of classes of ghost number different from zero vanish.

We will show that the BV action S of BF theories, to be introduced in (3.4.1), satisfies separately the equations

$$\Delta_{BV} S = 0 \quad \text{and} \quad (S \, , \, S) = 0,$$

which imply that S satisfies the quantum master equation.

3.3 The BV superformalism for BF theories

The aim of this section is to define a new type of BV antibracket, which will allow us to obtain the BV action for BF theories in a simple way and to write it in a compact form.

From now on we consider a new grading on the space of functionals $\mathcal{S}$ called the total degree, which is defined as the sum of the form degree and the ghost number; we will denote the total degree of a form α with degree $\deg\alpha$ and ghost number $\operatorname{gh}\alpha$ by $|\alpha| := \deg\alpha + \operatorname{gh}\alpha$; by homogeneous we will mean homogeneous w.r.t. the total degree.

We note now that all fields $\left\{c^+; a^+; B; \tau_1; \ldots; \tau_{m-2}\right\}$ have total degree $m-2$, while all remaining fields $\left\{\tau^+_{m-2}; \ldots; \tau^+_1; B^+; a; c\right\}$ have total degree equal to 1. Here a is the difference between A and a given background connection A_0 as in Assumption 1; for notational consistency, we denote by a^+ the associated antifield. We can cast all fields in two homogeneous superforms which we will denote by B and A:

$$\mathsf{B} := \sum_{k=1}^{m-2} (-1)^{\frac{k(k-1)}{2}} \tau_k + B + (-1)^m a^+ + c^+, \tag{3.3.1}$$

$$\mathsf{A} := (-1)^{m+1} c + A + (-1)^m B^+ + \sum_{k=1}^{m-2} (-1)^{\frac{k(k-1)}{2} + m(k+1)} \tau^+_k. \tag{3.3.2}$$

Further, we define $\mathsf{a} := \mathsf{A} - A_0$.

We refer to Appendix 8.1, for the definitions of the dot product $\cdot$ and of the dot Lie bracket $[\![\ ;\ ]\!]$. We only recall that the dot structures make the algebra $\mathcal{S}$ into a superalgebra w.r.t. the total degree. Analogously, we define the dot version $\langle\!\langle\ ;\ \rangle\!\rangle$ of the bilinear form $\langle\ ,\ \rangle$ on $\Omega^*(M, \operatorname{ad} P)$ by

$$\langle\!\langle\, \alpha\,;\, \beta \,\rangle\!\rangle := (-1)^{\operatorname{gh}\alpha \deg\beta} \left\langle\, \alpha\,,\, \beta\,\right\rangle.$$

It satisfies

$$\langle\!\langle\, \beta\,;\, \alpha \,\rangle\!\rangle = (-1)^{|\alpha||\beta|} \langle\!\langle\, \alpha\,;\, \beta \,\rangle\!\rangle.$$

3.3.1 The space of functionals $\mathcal{S}_{\mathsf{A},\mathsf{B}}$

As in the previous section, we consider the algebra generated by local functionals in the fields taking values in a commutative algebra $\mathfrak{A}$ or in a de Rham complex $\Omega^*(N; E)$. However, from now on we will restrict ourselves only to those functionals which depend on the linear combinations A and B (and not on the component fields). We will denote these algebras by $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$, resp. $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N; E)$, or generically by $\mathcal{S}_{\mathsf{A},\mathsf{B}}$.

We give $\mathcal{S}_{\mathsf{A},\mathsf{B}}$ a unique grading, by defining the degree of a monomial in the superfields A and B to be the sum of the total degrees of its factors (modulo e.g. integration).

Since the superform a has total degree 1 and lies in $\Omega^*(M, \operatorname{ad} P)$, we can consider A as a superconnection in the sense of Mathai and Quillen [42]. With the help of the dot Lie bracket (see Appendix 8.1), we can then define the covariant derivative of B w.r.t. the superconnection A and the supercurvature F_{A} by:

$$\begin{aligned} \mathrm{d}_{\mathsf{A}}\mathsf{B} &:= \mathrm{d}_{A_0}\mathsf{B} + [\![\mathsf{a}\,;\,\mathsf{B}]\!], \\ \mathsf{F}_{\mathsf{A}} &:= \mathrm{d}_{A_0}\mathsf{a} + \frac{1}{2}[\![\mathsf{a}\,;\,\mathsf{a}]\!]. \end{aligned}$$

Notice that the supercurvature would contain the extra term F_{A_0} if the background connection A_0 were not chosen to be flat. Note that in this new context the exterior and covariant derivatives are operators of total degree 1.

The super functional derivatives

We begin by introducing the super test forms ρ_{a} and ρ_{B}: the super test form ρ_{a} is defined to be the sum of the test forms corresponding to the fields that appear in the superform a, with the same sign convention as in (3.3.2); analogously we define the super test form ρ_{B}. By definition, the super test forms have then total degree 1, resp. $m-2$. We then define the super functional derivatives of an element F in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$ by:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a}+t\rho_{\mathsf{a}};\mathsf{B}) &= \int_M \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\mathsf{a}}\ ;\ \rho_{\mathsf{a}} \right\rangle\!\!\right\rangle = \int_M \left\langle\!\!\left\langle \rho_{\mathsf{a}}\ ;\ \frac{\overrightarrow{\partial}F}{\partial\mathsf{a}} \right\rangle\!\!\right\rangle,\\
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a};\mathsf{B}+t\rho_{\mathsf{B}}) &= \int_M \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\mathsf{B}}\ ;\ \rho_{\mathsf{B}} \right\rangle\!\!\right\rangle = \int_M \left\langle\!\!\left\langle \rho_{\mathsf{B}}\ ;\ \frac{\overrightarrow{\partial}F}{\partial\mathsf{B}} \right\rangle\!\!\right\rangle.
\end{aligned}
$$

It is then easy to determine the total degree of the super functional derivatives of F; in fact, the following identities hold:

$$
\begin{aligned}
\left|\frac{\overrightarrow{\partial}F}{\partial\mathsf{a}}\right| &= \left|\frac{F\overleftarrow{\partial}}{\partial\mathsf{a}}\right| = |F|+m-1,\\
\left|\frac{\overrightarrow{\partial}F}{\partial\mathsf{B}}\right| &= \left|\frac{F\overleftarrow{\partial}}{\partial\mathsf{B}}\right| = |F|+2.
\end{aligned}
\tag{3.3.3}
$$

It will be also useful to express the right derivative of the functional F in terms of the left one, and vice versa. The result of this computation is given by:

$$
\begin{aligned}
\frac{\overrightarrow{\partial}F}{\partial\mathsf{a}} &= (-1)^{|F|+m-1}\frac{F\overleftarrow{\partial}}{\partial\mathsf{a}},\\
\frac{\overrightarrow{\partial}F}{\partial\mathsf{B}} &= (-1)^{|F|m}\frac{F\overleftarrow{\partial}}{\partial\mathsf{B}}.
\end{aligned}
$$

We next define the super functional derivatives of an element F of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$ by:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a}+t\rho_{\mathsf{a}};\mathsf{B}) &= \tilde{\pi}_{2*}\left\langle\!\!\left\langle \tilde{\pi}_1^*\rho_{\mathsf{a}}\ ;\ \frac{\overrightarrow{\partial}F}{\partial\mathsf{a}} \right\rangle\!\!\right\rangle = \pi_{1*}\left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\mathsf{a}}\ ;\ \pi_2^*\rho_{\mathsf{a}} \right\rangle\!\!\right\rangle;\\
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a};\mathsf{B}+t\rho_{\mathsf{B}}) &= \tilde{\pi}_{2*}\left\langle\!\!\left\langle \tilde{\pi}_1^*\rho_{\mathsf{B}}\ ;\ \frac{\overrightarrow{\partial}F}{\partial\mathsf{B}} \right\rangle\!\!\right\rangle = \pi_{1*}\left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\mathsf{B}}\ ;\ \pi_2^*\rho_{\mathsf{B}} \right\rangle\!\!\right\rangle.
\end{aligned}
$$

Their total degrees are still given by (3.3.3).

The super BV antibracket

Let us pick two functionals F and G in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$; then the super BV antibracket is defined by:

$$(\!(\, F\,;\, G\,)\!) := \int_M \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{B}}\,;\,\frac{\overrightarrow{\partial}G}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle - (-1)^m \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\,;\,\frac{\overrightarrow{\partial}G}{\partial \mathsf{B}} \right\rangle\!\!\right\rangle. \tag{3.3.4}$$

Note that the BV antibracket of F and G is again a functional in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$.

Next we consider a functional F in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$ and a functional G in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$, with E an $\mathfrak{A}$-module; we define the BV antibracket of F and G by:

$$(\!(\, F\,;\, G\,)\!) = \tilde{\pi}_{2*} \left\langle\!\!\left\langle \tilde{\pi}_1^*\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{B}}\Big)\,;\,\frac{\overrightarrow{\partial}G}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle - (-1)^{\dim M}\tilde{\pi}_{2*} \left\langle\!\!\left\langle \tilde{\pi}_1^*\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\Big)\,;\,\frac{\overrightarrow{\partial}G}{\partial \mathsf{B}} \right\rangle\!\!\right\rangle. \tag{3.3.5}$$

In this case the BV antibracket of F and G is a functional in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$.

We finally define the BV antibracket of two functionals F and G in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$ by:

$$(\!(\, F\,;\, G\,)\!) := \pi_{13*} \left\langle\!\!\left\langle \pi_{12}^*\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{B}}\Big)\,;\,\pi_{23}^*\Big(\frac{\overrightarrow{\partial}G}{\partial \mathsf{a}}\Big) \right\rangle\!\!\right\rangle - (-1)^m\pi_{13}^* \left\langle\!\!\left\langle \pi_{12}^*\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\Big)\,;\,\pi_{23}^*\Big(\frac{G\overleftarrow{\partial}}{\partial \mathsf{B}}\Big) \right\rangle\!\!\right\rangle.$$

In this case we obtain that $(\!(\, F\,;\, G\,)\!)$ is a functional in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N,E)$.

The antibracket, in all the above cases, has total degree 1; i.e., if we denote generically by $\mathcal{S}_{\mathsf{A},\mathsf{B}}$ the space of functionals and by $\mathcal{S}^k_{\mathsf{A},\mathsf{B}}$ the subspace of homogeneous elements of total degree k, then

$$(\!(\;\,;\;\,)\!) : \mathcal{S}^k_{\mathsf{A},\mathsf{B}} \otimes \mathcal{S}^l_{\mathsf{A},\mathsf{B}} \to \mathcal{S}^{k+l+1}_{\mathsf{A},\mathsf{B}}.$$

From now on we will use the short notation given in (3.3.4) for all types of functionals that we have discussed until now, and we omit in all cases the specific notation, leaving to the reader the understanding of the real meaning of the formula.

3.3.2 Main properties of the super BV antibracket

One could now wonder if there is a relationship between the super BV antibracket defined in the previous subsection and the BV antibracket defined in 3.2.2 that we have discussed in the previous subsection. We begin by explaining this relationship for the case of functionals in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$.

Lemma 3.3.1. *Suppose that we have two functionals F and G in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$; then the following identity holds:*

$$(\!(\, F\,;\, G\,)\!) = (\, F\,,\, G\,)\,. \tag{3.3.6}$$

Proof. We prove the identity for homogeneous functionals; the general case follows by linearity. We begin by computing the functional derivatives of F and G:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a}+t\rho_{\mathsf{a}};\mathsf{B}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(a+t\rho_a; c+t\rho_c;\ldots;\mathsf{B}) = \int_M \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\,;\,\rho_{\mathsf{a}} \right\rangle\!\!\right\rangle.$$

Next, we note that the integral selects the part of the integrand whose form degree in M is equal to m, and that the super test form ρ_{a} is the sum of the usual test forms (with some signs to be considered). We write ρ_{a} as

$$\rho_{\mathsf{a}} = \sum_{i=0}^{m} \sigma_{\mathsf{a}_i} \rho_{\mathsf{a}_i},$$

where by ρ_{a_i} we denote the degree i part of ρ_{a}; i.e., $\rho_{\mathsf{a}_0} = \rho_c$, $\rho_{\mathsf{a}_1} = \rho_a$ and so on. The signs σ_{a_i} are the same as in the definition (3.3.2) of A; namely, $\mathsf{a} = \sum \sigma_{\mathsf{a}_i} \mathsf{a}_i$. Similarly we introduce signs σ_{B_j} as in $\mathsf{B} = \sum \sigma_{\mathsf{B}_j} \mathsf{B}_j$. We can then write:

$$\begin{aligned} \int_M \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{a}} \,;\, \rho_{\mathsf{a}} \right\rangle\!\!\right\rangle &= \int_M \left\langle \frac{F\overleftarrow{\partial}}{\partial c} \,,\, \rho_c \right\rangle + \int_M \left\langle \frac{F\overleftarrow{\partial}}{\partial a} \,,\, \rho_a \right\rangle + \dots \\ &= \sum_{j=0}^{m} \sigma_{\mathsf{a}_j} \int_M \left\langle\!\!\left\langle \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\right)_{m-j} \,;\, \rho_{\mathsf{a}_j} \right\rangle\!\!\right\rangle, \end{aligned} \tag{3.3.7}$$

where the subscript denotes the restriction to the term of the indicated form degree. We recall that $\operatorname{gh}\rho_{\mathsf{a}_j} = 1 - j$; then we obtain e.g. for the j-th term in the last expression of the above identity (recalling the definition of the total degree of the functional derivative of F w.r.t. a):

$$\int_M \left\langle\!\!\left\langle \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\right)_{m-j} \,;\, \rho_{\mathsf{a}_j} \right\rangle\!\!\right\rangle = (-1)^{(|F|+j-1)j} \int_M \left\langle \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\right)_{m-j} \,,\, \rho_{\mathsf{a}_j} \right\rangle$$

By confronting the two expressions in (3.3.7), and doing similar computations in the other cases, we obtain for $j = 0, \dots, m$:

$$\begin{aligned} \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\right)_{m-j} &= \sigma_{\mathsf{a}_j} (-1)^{(|F|+j-1)j} \frac{F\overleftarrow{\partial}}{\partial \mathsf{a}_j}, & \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{B}}\right)_{m-j} &= \sigma_{\mathsf{B}_j} (-1)^{(m-2-j)(m-j)} \frac{F\overleftarrow{\partial}}{\partial \mathsf{B}_j}, \\ \left(\frac{\overrightarrow{\partial} F}{\partial \mathsf{a}}\right)_{m-j} &= \sigma_{\mathsf{a}_j} (-1)^{(1-j)(m-j)} \frac{\overrightarrow{\partial} F}{\partial \mathsf{a}_j}, & \left(\frac{\overrightarrow{\partial} F}{\partial \mathsf{B}}\right)_{m-j} &= \sigma_{\mathsf{B}_j} (-1)^{(|F|-m+2+j)j} \frac{\overrightarrow{\partial} F}{\partial \mathsf{B}_j}, \end{aligned}$$

We cast then all these expressions in the definition of the super BV antibracket

$$\left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{B}} \,;\, \frac{\overrightarrow{\partial} G}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle.$$

Then we use the above expressions, and, after rewriting $\langle\!\langle \ ; \ \rangle\!\rangle$ as $\langle \ , \ \rangle$, we compute the products $\sigma_{\mathsf{B}_{m-j}} \sigma_{\mathsf{a}_j}$, separately for the case m even and m odd. In order for the superbracket to coincide with the ordinary bracket, these products must be

$$\sigma_{\mathsf{B}_{m-i}} \sigma_{\mathsf{a}_i} = \begin{cases} -1 & \text{if } i = 0, \\ 1 & \text{otherwise} \end{cases}$$

for m even, and

$$\sigma_{\mathsf{B}_{m-i}}\sigma_{\mathsf{a}_i} = \begin{cases} (-1)^i & \text{for } i=0,1, \\ (-1)^{i+1} & \text{otherwise} \end{cases}$$

for m odd. It can be readily computed that the choice of signs made in (3.3.1) and in (3.3.2) is consistent with the above rules; therefore, the proof then follows. ☐

For the general case of elements of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$, the above rule must be slightly modified. We begin by noting that any homogeneous F of total degree $|F|$ in this algebra can be written in the form

$$F = \sum_l F_l,$$

where F_l is an element of $\mathcal{S}^{|F|-l,l}(N;E)$. This is obtained by expanding the superfields in their components. We are now ready to state the following

Lemma 3.3.2. *Let F and G be homogeneous elements of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N;E)$. If we expand them according to the above rule*

$$F = \sum_k F_k \;\; \textit{and} \;\; G = \sum_l G_l,$$

then the following identity holds:

$$(\!(F\, ;\, G)\!) = \sum_{k,l} (-1)^{(|F|-k+1)l} \, (\, F_k \, , \, G_l \,)\,. \tag{3.3.8}$$

Proof. The proof of this identity is similar to the proof of Lemma 3.3.1; in fact, we have to compute the functional derivatives of F and G w.r.t. a and B, and express them via the functional derivatives w.r.t. the usual fields of the theory. We therefore recall the formulae for the functional derivatives, and we apply them to F, obtaining:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\mathsf{a}+t\rho_{\mathsf{a}};\mathsf{B}) = \pi_{1*}\left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\, ;\, \pi_2^*\rho_{\mathsf{a}} \right\rangle\!\!\right\rangle = \\
&= \sum_{j=0}^m \sigma_{\mathsf{a}_j}\pi_{1*}\left\langle\!\!\left\langle \left(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\right)_{m-j}\, ;\, \pi_2^*\rho_{\mathsf{a}_j} \right\rangle\!\!\right\rangle = \sum_l \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F_l(a+t\rho_a; c+t\rho_c;\dots) = \\
&\qquad = \sum_l\sum_{j=0}^m \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F_l(\mathsf{a}_j + t\rho_{\mathsf{a}_j}) = \sum_l\sum_{j=0}^m \pi_{1*}\left\langle \frac{F_l\overleftarrow{\partial}}{\partial \mathsf{a}_j}\, ,\, \pi_2^*\rho_{\mathsf{a}_j}\right\rangle = \\
&\qquad\qquad = \sum_{j=0}^m \pi_{1*}\left\langle \sum_l \frac{F_l\overleftarrow{\partial}}{\partial \mathsf{a}_j}\, ,\, \pi_2^*\rho_{\mathsf{a}_j}\right\rangle .
\end{aligned} \tag{3.3.9}$$

Then the following holds, if we go from the dot product to the ordinary product:

$$(-1)^{(|F|-l-1+j)j}\left\langle \frac{F_l\overleftarrow{\partial}}{\partial \mathsf{a}_j}\, ,\, \pi_2^*\rho_{\mathsf{a}_j}\right\rangle = \left\langle\!\!\left\langle \frac{F_l\overleftarrow{\partial}}{\partial \mathsf{a}_j}\, ;\, \pi_2^*\rho_{\mathsf{a}_j}\right\rangle\!\!\right\rangle .$$

By confronting the terms in (3.3.9), and operating similarly for the other cases, we obtain the following identities for $j = 0, \ldots, m$:

$$\begin{aligned}
\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{a}}\Big)_{m-j} &= \sum_l \sigma_{\mathsf{a}_j}(-1)^{(|F|-l-1+j)j}\frac{F_l\overleftarrow{\partial}}{\partial \mathsf{a}_j},\\
\Big(\frac{F\overleftarrow{\partial}}{\partial \mathsf{B}}\Big)_{m-j} &= \sum_l \sigma_{\mathsf{B}_j}(-1)^{(|F|-l-m+j)j}\frac{F_l\overleftarrow{\partial}}{\partial \mathsf{B}_j},\\
\Big(\frac{\overrightarrow{\partial}F}{\partial \mathsf{a}}\Big)_{m-j} &= \sum_l \sigma_{\mathsf{a}_j}(-1)^{(1-j)(l-m+j)}\frac{\overrightarrow{\partial}F_l}{\partial \mathsf{a}_j},\\
\Big(\frac{\overrightarrow{\partial}F}{\partial \mathsf{B}}\Big)_{m-j} &= \sum_l \sigma_{\mathsf{B}_j}(-1)^{(m-j)(l-m+j)}\frac{\overrightarrow{\partial}F_l}{\partial \mathsf{B}_j}.
\end{aligned}$$

We can finally cast all these expressions in the explicit formula for the super BV antibracket, and, by recalling the explicit values of the chosen signs σ_{a_j} and σ_{B_j}, we can finally obtain the desired identity (recall the form degree selection rule imposed by the pushforwards). $\square$

We notice that for the case $F \in \mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathfrak{A})$ and $G \in \mathcal{S}_{\mathsf{A},\mathsf{B}}(N; E)$ for an $\mathfrak{A}$-module E, the following identity holds:

$$(\!(\, F\,;\, G\,)\!) = \sum_l (-1)^{(|F|+1)l}\,(\, F\,,\, G_l\,)\,; \tag{3.3.10}$$

this formula will play a special rôle in later computations (we skip the proof of this identity, because it is in principle the same as for the two previous lemmata).

We now extend the super BV antibracket $(\!(\ ;\ )\!)$ to the whole of $\mathcal{S}$ by the following rule

$$(\!(\,\alpha\,;\,\beta\,)\!) := (-1)^{(\mathrm{gh}\,\alpha+1)\deg\beta}\,(\,\alpha\,,\,\beta\,)\,,$$

with α and β homogeneous elements of $\mathcal{S}$. Recalling the graded commutativity rule, the graded Leibniz rule and the graded Jacobi rule for $(\ ,\ )$, we can then show the following properties of the super BV antibracket $(\!(\ ;\ )\!)$:

- $(\!(\,\alpha\,;\,\beta\,)\!) = -(-1)^{(|\alpha|+1)(|\beta|+1)}(\!(\,\beta\,;\,\alpha\,)\!)$,
  whenever one of the two elements is central in $\mathcal{S}$.
- $(\!(\,\alpha\,;\,\beta\gamma\,)\!) = (\!(\,\alpha\,;\,\beta\,)\!)\,\gamma + (-1)^{(|\alpha|+1)|\beta|}\beta\,(\!(\,\alpha\,;\,\gamma\,)\!)$,
  whenever α or β or both are central in $\mathcal{S}$.
- $(-1)^{(|\alpha|+1)(|\gamma|+1)}(\!(\,\alpha\,;\,(\!(\,\beta\,;\,\gamma\,)\!)\,)\!) +$ cyclic permutations $= 0$,
  whenever two of the three elements are central in $\mathcal{S}$.

Here, we have used the previous notational convention for the total degree. In particular, if we restrict to $\mathcal{S}_{\mathsf{A},\mathsf{B}}$, by linearity the previous identities hold if we replace α, β and γ with elements F, G and H of $\mathcal{S}_{\mathsf{A},\mathsf{B}}$.

For central elements in $\mathcal{S}_{\mathsf{A,B}}$, we can take e.g. any functional F in $\mathcal{S}_{\mathsf{A,B}}(\mathfrak{A})$, while considering as more general elements in $\mathcal{S}_{\mathsf{A,B}}(N; E)$, for an $\mathfrak{A}$-module E. (We have omitted the products between elements in $\mathcal{S}_{\mathsf{A,B}}$, but it is understood that we are dealing with the shifted dot product.) We now pick a central element S of $\mathcal{S}_{\mathsf{A,B}}$ with even total degree; we then define an operator $\boldsymbol{\delta}$ on the superspace $\mathcal{S}_{\mathsf{A,B}}$ by

$$\boldsymbol{\delta} := (\!(S \,; \quad)\!) \,;$$

since S has even total degree, $\boldsymbol{\delta}$ is an odd derivation by the above identities. From Lemma 3.3.2, 3.2.3, 3.2.2, 3.2.4 and 3.2.5 we can derive the useful properties of δ_S:

Corollary 3.3.3. *We assume that the functional F lies in $\mathcal{S}_{\mathsf{A,B}}(N; E)$, and that we have a map h from a manifold H to the manifold N, then the following identity holds:*

$$\boldsymbol{\delta}[h^*(F)] = h^*(\boldsymbol{\delta} F).$$

Corollary 3.3.4. *We assume that we have a homogeneous functional F in $\mathcal{S}_{\mathsf{A,B}}(H; E)$, where E is a real or complex algebra and H is the total space H of a bundle over N with typical fiber B. The integration along fiber of the functional F gives a functional of the same type, defined on the manifold N and with total degree $|\pi_*(F)| = |F| - \dim B$. Then we obtain the following identity:*

$$\boldsymbol{\delta}[\pi_*(F)] = (-1)^{\dim B} \pi_*(\boldsymbol{\delta} F)$$

Corollary 3.3.5. *We pick a functional F in $\mathcal{S}_{\mathsf{A,B}}(N; E)$, for N and E as in the preceding Lemma. We denote by d the exterior derivative on the manifold N. Then the following identity holds:*

$$\boldsymbol{\delta}(\mathrm{d}F) = -\mathrm{d}(\boldsymbol{\delta} F).$$

Corollary 3.3.6. *We assume the functional F to belong to the superalgebra $\mathcal{S}_{\mathsf{A,B}}(N, \mathfrak{g})$; we assume additionally that we have a $\mathfrak{g}$-module V. The application of the trace to F gives an element of $\mathcal{S}_{\mathsf{A,B}}(N; \mathbb{R})$ (or of $\mathcal{S}_{\mathsf{A,B}}(N; \mathbb{C})$, depending on whether V is a real or complex module). Then we obtain the following identity:*

$$\boldsymbol{\delta}[\mathrm{Tr}_\rho(F)] = \mathrm{Tr}_\rho(\boldsymbol{\delta} F).$$

3.3.3 The super BV Laplacian

In analogy with what we have done for the BV antibracket, we introduce a "twisted" version $\boldsymbol{\Delta}$ of the BV Laplacian on the superalgebra $\mathcal{S}$, endowed with the two usual gradings (the form degree and the ghost number). We define the *super BV Laplacian* by

$$\boldsymbol{\Delta}\alpha := (-1)^{\deg \alpha} \Delta_{BV} \alpha,$$

for $\alpha \in \mathcal{S}$. Since the BV Laplacian is nilpotent, it follows immediately that the super BV Laplacian is nilpotent, too. We take α and β in $\mathcal{S}$, and we assume at least one of the two elements to be central in $\mathcal{S}$. It follows then from (3.2.6) that

$$\boldsymbol{\Delta}(\alpha \cdot \beta) = (\boldsymbol{\Delta}\alpha) \cdot \beta + (-1)^{|\alpha|} (\!(\alpha \,; \beta)\!) + (-1)^{|\alpha|} \alpha \cdot (\boldsymbol{\Delta}\beta), \tag{3.3.11}$$

where α or β must be central. Restricting to the super algebra $\mathcal{S}_{\mathsf{A},\mathsf{B}}$, it follows easily that the super BV Laplacian is a coboundary operator

$$\boldsymbol{\Delta} : \mathcal{S}^k_{\mathsf{A},\mathsf{B}} \to \mathcal{S}^{k+1}_{\mathsf{A},\mathsf{B}}$$

satisfying (3.3.11) with α and β in $\mathcal{S}_{\mathsf{A},\mathsf{B}}$. The BV operator Ω_{BV} defined in (3.2.7) is replaced in the superformalism by the operator

$$\boldsymbol{\Omega} = \boldsymbol{\delta} - \mathrm{i}\hbar\boldsymbol{\Delta}.$$

As a consequence of the general case, $\boldsymbol{\Omega}$ is a coboundary operator.

3.3.4 Twists

We assume O to be an even element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}$. We define the twisted BV coboundary operator by

$$\widetilde{\boldsymbol{\Omega}} = \exp\left(-\frac{\mathrm{i}}{\hbar}\mathsf{O}\right)\,\boldsymbol{\Omega}\,\exp\left(\frac{\mathrm{i}}{\hbar}\mathsf{O}\right) = \boldsymbol{\Omega} + \boldsymbol{\partial}_{\mathsf{O}} + \frac{\mathrm{i}}{\hbar}\Phi_{\mathsf{O}},$$

with $\boldsymbol{\partial}_{\mathsf{O}} := (\!(\,\mathsf{O}\,;\quad)\!)$, and

$$\Phi_{\mathsf{O}} = \boldsymbol{\Omega}\mathsf{O} + \frac{1}{2}(\!(\,\mathsf{O}\,;\,\mathsf{O}\,)\!)$$

as a multiplication operator.

Definition 3.3.7. We call *flat* an even functional O with $\Phi_{\mathsf{O}} = 0$, *flat observable* an $\boldsymbol{\Omega}$-closed flat functional, and *flat invariant observable* a $\boldsymbol{\delta}$-closed flat observable.

A basic fact that we will need in Sections 4.2 and 4.3 is expressed by the following

Lemma 3.3.8. *If O is a flat observable, then so is $\lambda\mathsf{O}$ for any constant λ; moreover, $\boldsymbol{\partial}_{\mathsf{O}}$ is a superdifferential (of degree $|\mathsf{O}|+1$) which anticommutes with $\boldsymbol{\Omega}$. If we also assume that O is invariant, then $\boldsymbol{\partial}_{\mathsf{O}}$ anticommutes with $\boldsymbol{\delta}$, so $\boldsymbol{\delta}_\lambda := \boldsymbol{\delta} + \lambda\boldsymbol{\partial}_{\mathsf{O}}$ is an odd differential for all λ.*

Proof. By definition, a flat observable O satisfies separately

$$\boldsymbol{\Omega}\mathsf{O} = 0 \qquad \text{and} \qquad (\!(\,\mathsf{O}\,;\,\mathsf{O}\,)\!) = 0.$$

This implies that $\Phi_{\lambda\mathsf{O}} = 0$ for all λ. The second equation above together with the Jacobi identity implies that $\boldsymbol{\partial}_{\mathsf{O}}$ is a coboundary operator. Since $\widetilde{\boldsymbol{\Omega}}$ and $\boldsymbol{\Omega}$ square to zero and $\Phi_{\mathsf{O}} = 0$, we obtain

$$\boldsymbol{\Omega}\boldsymbol{\partial}_{\mathsf{O}} + \boldsymbol{\partial}_{\mathsf{O}}\boldsymbol{\Omega} + \boldsymbol{\partial}_{\mathsf{O}}^2 = 0.$$

The second claim then follows since $\boldsymbol{\partial}_{\mathsf{O}}$ squares to zero. For O invariant, we also have $(\!(\,S\,;\,\mathsf{O}\,)\!) = 0$. So by Jacobi we obtain $\boldsymbol{\delta}\boldsymbol{\partial}_{\mathsf{O}} + \boldsymbol{\partial}_{\mathsf{O}}\boldsymbol{\delta} = 0$. $\square$

3.4 The BV action for BF theories

We have now at our disposal all tools needed to write down the correct BV action for BF theories. Namely, we claim that it is given by

$$S_{\mathrm{BV}} := \int_M \langle\!\langle\, \mathsf{B} \,;\, \mathsf{F_A} \,\rangle\!\rangle\,. \tag{3.4.1}$$

Remark 3.4.1. Earlier versions of this form for the BV action of BF theories can be found in [51, 37], where however proofs were not given and, in particular, there was no explicit treatment of the sign conventions (i.e., our "dot" structures). Special cases (with explicit signs) were also considered in [14, 16]. In particular, the structure of the BV action in terms of superfields is in agreement with the general pattern described in [27]. See also [55] and [2] for the case of Chern–Simons theory.

This form of the BV action holds not only for the BF theories described in the previous sections but also for the "canonical BF theories" pointed out in Remark 3.1.1 (observe that the two-dimensional case has already been considered in [22]).

We divide the proof, for the ordinary case, in two steps: $i)$ we show that the above functional is a solution of the master equation corresponding to the BF action (3.1.3) with infinitesimal symmetries (3.1.6) and (3.1.7) (subsection 3.4.1); $ii)$ we show that it is Δ_{BV}-closed (subsection 3.4.2). In subsection 3.4.3 we will then give the proof in the case of canonical BF theories.

3.4.1 The master equation

We begin with the statement of the main Theorem, and we devote the rest of the section to its proof and to some important consequences.

Theorem 3.4.2. *The following identity holds:*

$$(\!(\, S \,;\, S \,)\!) = 0.$$

Remark 3.4.3. By lemma 3.3.1, the above result is equivalent to the statement that the action S satisfies the usual ME w.r.t. the usual BV antibracket.

Proof. We begin by computing the left and right partial derivatives w.r.t. a and B; e.g. the left partial derivative of S w.r.t. a is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S(\mathsf{a}+t\rho_{\mathsf{a}};\mathsf{B}) = \int_M \langle\!\langle\, \mathsf{B} \,;\, \mathrm{d}_{\mathsf{A}}\rho_{\mathsf{a}} \,\rangle\!\rangle = \\ = (-1)^{m-1}\int_M \langle\!\langle\, \mathrm{d}_{\mathsf{A}}\mathsf{B} \,;\, \rho_{\mathsf{a}} \,\rangle\!\rangle = \int_M \langle\!\langle\, \rho_{\mathsf{a}} \,;\, \mathrm{d}_{\mathsf{A}}\mathsf{B} \,\rangle\!\rangle\,.$$

It follows that:

$$\frac{S\overleftarrow{\partial}}{\partial \mathsf{a}} = (-1)^{m-1}\frac{\overrightarrow{\partial} S}{\partial \mathsf{a}} = (-1)^{m-1}\mathrm{d}_{\mathsf{A}}\mathsf{B}; \tag{3.4.2}$$

similarly

$$\frac{S\overleftarrow{\partial}}{\partial \mathsf{B}} = \frac{\overrightarrow{\partial} S}{\partial \mathsf{B}} = \mathsf{F}_{\mathsf{A}}. \tag{3.4.3}$$

If we now insert the above functional derivatives in the formula for the BV antibracket, we obtain

$$(\!(S \, ; \, S)\!) = 2 \int_M \langle\!\langle \, \mathsf{F}_{\mathsf{A}} \, ; \, \mathrm{d}_{\mathsf{A}} \mathsf{B} \, \rangle\!\rangle = 2 \int_M \mathrm{d} \, \langle\!\langle \, \mathsf{F}_{\mathsf{A}} \, ; \, \mathsf{B} \, \rangle\!\rangle - 2 \int_M \langle\!\langle \, \mathrm{d}_{\mathsf{A}} \mathsf{F}_{\mathsf{A}} \, ; \, \mathsf{B} \, \rangle\!\rangle \, ,$$

by the invariance of $\langle \ \ , \ \ \rangle$ (A is a superconnection). The first term vanishes by Stokes' Theorem, and the second by the super Bianchi identity

$$\mathrm{d}_{\mathsf{A}} \mathsf{F}_{\mathsf{A}} = 0.$$

So the claim follows. □

Since S satisfies the ME, the Leibnitz rule and the Jacobi identity for the super BV antibracket imply that the operator

$$\boldsymbol{\delta} := (\!(S \, ; \quad)\!) \, ,$$

defined on $\mathcal{S}_{\mathsf{A},\mathsf{B}}(N; E)$, is an odd differential. In many of the forthcoming computations we need the following

Proposition 3.4.4. *The action of* $\boldsymbol{\delta}$ *on the superfields* a *and* B *is given by:*

$$\boldsymbol{\delta} \mathsf{a} = (-1)^m \mathsf{F}_{\mathsf{A}} \tag{3.4.4}$$

and

$$\boldsymbol{\delta} \mathsf{B} = (-1)^m \mathrm{d}_{\mathsf{A}} \mathsf{B}. \tag{3.4.5}$$

Proof. The above formulae follow from (3.3.5). We begin with $\boldsymbol{\delta}\mathsf{a}$:

$$\boldsymbol{\delta} \mathsf{a}(x) = \int_M \left\langle\!\!\left\langle \frac{S\overleftarrow{\partial}}{\partial \mathsf{B}} \, ; \, \frac{\overrightarrow{\partial} \mathsf{a}(x)}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle .$$

By definition however

$$\int_M \left\langle\!\!\left\langle \rho \, ; \, \frac{\overrightarrow{\partial} \mathsf{a}(x)}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} [\mathsf{a}(x) + t\rho(x)] = \rho(x),$$

provided ρ is a test form of total degree 1. Since $\frac{S\overleftarrow{\partial}}{\partial \mathsf{B}}$ has total degree 2, we cannot apply the above formula directly. We use then the following trick. Let ϵ be a scalar of total degree -1. Then

$$\epsilon \cdot \boldsymbol{\delta} \mathsf{a}(x) = (-1)^m \int_M \left\langle\!\!\left\langle \epsilon \cdot \frac{S\overleftarrow{\partial}}{\partial \mathsf{B}} \, ; \, \frac{\overrightarrow{\partial} \mathsf{a}(x)}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle = (-1)^m \epsilon \cdot \frac{S(x)\overleftarrow{\partial}}{\partial \mathsf{B}}.$$

Thus,

$$\boldsymbol{\delta}\mathsf{a}(x) = (-1)^m \frac{S\overleftarrow{\partial}}{\partial \mathsf{B}}(x) = (-1)^m \mathsf{F}_{\mathsf{A}}(x),$$

where we have used (3.4.3). Similarly, we have

$$\boldsymbol{\delta}\mathsf{B}(x) = -\frac{S\overleftarrow{\partial}}{\partial \mathsf{a}}(x) = (-1)^m \mathrm{d}_{\mathsf{A}}\mathsf{B},$$

by (3.4.2). □

Recalling the formula (3.3.10) expressing the super BV antibracket in term of the usual BV antibracket, we can now recover the action of the usual δ_{BV} operator defined by $(\,S\,,\quad)$. Namely,

$$\boldsymbol{\delta}\mathsf{a} = \sum_{j=0}^{m} \sigma_{\mathsf{a}_j}(-1)^j \delta_{BV}\mathsf{a}_j.$$

Decomposing the expression for $\boldsymbol{\delta}\mathsf{a}$ in its homogeneous components and by confronting the two expressions, we get the action of the BV operator δ_{BV} on the fields. Similarly, we can recover the action of δ_{BV} on the components of B. Setting the antifields to zero, we obtain then that δ_{BV} on the fields $\{A, B, c, \tau_1, \ldots, \tau_{m-2}\}$ coincides with the BRST operator given in (3.1.6) and (3.1.7). Moreover, it follows easily from the definition of S and of the superforms a and B that the action reduces to the classical BF action, if we set all antifields to 0. Thus, we have proved the following

Theorem 3.4.5. *S_{BV} is a solution of the master equation for the BF theory.*

3.4.2 The Δ_{BV}-closedness of the BV action

We now turn to the proof of the identity

$$\Delta_{BV} S = \boldsymbol{\Delta} S = 0. \tag{3.4.6}$$

First, we recall that $\mathfrak{g}$ is endowed by assumption with a nondegenerate, symmetric, invariant bilinear form $\langle\ ,\ \rangle$. We now choose a basis $\{e_k\}$ of $\mathfrak{g}$ such that $\langle\, e_i\,,\, e_j\,\rangle = s_i\delta_{ij}$, $s_i = \pm 1$; in this basis we have the structure constants f_{ij}^k given by the relation

$$[\,e_i\,,\,e_j\,] = \sum_{k=1}^{\dim\mathfrak{g}} f_{ij}^k e_k.$$

We then introduce the symbols $\tilde{f}_{ij}^k$ as $s_k f_{ij}^k$. Thus,

$$\tilde{f}_{ij}^k = \langle\,[\,e_i\,,\,e_j\,]\,,\,e_k\,\rangle\,.$$

From the non-degeneracy of $\langle\ ,\ \rangle$ we then get the useful relation

$$\tilde{f}_{ij}^k = -\tilde{f}_{ik}^j = -\tilde{f}_{kj}^i. \tag{3.4.7}$$

If we write the BV action as a sum of local terms in the fields, we see from the very definition of the BV Laplacian Δ_{BV} (see Remark 3.2.7) that the only terms in this sum which are not automatically 0 have the form

$$\langle\, \phi^*_\alpha\,,\, [\phi^\alpha, c]\,\rangle_{\mathrm{Hodge}}\,,$$

for all α in the index set of the fields (this the only way to pair a field and its antifield allowed by the integration over M); we can rewrite it in the form (up to signs)

$$\langle\, \phi^*_\alpha\,,\, [\phi^\alpha, c]\,\rangle_{\mathrm{Hodge}} = \tilde{f}^i_{jk}\big(\, \phi^{*,i}_\alpha\,,\, \phi^{\alpha,j} c^k\,\big)_{\mathrm{Hodge}}, \tag{3.4.8}$$

with

$$(\, \alpha\,,\, \beta\,)_{\mathrm{Hodge}} := \int_M \alpha \wedge \star\beta, \qquad \alpha,\ \beta \in \Omega^*(M), \tag{3.4.9}$$

and $\phi^\alpha = \sum \phi^{\alpha,i} X_i$ and similarly for ϕ^*_α and c. Now, by (3.4.7), one sees that in the above formula no field component is paired to the corresponding antifield component. So, by Remark 3.2.7, it is annihilated by the BV Laplacian.

3.4.3 Canonical BF theories

We start here a digression about the version of BF theories mentioned in Remark 3.1.1. The material covered in this subsection is not essential for the rest of the paper and can be safely skipped. Though, this kind of BF theories is interesting by itself (and appears in two-dimensions as a particular case of the Poisson sigma model [36, 43]).

We recall now the basic idea: since the curvature F_A is a tensorial form of the adjoint type, the most natural way to define a BF theory is to choose B of the coadjoint type and to use the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. We consider then B as a form in $\Omega^{m-2}(M, \mathrm{ad}^* P)$. Observe that since we do not introduce a bilinear form on $\mathfrak{g}$ anymore, Assumption 3 is in this case meaningless. For simplicity we will retain in this case as well Assumptions 1 and 2. We begin with some notations:

- *By* $\langle\ ,\ \rangle$ *we will denote in this subsection the canonical pairing between* $\mathfrak{g}^*$ *and* $\mathfrak{g}$; it can be naturally extended to a pairing between forms in $\Omega^p(M, \mathrm{ad}^* P)$ and forms in $\Omega^q(M, \mathrm{ad}\, P)$, and we will denote this pairing by the same symbol.

- By $\{e_i\}$ we denote a basis of $\mathfrak{g}$, while by $\{\varepsilon^j\}$ we denote its dual basis: $\big\langle\, \varepsilon^i\,,\, e_j\,\big\rangle = \delta^i_j$.

- By f^k_{ij} we denote the structure constants w.r.t. the basis $\{e_l\}$, i.e.

$$f^k_{ij} = \big\langle\, \varepsilon^k\,,\, [\, e_i\,,\, e_j\,]\,\big\rangle\,.$$

- By Ad^* we denote the coadjoint action of G on $\mathfrak{g}^*$; i.e. $\langle\, \mathrm{Ad}^*(g)\xi\,,\, X\,\rangle := \big\langle\, \xi\,,\, \mathrm{Ad}(g^{-1})X\,\big\rangle$.

- by ad^* we denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$; i.e., $\langle\, \mathrm{ad}^*(X)\xi\,,\, Y\,\rangle = -\,\langle\, \xi\,,\, \mathrm{ad}(X)Y\,\rangle$. The coadjoint action can be extended to an action of forms in $\Omega^p(M, \mathrm{ad}\, P)$ on

$\Omega^q(M, \operatorname{ad}^* P)$ in the usual way. We only notice the sign rules for this extended coadjoint action

$$\operatorname{ad}^*([\alpha\,,\,\beta])\gamma = \operatorname{ad}^*(\alpha)\operatorname{ad}^*(\beta)\gamma - (-1)^{\deg\alpha\deg\beta+\operatorname{gh}\alpha\operatorname{gh}\beta}\operatorname{ad}^*(\beta)\operatorname{ad}^*(\alpha)\gamma;$$
$$\langle\,\operatorname{ad}^*(\alpha)\gamma\,,\,\beta\,\rangle = -(-1)^{\deg\alpha\deg\gamma+\operatorname{gh}\alpha\operatorname{gh}\gamma}\langle\,\gamma\,,\,[\,\alpha\,,\,\beta\,]\,\rangle\,,$$

for $\alpha, \beta \in \Omega^*(M, \operatorname{ad} P)$ and $\gamma \in \Omega^*(M, \operatorname{ad}^* P)$, where we have implicitly supposed to consider forms with additional ghost number gradation.

- Finally, we denote (improperly) by d_A the covariant derivative acting on $\Omega^*(M, \operatorname{ad}^* P)$; it satisfies
$$\mathrm{d}\,\langle\,\alpha\,,\,\beta\,\rangle = \langle\,\mathrm{d}_A\alpha\,,\,\beta\,\rangle + (-1)^{\deg\alpha}\langle\,\alpha\,,\,\mathrm{d}_A\beta\,\rangle\,,$$
where $\alpha \in \Omega^*(M, \operatorname{ad}^* P)$ and $\beta \in \Omega^*(M, \operatorname{ad} P)$, and
$$\mathrm{d}_A(\mathrm{d}_A\alpha) = \operatorname{ad}^*(F_A)\alpha.$$

With these conventions in mind, we define the canonical BF action functional by

$$S^{\mathrm{can}} := \int_M \langle\,B\,,\,F_A\,\rangle\,.$$

The Euler–Lagrange equations are still given by (3.1.4), where now the covariant derivative is understood to operate on $\Omega^{m-2}(M, \operatorname{ad}^* P)$.

We let the group $\Omega^0(M, G) \rtimes \Omega^{m-3}(M, \operatorname{ad}^* P)$ operate (from the right) on $\mathcal{A} \times \Omega^{m-2}(M, \operatorname{ad}^* P)$ by the rule

$$(A, B)(g, \tau) := (A^g, \operatorname{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau).$$

It is then easy to verify that S^{can} is invariant under this action. The infinitesimal transformations then read

$$\delta A = \mathrm{d}_A c; \qquad \delta B = -\operatorname{ad}^*(c)B + \mathrm{d}_A\tau.$$

These symmetries present the same reducibility problems as in Section 3.1; therefore, we have to resort to the BV formalism here as well.

The BRST and the BV formalism

The BRST transformations corresponding to the reducible infinitesimal symmetries in this case read

$$\delta_{\mathrm{BRST}}A = \mathrm{d}_A c; \qquad \delta_{\mathrm{BRST}}B = -\operatorname{ad}^*(c)B + \mathrm{d}_A\tau_1;$$
$$\delta_{\mathrm{BRST}}c = -\frac{1}{2}[\,c\,,\,c\,]; \qquad \delta_{\mathrm{BRST}}\tau_k = -\operatorname{ad}^*(c)\tau_k + \mathrm{d}_A\tau_{k+1}, \quad k = 1, \ldots, m-3;$$
$$\delta_{\mathrm{BRST}}\tau_{m-2} = -\operatorname{ad}^*(c)\tau_{m-2}.$$

Here, c denotes the Faddeev–Popov ghost, i.e. a form on the space of fields with values in $\Omega^0(M, \operatorname{ad} P)$ with ghost number 1, and by τ_k we denote the ghosts for ghosts taking values in $\Omega^{m-2-k}(M, \mathrm{ad}^* P)$ and with ghost number k. These BRST transformations present the same problems as in Section 3.1, namely δ_{BRST} is a differential only modulo terms containing the curvature of A. We have therefore to switch to the BV formalism. We associate to each field $\phi^\alpha \in \{A, B, c, \tau_1, \ldots, \tau_{m-2}\}$ a canonical antifield ϕ^+_α following the rules

- if ϕ^α takes values in $\Omega^{p^\alpha}(M, \mathrm{ad}\, P)$, resp. $\Omega^{p^\alpha}(M, \mathrm{ad}^* P)$, then its canonical antifield takes values in $\Omega^{m-p^\alpha}(M, \mathrm{ad}^* P)$, resp. $\Omega^{m-p^\alpha}(M, \mathrm{ad}\, P)$;
- the ghost number of ϕ^+_α is set to be equal to $-1 - \mathrm{gh}\,\phi^\alpha$.

We define the total degree of a form α with degree $\deg\alpha$ and ghost number $\mathrm{gh}\,\alpha$ by

$$|\alpha| := \deg\alpha + \mathrm{gh}\,\alpha.$$

Accordingly to what we have done in Section 3.3, we define the *dot duality* by the rule

$$\langle\!\langle\, \alpha \,;\, \beta \,\rangle\!\rangle := (-1)^{\mathrm{gh}\,\alpha \deg\beta} \langle\, \alpha \,,\, \beta \,\rangle\,,$$

for α an element of $\Omega^*(M, \mathrm{ad}^* P)$ with ghost number $\mathrm{gh}\,\alpha$ and β in $\Omega^*(M, \mathrm{ad}\, P)$ with form degree $\deg\beta$. The dot Lie bracket $[\![\ ;\ ]\!]$ is defined analogously as in Appendix 8.1, and it enjoys the same sign rule. We define additionally the *super coadjoint action* of $\Omega^*(M, \mathrm{ad}\, P)$ on $\Omega^*(M, \mathrm{ad}^* P)$ by the rule

$$\mathsf{ad}^*(\alpha)\beta := (-1)^{\mathrm{gh}\,\alpha \deg\beta}\, \mathrm{ad}^*(\alpha)\beta.$$

Without proof we write down some useful formulae, which are analogous to the formulae displayed in Appendix 8.1

$$\begin{aligned}
\mathsf{ad}^*([\![\alpha\,;\beta]\!])\gamma &= \mathsf{ad}^*(\alpha)\,\mathsf{ad}^*(\beta)\gamma - (-1)^{|\alpha||\beta|}\,\mathsf{ad}^*(\beta)\,\mathsf{ad}^*(\alpha)\gamma,\\
\langle\!\langle\, \mathsf{ad}^*(\alpha)\gamma \,;\, \beta \,\rangle\!\rangle &= -(-1)^{|\alpha||\gamma|}\, \langle\!\langle\, \gamma \,;\, [\![\alpha\,;\beta]\!] \,\rangle\!\rangle\,,
\end{aligned}$$

for $\alpha, \beta \in \Omega^*(M, \mathrm{ad}\, P)$ and $\gamma \in \Omega^*(M, \mathrm{ad}^* P)$. If A is a connection on P, we also have

$$\mathrm{d}\,\langle\!\langle\, \gamma \,;\, \alpha \,\rangle\!\rangle = \langle\!\langle\, \mathrm{d}_A\gamma \,;\, \alpha \,\rangle\!\rangle + (-1)^{|\gamma|}\, \langle\!\langle\, \gamma \,;\, \mathrm{d}_A\alpha \,\rangle\!\rangle\,.$$

Finally, it is useful to write the duality pairing also in the opposite order; as usual, one defines $\langle\, X \,,\, \xi \,\rangle = \langle\, \xi \,,\, X \,\rangle$ for $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. When we extend the pairing to forms and then consider the dot version, we obtain the rule

$$\langle\!\langle\, \gamma \,;\, \alpha \,\rangle\!\rangle = (-1)^{|\gamma||\alpha|}\, \langle\!\langle\, \alpha \,;\, \gamma \,\rangle\!\rangle\,.$$

We then define the functional derivatives w.r.t. all fields of the theory and the BV antibracket $(\ \ ,\ \ )$ by the same formulae as in subsection 3.2.1 (where the invariant, nondegenerate bilinear form $\langle\ \ ,\ \ \rangle$ is replaced by the duality pairing). This antibracket enjoys all usual properties of a BV antibracket. Analogously, provided we have a solution S of the master equation $(\, S \,,\, S\,) = 0$, we define the BV differential δ by the rule $\delta := (\, S \,,\ \ )$. This operator has the same properties of the previously introduced BV differential (see subsection 3.2.3).

Finally, we define the Hodge duals of the fields, the Hodge functional derivatives and the BV Laplacian in this case by the same formulae as in Section 3.2.4 (with the only difference that by $\langle\ \ ,\ \ \rangle$ we mean here the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$). We will denote all these objects by the same symbols as in the previous sections.

The BV superformalism and the BV action

We choose a background flat connection A_0, and we write a general connection A as $A = A_0 + a$, with a in $\Omega^1(M, \mathrm{ad}\, P)$, with ghost number 0. We are now ready to define in this case the analogues of the superforms introduced in Section 3.3, namely

$$\mathsf{B} := \sum_{k=1}^{m-2} (-1)^{\frac{k(k-1)}{2}} \tau_k + B + (-1)^n a^+ + c^+,$$

$$\mathsf{A} := (-1)^{n+1} c + A + (-1)^n B^+ + \sum_{k=1}^{m-2} (-1)^{\frac{k(k-1)}{2} + n(k+1)} \tau_k^+.$$

We also define $\mathsf{a} := \mathsf{A} - A_0$. We notice that B is a superform of total degree $m-2$ with values in $\mathrm{ad}^* P$, while A can be interpreted once again as a superconnection on M. It is not difficult to see that the curvature of the superconnection A is given by the formula

$$\mathsf{F}_{\mathsf{A}} = \mathrm{d}_{A_0} \mathsf{a} + \frac{1}{2} [\![\mathsf{a} \,; \mathsf{a}]\!].$$

We go on, as in subsection 3.3.1, to define the functional derivatives w.r.t. B and a and the super BV antibracket; they enjoy the same properties as the previously introduced ones, and we will denote them by the same symbols.

Finally, we claim that the BV action for the canonical BF theory on M is given by the formula

$$S_{\mathrm{BV}} := \int_M \langle\!\langle\, \mathsf{B} \,;\, \mathsf{F}_{\mathsf{A}} \,\rangle\!\rangle \,.$$

In order to prove the claim, we show once again separately that S_{BV} satisfies the master equation and that it is (at least formally) Δ_{BV}-closed.

The master equation. The proof that S_{BV} satisfies the master equation is analogous to the proof of the corresponding claim in Section 3.4.1; we therefore omit it. We will only write down the action of the super BV differential $\boldsymbol{\delta} := (\!(\, S_{\mathrm{BV}} \,; \quad)\!)$ on the super fields a and B

$$\boldsymbol{\delta} \mathsf{a} = (-1)^m \mathsf{F}_{\mathsf{A}}, \tag{3.4.10}$$

$$\boldsymbol{\delta} \mathsf{B} = (-1)^m \left(\mathrm{d}_{A_0} \mathsf{B} + \mathsf{ad}^*(\mathsf{a}) \mathsf{B} \right); \tag{3.4.11}$$

the action of the usual BV differential on all fields (fields *and* antifields) is encoded in the two previous equations and, upon switching off the antifields, gives back the BRST operator defined at the beginning of subsubsection 3.4.3. It is also easy to verify that S_{BV} reduces to S^{can} if we set the antifields to zero.

The Δ_{BV}-closedness of the BV action. The proof that S_{BV} satisfies the equation

$$\Delta_{BV} S_{\mathrm{BV}} = 0$$

is a little bit different from the proof of the same identity in Section 3.4.2; it relies on a formal argument similar to that used in [22].

As noted before, the main property of the BV Laplacian lies in the fact that it contracts each field with the corresponding antifield at the same point (see Remark 3.2.7); therefore, the only terms in the BV action that are not trivially annihilated by the BV Laplacian are of the form

$$\int_M \langle\, \phi^+_\alpha \,,\, [\, c \,,\, \phi^\alpha \,]\,\rangle\,,$$

for some field ϕ^α. More precisely, they are (independently of the dimension of M) given by the combination

$$I = \frac{1}{2}\langle\, c^* \,,\, [\, c \,,\, c \,]\,\rangle_{\text{Hodge}} - \langle\, a^* \,,\, [\, c \,,\, a \,]\,\rangle_{\text{Hodge}} + \\ - \langle\, B^* \,,\, \mathrm{ad}^*(c)B \,\rangle_{\text{Hodge}} + \sum_{l=1}^{m-2}(-1)^{l+1}\langle\, \tau_l^* \,,\, \mathrm{ad}^*(c)\tau_l \,\rangle_{\text{Hodge}}.$$

This is obtained from the formula for the BV action after rewriting the dot duality, the super coadjoint action and the dot Lie bracket in terms the usual ones, and recalling that the integral selects only the top form degree part of the integrand.

