Transverse Signature-Type Change in Singular Semi-Riemannian Manifolds

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♡ *In memory of my father. Semper Desiderari.*

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Abstract

In the early eighties, Hartle and Hawking put forth that signature-type change may be conceptually interesting, paving the way to the so-called no-boundary proposal for the initial conditions for the universe. Such singularity-free universes have no beginning, but they do have an origin of time. This leads to considerations of signature-type changing spacetimes, wherein the "initially" Riemannian manifold, characterized by a positive definite metric, undergoes a signature-type change, ultimately transitioning into a Lorentzian universe without boundaries or singularities. A metric with such a signature-type change is inherently degenerate or discontinuous at the locus of the signature change.

We present a coherent framework for signature-type changing manifolds characterized by a degenerate yet smooth metric. We adapt well-established Lorentzian tools and results to the signature-type changing scenario. Subsequently, we explore global issues, specifically those related to the causal structure, in singular semi-Riemannian manifolds. We introduce new definitions that carry unforeseen causal implications. A noteworthy consequence is the presence of locally closed time-reversing loops through each point on the hypersurface. By imposing the constraint of global hyperbolicity on the Lorentzian region, we demonstrate that throughout every point in M, there always exists a pseudo-timelike loop. Or put another way, there always exists a closed pseudo-timelike path in M around which the direction of time reverses, and a consistent designation of future-directed and past-directed vectors cannot be defined.

Moreover, we present a method for converting any arbitrary Lorentzian manifold (M, g) into a transverse type-changing semi-Riemannian manifold $(M, \tilde{g})$. Then we establish the Transformation Theorem, asserting that, conversely under certain conditions, such a metric $(M, \tilde{g})$ can be obtained from a Lorentz metric g through the aforementioned transformation procedure.

Contents

1 Introduction

1.1 Historical background

According to popular ideas about quantum cosmology, classical cosmological models contain an initial Riemannian region of Euclidean signature joined to a final semi-Riemannian region with the usual Lorentzian signature [25, 26]. In 1983 Hartle and Hawking [42] proposed that signature-type change may be conceptually intriguing, leading to the so-called 'no boundary' proposal for the initial conditions for the universe. According to the Hartle–Hawking proposal the universe has no beginning because there is no singularity or boundary to the spacetime.[1] In such singularity-free universes, there is no distinct beginning, but they do possess an origin of time [25]. Put differently, the positive definite Riemannian space manifests as a closed surface without end, seamlessly connected to a Lorentzian region at the transitional surface where time commences [33, 41].

While the mechanism in its entirety behind the signature-type change in the early universe remains mysterious, Wick rotation is commonly considered a useful computational trick for joining two spaces of different signature.[2] Every process that can be analyzed by resorting to imaginary time, can be also studied by means of signature-type change [42, 65]. This is based on the fact that the signature, which represents the pattern of the signs of the eigenvalues of the matrix of metric coefficients, is a coordinate invariant. However, Hayward [45] stresses that one requires real coordinate charts covering the junction of signature change in order to describe the change of signature in a geometrically respectable way. Recent developments suggest that signature change appears naturally in loop quantum cosmology for the quantum tunneling approach [3, 9, 13].

While from a physical viewpoint, a signature-type change may be used to avoid the singularities of general relativity (e.g. the big bang can be replaced by a Riemannian domain prior to the birth of time), from a mathematical point of view, the 'no boundary' proposal can be taken as 'no boundary' condition for manifolds. This leads to considerations of signature-type changing spacetimes, wherein the

[1] Although singularities can be considered points where curves stop at finite parameter value, a general definition is still deemed to be difficult [30].

[2] The metric signature change from Lorentzian to Euclidean is usually performed by the so-called Wick rotation ($t \to -i\tau$), under which the line element for a Lorentzian spacetime transforms to one for a Riemannian space. This yields an analytic continuation of the time coordinate t to the complex plane.

'initially' Riemannian manifold, characterized by a positive definite metric, undergoes a signature-type change, ultimately transitioning into a Lorentzian universe (i.e. the matrix of metric coefficients, independent of coordinates, possesses precisely one negative eigenvalue) without boundaries or singularities. It is essential to note that a signature-changing metric is inherently degenerate or discontinuous at the boundary of the hypersurface [20]: If we require the eigenvalue switchover to be continuous, then this means that an eigenvalue has to vanish on the surface of signature change, and thus the metric is degenerate at that point (and consequently the inverse metric is singular). Otherwise, if we allow a discontinuous signature change, this necessitates a distributional metric so that the derivatives of the metric are well defined. A trivial example [21] for a signature changing metric is $ds^2 = tdt^2 + dx^2$. The locus of signature change is at $t = 0$, while for $t < 0$ the signature is Lorentzian and for $t > 0$ Riemannian.

The possibility of having different signatures in different regions of spacetime has been discussed in the literature. Extensive mathematical investigations and numerous papers [17, 18, 19, 21, 27, 44, 46, 52, 87] have emerged which elucidated the mathematical implications of signature-type change, spanning differential geometry, topology, mathematical relativity, and physics. For instance, the continuity and behavior across the interface between the Riemannian and Lorentzian domains were analyzed, or the general constraints posed by a possible signature flip somewhere in the manifold were put forward. Some work just focussed on conditions for the co-existence of Euclidean and Lorentzian domains. Meanwhile also in modern subfields of differential geometry, such as Finsler geometry, signature change gets investigated [5].

But for all that, the topic remains underexplored, presenting an understudied subject at the intersection of differential geometry, (differential) topology, mathematical relativity, physics and even spectral geometry. The latter comes into play if field theory in a background spacetime is considered: By crossing over from the Lorentzian to the Riemannian domain the d'Alembert operator changes into the Laplacian, and by that, the Klein-Gordon equation becomes an eigenvalue problem of the Laplacian (with the eigenvalues corresponding to the possible values of the rest mass). All in all, signature change offers not only interesting new territory in differential geometry but also the chance to find results challenging conventional wisdom about the origin and evolution of the universe and its matter content.

1.2 Summary

In signature-type changing metrics, either some eigenvalue of the metric goes through the value zero, resulting in metric degeneracy, or it undergoes a jump from a positive to a negative value, causing metric discontinuity [25]. Singular semi-Riemannian geometry is concerned with smooth manifolds endowed with a singular metric tensor of arbitrary signature. In this work we dispense with studying the discontinuous type of metrics. Instead, we focus exclusively on the continuous approach, specifically with a *transverse radical* where the metric constitutes a smooth $(0, 2)$-tensor field that becomes degenerate at the subset $\mathcal{H} \subset M$, representing a smoothly embedded hypersurface in M.[3] The bilinear type of the metric changes upon crossing $\mathcal{H}$.

The initial objective of this dissertation is to establish a coherent framework for signature-type changing manifolds. This framework will serve as a foundation for the subsequent analysis of global issues, with a specific focus on the causal structure, in transverse singular semi-Riemannian manifolds.

Although the compatibility of Riemannian and Lorentzian domains is assumed to be given, the behavior of curves across the interface between the Riemannian and Lorentzian sectors is still left to further studies. Moreover, in a manifold where the signature changes from $(+, +, \ldots, +)$ to $(-, +, \ldots, +)$, the conventional concept of timelike (or spacelike) curves does not exist anymore. This gives rise to a new notion of curves called pseudo-timelike and pseudo-spacelike curves. In order to define these curves we make a detour to draw upon the concept of the generalized affine parameter which we use as a tool to distinguish genuine pseudo-timelike (and pseudo-spacelike, respectively) curves from curves that asymptotically become lightlike as they approach the hypersurface of signature change.

Equipped with this information, we first make drastic simplifications by reducing the dimensionality of a manifold. We initially explore what happens in close proximity to the hypersurface $\mathcal{H}$ within a 2-dimensional toy model setting. The intricacies of the toy model prompt intriguing questions related to geodesics, particularly in the context of interpreting particle behavior when crossing the junction. This raises fundamental queries about the compatibility of geodesics across the

[3]The radical at $q \in \mathcal{H}$ is defined as the subspace $Rad_q := \{w \in T_qM \mid g(w, \mathbf{.}) = 0\}$. This means $g(v_q, \mathbf{.}) = 0$ for all $v_q \in Rad_q$. Note that Rad_q can be either transverse or tangent to the hypersurface $\mathcal{H}$. Please refer to Definition 3.2 and Section 3 and 7 for details on the radical.

transition surface. Furthermore, challenges emerge in defining the Levi-Civita connection on the entire manifold, and as a consequence, in constructing the curvature invariants. These issues arise due to the fact that the inverse components of the metric blow up at the hypersurface of signature change.
Nevertheless, we can demonstrate the existence of an isometric embedding of the 2-dimensional toy model into 3-dimensional Minkowski space. Leveraging this embedding, we utilize the Gaussian curvature to highlight that, despite the non-smooth nature of the affine connection, there exist non-trivial singularity-free models within the considered class of signature-type changing manifolds.

In order to generalize the 2-dimensional signature-type changing manifold to the n-dimensional scenario, we bring in radical-adapted Gauss-like coordinates. These coordinates can be viewed as time-orthogonal coordinates for an n-dimensional signature-type changing manifold. These coordinates not only greatly simplify matters, but also imply the existence of a unique, coordinate independent, natural absolute time function in the neighborhood of the hypersurface.

In the global context, we establish criteria to determine which geodesics, and more generally, curves, along with their associated parallel vector fields, possess the property of smoothly extending across the hypersurface, defining them on the entire manifold M. This lays the foundation for the subsequent construction [84] of objects such as the covariant derivative of differential forms, the lower Riemann curvature operator, the Ricci curvature tensor and scalar curvature. As a consequence we can single out the pairs of vector fields X, Y on M with the property that the covariant derivative of Y in the direction of X extends smoothly to all of M. The properties of the lower covariant derivative guarantee the definition [50] of the so-called natural fundamental tensor II_q, serving as a suitable tool to characterize the nature of the hypersurface of signature change. On the basis of II_q the following two pieces of information can be extracted:

(i) we can determine which pairs of vector fields $X, Y \in \mathfrak{X}(M)$ have the property that the associated covariant derivative extends smoothly to all of M,

(ii) given a curve γ, with $\gamma(0) = q \in \mathcal{H}$, then II_q determines which parallel vector fields along γ extend smoothly through the hypersurface at q.

In the second part of this dissertation, our main emphasis is on global considerations. We endeavor to adapt well-established Lorentzian tools and results to the signature-type changing case, as far as possible. This task proves to be less straightforward than anticipated, necessitating the introduction of new definitions with unexpected causal implications, reaching a critical juncture in our exploration. We draw upon the definition of pseudo-time orientability and the given absolute time function to decide whether a pseudo-timelike curve is future-directed. This establishes the definition for the pseudo-chronological past (and pseudo-chronological future) of an event. In addition, it becomes imperative to introduce a reasonable definition for the causal classification of hypersurfaces in a signature-type changing manifold.

In order to elucidate the global structure of non-orientable signature-type changing manifolds we produce manifolds modeled on the topology of the Möbius strip: It is a well-known fact that, given two manifolds with homeomorphic boundaries,[4] one can obtain a new manifold by "gluing", achieved by identifying the two boundaries. Similarly, two connected components of the boundary of a single manifold can be glued together. This cutting-gluing surgery on semi-Riemannian manifolds yields a new signature-type changing manifold where the gluing junction becomes the locus of signature change. Some of these examples do not belong to the class under consideration, as they inevitably result in a type-changing metric with a radical that is both transverse and tangent.

Motivated by these counterexamples, we present a theorem to determine whether a singular signature-type changing manifold under consideration belongs to the class of *transverse type-changing semi-Riemannian manifolds* with a *transverse radical:* Firstly, we present a procedure called the Transformation Prescription by means of which an arbitrary Lorentzian manifold can be transformed into a singular signature-type changing manifold. Subsequently, we employ the Transformation Prescription to establish the so-called Transformation Theorem saying that locally the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ with a transverse radical is equivalent to the metric obtained from a Lorentzian metric g via the Transformation Prescription $\tilde{g} = g + fV^\flat \otimes V^\flat$,[5] where $f : M \longrightarrow \mathbb{R}$ is a C^∞ function and V is one of the pair $\{V, —V\}$ of a global non-vanishing timelike line element field. By augmenting the assumption by certain constraints, mutatis mutandis, the global version of the Transformation Theorem can be proven as well. Furthermore, we argue that by adjusting some assumptions this theorem can be also shown for type-changing manifolds with a tangent radical.

[4]Granted that both manifolds are smooth with nonempty boundary, having a collar neighborhood. In addition, suppose there is a diffeomorphism between the collar neighborhoods.

[5]The index lowering morphism flat $\flat\colon T_pM \to T_p^*M$ for $p \in M$ corresponds to the Lorentzian metric g, and is given by $v \mapsto \flat(v) = v^\flat = g(v, \centerdot)$.

Ultimately, we utilize the Transformation Theorem to demonstrate that the induced metric on the hypersurface $\mathcal{H}$ is either Riemannian or a positive semi-definite pseudo metric.

In conclusion, we show that for signature-type change of the delineated type, all these considerations lead to a surprising theorem revealing the non-well-behaved nature of these manifolds: In a sufficiently small region near the junction of signature change $\mathcal{H}$ transverse signature-type changing manifolds with a transverse radical exhibit local anomalies: Specifically, each point on the junction facilitates a closed time-reversing loop, challenging conventional notions of temporal consistency. Or put another way, there always exists a closed pseudo-timelike path in M around which the direction of time reverses, and along which a consistent designation of future-directed and past-directed vectors cannot be defined. By imposing the constraint of global hyperbolicity on the Lorentzian region, the global analog can be proven by showing that through every point in M there always exists a pseudo-timelike loop.[6]

Another approach to identify closed pseudo-timelike curves or loops is to consider a certain class of non-smooth metrics which have an infinite discontinuity at $t = 0$ and apply a conformal transformation. Taking advantage of the fact that the causal structures of conformally equivalent manifolds is the same, yields immediately pseudo-timelike geodesics with loops. This could be tantamount to a region of particle-antiparticle origination incidents.

[6] In the global version a key notion is global hyperbolicity which is developed and which plays a role in the spirit of completeness for Riemannian manifolds.

1.3 Main results

Throughout this work, especially in Chapter 8, we adapt familiar Lorentzian tools and results to the signature-type changing setting. We introduce the definition of pseudo-time orientability to determine the future-directed nature of a pseudo-timelike curve, laying the foundation for defining the pseudo-chronological past (and pseudo-chronological future) of an event. Our rationale behind these new definitions stems from the challenge posed by the locus of signature change, where many concepts from Lorentzian geometry become problematic or nonsensical. For instance, the interpretation of 'spacelike' curves, even if they are geodesics, can vary. Chapter 4 introduces a novel concept of curves called pseudo-timelike curves, along with a tool to distinguish genuine pseudo-timelike (and pseudo-spacelike, respectively) curves from those that asymptotically become lightlike as they approach the locus of signature change.

In Chapter 5, we utilize the new definition introduced in Chapter 4 and examine a toy model example. Chapter 6 extends this exploration by demonstrating how a generalized form of the toy model is situated within 3-dimensional Minkowski space, establishing the existence of an isometric embedding $f = (\vartheta, \xi, x)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^{1,2}$. In this particular context we illuminate that despite the affine connection not being smooth, there exist non-trivial singularity free models within the class of signature-type changing manifolds in consideration.

In Chapter 7 we elucidate the properties of the radical, and we provide examples when the radical is one-dimensional and transversal with respect to $\mathcal{H}$. Furthermore, we illustrate with counter examples under what conditions the radical becomes tangent.

In Chapter 11 we elaborate on the fact that a transverse type-changing semi-Riemannian manifold (M, g) with a transverse radical is conformally equivalent to a particular type-changing semi-Riemannian manifold $(M, \bar{g})$. This manifold is equipped with a non-smooth metric, featuring an infinite discontinuity at $t = 0$. The latter one has pseudo-timelike geodesics with loops. We take advantage of the fact that the causal structures of $(M, \bar{g})$ and (M, g) are always the same. Consequently, we utilize what we know about the causality of $(M, \bar{g})$ to identify closed pseudo-causal curves or loops in (M, g). Summarized, we show that there is an arbitrary conformal factor $\Omega \in \mathscr{F}(M)$, defined by $\Omega(t) = f(t)\mathrm{sgn}(f(t))$, such that $(M, g) := (M, \Omega\bar{g})$ is a causality-violating, transverse type-changing singular semi-Riemannian manifold with a transverse radical, and the metric is given by

$g = \Omega\bar{g} = -f(t)(dt)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2.$

In Chapter 12 we address the problem arising when dealing with degenerate metrics: the definition of the Levi-Civita connection (on the whole of M) is no longer possible. The issues surrounding the extendability of geodesics across the hypersurface, the smooth extension of parallel transport to the entirety of M, and the existence of the Levi-Civita connection on the hypersurface are intricately interlinked. We clarify that geodesics cannot traverse a point $q \in \mathcal{H}$ in any arbitrary direction; rather, they can only follow a particular admissible direction.

Chapter 13 deals with the Transformation Prescription and the Transformation Theorem: We present a procedure (called the Transformation Prescription) designed to transform any arbitrary Lorentzian manifold into a singular signature-type changing manifold. Notably, under certain conditions, the Transformation Prescription yields a transverse, type-changing semi-Riemannian manifold $(M, \tilde{g})$ with a transverse radical. Then we prove the Transformation Theorem, which asserts that, conversely, such a metric $(M, \tilde{g})$ can be locally obtained from a Lorentzian metric g through the aforementioned transformation procedure. By augmenting the assumption by specific constraints, mutatis mutandis, the global version of the Transformation Theorem is proven as well.

Moving on to Chapter 14, we employ the Transformation Theorem to demonstrate that the induced metric on the hypersurface $\mathcal{H}$ is either Riemannian or a positive semi-definite pseudo-metric.

In Chapter 15, our objective is to illuminate the global structure of non-orientable signature-type changing manifolds. To achieve this, we construct manifolds modeled on the topology of the Möbius strip. We generate one set of examples through a process of "gluing", resulting in a new signature-type changing manifold with the topology of the crosscap, where the gluing junction serves as the locus of signature change. In another set of examples, we leverage the Transformation Theorem and apply it to the Möbius strip. Then we test whether the obtained metrics are of the type $\tilde{g} = g + f(V^\flat \otimes V^\flat)$, with an arbitrary smooth transformation function f that interpolates between the Lorentzian and Riemannian regions, which are separated by the hypersurface $\mathcal{H} = f^{-1}(1)$. This exploration provides valuable insights into the diverse possibilities and structures of non-orientable signature-type changing manifolds.

In Chapter 16, we elaborate on the observation that in a sufficiently small region near $\mathcal{H}$, transverse signature-type changing manifolds with a transverse radical exhibit a non-well-behaved nature. At each point on the hypersurface $\mathcal{H}$, a closed time-reversing loop is locally present. Or put another way, there always exists a closed pseudo-timelike path in M around which the direction of time reverses, and along which a consistent designation of future-directed and past-directed vectors cannot be defined. We then propose an interpretation of this result, suggesting that it could be analogous to a region where incidents of particle-antiparticle origination occur.

By augmenting the assumption by certain constraints, mutatis mutandis, a global analog can be proven as well. In the global version a key notion is global hyperbolicity which is developed and which plays a role in the spirit of completeness for Riemannian manifolds.
By imposing the constraint of global hyperbolicity on the Lorentzian region, we demonstrate that through every point in M, there always exists a pseudo-timelike loop. In simpler terms, a transverse, signature-type changing manifold with a transverse radical has always pseudo-timelike loops.

The *main original work* in this dissertation is presented in Chapter 6, Chapter 8, Chapter 13, Chapter 14, Chapter 15 and Chapter 16.

2 Notation, conventions and preliminaries

Unless otherwise specified, the considered manifolds, denoted as M with dimension $\dim(M) = n$, are assumed to be locally homeomorphic to $\mathbb{R}^n$. Furthermore, these manifolds are expected to be connected, second countable, and Hausdorff. This definition also implies that all manifolds have empty boundary. If not indicated otherwise, all associated structures and geometric objects (curves, maps, fields, differential forms etc.) are assumed to be smooth. Furthermore, we will usually assume that the manifolds we consider are smooth as well.

Definition 2.1. (Smooth vector field) A smooth vector field over M is a section of $T(M)$. The space of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Definition 2.2. (Parametrized curve) A (parametrized) curve in a manifold M is a smooth map $\gamma\colon [a,b] \longrightarrow M$ with $[a,b] \subset \mathbb{R}$, $-\infty < a < b < \infty$. We also allow for curves to be defined on non-compact subsets of the real numbers or on all of $\mathbb{R}$. In situations where we are interested in the image $\mathrm{im}_\gamma([a,b]) \subset M$ of a curve only, we will refer to that image as an *unparametrized curve*.

Given a smooth parametrized curve $\gamma : [a,b] \longrightarrow M$ and $t_0 \in [a,b]$, we define the **velocity** of γ at t_0, denoted by $\dot\gamma(t_0)$, to be the vector $\dot\gamma(t_0) = d\gamma\left(\frac{d}{dt}\mid_{t_0}\right) = \frac{d\gamma^\mu}{dt_0}\partial_\mu \in T_{\gamma(t_0)}M$, where $\frac{d}{dt}\mid_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$. This tangent vector acts on functions by $\dot\gamma(t_0)f = d\gamma\left(\frac{d}{dt}\mid_{t_0}\right)f = \frac{d}{dt}\mid_{t_0}(f\circ\gamma) = (f\circ\gamma)'(t_0)$.

For smooth curves defined on an arbitrary differentiable manifold M we want to point out the following

Definition 2.3. (Closed curve) A curve $\gamma\colon [a,b] \longrightarrow M$ is called *closed*, if there is a curve $\tilde\gamma\colon \mathbb{R} \longrightarrow M$ with $\tilde\gamma\mid_{[a,b]} = \gamma$ and $\tilde\gamma(s+b-a) = \tilde\gamma(s)$ for all $s \in \mathbb{R}$, where in particular $\gamma(a) = \gamma(b)$ and all derivatives match, i.e. $\gamma'(a) = \gamma'(b)$, $\gamma''(a) = \gamma''(b)$ et cetera. The lifted curve $\tilde\gamma$ is also called periodic [61, 86].

A closed curve γ is said to be *simply closed*, if $\gamma\mid_{[a,b)}$ is injective, i.e. if there are no double points for which $\gamma(s_1) = \gamma(s_2)$ for some $a \leq s_1 < s_2 < b$. Intuitively, a simply closed curve never intersects itself between its endpoints (except of course for the closing up at $s = a, b$). Alternatively, we could define a closed curve (or simply closed curve) as an immersion (or embedding, respectively) of the circle S^1 in M.

Definition 2.4. (Closed geodesic) A geodesic $\gamma\colon [a,b] \longrightarrow M$ is closed if there is a $c \in \mathbb{R}_+$ such that $\tilde{\gamma}(s+c) = \tilde{\gamma}(s)$ for every $s \in [a,b]$ and if γ satisfies the assumption that $\gamma(a) = \gamma(b)$ and all derivatives match: $\gamma'(a) = \gamma'(b)$, $\gamma''(a) = \gamma''(b)$ et cetera, $a, b \in I$. If the latter condition is dropped, the curve γ is called a *geodesic loop* [71].

In semi-Riemannian geometry, the geometrical structure of a smooth manifold is determined by the metric: A **metric tensor** g is a smooth symmetric tensor field of type $(0,2)$ on M whose components $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ in any coordinate system are C^∞ functions.[7] That is, the metric acts on elements of T_pM as a bilinear function; the metric tensor g smoothly assigns to each point p of M an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$.

Definition 2.5. A metric tensor g is **non-degenerate** if for any $p \in M$, $g(u,v) \mid_p = 0$ for all $v \in T_pM$, then $u = 0$. In local coordinates this is equivalent to g being non-degenerate if and only if $\det([g_{\mu\nu}]) \neq 0$ [4].

The **index** associated to a metric tensor g on a smooth manifold M is a nonnegative integer I for which $index(g,p) = I$ for all $p \in M$: The index of the inner product space is the number of minus signs in the signature. This provides an alternative tool by which one can define a number of various notions typically associated to the signature of g. The **signature** of the metric g at p is the number of positive eigenvalues of the symmetric matrix $[g_{\mu\nu}]$ at p, minus the number of negative ones, i.e. $(\dim(M) - I) - I = \dim(M) - 2I$.[8] The number of nonzero eigenvalues is the rank of the metric. If the metric is nondegenerate the rank is equal to the dimension $\dim(M)$ and if the metric is continuous, the signature and the rank of the metric tensor are the same at every point. Then (M,g) is said to have *constant signature.*

The signature is a well-defined invariant and classifies the metric up to a choice of basis. Thus, for any point p of the Lorentzian manifold (M,g) the tangent space T_pM admits a basis, called the orthonormal basis, such that the metric components

[7] More precisely, the manifold (M,g) is called semi-Riemannian if g is non-degenerate. If g is also positive definite, then (M,g) is called Riemannian manifold.

[8] Equivalently, the signature of a symmetric matrix $[g_{\mu\nu}]$ at p expresses the number of positive and negative eigenvalues, counting multiplicities. Hence, it can be also defined in terms of positive and negative entries of the diagonalized matrix.

at any point p can be brought to the canonical form

$$g_{\mu\nu} = \mathbf{diag}(-1, \underbrace{+1, \ldots, +1}_{(\dim(M)-1)}).$$

In other words, for the integer $index(g,p) = I$ at each point $p \in M$, there is a basis $e_0, \ldots, e_I, \ldots, e_{n-1} \in T_pM$ such that $g(e_\mu, e_\mu) = -1$ for $0 \leq \mu \leq I$, $g(e_\mu, e_\mu) = 1$ for $I+1 \leq \mu \leq n-1$ and $g(e_\mu, e_\nu) = 0$ for $\mu \neq \nu$.

A **Riemannian manifold** is an ordered pair (M, g), consisting of a smooth manifold M furnished with a metric tensor g with $I = 0$; i.e. $g(e_\mu, e_\mu) = 1$ for all $\mu = 0, 1, \ldots, n-1$. Each g_p is then a positive definite inner product on T_pM. If $I \neq 0$, then is M a **semi-Riemannian manifold**. A **Lorentzian manifold** is a special type of a semi-Riemannian manifold, defined as a pair (M, g) for which $\dim(M) \geq 2$ and for which $I = 1$, i.e. $g(e_0, e_0) = -1$ and $g(e_i, e_i) = 1$ for $i = 1, \ldots, n-1$. Alternatively it can be defined as a manifold M of dimension $\dim(M) \geq 2$, equipped with a tensor g of metric signature $(\dim(M) - 1) - 1 = \dim(M) - 2$. Henceforth, our convention for the Lorentzian signature is $(-, +, \ldots, +)$.

If we allow (M, g) to be of *variable signature*, or if notably the metric tensor field has singular points (i.e. points where g degenerates), then the pair (M, g) is called **singular semi-Riemannian manifold** and for any point $p \in M$ there exists a basis $\alpha_1, \ldots, \alpha_n \in T_pM$ such that

- $g(\alpha_i, \alpha_j) = 0$ for $i \neq j$,
- $g(\alpha_i, \alpha_i) = 0$ for $1 \leq i \leq \mu$,
- $g(\alpha_i, \alpha_i) = -1$ for $\mu + 1 \leq i \leq \mu + \nu$,
- $g(\alpha_i, \alpha_i) = 1$ for $\mu + \nu + 1 \leq i \leq \mu + \nu + \eta = n$,

where μ, ν, η are certain integers, and $\mu = \dim(Rad_p)$ (jump to Section 3 for more details). Then the triple (μ, ν, η) is called the type of g at p, and $\alpha_1, \ldots, \alpha_n \in T_pM$ is an orthonormal basis. If $\mu = 0$ then g is called a *non-degenerate inner product* of type (ν, η) [59]. The relations $\dim M = \mu + \nu + \eta$ and $\text{rank}(g) = \nu + \eta = n - \mu$ hold. In this context also recall the following three definitions:

Definition 2.6. (Singular semi-Riemannian manifold). A *singular semi-Riemannian manifold* is a generalization of a semi-Riemannian manifold. It is a differentiable manifold having on its tangent bundle a symmetric bilinear form which is allowed to become degenerate.

Definition 2.7. Let (M, g) be a singular semi-Riemannian manifold and let be $p \in M$. We say that the metric changes its signature at a point $p \in M$ if any neighborhood of p contains at least one point q where the metric's signature differs from that at p.

The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy.

Definition 2.8. A tangent vector $v \in T_pM$ is

- spacelike if $g(v, v) > 0$ or $v = 0$,
- null if $g(v, v) = 0$ and $v \neq 0$,
- timelike if $g(v, v) < 0$.

In Lorentzian geometry, a nonzero vector is causal if it is either timelike or null. Moreover, a (suitable parametrized) curve is said to be **chronological** when its tangent vector at any point is timelike, and **causal** when its tangent vector is anywhere timelike or null. Let J_g be the set of all causal vectors and I_g the set of all timelike vectors. Then clearly, both $I_g \cap T_pM$ and $J_g \cap T_pM$ have two connected components in any tangent space T_pM, $p \in M$. A time orientation of a manifold (M, g) is the specific choice at each $p \in M$ of one such component I_g^+, which is then named timelike future. The timelike past is consequently defined as its I_g-complement $I_g^- := -I_g^+$. The causal future J_g^+ is defined analogously.

Consequently, J_p is disconnected and consists of two connected components and timelike tangent vectors at each point $p \in M$ can be divided in two distinct classes, allowing the distinction between past- or future-pointing. A semi-Riemannian manifold is **time-orientable** if a continuous designation of future-pointing and past-pointing for non-spacelike vectors can be made over the entire manifold. The choice for past or future is arbitrary, as long as it is consistent within all points in spacetime. The value of the metric applied to two timelike vectors of the same class is strictly negative, while it can assume any non-zero value if one vector is past-pointing and the other is future-pointing.

Let $\gamma\colon [a,b] \longrightarrow M$ be a smooth curve. If the tangent vectors have a positive time component $\gamma'(s) \in I_g^+ \ \forall s \in [a,b]$ (respectively, negative) at all points, then the curve is **future-directed** (respectively past-directed).

Let there be given a semi-Riemannian manifold (M,g) with a connection ∇ that is compatible with the metric g. Recall that a vector field $X \in \mathfrak{X}(M)$ is **autoparallel** in an open set $U \subseteq M$ if and only if $\nabla_X X \mid_U \equiv 0$. Let ∇ denote the Levi-Civita connection for a non-degenerate g, then a *geodesic* is an integral curve of an autoparallel vector field.

3 Singular semi-Riemannian manifolds

Spacetimes within Einstein's General Theory of Relativity are conventionally considered to be 4-dimensional, connected, time-orientable Lorentzian manifolds of index one which are furnished with a non-degenerate metric tensor. However, in this work we relax the non-degeneracy assumption. Since a signature-type changing metric is necessarily either degenerate or discontinuous at the locus of signature change [20], this means we allow for the metric to become degenerate. Also, we expand our considerations to n-dimensional manifolds, and permit a manifold to be non-orientable as well as non-time-orientable.

Let (M, g) be a singular semi-Riemannian manifold of $\dim M = n \geq 2$ and let g be a smooth, symmetric, degenerate $(0, 2)$-tensor field on M. Furthermore, let $\mathcal{H}$ be the signature-change subset of M so as for every point $p \in M$ there exists a neighborhood U of p in M and a continuously differentiable function $\rho\colon U \longrightarrow \mathbb{R}$, such that $\mathcal{H} \cap U = \{q \in U \mid \rho(q) = 0\}$. Then $\rho(q) = 0$ is an equation for $\mathcal{H}$ and ρ is the *function* producing the surface of signature change.

We follow [2, 52] by defining transverse type-changing singular semi-Riemannian manifolds:

Definition 3.1. We call g a codimension-1 **transverse type-changing metric** if $d(\det([g_{\mu\nu}])_q) \neq 0$ for any $q \in \mathcal{H}$ and any local coordinate system $\xi = (x^0, \ldots, x^{n-1})$ centered at q. Then we call the pair (M, g) a **transverse type-changing singular semi-Riemannian manifold**.

This implies that the subset $\mathcal{H} \subset M$ is a smoothly embedded hypersurface in M, and the bilinear type of g changes upon crossing $\mathcal{H}$. Moreover, at every point $q \in \mathcal{H}$ there exists a one-dimensional subspace $Rad_q \subset T_qM$ that is orthogonal to all of T_qM.

Definition 3.2. (Radical) The radical at $q \in \mathcal{H}$ is a totally degenerate space, defined as the subspace $Rad_q := \{w \in T_qM \mid g(w, \centerdot) = 0\}$.[9]

All this is valid without any assumption on the radical. On account of this, we require that the radical is transverse qua the ususal

[9] This means $g(v_q, \centerdot) = 0$ for all $v_q \in Rad_q$. Note that Rad_q can be either transverse or tangent to the hypersurface $\mathcal{H}$.

Definition 3.3. We call the radical Rad_q **transverse** [55] if Rad_q and $T_q\mathcal{H}$ span T_qM for any $q \in \mathcal{H}$, i.e. $Rad_q + T_q\mathcal{H} = T_qM$.[10]

Henceforward, we assume throughout that (M, g) is a transverse type-changing singular semi-Riemannian manifold, unless explicitly stated otherwise. Moreover, with this convention we only consider – unless otherwise stated – transverse type-changing singular semi-Riemannian manifolds with a *transverse radical*. In this work we dispense with studying discontinuous type-changing semi-Riemannian manifolds as well as with transverse type-changing singular semi-Riemannian manifolds with a *tangent radical*. In addition, we assume that one connected component of $M \setminus \mathcal{H}$ is Riemannian and all other connected components $(M_{L_\alpha})_{\alpha \in I} \subseteq M_L \subset M$ are Lorentzian.
What is more, we assume throughout that the point set $\mathcal{H}$ where g becomes degenerate is not empty. This means $\mathcal{H}$ is the locus where the rank of g fails to be maximal. These delimitations narrow down our study of signature change to a geometrical analysis of embedded submanifolds.

Proposition 3.4. *Let (M, g) be a singular semi-Riemannian manifold whose metric is degenerate at the hypersurface $\mathcal{H} \coloneqq \{q \in M : g\mid_q \text{ is degenerate}\}$, and divides M into two open connected regions M_L and M_R with a common connected boundary $\mathcal{H}$. Then the following properties hold:*

$M = M_L \cup \mathcal{H} \cup M_R$,
$M_L \cap M_R = \emptyset$,
$\overline{M}_L \cap \overline{M}_R = \mathcal{H}$,
$\overline{M}_L \cup \overline{M}_R = M$,

where M_L denotes the Lorentzian component and M_R denotes the Riemannian one. These two components are separated by the hypersurface of signature change $\mathcal{H} = M \setminus (M_L \cup M_R)$.

Note that in the degenerate case the metric is smooth and both, the Lorentzian metric as well as the Riemannian metric, exist as continuous limits and agree on $\mathcal{H}$: $g_L\mid_{\mathcal{H}} = g_R\mid_{\mathcal{H}}$. In other words, the continuous metric

$$g = \begin{cases} g_L & \text{on } \overline{M}_L \\ g_R & \text{on } \overline{M}_R \end{cases}$$

[10] This means that Rad_q is not a subset of $T_q\mathcal{H}$, and Rad_q is obviously not tangent to $\mathcal{H}$ for any q.

exists on M. Note that if we allow for metrics that are discontinuous on some hypersurface, then the signature of the metric may still change at that surface, but the considerations in Proposition 3.4 are not valid anymore.

Definition 3.5. Let (M, g) be a singular semi-Riemannian manifold. Then a vector field $X \in \mathfrak{X}(M)$ on M is called **pseudo-timelike** if for each $p \in M_L$ the tangent vector X_p at p is timelike.[11]

[11] Thus the associated integral curves are also pseudo-timelike.

4 Pseudo-timelike curves

Let (M, g) be an n-dimensional Lorentzian manifold. Then a curve $\alpha\colon I \longrightarrow M$ such that $g(\alpha'(t), \alpha'(t)) < 0$ for every $\alpha'(t)$ is called timelike, a curve with $g(\alpha'(t), \alpha'(t)) > 0$ for every $\alpha'(t)$ is called a spacelike, while one with $g(\alpha'(t), \alpha'(t)) = 0$ for every $\alpha'(t)$ is called a null curve. The latter one represents a path through spacetime of a light signal or a particle moving at the speed of light. A timelike curve that loops back on itself is a closed timelike curve (CTC).

In an n-dimensional manifold (M, g) where the signature-type changes from $(+, +, \cdots, +)$ to $(-, +, \cdots, +)$, the conventional concept of a timelike curves does not make sense anymore. From a suitable given point in the Lorentzian region, the junction might be reached in finite proper time, but there is no time concept in the Riemannian region.[12] Hence, curves in the Riemannian domain are devoid of causal meaning and cannot be distinguished as timelike, spacelike or null. In signature-type changing manifolds this gives rise to a novel notion of curves. In order to define those curves we have to make a detour to draw upon the concept of the generalized affine parameter.

4.1 Properties of the generalized affine parameter

In this section we want introduce the notion of *pseudo-timelike curves* and *pseudo-spacelike curves*. However, we need a method to discern genuine pseudo-timelike (and pseudo-spacelike, respectively) curves and such curves that asymptotically become lightlike as they approach the hypersurface of signature change. The generalized affine parameter will prove useful to draw this distinction. We will utilize the definition of b-completeness in Lorentzian manifolds as ansatz and then use the general affine parameter in a signature-type changing setting in order to define pseudo-timelike and pseudo-spacelike curves.

In common parlance, singularities are boundary sets of a manifold where the manifold structure breaks down. Usually this is visualized as a flaw in the fabric of spacetime.[13] Geroch [30] gave the first elaborated disquisition on the difficulty of framing a reasonable definition of a singular spacetime. As maintained by the Penrose–Hawking singularity theorems, singularities arise in a broad variety of space-

[12] Kossowski and Kriele [52] show that the assumption of signature-type change in general relativity implies the existence of a uniquely determined natural time function which yields a reasonable time concept in the Riemannian region. This result will be addressed later.

[13] Note that the singularity itself does not belong to the manifold.

times. Strongly related to the idea of extensions, curve incompleteness is a crucial notion in order to characterize singularities in a spacetime. Moreover, curve incompleteness is the feature that is widely recognized as the most consensual account of spacetime singularities (see, for instance, Wald [90], Section 9.1).

Definition 4.1. (Complete and incomplete curves) A complete curve is a curve which can be extended in both directions for arbitrarily large values of a specified parameter. A curve which is not complete is incomplete.

It appears that we should look at physically significant curves, such as geodesics and curves with a bounded acceleration, so that we can identify singularities in a spacetime. Notably, inextensible curves of finite length could point to the existence of a spacetime singularity as we might think of these inextensible curves being finite because they "hit" the singularity. That is to say, freely moving observers or particles moving along such an incomplete curve would simply disappear after a finite amount of proper time.

To investigate a broader class of curves beyond geodesics (i.e., non-geodesic curves), we require a method to characterize their completeness [4, 43]. This is in view of the fact that we face the following problem whenever we choose the proper time as a parameter: Every spacetime contains inextensible timelike curves of finite proper length that asymptotically become lightlike (i.e. curves of unbounded acceleration that go to infinity).[14] Considering this reality we would have to classify Minkowski space as singular which clearly is unreasonable. What we need is some generalization of the concept of an affine parameter to all C^1 curves, geodesic or non-geodesic, to tell apart suitable finite and inextensible curves from unsuitable ones. Hence, we require a notion of completeness so that every C^1 curve of finite length as measured by such a parameter has an endpoint. Ehresmann [23] and later Schmidt [78] appear to have been the first ones to propose using so-called *generalized affine parameters* to define the completeness of general curves [78]. The generalized affine parameter turns out to be a useful quantity to probe singularities because it can be defined for an arbitrary curve which is not necessarily a geodesic.

Definition 4.2. (Generalized affine parameter) Let M be a manifold of $\dim(M) = n$, with an affine connection and $\gamma\colon J \to M$ a C^1 curve on M. Recall that a smooth vector field V along γ is a smooth map $V\colon J \to TM$ such that

[14] This also applies to our signature-type changing models; but in this case the timelike curves that asymptotically become lightlike actually are lightlike at the hypersurface of signature change.

$V(t) \in T_{\gamma(t)}M$ for all $t \in J$. Such a smooth vector field V along γ is said to be a parallel field along γ if V satisfies the differential equation $\nabla_{\gamma'}V(t) = 0$ for all $t \in J$, see [4] for further details.

Choose any $t_0 \in J$ and a C^1 curve $\gamma\colon J \longrightarrow M$ through $p_0 = \gamma(t_0)$. Let $\{e_1, e_2, \ldots, e_n\}$ be any basis for $T_{\gamma(t_0)}M$. Let E_i be the unique parallel field along γ with $E_i(t_0) = e_i$ for $1 \leq i \leq n$. Then $\{E_1(t), E_2(t), \ldots, E_n(t)\}$ forms a basis for $T_{\gamma(t)}M$ for each $t \in J$. We can now write $\dot{\gamma}(t)$, the vector tangent to γ at p_0, as a linear combination of the elements of the chosen basis with coefficients $V^i(t)$: $\dot{\gamma}(t) = \sum_{i=1}^{n} \underbrace{V^i(t)E_{\gamma(t)i}}_{V^i(t)E_i(t)}$ with $V^i\colon J \longrightarrow \mathbb{R}$ for $1 \leq i \leq n$. Then the generalized affine parameter $\mu = \mu(\gamma, E_1, \ldots, E_n)$ of $\gamma(t)$ associated with this basis is given by

$$\mu(t) = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} [V^i(t)]^2 dt} = \int_{t_0}^{t} \sqrt{\delta_{ij} V^i(t) V^j(t)} dt, \ t \in J. \tag{4.1}$$

The assumption that γ is C^1 is necessary in order to obtain the vector fields $\{E_1, E_2, \ldots, E_n\}$ by parallel translation. Furthermore, we have

Proposition 4.3. [43] *The curve γ has a finite arc-length in the generalized affine parameter $\mu = \mu(\gamma, E_1, \ldots, E_n)$ if and only if γ has finite arc-length in any other generalized affine parameter $\mu = \mu(\gamma, \bar{E}_1, \ldots, \bar{E}_n)$.*

The generalized affine parameter of a curve does depend on the chosen basis. In effect, one treats the parallel-transported basis of vectors as though they were the orthonormal basis of a Riemannian metric and then defines the "length" of $\gamma(t)$ accordingly. Consequently, we have the following

Proposition 4.4. *If γ is a geodesic curve then μ is an affine parameter on γ.*

Proof. Let the (reparametrized) curve $\hat{\gamma}$ be defined by $\hat{\gamma}(\mu(t)) = \gamma(t)$, then according to the chain rule its tangent vectors are given by

$$\hat{\gamma}'(\mu) = \hat{\gamma}'(\mu(t)) \cdot \frac{1}{\frac{d\mu(t)}{dt}} = \dot{\gamma}(t) \cdot \frac{1}{\frac{d\mu(t)}{dt}}.$$

We have to prove that $\frac{D}{d\mu}\hat{\gamma}'(\mu) = 0$ along the curve $\hat{\gamma}$. Let γ be a geodesic, then $\frac{D}{dt}\dot{\gamma}(t) = f(t) \cdot \dot{\gamma}(t)$ along the curve γ, where f is an arbitrary function of t. Hence, from the Definition in 4.2 it follows that

$$\frac{D}{dt}\dot{\gamma}(t) = \frac{D}{dt}\left(\sum_{i=1}^{n} V^i(t)E_i(t)\right) = \sum_{i=1}^{n}\left(\frac{d}{dt}V^i(t)E_i(t) + V^i(t)\underbrace{\frac{D}{dt}E_i(t)}_{0}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{d}{dt}V^i(t)E_i(t)\right) = f(t)\cdot\left(\sum_{i=1}^{n} V^i(t)E_i(t)\right) = f(t)\cdot\dot{\gamma}(t).$$

Comparison of coefficients yields $\frac{d}{dt}V^i(t) = f(t)\cdot V^i(t)$. By applying the chain rule in Leibniz notation to $\frac{D}{d\mu}\hat{\gamma}'(\mu)$ we get

$$\frac{D}{d\mu}\hat{\gamma}'(\mu) = \frac{1}{\frac{d\mu(t)}{dt}}\cdot\frac{D}{dt}\hat{\gamma}'(\mu(t)) = \frac{1}{\frac{d\mu(t)}{dt}}\cdot\frac{D}{dt}\left(\dot{\gamma}(t)\cdot\frac{1}{\frac{d\mu(t)}{dt}}\right)$$

$$= \frac{1}{\frac{d\mu(t)}{dt}}\cdot\frac{D}{dt}\left(\left(\sum_{i=1}^{n} V^i(t)E_i(t)\right)\cdot\frac{1}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}}\right)$$

$$= \frac{1}{\frac{d\mu(t)}{dt}}\cdot\sum_{i=1}^{n}\left(\frac{D}{dt}\frac{V^i(t)E_i(t)}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}}\right)$$

$$= \frac{1}{\frac{d\mu(t)}{dt}}\cdot\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{V^i(t)}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}}\cdot E_i(t) + \frac{V^i(t)}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}}\cdot\underbrace{\frac{D}{dt}E_i(t)}_{0}\right)$$

$$= \frac{1}{\frac{d\mu(t)}{dt}}\cdot\sum_{i=1}^{n}(\underbrace{\frac{d}{dt}\frac{V^i(t)}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}}}_{0}\cdot E_i(t)) = 0. \tag{4.2}$$

Note that in the last line 4.2, because of $\frac{d}{dt}V^i(t) = f(t)\cdot V^i(t)$, we get a vanishing derivative

$$\frac{d}{dt}\frac{V^i(t)}{\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}} = 0.$$

Therefore is μ an affine parameter if γ is a null geodesic. Furthermore, we have

$$\frac{d}{dt}g(\hat{\gamma}'(\mu), \hat{\gamma}'(\mu)) = \frac{d}{dt}\frac{g(\dot{\gamma}(t), \dot{\gamma}(t))}{\left(\frac{d\mu}{dt}\right)^2}$$

$$= \frac{d}{dt}\frac{g(\sum_{i=1}^{n} V^i(t)E_i(t), \sum_{i=1}^{n} V^i(t)E_i(t))}{\sum_{i=1}^{n}[V^i(t)]^2} = 0,$$

which shows that μ is also an affine parameter if γ is a timelike or spacelike geodesic.

□

Note that if the metric g is positive definite, the generalized affine parameter defined by an orthonormal basis is arc-length. This characterization of completeness manages to discern exactly what we wanted to get winnowed. The elegance of this definition lies in its applicability of μ to any C^1 curve; it is equally effective for null curves as well as for timelike or spacelike curves. Moreover, any curve of unbounded proper length automatically has an unbounded generalized affine parameter which gives rise to the following Definition [4].

Definition 4.5. (b-completeness) A Lorentzian manifold (M, g) is said to be b-complete if every C^1 curve of finite arc length as measured by a generalized affine parameter has an endpoint in M.[15]
Hence, b-completeness identifies all C^1 curves which are incomplete with respect to their path length as measured by g. This path length is named generalized affine parameter because it agrees with a choice of affine parameter when restricted to geodesics. These days b-incompleteness is still considered to provide a reasonable definition of spacetime singularities in Lorentzian manifolds.[16]

Since the Hopf-Rinow theorem does not hold for Lorentzian manifolds, it certainly does not hold for signature-type changing manifolds. Hence, we need to resort to other concepts if we want to talk about metric completeness of Lorentzian manifolds − and with it, signature-type changing manifolds. To make matters worse,

[15] Put another way, a Lorentzian manifold is said to be b-complete if all inextensible C^1 curves have infinite length as measured by the generalized affine parameter. And if the metric g is positive definite, then b-completion coincides with the Cauchy completeness.

[16] It became manifest that the b-boundary behaves pathologically in some spacetime situations. As a consequence, some relativists turned to the causal boundary instead [36].

in a signature-type changing manifold we do not have an everywhere smooth Levi-Civita connection (see Section 12) and that is why the notion of b-completeness does not make sense.[17] However, the notion of a generalized affine parameter still proves useful to discern genuine pseudo-timelike (and pseudo-spacelike, respectively) curves and curves that asymptotically become lightlike as they approach the hypersurface of signature change (as we are going to discuss in Section 4.2).

The fact that the parallel-transported basis of vectors is treated though it were the orthonormal basis of a Riemannian metric is of major significance. Moreover, the generalized affine parameter only depends on the n-bein basis chosen and the initial point p. And even if the value of the generalized affine parameter is different, b-completeness is still well defined: Note that if only one generalized affine parameter reaches finite value all of them do − and that's the only information we need with respect to completeness. This reasoning is based on the following estimate, see also Appendix A:

Proof. For any two basis of $T_{\gamma(t)}M$ which are parallel transported along γ, then in the case of a change of basis, the components $V^i(t)$ with respect to another basis are given by $\tilde{V}^j(t) = \sum_{i=1}^n A_i^j V^i(t)$. We then have $\dot{\gamma}(t) = \sum_{i=1}^n V^i(t)E_i(t) = \sum_{i=1}^n \tilde{V}^i(t)\tilde{E}_i(t)$. The constant items A_i^j are entries in a constant, non-degenerate $n \times n$ matrix A. Hence, there exists its inverse matrix A^{-1} such that $V^j(t) = \sum_{i=1}^n a_i^j \tilde{V}^i(t)$. Accordingly, the generalized affine parameters with respect to these basis are $\mu(t) = \int_{t_0}^t \sqrt{\sum_{i=1}^n [V^i(t)]^2}dt$ and $\tilde{\mu}(t) = \int_{t_0}^t \sqrt{\sum_{i=1}^n [\tilde{V}^i(t)]^2}dt$. From this it follows that

$$\left|\tilde{V}(t)\right| = \left|\sum_{i=1}^n A_i^j V^i(t)\right| \leq \sum_{i=1}^n \mid A_i^j \mid\mid V^i(t) \mid \leq \max_{ij} \mid A_i^j \mid \sum_{i=1}^n \mid V^i(t) \mid .$$

Then, by virtue of the Cauchy-Schwarz inequality:

$$\mid \tilde{V}^j(t) \mid^2 \leq \max_{ij} \mid A_i^j \mid^2 \underbrace{\left(\sum_{i=1}^n \mid V^i(t) \mid \right)^2}_{\left(\sum_{i=1}^n |V^i(t)|\cdot 1\right)^2}$$

[17]In fact, the process of b-completion is applicable to any manifold with a Levi-Civita connection.

$$\le \max_{ij} \mid A_i^j \mid^2 \left(\sum_{i=1}^{n} \mid V^i(t) \mid^2 \right) \cdot \left(\sum_{i=1}^{n} 1 \right) = n \cdot \max_{ij} \mid A_i^j \mid^2 \left(\sum_{i=1}^{n} \mid V^i(t) \mid^2 \right).$$

Thus we have

$$\sum_{j=1}^{n} \mid \tilde{V}^j(t) \mid^2 \le \sum_{j=1}^{n} \left(n \cdot \max_{ij} \mid A_i^j \mid^2 \left(\sum_{i=1}^{n} \mid V^i(t) \mid^2 \right) \right)$$

$$= n^2 \cdot \max_{ij} \mid A_i^j \mid^2 \left(\sum_{i=1}^{n} \mid V^i(t) \mid^2 \right).$$

On the other hand, we get $\sum_{j=1}^{n} \mid V^j(t) \mid^2 \le n^2 \cdot \max_{ij} \mid a_i^j \mid^2 \left(\sum_{i=1}^{n} \mid \tilde{V}^i(t) \mid^2 \right)$. Combining both estimates yields

$$\sum_{j=1}^{n} \mid \tilde{V}^j(t) \mid^2 \le n^2 \cdot \max_{ij} \mid A_i^j \mid^2 \left(\sum_{j=1}^{n} \mid V^i(t) \mid^2 \right)$$

$$\le n^2 \cdot \max_{ij} \mid A_i^j \mid^2 \left(n^2 \cdot \max_{ij} \mid a_i^j \mid^2 \left(\sum_{i=1}^{n} \mid \tilde{V}^i(t) \mid^2 \right) \right)$$

$$\iff \frac{1}{n^2 \cdot \max_{ij} \mid A_i^j \mid^2} \sum_{j=1}^{n} \mid \tilde{V}^j(t) \mid^2$$

$$\le \sum_{i=1}^{n} \mid V^i(t) \mid^2 \le n^2 \cdot \max_{ij} \mid a_i^j \mid^2 \left(\sum_{i=1}^{n} \mid \tilde{V}^i(t) \mid^2 \right),$$

$$\implies \underbrace{\frac{1}{\sqrt{n^2 \cdot \max_{ij} \mid A_i^j \mid^2}}}_{c_1} \sqrt{\sum_{j=1}^{n} \mid \tilde{V}^j(t) \mid^2}$$

$$\le \sqrt{\sum_{i=1}^{n} \mid V^i(t) \mid^2} \le \underbrace{\sqrt{n^2 \cdot \max_{ij} \mid a_i^j \mid^2}}_{c_2} \sqrt{\sum_{i=1}^{n} \mid \tilde{V}^i(t) \mid^2}$$

$$\Longrightarrow c_1 \cdot \tilde{\mu}(t) \leq \mu(t) \leq c_2 \cdot \tilde{\mu}(t). \tag{4.3}$$

□

4.2 Application of the generalized affine parameter in a type-changing manifold

Let $M = M_L \cup \mathcal{H} \cup M_R$ be an n-dimensional transverse type-changing singular semi-Riemannian manifold with a symmetric type-changing metric g, and $\mathcal{H} := \{q \in M \colon g\mid_q \ \textit{is degenerate}\}$ the locus of signature change. We further assume that one component, M_L, of $M \setminus \mathcal{H}$ is Lorentzian and the other one, M_R, is Riemannian.