W.r.t. the bases $\{e_i\}$ and $\{\varepsilon^j\}$, we can write a field ϕ^α with values in $\Omega^*(M, \mathrm{ad}\, P)$, resp. in $\Omega^*(M, \mathrm{ad}^*\, P)$, as $\phi^\alpha = \phi^{\alpha\; i} e_i$, resp. $\phi^\alpha = \phi^\alpha_j \varepsilon^j$. For any two real-valued forms on M with the same degree we define

$$(\, \alpha \,,\, \beta \,)_{\text{Hodge}} = \int_M \alpha \wedge \star\beta,$$

where $\star$ is the star Hodge operator w.r.t. a chosen metric on M. We therefore obtain

$$I = -\frac{1}{2} f^i_{jk}\big(\, c^*_i \,,\, c^k\, c^j \,\big)_{\text{Hodge}} - f^i_{jk}\big(\, a^*_i \,,\, a^k c^j \,\big)_{\text{Hodge}} + \\ + f^k_{ji}\big(\, (B^*)^i \,,\, B_k c^j \,\big)_{\text{Hodge}} + \sum_{l=1}^{m-2} f^k_{ji}\big(\, (\tau^*_l)^i \,,\, (\tau_l)_k c^j \,\big)_{\text{Hodge}};$$

we have used here the identity $\langle\, e_i \,,\, \mathrm{ad}^*(e_j)\varepsilon^k \,\rangle = -\langle\, \varepsilon^k \,,\, [\, e_j \,,\, e_i \,]\,\rangle = -f^k_{ji}$.

Finally, we apply the BV Laplacian to the above expression and get (independently of the dimension of M)

$$\Delta_{BV} S_{\text{BV}} = \Delta_{BV} I = \\ = C\int_M d\mathrm{vol}\, f^i_{ji} c^j \left[\binom{m}{m} - \binom{m}{m-1} + \binom{m}{m-2} - \cdots + (-1)^l \binom{m}{m-l} + \cdots\right] = \\ = C\int_M d\mathrm{vol}\, f^i_{ji} c^j (1-1)^m = 0,$$

where $d\mathrm{vol}$ is the Riemannian volume element and C is an infinite constant (explicitly, a Dirac distribution evaluated in 0). The binomial coefficients appear as the number of components of the forms ϕ^α_j; e.g. , B_k is an $m-2$ form on the m-dimensional manifold M, so it has $\binom{m}{m-2}$ components in local coordinates. The signs before the binomial coefficients are determined by the ghost numbers of the fields ϕ^α (recall the explicit definition (3.2.5) of the BV Laplacian).

Of course, the previous computation should be performed with a regularization in order to avoid the infinite constant C. If the regularization is such that the above formal manipulations still hold, then S_{BV} is BV harmonic.

3.4.4 Gauge fixing

We conclude this section giving a brief account on the gauge fixing necessary to start a perturbative expansion of the theory. (For simplicity we restrict ourselves to ordinary BF theories, the modifications needed for the canonical ones being obvious.)

The first step is to extend the space of fields by introducing antighosts and Lagrange multipliers. Along with the usual ghost c one introduces an antighost $\bar{c}$ (of ghost number -1) and a Lagrange multiplier λ (of ghost number 0); both are chosen to take values in the sections of $\mathrm{ad}\,P$. Similarly, along with the ghost τ_1 one introduces an antighost $\bar{\tau}_1$ and a Lagrange multiplier λ_1 with values in $\Omega^{m-3}(M, \mathrm{ad}\,P)$. As for the ghosts-for-ghosts τ_k, one needs an entire family of k antighosts and k Lagrange multipliers ([5]). Namely, we denote by $\bar{\tau}_{k,i}$ and $\lambda_{k,i}$ ($i = 1, \dots, k$) the antighosts and the Lagrange multipliers corresponding to τ_k, all of which take values in $\Omega^{m-2-k}(M, \mathrm{ad}\,P)$. As for the ghost number, one sets

$$\mathrm{gh}(\bar{\tau}_{k,i}) = 2i - k - 2, \qquad \mathrm{gh}(\lambda_{k,i}) = 2i - k - 1.$$

We will denote by Φ the collection of the fields including the newly introduced ones.

Next, one has to consider antifields for the antighosts and the Lagrange multipliers. They will be denoted by $\bar{c}^+$, λ^+, $\bar{\tau}_1^+$, $\bar{\lambda}_1^+$, $\bar{\tau}_{k,i}^+$ and $\lambda_{k,i}^+$ ($k = 2, \dots, m-3$; $i = 1, \dots, k$) with the usual rule; i.e., each antifield takes values in the space of forms of complementary degree of the corresponding field and its ghost number is minus the ghost number of the corresponding field, minus one. We will denote by Φ^+ the collection of all the antifields including the new ones. Finally, we extend the BV structure by declaring that each of the new antifield is BV-canonically conjugated to its field.

The newly introduced fields are there only to write down a gauge fixing fermion (see later). From the point of view of BV cohomology their complex must be trivial; i.e., one sets

$$\delta\bar{\tau}_{k,i} = \lambda_{k,i}, \quad \delta\lambda_{k,i} = 0, \qquad k = 2, \dots, m-3;\ i = 1, \dots, k.$$

This is achieved by defining the extended BV action:

$$S^{\mathrm{ext}} = S + \Sigma,$$

with S given in (3.4.1) and

$$\Sigma \colon = \int_M \left(-\left\langle \bar{c}^+ , \lambda \right\rangle - \left\langle \bar{\tau}_1^+ , \lambda_1 \right\rangle - \sum_{k=2}^{m-3} \sum_{i=1}^{k} (-1)^k \left\langle \bar{\tau}_{k,i}^+ , \lambda_{k,i} \right\rangle \right). \quad (3.4.12)$$

The gauge-fixed action, which is a function of Φ only, is then defined by

$$S^{\mathrm{g.f.}} = S^{\mathrm{ext}}\Big|_{\Phi^+ = \frac{\vec{\partial}\,\Psi}{\partial\Phi}},$$

where Ψ (the *gauge-fixing fermion*) is a function of Φ of ghost number -1 and has to be chosen so that the Hessian of $S^{\mathrm{g.f.}}$ at a critical point is not degenerate. In case one wants to expand around a given flat connection A_0, a suitable gauge-fixing fermion (in accordance with Assumption 1) is

$$\Psi = \int_M \bar{c}\ \mathrm{d}_{A_0} \star a + \bar{\tau}_1\ \mathrm{d}_{A_0} \star B + \sum_{k=1}^{m-4} \bar{\tau}_{k+1,1}\ \mathrm{d}_{A_0} \star \tau_k + \sum_{k=1}^{m-4} \sum_{i=2}^{k+1} \bar{\tau}_{k+1,i}\ \mathrm{d}_{A_0} \star \bar{\tau}_{k,k+2-i}, \tag{3.4.13}$$

where $\star$ is the Hodge star operator induced from a Riemannian metric on M. The BV formalism ensures in particular that the partition function and the expectation values of BV closed observables do not depend on the chosen metric.

Some comments on the family of antighosts for ghosts-for-ghosts

In this subsubsection, we comment briefly on the need of an entire family of antighosts for any ghost-for-ghost τ_i, $i \geq 2$.

We assume $m = 4$ for simplicity. Since we want the covariant derivative w.r.t. the flat connection A_0, d_{A_0}, to be nondegenerate, we have to restrict ourselves to particular forms on M with values in $\mathrm{ad}\, P$, where d_{A_0} is invertible; equivalently, we have to find a condition on B ensuring that any class $B + \mathrm{d}_{A_0}\tau_1$ has a unique representative fulfilling the given condition. Of course, since d_{A_0}, the addition to τ_1 of an exact 0-form $\mathrm{d}\tau_2$ does not modify the given representative. However, the reducibility of the symmetry stops here, since τ_2 has degree 0.

Since d_{A_0} is acyclic on the complex of forms on M with values in $\mathrm{ad}\, P$ by assumption, natural conditions on B and τ_1 are

$$\mathrm{d}_{A_0}^{\star} B = 0, \quad \mathrm{d}_{A_0}^{\star} \tau_1 = 0, \tag{3.4.14}$$

where $\mathrm{d}_{A_0}^{\star}$ is the adjoint of d_{A_0} w.r.t. the L_2-metric on $\Omega^*(M, \mathrm{ad}\, P)$ constructed by means of a chosen Riemannian metric on M:

$$\langle\, \eta\, ,\, \omega \,\rangle_{\mathrm{L}_2} :\, = \int_M \langle\, \eta\, ,\, \star\omega \,\rangle\, , \quad \forall\, \eta, \omega \in \Omega^*(M, \mathrm{ad}\, P)\, ,$$

and $\star$ denotes the Hodge star operator on $\Omega^*(M, \mathrm{ad}\, P)$ induced by the chosen Riemannian metric. Standard arguments of the theory of elliptic differential operators ensure that both conditions fix uniquely representatives of the class $B + \mathrm{d}_{A_0}\tau_1$, for B given.

We have then to integrate formally the BV action for the BF theory over such B and τ_1 satisfying (3.4.14). We have therefore to introduce antighosts $\overline{\tau}_1$, resp. $\overline{\tau}_2$, to τ_1, resp. τ_2, of degree 1 and ghost number -1, resp. of degree 0 and ghost number -2; we recall that τ_1, resp. τ_2, has ghost number 1, resp. 2. Corresponding Lagrange multipliers λ_1 and λ_2, of degree 1, resp. 0 and ghost number 0 and -1 have also to be taken into account. The action of the BV operator δ_{BV} on antighosts and corresponding Lagrange multipliers is given by

$$\delta_{BV} \overline{\tau}_i = \lambda_i, \quad \delta_{BV} \lambda_i = 0, \quad i = 1, 2.$$

As we have seen, these requirements on the BV operator on antighosts and corresponding Lagrange multipliers are fulfilled by the addition to the BV action of a quadratic piece coupling the BV antifield of any antighost to the corresponding Lagrange multiplier; the new action satisfies also the Quantum Master Equation.

The sense of this operation is the following: if we recall the formal rules of functional integration, the Lagrange multipliers are introduced in order to rewrite as pieces of the action formal δ-distributions recording the gauge-fixing conditions, and the antighosts to keep track of the formal determinants arising as volumes of the orbits specified by the gauge-fixing conditions.

Now, the gauge-fixing fermion corresponding to conditions (3.4.14) would be intuitively

$$\Psi :=\left\langle \overline{\tau}_1 , \mathrm{d}_{A_0}^\star B \right\rangle_{\mathrm{L}_2} + \left\langle \overline{\tau}_2 , \mathrm{d}_{A_0}^\star \tau_1 \right\rangle_{\mathrm{L}_2} .$$

By the flatness of d_{A_0}, it is not difficult to check that the addition to the antighost $\overline{\tau}_1$ of an exact 1-form of ghost number -1 is a symmetry of the first term of the gauge-fixing fermion. This implies degeneracy of the corresponding term in the gauge-fixed action, $\left\langle \lambda_1 , \mathrm{d}_{A_0}^\star B \right\rangle_{\mathrm{L}_2}$.

Therefore, we have to add a new term, so as to include a gauge-fixing condition for the antighost $\overline{\tau}_1$. Since the symmetry corresponds to the addition of exact terms, it makes sense to impose also on $\overline{\tau}_1$ the covariant gauge w.r.t. A_0

$$\mathrm{d}_{A_0}^\star \overline{\tau}_1 = 0.$$

Further, we introduce, following the prescriptions of BV framework, an antighost $\overline{\tau}_{2,2}$ and a corresponding Lagrange multiplier $\lambda_{2,2}$ for $\overline{\tau}_1$. Since the gauge-fixing fermion has to have ghost number -1, $\overline{\tau}_{2,2}$ must have degree 0 and ghost number 0, while $\lambda_{2,2}$ has degree 0 and ghost number 1. Analogously to the previous arguments, the BV operator on $\overline{\tau}_{2,2}$ and $\lambda_{2,2}$ are

$$\delta_{BV} \overline{\tau}_{2,2} = \lambda_{2,2}, \quad \delta_{BV} \lambda_{2,2} = 0.$$

The additional piece of the gauge-fixing fermion keeping track of the gauge-fixing condition on the antighost $\overline{\tau}_1$ is then

$$\left\langle \overline{\tau}_{2,2} , \mathrm{d}_{A_0}^\star \overline{\tau}_1 \right\rangle_{\mathrm{L}_2} .$$

The antighosts $\overline{\tau}_2$ and $\overline{\tau}_{2,2}$ have the same degree; their ghost numbers are respectively -2 and 0. If we rename $\overline{\tau}_2$ by $\overline{\tau}_{2,1}$, resp. λ_2 by $\lambda_{2,1}$, we see immediately that we get a special case of equation (5.4.13).

More generally, when the symmetry of the action is further reducible, we have to introduce antighosts and corresponding Lagrange multipliers for any ghost-for-ghost except for the one of lowest degree. Namely, for the ghost-for-ghost τ_2 we need the antighost $\overline{\tau}_3$ and the Lagrange multiplier λ_3, but the terms in the gauge-fixing fermion recording the covariant condition for $\overline{\tau}_1$ and τ_1 present both a symmetry, namely the addition to $\overline{\tau}_{2,1}$ and $\overline{\tau}_{2,2}$ of exact forms of the corresponding ghost numbers. These symmetries have also to be fixed, and since they are of the same form as the symmetries of the BF action, a natural condition to impose on $\overline{\tau}_{2,1}$ and $\overline{\tau}_{2,2}$ is

$$\mathrm{d}_{A_0}^\star \overline{\tau}_{2,1} = 0, \quad \mathrm{d}_{A_0}^\star \overline{\tau}_{2,2} = 0.$$

According to the BV formalism, we have to introduce two antighosts $\overline{\tau}_{3,3}$ and $\overline{\tau}_{3,2}$, one for $\overline{\tau}_{2,1}$ and $\overline{\tau}_{2,2}$, and corresponding Lagrange multipliers $\lambda_{3,3}$ and $\lambda_{3,2}$; the degrees of $\overline{\tau}_{3,3}$ and $\overline{\tau}_{3,2}$ are equal to the degrees of $\overline{\tau}_{2,1}$ and $\overline{\tau}_{2,2}$ minus 1, which is also the degree of the antighost $\overline{\tau}_3$ for σ_2, and their ghost numbers are resp. 1 and -1, while the ghost number of $\overline{\tau}_3$ is -3. The corresponding additional piece to the gauge-fixing fermion is

$$\left\langle \overline{\tau}_{3,3} \, , \, \mathrm{d}^{\star}_{A_0} \overline{\tau}_{2,1} \right\rangle_{\mathrm{L}_2} + \left\langle \overline{\tau}_{3,2} \, , \, \mathrm{d}^{\star}_{A_0} \overline{\tau}_{2,2} \right\rangle_{\mathrm{L}_2} + \left\langle \overline{\tau}_{3} \, , \, \mathrm{d}^{\star}_{A_0} \tau_{2} \right\rangle_{\mathrm{L}_2} . \tag{3.4.15}$$

If the degree of $\overline{\tau}_3$ (and also of $\overline{\tau}_{3,3}$ and $\overline{\tau}_{3,2}$) is not 0, we have additional symmetries in equation (3.4.15), namely the addition to $\overline{\tau}_{3,3}$, $\overline{\tau}_{3,2}$ and $\overline{\tau}_3$ of d_{A_0}-exact forms of the respective ghost numbers. Hence, we have to introduce, for any antighost $\overline{\tau}_{3,3}$, $\overline{\tau}_{3,2}$ or $\overline{\tau}_3$, a corresponding antighost and Lagrange multiplier, whose degree is equal to the degree of $\overline{\tau}_{3,3}$, $\overline{\tau}_{3,2}$ or $\overline{\tau}_3$ minus 1 respectively, and whose ghost number is specified by the requirement that the ghost number of the gauge-fixing fermion is -1 and that we have to add to it terms coupling the covariant condition for $\overline{\tau}_{3,3}$, $\overline{\tau}_{3,2}$ and $\overline{\tau}_3$ to the respective antighosts through the L_2-duality (5.4.12). We continue with this procedure, fixing the gauge for ghosts-for-ghosts via antighosts $\overline{\tau}_i$, and fixing also the gauge for antighosts in the gauge-fixing fermion at each step, until we arrive at degree 0, where no further symmetry is possible. This can be summarized in the following scheme:

	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$m-2$					B				
$m-3$				$\overline{\tau}_1 \equiv \overline{\tau}_{1,1}$	λ_1	τ_1			
$m-4$			$\overline{\tau}_2 \equiv \overline{\tau}_{2,1}$	$\lambda_{2,1}$	$\overline{\tau}_{2,2}$	$\lambda_{2,2}$	τ_2		
$m-5$		$\overline{\tau}_3 \equiv \overline{\tau}_{3,1}$	$\lambda_{3,1}$	$\overline{\tau}_{3,2}$	$\lambda_{3,2}$	$\overline{\tau}_{3,3}$	$\lambda_{3,3}$	τ_3	
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Each row in the table has the same form degree, while each column has the same ghost number. The "principal" antighost $\overline{\tau}_i \equiv \overline{\tau}_{i,1}$ to the ghost-for-ghost σ_{i-1} lies on the left of the row next to the one containing τ_i; the other antighosts $\overline{\tau}_{i,l}$, $2 \leq l \leq i$, are coupled to the antighosts of in the preceding row by the rule $\mathrm{gh}\, \overline{\tau}_{i,l} + \mathrm{gh}\, \overline{\tau}_{i-1,k} = -1$, for $1 \leq k \leq i-1$.

3.4.5 Superpropagator

The perturbative expansion of the partition function or of the expectation values of observables is obtained in terms of propagators, i.e., expectation values of the fields w.r.t. the quadratic part of the action S^{ext}. We will briefly describe this computation in the case of ordinary BF theories.

Since the interaction terms and the observables that we will introduce in the next chapters depend only on the superfields a and B, it is sufficient to compute the "superpropagator"

$$\mathrm{i}\hbar\, \eta = \langle\, \pi_1^* \mathsf{a}\, \pi_2^* \mathsf{B} \,\rangle_0 := \frac{1}{Z} \int_{\Phi^+ = \frac{\overrightarrow{\partial} \Psi}{\partial \Phi}} \exp\left(\int_M \langle\!\langle\, \mathsf{B} \,;\, \mathrm{d}_{A_0} \mathsf{a} \,\rangle\!\rangle + \Sigma \right)\, \pi_1^* \mathsf{a}\, \pi_2^* \mathsf{B},$$

where Z is the partition function, A_0 is the chosen background flat connection, Σ is the extension defined in (3.4.12), and π_1 and π_2 are the projections from $M \times M$ to

M. So, η is a distributional $(m-1)$-form on $M \times M$ with values in $\operatorname{ad} P \boxtimes \operatorname{ad} P$. This superpropagator with the gauge fixing (3.4.13) can be computed by generalizing Axelrod and Singer's construction [3] to higher dimensions. Another possibility is to formally compute the main properties of the superpropagator and then construct a form that satisfies them generalizing the construction of [11]. The first property relies on the symmetry $\mathsf{a} \leftrightarrow \mathsf{B}$ of the quadratic part of the action: $\int_M \langle\!\langle\, \mathsf{B} \,;\, \mathrm{d}_{A_0}\mathsf{a} \,\rangle\!\rangle$. This implies

$$T^*\eta = (-1)^m\,\eta \tag{3.4.16}$$

where T is the automorphism of $\operatorname{ad} P \boxtimes \operatorname{ad} P$ that acts on the basis by exchanging the points and at the same time exchanges the corresponding fibers (in a local trivialization $T(x,x';\xi,\xi') = (x',x;\xi',\xi)$, with $x,x' \in U \subset M$ and $\xi,\xi' \in \mathfrak{g}$). A subsequent computation shows that

$$\mathrm{i}\hbar\,\mathrm{d}_{A_0}\eta = (-1)^m\,\langle\, \boldsymbol{\delta}_0(\pi_1^*\mathsf{a}\;\pi_2^*\mathsf{B})\,\rangle_0,$$

where $\boldsymbol{\delta}_0$ is the linear part of $\boldsymbol{\delta}$. By the main properties of the BV formalism, one then gets the Ward identity

$$(-1)^m\,\mathrm{d}_{A_0}\eta = \langle\, \boldsymbol{\Delta}(\pi_1^*\mathsf{a}\;\pi_2^*\mathsf{B})\,\rangle_0.$$

By a straightforward computation similar to that leading to the BV harmonicity of the BV action for the BF theory, the right-hand side is a delta form localized on the diagonal Diag of $M \times M$ tensorized with the section ϕ of $\operatorname{ad} P \otimes \operatorname{ad} P \to \mathrm{Diag}$ determined by the invariant form $\langle\ ,\ \rangle$; that is, ϕ is the section induced by the constant equivariant map $\tilde\phi\colon P \to \mathfrak{g}\times\mathfrak{g}$, $p \mapsto \sum_i \sigma_i\, e_i \otimes e_i$, where $\{e_i\}$ is a pseudo-orthonormal basis of $\mathfrak{g}$: $\langle\, e_i\,,\, e_j\,\rangle = \sigma_i\delta_{ij}$, $\sigma_i = \pm 1$. Thus, if we define $\langle\ ,\ \rangle_{13}$ on $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ as acting on the first and third components and define consequently $\langle\!\langle\ ;\ \rangle\!\rangle_{13}$, we may interpret η as a distributional form such that the linear operator $\mathrm{P}\colon \Omega^*(M,\operatorname{ad} P) \to \Omega^{*-1}(M,\operatorname{ad} P)$,

$$\mathrm{P}\,\gamma := \pi_{2*}\,\langle\!\langle\, \eta\,;\, \pi_1^*\gamma\,\rangle\!\rangle_{13}\,, \qquad \gamma \in \Omega^*(M,\operatorname{ad} P),$$

satisfies the equation

$$\mathrm{d}_{A_0}\,\mathrm{P} + \mathrm{P}\,\mathrm{d}_{A_0} = \mathrm{id}\,. \tag{3.4.17}$$

A regularized version of η (which is needed since η is usually ill-defined on the diagonal) consists in finding a smooth form (which we will continue to denote by η) on the open configuration space $C_2^0(M)$ so that P defined as above (with the obvious understanding that the projections π_1 and π_2 are now from $C_2^0(M)$ to M) satisfies the same equation. We notice however that $C_2^0(M)$ is not compact; so one has to replace it by its compactification $C_2(M)$, which is obtained, following [30], from $M \times M$ by replacing the diagonal with its differential-geometric blowup. We notice also that $C_2(M)$ is a manifold with the spherical normal bundle SNDiag of Diag in $M \times M$ as boundary. Since we have removed the diagonal, we require now that the superpropagator should be an A_0-covariantly closed form $\eta \in \Omega^{m-1}(C_2(M)\,,\operatorname{ad} P \boxtimes \operatorname{ad} P)$, where, by abuse of notation, we have denoted by $\operatorname{ad} P \boxtimes \operatorname{ad} P$ the pulled-back bundle of $\operatorname{ad} P \boxtimes \operatorname{ad} P$ w.r.t. the projection $C_2(M) \to M \times M$. Next, the generalized Stokes Theorem implies that the left-hand side of (3.4.17) applied to a form γ is $\pi_*^\partial \left\langle\!\left\langle\, \iota^*\eta\,;\, \pi^{\partial *}\gamma \,\right\rangle\!\right\rangle_{13}$, with ι the inclusion of the unit normal bundle of the diagonal $\mathrm{SN\,Diag}$ in $\overline{C}_2(M)$ and π^∂ the projection $SN\,\mathrm{Diag} \to \mathrm{Diag}$. Thus, for Identity (3.4.17) to hold, one has to require that the restriction of η to the boundary equals

$$\iota^*\eta = \vartheta \otimes \pi^{\partial *}\phi + \pi^{\partial *}\beta \tag{3.4.18}$$

where ϑ is the global angular form of SNDiag and $\beta \in \Omega^{m-1}(\mathrm{Diag}, \mathrm{ad}\, P \otimes \mathrm{ad}\, P)$ is so far undetermined.

We recall (see also [13]) that a global angular form ϑ on a sphere bundle $\mathrm{S}\, S \xrightarrow{\pi} M$ over a smooth manifold B is a form on S satisfying $\pi_*\vartheta = 1$ and $\mathrm{d}\vartheta = -\pi^* e$, where e is a representative of the Euler class of the bundle. In our case, since $\mathrm{NDiag} \simeq \mathrm{T}M$, e will be a representative of the Euler class of M. The first property of ϑ is what we need for (3.4.17) to hold; the second property, of which one cannot dispose, implies $\mathrm{d}_{A_0}\beta = e \otimes \phi$.

This is a very strong constraint in even dimensions, as it requires the Euler class of M to be trivial. In fact, multiply both sides of the equation by ϕ and contract the first $\mathfrak{g}$-component with the third and the second with the fourth; this yields

$$\mathrm{d} \left\langle\!\left\langle\, \phi\,;\, \beta \,\right\rangle\!\right\rangle_{13,24} = e \,\dim \mathfrak{g}.$$

We finally notice that we also want η to satisfy (3.4.16), with T now the corresponding involution on $\mathrm{ad}\, P \boxtimes \mathrm{ad}\, P \to C_2(M)$. In particular, this implies that we have to choose ϑ to be even (odd) under the antipodal map on the fibers if m is even (odd); in odd dimensions this also implies that one must choose $e = 0$. Moreover, β has to be an element of $\Omega^{m-1}(\mathrm{Diag}, \mathfrak{S}^2\, \mathrm{ad}\, P)$ in even dimensions and of $\Omega^{m-1}(\mathrm{Diag}, \bigwedge^2 \mathrm{ad}\, P)$ in odd dimensions.

Such a form η can be obtained generalizing the construction of [11]:

Theorem 3.4.6. *Under Assumptions 1 and 3, there exists a covariantly closed element η of $\Omega^{m-1}(C_2(M)\,, \mathrm{ad}\, P \boxtimes \mathrm{ad}\, P)$, satisfying* (3.4.16) *and* (3.4.18)*. Moreover, in odd dimensions β will be automatically covariantly closed, while in even dimensions—where the above Assumptions imply $[e] = 0$—this will be true only if one chooses $e = 0$. Finally, β may be chosen to vanish if $H^{m-1}_{\mathrm{d}_{A_0}}(M, \bigwedge^2 \mathrm{ad}\, P)$ is trivial in odd dimensions and if $H^{m-1}_{\mathrm{d}_{A_0}}(M, \mathfrak{S}^2\, \mathrm{ad}\, P)$ is trivial in even dimensions.*

Proof. Following [10], we first build a global angular form ϑ on SNDiag with the correct behavior under the antipodal map on the fibers: we may construct it as in Section 2.7 using the Levi-Civita connection for a given Riemannian metric, which also allows to identify SNDiag with the unit sphere bundle $SO(\mathrm{Diag}) \times_{SO(m)} S^{m-1}$.

Next we extend ϑ to the complement of the zero section of NDiag and multiply it by a function ρ that is identically one in a neighborhood U_1 of the zero section and identically zero outside a second neighborhood $U_2 \supset U_1$. We then define $\eta_0 \in \Omega^{m-1}(C_2(M)\,, \mathrm{ad}\, P \boxtimes \mathrm{ad}\, P)$ as the extension by zero of $\rho\, \vartheta \otimes \pi^{\partial *}\phi$. Since $\mathrm{d}_{A_0}\phi = 0$, $\mathrm{d}_{A_0}\eta_0$ is the extension by zero of $\mathrm{d}\rho\, \vartheta \otimes \pi^{\partial *}\phi - \rho\, \pi^{\partial *}(e \otimes \phi)$. The last form may be extended to the zero section of NDiag; hence, the extension by zero of $\mathrm{d}_{A_0}\eta_0$ can be seen as a covariantly closed element of $\Omega^m(M \times M, \mathrm{ad}\, P \boxtimes \mathrm{ad}\, P)$. The general

Kuenneth theorem implies $H^*_{\mathrm{d}_{A_0}}(M \times M, \mathrm{ad}\, P \boxtimes \mathrm{ad}\, P) \cong H^*_{\mathrm{d}_{A_0}}(M, \mathrm{ad}\, P)^{\otimes 2}$. So Assumption 1 implies that there is a form $\alpha \in \Omega^{m-1}(M \times M, \mathrm{ad}\, P \boxtimes \mathrm{ad}\, P)$ such that $\mathrm{d}_{A_0}\pi^*\alpha = \mathrm{d}_{A_0}\eta_0$, with π the projection $C_2(M) \to M \times M$. We also observe that we may choose α to satisfy $T^*\alpha = (-1)^m \alpha$. Finally, we define $\eta := \eta_0 - \pi^*\alpha$. An easy check shows that it satisfies all properties above (with β determined by the restriction of α to the diagonal). □

Remark 3.4.7. There are a couple of interesting cases when M does not satisfy Assumption 1, but one can define the superpropagator anyway. First, when $M = \mathbb{R}^m$ all boils down to looking for (the higher-dimensional generalization of) Bott and Taubes's [12] tautological forms, as described in [20]. Second, when M is a rational homology sphere, one can generalize the construction of [10] (which does not yield a closed η, so that extra diagrams must be introduced to correct for it) or alternatively remove one point, as suggested in [40], and essentially reduce to the previous case.

3.5 Sigma-model interpretation

We end this chapter with a small digression about the Alexandrov–Kontsevich–Schwarz–Zaboronsky formalism, shortly the AKSZ formalism, introduced and discussed in [2]. Introducing some new notions in the study of supermanifolds, the authors provided a good mathematical framework for a geometric interpretation of the BV formalism; additionally, they provided a general method to construct solutions of the (Classical) Master Equation for some theories, e.g. the Chern–Simons theory and Witten's A- and B-model in Mirror Symmetry. The main fact about the AKSZ formalism is that first one constructs a solution of the Master Equation and then recovers the classical action; this is the way inverse to the BV method, where one constructs a BV action starting from a classical action. It is possible to recover the BV action for BF theories also with the AKSZ formalism; we want to sketch some arguments of the construction.

Main ingredients in the formulation of the results are supermanifolds; readers not familiar with supermanifolds may skip directly this section, as it is not essential for what follows. We therefore assume basic notions of superanalysis; we refer also to [26] for more details on supermanifolds. We recall only that a smooth supermanifold $\mathcal{M}$ of dimension $(m|n)$ is a ringed space $(M, \mathcal{O}_M)$, where M is a smooth manifold of dimension m and $\mathcal{O}_M$ is a sheaf of rings over M with the identity, such that, for any open subset U of M, the restriction of $\mathcal{O}_M$ to U is isomorphic to the sheaf $C^\infty(U) \otimes \bigwedge^n(\mathbb{R})$, and $\bigwedge^n(\mathbb{R})$ is the Grassmann algebra with n generators.

First of all, we take the *source* $\Sigma := \Pi\, \mathrm{T}M$, where M is our original m-dimensional manifold and Π indicates that the fiber has to be taken with reversed Grassmann parity; the target N has to be chosen among the following possibilities:

	ordinary BF	canonical BF
m odd	$\Pi\mathfrak{g} \times \Pi\mathfrak{g}$	$\Pi\mathfrak{g} \times \Pi\mathfrak{g}^*$
m even	$\Pi\mathfrak{g} \times \mathfrak{g}$	$\Pi\mathfrak{g} \times \mathfrak{g}^*$

where Π again reverses the Grassmann parity. To encompass all cases, we will write $N = V_1 \times V_2$ with V_1 and V_2 as in the above table. The superfields a and B are then *related to* the 1 and 2 components of a map $f\colon \Sigma \to N$. We also recall that there is a pairing $\langle\ ,\ \rangle$ of V_2 with V_1 which is induced from the bilinear form of Assumption 3 resp. from the canonical pairing in the case of ordinary resp. canonical BF theories. In the language of [26], the supermanifold $\Pi\,\mathrm{T}M$ can be viewed as the ringed space $(M, \Omega^*(M))$, where $\Omega^*(M)$ denotes in this case the sheaf of smooth forms on M. Analogously, $\Pi\mathfrak{g}$ can be viewed as the ringed space $\left(*, \bigwedge^* \mathfrak{g}^*\right)$, where $*$ denotes a point (the even part, or the body of $\Pi\mathfrak{g}$, is trivial).

We recall that a P-manifold is a supermanifold endowed with an odd non-degenerate closed 2-form (shortly, an odd symplectic form); a Q-manifold is a supermanifold endowed with an odd vector field Q that has vanishing Lie bracket with itself; finally, a QP-manifold is a supermanifold that has both structures in a compatible way, i.e., such that the odd symplectic form is Q-invariant. We notice that an odd symplectic form defines a BV bracket; moreover, an even solution of the master equation defines a compatible Q-structure and vice versa. For more details on Q-, P- and QP-manifolds, we refer to [44].

Following [2], we give the BV bracket and the BV action for BF theories (to begin with in the case when the background connection A_0 is trivial) a beautiful interpretation in terms of a natural QP-structure on the space $\mathcal{E}$ of maps $\Sigma \to N$, which we are now going to describe.

The P-structure on $\mathcal{E}$ is defined in terms of the following constant symplectic form on N:

$$
\begin{gathered}
\omega(v_1 \oplus v_2, w_1 \oplus w_2) := \langle\, v_2\,,\, w_1\,\rangle - (-1)^m \,\langle\, v_1\,,\, w_2\,\rangle\,,\\
v_1 \oplus v_2,\ w_1 \oplus w_2 \in T_{(\xi_1,\xi_2)}N \simeq V_1 \oplus V_2, \qquad \forall (\xi_1,\xi_2) \in N.
\end{gathered}
$$

Observe that ω is odd (even) when m is even (odd); i.e., ω defines an ordinary symplectic structure—though on an odd vector space—when m is odd and a P-structure when m is even. This induces the following constant *odd* symplectic form on $\mathcal{E}$:

$$
\begin{gathered}
\tilde{\omega}(\phi, \phi') := \int_\Sigma \omega(\phi, \phi')\, \mathrm{d}\mu,\\
\phi,\ \phi' \in T_f\mathcal{E} \simeq \Gamma(\Sigma, f^*\,\mathrm{T}N), \qquad \forall f \in \mathcal{E}.
\end{gathered}
$$

Here we have denoted by $\int_\Sigma \mathrm{d}\mu$ the canonical density associated to the supermanifold $\Pi\,\mathrm{T}M$. It is defined as follows: since $\Pi\,\mathrm{T}M = (M, \Omega^*(M))$, every function on $\Pi\,\mathrm{T}M$ can be identified with a sum of forms on M of all degrees, so a canonical density can be given by the usual integral of forms (which selects the top degree part of the integrand). Locally,

$$
\int_{\Sigma|_U} \mathrm{d}\mu = \int_U \mathrm{d}x^1 \cdots \mathrm{d}x^m\ ,
$$

where the x's are local coordinates on M.

Next we come to the Q-structure. Observe first that any flow on Σ or on N defines (by composition on the right resp. on the left) a flow on $\mathcal{E}$ and that flows of the two types commute. Infinitesimally, any vector field on Σ or on N defines a vector field on $\mathcal{E}$ and vector fields of the two types commute. Moreover, nilpotency is preserved. In conclusion, any Q-structures Q_Σ on Σ and Q_N on N determine Q-structures $\widetilde{Q}_\Sigma$ and

$\widetilde{Q}_N$ on $\mathcal{E}$; moreover, any linear combination of the two is still a Q-structure, since the flows obviously commute. On N we consider the Q-structure given by the Hamiltonian vector field associated to the function

$$\sigma(\xi_1, \xi_2) = \frac{1}{2} \langle \xi_2 \, , \, [\, \xi_1 \, , \, \xi_1 \,] \rangle \, .$$

Observe that this function is odd (even) for m odd (even), so the corresponding vector field is always odd. The corresponding Q-structure on $\mathcal{E}$ yields the following action on the superfields a and B:

$$\boldsymbol{\delta}_N \mathsf{a} = \frac{1}{2} [\![\mathsf{a} \, ; \mathsf{a}]\!], \qquad \boldsymbol{\delta}_N \mathsf{B} = \begin{cases} [\![\mathsf{a} \, ; \mathsf{B}]\!] & \text{ordinary } BF, \\ \mathsf{ad}^*(\mathsf{a})\mathsf{B} & \text{canonical } BF. \end{cases}$$

This Q-structure is automatically compatible with the P-structure defined above. On Σ we consider instead the canonical Q-structure which in local coordinates reads

$$Q_\Sigma = \theta^i \, \frac{\partial}{\partial x^i}.$$

The induced Q-structure on $\mathcal{E}$ acts on the superfields by

$$\boldsymbol{\delta}_\Sigma \mathsf{a} = \mathrm{d}\mathsf{a}, \qquad \boldsymbol{\delta}_\Sigma \mathsf{B} = \mathrm{d}\mathsf{B}.$$

This Q-structure is also compatible with the P-structure defined by $\tilde{\omega}$ as follows by an explicit computation: in fact, it is not difficult to check that the odd vector field Q_Σ has zero-divergence w.r.t. the density specified above. Since M has no boundary, this guarantees automatically that the P-structure on $\mathcal{E}$ is compatible with the Q-structure defined by Q_Σ.

Finally, we consider a linear combination with nonvanishing coefficients of the above vector fields. This yields an entire family of QP-structures on $\mathcal{E}$. We notice however that rescaling a with a parameter λ and B with $1/\lambda$ ($\lambda \neq 0$) is a canonical transformation. So, up to equivalence, we can always set the coefficients to have the same ratio as in (3.4.4) and (3.4.5) (or (3.4.10) and (3.4.11) for canonical BF theories). Given the P-structure, there is a unique (up to an additive constant) action functional generating the given Q-structure, i.e. such that its Hamiltonian vector field w.r.t. the P-structure on $\mathcal{E}$. Choosing the additive constant appropriately, the action functional is then a multiple of our S in (3.4.1). Finally, the remaining multiplicative constant can be absorbed in $\hbar$ (or taken as a definition thereof).

In order to take into account nontrivial background connections (or even nontrivial bundles $P \to M$), we have to modify a little bit the above construction. First we have to introduce a vector bundle $E \to \Sigma$ with fiber N, with Σ and N as above. If the original bundle P is trivial, so will be E (otherwise it will be constructed by using the transition functions of $\operatorname{ad} P$ and $\operatorname{ad}^* P$). The space $\mathcal{E}$ will be now the space of sections of E. The P- and Q_N-structures are introduced as above. The Q_Σ-structure instead requires the choice of a connection A_0 in order to lift to E the vector field on Σ; this connection has moreover to be flat to ensure the nilpotency of $\tilde{Q}_\Sigma$. The rest of the construction is the same as above.

Chapter 4

Generalized Wilson loops in the super BV formalism

In this chapter we deal with the BV quantum observables related to loops in M. We begin with a brief digression on the classical observables, introducing later the BV superversion of these. We then prove that such functionals are observables in BV framework, although we have to restrict to a particular class of loops, in order to avoid problems with the BV Laplacian.

4.1 Classical observables

Before going into the details concerning generalized Wilson loops, it is better to have a glimpse of the classical observables for BF theories related to loops in M.

We start by considering $\mathrm{Tr}_\rho \mathrm{H}(A)|_0^1$, where by $\mathrm{H}(A)|_0^1$ we denote here the inverse of the holonomy w.r.t. the connection A viewed as a G-valued function on $\mathrm{L}M$; we refer to Section 2.3.

Taking a representation (ρ, V), we get an $\mathrm{Aut}\, V$-valued function, which under the trace yields then an ordinary function. This function depends also on the choice of a connection A, but its very definition implies that it is invariant w.r.t. the action of the gauge group $\mathcal{G}$ on the space $\mathcal{A}$ of connections on P, so it defines a function on $\mathcal{A}/\mathcal{G} \times \mathrm{L}M$ ($\mathcal{A}/\mathcal{G}$ denotes the coarse moduli space of G-connections on the principal bundle P). We notice that in a local trivialization, the inverse of the holonomy possesses a representation in terms of a formal series of iterated integrals. Finally, if P is trivial, the holonomy becomes a function on $\mathrm{L}M$ with values in G, and so also its inverse.

Next, we define

$$h_{n,\rho}(A, B) := \mathrm{Tr}_\rho \left\{ \pi_{n*} \left[\widehat{B}_{1,n} \wedge \cdots \wedge \widehat{B}_{n,n} \right] \mathrm{H}(A)|_0^1 \right\} , \tag{4.1.1}$$

where the notations are borrowed from Section 2.3.

From now on, we will omit the wedge product between forms. We notice that we have already omitted to write the representation ρ before all forms in the definition of

$\widehat{B}_i$; for all i, $\widehat{B}_{i,n}$ is a form on $\mathrm{L}M \times \triangle_n$ with values in $\mathrm{End}(V)$. It follows from the definition that $h_{n,\rho}(A, B)$, for all n, is a differential form of degree $(m-3)n$ on $\mathrm{L}M$.

Proposition 4.1.1. *If A and B be on shell for BF theories, $h_{n,\rho}(A, B)$ is a closed form for m odd and for all n, while for m even and greater than 4, the $h_{2k+1,\rho}(A, B)$'s are closed.*

Proof (sketch). Since $F_A = 0$ and $\mathrm{d}_A B = 0$, as a consequence of Theorem 2.3.3 the following identities hold:

$$\mathrm{d}_{\pi_1^* \mathrm{ev}(0)^* A} \widehat{B} = \widehat{\mathrm{d}_A B} = 0, \ \forall i;$$
$$\mathrm{d}_{\mathrm{ev}(0)^* A} \, \mathrm{H}(A)|_0^1 = 0$$

as a consequence of 2.1.25. The cyclicity of Tr_ρ implies $\mathrm{Tr}_\rho \, \mathrm{d}_{\mathrm{ev}(0)^* A} = \mathrm{d} \, \mathrm{Tr}_\rho$.

In order to compute $\mathrm{d}h_{n,\rho}$, we have to apply the generalized Stokes Theorem (2.2.2). As we have already discussed, the boundary of the n-simplex decomposes as the union of $n+1$ $(n-1)$-simplices (the so-called faces of the n-simplex), corresponding to the collapse of successive points and two other faces, where the first point becomes 0, resp. where the last becomes 1. The faces describing the collapse of two consecutive points contribute to the exterior derivative trivially, because they yield terms containing $\widehat{B^2}_{i,n}$, which vanish for dimensional reasons. The remaining two faces give the following contributions

$$- (-1)^{m(n-1)} \, \mathrm{Tr}_\rho \left\{ \mathrm{ev}(0)^* B \pi_{n-1*} \left[\widehat{B}_{1,n-1} \ldots \widehat{B}_{n-1,n-1} \right] \mathrm{H}(A)|_0^1 \right\}$$
$$+ (-1)^{n+1} \, \mathrm{Tr}_\rho \left\{ \pi_{n-1*} \left[\widehat{B}_{1,n-1} \ldots \widehat{B}_{n-1,n-1} \right] \mathrm{H}(A)|_0^1 \, \mathrm{ev}(0)^* B \right\};$$

again, for $m = \dim M$ odd, the cyclicity of Tr_ρ implies that these terms cancel each other. This also works for m even, in case n is odd.

On the other hand, when both m and n are even, these two terms have the same sign, and therefore they do not cancel each other. $\square$

Similar computations show that the $h_{n,\rho}(A, B)$'s are observables on shell and modulo exact terms, either if m is odd and greater than 5 or if m is even and greater than 4 but n is odd.

The advantage of the BV formalism in the study of BF theories is that it allows to deal with observables which are BRST closed only on shell, upon extending them suitably. This will be explained in the next sections.

4.2 Generalized Wilson loops in odd dimensions

In this section we display some observables for odd-dimensional BF theories which in some sense generalize the classical observables (4.1.1), i.e. the iterated-integral expansions of Wilson loops. In the first subsection we construct a flat invariant observable (see Definition 3.3.7 on page 116) S_3 which represents a sort of "cosmological term" (although it does not have the correct ghost number, except for the case

$\dim M = 3$). We next define in subsection 4.2.2 a "generalized holonomy" constructed via iterated integrals by means of A and B, and we show that it defines a cohomology class w.r.t. the super BV coboundary operator twisted with S_3 which takes values in $H^*(\mathrm{Imb}_f(S^1, M))$. From this we then derive a true BV observable.

4.2.1 The "cosmological term"

We define the local functional

$$\mathsf{S}_3 := \frac{1}{6}\int_M \langle\!\langle\, \mathsf{B}\, ;\, [\![\mathsf{B}\, ;\mathsf{B}]\!]\,\rangle\!\rangle$$

which is an element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathbb{R})$ of total degree $2m-6$. We want to show that S_3 is a flat invariant observable in the sense of definition 3.3.7. This is expressed by the following

Lemma 4.2.1.

$$\boldsymbol{\delta}\mathsf{S}_3 = 0, \tag{4.2.1}$$

$$\boldsymbol{\Delta}\mathsf{S}_3 = 0, \tag{4.2.2}$$

$$(\!(\, \mathsf{S}_3\, ;\, \mathsf{S}_3\,)\!) = 0. \tag{4.2.3}$$

Proof. First of all, we write down the left partial derivatives of S_3:

$$\frac{\overrightarrow{\partial}\mathsf{S}_3}{\partial \mathsf{a}} = 0, \qquad \frac{\overrightarrow{\partial}\mathsf{S}_3}{\partial \mathsf{B}} = \frac{1}{2}[\![\mathsf{B}\, ;\mathsf{B}]\!].$$

With the help of (3.4.2) and by the definition of the super BV antibracket, we get

$$\boldsymbol{\delta}\mathsf{S}_3 = (\!(\, S\, ;\, \mathsf{S}_3\,)\!) = \frac{1}{2}\int_M \langle\!\langle\, \mathrm{d}_{\mathsf{A}}\mathsf{B}\, ;\, [\![\mathsf{B}\, ;\mathsf{B}]\!]\,\rangle\!\rangle\, ;$$

By the invariance of $\langle\ ,\ \rangle$ it follows

$$\frac{1}{2}\int_M \langle\!\langle\, \mathrm{d}_{\mathsf{A}}\mathsf{B}\, ;\, [\![\mathsf{B}\, ;\mathsf{B}]\!]\,\rangle\!\rangle = \frac{1}{6}\int_M \mathrm{d}\,\langle\!\langle\, \mathsf{B}\, ;\, [\![\mathsf{B}\, ;\mathsf{B}]\!]\,\rangle\!\rangle = 0$$

by Stokes' theorem. So we have proved (4.2.1).

Eqns. (4.2.2) and (4.2.3) follow from the definitions of the super BV antibracket and of the super BV Laplacian $\boldsymbol{\Delta}$ and from the fact that S_3 depends only on B. ☐

It follows from Lemma 3.3.8 that not only S_3 but any of its multiples is a flat observable. So we introduce the "cosmological constant" κ and consider a twisting by $\kappa^2\mathsf{S}_3$ (the reason for putting κ^2 instead of κ will be clear in the next subsection). We then define

$$\boldsymbol{\delta}_{\kappa^2} := \boldsymbol{\delta} + \kappa^2\,(\!(\, \mathsf{S}_3\, ;\ \ )\!)\, .$$

and, again by Lemma 3.3.8, $\boldsymbol{\delta}_{\kappa^2}$ is an odd differential for any κ. Its action on the fundamental superfields is easily computed:

$$\boldsymbol{\delta}_{\kappa^2}\mathsf{a} = -\mathsf{F}_{\mathsf{A}} - \frac{\kappa^2}{2}[\![\mathsf{B}\, ;\mathsf{B}]\!], \qquad \boldsymbol{\delta}_{\kappa^2}\mathsf{B} = -\mathrm{d}_{\mathsf{A}}\mathsf{B}. \tag{4.2.4}$$

4.2.2 The generalized Wilson loop in the BV superformalism

We want to define an object that generalizes the observable introduced in [15] for the 3-dimensional BF theory with cosmological term. We shall realize this proposal by introducing the new superform

$$\mathsf{C}_\kappa := \mathsf{a} + \kappa\mathsf{B}.$$

Observe that C_κ is not a homogeneous element in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(M, \operatorname{ad} P)$ w.r.t. the total degree, but it is homogeneous of degree one with respect to its reduction modulo 2. By recalling (4.2.4), it is easy to see that

$$\boldsymbol{\delta}_{\kappa^2}\mathsf{C}_\kappa = -\mathrm{d}_{A_0}\mathsf{C}_\kappa - \frac{1}{2}[\![\mathsf{C}_\kappa\,;\mathsf{C}_\kappa]\!].$$

The previous equation suggests that we may interpret the superform C_κ as a "variation" of the flat connection A_0, and therefore $\boldsymbol{\delta}_{\kappa^2}\mathsf{C}_\kappa$ can be interpreted as its curvature. Observe that, since C_κ is of odd degree, all formulae of Appendix 2.3 are basically the same as if C_κ were an ordinary variation of A_0. We exploit then this analogy to define the n-th iterated integral of C_κ as

$$\pi_{n*}\left(\widehat{\mathsf{C}_\kappa}_{1,n}\cdots\widehat{\mathsf{C}_\kappa}_{n,n}\right)\mathrm{H}(A_0)|_0^1.$$

We refer from now on to Appendix 2.3 for the main notations (simplices, evaluation maps, etc.). We recall the definition of $\widehat{\mathsf{C}_\kappa}$: We have written

$$\widehat{\mathsf{C}_\kappa} := \mathrm{H}(A)|_0^\bullet\,\mathrm{ev}_1^*\,\mathsf{C}_\kappa\,\left(\mathrm{H}(A)|_0^\bullet\right)^{-1},$$

and $\widehat{\mathsf{C}_\kappa}_{i,n} := \pi_{i,n}^*\widehat{\mathsf{C}_\kappa}$. We have suppressed ρ before all $\widehat{\mathsf{C}_\kappa}$'s in the above product; the forms considered in the n-th iterated integral take values in the associative algebra $\mathrm{End}(V)$. We then define the generalized holonomy of C_κ from 0 to 1 via the path-ordered exponential

$$\mathsf{Hol}(\mathsf{C}_\kappa) := \mathrm{H}(A_0)|_0^1 + \sum_{n\geq 1}\pi_{n*}\left(\widehat{\mathsf{C}_\kappa}_{1,n}\cdots\widehat{\mathsf{C}_\kappa}_{n,n}\right)\mathrm{H}(A_0)|_0^1;$$

it defines an element in $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathrm{L}M, \mathrm{End}(V))$, and since $\dim\triangle_n = n$, it follows that it has even total degree. We now pick a finite-dimensional representation ρ and define the generalized Wilson loop

$$\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B}) = \mathrm{Tr}_\rho\,\mathsf{Hol}(\mathsf{C}_\kappa). \tag{4.2.5}$$

From the previous considerations, it is an element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathrm{L}M, \mathbb{R})$, with even total degree. We are now ready to state the main Theorem of this Section.

Theorem 4.2.2. *The generalized Wilson loop is $(\boldsymbol{\delta}_{\kappa^2} + \mathrm{d})$-closed:*

$$(\boldsymbol{\delta}_{\kappa^2} + \mathrm{d})\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B}) = 0.$$

Proof. By above reasonings, we can consider C_κ as a variation of the (flat) connection A_0. The cyclicity of the trace allows to replace the exterior derivative by the covariant derivative $\mathrm{d}_{\mathrm{ev}(0)^* A_0}$. $\mathsf{Hol}(\mathsf{C}_\kappa)$ has the same form as $\mathrm{H}(A+a)|_0^1$ of Section 2.3, where we have set $A_0 = A$, and we have replaced a by C_κ and the wedge product by the dot product. Accordingly to the sign rules for the dot product and repeating almost verbatim the arguments used in the proof of Theorem 2.3.3, we get

$$\mathrm{d}\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B}) = \sum_{m\geq 1}\sum_{i=1}^{m}(-1)^{m+i}\,\mathrm{Tr}_\rho\left\{\pi_{m*}\left[\widehat{\mathsf{C}_\kappa}_{1,m}\cdots\left(\widehat{\boldsymbol{\delta}_{\kappa^2}\mathsf{C}_\kappa}\right)_{i,m}\cdots\right.\right.$$
$$\left.\left.\cdots\widehat{\mathsf{C}_\kappa}_{m,m}\right]\mathrm{H}(A_0)|_0^1\right\}.$$

Recalling Lemma 3.3.4, 3.3.3 and 3.3.6 and the Leibnitz rule, it is then not difficult to verify that

$$\boldsymbol{\delta}_{\kappa^2}\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B}) = \sum_{m\geq 1}\sum_{i=1}^{m}(-1)^{m+i+1}\,\mathrm{Tr}_\rho\left\{\pi_{m*}\left[\widehat{\mathsf{C}_\kappa}_{1,m}\cdots(\widehat{\boldsymbol{\delta}_{\kappa^2}\mathsf{C}_\kappa})_{i,m}\cdots\right.\right.$$
$$\left.\left.\cdots\widehat{\mathsf{C}_\kappa}_{m,m}\right]\mathrm{H}(A_0)|_0^1\right\}.$$

which yields the desired identity. ◻

We would like a stronger assertion than what we proved in the above Theorem; namely, that $\mathcal{H}_\rho$ is $(-\mathrm{i}\hbar\boldsymbol{\Delta} + \boldsymbol{\delta}_{\kappa^2} + \mathrm{d})$-closed. So we need

$$\boldsymbol{\Delta}\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B}) = 0.$$

If a loop has transversal self-intersections, the above identity is certainly false since on the two intersecting strands appear complementary components of a field and its anti-field. If the loop has non-transversal intersections or cusps, it is not even clear what the action of the BV Laplacian should be. However, even restricting to imbeddings might not be enough since in the computation of the BV Laplacian there are ill-defined terms coming from subsequent fields in the iterated integrals as the evaluation points come together. To establish the validity of the above identity, we can choose the following

Regularization procedure. We only consider elements of $\mathrm{Imb}_f(S^1, M)$, the space of framed imbeddings of S^1 into M. For each element we then consider a tubular neighborhood of the imbedding and use the framing to select a companion imbedding on the boundary. Finally we put each component of A appearing in the iterated integrals on the imbedding and each component of B on its companion (following a procedure introduced in [18]).

Since the cosmological term is a flat invariant observable we then obtain, under the above assumption, the following

Corollary 4.2.3.

$$(\boldsymbol{\Omega} + \mathrm{d})\left[\exp\left(\frac{\mathrm{i}}{\hbar}\kappa^2\mathsf{S}_3\right)\boldsymbol{\mathcal{H}}_\rho(\kappa;\mathsf{A},\mathsf{B})\right] = 0.$$

As a consequence, the d-cohomology class of the above functional are BV observables. This implies Theorem 2 of [20], which states that the above functional defines an $H^*(\mathrm{Imb}_f(S^1, M))$-valued BV observable.

Remark 4.2.4. We notice that the v.e.v.s of the generalized Wilson loops together with the cubic cosmological term do not depend on the representative of flat connection A_0. Let $g \in \mathcal{G}$ be a gauge transformation, viewed as a section of $\mathrm{Ad}\, P$. Then, by Lemma 2.1.15, $\mathrm{H}(A_0)|_{t_1}^{t_2}$ is sent to $g^{-1}(\gamma(t_1))\, \mathrm{H}(A_0)|_{t_1}^{t_2} g(\gamma(t_2))$. This implies that the superfields a and B in the generalized Wilson loops are acted upon by Ad_g (this is a consequence of the definition of the generalized Wilson loops and of the cyclicity of the trace). This can be compensated by a change of variables in the functional integral, whose formal measure is constructed upon using the bilinear form $\langle\ ,\ \rangle$ and hence formally Ad-invariant. Therefore, the v.e.v.s of the generalized Wilson loops are functions on the moduli space of flat connections.

4.3 Other loop observables in odd dimensions

We now generalize the ideas of the previous Section along two directions: i) consider variations of the connection which are not necessarily of odd degree; ii) introduce interaction terms with higher powers of B. Both generalizations require the following

Assumption 4. Throughout this section we work with a Lie algebra $\mathfrak{g}$, coming from an associative algebra endowed with a trace Tr (e.g., we may take $\mathfrak{g} = \mathfrak{gl}(N)$ with the usual trace of matrices). Furthermore, we define the ad-invariant symmetric bilinear form $\langle \eta\, ,\, \xi \rangle$ on $\mathfrak{g}$ by $\mathrm{Tr}\, \eta\xi$ and assume that it is nondegenerate (as required by Assumption 3). Finally, we will only consider representations ρ of $\mathfrak{g}$ as an associative algebra.