Definition 4.6. (Pseudo-lightlike curve) Given a continuous and differentiable curve $\gamma \colon [a, b] \longrightarrow M$, with $[a, b] \subset \mathbb{R}$, where $-\infty < a < b < \infty$. Then the curve $\gamma = \gamma^\mu(u) = x^\mu(u)$ is a pseudo-lightlike curve if

- its tangent vector field in the Lorentzian component M_L is null,
- its tangent vector field in the Riemannian component M_R is arbitrary.

Similarly for a **pseudo-causal curve**. Note that an analogous definiton for pseudo-timelike and pseudo-spacelike curves turns out to be problematic as the definition would also include curves that asymptotically become lightlike as they approach $\mathcal{H}$.

For example we may refer to the metric $g = t(dt)^2 + (dx)^2$ defined on $\mathbb{R}^2$, and the non-parametrized, non-geodesic curve γ given by $\tan x = \frac{2}{3}\sqrt{\mid t \mid^3} \cdot \operatorname{sgn}(t)$, with $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We rearrange this equation so that the variable x is by itself on one side:

$$\frac{3}{2}\tan x = \operatorname{sgn}(x) \cdot \mid \frac{3}{2}\tan x \mid = \operatorname{sgn}(t) \cdot \mid t \mid^{\frac{3}{2}}$$

$$\Longleftrightarrow \underbrace{\operatorname{sgn}(t) \cdot \mid t \mid}_{t} = \operatorname{sgn}(x) \cdot \left(\mid \frac{3}{2}\tan x \mid\right)^{\frac{2}{3}}$$

$$\Longleftrightarrow t = \operatorname{sgn}(x) \cdot \left(\mid \frac{3}{2}\tan x \mid\right)^{\frac{2}{3}}.$$

Reintroducing the coordinate transformation as suggested by Dray [21]

$$T = \int_0^t \sqrt{|\, \tilde{t} \,|} d\tilde{t} = \frac{2}{3}\sqrt{|\, t \,|}^3 \cdot \operatorname{sgn}(t).$$

This gives us the metric expression $g = \operatorname{sgn}(T)(dT)^2 + (dx)^2$, and for the curve γ we get $T = \tan x$. Hence, the curve in the $(T, x)-$coordinate system is just the $\tan-$function and its derivative is $\frac{1}{\cos^2(x)}$. As a result is γ in M_L timelike, approaching from timelike infinity the lightcone, and tangentially touches the light cone at $T = 0$ (where the derivative assumes $\frac{1}{\cos^2(0)} = 1$). These are the sort of curves we want to avoid in our definition.

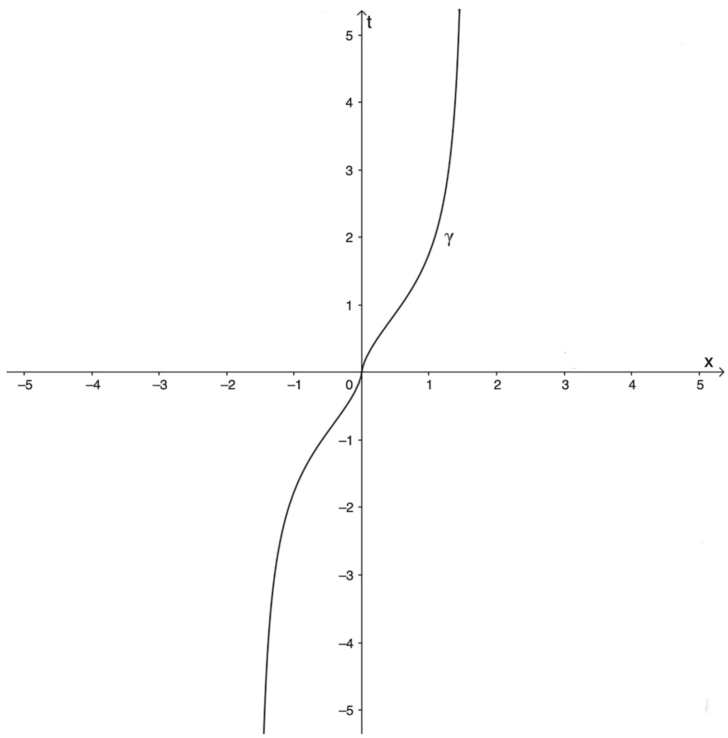

Figure 4.1: The curve defined by $t = \operatorname{sgn}(x) \cdot \left(|\, \frac{3}{2}\tan x \,|\right)^{\frac{2}{3}}$.

Moreover, if the curve $\gamma = (T(s), x(s))$ is parametrized by arc length s, then in the (t, x)-coordinate system both $\frac{dx}{ds}$ and $\frac{dt}{ds}$ diverge in M_L:

$$-1=-(\tfrac{dT}{ds})^2+(\tfrac{dx}{ds})^2=-(\tfrac{dT}{dx}\tfrac{dx}{ds})^2+(\tfrac{dx}{ds})^2$$

$$=\left(-(\tfrac{d\tan x}{dx})^2+1\right)(\tfrac{dx}{ds})^2=(-\tfrac{1}{\cos^4 x}+1)(\tfrac{dx}{ds})^2$$

$$\Longleftrightarrow (\tfrac{dx}{ds})^2=\tfrac{-1}{(-\frac{1}{\cos^4 x}+1)}$$

$$\Longrightarrow \lim_{x\to 0}\tfrac{dx}{ds}=\lim_{x\to 0}\pm\sqrt{\tfrac{-1}{(-\frac{1}{\cos^4 x}+1)}}=\pm\infty.$$

$$-1=-(\tfrac{dT}{ds})^2+(\tfrac{dx}{ds})^2=\left(-1+\tfrac{1}{(\frac{dT}{dx})^2}\right)(\tfrac{dT}{ds})^2=(-1+\underbrace{\cos^4 x}_{\frac{1}{(\frac{d\tan x}{dx})^2}})(\tfrac{dT}{dt})^2(\tfrac{dt}{ds})^2$$

$$=(-1+\tfrac{1}{(1+\tan^2 x)^2})\cdot\mid t\mid(\tfrac{dt}{ds})^2=(-1+\tfrac{1}{(1+T^2)^2})\cdot\mid t\mid(\tfrac{dt}{ds})^2$$

$$=(-1+\tfrac{1}{(1+\frac{4}{9}|t|^3)^2})\cdot\mid t\mid(\tfrac{dt}{ds})^2$$

$$\Longleftrightarrow (\tfrac{dt}{ds})^2=\tfrac{-1}{(-1+\frac{1}{(1+\frac{4}{9}|t|^3)^2})\cdot|t|}$$

$$\Longrightarrow \lim_{t\to 0}\tfrac{dt}{ds}=\lim_{t\to 0}\pm\sqrt{\tfrac{-1}{(-1+\frac{1}{(1+\frac{4}{9}|t|^3)^2})\cdot|t|}}=\pm\infty.$$

While the components of γ' do not diverge in the (T,x)-coordinate system, both $\frac{dx}{ds}$ and $\frac{dt}{ds}$ diverge in M_L in the (t,x)-coordinate system. Because of this dependency of coordinates the criterion of divergence is not useful for defining pseudo-timelike and pseudo-spacelike curves. That is where the coordinate-independent generalized affine parameter comes into play.

Definition 4.7. (Pseudo-timelike curve) Let $M=M_L\cup\mathcal{H}\cup M_R$ be an n-dimensional transverse type-changing singular semi-Riemannian manifold, g be a symmetric type-changing metric, and $\mathcal{H}:=\{q\in M\colon g\mid_q \textit{ is degenerate}\}$ the locus of signature change. We further assume that one component, M_L, of $M\setminus\mathcal{H}$ is Lorentzian and the other one, M_R, is Riemannian.
Given a continuous and differentiable curve $\gamma\colon[a,b]\to M$, with $[a,b]\subset\mathbb{R}$, where $-\infty<a<b<\infty$. Then the curve $\gamma=\gamma^\mu(u)=x^\mu(u)$ in M is called pseudo-timelike *(respectively, pseudo-spacelike)* if for every generalized affine parametrization of γ in M_L $\exists\,\varepsilon>0$ such that $g(\gamma',\gamma')<-\varepsilon$ (respectively, $g(\gamma',\gamma')>\varepsilon$).[18]

[18]Since Definition 4.2 is already independent of a choice of coordinates and instead refers to a (generally anholonomic) basis, the above Definition 4.7 is also coordinate independent. The independence of Definition 4.7 from the choice of this basis is a direct consequence of Proposition 4.3. In particular, in the case of a basis change we just relegate to the Estimate 4.3.

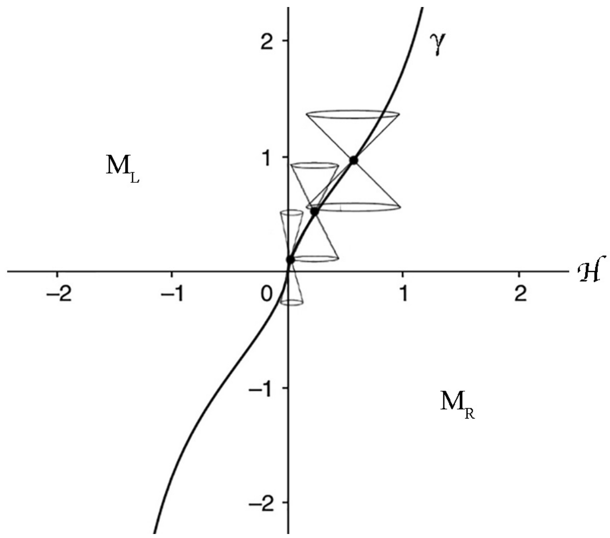

Figure 4.2: The curve γ is not a pseudo-timelike since it approaches a null vector at the locus of signature change. This curve is asymptotically lightlike.

Example 4.8. Revisiting the example from Section 4.2, we find that both coordinate vector fields, $\frac{\partial}{\partial T}$ and $\frac{\partial}{\partial x}$, are covariantly constant in M_L and M_R (this is because the Christoffel symbols all vanish in the (T, x)-coordinate system). Hence, we can parallel transport $\frac{\partial}{\partial T}$ and $\frac{\partial}{\partial x}$ along any curve in M_L and M_R.
Since we aim at parametrizing the curve γ by the generalized affine parameter μ with respect to the coordinate vector fields $\frac{\partial}{\partial T} = \frac{1}{\sqrt{|t|}} \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ we are able to start with an arbitrary parametrization. Hence, let $\gamma(t) = (T(t), x(t))$ be parametrized by t, and then $\dot{\gamma}(t) = \frac{dT}{dt} \frac{\partial}{\partial T} + \frac{dx}{dt} \frac{\partial}{\partial x}$.
By means of Definition 4.2 we immediately get

$$V^0(t) = \frac{dT}{dt} = \sqrt{\mid t \mid}$$

and

$$V^1(t) = \frac{dx}{dt} = \frac{d}{dt} \arctan \left(\frac{2}{3} \sqrt{\mid t \mid}^3 \operatorname{sgn}(t) \right).$$

And in M_L this yields

$$V^1(t) = \frac{\sqrt{\mid t \mid}}{1 + \frac{4}{9} \mid t \mid^3}.$$

Consider now $\tilde{\gamma}(t(s)) = \gamma(s)$, in which $\tilde{\gamma}$ is related to the curve γ by reparametrization of $\tilde{\gamma}$ by t. With this notation we have the basis fields $E_{\tilde{\gamma}(t),0} = \frac{1}{\sqrt{|t|}}\frac{\partial}{\partial t}$ and $E_{\tilde{\gamma}(t),1} = \frac{\partial}{\partial x}$ along $\tilde{\gamma}$. The reparametrized curve $\tilde{\gamma}(t(s))$ also gives

$$\dot{\tilde{\gamma}}(t) = V^i(t) E_{\tilde{\gamma}(t),i} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x}.$$

The Definition 4.2 for the generalized affine parameter gives

$$\frac{d\mu}{dt} = \sqrt{(V^0(t))^2 + (V^1(t))^2} = \sqrt{\mid t \mid + \frac{\mid t \mid}{(1 + \frac{4}{9} \mid t \mid^3)^2}}.$$

It now follows easily that for the reparametrization of $\hat{\gamma}(t)$ by the generalized affine parameter μ (i.e. $\hat{\gamma}(\mu(t)) = \tilde{\gamma}(t)$) we have in M_L:

$$g(\dot{\hat{\gamma}}(\mu(t)), \dot{\hat{\gamma}}(\mu(t))) = g(\frac{d\hat{\gamma}(\mu(t))}{d\mu}, \frac{d\hat{\gamma}(\mu(t))}{d\mu}) = g(\frac{1}{\frac{d\mu}{dt}}\dot{\hat{\gamma}}(t), \frac{1}{\frac{d\mu}{dt}}\dot{\hat{\gamma}}(t))$$

$$= \frac{g(\frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x}, \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x})}{(\frac{d\mu}{dt})^2} = \frac{t + (\frac{dx}{dt})^2}{\mid t \mid + \frac{|t|}{(1+\frac{4}{9}|t|^3)^2}} = \frac{t + \frac{|t|}{(1+\frac{4}{9}|t|^3)^2}}{\mid t \mid + \frac{|t|}{(1+\frac{4}{9}|t|^3)^2}}.$$

Taking the limit

$$\lim_{t \to 0^-} \frac{t + \frac{|t|}{(1+\frac{4}{9}|t|^3)^2}}{\mid t \mid + \frac{|t|}{(1+\frac{4}{9}|t|^3)^2}} = 0$$

reveals that the curve γ is not pseudo-timelike as it does not meet the $\varepsilon-$requirement of Definition 4.7.

In Section 4.1 we repeatedly vaguely referred to the concept of a timelike (or spacelike curve, respectively) curve that asymptotically becomes lightlike. The above example highlights how the notion of "asymptotically lightlike" should be understood. A timelike (or spacelike curve, respectively) curve in M_L that is not pseudo-timelike (or pseudo-spacelike, respectively) can be thus specified as asymptotically lightlike.

Example 4.9. Finally, if we slightly modify the previously discussed curve γ by keeping the t-coordinate but stating $x = 0$, we get the curve α. With the same notation as above, we then get $V^0(t) = \sqrt{|\ t\ |}$, $V^1(t) = 0$ and $\frac{d\mu}{dt} = \sqrt{|\ t\ |}$. Hence, this results in $g(\dot{\hat{\alpha}}(\mu(t)), \dot{\hat{\alpha}}(\mu(t))) = \frac{1}{|t|} g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \frac{t}{|t|} = -1$ in the Lorentzian region M_L. The curve α is pseudo-timelike as it obviously does meet the ε-requirement of Definition 4.7.

This disquisition makes it clear why the notion of the generalized affine parameter is necessary and useful in order to define pseudo-timelike and pseudo-spacelike curves: If we were to loosen the requirement for Definition 4.7 and replaced "*for every generalized affine parametrization of γ in M_L*" with "*for any affine parametrization of γ in M_L*", there wouldn't exist any timelike curve in M_L that would qualify to be pseudo-timelike in all of M. Similarly, any timelike curve in M_L would meet the requirements of a pseudo-timelike curve if we modified the definition by requesting "*for a suitable parametrization of γ in M_L*" instead of "*for every generalized affine parametrization of γ in M_L*". In this regard, the concept of the generalized affine parameter is the right tool to tell apart suitable from unsuitable curves for the definition of pseudo-timelike and pseudo-spacelike curves.

Interestingly, our rationale for the new definition of a pseudo-timelike curve is reminiscent of the analysis undertaken in [63]. In Section 2 of [63] the distinction between *causal curves*, *timelike almost everywhere curves* and *timelike curves* is introduced in which the latter one is defined as follows: A timelike curve is a causal curve $\gamma\colon I \longrightarrow M$ such that $g(\gamma', \gamma') < -\varepsilon$ almost everywhere for some $\varepsilon > 0$.
The authors illustrate the situation in Figure 1 which contrasts a timelike curve with a timelike almost everywhere curve. The latter one can not be viewed as a timelike curve because it approaches a null vector at its break point. Compared to our setting, however, the culprit here is that the curve is not differentiable at the breaking point. But even if we smoothed out the breaking point, the curve would still remain timelike almost everywhere.

Let us now conceive of the breaking point being located at a hypersurface of signature change and the Minkowski space replaced by a signature-type changing manifold. Similar to our toy model in (T, x)-coordinates (see Section 5) the "lower" part would be Lorentzian and the "upper" part Riemannian. Then the depicted "timelike almost everywhere" curve would be in fact just timelike in most of the Lorentzian sector, but it would approach a null vector at the locus of signature change. Therefore it could not be classified as a pseudo-timelike curve according to our Definition 4.7. This again reconfirms and justifies our reasoning to introduce such a definition for pseudo-timelike curves.

Now we can slightly modify the definition of a (simply) closed curve in order for it to correctly apply to signature-type changing singular semi-Riemannian manifolds M with a metric g:

Definition 4.10. (Chronology-violating curve) A smooth, *pseudo-timelike curve* $\gamma\colon I \longrightarrow M$ is said to be chronology-violating when there is a subset of $\gamma[I]$ homeomorphic to S^1 such that there are at least two parameters $s_1, s_2 \in I$ that satisfy $\gamma(s_1) = \gamma(s_2)$, and γ belongs to one of the following two classes:[19]

1. The pseudo-timelike curve γ is periodic, i.e the image $\gamma[I]$ is homeomorphic to S^1. Moreover, for s_1, $s_2 \in I$ the associated tangent vectors, $\gamma'(s_1)$ and $\gamma'(s_2)$, are timelike and positively proportional. We denote this type of curve as *closed pseudo-timelike curve.*

2. The curve γ intersects itself for s_1, $s_2 \in I$ and the associated tangent vectors, $\gamma'(s_1)$ and $\gamma'(s_2)$, are timelike whereas the tangent directions are not necessarily the same (i.e. they do not need to be positively proportional). This type of curve is said to contain a *loop.*

The following Definition leans on [71]:

Definition 4.11. A geodesic $\gamma\colon I \longrightarrow M$ for the metric g with $\gamma(s_1) = \gamma(s_2)$, for some $s_1, s_2 \in I$, is closed if the associated tangent vectors, $\gamma'(s_1)$ and $\gamma'(s_2)$, are timelike, the curve γ is periodic and all derivatives match: $\gamma'(s_1) = \gamma'(s_2)$, $\gamma''(s_1) = \gamma''(s_2)$ et cetera, $s_1, s_2 \in I$.
If the latter condition is dropped, the curve γ is called a *geodesic loop.*

[19]Note that this means that there must be at least one such subset to fulfill this definition.

5 Signature-type changing toy model

Equipped with the information from the preceding section, we want to explore what happens sufficiently close to the boundary hypersurface in a 2-dimensional toy model setting.

Consider on $\mathbb{R}^2$ the signature-type changing metric $ds^2 = t(dt)^2 + (dx)^2$, which becomes degenerate at $t = 0$, where it makes a transition from being Lorentzian to Riemannian [21]. Then the pair $(\mathbb{R}^2, g)$ is a flat type-changing singular semi-Riemannian manifold with the locus of signature change at $t = 0$.[20] Considering the canonical embedding $f\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $f(x) \coloneqq (0, x)$, the metric induced on the hypersurface of signature change has the form $f^*(t(dt)^2 + (dx)^2) = (dx)^2$. Since the induced metric is positive-definite the hypersurface is spacelike, hence Riemannian. Moreover, by Sylvester's law of inertia, the normal to the hypersuface $\mathcal{H}$ at each point $(0, x)$ is $\operatorname{span}(\{\frac{\partial}{\partial t}\})$, and therefore null.

5.1 General affinely parameterized geodesics

We want to take special care in understanding the behavior of the geodesics as they traverse the threshold between the domains of different signature. The geodesics in a signature-type changing universe bring up interesting questions. It is usually clear how geodesics evolve in the Riemannian region as well as in the Lorentzian region, but there are problems of interpretation concerning particles crossing the junction. This points to questions regarding the compatibility of geodesics across the transition surface: A careful analysis into whether geodesics on the Lorentzian domain can be matched to those of the Riemannian domain has to be carried out.

We are going to solve the general geodesic equations in both domains and then match them at the boundary. It is desired to introduce an affine parameter s which then gives the usual definitions of the Lagrange function $\mathcal{L}$ and the Euler-Lagrange equations as follows:

$$\mathcal{L} = \tfrac{1}{2}t\dot{t}^2 + \tfrac{1}{2}\dot{x}^2,$$

$$0 = \tfrac{\partial \mathcal{L}}{\partial t} - \tfrac{d}{ds}\big(\tfrac{\partial \mathcal{L}}{\partial \dot{t}}\big) = \tfrac{1}{2}\dot{t}^2 - \tfrac{d}{ds}(t\dot{t}) = \tfrac{1}{2}\dot{t}^2 - (\dot{t}^2 + t\ddot{t}) = -\tfrac{1}{2}\dot{t}^2 - t\ddot{t},$$

[20] Flat in the sense that the regions with $t < 0$ and $t > 0$ are flat.

$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{ds}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = -\ddot{x}$,

where the dots denote differentiation with respect to the affine parameter s. We take into account that the Lagrangian $\mathcal{L}$ is composed of two indepented components, $\frac{1}{2}t\dot{t}^2$ and $\frac{1}{2}\dot{x}^2$. This enables us to examine each of the components separately which makes our calculations simpler. Furthermore, it is a well known fact that an affine parameterization along a geodesic yields that the Lagrangian itself is a constant of motion, which gives us

$$\frac{d}{ds}(\frac{1}{2}t\dot{t}^2) = \frac{1}{2}\dot{t}^3 + t\dot{t}\ddot{t} = \dot{t}(\frac{1}{2}\dot{t}^2 + t\ddot{t}) = 0.$$

For the sake of simplicity we denote the constants $\frac{1}{2}t\dot{t}^2$ by L_0 and $\frac{1}{2}\dot{x}^2$ by L_1, respectively, with $\frac{d}{ds}L_0 = \frac{d}{ds}L_1 = 0$. In addition, the coefficients of the Levi-Civita connection ∇ for the metric $ds^2 = t(dt)^2 + (dx)^2$ are obviously given by $\Gamma^1_{11} = \Gamma^t_{tt} = \frac{1}{2t}$, while the rest of the Christoffel symbols are zero.

- Case $L_0 > 0$:

For $L_0 > 0 \Longleftrightarrow t > 0$ we are looking at the Riemannian region. In order to avoid confusion, let L_0 for $t > 0$ be denoted as L_{0+}. From $L_0 = \frac{1}{2}t\dot{t}^2 = \frac{1}{2}t\left(\frac{dt}{ds}\right)^2$ follows that

$$\left(\frac{dt}{ds}\right)^2 = \frac{2L_{0+}}{t}$$

$$\Leftrightarrow \sqrt{t}\frac{dt}{ds} = \pm\sqrt{2L_{0+}}. \tag{5.1}$$

This equation 5.1 can be integrated immediately to obtain

$$\frac{2}{3}\sqrt{t}^3 = \pm\sqrt{2L_{0+}}(s - s_0), \tag{5.2}$$

where s_0 is an arbitrary constant of integration. Then we square 5.2 which gives

$$\frac{4}{9}t^3 = 2L_{0+}(s - s_0)^2.$$

From this we obtain as part of the geodesic equation

$$t(s) = \sqrt[3]{\frac{9}{2}L_{0+}(s - s_0)^2}.$$

As expected, for $t < 0$ this equation cannot be satisfied. The transition surface at $t = 0$ is crossed when $s = s_0$. However, as t approaches 0 from above, $\left|\dot{t}\right| = \sqrt{\frac{2L_{0+}}{t}}$ for $t \to 0$ tends to infinity, and the geodesics appear to be almost veritical close to the junction at $t = 0$. Therefore the unit spheres become more narrow as they approach the junction and eventually degenerate at $t = 0$. These geodesics cannot be extended to $t < 0$ as we will show below.

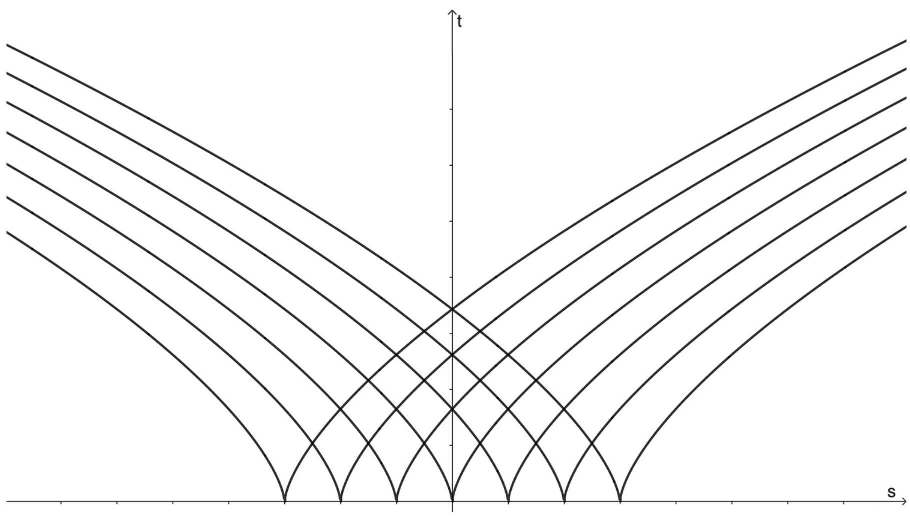

Figure 5.1: General geodesics in the Riemannian region for the case $L_0 > 0$ with L_0 fixed and arbitrary s_0. The coordinate t serves as a function of the affine parameter s. It is easy to see that $\left|\dot{t}\right| = \sqrt{\frac{2L_{0+}}{t}} \to \infty$ for $t \to 0$.

- Case $L_0 = 0$:

For the case $L_0 = \frac{1}{2}t\dot{t}^2 = 0$ the only relevant equation for general, affinely parameterized geodesics results from the condition $\dot{t} = 0 \wedge t \neq 0$:
This again means that $\dot{t}(s) = 0$ for all the parameter values s with $t(s) \neq 0$. If s_1 is such a parameter value, then because of t being continuous, on a sufficiently small interval $I \ni s_1$ we have $t(s) \neq 0$ for all $s \in I$. This in turn implies that $\dot{t}(s) = 0$ for $s \in I$, which means that $t(s)$ is constant on the interval I. Because t is locally Lipschitz continuous we can then conclude through analytic continuation that this solution is valid for all $s \in \mathbb{R}$.

Having said that, we assume that for the case $L_0 = \frac{1}{2}t\dot{t}^2 = 0$ we have an arbitrary $\dot{t}(s)$ if $t(s) = 0$. Let s_2 be such a parameter value for which $t(s_2) = 0$ and $\dot{t}(s_2) \neq 0$. Then similarly to the reasoning above, on a sufficiently small interval $J \ni s_2$ we have $\dot{t}(s) \neq 0$ for all $s \in J$. This means that t is not constant, i.e. $t(s_2) = 0$ and $t(s) \neq 0$ for all other $s \in J$. According to the above argument, however, $t(s) \neq 0$ for all other $s \in J$ would imply that $t(s)$ is constant on the interval J, which is a contradiction. Obviously, if $t(s) = 0$ then $\dot{t}(s) = 0$.
Moreover, this line of argument leads to the geodesic equation $t(s) = c_t$ for the initial condition given by $t(s_1)$ and s_1.

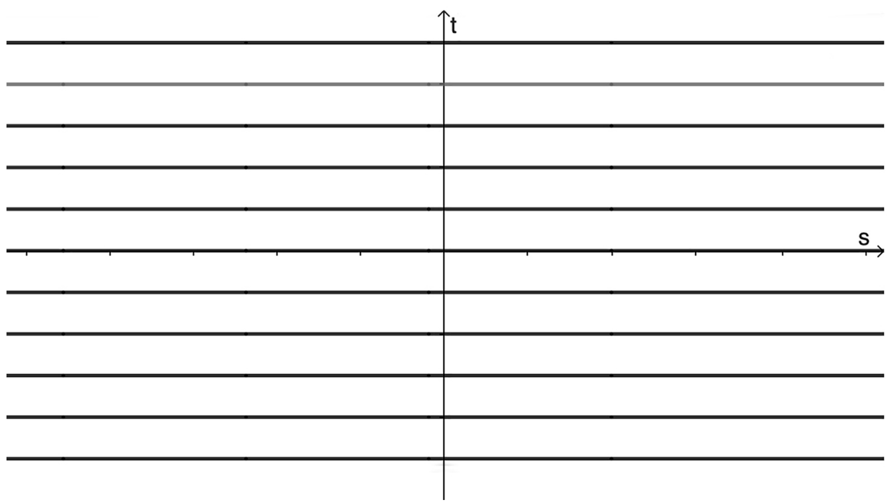

Figure 5.2: Geodesics for $L_0 = 0$ resulting from the condition $\dot{t} = 0 \wedge t \neq 0$, where the coordinate t is a function of the affine parameter s.

- Case $L_0 < 0$:

For $L_0 < 0 \Longleftrightarrow t < 0$ we are looking at the Lorenzian region. Let L_0 for $t < 0$ be denoted as L_{0-}. From $L_0 = \frac{1}{2}t\dot{t}^2 = \frac{1}{2}t\left(\frac{dt}{ds}\right)^2$ follows that

$$-t\left(\frac{dt}{ds}\right)^2 = -2L_{0-}$$

$$\Leftrightarrow \sqrt{-t}\frac{dt}{ds} = \pm\sqrt{-2L_{0-}}. \tag{5.3}$$

Once again, this equation 5.3 can be integrated and yields

$$-\frac{2}{3}\sqrt{-t}^3 = \pm\sqrt{-2L_{0-}}(s - s_0), \tag{5.4}$$

where s_0 is an arbitrary constant of integration. Then we square 5.4 which gives

$$\frac{4}{9}(-t)^3 = -2L_{0-}(s - s_0)^2.$$

This leads to the solution

$$t(s) = -\sqrt[3]{-\frac{9}{2}L_{0-}(s - s_0)^2}.$$

Analogously to the case $L_0 > 0$, this part of the geodesic equation cannot be satisfied for $t > 0$. The transition surface at $t = 0$ is crossed when $s = s_0$. However, as t approaches 0 from below, $\left|\dot{t}\right| = \sqrt{\frac{2L_{0-}}{t}}$ for $t \to 0$ tends to infinity and the geodesics appear to be almost vertical close to the junction at $t = 0$. Therefore, the light cones become more narrow as they approach the junction and eventually squeeze shut at $t = 0$. Also, these geodesics cannot be extended to $t > 0$.

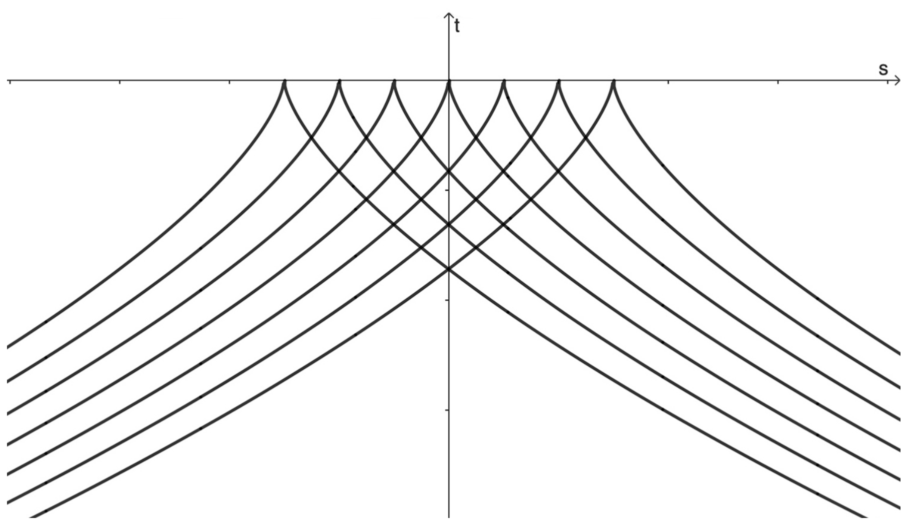

Figure 5.3: Geodesics in the Riemannian region for the case $L_0 < 0$, with L_0 fixed and arbitrary s_0. The coordinate t serves as as a function of the affine parameter s. It is easy to see that $\left|\dot{t}\right| = \sqrt{\frac{2L_{0-}}{t}} \to \infty$ for $t \to 0$.

- Case L_1:

Continuing in the same fashion as above for L_1 leads to the x-part of the geodesic equation. However, the expression $L_1 = \frac{1}{2}\dot{x}^2$ is obviously non-negative for all $x \in \mathbb{R}$, so while still following the previous approach we can drop the case distinction.

From $L_1 = \frac{1}{2}\dot{x}^2 = \frac{1}{2}\left(\frac{dx}{ds}\right)^2$ follows that

$$\begin{aligned} \left(\frac{dx}{ds}\right)^2 &= 2L_1 \\ \Leftrightarrow \frac{dx}{ds} &= \pm\sqrt{2L_1}. \end{aligned} \tag{5.5}$$

The equation 5.5 can be easily integrated to obtain the general solution

$$x(s) = \pm\sqrt{2L_1} \cdot s + c = c + \dot{x} \cdot s,$$

where c is an arbitrary (possibly zero) constant on integration. Denoting the constant $\dot{x}$ by w and substituting c with $x(s_0) - w \cdot s_0$ leads to

$$x(s) = (x(s_0) - w \cdot s_0) + w \cdot s = x(s_0) + w(s - s_0),$$

where s_0 is an arbitrary constant of integration. From this we obtain the x-part of the geodesic equation in terms of the parameter s:

$$x(s) = x(s_0) + w(s - s_0). \tag{5.6}$$

5.2 Matching geodesics

At this point we want to examine how to match geodesics at the junction at $t = 0$. Each geodesic on the Riemannian domain has to be matched to a geodesic on the Lorentzian domain. Obviously there are infinitely many geodesics at each parameter s_0. More precisely, from any fixed parameter s_{0-} infinitely many geodesics emanate into the Lorentzian domain, each associated to a particular choice of (L_{0i-}). Similarly on the Riemannian domain, from any fixed parameter s_{0+} infinitely many geodesics emanate into the Riemannian domain, each associated to a particular choice of (L_{0i+}). And it is not clear which geodesic on the Lorentzian domain can be matched to which one on the Riemannian domain (see Figure 5.4).

Also, questions regarding the compatibility of geodesics across the transition surface must be addressed.

This means that for any pair of parameters (L_{0i+}, s_{0+}) on the Riemannian region $(t > 0)$ we want to find a matching pair of parameters (L_{0i-}, s_{0-}) on the Lorentzian region $(t < 0)$. It is assumed that $t(s_{0-}) = t(s_{0+}) = 0$, and we clearly have $s_{0+} = s_{0-}$ because t is continuous in s_0. Hence, we have the corresponding limit:

$$s_{0+} = \lim_{\substack{t \to 0 \\ t<0}} t = \lim_{\substack{t \to 0 \\ t>0}} t = s_{0-}.$$

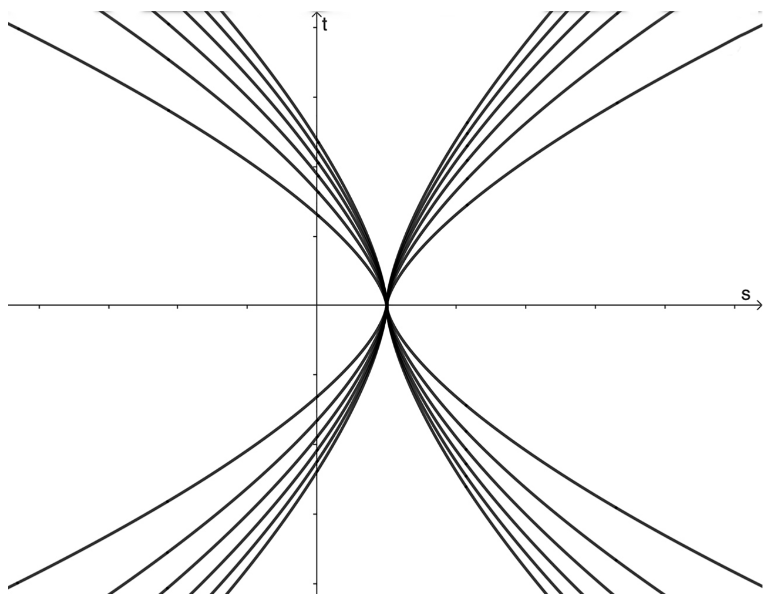

Figure 5.4: Geodesics with fixed s_0 and arbitrary L_{0i+}, L_{0i-}.

It remains to show that we can properly assign a L_{0i+} to a L_{0i-}. However, because of

$$\lim_{s \to s_0} \dot{t}(s) = \pm\infty$$

the first derivative of t is discontinuous (and has a vertical inflection point in s_0), hence the parameter value s_0 is associated with a cusp. So there is no (unique) continuous continuation from the Lorentzian region with $t < 0$ to the Riemannian region with $t > 0$. In a sense the geodesics all break down at the signature-type changing junction. From the analysis in the previous section, this issue is rooted in the choice of the coordinate t. To get around this, we introduce a suitable coordinate transformation as suggested by Dray [21]

$$T = \int_{0}^{t} \sqrt{|\tilde{t}|} d\tilde{t}, \tag{5.7}$$

from which we get the expression

$$(dT)^2 = |t|\,(dt)^2 = \underbrace{t \cdot \operatorname{sgn}(t)}_{|t|}(dt)^2 = t \cdot \operatorname{sgn}(T)(dt)^2 = \frac{t}{\operatorname{sgn}(T)}(dt)^2 \tag{5.8}$$

for $t \neq 0$. And 5.8 can be rewritten as

$$t(dt)^2 = \operatorname{sgn}(T) dT^2.$$

This yields $t(dt)^2 + (dx)^2 = \operatorname{sgn}(T)(dT)^2 + (dx)^2$ for $T \neq 0$. The metric

$$ds^2 = \operatorname{sgn}(T)(dT)^2 + (dx)^2 \tag{5.9}$$

is very different from

$$ds^2 = t(dt)^2 + (dx)^2 \tag{5.10}$$

as it is no longer degenerate at $t = 0$, but it is discontinuous and not well defined at $t = 0$. Note that the difference between the metric (5.9) and (5.10) is to some extend only a coordinate transformation. Hence, $\operatorname{sgn}(T) = \operatorname{sgn}(t)$ and $|T| = \frac{2}{3}\sqrt{|t|^3}$. Therefrom it suffices to focus on either approach because a straightforward coordinate transformation will translate any statement made in one setting into the other setting.
This stems from the fact that only the limits of quantities as computed within the embedding from either side are relevant for the Lagrangian point of view, but we never need to specify a particular value at the hypersurface of signature change (i.e., at $t = 0$), even if the metric is discontinuous. However, the two coordinates T and t are related to two different notions of differentiability: for any function f, the existence of $\partial_t f$ is not equivalent to the existence of $\partial_T f$ at $T = t = 0$ [27].

Considering again the standard embedding $f\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $f(x) \coloneqq (0, x)$, the metric induced on the hypersurface of signature change by the metric $ds^2 = \operatorname{sgn}(T)(dT)^2 + (dx)^2$ has the form $f^*(\operatorname{sgn}(T)(dT)^2 + (dx)^2) = (dx)^2$.

We consider the geodesics using the discontinuous metric

$$ds^2 = \operatorname{sgn}(T)(dT)^2 + (ds)^2$$

for which we get the following Lagrange function $\mathcal{L}$ and the associated Euler-Lagrange equations as follows:[21]

$$\mathcal{L} = \frac{1}{2}\operatorname{sgn}(T)\dot{T}^2 + \frac{1}{2}\dot{x}^2,$$

$$0 = \frac{\partial \mathcal{L}}{\partial T} - \frac{d}{ds}(\frac{\partial \mathcal{L}}{\partial \dot{T}})$$

$$= \underbrace{\frac{1}{2}\dot{T}^2\frac{\partial \operatorname{sgn}(T)}{\partial T}}_{0} - \frac{d}{ds}\operatorname{sgn}(T)\dot{T} = -\underbrace{\frac{d}{ds}\operatorname{sgn}(T)}_{0} \cdot \dot{T} - \operatorname{sgn}(T) \cdot \frac{d}{ds}\dot{T} = -\operatorname{sgn}(T)\ddot{T},$$

$$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{ds}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = \ddot{x},$$

where $\operatorname{sgn}(T)$ is a constant on each component away from the surface of signature change at $T = 0$, and the dots denote differentiation with respect to the affine parameter s. Keep in mind that $\mathcal{L}$ is actually not defined at $T = t = 0$. What is more, the coefficients of the Levi-Civita connection ∇ for the metric $ds^2 = \operatorname{sgn}(T)(dT)^2 + (dx)^2$ are all zero.

The Lagrangian $\mathcal{L}$ is again composed of two independent components, $\frac{1}{2}\operatorname{sgn}(T)\dot{T}^2$ and $\frac{1}{2}\dot{x}^2$. So we examine each of the components separately:

$$\frac{d}{ds}(\frac{1}{2}\operatorname{sgn}(T)\dot{T}^2) = \dot{T}\underbrace{\operatorname{sgn}(T)\ddot{T}}_{0} = 0,$$

[21] Note that the Euler-Lagrangian equations are only well-defined in the regions away from $T = 0$.

and we denote the constant $\frac{1}{2}\mathrm{sgn}(T)\dot{T}^2$ by L_0, with $\frac{d}{ds}L_0 = 0$. Now the geodesics can be immediately computed. From

$$L_0 = \frac{1}{2}\mathrm{sgn}(T)\dot{T}^2 = \frac{1}{2}\mathrm{sgn}(T)\left(\frac{dT}{ds}\right)^2$$

follows

$$\left(\frac{dT}{ds}\right)^2 = \frac{2L_0}{\mathrm{sgn}(T)}$$

$$\Leftrightarrow \frac{dT}{ds} = \pm\sqrt{\frac{2L_0}{\mathrm{sgn}(T)}}$$

$$\overset{\int}{\Rightarrow} T(s) = \pm\sqrt{\frac{2L_0(s-s_0)^2}{\mathrm{sgn}(T)}} = \pm\sqrt{\frac{2L_0}{\mathrm{sgn}(T)}}(s-s_0).$$

- Case $L_0 > 0$:

For $L_0 > 0 \iff T > 0 \iff t > 0 \iff \mathrm{sgn}(T) = 1$ we are looking at the Riemannian region with

$$T(s) = \pm\sqrt{2L_{0+}}(s-s_0),$$

where L_0 for $t > 0$ is denoted as L_{0+}.

- Case $L_0 < 0$:

For $L_0 < 0 \iff T < 0 \iff t < 0 \iff \mathrm{sgn}(T) = -1$ we are looking at the Lorentzian region with

$$T(s) = \pm\sqrt{-2L_0}(s-s_0),$$

where L_0 for $t < 0$ is denoted as L_{0-}.

Due to

$$\lim_{s\to s_0} \pm\sqrt{-2L_{0-}}(s-s_0) = \lim_{s\to s_0} \pm\sqrt{2L_{0+}}(s-s_0)$$

there obviously exists a continuous continuation from the Lorentzian region with $T < 0$ to the Riemannian region with $T > 0$. We can easily assign a L_{0i+} to a

L_{0i-}, given $L_{0i+} = -L_{0i-}$, which result in geodesics that are well behaved at the transition surface.

To further analyze the geodesics we want the coordinate x to be a function of t. We start with the case $t > 0$ and the geodesic equation 5.6 from case L_1 in Subsection 5.1

$$x(s) = x(s_0) + w(s - s_0).$$

Plugging in the term 5.2 $\pm\frac{2}{3}\frac{\sqrt{t}^3}{\sqrt{2L_{0+}}}$ from Subsection 5.1 (case $L_0 > 0$) for $(s - s_0)$ results in

$$x(s) = x(s_0) \pm \frac{2}{3}\frac{w\sqrt{t}^3}{\sqrt{2L_{0+}}}.$$

In an analogous manner we get for the case $t < 0$ the equation

$$x(s) = x(s_0) \mp \frac{2}{3}\frac{w\sqrt{-t}^3}{\sqrt{-2L_{0-}}}.$$

By taking the limit as t tends to zero,

$$\lim_{\substack{t\to 0\\ t>0}} x(s) = \lim_{\substack{t\to 0\\ t>0}} \left(x(s_0) \pm \frac{2}{3}\frac{w\sqrt{t}^3}{\sqrt{2L_{0+}}} \right) = \lim_{\substack{t\to 0\\ t<0}} \left(x(s_0) \mp \frac{2}{3}\frac{w\sqrt{-t}^3}{\sqrt{-2L_{0-}}} \right) = \lim_{\substack{t\to 0\\ t<0}} x(s) = x(s_0),$$

makes it obvious that we can merge both equation into one by setting $L_0 \coloneqq -L_{0-} = L_{0+}$. The new equation then becomes

$$x(s) = x(s_0) \pm \frac{2}{3} \cdot \mathrm{sgn}(t)\frac{w\sqrt{|t|}^3}{\sqrt{2L_0}}.$$

5.2.1 Null geodesics

Now we are jumping straight into calculating the null geodesics by imposing the null-like condition. For the toy model this condition is $2\mathcal{L} = \mathrm{sgn}(T)\dot{T}^2 + \dot{x}^2 = 0$.

For $t < 0$ this equation then, using that $L_{0-} = \frac{1}{2}\mathrm{sgn}(T)\dot{T}^2$ and $w = \dot{x}$, takes the form

$$0 = \mathcal{L} = L_{0-} + \frac{1}{2}w^2 = -L_0 + \frac{1}{2}w^2.$$

Rewriting the equation and taking the square root produces

$$\pm 1 = \frac{w}{\sqrt{2L_0}}.$$

Plugging this null condition into the geodesic equation

$$x(s) = x(s_0) \pm \frac{2}{3} \cdot \operatorname{sgn}(t) \frac{w\sqrt{|t|}^3}{\sqrt{2L_0}}$$

gives the null geodesics

$$x(s) = x(s_0) \pm \frac{2}{3} \cdot \operatorname{sgn}(t) \cdot \sqrt{|t|}^3.$$

This geodesic equation is obviously valid for all $t \in \mathbb{R}$.

6 Generalization of the toy model

Let (M, g) be a transverse type-changing singular semi-Riemannian manifold with $\dim(M) = 2$. We will consider the toy model (Section 5) as an example of this family, and analyze two classical problems in differential geometry. We show that the toy model possesses a smooth isometric embedding into Minkowski space. We also provide conditions on the Gaussian curvature K, which guarantee the existence of non-trivial singularity free models within the class of signature-type changing manifolds in consideration.

6.1 Isometric embedding into Minkowski space

In this section, we set forth that an isometric embedding of the 2-dimensional toy model (Section 5) into the 3-dimensional Minkowski space exist. Then we shed light on what such an embedding looks like. We recall that our toy model universe is modeled on $\mathbb{R}^2$ and is a 2-dimensional signature-type changing manifold. Furthermore, g is a symmetric type-changing metric with the metric tensor having the form $ds^2 = t(dt)^2 + (dx)^2$.

To make the question more precise, we can ask whether there exists an embedding $f = (\vartheta, \xi, x) : \mathbb{R}^2 \longrightarrow \mathbb{R}^{1,2}$, such that the Minkowski metric in $\mathbb{R}^{1,2}$ induces on the submanifold $f(\mathbb{R}^2)$ the given metric g? In other words, f is a smooth map, such that $df \cdot df = g$. The isometric embedding in question is given locally by functions $\vartheta(t,x), \xi(t,x), x(t,x)$, such that $-(d\vartheta)^2 + (d\xi)^2 + (dx)^2 = t(dt)^2 + (dx)^2$. This implies

$$t = -(\frac{\partial\vartheta(t,x)}{dt})^2 + (\frac{\partial\xi(t,x)}{dt})^2 + (\frac{\partial x(t,x)}{dt})^2,$$

$$1 = -(\frac{d\vartheta(t,x)}{dx})^2 + (\frac{d\xi(t,x)}{dx})^2 + (\frac{dx(t,x)}{dx})^2.$$

For the time being we can restrict our attention to the relevant embedding functions and ignore the x-coordinate which both manifolds have in common. Thus, without loss of generality, we only have to deal with

$$t = -(\frac{d\vartheta(t,x)}{dt})^2 + (\frac{d\xi(t,x)}{dt})^2.$$

It is reasonable to choose the initial values of $\vartheta = \xi = 0$ for $t = 0$, such that we have $(0 \leq \frac{d\vartheta}{dt}) \wedge (0 \leq \frac{d\xi}{dt})\ \forall t$. The first requirement ensures, nota bene, that the hypersurface of signature change goes through the origin.

A promising ansatz to solve this underdetermined equation lies in the fact that we are dealing with Minkowski space. We already know that for $t = 0$ we have $\vartheta = \xi = 0$. Furthermore, the embedded curve segment for $t < 0$ should lie within the light cone (i.e. the curve for $t < 0$ should be timelike). And the curve segment for $t > 0$ should lie outside of the light cone (i.e. the curve for $t > 0$ should be spacelike). So it suggests itself to consider the hyperbola $\vartheta^2 - \xi^2 = -1$ which lies inside the light cone and then rotate it by 45 degrees in clockwise direction and shift it such a way that it goes through the origin.

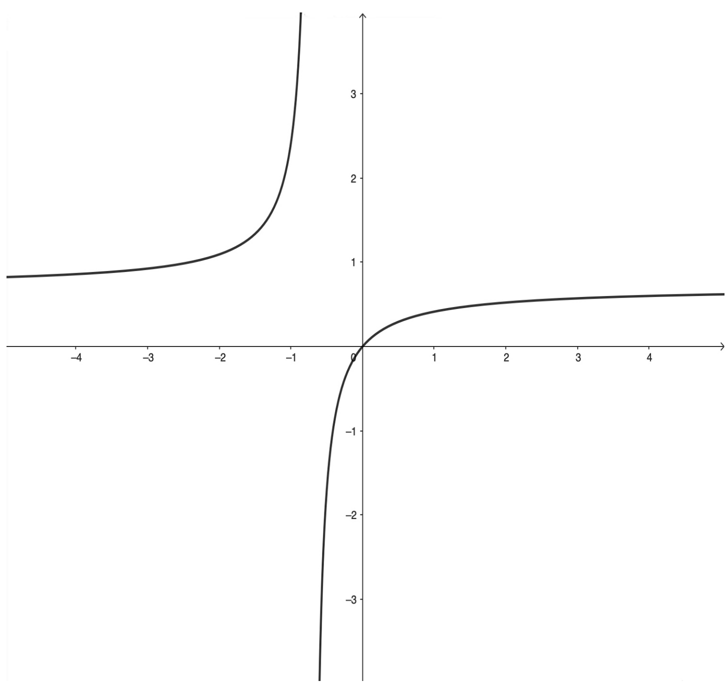

Figure 6.1: The hyperbola $\frac{1}{2}(2\xi + \sqrt{2})(\sqrt{2} - 2\vartheta) = 1$ obtained from $\vartheta^2 - \xi^2 = 1$ by rotating it by 45 degrees in clockwise direction and shift it such that it goes through the origin.

This procedure yields

$((\xi + (1/\sqrt{2}))\cos(\pi/4) - (\vartheta - (1/\sqrt{2}))\sin(\pi/4))^2 - ((\vartheta - (1/\sqrt{2}))\sin(\pi/4) + (\xi + (1/\sqrt{2}))\cos(\pi/4))^2 = 1$

$\iff \frac{1}{2}(2\xi + \sqrt{2})(\sqrt{2} - 2\vartheta) = 1$

$\iff \xi = \frac{\sqrt{2}\vartheta}{\sqrt{2}-2\vartheta}$

$\iff \vartheta = \frac{\sqrt{2}\xi}{2\xi+\sqrt{2}}$.

Plugging $\frac{d\xi}{d\vartheta} = \frac{2}{(\sqrt{2}-2\vartheta)^2}$ into

$$t = -(\frac{d\vartheta(t,x)}{dt})^2 + (\frac{d\xi(t,x)}{dt})^2 = \left(-1 + (\frac{d\xi(t,x)}{d\vartheta(t,x)})^2\right)\left(\frac{d\vartheta(t,x)}{dt}\right)^2$$

gives

$$\left(\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right)(\frac{d\vartheta}{dt})^2 = t \Leftrightarrow \left(\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right)(d\vartheta)^2 = t(dt)^2.$$

Note that we have

$\left(\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right) \geq 0$ if $t \geq 0$, hence $4 \geq (\sqrt{2} - 2\vartheta)^4$,
$\left(\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right) \leq 0$ if $t \leq 0$, hence $4 \geq (\sqrt{2} - 2\vartheta)^4$,
$4 = (\sqrt{2} - 2\vartheta)^4$ if $t = 0$, hence $(\vartheta = 0) \vee (\vartheta = \sqrt{2})$.

If we take into account the above symmetry with respect to $\vartheta = \frac{1}{\sqrt{2}}$ of the graph $\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1$, we can consider the absolute value of the equation:

$$\left|\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right|(d\vartheta)^2 = |t|\,(dt)^2.$$

Taking the square root of both sides of the equation gives us

$$\sqrt{\left|\frac{4}{(\sqrt{2}-2\vartheta)^4} - 1\right|}d\vartheta = \sqrt{|t|}dt$$

$$\iff \int_0^{\vartheta} \sqrt{\left|\frac{4}{(\sqrt{2}-2\tilde{\vartheta})^4}-1\right|}d\tilde{\vartheta} = \int_0^{t} \sqrt{|\tilde{t}|}d\tilde{t} = \frac{2}{3}\sqrt{|t|}^3\operatorname{sgn}(t),$$

which is an exact, but implicit solution for the isometric embedding. To illuminate how the function behaves, we first note that

$$\lim_{\vartheta\to\frac{1}{\sqrt{2}}} \int_0^{\vartheta} \sqrt{\left|\frac{4}{(\sqrt{2}-2\tilde{\vartheta})^4}-1\right|}d\tilde{\vartheta} = \infty.$$

Then based on the series expansion at $\vartheta = 0$, we have the following approximations:

For $-1 \ll \theta \ll \frac{1}{\sqrt{2}}$:

$$\int_0^{\vartheta} \sqrt{\left|\frac{4}{(\sqrt{2}-2\tilde{\vartheta})^4}-1\right|}d\tilde{\vartheta} \approx \int_0^{\vartheta} \sqrt{\left|4\sqrt{2}\tilde{\vartheta}\right|}d\tilde{\vartheta} = 2\sqrt[4]{2}\left(\int_0^{\vartheta} \sqrt{|\tilde{\vartheta}|}d\tilde{\vartheta}\right)$$

$$= 2\sqrt[4]{2}\left(\frac{2}{3}\sqrt{|\vartheta|}^3\operatorname{sgn}(\vartheta)\right) = \frac{4\sqrt[4]{2}}{3}\sqrt{|\vartheta|}^3\operatorname{sgn}(\vartheta)$$

$$\implies \frac{4\sqrt[4]{2}}{3}\sqrt{|\vartheta|}^3\operatorname{sgn}(\vartheta) \approx \frac{2}{3}\sqrt{|t|}^3\operatorname{sgn}(t)$$

$$\iff 2\sqrt[4]{2}\sqrt{|\vartheta|}^3\operatorname{sgn}(\vartheta) \approx \sqrt{|t|}^3\operatorname{sgn}(t)$$

$$\implies \vartheta \approx \frac{t}{2^{\frac{5}{6}}}.$$

For $\vartheta \ll -1$:

$$\int_0^{\vartheta} \underbrace{\sqrt{\left|\frac{4}{(\sqrt{2}-2\tilde{\vartheta})^4}-1\right|}}_{\approx 1}d\tilde{\vartheta} \approx \vartheta.$$

6.2 Gaussian curvature

We have figured out how the 2-dimensional toy model is situated within the 3-dimensional Minkowski space, and that an isometric embedding actually exists. Now we can turn to the Gaussian curvature K which completely characterizes the curvature of a surface. We restrict our considerations to the 2-dimensional transverse type-changing singular semi-Riemannian manifold.

Gaussian curvature is an intrinsic property of a manifold independent of the coordinate system used to describe it, and is usually defined in terms of the metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ as

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \frac{\frac{\partial G}{\partial u}}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{\frac{\partial E}{\partial v}}{\sqrt{EG}} \right), \tag{6.1}$$

where u, v denote an orthogonal coordinate system (i.e. $F = g(\partial_u, \partial_v) = 0$).

Applying this to a generalized toy model metric

$$g = \operatorname{sgn}(T)(dT)^2 + G(T,x)(dx)^2$$

in the (T,x)-coordinate system, makes it clear that K can only be calculated for the Riemannian region M_R. Plugging in $E = 1$ yields immediately

$$K = -\frac{1}{2\sqrt{G}} \left(\frac{\partial}{\partial T} \frac{\frac{\partial G}{\partial T}}{\sqrt{G}} \right) = -\frac{1}{2G} \frac{\partial^2 G}{\partial T^2} + \frac{1}{G^2} (\frac{\partial G}{\partial T})^2.$$

For the Lorentzian region M_L we resort to the formula that is based on the Riemannian curvature tensor, then the Gaussian curvature is given by

$$K = \frac{1}{EG} \left(\frac{1}{2} \frac{\partial^2 G}{\partial u^2} - \frac{1}{4} \frac{(\frac{\partial E}{\partial v})^2}{E} - \frac{1}{4} \frac{(\frac{\partial G}{\partial u})^2}{G} \right).$$

Plugging in $E = -1$ yields immediately $K = -\frac{1}{2G} \left(\frac{\partial^2 G}{\partial T^2} - \frac{1}{2G} (\frac{\partial G}{\partial T})^2 \right)$.

Hence, the Gaussian curvature satisfies $K = 0$ if and only if we have the following, respectively:

- In the Riemannian region, for $T > 0$:

$$K = -\frac{1}{2\sqrt{G}}\left(\frac{\partial}{\partial T}\frac{\frac{\partial G}{\partial T}}{\sqrt{G}}\right) = 0$$

$$\Longleftrightarrow \frac{\partial}{\partial T}\frac{\frac{\partial G}{\partial T}}{\sqrt{G}} = 0$$

$$\Longleftrightarrow \frac{\frac{\partial G}{\partial T}}{\sqrt{G}} = \kappa(x), \tag{6.2}$$

where $\kappa(x)$ is an arbitrary function that depends on x. The term (6.2) is equivalent to

$$\underbrace{\frac{1}{\sqrt{G}}dG}_{\int \frac{1}{\sqrt{G}}dG} = \underbrace{\kappa(x)dT}_{\int \kappa(x)dT} \Longleftrightarrow 2\sqrt{G} + c_1(x) = \kappa(x)T + c_2(x), \tag{6.3}$$

where $c_1(x), c_2(x)$ are arbitrary functions that depend upon x. The term (6.3) is equivalent to

$$2\sqrt{G} = \kappa(x)T + \underbrace{\phi(x)}_{(c_2(x)-c_1(x))},$$

from which $G = \frac{1}{4}\left(\kappa(x)T + \phi(x)\right)^2$ follows.

- In the Lorentzian region, for $T < 0$:

$$K = -\frac{1}{2G}\left(\frac{\partial^2 G}{\partial T^2} - \frac{1}{2G}(\frac{\partial G}{\partial T})^2\right) = 0$$

$$\Longleftrightarrow \left(\frac{\partial^2 G}{\partial T^2} - \frac{1}{2G}(\frac{\partial G}{\partial T})^2\right) = 0, \tag{6.4}$$

and from (6.4) follows $G = \frac{1}{4}(\tilde{\kappa}(x)T + \tilde{\phi}(x))^2$.

All things considered, the Gaussian curvature vanishes in the (T, x)-coordinate system in both – the Lorentzian and Riemannian – regions, if and only if $G(T, x) = \frac{1}{4}\left(\kappa(x)T + \phi(x)\right)^2$.

In the next step we want to investigate what occurs at the hypersurface of signature change $\mathcal{H}$, where the manifold goes from Lorentzian to Riemannian. The behavior of the Gaussian curvature at $\mathcal{H}$ becomes apparent when we switch to the (t, x)-coordinate system with the metric $g = t(dt)^2 + G(t, x)(dx)^2$.

With $E = t$ and $t > 0$ the Formula 6.1 yields

$$
\begin{aligned}
K &= -\frac{1}{2\sqrt{tG}}\left(\frac{\partial}{\partial t}\frac{\frac{\partial G}{\partial t}}{\sqrt{tG}}\right)\\
&= -\frac{1}{2\sqrt{tG}^3}\left(\frac{\partial^2 G}{\partial t^2}\sqrt{tG} - \frac{\partial G}{\partial t}\frac{\partial\sqrt{tG}}{\partial t}\right)\\
&= -\frac{1}{2}\frac{\frac{\partial^2 G}{\partial t^2}}{tG} + \frac{1}{4}\frac{\frac{\partial G}{\partial t}\frac{\partial (tG)}{\partial t}}{t^2G^2}\\
&= -\frac{1}{2}\frac{\frac{\partial^2 G}{\partial t^2}}{tG} + \frac{1}{4}\left(\frac{\frac{\partial G}{\partial t}}{t^2G} + \frac{(\frac{\partial G}{\partial t})^2}{tG^2}\right)\\
&= -\frac{2Gt \cdot \frac{\partial^2 G}{\partial t^2} - \frac{\partial G}{\partial t}(\frac{\partial G}{\partial t} \cdot t + G)}{4G^2t^2}.
\end{aligned}
\tag{6.5}
$$

In these coordinates the Gauss curvature K diverges for $t \to 0$, and in two dimensions this is obviously associated with the divergence of the Ricci scalar $R = 2K$. A similar behavior can be expected in the Lorentzian region. The divergence of the Ricci scalar implies that a spacetime itself is singular [24], with classic examples being scalar singularities that might exist at the beginning of the universe in general spacetimes.[22]

This begs the question how the function $G(t, x)$ should look like in order to avoid a curvature singularity at $\mathcal{H}$, i.e. at $t = 0$. This quest may seem related to the discussion in Hayward [44] where it is stated that *"the spatial metric and the Klein-Gordon field must be instantaneously stationary at the junction"*. Since in a 2-dimensional signature-type changing manifold only the two metric signs $(-, +)$ and $(+, +)$ occur, we only need to consider the case $G(t, x) > 0$. We can get such

[22]This obviously only applies to spacetimes without a removable singularity in K.

a $G(t,x)$ by picking a strictly positive real function or choosing any real function that is bounded from below and shifting it to the Riemannian area M_R $(t>0)$.

In order to figure out how to get a removable singularity in K, we have to find out how to cancel the singularity at $t=0$ in the denominator with the numerator. For this purpose we replace the expression 6.5 for K with the respective Taylor series at $t=0$:

$$K=-\frac{1}{2}\frac{\frac{\partial^2 G}{\partial t^2}}{tG}+\frac{1}{4}\left(\frac{\frac{\partial G}{\partial t}}{t^2G}+\frac{(\frac{\partial G}{\partial t})^2}{tG^2}\right)=\frac{-\frac{\partial^2 G(0,x)}{\partial t^2}}{2G(0,x)}\frac{1}{t}+\frac{\left(\frac{\partial G(0,x)}{\partial t}\frac{\partial^2 G(0,x)}{\partial t^2}-G(0,x)\frac{\partial^3 G(0,x)}{\partial t^3}\right)}{2G(0,x)^2}$$

$$+\frac{1}{4G(0,x)^3}(G(0,x)((\frac{\partial^2 G(0,x)}{\partial t^2})^2-G(0,x)\frac{\partial^4 G(0,x)}{\partial t^4})+2G(0,x)\frac{\partial^3 G(0,x)}{\partial t^3}\frac{\partial G(0,x)}{\partial t}$$

$$-2(\frac{\partial G(0,x)}{\partial t})^2\frac{\partial^2 G(0,x)}{\partial t^2})t+\frac{1}{12G(0,x)^4}(-6G(0,x)\frac{\partial^3 G(0,x)}{\partial t^3}(\frac{\partial G(0,x)}{\partial t})^2+6(\frac{\partial G(0,x)}{\partial t})^3\frac{\partial^2 G(0,x)}{\partial t^2}$$

$$+G(0,x)^2(4\frac{\partial^3 G(0,x)}{\partial t^3}\frac{\partial^2 G(0,x)}{\partial t^2}-G(0,x)\frac{\partial^5 G(0,x)}{\partial t^5})+3G(0,x)\frac{\partial G(0,x)}{\partial t}(G(0,x)\frac{\partial^4 G(0,x)}{\partial t^4}$$

$$-2(\frac{\partial^2 G(0,x)}{\partial t^2})^2))t^2+O(t^3)+\frac{\frac{\partial G(0,x)}{\partial t}}{4G(0,x)}\frac{1}{t^2}+\frac{\frac{\partial^2 G(0,x)}{\partial t^2}}{4G(0,x)}\frac{1}{t}+\frac{G(0,x)^2\frac{\partial^3 G(0,x)}{\partial t^3}-2(\frac{\partial G(0,x)}{\partial t})^3+G(0,x)\frac{\partial G(0,x)}{\partial t}\frac{\partial^2 G(0,x)}{\partial t^2}}{8G(0,x)^3}$$

$$+\frac{1}{24G(0,x)^4}(G(0,x)^3\frac{\partial^4 G(0,x)}{\partial t^4}+3G(0,x)^2(\frac{\partial^2 G(0,x)}{\partial t^2})^2+12(\frac{\partial G(0,x)}{\partial t})^4$$

$$+2G(0,x)^2\frac{\partial^3 G(0,x)}{\partial t^3}\frac{\partial G(0,x)}{\partial t}-18G(0,x)(\frac{\partial G(0,x)}{\partial t})^2\frac{\partial^2 G(0,x)}{\partial t^2})t$$

$$+\frac{1}{96G(0,x)^5}(-72(\frac{\partial G(0,x)}{\partial t})^5-36G(0,x)^2\frac{\partial^3 G(0,x)}{\partial t^3}(\frac{\partial G(0,x)}{\partial t})^2+156G(0,x)(\frac{\partial G(0,x)}{\partial t})^3\frac{\partial^2 G(0,x)}{\partial t^2}$$

$$+G(0,x)^3(G(0,x)\frac{\partial^5 G(0,x)}{\partial t^5}+14\frac{\partial^3 G(0,x)}{\partial t^3}\frac{\partial^2 G(0,x)}{\partial t^2})$$

$$+3G(0,x)^2\frac{\partial G(0,x)}{\partial t}(G(0,x)\frac{\partial^4 G(0,x)}{\partial t^4}-22(\frac{\partial^2 G(0,x)}{\partial t^2})^2))t^2+O(t^3).$$

Henceforth we shall write $G(0,x)$ just as G_0 in order to simplify the notation. Furthermore, we shall write $G^{(i)}$ for $\frac{\partial^i G(0,x)}{\partial t^i}$. Collecting and rearranging the coefficients of all the low powers of t of the above Taylor series expression for K yields

$$K=-\frac{G_0^{(2)}}{4G_0}\frac{1}{t}+\frac{G_0^{(1)}}{4G_0}\frac{1}{t^2}+\frac{1}{24G_0^4}(9G_0^2(G_0^{(2)})^2-5G_0^3G_0^{(4)}+14G_0^2G_0^{(3)}G_0^{(1)}$$

$$-30G_0(G_0^{(1)})^2G_0^{(2)}+12(G_0^{(1)})^4)t$$

$$+\frac{1}{96G_0^5}(-84G_0^2G_0^{(3)}(G_0^{(1)})^2+204G_0(G_0^{(1)})^3G_0^{(2)}+46G_0^3G_0^{(3)}G_0^{(2)}$$

$-7G_0^4G_0^{(5)} + 27G_0^3G_0^{(1)}G_0^{(4)} - 114G_0^2G_0^{(1)}(G_0^{(2)})^2 - 72(G_0^{(1)})^5)t^2+$

$\frac{1}{8G_0^3}(5G_0G_0^{(1)}G_0^{(2)} - 3G_0^2G_0^{(3)} - 2(G_0^{(1)})^3).$

If $G(0,x) = 0$, then all the derivatives, $\frac{\partial^i G(0,x)}{\partial t^i} = G^{(i)} = 0$, would also vanish, which in turn means that the Gaussian curvature must be zero. Hence, we have to assume that $G(0,x) = G_0 > 0$. Then the condition $\frac{\partial^1 G(0,x)}{\partial t^1} = \frac{\partial^2 G(0,x)}{\partial t^2} = 0$ is necessary and sufficient for avoiding a curvature singularity at $\mathcal{H}$. By satisfying the requirements $\frac{\partial^1 G(0,x)}{\partial t^1} = \frac{\partial^2 G(0,x)}{\partial t^2} = 0$ and $G_0 = G(0,x) > 0$ for $t = 0$, we are left with $K = -\frac{3}{8}\frac{G_0^{(3)}}{G_0}$.

This illuminates that despite the affine connection not being smooth, there exist non-trivial singularity free models (bearing in mind that G is actually also dependent on x in a not specified way) within the class of signature-type changing manifolds in consideration.

7 Radical of a singular semi-Riemannian manifold

Definition 7.1. Let V be a finite dimensional vector space, g be a symmetric bilinear form. The totally degenerate space $Rad \coloneqq V^{\perp} = \{u \in V : g(u,v) = 0 \ \forall v \in V\}$ is called the *radical* (also called *nullspace)* of V.

The symmetric bilinear form g is non-degenerate if and only if $Rad = \{0\}$ [71].

Let (V, g) be a vector space endowed with a bilinear form g (which does not need to be symmetric nor non-degenerate). Then we can define a natural map $\flat \colon V \longrightarrow V^*$, which associates to any $u \in V$ a 1-form $\flat(u)$, called *index lowering morphism*. The linear morphism $\flat$ is defined by $\flat(u)v \coloneqq g(u,v) \ \forall v \in V$.[23]
If g is non-degenerate, the index lowering map is invertible and we get an isomorphism of V where V^* is called the *musical isomorphism*. In this case, we will denote its inverse map by $\# \colon V^* \longrightarrow V$. From this it becomes obvious that $Rad \coloneqq V^{\perp} = \ker(\flat)$, moreover $\flat$ is an isomorphism if and only if g is non-degenerate.

Definition 7.2. [84] The vector space $V^{\flat} \coloneqq \operatorname{im}(\flat) \subseteq V^*$ is called *radical-annihilator*. It is the vector space of 1-forms ω that can be expressed as $\omega = u^{\flat} = \flat(u)$ for some $u \in V$.[24]

Definition 7.3. [84] On $V^{\flat} \subseteq V^*$ we define a canonical non-degenerate inner product by $g_{\flat}(\omega, \theta) \coloneqq g(u, v)$, with $\omega = u^{\flat}$ and $\theta = v^{\flat}$.

Proposition 7.4. *The inner product $g_{\flat}$ on $V^{\flat}$ in well-defined.*

Proof. Consider $u, v \in V$ with $\omega = u^{\flat}$ and $\theta = v^{\flat}$, and also $u', v' \in V$ with $\omega = (u')^{\flat}$ and $\theta = (v')^{\flat}$. Then we have $(u' - u) \in V^{\perp}$ and $(v' - v) \in V^{\perp}$. This yields $g(u', v') = g(u, v) + g(u' - u, v) + g(u, v' - v) + g(u' - u, v' - v) = g(u, v)$. $\square$

Proposition 7.5. [59, 84] *The inner product $g_{\flat}$ on $V^{\flat}$ is non-degenerate. And if g has the signature (r, s, t), then $g_{\flat}$ has the signature $(0, s, t)$.*

Definition 7.6. Let (M, g) be a singular semi-Riemannian manifold. The subset $TM^{\perp}$ of the tangent bundle TM is called the *radical* of TM and is defined by $TM^{\perp} \coloneqq \bigcup_{p \in M} (T_pM)^{\perp}$.

---

[23] Alternatively we will also use the notation $u^{\flat}$ for $\flat(u)$.

[24] In the non-degenerate case we have $V^{\flat} = \operatorname{im}(\flat) = V^*$. Moreover, with this definition it follows $n = \dim(V^{\perp}) + \dim(V^{\flat})$.

Note that $TM^{\perp}$ is a vector bundle if and only if the signature of g is constant on M. We denote the set of vector fields on M valued in $TM^{\perp}$, i.e. $V_p \in (T_pM)^{\perp}$, by $\mathfrak{X}^{\perp}(M) \subseteq \mathfrak{X}(M)$. Then $\mathfrak{X}^{\perp}(M)$ is a vector space over $\mathbb{R}$.

Definition 7.7. [84] Let (M, g) be a singular semi-Riemannian manifold. The subset $TM^{\flat}$ of the cotangent bundle T^*M, defined by $TM^{\flat} = \bigcup_{p\in M}(T_pM)^{\flat}$, is called the *annihilator* of TM, where $(T_pM)^{\flat} \subseteq (T_pM)^*$ is the annihilator space of the radical space.[25]

The *sections* of $TM^{\flat}$ are defined by $\mathcal{A}^{\flat}(M) := \{\omega \in \Omega^1(M) \mid \omega_p \in (T_pM)^{\flat}\ for\ p \in M\}$, where $\Omega^1(M)$ denotes the set of 1-forms on T^*M. Note that $(T_pM)^{\flat}$ is the annihilator space of the radical space $(T_pM)^{\perp}$, which yields for any $\omega_p \in (T_pM)^{\flat}$, $V_p \in (T_pM)^{\perp}$: $\omega_p(V_p) = 0$.

7.1 Radical and radical-annihilator tensors

This subsection is closedly based on [84], and deals with tensors which are radical in a contravariant slot and radical-annihilator in a covariant slot. A more detailed and rigorous exposition can be found in [59, 84, 85], so proofs will be omitted. Note that we can contract tensors in two covariant slots provided they are radical-annihilator.

Definition 7.8. Let T be a (r, s)-tensor. Then the tensor T is said to be *radical* in the k-th contravariant slot if $T \in \mathfrak{T}_0^{k-1}(M) \times TM^{\perp} \times \mathfrak{T}_s^{r-k}(M)$. T is called *radical-annihilator* in the l-th covariant slot if $T \in \mathfrak{T}_{l-1}^{r}(M) \times TM^{\flat} \times \mathfrak{T}_{s-l}^{0}(M)$.

Proposition 7.9. *A tensor $T \in \mathfrak{T}_s^r(M)$ is radical in the k-th contravariant slot if and only if the contraction $C_{s+1}^{k}(T \otimes \omega)$ with a radical-annihilator one-form $\omega \in \Omega^1(M)$ is zero.*
A tensor $T \in \mathfrak{T}_s^r(M)$ is radical-annihilator in the l-th covariant slot if and only if the l-th contraction with a radical vector field $X \in \mathfrak{X}^{\perp}(M)$ is zero.

Hence, based on the above, the metric tensor is radical-annihilator in both of its slots, i.e. $g \in \Omega^1(M) \otimes \Omega^1(M)$.

[25] $TM^{\flat}$ is a vector bundle if and only if the signature of the metric is constant.

Proposition 7.10. *The contraction of a tensor between a radical slot and a radical-annihilator slot is zero.*

Since a contraction is not always well defined for two covariant indices, Stoica [84] proposes to use $g^\flat$ in those cases (provided the tensors are radical-annihilator in the covariant slot).

Definition 7.11. The canonical *covariant contraction* is defined as follows: For a tensor $T \in V^\flat \otimes V^\flat$ the covariant contraction is given by $C_{12}T = g_\flat^{ab}T_{ab}$, with $g_\flat \in (V^\flat)^* \otimes (V^\flat)^*$.

Let $T \in \mathfrak{T}^r_s(V)$ be a (r,s)-tensor with $r \geq 0$ and $s \geq 2$, such that $T \in \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s-2} \otimes V^\flat \otimes V^\flat$.

This means that $T(\omega_1, \ldots, \omega_r, v_1, \ldots, v_s) = 0$ for $\omega_i \in V^*$, $i = 1, \ldots, r$, $v_j \in V$, $j = 1, \ldots, s$ whenever $v_{s-1} \in V^\perp$ or $v_s \in V^\perp$. The covariant contraction between the last two covariant slots is then defined as $C_{s-1,s} \coloneqq Id_{\mathfrak{T}^r_{s-2}(V)} \otimes C_{1,2}$: $\mathfrak{T}^r_{s-2}(V) \otimes V^\flat \otimes V^\flat \longrightarrow \mathfrak{T}^r_{s-2}(V)$, with $Id_{\mathfrak{T}^r_{s-2}(V)} \colon \mathfrak{T}^r_{s-2}(V) \longrightarrow \mathfrak{T}^r_{s-2}(V)$ the identity.

Let $T \in \mathfrak{T}^r_s(V)$ be a (r,s)-tensor with $r \geq 0$ and $s \geq 2$, such that $T \in \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{k-1} \otimes V^\flat \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l-k-1} \otimes V^\flat \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s-l}$, $1 \leq k < l \leq s$.

This means that $T(\omega_1, \ldots, \omega_r, v_1, \ldots, v_k, \ldots, v_l, \ldots, v_s) = 0$ for $\omega_i \in V^*$, $i = 1, \ldots, r$, $v_j \in V$, $j = 1, \ldots, s$ whenever $v_k \in V^\perp$ or $v_l \in V^\perp$. The contraction $C_{k,l} \colon \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{k-1} \otimes V^\flat \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l-k-1} \otimes V^\flat \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s-l} \longrightarrow \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s-2}$ is defined by $C_{k,l} \coloneqq C_{s-1,s} \circ P_{k,s-1;l,s}$, where $C_{s-1,s}$ is defined as above, and $P_{k,s-1;l,s} \colon \mathfrak{T}^r_s(V) \longrightarrow \mathfrak{T}^r_s(V)$, $T \mapsto T$ is the permutation isomorphism that shifts the k-th and l-th slots to the last to slots.[26]

Based on Definition 7.8 we can extend the definition for the covariant contraction in radical-annihilator slots to singular semi-Riemannian manifolds:

Definition 7.12. Let $T \in \mathfrak{T}^r_s(V)$ be a (r,s)-tensor field on M, with $s \geq 2$, that is radical-annihilator in the k-th and l-th covariant slot, $1 \leq k < l \leq s$. The linear operator $C_{k,l}$ defined by

[26]Stoica also presents how the components of the covariant contraction relative to a basis look like. Please refer to [84] for more details.

$$
\begin{aligned}
&C_{k,l}\colon \mathfrak{T}^r_{k-1}(M) \times \mathcal{A}^\flat(M) \times \mathfrak{T}^0_{l-k-1}(M) \times \mathcal{A}^\flat(M) \times \mathfrak{T}^0_{s-l}(M) \longrightarrow \mathfrak{T}^r_{s-2}(M)\\
&(C_{k,l}T)(p) = C_{k,l}(T(p)) = C(T(\omega_1, \ldots, \omega_r, X_1, \ldots, \cdot, \ldots, \cdot, \ldots, X_s))
\end{aligned}
$$

is called the *covariant contraction.*

Lemma 7.13. *Let $T \in \mathfrak{T}^r_s(M)$ be a (r, s)-tensor field on M, with $r \geq 0$ and $s \geq 1$, that is radical-annihilator in the k-th covariant slot, $1 \leq k < s$. Then the contraction of T with the metric tensor g yields again T:*

$$T(\omega_1, \ldots, \omega_{r,} X_1, \ldots, \cdot, \ldots, X_s)g(X_k, \cdot) = T(\omega_1, \ldots, \omega_{r,} X_1, \ldots, X_k, \ldots, X_s).$$

In particular, $g(X, \cdot)g(Y, \cdot) = g(X, Y)$.

Theorem 7.14. *Let (M, g) be a singular semi-Riemannian manifold with constant signature and $T \in \mathfrak{T}^r_s(M)$ a (r, s)-tensor field on M, $s \geq 2$, that is radical-annihilator in the k-th and l-th covariant slot, $1 \leq k < l \leq n$. If $E_1, \ldots, E_n$ is a frame field on M, such that $E_1, \ldots, E_{n-rank(g)} \in \mathfrak{X}^\perp(M)$, then*

$$
\begin{aligned}
&T(\omega_1, \ldots, \omega_{r,} X_1, \ldots, \cdot, \ldots, \cdot, \ldots, X_s)\\
&= \textstyle\sum_{i=n-rank(g+1)}^{n} \frac{1}{g(E_i,E_i)} T(\omega_1, \ldots, \omega_{r,} X_1, \ldots, E_i, \ldots, E_i, \ldots, X_s),
\end{aligned}
$$

for $X_1, \ldots, X_s \in \mathfrak{X}(M)$ and $\omega_1, \ldots, \omega_r \in {}^1(M)$.

7.2 The radical on the hypersurface of signature change

Let (M, g) be a transverse type-changing singular semi-Riemannian manifold with $\dim(M) = n \geq 2$, endowed with a smooth, symmetric transverse signature-type changing metric g, which changes bilinear type at the hypersurface $\mathcal{H}$. The hypersurface of signature change is defined as $\mathcal{H} := \{q \in M \colon g\mid_q \textit{ is degenerate}\}$. Furthermore, we assume that one connected component of $M \setminus \mathcal{H}$ is Riemannian and the other one Lorentzian, i.e. $M = M_L \cup \mathcal{H} \cup M_R$.

Then, for each point $p \in M$ and any local coordinate system, and for each open neighborhood of p we have

$$\triangle(p)\begin{cases} >0 & for \quad p\in M_R \\ =0 & for \quad p\in\mathcal{H} \\ <0 & for \quad p\in M_L \end{cases},$$

where $\triangle := \det([g_{\mu\nu}])$. Recall that because (M, g) is a transverse type-changing singular manifold, the function $\triangle := \det([g_{\mu\nu}])$ has a non-zero differential, $d\triangle \neq 0$, at any $q \in \mathcal{H}$ and for any local coordinate system. As is known, this condition implies that $\mathcal{H} \subset M$ is a smoothly embedded hypersurface in M. Also recall the following

Definition 7.15. The radical for $q \in \mathcal{H}$ with respect to g is a subspace Rad_q of T_qM, defined as $Rad_q := \{w \in T_qM \mid g(w, \centerdot) = 0, \forall v \in T_qM\}$.

Definition 7.16. [58] The manifold (M, g) has a transverse radical if Rad_q intersects $T_q\mathcal{H}$ transversally for all $q \in \mathcal{H}$. We say that (M, g) is *radical transverse* on $\mathcal{H}$ if $Rad_q \cap T_q\mathcal{H} = \{0\}$, and *radical tangent* on $\mathcal{H}$ if $Rad_q \subset T_q\mathcal{H}$, for all $q \in \mathcal{H}$.

The dimension of Rad_q is called the **nullity degree** of g, denoted by *null* T_qM. Clearly, a smooth, symmetric $(0, 2)$-tensor field g is degenerate on T_qM if and only if *null* $T_qM > 0$ [22]. Hence, if the the metric g is degenerate it fails to have maximal rank on the subset $\mathcal{H} \subset M$. Then at each $q \in \mathcal{H}$ there exists a nontrivial subspace $Rad_q \subset T_qM$ which is *orthogonal* to the whole T_qM.

The above premises support the following assertions:

Proposition 7.17. *Let (M, g) be transverse type-changing on a hypersurface $\mathcal{H}$, in other words $d\triangle \neq 0$ on $\mathcal{H}$. Then the subspace Rad_q is one-dimensional for all $q \in \mathcal{H}$.*

Proof. We examine the eigenvalue equation $g_{ij}v^j = \lambda\delta_{ij}v^j$ for the metric g restricted to the hypsersurface $\mathcal{H}$. Due to the requirement $\triangle(q) = \det([g_{ij}]) = 0$ for $q \in \mathcal{H}$, and since any symmetric matrix has an eigenvector, the eigenvalue equation has a non-zero solution v for $\lambda = 0$:

$$\det([g_{ij} - \lambda\delta_{ij}]) = \det([g_{ij}]) = 0 \Longleftrightarrow \lambda = 0.$$

From this follows that $\mathrm{span}(\{v_q\}) \subset Rad_q \Longrightarrow \dim(Rad_q) \geq 1$ for all $q \in \mathcal{H}$.
To finish the proof by reductio ad absurdum we make the assumption that

$\dim(Rad_q) \geq 2$ for $q \in \mathcal{H}$. As $[g_{ij}]$ is a real symmetric matrix, it is diagonalizable and there exists an orthonormal basis $\{b_1, \ldots, b_n\}$ for T_qM consisting of eigenvectors of $[g_{ij}]$. Hence, $g_{ij}b^j_{(\sigma)} = \lambda_{(\sigma)}\delta_{ij}v^j_{(\sigma)}$, with smooth $\lambda_{(\sigma)} \colon M \longrightarrow \mathbb{R}$.
The eigenspace $Eig(g_{ij}, 0)$ of $[g_{ij}]$ associated with $\lambda = 0$ is exactly the kernel (or nullspace) of the matrix $[g_{ij} - \lambda\delta_{ij}] = [g_{ij}]$. Therefore, if $\dim(Rad_q) \geq 2$ applies, then the dimension of the eigenspace $Eig(g_{ij}, 0)$ has to be at least two. According to that, there exist linearly independent vectors $v_q, w_q \in T_qM$ that satisfy $g(v_q, \centerdot) = g(w_q, \centerdot) = 0$. Without loss of generality we set $b_{(1)q} = v_q$ and $b_{(2)q} = w_q$. Assuming that g_{ij} is in diagonal form and the determinant is the product of the diagonal entries $\triangle = \lambda_{(1)} \cdot \cdots \cdot \lambda_{(n)}$, it follows

$$d\triangle = \lambda_{(2)} \cdot \cdots \cdot \lambda_{(n)} d\lambda_{(1)} + \lambda_{(1)} \cdot \lambda_{(3)} \cdot \cdots \cdot \lambda_{(n)} d\lambda_{(2)} + \cdots + \lambda_{(1)} \cdot \cdots \cdot \lambda_{(n-1)} d\lambda_{(n)}.$$

We have $\lambda_{(1)}(q) = \lambda_{(2)}(q) = 0$ because these two are eigenvalues associated with v_q, w_q, and this immediately yields $(d\triangle)\mid_q = 0$ which in turn contradicts the premise $d\triangle \neq 0$ for $q \in \mathcal{H}$. □

Proposition 7.18. *For each $q \in \mathcal{H}$ and a suitable $\varepsilon > 0$ there exists a curve $\gamma \colon (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = q$, and $\dot{\gamma}(0)$ is transverse with respect to $\mathcal{H}$. Furthermore, for $\varepsilon < 0$ the curve is Lorentzian, $\gamma(\varepsilon) \in M_L$, and for $\varepsilon < 0$ the curve is Riemannian, $\gamma(\varepsilon) \in M_R$. Simply put, the hypersurface of signature change $\mathcal{H}$ is located between M_L and M_R.*

Proof. We are assuming again that $[g_{ij}]$ is in diagonal form and the determinant is the product of the diagonal entries $\triangle = \lambda_{(1)} \cdot \cdots \cdot \lambda_{(n)}$ with smooth $\lambda_{(\sigma)} \colon M \longrightarrow \mathbb{R}$. On the junction surface $\mathcal{H}$ we set, without loss of generality, $\lambda_{(1)} = 0$ and $\lambda_{(\sigma)} > 0$ for $\sigma \geq 2$ (the latter choice is arbitrary because $\triangle(q) = \underbrace{\lambda_{(1)}}_{0} \cdot \cdots \cdot \lambda_{(n)} = 0$, $q \in \mathcal{H}$).

Then on the hypersurface $\mathcal{H}$ this yields

$$0 \neq d\triangle = \lambda_{(2)} \cdot \cdots \cdot \lambda_{(n)} d\lambda_{(1)} + \underbrace{\lambda_{(1)} \cdot \lambda_{(3)} \cdot \cdots \cdot \lambda_{(n)} d\lambda_{(2)}}_{0} + \cdots + \underbrace{\lambda_{(1)} \cdot \cdots \cdot \lambda_{(n-1)} d\lambda_{(n)}}_{0}$$

$$= \lambda_{(2)} \cdot \cdots \cdot \lambda_{(n)} d\lambda_{(1)}.$$

Since $d\lambda_{(1)}$ is a dual basis vector, we have $d\lambda_{(1)} \neq 0$ on $\mathcal{H}$.

By reason that $\lambda_{(\sigma)}$ is smooth, we conclude that the conditions $d\lambda_{(1)} \neq 0$ and

$\lambda_{(\sigma)} \neq 0$ for $\sigma \geq 2$ are also valid in a neighborhood $U(q)$ of q. For this it follows that we have $\triangle(p) \neq 0 \Longleftrightarrow \lambda_{(1)}(p) \neq 0$ for $p \in U(q)$, and $\mathcal{H} \cap U(q)$ is defined by $\lambda_{(1)}(p) = 0$.
Let $\tilde{\gamma}\colon I \longrightarrow U(q) \subset M$ be a transvere curve in $U(q)$ with $\tilde{\gamma}(0) = q$. Since we have $(d\lambda_{(1)}) \mid_q \neq 0$ and $\lambda_{(1)} = 0$ on the hypersurface $\mathcal{H}$, we get $(d\lambda_{(1)}) \mid_q (\dot{\tilde{\gamma}}(0)) = (\dot{\tilde{\gamma}}(0))(\lambda_{(1)}) \mid_q \neq 0$, which means that $\tilde{\gamma}$ is regular in 0. Then there exists a neighborhood $\bar{I}$ of $0 \in I$ such that $0 \in \bar{I} \subset I$, and a reparametrization $\hat{\gamma} = \tilde{\gamma}(\phi(s))$ of $\tilde{\gamma}$ by $\phi\colon (-\varepsilon, \varepsilon) \longrightarrow \bar{I}$ such that $\lambda_{(1)}$ is locally bijective and $\lambda_{(1)}(\hat{\gamma}(s)) = s$ for $s \in (-\varepsilon, \varepsilon)$. Next, by restricting $\hat{\gamma}$ to $(-\varepsilon, \varepsilon)$ we obtain $\gamma \coloneqq \hat{\gamma} \mid_{(-\varepsilon,\varepsilon)}$, which proves the claim because of $\operatorname{sgn}(s) = \operatorname{sgn}(\lambda_{(1)}(\gamma(s)))$. □

Remark 7.19. The proposition 7.18 is where the term 'transverse type-changing' originates from.

Example 7.20. Consider on $\mathbb{R}^2$ the metric $ds^2 = t(dt)^2 + (dx)^2$. This is a signature-type changing metric with

$$\begin{aligned} &\triangle \coloneqq \det([g_{ij}]) = t,\\ &d\triangle = \tfrac{\partial t}{\partial t}dt + \tfrac{\partial t}{\partial x}dx = dt \neq 0,\\ &\mathcal{H} = \{q \in M \mid \triangle(q) = t(q) = 0\}. \end{aligned}$$

Only for the eigenvalue $\lambda = t(q) \mid_{\mathcal{H}} = 0$ we have that the eigenspace

$$Eig((g_{ij}), 0) = \{w \in T_qM \mid g(w, \centerdot) = \underbrace{\lambda}_{0} \delta_{ij} w^j = 0\} = Rad_q$$

equals the radical. Then we get for the associated eigenvector: $g(\frac{\partial}{\partial t}, \centerdot) = 0 \Longrightarrow \frac{\partial}{\partial t} \in Rad_q$.
Hence, $Rad_q = \operatorname{span}(\{\frac{\partial}{\partial t}\}) \Longrightarrow \dim(Rad_q) = 1$ and Rad_q is transverse with respect to $\mathcal{H}$.

Example 7.21. Consider on $\mathbb{R}^2$ the metric $ds^2 = tx(dt)^2 + (dx)^2$. This is a signature-type changing metric with $\mathcal{H} = \{q \in M \mid \triangle(q) = t(q)x(q) = 0\}$, and $\triangle \coloneqq \det([g_{ij}]) = tx$. Then

$$d\triangle = d(tx) = \frac{\partial(tx)}{\partial t}dt + \frac{\partial(tx)}{\partial x}dx = xdt + tdx \begin{cases} = 0 & for\, t = x = 0 \\ \neq 0 & other \end{cases}.$$

The locus of signature change $\mathcal{H}$ consists of the two intersecting coordinate axes and is therefore not a smoothly embedded hypersurface.
The eigenvalue problem $g(w, \mathbf{.}) = \lambda\delta_{ij}w^j = 0$ is only valid for $\lambda = t(q)x(q)$, with $q \in \mathcal{H}$. For the associated eigenvector we have $g(\frac{\partial}{\partial t}, \mathbf{.}) = 0 \Longrightarrow \frac{\partial}{\partial t} \in Rad_q$, which in turn means that $Rad_q = \text{span}(\{\frac{\partial}{\partial t}\}) \Longrightarrow \dim(Rad_q) = 1$, for $q \in \mathcal{H}$.
However, Rad_q is only transverse with respect to $\mathcal{H}$ for $x(q) \neq 0$, otherwise (for $x(q) = 0$) is Rad_q tangent with respect to $\mathcal{H}$. This kind of metric does not fulfill our above proposed conditions and requirements as the hypersurface of signature change $\mathcal{H}$ is not located between M_L and M_R.

Example 7.22. Consider on $\mathbb{R}^2$ the metric $ds^2 = t^2(dt)^2 + (dx)^2$, which is clearly not a signature-type changing metric because $M_L = \emptyset$. We have

$$\begin{aligned}
&\triangle := \det([g_{ij}]) = t^2,\\
&d\triangle = dt^2 = \tfrac{\partial t^2}{\partial t}dt + \tfrac{\partial t^2}{\partial x}dx = 2tdt,\\
&\mathcal{H} = \{q \in M \mid \triangle(q) = t^2(q) = 0\} \Longleftrightarrow \{q \in M \mid \triangle(q) = t(q) = 0\}.
\end{aligned}$$

The differential $d\triangle = 2tdt = 0$ on $\mathcal{H}$, with $\mathcal{H}$ being a smoothly embedded hypersurface. The equation $g(w, \mathbf{.}) = \lambda\delta_{ij}w^j = 0$ is valid for $\lambda = t^2(q) = 0$, and yields for the associated eigenvector $\frac{\partial}{\partial t}$ again $Rad_q = \text{span}(\{\frac{\partial}{\partial t}\})$. The radical Rad_q is one-dimensional and transverse with respect to $\mathcal{H}$. But again, this metric does not fulfill our above proposed conditions and requirements as the hypersurface of signature change $\mathcal{H}$ is not located between M_L and M_R, simply because there is no Lorentzian region M_L.

Example 7.23. Consider on $\mathbb{R}^2$ the metric $ds^2 = t(dt)^2 + t^2(dx)^2$. This is a signature-type changing metric with

$$\begin{aligned}
&\triangle := \det([g_{ij}]) = t^3,\\
&d\triangle = \tfrac{\partial t^3}{\partial t}dt + \tfrac{\partial t^3}{\partial x}dx = 3t^2dt,\\
&\mathcal{H} = \{q \in M \mid \triangle(q) = t^3(q) = 0\} \Longleftrightarrow \{q \in M \mid \triangle(q) = t(q) = 0\}.
\end{aligned}$$

The differential is $d\triangle = 3t^3dt = 0$ on $\mathcal{H}$, with $\mathcal{H}$ being a smoothly embedded hypersurface. The eigenvalue problem $g(w, \mathbf{.}) = \lambda\delta_{ij}w^j = 0$ is valid for both eigenvalues, $\lambda_{(1)} = t(q)$ and $\lambda_{(2)} = t^2(q)$ respectively, with $q \in \mathcal{H}$. Hence, the geometric multiplicity is two and the associated eigenvectors, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, yield $g(\frac{\partial}{\partial t}, \mathbf{.}) = g(\frac{\partial}{\partial x}, \mathbf{.}) = 0$ $\Longrightarrow \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \in Rad_q$. Therefore, $Rad_q = T_qM \Longrightarrow \dim(Rad_q) = 2$, for $q \in \mathcal{H}$. This metric does not fulfill our above proposed conditions and requirements as the radical is not one-dimensional.

Proposition 7.24. *If the hypersurface of signature change $\mathcal{H}$ is smoothly immersed and located between M_L and M_R (such as postulated in Proposition 7.18), then $\mathcal{H}$ is a smoothly embedded submanifold.*

Proof. We know that for each $q \in \mathcal{H}$ and a suitable $\varepsilon > 0$ there exists a curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = q$ and $\dot{\gamma}(0)$ is transvere with respect to $\mathcal{H}$. By reason that γ is smooth, we can locally extend the vector $\dot{\gamma}$ to a transverse vector field Γ defined on a neighborhood of $q \in \mathcal{H} \subset M$. We now want to show that any $p \in \mathcal{H}$ has a neighborhood $U(q) \subset M$ such that $U(q) \cap \mathcal{H}$ is an embedded $(n-1)$-submanifold of $U(q)$. Then $\mathcal{H}$ is an embedded $(n-1)$-submanifold of M [61].

For this, we consider the local flow $\phi_{f(p)}\colon U(q) \longrightarrow M$ of Γ on a suitable neighborhood $U(q)$ of q. This local flow produces a C^∞ function $f\colon U(q) \longrightarrow \mathbb{R}$ by defining $\phi_{f(p)}(\tilde{q}) = p$, for a fixed $\tilde{q} \in \mathcal{H}$. This specifically yields $\phi_{f(0)}(\tilde{q}) = 0$ and for $f(p) = 0$ we have $\phi_0(\tilde{q}) = \phi(0, \tilde{q}) = \tilde{q}$ [16]. Consequently, we get $p \in M_L$ if $f(p) < 0$ and $p \in M_R$ if $f(p) > 0$. Thus we know that $U(q) \cap \mathcal{H}$ is given by $f(p) = 0$. The latter defines a regular surface of dimension $n-1$, which in turn is a submanifold of M. $\square$

8 Global structure of signature-type changing semi-Riemannian manifolds

First, let us revisit the definitions for the following two concepts of manifold orientability:

Definition 8.1. [10] A smooth n-dimensional manifold M is *orientable* if and only if it has a smooth global nowhere vanishing n-form (also called a top-ranked form).[27]

For a differentiable manifold to be orientable all that counts is that it admits a global top-ranked form; it is not important which specific top-ranked form is selected.

It is well-known that a manifold M of dimension n is defined to be parallelizable if there are n vector fields that are linearly independent at each point. The definition of parallelizability involves no metric and is therefore also valid for signature-type changing manifolds; we define similarly as in [11]:

Definition 8.2. A smooth n-dimensional manifold M is *parallelizable* if there exists a set of smooth vector fields $\{V, E_1, \ldots, E_{n-1}\}$ on M, such that at every point $p \in M$ the tangent vectors $\{V(p), E_1(p), \ldots, E_{n-1}(p)\}$ provide a basis of the tangent space T_pM. A specific choice of such a basis of vector fields on M is called an *absolute parallelism* of M.

Equivalently, a manifold M of dimension n is parallelizable if its tangent bundle TM is a trivial bundle, so that the associated principal bundle of linear frames has a global section on M, i.e. the tangent bundle is then globally of the form $TM \simeq M \times \mathbb{R}^n$.

It is worth pointing out that orientability and also parallelizability are *differential topological properties* which do not depend on the metric structure, only on the topological manifold with a globally defined differential structure.[28] The next three

[27] An orientation of M is the choice of a continuous pointwise orientation, i.e. the specific choice of a global nowhere vanishing n-form.

[28] It is worth mentioning that given an absolute parallelism of M, one can use these n vector fields to define a basis of the tangent space at each point of M and thus one can always get

definitions, however, depend not only on the underlying manifold but also on its specific type-changing metric g. For our purpose, let (M, g) be a smooth, signature-type changing manifold (possibly with boundary).

Definition 8.3. **(Pseudo-timelike)** A vector field V in a signature-type changing manifold (M, g) is *pseudo-timelike* if and only if V is timelike in M_L.