4.3.1 Higher-order B-interactions

We define, for $k \in \mathbb{N}$, the following even element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}$:

$$\mathsf{O}_{2k+1} = \frac{1}{2k+1} \int_M \mathrm{Tr}\, \mathsf{B}^{2k+1}.$$

Observe that even powers of B would vanish by the cyclicity of the trace

Lemma 4.3.1. *The following identities hold for the functional* O_{2k+1}*:*

$$\boldsymbol{\delta}\, \mathsf{O}_{2k+1} = 0, \forall k \in \mathbb{N} \tag{4.3.1}$$

$$\boldsymbol{\Delta}\, \mathsf{O}_{2k+1} = 0, \forall k \in \mathbb{N} \tag{4.3.2}$$

$$(\!(\, \mathsf{O}_{2k+1}\, ;\, \mathsf{O}_{2l+1}\,)\!) = 0, \quad \forall k, l \in \mathbb{N}. \tag{4.3.3}$$

It follows in particular that, $\forall k \in \mathbb{N}$*, the functional* O_{2k+1} *is a flat invariant observable (see subsection 3.3.4).*

Proof. From the definition of the super BV antibracket, we get

$$(\!(\, S\,;\, \mathsf{O}_{2k+1}\,)\!) = \int_M \mathrm{Tr}\left[\mathrm{d}_{\mathsf{A}}\mathsf{B}\cdot\mathsf{B}^{2k}\right] = \frac{1}{2k+1}\int_M \mathrm{d}\,\mathrm{Tr}\,\mathsf{B}^{2k+1} = 0.$$

(4.3.2) and (4.3.3) follow respectively from the definition of the BV Laplacian and of the super BV antibracket, and from the fact that the functionals O_{2k+1} do not depend on a. □

Let us now choose $n \in \mathbb{N}$ and a sequence of real numbers $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\lambda}) = \{\mu_2, \mu_4, \ldots, \mu_{4n+2}\}$. Then we define the following even element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}(\mathbb{R})$:

$$\mathsf{O}_{\boldsymbol{\mu}} := \sum_{k=1}^{2n+1} \mu_{2k}\,\mathsf{O}_{2k+1}\,.$$

From the Lemma it follows that $\mathsf{O}_{\boldsymbol{\mu}}$ is a flat invariant observable for any $\boldsymbol{\mu}$. So, as in subsection 3.3.4, we can introduce the following odd differential:

$$\boldsymbol{\delta}_{\boldsymbol{\mu}} = \boldsymbol{\delta} + \sum_{k=1}^{2n+1} \mu_{2k}\boldsymbol{\partial}_{\mathsf{O}_{2k+1}}.$$

Its action on the fundamental superfields is easily computed:

$$\boldsymbol{\delta}_{\boldsymbol{\mu}}\mathsf{a} = -\mathsf{F}_{\mathsf{A}} - \sum_{k=1}^{2n+1} \mu_{2k}\mathsf{B}^{2k}, \qquad \boldsymbol{\delta}_{\boldsymbol{\mu}}\mathsf{B} = -\mathrm{d}_{\mathsf{A}}\mathsf{B}. \tag{4.3.4}$$

4.3.2 Extended generalized Wilson loops

Let $\boldsymbol{\lambda} := \{\lambda_1, \lambda_3, \ldots, \lambda_{2n+1}\}$ be another sequence of real numbers with the same n as above. We then define the odd superform

$$\mathsf{C}_{\boldsymbol{\lambda}} := \mathsf{a} + \sum_{k=0}^{n} \lambda_{2k+1}\mathsf{B}^{2k+1}.$$

If the sequences $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are related by

$$\mu_{2k} := \sum_{\substack{0\le i,j\le n\\ i+j=k-1}} \lambda_{2i+1}\lambda_{2j+1}, \tag{4.3.5}$$

then (4.3.4) implies

$$\boldsymbol{\delta}_{\boldsymbol{\mu}}\,\mathsf{C}_{\boldsymbol{\lambda}} = -\mathrm{d}_{A_0}\,\mathsf{C}_{\boldsymbol{\lambda}} - \frac{1}{2}[\![\mathsf{C}_{\boldsymbol{\lambda}}\,;\mathsf{C}_{\boldsymbol{\lambda}}]\!].$$

The above expression has again the form of a curvature; we can therefore view the superform $\mathsf{C}_{\boldsymbol{\lambda}}$ as a variation of the connection A_0. So, analogously to what we did in subsection 4.2.2, we define the path-ordered integral

$$\mathsf{Hol}(\mathsf{C}_{\boldsymbol{\lambda}}) = \mathrm{H}(A_0)|_0^1 + \sum_{m\ge 1}\pi_{m*}\left(\widehat{\mathsf{C}_{\boldsymbol{\lambda}}}_{1,m}\cdots\widehat{\mathsf{C}_{\boldsymbol{\lambda}}}_{m,m}\right)\mathrm{H}(A_0)|_0^1.$$

We next define accordingly

$$\boldsymbol{\mathcal{H}}_\rho(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B}) := \mathrm{Tr}_\rho\, \mathsf{Hol}(\mathsf{C}_{\boldsymbol{\lambda}}).$$

Repeating the arguments used in the proof of (4.2.2), we can state the following

Theorem 4.3.2. *If $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are related by* (4.3.5)*, then*

$$(\boldsymbol{\delta}_{\boldsymbol{\mu}} + \mathrm{d})\boldsymbol{\mathcal{H}}_\rho(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B}) = 0.$$

Since $\mathsf{O}_{\boldsymbol{\mu}}$ is a flat invariant observable, this implies the following

Corollary 4.3.3. *With the same hypothesis as above and with the regularization procedure defined on page 138, we obtain*

$$(\boldsymbol{\Omega} + \mathrm{d}) \left[\exp\left(\frac{\mathrm{i}}{\hbar}\, \mathsf{O}_{\boldsymbol{\mu}} \right) \boldsymbol{\mathcal{H}}_\rho(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B}) \right] = 0.$$

Again this implies that the d-cohomology class of the above functional is a BV observable; from this Theorem 4 of [20] follows.

Remark 4.3.4. The same reasonings sketched in Remark 4.2.4 do hold in this case as well; therefore, we may conclude that the v.e.v.s of the generalized Wilson loops with higher-order B-interactions depend only the class $[A_0]$ in $\{A \in \mathcal{A} \colon F_A = 0\}\,/\mathcal{G}$.

4.4 The even-dimensional case

We now turn to the problem of defining generalized Wilson loop observables for the case $\dim M$ even. Observe that in even-dimensional BF theories B has even total degree; so $[\![\mathsf{B}\,;\mathsf{B}]\!] = 0$. This implies that it is not possible to define a generalized cosmological term as in Section 4.2 because we cannot anymore rely on the dot Lie bracket to construct this functional. Therefore, in order to define products of B with itself (either cubic or not) we must do as in the preceding subsection and, in particular, we need Assumption 4 on page 139.

4.4.1 B-interactions

For a given $k > 1$ we define the following even element of $\mathcal{S}_{\mathsf{A},\mathsf{B}}$:

$$\mathsf{O}_k = \frac{1}{k} \int_M \mathrm{Tr}\, \mathsf{B}^k.$$

We now state the following

Lemma 4.4.1. *The functionals O_k satisfy the identities*

$$\boldsymbol{\delta}\mathsf{O}_k = 0, \forall k > 1, \tag{4.4.1}$$

$$\boldsymbol{\Delta}\mathsf{O}_k = 0, \forall k > 1, \tag{4.4.2}$$

$$(\!(\, \mathsf{O}_k\,;\, \mathsf{O}_l \,)\!) = 0, \forall k, l > 1. \tag{4.4.3}$$

Proof. By definition of the super BV antibracket, we can write

$$\frac{1}{k}\boldsymbol{\delta}\int_M \operatorname{Tr}\mathsf{B}^k = \int_M \operatorname{Tr}\left[\mathrm{d}_{\mathsf{A}}\mathsf{B}\cdot\mathsf{B}^{k-1}\right] = \frac{1}{k}\int_M \mathrm{d}\operatorname{Tr}\mathsf{B}^k = 0.$$

(4.4.2) and (4.4.3) are consequences of the fact that the sfO_ks do not depend on a and of the definitions of the super BV antibracket and of the super BV Laplacian. ☐

Again it follows that each linear combination of O_ks is a flat invariant observable (see subsection 3.3.4). So, for a given positive integer n, we take a sequence of real numbers $\boldsymbol{\mu} := \{\mu_2, \mu_3, \ldots, \mu_{2n}\}$ and define

$$\mathsf{O}_{\boldsymbol{\mu}} = \sum_{i=2}^{2n} \mu_i \mathsf{O}_{i+1}.$$

Therefore, Lemma 3.3.8 implies that

$$\boldsymbol{\delta}_{\boldsymbol{\mu}} := \boldsymbol{\delta} + \boldsymbol{\partial}_{\boldsymbol{\mu}} := \boldsymbol{\delta} + \sum_{i=2}^{2n} \mu_i \left(\!\left(\mathsf{O}_{i+1}\,;\ \ \right)\!\right)$$

is an odd differential for any sequence $\boldsymbol{\mu}$. Moreover we have

$$\boldsymbol{\delta}_{\boldsymbol{\mu}}\mathsf{a} = \mathsf{F}_{\mathsf{A}} + \sum_{i=2}^{2n}\mu_i\mathsf{B}^i, \qquad \boldsymbol{\delta}_{\boldsymbol{\mu}}\mathsf{B} = \mathrm{d}_{\mathsf{A}}\mathsf{B}, \tag{4.4.4}$$

using once again arguments similar to those introduced in the proof of (3.4.4) and (3.4.5).

4.4.2 The generalized Wilson loop

For the same n as above, we consider a sequence $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ so that

$$\mu_i := \sum_{\substack{1\le k,l\le n\\ k+l=i}} \lambda_k\lambda_l \tag{4.4.5}$$

for the previously introduced subsequence $\boldsymbol{\mu}$. Then we define

$$\mathsf{B}_{\boldsymbol{\lambda}} = \sum_{i=1}^{n}\lambda_i\mathsf{B}^i.$$

From (4.4.4) it follows that

$$\boldsymbol{\partial}_{\boldsymbol{\mu}}\mathsf{a} = \mathsf{B}_{\boldsymbol{\lambda}}\cdot\mathsf{B}_{\boldsymbol{\lambda}}, \qquad \boldsymbol{\partial}_{\boldsymbol{\mu}}\mathsf{B}_{\boldsymbol{\lambda}} = 0. \tag{4.4.6}$$

Then, in analogy with the superform $\widehat{\mathsf{C}_{\boldsymbol{\lambda}}}$ of the previous subsection, we define

$$\widehat{\mathsf{B}_{\boldsymbol{\lambda}}} := \mathrm{H}(A_0)|_0^{\bullet}\ \mathrm{ev}_1^*\,\mathsf{B}_{\boldsymbol{\lambda}}\left(\mathrm{H}(A_0)|_0^{\bullet}\right)^{-1}, \qquad \widehat{\mathsf{a}} := \mathrm{H}(A_0)|_0^{\bullet}\ \mathrm{ev}_1^*\,\mathsf{a}\left(\mathrm{H}(A_0)|_0^{\bullet}\right)^{-1},$$

and, accordingly with the notations of Section 2.3, $\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{i,n}$ and $\widehat{\mathsf{a}}_{i,n}$, which we will write as $\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}$ and $\widehat{\mathsf{a}}_{t_i}$. We then define

$$\mathsf{h}_{m,\rho}(\boldsymbol{\lambda};\mathsf{A},\mathsf{B}) := \mathrm{Tr}_\rho\, \pi_{m*}\left[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdot\mathrm{H}(\widehat{\mathsf{a}})|_{t_1}^{t_2}\cdots\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_m}\cdot\mathrm{H}(\widehat{\mathsf{a}})|_{t_m}^1\right]\mathrm{H}(A_0)|_0^1,$$

where we have written

- $\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} := \pi_{1,m}^*\,\mathrm{H}(A_0+\mathsf{a})|_0^{\bullet}$;
- $\mathrm{H}(\widehat{\mathsf{a}})|_{t_m}^1 := \pi_{m,m}^*\,\mathrm{H}(A_0+\mathsf{a})|_{\bullet}^1$;
- $\mathrm{H}(\widehat{\mathsf{a}})|_{t_i}^{t_{i+1}} := \pi_{i,i+1,m}^*\,\mathrm{H}(A_0+\mathsf{a})|_{\bullet}^{\bullet}$,

using the notations of Remark 2.3.2, where we have set again $A_0 = A$, and we have replaced a by a and wedge products by dot products; $\pi_{i,i,+1,m}(\gamma;t_1,\ldots,t_m) := (\gamma;t_i,t_{i+1})$, for $i\in\{1,\ldots,m-1\}$. We finally define

$$\boldsymbol{\mathcal{H}}_\rho^o(\boldsymbol{\lambda};\mathsf{A},\mathsf{B}) = \sum_{m=0}^{\infty}\mathsf{h}_{2m+1,\rho}(\boldsymbol{\lambda};\mathsf{A},\mathsf{B}).$$

We can now state the main theorem of this subsection

Theorem 4.4.2. *The following identity holds:*

$$(\mathrm{d}-\boldsymbol{\delta}_{\boldsymbol{\mu}})\boldsymbol{\mathcal{H}}_\rho^o(\boldsymbol{\lambda};\mathsf{A},\mathsf{B}) = 0.$$

Proof. We begin by computing the exterior derivative of one of the factors of the above sum. With the help of the generalized Stokes Theorem we obtain

$$\begin{aligned}\mathrm{d}\mathsf{h}_{2m+1,\rho}(\boldsymbol{\lambda};\mathsf{A},\mathsf{B}) = \mathrm{Tr}_\rho(-1)^{2m+1}\pi_{2m+1*}\left\{\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\left[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\right]\right\}+\\ +\mathrm{Tr}_\rho(-1)^{2m}\pi_{\partial_{2m+1}*}\left[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\right]\end{aligned}\tag{4.4.7}$$

We consider the first term on the right-hand side of (4.4.7); the Leibnitz rule for the dot product implies

$$\begin{aligned}\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\left[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\right] = \sum_{i=1}^{2m+1}\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\\ \mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\left[\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i}\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}\right]\cdots+\\ +\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdots\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_{2m+1}}\cdot\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\mathrm{H}(\widehat{\mathsf{a}})|_{t_{2m+1}}^1.\end{aligned}$$

We recall that $\mathrm{d}_{\mathrm{ev}(0)^*A_0}\mathrm{H}(A_0)|_0^1 = 0$ by (2.1.22).

We compute the following expression

$$\begin{aligned}\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\left[\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i}\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}\right] = \left[\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i}\right]\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}+\\ +\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i}\cdot\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i};\end{aligned}$$

For the second term on the right-hand side of the above equation, we obtain, repeating (almost) the same arguments used in the proof of Theorem 2.3.3, $\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i} \cdot \widehat{\mathrm{d}_{A_0}\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}$; for the first term, we obtain analogously

$$\Big[\boldsymbol{\delta}\,\mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i} - \widehat{\mathsf{a}}_{t_{i-1}} \cdot \mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i} + \mathrm{H}(\widehat{\mathsf{a}})|_{t_{i-1}}^{t_i} \cdot \widehat{\mathsf{a}}_{t_i}\Big] \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_i}.$$

Summing up all these contributions with the right signs and using repeatedly (2.2.2), we obtain, for the first term on the right-hand side of (4.4.7), the result

$$\begin{aligned} - \mathrm{ev}(0)^*\mathsf{a} \cdot \pi_{2m+1*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big] - \pi_{2m+1*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big] \cdot \mathrm{ev}(0)^*\mathsf{a} \\ + \boldsymbol{\delta}\Big\{\pi_{2m+1*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big]\Big\}. \end{aligned}$$

By the invariance of Tr_ρ, and since a and $\pi_{2m+1*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big]$ have odd total degree, we get

$$(-1)^{2m+1}\,\mathrm{Tr}_\rho\,\pi_{2m+1*}\Big\{\mathrm{d}_{\pi_{2m+1}^*\mathrm{ev}(0)^*A_0}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\Big]\Big\} = \boldsymbol{\delta}\Big\{\,\mathrm{Tr}_\rho\,\Big[\mathrm{H}(\mathsf{a})|_0^{t_1}\cdot\widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1}\cdots\Big]\Big\}.$$

We now consider the second term on the right hand side of (4.4.7). We recall first the orientation choices for the m-simplex made in Section 2.3; with these in mind we obtain (once again with the same arguments of the proof of Theorem 2.3.3)

$$\begin{aligned} - \mathrm{Tr}_\rho\,\mathrm{ev}(0)^*\mathsf{B}_{\boldsymbol{\lambda}} \cdot \Big\{\pi_{2m*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_i} \cdot \widehat{\mathsf{B}}_{t_1} \cdots\Big]\Big\} + \\ + \mathrm{Tr}_\rho\Big\{\pi_{2m*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big] \cdot \mathrm{ev}(0)^*\mathsf{B}_{\boldsymbol{\lambda}}\Big\} + \\ + \sum_{j=1}^{2m}(-1)^{2m+j+1}\,\mathrm{Tr}_\rho\,\pi_{2m*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}}_{t_1} \cdots \big(\widehat{\mathsf{B}_{\boldsymbol{\lambda}} \cdot \mathsf{B}_{\boldsymbol{\lambda}}}\big)_{t_i} \cdots\Big]. \end{aligned} \tag{4.4.8}$$

Since the trace is cyclic in the arguments and $\mathsf{B}_{\boldsymbol{\lambda}}$ has even total degree, the first two terms in the above expression cancel each other.

In summary, we have obtained

$$\begin{aligned} (-1)^{2m}\,\mathrm{Tr}_\rho\,\pi_{\partial_{2m+1}*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots\Big] = \\ = \sum_{j=1}^{2m}(-1)^{2m+j+1}\,\mathrm{Tr}_\rho\,\pi_{2m*}\Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdots \big(\widehat{\mathsf{B}_{\boldsymbol{\lambda}} \cdot \mathsf{B}_{\boldsymbol{\lambda}}}\big)_{t_i} \cdots\Big]. \end{aligned}$$

Recalling formulae (4.4.6), we apply $\boldsymbol{\partial}_{\boldsymbol{\mu}}$ to $\mathrm{H}(\widehat{\mathsf{a}})|_{t_i}^{t_{i+1}}$. Repeating (almost) verbatim the arguments in the proof of Theorem 2.3.3, we obtain

$$\boldsymbol{\partial}_{\boldsymbol{\mu}}\,\mathrm{H}(\widehat{\mathsf{a}})|_{t_i}^{t_{i+1}} = -\int_{t_i \le t \le t_{i+1}} \mathrm{H}(\widehat{\mathsf{a}})|_{t_i}^{t} \cdot \big(\widehat{\mathsf{B}_{\boldsymbol{\lambda}} \cdot \mathsf{B}_{\boldsymbol{\lambda}}}\big)_t \cdot \mathrm{H}(\widehat{\mathsf{a}})|_t^{t_{i+1}},$$

with the same unifying notation of Remark 2.3.5.

We notice the following commutative square

$$
\begin{array}{ccc}
\mathrm{L}M \times \triangle_{n+1} & \xrightarrow{\pi_{i-1,i,i+1,n+1}} & \mathrm{L}M \times \triangle_3 \\
\pi_{\widehat{i},n+1}\Big\downarrow & & \Big\downarrow \pi_{\widehat{2},3} \\
\mathrm{L}M \times \triangle_n & \xrightarrow{\pi_{i-1,i,n}} & \mathrm{L}M \times \triangle_2,
\end{array}
\tag{4.4.9}
$$

where $\pi_{\widehat{i},n+1}(\gamma; t_1, \ldots, t_{n+1}) : = \left(\gamma; t_1, \ldots, \widehat{t_i}, \ldots, t_{n+1}\right)$ and $\pi_{i-1,i,i+1,n+1}(\gamma; t_1, \ldots, t_{n+1}) : = (\gamma; t_{i-1}, t_i, t_{i+1})$. We have borrowed notations from Section 2.3.

We make also use of the following identity, which is a consequence of Fubini's Theorem and of our orientation conventions for the n-simplex $\triangle_n$:

$$\pi_{n+1*} = (-1)^{i-1} \pi_{n*} \circ \pi_{\widehat{i},n*}, \tag{4.4.10}$$

where $\pi_{\widehat{i},n}(\gamma; t_1, \ldots, t_{n+1}) : = \left(\gamma; t_1, \ldots, \widehat{t_i}, \ldots, t_{n+1}\right)$, where the hat means that the coordinate t_i has to be omitted; clearly, $t_{i-1} \leq t_i \leq t_{i+1}$.

Using equation (4.4.10), Lemma 2.2.1 and equations (2.2.2), along with the commutative square (4.4.9), we get the following result:

$$
\begin{aligned}
\boldsymbol{\partial_\mu} \mathsf{h}_{2m-1,\rho}(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B}) &= \sum_{k=1}^{2m} (-1)^{2m+k+1} \operatorname{Tr}_\rho \pi_{2m*} \Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots \big(\widehat{\mathsf{B}_{\boldsymbol{\lambda}} \cdot \mathsf{B}_{\boldsymbol{\lambda}}}\big)_{t_k} \cdots \Big] \\
&= (-1)^{2m} \operatorname{Tr}_\rho \pi_{\partial_{2m+1}*} \Big[\mathrm{H}(\widehat{\mathsf{a}})|_0^{t_1} \cdot \widehat{\mathsf{B}_{\boldsymbol{\lambda}}}_{t_1} \cdots \Big] ;
\end{aligned}
$$

so the claim follows. ☐

Remark 4.4.3. Observe that the statement of Theorem 4.4.2 does not extend to $\mathsf{h}_{2i,\rho}$. The problem in this case arises in (4.4.8) in which the first two terms sum up instead of canceling each other. This reflects what was already noted in subsection 4.1 about the classical versions of these observables in even dimensions.

Since $\mathsf{O}_{\boldsymbol{\mu}}$ is a flat invariant observable, the results of subsection 3.3.4 together with Theorem 4.4.2 imply

Corollary 4.4.4. *If $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are related by* (4.4.5), *then*

$$(\boldsymbol{\Omega} - \mathrm{d}) \left[\exp\left(\frac{\mathrm{i}}{\hbar} \mathsf{O}_{\boldsymbol{\mu}} \right) \boldsymbol{\mathcal{H}}^o_\rho(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B}) \right] = 0, \tag{4.4.11}$$

under the assumptions of the regularization procedure on page 138.

We notice that this implies Theorem 3 and Theorem 4 (for M even-dimensional) of [20].

Remark 4.4.5. Let us finally note that, following the same arguments sketched in Remark 4.2.4, we may prove that the v.e.v.s of $\boldsymbol{\mathcal{H}}^o_\rho(\boldsymbol{\lambda}; \mathsf{A}, \mathsf{B})$ together with (the exponential of) the polynomial B-terms depend only on the $\mathcal{G}$-equivalence class of the flat connection A_0.

4.4.3 The Δ_{BV}-exactness of the polynomial observables

We end with a digression devoted to proving the identity

$$\mathsf{O}_n = \Delta_{BV}\left(\frac{1}{n}\,\mathsf{O}_n\,\mathsf{s}\right), \tag{4.4.12}$$

where we have used the following notation:

$$\mathsf{s} := \int_M \langle\!\langle\, \mathsf{a}\,;\,\mathsf{B}\,\rangle\!\rangle\,.$$

Of course, the functional s depends implicitly on a chosen background flat connection A_0, because the superfield a is seen as a supervariation of the superconnection A, constructed via A_0; we do not indicate the dependence on A_0 in order to avoid cumbersome notation. It is immediate to verify that s is an element of $\mathcal{S}$ of ghost number -1. The validity of (4.4.12) relies on the important identity satisfied by the BV antibracket and by the BV Laplacian, namely the failure of the BV Laplacian Δ_{BV} to satisfy the Leibniz rule (3.2.6). We already know that, for all n, $\int_M \mathrm{Tr}\,\mathsf{B}^n$ is Δ_{BV}-closed (since it does depend only on B). We want to prove separately the following identities:

$$(\,\mathsf{O}_n\,,\,\mathsf{s}\,) = n\,\mathsf{O}_n, \qquad \Delta_{BV}\mathsf{s} = 0. \tag{4.4.13}$$

If we assume the validity of the two previous identities, we can then derive (4.4.12) from (3.2.6). We begin with the first identity:

Theorem 4.4.6. *The following identity holds*

$$(\,\mathsf{O}_n\,,\,\mathsf{s}\,) = n\,\mathsf{O}_n \tag{4.4.14}$$

for all $n \in \mathbb{N}$.

Proof. Since O_n does not depend on a, we have

$$(\,\mathsf{O}_n\,,\,\mathsf{s}\,) = (\!(\,\mathsf{O}_n\,;\,\mathsf{s}\,)\!) = \int_M \left\langle\!\!\left\langle \frac{\mathsf{O}_n \overleftarrow{\partial}}{\partial \mathsf{B}}\,;\,\frac{\overrightarrow{\partial}\mathsf{s}}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle.$$

We compute the right super functional derivative of s w.r.t. a getting

$$\frac{\overrightarrow{\partial}\mathsf{s}}{\partial \mathsf{a}} = \mathsf{B}.$$

The left super functional derivative w.r.t. B of O_n reads

$$\frac{\mathsf{O}_n \overleftarrow{\partial}}{\partial \mathsf{B}} = \mathsf{B}^{n-1}.$$

So it follows by

$$\int_M \left\langle\!\!\left\langle \frac{\mathsf{O}_n \overleftarrow{\partial}}{\partial \mathsf{B}}\,;\,\frac{\overrightarrow{\partial}\mathsf{s}}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle = \int_M \langle\!\langle\, \mathsf{B}^{n-1}\,;\,\mathsf{B}\,\rangle\!\rangle = \int_M \mathrm{Tr}\,\mathsf{B}^{n-1}\cdot\mathsf{B} = \int_M \mathrm{Tr}\,\mathsf{B}^n,$$

that the claim is true. $\square$

We want now to prove the second identity in (4.4.13). Since $\langle\ ,\ \rangle$ is nondegenerate by assumption, we can find a basis e_i of $\mathfrak{g}$, $i = 1, \ldots, \dim \mathfrak{g}$ satisfying $\langle e_i\,,\, e_j \rangle = \delta_{ij}\sigma_i$, where $\sigma_i = \pm 1$. We can then write

$$\phi^\alpha = \phi^{\alpha i} e_i, \qquad \phi^*_\alpha = \phi^{j*}_\alpha e_j,$$

where the coefficients $\phi^{\alpha i}$ and ϕ^{i*}_α are forms on M (of course, sum over repeated indices is understood here). By recalling the formulae defining the Hodge dual antifields and the definition of $\langle\!\langle\ ;\ \rangle\!\rangle$ for forms with ghost number, we may write, despite of the dimension of M,

$$\begin{aligned}\mathsf{s} &= -(\,c^*\,,\,c\,)_{\mathrm{Hodge}} - (\,a^*\,,\,a\,)_{\mathrm{Hodge}} + (\,B^*\,,\,B\,)_{\mathrm{Hodge}} + \sum_{k=1}^{m-2} (\,\tau^*_k\,,\,\tau_k\,)_{\mathrm{Hodge}} = \\ &= \sum_{i=1}^{\dim\mathfrak{g}} \sigma_i \left[-(\,c^*_i\,,\,c_i\,)_{\mathrm{Hodge}} - (\,a^*_i\,,\,a_i\,)_{\mathrm{Hodge}} + (\,B^*_i\,,\,B_i\,)_{\mathrm{Hodge}} + \sum_{k=1}^{m-2} (\,\tau^*_i\,,\,\tau_i\,)_{\mathrm{Hodge}} \right],\end{aligned}$$

where $(\ ,\ )_{\mathrm{Hodge}}$ is defined in (3.4.9). We now apply the BV Laplacian to the above expression, and we get the following result

$$\begin{aligned}\Delta_{BV}\mathsf{s} &= \sum_{i=1}^{\dim\mathfrak{g}} \sigma_i C \left[\binom{m}{m} - \binom{m}{m-1} + \binom{m}{m-2} - \cdots + (-1)^l \binom{m}{m-l} + \ldots \right] = \\ &= \sum_{i=1}^{\dim\mathfrak{g}} \sigma_i C\,(1-1)^m = 0,\end{aligned}$$

where C is an infinite constant (in fact, it is the Dirac δ distribution evaluated in 0, multiplied by the volume of the manifold M). This argument is very similar to that used in the proof of the Δ_{BV}-closedness of the BV action for canonical BF theories (see subsection 3.4.3). The binomial coefficients take into account the number of components of ϕ^α_i (recall that they are forms on M), while the signs come from the ghost numbers of the fields. In an appropriate regularization procedure the above expression vanishes. So the claim follows.

Chapter 5

An observable for BF theories related to imbeddings of codimension 2

We define in this chapter an observable related to imbeddings of codimension 2 into a general compact, closed, oriented smooth manifold M, or into $\mathbb{R}^m$, in the setting of canonical BF theories (we refer to Subsection 3.4.3 for more details). First, we define the observable through a the partition function of a special action, the $\mathcal{I}$ action, and we show, at a formal level, that it is indeed gauge-invariant and homotopy-invariant. The next step is to give a precise meaning to the functional integral; this can be done via perturbative expansion. To perform a perturbative expansion of the functional integral corresponding to the $\mathcal{I}$ action, we have in general to resort to the BV formalism, since there are reducible symmetries to be fixed, as we will see later in details. Finally, after having written the perturbative expansion explicitly, we give expressions for the BV superversion of the partition function of the $\mathcal{I}$ action in the setting of super BV formalism for canonical BF theories.

5.1 The $\mathcal{I}$ action related to imbeddings of codimension 2

From now on, M will denote an oriented, compact, closed manifold of dimension m.

Definition 5.1.1. An imbedding f of codimension 2 into M is a smooth, injective map from S^{m-2} into M, such that, at any point in S^{m-2}, the tangent map of f is injective.

In particular, we will be interested in imbeddings of codimension 2 into $\mathbb{R}^m$, which is clearly not a compact, closed manifold; in this case, there are some modifications to be made.

It is a well-known fact $\mathbb{R}^m$ is diffeomorphic to the m-dimensional sphere S^m with a point removed (S^m may be seen as a "one-point-compactification" of $\mathbb{R}^m$, and the added point is viewed as the point at infinity and is accordingly denoted as ∞); analogously, we may also view S^{m-2} as the one-point-compactification of $\mathbb{R}^{m-2}$ and the added point will be also denoted by ∞. The action of the group $\mathrm{Diff}(S^m)$ (which contains $SO(m+1)$) allows to deform a given imbedding of S^{m-2} into S^m (a knot in S^m) to a base-point-preserving imbedding (base-point-preserving means that it maps ∞ to ∞); moreover, again by the action of Diff (S^m), it is possible to deform the base-point-preserving imbedding such that the tangent maps in a neighbourhood of $\infty \in S^{m-2}$ take all a given shape. Therefore, it makes sense to consider *long knots* in the sense of definition (2.4.17). We will mostly consider long knots in $\mathbb{R}^m$ since, by the above reasonings, invariants of long knots are also invariants of higher-dimensional knots in $\mathbb{R}^m$ (i.e. imbeddings of S^{m-2} into $\mathbb{R}^m$).

5.1.1 The $\mathcal{I}$-action

The aim of this section is to construct an observable related to imbeddings of codimension 2 into M (oriented, compact, closed) or into $\mathbb{R}^m$. The first step is to define a functional $S_{\mathcal{I}}$ depending on the classical fields of canonical BF theories and on imbeddings of codimension 2, which is a function of two additional fields α and β; this functional represents an action, and the observable we are looking will be defined as its partition function w.r.t. α and β. Of course, the partition function of a given action is represented as a functional integral, which is a mathematically ill-defined quantity, as we lack yet of sufficient notions on integration on infinite-dimensional spaces. However, as we will see, it is possible to evaluate such an integral by means of perturbative expansion. By means of the BV formalism, we will evaluate explicitly the partition function of $S_{\mathcal{I}}$, and we will then show that the result represents a BV observable, and that the expectation value of this functional w.r.t. canonical BF theories yield possible invariants of imbeddings of codimension 2 in M or $\mathbb{R}^m$.

First of all, we choose a particular imbedding f of codimension 2 into M. We consider the pull-back of the principal bundle P w.r.t. f, which is also a principal bundle over S^{m-2}:

$$\begin{array}{ccc} f^*P & \xrightarrow{\widetilde{f}} & P \\ {\scriptstyle\widetilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ S^{m-2} & \xrightarrow{f} & M \end{array}$$

In the above commutative square, we have written $\widetilde{\sigma}(x,p) := p$ and $\widetilde{\pi}(x,p) := x$. It is well-known that there is a canonical vector bundle isomorphism $\mathrm{ad}\, f^*P \cong f^*(\mathrm{ad}\, P)$. We denote by α, resp. β, a general section of $\mathrm{ad}\, f^*P$, resp. $\bigwedge^{m-3} \mathrm{T}S^{m-2} \otimes \mathrm{ad}^* \, (f^*P)$. We define the $S_{\mathcal{I}}$-action by the formula

$$S_{\mathcal{I}} = S_{\mathcal{I}}(A, B; \alpha, \beta) := \int_{S^{m-2}} \langle\, \alpha \,,\, \mathrm{d}_{f^*A}\beta \,\rangle + \langle\, \alpha \,,\, f^*B \,\rangle \,; \tag{5.1.1}$$

here we have used the notation f^*A for the pull-back connection A w.r.t. the bundle map $\widetilde{f}$.

If we consider long knots in $\mathbb{R}^m$, the bundle P is then trivial; hence, any pull-back bundle of P will also be trivial. Therefore, α will simply be a function on $\mathbb{R}^{m-2}$ with values in $\mathfrak{g}$, while β is a form on $\mathbb{R}^{m-2}$ with values in $\mathfrak{g}^*$ of degree $m-3$. The action $S_{\mathcal{I}}$ takes then the same form as in (5.1.1).

5.1.2 Formal properties of the partition function of the $S_{\mathcal{I}}$ action

In this subsection, we want to exploit some formal properties of the partition function of the $\mathcal{I}$ action, in view of future applications.

Gauge-invariance of the partition function of the $\mathcal{I}$ action

We consider first the partition function of the sum of the BF action and of the $\mathcal{I}$ action:

$$\int \mathcal{D}A\mathcal{D}B\mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar}\left(S_{BF} + S_{\mathcal{I}}\right). \tag{5.1.2}$$

We notice that the (formal) functional measure $\mathcal{D}\alpha\mathcal{D}\beta$ on the space of sections of the bundles $f^*(\mathrm{ad}\, P)$ and $\bigwedge^{m-3} \mathrm{T}S^{m-2} \otimes f^*(\mathrm{ad}^*\, P)$ depends on the chosen imbedding f, if the bundle P is nontrivial.

We want to examine more carefully the quantity (5.1.2). Since the pure BF action does not depend neither on f nor on α or β, we may apply formally Fubini's theorem, and we may therefore rewrite the above functional integral in the form

$$\int \mathcal{D}A\mathcal{D}B \exp \frac{\mathrm{i}}{\hbar} S_{BF} \left(\int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}\right) := \int \mathcal{D}A\mathcal{D}B \exp \frac{\mathrm{i}}{\hbar} S_{BF}\, \mathcal{T}_{\mathcal{I}},$$

where we denote by $\mathcal{T}_{\mathcal{I}}$ the following functional integral

$$\mathcal{T}_{\mathcal{I}} :\, = \int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}.$$

Thus, we may interpret the partition function of the $BF + \mathcal{I}$-action (5.1.2) as the unnormalized v.e.v. of the functional $\mathcal{T}_{\mathcal{I}}$, which depends clearly only on A, B and the imbedding σ.

First, we show that $\mathcal{T}_{\mathcal{I}}$ is invariant w.r.t. the classical symmetries of the canonical BF action.

We take an element (g, τ_1) of the symmetry group of canonical BF theories, i.e. g is an element of $\mathcal{G}$ and τ_1 is a form of degree $m-3$ with values in $\mathrm{ad}^*\, P$. We denote by $\widetilde{g}$ the associated element of the restricted gauge group $\mathrm{Aut}\,(f^*P)$ of f^*P defined by the equation

$$\widetilde{g}(x,p) := (x, g(p)).$$

Equivalently, $\widetilde{g}$ is characterized by the commutative diagram

$$\begin{array}{ccc} f^*P & \xrightarrow{\widetilde{f}} & P \\ {\scriptstyle\widetilde{g}}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ f*P & \xrightarrow{\widetilde{f}} & P \end{array} \tag{5.1.3}$$

(Clearly, $\widetilde{g}$ induces a section of the associated bundle $\mathrm{Ad}\, f^*P$, which we denote by the same symbol.)

We let (g, τ_1) act on (A, B) in the $\mathcal{I}$ action

$$S_{\mathcal{I}}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1; \alpha, \beta) = \\ = \int_{S^{m-2}} \left\langle \alpha\,,\, \mathrm{d}_{\widetilde{f^*(A^g)}}\beta + f^*(\left[\mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1\right] \right\rangle. \tag{5.1.4}$$

We state the following facts (which are all easy to prove):

- $\mathrm{d}_{f^*(A^g)}\beta = \mathrm{Ad}^*(\widetilde{g}^{-1})\,[\mathrm{d}_{f^*A}\,(\mathrm{Ad}^*(\widetilde{g})\beta)]$, by the definition of the action of $\mathcal{G}$ on $\mathcal{A}$ and on $\Omega^*(M, \mathrm{ad}\, P)$ and by (5.1.3).
- $f^*\left[\mathrm{Ad}^*(g^{-1})B\right] = \mathrm{Ad}^*(\widetilde{g}^{-1})f^*B$, again by (5.1.3) and by the definition of the action of the gauge group on $\Omega^*(M, \mathrm{ad}\, P)$.
- $f^*\left(\mathrm{d}_{A^g}\tau_1\right) = \mathrm{Ad}^*(\widetilde{g}^{-1})\left[\mathrm{d}_{\widetilde{f^*A}}\left(\mathrm{Ad}^*(\widetilde{g})f * \tau_1\right)\right]$, by the same arguments used in the two preceding cases.

(It is not difficult to see that all these equalities hold also when we consider $\mathbb{R}^{m-2}$ instead of S^{m-2} and forms on $\mathbb{R}^{m-2}$ with values in trivial bundles.)

Putting these identities in the expression on the right-hand side of (5.1.4), we get

$$\begin{aligned} &S_{\mathcal{I}}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1; \alpha, \beta) = \\ &= \int_{S^{m-2}} \left\langle \alpha\,,\, \mathrm{Ad}(\widetilde{g}^{-1})\left\{\mathrm{d}_{\widetilde{f^*A}}\left[\mathrm{Ad}^*(\widetilde{g})\,(\beta + f^*\tau_1)\right] + f^*B\right\} \right\rangle = \\ &= \int_{S^{m-2}} \left\langle \mathrm{Ad}(\widetilde{g})\alpha\,,\, \mathrm{d}_{\widetilde{f^*A}}\left[\mathrm{Ad}^*(\widetilde{g})\,(\beta + f^*\tau_1)\right] + f^*B \right\rangle. \end{aligned}$$

We notice that there is a left action of the semidirect product $\mathcal{G} \rtimes_{\mathrm{Ad}^*} \Omega^{m-3}(M, \mathrm{ad}^*\, P)$ (with multiplication given by the rule $(g, \sigma)(h, \tau) := (gh, \mathrm{Ad}^*(h^{-1})\sigma + \tau))$ on the space of sections of the bundles $f * (\mathrm{ad}\, P)$ and $\bigwedge^{m-3} \mathrm{T}S^{m-2} \otimes f * (\mathrm{ad}^*\, P)$:

$$(g, \tau)(\alpha, \beta) := (\mathrm{Ad}(\widetilde{g})\alpha; \mathrm{Ad}^*(\widetilde{g})\,(\beta + \sigma^*\tau)).$$

(More precisely, there is a left action of the semidirect product $\Gamma\left(S^{m-2}, \mathrm{Ad}\, f^*P\right) \rtimes_{\mathrm{Ad}^*} \Omega^{m-3}\left(S^{m-2}, \mathrm{ad}^*\,(f^*P)\right)$ on $\Omega^0\left(S^{m-2}, \mathrm{ad}\, f^*P\right) \times \Omega^{m-3}\left(S^{m-2}, \mathrm{ad}^*\,(f^*P)\right)$. The action described above is just the restriction of the latter one to the set $\mathcal{G} \rtimes_{\mathrm{Ad}^*} \Omega^{m-3}(M, \mathrm{ad}^*\, P)$, which via f embeds as a subgroup, since $\widetilde{g}\widetilde{h} = \widetilde{gh}$ by definition and since we take restrictions on S^{m-2} via pull-backs.

Of course, all this hold also when we replace S^{m-2} by $\mathbb{R}^{m-2}$ and all forms on S^{m-2} by forms on $\mathbb{R}^{m-2}$ with values in trivial bundles.)

Summarizing all these results, if we change (A, B) by a transformation (g, τ_1), the $\mathcal{I}$ action changes in the way described by the formula

$$S_{\mathcal{I}}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1; \alpha, \beta) = S_{\mathcal{I}}(A, B; \mathrm{Ad}(\widetilde{g})\alpha, \mathrm{Ad}^*(\widetilde{g})\,(\beta + f^*\tau_1)).$$

By the very definition of $\mathcal{T}_\mathcal{I}$, we obtain

$$\begin{aligned}
&\mathcal{T}_\mathcal{I}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1) = \\
&= \int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_\mathcal{I}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1; \alpha, \beta) = \\
&= \int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_\mathcal{I}(A, B; \mathrm{Ad}(\widetilde{g})\alpha, \mathrm{Ad}^*(\widetilde{g})\,(\beta + f^*\tau_1)).
\end{aligned}$$

If we choose the functional measure $\mathcal{D}\alpha\mathcal{D}\beta$ to be $\mathrm{Ad} \oplus \mathrm{Ad}^*$-invariant, the coordinate transformation $(\widetilde{g}, \sigma^*\tau_1)$ is formally volume-preserving; in fact, we may construct the measure via the $\mathrm{Ad} \oplus \mathrm{Ad}^*$-invariant duality.

So, the formal $\mathrm{Ad} \oplus \mathrm{Ad}^*$-invariance and the obvious translation-invariance of the formal measure $\mathcal{D}\alpha\mathcal{D}\beta$ ensure

$$\begin{aligned}
&\int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_\mathcal{I}(A, B; \mathrm{Ad}(\widetilde{g})\alpha, \mathrm{Ad}^*(\widetilde{g})\,(\beta + f^*\tau_1)) = \\
&= \int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_\mathcal{I}(A, B; \alpha, \beta).
\end{aligned}$$

We finally get $\mathcal{T}_\mathcal{I}(A^g, \mathrm{Ad}^*(g^{-1})B + \mathrm{d}_{A^g}\tau_1) = \mathcal{T}_\mathcal{I}(A, B)$; hence, $\mathcal{T}_\mathcal{I}$ is a classical observable at a formal level.

5.1.3 Formal isotopy invariance of $\mathcal{T}_\mathcal{I}$

In order to simplify the computations, we assume P to be trivial also in the case of a compact manifold M. Since any pull-back of P is also trivial, α, resp. β, is a function on S^{m-2} with values in $\mathfrak{g}$, resp. a form on S^{m-2} of degree $m-3$ taking value in $\mathfrak{g}^*$; the connection A can be written as the sum of the trivial connection A_0 on P and of a 1-form a on M with values in $\mathfrak{g}$ (choosing the trivialization $x \mapsto (x, e)$ of P, ,where e is the identity if G, we may additionally set $A_0 = 0$), and B is an $m-2$-form on M with values in $\mathfrak{g}$. By the triviality of P, $\widetilde{f} = f \times \mathrm{id}$, and therefore $f^*A = f^*a$. (Obvious modifications have to be made when we consider $\mathbb{R}^{m-2}$ instead of S^{m-2}.)

Until now, we have considered α and β to be sections of $f^*\,\mathrm{ad}\,P$ and $\bigwedge^{m-3}\mathrm{T}S^{m-2} \otimes \mathrm{ad}^*\,(f^*P)$. When we consider isotopies of the imbedding f, the formal measure w.r.t. α and β has to be modified according to the isotopies we consider: namely, isotopies modify the bundle on which α and β are defined. It is well-known that, if f and $\overline{f}$ are isotopic, the pull-back bundles f^*P and $\overline{f}^*P$ are isomorphic; nonetheless, the isomorphism relating α and β taking values on $f^*\,\mathrm{ad}\,P$ and $\bigwedge^{m-3}\mathrm{T}S^{m-2} \otimes f^*\,(\mathrm{ad}^*\,P)$ and $\overline{\alpha}$ and $\overline{\beta}$ taking values in $\overline{f}^*\,\mathrm{ad}\,P$ and $\bigwedge^{m-3}\mathrm{T}S^{m-2} \otimes \overline{f}^*\,(\mathrm{ad}^*\,P)$ may take a complicated form. Therefore, we assume P trivial in order to simplify the computations.

$\mathrm{Diff}(M)$-invariance of $\mathcal{T}_\mathcal{I}$

In this sububsection, we prove that the v.e.v. of $\mathcal{T}_\mathcal{I}\,(A, B)$ w.r.t. the BF action is invariant w.r.t. the action of $\mathrm{Diff}(M)$ on $\mathrm{Imb}\big(S^{m-2}, M\big)$. In order to avoid cumbersome notations, we will simply write

$$\langle \mathcal{T}_{\mathcal{I}} \rangle_{BF} :\, = \int \mathcal{D}A\mathcal{D}B\ \mathcal{T}_{\mathcal{I}} \exp \frac{\mathrm{i}}{\hbar} S_{BF}.$$

We want to show the following identity:

$$\langle \mathcal{T}_{\mathcal{I}} \rangle_{BF} \left(\varphi \circ f\right) = \langle \mathcal{T}_{\mathcal{I}} \rangle_{BF} \left(f\right), \quad \forall \varphi \in \mathrm{Diff}(M), f \in \mathrm{Imb}\big(S^{m-2}, M\big)\,.$$

We consider the observable $\mathcal{T}_{\mathcal{I}}\left(\varphi \circ f\right)$; it takes the explicit form

$$\mathcal{T}_{\mathcal{I}}\left(\varphi \circ f\right) = \int \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}(\alpha, \beta; A, B; \varphi \circ f),$$

where

$$\begin{aligned} S_{\mathcal{I}}\left(\alpha, \beta; A, B; \varphi \circ f\right) &= \int_{S^{m-2}} \left\langle\, \alpha\,,\, \mathrm{d}_{(\varphi\circ f)^* A}\beta + \left(\varphi \circ f\right)^* B \,\right\rangle = \\ &= \int_{S^{m-2}} \left\langle\, \alpha\,,\, \mathrm{d}_{f^*(\varphi^* A)}\beta + f^*\left(\varphi^* B\right) \,\right\rangle = \\ &= S_{\mathcal{I}}(\alpha, \beta; \varphi^* A, \varphi^* B; f). \end{aligned}$$

It follows

$$\mathcal{T}_{\mathcal{I}}\left(A, B; \varphi \circ f\right) = \mathcal{T}_{\mathcal{I}}\left(\varphi^* A, \varphi^* B; f\right).$$

Since the canonical BF action is topological, it follows that

$$S_{BF}\left(A, B\right) = S_{BF}\left(\varphi^* A, \varphi^* B\right), \quad \forall \varphi \in \mathrm{Diff}(M).$$

Thus, we obtain

$$\begin{aligned} \langle \mathcal{T}_{\mathcal{I}} \rangle_{BF}\left(\varphi \circ f\right) &= \int \mathcal{D}A\mathcal{D}B\ \mathcal{T}_{\mathcal{I}}\left(A, B; \varphi \circ f\right) \exp \frac{\mathrm{i}}{\hbar} S_{BF}\left(A, B\right) = \\ &= \int \mathcal{D}A\mathcal{D}B\ \mathcal{T}_{\mathcal{I}}\left(\varphi^* A, \varphi^* B; f\right) \exp \frac{\mathrm{i}}{\hbar} S_{BF}\left(\varphi^* A, \varphi^* B\right). \end{aligned}$$

The functional measure $\mathcal{D}A\mathcal{D}B$ is chosen to be formally invariant w.r.t. the action of the group $\mathrm{Diff}(M)$. We finally get the result

$$\langle \mathcal{T}_{\mathcal{I}} \rangle_{BF}\left(\varphi \circ f\right) = \langle \mathcal{T}_{\mathcal{I}} \rangle_{BF}\left(f\right), \quad \forall \varphi \in \mathrm{Diff}(M).$$

So, the claim follows.

There is also a right group action on the space on $\mathrm{Imb}\big(S^{m-2}, M\big)$, namely the action of the diffeomorphism group of the sphere S^{m-2}. We briefly comment how this group affects the observable $\mathcal{T}_{\mathcal{I}}$. Clearly, we want to characterize first the right action of $\mathrm{Diff}(S^{m-2})$ on the $\mathcal{I}$ action:

$$\begin{aligned} S_{\mathcal{I}}\left(A, B : \alpha, \beta; f \circ \psi^{-1}\right) &= \int_{S^{m-2}} \Big\langle\, \alpha\,,\, \mathrm{d}_{(f\circ\psi^{-1})^* A}\beta + \left(f \circ \psi^{-1}\right)^* B \,\Big\rangle = \\ &= \int_{S^{m-2}} \Big\langle\, \alpha\,,\, \mathrm{d}_{(\psi^{-1}) * f^* A}\beta + \left(\psi^{-1}\right)^* f^* B \,\Big\rangle = \\ &= \int_{S^{m-2}} \left\langle\, \psi^* \alpha\,,\, \mathrm{d}_{f^* A}\psi^* \beta + f^* B \,\right\rangle, \end{aligned}$$

if ψ is an orientation-preserving diffeomorphism of S^{m-2}; otherwise, we have to take into account an additional sign before $S_{\mathcal{I}}$, coming from the orientation of ψ. Therefore, we get

$$S_{\mathcal{I}}\left(A, B; \alpha, \beta; f \circ \psi^{-1}\right) = S_{\mathcal{I}}\left(A, B; \psi^{*}\alpha, \psi^{*}\beta; f\right),$$

for all orientation-preserving diffeomorphisms of S^{m-2}.

Hence, the action of the (infinite-dimensional) Lie-group of orientation-preserving diffeomorphisms on $S_{\mathcal{I}}$ may be reabsorbed in a transformation of the fields α and β. Heuristically, the formal measure $\mathcal{D}\alpha\mathcal{D}\beta$ in the observable $\mathcal{T}_{\mathcal{I}}$ should be invariant w.r.t. diffeomorphisms of S^{m-2}. Hence, at a formal level, the observable $\mathcal{T}_{\mathcal{I}}$ is invariant w.r.t. the action of $\mathrm{Diff}\, S^{m-2}$.

We have therefore constructed a function $\langle \mathcal{T}_{\mathcal{I}} \rangle_{BF}$ on the space $\mathrm{Imb}\big(S^{m-2}, M\big)$, which is invariant w.r.t. the action of $\mathrm{Diff}(M)$ and w.r.t. the action of orientation-preserving diffeomorphisms of S^{m-2}.

We notice that the above computations also hold replacing S^{m-2} by $\mathbb{R}^{m-2}$ with the obvious modifications.

We shall show in the next Subsubsection that this implies that $\mathcal{T}_{\mathcal{I}}$ is also isotopy invariant.

Tubular neighbourhoods and extensions of vector fields

We consider an imbedding f. We may then identify in a natural way the tangent space at f of $\mathrm{Imb}\big(S^{m-2}, M\big)$ with the space of sections of the pull-back bundle $f^{*}\,\mathrm{T}M$:

$$\mathrm{T}_f \mathrm{Imb}\big(S^{m-2}, M\big) \cong \Gamma\left(S^{m-2}, f^{*}\,\mathrm{T}M\right).$$

We take a vector field $X \in \Gamma\left(\mathrm{T}M\right)$. To X belongs a one-parameter family of diffeomorphisms of M, which we denote by ϕ_t^X, $t \in \mathbb{R}$. By acting on $f \in \mathrm{Imb}\big(S^{m-2}, M\big)$ by ϕ_t on the left, we get a curve in $\mathrm{Imb}\big(S^{m-2}, M\big)$ starting at f and ending at $f_1 := \phi_1 \circ \sigma$ (provided the flow of X can be extended to 1). Analogously, a vector field Y on S^{m-2} generates a one-parameter group of diffeomorphisms of S^{m-2}, which we denote by ψ_t, which induces in turn a curve in $\mathrm{Imb}\big(S^{m-2}, M\big)$.

By definition, a continuous isotopy between two elements f and $\overline{f}$ of $\mathrm{Imb}\big(S^{m-2}, M\big)$ is a continuous map

$$\begin{aligned} F\colon S^{m-2} \times [0,1] &\to S^{m-2}, \\ (x,t) &\mapsto F\left(x,t\right) =: F_t(x), \end{aligned}$$

such that $F_0 = f$ and $F_1 = \overline{f}$. For X as above, we take $\phi_1^X \circ f =: \overline{f}$. Then $F(x,t) := \left[\phi_t^X \circ f\right](x)$ is a smooth isotopy between f and $\overline{f}$.

We construct an extension of vector fields on the sphere S^{m-2} in a way compatible with the restriction induced by a given imbedding $f \in \mathrm{Imb}\big(S^{m-2}, M\big)$. A pivotal rôle is played by a tubular neighbourhood of the imbedding f.

Definition 5.1.2. We are given a higher-dimensional knot in $\mathbb{R}^m$. A tubular neighbourhood of f consists of a pair $(\mathcal{N}, g)$, where $\mathcal{N} \xrightarrow{\pi} S^{m-2}$ is a vector bundle over S^{m-2}, and g is an imbedding from $\mathcal{N}$ into M, enjoying the following properties:

- the image of $\mathcal{N}$ w.r.t. g is an open neighbourhood of $f\left(S^{m-2}\right)$ in M;

- If we denote by $0_{\mathcal{N}}$ the zero section of the bundle $\mathcal{N}$, the following identity holds

$$g \circ 0_{\mathcal{N}} = f.$$

We denote by $\mathcal{N}_g$ the image of $\mathcal{N}$ w.r.t. g.

The manifold M is assumed to be compact and closed, or, if not, it is the Euclidean m-dimensional space $\mathbb{R}^m$. In both cases, let ∇ be the Levi–Civita connection.

Given a smooth vector field X on the sphere S^{m-2}, our aim is to construct an extension Y in the following sense: we want to construct a vector field Y on M, such that the push-forward w.r.t. f of the restriction of Y to $f\left(S^{m-2}\right)$ equals X.

We pick $y \in \mathcal{N}_g$. There exists exactly one $v \in \mathcal{N}$ such that $g(v) = y$. We denote by x the projection onto S^{m-2} of $v \in \mathcal{N}$. To v we associate a curve $\overline{\gamma}$ in $\mathcal{N}$ in the following way

$$\overline{\gamma}(t) \colon = tv, \quad t \in [0,1].$$

Clearly, $\overline{\gamma}(0) = 0_{\mathcal{N}}(x)$, while $\overline{\gamma}(1) = v$, as $\overline{\gamma}$ lies in the fiber over x. The image of $\overline{\gamma}$ w.r.t. g will be denoted by γ; it is a curve in M, such that

$$\gamma(0) = g \circ \overline{\gamma}(0) = g(0_{\mathcal{N}}(x)) = f(x) = \left(f \circ \pi \circ g^{-1}\right)(y), \quad \gamma(1) = y.$$

We notice that the curve $\gamma = \gamma_y$ depends smoothly on the point y. Via the connection ∇ on $\mathrm{T}M$, we construct the parallel transport w.r.t. γ from 0 to 1, which we denote by $\mathcal{P}_\gamma$. $\mathcal{P}_\gamma$ is a linear isomorphism from $\mathrm{T}_{f(x)}M$ into T_yM.

Finally, we associate to $\mathcal{N}_g$ a smooth function $\rho_{\mathcal{N}}$ on M, such that its support is contained in $\mathcal{N}_g$ and such that it is identically 1 on $f\left(S^{m-2}\right)$.