Definition 8.4. **(Pseudo-time orientable)** A signature-type changing manifold (M, g) is *pseudo-time orientable* if and only if the Lorentzian region M_L is time orientable.[29]

Lemma 8.5. *A singular semi-Riemannian manifold (M, g) is pseudo-time orientable if and only if there exists a vector field $X \in \mathfrak{X}(M)$ that is pseudo-timelike.*

Proof.

" $\Longrightarrow$ " A singular semi-Riemannian manifold (M, g) is pseudo-time orientable. This means the Lorentzian region M_L is time orientable. A Lorentzian manifold is time-orientable if there exists a continuous timelike vector field. Accordingly there must exist a continuous timelike vector field $X \in \mathfrak{X}(M_L)$ in the Lorentzian region. As per Definition 8.3, a vector field X in a signature-type changing manifold is pseudo-timelike if and only if X is timelike in M_L; this means that X is allowed to vanish on M_R. Hence, we can extend the vector field X arbitrarily to all of M, and per definition $X \in \mathfrak{X}(M)$ is pseudo-timelike.

" $\Longleftarrow$ " Let $X \in \mathfrak{X}(M)$ be a pseudo-timelike vector field in a singular semi-Riemannian manifold (M, g). Hence, as per Definition 8.3, X is timelike in M_L. A Lorentzian manifold is time-orientable if and only if there exists a timelike vector field. Since X is a timelike vector field on M_L, the Lorentzian region M_L is time-orientable. Then, according to Definition 8.4, the signature-type changing manifold (M, g) is pseudo-time orientable. $\square$

According to that, such a definition of a pseudo-time orientation is possible if M_L admits a globally consistent sense of time, i.e. if in M_L we can continuously define a division of non-spacelike vectors into two classes. For a transverse, signature-type

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a frame-dependent metric g by defining the frame to be orthonormal. Moreover, the special orthogonal group, denoted $SO(n, \mathbb{R})$, acts naturally on each tangent space via a change of basis, it is then possible to obtain the set of all orthonormal frames for M at each point qua the oriented orthonormal frame bundle of M, denoted $F_{SO}(M)$, associated to the tangent bundle of M.

[29] A pseudo-time orientation of such a manifold (M, g) corresponds to the specific choice of a continuous non-vanishing pseudo-timelike vector field V on M.

changing manifold (with a transverse radical) this definition arises naturally due to the fact that in M_R all vectors can be considered spacelike. And because of $Rad_q \cap T_q\mathcal{H} = \{0\}\ \forall q \in \mathcal{H}$, on $\mathcal{H}$ all non-spacelike vectors are lightlike, which are naturally divided into two classes: vectors that are pointing towards M_L and the ones pointing towards M_R.

Example 8.6. Consider the classic type of a spacetime M with signature-type change which is obtained by cutting an S^4 along its equator and joining it to the corresponding half of a de Sitter space, Figure 8.1. The de Sitter spacetime is time-orientable [43], so M is pseudo-time orientable.

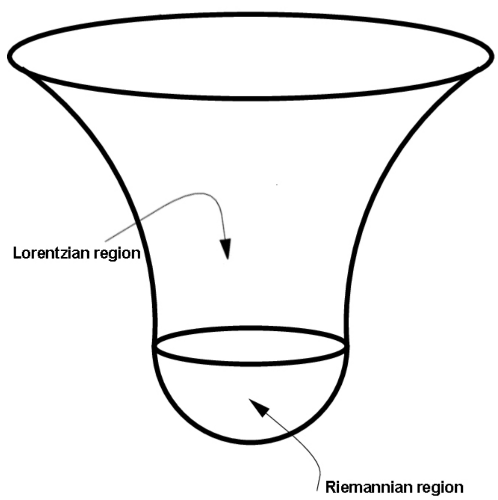

Figure 8.1: Riemannian and Lorentzian region in the Hartle-Hawking no-boundary model.

Definition 8.7. (Pseudo-space orientable) A signature-type changing manifold (M, g) of dimension n is *pseudo-space orientable* if and only if it admits a continuous non-vanishing $(n-1)$-frame field, that is a set of $n-1$ pointwise orthonormal spacelike vector fields on M.[30]

Example 8.8. Consider the 2-dimensional signature-type changing crosscap manifold $\overline{\mathbb{D}}^2 \cup_h (M \cup \partial M)$, which is obtained by sewing a Möbius strip to the edge of

[30]A pseudo-space orientation of a manifold (M, g) corresponds to the specific choice of a continuous non-vanishing field of orthonormal spacelike $(n-1)-$beins on M.

a disk (the Riemannian sector), equipped with the metric $g = (1-t^2)(dt)^2 + 2txdtdx + (1-x^2)(dx)^2$, see Subsection 15.3.3. According to Definition 8.7 we would need a set of $2-1=1$ pointwise orthonormal spacelike vector fields on all of $\overline{\mathbb{D}}^2 \cup_h (M \cup \partial M)$ for the crosscap to be *pseudo-space orientable.* However, the crosscap is closed and has the Euler characteristic $\chi = 1$.[31] Since a closed, connected manifold admits a non-vanishing vector field if and only if the Euler characteristic of the manifold is zero, there is no global non-vanishing vector field on the crosscap. Hence, the manifold (15.3.3) is not pseudo-space orientable, but it is indeed pseudo-time orientable.

Proposition 8.9. [61] *Every parallelizable manifold M is orientable.*

In Lorentzian geometry the fact of M being time-orientable and space-orientable implies that M is orientable [40]. The proposition below illustrates that this result from Lorentzian geometry cannot be applied to signature-type changing manifolds.

Proposition 8.10. *Even if a transverse, signature-type changing manifold (M, g) with a transverse radical is pseudo-time orientable and pseudo-space orientable, it is not necessarily orientable.*

Proof. Consider the 2-dimensional Möbius strip $M_\infty := (M/\sim, g)$ equipped with the standard topology and the smooth metric $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$, $\varphi = \pi x$, see Section 15.2.2. As a time-orientable Lorentzian manifold, the Möbius strip M_∞ admits a global, non-vanishing, timelike vector field $V = (\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}$, satisfying $g(V, V) = -1$. Hence we can extend V to an "orthonormal" basis $\{V, E_1\}$ of T_pM, for all $p \in M$, such that $g(E_1, E_1) = 1$ and $g(V, E_1) = 0$, with E_1 being a frame field.

Then by applying the Transformation Prescription (Section 15.2.1) we transform M_∞ into the signature-type changing Möbius strip $\tilde{M}_\infty = (M/\sim, \tilde{g})$, with $\tilde{g} = (f\cdot\cos^2(\varphi) - \cos(2\varphi))dt^2 + (2-f)\sin(2\varphi)dtdx + (f\cdot\sin^2(\varphi) + \cos(2\varphi))dx^2$, where the quotient map identifies (t, x) with $(\tilde{t}, \tilde{x}) = ((-1)^k t, x+k)$, $k \in \mathbb{Z}$, and f is an arbitrary C^∞ function that also accounts for the hypersurface of signature-change $\mathcal{H} = f^{-1}(1)$.
The signature-type changing manifold $\tilde{M}_\infty$ is then pseudo-time orientable with $V = (\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}$ being a continuous, non-vanishing, timelike vector field

[31] Here “closed” is meant in a manifold sense of “a manifold without boundary that is compact” not in the topology sense of "the complement of an open subset of $\mathbb{R}^n$”.

V on M_L. Also with $\{E_1\}$ spacelike, $\tilde{M}_\infty$ is pseudo-space orientable. But the Möbius strip $\tilde{M}_\infty$ is not orientable. $\square$

In place of the above specific example, one can consider an arbitrary manifold of $\dim(M) = 2$ with a change of signature, for which the conditions of proposition 8.10 are given (in higher dimensions, the same idea can be carried out through a trivial augmentation of dimensions). In case this manifold is non-orientable, there is nothing to show. However, if it is orientable, cut out a disk from the Riemannian sector and replace it with a crosscap, equipped with any Riemannian metric. In a tubular neighborhood of the cutting line, construct a Riemannian metric that mediates between the metrics of the crosscap and the rest (this is possible due to the convexity of the space formed by all Riemannian metrics). This surgical intervention results in the transition to a non-orientable manifold with a change of signature. Since the intervention is limited to the Riemannian sector, the conditions of the proposition remain unaffected. Thus, the proposition 8.10 is proven.

Remark 8.11. One can always "switch" between non-orientability and orientability using the crosscap. Starting with an orientable manifold, one transitions to non-orientable by replacing a cross cap (if already present) with a disk. If no cross cap is present, such a transition occurs by replacing a disk with a cross cap

Example 8.12. The Möbius strip $\mathbb{M}$ has a non-trivial vector bundle over S^1 with two co-cycles.[32] Hence, $\mathbb{M}$ is not parallelizable, and not orientable.
To see this, consider the Möbius strip $\mathbb{M} = \mathbb{R}\times\mathbb{R}/\sim$ with the identification $(t, x) \sim (\tilde{t}, \tilde{x}) \iff (\tilde{t}, \tilde{x}) = ((-1)^k t, x+k), k \in \mathbb{Z}$, as introduced in Subsection 15.2.2. Notice first that the identification has no bearing on proper subsets of $((-1)^k t, x+k), k \in \mathbb{Z}$ and the fibre $\mathbb{R}$ is a vector space.
As $\mathbb{M}$ is a fiber bundle over the base space S^1, then a section of that fiber bundle must be a continuous map $\sigma : S^1 \longrightarrow \mathbb{M}$ such that $\sigma(x) = (h(x), x) \in \mathbb{M}$. But for σ to be continuous we must require that h satisfies $-h(0) = h(k)$. Then the intermediate value theorem guarantees that there is a some $\tilde{x} \in [0, k]$ such $h(\tilde{x}) = 0$. This means that every section of $\mathbb{M}$ intersects the zero section and the sections that form a basis for the fibre are not non-zero everywhere.

Definition 8.13. A **pseudo-spacetime** is a 4-dimensional, pseudo-time oriented, semi-Riemannian manifold with a type-changing metric.

[32]The Möbius strip is insofar interesting that we always can find a Möbius strip $\mathbb{M}$ on any arbitrary non-orientable surface. And any Lorentzian manifold $\mathbb{M} \times \mathbb{R}^n$ based on the Möbius strip crossed with $\mathbb{R}^n$ either fails to be time-orientable or space-orientable [32].

In contrast to the definition of a spacetime for the Lorentzian case, the manifold can be either connected or disconnected. The latter one arises in a signature-type changing manifold where the metric diverges at the surface of signature change. In such a situation the surface does not belong to the manifold.

Proposition 8.14. *Let $(\mathbb{R}^n, g)$ be a transverse, signature-type changing n-manifold with a transverse radical, and let $\mathcal{H} \subset \mathbb{R}^n$ be a codimension one closed hypersurface of signature change without boundary.*[33] *Then is $\mathcal{H}$ always orientable.*

Proof. This can be shown by a purely topological argument, as in [74]. □

Proposition 8.15. *Let (M, g) be a transverse, signature-type changing, oriented, n-dimensional manifold with a transverse radical, and let $\mathcal{H} \subset M$ be the hypersurface of signature change. Then $\mathcal{H}$ is also oriented.*

Proof. The hypersurface of signature change as a closed submanifold $\mathcal{H} \subset M$ of codimension one is the inverse image of a regular value of a smooth transformation map $f : M \longrightarrow \mathbb{R}$, hence it has a trivial normal bundle $T\mathcal{H}$. Then, per definition $\mathcal{H}$ is parallelizable. Based on Proposition 8.9 $\mathcal{H}$ is orientable.
Moreover, recall that $\mathcal{H} \coloneqq f^{-1}(1) = \{p \in M : f(p) = 1\}$ is a submanifold of M of dimension $n-1$. And for every $q \in \mathcal{H}$ the tangent space $T_q\mathcal{H} = T_q(f^{-1}(1))$ to $\mathcal{H}$ at q is the kernel $\ker(df_q)$ of the map $df_q : T_qM \longrightarrow T_1\mathbb{R}$. Then $T_q\mathcal{H} = \langle \mathrm{grad} f_q \rangle^{\perp}$, and therefore the gradient $\mathrm{grad} f$ yields an orientation of $\mathcal{H}$. □

Provided a transverse, signature-type changing manifold (M, g) with a transverse radical is pseudo-time orientable, then we can choose one of the two possible time orientations at any point in each connected component M_L, and thus designating the future direction of time in the Lorentzian regime. On $\mathcal{H}$ all non-spacelike vectors are *lightlike*, and smoothly divided into two classes in a natural way: the vectors located at an initial base point on $\mathcal{H}$ are either pointing towards M_L or towards M_R. This together with the existent absolute time function (that establishes a time concept [52] in the Riemannian region) can be considered as arrow of time on M.

Definition 8.16. (Natural time direction) Let (M, g) be a pseudo-time orientable, transverse, signature-type changing n-dimensional manifold with a transverse radical. Then in the neighborhood of $\mathcal{H}$ the absolute time function $h(t, \hat{x}) \coloneqq t$

[33] Here “closed” is meant in the topology sense of "the complement of an open subset of $\mathbb{R}^n$” and not in a manifold sense of “a manifold without boundary that is compact”.

imposes *a natural time direction* by postulating that the future corresponds to the increase of the absolute time function. In this way, the time orientation is determined in M_L.

Remark 8.17. Note that ∂_t with an initial point on $\mathcal{H}$ points in the direction in which $t = h(t, \hat{x})$ increases and x_i is constant. Away from the hypersurface the future direction is defined *relative to* $\mathcal{H}$ by the accordant time orientation of M_L. Recall that functions of the type like the *absolute time function* lead to metric splitings by default.

Definition 8.18. (Future-Directed) A pseudo-timelike curve (see Definition 4.7) in (M, g) is *future-directed* (in the sense of Definiton 8.16 and Remark 8.17) if for every point in the curve

(i) within M_L the tangent vector is future-pointing, and

(ii) on $\mathcal{H}$ the associated tangent vector with an initial base point on $\mathcal{H}$ is future-pointing, if applicable.

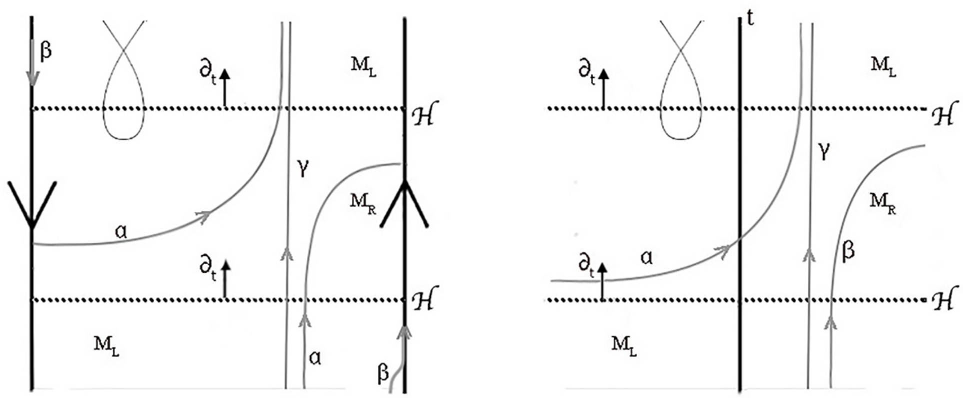

Figure 8.2: In the left example the curves α and γ are both future-directed. The curve β runs within the edge that is twisted and identified with the left edge; therefore β is neither future-directed nor past-directed. In the right example the curves α, β and γ are future-directed. In both examples the loops around $\mathcal{H}$ are neither future-directed nor past-directed.

Respective *past-directed curves* are defined analogously. Notice that per assumption one connected component of $M \setminus \mathcal{H}$ is Riemannian and all other connected components $(M_{L_\alpha})_{\alpha \in I} \subseteq M_L \subset M$ are Lorentzian. This could allow (at least locally) for a $M_L - M_R - M_L-$sandwich-like structure of M, where $\mathcal{H}$ consists of two connected components $(\mathcal{H}_\alpha)_{\alpha\in\{1,2\}}$, and to that effect we would also have two absolute time functions.

Definition 8.19. (Pseudo-chronological past and future) Let (M, g) be a pseudo-time orientable, transverse, signature-type changing n-dimensional manifold with a transverse radical.

$\mathcal{I}^-(p) = \{q \in M : q \ll p\}$ is the *pseudo-chronological past* of the event $p \in M$. In other words, there is a future-directed pseudo-timelike curve from q to p in M.

$\mathcal{I}^+(p) = \{q \in M : p \ll q\}$ is the *pseudo-chronological future* of the event $p \in M$. In other words, there is a future-directed pseudo-timelike curve from p to q in M.

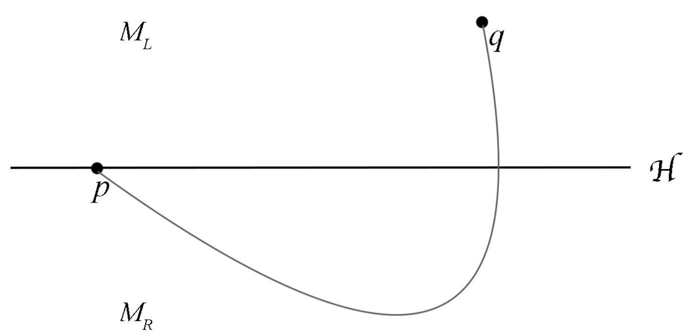

Figure 8.3: For an event $p \in \mathcal{H}$ there exists a future-directed pseudo-timelike curve (as depicted) that connects the points p and q in M. Similarly any point in M can be reached by such a future-directed pseudo-timelike curve from p. That is why for the pseudo-chronological future we have $\mathcal{I}^+(p) = \{q \in M : p \ll q\} = M$.

Remark 8.20. Interestingly, this definition leads to the following peculiar situation: Recall that any curve is denoted pseudo-timelike if its M_L-segment is timelike. To that effect, all curves that steer clear of M_L (and do not have a M_L-segment) are also considered pseudo-timelike. When $p \in \mathcal{H} \cup M_R$ then the pseudo-chronological past of p is $\mathcal{I}^-(p) = M \setminus M_L$ and the pseudo-chronological future of p is $\mathcal{I}^+(p) = M$, see Figure 8.3.

9 Radical-adapted Gauss-like coordinates

Kriele and Martin [58] expound that signature-type change can not be used to avoid singularities of solutions of Einstein's equations. Thus they look into imposing a continuous change of signature at the event horizon of a black hole. Given that the event horizon is a null surface, they propose that signature change at a null hypersurface $\mathcal{H}$ implicates the existence of curvature singularities [53].

Example 9.1. Consider the 2-dimensional toy model $g = x(dt)^2 + (dx)^2$ as presented in [58]. This is a transverse, type-changing metric with a smoothly embedded hypersurface of signature-type change located at $\mathcal{H} = \{q \in M \mid x(q) = 0\}$. However, the radical Rad_q is tangent and not transverse with respect to $\mathcal{H}$. Hence, according to Definition 14.1, the hypersurface $\mathcal{H}$ is a *null* surface. The calculation of the Gauss curvature yields $K = \frac{1}{x(q)^2}$ which obviously diverges at $\mathcal{H}$.

From a cosmological point of view it is justified to exclusively focus on spacelike surfaces of signature-type change. On this account recall that a manifold (M, g) has a spacelike hypersurface if and only if the radical $Rad_q = \{w \in T_qM \mid g(w, \centerdot) = 0\}$ intersects $T_q\mathcal{H}$ transversally for all $q \in \mathcal{H}$.

Theorem 9.2. [2, 58] *Let M be a singular semi-Riemannian manifold endowed with a $(0,2)$-tensor field g and the surface of signature change defined as $\mathcal{H} :=$ $\{q \in M \colon g\mid_g$ is degenerate$\}$. Then is (M, g) a transverse, signature-type changing manifold with a transverse radical if and only if for every $q \in \mathcal{H}$ there exist a neighborhood $U(q)$ and smooth coordinates $(t, x^1, \ldots, x^{n-1})$ such that $g = -t(dt)^2 + g_{ij}(t, x^1, \ldots, x^{n-1})dx^i dx^j$.*

In [52, 54] Kossowski and Kriele supplement this by

Theorem 9.3. *There exists a unique family $\{\gamma_q\}$, $q \in \mathcal{H}$, of smoothly immersed pre-geodesics such that $\gamma_q(0) = q$ and $g(\dot\gamma_q(s), \dot\gamma_q(s)) = -s$.*[34]
Moreover, for a transverse, signature-type changing manifold (M, g) with a transverse radical, there exist smooth coordinates $(t, x^1, \ldots, x^{n-1})$ such that the metric takes the form $g = -t(dt)^2 + g_{ij}dx^i dx^j$, for $i, j \in \{1, \ldots, n-1\}$, and $\dot\gamma_q = \partial_t$ holds (see also Section 12 for details).

[34] Recall that a pregeodesic in M is a smooth curve $\gamma \colon I \longrightarrow M$, which can be reparametrized to a geodesic.

Interestingly Hayward [45] refers to this kind of coordinates as *normal coordinates*. But this choice of name holds a likelihood of confusion with Riemannian normal coordinates. Hence, in the style of time-orthonormal coordinates in Lorentzian geometry we rather denote the coordinates in Theorem 9.2 as **radical-adapted Gauss-like coordinates**. It is now possible to simplify matters by using radical-adapted Gauss-like coordinates whenever dealing with a transverse, signature-type changing manifold with a transverse radical.[35]

Notably, signature-type change and the radical-adapted Gauss-like coordinates (such that the metric can be expressed in the form $g = -t(dt)^2 + g_{ij}dx^i dx^j$) imply the existence of an uniquely determined, coordinate independent, natural *absolute time function* $h(t, \hat{x}) := t$ (see Definition 8.16) in the neighborhood of the hypersurface [52]. This allows us to introduce a time concept locally around $\mathcal{H}$, even in the Riemannian region. We will study the properties of the absolute time function in Section 13.

Remark 9.4. Note that in general, a time function is defined as follows: A continuous function $f\colon M \longrightarrow \mathbb{R}$ which satisfies $p < \tilde{p} \Longrightarrow f(p) < f(\tilde{p})$ is a *time function*. In general relativity a function f is called time function, if ∇f is timelike past-pointing. Observe that unlike the absolute time function in our case, the generic time function f is not unique.

With the above we have established how the topological structure of the unit sphere bundles (for a transverse, signature-type changing manifold with a transverse radical) evolve when they approach and pass through the hypersurface. A **unit sphere bundle** T^1M is the submanifold of TM defined a the regular level set of the smooth function $f\colon TM \longrightarrow \mathbb{R}$, $w \mapsto g(w, w)$. For the 2-dimensional case we have therefore the following visualization:

[35] If we were to deflect our focus from the case with a transverse radical and considered a tangent radical instead, then the associated radical-adapted coordinates would have a much more complicated form. Please refer to [1], equation (10) in Remark 3 for more details.

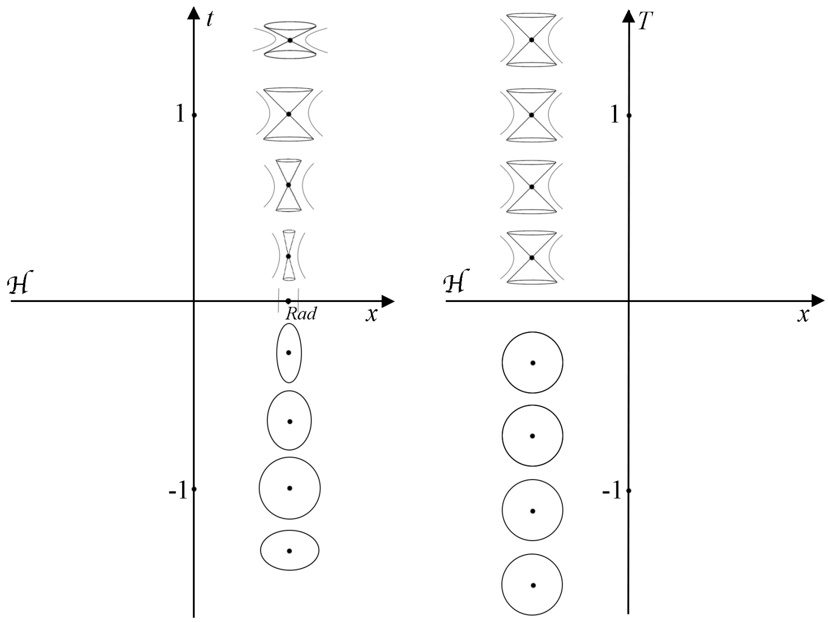

Figure 9.1: The evolution of the unit spheres $\{v \in T_qM : g(v,v) = 1\}$ across $\mathcal{H}$: The unit spheres change their *topological structure* when the bilinear type of g changes upon crossing $\mathcal{H}$. With respect to the light cone structure we can also visualize all timelike vectors with the squared length -1 (hyperbola that belongs to the timelike coordinate) and all spacelike vectors with the squared length $+1$. On $\mathcal{H}$ we have the vectors with squared length 1, and the vectors that yield zero (radical).

Since (M, g) is a singular semi-Riemannian manifold with a smooth metric, we can always choose a non-holonomic basis field (independently of the choice of a coordinate chart) in each tangent space. While each tangent space of a Lorentzian manifold is isometric to Minkowski space, we can consider a transverse type-changing singular semi-Riemannian manifold to be infinitesimally modeled on a local C^∞ frame $e_0, \ldots, e_{n-1}$, such that $e_0(q) \in Rad_q$ for $q \in \mathcal{H}$. Therefore g is *infinitesimally* represented by

$$(g_{ij})(p) = \begin{pmatrix} g_{00}(p) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $g_{00}(p) = g(e_0, e_0)(p)$ is smooth and $g_{00}(q) = 0$, $g_{00}(p) \mid_{M_R} = -1$, $g_{00}(p) \mid_{M_L} = 1$. We refer to such a frame as a **radical-adapted orthonormal frame** [51].

If the 1-dimensional subspace Rad_q (which is orthogonal to all of T_qM) and $T_q\mathcal{H}$ split T_qM then the metric singularity is called *radical transverse* at q:

Proposition 9.5. [50] *Let* (M, g) *be a singular semi-Riemannian manifold,* $\dim M = n \geq 2$*, with a smooth, symmetric, degenerate metric* g*, and* $\mathcal{H}$ *a metric singularity of co-dimension one. Then* (M, g) *has a transverse metric singularity at* $q \in \mathcal{H}$ *if and only if* $d(\det([g_{\mu\nu}])_q) \neq 0$ *for some local* C^∞ *frame* $e_0, \ldots, e_{n-1}$ *centered at* q*.*

10 Generalization

Recall that here and in the sequel of this work, we side with [52] in demanding that the metric should be a smooth, symmetric transverse type-changing $(0,2)$-tensor field with a transverse radical (see Section 9).

10.1 Generalization in 2D

In Section 6.2 we examined a 2-dimensional signature-type changing manifold in the (t,x)-coordinate system where we introduced the metric coefficient $G(t,x)$ in order to obtain the generalized metric $g = t(dt)^2 + G(t,x)(dx)^2$.

We can obviously choose an arbitrary strictly positive real function for $G(t,x)$ (or any real function that is bounded from below and shift it to the Riemannian area M_R). According to Section 9 there always exist radical-adapted Gauss-like coordinates (Theorem 9.2) such that the metric has actually the above form $g = t(dt)^2 + G(t,x)(dx)^2$ with the coefficient function $E(t,x) = t$.

Since we restrict our discussion in this work to the class of transverse, signature-type changing manifolds with a transverse radical, the metric $g = t(dt)^2 + G(t,x)(dx)^2$ is the (only possible) generalized form of the 2-dimensional toy model metric. Any other metric in this class can be thus obtained by a coordinate transformation from this general metric represented in radical-adapted Gauss-like coordinates.

Suppose however, we consider a new metric $\bar{g}$ which is related to g by a conformal transformation with conformal factor $\Omega = \frac{1}{t}$. This Ω seems to be a natural choice and uniquely determined by the absolute time function t.[36] In this way we obtain a metric representative of a different class: $\bar{g} = \Omega g = 1(dt)^2 + \frac{1}{t}G(t,x)(dx)^2$. As the conformal factor $\mathit{\Omega} = \frac{1}{t}$ constitutes an infinite discontinuity and blows up at the events with $t = 0$, the metric there is neither differentiable nor continuous.

With $E = 1$ and $t > 0$ the formula for the Gauss curvature 6.1 yields

$$K = -\frac{1}{2\sqrt{\frac{G}{t}}}\left(\frac{\partial}{\partial t}\frac{\frac{\partial \frac{G}{t}}{\partial t}}{\sqrt{\frac{G}{t}}}\right)$$

[36] Recall that signature-type change implies the existence of an uniquely determined absolute time function in a neighborhood of the hypersurface [52].

$$= -\frac{t}{2G\sqrt{\frac{G}{t}}}\left(\frac{\partial^2 \frac{G}{t}}{\partial t^2}\sqrt{\frac{G}{t}} - \frac{\partial \frac{G}{t}}{\partial t}\frac{\partial\sqrt{\frac{G}{t}}}{\partial t}\right)$$

$$= -\frac{t}{2G}\frac{\partial^2 \frac{G}{t}}{\partial t^2} + \frac{t}{2G\sqrt{\frac{G}{t}}}\frac{\partial \frac{G}{t}}{\partial t}\frac{\partial\sqrt{\frac{G}{t}}}{\partial t}$$

$$= \frac{G^{\frac{3}{2}}}{2t^{\frac{7}{2}}} - \frac{\sqrt{G}\frac{\partial G}{\partial t}}{t^{\frac{5}{2}}} + \frac{(\frac{\partial G}{\partial t})^2}{2\sqrt{G}t^{\frac{3}{2}}}.$$

The Gauss curvature K diverges for $t \to 0^+$, and in two dimensions this is associated with the divergence of the Ricci scalar $R = 2K$. Hence, $t = 0$ is not simply a coordinate singularity, but a curvature singularity and the metric $\bar{g} = \Omega g$ is not defined at $t = 0$.[37]

In the generic case, the determinant $\triangle := \det([g_{ij}]) = \frac{1}{t}G(t,x)$ obviously also diverges at $t = 0$ which prevents us from defining the hypersurface of signature change as proposed in the Definition 7.2. As a result we are also not able to define the radical on the hypersurface. Metrics of the latter type (i.e. metrics that are not differentiable on the hypersurface of signature change) are explored in [29].

Example 10.1. We apply the conformal transformation to our toy model metric $ds^2 = t(dt)^2 + (dx)^2$ which immediately yields $ds^2 = 1(dt)^2 + \frac{1}{t}(dx)^2$. This gives us apparently the *metric A* as described in [29], but with reversed Riemannian and Lorentzian sectors (the upper half plane of the spacetime, $t > 0$, has here Riemannian structure and the lower part, $t < 0$, is Minkowski).

The corresponding Lagrangean $L_{\bar{A}} = \frac{1}{2}(\dot{t}^2 + \frac{1}{t}\dot{x}^2)$ determines the geodesics

$$\begin{aligned}
&\bar{t}(\lambda) = -t(\lambda) = -\tfrac{c^2}{4}\lambda^2 + \tfrac{1}{c^2}2L_{\bar{A}}\\
&\bar{x}(\lambda) = -x(\lambda) = -\tfrac{c^3}{12}\lambda^3 + \tfrac{2L_{\bar{A}}}{c}\lambda - x(0),
\end{aligned}$$

which are also just the geodesic equations (16.6) in [29], but multiplied by (-1).

[37] In exceptional cases, a curvature singularity can be possibly avoided by picking a suitable $G(t,x)$.

10.2 Generalization to the n-dimensional case

We can refer again to radical-adapted Gauss-like coordinates in order to generalize the 2-dimensional signature-type changing manifold to the n-dimensional situation. Hence, the canonical metric for an n-dimensional signature-type changing manifold has the form $g = -t(dt)^2 + G_{ij}dx^i dx^j$, $i, j \in \{1, \ldots, n-1\}$. This result is insofar intriguing as the metric components in radical-adapted Gauss-like coordinates are smooth, even across the hypersurface of signature change. Moreover, these radical-adapted Gauss-like coordinates can be considered as time-orthogonal coordinates for an n-dimensional signature-type changing manifold.[38]

The difference between the two established notions of classical signature-type change, denominated as the *"continuous"* and the *"discontinuous"* approaches, is to some extend only a coordinate transformation [27]: By means of a rather simple coordinate transformation we can translate any statement about the "discontinuous" approach into one in the "continuous" approach.[39] Furthermore, by means of a conformal transformation, we can translate the above findings to the signature-type changing class of metrics with an infinite discontinuity at $t = 0$ as described in [29].

Considering all this, we shall henceforth restrict ourselves to the continuous approach where the metric g constitutes a smooth, symmetric $(0, 2)$-tensor field that is degenerate at $\mathcal{H}$. Furthermore, we stick with transverse, signature-type changing manifolds with a transverse radical. And we assume that one connected component of $M \setminus \mathcal{H}$ is Riemannian and the other ones are Lorentzian. However, in the next chapter we relax that restriction in order to explore a signature-type changing class of metrics with an infinite discontinuity at $t = 0$. From this class of metrics we can again extract information about transverse, signature-type changing manifolds qua a conformal transformation.

[38] In general, for dimensions $n \geq 4$ an orthogonal coordinate frame does not exist, so a time-orthogonal coordinate frame seems to be the best choice.

[39] If, however, we are pedantic, then the relationship between the two "approaches" does not exist by means of a coordinate transformation, but rather by means of a switch to another maximal atlas, with which the topological manifold is turned into a differentiable manifold. It should be added that the two different maximal atlases are not considered as different differentiable structures as there is a diffeoemorphism from one atlas to the other. Hence, in the category of differentiable manifolds, they are the same object.

11 Conformal transformation

A transverse type-changing singular semi-Riemannian manifold (M, g) with a transverse radical, as specified in Section 3, will be shown to be conformally equivalent to a signature-type changing semi-Riemannian manifold $(M, \bar{g})$ which has pseudo-timelike geodesics with loops. The latter one, the class of signature-type changing semi-Riemannian manifolds endowed with a non-smooth metric which has an infinite discontinuity at $t = 0$, is analyzed extensively in the work of Faustmann et al. [29]. Note that the Faustmann models are neither smooth nor have they a radical defined on their hypersurface. Here we want to show the relationship between the Faustmann models and the class of transverse type-changing singular semi-Riemannian manifolds with a transverse radical and, in so doing, how to easily identify its closed pseudo-timelike curves.

We take advantage of the fact that the causal structures of $(M, \bar{g})$ and $(M, \Omega\bar{g})$ are always the same, with Ω being a conformal factor. Admittedly, the geodesics of $\bar{g}$ and $g = \Omega\bar{g}$ are usually quite different, but we can utilize what we know about the causality of $(M, \bar{g})$ to identify closed pseudo-causal curves or loops in $(M, \Omega\bar{g})$, and vice versa. This insight elucidates the relationship between two significantly different signature-type changing manifolds and their (geodesic) curves.

Closed pseudo-timelike and closed pseudo-null curves in a transverse type-changing singular semi-Riemannian manifold (M, g), with a metric of the form

$$ds^2 = -g_{00}(dt)^2 + \sum_{i=1}^{n-1} (dx^i)^2,$$

can be obtained by a conformal transformation from a specific class of causality violating type-changing semi-Riemannian manifolds:

Consider the signature-type changing semi-Riemannian manifold $(M, \bar{g})$ with a metric of the form

$$\bar{g} = \operatorname{sgn}(f(t))[-(dt)^2 + \frac{1}{f(t)}(dx^1)^2 + \cdots + \frac{1}{f(t)}(dx^{n-1})^2], \tag{11.1}$$

where $f(t) \in \mathbb{R}$, $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, and $f'(0) > 0$. Moreover, without loss of generality, we assume that $f(0) = 0$.[40] The locus of signature change is given by the set

[40] Note that this means that there is an infinite discontinuity at $t = 0$ and thus some of the metric coefficients diverge there. According to this, the metric $\bar{g}$ is not defined for $t = 0$.

$\mathcal{H} = \{(t, x^1, \dots, x^{n-1}) \in \mathbb{R}^n \mid f(t) = 0\} = \{(0, x^1, \dots, x^{n-1}) \in \mathbb{R}^n\} \subset M$, although $\bar{g}$ is not defined on $\mathcal{H}$. Hence, in order for the signature of the metric $\bar{g}$ to change, we force $f(t)$ to go through the value zero; and at those events where $f(t) = 0$ the metric is "singular". This type of signature-type changing semi-Riemannian manifold and the behavior of the corresponding geodesics was extensively studied in the 2-dimensional setting by [29].
Also, there is a continuous, non-negative function $\Omega \in \mathscr{F}(M)$, defined by $\Omega(t) = f(t) \cdot \mathrm{sgn}(f(t))$, such that $g(X, Y) = \Omega \bar{g}(X, Y)$, which yields the metric $g = \Omega \bar{g} = -f(t)(dt)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2$. Note that there is a unique bilinear and continuous continuation to all of $\mathcal{H}$ and therefore g is continuous for all $t \in \mathbb{R}$.

Considering the above requirements, it is again possible by means of a transformation of the t-coordinate to get local coordinates such that $f(t) = t$ for all $t \in \mathbb{R}$. Then the metric g is in radical-adapted Gauss-like coordinates, i.e. with $g_{00} = -t$, and thus the coordinate independent *absolute time function* allows us to construct a *characteristic conformal factor* $\Omega = t \cdot \mathrm{sgn}(t)$.

Revisiting the signature-type changing semi-Riemannian manifold $(M, \bar{g})$ with the metric $\bar{g}$ defined by Equation (11.1),

$$ds^2 = \mathrm{sgn}(f(t))[-(dt)^2 + \frac{1}{f(t)}(dx^1)^2 + \cdots + \frac{1}{f(t)}(dx^{n-1})^2].$$

Without loss of generality, the surface of signature-change is determined by $f(0) = 0$ and located at $\mathcal{H} = \{(t, x^1, \dots, x^{n-1}) \in \mathbb{R}^n \mid f(t) = 0\}$. The lower half space $(t < 0)$ of the manifold is Riemannian and the upper part of the manifold $(t > 0)$ is Lorentzian.

The corresponding Lagrangian and Euler-Lagrange equations are given by

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(-\mathrm{sgn}(f(t))\dot{t}^2 + \frac{\mathrm{sgn}(f(t))}{f(t)} \sum_{i=1}^{n-1} (\dot{x}^i)^2\right), \\
0 = \frac{\partial \mathcal{L}}{\partial t} - \frac{d}{ds}(\frac{\partial \mathcal{L}}{\partial \dot{t}}) &= -\frac{1}{2}\mathrm{sgn}(f(t)) \cdot \frac{f'(t)}{f(t)^2} \sum_{i=1}^{n-1} (\dot{x}^i)^2 + \mathrm{sgn}(f(t))\ddot{t}, \\
0 = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{ds}(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}) &= -\frac{d}{ds}\frac{\mathrm{sgn}(f(t))}{f(t)}\dot{x}^i, \qquad (11.2)
\end{aligned}$$

and 11.2 implies that $\frac{\mathrm{sgn}(f(t))}{f(t)}\dot{x}^i = c_{x^i}$ is a constant.

Note that in the strict sense the Lagrangian is not defined at the hypersurface $\mathcal{H}$. Thence, it ought to be calculated for each region separately, and the resultant two constants should then be matched accordingly. However, in this particular case it is feasible to simplify matters and consider the above unified Lagrangian.[41]

Then the geodesics are determined by

$$\left(\int \frac{df(t)}{dt}ds\right)^2 = \frac{4f(t)}{\sum\limits_{i=1}^{n-1}(c_{x^i})^2} - \frac{8\mathcal{L}}{\mathrm{sgn}(f(t))\left(\sum\limits_{i=1}^{n-1}(c_{x^i})^2\right)^2},$$

$$x^i(s) = \frac{2\mathcal{L}c_{x^i}}{\sum\limits_{j=1}^{n-1}(c_{x^j})^2}s + \frac{\mathrm{sgn}(f(t))c_{x^i}}{4}\left(\sum_{j=1}^{n-1}(c_{x^j})^2\right)\int\left(\int \frac{df(t)}{dt}ds\right)^2 ds.$$

For $2\mathcal{L} = -1$, a pseudo-timelike geodesic starts in the Lorentzian region. It enters the Riemannian sector at $t = 0$, where accordingly $2\mathcal{L}$ =1 applies. The pseudo-timelike geodesic then reaches its turning point within the Riemannian region, at $\dot{t}(s) = 0$. This means

$$\begin{aligned}
\dot{t}(s) &= \frac{1}{2}\sum_{i=1}^{n-1}(c_{x^i})^2\int \frac{df(t)}{dt}ds = 0 \\
&\Longleftrightarrow \int \frac{df(t)}{dt}ds = 0 \\
&\Longleftrightarrow \frac{4f(t)}{\sum\limits_{i=1}^{n-1}(c_{x^i})^2} - \frac{8\mathcal{L}}{\mathrm{sgn}(f(t))\left(\sum\limits_{i=1}^{n-1}(c_{x^i})^2\right)^2} = 0 \\
&\Longleftrightarrow f(t) = \frac{2\mathcal{L}}{\mathrm{sgn}(f(t))\sum\limits_{i=1}^{n-1}(c_{x^i})^2} = \frac{1}{\mathrm{sgn}(f(t))\sum\limits_{i=1}^{n-1}(c_{x^i})^2},
\end{aligned}$$

[41]The full calculations can be found in Appendix B.

after which the geodesic turns back to the Lorentzian region. This pseudo-timelike geodesic intersects itself (in the Lorentzian region) at

$$-2f(t) = \frac{-4\mathcal{L}}{\operatorname{sgn}(f(t))\left(\sum\limits_{i=1}^{n-1}(c_{x^i})^2\right)^2} \cdot \sum_{i=1}^{n-1}(c_{x^i})^2 = \frac{2}{\operatorname{sgn}(f(t))\left(\sum\limits_{i=1}^{n-1}(c_{x^i})^2\right)^2} \cdot \sum_{i=1}^{n-1}(c_{x^i})^2.$$

Lightlike geodesics

$$\left(\int \frac{df(t)}{dt} ds\right)^2 = \frac{4f(t)}{\sum\limits_{i=1}^{n-1}(c_{x^i})^2},$$

$$x^i(s) = \frac{\operatorname{sgn}(f(t))\left(\sum\limits_{i=1}^{n-1}(c_{x^i})^2\right) c_{x^i}}{4} \int \left(\int \frac{df(t)}{dt} ds\right)^2 ds$$

have their turning point at $f(t) = 0$ and are therefore confined to the Lorentzian sector. In other words, lightlike geodesics do not cross the surface of signature change and do not have a point of intersection. The above reasoning can be summarized as

Proposition 11.1. *Let $(M, \bar{g})$ be a signature-type changing semi-Riemannian manifold such as introduced above in Equation 11.1. Then there is an arbitrary conformal factor $\Omega \in \mathscr{F}(M)$, defined by $\Omega(t) = f(t) \cdot sgn(f(t))$, such that $(M, g) := (M, \Omega\bar{g})$ is a causality-violating, transverse type-changing singular semi-Riemannian manifold with a transverse radical, and the metric is given by $g = \Omega\bar{g} = -f(t)(dt)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2$.*

*The manifold $(M, \bar{g})$ will be called the **causality-violating germ**.*

Denote the matrix representation of g as $[g_{\mu\nu}]$, and be $\mathcal{H} = \{q \in M \mid f(t(q)) = 0\} = \{q \in M \mid t(q) = 0\}$ the locus of signature change. Then we have $\triangle = \det([g_{\mu\nu}]) = -f(t)$, and the differential $d\triangle = -f'(t)dt$. Because of the precondition $f'(0) \neq 0$ we have $d\triangle \neq 0$ for all $q \in \mathcal{H}$. Thus it appears that $Rad_q = \operatorname{span}(\{\frac{\partial}{\partial t}\})$ is transverse with respect to $\mathcal{H}$, and $\dim(Rad_q) = 1$.

Corollary 11.2. *Let $(M, \Omega\bar{g})$ be a transverse type-changing singular semi-Riemannian manifold that is conformally equivalent to the causality-violating germ $(M, \bar{g})$, and γ an affinely parametrized pseudo-timelike geodesic on M with respect to a metric $\bar{g}$. Then γ is a closed pseudo-timelike curve on M with respect to the metric $g = \Omega\bar{g}$.*

Example 11.3. Let M be a 2-dimensional manifold, $f(t) = t$, and $\bar{g} = \operatorname{sgn}(t)[-(dt)^2 + \frac{1}{t}(dx)^2]$, which is similar to *metric A* in [29]. Note that the Lagrangian for the metric $\bar{g}$ is given by $\mathcal{L}_{\bar{g}} = \frac{-\operatorname{sgn}(t)}{2}\left(\dot{t}^2 - \frac{1}{t}\dot{x}^2\right)$; and the interrelation between the Lagrangian L_A for *metric A* and the Lagrangian $\mathcal{L}$ for $\bar{g}$ is $\mathcal{L} = -\operatorname{sgn}(t)L_A$. By picking the conformal factor $\Omega = t \cdot \operatorname{sgn}(t)$ and then, by applying the conformal transformation we get the familiar toy model metric $g = \Omega\bar{g} = \Omega \cdot \operatorname{sgn}(t)[-(dt)^2 + \frac{1}{t}(dx)^2] = -t(dt)^2 + (dx)^2$, see Section 5. The metric $g = \Omega\bar{g}$ is here represented in radical-adapted Gauss-like coordinates. Therefore we have the coordinate independent *absolute time function* which yields the above characteristic conformal factor.

As the causal structure of the spacetime $(M, \bar{g})$ is preserved, we can infer that (M, g) is causality violating and the closed pseudo-timelike curves coming from geodesics in $(M, \bar{g})$ are of the form (see also (16.6.) in [29])

$$t(s) = \frac{c^2}{4}s^2 + \frac{2\mathcal{L}}{\operatorname{sgn}(t(s))c^2} = \frac{c^2}{4}s^2 - \frac{1}{c^2},$$

with $2\mathcal{L} = -1$ for $t > 0$ and $2\mathcal{L} = 1$ for $t < 0$, and

$$x(s) = \frac{c^3}{12}s^3 + \frac{\operatorname{sgn}(t)2\mathcal{L}}{c}s + x(0),$$

with $2\mathcal{L} = -1$ for $t > 0$ and $2\mathcal{L} = 1$ for $t < 0$.

Example 11.4. Let M be a 2-dimensional manifold, $f(t) = \tanh(t)$, and $\bar{g} = \operatorname{sgn}(\tanh(t))[-(dt)^2 + \frac{1}{\tanh(t)}(dx)^2]$. This metric is obviously closely related to *metric C* in [29]. Unlike the manifold in the previous Example 11.3, this manifold $(M, \bar{g})$ is asymptotically flat insofar as for $t \longrightarrow \pm\infty$ the geometry in the Lorentzian region becomes indistinguishable from that of Minkowski spacetime.
The Lagrangian is given by

$$\mathcal{L}_{\bar{g}} = \frac{1}{2}\left(-\operatorname{sgn}(\tanh(t))\dot{t}^2 + \frac{\operatorname{sgn}(\tanh(t))}{\tanh(t)}\dot{x}^2\right).$$

By applying the conformal transformation with the conformal factor

$$\Omega = \tanh(t)\operatorname{sgn}(\tanh(t)),$$

we get

$$g = \Omega\bar{g} = -\tanh(t)(dt)^2 + (dx)^2.$$

Note that due to radical-adapted Gauss-like coordinates, the metric $g = \Omega\bar{g}$ can locally be expressed in the form $ds^2 = -t(dt)^2 + (dx)^2$.

The above delineated procedure shows how to get closed pseudo-timelike curves in a signature-type changing singular semi-Riemannian manifold (with has a smooth metric) when geodesic loops in a signature-type changing manifold with a non-smooth metric which has an infinite discontinuity, are placed at our disposal. On the other hand, we could refer to the same procedure to transform some non-geodesic curves into geodesics by means of a tailor-made conformal transformation.

12 Pseudo-Levi-Civita connection

When the metric is non-degenerate (which means the radical consists only of $\{0\}$) there exists a canonical invariant, called the Levi-Civita connection, which is characterized by the Koszul formula [71]. Furthermore, we can construct the Riemannian, Ricci and scalar curvatures.
But in our setting, when the metric is allowed to be degenerate and fails to have maximal rank on $\mathcal{H}$ (id est, M is a singular semi-Riemannian manifold with $\dim(Rad_q) \geq 1$ for $q \in \mathcal{H}$), the definition of the Levi-Civita connection (on the whole of M) and, as a consequence, the construction of the curvature invariants are no longer possible. This stems from the fact that the inverse components of the metric blow up at the hypersurface of signature change. In particular,

Proposition 12.1. *In signature-type changing singular semi-Riemannian manifold with a transverse radical, the canonical construction of the Levi-Civita connection fails to be compatible with the metric.*

Proof. Assuming again that the degenerate signature-type changing metric is a smooth, symmetric and transverse $(0,2)$-tensor field, we can refer to radical-adapted Gauss-like coordinates as introduced in 9. The canonical metric on this n-dimensional signature-type changing manifold (M, g) takes then the form $g = -t(dt)^2 + G_{ij}dx^i dx^j$, with $G_{00} = -t$, $i, j \in \{1, \dots, n-1\}$, in the neighborhood of $\mathcal{H}$.

The condition for a connection to be compatible with g is

$$\partial_k g_{ij} = g_{jm}\Gamma^m_{ki} + g_{im}\Gamma^m_{kj}.$$

Taking $0 = k = i = j$, applied to g in radical-adapted Gauss-like coordinates coordinates, we get

$$-1 \equiv \partial_0 g_{00} = \partial_t g_{tt} = -\partial_t t = g_{0m}\Gamma^m_{00} + g_{0m}\Gamma^m_{00} = 2(g_{0m}\Gamma^m_{00})$$

$$= 2(\underbrace{g_{00}}_{-t}\Gamma^0_{00} + \underbrace{g_{01}}_{0}\Gamma^1_{00} + \underbrace{g_{02}}_{0}\Gamma^2_{00} + \cdots + \underbrace{g_{0n}}_{0}\Gamma^n_{00}) = -2t\Gamma^0_{00}.$$

Since the right-hand side vanishes when $t = 0$, this is impossible no matter what connection we choose. Hence, in general a connection that is compatible with a degenerate signature-type changing metric does not exist.[42] □

[42] In order to deal with a connection that blows up at the hypersurface, we could only consider

In face of this result, one course of action would be to find a way to construct a surrogate for the covariant derivative (and other invariants) for the case when the metric is degenerate. Another way to deal with the above result would be to restrict our considerations to metric tensors for which the appropriate (i.e. problematic) components vanish at the signature-type changing locus $\mathcal{H}$.

However, in the following discussion we establish criteria that determine which geodesics and curves with the associated parallel vector fields have the property to extend across the hypersurface smoothly − and are such defined on all of M. It is a well-known fact that in semi-Riemannian geometry we can transform a vector field into a one-form via the linear morphism $\flat$ which is defined by $\flat(u)v := g(u,v)\ \forall v \in V$, see Section 7. If g is non-degenerate, this map is invertible and we get an isomorphism of V where V^* is called the musical isomorphism. Corresponding pairs $\flat(u) \longleftrightarrow u$ contain exactly the same information and are metrically equivalent [71]. This duality between one-forms and vector fields indicates where we have to look for a solution.