We define the extension Y of $X \in \Gamma(\mathrm{T}S^{m-2})$ as follows

$$Y(y) \colon = \begin{cases} \rho_{\mathcal{N}}(y)\, \mathcal{P}_{\gamma_y} \left\{ \left(\mathrm{T}_{(\pi \circ g^{-1})(y)} f\right) \left[X\left(\left(\pi \circ g^{-1}\right)(y)\right)\right] \right\}, & y \in \mathcal{N}_g \\ 0\,, & y \in \mathcal{N}_g^c. \end{cases}$$

We prove that the push-forward w.r.t. f of the restriction of Y on $f(S^{m-2})$ equals X. We consider the constant path $\gamma_{f(x)}$: its parallel transport is the identity. Since $f(x)$ lies in $f\left(S^{m-2}\right)$, the function $\rho_{\mathcal{N}}$ takes the value 1 in $f(x)$; $\left(\pi \circ f^{-1}\right)(f(x)) = \pi \circ 0_{\mathcal{N}}(x)$, since $g \circ 0_{\mathcal{N}} = f$. Clearly, $\pi \circ 0_{\mathcal{N}}(x) = x$, since $0_{\mathcal{N}}$ is a section of $\mathcal{N}$.

Putting all these facts together, we obtain

$$\begin{aligned} \left[f_*^{-1} Y|_{f(S^{m-2})}\right](x) &= \left(\mathrm{T}_{f(x)} g^{-1}\right) Y\left[g(x)\right] = \\ &= \left(\mathrm{T}_{f(x)} f^{-1}\right) \left(\mathrm{T}_x f\right) X(x) = \\ &= X(x), \end{aligned}$$

hence the claim follows.

We notice that the vector field Y depends smoothly on y because the curve γ_y depends smoothly on y and so does its parallel transport. If we denote by $\varepsilon_{\nabla,\mathcal{N}}(X)$ the vector field Y on M constructed from X, the vector field $\varepsilon_{\nabla,\mathcal{N}}(X)$ depends linearly on the vector field X on $\mathrm{T}S^{m-2}$ because $\Gamma\left(\mathrm{T}M\right)$ is a $C^\infty(M)$-module, and tangent maps and parallel transports are linear. Summarizing all these ideas, we have constructed a linear morphism

$$\varepsilon_{\nabla,\mathcal{N}}\colon \Gamma\left(\mathrm{T}S^{m-2}\right) \longrightarrow \Gamma_c\left(\mathrm{T}M\right),$$

where the subscript c means "with compact support", since extended vector fields by means of $\varepsilon_{\nabla,\mathcal{N}}$ are localized near the imbedded sphere. This map enjoys the property $f_*^{-1}\circ\varepsilon_{\nabla,\mathcal{N}} = \mathrm{id}$, where we have denoted by f_*^{-1} the composition of the push-forward w.r.t. f^{-1} with restriction on $f\left(S^{m-2}\right)$.

The consider an isotopy f_t between f and $\overline{f}$, both in $\mathrm{Imb}\big(S^{m-2},M\big)$ and a point x in S^{m-2}, $f_t(x)$ defines a smooth curve on M. The derivative w.r.t. t at 0 defines an element X_f of $\Gamma\left(S^{m-2}, f^*\,\mathrm{T}M\right)$.

We take a connection ∇ (e.g. the Levi–Civita connection w.r.t. a given Riemannian metric) on $\mathrm{T}M$, as well as a tubular neighbourhood of $f\left(S^{m-2}\right)$. By a slight modification of the construction of the morphism $\varepsilon_{\nabla,\mathcal{N}}$, we can find a vector field X on M, such that the following identity holds:

$$X(f(x)) = X_f(x), \quad \forall x \in S^{m-2}.$$

We denote by ϕ_t^X the flow of X; it follows by definition that

$$\phi_0^X\left(f(x)\right) = f_0(x) = f(x), \quad \forall x \in S^{m-2}.$$

Another important consequence is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi_t^X\left(f(x)\right) = X\left(f(x)\right) = X_f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_t(x).$$

It follows that, for t near 0, $f_t = \phi_t^X \circ f$. We have shown that infinitesimal diffeomorphisms of M are in one-to-one correspondence with infinitesimal isotopies of imbeddings of imbeddings of codimension 2. The same results can be also obtained considering long-knots in $\mathbb{R}^m$ instead of higher-dimensional knots in $\mathbb{R}^m$ with some slight modifications; in fact, to a long knot f corresponds a base-point preserving knot $\widetilde{f}$ in S^m with a prescribed linear behavior in a small neighbourhood of $\infty \in S^m$ (base-point preserving means here that it sends $\infty \in S^{m-2}$ to $\infty \in S^m$). To $\widetilde{f}$ we can associate a tubular neighbourhood as in Definition 5.1.2; restricting then the bundle $\mathcal{N}$ to $S^{m-2}\setminus\{\infty\}$ and pulling it back via the inverse of the stereographic projection from $S^{m-2}\setminus\{\infty\}$ onto $\mathbb{R}^{m-2}$, and further modifying the composition of the restriction of the map g with the stereographic projection from $S^m\setminus\{\infty\}$ onto $\mathbb{R}^m$, we get then a tubular neighbourhood of the corresponding long-knot f. The subsequent arguments for the extension of vector fields on S^{m-2} to vector fields on M via an imbedding of S^{m-2} into M apply then also for the extension of a vector field on $\mathbb{R}^{m-2}$ to a vector field on $\mathbb{R}^m$ via a given long knot.

From all these facts it follows that $\langle\mathcal{T}_{\mathcal{I}}\rangle_{BF}$ is invariant w.r.t. infinitesimal isotopies of $\mathrm{Imb}\big(S^{m-2},M\big)$.

5.2 Evaluation of the observable $\mathcal{T}_{\mathcal{I}}$

The main problem now is to give an explicit expression for the observable $\mathcal{T}_{\mathcal{I}}$; in fact, all the properties we have derived so far are formal, relying only on an ill-defined quantity such as a functional integral.

First, if we consider the connection A to be flat and B to be A-covariantly closed (corresponding to taking A and B to be solutions of the Euler–Lagrange equations of motion for the classical BF theory), the $\mathcal{I}$ action has an obvious symmetry, if $m \geq 4$,

$$\alpha \mapsto \alpha, \quad \beta \mapsto \beta + \mathrm{d}_{f^*(A)}\sigma_1, \tag{5.2.1}$$

where σ_1 is a form on S^{m-2} of degree $m-4$ with values in $\mathfrak{g}^*$.

Being A flat, the symmetry (5.2.1) is clearly reducible for $m > 4$: in fact, we see immediately that, once fixed a representative of the class $\beta + \mathrm{d}_{f^*(A)}\sigma_1$, by imposing conditions on β, there is a residual symmetry for σ_1, which does not modify the given representative:

$$\sigma_1 \mapsto \sigma_1 + \mathrm{d}_{f^*(A)}\sigma_2, \quad \sigma_2 \in \Omega^{m-5}\left(S^{m-2}, \mathfrak{g}^*\right).$$

Therefore, once we have fixed the symmetry for β, we have to fix also the symmetry for σ_1, i.e. we have to fix a representative of the class $\sigma_1 + \mathrm{d}_{f^*(A)}\sigma_2$, for σ_1 given. We see immediately that σ_2 presents also a residual symmetry, for $m > 5$,

$$\sigma_2 \mapsto \sigma_2 + \mathrm{d}_{f^*(A)}\sigma_3, \quad \sigma_3 \in \Omega^{m-6}\left(S^{m-2}, \mathfrak{g}^*\right),$$

and so on, until we arrive to degree 0, where one cannot add any $f^*(A)$-exact form.

(We notice that we have assumed $f^*(A)$ to be acyclic; otherwise, the residual symmetry for σ_1 would also contain $f^*(A)$-harmonic forms on S^{m-2} with values in $\mathfrak{g}^*$ of degree $m-3$, besides $f^*(A)$-exact forms, and so on at all degrees.)

Because of this reducible symmetry, even saddle point approximation of the functional integral is ill-defined: we get infinitely many contributions from any orbit w.r.t. the symmetry (5.2.1) in the integration.

For (A, B) on shell and such that $f^*(A)$ is an acyclic connection on $\Omega^*\left(S^{m-2}, \mathfrak{g}^*\right)$, we are led to think that it suffices to fix consistently all symmetries, and by the reducibility problem, we have to resort to the extended BRST formalism. Without going into the details yet, we have to introduce so-called "ghosts for ghosts" σ_i (keeping track of the reducibility problem) and a differential δ_{BRST} of degree 1 w.r.t. a new gradation, the ghost number, assigned to each ghost for ghost. Choosing a gauge-fixing condition (allowing to fix all symmetries for β and for the σ's), the functional yields then the torsion of the complex $\left(\Omega^*(S^{m-2}, \mathfrak{g}^*), \mathrm{d}_{\sigma^*(A)}\right)$, because the linear term in α does not contribute to the functional integral (a source for β is in fact absent).

But this is not true for more general critical points of $S_{\mathcal{I}}$.

Fro simplicity, we shall consider from now only the trivial solution $(0, 0)$ of the equation of motion of canonical BF theory, assuming $M = \mathbb{R}^m$. The $\mathcal{I}$ action takes then the simple form:

$$S_{\mathcal{I}}(\alpha, \beta) = \int_{S^{m-2}} \langle\, \alpha\,,\, \mathrm{d}\beta \,\rangle\,.$$

The equations of motion of the $\mathcal{I}$ action simplify in this case

$$\mathrm{d}\alpha = 0, \quad \mathrm{d}\beta = 0.$$

Furthermore, the $\mathcal{I}$ action presents also 0-modes:

$$\alpha \mapsto \alpha + \alpha_\infty, \quad \alpha \in \mathfrak{g}. \tag{5.2.2}$$

We have to take into account these 0-modes for α. Therefore, we choose a base point in S^{m-2}, denoted by ∞, and an element $\alpha_\infty \in \mathfrak{g}$. We integrate over all such α's, which take at ∞ the value α_∞.

For a given α_∞, it remains only to fix the reducible symmetry $\beta \mapsto \beta + \mathrm{d}\sigma_1$.

More generally, we take a "formal" neighbourhood of the trivial solution $(0,0)$ of the equations of motion of the classical BF theory, where any element (a, B) has the property that the cohomology of the flat covariant derivative $\mathrm{d} + \mathrm{ad}^*\left(f^*(a)\right)$ is acyclic at all degrees, except (perhaps) at degree 0 and $m-2$ (we call such a neighbourhood formal, because we solve the equations of motion formally in such a neighbourhood), and an element $\alpha_\infty \in \mathfrak{g}$.

We then define

$$\widetilde{\mathcal{T}}_{\mathcal{I}}(A, B, \alpha_\infty) \colon = \int_{\alpha(\infty)=\alpha_\infty} \mathcal{D}\alpha\mathcal{D}\beta \ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{S^{m-2}}\left\langle\, \alpha\,,\, \mathrm{d}_{f^*(a)}\beta + f^*(B)\,\right\rangle}. \tag{5.2.3}$$

We see immediately that, if we choose $(a, B) = (0, 0)$, we recover the partition function for the $\mathcal{I}$ action with the condition for fixing the 0-modes of α.

We notice at this point that the choice of the condition $\alpha(\infty) = \alpha_\infty$ spoils the invariance of $\widetilde{\mathcal{T}}_{\mathcal{I}}$ w.r.t. the action of the gauge group $\mathcal{G}$: in fact, we see that a gauge transformation g acting on $\widetilde{\mathcal{T}}_{\mathcal{I}}$ can be rewritten as a formal coordinate transformation on the fields α and β. Once again, we assume that the formal measure $\mathcal{D}\alpha\mathcal{D}\beta$ is $\mathrm{Ad} \oplus \mathrm{Ad}^*$-invariant, and this suffices to get rid of the coordinate transformation; but α is fixed at the base point ∞, hence a gauge transformation acts on α_∞ as

$$\alpha_\infty \mapsto \mathrm{Ad}\left((\sigma^* g)(\infty)\right)\alpha_\infty,$$

whence it follows

$$\widetilde{\mathcal{T}}_{\mathcal{I}}\left(A^g, B^g; \alpha_\infty\right) = \widetilde{\mathcal{T}}_{\mathcal{I}}\left(A, B; \mathrm{Ad}\left((\sigma^* g)(\infty)\right)\alpha_\infty\right).$$

In order to reabsorb the transformation for α_∞, we have to integrate $\widetilde{\mathcal{T}}_{\mathcal{I}}$ w.r.t. an Ad-invariant measure on $\mathfrak{g}$.

Alternatively, without introducing any measure on $\mathfrak{g}$, we consider the projection π_G from the completion $\widehat{S}\left(\mathfrak{g}^*\right)$ of the symmetric algebra of $\mathfrak{g}^*$ onto the G-invariant subalgebra of $\widehat{S}\left(\mathfrak{g}^*\right)$

$$\pi_G \colon \widehat{S}\left(\mathfrak{g}^*\right) \to \widehat{S}\left(\mathfrak{g}^*\right)_G.$$

We may then take the projection of $\widetilde{\mathcal{T}}_{\mathcal{I}}$, $\pi_G\left(\widetilde{\mathcal{T}}_{\mathcal{I}}\right)$, hence getting a gauge-invariant functional.

All these arguments can be also derived when considering the $\mathcal{I}$ action for long knots; the presence of zero-modes also appear. We can fix zero-modes by an analogous trick of fixing a value "at infinity"; once fixed such a value, it is better (also for later computations for the superpropagator) to write α as $\alpha_\infty + \overline{\alpha}$, where now $\overline{\alpha}$ is a function on $\mathbb{R}^{m-2}$ with values in $\mathfrak{g}$ of rapid decrease. The form β has to be of rapid decrease, in order to avoid problems with the superpropagator.

5.3 The case $m = 3$

In this section we discuss the functional integral quantization of the $\mathcal{I}$ action for $m = 3$. We discuss this special case separately, because it presents some peculiarities: in fact, the form β has degree 0, hence we cannot add to it any exact form w.r.t. the pull-back connection f^*A, so that no BRST procedure and gauge-fixing is required.

It is better to consider f as an imbedding of $\mathbb{R}$ into $\mathbb{R}^3$. With these prescriptions in mind, we rewrite the $\mathcal{I}$ action as follows

$$S_{\mathcal{I}} = \int_{\mathbb{R}} \left\langle \, \alpha \, , \, \mathrm{d}_{f^*(A)}\beta + f^*(B) \, \right\rangle, \tag{5.3.1}$$

with the usual notations. We notice that α is a function on $\mathbb{R}$ with values in $\mathfrak{g}$; the same holds for β, which takes value in $\mathfrak{g}^*$.

If we put $B = 0$ assuming A to be trivial, the $\mathcal{I}$ action takes now the form

$$\int_{\mathbb{R}} \langle \, \alpha \, , \, \mathrm{d}\beta \, \rangle \, .$$

We see immediately that this action has 0-modes: namely, we may add to α, resp. β constant functions with values in $\mathfrak{g}$, resp. $\mathfrak{g}^*$. One way to fix this symmetry is to choose the following conditions on α and β:

$$\alpha(\infty) = \alpha_\infty, \quad \beta(\infty) = \beta_\infty, \tag{5.3.2}$$

where α_∞, resp. β_∞, is a given element of $\mathfrak{g}$, resp. $\mathfrak{g}^*$. This means that we give a prescribed behavior of α and β at infinity (infinity here means $+\infty$ and $-\infty$). Therefore, we may now set

$$\widetilde{\mathcal{T}}_{\mathcal{I}}(A, B; \alpha_\infty, \beta_\infty; f) = \int_{\substack{\alpha(\infty)=\alpha_\infty \\ \beta(\infty)=\beta_\infty}} \mathcal{D}\alpha\mathcal{D}\beta \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}(A, B; \alpha, \beta; f).$$

The choice of the conditions (5.3.2) makes the quadratic part of the $\mathcal{I}$ action with the derivative w.r.t. t nondegenerate, allowing to perform a Gaussian integration around α_∞ and β_∞. On the other hand, the choice of the behavior for α and β at infinity spoils the gauge-invariance w.r.t. the action of the gauge group $\mathcal{G}$ of both fields α and β; this is analogous to the phenomenon already observed in the previous subsection. This problem may be cured by choosing an invariant measure on $\mathfrak{g}$ w.r.t. the adjoint action of G and an invariant measure on $\mathfrak{g}^*$ w.r.t. the coadjoint action of G.

However, the condition (5.3.2) on β spoils also the invariance w.r.t. the symmetry $B \mapsto B + \mathrm{d}_A\tau_1$, where now τ_1 is a 0-form on $\mathbb{R}^3$ of rapid decrease with values in $\mathfrak{g}^*$. Namely, as we have previously seen, such a transformation on the field B in the σ action may be reabsorbed into a translation of the field β, which in turn causes a translation by $\sigma^*(\tau_1)(\infty)$ of β_∞. Hence, the measure on $\mathfrak{g}^*$ against which we integrate $\widetilde{\mathcal{T}}_{\mathcal{I}}(A, B; \alpha_\infty, \beta_\infty; f)$ has to be additionally translation-invariant. Since τ_1 has rapid decrease, we can consider f to have the property that f maps the point at infinity in $\mathbb{R}$ to the point at infinity in $\mathbb{R}^3$ (alternatively, one may view $\mathbb{R}^3$ as S^3 with one point removed, which is usually denoted by ∞; in this setting, f maps S^1 to S^3, and e.g. the north pole of S^1, corresponding to the point at infinity in S^1, to ∞).

For simplicity, we will consider only the case $\beta_\infty = 0$, which is the only analogue of the case $m > 3$.

We will give only a glimpse of what happens in dimension 3, giving the main ideas behind the perturbative expansion and computing explicitly the propagator. In fact, many perturbative invariants of Knots in $\mathbb{R}^3$ are known; therefore, we prefer to address to the more general situation when the dimension is bigger that 3, which lacks, at the moment being, of a way to construct perturbative invariants.

If we write $\alpha = \alpha_\infty + \overline{\alpha}$, and $\beta = \overline{\beta}$, where $\overline{\alpha}$ and $\overline{\beta}$ are functions of rapid decrease on $\mathbb{R}$ (this assumption is compatible with the condition for α and β at infinity, and makes also the propagator particularly simple, as we will soon see), the $\mathcal{I}$ action takes the form

$$\begin{aligned}S_{\mathcal{I}}(A,B;\alpha,\beta;f) =& \frac{1}{2}\left\langle \begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix}, \begin{pmatrix}-\frac{\mathrm{d}}{\mathrm{d}t} & 0\\ 0 & \frac{\mathrm{d}}{\mathrm{d}t}\end{pmatrix}\begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix}\right\rangle + \\ &+ \frac{1}{2}\left\langle \begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix}, \begin{pmatrix}-\operatorname{ad}(f^*(a)) & 0\\ 0 & \operatorname{ad}^*(f^*(a))\end{pmatrix}\begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix}\right\rangle + \\ &+\left\langle \begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix}, \begin{pmatrix}-\operatorname{ad}(f^*(a))\,\alpha_\infty\\ f^*(B)\end{pmatrix}\right\rangle + \int_0^1 \langle\, \alpha_\infty\,,\, f^*(B)\,\rangle\,\mathrm{d}t,\end{aligned} \tag{5.3.3}$$

where we have adopted the following notations:

i) $f^*(a) \colon = f^*(a_i)\frac{\mathrm{d}f_i}{\mathrm{d}t}$, where $a = a_i\mathrm{d}x_i$, and f_i is the i-th component of the imbedding f as a function from $\mathbb{R}$ to $\mathbb{R}^3$;

ii) $f^*(B) \colon = f^*(B_i)\frac{\mathrm{d}f_i}{\mathrm{d}t}$;

iii) $\left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\beta}\end{pmatrix}\right\rangle \colon = \int_{\mathbb{R}}\left[\left\langle\, \alpha\,,\, \widetilde{\beta}\,\right\rangle + \langle\, \widetilde{\alpha}\,,\, \beta\,\rangle\right]\mathrm{d}t$, where α, $\widetilde{\alpha}$ are functions of rapid decrease on $\mathbb{R}$; analogously, β and $\widetilde{\beta}$ are functions of rapid decrease on $\mathbb{R}$ with values in $\mathfrak{g}^*$.

Since, by assumption, $\overline{\alpha}$ and $\overline{\beta}$ are both functions of rapid decrease, the operator $\begin{pmatrix}-\frac{\mathrm{d}}{\mathrm{d}t} & 0\\ 0 & \frac{\mathrm{d}}{\mathrm{d}t}\end{pmatrix}$ is invertible. Namely, the inverse of the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ is simply

$$\overline{\alpha}(t) \mapsto \int_{\mathbb{R}} \eta(t-s)\overline{\alpha}(s)\mathrm{d}s, \tag{5.3.4}$$

where $\eta(s)$ denotes the distribution

$$\eta(s) \colon = \begin{cases}\frac{1}{2}, & s > 0\\ -\frac{1}{2}, & s < 0.\end{cases}$$

We denote the operator defined by equation (5.3.4) by θ. Hence, the inverse of the operator $\begin{pmatrix}-\frac{\mathrm{d}}{\mathrm{d}t} & 0\\ 0 & \frac{\mathrm{d}}{\mathrm{d}t}\end{pmatrix}$ is given by $\begin{pmatrix}-\theta & 0\\ 0 & \theta\end{pmatrix}$.

The quadratic part in equation (5.3.3) is the sum of two operators, namely $\mathsf{A}\colon = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix}$ and $\mathsf{B}\colon = \begin{pmatrix} -\operatorname{ad}\left(f^*(a)\right) & 0 \\ 0 & \operatorname{ad}^*\left(f^*(a)\right) \end{pmatrix}$; the former is invertible, as we have already seen, while the latter is in general not invertible. However, it is possible to invert at a formal level the sum $\mathsf{A}+\mathsf{B}$, since A is invertible and B is, by our previous assumptions on a, formal.

(Of course, it is not always possible to invert such a sum of operators; namely, a sufficient condition for the invertibility of such an operator is that the operator norm of the product $\mathsf{A}^{-1}\mathsf{B}$ is less than 1; the operator norm has to be defined via a norm on the space where A and B operate, making this space into a Banach space.)

Formally, the inverse of $\mathsf{A}+\mathsf{B}$ is simply given by the power series

$$(\mathsf{A}+\mathsf{B})^{-1} = \sum_{n\geq 0}(-1)^n\left(\mathsf{A}^{-1}\mathsf{B}\right)^n\mathsf{A}^{-1}. \tag{5.3.5}$$

We compute the operator $\mathsf{A}^{-1}\mathsf{B}$, with A and B defined as above

$$\begin{aligned}\mathsf{A}^{-1}\mathsf{B}\begin{pmatrix}\overline{\alpha}\\ \overline{\beta}\end{pmatrix} &= \begin{pmatrix}\int_{\mathbb{R}}\eta(t-s)\operatorname{ad}\left(f^*(a)(s)\right)\overline{\alpha}(s)\mathrm{d}s\\ \int_{\mathbb{R}}\eta(t-s)\operatorname{ad}^*\left(f^*(a)(s)\right)\overline{\beta}(s)\mathrm{d}s\end{pmatrix} = \\ &= \begin{pmatrix}\frac{1}{2}\int_{-\infty}^{t}\operatorname{ad}\left(f^*(a)(s)\right)\overline{\alpha}(s)\mathrm{d}s - \frac{1}{2}\int_{t}^{+\infty}\operatorname{ad}\left(f^*(a)(s)\right)\overline{\alpha}(s)\mathrm{d}s\\ \frac{1}{2}\int_{-\infty}^{t}\operatorname{ad}^*\left(f^*(a)(s)\right)\overline{\beta}(s)\mathrm{d}s - \frac{1}{2}\int_{t}^{+\infty}\operatorname{ad}^*\left(f^*(a)(s)\right)\overline{\beta}(s)\mathrm{d}s\end{pmatrix}.\end{aligned} \tag{5.3.6}$$

At a formal level the functional integral $\widetilde{\mathcal{T}}_{\mathcal{I}}$ is of Gaussian type, with a formally nondegenerate quadratic form and with a source term, namely $\begin{pmatrix}-\operatorname{ad}^*\left(f^*(a)(t)\right)\\ f^*(B)(t)\end{pmatrix}$.

Therefore, $\widetilde{\mathcal{T}}_{\mathcal{I}}$ can be computed explicitly via formal Gaussian integration by inverting the operator $\mathsf{A}+\mathsf{B}$ with the help of Formula (5.3.5) and computing the quadratic form corresponding to the inverse of $\mathsf{A}+\mathsf{B}$ at the given source. Formal Gaussian integration produces also the inverse of the determinant of $\mathsf{A}+\mathsf{B}$; since the operator B depends explicitly on the field A, the determinant of $\mathsf{A}+\mathsf{B}$ has to be explicitly computed.

5.4 The BV formalism for the $\mathcal{I}$ action

As we have seen in the previous section, the $\mathcal{I}$ action has reducible symmetries, and we have to resort to the BV quantization procedure to fix them. Therefore, we have first to discuss the BRST formalism and then we have to construct all main ingredients of the BV formalism for the $\mathcal{I}$ action.

5.4.1 The extended BRST formalism for $\mathcal{I}$ action

In order to fix all symmetries of the $\mathcal{I}$ action consistently, we have to adopt the extended BRST formalism, where we introduce a hierarchy of ghosts for ghosts, the σ's. First of all, we assign to α and β an additional gradation, the ghost number, as follows: the ghost number of α and β is set to be 0. Next, we introduce the BRST operator for α and β:

$$\delta_{\mathrm{BRST}}\alpha\colon = 0, \quad \delta_{\mathrm{BRST}}\beta\colon = \mathrm{d}_{f^*A}\sigma_1; \tag{5.4.1}$$

where σ_1 belongs to $\Omega^{m-4}(S^{m-2}, \mathrm{ad}^* f^*P)$, with ghost number 1. The ghost number grading is consistent with the fact that the BRST operator δ_{BRST} is a derivation of ghost number 1. Since the symmetries are reducible, we have to introduce more ghosts for ghosts, namely a hierarchy of σ's as follows: σ_j is a form on the space of fields with values in $\Omega^{m-3-j}(S^{m-2}, \mathrm{ad}^* f^*P)$ with ghost number j ($j = 1, \dots, m-3$).

Having introduced all necessary fields of the canonical $\mathcal{I}$ action, we define the extended BRST operator on the ghosts for ghosts σ_j:

$$\begin{aligned} \delta_{\mathrm{BRST}}\sigma_j &= \mathrm{d}_{f^*A}\sigma_{j+1}, \quad j = 1, \dots, m-4; \\ \delta_{\mathrm{BRST}}\sigma_{m-3} &= 0. \end{aligned} \tag{5.4.2}$$

We compute e.g. the square of the BRST operator on β:

$${\delta_{\mathrm{BRST}}}^2\beta = \mathrm{ad}^*\left(F_{f^*A}\right)\beta = 0,$$

if A is flat. It is not difficult to show that the BRST operator squares to zero modulo the equations of motion of the canonical BF action(3.1.4). The equations (5.4.2) admit obvious analogues when considering long knots in $\mathbb{R}^m$.

From these computations we see that the extended BRST operator squares to 0, when (A, B) are "on shell" w.r.t. the canonical BF action.

5.4.2 The BV formalism for the $\mathcal{I}$ action

We consider a connection A on P and an $(m-2)$-form on M with values in $\mathrm{ad}^* P$. If $M = \mathbb{R}^m$, P is trivial, and we write $A = A_0 + a$, where A_0 is the trivial connection, resp. B, is a form of degree 1 with values in $\mathfrak{g}$, resp. a form of degree $m-2$ with values in $\mathfrak{g}^*$. We assume A to be flat and the cohomology with values in $\mathrm{ad}\, P$, resp. $\mathrm{ad}^* P$, w.r.t. the differential d_A trivial at all degrees but the 0-th.

We take the set of forms $\{\alpha, \beta, \sigma_1, \dots, \sigma_{m-3}\}$, where σ_i is a form on the space of fields of the $\mathcal{I}$ action with values in $\Omega^{m-3-i}\left(S^{m-2}, \mathrm{ad}^* f^*P\right)$, with ghost number i, for $i = 1, \dots, m-3$ as the fields of the $\mathcal{I}$ action in the BV formalism; we will denote a general field by θ^α, for an index α.

We associate to each field θ^α a canonical antifield, denoted by θ^+_α, which is a form on the space of fields with values in $\Omega^*\left(S^{m-2}, \mathrm{ad}\, f^*P\right)$ and has to obey the following rules:

- $\deg\theta^+_\alpha = m - 2 - \deg\theta^\alpha$, for all α;
- $\mathrm{gh}\,\theta^+_\alpha = -1 - \mathrm{gh}\,\theta^\alpha$, for all α.

We will denote globally fields and antifields of the $\mathcal{I}$ action by the symbol η_α.

Next, we denote by $\mathcal{S}_{\mathcal{I}}$ the algebra generated by local functionals with values in $\mathbb{R}$; such functionals can be polynomial or formal power series in the fields and antifields. We take a local monomial in the fields: we can assign it canonically a ghost number by summing up the ghost numbers of the fields present in it. A homogeneous functional is a power series of monomials of the same ghost number. So, we may assign to $\mathcal{S}_{\mathcal{I}}$ a grading, namely the ghost number. Hence, $\mathcal{S}_{\mathcal{I}}$ becomes a supercommutative algebra w.r.t. the usual product in $\mathbb{R}$ and the ghost number.

We define the BV antibracket for the $\mathcal{I}$ action as

$$(F \,,\, G)_{\mathcal{I}} := \int_{S^{m-2}} \left\langle \frac{F\overleftarrow{\partial}}{\partial\theta^{\alpha}} \,,\, \frac{\overrightarrow{\partial}G}{\partial\theta^{+}_{\alpha}} \right\rangle - (-1)^{m\deg\theta^{\alpha}} \left\langle \frac{F\overleftarrow{\partial}}{\partial\theta^{+}_{\alpha}} \,,\, \frac{\overrightarrow{\partial}G}{\partial\theta^{\alpha}} \right\rangle , \qquad (5.4.3)$$

for any two functionals F, G in $\mathcal{S}_{\mathcal{I}}$. The left and right functional derivatives are defined by the formulae:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(\eta_{\alpha} + t\rho_{\alpha}) = \int_{S^{m-2}} \left\langle \frac{F\overleftarrow{\partial}}{\partial\eta_{\alpha}} \,,\, \rho_{\alpha} \right\rangle = \int_{S^{m-2}} \left\langle \rho_{\alpha} \,,\, \frac{\overrightarrow{\partial}F}{\partial\eta_{\alpha}} \right\rangle ,$$

and the form ρ_{α} must have the same degree and ghost number of the field η_{α}. It follows immediately from the definition of the functional derivatives that, for F, G homogeneous in $\mathcal{S}_{\mathcal{I}}$, the BV antibracket possesses ghost number $\mathrm{gh}\,F + \mathrm{gh}\,G + 1$, and it is again an element of $\mathcal{S}_{\mathcal{I}}$.

It is also not difficult to verify the following properties enjoyed by the BV antibracket, which we state without proof:

- $(F \,,\, G)_{\mathcal{I}} = -(-1)^{(\mathrm{gh}\,F+1)(\mathrm{gh}\,G+1)} \, (G \,,\, F)_{\mathcal{I}}$ (graded commutativity);
- $(F \,,\, GH)_{\mathcal{I}} = (F \,,\, G)_{\mathcal{I}} H + (-1)^{(\mathrm{gh}\,F+1)\,\mathrm{gh}\,G} G \, (F \,,\, H)_{\mathcal{I}}$ (graded Leibnitz rule);
- $(-1)^{(\mathrm{gh}\,F+1)(\mathrm{gh}\,H+1)} \, (F \,,\, (G \,,\, H)_{\mathcal{I}})_{\mathcal{I}} + \mathrm{cycl.} = 0$ (graded Jacobi identity),

for any three homogeneous functionals F, G and H in $\mathcal{S}_{\mathcal{I}}$.

We come to the definition of the BV Laplacian for the $\mathcal{I}$ action. We need a Riemannian metric on S^{m-2}; since $\mathbb{R}^{m-2}$ is a submanifold of M, we can pull back the already chosen metric on M, with which we have defined the BV Laplacian in subsubsection (3.2.4). (When we consider $\mathbb{R}^{m-2}$ instead of S^{m-2}, we may also pick a Riemannian metric, e.g. the standard Euclidean metric.) By means of the chosen metric, we can construct a Hodge operator $\star\colon \Omega^{p}(S^{m-2}, \mathrm{ad}\,\Sigma^{*}P) \to \Omega^{m-2-p}(S^{m-2}, \mathrm{ad}\,\Sigma^{*}P)$.

To each field θ^{α} we associate its *Hodge dual antifield* θ^{*}_{α}, defined by the equation $\theta^{*}_{\alpha} := \star\theta^{+}_{\alpha}$; it is a form on the space of fields with values in $\Omega^{*}(S^{m-2}, \mathrm{ad}^{*} f^{*}P)$, resp. $\Omega^{*}(S^{m-2}, \mathrm{ad}\, f^{*}P)$, of degree $\deg\theta^{\alpha}$ and ghost number $-1 - \mathrm{gh}\,\theta^{\alpha}$, if θ^{α} takes value in the bundle $\mathrm{ad}\, f^{*}P$, resp. $\mathrm{ad}^{*} f^{*}P$. We denote globally by η_{α} any field or Hodge dual antifield of the $\mathcal{I}$ action.

We take η, ω in $\Omega^{p}(S^{m-2}, \mathrm{ad}\, f^{*}P)$, $\Omega^{p}(S^{m-2}, \mathrm{ad}^{*} f^{*}P)$; we then define (analogously to what we have done in subsubsection (3.2.4))

$$\langle\, \eta \,,\, \omega \,\rangle_{\mathrm{Hodge}} := \int_{S^{m-2}} \langle\, \eta \,,\, \star\omega \,\rangle \,.$$

By means of the duality $\langle\ ,\ \rangle_{\mathrm{Hodge}}$ we define the $\mathcal{I}$ BV Laplacian on $\mathcal{S}_{\mathcal{I}}$ by the formula

$$\Delta_{\mathcal{I}} F := \sum_{\alpha} (-1)^{\mathrm{gh}\,\theta^{\alpha}} \left\langle \frac{\overrightarrow{\delta}}{\delta\theta^{\alpha}} \,,\, \frac{\overrightarrow{\delta}\, F}{\delta\theta^{*}_{\alpha}} \right\rangle_{\mathrm{Hodge}} . \tag{5.4.4}$$

The functional derivatives are defined in this context by the rules

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(\eta_{\alpha} + t\rho_{\alpha}) = \left\langle \frac{F\overleftarrow{\delta}}{\delta\eta_{\alpha}} \,,\, \rho_{\alpha} \right\rangle_{\mathrm{Hodge}} = \left\langle \rho_{\alpha} \,,\, \frac{\overrightarrow{\delta}\, F}{\delta\eta_{\alpha}} \right\rangle_{\mathrm{Hodge}} ,$$

and the test form ρ_{α} must have the same degree and ghost number as the field η_{α}. The BV Laplacian formally satisfies the identities

- ${\Delta_{\mathcal{I}}}^2 = 0$, i.e. the BV Laplacian is a coboundary operator of ghost number 1 (this last statement follows from its definition);
- $\Delta_{\mathcal{I}}(FG) = (\Delta_{\mathcal{I}} F)G + (-1)^{\mathrm{gh}\,F} \left(F \,,\, G \right)_{\mathcal{I}} + (-1)^{\mathrm{gh}\,F} F \Delta_{\mathcal{I}} G$;
- $\Delta_{\mathcal{I}} \left(F \,,\, G \right)_{\mathcal{I}} = \left(\Delta_{\mathcal{I}} F \,,\, G \right)_{\mathcal{I}} + (-1)^{\mathrm{gh}\,F+1} \left(F \,,\, \Delta_{\mathcal{I}} G \right)_{\mathcal{I}}$,

for any two functionals in $\mathcal{S}_{\mathcal{I}}$.

Remark 5.4.1. As was already noted in Remark 3.2.7, the BV Laplacian is not well-defined on $\mathcal{S}_{\mathcal{I}}$: in fact, the BV Laplacian on $\mathcal{S}_{\Sigma}$ is usually the restriction on the diagonal of a distribution, and it therefore requires an appropriate regularization procedure in order to get rid of possible singularities on the diagonal.

Remark 5.4.2. A formal property of the BV Laplacian, which is useful for the proofs of most statements, is that it operates trivially on any functional such that any field is not coupled to its canonical Hodge dual antifield. In fact, we see directly from equation (5.4.4), the BV Laplacian couples any field with the corresponding Hodge dual antifield.

Remark 5.4.3. We notice briefly that the BV antibracket for the $\mathcal{I}$ action can be written in the form

$$\left(F \,,\, G \right)_{\mathcal{I}} = \left\langle \frac{F\overleftarrow{\delta}}{\delta\theta^{\alpha}} \,,\, \frac{\overrightarrow{\delta}\, G}{\delta\theta^{*}_{\alpha}} \right\rangle_{\mathrm{Hodge}} - \left\langle \frac{F\overleftarrow{\delta}}{\delta\theta^{*}_{\alpha}} \,,\, \frac{\overrightarrow{\delta}\, G}{\delta\theta^{\alpha}} \right\rangle_{\mathrm{Hodge}} .$$

This formula does not contain any signs, but depends explicitly on the Hodge operator; this is in complete analogy with the arguments in Remark 3.2.9.

Finally, we take a functional S of ghost number 0 satisfying the Quantum Master Equation w.r.t. the canonical $\mathcal{I}$-action:

$$-2\mathrm{i}\hbar \Delta_{\mathcal{I}} S + \left(S \,,\, S \right)_{\mathcal{I}} = 0;$$

by means of S, we define the following ghost-number-1 operator on $\mathcal{S}_{\mathcal{I}}$

$$\Omega_{\mathcal{I}} := -\mathrm{i}\hbar \Delta_{\mathcal{I}} + \left(S \,,\ \right)_{\mathcal{I}} .$$

Since S satisfies the quantum master equation, the operator $\Omega_{\mathcal{I}}$ satisfies ${\Omega_{\mathcal{I}}}^2 = 0$. We want the functional S to enjoy the following additional requirement: $S\left[\theta^{+}_{\alpha} = 0\right] = S_{\mathcal{I}}$, for all α. If this holds also, S is a BV action for the $\mathcal{I}$ action.

The super BV formalism for the $\mathcal{I}$ action

We denote by $|\alpha|$ the total degree of a form $\alpha \in \Omega^*\left(S^{m-2}, \operatorname{ad} f^*P\right)$ or in $\Omega^*\left(S^{m-2}, \operatorname{ad}^* f^*P\right)$ with a given degree and ghost number. In other words,

$$|\alpha| := \deg \alpha + \operatorname{gh} \alpha.$$

We consider all fields and antifields of the $\mathcal{I}$ action. It follows from the assumptions in Subsubsection (5.4.1) that the fields $\left\{\alpha, \beta^+, \sigma_1^+, \ldots, \sigma_{m-3}^+\right\}$ all have total degree 0, while $\{\sigma_{m-3}, \ldots, \sigma_1, \beta, \alpha^+\}$ all have total degree $m-3$.

We may cast both sets separately in two superforms in $\Omega^*\left(S^{m-2}, \operatorname{ad} \Sigma^* P\right)$, resp. $\Omega^*\left(S^{m-2}, \operatorname{ad}^* \Sigma^* P\right)$, denoted by $\boldsymbol{\alpha}$, resp. $\boldsymbol{\beta}$, of total degree 0, resp. $m-3$, defined by

$$\boldsymbol{\alpha} \colon = \alpha + (-1)^{m+1}\beta^+ + \sum_{k=1}^{m-3} (-1)^{\frac{k(k+1)}{2} + (m+1)(k+1)} \sigma_k^+, \tag{5.4.5}$$

$$\boldsymbol{\beta} \colon = \sum_{k=1}^{m-3} (-1)^{\frac{k(k+1)}{2}} \sigma_k + \beta - \alpha^+. \tag{5.4.6}$$

We still denote by $\langle\!\langle \ ; \ \rangle\!\rangle$ the shifted extension of the duality $\langle \ , \ \rangle$ for forms on S^{m-2} with values in $\operatorname{ad} f^*P$, resp. $\operatorname{ad}^* f^*(P)$, i.e.

$$\langle\!\langle \epsilon \, ; \, \xi \rangle\!\rangle := (-1)^{\operatorname{gh}\epsilon \deg \xi} \langle \epsilon \, , \, \xi \rangle, \quad \epsilon \in \Omega^*(S^{m-2}, \operatorname{ad} f^*P), \quad \xi \in \Omega^*(S^{m-2}, \operatorname{ad}^* f^*P).$$

We define analogously the dot Lie bracket $[\![\ ; \ ]\!]$, and we still denote it by the same symbol as for BF theories.

We are going to consider mainly (products of) local functionals, which may be polynomials or formal power series in the superfields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We denote the algebra generated by such local functionals by $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Analogously to what we have done in subsubsection (3.3.1), we can give this algebra a grading by means of the total degree (which is equal to the ghost number for local functionals and product of them). It is easy to verify that $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a subalgebra of $\mathcal{S}_{\mathcal{I}}$.

(As before, we do not write explicitly the dot product between any two functionals.)

To the superconnection A we assign a covariant derivative on forms on S^{m-2} with values in $\operatorname{ad} f^*P$ and $\operatorname{ad}^* f^*P$ by the rules

$$\begin{aligned}
\mathrm{d}_{\widetilde{f^*\mathsf{A}}}\epsilon \colon &= \mathrm{d}_{\widetilde{f^*A_0}}\epsilon + [\![f^*\mathsf{a} \, ; \epsilon]\!], \quad \forall \epsilon \in \Omega^*(S^{m-2}, \operatorname{ad} f^*P),\\
\mathrm{d}_{\widetilde{f^*\mathsf{A}}}\epsilon \colon &= \mathrm{d}_{\widetilde{f^*A_0}}\epsilon + \operatorname{ad}^*(f^*\mathsf{a})\,\epsilon, \quad \forall \epsilon \in \Omega^*(S^{m-2}, \operatorname{ad} f^*P),
\end{aligned}$$

if ε takes value in $\operatorname{ad} f^*P$, resp. $\operatorname{ad}^* f^*P$. The notation ad^* means the usual ad^*-representation on forms with values in $\operatorname{ad}^* f^*P$ with the degree shifted as in the definition of the super Lie bracket $[\![\ ; \ ]\!]$.) Its supercurvature is given by (recall that A_0 is flat by assumption)

$$\begin{aligned}
[\![F_{\widetilde{f^*\mathsf{A}}} \, ; \epsilon]\!] &= \mathrm{d}_{\widetilde{f^*\mathsf{A}}}\left(\mathrm{d}_{\widetilde{f^*\mathsf{A}}}\epsilon\right) = [\![\mathrm{d}_{\widetilde{f^*A_0}} f^*\mathsf{a} + \frac{1}{2} f^*[\![\mathsf{a} \, ; \mathsf{a}]\!] \, ; \epsilon]\!],\\
\operatorname{ad}^*\left(F_{\widetilde{f^*\mathsf{A}}}\right)\epsilon &= \mathrm{d}_{\widetilde{f^*\mathsf{A}}}\left(\mathrm{d}_{\widetilde{f^*\mathsf{A}}}\epsilon\right) = \operatorname{ad}^*\left(\mathrm{d}_{\widetilde{f^*A_0}} f^*\mathsf{a} + \frac{1}{2} f^*[\![\mathsf{a} \, ; \mathsf{a}]\!]\right)\epsilon,
\end{aligned}$$

in analogy with the formula obtained in subsubsection (3.3.1).

We are now ready to define the super BV antibracket for the $\mathcal{I}$ action. We take two functionals F, G in $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$; then we define

$$(\!(F \,;\, G)\!)_{\mathcal{I}} := \int_{S^{m-2}} \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}} \,;\, \frac{\overrightarrow{\partial}G}{\partial\boldsymbol{\beta}} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\boldsymbol{\beta}} \,;\, \frac{\overrightarrow{\partial}G}{\partial\boldsymbol{\alpha}} \right\rangle\!\!\right\rangle, \tag{5.4.7}$$

where the super functional derivatives are defined by the formulae

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\boldsymbol{\alpha} + t\rho_{\boldsymbol{\alpha}}) = \int_{S^{m-2}} \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}} \,;\, \rho_{\boldsymbol{\alpha}} \right\rangle\!\!\right\rangle = \int_{S^{m-2}} \left\langle\!\!\left\langle \rho_{\boldsymbol{\alpha}} \,;\, \frac{\overrightarrow{\partial}F}{\partial\boldsymbol{\alpha}} \right\rangle\!\!\right\rangle$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\boldsymbol{\beta} + t\rho_{\boldsymbol{\beta}}) = \int_{S^{m-2}} \left\langle\!\!\left\langle \frac{F\overleftarrow{\partial}}{\partial\boldsymbol{\beta}} \,;\, \rho_{\boldsymbol{\beta}} \right\rangle\!\!\right\rangle = \int_{S^{m-2}} \left\langle\!\!\left\langle \rho_{\boldsymbol{\beta}} \,;\, \frac{\overrightarrow{\partial}F}{\partial\boldsymbol{\beta}} \right\rangle\!\!\right\rangle,$$

where $\rho_{\boldsymbol{\alpha}}$, resp. $\rho_{\boldsymbol{\beta}}$, denotes a superform on S^{m-2} with values in $\mathrm{ad}\, f^*P$ or in $\mathrm{ad}^*\, f^*P$ of the same total degree as $\boldsymbol{\alpha}$, resp. $\boldsymbol{\beta}$.

The main argument about super BV antibracket for the $\mathcal{I}$ action in any dimension is analogous to the proof of Theorem already proved in [20] for the super BV antibracket for canonical BF theories, namely the super BV antibracket is equal to the usual BV antibracket on $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Theorem 5.4.4. *On $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \otimes_{\mathbb{R}} \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the following identity holds:*

$$(\!(\ ;\ )\!)_{\mathcal{I}} = (\ ,\ )_{\mathcal{I}}\,. \tag{5.4.8}$$

Proof. The main argument of the proof relies, as it was the case in the proof of Theorem 3.3.1, on the definition of the super functional derivatives, resp. the usual functional derivatives, and on the definition of the dot invariant bilinear form $\langle\ ,\ \rangle$, as defined previously. As for canonical BF theories, the choice of signs in the definitions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are crucial for the validity of Identity (5.4.8). ☐

Remark 5.4.5. We have only considered the algebra $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, but we could have defined more general algebras of functionals analogous to the algebras introduced at the beginning of the construction of the super BV formalism for BF theories in [20]. Of course, all the constructions therein can be applied to the super BV formalism for the $\mathcal{I}$ action, and the theorems therein are still valid with the due modifications.

The above theorem is the most important feature of the super BV antibracket for the $\mathcal{I}$ action; namely, we can deduce all important properties of the super BV antibracket, e.g. the graded supercommutativity, the graded Leibnitz rule and the graded Jacobi identity w.r.t. the total degree on $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, from the analogous properties of the BV antibracket for the $\mathcal{I}$ action.

We briefly notice that we can define the super version of the BV Laplacian for the $\mathcal{I}$ action, denoted by $\boldsymbol{\Delta}_{\mathcal{I}}$ analogously to the super BV Laplacian for canonical BF

theories $\boldsymbol{\Delta}$. In fact, on $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, the super BV Laplacian for the $\mathcal{I}$ action is defined by the rule

$$\boldsymbol{\Delta}_{\mathcal{I}} F := \Delta_{\mathcal{I}} F, \quad \forall F \in \mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Of course, if we had considered more general functionals, then we should have modified the above definition (see e.g. [20]).

It is obviously nilpotent, it has total degree 1 and the super BV antibracket $(\!(\ ;\ )\!)_{\mathcal{I}}$ measures the failure of $\boldsymbol{\Delta}_{\mathcal{I}}$ to be a differential.

(We have adopted a different notation for the super BV Laplacian for the $\mathcal{I}$ action in order to distinguish it from the usual BV Laplacian; in fact, the usual BV Laplacian has ghost number 1, while the super BV Laplacian has total degree 1.)

The "on-shell" BV action for the $\mathcal{I}$ action

We work under the assumption (A, B) on shell. We are going to prove that, under this assumption, there exists a solution of the Quantum Master Equation for the $\mathcal{I}$ action w.r.t. the BV antibracket $(\ ,\ )_{\mathcal{I}}$ and for the BV Laplacian $\Delta_{\mathcal{I}}$.

In analogy with computations already done in [20] for canonical BF theories, we define the following element of $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, depending on the "on shell" pair (A, B)

$$S_{\mathrm{BV}\mathcal{I}} :\, = \int_{S^{m-2}} \langle\!\langle\, \boldsymbol{\alpha} \,;\, \mathrm{d}_{f^*A}\boldsymbol{\beta} + f^*(B) \,\rangle\!\rangle \,. \tag{5.4.9}$$

The form B has total degree $m-2$, as $\mathrm{d}_{f^*A}\boldsymbol{\beta}$, while $\boldsymbol{\alpha}$ has total degree 0; the integration on S^{m-2} has total degree $2-m$, hence $S_{\mathrm{BV}\mathcal{I}}$ has total degree 0, which also equals the ghost number.

Theorem 5.4.6. *If we consider the "on shell" pair (A, B), $S_{\mathrm{BV}\mathcal{I}}$ is a solution of the Master Equation*

$$(\, S_{\mathrm{BV}\mathcal{I}} \,,\, S_{\mathrm{BV}\mathcal{I}} \,)_{\mathcal{I}} = 0.$$

Proof. Since $S_{\mathrm{BV}\mathcal{I}}$ is a local functional in $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, it follows

$$(\, S_{\mathrm{BV}\mathcal{I}} \,,\, S_{\mathrm{BV}\mathcal{I}} \,)_{\mathcal{I}} = (\!(\, S_{\mathrm{BV}\mathcal{I}} \,;\, S_{\mathrm{BV}\mathcal{I}} \,)\!)_{\mathcal{I}} = 2\int_{S^{m-2}} \left\langle\!\!\left\langle \frac{S_{\mathrm{BV}\mathcal{I}}\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}} \,;\, \frac{\overrightarrow{\partial} S_{\mathrm{BV}\mathcal{I}}}{\partial\boldsymbol{\beta}} \right\rangle\!\!\right\rangle .$$

We therefore compute the functional derivatives of $S_{\mathrm{BV}\mathcal{I}}$ w.r.t. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. By their very definition, we get

$$\begin{aligned} \frac{S_{\mathrm{BV}\mathcal{I}}\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}} &= \frac{\overrightarrow{\partial} S_{\mathrm{BV}\mathcal{I}}}{\partial\boldsymbol{\alpha}} = \mathrm{d}_{f^*A}\boldsymbol{\beta} + f^*(B), \\ \frac{S_{\mathrm{BV}\mathcal{I}}\overleftarrow{\partial}}{\partial\boldsymbol{\beta}} &= (-1)^{m-1}\frac{\overrightarrow{\partial} S_{\mathrm{BV}\mathcal{I}}}{\partial\boldsymbol{\beta}} = -\mathrm{d}_{f^*A}\boldsymbol{\alpha}. \end{aligned} \tag{5.4.10}$$

Putting equations (5.4.10) in the super BV-antibracket $(\!(\ ;\ )\!)_{\mathcal{I}}$, we get

$$\begin{aligned} &-\int_{S^{m-2}} \langle\!\langle\, \mathrm{d}_{f^*A}\boldsymbol{\beta} + f^*(B) \,;\, \mathrm{d}_{f^*A}\boldsymbol{\alpha} \,\rangle\!\rangle = \\ &= (-1)^m \int_{S^{m-2}} \langle\!\langle\, \boldsymbol{\alpha} \,;\, \mathrm{ad}^*\left(F_{f^*A}\right)\boldsymbol{\beta} + f^*(\mathrm{d}_A B) \,\rangle\!\rangle = 0, \end{aligned}$$

where the first equality is a consequence of Stokes' Theorem, and the second one is a consequence of A being flat and B being d_A-covariantly closed, since (A, B) is "on shell".

Hence, the claim follows. $\square$

In order to prove that $S_{\mathrm{BV}\mathcal{I}}$ satisfies also the Quantum Master Equation, we use the formal argument that the BV-Laplacian annihilates any functional in $\mathcal{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, where any field is not coupled with its canonical Hodge dual antifield. The Hodge dual antifield of a given field is a modification of the usual antifield.

The only term in the BV action $S_{\mathrm{BV}\mathcal{I}}$ which may couple antifields with their respective canonical antifields is

$$\int_{S^{m-2}} \langle\!\langle\, \boldsymbol{\alpha} \,;\, \mathrm{d}_{f^*A}\boldsymbol{\beta} \,\rangle\!\rangle \,.$$

Since the canonical antifield θ^+_α to a field θ^α has degree equal to $m - 2 - \deg\theta^\alpha$, and since the covariant derivative d_{f^*A} has degree 1, no field can be coupled to its canonical antifield. Hence, $S_{\mathrm{BV}\mathcal{I}}$ satisfies $\Delta_{\mathcal{I}} S_{\mathrm{BV}\mathcal{I}} = 0$.

Moreover, since $\boldsymbol{\alpha}$ contains all antifields and α, if we set all antifields to 0, it follows that the only surviving term in $S_{\mathrm{BV}\mathcal{I}}$ is

$$\int_{S^{m-2}} \langle\!\langle\, \alpha \,;\, \mathrm{d}_{f^*A}\beta + f^*(B) \,\rangle\!\rangle \,,$$

because the only component of $\boldsymbol{\beta}$ of degree $m-3$ is β. Hence, putting all antifields in $S_{\mathrm{BV}\mathcal{I}}$ to 0, we recover the classical $\mathcal{I}$ action.

We have produced, at the end, for any given "on shell" pair (A, B), a BV action for the classical $\mathcal{I}$ action.

We finally notice that all definitions and computations of this Subsection admit obvious analogues when considering long knots in $\mathbb{R}^m$ with some slight modifications (e.g. forms of rapid decrease on $\mathbb{R}^{m-2}$ instead of forms on S^{m-2}).

5.4.3 The Gauge-fixing for the $\mathcal{I}$ action and the superpropagator

In the previous subsection we have found a BV action for the $\mathcal{I}$ action. In order to start a perturbative expansion of the theory, we need the quadratic part of the gauge-fixed $\mathcal{I}$ action to be nondegenerate. The BV formalism allows us to take into account the reducibility of the symmetries, which have now to be fixed.

The first step is to extend the space of fields of the $\mathcal{I}$ action by introducing antighosts and Lagrange multipliers in the following way: to the ghost σ_1 we associate an antighost $\overline{\sigma}_1$ and a Lagrange multiplier λ_1, such that $\overline{\sigma}_1$, resp. λ_1, is a form of degree $m-4$ with values in $\mathrm{ad}\, f^*P$ and ghost number -1, resp. of degree $m-4$ with values in $\mathrm{ad}\, f^*P$ and ghost number 0. For the ghosts-for-ghosts, keeping track of the reducibility problem of the $\mathcal{I}$ action, the prescription is more complicated: namely, to the ghost-for-ghost σ_i, $2 \leq i \leq m-3$, we have to associate a family of i antighosts $\overline{\sigma}_{i,l}$ and of i Lagrange multipliers $\lambda_{i,l}$, where $1 \leq l \leq i$, such that

- $\overline{\sigma}_{i,k}$ is a form of degree $m-3-i$ with values in $\mathrm{ad}\, f^*P$ and ghost number $-i+2l-2$;

- $\lambda_{i,k}$ is a form of degree $m-3-i$ with values in $\mathrm{ad}\, f^*P$ and ghost number $-i+2l-1$.

We consider, along with the antighosts $\overline{\sigma}_{i,l}$ and Lagrange multipliers $\lambda_{i,l}$, their corresponding BV-antifields, and we denote by Θ, resp. Θ^+, the set of all fields of the $\mathcal{I}$ theory (i.e. fields and antighosts and Lagrange multipliers), resp. of their BV-antifields. We can cast all these fields in the following table

	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$m-3$					β				
$m-4$				$\overline{\sigma}_1 \equiv \overline{\sigma}_{1,1}$	λ_1	σ_1			
$m-5$			$\overline{\sigma}_2 \equiv \overline{\sigma}_{2,1}$	$\lambda_{2,1}$	$\overline{\sigma}_{2,2}$	$\lambda_{2,2}$	σ_2		
$m-6$		$\overline{\sigma}_3 \equiv \overline{\sigma}_{3,1}$	$\lambda_{3,1}$	$\overline{\sigma}_{3,2}$	$\lambda_{3,2}$	$\overline{\sigma}_{3,3}$	$\lambda_{3,3}$	σ_3	
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The purpose of the antighosts and Lagrange multipliers is to write down explicitly a gauge-fixing fermion, i.e. an element Ψ of $\mathcal{S}_{\mathcal{I}}$ of ghost number -1, depending only on Θ and incorporating the gauge-fixing condition for the symmetries of the $\mathcal{I}$ action. We extend the BV operator δ_{BV} on antighosts and Lagrange multipliers as follows:

$$\begin{aligned} \delta_{BV}\overline{\sigma}_1 = \lambda_1, \quad \delta_{BV}\lambda_1 = 0; \\ \delta_{BV}\overline{\sigma}_{i,l} = \lambda_{i,l}, \quad \delta_{BV}\lambda_{i,l} = 0, \quad 1 \leq i \leq m-3, \quad 1 \leq l \leq i. \end{aligned} \tag{5.4.11}$$

This is clearly equivalent to adding to the BV action $S_{\mathcal{I}}$ an additional piece

$$S_{\text{Lagr.}} := -\int_{S^{m-2}} \langle \overline{\sigma}_1^+ , \lambda_1 \rangle + \sum_{i=2}^{m-3}\sum_{l=1}^{i} (-1)^i \int_{S^{m-2}} \left\langle \overline{\sigma}_{i,l}^+ , \lambda_{i,l} \right\rangle,$$

such that the Quantum Master Equation for $S_{\mathcal{I}}^{\text{ext.}} := S_{\mathcal{I}} + S_{\text{Lagr.}}$ is satisfied.

We pick an explicit metric on S^{m-2}. By means of it, we can construct the corresponding Hodge $\star$-operator, which maps linearly forms on $\mathbb{R}^{m-2}$ of degree k to forms of degree $m-2-k$; moreover, we may define the L_2-duality between forms on S^{m-2} with values in $\mathfrak{g}$ and $\mathfrak{g}^*$ as follows:

$$\langle \eta , \omega \rangle_{\mathrm{L}_2} := \int_{S^{m-2}} \langle \omega , \star\eta \rangle , \tag{5.4.12}$$

where the operator $\star$ acts on the form part of η, and $\langle \ , \ \rangle$ denotes the usual duality between $\mathfrak{g}$ and $\mathfrak{g}^*$. The operator $\mathrm{d}^\star$ is the adjoint of the usual exterior derivative w.r.t. the L_2-duality, i.e.

$$\langle \omega , \mathrm{d}\eta \rangle_{\mathrm{L}_2} = \langle \mathrm{d}^\star\omega , \eta \rangle_{\mathrm{L}_2} , \quad \forall \omega \in \Omega^*\left(S^{m-2}, \mathfrak{g}^*\right), \eta \in \Omega^*\left(S^{m-2}, \mathfrak{g}\right).$$

We have then to choose a gauge-fixing fermion Ψ in $\mathcal{S}_{\mathcal{I}}$ of ghost number -1, such that the functional

$$S_{\mathcal{I}_{\text{g.f.}}} := S_{\mathcal{I}}^{\text{ext.}}\left[\theta^\alpha, \theta^+_\alpha = \frac{\overrightarrow{\partial}\Psi}{\partial\theta^\alpha}\right]$$

has a nondegenerate Hessian at a critical point. Since the quadratic part of the BV action $S_{\mathcal{I}}$ contains the exterior derivative d, a gauge-fixing fermion which takes into account the acyclicity of the exterior derivative is the following generalization of usual covariant gauge-fixing:

$$\Psi :\ = \langle\, \overline{\sigma}_1\,,\, \mathrm{d}^\star\beta \,\rangle_{\mathrm{L}_2} + \sum_{i=1}^{m-4} \langle\, \overline{\sigma}_{i+1,1}\,,\, \mathrm{d}^\star\sigma_i \,\rangle_{\mathrm{L}_2} + \sum_{i=1}^{m-4}\sum_{l=2}^{i} \langle\, \overline{\sigma}_{i+1,k+2-l}\,,\, \mathrm{d}^\star\overline{\sigma}_{i,l} \,\rangle_{\mathrm{L}_2}\,. \tag{5.4.13}$$

All these computations can be slightly modified when we consider long knots in $\mathbb{R}^m$; we notice that we have to consider forms on $\mathbb{R}^{m-2}$ with rapid decrease.

5.4.4 Feynman rules and explicit expression of $\mathcal{T}_{\mathcal{I}}$

In the previous subsection we have constructed a gauge-fixing fermion for the $\mathcal{I}$ action, which makes the Hessian of the gauge-fixed action nondegenerate; this is the starting point for the perturbative expansion of $\mathcal{T}_{\mathcal{I}}$. For the sake of simplicity, we consider from now on the case of long knots in $\mathbb{R}^m$; hence, fields and antifields for the BV formalism for the $\mathcal{I}$ action have all rapid decrease. In order to make explicit computations, it remains to write down the Feynman rules.

The BV action for the canonical $\mathcal{I}$ theory may be written as the sum of a (after a choice of a gauge-fixing condition) nondegenerate quadratic part and other four pieces; we consider these other pieces as part of the interaction. One of these is constant in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, namely $\int_{S^{m-2}} \langle\, \Xi\,,\, f^*(B) \,\rangle$. We then choose a basis $\{e_1, \dots\}$ of $\mathfrak{g}$, and by $\left\{\varepsilon^1\right\}$ we denote the corresponding dual basis. Hence, we may write $\boldsymbol{\alpha} = \boldsymbol{\alpha}^\alpha e_\alpha$ and $\boldsymbol{\beta} = \boldsymbol{\beta}_\alpha \varepsilon^\alpha$. Accordingly, we write the three remaining pieces of the interaction terms of the BV action for the canonical $\mathcal{I}$ theory as follows:

$$\int_{\mathbb{R}^{m-2}} \langle\, \boldsymbol{\alpha}\,,\, \mathrm{ad}^*\left(f^*(a)\right)\boldsymbol{\beta} \,\rangle = \left[\int_{\mathbb{R}^{m-2}} \boldsymbol{\alpha}^\alpha f^*\left(a^\beta\right)\boldsymbol{\beta}_\gamma\right]\ \langle\, e_\alpha\,,\, \mathrm{ad}^*\left(e_\beta\right)\varepsilon^\gamma \,\rangle\,, \tag{5.4.14}$$

$$\int_{\mathbb{R}^{m-2}} \langle\, \boldsymbol{\alpha}\,,\, f^*\left(B\right) \,\rangle = \left[\int_{\mathbb{R}^{m-2}} \boldsymbol{\alpha}^\alpha f^*\left(B_\beta\right)\right]\ \left\langle\, e_\alpha\,,\, \varepsilon^\beta \,\right\rangle, \tag{5.4.15}$$

$$\int_{\mathbb{R}^{m-2}} \langle\, \Xi\,,\, \mathrm{ad}^*\left(f^*(a)\right)\boldsymbol{\beta} \,\rangle = \left[\int_{\mathbb{R}^{m-2}} f^*\left(a^\alpha\right)\boldsymbol{\beta}_\beta\right]\ \left\langle\, \Xi\,,\, \mathrm{ad}^*\left(e_\alpha\right)\varepsilon^\beta \,\right\rangle. \tag{5.4.16}$$

Therefore, the interaction term consists of a quadratic piece (5.4.14), of two linear terms (5.4.15) and (5.4.16) and of a constant term.

Summarizing all these results, we can write the exponential of the interaction term without the constant term as follows:

$$\begin{aligned}
&\exp\frac{\mathrm{i}}{\hbar}\left[\int_{\mathbb{R}^{m-2}} \langle\, \boldsymbol{\alpha}\,,\, \mathrm{ad}^*\left(f^*(a)\right)\boldsymbol{\beta} \,\rangle + \int_{S^{m-2}} \langle\, \boldsymbol{\alpha}\,,\, f^*\left(B\right) \,\rangle + \int_{S^{m-2}} \langle\, \Xi\,,\, \mathrm{ad}^*\left(f^*(a)\right)\boldsymbol{\beta} \,\rangle\right] = \\
&= \sum_{p,q,r\geq 0}\prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{k=1}^{r}\left(\frac{\mathrm{i}}{\hbar}\right)^{p+q+r}\frac{1}{p!q!r!}\left[\int_{\mathbb{R}^{m-2}} \boldsymbol{\alpha}^{\alpha_i} f^*\left(a^{\beta_i}\right)\boldsymbol{\beta}_{\gamma_i}\right]\left[\int_{\mathbb{R}^{m-2}} \boldsymbol{\alpha}^{\delta_j} f^*\left(B_{\varepsilon_j}\right)\right] \\
&\quad\left[\int_{\mathbb{R}^{m-2}} f^*\left(a^{\eta_k}\right)\boldsymbol{\beta}_{\theta_k}\right] \langle\, e_{\alpha_i}\,,\, \mathrm{ad}^*\left(e_{\beta_i}\right)\varepsilon^{\gamma_i} \,\rangle \left\langle\, e_{\delta_j}\,,\, \varepsilon^{\varepsilon_j} \,\right\rangle \left\langle\, \Xi\,,\, \mathrm{ad}^*\left(e_{\eta_k}\right)\varepsilon^{\theta_k} \,\right\rangle.
\end{aligned} \tag{5.4.17}$$

We begin by looking at a condition on p, q and r characterizing the (possibly) nontrivial Feynman diagrams coming from the perturbative expansion of (5.4.17). For given p, q and r, the total number of $\boldsymbol{\alpha}$-superfields equals $p+q$, while the total number of $\boldsymbol{\beta}$-superfields equals $p+r$; a necessary condition for nonvanishing Feynman diagrams is that the numbers of $\boldsymbol{\alpha}$-superfields and $\boldsymbol{\beta}$-superfields have to be equal:

$$p + q = p + r \Longleftrightarrow q = r. \tag{5.4.18}$$

Therefore, the only condition to get (possibly) nonvanishing Feynman diagrams is that the number of linear interaction terms with $\boldsymbol{\alpha}$-superfield has to be equal to the number of linear interaction terms with $\boldsymbol{\beta}$-superfields.

First of all, we need the superpropagator between α and β (which can be computed using the same arguments of Subsection 3.4.5 of Chapter 3):

$$\left\langle \pi_1^* \left(\boldsymbol{\alpha}^\alpha\right) \pi_2^* \left(\boldsymbol{\beta}_\beta\right)\right\rangle_{\text{g.f.}} : = \eta_{12}\, \delta^\alpha_\beta, \tag{5.4.19}$$

where η_{12} is the pull-back via the map (2.4.5) of the normalized, $SO(m-2)$-invariant top-form w on S^{m-3}; it follows that w is even, resp. odd, w.r.t. the antipodal map on S^{m-3} if m is even, resp. odd. In equation (5.4.19), π_i denotes the projection from $\mathbb{R}^{m-2} \times \mathbb{R}^{m-2}$ onto the i-th component. The superpropagator is clearly ill-defined on the diagonal of $\mathbb{R}^{m-2} \times \mathbb{R}^{m-2}$, hence it requires a regularization, which can be achieved simply by viewing the superpropagator as a form on the compactified configuration space $C_2\left(\mathbb{R}^{m-2}\right)$ (in fact, the form η_{12} extends smoothly from $C_2^0\left(\mathbb{R}^{m-2}\right)$ to $C_2\left(\mathbb{R}^{m-2}\right)$. The superpropagator (5.4.19) is depicted as an oriented dashed line; the orientation is directly linked to the parity of the form w w.r.t. the antipodal map on S^{m-3}, in particular, if m is odd, we need not indicate an orientation for (5.4.19).

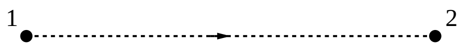

Figure 5.1: The η-propagator

The presence of tadpoles

As an example, when we choose $p = 1$ and $q = r = 0$, equation (5.4.17) yields the term

$$\frac{\mathrm{i}}{\hbar} \left[\int_{\mathbb{R}^{m-2}} \boldsymbol{\alpha}^\alpha f^* \left(a^\beta\right) \boldsymbol{\beta}_\gamma \right] \left\langle\, e_\alpha \,,\, \mathrm{ad}^* \left(e^\beta\right) \varepsilon^\gamma \,\right\rangle .$$

If we couple the $\boldsymbol{\alpha}$-superfield to the $\boldsymbol{\beta}$-superfield through the superpropagator (5.4.19), we get an ill-defined quantity, since the arguments of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equal. This term can be regularized as follows: we take a normalized vector field μ on $\mathbb{R}^{m-2}$ (which may be also viewed as a map from $\mathbb{R}^{m-2}$ to S^{m-3}), and define the regularized η-superfield, which we denote by $\overline{\eta}_\mu$, by the following rule

$$\overline{\eta}_\mu(x) : = \lim_{\varepsilon \downarrow 0} \eta_{12}(x + \varepsilon\mu(x), x)\,, \quad x \in \mathbb{R}^{m-2}. \tag{5.4.20}$$

Alternatively, $\overline{\eta}_\mu$ may be seen as the pull-back w.r.t. the map from $\mathbb{R}^{m-2}$ to S^{m-3} induced by μ of the form w. Hence, with the regularization of the superpropagator given by (5.4.20), the term at $p = 1$ takes the form

$$\begin{aligned} \tau_1 :&= \left[\int_{\mathbb{R}^{m-2}} \left(f^* a^\beta \right) \overline{\eta}_\mu \right] \left\langle e_\alpha \, , \, \mathrm{ad}^* \left(e^\beta \right) \varepsilon^\alpha \right\rangle = \\ &= \int_{\mathbb{R}^{m-2}} \mathrm{Tr} \left[\mathrm{ad}^* \left(f^* a \right) \right] \overline{\eta}_\mu = \\ &= - \int_{\mathbb{R}^{m-2}} \mathrm{Tr} \left[\mathrm{ad} \left(f^* a \right) \right] \overline{\eta}_\mu . \end{aligned} \tag{5.4.21}$$

More generally, tadpoles appear when we couple the $\boldsymbol{\alpha}$-superfield in the quadratic interaction (5.4.14) with the $\boldsymbol{\beta}$-superfield in the same term, and it is clear that equation (5.4.21) describes the general situation where tadpoles appear, since tadpoles are clearly isolated.

This term may be eliminated in two different ways: one way is to choose e.g. a unimodular Lie algebra $\mathfrak{g}$, such that the adjoint action is trace-free, and therefore the Lie-algebraic coefficient of (5.4.21) vanishes.

The second way is to add a counterterm to the BV action, whose sole purpose is to eliminate the term (5.4.21) at the perturbative level. The explicit form of the counterterm, which is added to the BV action for the $\mathcal{I}$ action, is given by

$$S_{\mathcal{I}\text{c.t.}} : = \int_{\mathbb{R}^{m-2}} \mathrm{Tr} \left[\mathrm{ad} \left(f^* a \right) \right] \overline{\eta}_\mu . \tag{5.4.22}$$

It is clear that $S_{\mathcal{I}} + S_{\mathcal{I}\text{c.t.}}$ satisfies also the Quantum Master Equation, since $S_{\mathcal{I}\text{c.t.}}$ does not depend neither on $\boldsymbol{\alpha}$ nor on $\boldsymbol{\beta}$.

From now on, we neglect Feynman diagrams containing isolated tadpoles.

The sum of connected diagrams when $q = r = 0$: the torsion

We discuss first the case $q = r = 0$; this corresponds to the absence of linear terms in the superfields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

We choose $p \geq 2$, and we consider only connected Feynman diagrams. Since $q = r = 0$, the only interaction term appearing in this situation is the quadratic term (5.4.14); moreover, recalling the renormalization (5.4.22) in the interaction term, Feynman diagrams with tadpoles are annihilated. Therefore, when $p \geq 2$, the only way of getting a connected Feynman diagram out of a product of p quadratic interaction terms (5.4.14) is to couple every $\boldsymbol{\alpha}$-superfield of a quadratic interaction term to an $\boldsymbol{\beta}$-superfield of a different quadratic interaction term. The most natural way to do so is to write a chain of p quadratic interaction terms and to couple the $\boldsymbol{\beta}$-superfield of the first interaction term to the $\boldsymbol{\alpha}$-superfield of the second interaction term, then to couple the $\boldsymbol{\beta}$-superfield of the second interaction term to the $\boldsymbol{\alpha}$-superfield of the third interaction term, and so on until we arrive at the last interaction term; at his point, we have to couple the $\boldsymbol{\alpha}$-superfield of the first interaction term to the $\boldsymbol{\beta}$-superfield of the last interaction term.

Therefore, it is immediate to see that the sum of the possibly nonvanishing connected Feynman diagrams without tadpole for $q = r = 0$ is given by

$$\sum_{n\geq 2}(-1)^{nm}\frac{1}{n}\int_{C_n^0(\mathbb{R}^{m-2})} \mathrm{Tr}\left[\mathrm{ad}^*\left(f^*a_1\right)\eta_{12}\cdots \mathrm{ad}^*\left(f^*\mathsf{a}_n\right)\eta_{n1}\right], \tag{5.4.23}$$

where the notations are as follows: $C_n^0\left(\mathbb{R}^{m-2}\right)$ the open configuration space of n points in $\mathbb{R}^{m-2}$. It is not clear if the integrals in (5.4.23) converge; a way of proving the convergence of these integrals is to resort to the FMcPAS compactification of $C_n^0\left(\mathbb{R}^{m-2}\right)$, noting that the superpropagator (5.4.19) extends smoothly to this compactification. By f^*a_i, for $1 \leq i \leq n$, we denote the pull-back w.r.t. the (smooth extension to the compactification $C_n\left(\mathbb{R}^{m-2}\right)$) projection from $C_n^0\left(\mathbb{R}^{m-2}\right)$ to the i-th component of f^*a. We recall the orientation convention for $C_n^0\left(\mathbb{R}^{m-2}\right)$: the orientation is induced by restriction of the volume-form $\pi_1^*\left(d\mathrm{vol}\right)\wedge\cdots\pi_n^*\left(d\mathrm{vol}\right)$, where $d$vol denotes the volume form of $\mathbb{R}^{m-2}$, and π_i denotes the projection from $C_n^0\left(\mathbb{R}^{m-2}\right)$ onto the i-th component; the convention has to be taken into account, and it produces some of the signs appearing in (5.4.23). The combinatorial factor $\frac{1}{n}$ corresponds to the number of different ways to couple $\boldsymbol{\alpha}$-superfields to $\boldsymbol{\beta}$-superfields through n superpropagators, such that the corresponding diagram is connected; the sign $(-1)^{nm}$ appears as a consequence of the behavior of the superpropagator w.r.t. the antipodal map on S^{m-3}.

It is better to rewrite the torsion with the adjoint action instead of the coadjoint action. Therefore, we compute

$$\begin{aligned}
&\mathrm{Tr}\left[\mathrm{ad}^*\left(f^*a_1\right)\eta_{12}\cdots \mathrm{ad}^*\left(f^*\mathsf{a}_n\right)\eta_{n1}\right] = \\
&= \left\langle\, e_\alpha\,,\, \mathrm{ad}^*\left(f^*a_1\right)\eta_{12}\cdots \mathrm{ad}^*\left(f^*\mathsf{a}_n\right)\eta_{n1}\varepsilon^\alpha \,\right\rangle = \\
&= (-1)^{\frac{n(n-1)}{2}m+n}\left\langle\, \varepsilon^\alpha\,,\, \mathrm{ad}\left(f^*a_n\right)\eta_{n1}\cdots \mathrm{ad}\left(f^*\mathsf{a}_1\right)\eta_{12}e_\alpha \,\right\rangle = \\
&= (-1)^{\frac{n(n-1)}{2}m+n}\,\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_n\right)\eta_{n1}\cdots \mathrm{ad}\left(f^*\mathsf{a}_1\right)\eta_{12}\right],
\end{aligned}$$

whence it follows

$$\begin{aligned}
&(-1)^{nm}\int_{C_n(\mathbb{R}^{m-2})} \mathrm{Tr}\left[\mathrm{ad}^*\left(f^*a_1\right)\eta_{12}\cdots \mathrm{ad}^*\left(f^*\mathsf{a}_n\right)\eta_{n1}\right] = \\
&\qquad = (-1)^{\left[\frac{n(n-1)}{2}+n\right]m+n}\int_{C_n(\mathbb{R}^{m-2})} \mathrm{Tr}\left[\mathrm{ad}\left(f^*a_n\right)\eta_{n1}\cdots \mathrm{ad}\left(f^*\mathsf{a}_1\right)\eta_{12}\right].
\end{aligned}$$

We consider the following permutation of n elements:

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix};$$

its sign is $(-1)^{\frac{n(n-1)}{2}}$. This permutation induces an involution on $C_n\left(\mathbb{R}^{m-2}\right)$, since permutation groups operate freely on configuration spaces and their action extend smoothly to FMcPAS compactifications; the sign of the involution is $(-1)^{\frac{n(n-1)}{2}m}$ w.r.t. the orientation of $C_n\left(\mathbb{R}^{m-2}\right)$. The action of the involution on the n-th term of the torsion is therefore

$$
\begin{aligned}
&(-1)^{\left[\frac{n(n-1)}{2}+n\right]m+n}\int_{C_n(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_n\right)\eta_{n1}\cdots\mathrm{ad}\left(f^*\mathsf{a}_1\right)\eta_{12}\right]=\\
&=(-1)^{n(m-1)}\int_{C_n(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_1\right)\eta_{1n}\cdots\mathrm{ad}\left(f^*\mathsf{a}_n\right)\eta_{n,n-1}\right]=\\
&=(-1)^{nm+(m-1)}\int_{C_n(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_1\right)\eta_{12}\cdots\mathrm{ad}\left(f^*\mathsf{a}_n\right)\eta_{n1}\right].
\end{aligned}
$$

The last identity follows by switching all η-propagators but η_{1n} of one position to the left and by the parity of η-propagators w.r.t. the antipodal map.

Finally, the torsion takes the form

$$
(-1)^{m-1}\sum_{n\geq 2}(-1)^{nm}\int_{C_n(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_1\right)\eta_{12}\cdots\mathrm{ad}\left(f^*a_n\right)\eta_{n1}\right].
$$

There is another subtlety concerning the sign $(-1)^{nm}$: in fact, we may reabsorb the sign $(-1)^m$ by in the superpropagator η_{12}. In other words, if we take η_{12} to be the pull-back w.r.t. the map (2.4.5) of the form $(-1)^m w$, the torsion may be written as

$$
\tau:=(-1)^{m-1}\sum_{n\geq 2}\int_{C_n(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_1\right)\eta_{12}\cdots\mathrm{ad}\left(f^*a_n\right)\eta_{n1}\right]. \tag{5.4.24}
$$

The top-form $\overline{w}:=(-1)^m w$ is clearly $SO(m-2)$-invariant and has the same parity w.r.t. the antipodal map as w; the only difference is that the normalization condition reads now

$$
\int_{S^{m-3}}\overline{w}=(-1)^m.
$$

We finally notice that the torsion starts in reality by the term with three a-fields; namely, the term with two a-fields takes the form (up to signs and combinatorial factors)

$$
\int_{C_2(\mathbb{R}^{m-2})}\mathrm{Tr}\left[\mathrm{ad}\left(f^*a_1\right)\eta_{12}\,\mathrm{ad}\left(f^*a_2\right)\eta_{21}\right],
$$

which vanishes immediately, since it contains two η-superpropagators η_{12}, and their product is zero.

The sum of connected Feynman diagrams when $r=q=1$: the parallel transport

We consider the second special situation, namely when $r=q=1$. In this case, as $p\geq 1$, we have to consider the three interaction terms (5.4.14) (the quadratic interaction) plus the two linear interaction terms (5.4.15) and (5.4.16).

Since we are interested only in connected diagrams and since there are exactly one linear interaction with a $\boldsymbol{\alpha}$-superfield and exactly one interaction term with an $\boldsymbol{\beta}$-superfield, taking $p\geq 1$, the only way of getting a connected diagram with p quadratic interactions is to couple the $\boldsymbol{\alpha}$-superfield in the linear term (5.4.15) to the final isolated

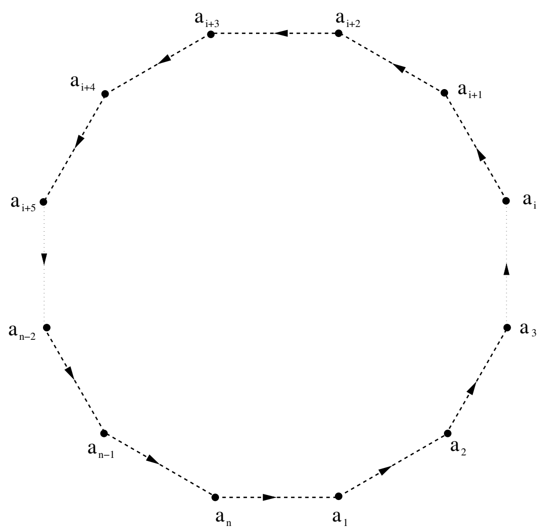

Figure 5.2: The n-th term of the torsion

$\boldsymbol{\beta}$-superfield in the connected chain with p quadratic interactions (5.4.14), and then to couple the isolated $\boldsymbol{\alpha}$-superfield in the first quadratic interaction of the chain with the $\boldsymbol{\beta}$-superfield in the linear interaction term (5.4.16) through superpropagators. The most natural way of seeing how the procedure works is to couple the $\boldsymbol{\alpha}$-superfield of the linear interaction term (5.4.15) to the $\boldsymbol{\beta}$-superfield of the last interaction term; we then couple the $\boldsymbol{\alpha}$-superfield of the last interaction term to the $\boldsymbol{\beta}$-superfield of the preceding interaction term. If $p = 1$, the preceding interaction term is linear w.r.t. the $\boldsymbol{\beta}$-superfield, and we are done; otherwise, the procedure continues by coupling each $\boldsymbol{\alpha}$-superfield in a quadratic interaction term to the $\boldsymbol{\beta}$-superfield of the preceding quadratic interaction term, until we arrive to the linear term (5.4.16).

Taking into account orientation conventions for configuration spaces, the parity of the superpropagator (5.4.19), the sum of all connected diagrams with exactly one linear interaction (5.4.14) and (5.4.16), and the constant term $\int_{\mathbb{R}^{m-2}} \langle\, \Xi\,,\, f^*B \,\rangle$, the above procedure leads to the result

$$\sum_{n\geq 0}(-1)^{nm}\int_{C_{n+1}(\mathbb{R}^{m-2})}\langle\, \Xi\,,\, \mathrm{ad}^*\,(f^*a_1)\,\eta_{12}\cdots \mathrm{ad}^*\,(f^*a_n)\,\eta_{n,n+1}f^*B\,\rangle\,.$$

As for the torsion, the sign $(-1)^{nm}$ may be reabsorbed in the superpropagator, if we define the new superpropagator by the top-form $\overline{w} = (-1)^m w$, instead of w.

Hence, with this convention on the superpropagator in mind, the sum of the connected Feynman diagrams when $q = r = 1$ takes the form

$$\sigma\colon\ = \sum_{n\geq 0}\int_{C_{n+1}(\mathbb{R}^{m-2})}\langle\, \Xi\,,\, \mathrm{ad}^*\,(f^*a_1)\,\eta_{12}\cdots \mathrm{ad}^*\,(f^*a_n)\,\eta_{n,n+1}f^*B\,\rangle\,. \quad (5.4.25)$$

We will usually refer to σ as in equation (5.4.25) as to the parallel transport w.r.t. $f \in \mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$.

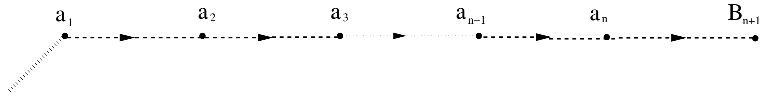

Figure 5.3: The $n+1$-th term of the parallel transport; the dotted line coming out from the first a-form denotes pairing with Ξ

The general case: the sum of all Feynman diagrams without tadpoles for $q = r \geq 0$

Having now discussed the shape of the connected Feynman diagrams when $q = r = 0$ and $q = r = 1$, which lead to well-defined expressions, namely the torsion (5.4.24) and the parallel transport (5.4.25) , we finally come to the more general case, when we consider also non-connected Feynman diagrams without tadpoles for general $q = r \leq 0$.

Tt turns out that the explicit computations yielding the torsion and the parallel transport are sufficient to characterize the sum of all Feynman diagrams under the necessary condition $q = r \geq 0$. In fact, a general theorem about the combinatoric of Feynman diagrams for a general field theory implies immediately that the sum of all Feynman diagrams coming from the perturbative expansion of the partition function $\widetilde{\mathcal{T}}_{\mathcal{I}}$ may be written as

$$\widetilde{\mathcal{T}}_{\mathcal{I}}\,(A, B; \Xi) :\ = \exp \Sigma, \quad \Sigma \quad : \ = \frac{\mathrm{i}}{\hbar}\sigma - (-1)^m \tau. \tag{5.4.26}$$

We refer to [56] for a more detailed discussion of the whole combinatoric of perturbative expansions and Feynman diagrams.

5.4.5 The product BV structure for the $BF + \Sigma$ theories

In Subsubsection 5.4.2, we have proved that there exists a BV action for the $\mathcal{I}$ action, provided (A, B) are solutions of the equations of motion of the BF action. We can perform a perturbative expansion of the partition function using the gauge-fixed BV action for the $\mathcal{I}$ action for any on-shell pair (A, B). We can then provide a functional in the super BV formalism simply by replacing A and B by their super BV versions A and B, proving then that such a functional is indeed a BV observable in the sense of (2.8.15).

We briefly sketch in this subsection another way to get the super BV version of the perturbative expansion of $\mathcal{T}_{\mathcal{I}}$ directly in a perturbative way, without restricting to “on-shell” conditions.

Product BV structure

We begin by introducing the algebra generated by local functionals determined by formal power series in all fields and antifields of the BF action and of the $\mathcal{I}$ action, which we denote by the symbol $\mathcal{S}_{BF,\mathcal{I}}$. This algebra can be given a grading in a standard way. The super algebra structure underlying this grading is given by the dot product.

We define the product BV antibracket on $\mathcal{S}_{(\mathsf{A},\mathsf{B};\boldsymbol{\alpha},\boldsymbol{\beta})}$ by the formula

$$(\, F \,,\, G \,)_{\mathrm{pr.}} : \, = (\, F \,,\, G \,) + (\, F \,,\, G \,)_{\mathcal{I}} \,, \tag{5.4.27}$$

for any two functionals F, G in $\mathcal{S}_{BF,\mathcal{I}}$. As we see immediately from the above formula, the bracket $(\ ,\ )_{\mathrm{pr.}}$ is reminiscent of the product Poisson structure on the product of two Poisson manifolds. This is why it is called the product BV antibracket. Since $(\ ,\ )_{\mathrm{pr.}}$ is a sum of two BV antibrackets, it is certainly an antibracket, i.e. it satisfies the graded commutativity, the graded Leibnitz rule and the graded Jacobi identity.

Since $(\ ,\ )$ and $(\ ,\ )_{\mathcal{I}}$ are both antibrackets, they measure separately the failures of the corresponding Laplacians to be differentials. Therefore, a natural choice of a Laplacian belonging to the product antibracket would be

$$\Delta_{\mathrm{pr.}} : \, = \Delta_{BV} + \Delta_{\mathcal{I}}. \tag{5.4.28}$$

In fact, it is easy to see that this operator satisfies the identity

$$\Delta_{\mathrm{pr.}}(F\, G) = \Delta_{\mathrm{pr.}} F\, G + (-1)^{|F|}\,(\, F \,,\, G \,)_{\mathrm{pr.}} + (-1)^{|F|} F\, \Delta_{\mathrm{pr.}} G$$

The operator $\Delta_{\mathrm{pr.}}$ squares to 0, and satisfies a shifted graded Leibnitz rule w.r.t. the product BV antibracket.

The BV action for the product BV structure

Now that we have introduced the product BV structure, we are ready to state the main theorem of the section. We define the local functional

$$S^{\mathrm{pr.}} := \int_{\mathbb{R}^m} \langle\!\langle\, \mathsf{F}_{\mathsf{A}} \,;\, \mathsf{B} \,\rangle\!\rangle + \int_{S^{m-2}} \langle\!\langle\, \boldsymbol{\alpha} \,;\, \mathrm{d}_{f^*\mathsf{A}} \boldsymbol{\beta} + f^* \mathsf{B} \,\rangle\!\rangle \,.$$

It follows immediately that $S^{\mathrm{pr.}}$ belongs to $\mathcal{S}_{BF,\mathcal{I}}$, and by construction it has total degree 0. We want to prove that $S^{\mathrm{pr.}}$ is a solution of the master equation w.r.t. the product BV antibracket.

Theorem 5.4.7. *The functional $S^{\mathrm{pr.}}$ satisfies the Quantum Master Equation w.r.t. the BV antibracket $(\ ,\ )_{\mathrm{pr.}}$ and the BV Laplacian $\Delta_{\mathrm{pr.}}$, i.e.*

$$-\mathrm{i}\hbar \Delta_{\mathrm{pr.}} S^{\mathrm{pr.}} + \frac{1}{2}\,(\, S^{\mathrm{pr.}} \,,\, S^{\mathrm{pr.}} \,)_{\mathrm{pr.}} = 0. \tag{5.4.29}$$

We skip the explicit proof, mentioning only a few facts hinting to the proof. As it was already the case for the proof of the Quantum Master Equation for the BV action for BF theories, the proof of the Quantum Master Equation is divided into two steps, namely the Master Equation for the BV antibracket $(\ ,\ )_{\mathrm{pr.}}$ and the harmonicity w.r.t. the BV Laplacian $\Delta_{\mathrm{pr.}}$.

The main ingredients in the proof of the Master Equation are the functional derivatives w.r.t. the superfields; in order to compute them, we need a distributional analogue of the Poincaré dual of the imbedding f and a way to extend forms on $\mathbb{R}^{m-2}$ to $\mathbb{R}^m$ via f. A distributional Poincaré dual can be obtained from the distributional kernel of the identity by pulling back w.r.t. f and then taking the push-forward of the result on $\mathbb{R}^{m-2}$; such a form is localized exactly on the image of f. The extension of forms from $\mathbb{R}^{m-2}$ to $\mathbb{R}^m$ via f is obtained via a tubular neighbourhood of f. The validity of the Master Equation is then a simple consequence of the definition of the product BV antibracket, of Stokes' Theorem and Bianchi identity.

As for the harmonicity w.r.t. the BV Laplacian $\Delta_{\mathrm{pr.}}$, we use the same arguments already used for the BV action for BF theories.

We can rewrite identity (5.4.29) in a slightly different form:

$$-\mathrm{i}\hbar\Delta_{BV}S_{\mathcal{I}} + \frac{1}{2}\,(\,S_{\mathcal{I}}\,,\,S_{\mathcal{I}}\,) + (\,S\,,\,S_{\mathcal{I}}\,) - \mathrm{i}\hbar\Delta_{\mathcal{I}}S_{\mathcal{I}} + \frac{1}{2}\,(\,S_{\mathcal{I}}\,,\,S_{\mathcal{I}}\,)_{\mathcal{I}} = 0. \quad (5.4.30)$$

Equation (5.4.30) is the basic groundstone for the next computations.

The partition function of the functional $S_{\mathcal{I}}$

We define the gauge-fixed partition function of the $\mathcal{I}$ action, computed at a given $f \in \mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, in the BV formalism:

$$\mathcal{Z}_{\Psi}^{f} :\, = \int \mathcal{D}\theta^{\alpha} \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}^{\mathrm{g.f.}}\,(\theta^{\alpha})\,, \quad (5.4.31)$$

where Ψ is a gauge-fixing fermion for the BV action $S_{\mathcal{I}}$. (We recall that the notation $S_{\mathcal{I}}^{\mathrm{g.f.}}$ means that we have replaced any antifield θ_{α}^{+} in $S_{\mathcal{I}}$ by the functional derivative of the gauge-fixing fermion Ψ w.r.t. the corresponding field.)

If we consider the functional integral $\mathcal{Z}_{\Psi}^{f}$ as a functional depending on the superfields a and B (we integrate only on fields and antifields of the $\mathcal{I}$ action), equation (5.4.30) implies that the gauge fixed partition function (5.4.31) is a BV observable,

$$\Omega_{BV}\ \mathcal{Z}_{\Psi}^{\Sigma} = 0,$$

if the gauge fixing fermion Ψ does not depend on any field or antifield for the BF action. We give a sketch of the proof of the above statement. The BV operator Ω_{BV} commutes formally with the integral in (5.4.31) as the formal integration measure does not depend on fields and antifields of pure BF theory. If we moreover assume the gauge-fixing fermion independent of the superfields A and B, we get

$$\begin{aligned}\Omega_{BV} \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}^{\text{g.f.}} &= \frac{\mathrm{i}}{\hbar}\Big[- \mathrm{i}\hbar\Delta_{BV} S_{\mathcal{I}}^{\text{g.f.}} + \frac{1}{2}\left(S_{\mathcal{I}}^{\text{g.f.}} \, , \, S_{\mathcal{I}}^{\text{g.f.}} \right) + \\ &\quad + \frac{\mathrm{i}}{\hbar}\left(S \, , \, S_{\mathcal{I}}^{\text{g.f.}} \right) \Big] \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}^{\text{g.f.}} = \\ &= -\frac{\mathrm{i}}{\hbar}\left[-\mathrm{i}\hbar\Delta_{\mathcal{I}} S_{\mathcal{I}} + \frac{1}{2}\left(S_{\mathcal{I}} \, , \, S_{\mathcal{I}} \right)_{\mathcal{I}} \right]_{\theta^+_\alpha = \frac{\vec{\partial}\Psi}{\partial\theta^\alpha}} \exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}}^{\text{g.f.}} = \\ &= \mathrm{i}\hbar \left[\Delta_{\mathcal{I}} \left(\exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}} \right) \right]_{\theta^+_\alpha = \frac{\vec{\partial}\Psi}{\partial\theta^\alpha}} ,\end{aligned}$$

since the gauge-fixing fermion is not acted by the BV coboundary as it does not depend on the BV superfields A and B of BF theories. Hence, the BV operator Ω_{BV} applied to (5.4.31) gives

$$\Omega_{BV} \, \mathcal{Z}_{\Psi}^{\Sigma} = \pm \mathrm{i}\hbar \int \mathcal{D}\theta^\alpha \left[\Delta_{\mathcal{I}} \left(\exp \frac{\mathrm{i}}{\hbar} S_{\mathcal{I}} \right) \right]_{\theta^+_\alpha = \frac{\vec{\partial}\Psi}{\partial\theta^\alpha}} ;$$

the term on the right-hand side vanishes by the formal properties of the BV Laplacian $\Delta_{\mathcal{I}}$ (see e.g. [45]).

Moreover, similar arguments show that the infinitesimal variation of a family of gauge-fixing fermions Ψ_t, not depending on fields and antifields of pure BF theory, changes (5.4.31) by an Ω_{BV}-exact term; in other words,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{Z}^f_{\Psi_t} = \Omega_{BV} \widetilde{\mathcal{Z}}\left(f; \Psi_0\right),$$

for $\widetilde{\mathcal{Z}}\left(f; \Psi_0\right)$ a functional on fields and antifields of pure BF theory, depending also on Ψ_0 and f.

Obvious modifications have to be made when we consider long knots in $\mathbb{R}^m$, but the computations still hold.

5.5 Quantum observables related to imbeddings of codimension 2

In this section we display the superversions of the classical observables obtained by computing explicitly the perturbative expansion of the partition function of the $\mathcal{I}$-action containing special imbeddings of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$; we also prove that these functional in the superfields of the canonical BF-theory are really BV observables.

The super BV version of the exponential of the parallel transport τ and of the the torsion τ are obtained simply by replacing the A and B fields by their super BV counterparts; the same results can be obtained directly by performing the perturbative expansion of gauge-fixed partition function (5.4.31) of Subsection 5.4.5 of the $\mathcal{I}$-piece of the product BV action for $BF + \mathcal{I}$ theory. The advantage of BV formalism is that the proof of homotopy-invariance of the exponential of the super BV versions of σ and

τ can be incorporated in a slight variant of the proof of the Ω_{BV}-closedness. This is achieved by replacing the map f by the evaluation map ev from $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times \mathbb{R}^{m-2}$ to $\mathbb{R}^m$ and the integrals by the corresponding push-forwards from the trivial fibration $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times \mathbb{R}^{m-2}$ over $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$; the exponential of σ and τ becomes then a sum of forms on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ at all degrees, but we are interested only in the piece of ghost number 0, which, as we will see, is simply a function on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$. It suffices to show that the piece of ghost number 0 of the exponential of σ and τ is Ω_{BV}-closed and that its exterior derivative gives an Ω_{BV}-exact term; both results imply together that the v.e.v. of the piece of ghost number 0 of the exponential of σ and τ w.r.t. the BF action is well-defined and has vanishing exterior derivative, hence giving at a perturbative level possible invariants of higher-dimensional knots.

5.5.1 Parallel transport and supertorsion: definition

The functional that we are going to define is somehow reminiscent of the generalized holonomy, that we introduced in [20]; we will therefore call this functional *parallel transport for imbeddings*.

We take the BV superfields a and B, and we denote by a_i, resp. B_i, the (super-)form on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ with values in $\mathfrak{g}$, resp. $\mathfrak{g}^*$, defined by

$$\mathsf{a}_i :\ = (\pi_i \circ \mathrm{ev}_n)^* \,\mathsf{a}; \quad \mathsf{B}_i :\ = (\pi_i \circ \mathrm{ev}_n)^* \,\mathsf{B}.$$

Next, we pick the form w of highest degree on S^{m-3}, which enjoys the following requirements:

a) w is invariant w.r.t. the standard action of $SO(m-2)$ on S^{m-3};

b) w is normalized in the following sense:

$$\int_{S^{m-3}} w = (-1)^m.$$

We denote by η_{ij}, for given $i \neq j \in \{1,\ldots,n\}$, the form of (total) degree $m-3$ on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times C_n(\mathbb{R}^{m-2})$, defined by

$$\eta_{ij} :\ = (\varphi_{12} \circ \pi_{ij})\, w.$$

Finally, we denote by Ξ a given element of the Lie algebra $\mathfrak{g}$.

Definition 5.5.1. The *parallel transport* σ is defined as a formal series $\sigma = \sum_{n\geq 0} \sigma_n$, where the n-th summand has the form:

$$\sigma_n(\mathsf{a},\mathsf{B};\Xi) :\ = \pi_*^{n+1} \left\langle \mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\, \mathrm{ad}^*(\mathsf{a}_2)\eta_{23} \cdots \mathrm{ad}^*(\mathsf{a}_n)\eta_{n,n+1} \mathsf{B}_{n+1}\,,\, \Xi \right\rangle.$$

By definition, σ_n is a sum of forms on $\mathrm{Imb}_{\infty}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of total degree 0; hence, the 0-ghost number part of σ_n defines a function on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$.

Definition 5.5.2. The *torsion* τ is defined as a formal power series $\tau = \sum_{n\geq 0} \tau_n$, where τ_n takes the form

$$\tau_n(\mathsf{a}) \colon = \frac{1}{n}\pi_*^n \operatorname{Tr}\left[\operatorname{ad}(\mathsf{a}_1)\eta_{12}\operatorname{ad}(\mathsf{a}_2)\eta_{23}\cdots\eta_{n-1,n}\operatorname{ad}(\mathsf{a}_n)\eta_{n1}\right].$$

τ_n has also total degree 0. In the above formula, the trace is taken in the adjoint representation.

Finally, we define the functional Σ as the exponential of $\Sigma\colon = \frac{\mathrm{i}}{\hbar}\sigma - (-1)^m\tau$.

5.5.2 The BV variation of the parallel transport and the torsion

The main feature of the functional Σ is encoded in the following

Theorem 5.5.3. *The functional Σ satisfies the equation*

$$(\boldsymbol{\delta} - (-1)^m \mathrm{d})\,\Sigma = 0. \tag{5.5.1}$$

Proof. In order to show that identity (5.5.1) holds, we show separately the two identities

$$(\boldsymbol{\delta} - (-1)^m \mathrm{d})\,\sigma = 0 \quad \text{and} \quad (\boldsymbol{\delta} - (-1)^m \mathrm{d})\,\tau = 0.$$

We prove the first identity.

First, we apply the exterior derivative to σ; we then use the following formula concerning the push-forward:

$$\begin{aligned}\mathrm{d}\sigma_n =& (-1)^{(n+1)(m-2)}\pi_*^{n+1}\left[\mathrm{d}\left\langle \operatorname{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\operatorname{ad}^*(\mathsf{a}_n)\eta_{n,n+1}\mathsf{B}_{n+1}\,,\,\Xi\right\rangle\right] - \\ &- (-1)^{(n+1)(m-2)}\pi_{\partial *}^{n+1}\iota_{\partial,n+1}^*\left[\left\langle \operatorname{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\operatorname{ad}^*(\mathsf{a}_n)\eta_{n,n+1}\mathsf{B}_{n+1}\,,\,\Xi\right\rangle\right],\end{aligned} \tag{5.5.2}$$

where $\pi_{\partial *}^{n+1}$ is the projection from $\operatorname{Imb}_{\infty,\sigma}\!\left(\mathbb{R}^{m-2},\mathbb{R}^m\right) \times \partial C_n(\mathbb{R}^{m-2})$ forgetting the boundary of the fiber, and $\iota_{\partial,n+1}$ is the inclusion of $\operatorname{Imb}_{\infty,\sigma}\!\left(\mathbb{R}^{m-2},\mathbb{R}^m\right) \times \partial C_n(\mathbb{R}^{m-2})$ into $\operatorname{Imb}_{\infty,\sigma}\!\left(\mathbb{R}^{m-2},\mathbb{R}^m\right) \times C_n(\mathbb{R}^{m-2})$.

By the Leibniz rule, we get for the first term on the right-hand side of (5.5.2):

$$\begin{aligned}&\sum_{i=1}^{n}(-1)^{(n+i)(m-2)}\pi_*^{n+1}\left[\left\langle \operatorname{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\eta_{i-1,i}\operatorname{ad}^*((\mathrm{d}\mathsf{a})_i)\eta_{i,i+1}\cdots\mathsf{B}_{n+1}\,,\,\Xi\right\rangle\right] + \\ &+ (-1)^{m-2}\pi_*^{n+1}\left[\left\langle \operatorname{ad}^*(\mathsf{a}_1)\eta_{12}\cdots(\mathrm{d}\mathsf{B})_{n+1}\,,\,\Xi\right\rangle\right].\end{aligned}$$

We consider the boundary term (i.e. the second term on the right-hand side of (5.5.2). In order to compute it, we need a technical Lemma, whose statement and proof we postpone.

Thanks to Lemma 5.5.4, we need only consider the principal faces of $C_{n+1}(\mathbb{R}^{m-2})$ corresponding to the collapse of two consecutive points. The convention (2.4.23) for the orientations of principal faces of configuration spaces implies that, for $i \in$

$\{1,\ldots,n\}$, the orientation of the face where i and $i+1$ collide is given by $(-1)^{(i+1)(m-2)}$. Hence, we get

$$\sum_{i=1}^{n-1}(-1)^{(n-1+i)(m-2)}\pi_*^n\left[\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\eta_{i-1,i}\,\mathrm{ad}^*\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_i\eta_{i,i+1}\cdots\mathsf{B}_n\right]+ \\ +(-1)^{m-2}\pi_*^n\left[\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\mathrm{ad}^*(\mathsf{a}_n)\mathsf{B}_n\right],$$

where we have used the normalization of the form w.

Summarizing all the computations so far, we get

$$\begin{aligned}\mathrm{d}\sigma_n=&\sum_{i=1}^{n}(-1)^{(n+i)(m-2)}\pi_*^{n+1}\left[\langle\,\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\eta_{i-1,i}\,\mathrm{ad}^*((\mathrm{d}\mathsf{a})_i)\eta_{i,i+1}\cdots\mathsf{B}_{n+1}\,,\,\Xi\,\rangle\right]+\\&+(-1)^{m-2}\pi_*^{n+1}\left[\langle\,\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots(\mathrm{d}\mathsf{B})_{n+1}\,,\,\Xi\,\rangle\right]+\\&+\sum_{i=1}^{n-1}(-1)^{(n-1+i)(m-2)}\pi_*^n\left[\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\eta_{i-1,i}\,\mathrm{ad}^*\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_i\eta_{i,i+1}\cdots\mathsf{B}_n\right]+\\&+(-1)^{m-2}\pi_*^n\left[\mathrm{ad}^*(\mathsf{a}_1)\eta_{12}\cdots\mathrm{ad}^*(\mathsf{a}_n)\mathsf{B}_n\right].\end{aligned}$$

If we now recall the rules for the super BV differential $\boldsymbol{\delta}$ and its action on the superfield a and B, we get the first equation.

As for the second one, we compute first the differential of τ_n:

$$\begin{aligned}\mathrm{d}\tau_n=&(-1)^{n(m-2)}\pi_*^n\,\mathrm{Tr}\,\mathrm{d}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n1}\right]-\\&-(-1)^{n(m-2)}\pi_{\partial*}^n\iota_{\partial,n}^*\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n1}\right].\end{aligned}\tag{5.5.3}$$

We compute explicitly the first term on the right-hand side of (5.5.3) by means of the Leibnitz rule:

$$\sum_{k=1}^{n}\frac{1}{n}(-1)^{(n-k-1)(m-2)}\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(\mathrm{d}\mathsf{a}_k)\,\eta_{k,k+1}\cdots\right]=$$

$$=\sum_{k=1}^{n}\frac{1}{n}(-1)^{(n(m-2)+(k-1)(n-k+1)(m-2)}$$

$$\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathrm{d}\mathsf{a}_k)\,\eta_{k,k+1}\cdots\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n1}\,\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\right]=$$

$$=\sum_{k=1}^{n}\frac{1}{n}(-1)^{n(m-2)}\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathrm{d}\mathsf{a}_1)\,\eta_{12}\,\mathrm{ad}\,(\mathsf{a}_2)\,\eta_{23}\cdots\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n1}\right]=$$

$$=(-1)^{n(m-2)}\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathrm{d}\mathsf{a}_1)\,\eta_{12}\,\mathrm{ad}\,(\mathsf{a}_2)\,\eta_{23}\cdots\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n1}\right].$$

In order to get the third equality, we have used the obvious action of the permutation group $\mathfrak{S}_n$ on $C_n\left(\mathbb{R}^{m-2}\right)$.

We consider the boundary term on the right-hand side of (5.5.3); again, we make use of Lemma 5.5.4, and therefore we need only consider the principal faces of $C_n\left(\mathbb{R}^{m-2}\right)$ corresponding to the collapse of two consecutive points and when the first and the last point collapse:

$$\sum_{k=1}^{n-1}\frac{1}{n}(-1)^{(n-k)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}(\mathsf{a}_1)\,\eta_{12}\cdots\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_k\eta_{k,k+1}\cdots\right]-$$
$$-\frac{1}{n}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}(\mathsf{a}_1)\,\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_1)\right]=$$
$$=\sum_{k=1}^{n-1}\frac{1}{n}(-1)^{(n-1)(m-2)+(k-1)(n-k)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_k\eta_{k,k+1}\cdots\right.$$
$$\left.\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\operatorname{ad}(\mathsf{a}_1)\,\eta_{12}\cdots\right]+$$
$$+\frac{1}{n}(-1)(n-1)(m-2)+1\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_1\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\right]=$$

$$=\sum_{k=1}^{n-1}\frac{1}{n}(-1)^{(n-1)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_1\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\right]+$$
$$+\frac{1}{n}(-1)^{(n-1)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_1\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\right]=$$
$$=(-1)^{(n-1)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_1\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\right].$$

Summarizing the results so far, the explicit expression for the differential for τ_n is

$$\begin{aligned}\mathrm{d}\tau_n&=\frac{1}{n}(-1)^{n(m-2)}\pi_*^{n}\operatorname{Tr}\left[\operatorname{ad}(\mathrm{d}\mathsf{a}_1)\,\eta_{12}\operatorname{ad}(\mathsf{a}_2)\,\eta_{23}\cdots\operatorname{ad}(\mathsf{a}_n)\,\eta_{n1}\right]+\\&\quad+(-1)^{(n-1)(m-2)+1}\pi_*^{n-1}\operatorname{Tr}\left[\operatorname{ad}\left(\frac{1}{2}[\![\mathsf{a}\,;\mathsf{a}]\!]\right)_1\eta_{12}\cdots\operatorname{ad}(\mathsf{a}_{n-1})\,\eta_{n-1,1}\right].\end{aligned}$$

It remains to sum over $n\geq 3$ and to recall again the rules of the super BV differential $\boldsymbol{\delta}$ and its action on the superfields a and B to get the result. □

Here we come to the quoted

Lemma 5.5.4 (Vanishing Lemma for parallel transport and torsion). *The only non-trivial contributions to the boundary term of (5.5.2), resp. (5.5.3), come from the principal faces of $C_{n+1}(\mathbb{R}^{m-2})$, when two consecutive vertices collapse together, resp. from the principal faces of $C_n\left(\mathbb{R}^{m-2}\right)$, when two consecutive vertices or the vertices 1 and n collapse together.*

Proof of Lemma 5.5.4. We consider, for a given $n\geq 3$, a subset $S\subset\{1,\dots,n\}$ of cardinality $3\leq|S|\leq n$, labeling the vertices collapsing together.

We notice that the superfields a and B are basic in the boundary fibration corresponding to the collapse of the vertices labeled by S. Therefore, the only contributions coming from the corresponding boundary face come from the η-propagators. It is also obvious that an η-propagator is nonbasic only if both its endpoints lie in S.

We consider both the parallel transport σ and the torsion τ. It is easy to see that, for any $S \subset \{1, \ldots, n\}$ of cardinality $3 \leq |S| \leq n$, the nonbasic piece of the integrand in both σ and τ has either a zerovalent vertex or a univalent vertex joined by an η-propagator or a bivalent vertex joined by two η-propagators. In all cases, Lemma 6.5.1 and Lemma 6.5.2 yield the claim.

If the cardinality of S equals 2, but S does not contain consecutive vertices or the vertices 1 and n, the corresponding contribution also vanishes by Lemma 6.5.1, since no η-propagator is nonbasic.

Finally, we consider the case when at least one vertex escapes to infinity. When at least one vertex escapes to infinity, all forms in the parallel transport as well as in the torsion with at least one vertex in the set S labeling the vertices escaping to infinity are nonbasic. This means that at least an a- or B-superfield has its argument escaping to infinity; this yields immediately the vanishing of the corresponding contribution, since a and B have both rapid decrease. $\square$

5.5.3 The BV Laplacian of the parallel transport and the torsion

In order to prove that Σ is a BV-observable, it remains to compute the action of the BV-Laplacian Δ_{BV} on it.

It is sufficient to consider only the pieces of Σ of ghost number 0 and -1 of Σ, which are respectively a function and a 1-form on $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$. It is not difficult to check that the piece of Σ of ghost number 0, resp. -1, is the exponential of the piece og shot number 0 of $\sigma - (-1)^m\tau$, resp. the piece of ghost number -1 of $\sigma - (-1)^m\tau$ multiplied by the piece of ghost number of Σ:

$$\Sigma^0 = \exp\left[\frac{\mathrm{i}}{\hbar}\sigma^0 - (-1)^m\tau^0\right] \quad \text{and} \quad \Sigma^{-1} = \left[\frac{\mathrm{i}}{\hbar}\sigma^{-1} - (-1)^m\tau^{-1}\right]\Sigma^0,$$

where the superscripts 0 and -1 refer to the respective ghost numbers. By its very definition, if we consider a given $f \in \mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$, Σ^0 evaluated at f is obtained simply by replacing the pull-backs w.r.t. evaluation maps in Σ by pull-backs w.r.t. f.