12.1 The dual connection

In order to cover the degenerate case in singular semi-Riemannian geometry a generalization of some standard constructions in semi-Riemannian geometry is studied by Stoica [84], where the assumption of a constant signature of the metric is dropped. For the class of so-called semi-regular singular semi-Riemannian manifolds Stoica (Definition 6.20, [84]) presents a way to construct objects such as the covariant derivative of differential forms, the lower Riemannian curvature operator, the Ricci curvature tensor and the scalar curvature − even if the metric is degenerate. In order to avoid the problem of inverse components of the metric to blow up at the hypersurface, Stoica suggests to use the Koszul form, which is the right side of the Koszul formula (in [71], p.61), as baseline:[43]

Definition 12.2. The **Koszul form** is defined [84] as
$\mathcal{K}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{F}(M)$,
$\mathcal{K}(X,Y,Z) :=$
$\frac{1}{2}\{Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]\}$.

metric tensors with certain appropriate components that vanish at the signature-changing locus. However, as delineated below, we will choose a different approach.

[43]In other literature, e.g. in [2], the Koszul form is called the Koszul-like formula.

When the metric is non-degenerate, we get the usual Levi-Civita connection $\nabla_X Y = \mathcal{K}(X, Y, \_)^\#$ by raising the 1-form $\mathcal{K}(X, Y, \_)$.[44] In this case, the Koszul form coincides with the Koszul formula, i.e. $\mathcal{K}(X, Y, Z) = g(\nabla_X Y, Z)$.[45] In this way the Riemannian curvature can be defined and it coincides with the usual Riemannian curvature tensor if the metric tensor is non-degenerate.

The properties derived from the Koszul form correspond to the key properties of the Levi-Civita connection for the non-singular semi-Riemannian case, and are axiomatized as follows:

(1) $\mathcal{K}(X, Y, Z)$ is additive and $\mathbb{R}$-linear in each argument.

(2) $\mathcal{K}(X, Y, Z)$ is $\mathfrak{F}(M)$-linear in X:
$\mathcal{K}(fX, Y, Z) = f\mathcal{K}(X, Y, Z)$ for $f \in \mathfrak{F}(M)$.

(3) $\mathcal{K}(X, fY, Z) = f\mathcal{K}(X, Y, Z) + X(f)g(Y, Z)$ for $f \in \mathfrak{F}(M)$.

(4) $\mathcal{K}(X, Y, Z)$ is $\mathfrak{F}(M)$-linear in Z:
$\mathcal{K}(X, Y, fZ) = f\mathcal{K}(X, Y, Z)$ for $f \in \mathfrak{F}(M)$.

(5) $\mathcal{K}(X, Y, Z)$ is compatible with the metric g:
$\mathcal{K}(X, Y, Z) + \mathcal{K}(X, Z, Y) = Xg(Y, Z)$.

(6) $\mathcal{K}(X, Y, Z)$ is torsion-free: $g([X, Y], Z) = \mathcal{K}(X, Y, Z) - \mathcal{K}(Y, X, Z)$.

Note that if $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ are the coordinate vector fields, then

$$\mathcal{K}(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}, \frac{\partial}{\partial x_c}) = \frac{1}{2}(\frac{\partial}{\partial x_a} g_{bc} + \frac{\partial}{\partial x_b} g_{ca} - \frac{\partial}{\partial x_c} g_{ab}),$$

with $a, b, c \in \{1, \ldots, n\}$, yields the Christoffel symbols of the first kind.

[44]This differential one-form is defined as $\mathcal{K}(X, Y, \_)\colon \mathfrak{X}(M) \longrightarrow \mathfrak{F}(M)$, $\mathcal{K}(X, Y, \_)(Z) := \mathcal{K}(X, Y, Z)$.

[45]Note that away from the singular locus we have $\Omega^1(M) \ni 0 = \mathcal{K}(\dot{x}, \dot{x}, \_) = g(\nabla_{\dot{x}}\dot{x}, \_)$, which is obviously equivalent to the geodesic equation $\nabla_{\dot{x}}\dot{x} = 0$ because of
$\mathfrak{X}(M) \ni 0 = \mathcal{K}(\dot{x}, \dot{x}, \_)^\# = \nabla_{\dot{x}}\dot{x}$. Hence, $K(\dot{x}, \dot{x}, \_) = 0$ can be considered as a different way to express the Euler-Lagrange equations.

Corollary 12.3. [84] *Let $X, Y \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}^{\perp}(M)$ (see Section 7), then* $\mathcal{K}(X,Y,W) = \mathcal{K}(Y,X,W) = -\mathcal{K}(X,W,Y) = -\mathcal{K}(Y,W,X)$.

There exists another well-defined map, called a C^{∞} dual connection [50] on M. Given (M,g), a dual connection $\square_X Y \in \Omega^1(M)$ on M is the differential 1-form defined by

Definition 12.4. $\square\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Omega^1(M)$,
$X, Y \longrightarrow \square_X Y$,

which satisfies the following properties for $f \in \mathfrak{F}(M)$, $X, Y \in \mathfrak{X}(M)$

(1) $\square$ is additive and $\mathbb{R}$-linear in each argument,
(2) $\square_X fY = X(f)g(Y, \centerdot) + f\,\square_X Y$,
(3) $\square_{fX} Y = f\,\square_X Y$,

the *torsion* of $\square$ is the $(0,3)$-tensor $T(X,Y,Z) = \square_X Y(Z) - \square_Y X(Z) - g([X,Y],Z)$, and $\square$ is metric compatible if $Xg(Y,Z) = \square_X Y(Z) + \square_X Z(Y)$.[46]

Lemma 12.5. *Let (M,g) be a transverse type-changing singular semi-Riemannian manifold. Given (M,g), there exists a unique torsion-free dual connection that is compatible with the metric.*

If $X, Y, Z \in \mathfrak{X}(M)$ are vector fields on M, then we denote this unique metric-compatible and torsion-free dual connection by $\nabla^{\flat}_X Y$ and call it the **lower covariant derivative of** Y in the direction of X. Based on the aforesaid Koszul form $\mathcal{K}$, the lower covariant derivative is just defined [84] as

$$(\nabla^{\flat}_X Y)(Z) \coloneqq \mathcal{K}(X,Y,Z)$$

$$= \frac{1}{2}\{Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]\}.$$

The **lower covariant derivative operator** is the map

$$\nabla^{\flat}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Omega^1(M)$$

[46] We may describe $\square$ in arbitrary local coordinates as $\square_X Y(Z) = (g_{\mu\nu} Y^{\mu}_{,\alpha} + \Gamma_{\nu\alpha\beta} Y^{\beta}) X^{\alpha} Z^{\nu}$.

that spits out the differential one-form $\nabla^\flat_X Y$.

Since the lower covariant derivative is equivalent to the Koszul form, the key properties $(1) - (6)$ in 12.1 for the former one correspond directly to the latter one. Moreover, for the non-singular semi-Riemannian case the concepts derived from the lower covariant derivative coincide to the ones derived from the Levi-Civita connection. When the metric is non-degenerate we get the natural covariant derivative $\nabla_X Y = (\nabla^\flat_X Y)^\# \in \mathfrak{X}(M)$ simply by raising the one-form $\nabla^\flat_X Y$.[47] The beauty of the lower covariant derivative lies in the fact that it is also defined for singular semi-Riemannian manifolds and is not restricted to a non-degenerate metric.

Based on the above reasoning, one might be tempted to speculate that the lower covariant derivative consistently results in parallel transport, as suggested by the following

Claim 12.6. $\nabla^\flat_U V := \mathcal{K}(U, V, \centerdot) = 0$ if and only if the vector field V is parallel transported along the vector U.

Proof. The presumption can be easily rebutted by testing it for $\nabla^\flat_U U := \mathcal{K}(U, U, \centerdot) = 0$ and by trying to define geodesics using the stated formula:

Away from the singular locus (i.e. the hypersurface which is a codimension-one submanifold), we have the Levi-Civita connection which corresponds to the Koszul form. We want to know whether we can define geodesics by the equation $K(\dot\gamma, \dot\gamma, \centerdot) = 0 \in \Omega^1(M)$ for a parametrized curve $\gamma\colon I \longrightarrow M$.

Away from the hypersurface we indeed have $\Omega^1(M) \ni 0 = \mathcal{K}(\dot\gamma, \dot\gamma, \centerdot) = g(\nabla_{\dot\gamma}\dot\gamma, \centerdot)$, which is obviously equivalent to the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ because of $\mathfrak{X}(M) \ni 0 = \mathcal{K}(\dot\gamma, \dot\gamma, \centerdot)^\# = \nabla_{\dot\gamma}\dot\gamma$.

Hence, our considerations pertain to the region around the singular locus and should address whether geodesics can be matched on both sides of the hypersurface. There are two possible ways to answer the question.

i) The zero in the equation $\Omega^1(M) \ni 0 = \mathcal{K}(\dot\gamma, \dot\gamma, \centerdot)$ is a zero 1-form field that is defined *everywhere* on M. Then the vector fields $\dot\gamma$ also must be defined everywhere on M. Moreover, away from the singular locus the integral curves $\gamma\colon I \longrightarrow M$ are

[47] In that case we also have $\nabla^\flat_X Y(Z) = g(\nabla_X Y, Z)$ where ∇ denotes the Levi-Civita connection [16].

affinely parametrized geodesics.
However, such vector fields can only be obtained as a special case. In the toy model example 5 the geodesics for L_0 are of the form $\gamma(s) = (t(s), x(s)) = (c_t, x(s))$, with $t(s) = c_t$ being a constant. The corresponding vector fields then have the form $\dot{\gamma}(s) = (\dot{t}(s), \dot{x}(s)) = (0, \dot{x}(s))$.
In the toy model these special vector fields are indeed defined everywhere on M. However, if the vector fields were of the form $\dot{\gamma}(s) = (\dot{t}(s), \dot{x}(s))$ with $\dot{t} \neq 0$, then we already know (based on the discussion about the toy model in Subsection 5.2) that $\dot{t}$ is discontinuous at the singular locus and there is no continuous continuation across the hypersurface. So in general we won't be able to find vector fields that meet our requested conditions.

ii) If we do not require that the zero in the equation $\Omega^1(M) \ni 0 = \mathcal{K}(\dot{\gamma}, \dot{\gamma}, \centerdot)$ is a 1-form field that is defined *everywhere* on M, then we can consider vector fields $\dot{\gamma}$ along their own integral curves. In that case, the zero in the equation $\Omega^1(M) \ni 0 = \mathcal{K}(\dot{\gamma}, \dot{\gamma}, \centerdot)$ is a 1-form field along these integral curves. Away from the singular locus the integral curves γ are again affinely parametrized geodesics. The zero 1-form field may be smooth, even across the hypersurface. But as we have pointed out before, at the singular locus there are no general matching conditions available for vector fields and geodesics. Based on our calculations for the toy model (see Section 5) the tangent vectors of the integral curves diverge at the hypersurface of signature change.

Considering all of this, we are not able to define geodesics by the equation $K(\dot{\gamma}, \dot{\gamma}, \centerdot) = 0 \in \Omega^1(M)$ for a parametrized curve $\gamma\colon I \longrightarrow M$. $\square$

While the lower covariant derivative is a sensible and beneficial modification of the natural covariant derivative, it does not inherently establish a default definition of parallel transport. Furthermore, it does not serve as a surrogate for an affine connection in the degenerate case.

As a consequence we have to single out the pairs of vector fields X, Y on M that have the property that the covariant derivative $\nabla_X Y$ of Y in the direction of X extends smoothly to all of M. Then we are able to determine which curves allow for parallel transport, and we can also apply those findings to classify all geodesics that smoothly cross the hypersurface.

12.2 The natural fundamental tensor

The properties of $\nabla^{\flat}_X Y$ guarantee that we have a natural tensor defined [50] by

$$II_q\colon T_qM \times T_qM \times Rad_q \longrightarrow \mathbb{R},$$

$$II_q(X,Y,R) = \nabla^{\flat}_{X_q} Y(R),$$

with $R_q \in Rad_q$ and $X_q, Y_q \in T_qM$ for all $q \in \mathcal{H}$.

Proof. Note that for $f \in \mathfrak{F}(M)$ we have $\nabla^{\flat}_{X_q} fY(R) = X(f)g(q)(Y,R) + f\nabla^{\flat}_{X_q}Y(R)$ and $g(q)(Y,R) = 0$. □

Lemma 12.7. [50] $II_q(X,Y,R) = II_q(Y,X,R)$.

Proof. Because $\nabla^{\flat}_X Y$ is compatible with the metric, we get
$0 = g(q)(R,[X,Y]) = \nabla^{\flat}_{X_q} Y(R) - \nabla^{\flat}_{Y_q} X(R)$. □

Observe that, since II_q is a tensor, we may write II_q relative to arbitrary coordinates as $II_q(X,Y,R) = \Gamma_{\nu\alpha\beta}X^{\alpha}Y^{\beta}R^{\nu}$. The term $\Gamma_{\nu\alpha\beta}$ represents the Christoffel symbols of the first kind, which can be also expressed by means of the Koszul form as $\mathcal{K}(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}, \frac{\partial}{\partial x_c}) = \frac{1}{2}(\frac{\partial}{\partial x_a}g_{bc} + \frac{\partial}{\partial x_b}g_{ca} - \frac{\partial}{\partial x_c}g_{ab})$, when $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ are the coordinate vector fields.

Example 12.8. Consider on $\mathbb{R}^2$ our toy model spacetime 5 with the signature-type changing metric $ds^2 = t(dt)^2 + (ds)^2$. The hypersurface of signature-change is located at $\mathcal{H} = \{(t,x) \in \mathbb{R}^2 \mid t = 0\}$, and the transverse radical at $\mathcal{H}$ is given by $\mathrm{span}(\{\frac{\partial}{\partial_t}\})$, see Example 7.20 in Subsection 7.2. The only non-zero tensor component with respect to this basis is $II_q(\frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}) = \mathcal{K}(\frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}) = \frac{1}{2}(\frac{\partial}{\partial_t}g_{tt}) = \frac{1}{2}$, hence the tensor is given by $II_q = \frac{1}{2}dt \otimes dt \otimes dt$.

We have now a suitable tool, namely the tensor II_q, at our disposal to analyze the geometry of the hypersurface. It can be used to characterize the nature of the hypersurface of signature change:

Theorem 12.9. [50] *Let (M,g) be a singular semi-Riemannian manifold of $\dim M = n \geq 2$, with a smooth, symmetric, degenerate metric g, and $\mathcal{H}$ a metric singularity of co-dimension one. Then (M,g) has a transverse metric singularity at any $q \in \mathcal{H}$ if and only if there exists a $X_q \in T_qM$ with $II_q(X,R,R) \neq 0$, with $R_q \in Rad_q \backslash \{0\}$. Moreover, the radical is transverse if and only if $II_q(R,R,R) \neq 0$.*

Proof. We pick a radical adapted orthonormal frame $e_0, \ldots, e_{n-1}$ (see Section 9), assuming that $e_0(q) = R_q \in Rad_q$. Thus the metric has the following matrix representation

$$[g_{ij}] = \begin{pmatrix} g_{00} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

From this obviously follows $\det([g_{ij}]) = g(e_0, e_0) = g_{00}$. Because of Proposition 9.5 we have that (M, g) has a transverse metric singularity at $q \in \mathcal{H}$ if and only if $d(\det([g_{ij}])_q) = d(g(e_0, e_0)_q) \neq 0$ for some local frame $(e_0, \ldots, e_{n-1})$ centered at q. Hence, this equivalence is fulfilled when there exists a $X_q \in T_qM$ such that

$$0 \neq 2II_q(X, R, R) = 2(\nabla^\flat_{X_q} R)(R) := 2\mathcal{K}(X, R, R) = Xg(e_0, e_0) \mid_q .$$

Therefore, based on Sections 3 and 7.2, $T_q\mathcal{H} = \ker(dg(e_0, e_0)_q)$ is the kernel of the one-form $d(g(e_0, e_0)_q) \in T_qM^*$. And the radical is transverse if and only if $= Rg(e_0, e_0) \mid_q = II_q(R, R, R) \neq 0$. □

Proposition 12.10. [51] *Given a singular semi-Riemannian manifold (M, g) with a smooth co-dimension-1, degenerate signature-type changing metric g, then the orthogonal complement of Rad_q relative to $II_q(\boldsymbol{\cdot}, \boldsymbol{\cdot}, R)$ is $T_q\mathcal{H}$.*

Proof. Choose at $q \in \mathcal{H}$ a radical adapted orthonormal frame $e_0, \ldots, e_{n-1}$ (see Subsection 9) and $w_q \in T_q\mathcal{H}$. Furthermore, we assume that $e_0(q) = R_q \in Rad_q$. Because of $II_q(X, Y, R) = II_q(Y, X, R)$ it is sufficient to only consider $II_q(w, R, R) = \nabla^\flat_{w_q} e_0(e_0) = \frac{1}{2} w_q g(e_0, e_0) = 0$, because $g(e_0, e_0) = g_{00} = 0$ on $\mathcal{H}$. □

12.3 Parallel transport

On the basis of the tensor II_q the following two pieces of information can be extracted:

(i) We can determine which pairs of vector fields $X, Y \in \mathfrak{X}(M)$ have the property that the associated covariant derivative $\nabla_X Y = (\nabla^\flat_X Y)^\#$ extends smoothly to all of M,

(ii) Given a curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$, with $\gamma(0) = q \in \mathcal{H}$, then II_q determines which parallel vector fields along γ extend smoothly through the hypersurface at the point q.

We want to bring to mind that (M, g) is again a transverse type-changing singular semi-Riemannian manifold with a transverse radical. This means that (M, g) has a transverse metric singularity at any $q \in \mathcal{H}$, so the hypersurface $\mathcal{H}$ is an $(n-1)$-dimensional submanifold.

Theorem 12.11. [51] *Let (M, g) be a transverse type-changing singular semi-Riemannian manifold with a transverse radical. Given two vector fields $X, Y \in \mathfrak{X}(M)$, then the associated covariant derivative $\nabla_X Y = (\nabla^\flat_X Y)^\#$ extends smoothly to all of M if and only if $II_q(X, Y, R) = 0$ for all $q \in \mathcal{H}$.*

Proof. Choose at $q \in \mathcal{H}$ a radical adapted orthonormal frame $e_0, \ldots, e_{n-1}$ (see Subsection 9), assuming that $e_0(q) = R_q \in Rad_q$ spans the one-dimensional radical.

" $\Longrightarrow$ " In the case that $\nabla_X Y$ extends smoothly to all of M we already established that $\nabla^\flat_X Y(e_i) = g(\nabla_X Y, e_i)$, $i \in \{0, \ldots, n-1\}$ with $\nabla^\flat_X Y(e_i)$ being smooth. The vector $\nabla_X Y$ can be described relative to the radical adapted orthonormal frame as

$$\nabla_X Y = \frac{g(\nabla_X Y, e_0)}{g(e_0, e_0)} e_0 + \sum_{i=1}^{n-1} g(\nabla_X Y, e_i).$$

At $q \in \mathcal{H}$ we have $g(e_0, e_0)_q = g_{00}(q) = 0$, hence the term $\frac{g(\nabla_X Y, e_0)}{g(e_0, e_0)}$ blows up at the hypersurface. Since the term $\frac{g(\nabla_X Y, e_0)}{g(e_0, e_0)}$ extends smoothly to all of M, the numerator $g(\nabla_X Y, e_0) = \nabla^\flat_X Y(e_0)$ must vanish on $\mathcal{H}$. And because $e_0 \in Rad_q$, this just means $g(\nabla_X Y, e_0) = \nabla^\flat_{X_q} Y(e_0) = II_q(X, Y, R) = 0$.

" $\Longleftarrow$ " If we are given that $II_q(X, Y, R) = 0$ for all $q \in \mathcal{H}$, then $II_q(X, Y, R) = \nabla^\flat_{X_q} Y(R) = 0$, with $R_q \in Rad_q$ and $X_q, Y_q \in T_q M$ for all $q \in \mathcal{H}$. Since $0 = (\nabla^\flat_X Y)(R) \Longrightarrow g((\nabla^\flat_X Y)^\#, R_q) = g((\nabla^\flat_X Y)^\#, e_0) = 0$ for all $R_q \in Rad_q$. $\square$

This theorem implies that the covariant derivative $\nabla_X Y = (\nabla^\flat_X Y)^\#$ extends smoothly across $\mathcal{H}$ if and only if either X or Y is tangent to $\mathcal{H}$ at every point $q \in \mathcal{H}$. This gives rise to a new name for singular semi-Riemannian manifolds for which *any* set of vector fields $X, Y \in \mathfrak{X}(M)$ have the property that the associated covariant derivative $\nabla_X Y = (\nabla^\flat_X Y)^\#$ extends smoothly to all of M.

Definition 12.12. [59, 84] A singular semi-Riemannian manifold (M, g) is called **radical-stationary** if for any two vector fields $X, Y \in \mathfrak{X}(M)$ the following condition is satisfied:

$$\nabla^\flat_X Y \coloneqq \mathcal{K}(X,Y,\centerdot) \in \mathcal{A}^\flat(M),$$

where $(T_qM)^\flat$ is the *radical-annihilator space* of the radical space $(T_qM)^\perp = Rad_q$, and $\mathcal{A}^\flat(M) \coloneqq \{\omega \in \Omega^1(M) \mid \omega_p \in (T_pM)^\flat \ for\ p \in M\}$.

Proposition 12.13. *Let (M,g) be a radical-stationary singular semi-Riemannian manifold. Then the covariant derivative $\nabla_X Y$ extends smoothly to all of M for any $X, Y \in \mathfrak{X}(M)$.*

Proof. Since (M,g) is radical-stationary we have $\nabla^\flat_X Y \coloneqq \mathcal{K}(X,Y,\centerdot) \in \mathcal{A}^\flat(M)$ for any $X, Y \in \mathfrak{X}(M)$. Since $\mathcal{A}^\flat(M)$ is the radical-annihilator this yields $\nabla^\flat_{X_q} Y(R) \coloneqq \mathcal{K}(X,Y,R) = II_q(X,Y,R) = 0$ for all $q \in \mathcal{H}$. And Theorem 12.11 implies that the covariant derivative $\nabla_X Y$ extends smoothly across $\mathcal{H}$. $\square$

The opposite is in general not true.

Keep in mind that the *unit tangent bundle* [77]

$$UM = \bigcup_{p \in M} U_pM,$$

also denoted as T^1M, is a $(2m-1)$-dimensional submanifold of the tangent bundle TM. Also, T^1M is the *sphere bundle* of the canonical projection $\pi_M \colon TM \longrightarrow M$ of the tangent bundle [76]. If there exists the Levi-Civita connection, then we may naturally assign a complementary *horizontal subspace* H_u to the *vertical space* V_u in T_uTM, and $H_u \subset T_uUM$ must apply for $u \in UM$.

Theorem 12.14. [51] *Let (M,g) be a transverse type-changing singular semi-Riemannian manifolds with a transverse radical. Given $c : (-\varepsilon, \varepsilon) \longrightarrow M$ a smooth curve transverse to the hypersurface $\mathcal{H}$ at $q = c(0)$. Let $II^\perp(\dot{c}) \coloneqq \{v \in T_qM \mid II_q(\dot{c}, v, R) = 0\}$. Then for each $v \in II^\perp$ there exists a unique smooth vector field $P(t)$ along c such that $\frac{DP}{dt} = 0$ for $t \neq 0$ and $P(0) = v$.*

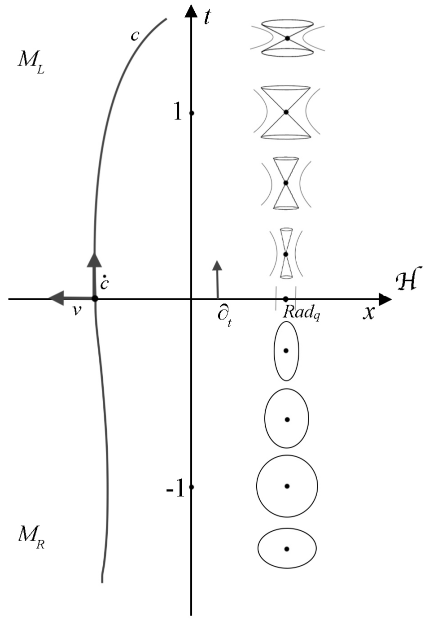

Figure 12.1: A smooth curve c transverse to the hypersurface $\mathcal{H}$ at $q = c(0)$.

This theorem follows from Theorem 12.11 and is proven in [51]. The proof is very technical, but relies on the idea that for $\dot{c}(0) \in T_qM$ the horizontal lift $\dot{c}_v \in H_v \subset T_vTM$ of $\dot{c}(0)$ at $v = P(0) \in TM$ is defined as the tangent vector at $t = 0$ to the curve $P(t)$ in TM obtained by parallel transporting $v = P(0) \in UM$, which remains in UM.

In other words, for the specified set of tangent vectors $II^{\perp}(\dot{c})$, $\dot{c}(0) \in T_qM = T_{c(0)}M$ and a given smooth curve $c(t)$ (transverse to $\mathcal{H}$) which is trivially tangent to $\dot{c}(t)$ at $t = 0$ the following has to be shown:
The vector field $P(t)$ along $c(t)$ obtained by parallel translating $v = P(0)$ along $c(t)$ gets regarded as a curve in the tangent bundle TM through $P(0) = v$, then we get $\dot{c}_v \in T_vTM$ the tangent vector to $P(t)$ at $t = 0$. This tangent vector $\dot{c}_v \in T_vTM$ is the horizontal lift of $\dot{c}(0)$ at $v \in TM$ and part of the space H_v of all horizontal lifts of tangent vectors in T_qM at $v = P(0)$ (which is an n-dimensional horizontal space of T_vTM at v).[48]

[48]Note that for $v \in TM$ and $\dot{c}(t) \in \mathfrak{X}(M)$ we have then $K(D\dot{c}(v)) = \nabla_v \dot{c}$, where D is the derivation of a vector field $\dot{c}$ as a map and K the connection map.

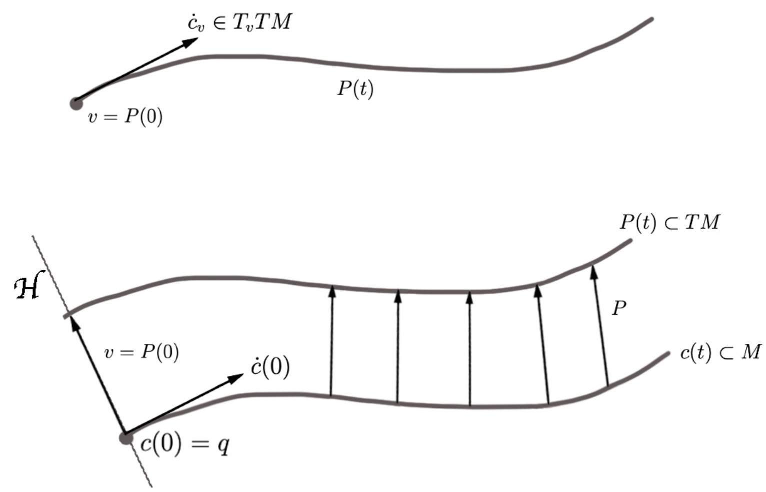

Figure 12.2: For $\dot{c}(0) \in T_qM$ the horizontal lift $\dot{c}_v \in H_v \subset T_vTM$ of $\dot{c}(0)$ at $v = P(0) \in TM$ is defined as the tangent vector at $t = 0$ to the curve $P(t)$ in TM obtained by parallel transporting $v = P(0) \in UM$, which remains in UM. Note that the curve $P(t) \subset TM$ is a section in TM.

Moreover, the proof in [51] takes advantage of Proposition 12.10 that yields a decomposition of the tangent space $T_qM = II^{\perp}(\dot{c}) \oplus Rad_q$. Thus a radical adapted orthonormal frame along the curve c can be constructed such that it spans $II^{\perp}(\dot{c})$ and Rad_q, respectively.

Recall that Theorem 12.11 implies that $\nabla_{\dot{c}} v$ for $v \in II^{\perp}(\dot{c})$ extends smoothly across $\mathcal{H}$ if and only if either $\dot{c}$ or v is tangent to $\mathcal{H}$ at every point $q \in \mathcal{H}$. Since we require the curve c to be transverse to the hypersurface $\mathcal{H}$ at $q = c(0)$, the tangent vector $v = P(0) \in TM$ must be tangent to $\mathcal{H}$. This gives us enough information to set up the Schild's ladder for the singular semi-Riemannian case.

12.3.1 Schild's ladder

The Schild's ladder is a well-known first-order method in Lorentzian geometry for approximating parallel transport of a vector along a curve. Here we want to use the Schild's ladder construction in the singular semi-Riemannian case for parallel transport along the curve c in order to provide some geometric intuition for the preceding Theorem. The tangent vector $v = P(0) \in II^{\perp}(\dot{c})$ gets propagated parallel to itself along the curve c which crosses $\mathcal{H}$ transversally at $c(0) = q$.

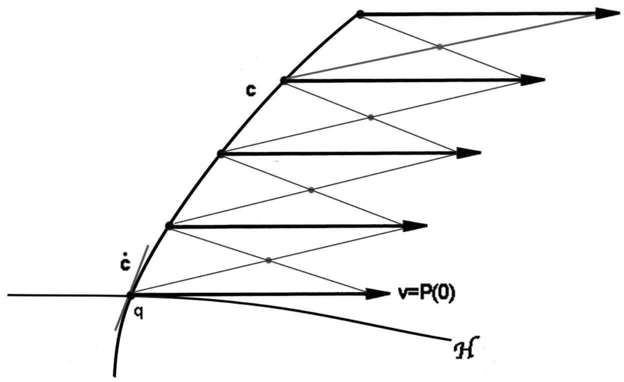

Figure 12.3: The Schild's ladder shows how a tangent vector $v = P(0) \in II^{\perp}(\dot{c})$ gets propagated parallel to itself along the curve c which crosses $\mathcal{H}$ transversally at $c(0) = q$.

12.4 Geodesics

In 1992 Larsen [60] addressed already the question whether one can suitably extend geodesics through noncritical singular points, which are defined as the hypersurface of signature change (this is because of the requirement $d(\det([g_{\mu\nu}])_q) \neq 0$). Based on [60] Kossowski and Kriele [54] developed this question further and could prove existence and uniqueness conditions for geodesics and pre-geodesics to cross the hypersurface. The result shows that geodesics cannot pass through a point $q \in \mathcal{H}$ in an arbitrary direction, but only in a particular admissible direction.

The questions about the extendability of geodesics across the hypersurface, the parallel transport that extends smoothly to all of M and the existence of the covariant derivative on the hypersurface are very closely related:

In the full geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ are only present in the term $\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$, where the $\frac{dx^i}{ds} = X^i$ are the components of the tangent vector $\gamma'(s) = (X_1(s), \ldots, X_n(s))$ of the geodesic γ. Hence, for each μ the term

$$\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = \Gamma^{\mu}_{\alpha\beta}X^{\alpha}X^{\beta}$$

constitutes a quadratic form

$$Q_{\mu}(X) \coloneqq \Gamma^{\mu}_{\alpha\beta}X^{\alpha}X^{\beta}.$$

On the other hand, given

$$X = \sum_{\alpha} x_{\alpha}\frac{\partial}{\partial x_{\alpha}},$$

$$Y = \sum_{\beta} y_{\beta}\frac{\partial}{\partial x_{\beta}},$$

the Christoffel symbols in the equation

$$\nabla_X Y = \sum_{\mu}(x^{\alpha}y^{\beta}\Gamma^{\mu}_{\alpha\beta} + X(y_{\mu}))\frac{\partial}{\partial x_{\mu}}$$

for the covariant derivative are only present in the term $\Gamma^{\mu}_{\alpha\beta}x^{\alpha}y^{\beta}$.[49] Here the x^{α} are the components of a vector that indicates the directions in which the covariant derivative is taken, and y^{β} are the components of the vector along a curve γ with the tangent vector X. Hence, for each μ the term $\Gamma^{\mu}_{\alpha\beta}x^{\alpha}y^{\beta}$ constitutes a bilinear form $B_{\mu}(X,Y) = \Gamma^{\mu}_{\alpha\beta}x^{\alpha}y^{\beta}$. The bilinear form $B_{\mu}(X,Y)$ is symmetric because we assume the connection to be torsion free. For every quadratic form $Q_{\mu}(x)$, there exists a unique symmetric bilinear form B_{μ} such that $Q_{\mu}(x) = B_{\mu}(x,x)$. This implies that all the information we need about parallel transport and the covariant derivative (and in particular the Levi-Civita connection) can be extracted from the information we have about geodesics, and vice versa. Therefore, it is sufficient to focus on either approach. According to this, we just summarize (without proof) the results about the extendability of geodesics across $\mathcal{H}$ below.

Example 12.15. In a signature-type changing setting not all of the quadratic forms and bilinear forms are well-defined on $\mathcal{H}$. Consider on $\mathbb{R}^2$ our toy model spacetime (Section 5) with the signature-type changing metric $ds^2 = t(dt)^2 + (ds)^2$.

[49]Recall that parallel transport is defined as the special case $\nabla_X Y = \sum_{\mu}(x^{\alpha}y^{\beta}\Gamma^{\mu}_{\alpha\beta} + X(y_{\mu}))\frac{\partial}{\partial x_{\mu}} = 0$.

Then the quadratic forms for $\mu = 0, 1$ are $Q_0(X) = \Gamma^0_{\alpha\beta}X^\alpha X^\beta = \Gamma^0_{00}(X^0)^2 = \frac{1}{2t}(X^0)^2$ and $Q_1(X) = \Gamma^1_{\alpha\beta}X^\alpha X^\beta = 0$. The associated bilinear forms are thus $B_0(x, y) = \Gamma^0_{\alpha\beta}x^\alpha y^\beta = \frac{1}{2t}x^0 y^0$ and $B_1(x, y) = \Gamma^1_{\alpha\beta}x^\alpha y^\beta = 0$.

Theorem 12.16. [54] *Let (M, g) be a transverse type-changing singular semi-Riemannian manifold, $q \in \mathcal{H}$ and $v_q \in T_qM$ not tangent to $\mathcal{H}$.*[50] *There exists a geodesic γ with $\dot{\gamma}(0) = v_q$ if and only if $II_q(V, V, R) = \nabla^\flat_{v_q}V(R) = 0$, for some non-vanishing $r_q \in Rad_q$. Furthermore, γ is smoothly immersed and unique. Note that V, R are smooth vector fields such that $V(q) = v_q$ and $R(q) = r_q$.*

Theorem 12.17. [54] *Let (M, g) be a transverse type-changing singular semi-Riemannian manifold, with a transverse radical at $q \in \mathcal{H}$. When $v_q \in Rad_q$, then there exists a smoothly immersed pre-geodesic γ with $\dot{\gamma}(0) = v_q$.*

Proposition 12.18. [54] *Given any immersed C^1 pre-geodesic γ with $\gamma(0) = q \in \mathcal{H}$ such that $\dot{\gamma}(0)$ is not tangent to $\mathcal{H}$. Then we have either $\dot{\gamma}(0) \in Rad_q$ or $II_q(\dot{\gamma}(0), \dot{\gamma}(0), r_q) = 0$, and γ can be parametrized as smoothly immersed geodesic.*

Example 12.19. Consider on $\mathbb{R}^2$ our toy model spacetime (Section 5) with the signature-type changing metric $ds^2 = t(dt)^2 + (ds)^2$. The hypersurface of signature-change is at $\mathcal{H} = \{(t, x) \in \mathbb{R}^2 \mid t = 0\}$, and the transverse radical at $\mathcal{H}$ is defined by span$(\{\frac{\partial}{\partial_t}\})$. Revisiting Example 12.8 we know that the natural tensor is given by $II_q = \frac{1}{2}dt \otimes dt \otimes dt$ and $II_q(\frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}, \frac{\partial}{\partial_t}) = \frac{1}{2} \neq 0$, with $v_q \in$ span$(\{\frac{\partial}{\partial_t}\})$. According to Theorem 12.16 there does not exist a geodesic that crosses $\mathcal{H}$ transversally with the initial condition $\dot{\gamma}(0) = v_q$. This result confirms our discussion in Subsection 5.2.

[50] Observe that the radical is not required to be transverse.

13 The transformation equivalence theorem

We present a procedure, called the *Transformation Prescription*, to transform an arbitrary Lorentzian manifold into a signature-type changing manifold.[51] Then we prove the so-called *Transformation theorem* saying that locally the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is equivalent to the metric obtained from a Lorentzian metric g via the aforementioned transformation prescription. By augmenting the assumptions by an additional constraint, mutatis mutandis, the global version of the Transformation theorem is proven as well.

13.1 Transformation Prescription

Let M be an n-dimensional (not necessarily time-orientable) Lorentzian manifold (M, g) with a smooth Lorentzian metric g. Since any Lorentzian manifold is locally time-orientable, there exists a locally defined, smooth, non-vanishing timelike C^∞ vector field.[52] While, in general, the existence of a global non-vanishing, timelike C^∞ vector field is not guaranteed, it does exist under ideal circumstances. However, we can alternatively depend on the following fact [64, 83]:

A line element field $\{V, —V\}$ over M is like a vector field with undetermined sign (i.e. determined up to a factor of ± 1) at each point p of M; this is a *smooth* assignment to each $p \in M$ of a zero-dimensional sub-bundle of the tangent bundle.[53] For any smooth manifold M the existence of a Lorentzian metric is equivalent to the existence of a global, smooth non-vanishing line element field [64, 83] on M. The set of all non-vanishing line element fields $\{V, -V\}$ on M is denoted by $\mathcal{L}(M)$. And henceforth, let $\mathfrak{F}(M)$ be the set of all smooth real-valued functions on M.

Remark 13.1. The existence of a global non-vanishing line element field is equivalent to the existence of a one-dimensional distribution. Also, the latter condition is equivalent to the manifold M admitting a C^∞ Lorentzian metric [40].

Proposition 13.2. ***(Transformation Prescription)*** *Let (M, g) be an n-dimensional (not necessarily time-orientable) Lorentzian manifold. Then we obtain a signature-type changing metric $\tilde{g}$ via the Transformation Prescription*

[51] A special case of this transformation can be found in [56, 57].

[52] One may say that Lorentzian manifolds are infinitesimally modeled on Minkowski space. Since Minkowski space is time-orientable, locally there always exists a Lorentz frame, i. e. a field of Lorentz bases. And a Lorentz basis contains a non-vanishing timelike vector field.

[53] Analogously, we can obtain a one-form field $(q, -q)$ up to sign, if we use g to lower indices. Then $w = q \otimes q$ is a well-defined 2-covariant symmetric tensor [62].

$\tilde{g} = g + fV^\flat \otimes V^\flat$, where $f\colon M \longrightarrow \mathbb{R}$ is a smooth transformation function and V is one of the unordered pair $\{V, —V\}$ of a global smooth non-vanishing line element field.

Proof. Since a smooth non-vanishing line element field exists for any Lorentzian manifold and is defined as an assignment of a non-ordered pair of equal and opposite vectors $\{V, -V\}$ at each point $p \in M$, the existence of a smooth global non-vanishing line element field is always ensured (even in the case of a non time-orientable Lorentzian manifold). Hawking and Ellis [43], p.40, elucidate how to construct a smooth global non-vanishing *timelike* line element field from a smooth global non-vanishing line element field. And since V is a line element field that is non-zero at every point in M, it can be normalized by $g_{\mu\nu}V^\mu V^\nu = -1$.[54]

Now consider an arbitrary C^∞ function $f\colon M \to \mathbb{R}$, and define tensor fields on M of the form $\tilde{g} \coloneqq g + f(V^\flat \otimes V^\flat)$, where $\flat$ denotes the musical isomorphism.[55] Locally there exist vector fields E_i that form an orthonormal frame, such that we can express the metric in terms of coordinates:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + f(V_\mu V_\nu) = g_{\mu\nu} + f(g_{\mu\alpha}V^\alpha E^\mu \otimes g_{\nu\beta}V^\beta E^\nu)$$

$$= g_{\mu\nu} + f(g_{\mu\alpha}V^\alpha g_{\nu\beta}V^\beta(E^\mu \otimes E^\nu)) = g_{\mu\nu} + f(g_{\mu\alpha}V^\alpha g_{\nu\beta}V^\beta).$$

Moreover, locally there exist vector fields E_j such that $\{V, E_2, \ldots, E_n\}$ is a Lorentzian frame field relative to g. Then

$$\tilde{g}(E_i, E_j) = g(E_i, E_j) + f(V^\flat \otimes V^\flat)(E_i, E_j) = \delta_{ij} + f(\underbrace{g(V, E_i)}_{0} \cdot \underbrace{g(V, E_j)}_{0}) = \delta_{ij},$$

[54]Here "timelike" refers to the Lorentzian metric g.

[55]Here the metric g induces the musical isomorphisms between the tangent bundle TM and the cotangent bundle T^*M: Recall that *flat* $\flat$ is the vector bundle isomorphism $\flat\colon TM \to T^*M$ induced by the isomorphism of the fibers
$\flat\colon T_pM \to T_p^*M$, given by $v \mapsto v^\flat$, where $\underbrace{\flat(v)}_{v^\flat}(w) = v^\flat(w) = g(v, w)\ \forall\, v, w \in T_pM$.

We thus obtain for the vector $v = v^j e_j$ and the associated 1-form $v^\flat = v_k e^k$ the expression for the musical isomorphism in local coordinates:
$v^\flat(w) = v_k e^k(w^i e_i) = v_k w^i \delta_i^k = v_i w^i = g(v^j e_j, w^i e_i) = g_{ij}v^j w^i$, where $w = w^i e_i$ is an arbitrary vector. This calculation yields $v_i = g_{ij}v^j \Longrightarrow v^\flat = v_i e^i = g_{ij}v^j e^i$.

$$\tilde{g}(V, E_j) = g(V, E_j) + f(\underbrace{g(V, V)}_{-1} \cdot \underbrace{g(V, E_j)}_{0}) = 0,$$

$$\tilde{g}(V, V) = g(V, V) + f(\underbrace{g(V, V)}_{-1} \cdot \underbrace{g(V, V)}_{-1}) = f - 1.$$

Consequently, because of $0 > \tilde{g}(V, V) = f - 1 \Leftrightarrow 1 > f$, $\tilde{g}$ is a Lorentzian metric on M in the region with $1 > f(p)$. Analogously, in the region with $1 < f(p)$, we have that $\tilde{g}$ is a Riemannian metric, and for $f(p) = 1$ the metric $\tilde{g}$ is degenerate. $\square$

If 1 is a regular value of $f\colon M \to \mathbb{R}$, then $\mathcal{H} := f^{-1}(1) = \{p \in M \colon f(p) = 1\}$ is a hypersurface in M.[56] Moreover, for every $q \in \mathcal{H}$, the tangent space $T_q\mathcal{H}$ is the kernel of the map $df_q\colon T_qM \longrightarrow T_1\mathbb{R}$. According to this, $(M, \tilde{g})$ represents a signature-type changing manifold with the locus of signature change at $\mathcal{H}$.

13.1.1 Representation of $\tilde{g}$

The representation of $\tilde{g}$ as $\tilde{g} = g + fV^\flat \otimes V^\flat$, as introduced in the *Transformation prescription 13.2* above, is ambiguous. More precisely, different triples (g, V, f) can yield the same metric $\tilde{g}$. To see this, notice that for $(M, \tilde{g})$ with $\dim(M) = n$, the signature-type changing metric $\tilde{g}$ and the Lorentzian metric g are determined pointwise by $\frac{n(n+1)}{2}$ metric coefficients. Aside from that, for the former we have the n components of V and the value of f (besides 1, because of $g(V, V) = -1$) at each point in M.

In the following, we demonstrate that one can choose either V or f arbitrarily, with the fixed condition that $f(q) = 1\ \forall q \in \mathcal{H}$ on the hypersurface, but not both independently of each other.

Proposition 13.3. *Given a signature-type changing metric $\tilde{g}$, then in particular, the line element field V can be chosen arbitrarily (subject to temporal causality constraints in M_L with respect to $\tilde{g}$). This choice subsequently allows for the determination of the Lorentzian metric g and a C^∞ function f, with the fixed condition $f(q) = 1\ \forall q \in \mathcal{H}$. Then the triple (g, V, f) constitutes the representation of $\tilde{g}$ through $\tilde{g} = g + fV^\flat \otimes V^\flat$, as introduced in the Transformation Prescription 13.2*

[56] This is to say, $\mathcal{H}$ is a submanifold of dimension $n - 1$ in M, constituting the locus where the signature change occurs.

Proof. Let $\psi\colon M_R \cup M_L \longrightarrow \mathbb{R}$ be chosen such that $\tilde{V} := \psi V$ is normalized with respect to $\tilde{g}$: that is, $\tilde{g}(\tilde{V}, \tilde{V}) = 1$ and $\tilde{g}(\tilde{V}, \tilde{V}) = -1$ in M_R and M_L, respectively. In the following, only the relationships in M_L will be examined, the argumentation in the Riemannian sector M_R is carried out analogously with corresponding changes in sign. Consider the normalized line element field $\tilde{V}$, with

$$-1 = \tilde{g}(\tilde{V}, \tilde{V}) = g(\tilde{V}, \tilde{V}) + f[g(V, \tilde{V})]^2$$

$$= \psi^2 g(V, V) + f\psi^2 [g(V, V)]^2 = \psi^2 \cdot (-1 + f)$$

$$\Longrightarrow f - 1 = -\frac{1}{\psi^2}. \tag{13.1}$$

Then, we extend $\tilde{V}$ to a Lorentzian basis $\{\tilde{V}, \tilde{E}_1, \ldots, \tilde{E}_{n-1}\}$ relative to $\tilde{g}$, and we get

$$0 = \tilde{g}(\tilde{V}, \tilde{E}_i) = g(\tilde{V}, \tilde{E}_i) + f g(V, \tilde{V}) g(V, \tilde{E}_i)$$

$$= \psi[g(V, \tilde{E}_i) + f g(V, V) g(V, \tilde{E}_i)] = \psi \cdot (1 - f) g(V, \tilde{E}_i)$$

$$\Longrightarrow g(V, \tilde{E}_i) = 0, \tag{13.2}$$

because of $\psi \cdot (1 - f) \neq 0$ on $M_R \cup M_L$. Furthermore,

$$\delta_{ij} = \tilde{g}(\tilde{E}_i, \tilde{E}_j) = g(\tilde{E}_i, \tilde{E}_j) + f \underbrace{g(V, \tilde{E}_i)}_{0} \underbrace{g(V, \tilde{E}_j)}_{0}$$

$$\Longrightarrow g(\tilde{E}_i, \tilde{E}_j) = \delta_{ij}. \tag{13.3}$$

Since $\{V, \tilde{E}_1, \ldots, \tilde{E}_{n-1}\}$ is also a basis, and in addition $g(V, V) = -1$ holds, the metric g is uniquely determined. Moreover, for the function $\psi\colon M_R \cup M_L \longrightarrow \mathbb{R}$ we have established the relation $\tilde{g}(V, V) = \frac{1}{\psi^2}\tilde{g}(\tilde{V}, \tilde{V}) = -\frac{1}{\psi^2}$, and therefore is the C^∞ function f, based on Equation 13.1, also uniquely determined by $f = 1 - \frac{1}{\psi^2}$. Note that a change in length defined by $\phi\colon M_R \cup M_L \longrightarrow \mathbb{R}$, $V \mapsto \phi V$, entails the change $f \mapsto 1 + \phi^2 \cdot (f - 1)$. □

According to the above proposition, these triples (g, V, f) form equivalence classes, where all triples within an equivalence class yield the same metric $\tilde{g}$. If one perturbs a triple, especially at V but not at g and f, the new triple belongs to a different equivalence class and thus yields a different $\tilde{g}$. However, within the new equivalence class, there is also a triple with the original V that similarly yields the 'new' $\tilde{g}$. On the other hand, one can 'simultaneously' perturb V, g, and f in such a way that the original equivalence class is maintained, and hence, the original $\tilde{g}$ is preserved. This insight suggests the following proposition.

Proposition 13.4. *Let X be the set of all triples (g, V, f) with $g \in Lor(M)$, $f \in \mathfrak{F}(M)$, $V \in \mathcal{L}(M)$, where $Lor(M)$ denotes the set of all Lorentzian metrics on M, $\mathfrak{F}(M)$ the set of all smooth real-valued functions on M, and $\mathcal{L}(M)$ the set of all non-vanishing line element fields $\{V, -V\}$ on M. The equivalence relation $\sim$ on X is defined by: $(g, V, f) \sim (\bar{g}, \bar{V}, \bar{f})$ if and only if $\tilde{g} = g + fV^\flat \otimes V^\flat = \bar{g} + \bar{f}\bar{V}^\flat \otimes \bar{V}^\flat$. Then the partition X of the set of all triples (g, V, f) is given by $X/\sim = \{[\tilde{g}]_\sim = [(g, V, f)]_{\tilde{g}} \colon (g, V, f) \in X\}$, where $\tilde{g}$ is to be interpreted in such a way that it can be regarded as a representative of the equivalence class of triples corresponding to $\tilde{g}$.*

Proof. First recall that a relation $\sim$ on X is called an equivalence relation if it is reflexive, symmetric, and transitive. Moreover, a partition of the set X is then defined as a collection of all disjoint non-empty subsets X_i of X, where $i \in I$ (I is the index set), such that

$X_i \neq 0 \ \forall i \in I$,
$X_i \cap X_j = \emptyset$, when $i \neq j$,
$\bigcup_{i \in I} X_i = X$.

It is straightforward to demonstrate that the relation $\sim$ satisfies all three conditions for an equivalence relation. Specifically, two triples $(g, V, f) \sim (\bar{g}, \bar{V}, \bar{f})$ are equivalent if and only if $\tilde{g} = g + fV^\flat \otimes V^\flat = \bar{g} + \bar{f}\bar{V}^\flat \otimes \bar{V}^\flat$. Given the relation $\sim$, we can define the equivalence class $[\tilde{g}]_\sim = [(g, V, f)]_{\tilde{g}} = \{(g, V, f) \in Lor(M) \times \mathcal{L}(M) \times \mathfrak{F}(M) \colon \tilde{g} = g + fV^\flat \otimes V^\flat\}$, where $\tilde{g}$ can be viewed as a class representative of the equivalence class of triples corresponding to $\tilde{g}$. And we can establish $X/\sim = \{[\tilde{g}]_\sim = [(g, V, f)]_{\tilde{g}} \colon (g, V, f) \in X\}$, which is a pairwise disjoint partition of X. Note that the set of class representatives $\tilde{g}$ is a subset of X which contains exactly one element from each equivalence class $[\tilde{g}]_\sim = [(g, V, f)]_{\tilde{g}}$, this is the set of all signature-type changing metrics $\tilde{g}$ on M. □

Corollary 13.5. *There is a bijection between the partition of the set of all triples (g, V, f) and the set of all signature-type changing metrics $\tilde{g}$ on M.*

13.2 Local transformation theorem

Note that in the *Transformation Prescription 13.2* the locus of signature-change is not necessarily an embedded hypersurface in M. Recall that this is only the case if 1 is a regular value of $f\colon M \longrightarrow \mathbb{R}$, and then $\mathcal{H} :=f^{-1}(1)$ is a smoothly embedded hypersurface in M. Also, the signature-type changing manifold $(M, \tilde{g})$ has a spacelike hypersurface if and only if the radical Rad_q intersects $T_q\mathcal{H}$ transversally for all $q \in \mathcal{H}$, see Section 9.