We can then prove

Theorem 5.5.5. *The functionals Σ^i, for $i \in \{-1, 0\}$, satisfy the equation*

$$\Delta_{BV}\Sigma^i = 0.$$

Proof. Using the main properties of the super BV Laplacian $\boldsymbol{\Delta}$, we obtain

$$\begin{aligned}\boldsymbol{\Delta}\Sigma = 0 \Longleftrightarrow & \frac{1}{2}\left(\left(\frac{\mathrm{i}}{\hbar}\sigma - (-1)^m\tau\,;\,\frac{\mathrm{i}}{\hbar}\sigma - (-1)^m\tau\right)\right) + \\ & + \frac{\mathrm{i}}{\hbar}\boldsymbol{\Delta}\sigma - (-1)^m\boldsymbol{\Delta}\tau = 0.\end{aligned}$$

Clearly, $(\!(\tau \,;\, \tau)\!) = 0$ and $\boldsymbol{\Delta}\tau = 0$, as the torsion τ depends only on a.

It is enough to show the following identities:

- $\Delta_{BV}\sigma^0 = 0$,
- $(\sigma^0 \,,\, \sigma^0) = (\sigma^0 \,,\, \tau^0) = 0$,
- $\Delta_{BV}\sigma^{-1} = 0$,
- $(\sigma^0 \,,\, \sigma^{-1}) = (\sigma^0 \,,\, \tau^{-1}) = (\sigma^{-1} \,,\, \tau^0) = 0$

First of all, we give the explicit formulae for the functional derivatives of σ_n w.r.t. a and B and the functional derivative of τ_n w.r.t. a:

$$\frac{\sigma_n \overleftarrow{\partial}}{\partial \mathsf{B}} = (-1)^{\left[\frac{n(n-1)}{2}+1\right]m+1}\pi_*^{n+1}\left\langle\!\!\left\langle \mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi \,;\, \frac{\mathsf{B}_{n+1}\overleftarrow{\partial}}{\partial \mathsf{B}}\right\rangle\!\!\right\rangle; \tag{5.5.4}$$

$$\frac{\overrightarrow{\partial}\sigma_n}{\partial \mathsf{a}} = \sum_{k=1}^{n}(-1)^{\frac{n(n-1)}{2}m+n}\pi_*^{n+1}\left\langle\!\!\left\langle \mathrm{ad}\,(\mathsf{a}_{n+1})\,\eta_{n,n+1}\cdots\mathrm{ad}\left(\frac{\overrightarrow{\partial}\mathsf{a}_k}{\partial \mathsf{a}}\right)\eta_{k,k+1}\cdots\Xi \,;\, \mathsf{B}_{n+1}\right\rangle\!\!\right\rangle; \tag{5.5.5}$$

$$\frac{\overrightarrow{\partial}\tau_n}{\partial \mathsf{a}} = \pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\left(\frac{\overrightarrow{\partial}\mathsf{a}_n}{\partial \mathsf{a}}\right)\eta_{n1}\right]. \tag{5.5.6}$$

The computations leading to these formulae are straightforward, using the definitions of the functional derivatives w.r.t. a and B; one has to be careful in regard to the sign rules involved, keeping track of the total degrees involved. E.g. we compute the left functional derivative of τ_n w.r.t. a:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\tau_n\left(\mathsf{a}+t\rho_{\mathsf{a}}\right) &= \sum_{k=1}^{n}\frac{1}{n}\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(\rho_{\mathsf{a}})_k\,\eta_{k,k+1}\cdots\right] = \\
&= \pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(\rho_{\mathsf{a}})_n\,\eta_{n1}\right] = \\
&= \pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\left(\int_{\mathbb{R}^m}\left\langle\!\!\left\langle \rho_{\mathsf{a}}\,;\,\frac{\overrightarrow{\partial}\mathsf{a}_n}{\partial \mathsf{a}}\right\rangle\!\!\right\rangle\right)\eta_{n1}\right] = \\
&= \int_{\mathbb{R}^m}\left\langle\!\!\left\langle \rho_{\mathsf{a}}\,;\,\pi_*^n\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\left(\frac{\overrightarrow{\partial}\mathsf{a}_n}{\partial \mathsf{a}}\right)\eta_{n1}\right]\right\rangle\!\!\right\rangle.
\end{aligned}$$

The second equality is the result of a cyclic permutation of the points in $C_n\left(\mathbb{R}^{m-2}\right)$; the third follows directly by the definition of functional derivative.

Recalling the main properties of the super BV Laplacian $\mathbf{\Delta}$, we compute $\mathbf{\Delta}\sigma_n$:

$$
\begin{aligned}
\mathbf{\Delta}\sigma_n &= (-1)^{\frac{n(n-1)}{2}m+n}\mathbf{\Delta}\left[\pi_*^{n+1}\langle\!\langle\, \mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots \mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi\,;\,\mathsf{B}_{n+1}\,\rangle\!\rangle\right] = \\
&= (-1)^{\left[\frac{n(n-1)}{2}+(n+1)\right]m+n}\pi_*^{n+1}\left\{\mathbf{\Delta}\left[\langle\!\langle\, \mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots \mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi\,;\,\mathsf{B}_{n+1}\,\rangle\!\rangle\right]\right\} = \\
&= (-1)^{\frac{n(n-1)}{2}m+n}\pi_*^{n+1}\left[\int_{\mathbb{R}^m}\left\langle\!\!\left\langle \frac{\mathsf{B}_{n+1,\alpha}\overleftarrow{\partial}}{\partial\mathsf{B}}\,;\,\frac{\overrightarrow{\partial}\langle\!\langle\, \mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots\Xi\,;\,\xi^\alpha\,\rangle\!\rangle}{\partial\mathsf{a}}\right\rangle\!\!\right\rangle\right] = \\
&= \sum_{k=1}^n(-1)^{\left[\frac{n(n-1)}{2}+(n-k)\right]m+n}\pi_*^{n+1}\left\{\left[\int_{\mathbb{R}^m}\left\langle\!\!\left\langle \frac{\mathsf{B}_{n+1,\alpha}\overleftarrow{\partial}}{\partial\mathsf{B}}\,;\,\frac{\overrightarrow{\partial}\mathsf{a}_k^\beta}{\partial\mathsf{a}}\right\rangle\!\!\right\rangle\right]\right. \\
&\qquad\qquad\qquad \left.\langle\!\langle\, \mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots \mathrm{ad}\,(X_\beta)\,\eta_{k,k+1}\cdots\Xi\,;\,\xi^\alpha\,\rangle\!\rangle\right\} = \\
&= \sum_{k=1}^n(-1)^{\left[\frac{n(n-1)}{2}+(n-k+1)\right]m+(n+1)}\pi_*^{n+1}\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{n+1}\,;\,\mathsf{a}_k\,\rangle\!\rangle\right)\right. \\
&\qquad \left.\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_n)\,\eta_{n,n+1}\cdots \mathrm{ad}\,(\mathsf{a}_{k+1})\,\eta_{k+1,k+2}\,\mathrm{ad}\,(\eta_{k,k+1}\,\mathrm{ad}\,(\mathsf{a}_{k-1})\,\eta_{k-1,k}\cdots\Xi)\right]\right\}.
\end{aligned}
\tag{5.5.7}
$$

By e^α, resp. ε_α, we have denoted the α-th element of a chosen basis of $\mathfrak{g}$, resp. the corresponding dual basis vector; in the above computations, we have used that the super BV Laplacian formally vanishes on functionals which do not contain both a field and the corresponding antifield.

On the other hand, we consider the super BV antibracket of σ_p and τ_q:

$$
\begin{aligned}
\langle\!\langle\,\sigma_p\,;\,\tau_q\,\rangle\!\rangle &= (-1)^{\left[\frac{p(p-1)}{2}+(p+1)(q+1)+1\right]m+p}\pi_*^{p+1}\pi_*^q\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+1}\,;\,\mathsf{a}_q\,\rangle\!\rangle\right)\right. \\
&\qquad \left.\langle\!\langle\,\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+1}\cdots\Xi\,;\,\xi^\alpha\,\rangle\!\rangle\,\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\mathrm{ad}\,(X_\alpha)\,\eta_{q1}\right]\right\} = \\
&= (-1)^{\left[\frac{p(p-1)}{2}+q\right]m+p}\pi_*^{p+1}\pi_*^q\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+1}\,;\,\mathsf{a}_q\,\rangle\!\rangle\right)\right. \\
&\qquad \left.\mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\cdots\eta_{q-1,q}\,\mathrm{ad}\,(\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+1}\cdots\Xi)\,\eta_{q1}\right]\right\} = \\
&= (-1)^{\left[\frac{p(p-1)}{2}+pq\right]m+p}\pi_*^{p+q+1}\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+1}\,;\,\mathsf{a}_{p+q+1}\,\rangle\!\rangle\right)\right. \\
&\qquad \mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_{p+2})\,\eta_{p+2,p+3}\cdots\mathrm{ad}\,(\mathsf{a}_{p+q})\,\eta_{p+q,p+q+1}\eta_{p+q+1,p+2}\right. \\
&\qquad\qquad \left.\left.\mathrm{ad}\,(\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+1}\cdots\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi)\right]\right\} = \\
&= (-1)^{\left[\frac{p(p-1)}{2}+pq+\frac{(q-1)(q-2)}{2}\right]m+p}\pi_*^{p+q+1}\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+1}\,;\,\mathsf{a}_{p+q+1}\,\rangle\!\rangle\right)\right. \\
&\qquad \mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_{p+q})\,\eta_{p+q,p+q-1}\cdots\mathrm{ad}\,(\mathsf{a}_{p+2})\,\eta_{p+2,p+q+1}\eta_{p+q+1,p+q}\right. \\
&\qquad\qquad \left.\left.\mathrm{ad}\,(\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+1}\cdots\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi)\right]\right\} = \\
&= (-1)^{\left[\frac{p(p-1)}{2}+(pq+1)+\frac{(q-1)(q-2)}{2}\right]m+p+q+1}\pi_*^{p+q+1}\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+1}\,;\,\mathsf{a}_{p+q+1}\,\rangle\!\rangle\right)\right. \\
&\qquad \mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_{p+q})\,\eta_{p+q,p+q+1}\cdots\mathrm{ad}\,(\mathsf{a}_{p+2})\,\eta_{p+2,p+3}\eta_{p+q+1,p+2}\right. \\
&\qquad\qquad \left.\left.\mathrm{ad}\,(\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+1}\cdots\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi)\right]\right\} = \\
&= (-1)^{\left[\frac{(p+q)(p+q+1)}{2}+q+1\right]m+(p+q+1)}\pi_*^{p+q+1}\left\{\mathbf{\Delta}\left(\langle\!\langle\,\mathsf{B}_{p+q+1}\,;\,\mathsf{a}_{p+1}\,\rangle\!\rangle\right)\right. \\
&\qquad \mathrm{Tr}\left[\mathrm{ad}\,(\mathsf{a}_{p+q})\,\eta_{p+q,p+1}\cdots\mathrm{ad}\,(\mathsf{a}_{p+2})\,\eta_{p+2,p+3}\eta_{p+1,p+2}\right. \\
&\qquad\qquad \left.\left.\mathrm{ad}\,(\mathrm{ad}\,(\mathsf{a}_p)\,\eta_{p,p+q+1}\cdots\mathrm{ad}\,(\mathsf{a}_1)\,\eta_{12}\Xi)\right]\right\}.
\end{aligned}
$$

The first equality follows from the definition of the super BV antibracket and of the super BV Laplacian, from equation (5.5.4) and (5.5.6), from equation (3.3.11) and once again since the BV Laplacian formally vanishes on functionals not containing a field and the corresponding antifield. We obtain the second identity by inserting the functional $\langle\!\langle\, \mathrm{ad}\,(\mathsf{a}_p)\, \eta_{p,p+1} \cdots \Xi\, ;\, \xi^\alpha \,\rangle\!\rangle$ in the trace at the q-th position. The third identity follows by casting together all integrations over $C_{p+1}\big(\mathbb{R}^{m-2}\big)$ and $C_q\big(\mathbb{R}^{m-2}\big)$, recalling the orientation conventions on configuration spaces. The fourth identity is a result of the application of the permutation

$$\begin{pmatrix} 1 & \cdots & p+1 & p+2 & p+3 & \cdots & p+q-1 & p+q & p+q+1 \\ 1 & \cdots & p+1 & p+q & p+q-1 & \cdots & p+3 & p+2 & p+q+1 \end{pmatrix}$$

in $\mathfrak{S}_{p+q+1}$ on $C_{p+q+1}\big(\mathbb{R}^{m-2}\big)$. The fifth identity is a consequence of moving all η-propagators in the middle expression by one position and successively permuting the positions of their arguments. Finally, the last identity is obtained by applying the permutation exchanging the points $p+q+1$ and $p+1$.

The crucial fact now is the presence of the (pull-back of the) super BV Laplacian of $\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle$, where π_i, $i=1,2$, are the natural projections from $C_2(\mathbb{R}^m)$ onto its factors. It is possible to obtain an explicit expression for $\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle$ by straightforward computations similar to those for the BV Laplacian of the canonical BF action in Subsection 3.4.3:

$$\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle\right) = (-1)^{m-1} \dim \mathfrak{g}\; \delta(x-y) \left[\sum_{k=0}^{m} \sum_{\substack{1\leq\mu_1<\dots \\ <\mu_{m-k}\leq m}} (-1)^k \mathrm{d}x_{\mu_1,\dots,\mu_{m-k}} \wedge \star\left(\mathrm{d}y_{\mu_1\dots\mu_{m-k}}\right) \right], \tag{5.5.8}$$

where $\mathrm{d}x_{\mu_1,\dots,mu_k} :\, = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$. In equation (5.5.8), δ denotes the usual Dirac δ-distribution concentrated at 0.

Thus, $\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle\right)$ is a distributional form on $\mathbb{R}^m \times \mathbb{R}^m$; moreover, a simple computation shows that (up to the multiplicative constant $(-1)^{m-1}\dim\mathfrak{g}$) (5.5.8) is the distributional kernel of the identity operator on forms with compact support or rapid decrease on $\mathbb{R}^m$, i.e.

$$\pi_{2*}\left[\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle\right) \wedge \pi_1^*\omega\right] = (-1)^{m-1} \dim\mathfrak{g}\, \omega, \quad \forall \omega \in \Omega^*\left(\mathbb{R}^m\right), \tag{5.5.9}$$

where now $\Omega^*\left(\mathbb{R}^m\right)$ means forms on $\mathbb{R}^m$ with compact support or rapid decrease. Clearly, the distributional forms $\boldsymbol{\Delta}\left(\langle\!\langle\, \mathsf{B}_{n+1}\, ;\, \mathsf{a}_k \,\rangle\!\rangle\right)$ and $\boldsymbol{\Delta}\left(\langle\!\langle\, \mathsf{B}_{p+q+1}\, ;\, \mathsf{a}_{p+1} \,\rangle\!\rangle\right)$ are pull-backs resp. of $\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle\right)$ w.r.t. the composition of the evaluation map on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times \mathbb{R}^{m-2}$ with the projections from $C_{n+1}\big(\mathbb{R}^{m-2}\big)$ and $C_{p+q+1}\big(\mathbb{R}^{m-2}\big)$ onto $C_2\big(\mathbb{R}^{m-2}\big)$ on the $n+1$-th and k-th point, resp. on the $p+q+1$-th and $p+1$-th point.

The main idea is that, if we consider Σ^i for $i = in\,\{0,-1\}$, which correspond to a function, resp. a 1-form, on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, the (pull-back w.r.t. evaluation map and respective projections of) $\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\, ;\, \pi_2^*\mathsf{a} \,\rangle\!\rangle\right)$ has to be a form of degree at most one in $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ in both cases by the presence of push-forwards along the configuration spaces.

Since the pull-back of $\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\ ;\ \pi_2^*\mathsf{a}\,\rangle\!\rangle\right)$ w.r.t. the composition of the evaluation map and projections from some $C_n\left(\mathbb{R}^{m-2}\right)$ into any two factors is a form of degree m on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \times C_n\left(\mathbb{R}^{m-2}\right)$, with at most a form component in $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ and the presence of the Dirac distribution localizes the two $\mathbb{R}^{m-2}$-arguments of the distributional form on the diagonal of $\mathbb{R}^{m-2} \times \mathbb{R}^{m-2}$, which is isomorphic to $\mathbb{R}^{m-2}$, the pull-back of $\boldsymbol{\Delta}\left(\langle\!\langle\, \pi_1^*\mathsf{B}\ ;\ \pi_2^*\mathsf{a}\,\rangle\!\rangle\right)$ automatically vanishes since it has degree bigger than the dimension of the space where it is defined. The claim follows now by the relationship between super BV antibracket and usual BV antibracket, and super BV Laplacian and usual BV Laplacian.

As for the antibracket of σ with itself, we write the result of the computations, which are analogous to the computations leading to the explicit expression for the super BV antibracket of σ_p with τ_q, recalling equations (5.5.4) and (5.5.5):

$$
\begin{aligned}
\int_{\mathbb{R}^m} \left\langle\!\!\left\langle \frac{\sigma_p \overleftarrow{\partial}}{\partial \mathsf{B}}\ ;\ \frac{\overrightarrow{\partial}\sigma_q}{\partial \mathsf{a}} \right\rangle\!\!\right\rangle = \sum_{k=1}^{q} &(-1)^{\left[\frac{p(p-1)}{2}+\frac{q(q-1)}{2}+(p+1)q+k\right]m+(p+q)} \\
&\pi_*^{p+q+2}\,\langle\!\langle \boldsymbol{\Delta}\left(\langle\!\langle\, \mathsf{B}_{p+q+2}\ ;\ \mathsf{a}_{p+k}\,\rangle\!\rangle\right) \mathrm{ad}\left(\mathsf{a}_{p+q}\right) \eta_{p+q,p+q+1} \cdots \\
&\cdots \mathrm{ad}\left(\mathsf{a}_{k+p+1}\right) \eta_{k+p+1,k+p+2}\eta_{k+p,k+p+1} \\
&[\![\mathrm{ad}\left(\mathsf{a}_{k+p-1}\right) \eta_{k+p-1,p+q+2} \cdots \mathrm{ad}\left(\mathsf{a}_{k}\right) \eta_{k,k+1}\Xi; \\
&\mathrm{ad}\left(\mathsf{a}_{k-1}\right) \eta_{k-1,k+p} \cdots \mathrm{ad}\left(\mathsf{a}_{1}\right) \eta_{12}\Xi]\!];\mathsf{B}_{p+q+1}\rangle\!\rangle\,.
\end{aligned}
\tag{5.5.10}
$$

Restricting to the pieces of σ_p and σ_q of ghost number 0 and -1, we can repeat the same formal argument we have used above to evaluate the super BV Laplacian of σ and the super BV antibracket of σ with τ. ☐

The two preceding theorems imply immediately that the piece of $\exp \frac{\mathrm{i}}{\hbar}\sigma - \tau$ of ghost number 0 is a BV-observable modulo d-exact forms, or, alternatively, that its expectation value is a locally constant function (up to anomalies) on the space $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, yielding potential invariants of long knots in $\mathbb{R}^m$.

Chapter 6

The perturbative expansion

6.1 The perturbative expansion

In the previous section we have introduced the BV superversion of the observable Σ, which was the correct way to interpret, as a perturbative series, partition function of the $\mathcal{I}$-action for special imbeddings from $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$. As we have already seen, the piece of ghost number 0 of Σ is a BV observable; therefore, we may explicitly evaluate the v.e.v. of the exponential of Σ w.r.t. the canonical BF-action as a perturbative series, and since the exponential of Σ is a BV observable, the perturbative series (which needs a gauge-fixing condition in order to invert the quadratic part of the BF-action) does not depend on the gauge-fixing.

We begin by rewriting the exponential of Σ in another shape. Since a takes value in $\mathfrak{g}$ and B in $\mathfrak{g}^*$, we may choose a basis of $\mathfrak{g}$ and its dual basis, which we denote by $\{X_1, \dots, X_d\}$, resp. $\{\xi^1, \dots, \xi^d\}$ (d is the dimension of $\mathfrak{g}$), and hence we may write a and B as

$$\mathsf{a} = \mathsf{a}^\alpha X_\alpha, \quad \mathsf{B} = \mathsf{B}_\alpha \xi^\alpha.$$

As always, a summation over repeated indices is understood.

We may then rewrite σ_n and τ_n as follows:

$$\sigma_n = (-1)^{(m-1)\frac{n(n-1)}{2}+n} \pi_*^{n+1} \left(\mathsf{a}_1^{\alpha_1} \cdots \mathsf{a}_n^{\alpha_n} \mathsf{B}_\beta^{n+1} \prod_{i=1}^{n} \eta_{i,i+1} \right)$$
$$\left\langle \xi^\beta , \operatorname{ad}(X_{\alpha_n}) \cdots \operatorname{ad}(X_{\alpha_1}) \Xi \right\rangle;$$
$$\tau_n = \frac{1}{n}(-1)^{(m-3)\frac{n(n-1)}{2}} \pi_*^n \left(\mathsf{a}_1^{\alpha_1} \cdots \mathsf{a}_n^{\alpha_n} \prod_{\substack{i=1: \\ n+1\equiv 1}}^{n} \eta_{i,i+1} \right) \operatorname{Tr}\left[\operatorname{ad}(X_{\alpha_1}) \cdots \operatorname{ad}(X_{\alpha_n})\right].$$

Analogously, the trivalent part S_{triv} of the BV action for canonical BF theories may be written as

$$S_{\text{triv}} := \frac{1}{2} \int_{\mathbb{R}^m} \langle\!\langle \mathsf{B} \,;\, [\![\mathsf{a} \,;\, \mathsf{a}]\!] \rangle\!\rangle = \frac{1}{2} \left(\int_{\mathbb{R}^m} \mathsf{B}_\varepsilon \mathsf{a}^\eta \mathsf{a}^\theta \right) \langle \xi^\varepsilon , [X_\eta , X_\theta] \rangle .$$

For computational reasons, it is better to rescale the superfields a and B by the square root of $\frac{\mathrm{i}}{\hbar}$; the rescaling modifies σ, τ and the trivalent part of the BV action as follows

$$\frac{\mathrm{i}}{\hbar}\,\sigma_n \mapsto \left(\frac{\hbar}{\mathrm{i}}\right)^{\frac{n-1}{2}}\,\sigma_n; \quad \tau_n \mapsto \left(\frac{\hbar}{\mathrm{i}}\right)^{\frac{n}{2}}\,\tau_n; \quad S_{\text{triv}} \mapsto \left(\frac{\hbar}{\mathrm{i}}\right)\,S_{\text{triv}}.$$

With all these preliminaries, we may write the exponential of $\frac{\mathrm{i}}{\hbar}\,S_{\text{triv}}$ and Σ as follows:

$$\begin{aligned}
&\exp\frac{\mathrm{i}}{\hbar}\,S_{\text{triv}}\,\exp\Sigma = \sum_{p,q,r\geq 0}\prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{k=1}^{r}\sum_{n_i\geq 0}\sum_{m_k\geq 3}\frac{1}{p!q!r!}\frac{1}{2^q}\frac{1}{m_k}\left(\frac{\hbar}{\mathrm{i}}\right)^{\frac{-p+q+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k}{2}}\\
&(-1)^{\left[\sum_{i=1}^{p}\frac{n_i(n_i-1)}{2}+\sum_{k=1}^{r}\frac{m_k(m_k-1)}{2}+r\right](m-1)+\sum_{i=1}^{p}n_i}\left[\int_{\mathbb{R}^m}\mathsf{B}_{\varepsilon_j}\,\mathsf{a}^{\eta_k}\,\mathsf{a}^{\theta_k}\right]\\
&\left[\pi_*^{n_i+1}\left(\mathsf{a}_1^{\alpha_1^i}\cdots\mathsf{B}_{\beta_i}^{n_i+1}\prod_{l=1}^{n_i}\eta_{l,l+1}\right)\right]\left[\pi_*^{m_k}\left(\mathsf{a}_1^{\alpha_1^k}\cdots\mathsf{a}_{m_k}^{\alpha_{m_k}^k}\prod_{\substack{k=1\\ m_k+1\equiv 1}}^{m_k}\eta_{j,j+1}\right)\right]\\
&\langle\,\xi^{\varepsilon_j}\,,\,[\,X_{\eta_k}\,,\,X_{\theta_k}\,]\,\rangle\left\langle\,\xi^{\beta_i}\,,\,\mathrm{ad}\left(X_{\alpha_{n_i}^i}\right)\cdots\mathrm{ad}\left(X_{\alpha_1^i}\right)\Xi\right\rangle\mathrm{Tr}\left[\mathrm{ad}\left(X_{\alpha_1^k}\right)\cdots\mathrm{ad}\left(X_{\alpha_{m_k}^k}\right)\right].
\end{aligned} \tag{6.1.1}$$

First of all, we give a closer look to the term containing powers of $\frac{\hbar}{\mathrm{i}}$: the exponent is $\frac{-p+q+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k}{2}$. Since we are looking for nontrivial Feynman diagrams and since the number of a and B superfields are directly linked to p, q, r, n_i and m_k, we may establish a condition for possibly nontrivial Feynman diagrams depending on p, q, r, n_i and m_k.

Namely, the number of B-superfields equals $q+p$, while the number of a-superfields equals $2q+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k$. Therefore, possible nontrivial Feynman diagrams are characterized by the following equation

$$q+p = 2q+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k \Longrightarrow p = q+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k. \tag{6.1.2}$$

Therefore, if p, q, r, n_i and m_k do not satisfy equation (6.1.2), the corresponding v.e.v. vanishes. Moreover, it is immediate to see that the corresponding power of $\frac{\hbar}{\mathrm{i}}$ vanishes.

We refer to the order of a possibly nontrivial Feynman diagram as to p.

6.1.1 Feynman rules

Now that we have established a condition for possibly nontrivial Feynman diagrams coming from the perturbative expansion of the exponential of Σ w.r.t. the canonical BF-theory, we want to perform explicit computations. In order to do so, we have first to display the Feynman rules for the superpropagator and for the vertices.

As we have discussed previously, the v.e.v. of the exponential of Σ w.r.t. the BF-action does not depend on the gauge-fixing condition we have chosen to evaluate the quadratic part of the BV action. We have chosen a generalization of the covariant gauge-fixing condition in 3 dimensions; as we have already seen, with this choice of gauge-fixing, the superpropagator on $\mathbb{R}^m$ is simply

$$\left\langle \pi_1^* \left(\mathsf{B}_\alpha\right)\ \pi_2^* \left(\mathsf{a}^\beta\right)\right\rangle_{\text{g.f.}} = \theta_{12}\ \delta_\alpha^\beta, \tag{6.1.3}$$

where π_i denotes the projection from $\mathbb{R}^m \times \mathbb{R}^m$ onto the i-th component; δ_α^β denotes the usual Kronecker δ-function. By θ_{12} we have denoted the pull-back on the configuration space $C_2^0(\mathbb{R}^m)$ of two points in $\mathbb{R}^m$ of the normalized, $SO(m)$-invariant top-form v on S^{m-1} by the map (2.4.5); v is even, resp. odd, w.r.t. the action of the antipodal map on S^{m-1}. The superpropagator $\left\langle \pi_1^* \left(\mathsf{B}_\alpha\right)\ \pi_2^* \left(\mathsf{a}^\beta\right)\right\rangle_{\text{g.f.}}$ is clearly ill-defined on the diagonal of $\mathbb{R}^m \times \mathbb{R}^m$, thus it needs a regularization. The regularization we have chosen is to replace the open configuration space $C_2^0(\mathbb{R}^m)$ by its Fulton-MacPherson–Axelrod–Singer compactification $C_2(\mathbb{R}^m)$; since, by Theorem 2.4.6, the map (2.4.5) extends smoothly from $C_2^0(\mathbb{R}^m)$ to $C_2(\mathbb{R}^m)$, the BF theory can be regularized (a posteriori) by passing from the (at first sight) obvious space $\mathbb{R}^m \times \mathbb{R}^m$ on which the superpropagator lives to the compactification $C_2(\mathbb{R}^m)$.

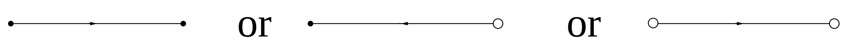

Figure 6.1: The θ-propagator

In Figure 6.1.1 we have included the three cases when the θ-propagator has $a)$ two internal vertices, $b)$ an internal and an external vertex and $c)$ two external vertices. An internal vertex is depicted as a small dot, while an external vertex is depicted as a bigger circle.

After having defined the superpropagator, we now give the Feynman rules for the vertices.

In general, points in $\mathbb{R}^{m-2}$ and $\mathbb{R}^m$ will be called from now on *vertices*. Vertices in $\mathbb{R}^{m-2}$, resp. in $\mathbb{R}^m$, are called *internal*, resp. *external*. From now on, we will call the superpropagator θ-propagator. It is clear from equation (6.1.1) that nontrivial Feynman diagrams are products of θ- and η-propagators.

The *valence* of a vertex (be it internal or external) is defined as the number of θ- and η-propagators meeting at the given vertex.

We can read directly from equation (6.1.1) that an internal vertex can only be:

i) univalent joined by a θ-form;

ii) bivalent joined by exactly one θ-form and exactly one η-form;

iii) trivalent joined by exactly one θ-form and two η-propagators.

On the other hand, an inspection of equation (6.1.1) gives the rule for external vertices:

i) An external vertex can only be trivalent joined by three θ-propagators.

These are the Feynman rules that we need to start an explicit evaluation of the v.e.v. of the exponential of Σ.

Pictorially, an internal vertex is denoted by a dot, while an external one is denoted by a bigger circle. As we have seen before, η-propagators are depicted by dashed (oriented) lines, while θ-propagators are depicted by solid (oriented) lines. The orientation of a propagator is directly linked to the dimension m: a propagator has to be oriented if and only m is odd (this depends on the parity of the propagators w.r.t. the antipodal map on S^{m-3} and S^{m-1}).

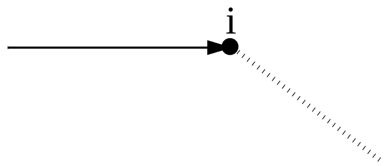

Figure 6.2: A univalent internal i vertex joined by a θ-propagator; the dashed line coming out from the internal vertex i denotes the Lie-algebraic coefficient Ξ

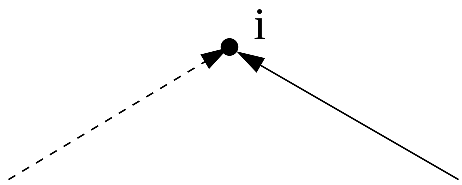

Figure 6.3: A bivalent internal vertex i joined by a θ- and an η-propagator

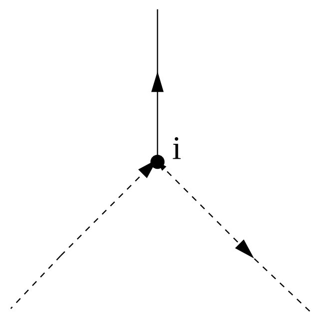

Figure 6.4: A trivalent internal vertex i joined by one θ- and two η-propagators

The order of a possibly nontrivial Feynman diagram is given by equation (6.1.2). For p, q, r, n_i and m_k given, satisfying equation (6.1.2), the number of η-propagators is exactly $\sum_{i=1}^{p} n_i + \sum_{k=1}^{r} m_k$. Since there must be exactly $p+q$ θ-propagators, it follows that the total number number of θ- and η-propagators for a possibly nontrivial

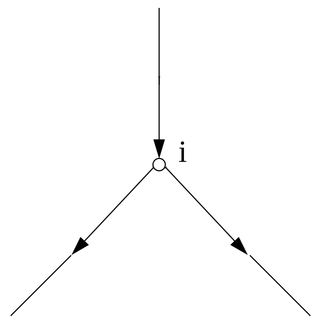

Figure 6.5: A trivalent external vertex i joined by three θ-propagators

Feynman diagram of order p is exactly

$$(p+q)+\sum_{i=1}^{p} n_i + \sum_{k=1}^{r} m_k \overset{\text{By equation (6.1.2)}}{=} 2p.$$

Therefore, a possibly nontrivial Feynman diagram of order p has exactly $2p$ θ- and η-propagators.

We discuss next two particular situations which cannot occur in the perturbative expansion. A crucial rôle in the derivation of these obstructions is played by the Lie-algebraic coefficients of S_{triv}, σ and τ.

First, we consider equation (6.1.1), and we single out two particular terms, namely $\pi_*^1\left(\mathsf{B}_\beta^1\right)$ and $\pi_*^{n_i+1}\left(\mathsf{a}_1^{\alpha_1^i}\cdots\mathsf{B}_{\beta_i}^{n_i+1}\prod_{l=1}^{n_i}\eta_{l,l+1}\right)$, for $n_i \geq 1$. We assume that we couple through the superpropagator (6.1.3) B_β^1 to $\mathsf{a}_1^{\alpha_1^i}$; because of the definition of the superpropagator, the index β must be equal to α_1^i. We inspect the corresponding Lie-algebraic coefficients, under the assumptions $\beta=\alpha_1^i$

$$\left\langle \xi^\beta\,,\,\Xi\right\rangle\left\langle \xi^{\beta_i}\,,\,\mathrm{ad}\left(X_{\alpha_{n_i}^i}\right)\cdots\mathrm{ad}\left(X_\beta\right)\Xi\right\rangle=\left\langle \xi^{\beta_i}\,,\,\mathrm{ad}\left(X_{\alpha_{n_i}^i}\right)\cdots\mathrm{ad}\left(\Xi\right)\Xi\right\rangle=0,$$

by the antisymmetry of the Lie-bracket.

Similarly we consider again equation (6.1.1). This time, we single out two terms $\pi_*^1\left(\mathsf{B}_{\beta_1}^1\right)$ and $\pi_*^1\left(\mathsf{B}_{\beta_2}^1\right)$ and $\int_{\mathbb{R}^m}\mathsf{B}_{\varepsilon_j}\mathsf{a}^{\eta_k}\mathsf{a}^{\theta_k}$ (of course, the index 1 in the first two integrals are not the same). We assume that we have coupled e.g. $\mathsf{B}_{\beta_1}^1$ to a_η and $\mathsf{B}_{\beta_2}^1$ to a_θ; hence, the indices β_1 and η, and β_2 and θ have to be equal. Hence, the corresponding Lie-algebraic coefficients become

$$\left\langle \xi^{\beta_1}\,,\,\Xi\right\rangle\left\langle \xi^{\beta_2}\,,\,\Xi\right\rangle\left\langle \xi^{\varepsilon_j}\,,\,[\,X_{\beta_1}\,,\,X_{\beta_2}\,]\right\rangle=\left\langle \xi^{\varepsilon_j}\,,\,[\,\Xi\,,\,\Xi\,]\right\rangle=0,$$

again by antisymmetry of the Lie-bracket.

If we translate these two results in a more formal language, the corresponding Feynman rules are as follows

i) No Feynman diagram can contain a bivalent internal vertex with an η-form and a θ-form meeting there, such that the θ-form connects the given vertex to a univalent internal vertex.

ii) No Feynman diagram can contain a trivalent external vertex, such that two of the three θ-propagators meeting there connect the external vertex to two distinct univalent internal vertices.

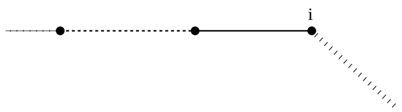

Figure 6.6: The Feynman rule $i)$; the dashed line coming out from the internal vertex i means the presence of the Lie-algebraic term Ξ

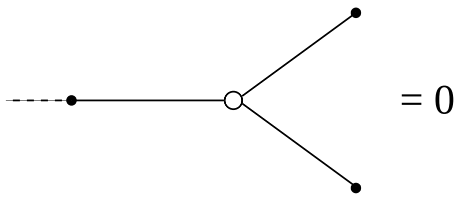

Figure 6.7: The Feynman rule $ii)$

Remark 6.1.1. In fact, the second Feynman rule has also another explanation: if we consider the involution exchanging the two univalent internal vertices, we see that this involution (which is orientation-preserving, resp. -reversing, if m is even, resp. odd) maps to corresponding Feynman diagram to minus itself, in the same spirit of the vanishing lemmata of Subsection 6.5.1.

6.1.2 The presence of "tadpoles"

As one can immediately get from equation (6.1.1) and from the Feynman rules for the superpropagator and the vertices, there is another situation which can occur, which we have not yet discussed, namely the appearance of so-called "tadpoles". We consider an interaction term coming from the trivalent piece of the BV action $\frac{1}{2}\int_{\mathbb{R}^m} \mathsf{B}_\varepsilon \mathsf{a}^\eta \mathsf{a}^\theta$; the corresponding Lie algebraic coefficient is simply $\langle\, \xi^\varepsilon\,,\, [\, X_\eta\,,\, X_\theta\,]\,\rangle$.

A tadpole is the result of the operation connecting the B-form in the given interaction term to one of the two a-forms in the same interaction term. Clearly, the result is ill-defined, as the arguments of the superpropagator coincide. The corresponding Lie-algebraic coefficient takes the form

$$\mathrm{Tr}\left[\mathrm{ad}\left(X_\gamma\right)\right],$$

where γ is the renamed index connected to the surviving a-form. We may regularize the result in the following way: we take a smooth, normalized vector field ν on $\mathbb{R}^m$ (hence, ν is everywhere nonvanishing), and then consider the limit

$$\overline{\theta}_\nu(x) \colon = \lim_{\varepsilon \downarrow 0} \theta_{12}(x + \varepsilon\nu(x), x). \tag{6.1.4}$$

If we view ν as a map from $\mathbb{R}^m$ to itself, then $\overline{\theta}_\nu$ is simply the pull-back w.r.t. ν of v. Of course, the regularized θ-form $\overline{\theta}_\nu$ depends clearly explicitly on a choice of a normalized vector field ν.

There are two ways of eliminating tadpoles. One is to take e.g. a unimodular Lie algebra $\mathfrak{g}$, which has the property that the adjoint action is trace-free. Therefore, the Lie-algebraic coefficient of a regularized Feynman diagram containing a tadpole is zero, hence also the Feynman diagram.

Another way, which does not require particular assumptions on the Lie algebra $\mathfrak{g}$, is to add to the BV action an additional piece, whose presence is to provide counterterms for tadpoles. Since the B-form may be connected to any of the two a-forms, the coefficient $\frac{1}{2}$ before the interaction is cancelled when we consider the tadpole. It turns out that the counterterm that does the job is given by

$$S_{\text{c.t.}} \colon = \int_{\mathbb{R}^m} \operatorname{Tr}\left[\operatorname{ad}(\mathsf{a})\,\overline{\theta}_\nu\right] = -\left(\int_{\mathbb{R}^m} \mathsf{a}^\alpha \overline{\theta}_\nu\right) \operatorname{Tr}\left[\operatorname{ad}(X_\alpha)\right], \tag{6.1.5}$$

where $\overline{\theta}_\nu$ is the same as in equation (6.1.4).

We add this term to the BV action; we have only to check that the exponential of $S_{\text{c.t.}}$ is a BV quantum observable. Clearly, the BV Laplacian of $S_{\text{c.t.}}$ vanishes as well as the BV antibracket of $S_{\text{c.t.}}$ with itself, since $S_{\text{c.t.}}$ depends only on a, hence it cannot contain a field and the corresponding antifield.

On the other hand, we compute the super BV differential of $S_{\text{c.t.}}$:

$$\begin{aligned}
\boldsymbol{\delta} S_{\text{c.t.}} &= \int_{\mathbb{R}^m} \operatorname{Tr}\left[\operatorname{ad}(\mathsf{F_A})\,\overline{\theta}_\nu\right] = \\
&= \int_{\mathbb{R}^m} \operatorname{Tr}\left[\operatorname{ad}(\mathrm{d}\mathsf{a})\,\overline{\theta}_\nu\right] + \frac{1}{2}\int_{\mathbb{R}^m} \operatorname{Tr}\left[\operatorname{ad}(\llbracket \mathsf{a} \,;\, \mathsf{a} \rrbracket)\,\overline{\theta}_\nu\right] = \\
&= \int_{\mathbb{R}^m} \mathrm{d}\operatorname{Tr}\left[\operatorname{ad}(\mathsf{a})\,\overline{\theta}_\nu\right] + \frac{1}{2}\int_{\mathbb{R}^m} \operatorname{Tr}\left[\operatorname{ad}(\mathsf{a})\operatorname{ad}(\mathsf{a})\,\overline{\theta}_\nu\right] = \\
&= 0,
\end{aligned}$$

since $\overline{\theta}_\nu$ is closed and by Stokes' Theorem and by the cyclicity of the trace.

Hence, the corrected BV action $S + S_{\text{c.t.}}$ satisfies also the Quantum Master Equation, and we then take the v.e.v. of the piece of ghost number 0 of the exponential of Σ w.r.t. the same gauge-fixing condition as before, the only difference being that we have to include in the interaction term also the counterterm for tadpoles. Hence, equation

(6.1.1 becomes

$$\exp\left(\frac{\mathrm{i}}{\hbar}\,S_{\text{triv}}-(-1)^m S_{\text{c.t.}}\right)\,\exp\Sigma=\sum_{p,q,r,s\geq 0}\prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{k=1}^{r}\prod_{l=1}^{s}\sum_{n_i\geq 0}\sum_{m_k\geq 3}\frac{1}{p!q!r!s!}\frac{1}{2^q}\frac{1}{m_k}$$
$$\left(\frac{\hbar}{\mathrm{i}}\right)^{\frac{-p+q+s+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k}{2}}(-1)^{\left[\sum_{i=1}^{p}\frac{n_i(n_i-1)}{2}+\sum_{k=1}^{r}\frac{m_k(m_k-1)}{2}+r+s\right](m-1)+\sum_{i=1}^{p}n_i}$$
$$\left[\int_{\mathbb{R}^m}\mathsf{B}_{\varepsilon_j}\,\mathsf{a}^{\eta_k}\mathsf{a}^{\theta_k}\right]\left[\int_{\mathbb{R}^m}\mathsf{a}^{\gamma_l}\overline{\theta}_\nu\right]\left[\pi_*^{n_i+1}\left(\mathsf{a}_1^{\alpha_1^i}\cdots\mathsf{B}_{\beta_i}^{n_i+1}\prod_{l=1}^{n_i}\eta_{l,l+1}\right)\right]$$
$$\left[\pi_*^{m_k}\left(\mathsf{a}_1^{\alpha_1^k}\cdots\mathsf{a}_{m_k}^{\alpha_{m_k}^k}\prod_{\substack{k=1\\ m_k+1\equiv 1}}^{m_k}\eta_{j,j+1}\right)\right]\operatorname{Tr}\left[\operatorname{ad}\left(X_{\gamma_l}\right)\right]\left\langle\xi^{\varepsilon_j}\,,\,\left[\,X_{\eta_k}\,,\,X_{\theta_k}\,\right]\right\rangle$$
$$\left\langle\xi^{\beta_i}\,,\,\operatorname{ad}\left(X_{\alpha_{n_i}^i}\right)\cdots\operatorname{ad}\left(X_{\alpha_1^i}\right)\Xi\right\rangle\operatorname{Tr}\left[\operatorname{ad}\left(X_{\alpha_1^k}\right)\cdots\operatorname{ad}\left(X_{\alpha_{m_k}^k}\right)\right]. \tag{6.1.6}$$

Clearly, equation (6.1.2) has also to be modified:

$$p=q+s+\sum_{i=1}^{p}n_i+\sum_{k=1}^{r}m_k. \tag{6.1.7}$$

Hence, possible nontrivial Feynman diagrams are characterized by p, q, r, s, n_i and m_k satisfying equation (6.1.7), and we still refer to p as to the order of such a diagram. As before, for such diagrams, the exponent of $\frac{\hbar}{\mathrm{i}}$ vanishes also.

We may summarize all the above computations in the following

Proposition 6.1.2. *The v.e.v. of the exponential of Σ w.r.t. the renormalized BF action by the counterterm $S_{\text{c.t.}}$ does not contain tadpoles.*

6.2 Feynman diagrams of order 1

We want to compute all possibly nontrivial Feynman diagrams of order 1. equation (6.1.2) takes the form, for $p=1$,

$$q+n_1+\sum_{k=1}^{r}m_k=1.$$

Since all m_k's all bigger than 3, r has to zero, if equation (6.1.2) is satisfied. Since by Proposition 6.1.2 we can discard tadpoles, it is not difficult to see that there is only one possible solutions, namely $(1;0;0)$.

Recalling equation (6.1.1), the solution $(1;0;0)$ corresponds to

$$-\pi_*^2\left(\mathsf{a}_1^{\alpha_1}\mathsf{B}_\beta^2\eta_{12}\right)\,\left\langle\xi^\beta\,,\,\operatorname{ad}\left(X_{\alpha_1}\right)\Xi\right\rangle;$$

taking the superpropagator between B^2_β and $\mathsf{a}_1^{\alpha_1}$, recalling equation (6.1.3), we get

$$\left(\int_{C_{2,0}} \theta_{12}\eta_{12}\right) \, \mathrm{Tr}\left[\mathrm{ad}\,(\Xi)\right]. \tag{6.2.1}$$

We see immediately that a choice of an e.g. unimodular Lie algebra $\mathfrak{g}$ is not good, because such a choice eliminates automatically (6.2.1).

The integral $\int_{C_{2,0}} \theta_{12}\eta_{12}$ is represented by the Feynman diagram

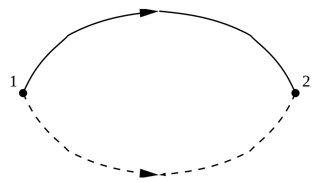

Figure 6.8: The Feynman diagram of order 1

First of all, we consider on $C_{2,0}$ the permutation of the two internal vertices:

$$\int_{C_{2,0}} \theta_{12}\eta_{12} \overset{1\leftrightarrow 2}{=} (-1)^m \int_{C_{2,0}} \theta_{21}\eta_{21} = (-1)^m \int_{C_{2,0}} \theta_{12}\eta_{12},$$

where we made use of the parity of v and w w.r.t. the antipodal maps on S^{m-1} and S^{m-3}. Hence, if m is odd, the function Θ_1 vanishes automatically; we assume therefore m even.

We compute the exterior derivative of Θ_1. Since η- and θ-propagators are closed, it follows

$$\mathrm{d}\Theta_1 = -\int_{\partial C_{2,0}} \overline{\theta}_{12}\overline{\eta}_{12},$$

where $\overline{\eta}$ and $\overline{\theta}$ denote the restriction of η- and θ-propagators to the codimension-1 boundary $\partial C_{2,0}$ of $C_{2,0}$.

The boundary $\partial C_{2,0}$ has three codimension-1 faces, namely $i)$ when both internal vertices collapse together, $ii)$ when one of the two vertices escapes to infinity and $iii)$ when both vertices escape to infinity.

We will deal first with the three boundary faces at infinity. When one of the two vertices escapes to infinity, the corresponding contribution vanishes by means of Lemma 6.5.9, since in both cases the θ-propagator has exactly one vertex escaping to infinity. When both vertices escape to infinity, Lemma 6.5.9 yields the vanishing of the corresponding contribution.

Finally, by Lemma 6.5.13, we consider the $m-1$-form on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ corresponding to the contribution of the boundary face where both internal vertices collapse together:

$$\widehat{\Theta}_1 := \int_{S^{m-3}} \widehat{\theta}\, w, \tag{6.2.2}$$

where $\widehat{\theta}$ denotes the pull-back of v w.r.t. the map from $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \times S^{m-3}$ to S^{m-1}

$$(\alpha; x) \mapsto \frac{\alpha(x)}{\|\alpha(x)\|}.$$

(the *normalized tangent map*, borrowing notation from [17]). Since $\widehat{\theta}$ and w are both closed, $\widehat{\Theta}_1$ is also closed by the generalized Stokes Theorem.

6.3 Feynman diagrams of 2: the Bott invariant

In this subsection, we compute and discuss the term of order 2 of the perturbative expansion. We will obtain a function on $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$, which is the sum of three Feynman diagrams. We will omit the explicit computations leading to the three Feynman diagrams, referring to the Feynman rules. We will but notice that we do not consider neither diagrams with tadpoles (which do not contribute to the perturbative expansion thanks to the counterterm (6.1.5)) nor non-connected diagrams; also, we will consider only the configuration space integrals corresponding to the given Feynman diagrams with their signs and numerical coefficients without the Lie-algebraic coefficient, which for all connected diagrams without tadpoles of order 2, equals $\mathrm{Tr}\left[\mathrm{ad}\,(\Xi)^2\right]$.

We show first that the diagrams in question vanish in even dimension, as a consequence of the action of the permutation groups $\mathfrak{S}_3$ and $\mathfrak{S}_4$ on configuration spaces. Furthermore, using the action of $\mathfrak{S}_4$ on configuration spaces, the sum of the three diagrams may be rewritten as a sum of two integrals containing cyclic sums of η-propagators, so that we can identify it with the invariant constructed by Bott in [9] for $2m-1$-spheres imbedded in $\mathbb{R}^{2m+1}$. Finally, we prove rigorously that this function is indeed an invariant, making use of various vanishing lemmata.

6.3.1 The explicit Feynman diagrams of order 2

We give first explicit expressions for the three Feynman diagrams at order 2:

$$\mathcal{D}_{2,\text{tr.}} := \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23}; \tag{6.3.1}$$

$$\mathcal{D}_{2,\text{sq.}} := \frac{1}{2}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34}; \tag{6.3.2}$$

$$\mathcal{D}_{2,\text{sq.ext.}} := \int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12}. \tag{6.3.3}$$

We first show that all functions (6.3.1), (6.3.2) and (6.3.3) vanish if m is even.

Taking (6.3.1), we consider the permutation of the points labeled by 1 and 3:

$$\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23} = (-1)^m \int_{C_{4,0}} \theta_{31}\theta_{24}\eta_{32}\eta_{21} = (-1)^{m+3m+(m-3)} \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23} =$$
$$= (-1)^{m-1} \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23}.$$

As for (6.3.2), we consider the permutation exchanging the point labeled by 1 with the one labeled by 2 and the point labeled by 3 with the one labeled by 4. This permutation obviously preserves orientation for m even and odd. Therefore, we get

$$\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} = \int_{C_{4,0}} \theta_{24}\theta_{13}\eta_{21}\eta_{43} = (-1)^{(m-1)+4m}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} = \\ = (-1)^{m-1}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34}.$$

Finally, taking (6.3.3), we consider the cyclic permutation exchanging the points labeled by 1 and 2; it is orientation-preserving, resp. -reversing, for m even, resp. odd. It follows

$$\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12} = (-1)^m\int_{C_{3,1}} \theta_{24}\theta_{14}\theta_{34}\eta_{21} = (-1)^{m-1}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12}.$$

We consider the sum of (6.3.1), (6.3.2) and (6.3.3)

$$\Theta_2 \colon = \int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23} + \frac{1}{2}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} + \int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12}. \tag{6.3.4}$$

Hence, the function Θ_2 is nonzero only in odd dimensions. Using again the action of $\mathfrak{S}_4$ and $\mathfrak{S}_3$ on the configuration spaces C_4 and $C_{3,1}$, we may rewrite (6.3.4) in the form

$$\Theta_2 = \frac{1}{8}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{1234}^2 + \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{123}, \tag{6.3.5}$$

where η_{1234} and η_{123} are the following cyclic sums:

$$\eta_{1234} \colon = \eta_{12} + \eta_{23} + \eta_{34} + \eta_{41}; \quad \eta_{123} \colon = \eta_{12} + \eta_{23} + \eta_{31}.$$

The function (6.3.5) may be then viewed as a function on the space $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, for m odd.

The function (6.3.5) takes the diagrammatic form

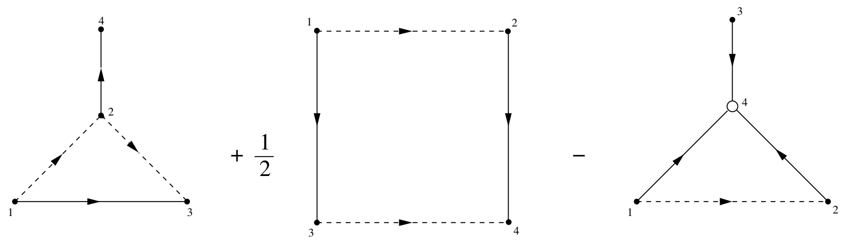

Figure 6.9: The Feynman diagram corresponding to the Bott invariant

6.3.2 The function Θ_2 is an invariant

The task now is to compute the exterior derivative of (6.3.4).

At this point, it is better to make a change in the tautological form θ_{12} on $C_2(\mathbb{R}^m)$. By definition, θ_{12} is the pull-back to $C_2(\mathbb{R}^m)$ w.r.t. the map (2.4.5) of a normalized, $SO(m)$-invariant top-form v on S^{m-1}, which is even, resp. odd, w.r.t. the antipodal map on S^{m-1}. Instead of v, we will consider the top-form $\overline{v}\colon = (-1)^m v$; clearly, $\overline{v}$ is again $SO(m)$-invariant and has the same parity w.r.t. the antipodal map on S^{m-1}; since m has to be odd, we replace v by minus itself in Θ_2. We get with this change

$$\begin{aligned}\Theta_2 &= \frac{1}{8}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{1234}^2 - \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{123} = \\ &= \int_{\partial C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{23} + \frac{1}{2}\int_{\partial C_{4,0}} \theta_{13}\theta_{24}\eta_{12}\eta_{34} - \int_{\partial C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{12}.\end{aligned}$$

By the generalized Stokes Theorem, the exterior derivative of Θ_2 equals:

$$\begin{aligned}\mathrm{d}\Theta_2 &= -\frac{1}{8}\int_{\partial C_{4,0}} \iota_{\partial_{4,0}}^*\left(\theta_{13}\theta_{24}\eta_{1234}^2\right) + \frac{1}{3}\int_{\partial C_{3,1}} \iota_{\partial_{3,1}}^*\left(\theta_{14}\theta_{24}\theta_{34}\eta_{123}\right) = \\ &= -\int_{\partial C_{4,0}} \iota_{\partial_{4,0}}^*\left(\theta_{13}\theta_{24}\eta_{12}\eta_{23}\right) - \frac{1}{2}\int_{\partial C_{4,0}} \iota_{\partial_{4,0}}^*\left(\theta_{13}\theta_{24}\eta_{12}\eta_{34}\right) + \\ &\quad + \int_{\partial C_{3,1}} \iota_{\partial_{3,1}}^*\left(\theta_{14}\theta_{24}\theta_{34}\eta_{12}\right);\end{aligned}$$

here, $\partial C_{4,0}$, resp. $\partial C_{3,1}$, denotes the codimension-1 boundary of $C_{4,0}$, resp. $C_{3,1}$. re

The contribution of the principal faces

By a principal face, we mean a codimension-1 boundary face of $C_{4,0}$, resp. $C_{3,1}$, describing the collapse of two internal vertices, resp. of two internal vertices or of the only external vertex to one of the three internal vertices.

Lemma 6.3.1. *The sum of the contribution of the principal faces coming from the exterior derivative of (6.3.4) vanishes.*

Proof. By Lemma 6.5.1, the only possibly nontrivial contributions from principal faces of $C_{4,0}$, resp. $C_{3,1}$, are associated to the collapse of two internal vertices, resp. of two internal vertices or of the only external vertex to exactly one of the three internal vertices, connected through a θ- or η-form.

Using the normalization and the parity w.r.t. the antipodal map on S^{m-3} and S^{m-1} of the θ- and η-propagator, Convention (2.4.23) and the action of permutation groups on $C_{3,0}$, the result is

- $2\int_{C_{3,0}} \theta_{12}\theta_{13}\eta_{12} + \int_{C_{3,0}} \overline{\theta}_1\theta_{23}\eta_{12}\eta_{13}$, coming from the integral (6.3.1);
- $\int_{C_{3,0}} \theta_{12}\theta_{13}\eta_{23} - \int_{C_{3,0}} \overline{\theta}_1\theta_{23}\eta_{12}\eta_{13}$, coming from the integral (6.3.2);
- $-\int_{C_{3,0}} \theta_{12}\theta_{13}\eta_{23} - 2\int_{C_{3,0}} \theta_{12}\theta_{13}\eta_{12}$, coming from the integral (6.3.3.

It is not difficult to verify that the sum of the three contributions is 0.

We need to explain what is $\overline{\theta}_1$ in the above formulae. We consider, borrowing the notations from Subsubsection 6.5.4, the following form on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$:

$$\int_{S^{m-3}} \widehat{\theta}. \tag{6.3.6}$$

By Lemma 6.5.13, this form descends to an invariant form on $\mathrm{Gr}_{m,m-2}$ of degree 2; such forms exist in any dimension m. We then denote by $\overline{\theta}_1$ the pull-back to $C_{3,0}$ of (6.3.6) w.r.t. the composition of the normalized tangent map with projection onto the first component of $C_{3,0}$. $\square$

However, there are other codimension-1 contributions to consider, coming from so-called "hidden faces". These are boundary faces corresponding to the collapse of more than 2 vertices (be they internal or external or both) or to the escape of one or more internal and/or external vertices to infinity. It may be shown that these contributions all vanish, hence making the function (6.3.5) a true invariant. The proof of the vanishing of the hidden faces will be postponed in Section 6.5.1.

6.4 Feynman diagrams of order 3

In this section we give an explicit expression for the term of order 3 coming from the perturbative expansion of the v.e.v. of Σ w.r.t. BF-theories. As for the Feynman diagrams of order 2, we do not write the computations leading to the result; the computations are easy consequences of the displayed Feynman rules. However, it is worth noticing that we consider here connected diagrams without tadpoles, and that the Lie-algebraic coefficients of all these diagrams equal $\mathrm{Tr}\left[\mathrm{ad}\,(\Xi)^3\right]$. Again, the reason for not considering diagrams with tadpoles lies in the presence of (6.1.5).

The section is divided in two subsections: in the first one, we write down the connected configuration space integrals without tadpoles coming from the Feynman diagrams of order 3 and we show that they are (possibly) nontrivial only in even dimension, and we rewrite their sum so that it contains cyclic alternating sums of η-propagators. In the second subsection, we show that the sum of the contributions coming from principal faces vanishes.

6.4.1 The eight connected Feynman diagrams without tadpoles of order 3

We get by a direct computation, recalling the Feynman rules, the following eight Feynman diagrams, with their signs and combinatorial coefficients:

$$\mathcal{D}_{3,\text{hex.}} := \frac{1}{3}\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{34}\eta_{56}; \tag{6.4.1}$$

$$\mathcal{D}_{3,\text{pent.}} := \int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{45}; \tag{6.4.2}$$

$$\mathcal{D}_{3,\text{sq.}} := -\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{34}; \tag{6.4.3}$$

$$\mathcal{D}_{3,\text{tr.}} := \frac{1}{3}\int_{C_{6,0}} \theta_{14}\theta_{25}\theta_{36}\eta_{12}\eta_{23}\eta_{31}; \tag{6.4.4}$$

$$\mathcal{D}_{3,\text{sq.ext}} := -\int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{23}; \tag{6.4.5}$$

$$\mathcal{D}_{3,\text{pent.ext.}} := \int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{34}; \tag{6.4.6}$$

$$\mathcal{D}_{3,\text{sq.2 ext.}} := -\int_{C_{4,2}} \theta_{16}\theta_{36}\theta_{56}\theta_{25}\theta_{45}\eta_{12}; \tag{6.4.7}$$

$$\mathcal{D}_{3,\text{tr.3 ext.}} := \frac{1}{3}\int_{C_{3,3}} \theta_{14}\theta_{25}\theta_{36}\theta_{45}\theta_{46}\theta_{56}. \tag{6.4.8}$$

We consider the sum of the eight connected Feynman diagrams (6.4.1), (6.4.2), (6.4.3), (6.4.4), (6.4.5), (6.4.6), (6.4.7) and (6.4.8) :

$$\begin{aligned}\Theta_3 := \frac{1}{3}\int_{C_{6,0}} &\theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{34}\eta_{56} + \int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{45} - \\ &- \int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{34} + \frac{1}{3}\int_{C_{6,0}} \theta_{14}\theta_{25}\theta_{36}\eta_{12}\eta_{23}\eta_{31} - \\ &- \int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{23} + \int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{34} - \\ &- \int_{C_{4,2}} \theta_{16}\theta_{36}\theta_{56}\theta_{25}\theta_{45}\eta_{12} + \frac{1}{3}\int_{C_{3,3}} \theta_{14}\theta_{25}\theta_{36}\theta_{45}\theta_{46}\theta_{56}.\end{aligned} \tag{6.4.9}$$

We show that all configuration space integrals in (6.4.9) vanish in even dimension; for this purpose, we use the action of permutation groups on configuration spaces. We write down explicitly the permutations which we use in order to show the vanishing of the integrals; the signs are consequences of the parity rules of θ- and η-propagators and of the fact that v, resp. w, has degree $m-1$, resp. $m-3$.

Taking the first integral, we consider the permutation exchanging 1 with 2, 4 with 6 and 3 with 5; such an involution of $C_{6,0}$ is orientation preserving, resp. reversing, if m is even, resp. odd.

For the second integral, we consider the permutation exchanging 1 with 3 and 4 with 5, which is orientation-preserving in both cases m even and odd.

For the third integral, we consider the permutation exchanging 1 with 4, 2 with 3 and 5 with 6, which has the same orientation behavior as the permutation for the first integral.

Taking the fourth integral, we consider the orientation-preserving permutation exchanging 1 with 3 and 4 with 6.

Taking the fifth integral, we consider the orientation-preserving permutation exchanging 1 with 3 and 2 with 4.

For the sixth integral, we consider the permutation exchanging 1 with 3; such a permutation is orientation-preserving, resp. -reversing, if m is even, resp. odd.

For the seventh integral, we consider the permutation exchanging 1 with 2, 3 with 4 and 5 and 6, which has clearly the same orientation behavior as the permutation used for the first integral, since we consider imbeddings of codimension 2.

Taking the eighth integral, we consider the orientation-preserving permutation exchanging 2 with 3 and 5 with 6.

Finally, the function Θ_3 is represented by the following Feynman diagram:

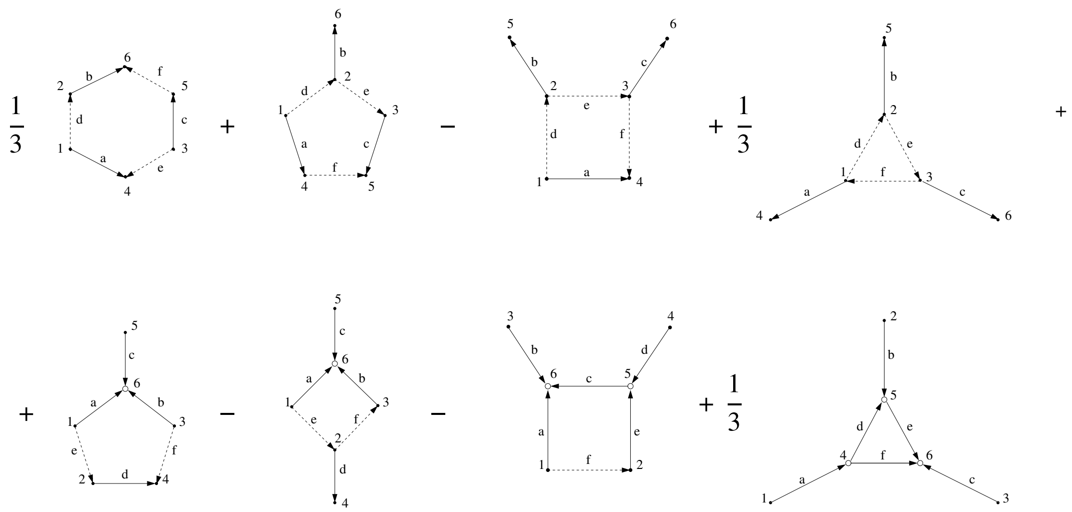

Figure 6.10: The Feynman diagrams of order 3

6.4.2 The contribution of the principal faces to the exterior derivative of θ_3

In this subsection, we compute the contributions coming from the principal faces of compactified configuration spaces to the exterior derivatives of (6.4.9).

Lemma 6.4.1. *The sum of the contributions to the exterior derivative of (6.4.9) coming from principal faces vanishes.*

Proof. By Lemma 6.5.1, we have to consider only the principal faces where two vertices collapse together only if they are connected by a θ- or η-form.

We compute the sum of the contributions for each integral separately, recalling the orientation convention (2.4.23) and the normalization conditions on v and w:

- The first integral

$$-\int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{23}\eta_{45} + \int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{13}\eta_{45};$$

- the second integral

$$- 2\int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{12}\eta_{35} - \int_{C_{5,0}} \theta_{14}\theta_{25}\theta_{34}\eta_{12}\eta_{23}+$$
$$+ 2\int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{23}\eta_{15} - \int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{13}\eta_{45};$$

- the third integral

$$2\int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{12}\eta_{23} - \int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{23}\eta_{31}-$$
$$- 2\int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{23}\eta_{15};$$

- the fourth integral

$$\int_{C_{5,0}} \overline{\theta}_1\theta_{24}\theta_{35}\eta_{12}\eta_{23}\eta_{31}$$

- the fifth integral

$$- 2\int_{C_{4,1}} \theta_{15}\theta_{25}\theta_{45}\theta_{13}\eta_{12} + \int_{C_{4,1}} \theta_{15}\theta_{35}\theta_{45}\overline{\theta}_2\eta_{12}\eta_{23}-$$
$$- 2\int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{12}\eta_{23} + \int_{C_{5,0}} \theta_{14}\theta_{25}\theta_{34}\eta_{12}\eta_{23};$$

- the sixth integral

$$2\int_{C_{4,1}} \theta_{15}\theta_{25}\theta_{45}\theta_{13}\eta_{23} - \int_{C_{4,1}} \theta_{15}\theta_{35}\theta_{45}\overline{\theta}_2\eta_{12}\eta_{23}+$$
$$+ 2\int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{12}\eta_{35} + \int_{C_{5,0}} \theta_{13}\theta_{14}\theta_{25}\eta_{23}\eta_{45};$$

- the seventh integral

$$\int_{C_{3,2}} \theta_{15}\theta_{25}\theta_{45}\theta_{14}\theta_{34} + 2\int_{C_{4,1}} \theta_{15}\theta_{25}\theta_{45}\theta_{13}\eta_{12}-$$
$$- 2\int_{C_{4,1}} \theta_{15}\theta_{25}\theta_{45}\theta_{13}\eta_{23};$$

- the eighth integral

$$-\int_{C_{3,2}} \theta_{15}\theta_{25}\theta_{45}\theta_{14}\theta_{34}.$$

(We notice that we have used the same notations as in the proof of Lemma 6.3.1.)

It is immediate to verify that the sum of all these contributions vanishes. ☐

6.5 Vanishing lemmata for hidden faces of Θ_2 and Θ_3

In this section, we will give an explicit proof of the vanishing of all hidden faces of the Bott invariant Θ_2; for this purpose, we need some useful well-known vanishing lemmata, which we quote from [17], plus other (as far as we know) new vanishing lemmata, which we state and prove in the next subsection.

6.5.1 Vanishing lemmata for hidden faces

If we take a diagram Γ and a boundary face labeled by $S \subset \{1,2,3,4\}$, we denote by $\Gamma|_{(S)}$ the push-forward along the fiber of the boundary fibration corresponding to the face S of those propagators which are not basic w.r.t. the boundary fibration (i.e. those propagators with both endpoints in S). Sometimes, we will call $\Gamma|_{(S)}$, for a given set S, the *collapsing diagram* w.r.t. S.

Vanishing lemmata for the collapse of vertices

The following vanishing lemmata deal with boundary faces corresponding to the collapse of vertices labeled by elements of S; the situation of vertices escaping to infinity has to be dealt with in a different way in the next subsubsection.

By the arguments of Subsection 2.4.4 of Chapter 2, the interior of the boundary face corresponding to the collapse of the vertices labeled by S is given by a pull-back of the trivial bundle $\mathbb{R}^{m-2} \times \left(\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}_S\right)$, where $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}_S$ is defined in (2.4.10) of Subsection 2.4.4. Using Lemma 6.5.13, $\Gamma|_{(S)}$ is the pull-back w.r.t. the composition of the projection from the base of the boundary fibration onto the vertex, where the vertices labeled by elements of S collapse, with the map (2.4.13) of a form on $\mathbb{R}^{m-2} \times \mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$. This form is in turn a pull-back w.r.t. the projection on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ of the push-forward w.r.t. the projection from $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}_S$ onto the space $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ of a product of tautological forms $\overline{\theta}_{ij}$ and $\overline{\eta}_{ij}$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}_S$, as defined in Subsubsection 6.5.4.

In the proofs of some of the following lemmata, $\Gamma|_{(S)}$ denotes the form on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, whose pull-back w.r.t. the above map yields the true form denoted by $\Gamma|_{(S)}$.

With these prescriptions in mind, we state three basic general vanishing lemmata.

Lemma 6.5.1. *If $\Gamma|_{(S)}$ has a zerovalent internal or external vertex (in the second case, we assume that the labeled vertex is not the point at infinity), then it vanishes.*

Lemma 6.5.2. *If $\Gamma|_{(S)}$ has $a)$ either a univalent internal vertex joined by an η-propagator or $b)$ a univalent external vertex joined by a θ-propagator, then it vanishes.*

Lemma 6.5.3. *If $\Gamma|_{(S)}$ has $a)$ either a bivalent internal vertex joined by two η-propagators or $b)$ a bivalent external vertex joined by two θ-propagators, then it vanishes.*

The proofs of the preceding lemmata are completely equivalent to the proofs of Lemma A.7, Lemma A.8 and Lemma A.9 in Appendix A of [17], where these lemmata were proved only for external points with θ-propagators. The case of internal points with η-propagators is identical, because it represents the same situation.

Remark 6.5.4. Lemma 6.5.3 is Kontsevich's Lemma from [40], or, to be more precise, a variant thereof adapted to the situation at hand. What is borrowed from the original version is Kontsevich's involution, which is the main ingredient of the proof.

We provide now new vanishing lemmata, which concern certain "bad" faces of the Bott invariant and of function (6.4.9). The idea behind these lemmata lies mainly in a generalization of Kontsevich's Lemma (or, to be more precise, in a generalization of the involution displayed by Kontsevich in order to deal with trivalent vertices in his construction of invariant of knots and 3-manifolds).

Lemma 6.5.5. *If the integrand of* $\Gamma|_{(S)}$ *contains a product of forms of the type* $a)$ *either* $\theta_{ij}\eta_{jk}\theta_{kl}$ *or* $b)$ $\eta_{ij}\theta_{jk}\eta_{kl}$*, where* i, j, k, l *are all distinct internal vertices and* j *and* k *are exactly bivalent, then it vanishes.*

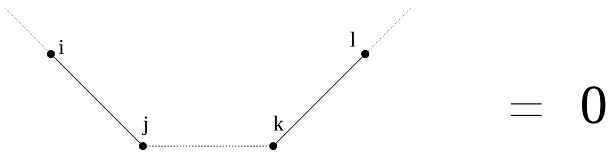

Figure 6.11: The pictorial form of Lemma 6.5.5 for the case $a)$

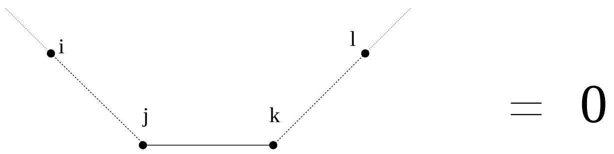

Figure 6.12: The pictorial form of Lemma 6.5.5 for the case $b)$

Proof. We notice that in the case of η-propagators, i and j must be both internal.

We consider e.g. the situation $a)$. In this case, the integrand corresponding to $\Gamma|_{(S)}$ contains, by assumption, a product of the form

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl},$$

where the four points i, j, k and l are all distinct and internal.

The interior of the fiber corresponding to the face labeled by S is the space $\widehat{C}^0_{p,q}$ introduced in Subsection 2.4.4; we denote a point in this quotient by

$$[(\alpha; \dots, x_j, \dots, x_k, \dots,)], \tag{6.5.1}$$

where we have ordered the points, and we have labeled only two of the four special points x_i, x_j, x_k and x_l, which are all internal; α denotes an element of $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$, corresponding to the tangent map of a given imbedding at the point in $\mathbb{R}^{m-2}$, where the vertices labeled by S have collapsed.

We may define on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$ the following map

$$\phi_{ijkl}\colon\ (\alpha;\dots,x_j,\dots,x_k,\dots) \longmapsto (\alpha;\dots,x_i+x_l-x_k,\dots,x_i+x_l-x_j,\dots)\,, \tag{6.5.2}$$

and all other points remain unaltered. The map (6.5.2) is clearly an involution, and the sign of its orientation is simply $(-1)^{m-2}=(-1)^m$. It is also not difficult to see that it descends to an involution of the quotient $\widehat{C}^0_{p,r}$ of $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes C^0_{p,q}$, since (6.5.2) is equivariant w.r.t. the action of scalings and global translations. In fact, the action of scalings and global translations on all internal points is given by

$$x \longmapsto \lambda x + \xi, \quad \forall \lambda \in \mathbb{R}^+, \xi \in \mathbb{R}^k.$$

(The action of global translations by vectors in $\mathbb{R}^{m-2}$ on external points is a little bit different, as it involves translations by $\alpha(\xi)$.) Therefore, we see

$$x_j \longmapsto \lambda x_j + \xi \longmapsto (\lambda x_i+\xi)+(\lambda x_l+\xi)-(\lambda x_k+\xi) = \lambda(x_i+x_l-x_k)+\xi,$$

and an analogous computation for x_k shows that the map (6.5.2) is equivariant w.r.t. the action of global translations and scalings. Hence, (6.5.2) descends to a smooth involution on the fiber of the face S, with orientation sign $(-1)^{m-2}$.

We have to compute its action on the integrand of $\Gamma|_{(S)}$. Since ϕ_{ijkl} affects only the j-th and k-th point, it does affect only those factors of the integrand which depend explicitly on x_j and x_k. Such factors are by assumption exactly $\overline{\theta}_{ij}$, $\overline{\eta}_{jk}$ and $\overline{\theta}_{kl}$. The action of ϕ_{ijkl} on these forms is computed by seeing how ϕ_{ijkl} affects the maps by which we pull-back the forms θ and η.

The form $\overline{\theta}_{ij}$ is realized via the (smooth lift of the) map

$$[(\alpha;\dots,x_j,\dots,x_k,\dots)] \longmapsto \frac{\alpha\,(x_i-x_j)}{\|\alpha\,(x_i-x_j)\|}.$$

Analogously, $\overline{\theta}_{kl}$ is realized via

$$[(\alpha;\dots,x_j,\dots,x_k,\dots)] \longmapsto \frac{\alpha\,(x_k-x_k)}{\|\alpha\,(x_k-x_l)\|}.$$

On the other hand, the form $\overline{\eta}_{jk}$ is realized via the (smooth lift of the) map

$$[(\alpha;\dots,x_j,\dots,x_k,\dots)] \longmapsto \frac{x_j-x_k}{\|x_j-x_k\|}.$$

The first map is transformed via the map (6.5.2) as follows:

$$\begin{aligned}
[(\alpha;\dots,x_j,\dots,x_k,\dots)] \longmapsto [(\alpha;\dots,x_i+x_l-x_k,\dots,x_i+x_l-x_j,\dots)] \longmapsto \\
\longmapsto \frac{\alpha\,(x_i-(x_i+x_l-x_k))}{\|\alpha\,(x_i-(x_i+x_l-x_k))\|} = \\
= \frac{\alpha\,(x_k-x_l)}{\|\alpha\,(x_k-x_l)\|}.
\end{aligned}$$

Analogously, we get for the second map:

$$[(\alpha;\ldots,x_j,\ldots,x_k,\ldots)]\longmapsto\frac{\alpha\,(x_i-x_j)}{\|\alpha\,(x_i-x_j)\|}.$$

On the other hand, the third map is transformed as follows:

$$\begin{aligned}[(\alpha;\ldots,x_j,\ldots,x_k,\ldots)]&\longmapsto[(\alpha;\ldots,x_i+x_l-x_k,\ldots,x_i+x_l-x_j,\ldots)]\longmapsto\\&\longmapsto\frac{(x_i+x_l-x_k)-(x_i+x_l-x_j)}{\|(x_i+x_l-x_k)-(x_i+x_l-x_j)\|}=\\&=\frac{x_j-x_k}{\|x_j-x_k\|}.\end{aligned}$$

Accordingly, the product $\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl}$ transforms as follows:

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl}\overset{\text{By (6.5.2)}}{\longmapsto}\overline{\theta}_{kl}\overline{\eta}_{jk}\overline{\theta}_{ij}=(-1)^{m-1}\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl};$$

we have used here the fact that θ-propagators have degree $m-1$, and the η-propagators have degree $m-3$.

Therefore, the form $\Gamma|_{(S)}$ is transformed by (6.5.2) as follows:

$$\Gamma|_{(S)}\longmapsto(-1)^{m+m-1}\Gamma|_{(S)}=-\Gamma|_{(S)}.$$

Hence, the involution (6.5.2) transforms the integral into its opposite, whence it follows $\Gamma|_{(S)}$.

Analogous arguments work for situation $b)$. □

Lemma 6.5.6. *If the integrand of $\Gamma|_{(S)}$ contains a product $\theta_{ij}\theta_{kl}$, where the four labeled vertices are all distinct and internal, and if at least two of them are univalent and do not belong to the same θ-propagator, then it vanishes.*

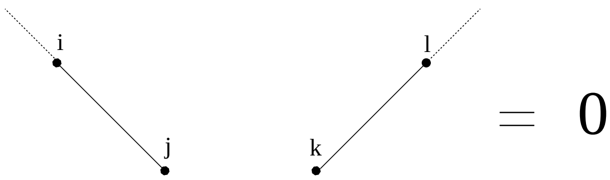

Figure 6.13: The diagrammatic version of Lemma 6.5.6

Proof. We use the same notations as in the previous proof.

By assumption, there are at least four internal points in the fiber of the boundary face S, and the integrand in $\Gamma|_{(S)}$ contains a factor of the form

$$\overline{\theta}_{ij}\overline{\theta}_{kl}.$$

The points x_i, x_j, x_k and x_l are all distinct. We assume that the vertices labeled by i and k are univalent.

On the interior of the boundary face $\widehat{C}^0_{p,q}$, corresponding to the collapsing of all vertices labeled by S, we consider the map

$$[(\alpha; \dots, x_i, \dots, x_k, \dots)] \longmapsto [(\alpha; \dots, x_j + x_l - x_k, \dots, x_j + x_l - x_i)], \quad (6.5.3)$$

where the other points remain unaltered. The map (6.5.3) is an involution of the fiber of the hidden face S, it has orientation sign $(-1)^m$, and, since the only points changed by (6.5.3) are the i-th and k-th point, the only factors in the integrand affected by (6.5.3) are $\overline{\theta}_{ij}$ and $\overline{\theta}_{kl}$. Computations similar to those used at the end of the proof of the preceding Lemma yield:

$$\overline{\theta}_{ij}\overline{\theta}_{kl} \overset{\text{By (6.5.3)}}{\longmapsto} \overline{\theta}_{lk}\overline{\theta}_{ji} = (-1)^{m-1}\overline{\theta}_{ij}\overline{\theta}_{kl}.$$

We have used the fact that θ-propagators have degree $m-1$, and that θ is even,resp. odd, w.r.t. the antipodal map if m is even, resp. odd.

The total action of the involution (6.5.3) on the form $\Gamma|_{(S)}$ is therefore

$$\Gamma|_{(S)} \longmapsto (-1)^{m+m-1}\Gamma|_{(S)} = -\Gamma|_{(S)}.$$

The claim then follows. □

Lemma 6.5.7. *If the integrand of $\Gamma|_{(S)}$ contains of product of forms of type $a)$ $\theta_{ij}\eta_{kj}\eta_{lj}$, where all four vertices $\{i,j,k,l\}$ are internal and the vertex labeled by i is univalent, or $b)$ $\theta_{ie}\theta_{je}\theta_{ke}$, where the vertices labeled by $\{i,j,k,\}$ are internal and the vertex labeled by e is external, and moreover the vertex i is univalent, then $\Gamma|_{(S)}$ vanishes.*

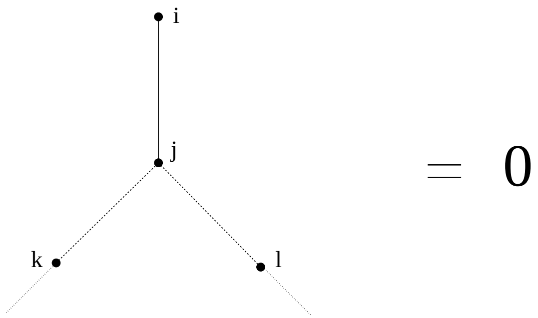

Figure 6.14: The diagrammatic version of Lemma 6.5.7 for case $a)$

Proof. We consider first the case $a)$. We make use again of an involution, reminiscent of Kontsevich's one, discovered by D. Thurston in [49], which we have modified and adapted to a more general situation.

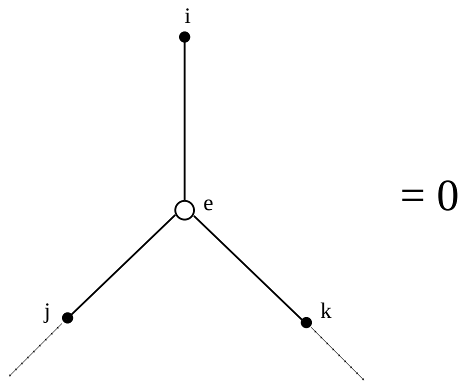

Figure 6.15: The diagrammatic version of Lemma 6.5.7 for case $b)$

In fact, assuming the vertices $\{i, j, k, l\}$ to be lexicographically ordered, we consider the map on the interior of the boundary face $\widehat{C}^0_{p,q}$:

$$\psi_{ijkl}\left([(\alpha; \dots, x_i, \dots, x_j, \dots)]\right) := [(\alpha; \dots, x_k + x_l - x_i, \dots, x_k + x_l - x_j, \dots)],$$

leaving all other vertices unaltered. It is immediate to check that the map ψ_{ijkl} is a well-defined orientation preserving involution.

It remains to compute the action of ψ_{ijkl} on the integrand of $\Gamma|_{(S)}$; since ψ_{ijkl} affects only the vertices x_i and x_j, it is sufficient to compute its action on the product $\theta_{ij}\eta_{kj}\eta_{lj}$. The form θ_{ij} is the pull-back w.r.t. the map

$$[(\alpha; \dots, x_i, \dots, x_j, \dots)] \mapsto \frac{\alpha\,(x_j - x_i)}{\|\alpha\,(x_j - x_i)\|}.$$

The composition of the preceding map with ψ_{ijkl} gives

$$\begin{aligned}
[(\alpha; \dots, x_i, \dots, x_j, \dots)] &\mapsto [(\alpha; \dots, x_k + x_l - x_i, \dots, x_k + x_l - x_j, \dots)] \mapsto \\
&\mapsto \frac{\alpha\,(x_k + x_l - x_j - x_k - x_l + x_i)}{\|\alpha\,(x_k + x_l - x_j - x_k - x_l + x_i)\|} = \\
&= \frac{\alpha\,(x_i - x_j)}{\|\alpha\,(x_i - x_j)\|} \Longrightarrow \\
\theta_{ij} &\mapsto \theta_{ji}.
\end{aligned}$$

As for the other two forms, η_{kj} and η_{lj}, the action of ψ_{ijkl} on them is similarly computed:

$$\begin{aligned}
\frac{x_j - x_k}{\|x_j - x_k\|} &\mapsto \frac{x_l - x_j}{\|x_l - x_j\|} \Longrightarrow \eta_{kj} \mapsto \eta_{jl}, \\
\frac{x_j - x_l}{\|x_j - x_l\|} &\mapsto \frac{x_k - x_j}{\|x_k - x_j\|} \Longrightarrow \eta_{lj} \mapsto \eta_{jk}.
\end{aligned}$$

Resuming all these computations, we get

$$\theta_{ij}\eta_{kj}\eta_{lj} \overset{\psi_{ijkl}}{\longmapsto} \theta_{ji}\eta_{jl}\eta_{jk} = (-1)^{(m-1)+3m}\theta_{ij}\eta_{kj}\eta_{lj} = -\theta_{ij}\eta_{kj}\eta_{lj},$$

by the parity of forms of type θ and η w.r.t. the exchange of their arguments. Finally, we get

$$\Gamma|_{(S)} \overset{\psi_{ijkl}}{\longmapsto} -\Gamma|_{(S)},$$

whence the claim follows.

As for case $b)$, we consider the map $\psi_{e,ijk}$

$$\begin{aligned}&\psi_{e,ijk}\left([(\alpha;\dots,x_i,\dots;\dots,y_e,\dots)]\right) :=\\ &:= [(\alpha;\dots,x_j+x_k-x_i,\dots;\dots,\alpha(x_j+x_l)-y_e,\dots)],\end{aligned}$$

and all other vertices remain unchanged; it is not difficult to check that $\psi_{e,ijk}$ is a well-defined, orientation preserving involution of $\widehat{C}^0_{p,q}$. The computation of the action of $\psi_{e,ijk}$ on the integrand can be performed along the same lines as for the computations for $a)$, leading to the claim. $\square$

Lemma 6.5.8. *If the face S has at least three internal vertices, and the integrand of $\Gamma|_{(S)}$ contains a product of forms of the type $\theta_{ij}\theta_{ik}$, where i is an external vertex and j and k are internal and univalent, then $\Gamma|_{(S)}$ vanishes.*

Proof. We use the notations as in the proofs of the previous lemmata. We label by j and k the univalent internal vertices.

On the interior of the boundary face we define the map

$$[(\alpha;\dots,x_j,\dots,x_k,\dots)] \longmapsto [(\alpha;\dots,x_k,\dots,x_j,\dots)], \tag{6.5.4}$$

and all other points remain unchanged. The map (6.5.4) is an involution, and it has orientation sign $(-1)^m$ (as both x_j and x_k are internal). Clearly, the involution (6.5.4) affects only the points x_j and x_k, hence the only factors in the integrand of $\Gamma|_{(S)}$ acted on nontrivially by (6.5.4) are $\overline{\theta}_{ij}$ and $\overline{\theta}_{ik}$. These forms are transformed clearly by

$$\overline{\theta}_{ij}\overline{\theta}_{ik} \overset{\text{By (6.5.4)}}{\longmapsto} \overline{\theta}_{ik}\overline{\theta}_{ij} = (-1)^{m-1}\overline{\theta}_{ij}\overline{\theta}_{ik}.$$

Again, we have used that θ-propagators have degree $m-1$.

Hence, the collapsing form $\Gamma|_{(S)}$ is acted on by the involution (6.5.4) by

$$\Gamma|_{(S)} \overset{\text{By (6.5.4)}}{\longmapsto} (-1)^{m+m-1}\Gamma|_{(S)} = -\Gamma_{(S)}.$$

The claim follows. $\square$

Vanishing lemma for faces "at infinity"

We state and prove now a vanishing lemma for all faces at infinity of our invariants for $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$; in fact, besides the faces where $1 \leq q \leq t$ points in $\mathbb{R}^m$ tend to infinity, we have also to deal with faces where $1 \leq p \leq s$ points in $\mathbb{R}^{m-2}$ tend to infinity or where $1 \leq p \leq s$ points in $\mathbb{R}^{m-2}$ and $1 \leq q \leq t$ points in $\mathbb{R}^m$ tend to infinity.

We consider an ordered subset $S = S_1 \cup S_2$ of $\{1, \ldots, s\} \cup \{1, \ldots, t\}$ labeling the vertices escaping to infinity; clearly, if S_2 has no elements, $S = S_1$ labels a face where $1 \leq p \leq s$ internal points tend to infinity, if not, the internal and external vertices labeled by S tend to infinity. If we consider a given diagram Γ, the push-forward along the fiber of the boundary face labeled by S of the restriction to the given boundary face of η- and θ-propagators not basic w.r.t. the boundary fibration is denoted by $\Gamma|_{(S)}$.

Lemma 6.5.9. *For any subset S of $\{1, \ldots, s\} \cup \{1, \ldots, t\}$, labeling vertices (be they internal or external) escaping to infinity (if S contains internal and external vertices), $\Gamma|_{(S)}$ vanishes.*

Proof. If $S = S_1 \cup S_2$, we denote by t_i the cardinality of S_1 and by t_e the cardinality of S_2, i.e. the number of internal, resp. external vertices escaping to infinity. We denote by Γ_S the product of all η- and θ-propagators with at least one vertex in S. By the definition of the fiber of the face at infinity corresponding to set S, it is clear that Γ_S is the piece of the integrand of Γ not basic w.r.t. the boundary fibration; clearly, $\Gamma|_{(S)}$ is the push-forward of Γ_S along the fiber of the boundary fibration.

Since we consider nontrivial diagrams, the following inequality must hold:

$$\deg \Gamma_S \geq t_i(m-2) + t_e m. \tag{6.5.5}$$

Inequality (6.5.5) is a consequence of the following argument: we consider the whole diagram containing $\Gamma|_{(S)}$ as an integral over the interior of a compactified configuration space. If we integrate Γ_S over all vertices appearing in this subdiagram (t_i internal and t_e external vertices) and if the degree of Γ_S is less than the sum of the dimensions of the vertices over which we integrate, the whole diagram is automatically zero by the definition of the push-forward. Hence, if we consider (possibly) nontrivial diagrams, inequality (6.5.5) must hold.

As we have already noted before, Γ_S is the nonbasic piece of the integrand of the diagram w.r.t. the boundary fibration corresponding to the escape to infinity of the vertices labeled by S; this is clear since we are considering only those imbeddings of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$ with a prescribed behavior at infinity. In case $S_2 \neq \varnothing$, all vertices (internal and external) escape to infinity; hence the fiber has dimension $t_i(m-2) + t_e m - 1$. If $S_2 = \varnothing$, then $t_e = 0$ and the dimension of the fiber is $t_i(m-2) - 1$. If $S_1 = \varnothing$, then $t_i = 0$ and the fiber has dimension $t_e m - 1$.

In all cases, the following strict inequality holds

$$\deg \Gamma_S > t_i(m-2) + t_e m - 1,$$

in virtue of (6.5.5).

Hence, the integrand, which lives only on the fiber since we are considering long knots in $\mathbb{R}^m$ (which have a prescribed behavior at infinity), has degree strictly exceeding the dimension of the fiber, hence it vanishes. $\square$

6.5.2 The vanishing of the contribution of hidden faces to the exterior derivative of the Bott invariant

With the help of the vanishing lemmata displayed in the previous subsections, we prove now that the contributions to the exterior derivative of Θ_2 (the Bott invariant) coming from hidden faces of configuration spaces vanish.

We begin by studying the hidden faces of the diagram (6.3.1) corresponding to the collapse of S vertices, where S is a subset of $\{1, 2, 3, 4\}$ of cardinality $|S| = 3$. Hence, we have to consider the case where exactly three internal vertices collapse together. There are clearly four possibilities: $a)$ $S = \{1, 2, 3\}$, $b)$ $S = \{1, 2, 4\}$, $c)$ $S = \{1, 3, 4\}$ and $d)$ $S = \{2, 3, 4\}$. We first consider $a)$. In this case, the integrand along the fiber presents a bivalent internal vertex joined by two η-propagators, namely the vertex labeled by 2. Lemma 6.5.3 yields the claim, namely that the corresponding contribution vanishes.

We next consider $b)$. The integrand is the product of a θ- and an η-propagator. Moreover, the vertex labeled by 1 is univalent, joined by an η-propagator. Lemma 6.5.2 yields the claim.

We then consider $c)$. In this case, the vertex labeled by 4 is zerovalent, as it is the endpoint of a θ-propagator labeled also by 2, which does not belong to the set of collapsing points. Hence, Lemma 6.5.1 implies the vanishing of the corresponding contribution.

Finally, we consider $d)$. This case is symmetric to case $b)$, and therefore the claim holds.

Second, we consider hidden faces of the diagram (6.3.2) corresponding to a subset S of $\{1, 2, 3, 4\}$ with $|S| = 3$. There are also four cases to be considered, the same as for the diagram (6.3.1). We begin by considering $a)$. It is immediate to see that the vertex labeled by 2 is univalent and is joined by an η-propagator. Therefore, Lemma 6.5.2 yields the claim.

We then consider $b)$. This situation presents also a univalent vertex joined by an η-propagator, namely the vertex labeled by 1. The claim follows.

We consider the case $c)$. As one can easily see, the vertex labeled by 4 is univalent with an η-propagator.

Finally, we consider the case $d)$. The vertex labeled by 3 is clearly univalent, with an η-propagator.

We consider the diagram (6.3.3). We consider the case when the three internal vertices collapse together. The three θ-propagators with vertices labeled by 1, 2 and 3 are clearly basic w.r.t. the boundary fibration. Moreover, the vertex labeled by 1 (and by symmetry the vertex labeled by 2) is univalent with exactly one η-propagator. Therefore, Lemma 6.5.2 implies the claim.

We next consider the hidden face where exactly two internal vertices and the external vertex in (6.3.3) collapse together in $\mathbb{R}^{m-2}$. As always, an η- or θ-propagator is basic, if at most one of its endpoints lie in the set of collapsing points. Therefore, due to the form of the diagram, the vertex labeled by 4 (i.e. the external vertex) is always bivalent, joined by two θ-propagators. Therefore, Lemma 6.5.3 yields the vanishing of boundary face.

We consider now only the diagrams (6.3.1) and (6.3.2). For these two diagrams, we take the most degenerate face where all four internal vertices collapse together. First, there are no dimensional reasons as to why the boundary contributions must vanish, as the integrands depend on the space of imbeddings through the θ-propagators. The push-forwards along the fiber of the boundary face are pull-backs of invariant forms of degree $m-1$ on the space $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ by Lemma 6.5.13. Since m is odd, $m-1$ is even.

We need only prove the vanishing of the boundary face $S=\{1,2,3,4\}$ for the diagrams (6.3.1) and (6.3.2); for this purpose, we can apply Lemma 6.5.5 to the diagrams (6.3.1) and (6.3.2). Namely, for what concerns the diagram (6.3.1), we consider the product $\theta_{13}\eta_{12}\eta_{23}$. We notice that the vertices labeled by 1 and 3 are bivalent and therefore play the rôle of j and k, and that we must consider $i=l=2$. For the diagram (6.3.2), we consider the product of factors $\eta_{12}\theta_{24}\eta_{34}$; the vertices 2 and 4 are bivalent, hence play the rôles of j and k.

We then consider the face, where the three internal vertices and the external vertex collapse together in $\mathbb{R}^{m-2}$ for the diagram (6.3.3). The push-forward along the fiber of the boundary face of the integrand is again the pull-back to the basis of the boundary face of a form on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ again by Lemma 6.5.13. In this case, we can apply Lemma 6.5.7, taking the univalent internal vertex labeled by 3 as i and the external vertex labeled by 4 as α.

Finally, we consider the hidden faces at infinity, i.e. when some internal or external vertices escape to infinity, or when both types of vertices tend to infinity. Lemma 6.5.9 yields also the vanishing of these contributions.

6.5.3 The vanishing of the contribution of hidden faces to the exterior derivative of Θ_3

We prove now, again with the help of all the vanishing lemmata displayed in subsection 6.5.1, that all contributions to the exterior derivative of Θ_3 coming from hidden faces of configuration spaces vanish except for the most degenerate face of exactly two diagrams, when all vertices collapse together in $\mathbb{R}^{m-2}$. We will deal with this particular face in the next subsection.

First of all, we recall that the function Θ_3 consists of 8 connected diagrams. We will show that the contributions of all hidden boundary faces vanish, separately for each integral in equation (6.4.9), except for the hidden face where all vertices, internal and external collapse together in $\mathbb{R}^{m-2}$, where the vanishing lemmata do not apply.

The first integral of (6.4.9)

We consider the integral

$$\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{34}\eta_{56}.$$

Writing down the corresponding diagram, we see that all its vertices are internal and bivalent with an η- and a θ-propagator. Since the vertices are all internal, the hidden faces correspond to the collapse of at least three vertices. Such hidden faces are associated to fibrations of the configuration space $C_{6,0}$; namely, if S denotes the set of collapsing points (which, by the free action of $\mathfrak{S}_6$, may be chosen to be $S = \{1, \ldots, k\}$, with $k \leq 6$), the fibration is the pull-back of $\widehat{C}_S(\mathbb{R}^{m-2})$ w.r.t. the projection from C_{6-S+1} onto $\mathbb{R}^{m-2}$, mapping onto to first point.

If the set of collapsing vertices contains an isolated vertex, the corresponding collapsing subdiagram vanishes. In fact, an isolated vertex corresponds to the situation of a collapsing subdiagram with a zerovalent internal vertex, which vanishes by Lemma 6.5.1.

If we consider the situation where 3 vertices collapse together, such that none of the three vertices is isolated, then the three vertices, which we denote by i, j and k appear only in the expressions:

$$\overline{\theta}_{ij}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\theta}_{jk},$$

where the bar over θ- or η-forms denotes restriction to the boundary fibration. Only these two expressions are not basic in the boundary fibration; the corresponding collapsing diagram is the push-forward of such expressions w.r.t. the boundary fibration. In both expressions, we see immediately that there is a univalent internal vertex joined by an η propagator; such a collapsing diagram vanishes therefore in virtue of Lemma 6.5.2.

If 4 vertices collapse together, and no vertex is isolated, then the vertices i, j, k and l appear only in the three expressions

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\theta}_{jk}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{kl}.$$

The push-forward of the second and the third expression vanishes immediately by Lemma 6.5.2, as the vertex i, resp. k) is univalent and is joined by an η-propagator. The collapsing diagram corresponding to the first expression vanishes also by Lemma 6.5.5.

We consider now the collapse of 5 vertices; since the diagram has 6 vertices, there cannot be any isolated vertex among the collapsing ones. The piece of the diagram, which is not basic in the boundary fibration, is of the form

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl}\overline{\eta}_{lm},$$

hence $S = \{i, j, k, l, m\}$. The corresponding collapsing diagram vanishes immediately, since the vertex m is univalent and is joined by an η-propagator.

The most degenerate face of this diagram vanishes also in virtue of Lemma 6.5.5.

The second piece of (6.4.9)

The second integral appearing in Θ_2 has the form

$$\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{45}.$$

The corresponding diagram has only internal vertices, one of which (the vertex labeled by 2) is trivalent, and one (the vertex labeled by 6) is univalent; all other vertices are bivalent.

Again, if the set of collapsing vertices contain an isolated vertex, then, in virtue of Lemma 6.5.1, the corresponding collapsing diagram vanishes.

We consider the case of 3 collapsing vertices, none of them isolated. The non-basic piece of the integrand w.r.t. the boundary fibration has the possible forms

$$\overline{\theta}_{ij}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{jk}$$

(because there is a trivalent internal vertex). Both corresponding collapsing subdiagrams vanish in virtue of Lemma 6.5.1.

In the case where 4 nonisolated vertices collapse, we are faced with four possible expressions:

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\theta}_{jk}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{kl}.$$

All corresponding collapsing subdiagrams vanish, the first one in virtue of Lemma 6.5.5, the remaining three by Lemma 6.5.2.

When 5 nonisolated vertices collapse together, we have to consider two possible expressions for the nonbasic piece of the integrand

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl}\overline{\theta}_{lm} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{ik}\overline{\theta}_{jl}\overline{\theta}_{km}\overline{\eta}_{lm}.$$

The first corresponding collapsing subdiagram vanishes by Lemma 6.5.2, while the second one vanishes by Lemma 6.5.3, since the vertex i is bivalent with two η-propagators.

The most degenerate face of this diagram vanishes also by Lemma 6.5.7 or Lemma 6.5.5.

The third piece of (6.4.9)

The third integral in (6.4.9) has the form

$$\int_{C_{6,0}} \theta_{14}\theta_{26}\theta_{35}\eta_{12}\eta_{23}\eta_{34}.$$

As before, the set S of collapsing vertices cannot contain an isolated vertex. Therefore, if we consider the case of three nonisolated points collapsing together, the nonbasic piece of the integrand may be of two types:

$$\overline{\theta}_{ij}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{jk}.$$

Both corresponding collapsing diagrams vanish by Lemma 6.5.2.

When 4 nonisolated vertices collapse together, the nonbasic piece of the integrand has one of the following five shapes:

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\theta}_{jk}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{ik}\overline{\eta}_{kl}\overline{\eta}_{jl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\eta}_{jl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\eta}_{kl}.$$

The collapsing subdiagram corresponding to the first integrand vanishes by Lemma 6.5.5; the second, the fourth and the fifth vanish in virtue of Lemma 6.5.2. The third one vanishes by Lemma 6.5.3, since the vertex labeled by k is bivalent with two η-propagators.

Finally, when 5 nonisolated vertices collapse together, we are faced with two different shapes for the nonbasic piece of the integrand:

$$\overline{\theta}_{ij}\overline{\theta}_{kl}\overline{\eta}_{jl}\overline{\eta}_{jm} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\theta}_{km}\overline{\eta}_{jk}\overline{\eta}_{jl}\overline{\eta}_{lm}.$$

The subdiagram corresponding to the first possibility vanishes by Lemma 6.5.6, since i and k are both univalent. The subdiagram corresponding to the second possibility vanishes in virtue of Lemma 6.5.3, as l is a bivalent internal vertex, joined by two η-propagators.

The most degenerate face of this diagram vanishes, because we can apply Lemma 6.5.7, Lemma 6.5.5 or Lemma 6.5.6.

The fourth piece of (6.4.9)

We consider the integral

$$\int_{C_{6,0}} \theta_{14}\theta_{25}\theta_{36}\eta_{12}\eta_{23}\eta_{31}.$$

We consider the case of 3 nonisolated vertices collapsing together. The nonbasic piece of the integrand can be

$$\overline{\theta}_{ij}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{jk}\overline{\eta}_{ik}.$$

The collapsing diagram corresponding to the first situation vanishes by Lemma 6.5.2, while the second collapsing diagram is zero in virtue of Lemma 6.5.3, as the vertex i is bivalent with two η-propagators.

If 4 nonisolated vertices collapse together, we find for the nonbasic piece of the integrand corresponding to the given collapse

$$\overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk}\overline{\eta}_{jl}\overline{\eta}_{kl}.$$

The first collapsing diagram vanishes by Lemma 6.5.5; the second one vanishes by Lemma 6.5.3.

When 5 nonisolated vertices collapse together, the nonbasic piece of the integrand

$$\overline{\theta}_{ij}\overline{\theta}_{kl}\overline{\eta}_{jm}\overline{\eta}_{jl}\overline{\eta}_{lm}.$$

The corresponding collapsing diagram vanishes by Lemma 6.5.3, as the vertex labeled by m is bivalent with two η-propagators.

The most degenerate face of this diagram vanishes in virtue of Lemma 6.5.7 or Lemma 6.5.6.

The fifth piece of (6.4.9)

We consider the following integral

$$\int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{23}.$$

Again, the set of collapsing vertices cannot contain any isolated vertex, be it internal or external.

If 3 nonisolated vertices collapse together, the nonbasic piece of the integrand may take one of the following shapes

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}, \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\eta}_{ij} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{jk}.$$

The corresponding collapsing diagrams all vanish: the first one by Lemma 6.5.3, since α is bivalent with two θ-propagators, the second one by Lemma 6.5.2, as α is univalent with a θ-propagator, the third one in virtue of Lemma 6.5.2, since k is univalent with an η-propagator, and the last one also by Lemma 6.5.2, as i is univalent with an η-propagator.

When 4 nonisolated vertices collapse, we get the following expressions for the nonbasic piece of the integrand

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\alpha k} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\eta}_{ij}\overline{\theta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{ik}\overline{\eta}_{jk}$$
$$\text{or} \quad \overline{\eta}_{ij}\overline{\eta}_{ik}\overline{\theta}_{il}.$$

All corresponding collapsing diagrams vanish. In fact, the first one vanishes by Lemma 6.5.2, as the vertex k is univalent and is joined by an η-propagator; the second one by Lemma 6.5.8; the third one by Lemma 6.5.2, as the vertex α is univalent with a θ-propagator; the fourth one by Lemma 6.5.2, again since α is univalent with a θ-propagator; the fifth one by Lemma 6.5.3, as α is bivalent with two θ-propagators. Finally, the last one also vanishes by Lemma 6.5.2, since the vertex j is univalent with an η-propagator.

When 5 nonisolated vertices collapse together, the nonbasic piece of the integrand must of the form

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\alpha k}\overline{\eta}_{jl}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{ik}\overline{\eta}_{jk}\overline{\theta}_{kl}.$$

The corresponding collapsing diagrams vanish, the first one in virtue of Lemma 6.5.3, as α is bivalent with two θ-propagators, the second one also by Lemma 6.5.3, since l is bivalent with two η-propagators, and the last one again by Lemma 6.5.3, since α is bivalent with two θ-propagators.

The most degenerate face of this diagram vanishes by Lemma 6.5.7.

The sixth integral of (6.4.9)

We consider the sixth piece of Θ_3

$$\int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{12}\eta_{34};$$

The possible hidden faces we have to consider for this integral are more complicated: in fact, we have to consider the hidden faces where internal vertices collapse together, where the external point and some internal vertices collapse and where the external point escape to infinity.

First of all, when the set of collapsing vertices contains an isolated vertex (be it internal or external), the corresponding collapsing subdiagram vanishes in virtue of Lemma 6.5.1.

We begin by considering the collapse of 3 nonisolated vertices (one of the three vertices may be the external one). The nonbasic piece of the integrand in this situation may take one of the following shapes:

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\eta}_{ij} \quad \text{or} \quad \overline{\theta}_{ij}\overline{\eta}_{jk},$$

where in this case we denote by α the external vertex. The collapsing diagram corresponding to the first expression vanishes by Lemma 6.5.3, as the external vertex is bivalent with two θ-propagators. The collapsing diagram corresponding to the second one vanishes also in virtue of Lemma 6.5.2, as the internal vertex j is univalent, joined by an η-propagator. The collapsing diagram corresponding to the third one vanishes also by Lemma 6.5.2, as the vertex k is univalent with η-propagator.

The case of 4 nonisolated collapsing vertices presents the following nonbasic pieces of the integrand

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\alpha k} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\eta}_{ij}\overline{\theta}_{jk} \quad \text{or} \quad \overline{\eta}_{ij}\overline{\theta}_{jk}\overline{\eta}_{kl}.$$

The corresponding collapsing diagrams all vanish: the first one by Lemma 6.5.8, as the vertex α is trivalent, the second one vanishes by Lemma 6.5.2, as k is univalent with η-propagator, the third one by Lemma 6.5.2, since α is univalent with θ-propagator, the fourth one again by Lemma 6.5.2, since α is again univalent with θ-propagator, and the fifth one by Lemma 6.5.2, as i is univalent with η-propagator.

When 5 nonisolated vertices collapse together, we are faced with the following nonbasic piece of the integrand

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\alpha k}\overline{\eta}_{kl} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{jk}\overline{\theta}_{kl} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{ik}\overline{\eta}_{jl}\overline{\theta}_{kl}.$$

The corresponding collapsing diagrams all vanish. Namely, the first one vanishes by Lemma 6.5.2, since l is univalent with η-propagator; the second one vanishes by Lemma 6.5.3, as α is bivalent with two θ-propagators, and the third one again by Lemma 6.5.3, as α is bivalent with two θ-propagators.

In this case also, the most degenerate face of this diagram vanishes by Lemma 6.5.7 or Lemma 6.5.5.

The seventh integral of (6.4.9)

We consider the following integral

$$\int_{C_{4,2}} \theta_{16}\theta_{36}\theta_{56}\theta_{25}\theta_{45}\eta_{12}.$$

Again, if the set of collapsing vertices contain an isolated vertex, the corresponding collapsing diagram vanishes.

Therefore, we consider the collapse of 3 nonisolated vertices. The nonbasic piece of the integrand is given by

$$\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j} \quad \text{or} \quad \overline{\theta}_{\alpha\beta}\overline{\theta}_{\beta i} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\eta}_{ij}.$$

All corresponding collapsing diagrams vanish: the first one in force of Lemma 6.5.3, as α is bivalent with two θ-propagators, the second one by Lemma 6.5.3, since β is bivalent with two θ-propagators, and the last one by Lemma 6.5.2, since α is univalent with a θ-propagator.

In the situation where 4 nonisolated vertices collapse together, we get the following possible expressions for the nonbasic piece of the integrand

$$\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i}\overline{\theta}_{\beta j} \quad \text{or} \quad \overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\eta}_{ik} \quad \text{or} \quad \overline{\theta}_{\alpha\beta}\overline{\theta}_{\beta i}\overline{\theta}_{\beta j} \quad \text{or} \quad \overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i}\overline{\theta}_{\beta j}\overline{\eta}_{ij}.$$

All corresponding collapsing diagrams vanish, the first one in virtue of Lemma 6.5.3, as the vertex α is bivalent with two θ-propagators, the second one again by Lemma 6.5.3, taking into account the vertex α, the third one by Lemma 6.5.2, considering the vertex α, and the last one by Lemma 6.5.3, if we consider the the vertex α.

When 5 nonisolated vertices collapse together, we have to deal with the following possibilities for the nonbasic piece of the integrand

$$\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\beta k}\overline{\eta}_{jk} \quad \text{or} \quad \overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i}\overline{\theta}_{\alpha j}\overline{\theta}_{\beta k}.$$

The two corresponding collapsing diagrams vanish, both in virtue of Lemma 6.5.3, if we take into account the bivalent vertex β.

The eighth integral of (6.4.9)

The last integral that we are going to consider is

$$\int_{C_{3,3}} \theta_{14}\theta_{25}\theta_{36}\theta_{45}\theta_{46}\theta_{56}$$

If the set of collapsing points contain an isolated vertex, be it internal or external, then the corresponding collapsing diagram vanishes.

We consider the case when 3 nonisolated vertices collapse together. The nonbasic piece of the integrand may take the form

$$\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i} \quad \text{or} \quad \overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha\gamma}\overline{\theta}_{\beta\gamma}.$$

Since the corresponding collapsing diagrams contain both a bivalent external vertex with two θ-propagators, Lemma 6.5.3 yields their vanishing.

If 4 nonisolated points collapse together, then one of them must be internal. Hence, the nonbasic piece of the integrand must be of the form

$$\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha i}\overline{\theta}_{\beta j} \quad \text{or} \quad \overline{\theta}_{i\alpha}\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha\gamma}\overline{\theta}_{\beta\gamma}.$$

In the corresponding collapsing diagrams, there is a always a bivalent external vertex with two θ-propagators. Hence, Lemma 6.5.3 implies their vanishing.

In the case of 5 nonisolated collapsing points, since at least two internal vertices must be in the collapsing set, the nonbasic piece of the integrand must be

$$\overline{\theta}_{\alpha\beta}\overline{\theta}_{\alpha\gamma}\overline{\theta}_{\beta\gamma}\overline{\theta}_{\beta i}\overline{\theta}_{\gamma j}.$$

The vertex α is bivalent with two θ-propagators; hence, the corresponding collapsing diagram vanishes by Lemma 6.5.3.

Finally, the contributions to the exterior derivative of Θ_3 coming from hidden faces where at least 1 vertex (be it internal or external) escapes to infinity all vanish in virtue of Lemma 6.5.9.

The most degenerate face of this diagram vanishes by the following argument (which will be made more precise in the next Subsection): the corresponding form on the space $\mathcal{I}\big(\mathbb{R}^d,\mathbb{R}^m\big)$ descends to a form on the Grassmann manifold $\mathrm{Gr}_{m,m-2}$, and it is additionally $SO(m)$-invariant. Since this form has degree $m-1$, and m is odd, it vanishes automatically, as there are no $SO(m)$-invariant forms on $\mathrm{Gr}_{m,m-2}$ of odd degree (we refer also to Subsection 2.6.6).