Remark 13.6. Indeed, if $(M, \tilde{g})$ is a transverse type-changing and the radical is *transverse*,[57] then for each $q \in \mathcal{H}$ there exists a neighborhood $U(q)$ allowing for smooth radical-adapted Gauss-like coordinates (see Theorem 9.2), such that the metric takes the form $\tilde{g} = -t(dt)^2 + \tilde{g}_{ij}dx^i dx^j$, for $i, j \in \{1, \dots, n-1\}$. The subset $\mathcal{H} \cap U(q) \subset U(q)$ is closed in $U(q)$ and has the representation $(0, x^1, \dots, x^{n-1})$. So on each $\mathcal{H} \cap U(q)$ the transformation function is smooth and on $U(q)$ it takes the special form $f_q(t, \hat{x}) = 1 - t$,[58] such that $f_q \mid_{\mathcal{H}} = 1$. Consequently there is a smooth function $f\colon U(q) \longrightarrow \mathbb{R}$ such that $f \mid_{\mathcal{H}\cap U(q)} = f_q \mid_{\mathcal{H}\cap U(q)}$. Due to that we can use $t =: h(t, \hat{x})$ as the absolute time function (see 8.16) by means of which we can define $f(t, \hat{x}) := 1 - h(t, \hat{x})$ on $U := \bigcup_{q\in\mathcal{H}} U(q)$.

Lemma 13.7. *Let $(M, \tilde{g})$ be an n-dimensional transverse, signature-type changing manifold with an transvere radical. Furthermore is $f\colon M \longrightarrow \mathbb{R}$ a C^∞ function, such that 1 is a regular value of f, and $\mathcal{H} :=f^{-1}(1) = \{p \in M\colon f(p) = 1\}$ an embedded hypersurface in M. For any $q \in \mathcal{H}$ and any local coordinate system $\xi = (t, \hat{x}) = (t, x^1, \dots, x^{n-1})$ centered at q the following holds: $d(\det([\tilde{g}_{\mu\nu}])) \neq 0 \Longrightarrow df \neq 0$.*

Proof. In a neighborhood of the hypersurface of signature change $\mathcal{H}$ there exist radical-adapted Gauss-like coordinates, and the matrix represenation of $\tilde{g}$ takes the form

$$[\tilde{g}_{\mu\nu}] = \left(\begin{array}{c|ccc} -t & 0 & \cdots & 0 \\ \hline 0 & \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{array}\right) = \left(\begin{array}{c|ccc} -t & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{G} & \\ 0 & & & \end{array}\right),$$

[57] Recall that this definition implies that $\mathcal{H}$ is a smoothly embedded hypersurface in M.

[58] Note that the t-coordinate of the argument in $f_q(t, \hat{x}) = 1 - t$ can be different for different coordinate patches U_q, but the function values $f_q(t, \hat{x})$ for each $q \in \mathcal{H}$ are identical.

and the determinant of this block diagonal matrix yields

$$\det([\tilde{g}_{\mu\nu}]) = \det(-t)\det(\tilde{G}) = -t \cdot \prod_{i-1}^{n-1} \lambda_i,$$

where λ_i are the positive eigenvalues of $\tilde{G}$ corresponding to the signature of the metric.[59] Then locally, for any $q \in \mathcal{H}$ and $f = 1 - t$, we have

$$\det([\tilde{g}_{\mu\nu}]) = (f-1) \cdot \prod_{i-1}^{n-1} \lambda_i,$$

so as to the metric $\tilde{g}$ is exactly degenerate for $f(q) = 1$. Since 1 is supposed to be a regular value, we impose the restriction that

$$d(\det([\tilde{g}_{\mu\nu}])) = d\left((f-1) \cdot \prod_{i-1}^{n-1} \lambda_i\right) = \prod_{i-1}^{n-1} \lambda_i \cdot d(f-1) \neq 0$$

for any $q \in \mathcal{H}$ and any local coordinate system $\xi = (t, \hat{x}) = (t, x^1, \ldots, x^{n-1})$ centered at q. Since

$$\prod_{i-1}^{n-1} \lambda_i > 0$$

always holds, this implies $d(f-1) \neq 0$ for any $q \in \mathcal{H}$. Hence, we obtain for f that

$$d(f-1) = \frac{\partial f(t,\hat{x})}{\partial t} dt + \frac{\partial f(t,\hat{x})}{\partial x_1} dx_1 + \cdots + \frac{\partial f(t,\hat{x})}{\partial x_{n-1}} dx_{n-1} = df \neq 0$$

for any $q \in \mathcal{H}$. $\square$

Corollary 13.8. *Let M be an n-dimensional manifold with a signature-type changing metric $\tilde{g}$, obtained via the Transformation Prescription $\tilde{g} = g + fV^\flat \otimes V^\flat$, where $f \colon M \longrightarrow \mathbb{R}$ is a C^∞ function such that 1 is a regular value, and $\mathcal{H} := f^{-1}(1) = \{p \in M \colon f(p) = 1\}$. Then in the case that the radical is transverse, and due to*

[59]The signature of a symmetric matrix expresses the number of positive and negative eigenvalues, counting multiplicities.

$d(\det([\tilde{g}_{\mu\nu}])) \neq 0$, there exists in a neighborhood $U := \bigcup_{q\in\mathcal{H}} U(q)$ of $\mathcal{H}$ the absolute time function $h(t,\hat{x}) = t$. This time function establishes a foliation [33, 41] in U, such that $\mathcal{H}$ is a level surface of that decomposition. And the transformation function $f(t,\hat{x})\mid_U := 1 - h(t,\hat{x}) = 1 - t$ interpolates between the Lorentzian region and the Riemannian region. These two regions are separated by $\mathcal{H}$ at $f(0,\hat{x}) = 1$.

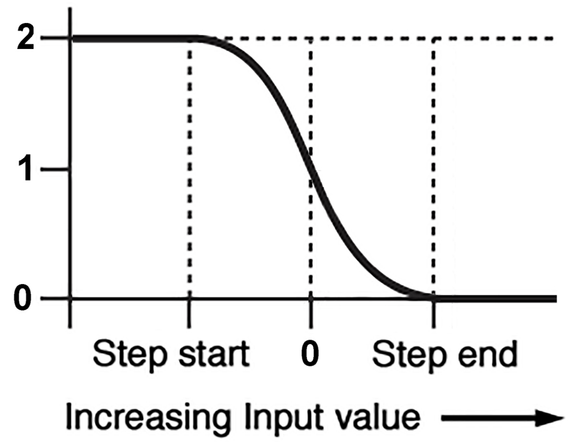

Figure 13.1: A typical smoothstep function that ensures a smooth transition from 2 to $1-t$ to 0 over a set interval.

Remark 13.9. In light of Proposition 13.3, we understand that, due to arbitrary rescaling, there are no distinguished values for f.[60]However, when considering compelling examples, especially in a physical context, where a Lorentzian manifold transforms into a signature-changing manifold, it is reasonable to assume that the transformation is imperceptible far away from the hypersurface. Therefore, certain specific choices of f hold more physical significance than others. To achieve this, we can, without loss of generality, choose a transformation function as follows:
The transformation function $f(t,\hat{x})\mid_U := 1 - h(t,\hat{x}) = 1 - t$ interpolates between the Lorentzian region $U \cap M_L$ at $f(t,\hat{x}) = 0$ for certain $t > 0$ and the Riemannian region $U \cap M_R$ at $f(t,\hat{x}) = 2$ for certain $t < 0$. These two regions are separated by $\mathcal{H}$ at $f(0,\hat{x}) = 1$. And in order to achieve a smooth transition with a continuous and differentiable transformation function $f(t,\hat{x})$, we can use a smoothstep

[60]Although the Lorentzian metric can be significantly deformed by arbitrary transformation functions f without a 'smoothstep' as introduced below, this deformation does not affect the metric's signature.

function (cubic Hermite interpolation) that smoothly interpolates between the different segments. Thus the smoothstep function ensures a smooth transition from 2 to $1-t$ to 0 over a set interval, and it smoothly interpolates between the values, see Figure 13.1. We can adjust the transition points as needed, dependent on the manifold.

Remark 13.10. Also by the same token: For the reverse direction of the equivalence proof for the *Transformation Theorem* (see below) a transverse, signature-type changing manifold with a transverse radical is given. In that case we need to *find* a suitable triple (V, g, f) with certain properties in $U := \bigcup_{q \in \mathcal{H}} U(q)$. Also in this case we understand that different triples (g, V, f) can yield the same metric $\tilde{g}$, and due to arbitrary rescaling, there are no distinguished values for f (Subsection 13.1.1). Because we know that the transformation function must take the value $1 = f(0, \hat{x})$ for $t = 0$, so without loss of generality we can choose $f(t, \hat{x}) := 1 - h(t, \hat{x})$ in the neighborhood of $\mathcal{H}$, and start from there to find an appropriate extension.
Obviously the manifold is built from more coordinate patches besides U. This means that the *transformation function* f in $\tilde{g} = g + fV^\flat \otimes V^\flat$ is defined by $f(t, \hat{x}) := 1 - h(t, \hat{x})$ locally (i.e. in a neighborhood of the hypersurface of signature change $\mathcal{H}$). However, since the manifold M is assumed to be paracompact, we can construct the extension of the transformation function $f(t, \hat{x})$ by using the properties of a partition of unity subordinate to the collection of chart domains in order to patch the local functions f_i on different open subsets U_i together (such that the global function agrees on overlaps):
The sum of the form $\sum\limits_i f_i \psi_i =: f$ then reduces to a global smooth function on the manifold M. With this convention, which we assume throughout, we just require that the *extended* transformation function $f(t, \hat{x})$ is well-defined.[61]

To sum up, for every $q \in \mathcal{H}$ the tangent space $T_q\mathcal{H}$ to $\mathcal{H}$ at p is the kernel $T_q(f^{-1}(1)) = \ker df_q$ of the map $df_q \colon T_qM \longrightarrow T_1\mathbb{R}$. And although locally near $\mathcal{H}$, the transformation function can take the form $f(t, \hat{x}) := f(t, x^1, \ldots, x^{n-1}) = (1-t)$, globally (dependent on the large scale causal structure of the manifold) it can look

[61]A a partition of unity is a collection of nonnegative functions ψ_i, such that the support of the partition function ψ_i is contained in just one open subset U_i each, and $\sum\limits_i \psi_i = 1$. Hence, at any one point in M, only one of the partition functions has nonzero value, so the sum $\sum\limits_i f_i \psi_i =: f$ reduces to a global smooth function on the manifold M. In other words, the approach would be to construct f_i locally and use a partition of unity to glue the local functions together.

a lot different. Take for instance an asymptotically flat spacetime, then one could choose $f(t,\hat{x}) = -\arctan(t-1)$ as *transformation function* for which the first-order Taylor polynomial at $t = 1$ is given by $p_1(t,\hat{x}) = -(t-1) = 1-t$. So the function $f(t,\hat{x}) = -\arctan(t-1)$ looks in a neighborhood of $\mathcal{H}$ like $1-t$ but away from the hypersurface it certainly looks a lot different. We obtain $\tilde{g} = g - \arctan(t-1)V^\flat \otimes V^\flat$, which is a signature-changing manifold $(M,\tilde{g})$ with a transverse, type-changing metric. The locus of signature change is again at $t = 0$, with the Lorentzian sector determined by $t > 0$ and the Riemannian sector set by $t < 0$.

Lemma 13.11. *Let $(M,\tilde{g})$ be a transverse, signature-type changing manifold with a transverse radical. Then in the Lorentzian sector M_L there always exists a smooth non-vanishing timelike line element field $\{V,-V\}$ on M_L with $\frac{\partial}{\partial t} = V$ in $M_L \cap U$, where $U \coloneqq \bigcup_{q\in\mathcal{H}} U(q)$ is the neighborhood of $\mathcal{H}$ (see Remark 13.6) with respect to smooth radical-adapted Gauss-like coordinates.*

Proof. Since a smooth non-vanishing timelike line element field exists for any Lorentzian manifold, the existence of such a line element field $\{V,-V\}$ in M_L is always ensured.

In $U \coloneqq \bigcup_{q\in\mathcal{H}} U(q)$ there is a unique absolute time function defined by $h(t,\hat{x}) \coloneqq t$, and in the Lorentzian sector we have $-h\mid_{M_L\cap U}(t,\hat{x}) = \tilde{g}\mid_{M_L\cap U}(\frac{\partial}{\partial_t},\frac{\partial}{\partial_t}) = -t < 0$. Consider now a neighborhood $U(q)$ for any $q \in \mathcal{H}$, where the vector field $\frac{\partial}{\partial t}$ is given and is timelike in $M_L \cap U(q)$.

So locally $\frac{\partial}{\partial t}$ can be chosen as timelike line element field $W = \{\frac{\partial}{\partial t}, -\frac{\partial}{\partial t}\}$ with determined sign. In the case that $W \notin \{V,-V\}$ within $M_L \cap U(q)$, we choose another, suitable slightly larger neighborhood $U^+(q)$ such that $U(q) \subset U^+(q)$.[62] Note that W should also be a line element field on the extended neighborhood $U^+(q)$ and can by arbitrary (besides the requirement of being smooth everywhere) outside of $U^+(q)$.

[62]In order for the transition zone $(U^+(q) \setminus \overline{(U(q))} \cap M_L)$ not to become "too narrow," every point on the boundary of $U(q)$ must have an open neighborhood that is a subset of $U^+(q)$.

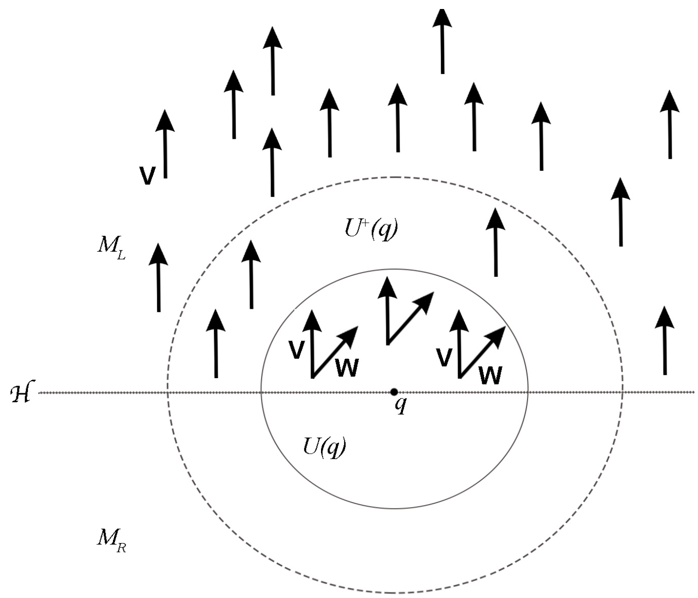

Figure 13.2: Locally $\frac{\partial}{\partial t}$ can be chosen as timelike line element field $W = \{\frac{\partial}{\partial t}, -\frac{\partial}{\partial t}\}$ with determined sign. In the case that $W \notin \{V, -V\}$ within $M_L \cap U(q)$, we choose another, suitable slightly larger neighborhood $U^+(q)$ such that $U(q) \subset U^+(q)$.

Then, due to the convexity of both components, $C^+(p)$ and $C^-(p)$, of the light cone in each point $p \in M_L$, we construct for $V, W \in C^+(p)$, $\forall p \in M_L$ the timelike vectors $(1-a)V + aW =: \tilde{V} \in C^+(p)$,[63] where a is a continuous function with $a = 0$ in $M_L \setminus U^+(q)$, $a = 1$ in $\overline{U(q)} \cap M_L$, and $0 < a < 1$ in $(U^+(q) \setminus \overline{(U(q))} \cap M_L)$. This yields a smooth non-vanishing timelike line element field $\{\tilde{V}, -\tilde{V}\}$ in M_L corresponding to $\frac{\partial}{\partial_t} =: \tilde{V}$ in $M_L \cap U$. $\square$

Theorem 13.12. ***Local Transformation theorem (transverse radical)*** *For every $q \in M$ there exists a neighborhood $U(q)$, such that the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is a transverse, type-changing metric with a transverse radical if and only if $\tilde{g}$ is locally obtained from a Lorentzian metric g via the Transformation Prescription 13.2 $\tilde{g} = g + fV^\flat \otimes V^\flat$, where $df \neq 0$ and $(V(f))(q) = ((df)(V))(q) \neq 0$ for every $q \in \mathcal{H} := f^{-1}(1) = \{p \in M : f(p) = 1\}$.*[64]

[63] As the affine combination of two timelike vectors V, W belonging to the same component of the light cone, the vector $\tilde{V}$ is also timelike. and also belongs to that component.

[64] Recall that for every $q \in \mathcal{H}$, $T_q\mathcal{H}$ is the kernel of the map $df_q : T_qM \longrightarrow T_1\mathbb{R}$. Therefore the condition $V(f) = (df)(V) \neq 0$, $\forall q \in \mathcal{H}$ ensures that $V \notin T_q\mathcal{H}$ and thus V is not tangent to $\mathcal{H}$.

Proof. In the subsequent proof, we only consider the scenario where $q \in \mathcal{H}$. If q is within the Lorentzian sector M_L, then $U(q)$ can be chosen to be sufficiently small, ensuring that $U(q)$ is entirely contained within M_L. Consequently, in this scenario, the theorem's assertion becomes trivial, as $\tilde{g}$ already represents a Lorentzian metric there, and thus, $f = 0$ satisfies all the stated conditions. Similarly, we can select a neighborhood in the Riemannian sector M_R where, for instance, $f = 2$ is trivially applicable.

To begin, consider that, according to Proposition 13.3 and Proposition 13.4 any triple (g, V, f), where $g \in Lor(M)$, $V \in \mathcal{L}(M)$ with $g(V, V) = -1$ and $f \in \mathfrak{F}(M)$, yields a signature-type changing metric $\tilde{g} = g + fV^\flat \otimes V^\flat = g + fg(V, \centerdot)g(V, \centerdot)$, which is defined over the entire manifold M (see *Transformation Prescription* 13.2). Conversely, if we have a signature-type changing metric $\tilde{g}$ we can always single out the associated triple (g, V, f) belonging to the equivalence class of $[\tilde{g}]$, such that $\tilde{g}$ locally takes the form $\tilde{g} = g + fV^\flat \otimes V^\flat$. In either case, we can initiate the proof by assuming that locally $\tilde{g} = g + f \cdot V^\flat \otimes V^\flat$ is given, and normalized with $g(V, V) = -1$.

In order to simplify the problem as much as possible we adopt co-moving coordinates (refer to [28, 48, 79, 73] for more details), that is $g_{00}(V^0)^2 = g(V, V) = -1$ and $V^i = 0$ for $i \neq 0$. Then the metric $\tilde{g}$ relative to these coordinates is given by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + fg_{\mu\alpha}V^\alpha g_{\nu\beta}V^\beta = g_{\mu\nu} + fg_{\mu 0}g_{\nu 0}(V^0)^2 = g_{\mu\nu} - f\frac{g_{\mu 0}g_{\nu 0}}{g_{00}}. \tag{13.4}$$

The last equality follows from the aforementioned condition

$$g_{00}(V^0)^2 = g(V, V) = -1 \Longleftrightarrow (V^0)^2 = -\frac{1}{g_{00}}.$$

According to Equation 13.4 the components of the metric in co-moving coordinates are determined by

$$\begin{aligned}
&\tilde{g}_{00} = g_{00} - f\tfrac{(g_{00})^2}{g_{00}} = (1 - f)g_{00}, \\
&\tilde{g}_{ij} = g_{ij} - f\tfrac{g_{i0}g_{j0}}{g_{00}}, \\
&\tilde{g}_{0i} = g_{0i} - fg_{i0} = (1 - f)g_{0i}.
\end{aligned}$$

And the associated matrix representation of $\tilde{g}$ is given by

$$
[\tilde{g}_{\mu\nu}] = \left(\begin{array}{c|ccc} (1-f)g_{00} & (1-f)g_{01} & \cdots & (1-f)g_{0n-1} \\ \hline (1-f)g_{10} & \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \vdots & & \vdots \\ (1-f)g_{n-10} & \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{array}\right)
$$

$$
= \left(\begin{array}{c|ccc} (1-f)g_{00} & (1-f)g_{01} & \cdots & (1-f)g_{0n-1} \\ \hline (1-f)g_{10} & & & \\ \vdots & & \tilde{G} & \\ (1-f)g_{n-10} & & & \end{array}\right).
$$

Then take the determinant $\det([\tilde{g}_{\mu\nu}]) = (1-f)\det(G_{\mu\nu})$, where

$$
G_{\mu\nu} = \left(\begin{array}{c|ccc} g_{00} & g_{01} & \cdots & g_{0n-1} \\ \hline (1-f)g_{10} & & & \\ \vdots & & \tilde{G} & \\ (1-f)g_{n-10} & & & \end{array}\right).
$$

Now consider $d(\det([\tilde{g}_{\mu\nu}])) = d[(1-f)\det(G_{\mu\nu})] = (1-f)d(\det(G_{\mu\nu})) - df\det(G_{\mu\nu})$. Since $q \in \mathcal{H} := f^{-1}(1) = \{p \in M : f(p) = 1\}$ is a regular point for f, the term $1-f = 0$ is zero on the hypersurface $\mathcal{H}$. Hence, on $\mathcal{H}$ we are left with $d(\det([\tilde{g}_{\mu\nu}])_q) = -df \cdot \det(G_{\mu\nu})$. On $\mathcal{H}$ we have $f \equiv 1$, and therefore it remains to show that on $\mathcal{H}$ the following is true

$$
0 \neq \det(G_{\mu\nu}) = \det \begin{pmatrix} g_{00} & g_{01} & \cdots & g_{0n-1} \\ 0 & g_{11} - \frac{g_{01}g_{01}}{g_{00}} & \cdots & g_{1n-1} - \frac{g_{01}g_{0n-1}}{g_{00}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n-11} - \frac{g_{0n-1}g_{01}}{g_{00}} & \cdots & g_{n-1n-1} - \frac{g_{0n-1}g_{0n-1}}{g_{00}} \end{pmatrix}.
$$

Notice that $\det(G_{\mu\nu})$ is the determinant of a block matrix with the block $[g_{00}] \neq 0$ being invertible on $\mathcal{H}$, hence

$$
\det(G_{\mu\nu}) = \det([g_{00}]) \cdot \det([g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}}]) = g_{00} \cdot \det(\tilde{G}).
$$

Because of the condition $g_{00}(V^0)^2 = -1$ we have

$$g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}} = g_{ij} + g_{0i}g_{0j}(V^0)^2 =: h_{ij},$$

which are the metric coefficients of the degenerate metric $h = g + fV^\flat \otimes V^\flat$, i.e. the coefficients of $[\tilde{g}_{\mu\nu}]$ for $f = 1$. More precisely, $h_{\mu\nu} = g_{\mu\nu} + g_{\mu\alpha}V^\alpha g_{\nu\beta}V^\beta = g_{\mu\nu} + g_{\mu 0}g_{\nu 0}(V^0)^2$. Since in co-moving coordinates, the vector V is orthogonal to the spatial coordinates [48], the metric h, restricted to the g-orthogonal complement $V^\perp$ of V in each tangent space, results in the non-degenerate "spatial metric" with metric coefficients h_{ij}. Finally, we may show that from $\det(G_{\mu\nu}) = g_{00} \cdot \det(\tilde{G}) = g_{00} \cdot \det([h_{ij}])$ on $\mathcal{H}$, and the fact that the "spatial metric" h is non-degenerate, follows that $\det(G_{\mu\nu}) \neq 0$ on $\mathcal{H}$. Consequently, from $d(\det([\tilde{g}_{\mu\nu}])) = -df \cdot \det(G_{\mu\nu})$ on $\mathcal{H}$ we get the biconditional statement

$$d(\det([\tilde{g}_{\mu\nu}])) \neq 0 \ \forall q \in \mathcal{H} \iff df \neq 0 \ \forall q \in \mathcal{H}.$$

By additionally requiring that $(V(f))(q) = ((df)(V))(q) \neq 0$, the radical is guaranteed to be transverse.

□

Remark 13.13. Remember that the triples (g, V, f) form equivalence classes (see Proposition 13.4), where all triples within an equivalence class yield the same metric $\tilde{g}$. Hence, by picking an arbitrary triple (g, V, f) in co-moving coordinates, we have shown that the relation $d(\det([\tilde{g}_{\mu\nu}])) \neq 0 \ \forall q \in \mathcal{H} \iff df \neq 0 \ \forall q \in \mathcal{H}$ holds independently of a choice of coordinates. This result is coordinate-independent due to the nature of f as a scalar, even though the determinant depends on coordinates. However, whether the differential of the determinant on the hypersurface is zero or not is not contingent on the choice of coordinates.[65] The equivalence $d(\det([\tilde{g}_{\mu\nu}])) \neq 0 \iff df \neq 0$ is, in this sense, a local statement, as it only holds on the hypersurface. On the other hand, the statement's coordinate independence on the hypersurface $\mathcal{H}$ implies that it remains unaffected by the choice of coordinates, ensuring its validity across the entire hypersurface. Thus, it possesses a global character in this regard - this aspect could be called $\mathcal{H}$-*global*.

[65] This follows from the nonsingularity of the Jacobian matrix associated with a coordinate transformation, in conjunction with the multiplicativity of the determinant in matrix multiplication, and the vanishing of the determinant on the hypersurface.

An alternative proof for the " $\Longrightarrow$ " direction of the local Transformation theorem, using radical-adapted Gauss-like coordinates, is provided in Appendix C.

Recall that for every $q \in \mathcal{H}$, the tangent space $T_q\mathcal{H}$ is the kernel of the map $df_q \colon T_qM \longrightarrow T_1\mathbb{R}$. Therefore the condition $V(f_q) = df_q(V) \neq 0$, $\forall q \in \mathcal{H}$ ensures that $V \notin T_q\mathcal{H}$ and thus V is not tangent to $\mathcal{H}$. This guarantees that the radical in $(M, \tilde{g})$ is transverse. If we are ready to relax our constraints and do not impose this restriction on V, then we get a slightly modified version of the Transformation theorem, such that the following corollary holds.

Corollary 13.14. *For every $q \in M$ there exists a neighborhood $U(q)$, such that the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is a transverse, type-changing metric with a transverse radical only if $\tilde{g}$ is locally obtained from a Lorentzian metric g via the Transformation Prescription $\tilde{g} = g + fV^\flat \otimes V^\flat$, as introduced in Proposition 13.2, where $df \neq 0$ for every $q \in \mathcal{H} \coloneqq f^{-1}(1) = \{p \in M \colon f(p) = 1\}$.*

Contrariwise, if the additional constraints, $(V(f))(q) = ((df)(V))(q) = 0$ and $df = 0$ for every $q \in \mathcal{H} \coloneqq f^{-1}(1) = \{p \in M \colon f(p) = 1\}$, are imposed on the Transformation Prescription 13.2, then we get an alternative version of the Transformation Theorem:

Conjecture 13.15. ***Local Transformation theorem (tangent radical)*** *For every $q \in M$ there exists a neighborhood $U(q)$, such that the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is a type-changing metric with a tangent radical if and only if $\tilde{g}$ is locally obtained from a Lorentzian metric g via the Transformation Prescription 13.2 $\tilde{g} = g + f \cdot V^\flat \otimes V^\flat$, where $df = 0$ and $(V(f))(q) = ((df)(V))(q) = 0$ for every $q \in \mathcal{H} \coloneqq f^{-1}(1) = \{p \in M \colon f(p) = 1\}$.*[66]

Proof. If we want to deflect our focus from the case with a transverse radical and consider a tangent radical instead, then the associated radical-adapted coordinates will have a more complicated form. Please refer to [1], equation (10) in Remark 3 for more details. Referring to those coordinates, we presume the conjecture can be proven. □

[66] Recall that for every $q \in \mathcal{H}$, $T_q\mathcal{H}$ is the kernel of the map $df_q \colon T_qM \longrightarrow T_1\mathbb{R}$. Therefore the condition $V(f) = (df)(V) = 0$, $\forall q \in \mathcal{H}$ ensures that $V \in T_q\mathcal{H}$ and thus V is tangent to $\mathcal{H}$.

Example 13.16. Consider on $\mathbb{R}^2$ the metric $ds^2 = x(dt)^2 + (dx)^2$ ([58], page 9). This is a signature-type changing metric with

$$\begin{aligned}
&\triangle := \det([g_{ij}]) = x,\\
&d\triangle = \tfrac{\partial x}{\partial t}dt + \tfrac{\partial x}{\partial x}dx = dx,\\
&\mathcal{H} = \{q \in M \mid x(q) = 0\}.
\end{aligned}$$

The differential is $d\triangle = dx \neq 0$ on $\mathcal{H}$, with $\mathcal{H}$ being a smoothly embedded hypersurface. The 1-dimensional radical is given by $Rad_q = \operatorname{span}(\{\frac{\partial}{\partial t}\})$ for $q \in \mathcal{H}$, and it is tangent with respect to $\mathcal{H}$.

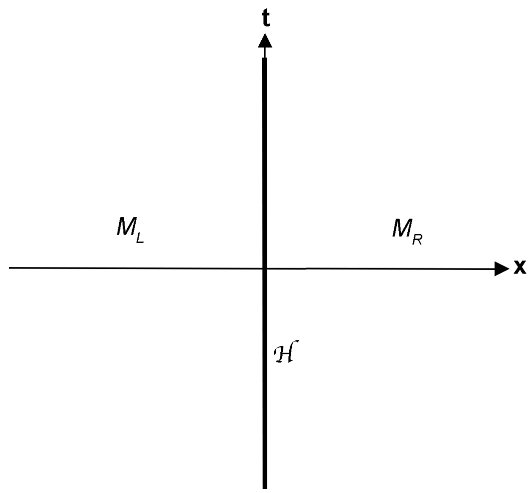

Figure 13.3: The radical is tangent with respect to the hypersurface.

Here the transverse, signature-type changing metric $ds^2 = x(dt)^2 + (dx)^2$ is given. So we need to *find* a suitable Lorentzian metric g as well as a global smooth function $f\colon M \longrightarrow \mathbb{R}$ and a non-vanishing line element field V, both with the properties mentioned in 13.14.

By an educated guess with start with $g = -(dt)^2 + (dx)^2$, from which follows $x(dt)^2 = -(dt)^2 + fV^\flat \otimes V^\flat$. As non-vanishing line element field we pick $V = \frac{\partial}{\partial t}$, with $V^\flat = g(V, \centerdot) = g(\frac{\partial}{\partial t}, \centerdot) = -(dt \otimes dt)(\frac{\partial}{\partial t}, \centerdot) = -dt$. Hence, $x(dt)^2 = -(dt)^2 + f \cdot (dt)^2 \Longrightarrow f(t,x) = 1 + x$.
Consequently this yields $\tilde{g} = g + (1+x)(dt)^2 = -(dt)^2 + (dx)^2 + (1+x)(dt)^2$

with $f(t,x) = 1 + x$. Then on $\mathcal{H}$ for $x = 0 \Longleftrightarrow f(t,0) = 1$ we have $\tilde{g}(V,X) = (dx)^2(V,X) = 0 \ \forall X$, and this means $V \in Rad_{p(t,0)} \ \forall t$. And since $Rad_{q(t,0)} = \text{span}\{V\} = \text{span}(\{\frac{\partial}{\partial t}\})$ the radical is tangent with respect to $\mathcal{H}$.

Example 13.17. Let g be an arbitrary Lorentzian metric. By continuity, throughout any sufficiently small neighborhood $U(p)$ of any arbitrary point $p \in M$, it is always possible to choose Gaussian coordinates (synchronous coordinates) with $g_{00} = -1$, hence we can express g with respect to these coordinates as $g = -(dt)^2 + g_{ij}dx^i dx^j$, for $i,j \in \{1,\ldots,n-1\}$. Since (M,g) is Lorentzian there exists, particularly within the aforementioned chart $U(p)$, a timelike line element field. In this case we can pick in $U(p)$ one of the pair $V = \{\frac{\partial}{\partial t}, -\frac{\partial}{\partial t}\}$, such that V can be considered a timelike line element field with a determined sign, i.e. V is thus locally a timelike vector field satisfying $g(V,V) = -1$ and $V^\flat = -dt$.

Then, without loss of generality, we choose the C^∞ function $f\colon U \longrightarrow \mathbb{R}$, $f(t,\hat{x}) = 1 - t$, such that $df \neq 0 \ \forall\, q = (t,\hat{x}) = (t, x^1, \ldots, x^{n-1}) \in \mathcal{H} := f^{-1}(1)$. Applying the *Transformation Prescription* 13.2 we obtain the signature-type changing metric

$$\tilde{g} = g + (1-t)V^\flat \otimes V^\flat = g + (1-t)\left(g(\frac{\partial}{\partial t}, \cdot) \otimes g(\frac{\partial}{\partial t}, \cdot)\right),$$

with the associated matrix representation

$$\tilde{g}_{\mu\nu} = \left(\begin{array}{c|ccc} -t & 0 & \cdots & 0 \\ \hline 0 & \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{array}\right) = \left(\begin{array}{c|ccc} -t & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{G} & \\ 0 & & & \end{array}\right).$$

Ultimately, because of Theorem 9.2 $(M,\tilde{g})$ is a transverse, signature-type changing manifold with a transverse radical.

13.3 Global transformation theorem

By imposing an extra constraint on the local variant of the Transformation Theorem 13.12, we can establish the validity of the global version of the Transformation Theorem: The Riemannian sector with boundary $M_R \cup \mathcal{H}$ is required to possess a smoothly defined non-vanishing line element field that is transverse to the boundary

$\mathcal{H}$.[67] In the subsequent discussion, we explore the means by which this supplementary constraint can be guaranteed.

13.3.1 Non-vanishing vector field in $M_R \cup \mathcal{H}$ transverse to $\mathcal{H}$

There are certain results and theorems in differential topology that provide conditions under which a non-vanishing vector field exists on a manifold [47, 61, 81, 82]. For more general manifolds, the existence of a non-vanishing vector field is related to the topology of the manifold:

(i) If $M_R \cup \mathcal{H}$ is a noncompact connected manifold with boundary, then it admits a nowhere vanishing vector field.

(ii) If $M_R \cup \mathcal{H}$ is compact and connected with boundary, then $M_R \cup \mathcal{H}$ admits a nowhere vanishing vector field if $\chi(M_R \cup \mathcal{H}) = 0$, where χ is the Euler characteristic.

For a manifold with boundary, a non-vanishing vector field *transverse* to the boundary may or may not exist, depending on the specific characteristics of the manifold and its boundary. The condition for the existence of a non-vanishing vector field on a differentiable manifold with boundary such that the vector field is transverse to the boundary involves the notion of a vector field being "outward-pointing" along the boundary [89] (in particular, for each $q \in \mathcal{H}$, $V(q) \notin T_q\mathcal{H}$ for such a vector field V).

First, let us recall some general definitions [49, 61, 89].

Definition 13.18. Let $M_R \cup \mathcal{H}$ be a manifold with boundary $\partial M_R = \mathcal{H}$ and $q \in \mathcal{H}$. A tangent vector $V_q \in T_q(M_R \cup \mathcal{H})$ is said to be inward-pointing if $V_q \notin T_q(\mathcal{H})$ and there is an $\epsilon > 0$ and an associated curve $\gamma\colon [0, \epsilon) \longrightarrow M_R \cup \mathcal{H}$ such that $\gamma(0) = q$, $\gamma((0, \epsilon)) \subset M^\circ$, with $\gamma'(0) = V_q$. Correspondingly, we say a vector field $V_q \in T_q(M_R \cup \mathcal{H})$ is outward-pointing if $-V_q$ is inward-pointing.

Definition 13.19. A collar of a manifold M_R with boundary $\partial M = \mathcal{H}$ is a diffeomorphism $\phi = (\phi_1, \phi_2)$ from an open neighborhood $U(\mathcal{H})$ of $\mathcal{H}$ to the product $\mathbb{R}^+ \times \mathcal{H}$ such that $\phi_2 \mid_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$. In particular, $\phi(\mathcal{H}) = \{0\} \times \mathcal{H}$.[68]

[67] The hypersurface $\mathcal{H}$ can be viewed as common connected boundary of the Riemannian region M_R and the Lorentzian region M_L. Recall that a manifold with boundary is a topological space in which, near each point, the space looks like the half-space $\{(x_1, x_2, \dots, x_n)\} \in \mathbb{R}^n : x_n \geq 0\}$ for some n.

[68] Lee [61] refers to a different but equivalent definition: A neighborhood of $\mathcal{H}$ is called a collar neighborhood if it is the image of a smooth embedding $[0, \varepsilon) \times \mathcal{H} \longrightarrow M_R \cup \mathcal{H}$ with $\varepsilon > 0$.

According to the Brown's collaring theorem [12, 61], the boundary $\mathcal{H}$ has a collar neighborhood $U(\mathcal{H})$.

Given a vector field V on $M_R \cup \mathcal{H}$ with a collar ϕ we define on the open neighborhood $U(\mathcal{H})$ the tangent and transverse components of V with respect to ϕ:

$$V_{\parallel} := T\phi_2 \circ V \colon U(\mathcal{H}) \longrightarrow T(\mathcal{H})$$

$$V_{\pitchfork} := T\phi_1 \circ V \colon U(\mathcal{H}) \longrightarrow \mathbb{R}.$$

Moreover, a vector field is termed 0-transverse if it is transverse to the zero section of the tangent bundle.

If $M_R \cup \mathcal{H}$ is a noncompact and connected manifold with boundary:
In the case where $M_R \cup \mathcal{H}$ is connected and noncompact with a boundary, a nonvanishing vector field can always be constructed using the following procedure: If $M_R \cup \mathcal{H}$ is noncompact, we can establish a compact exhaustion, denoted as $\varnothing = K_0 \subset K_1 \subset K_2 \subset \ldots \subseteq M = \bigcup_i K_i$. Zeros of a vector field existing in $K_i \setminus K_{i-1}$ are systematically pushed to $K_{i+1} \setminus K_i$. Notably, this process leaves the vector fields defined on K_{i-1} unchanged. By continuously pushing all zeros of a vector field towards infinity, we obtain a well-defined nonvanishing vector field on $T(M_R \cup \mathcal{H})$, where $T(M_R \cup \mathcal{H})$ represents a tangent bundle on $M_R \cup \mathcal{H}$.
This vector field can always be made transverse to the boundary by perturbing it within a collar neighborhood $U(\mathcal{H})$ with a small vector field that is transverse to the boundary [37, 88].

Then again, assuming $\mathcal{H}$ possesses a transverse vector field without zeros, then it suffices if there exists a vector field on $M_R \cup \mathcal{H}$ that is transverse to $\mathcal{H}$ and has isolated zeros outside of $\mathcal{H}$. In such a case, we can apply the same procedure as described above. This essentially boils down to determining the existence of a non-trivial section of the normal bundle of $\mathcal{H}$.

If $M_R \cup \mathcal{H}$ is a compact and connected manifold with boundary:
In the case where $M_R \cup \mathcal{H}$ is connected and compact with a boundary, the situation is a bit more complicated. The existence of a non-vanishing vector field transverse to the boundary of a compact manifold $M_R \cup \mathcal{H}$ depends on the topology and geom-

etry of the manifold. One key result that addresses this question is the *generalized* Poincaré-Hopf theorem [49, 61, 66, 81, 83]:[69]

> Let M be a compact oriented manifold with boundary, and let V be a smooth vector field on M such that:
>
> V is transverse to the boundary ∂M.
> The vector field V has isolated zeros in the interior of M, and the zeros on the boundary are assumed to be pointing outward.
>
> Then, the sum of the indices of the isolated zeros of V (counted with signs) is equal to the Euler characteristic $\chi(M)$ of M. The index of a zero is defined using the orientation of the manifold and measures how many times the vector field winds around the zero.[70]

However, having zero Euler characteristic does not provide a direct guarantee of the existence of a non-vanishing vector field transverse to the boundary. The Poincaré-Hopf theorem provides a necessary condition for the existence of a nowhere vanishing vector field. It states that if $M_R \cup \mathcal{H}$ is a compact manifold with boundary and admits a nowhere vanishing vector field, then the Euler characteristic $\chi(M_R \cup \mathcal{H})$ must be zero. So, while $\chi(M_R \cup \mathcal{H}) = 0$ is a necessary condition, it is not a sufficient condition for the existence of a nowhere vanishing vector field on a compact manifold with boundary.

Moreover, compact manifolds with boundary and Euler characteristic zero do not naturally exist.[71] The construction of such manifolds involves combining closed manifolds in a specific manner - primarily by attaching handles to well$-$known manifolds, such as a torus with handles, real projective space with handles, and handlebodies with boundaries.

[69]This version of the Poincaré-Hopf theorem accounts for the presence of the boundary and ensures that the vector field is transverse to it, taking into consideration the orientation of the manifold. The orientation of the zeros on the boundary is also specified to maintain consistency.

[70]In the context of a vector field X on a manifold M, a zero of X refers to a point p in the manifold where the vector field vanishes, meaning $X(p) = 0$. So if the sum of the indices of the zeros of a vector field X on a compact manifold M is 0, then the Poincaré-Hopf Index Theorem implies that the Euler characteristic $\chi(M) = 0$.

[71]The term "naturally" suggests that such manifolds are not commonly encountered without intentional construction or modification.

Example 13.20. Consider a torus with handles, which can be thought of as a higher-genus surface obtained by attaching handles to a torus. If the handles are attached in a way that preserves the orientation of the torus, it's possible to construct a non-vanishing vector field transverse to the boundary. However, if the handles are attached in a way that reverses the orientation of the torus, then constructing such a vector field becomes impossible due to the Poincaré-Hopf theorem. Since the Euler characteristic of the torus with handles is not zero, there must be points where a non-vanishing vector field is tangent to the boundary.
Therefore, a non-vanishing vector field transverse to the boundary is not guaranteed in the general case. And in summary, the existence of a non-vanishing vector field transverse to the boundary for a torus with handles depends on the specific way the handles are attached and whether the resulting orientation is consistent with the Poincaré-Hopf theorem.

Example 13.21. Consider the unit disk B^2. To be transverse at the boundary, a non-vanishing vector field should consistently point outward. But we know [68, 69] that given a non-vanishing vector field V on B^2, there exists a point of S^1 where the vector field V points directly inward and a point of $S^1$where it points directly outward. Thus we arrive at a contradiction with the condition for transversality. The reason is as follows:
If the vector field $(x, V(x))$ on B^2 (written as an ordered pair) changes direction between pointing "outward" and "inward" along the boundary of a differentiable manifold, it means that the field is not consistently transverse to the boundary, but becomes tangent at some boundary points. In mathematical terms this means that for some $x \in S^1$ we have $V(x) = ax$ for some $a < 0$, where $V(x) = ax$ for some $a > 0$ means pointing directly outward [89]. Hence, the vector field V vanishes if $V(x) = 0$. Hence, a vector field on B^2 that is nowhere-vanishing cannot point inward everywhere or outward everywhere, so it has to be tangent to the boundary somewhere by continuity.
In other words, if the vector field fails to be transverse to the boundary this includes inconsistency in direction across the boundary, violating the desired conditions for a well-defined transverse vector field. Analogously this also applies to the associated line element field $\{V, -V\}$.
This example reflects the situation for the 2-dimensional case of the "no-boundary proposal" spacetime.

In conclusion, when $M_R \cup \mathcal{H}$ is compact with a boundary, the relationship between the Euler characteristic $\chi(M_R \cup \mathcal{H})$ and the existence of a nowhere vanishing vector field on a compact manifold with boundary becomes more subtle. There are no clear conditions that we can impose on $M_R \cup \mathcal{H}$ for the existence of a nowhere vanishing vector field that is also transverse to the boundary. As a consequence, we will either explicitly require the existence of a non-vanishing vector field transverse to the boundary, or limit our analysis exclusively to the non-compact case where constructing such a vector field transverse to the boundary is always feasible. Analogously, this also applies to the associated line element field $\{V, -V\}$.

13.3.2 Transformation theorem

In the context of the global Transformation theorem, as opposed to the local version, the emphasis is also placed on the region in the Riemannian sector that is "distant" from the hypersurface. The relationship between the Riemannian sector and the hypersurface is crucial in determining whether the equivalence statement in the local Transformation theorem 13.12 is applicable in the desired manner.

Theorem 13.22. ***Global Transformation theorem (transverse radical)*** *Let M be an transverse, signature-type changing manifold of $\dim(M) = n \geq 2$, which admits in $M_R \cup \mathcal{H}$ a smoothly defined non-vanishing line element field that is transverse to the boundary $\mathcal{H}$. Then the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is a transverse, type-changing metric with a transverse radical if and only if $\tilde{g}$ is obtained from a Lorentzian metric g via the Transformation Prescription $\tilde{g} = g + fV^\flat \otimes V^\flat$, where $df \neq 0$ and $(V(f))(q) = ((df)(V))(q) \neq 0$ for every $q \in \mathcal{H} := f^{-1}(1) = \{p \in M \colon f(p) = 1\}$.*

Proof.
" $\Longleftarrow$ " Since (M, g) is Lorentzian, there always exist a well-defined smooth timelike line element field $\{V, -V\}$ on all of M. Consequently, any triple (g, V, f), where g is a Lorentzian metric, $V \in \{V, —V\}$ a non-vanishing line element field on M with $g(V, V) = -1$ and $f \colon M \longrightarrow \mathbb{R}$ a smooth function, yields a signature-type changing metric $\tilde{g} = g + fV^\flat \otimes V^\flat$, which is defined over the entire manifold M (see *Transformation Prescription* 13.2). Hence, the local version of the Transformation theorem 13.12 entails trivially also the global version.

" $\Longrightarrow$ " Let $(M, \tilde{g})$ be a signature-type changing manifold with a transverse radical, and $\tilde{g}$ the associated transverse, type-changing metric. Following the precondition, we select (independently of choosing specific coordinates) a smooth,

non-vanishing line element field $V = (1, 0, 0, \dots, 0)$defined across the entire manifold M, such that $g(V, V) < 0$ in M_L. Such a globally defined vector field exists per assumption. Moreover, V can be specified as an arbitrary, time-like vector field with respect to g, and normalized with $g(V, V) = -1$.[72]
According to Proposition 13.3, this yields already a triple (g, V, f), with a uniquely determined Lorentzian metric g and a smooth function $f\colon M \longrightarrow \mathbb{R}$ obeying the fixed condition $f(q) = 1 \ \forall q \in \mathcal{H}$, constituting the representation of $\tilde{g}$ through $\tilde{g} = g + fV^\flat \otimes V^\flat = g + fg(V, \centerdot)g(V, \centerdot)$. The insertion of V into the transformation prescription yields

$$\tilde{g}(V, V) = \underbrace{g(V, V)}_{-1} + f\underbrace{[g(V, \centerdot)]^2}_{1} = -1 + f,$$

which is negative in M_L for $1 > f(p)$ and positive in M_R for $1 < f(p)$.

In order to write down the representation matrix of $\tilde{g}$, a coordinate chart on the manifold needs to be chosen. Then, every vector field X on M can be written down as a linear combination of the coordinate basis vector fields, i.e. $X = X^\mu \partial_{x^\mu}$. In particular we have $V = 1\partial_t + 0\partial_{x^1} + \dots + 0\partial_{x^{n-1}} = \partial_t$. This means, that in a change of coordinates, the vector field V changes because the components $(1, 0, 0, \dots, 0)$ are fixed while the Gaussian basis vector fields are changed.[73]

In these coordinates,[74] the matrix representation of $\tilde{g}$ is given by

$$[\tilde{g}_{\mu\nu}] = \left(\begin{array}{c|ccc} f-1 & g_{01} - fg_{01} & \cdots & g_{0n-1} - fg_{0n-1} \\ \hline g_{10} - fg_{10} & g_{11} + fg_{01}g_{01} & \cdots & g_{1n-1} + fg_{01}g_{0n-1} \\ \vdots & \vdots & & \vdots \\ g_{n-10} - fg_{n-10} & g_{n-11} + fg_{0n-1}g_{01} & \cdots & g_{n-1n-1} + fg_{0n-1}g_{0n-1} \end{array}\right)$$

[72]In the procedure 13.1.1 the metric g is not determined before V is chosen. Moreover g is determined not only by $\tilde{g}$ but also by the normalization condition imposed on V. This normalization condition is to be used in the determinant calculation below.

[73]Keep in mind that the coordinates are associated with the corresponding Gaussian basis vector fields, with which the metric coefficients are defined by substituting them into the metric.

[74]This is to say, for instance, $\tilde{g}_{01} = g(\partial_t, \partial_{x^1}) + fg\underbrace{(V, \partial_t)}_{(V,V)}g(V, \partial_{x^1}) = g_{01} + f\underbrace{g_{00}}_{-1}g_{01} = g_{01} - fg_{01}$.

$$= \left(\begin{array}{c|ccc} -(1-f) & g_{01}(1-f) & \cdots & g_{0n-1}(1-f) \\ \hline g_{10}(1-f) & \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \vdots & & \vdots \\ g_{n-10}(1-f) & \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{array}\right).$$

Now take the determinant $\det([\tilde{g}_{\mu\nu}]) = \det(1-f)\det(\tilde{G}_{\mu\nu}) = (1-f)\det(\tilde{G}_{\mu\nu})$, where

$$\tilde{G}_{\mu\nu} = \left(\begin{array}{c|ccc} -1 & g_{01} & \cdots & g_{0n-1} \\ \hline (1-f)g_{10} & & & \\ \vdots & & \tilde{G} & \\ (1-f)g_{n-10} & & & \end{array}\right).$$

Then take the differential $d(\det([\tilde{g}_{\mu\nu}])) = d((1-f)\det(\tilde{G}_{\mu\nu})) = (1-f)\cdot d(\det(\tilde{G}_{\mu\nu})) + d(1-f)\cdot\det(\tilde{G}_{\mu\nu}) = (1-f)\cdot d(\det(\tilde{G}_{\mu\nu})) - df\cdot\det(\tilde{G}_{\mu\nu})$.

Because of the condition $f(q) = 1\ \forall q \in \mathcal{H} := f^{-1}(1) = \{p \in M\colon f(p) = 1\}$, we get $(1-f)\cdot d(\det(\tilde{G}_{\mu\nu})) = 0$, hence $d(\det([\tilde{g}_{\mu\nu}])) = -df\cdot\det(\tilde{G}_{\mu\nu})$ on $\mathcal{H}$. Accordingly it remains to show that we indeed have $-df\cdot\det(\tilde{G}_{\mu\nu}) \neq 0$ on $\mathcal{H}$.

First notice that $\det(\tilde{G}_{\mu\nu})$ is the determinant of a block matrix with the block $[-1]$ being invertible, hence

$$\det(\tilde{G}_{\mu\nu}) = \det(-1)\cdot\det(\tilde{G} - [(g_{10}(1-f),\ldots,g_{n-10}(1-f))^T(-1)(g_{01}\ldots g_{0n-1})]$$

$$= -\det\left(\begin{bmatrix} \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \ddots & \vdots \\ \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{bmatrix} - \begin{bmatrix} -g_{10}g_{01}(1-f) & \cdots & -g_{10}g_{0n-1}(1-f) \\ \vdots & \ddots & \vdots \\ -g_{n-10}g_{01}(1-f) & \cdots & -g_{n-10}g_{0n-1}(1-f) \end{bmatrix}\right)$$

$$= -\det\left(\begin{bmatrix} \tilde{g}_{11} + g_{10}g_{01}(1-f) & \cdots & \tilde{g}_{1n-1} + g_{10}g_{0n-1}(1-f) \\ \vdots & \ddots & \vdots \\ \tilde{g}_{n-11} + g_{n-10}g_{01}(1-f) & \cdots & \tilde{g}_{n-1n-1} + g_{n-10}g_{0n-1}(1-f) \end{bmatrix}\right).$$

On the hypersurface $\mathcal{H}$ this yields

$$\det(\tilde{G}_{\mu\nu}) = -\det\left(\begin{bmatrix} \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \ddots & \vdots \\ \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{bmatrix}\right) = -\det\tilde{G}.$$

The sub-determinant $\det\tilde{G}$, however, is precisely the determinant of the coefficients of the metric $\tilde{g}$ reduced to the surfaces $t = const$ (this is a "purely spatial" metric, induced by projection). This "spatial metric" $\tilde{G}$ is non-degenerate, and thus, $-\det\tilde{G} \neq 0$ on the hypersurface.[75] Consequently, we get that $-df \cdot \det(\tilde{G}_{\mu\nu}) \neq 0 \Longleftrightarrow df \neq 0$ on $\mathcal{H}$.

And on $\mathcal{H}$ we have imposed the condition

$$0 \neq d(\det([\tilde{g}_{\mu\nu}])) = \underbrace{(1-f)\cdot d(\det(\tilde{G}_{\mu\nu}))}_{0} - df \cdot \underbrace{\det(\tilde{G}_{\mu\nu})}_{\neq 0},$$

so it follows from the above calculation that $0 \neq d(\det([\tilde{g}_{\mu\nu}])) \Longrightarrow df \neq 0\ \forall q \in \mathcal{H}$. $\square$

Remember that the triples (g, V, f) form equivalence classes (see Proposition 13.4), where all triples within an equivalence class yield the same metric $\tilde{g}$. If an element's properties change in a way that aligns with a different equivalence class, it may move accordingly to a different equivalence class, as elucidated in Subsection13.1.1.[76] Hence, by picking $V = (1, 0, \ldots, 0)$ in a coordinate independent manner, we have shown that the *Transformation Prescription* also holds for arbitrary triples (g, V, f) with $df \neq 0$ on the hypersurface, indicating that the differential of the determinant of $\tilde{g}$ is nonzero, independet of the choice of coordinates.

[75] Recall that he signature of a symmetric matrix expresses the number of positive and negative eigenvalues, counting multiplicities. Accordingly, the matrix G has only positive eigenvalues. Since the determinant of the representation matrix of a Riemannian metric G is the product of the eigenvalues of G (counted with multiplicity), the determinant is positive.

[76] Furthermore, it is ruled out that, amid all these (described) perturbations within an equivalence class, a transformation from an f with $df \neq 0$ to an f with $df = 0$ can occur. This is evident from the fact that $g_{00} = -1$ (this is due to the specific choice of $V = (1, 0, 0, \ldots, 0)$), resulting in $\tilde{g}_{00} = -1 + f$ because of the transformation prescription. Since $\tilde{g}$ remains unchanged within an equivalence class, and thus (with initially arbitrary coordinates, but perturbed in such a way as to preserve the $(1, 0, 0, \ldots, 0)$-shape of V); especially $\tilde{g}_{00}$ remains unchanged as a function of the respective coordinates, and f also does not change as a function of the respective coordinates (although $\tilde{g}_{00}$ and f do change as functions on M, but not as functions of the respective coordinates).

Remark 13.23. An alternative proof for the " $\Longrightarrow$ " direction of the global Transformation theorem can be obtained by revisiting the local Transformation theorem. This involves assuming that the Riemannian region with “boundary” $M_R \cup \mathcal{H}$ admits a smoothly defined non-vanishing line element field that is transverse to the boundary $\mathcal{H}$. This approach ensures the local version can get extended a global manner to the entire manifold M.

Instead of requiring the existence of a non-vanishing vector field transverse to the boundary, we can limit our analysis exclusively to the non-compact case where constructing such a vector field transverse to the boundary is always feasible. This yields the following

Corollary 13.24. *Let M be an n-dimensional transverse, signature-type changing manifold with $M_R \cup \mathcal{H}$ non-compact. Then the metric $\tilde{g}$ associated with a signature-type changing manifold $(M, \tilde{g})$ is a transverse, type-changing metric with a transverse radical if and only if $\tilde{g}$ is obtained from a Lorentzian metric g via the Transformation Prescription $\tilde{g} = g + fV^\flat \otimes V^\flat$, where $df \neq 0$ and $(V(f))(q) = ((df)(V))(q) \neq 0$ for every $q \in \mathcal{H} := f^{-1}(1) = \{p \in M \colon f(p) = 1\}$.*

Example 13.25. Consider again the classic type of a spacetime M with signature-type change which is obtained by cutting an S^4 along its equator and joining it to the corresponding half of a de Sitter space, see Figure 8.1. This is the universe model obeying the 'no-boundary' condition. The 4-dimensional half sphere is homeomorphic to a disk (of corresponding dimension) and there exists a non-vanishing line element field. However, any such non-vanishing line element field V will be tangent to the equator $\mathcal{H}$ (which is the surface of signature-type change) at some point $q \in \mathcal{H}$, see Example 13.21. Hence, we cannot extend V smoothly across the equator to the Lorentzian sector because $V \in T_q\mathcal{H}$ and thus $\exists\, q \in \mathcal{H}$ such that $V(f_q) = df_q(V) = 0$. Therefore, the radical is tangent at some $q \in \mathcal{H}$, and the *global version* of the Transformation Theorem cannot apply.

14 The hypersurface of signature change

14.1 Causal character of the hypersurface

In Lorentzian geometry there are three categories of hypersurfaces due to the causal character (spacelike, timelike and lightlike) of the occurring vector fields in an n-dimensional manifold M:

- A hypersurface $\mathcal{H}$ is called spacelike, if the normal N_q at each point $q \in \mathcal{H}$ is timelike. In this case, $g\mid_{T_q\mathcal{H}}$ is positive-definite (i.e. $\mathcal{H}$ is a Riemannian manifold).
- A hypersurface $\mathcal{H}$ is called null, if the normal N_q at each point $q \in \mathcal{H}$ is null. In this case, $g\mid_{T_q\mathcal{H}}$ is degenerate.
- A hypersurface $\mathcal{H}$ is called timelike, if the normal N_q at each point $q \in \mathcal{H}$ is spacelike. In this case, $g\mid_{T_q\mathcal{H}}$ has signature $(-,+,\ldots,+)$.

In contrast, for our setting the signature-type changing metric $\tilde{g}$ fails to produce timelike normal vectors with a base point on $\mathcal{H}$.[77] Aside from that, the classification is not unambiguous for manifolds with a radical that have a transverse as well as a tangential component with respect to the hypersurface. See, for instance, Example *7.21* in Section 7. On this account it seems reasonable to introduce a new definition for the causal classification of hypersurfaces in signature-type changing manifolds.

Definition 14.1. There are two categories of hypersurfaces $\mathcal{H}$ in a signature-type changing manifold M with a smooth, symmetric type changing metric $\tilde{g}$:

- A hypersurface $\mathcal{H}$ is called *spacelike*, if all tangent vectors to $\mathcal{H}$ are spacelike. In this case is $\mathcal{H}$ spacelike if and only if the radical is transverse with respect to $\mathcal{H}$ (i.e. $\mathcal{H}$ is a Riemannian manifold).
- A hypersurface $\mathcal{H}$ is called *null*, if there exist a tangent null vector to $\mathcal{H}$. In this case is $\mathcal{H}$ null if and only if the radical is tangent with respect to $\mathcal{H}$.

[77]Note that here $\mathcal{H}$ always refers to the hypersurface of signature change.

14.2 The induced metric on $\mathcal{H}$

In the following we demonstrate that in general the induced metric on the hypersurface $\mathcal{H}$ is either Riemannian or a positive semi-definite pseudo metric.

Let $(M, \tilde{g})$ be an n-dimensional signature-type changing manifold as introduced in Section 13 and $\mathcal{H} \coloneqq \{q \in M \colon \tilde{g} \mid_q \textit{ is degenerate}\}$ the hypersurface of signature change. Furthermore, we assume again that one component of $M \setminus \mathcal{H}$ is Riemannian and the other one Lorentzian. Hence, $M \setminus \mathcal{H}$ is a union of two semi-Riemannian manifolds with constant signature. As suggested by [52] we consider the one-dimensional subspace of T_qM defined as $Rad_q \coloneqq \{w \in T_qM \colon \tilde{g}(w, .) = 0\}$, for all $q \in \mathcal{H}$.[78] If $v \in Rad_q$, then $\tilde{g}(v, \centerdot) = 0$ must apply. Indeed, we have

$$\tilde{g}(v, \centerdot) = g(v, \centerdot) + \underbrace{f(q)}_{1}(\underbrace{v^\flat(v)}_{g(v,v)} \cdot \underbrace{v^\flat(\centerdot)}_{g(v,\centerdot)}) = g(v, \centerdot) - g(v, \centerdot) = 0.$$

And because Rad_q is a one-dimensional subspace of T_qM, we have $Rad_q = \operatorname{span}\{v\}$.

Corollary 14.2. *Let (M, g) be an n-dimensional signature-type changing manifold and $\mathcal{H} \subset M$ the hypersurface of signature change. Then the set $\mathcal{H}$ is closed. Furthermore, the set $M \setminus \mathcal{H}$ is dense in M and open.*

Proof. The set $\mathcal{H}$ is closed follows directly from Definition 2.7. □

Theorem 14.3. *If $q \in \mathcal{H}$ and $x \notin Rad_q$, then $\tilde{g}(x, x) > 0$ holds for all $x \in T_qM$.*

Proof. We start by decomposing the vector x into the sum of two vector components with respect to a non-degenerate metric g,[79] where one component is parallel to v and the other one is perpendicular to v: $x = v^{||}(x) + v^{\perp}(x)$ with $v^{||}(x) = \frac{g(x,v)}{g(v,v)}v = -g(x, v)v$ [71, p.50]. Substituting for $v^{||}(x)$ and rearranging the vector decomposition, produces $v^{\perp}(x) = x + g(v, x)v$. Plugging $v^{\perp}(x)$ into the metric gives

$$g(v^{\perp}(x), v^{\perp}(x)) = g(x + g(x, v)v, x + g(v, x)v)$$

$$= g(x, x) + 2g(x, g(x, v)v) + g(x, v)^2 g(v, v)$$

[78] Alternatively we can take the radical as the kernel of the linear map $T_qM \longrightarrow T_q^*M \colon w \longmapsto (v \mapsto g(w, v))$.

[79] For $f(q) = 1$ we have $\tilde{g}(\centerdot, \centerdot) = g(\centerdot, \centerdot) + g(v, \centerdot)g(v, \centerdot)$ on T_qM.

$$= g(x,x) + 2g(x,v)^2 + g(x,v)^2 \underbrace{g(v,v)}_{-1} = g(x,x) + g(x,v)^2 = \tilde{g}(x,x).$$

Note that because of $g(v,v) = -1$, the vector v is timelike and g is a Lorentzian metric. Hence, $v^{\perp}(x)$ is spacelike.[80] Moreover, as $x \notin Rad_q = \text{span}\{v\}$ we know that $v^{\perp}(x) \neq 0$, and therefore $\underbrace{g(v^{\perp}(x), v^{\perp}(x))}_{\tilde{g}(x,x)} > 0$. □

As mentioned in Section 13, for every $q \in \mathcal{H}$, the tangent space $T_q\mathcal{H}$ is the kernel of the map df_q. Hence, $\ker(df_q) = T_q\mathcal{H}$ is a vector subspace of T_qM. Provided that the radical is not a vector subspace of $T_q\mathcal{H}$, that is $Rad_q \nsubseteq T_q\mathcal{H}$, then the induced metric on $\mathcal{H}$ is Riemannian. This follows directly from the proof of Theorem 14.3 because for all $x \in T_q\mathcal{H} \setminus \{0\}$ the restriction of the metric on T_qM to the subspace $T_q\mathcal{H}$ is positive definite. In the event of the radical being a vector subspace of $T_q\mathcal{H}$, that is $Rad_q \subseteq T_q\mathcal{H} \subseteq T_qM$, then according to the definition of Rad_q the induced metric on $\mathcal{H}$ is degenerate. But then based on Proof 14.3 we also have $\tilde{g}(x,x) \geq 0$ for all $x \in T_q\mathcal{H}$. Ultimately this leads to the conclusion that the induced metric on $\mathcal{H}$ is a positive semi-definite pseudo metric with the signature $(0, \underbrace{+, \ldots, +}_{(n-2)\,times})$.

Dependent on whether $Rad_q \subset T_q\mathcal{H}$, or alternatively whether $Rad_q \subset \ker(df_q) = T_q\mathcal{H}$, the induced metric on the hypersurface of signature change $\mathcal{H}$ can be either Riemannian (and non-degenerate) or a positive semi-definite pseudo-metric with signature $(0, \underbrace{+, \ldots, +}_{(n-2)\,times})$. The latter one is degenerate if $\ker(df_q) = T_q\mathcal{H} = Rad_q$.[81]

Proof. In our example, we have $Rad_q = \text{span}\{v\}$. And therefore,

$$\begin{aligned}
&Rad_q = \text{span}\{v\} \subset T_q\mathcal{H} \\
&\iff \text{span}\{v\} \subset \ker(df_q) \\
&\iff \text{span}\{v\} \subset \{w \in T_qM : df_q(w) = 0\} \\
&\iff df_q(v) = 0 \\
&\iff v(f) = 0.
\end{aligned}$$

□

[80] If a nonzero vector in M is orthogonal to a timelike vector, then it must be spacelike [70].

[81] A pseudo metric is considered to be a field of symmetric bilinear forms which don't need to be everywhere non-degenerate. The main point of pseudo-metric spaces is that we cannot use our concept of distance to distinguish between different points, forcing us to think of things in terms of equivalence classes where points declared to have zero distance are considered equivalent.

15 Classes of signature-type changing metrics

Due to our method introduced in Section 13 we can transform any Lorentzian manifold (M, g) into a signature-type changing manifold with a metric $\tilde{g}$ by virtue of a non-vanishing timelike line element field $\{V, -V\}$ and a smooth function $f \colon M \longrightarrow \mathbb{R}$. In ideal circumstances there even exists a smooth, non-vanishing timelike vector field V. Motivated by this, we want to analyze some classes of signature-type changing metrics, and test whether they are of the type $\tilde{g} = g + f(V^\flat \otimes V^\flat)$, with some smooth transformation function f that interpolates between the Lorentzian and Riemannian regions, which are separated by the hypersurface $\mathcal{H} = f^{-1}(1)$.

15.1 Rotating Minkowski metric

Consider $M = \mathbb{R}^2$ equipped with the standard topology and the canonical Minkowski metric $\eta = -(dt)^2 + (dx)^2$, where (t, x) are Cartesian coordinates and E_0, E_1 are the Gaussian basis vector fields.[82] In order to rotate E_0 and E_1 clockwise by an angle φ we define a new, nonholonomic basis

$$\begin{aligned} F_0 &= -(\sin\varphi)E_1 + (\cos\varphi)E_0, \\ F_1 &= (\cos\varphi)E_1 + (\sin\varphi)E_0, \end{aligned}$$

with φ denoting an arbitrary smooth function depending on the arguments t and x. We furthermore require that the metric $g = g_{00}(dt)^2 + 2g_{01}dtdx + g_{11}(dx)^2$ satisfies $g(F_1, F_1) = -g(F_0, F_0) = 1$, and $g(F_0, F_1) = 0$.

The above requirements yield an nonhomogeneous linear equation system with the unique solution

$$\begin{aligned} g_{11} &= -g_{00} = \cos^2\varphi - \sin^2\varphi = \cos(2\varphi), \\ g_{01} &= 2\cos\varphi\sin\varphi = \sin(2\varphi). \end{aligned}$$

This yields the smooth metric $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$ on $\mathbb{R}^2$ that rotates clockwise.[83]

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[82] We consider here only two-dimensional toy models. But we could alternatively add two dimensions that correspond to a flat plane with a vanishing Riemannian curvature tensor.

[83] A similar result can be achieved by starting with a non-symmetric tensor $g = -[(\cos\varphi)dx + (\sin\varphi)dt] \otimes [-(\sin\varphi)dx + (\cos\varphi)dt] = (\sin\varphi)(\cos\varphi)dx \otimes dx - (\cos^2\varphi)dx \otimes dt + (\sin^2\varphi)dt \otimes dx - (\sin\varphi)(\cos\varphi)dt \otimes dt$, and then by producing a new symmetric tensor $g_{(ab)}$ from the old one based

Note that we use the term "rotating Minkowski" metric quite loosely. Unlike the Minkowski manifold, the manifold (M, g) introduced above is not flat.

The associated determinant $\triangle = \det([g_{\mu\nu}]) = -\cos^2(2\varphi) - \sin^2(2\varphi) = -1$ is non-vanishing and negative, and therefore the metric g is Lorentzian. By setting $\varphi = \pi x$ we get a rather natural choice where the light cones rotate clockwise through an angle φ in the tx-plane when moving in $x-$direction; the metric reaches the canonical form $g = -1(dt)^2 + 1(dx)^2$ for $x = k$, $k \in \mathbb{Z}$.
However, compared to a choice of time-orientation that has been specified at the initial value of x, the past and future of the light cones are swapped at $x \in \{2k - 1; k \in \mathbb{Z}\}$. Only after a further displacement of the absolute value 1 along the $x-$direction, a full rotation of the light cone takes place (with respect to the basis vectors E_0 and E_1). Hence, only at $x \in \{2k; k \in \mathbb{Z}\}$ a full rotation takes place such that the past and future of the light cones conform with their orientation at $x = 0$, i.e. along the $t-$axis. As we shall see, (M, g) is an orientable and time-orientable 2-dimensional Lorentzian manifold (also see Subsection 15.2.1 for the definition of the global non-vanishing, timelike vector field V).

The causal structure can be elucidated by means of

$$g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2 \stackrel{!}{=} 0$$

$$\stackrel{\cdot\frac{1}{(dx)^2}}{\Longleftrightarrow} g = -\cos(2\varphi)\frac{(dt)^2}{(dx)^2} + 2\sin(2\varphi)\frac{dt}{dx} + \cos(2\varphi) \stackrel{!}{=} 0,$$

note that this is a quadratic equation of the form $ax^2 + bx + c = 0$ with solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore,

$$\frac{dt}{dx} = \frac{-(2\sin(2\varphi)) \pm \sqrt{(2\sin(2\varphi))^2 + 4\cos^2(2\varphi)}}{-2\cos(2\varphi)} = \frac{-\sin(2\varphi) \pm 1}{-\cos(2\varphi)},$$

$$\Longrightarrow \frac{dt}{dx} = \begin{cases} \frac{-1+\sin(2\varphi)}{\cos(2\varphi)} = \frac{\sin(\varphi)-\cos(\varphi)}{\sin(\varphi)+\cos(\varphi)} \\ \frac{1+\sin(2\varphi)}{\cos(2\varphi)} = \frac{\sin(\varphi)+\cos(\varphi)}{\cos(\varphi)-\sin(\varphi)} \end{cases},$$

---

on $\underbrace{g_{ab}}_{g} = \underbrace{g_{(ab)}}_{symmetric} + \underbrace{g_{[ab]}}_{skew-symmetric} = \frac{1}{2}(g_{ab} + g_{ba}) + \frac{1}{2}(g_{ab} - g_{ba})$.

$$\iff dt = \begin{cases} \frac{\sin(\varphi)-\cos(\varphi)}{\sin(\varphi)+\cos(\varphi)}dx \\ \frac{\sin(\varphi)+\cos(\varphi)}{\cos(\varphi)-\sin(\varphi)}dx \end{cases}.$$

For $\varphi = \pi x$ we get

$$dt = \begin{cases} \frac{\sin(\pi x)-\cos(\pi x)}{\sin(\pi x)+\cos(\pi x)}dx \\ \frac{\sin(\pi x)+\cos(\pi x)}{\cos(\pi x)-\sin(\pi x)}dx \end{cases}$$

$$\iff \begin{array}{l} t(x) = -\frac{1}{\pi}\log|\sin(\pi x)+\cos(\pi x)| + const \\ t(x) = -\frac{1}{\pi}\log|\cos(\pi x)-\sin(\pi x)| + const \end{array}.$$

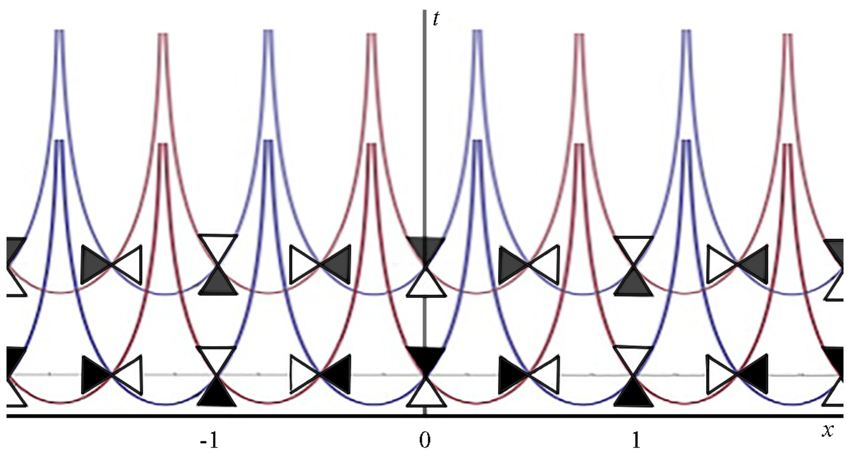

Figure 15.1: The rotating "Minkowski" metric $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$ on $\mathbb{R}^2$ that rotates clockwise.

At first glance the above Lorentzian manifold (M, g) looks deceivingly innocent. As the underlying 2-manifold is $M = \mathbb{R}^2$ with the standard topology, we are able to choose an atlas that consists of a single chart equipped with the identity map. The metric is smooth, and the non-vanishing Christoffel symbols are given by

$$\begin{array}{l} \Gamma^t_{tt} = -\frac{1}{2}g^{tx}\frac{d}{dx}g_{tt} = -\pi\sin^2(2\varphi), \\ \Gamma^x_{xx} = -\Gamma^t_{tx} = -\Gamma^x_{tt} = \pi\sin(2\varphi)\cos(2\varphi) = \frac{\pi}{2}\sin(4\varphi), \\ \Gamma^x_{tx} = -\Gamma^t_{tt} = \pi\sin^2(2\varphi) = \frac{\pi}{2}(1-\cos(4\varphi)) \\ \Gamma^t_{xx} = -\pi(1+\cos^2(2\varphi)) = -\frac{\pi}{2}(3+\cos(4\varphi)). \end{array}$$

The curvature scalar R (which in two dimensions completely characterizes the curvature) is bounded. A short calculation gives $R = 2K = 2(\varphi'' \sin(2\varphi) + 2\varphi'^2 \cos^3(2\varphi))$, where K denotes the Gaussian curvature.

However, the peculiar causal structure is revealed by the Killing vector fields. The metric is independent of the coordinate t, hence $W = \partial_t$ is a Killing field of g (note that Figure 15.1 gives a hint at how the Killing vector fields must look like). However, the Killing vector field ∂_t periodically changes its causal character along the x-axis: it is timelike for $\frac{1}{4}(4\pi k - \pi) < \varphi < \frac{1}{4}(4\pi k + \pi) \iff \frac{1}{4}(4k-1) < x < \frac{1}{4}(4k+1)$, $k \in \mathbb{Z}$, becomes null at $x = \frac{k}{2} - \frac{1}{4}$, $k \in \mathbb{Z}$ and spacelike for $\frac{1}{4}(4k+1) < x < \frac{1}{4}(4k+3), k \in \mathbb{Z}$. Accordingly, the manifold (M, g) is stationary when the Killing vector field is timelike, that is in the regions $M(k) \coloneqq \{(t, x) \in M; \frac{1}{4}(4k-1) < x < \frac{1}{4}(4k+1)\}$, $k \in \mathbb{Z}$. The stationary regions $M(k)$ are separated by non-stationary regions, which are given by $\{(t, x) \in M; \frac{1}{4}(4k+1) < x < \frac{1}{4}(4k+3)\}$ for $k \in \mathbb{Z}$. So the manifold (M, g) has a "stripe-like pattern" consisting of alternating stationary and non-stationary stripes (regions), such that adjoint stripes are separated by lightlike curves. Furthermore, the curvature scalar R is positive in the stationary regions and negative in the non-stationary ones.

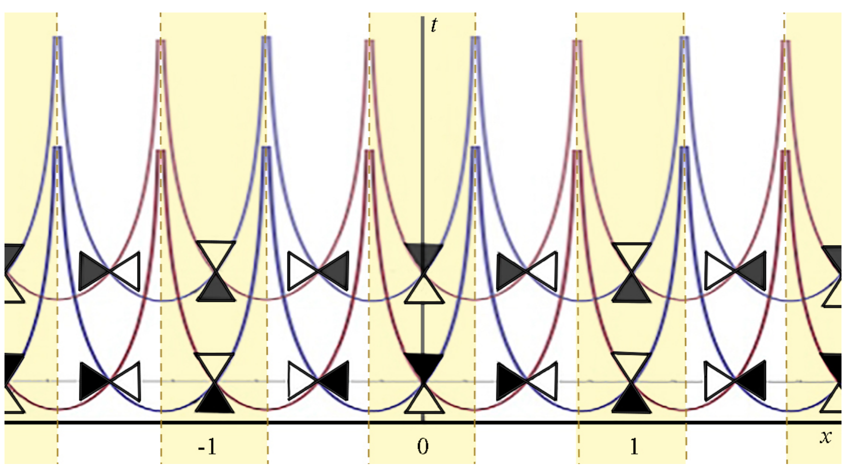

Figure 15.2: The stationary regions are separated by non-stationary regions. So the manifold has a "stripe-like pattern" consisting of alternating stationary and non-stationary stripes (regions), such that adjoint stripes are separated by lightlike curves.

In the Figure 15.2, we can see how (M, g) has a alternating pattern structure, where the yellow stripes indicate stationary areas and the dashed lines stand for the lightlike curves.

Assuming ∂_t to be future-pointing, causal curves in (M, g) that enter the stationary regions $M(2k)$, $k \in \mathbb{Z}$, get trapped in these regions, whereas no causal curve can enter the stationary regions $M(2k-1)$, $k \in \mathbb{Z}$.

15.2 Signature-type change in non-compact manifold

15.2.1 Transformation prescription applied to orientable and time-orientable manifold

As we have established in the preceding Section 15.1, our example (M, g) is a non-compact, orientable and time-orientable Lorentzian manifold. As such, this manifold admits a global, non-vanishing, timelike vector field. By means of making an educated guess we can indeed pick $V = (\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}$ as a global timelike vector field with respect to $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$, $\varphi = \pi x$. A short calculation confirms that $g(V, V) = -1$.

Given this vector field we can now consider the following class of signature-type changing metrics:
$\tilde{g} = g + f(V^\flat \otimes V^\flat)$, where f is an arbitrary smooth function and $V^\flat = g(V, \cdot)$ the index lowering morphism for $V \in \mathfrak{X}(M)$. Note that by picking a particular vector field V the resulting signature-type changing metric does depend on this choice. However, here the vector field V can be considered a canonical choice for the (M, g) in question.

$$\tilde{g} = g + f \cdot \left(g((\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}, \cdot) \otimes g((\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}, \cdot) \right)$$

$$\Longleftrightarrow \tilde{g} = (-\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2)$$

$$+f \cdot \left(\cos^2(\varphi)dt^2 - 2\sin(\varphi)\cos(\varphi)dtdx + \sin^2(\varphi)dx^2\right)$$

$$\iff \tilde{g} = (f \cdot \cos^2(\varphi) - \cos(2\varphi))dt^2 + (2-f)\sin(2\varphi)dtdx + (f \cdot \sin^2(\varphi) + \cos(2\varphi))dx^2$$

$$\iff [\tilde{g}_{\mu\nu}] = \begin{pmatrix} f \cdot \cos^2(\varphi) - \cos(2\varphi) & (1 - \frac{f}{2})\sin(2\varphi) \\ (1 - \frac{f}{2})\sin(2\varphi) & f \cdot \sin^2(\varphi) + \cos(2\varphi) \end{pmatrix}.$$

It is not surprising that the determinant yields $\triangle = \det([\tilde{g}_{\mu\nu}]) = f - 1$, so as to the metric $\tilde{g}$ is degenerate for $f(p) = 1$. Thus the hypersurface of signature change is the $(n-1)$-dimensional submanifold $\mathcal{H} = f^{-1}(1) = \{p \in M \colon f(p) = 1\}$.

Recall that we only want to consider transverse type-changing singular semi-Riemannian manifolds with a transverse radical (Section 3). Hence we need to impose the restrictions that $d(\det([\tilde{g}_{\mu\nu}])) = d(f-1) \neq 0$ for any $q \in \mathcal{H}$ and any local coordinate system $\xi = (x^0, \ldots, x^{n-1})$ centered at q, furthermore Rad_q is not tangent to $\mathcal{H}$ for any q. Based on the above calculations we require for f that $d(f-1) = \frac{\partial f(t,x)}{\partial t}dt + \frac{\partial f(t,x)}{\partial x}dx = df \neq 0$ for any $q := (t,x) \in \mathcal{H}$.

15.2.2 Open infinite time-orientable Möbius strip

Start again with $M = \mathbb{R}^2$ equipped with the standard topology and the smooth metric $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$, $\varphi = \pi x$, see Section 15.1.

The Möbius strip does not need to be time-orientable, because it depends on how the two opposite edges get identified. To achieve a time-orientable model, let $M = \mathbb{R} \times \mathbb{R}$ be the infinite "strip" with width $(-\infty, \infty)$, equipped with the standard topology and the metric g. Then we fold the manifold by defining the quotient map $q \colon \mathbb{R} \times \mathbb{R} \to (\mathbb{R} \times \mathbb{R})/\sim$ as the identification $(t,x) \sim (\tilde{t}, \tilde{x}) \iff (\tilde{t}, \tilde{x}) = ((-1)^k t, x + k)$, $k \in \mathbb{Z}$. The resulting Lorentzian manifold $M_\infty := (M/\sim, g)$ is non-compact, non-orientable and time-orientable, it has the topology of the open infinite Möbius strip, which is for sure different from Minkowski space in a global sense.

As a time-orientable Lorentzian manifold, the Möbius strip $M_\infty = (M/\sim, g)$ admits a global, non-vanishing, timelike vector field $V = (\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}$. Therefore, we can refer to the Transformation Trescription from Section 15.2.1

in order to transform M_∞ into the signature-type changing Möbius strip $\tilde{M}_\infty = (M/\sim, \tilde{g})$, with the metric given by

$$\tilde{g} = (f \cdot \cos^2(\varphi) - \cos(2\varphi))dt^2 + (2 - f)\sin(2\varphi)dtdx + (f \cdot \sin^2(\varphi) + \cos(2\varphi))dx^2,$$

where the quotient map identifies (t, x) with $(\tilde{t}, \tilde{x}) = ((-1)^k t, x + k)$, $k \in \mathbb{Z}$, and f is an arbitrary C^∞ function that also accounts for the hypersurface of signature-change $\mathcal{H} = f^{-1}(1)$.

Remark 15.1. Since only the x-dimension of M got compactified (and the t-dimension remains untouched), the hypersurface of signature change $\mathcal{H}$ can be either compact or non-compact. Dependent on the alignment of $\mathcal{H}$, it will be either compact or non-compact (see Figure 15.3).

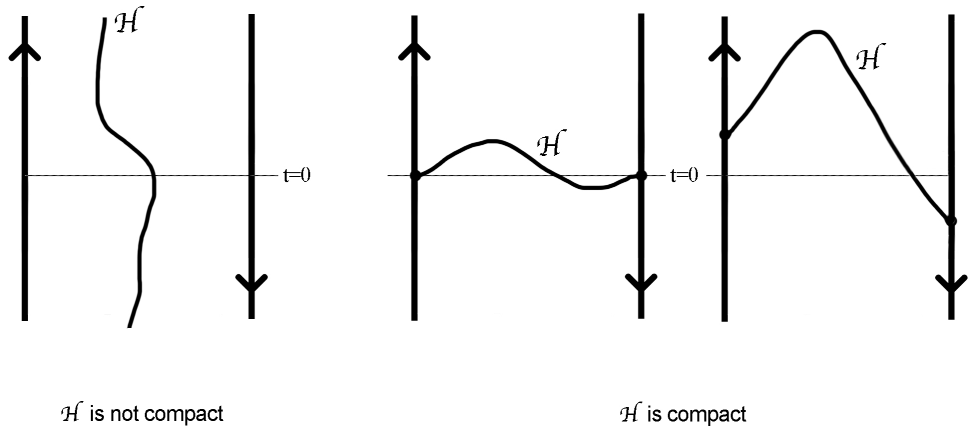

Figure 15.3: The hypersurface of signature change can be either compact or non-compact: The very left example depicts a hypersurface that is not compact. The two right examples show a hypersurface that is compact. For the latter two, it should be noted that the inclinations of the tangents at $\mathcal{H}$ must also align with the identification.

15.3 Signature-type change in compact manifold

It is a well known fact that all non-compact smooth manifolds admit a Lorentzian metric. However, this is in general no longer the case for compact manifolds.[84] We want to examine two compact 2-manifolds, namely the compact Möbius strip (which indeed admits a Lorentzian metric) and the real projective plane which does not.
In this subsection, all considered manifolds are smooth with nonempty boundary. In line with Brown's collaring theorem [12, 61], the boundary $\mathcal{H}$ has a collar neighborhood $U(\mathcal{H}) := [0, \varepsilon) \times \mathcal{H}$, with $\varepsilon > 0$. In addition, suppose there exists a diffeomorphism between the collar neighborhoods.

15.3.1 Time-orientable Möbius strip with boundary (non-simply connected M_L)

We can now consider the quotient manifold with boundary $M = ([0,1] \times [0,1])/\sim$ which is obtained by the identification $(t, 0) \sim (1-t, 1)$ of the two opposite sides of the unit square. This quotient manifold is homeomorphic to the upper half space $\mathbb{H}^2 = \{(t, x) : t \leq 0\}$ with the inherited subspace topology of $\mathbb{R}^2$, and is equipped with the rotating metric $g = -\cos(2\varphi)(dt)^2 + 2\sin(2\varphi)dtdx + \cos(2\varphi)(dx)^2$, $\varphi = \pi x$ (see Section 15.1) induced from the metric of M_∞, where the quotient map identifies $(t, 0)$ with $(1-t, 1)$.

The resulting manifold $M_c = (([0,1] \times [0,1])/\sim, g)$ is compact, not simply connected, non-orientable and time-orientable; and it has again the topology of the Möbius strip.

As a time-orientable Lorentzian manifold, the compact Möbius strip M_c admits a global non-vanishing vector field $V = (\cos\varphi)\frac{\partial}{\partial t} - (\sin\varphi)\frac{\partial}{\partial x}$. Therefore, we can refer to the transformation prescription from Section 15.2.1 in order to transform $M_c = (([0,1] \times [0,1])/\sim, g)$ into the signature-type changing Möbius strip $\tilde{M}_c = (([0,1] \times [0,1])/\sim, \tilde{g})$, where again $\tilde{g} = (f \cdot \cos^2(\varphi) - \cos(2\varphi))dt^2 + (2 - f)\sin(2\varphi)dtdx + (f \cdot \sin^2(\varphi) + \cos(2\varphi))dx^2$ and f is an arbitrary C^∞ function that again accounts for the hypersurface of signature-change $\mathcal{H} = f^{-1}(1)$.

[84] Also, compact time-oriented manifolds have a non-empty chronology violating set. This means that such manifolds contain closed timelike curves, which is considered pathological.

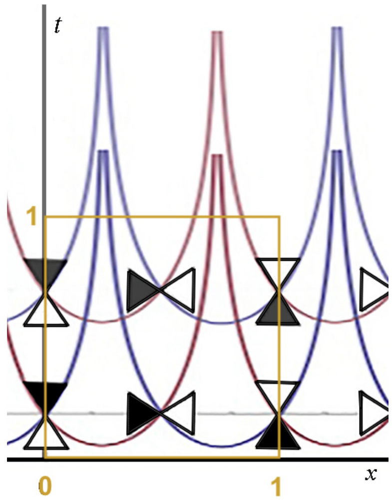

Figure 15.4: The yellow marked area represents the quotient manifold with boundary $M = ([0,1] \times [0,1])/\sim$ which is obtained by the identification $(t,0) \sim (1-t,1)$ of the two opposite sides of the unit square.

15.3.2 The Cross-Cap

Consider again the compact Möbius strip $M = ([0,1] \times [0,1])/((t,0) \sim (1-t,1))$ with nonempty boundary ∂M. Note that $M_{\frac{1}{2}} = (\{\frac{1}{2}\} \times [0,1])/((\frac{1}{2},0) \sim (\frac{1}{2},1)) \subset M$ is the center line of the Möbius strip and the boundary (a 1-dimensional manifold) is given by $\partial M = \{(0,x) \colon x \in [0,1]\} \cup \{(1,x) \colon x \in [0,1]\}$, where $(0,1) \sim (1,0)$ and $(0,0) \sim (1,1)$.

Then ∂M can be identified with S^1 which enables us to define the map $\phi \colon [0,\frac{1}{2}] \times S^1 \longrightarrow M$, such that $\phi(0,\tilde{x}) \in \partial M$ and $\phi(\frac{1}{2},\tilde{x}) \in M_{\frac{1}{2}}$. Given a point $(t,\tilde{x}) \in [0,\frac{1}{2}] \times S^1$, the point $\phi(t,\tilde{x}) \in M$ is obtained after an orthogonal displacement of length t into the Möbius strip starting from $\tilde{x} \in \partial M$. In the case at hand, it is clear that the map ϕ is continuous and defined over the entire Möbius strip and thus surjective. But because $\tilde{x}$ and $-\tilde{x} \in S^1$ are antipodal points we have $\phi(\frac{1}{2},\tilde{x}) = \phi(\frac{1}{2},-\tilde{x})$ for all $\tilde{x} \in S^1$. This yields the quotient manifold $C \coloneqq ([0,\frac{1}{2}] \times S^1)/\sim$ called crosscap where points are identified by $(\frac{1}{2},\tilde{x}) \sim (\frac{1}{2},-\tilde{x})$. Note that we can now define the boundary as $\partial M \coloneqq \phi(\{0\} \times S^1)$. Hence, M and C are homeomorphic due to the fact that after the identification of antipodal points we get a map $\tilde{\phi} \colon C \longrightarrow M$ which is continuous and bijective, and the inverse map $\tilde{\phi}^{-1}$ is also continuous.

The crosscap is closed, but compact manifolds carry a Lorentzian metric only if their Euler characteristic is zero.[85] It is a well-known fact that, unlike the closed Möbius strip, the crosscap does not admit a globally defined Lorentzian metric. This means we cannot resort to our prescription from Section 13 to create a signature-type changing metric starting with the crosscap metric. However, since the crosscap is obtained by sewing a Möbius strip to the edge of a disk, we could refer to the compact Möbius strip manifold M_c (15.3.1) and utilize its Lorentzian metric g for the Lorentzian portion of the crosscap. Then the locus of signature change should be at ∂M and the closed unit disk $\overline{\mathbb{D}}^2$ constitutes the Riemannian sector. The boundaries of the disk and the Möbius strip are both nonempty, therefore (according to Brown's collaring theorem [12, 61]) both boundaries have a collar neighborhood.

In order to elucidate the partition of the crosscap into those three areas we define the crosscap C as the *adjunction space* [61]

$$C := \overline{\mathbb{D}}^2 \cup_\psi \mathbb{M},$$

where $\overline{\mathbb{D}}^2 = \{(\theta, r) \in \mathbb{R}^2 \colon r \leq 1\}$with $\partial\overline{\mathbb{D}}^2 = \{(\theta, 1) \colon 0 \leq \theta \leq 2\pi\}$ and

$$\psi(\theta, 1) := \begin{cases} (1, \frac{\theta}{\pi}) & 0 \leq \theta \leq \pi \\ (0, \frac{\theta}{\pi} - 1) & \pi \leq \theta \leq 2\pi \end{cases} .$$

The map $\psi \colon \partial\mathbb{D}^2 \longrightarrow \partial M$ goes from the boundary circle of the disk to the boundary of the Möbius strip. Then the locus of signature change is defined by the set $\mathcal{H} := \partial M = \{(0, x) \colon x \in [0, 1]\} \cup \{(1, x) \colon x \in [0, 1]\}$ where $(0, 1) \sim (1, 0)$ and $(0, 0) \sim (1, 1)$.

If we require the signature-change to take place at $\partial M \approx S^1$ we have to pick $f(t, x)$ in the expression $\tilde{g} = (f \cdot \cos^2(\varphi) - \cos(2\varphi))dt^2 + (2 - f)\sin(2\varphi)dtdx + (f \cdot \sin^2(\varphi) + \cos(2\varphi))dx^2$ such that $\mathcal{H} = \partial M = f^{-1}(1) = \{(t, x) \colon t^2 + x^2 = 1\}$. This simply means that $\mathcal{H} = S^1$, hence without loss of generality, $f(t, x) = t^2 + x^2$ is the natural choice of a smooth function f that yields a signature-type change at the boundary ∂M of $\tilde{M}_c = (([0, 1] \times [0, 1])/ \sim, \tilde{g})$, see 15.3.1.

[85] In this context, a closed manifold is a manifold without boundary that is compact.

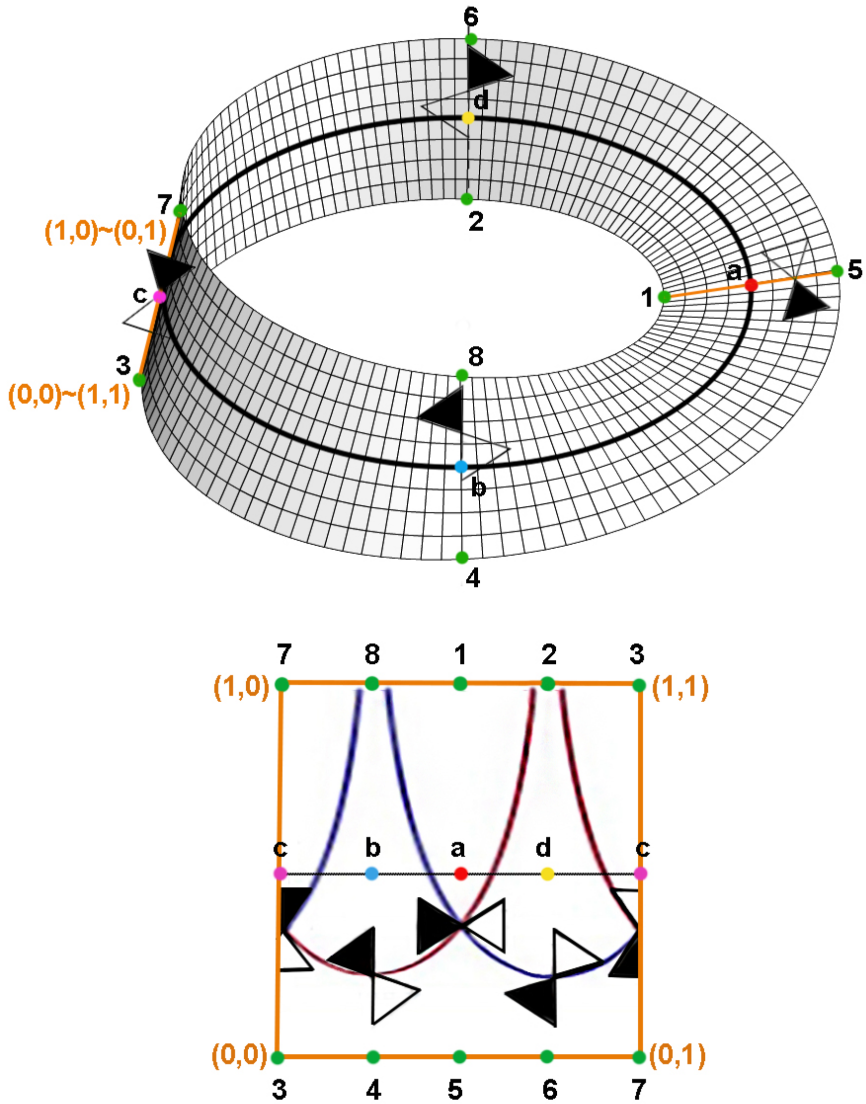

Figure 15.5: The light cone structure in the Möbius strip.

Given $f(t,x) = t^2 + x^2$, this yields

$\tilde{g} = (f \cdot \cos^2(\varphi) - \cos(2\varphi))dt^2 + (2 - f)\sin(2\varphi)dtdx + (f \cdot \sin^2(\varphi) + \cos(2\varphi))dx^2$
$= ((t^2 + x^2) \cdot \cos^2(\pi x) - \cos(2\pi x))dt^2 + (2 - t^2 - x^2))\sin(2\pi x)dtdx + ((t^2 + x^2) \cdot \sin^2(\pi x) + \cos(2\pi x))dx^2$

$= ((t^2+x^2)\cdot\cos^2(\pi x) - \cos(2\pi x))dt^2 - 2(t^2+x^2-2)\sin(\pi x)\cos(\pi x)dtdx + ((t^2+x^2)\cdot\sin^2(\pi x)+\cos(2\pi x))dx^2$,

with the associated matrix representation

$$[\tilde{g}] = \begin{pmatrix} (t^2+x^2)\cdot\cos^2(\pi x) - \cos(2\pi x) & -(t^2+x^2-2)\sin(\pi x)\cos(\pi x) \\ -(t^2+x^2-2)\sin(\pi x)\cos(\pi x) & (t^2+x^2)\cdot\sin^2(\pi x)+\cos(2\pi x) \end{pmatrix}.$$

This metric reaches its canonical form $g = -1(dt)^2 + 1(dx)^2$ only for $x = t = 0$. The determinant $\triangle = \det([\tilde{g}]) = t^2 + x^2 - 1$ reveals that the metric $\tilde{g}$ is Lorentzian for $t^2 + x^2 = f(t,x) < 1$ and Riemannian for $t^2 + x^2 = f(t,x) > 1$. Moreover, $d(\det([\tilde{g}])) = 2tdt + 2xdx = df \neq 0$ for any $(t,x) \in \mathcal{H}$.

The causal structure can be elucidated by means of

$0 = ((t^2+x^2)\cdot\cos^2(\pi x) - \cos(2\pi x))(\frac{dt}{dx})^2 - 2(t^2+x^2-2)\sin(\pi x)\cos(\pi x)\frac{dt}{dx} + ((t^2+x^2)\cdot\sin^2(\pi x)+\cos(2\pi x))$

$\Longleftrightarrow \frac{dt}{dx} = \frac{(t^2+x^2-2)\sin(\pi x)\cos(\pi x)\pm\sqrt{-t^2-x^2+1}}{((t^2+x^2)\cdot\cos^2(\pi x)-\cos(2\pi x))}$.

The causal structure as well as the fact that the determinant $\triangle = \det([\tilde{g}]) = t^2 + x^2 - 1 < 0 \Longleftrightarrow t^2 + x^2 < 1$ reveal that the Lorentzian portion of the metric $\tilde{g}$ is located on the disk $\overline{\mathbb{D}}^2$.[86] This is a contradiction!

Even if we instead consider the signature transformation

$$\tilde{g} = (-t(dt)^2 + (dx)^2) + f\cdot\left(g(\frac{\partial}{\partial t},\boldsymbol{\cdot})\otimes g(\frac{\partial}{\partial t},\boldsymbol{\cdot})\right) = (f-1)dt^2 + dx^2,$$

with the global, non-vanishing, timelike vector field $V = \frac{\partial}{\partial t}$ with respect to $g = -t(dt)^2 + (dx)^2$, then with $f(t,x) = t^2 + x^2$, this yields

$$\tilde{g} = (f-1)dt^2 + dx^2 = (t^2+x^2-1)dt^2 + dx^2.$$

The causal structure as well as the fact that the determinant yields $\triangle = \det([\tilde{g}]) = t^2 + x^2 - 1 < 0 \Longleftrightarrow t^2 + x^2 < 1$ reveal that the Lorentzian portion of the metric $\tilde{g}$ is located on the disk $\overline{\mathbb{D}}^2$. This is again a contradiction!

[86]If A is an $n \times n$ square matrix and n is odd, then $\det(-A) = -\det(A)$. However, since $\tilde{g}$ is a 2×2 matrix which is not odd, a slight modification of the metric in oder to get $-\det([\tilde{g}]) = -(t^2 + x^2 - 1) < 0 \Longleftrightarrow 1 < t^2 + x^2$ is also not possible.

These counter examples illustrate that $\tilde{g}$ is not of the class "transverse, type-changing metric with a transverse radical" because the radical is both, transverse and tangent. Hence, there is no *transformation function* f that would produce a signature-type change at ∂M with only a transverse and orthogonal radical. Moreover, we cannot even locally apply the transformation prescription from Section 13 in a reasonably way as the obtained $\tilde{g}$ is not a transverse, type-changing metric with a transverse radical.

15.3.3 Non-time orientable Cross-Cap

Here we are going to construct a signature-type changing metric defined on the real projective plane $\mathbb{R}P^2$. Let $\overline{\mathbb{M}} = ([-\sqrt{2},\sqrt{2}] \times [-\sqrt{2},\sqrt{2}])/((t,-\sqrt{2}) \sim (-t,\sqrt{2}))$ be again the Möbius strip with boundary $\partial\mathbb{M} = \{(-\sqrt{2},x)\colon x \in [-\sqrt{2},\sqrt{2}]\} \cup \{(\sqrt{2},x)\colon x \in [-\sqrt{2},\sqrt{2}]\}$ where $(\sqrt{2},-\sqrt{2}) \sim (-\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2}) \sim (\sqrt{2},\sqrt{2})$, and $\mathbb{M}_0 = (\{0\} \times [-\sqrt{2},\sqrt{2}])/((0,-\sqrt{2}) \sim (0,\sqrt{2})) \subset \mathbb{M}$ the center line. This quotient manifold is equipped with the subspace topology of $\mathbb{R}^2$ and Cartesian coordinates (t,x). Note that the punctured projective plane is topologically equivalent to the <u>open</u> Möbius strip $\mathbb{M} \cong \mathbb{R}P^2 \setminus \{(t,x)\colon t^2+x^2 \leq 1\} = \mathbb{R}P^2 \setminus \overline{\mathbb{D}}^2$. On $\mathbb{R}P^2 \setminus \overline{\mathbb{D}}^2$,[87] with respect to the canonical coordinates $(t,x) \in ((-\sqrt{2},\sqrt{2}) \times [-\sqrt{2},\sqrt{2}]) \setminus \overline{\mathbb{D}}^2$, the Lorentzian 2-manifold is then defined by the smooth metric

$$g = (1-t^2)(dt)^2 + 2txdtdx + (1-x^2)(dx)^2,$$

where the quotient map identifies $(t,-\sqrt{2}) \sim (-t,\sqrt{2})$ and $(-\sqrt{2},x) \sim (\sqrt{2},-x)$.