6.5.4 The most degenerate face of Θ_3

A characterization of contributions from hidden faces

In this subsection, we describe the contribution to the exterior differential of the function Θ_3 coming from the most degenerate boundary face, i.e. when all vertices of the diagrams collapse together in $\mathbb{R}^{m-2}$.

For these purposes, we give a general criterion for the classification of contributions coming from hidden faces; we recall first some facts from Subsection 2.4.4 of Chapter 2.

We consider, for $1 \leq p$ and $1 \leq q$ such that $2 \leq p+q$, the set $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes C^0_{p,q}$, introduced in (2.4.10). We define two maps, denoted resp. by Φ_{ij} and ϕ_{ij}, as follows

$$\Phi_{ij}\left((\alpha; x_1, \dots; y_1, \dots)\right) :=\begin{cases} \frac{\alpha(x_i - x_j)}{\|\alpha(x_i - x_j)\|}, & 1 \leq i \neq j \leq p \\ \frac{\alpha(x_i) - y_j}{\|\alpha(x_i) - y_j\|}, & 1 \leq i \leq p, \quad 1 \leq j \leq p \\ \frac{y_i - y_j}{\|y_i - y_j\|}, & 1 \leq i \neq j \leq q \end{cases} \tag{6.5.6}$$

$$\phi_{ij}\left((\alpha; x_1, \dots; y_1, \dots)\right) := \frac{x_i - x_j}{\|x_i - x_j\|}, \quad 1 \leq i \neq j \leq p. \tag{6.5.7}$$

Φ_{ij} maps $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes C^0_{p,q}$ to S^{m-1}, while ϕ_{ij} maps $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes C^0_{p,q}$ into S^{m-3}. It is not difficult to check that Φ_{ij} and ϕ_{ij} are invariant w.r.t. the action of $\mathbb{R}^+ \ltimes \mathbb{R}^{m-2}$; hence, any pull-back of a form on S^{m-1}, resp. S^{m-3}, w.r.t. the maps Φ_{ij}, resp. ϕ_{ij}, to $V_{m,m-2} \ltimes C^0_{p,q}$ is basic w.r.t. the action of $\mathbb{R}^+ \ltimes \mathbb{R}^{m-2}$, hence it descends to $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}^0_{p,q}$. In particular, we will write $\overline{\theta}_{ij}$, resp. $\overline{\eta}_{ij}$, for the pull-back w.r.t. Φ_{ij}, resp. ϕ_{ij}, of the normalized $SO(m)$-invariant volume form v on S^{m-1}, resp. the normalized $SO(m-2)$-invariant volume form w on S^{m-3}.

There is a right action of $SO(m-2)$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes C^0_{p,q}$

$$\left((\alpha; x_1, \dots; y_1, \dots); h\right) \mapsto \left(\alpha \circ h^{-1}; h(x_1), \dots; y_1, \dots\right), \quad h \in SO(m-2). \tag{6.5.8}$$

The above action of $SO(m-2)$ descends clearly to the quotient $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}^0_{p,q}$; moreover, since the action of $SO(m-2)$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ is free, the action of $SO(m-2)$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}^0_{p,q}$ is also free. One can also prove that the maps Φ_{ij} are invariant w.r.t. the right action of $SO(m-2)$, whence it follows that any pull-back w.r.t. Φ_{ij} of a form on S^{m-1} is basic w.r.t. this action.

On the other hand, there is a left action of $SO(m)$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes C^0_{p,q}$, which descends to the quotient $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big) \ltimes \widehat{C}^0_{p,q}$:

$$\left(g; (\alpha; x_1, \dots; y_1, \dots)\right) \mapsto \left(g \circ \alpha; x_1, \dots; g(y_1), \dots\right), \quad g \in SO(m); \tag{6.5.9}$$

notice that this action is in general not free.

Remark 6.5.10. The actions of $SO(m-2)$ and $SO(m)$ descends from $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ to actions on the submanifold $V_{m,m-2}$ of linear isometries, the Stiefel manifold, defined by analogous formulae.

The group $SO(m)$ operates in an obvious way on S^{m-1}. It is not difficult to verify that the maps ϕ_{ij} are invariant w.r.t. the action of $SO(m)$; on the other hand, the maps Φ_{ij} intertwine the action (6.5.9) with the action on S^{m-1}.

The properties of the maps (6.5.6) and (6.5.7) w.r.t. the actions (6.5.8) and (6.5.9) of $SO(m-2)$ and $SO(m)$ may be summarized in

Lemma 6.5.11. *If v is an $SO(m)$-invariant form of degree $m-1$ on S^{m-1} and w is an $SO(m-2)$-invariant form of degree $m-3$ on S^{m-3}, we denote by $\overline{\theta}_{ij}$, resp. $\overline{\eta}_{ij}$, the pull-back w.r.t. Φ_{ij} of v, resp. ϕ_{ij} of w.*

Then

a) $\overline{\theta}_{ij}$ is a form of degree $m-1$ on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$, which is basic w.r.t. the action (6.5.8) and invariant w.r.t. the action (6.5.9).

b) $\overline{\eta}_{ij}$ is a form of degree $m-3$ on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$, which is basic w.r.t. the action (6.5.9) and invariant w.r.t. the action (6.5.8).

We consider now the forgetful projection $\pi_{p,q}$ from $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$ onto $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$. It follows directly that $\pi_{p,q}$ intertwines the actions (6.5.9) and (6.5.8) with the actions of $SO(m)$ and $SO(m-2)$ on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$:

$$\pi_{p,q} \circ \mathrm{L}_g = \mathrm{L}_g \circ \pi_{p,q}, \quad \pi_{p,q} \circ \mathrm{R}_h = \mathrm{R}_h \circ \pi_{p,q}, \quad \forall\, g \in SO(m), h \in SO(m-2), \tag{6.5.10}$$

where we have denoted by L_g, resp. R_h, the (lift of the) action (6.5.9), resp. (6.5.8), on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}_{p,q}$ and $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$.

Remark 6.5.12. It is possible to define a similar right action of $GL(m-2)$ on the quotient $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$ and $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$; this action is compatible with the projection from $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m) \ltimes \widehat{C}^0_{p,q}$ onto $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$. We recall that the quotient of $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ by the right action of $GL(m-2)$ is the Grassmann manifold $\mathrm{Gr}_{m,m-2}$.

We consider a product of $\overline{\theta}$- and $\overline{\eta}$-forms, which we denote by $\overline{\gamma}$, such that the degree of this form is bigger or equal to $(p-1)(m-2)+qm-1$. The following Lemma characterizes precisely the contributions coming from boundary faces of configuration space integrals such as (6.3.5) and (6.4.9) as biinvariant forms on spaces $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ (see also Section 2.6 of Chapter 2).

Lemma 6.5.13. *The push-forward of $\overline{\gamma}$ w.r.t. to $\pi_{p,q}$, which we denote by $\int_{\widehat{C}_{p,q}} \overline{\gamma}$, is a biinvariant form on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$. If $\overline{\gamma}$ contains only $\overline{\theta}$-forms, $\int_{\widehat{C}_{p,q}} \overline{\gamma}$ is an $SO(m)$-invariant form on $\mathrm{Gr}_{m,m-2}$.*

Proof. The proof is a consequence of Lemma 6.5.11, equation (6.5.10) and Lemma 2.2.3 and 2.2.4. □

As a corollary, a form $\int_{\widehat{C}_{p,q}} \overline{\gamma}$ vanishes, if the integrand contains only $\overline{\theta}$-forms and has odd degree.

Remark 6.5.14. Lemma 6.5.13 gives analogous results, when we restrict ourselves to the Stiefel manifold $V_{m,m-2}$: in this case, the form $\int_{\widehat{C}_{p,q}} \overline{\gamma}$ is a biinvariant form on $V_{m,m-2}$, and similarly, if $\overline{\gamma}$ does not contain any $\overline{\eta}$-form, $\int_{\widehat{C}_{p,q}} \overline{\omega}$ descends to an $SO(m)$-invariant form on $\mathrm{Gr}_{m,m-2}$.

We recall at this point some facts from Section 2.5 of Chapter 1: there is a deformation retraction of $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ to the Stiefel manifold $V_{m,m-2}$, denoted by $\Lambda_{m,m-2}$. The homotopy between the identity and the composition of $\Phi_{m,m-2}$ with the inclusion of $V_{m,m-2}$ into $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ will be denoted simply $\widehat{\Lambda}_{m,m-2}$. Now assume that we have a closed form ω on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of degree p. We then define the following form of degree $p-1$ on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$:

$$\widehat{\omega}\colon=\int_0^1 \widehat{\Lambda}^*_{m,m-2}\omega;$$

here, the integral denotes push-forward w.r.t. the projection from $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)\times[0,1]$ onto $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$. We take the exterior derivative of $\widehat{\omega}$: the generalized Stokes Theorem gives

$$\mathrm{d}\widehat{\omega}=\iota_1^*\widehat{\Lambda}^*_{m,m-2}\omega-\iota_0^*\widehat{\Lambda}^*_{m,m-2}\omega,$$

where ι_i, $i=0,1$, denotes the inclusion

$$\alpha\hookrightarrow(\alpha;i)$$

of $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ into $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)\times[0,1]$. Recalling the computations of Section 2.5, $\widehat{\Lambda}_{m,m-2}\circ\iota_1=\iota_{m,m-2}\circ\Lambda_{m,m-2}$ and $\widehat{\Lambda}_{m,m-2}\circ\iota_0=\mathrm{id}$, whence

$$\mathrm{d}\widehat{\omega}=\Lambda^*_{m,m-2}\iota^*_{m,m-2}\omega-\omega.$$

The form $\iota^*_{m,m-2}\omega$ is simply the restriction of ω to the Stiefel manifold; hence, a closed form ω on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$ can be written as the sum of the pull-back of a form on the Stiefel manifold and an exact form. We will make use in the next Subsubsection in order to characterize the contribution of the most degenerate face of Θ_3.

The vanishing of the most degenerate face of (6.4.9) in 4 dimensions

With the help of the preceding lemmata, we are now ready to characterize the contribution from the most degenerate face of (6.4.9) in 4 dimensions.

We denote by $\widehat{\Theta}_3$ the form of degree $m-1$ on $\mathcal{I}(\mathbb{R}^{m-2},\mathbb{R}^m)$, for m even, given by

$$\begin{aligned}\widehat{\Theta}_3\colon=&\frac{1}{3}\int_{\widehat{C}_{6,0}}\overline{\theta}_{14}\overline{\theta}_{26}\overline{\theta}_{35}\overline{\eta}_{12}\overline{\eta}_{34}\overline{\eta}_{56}+\int_{\widehat{C}_{6,0}}\overline{\theta}_{14}\overline{\theta}_{26}\overline{\theta}_{35}\overline{\eta}_{12}\overline{\eta}_{23}\overline{\eta}_{45}-\\&-\int_{\widehat{C}_{6,0}}\overline{\theta}_{14}\overline{\theta}_{26}\overline{\theta}_{35}\overline{\eta}_{12}\overline{\eta}_{23}\overline{\eta}_{34}+\frac{1}{3}\int_{\widehat{C}_{6,0}}\overline{\theta}_{14}\overline{\theta}_{25}\overline{\theta}_{36}\overline{\eta}_{12}\overline{\eta}_{23}\overline{\eta}_{31}-\\&-\int_{\widehat{C}_{5,1}}\overline{\theta}_{16}\overline{\theta}_{36}\overline{\theta}_{56}\overline{\theta}_{24}\overline{\eta}_{12}\overline{\eta}_{23}+\int_{\widehat{C}_{5,1}}\overline{\theta}_{16}\overline{\theta}_{36}\overline{\theta}_{56}\overline{\theta}_{24}\overline{\eta}_{12}\overline{\eta}_{34}-\\&-\int_{\widehat{C}_{4,2}}\overline{\theta}_{16}\overline{\theta}_{36}\overline{\theta}_{56}\overline{\theta}_{25}\overline{\theta}_{45}\overline{\eta}_{12}+\frac{1}{3}\int_{\widehat{C}_{3,3}}\overline{\theta}_{14}\overline{\theta}_{25}\overline{\theta}_{36}\overline{\theta}_{45}\overline{\theta}_{46}\overline{\theta}_{56},\end{aligned}\tag{6.5.11}$$

i.e. the contribution coming from the most degenerate face of Θ_3; notice that the integrals are already computed on the compactifications à la FMcPAS of the spaces $\widehat{C}^0_{p,q}$, which we have discussed in Subsection 2.4.4 of Chapter 2.

In order to apply the arguments at the end of Subsubsection 6.5.4, we need two technical lemmata.

Lemma 6.5.15. *The exterior derivative of the form $\widehat{\Theta}_3$ vanishes.*

Proof. We notice first that the first six integrals and the last one vanish already by arguments exhibited in Subsection 6.5.3; hence, it remains to compute the exterior derivative of the seventh integral.

The computation of exterior derivative of the seventh integral is performed via the generalized Stokes Theorem; since the integrand is clearly closed, we have only to compute the corresponding contribution coming from the boundary. Recalling the characterization of the compactification of $\widehat{C}^0_{4,2}$ sketched at the end of Subsection 2.4.4, the boundary of $\widehat{C}_{4,2}$ is the union of different boundary faces, which present the same features as the boundary faces of $C_{4,2}$ corresponding *only* to the collapse together of some internal vertices, of some external vertices in $\mathbb{R}^m$ or of both types of vertices, *with the exception of the case, when all vertices collapse together* (we recall that this possibility is automatically ruled out by the explicit shape of $\widehat{C}^0_{4,2}$). We can once again distinguish boundary faces of $\widehat{C}_{4,2}$ between *principal* and *hidden* faces, the former corresponding to the collapse of exactly two vertices, the latter to the collapse of more than 2 vertices.

Since boundary faces of $\widehat{C}_{4,2}$ admit a description which mimics that of boundary faces of $C_{4,2}$, the contributions coming from hidden faces of $\widehat{C}_{4,2}$ vanish by a mild modification of the arguments used in Subsection 6.5.3 to show the vanishing of the hidden faces of the seventh integral of Θ_3.

The contributions coming from principal faces are of four types, which, for the sake of simplicity, are illustrated in Figure (6.5.4), (6.5.4), (6.5.4) and (6.5.4), which correspond respectively to the collapse of the vertices labeled by 1 and 2, those labeled by 1 (or 2) and 6 (or 5), those labeled by 3 (or 4) and 6 (or 5) and those labeled by 5 and 6. From the diagrammatic representations of the diagrams, it is easy to see that the integrals corresponding to (6.5.4), (6.5.4) and (6.5.4) vanish, the former two by Lemma 6.5.7 and the latter by Lemma 6.5.8. The integral corresponding to (6.5.4) represents a form of degree m on the Grassmann manifold $\mathrm{Gr}_{m,m-2}$, as it contains only $\overline{\theta}$-forms. It vanishes, because the integral represented by the diagram (6.5.4) is the exterior derivative of the eighth integral of $\widehat{\Theta}_3$. In fact, it is possible to compute the exterior derivative of the eighth integral of $\widehat{\Theta}_3$ using the generalized Stokes Theorem, reducing the problem to the computation of the boundary contribution; again, we have to compute all contributions coming from principal and hidden boundary faces of $\widehat{C}_{3,3}$; the hidden faces vanish all by some mild modifications of the arguments used in Subsection 6.5.3 for the eighth integral of Θ_3. There are only two types of contributions coming from principal faces: the former is depicted graphically in (6.5.4), the latter vanishes, as it contains a square of a $\overline{\theta}$-form. Hence, the exterior derivative of the eighth integral of $\widehat{\Theta}_3$ equals (up to some factors) the exterior derivative of the seventh integral of $\widehat{\Theta}_3$; since the former differential form is zero, so is its exterior derivative, whence it follows that the form represented by the seventh integral of $\widehat{\Theta}_3$ is closed, and so is also $\widehat{\Theta}_3$. $\square$

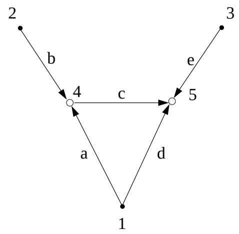

Figure 6.16: The contribution to the seventh integral of $\widehat{\Theta}_3$ coming from the collapse of the vertices labeled by 1 and 2

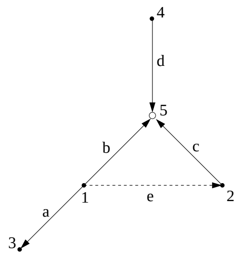

Figure 6.17: The contribution to to the seventh integral of $\widehat{\Theta}_3$ coming from the collapse of the vertices labeled by 1 and 6

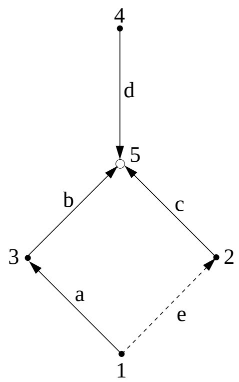

Figure 6.18: The contribution to the seventh integral of $\widehat{\Theta}_3$ coming from the collapse of the vertices labeled by 3 and 6

We consider now the special case $m = 4$, and we take the restriction of $\widehat{\Theta}_3$ to the Stiefel manifold $V_{4,2}$.

Lemma 6.5.16. *The restriction to $V_{4,2}$ of the 3-form $\widehat{\Theta}_3$ on $\mathcal{I}\left(\mathbb{R}^2,\mathbb{R}^4\right)$ in equation (6.5.11) is basic w.r.t. the right $SO(2)$-action on $V_{4,2}$, and hence descends to $\mathrm{Gr}_{4,2}$; it*

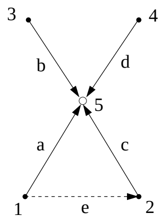

Figure 6.19: The contribution to the seventh integral of $\widehat{\Theta}_3$ coming from the collapse of the vertices labeled by 3 and 6

is also $SO(4)$-invariant.

Proof. We have previously shown that the first six integrals and the last one vanish; the only nonvanishing integral is the seventh one, whose restriction to $V_{4,2}$ represents, by the results in the previous Subsubsection, a biinvariant form on $V_{4,2}$ of degree 3. If we are able to show that it is moreover $SO(2)$-horizontal, then we are done, since, in this case, $\widehat{\Theta}_3$ descends to a form of degree 3 on the Grassmann manifold $\mathrm{Gr}_{4,2}$, and so it has to vanish.

We consider the fundamental vector field X_ξ associated to the generator of $\mathfrak{so}(2)$, and we compute the contraction by X_ξ of the seventh term in (6.5.11). By Lemma 6.5.11 any θ-form is already $SO(2)$-basic, hence the possible contributions to contraction of $\widehat{\Theta}_3$ by X_ξ come from contraction by X_ξ of any η-form. Moreover, as the maps ϕ_{ij} (by which we pull-back the volume form on S^1) are $SO(2)$-equivariant, contraction by X_ξ of any η-form equals a nonzero constant λ, as any η-form is constructed via the normalized top-form on S^1.

Contraction by X_ξ of the seventh integral vanish by means of Lemma 6.5.8 (see also Figure 6.5.4 for a pictorial description). $\square$

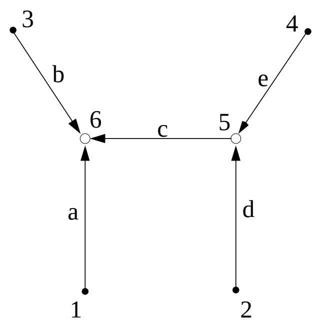

Figure 6.20:

We have thus proved (Lemma 6.5.15) that the form $\widehat{\Theta}_3$, the contribution to the exterior derivative of Θ_3 coming from the most degenerate face, viewed as a form on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$, for m even, is closed; the existence of a deformation retraction of $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ to the Stiefel manifold $V_{m,m-2}$ implies, by what we have proved at the end of Subsubsection 6.5.4, that $\widehat{\Theta}_3$ differs from (the pull-back w.r.t. the map $\Lambda_{m,m-2}$ of) its restriction to $V_{m,m-2}$ by an exact $m-1$-form. Moreover, in the special case $m=4$, the restriction to $V_{4,2}$ of $\widehat{\Theta}_3$ vanishes by Lemma 6.5.16.

Hence, we consider first the following form of degree $m-2$ on $\mathcal{I}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$

$$\int_0^1 \widehat{\Lambda}^*_{m,m-2}\widehat{\Theta}_3,$$

where the notations are as at the end of Subsubsection 6.5.4. We then consider its pull-back w.r.t. the "tangential evaluation map" $(x;f)\mapsto \mathrm{T}_x f$ (denoted by the same symbol), which yields a form of degree $m-2$ on $C_1\big(\mathbb{R}^{m-2}\big)\times \mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ (where $C_1 = C_1\big(\mathbb{R}^{m-2}\big)$ denotes $\mathbb{R}^{m-2}$ with the point at infinity blown up). We consider finally the function

$$\widetilde{\Theta}_3 :\; = \int_{C_1}\left(\int_0^1 \widehat{\Lambda}^*_{m,m-2}\widehat{\Theta}_3\right). \tag{6.5.12}$$

We compute the exterior derivative of (6.5.12) by means of the generalized Stokes Theorem:

$$\begin{aligned} \mathrm{d}\widetilde{\Theta}_3 &= \int_{C_1}\mathrm{d}\left(\int_0^1 \widehat{\Lambda}^*_{m,m-2}\widehat{\Theta}_3\right) = \\ &= \int_{C_1}\Lambda^*_{m,m-2}\iota^*\widehat{\Theta}_3 - \int_{C_1}\widehat{\Theta}_3. \end{aligned} \tag{6.5.13}$$

In fact, we have also to consider the only boundary contribution, which corresponds in this situation to the vertex in $\mathbb{R}^{m-2}$ escaping to infinity. Such a boundary face vanishes by dimensional reasons: the fiber of the boundary is diffeomorphic to S^{m-3} (the sphere "at infinity"), and the integrand, which is clearly nonbasic w.r.t. the boundary fibration, is a form of degree $m-2$, depending only on the fiber, since any imbedding in $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$ at infinity takes a given linear behavior, which does not depend neither on the imbedding nor on the point in $\mathbb{R}^{m-2}$. Hence, we have only to consider the exterior derivative of the integrand; since $\widehat{\Theta}_3$ is closed, the result follows immediately from the computations at the end of Subsubsection 6.5.4.

We consider the function on $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^{m-2},\mathbb{R}^m\big)$

$$\overline{\Theta}_3 :\; = \Theta_3 - \widetilde{\Theta}_3. \tag{6.5.14}$$

We can finally formulate the following

Theorem 6.5.17. *In the special case $m=4$, the exterior derivative of (6.5.14) vanishes, hence giving a true (isotopy-)invariant of* $\mathrm{Imb}_{\infty,\sigma}\big(\mathbb{R}^2,\mathbb{R}^4\big)$.

Proof. The proof is an immediate consequence of equations (6.5.14) and (6.5.13), noting that, by Lemma 6.5.16 holding for the special case $m=4$, the first term in the second row of (6.5.13) vanishes, as it is the pull-back of a biinvariant form on $V_{4,2}$ of degree 3, therefore descending to an invariant form on the Grassmann manifold $\mathrm{Gr}_{4,2}$, which is necessarily 0 by the arguments of Subsection 2.6.6. □

Remark 6.5.18. In the more general case m even and strictly bigger than 4, the function (6.5.14) is a *quasi-invariant*, in the sense that its exterior derivative (which, for Θ_3, equals the boundary face coming from the most degenerate face, where all vertices collapse together) can be written as a linear combination of a finite number of terms, each of which is a pull-back of a biinvariant form on the Stiefel manifold $V_{m,m-2}$.

Chapter 7

Conclusions and outlook

In this final section we briefly discuss two main problems:

i) the functions (6.3.5) and (6.4.9) are defined on the space $\mathrm{Imb}_{\infty,\sigma}(\mathbb{R}^{m-2},\mathbb{R}^m)$ of long knots in $\mathbb{R}^m$, but it is nonetheless possible to adjust them so as to obtain (quasi-)invariants of imbedded spheres of codimension 2 in $\mathbb{R}^m$;

ii) other possible invariants for long knots in $\mathbb{R}^m$ can be obtained by considering terms of higher order of the perturbative expansion of the v.e.v. of the partition function of the $\mathcal{I}$ action.

However, as for $i)$, our invariants (6.3.5) and (6.4.9) for long knots in $\mathbb{R}^m$ yield also true invariants of higher-dimensional knots in $\mathbb{R}^m$: in fact, any higher-dimensional knot in $\mathbb{R}^m$ can be made in an obvious way into a higher-dimensional knot in S^m (i.e. an imbedding of S^{m-2} into S^m) by choosing a point at infinity ∞ in S^m (e.g. the north pole). Moreover, the action of the group of diffeomorphisms of S^m (which contains $SO(m+1)$) permits, choosing also a point at infinity ∞ in S^{m-2}, to deform a higher-dimensional knot in S^m, so that the point at infinity in S^{m-2} is mapped to the point at infinity in S^m (this can be done by the action of $SO(m+1)$). At this point, we have obtained from a higher-dimensional knot in $\mathbb{R}^m$ a base-point preserving imbedding of S^{m-2} into S^m. Furthermore, since we are considering imbeddings of S^{m-2} into S^m, the action of $\mathrm{Diff}(S^m)$ enables one to deform a base-point preserving knot in S^m in a small neighbourhood of ∞ in S^m such that its behavior in this small neighbourhood is linear and governed moreover by a fixed linear imbedding of $\mathbb{R}^{m-2}$ into $\mathbb{R}^m$, which corresponds to the tangent map of the given imbedding at ∞. Hence, it becomes a long knot in $\mathbb{R}^m$ in the sense of (2.4.17) of Subsection 2.4.5 of Chapter 2. This shows that in every connected component of the space of higher-dimensional knots in S^m there is a long knot in $\mathbb{R}^m$. Clearly, isotopy between higher-dimensional knots in $\mathbb{R}^m$ translates into isotopy of long knots in $\mathbb{R}^m$. This proves that (6.3.5) and (6.4.9) define also invariants of higher-dimensional knots in $\mathbb{R}^m$.

7.0.5 The Bott invariant and the function (6.4.9) for imbedded spheres of codimension 2

The Bott invariant (6.3.5) is the sum of three Feynman diagrams, which can also be written as the sum of two configuration space integrals whose integrands contain cyclic sums of η-forms; in fact, this was the way Bott originally wrote its invariant. This way of writing the Bott invariant is not only aesthetic, but has to do with the fact that Bott originally intended to construct an invariant for imbedded spheres of codimension 2 in $\mathbb{R}^m$ instead of long knots in $\mathbb{R}^m$.

Namely, the main difference between invariants for long knots in $\mathbb{R}^m$ and invariants for higher-dimensional knots in $\mathbb{R}^m$ obtained by configuration space integrals is that we have to replace configuration spaces $C_{s,t} \equiv C_{s,t}\left(\mathbb{R}^m, \mathbb{R}^{m-2}\right)$ by $C_{s,t}\left(\mathbb{R}^m, S^{m-2}\right)$ and the superpropagator of the exterior differential on $\mathbb{R}^{m-2}$ (which corresponds to the η-form) by the superpropagator of the exterior differential on S^{m-2}; this is the most difficult point.

For the construction of the superpropagator of the exterior differential on S^{m-2} we have to resort to the notion of *parametrix of the exterior derivative*. We briefly introduce some basic notions. Given a smooth, compact manifold M, a parametrix for the exterior derivative on M is an operator P on $\Omega^*(M)$ decreasing the degree by 1 and satisfying the equation

$$\mathrm{d} \circ \mathrm{P} + \mathrm{P} \circ \mathrm{d} = \mathrm{id} + \mathrm{S}, \tag{7.0.1}$$

where S is a smooth operator of degree 0. The parametrix of d is not unique, since any parametrix P defines another parametrix by setting $\mathrm{P} \mapsto \mathrm{P} + \mathrm{d} \circ Q - Q \circ \mathrm{d}$, for any smooth operator decreasing by 2 the degree. Usually, we represent P by a convolution, i.e. we want to associate to P a form η on the product $M \times M$ realizing P as

$$\mathrm{P}\left(\alpha\right) = \pi_2^*(\eta \wedge \pi_1^* \alpha)\,;$$

due to the presence of the identity in (7.0.1), the form η cannot be smooth on $M \times M$. A way of dealing with this problem is to consider η to be a form on $C_2(M)$, which projects down in an obvious way onto $M \times M$, with projection π. The boundary of $C_2(M)$ is a spherical bundle over the diagonal $M \cong \mathrm{Diag}(M \times M)$. Therefore, as a consequence of Section 2.7 of Chapter 2, the bundle $\partial C_2(M) \overset{\pi_\partial}{\to} \mathrm{Diag}(M \times M)$ is endowed with a global angular form $\widehat{\eta} \in \Omega^{m-1}\left(\partial C_2(M)\right)$ (for the definition of a global angular form on a sphere bundle, see Section 2.7 of Chapter 2); the definition of a global angular form on a sphere bundle over M needs an explicit representative e of the Euler class of M.

We denote by $\chi_\Delta \in \Omega^m(M \times M)$ a representative of the Poincaré dual of the diagonal $\mathrm{Diag} \equiv \mathrm{Diag}\left(M \times M\right)$. We denote further by T the smooth map on $C_2(M)$ induced by the exchange of the two factors in $C_2^0(M)$.

Generalizing Proposition 3.2 in [10] by similar arguments to the case of a general compact manifold M, we get the following

Theorem 7.0.19. *There exists a form $\eta \in \Omega^{m-1}(C_2(M))$ and a form $\beta \in \Omega^{m-1}(\mathrm{Diag})$, for given representatives e, resp. χ_Δ, of the Euler class of M, resp. of the Poincaré dual of Diag, such that*

- $\mathrm{d}\eta = \pi^*\chi_\Delta$,
- $\iota_\partial^*\eta = -\widehat{\eta} + \pi_\partial^*\beta$,
- $T^*\eta = (-1)^m\eta$,

provided m is even and $H^{m-1}(M)$ is trivial or m is odd.

The map ι_∂ is the inclusion of the boundary $\partial C_2(M)$ into $C_2(M)$.

From Theorem 7.0.19 we get a parametrix of the exterior derivative on M by the following

Theorem 7.0.20. *Denoting by π_i the smooth lifts to $C_2(M)$ of the two natural projections from $C_2^0(M)$ onto M, and by p_i the two natural projections from $M \times M$ onto M, the operator P defined by*

$$\mathrm{P}\,(\alpha) :\, = -\pi_{2*}(\eta \wedge \pi_1^*\alpha) \tag{7.0.2}$$

is a parametrix, and the smooth operator S is defined as

$$S\,(\alpha) :\, = (-1)^m p_{2*}(\chi_\Delta \wedge p_1^*\alpha)\,.$$

The proof is a simple application of the generalized Stokes Theorem.

We return back to $M = S^{m-2}$; Theorem 7.0.19 and 7.0.20 yield a superpropagator $\eta \in \Omega^{m-3}\big(C_2\big(S^{m-2}\big)\big)$ for the exterior derivative on S^{m-2}. Quite differently from the η-form on $C_2\big(\mathbb{R}^{m-2}\big)$, the η-form is *not* closed; namely, if we choose a generator v of the highest-degree cohomology of S^{m-2} (i.e. a top-form on S^{m-2}), its exterior derivative equals

$$\mathrm{d}\eta = v_2 + (-1)^m v_1 \neq 0. \tag{7.0.3}$$

where $v_i :\, = p_i^* v$, $i = 1, 2$. If we consider the compactified configuration spaces $C_{s,t}\big(\mathbb{R}^m, S^{m-2}\big)$ à la FMcPAS and projections π_{ij} on $C_2\big(S^{m-2}\big)$, $1 \leq i \neq j \leq s$, we denote by η_{ij} the pull-backs of η by π_{ij}; Equation (7.0.3) shows immediately that any cyclic alternate sum, resp. cyclic sum, of η-forms η_{ij} is closed, if m is even, resp. odd.

Therefore, we can write the following two functions Θ_2 and Θ_3 on the space of higher-dimensional knots in $\mathbb{R}^m$:

$$\Theta_2 :\, = \frac{1}{8}\int_{C_{4,0}} \theta_{13}\theta_{24}\eta_{1234}^2 - \frac{1}{3}\int_{C_{3,1}} \theta_{14}\theta_{24}\theta_{34}\eta_{123} \tag{7.0.4}$$

and

$$\begin{aligned}\Theta_3 :\, = -\frac{1}{24}\int_{C_{6,0}} \theta_{14}\theta_{25}\theta_{36}\eta_{1245}\eta_{1346}\eta_{2356} - \frac{1}{6}\int_{C_{5,1}} \theta_{16}\theta_{36}\theta_{56}\theta_{24}\eta_{1234}\eta_{2345} - \\ -\frac{1}{4}\int_{C_{4,2}} \theta_{16}\theta_{36}\theta_{56}\theta_{25}\theta_{45}\eta_{1234} + \frac{1}{3}\int_{C_{3,3}} \theta_{14}\theta_{25}\theta_{36}\theta_{45}\theta_{46}\theta_{56}.\end{aligned} \tag{7.0.5}$$

In (7.0.4), $\eta_{123} :\, = \eta_{12} + \eta_{23} + \eta_{31}$ and η_{1234} is defined accordingly. In (7.0.5), $\eta_{ijkl} :\, = \eta_{ij} - \eta_{jk} + \eta_{kl} - \eta_{li}$, for any 4-tuple of distinct indices. If we write explicitly the cyclic (alternate) sums of the η-forms in both expressions (7.0.4) and (7.0.5) and we use the parity of η- and θ-forms w.r.t. involution T, resp. the antipodal map on the sphere S^{m-1} together with the action of permutation groups on configuration spaces, we find the same integrals as in (6.3.5) and (6.4.9), where now the η-forms are pull-backs of the parametrix of Theorem 7.0.20 and the compactified configuration spaces are $C_{s,t}\left(\mathbb{R}^m, S^{m-2}\right)$.

Again, the parity of the parametrix w.r.t. the map exchanging the arguments of $C_2\bigl(S^{m-2}\bigr)$ implies that (7.0.4) vanishes in even dimensions, while (7.0.5) vanishes in even dimensions. The generalized Stokes Theorem implies that the exterior derivative of both (6.3.5) and (6.4.9) is given by their respective boundary contributions.

The parity of the parametrix η w.r.t. the involution exchanging the two points in $C_2\bigl(S^{m-2}\bigr)$ and the fact that η restricts to a global angular form on $\partial C_2\bigl(S^{m-2}\bigr)$ allow us to repeat almost the same arguments and vanishing lemmata of Section 6.3, 6.4 and 6.5 to evaluate principal and hidden boundary contributions, hence proving that Θ_2 is an invariant for imbedded spheres of codimension 2 into odd-dimensional Euclidean spaces and that Θ_3 is a *quasi-invariant for imbedded spheres of codimension* 2 *into even-dimensional Euclidean spaces* in the sense that all boundary contributions to the exterior derivative of Θ_3 vanish by the same vanishing lemmata of Subsection 6.5.1, except the most degenerate face where all vertices collapse together. This contribution can be shown to be the (pull-back of) an $SO(m)$-invariant form on the fiber bundle $\mathrm{SO}\bigl(S^{m-2}\bigr)\times_{SO(m-2)}\mathcal{I}\bigl(\mathbb{R}^{m-2},\mathbb{R}^m\bigr)$. Since the restriction to each fiber of $\mathrm{SO}\bigl(S^{m-2}\bigr)\times_{SO(m-2)}\mathcal{I}\bigl(\mathbb{R}^{m-2},\mathbb{R}^m\bigr)$ of this form yields an $SO(m)$-invariant form on the Stiefel manifold $\mathcal{I}\bigl(\mathbb{R}^{m-2},\mathbb{R}^m\bigr)$, it follows that there is a finite number of such forms at each degree by the arguments of Section 2.6 of Chapter 2. The most degenerate boundary contribution to the exterior derivative of Θ_3 can then be written as (the pull-back of) a linear combination of a finite-dimensional basis of $SO(m)$-invariant forms on $\mathcal{I}\bigl(\mathbb{R}^{m-2},\mathbb{R}^m\bigr)$.

We see immediately that the construction configuration space integral invariants of higher-dimensional knots in $\mathbb{R}^m$ (as was the idea of Bott in [9]) presents more technical difficulties than the construction of similar invariants for long knots.

7.0.6 Higher-order invariants

A more difficult task is the computation of terms of higher order coming from the perturbative expansion of the partition function of the $\mathcal{I}$ action; as we have remarked before, we could get possible invariants of long knots in $\mathbb{R}^m$. However, it is possible to give an idea of how other possible perturbative invariants of long knots can be obtained.

From an inspection of (6.1.1) in Section 6.1, we see immediately that, at any order $p > 3$, there is a connected Feynman diagram with weight the ratio of p and Lie-algebraic term $\mathrm{Tr}\left(\Xi^p\right)$:

$$\int_{C_{2p,0}} \theta_{1,p+1}\theta_{2,p+2}\cdots\theta_{p,2p}\eta_{12}\eta_{23}\cdots\eta_{p1}. \tag{7.0.6}$$

The exterior derivative of (7.0.6) can be computed by the generalized Stokes Theorem. We need only consider principal faces, where either two consecutive vertices

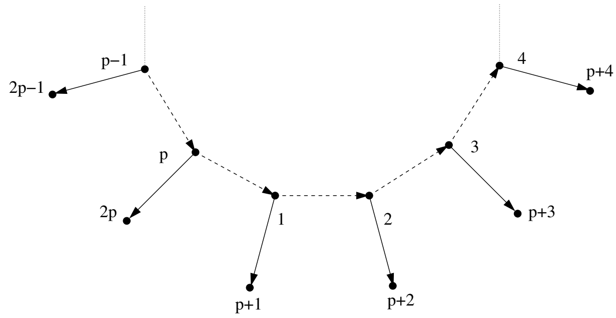

Figure 7.1: The configuration space integral (7.0.6) in diagrammatic form

labeled elements of $\{1,\dots,p\}$ or two vertices labeled by $\{k,k+p\}$, $k\in\{1,\dots,p\}$ since the vanishing lemmata from Subsection 6.5.1 imply that hidden faces give trivial contributions. When two consecutive vertices in $\{1,\dots,p\}$ collapse together, we also find a trivial contribution: namely, the resulting configuration space integral contains

$$\theta_{i,i+p}\theta_{i,i+p+1},$$

where i is bivalent internal, and $i+p$ and $i+p+1$ are univalent internal. We consider the involution $\sigma_{i+p,i+p+1}$ of $C_{2p-1,0}$ exchanging $i+p$ and $i+p+1$; such an involution is orientation-preserving, resp. -reversing, if m is even, resp. odd. On the other hand, the action of $\sigma_{i+p,i+p+1}$ on the integrand affects only the above term, yielding

$$\theta_{i,i+p}\theta_{i,i+1+p}\mapsto\theta_{i,i+p+1}\theta_{i,i+p}=(-1)^{m-1}\theta_{i,i+p}\theta_{i,i+1+p}.$$

Hence, under the involution $\sigma_{i+p,i+p+1}$, the configuration space integral corresponding to the boundary contribution coming from the collapse of i and $i+1$ is mapped into its opposite, and hence vanishes.

Using the action of the permutation group $\mathfrak{S}_{2p-1}$ on $C_{2p-1,0}$, the only boundary contribution to the exterior derivative of (7.0.6) is given by (up to signs)

$$\int_{C_{2p-1,0}}\overline{\theta}_1\theta_{2,p+1}\cdots\theta_{p,2p-1}\eta_{12}\eta_{23}\cdots\eta_{p1},$$

where $\overline{\theta}_1$ is as in (6.3.6).

One can show that it is possible to construct a (quasi-)invariant Θ_p on the space of long knots in $\mathbb{R}^m$, for any $m>3$, of the form of a finite sum of configuration space integrals, starting from the configuration space integral (7.0.6): the idea, motivated by the above computations, is to sum to (7.0.6) a configuration space integral, whose exterior derivative cancels the exterior derivative of (7.0.6). Diagrammatically (and up to numerical coefficients),

As one can immediately see, the exterior derivative of the added counterterm produces also other boundary contributions, which need corresponding counterterms to be added along the same lines. It turns out that one needs only a finite number of

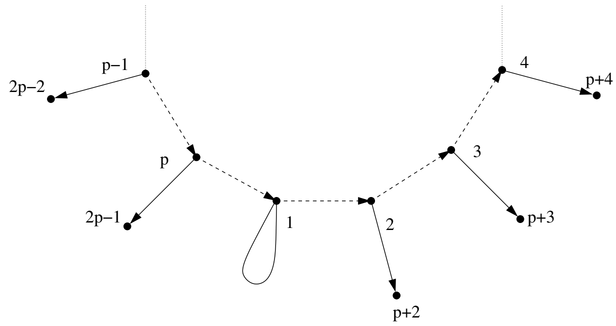

Figure 7.2: The exterior derivative of (7.0.6) in diagrammatic form; the contribution $\overline{\theta}_1$ is indicated by the closed arc at the vertex labeled by 1

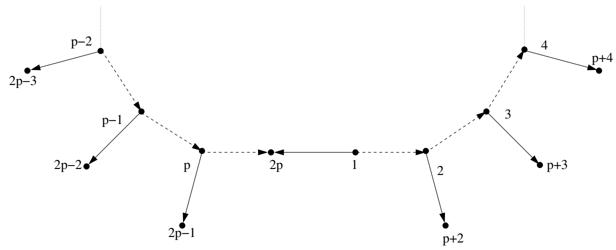

Figure 7.3: The diagrammatic form of the counterterm added to (7.0.6); the principal face corresponding to the collapse of the internal vertices labeled by 1 and $2p$ yields clearly the counterterm to the diagram (7.0.6)

such configuration space integrals in this procedure. In fact, the function Θ_p contains configuration space integrals, whose associated diagrams via the Feynman rules of Section 6.1 are generalizations of the diagrams present in (6.3.5) and (6.4.9) with the Feynman rules of Section 6.1, in the sense that they take the form of regular k-gons (with $p \leq k \leq 2p$) with bivalent or trivalent vertices and each trivalent vertex (be it internal or external) is connected to exactly one univalent internal vertex by a θ form; the edges of such k-gons correspond to η- or θ-forms. Moreover, in the computation of the exterior derivative of Θ_p, again by the generalized Stokes Theorem, we are reduced to compute contributions coming from terms corresponding to faces of codimension 1 of compactified configuration spaces. It is not difficult to see that for the configuration space integrals of Θ_p hold all vanishing lemmata displayed in Section 6.5; hence, we need only consider contributions coming from principal faces and, in even dimensions, the contribution coming from the most degenerate boundary face, where all points collapse together. The sum of the contributions coming from principal faces vanish by the signs coming from (2.4.23) of Subsection 2.4.6 of Chapter 2 and by the action of permutation groups on configuration spaces. The contribution to the exterior derivative of Θ_p coming from the most degenerate boundary face in even dimensions is given (as was the case for (6.4.9)) by (pull-backs of) of forms on the space $\mathcal{I}\left(\mathbb{R}^{m-2}, \mathbb{R}^{m}\right)$ of injective linear maps from $\mathbb{R}m-2$ into $\mathbb{R}^m$, for m even. A generalization of the arguments used in Subsubsection 6.5.4 shows that, in dimension 4, it is possible to correct this "bad" behavior by the addition of a term: the most degenerate contribution of this new function on $\mathrm{Imb}_{\infty,\sigma}\left(\mathbb{R}^2, \mathbb{R}^4\right)$ comes now from an invariant form of degree 3 on $\mathrm{Gr}_{4,2}$ (similarly to Lemma 6.5.16) and hence vanishes by the arguments of Subsection 2.6.6 of Chapter 2.

Finally, we may summarize all these facts by saying that, using the generalized Stokes Theorem, the vanishing lemmata of Subsection 6.5.1 and the arguments of Subsection 6.5.4 and Subsubsection 6.5.4,

a) it is in principle possible to construct, at any order $p \geq 2$, functions Θ_p on the space of long knots in $\mathbb{R}^m$, for $m > 3$, in the form of sums of configuration space integrals in the spirit of Bott and Taubes;

b) it is in principle possible to prove that

    i) the functions Θ_p are d-closed, hence yielding invariants of long knots in $\mathbb{R}^m$, for m odd;

    ii) Θ_p can be modified, in dimension $m = 4$, so as to give a true invariant of long knots;

    iii) Θ_p is, by the addition of an explicit term, a *quasi-invariant for long knots in* $\mathbb{R}^m$, for m even, in the sense that the contribution of the most degenerate is given by the pull-back of a finite sum of biinvariant forms on the Stiefel manifold $V_{m,m-2}$. Hence, in the spirit of [12], taking a sufficiently large number of quasi-invariants Θ_p, for m even and strictly bigger than 4, it is possible to take linear combinations of them in such a way as to get a true invariant of long knots.

Chapter 8

Appendix

8.1 Sign rules

To introduce the dot product, let us for a moment suppose that we have a $\mathbb{Z}$-graded superalgebra E, and let us consider $\Omega^*(M;E)$ with differential

$$\mathrm{d}(\omega \otimes e) := \mathrm{d}\omega \otimes e.$$

Let us pick an element $\omega \otimes e$ in $\Omega^*(M;E)$; we can assign to it two gradings, namely its degree as a form on M and the degree of its E-part; from now on, we will call the degree in E "ghost number." By "homogeneous" in $\Omega^*(M;E)$ we mean from now on any element α of given degree *and* ghost number. We then define the product of homogeneous elements in $\Omega^*(M;E)$ by the rule

$$(\omega \otimes e)\,(\omega' \otimes e') := \omega \wedge \omega' \otimes ee'.$$

The graded Leibniz rule reads

$$\mathrm{d}(\alpha\,\beta) = \mathrm{d}\alpha\,\beta + (-1)^{\deg\alpha}\alpha\,\mathrm{d}\beta, \quad \forall\alpha,\ \beta \in \Omega^*(M;E).$$

In the case when E is supercommutative, it also follows that

$$\alpha\,\beta = (-1)^{\deg\alpha\deg\beta + \mathrm{gh}\,\alpha\,\mathrm{gh}\,\beta}\beta\,\alpha.$$

In case E is associative, we define the super Lie bracket of two homogeneous elements a, b by

$$[\,a\,,\,b\,] := a\,b - (-1)^{\mathrm{gh}\,a\,\mathrm{gh}\,b}b\,a,\ \forall a,b \in E;$$

it satisfies the graded antisymmetry

$$[\,a\,,\,b\,] = -(-1)^{\mathrm{gh}\,a\,\mathrm{gh}\,b}[\,b\,,\,a\,]$$

and the graded Jacobi identity

$$[\,a\,,[\,b\,,\,c\,]] = [[\,a\,,\,b\,]\,,\,c\,] + (-1)^{\mathrm{gh}\,a\,\mathrm{gh}\,b}[\,b\,,[\,a\,,\,c\,]],$$

for all homogeneous $a, b, c \in E$. The super Lie bracket on E can be extended to $\Omega^*(M; E)$ with the help of the wedge product by the rule

$$[\,\alpha \otimes a\,,\,\beta \otimes b\,] := \alpha \wedge \beta \otimes [\,a\,,\,b\,].$$

The graded antisymmetry and the graded Jacobi identity imply

- $[\,\alpha\,,\,\beta\,] = -(-1)^{\deg \alpha \deg \beta + \mathrm{gh}\,\alpha\,\mathrm{gh}\,\beta}[\,\beta\,,\,\alpha\,]$;
- $[\,\alpha\,,\,[\,\beta\,,\,\gamma\,]\,] = [\,[\,\alpha\,,\,\beta\,]\,,\,\gamma\,] + (-1)^{\deg \alpha \deg \beta + \mathrm{gh}\,\alpha\,\mathrm{gh}\,\beta}[\,\beta\,,\,[\,\alpha\,,\,\gamma\,]\,]$,

for all homogeneous forms $\alpha, \beta, \gamma \in \Omega^*(M; E)$.

Remark 8.1.1. It is possible to start directly with a super Lie algebra $\mathfrak{H}$ instead of E. The graded antisymmetry and the graded Jacobi identity in $\Omega^*(M; \mathfrak{H})$ hold as in the previous formulae.

8.1.1 Dot products

Since $\Omega^*(M; E)$ has two gradings, each homogeneous element α in the degree and in the ghost number inherits a new grading, the *total degree*, which is defined by $|\alpha| := \deg \alpha + \mathrm{gh}\,\alpha$.

With the help of the total degree, we can define the *dot product* of two homogeneous forms α, β in $\Omega^*(M; E)$ by the rule

$$\alpha \cdot \beta := (-1)^{\mathrm{gh}\,\alpha \deg \beta} \alpha\, \beta,$$

and accordingly the *dot Lie bracket*

$$[\![\alpha\,;\beta]\!] := (-1)^{\mathrm{gh}\,\alpha \deg \beta}[\,\alpha\,,\,\beta\,].$$

We now list some obvious properties: Let us suppose that E is supercommutative; then

$$\alpha \cdot \beta = (-1)^{|\alpha||\beta|}\beta \cdot \alpha, \qquad \text{(graded commutativity).}$$

For the dot Lie bracket holds in general

$$[\![\alpha\,;\beta]\!] = -(-1)^{|\alpha||\beta|}[\![\beta\,;\alpha]\!] \qquad \text{(graded antisymmetry),}$$
$$[\![\alpha\,;[\![\beta\,;\gamma]\!]]\!] = [\![[\![\alpha\,;\beta]\!]\,;\gamma]\!] + (-1)^{|\alpha||\beta|}[\![\beta\,;[\![\alpha\,;\gamma]\!]]\!], \qquad \text{(graded Jacobi identity),}$$

for all homogeneous forms α, β, γ in $\Omega^*(M; E)$.

Next, we notice that the exterior derivative satisfies the following graded Leibniz rule

$$\mathrm{d}(\alpha \cdot \beta) = \mathrm{d}\alpha \cdot \beta + (-1)^{|\alpha|}\alpha \cdot \mathrm{d}\beta.$$

If we consider an (ungraded) algebra bundle (or more generally, a Lie algebra bundle) $\mathcal{B} \to M$, we can consider the space $\Omega^*(M, \mathcal{B}) \otimes E$, instead of $\Omega^*(M; E)$; we define accordingly the total degree ($\mathcal{B}$ is ungraded and each fiber is an algebra) and the dot product (and the dot Lie bracket).

We next consider a covariant derivative d_A, coming from a connection A on $\mathcal{B}$, and define its action on $\Omega^*(M,\mathcal{B})\otimes E$ by the rule

$$\mathrm{d}_A(\alpha\otimes a) := \mathrm{d}_A\alpha\otimes a.$$

Then, the Leibniz rule for d_A w.r.t. the dot product and the dot Lie bracket follows easily.

8.1.2 Superderivations

We can also consider in this setting the BV operator δ defined by the BV action as a graded derivation on the superalgebra E, which we extend to $\Omega^*(M;E)$ by the rule

$$\delta(\alpha\otimes a) := \alpha\otimes\delta a.$$

It follows:

- $\delta(\alpha\,\beta)=\delta\alpha\,\beta+(-1)^{\mathrm{gh}\,\alpha}\alpha\,\delta\beta$ for homogeneous α,β in $\Omega^*(M;E)$;
- $\delta\circ\mathrm{d}=\mathrm{d}\circ\delta$ on $\Omega^*(M;E)$.

Let us next define $\boldsymbol{\delta}:=(-1)^{\deg}\,1\otimes\delta$, where $(-1)^{\deg}$ is the operator which multiplies each homogeneous form on M by the parity of its degree.

From its very definition, it follows

- $\boldsymbol{\delta}(\alpha\cdot\beta)=\boldsymbol{\delta}\alpha\cdot\beta+(-1)^{|\alpha|}\alpha\cdot\boldsymbol{\delta}\beta$ for homogeneous α,β in $\Omega^*(M;E)$;
- $\boldsymbol{\delta}\circ\mathrm{d}=-\mathrm{d}\circ\boldsymbol{\delta}$ on $\Omega^*(M;E)$.

Remark 8.1.2. The same identities can be proved even when we substitute $\Omega^*(M;E)$ with $\Omega^*(M,\mathcal{B})\otimes E$, for an ungraded algebra bundle $\mathcal{B}\to M$ and the exterior derivative with a covariant one, or if we replace E by a super Lie algebra $\mathfrak{H}$.

We can then define the operator

$$\mathcal{D}:=\mathrm{d}\otimes 1+(-1)^{m+1}\,\boldsymbol{\delta},\qquad m=\dim M;$$

it follows easily from all the above results that it is a superderivation w.r.t. the total degree. Moreover, if δ is nilpotent, then so is $\mathcal{D}$, and consequently a differential on $\Omega^*(M;E)$. If we are dealing with $\Omega^*(M,\mathcal{B})\otimes E$, we can replace d by a covariant derivative d_A and define $\mathcal{D}_A:=\mathrm{d}_A\otimes 1+\boldsymbol{\delta}$, which is then a superderivation. Moreover, if A is flat, $\mathcal{D}_A$ is a superdifferential, too. (Of course any linear combination of $\mathrm{d}\otimes 1$ and $\boldsymbol{\delta}$ has these properties. The conventional choice of the factor $(-1)^{m+1}$ is consistent with the choices made in the rest of the paper.)

In the paper, we also consider a flat background connection A_0 and its relative covariant derivative, along with a sum of forms, which we denote by a, of total degree 1. Then $\mathrm{d}_{\mathsf{A}}=\mathrm{d}_{A_0}+[\![\mathsf{a}\,;\ ]\!]$ defines a superconnection on $\Omega^*(M,\mathrm{ad}\,P)$. In the setting of this Appendix, this is tantamount to choosing forms on $\Omega^*(M,\mathrm{ad}\,P)\otimes E$ of total degree 1; we sum all these forms and obtain a variation of the superconnection A_0. We define accordingly $\mathcal{D}_{\mathsf{A}} := \mathrm{d}_{\mathsf{A}} + (-1)^{m+1}\boldsymbol{\delta}$; it is clear that $\mathcal{D}_{\mathsf{A}}$ is a derivation, and its curvature is given by

$$\mathcal{D}_{\mathsf{A}}^2 = [\![(-1)^{m+1}\boldsymbol{\delta}\mathsf{a} + \mathsf{F}_{\mathsf{A}}\,;\ ]\!] =: [\![\mathcal{F}_{\mathsf{A}}\,;\ ]\!];$$

so (3.4.4) can be interpreted as the vanishing of the curvature $\mathcal{F}_{\mathsf{A}}$ of A on $\Omega^*(M, \operatorname{ad} P)\otimes E$; thus, A is formally "superflat." Similarly, (3.4.5) implies that the superform B (seen as an element of $\Omega^*(M, \operatorname{ad} P) \otimes E$ of total degree $m-2$) is $\mathcal{D}_{\mathsf{A}}$-closed.

8.1.3 Pull-backs and push-forwards

Finally, let $\pi : \mathcal{E} \to M$ be a fiber bundle. We then define the pullback, resp. push-forward, w.r.t. π by the rules

- $\pi^*(\omega \otimes e) := \pi^*\omega \otimes e$, for $\omega \otimes e \in \Omega^*(M; E)$;
- $\pi_*(\eta \otimes e) := \pi_*\eta \otimes e$, for $\eta \otimes e \in \Omega^*(\mathcal{E}, E)$.

It follows immediately that

- $\delta \circ \pi^* = \pi^* \circ \delta$;
- $\delta \circ \pi_* = \pi_* \circ \delta$.

It is then not difficult to verify that

- $\boldsymbol{\delta} \circ \pi^* = \pi^* \circ \boldsymbol{\delta}$;
- $\boldsymbol{\delta} \circ \pi_* = (-1)^{\operatorname{rk}\mathcal{E}}\pi_* \circ \boldsymbol{\delta}$.

By definition of the dot product, it follows (in analogy with the first two equations in (2.2.2))

- $\pi_*(\pi^*\alpha \cdot \beta) = (-1)^{\operatorname{rk}\mathcal{E}|\alpha|}\alpha \cdot \pi_*\beta$;
- $\pi_*(\alpha \cdot \pi^*\beta) = \alpha \cdot \pi_*\beta$.

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