The associated determinant $\triangle = \det([g_{\mu\nu}]) = (1-x^2)(1-t^2)-(tx)^2 = 1-t^2-x^2 <$

[87] Recall that the real projective plane $\mathbb{R}P^2$ is given as the unit sphere S^2 in the 3-dimensional Euclidean space with identification of antipodes – in other words it is the quotient space obtained from S^2. The antipodal map $A\colon S^2 \to S^2$ is an isometry of S^2 given by $A(p) = -p$. Furthermore, one can make use of the natural projection $\pi\colon S^2 \to \mathbb{R}P^2$ as quotient map that identifies all pairs of antipodal points. Observe that T_pS^2 and $T_{A(p)}S^2 = T_{-p}S^2$ are then identified as well.

As the sphere cannot carry a globally defined Lorentzian metric, there is no Lorentzian metric on the projective space either. However, we can utilize the quotient map π to introduce a Riemannian metric on $\mathbb{R}P^2$ in the following way: $<(d\pi)_p(v),(d\pi)_p(w)>_{\pi(p)} = <v,w>_p$, where $p \in S^2$, $\pi(p) \in \mathbb{R}P^2$. This also means that π is a local isometry.

Alternatively, we can assume further that the general metric is given by $ds^2 = g(p)d\rho^2$, and $d\rho^2$ being the line element of the unit sphere taken from the embedding in the Euclidean space; $g(p) > 0$ for any point p on the manifold.

$0 \Longleftrightarrow t^2 + x^2 > 1$ is non-vanishing for $(t, x) \in ((-\sqrt{2}, \sqrt{2}) \times [-\sqrt{2}, \sqrt{2}]) \setminus \overline{\mathbb{D}}^2$, and therefore is $(\mathbb{R}P^2 \setminus \overline{\mathbb{D}}^2, g)$ Lorentzian. Notice that the metric g reaches its canonical form $g = -(dt)^2 + (dx)^2$ when $x = 0$ and $t = \pm\sqrt{2}$.[88]

Taking into consideration that the real projective plane is topologically equivalent to the crosscap (which is obtained by sewing a Möbius strip to the edge of a disk) we are going to construct a signature-type changing manifold defined on the crosscap $C := \overline{\mathbb{D}}^2 \cup_h \overline{\mathbb{M}} = \overline{\mathbb{D}}^2 \cup_h (\mathbb{M} \cup \partial\mathbb{M})$, where h is a diffeomorphism between the collar neighborhoods, and then extend the metric across the boundary $\partial\mathbb{M} = \{(-\sqrt{2}, x) \colon x \in [-\sqrt{2}, \sqrt{2}]\} \cup \{(\sqrt{2}, x) \colon x \in [-\sqrt{2}, \sqrt{2}]\}$. Hence, after introducing a metric tensor, the adjunction space $\overline{\mathbb{D}}^2 \cup_h \overline{\mathbb{M}}$ becomes a semi-Riemannian manifold whose underlying topological structure is the projective plane.

Let us start with the square-shaped topological state $[-\sqrt{2}, \sqrt{2}] \times [-\sqrt{2}, \sqrt{2}]$. Then glue the top edge to the bottom edge in antiparallel sense and also glue the right edge to the left edge in antiparallel sense similarly:

$$(t, -\sqrt{2}) \sim (-t, \sqrt{2}) \text{ for } -\sqrt{2} \leq t \leq \sqrt{2},$$
$$(-\sqrt{2}, x) \sim (\sqrt{2}, -x) \text{ for } -\sqrt{2} \leq x \leq \sqrt{2}.$$

This yields the non-orientable, compact real projective plane $\mathbb{R}P^2$ that has no boundary. By a continuous deformation we can turn $\mathbb{R}P^2$ into the crosscap $C := \overline{\mathbb{D}}^2 \cup_h \overline{\mathbb{M}} = \overline{\mathbb{D}}^2 \cup_h (\mathbb{M} \cup \partial\mathbb{M})$, with $\partial\mathbb{M} = \{(t, x) \colon t^2 + x^2 = 1\}$.

We denote by (C, g) the resulting 2-dimensional signature-type changing manifold defined as $C = \overline{\mathbb{D}}^2 \cup_h (\mathbb{M} \cup \partial\mathbb{M})$ in the $t - x$-plane with the following metric:

$$g = (1 - t^2)(dt)^2 + 2txdtdx + (1 - x^2)(dx)^2.$$

[88]Choosing $x = \pm\sqrt{2}$ and $t = 0$ has the effect of interchanging the role of space and of time, and we get $g = (dt)^2 - (dx)^2$.

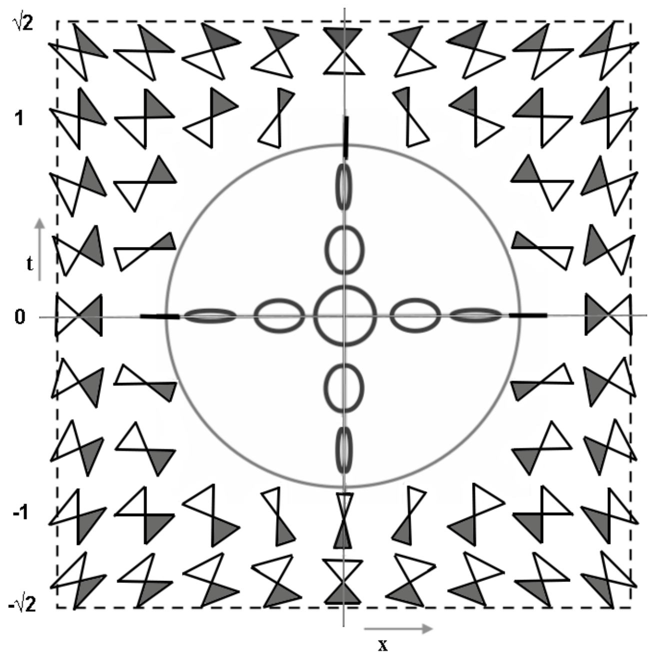

Figure 15.6: Light cone structure for the metric $g = (1-t^2)(dt)^2 + 2txdtdx + (1-x^2)(dx)^2$.

The determinant becomes degenerate for $(t, x) \in \partial\mathbb{M} \cong \mathbb{S}^1 \subset C$ because of $\triangle = \det([g_{\mu\nu}]) = (1-x^2)(1-t^2) - (tx)^2 = 0 \iff t^2 + x^2 = 1$. This implies that $\mathcal{H} := \{(t, x) \in C \colon t^2 + x^2 = 1\} \subset C$ is the compact locus of signature change. Hence, $C \setminus \mathcal{H} = \mathbb{D}^2 \cup \mathbb{M}$, where $\mathbb{D}^2$ has Riemannian signature and $\mathbb{M}$ has Lorentzian signature.

The differential

$$d\triangle = d(\det([g_{\mu\nu}])) = -2tdt - 2xdx \begin{cases} = 0 & if\, x = t = 0 \\ \neq 0 & otherwise \end{cases}$$

only vanishes in the origin and is otherwise non-zero on $\mathcal{H}$.

Furthermore, the calculation of the radical Rad_q, for $q \in \mathcal{H}$ reveals that Rad_q is transverse as well as tangent, and has $\dim(Rad_q) = 1$:

We are looking for all vectors $w = (w_1, w_2)^T \in T_qM$ such that $g(w, \cdot) = 0$. Hence,

$$
\begin{aligned}
&\begin{pmatrix} 1-t^2 & tx \\ tx & 1-x^2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \\
\Longleftrightarrow &\begin{cases} (1-t^2)w_1 + (tx)w_2 & = 0 \\ (tx)w_1 + (1-x^2)w_2 & = 0 \end{cases} \\
\Longleftrightarrow &\ (1-t^2+tx)w_1 + (1-x^2+tx)w_2 = 0
\end{aligned}
$$

$\Longleftrightarrow (1+t(x-t))w_1 + (1+x(t-x))w_2 = 0$, and since we have $\mathcal{H} := \{(t,x) \in C \colon t^2+x^2 = 1\}$ we get

$$
\begin{aligned}
&((t^2+x^2)+t(x-t))w_1 + ((t^2+x^2)+x(t-x))w_2 = 0 \\
\Longleftrightarrow &\ (x(t+x))w_1 + (t(t+x))w_2 = 0 \\
\Longleftrightarrow &\ (x+t)(xw_1+tw_2) = 0 \\
\Longleftrightarrow &\ (x=-t) \vee (xw_1+tw_2=0), \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
t &= \pm\sqrt{1-x^2}, \\
x &= \pm\sqrt{1-t^2},
\end{aligned}
$$

and this yields

$$
\begin{aligned}
&\left(w \in \left\{\alpha \begin{pmatrix} 1 \\ \pm\frac{\sqrt{1-t^2}}{t} \end{pmatrix} : \alpha \in \mathbb{R} \wedge (t \neq 0)\right\}\right) \vee \left(w \in \left\{\beta \begin{pmatrix} \pm\frac{\sqrt{1-x^2}}{x} \\ 1 \end{pmatrix} : \beta \in \mathbb{R} \wedge (x \neq 0)\right\}\right) \\
\Longleftrightarrow &\ w \in \left\{\alpha \begin{pmatrix} 1 \\ \pm\frac{\sqrt{1-t^2}}{t} \end{pmatrix} : \alpha \in \mathbb{R} \wedge (t \neq 0)\right\} \cup \left\{\beta \begin{pmatrix} \pm\frac{\sqrt{1-x^2}}{x} \\ 1 \end{pmatrix} : \beta \in \mathbb{R} \wedge (x \neq 0)\right\}.
\end{aligned}
$$

Note that these two eigenvectors are linearly dependent, so we need just one of these to form a basis

$$
[b] = \operatorname{span}\left\{\begin{pmatrix} 1 \\ \frac{\sqrt{1-t^2}}{t} \end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix} \frac{\sqrt{1-x^2}}{x} \\ 1 \end{pmatrix}\right\}
$$

of Rad_q.

The radical Rad_q is obviously transverse for $(t=1) \vee (x=1)$.

The manifold (C,g) is compact, and neither pseudo-time orientable nor orientable. Time begins at the surface of transition $\mathcal{H}$, given by $t^2+x^2=1$, but obviously the spacetime does not begin there.

For example (see Figure 15.7) a pseudo-timelike curve γ (blue) passes through r on $\mathcal{H}$, then goes through the disk which is the Riemannian regime, and finally re-emerges in the Lorentzian regime through s on $\mathcal{H}$. This curve is future-directed. Whereas the curve $\tilde{\gamma}$ (red) enters the Riemannian regime at $p \in \mathcal{H}$ and experiences a time reversal upon re-entering the Lorentzian regime at $q \in \mathcal{H}$. The latter curve is not future-directed.

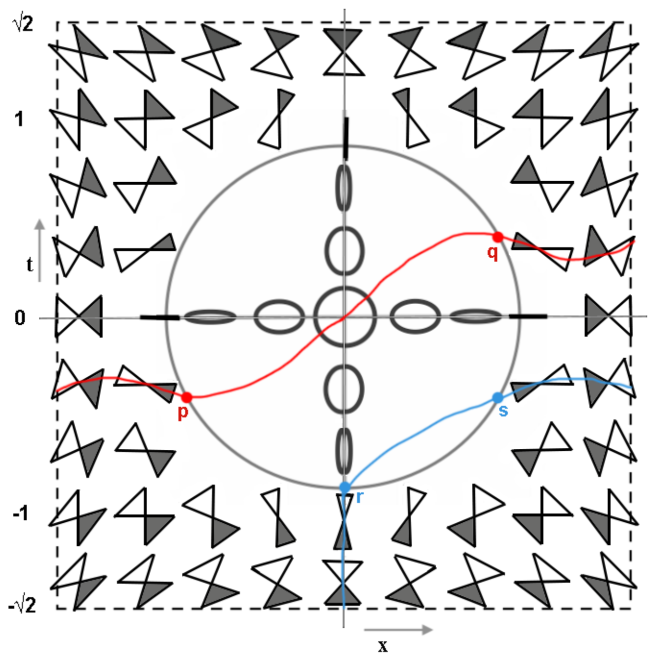

Figure 15.7: Pseudo-timelike curves pass through $\mathcal{H}$, then go through the disk which is the Riemannian regime, and finally re-emerge in the Lorentzian regime through $\mathcal{H}$ again.

The metric $g = (1-t^2)(dt)^2 + 2txdtdx + (1-x^2)(dx)^2$ is not a transverse, type-changing metric with a transverse radical because the radical is both, transverse and tangent. According to this, we did not introduce a transformation function f to obtain the said metric from a given Lorentzian metric. The transformation prescription 13 cannot get applied here.

Remark 15.2. The 2-dimensional versions of the "no-boundary proposal" spacetimes (which are obtained by cutting a sphere along its equator and joining it to the corresponding half of a de Sitter space) have the property that the radical is always transverse at the locus of signature change. Nevertheless we also cannot get these types of manifolds via the Transformation Prescription 13 from a Lorentzian manifold (M, g).

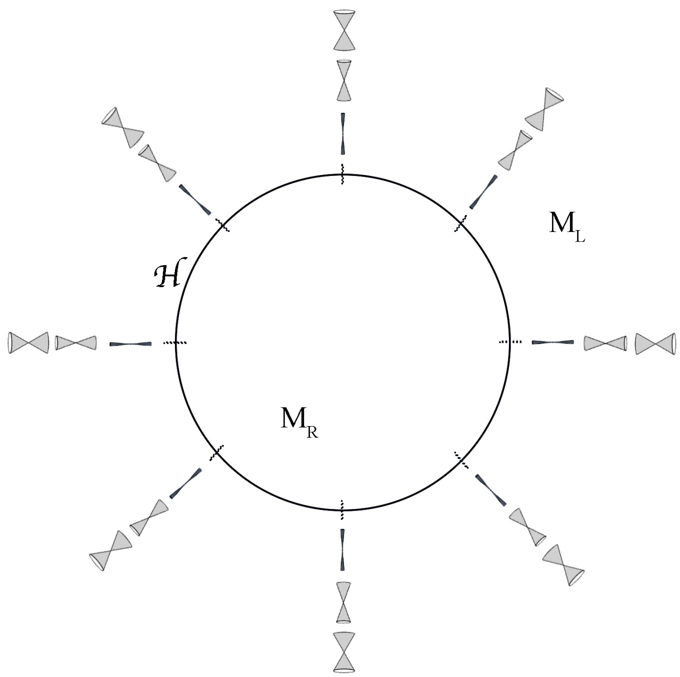

Figure 15.8: The causal structure of the 2-dimensional "no boundary" proposal spacetime.

16 Chronology violating pseudo-timelike loops

In Section 4 we introduced the notion of closed pseudo-timelike curves on a signature-type changing background, and we have shown how they have to be defined to ensure that the concept of causality still makes sense.

16.1 Local pseudo-timelike loops

In this subsection we are going to reveal the non-well-behaved nature of transverse, signature-type changing, n-dimensional manifolds with a transverse radical. In a sufficiently small region near the junction of signature change, these manifolds exhibit local anomalies. Specifically, each point on the junction gives rise to the existence of closed time-reversing loops, challenging conventional notions of temporal consistency.

Theorem 16.1. *Let $(M, \tilde{g})$ be a transverse, signature-type changing, n-dimensional ($n \geq 2$) manifold with a transverse radical. Then in each neighborhood of each point $q \in \mathcal{H}$ there always exists a pseudo-timelike loop.*

Proof. Let $\tilde{g} = -t(dt)^2 + \tilde{g}_{jk}(t, x^1, \dots, x^{n-1})dx^i dx^k,\ j,k \in \{1, \dots, n-1\}$, be a transverse, signature-type changing metric with respect to a radical-adapted Gauss-like coordinate patch (U_φ, φ) with $U_\varphi \cap \mathcal{H} \neq \emptyset$.[89] Choose smooth coordinates $(t_0, x_0^1, \dots, x_0^{n-1})$ with $t_0 > 0$ and $\xi_0 > 0$, such that

$$C_0 := [0, t_0] \times B_{\xi_0}^{n-1} = [0, t_0] \times \{x \in \mathbb{R}^{n-1} \mid \sum_{k=1}^{n-1} (x^k)^2 \leq \xi_0^2\} \subset \mathbb{R}^n$$

is contained in the domain of the coordinate chart (open neighborhood) U_φ. Then

$$C_0 \times \mathbb{S}^{n-2} = C_0 \times \{v \in \mathbb{R}^{n-1} \mid \sum_{k=1}^{n-1} (v^k)^2 = 1\}$$

as a product of two compact sets is again compact.

Next, consider the function

[89] This is, U_φ is sufficiently small to be expressed in the adapted radical-adapted Gauss-like coordinate system $\xi(U_\varphi)$.

$$\tilde{G}\colon C_0 \times \mathbb{S}^{n-2} \longrightarrow \mathbb{R},$$

$$(t, x^1, \ldots, x^{n-1}, v^1, \ldots, v^{n-1}) \mapsto \tilde{g}_{jk}(t, x^1, \ldots, x^{n-1})v^j v^k.$$

As $\tilde{G}$ is a smooth function defined on the compact domain $C_0 \times \mathbb{S}^{n-2}$, by the Extreme Value Theorem it has an absolute minimum G_0. Hence, on (U_φ, φ) we can uniquely define $\tilde{g}_0 = -t(dt)^2 + G_0\delta_{jk}dx^j dx^k$, $j, k \in \{1, \ldots, n-1\}$.

By this definition, for all nonzero lightlike vectors $X \in T_pM$, $p \in C_0$ with respect to $\tilde{g}_0$ we have $\tilde{g}_0 = -t(X^0)^2 + G_0\delta_{jk}X^j X^k = 0 \Longleftrightarrow -t(X^0)^2 = -G_0\delta_{jk}X^j X^k$, then

$$\tilde{g}(X, X) = -t(X^0)^2 + \tilde{g}_{jk}(t, x^1, \ldots, x^{n-1})X^j X^k$$

$$= -G_0\delta_{jk}X^j X^k + \tilde{g}_{jk}(t, x^1, \ldots, x^{n-1})X^j X^k$$

$$= \delta_{jk}X^j X^k \cdot (-G_0 + \tilde{g}_{rs}(t, x^1, \ldots, x^{n-1})\frac{X^r}{\sqrt{\delta_{ab}X^a X^b}}\frac{X^s}{\sqrt{\delta_{cd}X^c X^d}}) \geq 0.$$

Clearly, $\tilde{g}(X, X) \geq 0$ because $G_0 > 0$ per definition and $\delta_{jk}X^j X^k = \frac{t(X^0)^2}{G_0} \geq 0$. Therefore, the vector $X \in T_pM$, $p \in C_0$ is not timelike with respect to $\tilde{g}$. This means, within C_0 the $\tilde{g}$-light cones always reside inside of the $\tilde{g}_0$-light cones, i.e. $\tilde{g} \leq \tilde{g}_0$ in C_0. The cull cones of $\tilde{g}_0$ are more opened out than those of the metric $\tilde{g}$. Denote $p_0 \in C_0$ by $(t(p_0), x^1(p_0), \ldots, x^{n-1}(p_0)) = (t_0, x_0^1, \ldots, x_0^{n-1})$.

As $(M, \tilde{g})$ is an n-dimensional manifold for which in the neighborhood of $\mathcal{H}$ radical-adapted Gauss-like coordinates exist, we can single out the time coordinate that defines the real-valued and smooth absolute time function t whose gradient in M_L is everywhere non-zero and timelike. Hence, $(M, \tilde{g}) \mid_{U_\varphi}$ can be decomposed into spacelike hypersurfaces $\{(U_\varphi)_{t_i}\}$ of dimension $(n-1)$ which are specified as the level sets $(U_\varphi)_{t_i} = t^{-1}(t_i)$ of the time function.[90] The restriction $(\tilde{g}_0)_{t_i}$ of the metric $\tilde{g}_0$ to each spacelike slice makes the pair $((U_\varphi)_{t_i}, (\tilde{g}_0)_{t_i})$ a Riemannian manifold.

[90] This collection of space-like slices $\{(U_\varphi)_t\}$ should be thought of as a foliation of U_φ into disjoint $(n-1)$-dimensional Riemannian manifolds.

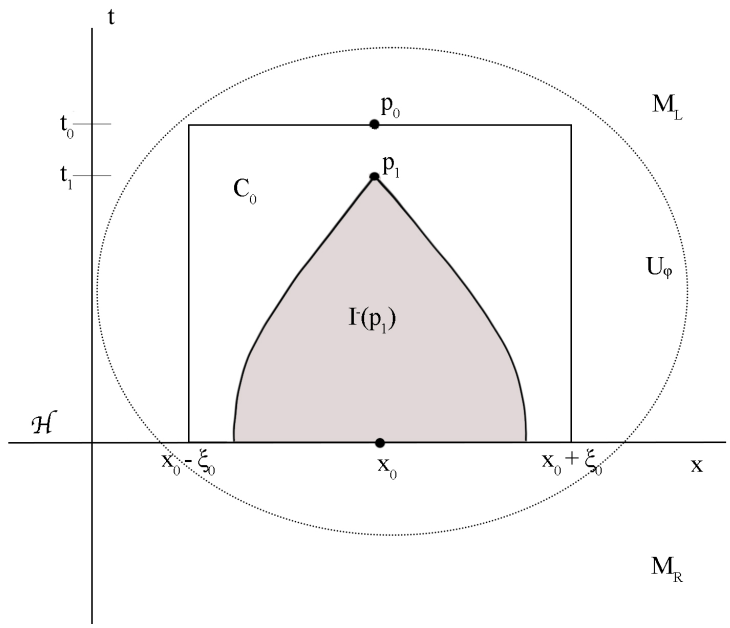

Figure 16.1: The chronological past $I^-(p_1)$ of a point $p_1 \in U_\varphi$.

For a lightlike curve $\alpha(t) \colon I \longrightarrow U_\varphi$ with starting point p_0, we have $\delta_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} > 0$ for each slice $(U_\varphi)_{t_i}$ with $t \neq 0$. Therefore we may choose σ as arc length parameter in $B^{n-1}_{\xi_0}$. In other words, lightlike curves with starting point p_0 can be parametrized in $B^{n-1}_{\xi_0}$ by arc length σ,[91] that is $(\tilde{g}_0)_t(\dot{\alpha}(\sigma), \dot{\alpha}(\sigma)) = \delta_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} = 1$, $\forall\, \sigma \in I$, where I is some interval in $\mathbb{R}$. Consequently we get

$$0 = \tilde{g}_0(\dot{\alpha}(\sigma), \dot{\alpha}(\sigma)) = -t(\dot{\alpha}^0)^2 + G_0 \delta_{ik} \dot{\alpha}^j \dot{\alpha}^k$$

$$= -t(\frac{dt}{d\sigma})^2 + G_0 \underbrace{\delta_{ik} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma}}_{1} = -t(\frac{dt}{d\sigma})^2 + G_0,$$

and this implies

$$\frac{d\sigma}{dt} = \pm \sqrt{\frac{t}{G_0}} \Longrightarrow \sigma(t) = \pm \int \sqrt{\frac{t}{G_0}} dt = \pm \frac{2}{3} t \sqrt{\frac{t}{G_0}} + const.$$

[91]More precisely, σ can be considered as arc length in terms of some auxiliary Riemannian metrics, each defined on a hypersurface with $t = const$.

Since σ is given as a function of t, it represents the arc length from the starting point at $t(p_0) = t_0$ to $t(0) = 0$. Then past-directed $\tilde{g}_0$-lightlike curves emanating from p_0 reach the hypersurface at $t = 0$ after passing through the arc length distance

$$\triangle\sigma = \pm \int_0^{t_0} \sqrt{\frac{t}{G_0}} dt = \pm\frac{2}{3} t_0 \sqrt{\frac{t_0}{G_0}} + const. = \pm\frac{2}{3}\sqrt{\frac{t_0^3}{G_0}} + const$$

along the said section of the curve from the fixed starting point p_0.

Provided this arc length distance satisfies $\triangle\sigma \leq \xi_0$, then the past-directed lightlike curves $\alpha(t)$ (emanating from p_0) reach the hypersurface at $t = 0$ while remaining within C_0. Accordingly this is also the case for $\tilde{g}-$lightlike curves emanating from p_0. Conversely, if $\triangle\sigma > \xi_0$ then there exist past-directed $\tilde{g}_0$-lightlike curves emanating from p_0 that reach the hypersurface outside of C_0.
In this case, we have $\triangle\sigma = \frac{2}{3}\sqrt{\frac{t_0^3}{G_0}} > \xi_0 \Longleftrightarrow t_0 > \sqrt[3]{\frac{9}{4}\xi_0^2 \cdot G_0}$ and we must adjust the new starting point $p_1 = (t_1, x_0)$ accordingly by setting $t_1 \leq \sqrt[3]{\frac{9}{4}\xi_0^2 \cdot G_0} < t_0$. Thereby we make sure that all past-directed $\tilde{g}$-lightlike curves emanating from p_1 hit the hypersurface $\mathcal{H}$ without leaving C_0. That is $\overline{I^-_{\tilde{g}}(p_1)} \subset C_0 \subset U_\varphi \subset M$, where $\overline{I^-_{\tilde{g}}(p_1)}$ is the $\tilde{g}$-*chronological past* of the event $p_1 \in M_L$, restricted to $M_L \cup \mathcal{H}$.

It now suffices to connect two of such points $\hat{x}_1, \hat{x}_2 \in \overline{I^-_{\tilde{g}}(p_0)} \cap \mathcal{H}$ (or, if need be $\overline{I^-_{\tilde{g}}(p_1)} \cap \mathcal{H}$) in an arbitrary fashion within the Riemannian sector M_R. By what a pseudo-timelike loop gets generated, if U_φ was chosen small enough. □

Summarized, for each neighborhood $U(q)$ that admits radical-adapted Gauss-like coordinates $\xi = (t, \hat{x}) = (t, x^1, \ldots, x^{n-1})$ centered at some $q \in \mathcal{H}$, and $U(q) \cap \mathcal{H} \neq \emptyset$, we are able to pick a point $p_0 \in U(q)$ and an associated compact set $C_0 \subset U(q)$. For the metric $\tilde{g}$ there exists a corresponding uniquely defined metric $\tilde{g}_0$ with $\tilde{g} \leq \tilde{g}_0$ within C_0. Then we must distinguish between two cases, that is

i) with respect to the metric $\tilde{g}_0$ we have $I_0^-(p_0) \subset C_0$, then also $I^-(p_0) \subset C_0$ with respect to $\tilde{g}$,

ii) with respect to the metric $\tilde{g}_0$ we have the situation $I_0^-(p_0) \nsubseteq C_0$, then there exists a point $p_1 = (t_1, x_0) \in C_0 \setminus \mathcal{H}$ with $t_1 < t_0$, such that $I_0^-(p_1) \subset C_0$, hence also $I^-(p_1) \subset C_0$ with respect to $\tilde{g}$.

Thus, for any point $q \in \mathcal{H}$ we can find a sufficiently small neighborhood $\tilde{U} \subset U(q)$ containing a point $p \in M_L$, such that all past-directed, causal curves emanating from that point, reach the hypersurface within a sufficiently small set C_0.

Corollary 16.2. *Let $(M, \tilde{g})$ be a transverse, signature-type changing, n-dimensional manifold with a transverse radical. Then in each neighborhood of each point $q \in \mathcal{H}$ there always exists a pseudo-lightlike curve.*

Corollary 16.3. *A transverse, signature-type changing manifold $(M, \tilde{g})$ with a transverse radical has always time-reversing pseudo-timelike loops.*

As a matter of course, in the Lorentizan region the tangent space at each point is isometric to Minkowski space which is time orientable. Hence, a Lorentzian manifold is always infinitesimally time- and space-orientable, and a continuous designation of future-directed and past-directed for non-spacelike vectors can be made.[92]
Having said that, the infinitesimal properties of a manifold with a signature change exhibit similarities to those of a Lorentzian manifold within the Lorentzian sector. However, when examining the Riemannian sector and the hypersurface, specific distinctions arise. The Riemannian sector and the hypersurface are not infinitesimally modelable by a Minkowski space. While the Riemannian sector reveals an absence of a meaningful differentiation between past- and future-directed vectors. Conversely, on the hypersurface, one has the flexibility to make arbitrary assignments of such distinctions at the infinitesimal level. If one now determines on the hypersurface whether the direction towards the Lorentzian sector is the future or past direction, it is not only a reference to the tangent space at a point. Rather, it is a local consideration.

In the context of local considerations, in a Lorentzian manifold the existence of a timelike loop that flips its time orientation (i.e. the timelike tangent vector switches between the two designated components of the light cone) is a sufficient condition for the absence of time orientability. Based on the previous Theorem 16.1, this is also true for a transverse, signature-type changing manifold $(M, \tilde{g})$ with a transverse radical:

[92] In case the Lorentzian manifold is time-orientable, a continuous designation of future-directed and past-directed for non-spacelike vectors can be made allover.

As we have proved above, through each point on the hypersurface $\mathcal{H}$ we have locally a closed time-reversing loop. That is, there always exists a closed pseudo-timelike path in M around which the direction of time reverses, and along which a consistent designation of future-directed and past-directed vectors cannot be defined.

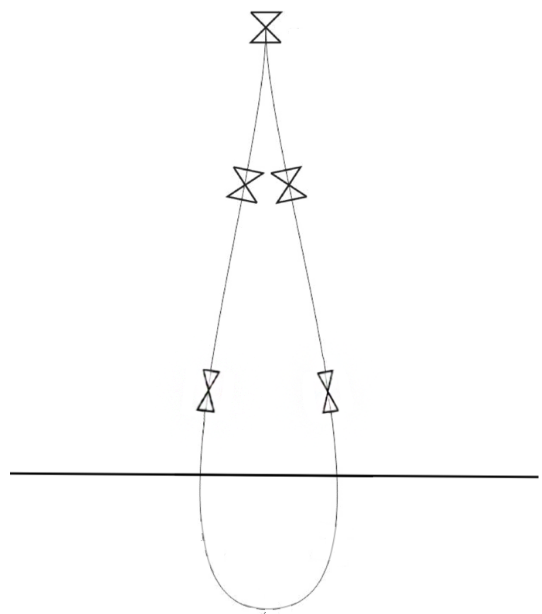

Figure 16.2: A closed time-reversing loop.

An observer in the region M_L near $\mathcal{H}$ perceives these locally closed time-reversing loops as the creation of a particle and an antiparticle at two different points $\hat{q}, q \in \mathcal{H}$.[93] This could be taken as an object entering the Riemannian region, then resurfacing in the Lorentzian region and proceeding to move backwards in time.

So in a transverse, signature-type changing manifold $(M, \tilde{g})$, the hypersurface with its time-reversing loops could be tantamount to a region of particle-antiparticle

[93]Such locally closed time-reversing loops around $\mathcal{H}$ obviously do not satisfy the causal relations $\ll$ as introduced above.

origination incidents. Moreover, Hadley [39] shows for Lorentzian spacetimes that a failure of time-orientability of a spacetime region is indistinguishable from a particle-antiparticle annihilation event. These are then considered equivalent descriptions of the same phenomena. It would be interesting to explore how this interpretation can be carried over to signature-type changing manifolds.
For fields take the conjugate $\psi_t^A = e^{-i\hat{H}t}\psi^*$ of $\psi_t = e^{i\hat{H}t}\psi$: The unitary temporal evolution of the field operator for antiparticles arises from the temporal evolution of the field operator for particles by applying the same Hamiltonian operator to the adjoint field operator under time reversal.

Some literature [38] points to the idea that concepts in quantum field theory are predicated on acausal properties derived from general relativity. In this context, Blum et al. [8] stress the importance of the CPT theorem (quoting verbatim):

> "CPT theorem is the statement that nothing would change - nobody would notice and the predictions of physics would not be altered − if we simultaneously replace particles by antiparticles and vice versa. Replace everything by its mirror image or more exactly: exchange left and right, up and down, and front and back, and reverse the flow of time. We call this simultaneous transformation CPT, where C stands for Charge Conjugation (exchanging particles and antiparticles), P stands for parity (mirroring), and T stands for time reversal."

16.2 Global pseudo-timelike loops

The existence of such pseudo-timelike curves locally near the hypersurface that loop back to themselves, gives rise to the question whether this type of curves also occur globally. We want to elucidate this question in the following.[94]

Definition 16.4. (Stably causal) [67] A connected time-orientable Lorentzian manifold (M, g) is said to be *stably causal* if there exists a nowhere-vanishing timelike vector field V_a such that the Lorentzian metric on M given by $g' := g_{ab} - V_a V_b$ admits no closed timelike curves. In other words, if (M, g) is stably

[94] A spacetime is a Lorentzian manifold that models space and time in general relativity and physics. This is conventionally formalized by saying that a spacetime is a smooth connected time-orientable Lorentzian manifold (M, g) with $\dim M = 4$. But in what follows we want to study the n-dimensional $(n \geq 2)$ case.

causal then, for some timelike V_a, the metric $g' := g_{ab} - V_a V_b$ on M is causal.[95]

Lemma 16.5. [75] *Stable causality is the necessary and sufficient condition for the existence of a smooth global time function, i.e. a differentiable map $T \colon M \to \mathbb{R}$ such that whenever $p << q \implies T(p) < T(q)$.*

Definition 16.6. (Globally hyperbolic) [7, 43] A connected time-orientable Lorentzian manifold (M, g) is called *globally hyperbolic* if and only if it is diamond-compact and causal, i.e., $p \notin J^+(p)\, \forall p \in M$.[96]

An equivalent condition for global hyperbolicity is as follows [31].

Definition 16.7. A connected time-orientable Lorentzian manifold (M, g) is called *globally hyperbolic* if and only if M contains a Cauchy surface. A Cauchy hypersurface in M is a subset S that is intersected exactly once by every inextendible timelike curve in M.[97]

In 2003, Bernal and Sánchez [6] showed that any globally hyperbolic Lorentzian manifold M admits a smooth spacelike Cauchy hypersurface S, and thus is diffeomorphic to the product of this Cauchy surface with $\mathbb{R}$, i.e. M splits topologically as the product $\mathbb{R} \times S$. Specifically, a globally hyperbolic manifold is foliated by Cauchy surfaces.

Remark 16.8. If M is a smooth, connected time-orientable Lorentzian manifold with boundary, then we say it is globally hyperbolic if its interior is globally hyperbolic.

[95] A partial ordering $<$ is defined in the set of all Lorentzian metrics $Lor(M)$ on M in the following way: $g < g'$ iff all causal vectors for g are timelike for g'. Then the metric $g_\lambda = g + \lambda(g' - g)$, $\forall\, \lambda \in [0, 1]$ is a Lorentzian metric on M, as well. Also, recall that $g < g'$ means that the causal cones of g are contained in the timelike cones of g'. A connected time-orientable Lorentzian manifold (M, g) is stably causal if there exists $g' \in Lor(M)$, such that $g' > g$, with g' causal.

[96] Diamond-compact means $J(p, q) := J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. Note that $J(p, q)$ is possibly empty.

[97] An inextensible curve is a curve with no ends; it either extends infinitely, remaining timelike or null, or it closes in on itself to form a circle - a closed, non-spacelike curve. In mathematical terms, a map $\alpha \colon (a, b) \to M$ is an inextensible timelike curve in (M, g) if $\alpha(t)$ does not approach a limit as t increases to b or as t decreases to a.

Theorem 16.9. *Let $(M, \tilde{g})$ be a pseudo-time orientable, transverse, signature-type changing, n-dimensional ($n \geq 2$) manifold with a transverse radical, where $M_L = M \setminus (M_R \cup \mathcal{H})$ is globally hyperbolic. Assume that a Cauchy surface S is a subset of the neighborhood $U = \bigcup_{q \in \mathcal{H}} U(q)$ of $\mathcal{H}$, i.e. $S \subseteq (U \cap M_L) = \bigcup_{q \in \mathcal{H}} (U(q) \cap M_L)$, with $U(q)$ being constructed as in Theorem 16.1. Then for every point $p \in M$, there exists a pseudo-timelike loop such that the intersection point is p.*

Proof. Let $(M, \tilde{g})$ be a *pseudo-time orientable* transverse, signature-type changing, n-dimensional ($n \geq 2$) manifold with a transverse radical, where M_L is globally hyperbolic with $\tilde{g}\mid_{M_L} = g$. Moreover, there is a neighborhood $U = \bigcup_{q \in \mathcal{H}} U(q)$ of $\mathcal{H}$ sufficiently small to satisfy the conditions for Theorem 16.1, and per assumption there exists a Cauchy surface $S_\varepsilon \subseteq (U \cap M_L)$, $\varepsilon > 0$.

Due to [6] we know that M_L admits a splitting $M_L = (\mathbb{R}_{>0})_t \times S_t = \bigcup_{t \in \mathbb{R}_{>0}} S_t$, such that the Lorentzian sector M_L is decomposed into hypersurfaces (of dimension $n-1$), specified as the level surfaces $S_t = \mathcal{T}^{-1}(t) = \{p \in M_L \colon \mathcal{T}(p) = t\}, t \in \mathbb{R}_{>0}$, of the real-valued smooth temporal function $\mathcal{T} \colon M_L \longrightarrow \mathbb{R}_{>0}$ whose gradient $\mathrm{grad}\mathcal{T}$ is everywhere non-zero and $d\mathcal{T}$ is an exact 1-form. Within the neighborhood $U = \bigcup_{q \in \mathcal{H}} U(q)$ this foliation $\bigcup_{t \in \mathbb{R}_{>0}} S_t$ can be chosen in such a way that it agrees with the natural foliation given by the absolute time function $h(t, \hat{x}) := t$, see Remark 8.17 and Corollary 13.8. For the calculation of the coordinates adapted to this foliation, see Appendix D.[98]

Moreover, the level surfaces $(S_t)_{t \in \mathbb{R}}$ are Cauchy surfaces and, accordingly, each inextendible pseudo-timelike curve in M_L can intersect each level set S_t exactly once as $\mathcal{T}$ is strictly increasing along any future-pointing pseudo-timelike curve.[99] Then, these level-sets S_t are all space-like hypersurfaces which are orthogonal to a timelike and future-directed unit normal vector field n.[100]

[98]Recall that a smooth function $T \colon M \longrightarrow \mathbb{R}$ on a connected time-orientable Lorentzian manifold (M, g) is a global time function if T is strictly increasing along each future-pointing non-spacelike curve. Moreover, a temporal function is a time function T with a timelike gradient $\mathrm{grad}T$ everywhere.

Since M_L is globally hyperbolic it admits a smooth global time function T and consequently it admits [4, 43] a temporal function $\mathcal{T}$. Hence, in the Lorentzian sector M_L there exists a global temporal function $\mathcal{T} \colon M_L \longrightarrow \mathbb{R}_{>0}$, and $\mathrm{grad}\mathcal{T}$ is orthogonal to each of the level surfaces $S_t = \mathcal{T}^{-1}(t) = \{p \in M_L \colon \mathcal{T}(p) = t\}, t \in \mathbb{R}_{>0}$, of $\mathcal{T}$. Note that $\mathcal{T} = t$ is a scalar field on M_L, hence $\mathrm{grad}\mathcal{T} = \mathrm{grad}t = dt$.

[99]Since $\mathcal{T}$ is regular the hypersurfaces S_t never intersect, i.e. $S_t \cap S_{t'} = \emptyset$ for $t \neq t'$.

[100]In other words, the unit vector n is normal to each slice S_t, and g restricted to S_t is Riemannian.

For ε sufficiently small, the level Cauchy surface

$$S_\varepsilon = \mathcal{T}^{-1}(\varepsilon) = \{p \in M_L \colon \mathcal{T}(p) = \varepsilon\}, \varepsilon \in \mathbb{R}_{>0}$$

is contained in the neighborhood $U \cap M_L = \bigcup_{q \in \mathcal{H}}(U(q) \cap M_L)$ of $\mathcal{H}$.[101]

Therefore, based on Theorem 16.1, for any $p = (\varepsilon, \hat{x}) \in S_\varepsilon \subseteq (U \cap M_L)$ all past-directed and causal curves emanating from that point reach the hypersurface $\mathcal{H}$.

The global hyperbolicity of M_L implies that every non-spacelike curve in M_L meets each S_t once and exactly once if S_t is a Cauchy surface. In particular is the spacelike hypersurface S_ε a Cauchy surface in the sense that for any $\bar{p} \in M_L$ in the future of S_ε, all past pseudo-timelike curves from $\bar{p}$ intersect S_ε. The same holds for all future directed pseudo-timelike curves from any point $\bar{\bar{p}} \in M_L$ in the past of S_ε.

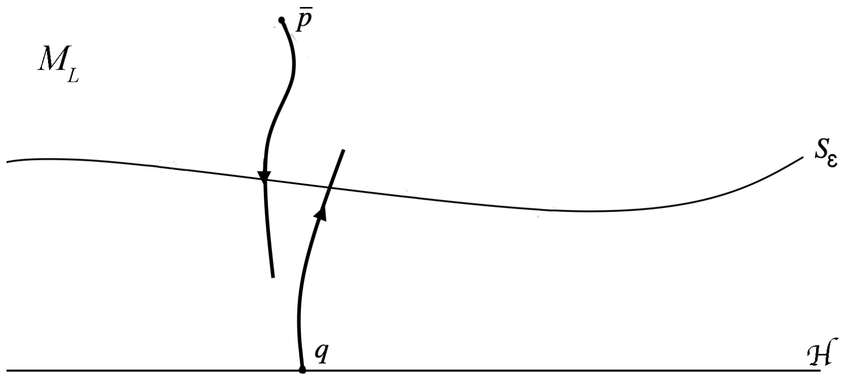

Figure 16.3: For any $\bar{p} \in M_L$ in the future of S_ε, all past pseudo-timelike curves from $\bar{p}$ intersect the Cauchy surface S_ε. Similarly, for any point $(t, \hat{x}) = \bar{p} \in M_L$ with $t > \varepsilon$ there exists a suitable point $q \in \mathcal{H}$, such that S_ε can be reached by a future-directed pseudo-timelike curve starting at $q \in \mathcal{H}$.

Consequently, by virtue of Theorem 16.1 and the above argument, all past-directed

[101]This is true because all neighborhoods $U(q)$ with $q \in \mathcal{H}$ can be chosen in such a way that the sets $U(q)$ have a compact closure. Thus, the $\overline{U(q)}$ are not "infinitely wide", and there exists a strictly positive value $\varepsilon_{\max}$, such that for all $\varepsilon < \varepsilon_{\max}$ the level Cauchy surface S_ε is contained in the neighborhood $U \cap M_L$ of $\mathcal{H}$.

pseudo-timelike curves emanating from any $\bar{p} \in M_L$ reach the hypersurface $\mathcal{H}$. Analogously we can conclude that any point $\bar{p} \in M_L$ can be reached by a future-directed pseudo-timelike curve starting at some suitable point in $\mathcal{H}$. Recall that based on Remark 8.20 we also know that $\mathcal{I}^+(q) = \{p \in M : q \ll p\} = M$, that is, any point in $M = M_R \cup \mathcal{H} \cup M_L$ can be reached by a future-directed pseudo-timelike curve from $q \in \mathcal{H}$.

We now obtain a loop with intersection point p in M_L if, for sufficiently small ε, we first prescribe the intersection point $p = (\varepsilon, \hat{x}) \in S_\varepsilon$. And then we connect the two points lying in $\mathcal{H}$ of the intersecting curve sections S_ε through an arbitrary curve segment in the Riemannian sector M_R (through a suitable choice of the two curve segments, we can ensure that different points on $\mathcal{H}$ are obtained). $\square$

Remark 16.10. The Theorem 16.9 explicitly states that through every point in M, there always exists a pseudo-timelike loop. Therefore, this assertion holds also true for points located both on the hypersurface and within the Riemannian region. In this case, the situation is as follows:

(i) If the given point lies on the hypersurface $p \in \mathcal{H}$, choose a timelike curve segment that connects it to S_ε (with ε sufficiently small), then proceed from there along another timelike curve segment to another point on the hypersurface, and connect both points in the Riemannian sector.

(ii) If the given point lies in the Riemann sector $p \in M_R$, choose an arbitrary loop of the form similar to those loops constructed in the proof of Theorem 16.1, and modify this loop within the Riemannian sector such that it passes through the specified point there.

Example 16.11. The prototype of a spacetime M with signature-type change is obtained by cutting an S^4 along its equator and joining it to the corresponding half of a de Sitter space. The Lorentzian sector is half de Sitter space which is globally hyperbolic [14], and therefore there are chronology-violating pseudo-timelike loops through each point in M.

Corollary 16.12. *Let $(M, \tilde{g})$ be a pseudo-time orientable, transverse, signature-type changing, n-dimensional ($n \geq 2$) manifold with a transverse radical, where M_L is globally hyperbolic, and $S \subseteq (U \cap M_L) = \bigcup_{q \in \mathcal{H}} (U(q) \cap M_L)$ for a Cauchy surface S. Then through every point there exists a path on which a pseudo-time orientation cannot be defined.*

The intriguing facet of the potential existence of closed timelike curves within the framework of Einstein's theory lies in the physical interpretation that CTCs, serving as the worldlines of observers, fundamentally permit an influence on the causal past.

This can also be facilitated through a causal curve in the form of a loop, i.e., the curve intersects itself. In the case of a non-time-orientable manifold, there would then be the possibility that at the intersection, the two tangent vectors lie in different components of the light cone. Thus, the "time traveler" at the encounter with himself, which he experiences twice, may notice a reversal of the past and future time directions in his surroundings during the second occurrence. Regardless of whether this effect exists or not, during the second experience of the encounter, which he perceives as an encounter with a younger version of himself, the traveler can causally influence this younger version and its surroundings.

Appendix

A Generalized affine parameter

A.1 Alternative proof for Estimate 4.3

Proof. For any two basis of $T_{\gamma(t)}M$ which are parallel transported along γ, then in the case of a change of basis, the components $V^i(t)$ with respect to another basis are given by

$$\tilde{V}^j(t) = \sum_{i=1}^{n} A_i^j V^i(t).$$

The constant items A_i^j are entries in a constant, non-degenerate $n \times n$ matrix A. From this it follows that

$$q(t) := \frac{\delta_{\tau\sigma}\tilde{V}^\tau(t)\tilde{V}^\sigma(t)}{\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)} = \frac{\left(\sum_{i=1}^{n} A_i^\tau V^i(t)\right)\left(\sum_{i=1}^{n} A_j^\sigma V^j(t)\right)}{\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)}.$$

Furthermore, we can view $\delta_{\tau\sigma}A_i^\tau A_j^\sigma$ and $\delta_{\alpha\beta}$ as elements of a matrix $M_{\tilde{B}} = (\delta_{\tau\sigma}A_i^\tau A_j^\sigma)$ and $E_N = (\delta_{\alpha\beta})$, respectively, in $\mathbb{R}^N$ with a basis $\mathcal{B}$. Then $(\delta_{\tau\sigma}A_i^\tau A_j^\sigma)$ and $(\delta_{\alpha\beta})$ define symmetric, positive definite bilinear forms, $\tilde{B}$ and B, in $\mathbb{R}^N$.

It can be easily seen that the quotient

$$\frac{\tilde{B}(X,X)}{B(X,X)} =: Q(X)$$

is scale invariant, i.e. $\forall\ X \in \mathbb{R}^N \setminus \{0\}$:

$$\frac{\tilde{B}(\frac{X}{|X|}, \frac{X}{|X|})}{B(\frac{X}{|X|}, \frac{X}{|X|})} = \frac{(\frac{1}{|X|})^2\tilde{B}(X,X)}{(\frac{1}{|X|})^2 B(X,X)} = \frac{\tilde{B}(X,X)}{B(X,X)}.$$

Therefore, without loss of generality, we can restrict $\frac{\tilde{B}(X,X)}{B(X,X)}$ to the unit sphere $\left\{X \in \mathbb{R}^N \colon \|X\| = 1\right\}$. Then as a continuous, real-valued function on a compact set, the quotient $Q(X)$ is bounded. Based on the extreme value theorem (where each continuous function on a compact set attains its maximum and minimum) there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 < \lambda_1 \leq \lambda_2 < \infty$, such that $\lambda_1 \leq \frac{\tilde{B}(X,X)}{B(X,X)} \leq \lambda_2$ $\forall\ X \in \mathbb{R}^N \setminus \{0\}$.

Define now

$$C := \begin{cases} \lambda_1 \; if \; \lambda_1\lambda_2 \leq 1 \\ \frac{1}{\lambda_2} \; if \; \lambda_1\lambda_2 \geq 1 \end{cases},$$

then we have the following equation:

$$C \leq \frac{\tilde{B}(X,X)}{B(X,X)} \leq \frac{1}{C} \; \forall \; X \in \mathbb{R}^N \setminus \{0\}.$$

We first consider the case $\lambda_1\lambda_2 \leq 1$ for which we have $C = \lambda_1 \leq Q(X) \leq \lambda_2 \leq \frac{1}{\lambda_1} = \frac{1}{C}$. The case $\lambda_1\lambda_2 \geq 1$ is handled similarly: $C = \frac{1}{\lambda_2} \leq \lambda_1 \leq Q(X) \leq \lambda_2 = \frac{1}{C}$.

Using this result, it follows that there exists a $C \in [0,1]$, such that $\forall t$: $C \leq q(t) \leq \frac{1}{C}$.

(i) If γ is a (possibly not affinely parametrized) geodesic, then according to proposition 4.4 are

$$\mu(t) = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} [V^i(t)]^2 dt}$$

and

$$\tilde{\mu}(t) = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} [\tilde{V}^i(t)]^2 dt}$$

affine parameters. The change of basis in T_pM which yields

$$\tilde{V}^j(t) = \sum_{i=1}^{n} A_i^j V^i(t)$$

also induces an affine reparametrization $\tilde{\mu} = a\mu + b$.

(ii) If γ is not a (possibly not affinely parametrized) geodesic, then we have

$$\frac{\frac{d\tilde{\mu}}{dt}}{\frac{d\mu}{dt}} = \frac{d\tilde{\mu}}{d\mu} = \sqrt{q(t)} = \sqrt{Q(V)} = \sqrt{Q(V^1(t), \ldots, V^N(t))},$$

which immediately yields

$$\sqrt{C} \leq \underbrace{\frac{d\tilde{\mu}}{d\mu}}_{\sqrt{q(t)}} \leq \frac{1}{\sqrt{C}} \forall\ t \in J.$$

On the other hand we have

$$\frac{\frac{d\tilde{\mu}}{dt}}{\frac{d\mu}{dt}} = \sqrt{q(t)} \Longleftrightarrow \frac{d\tilde{\mu}}{dt} = \sqrt{q(t)}\frac{d\mu}{dt} = \sqrt{q(t)}\sqrt{\sum_{i=1}^{n}[V^i(t)]^2}$$

$$\Longrightarrow \tilde{\mu} = \int_{t_0}^{t} \sqrt{q(t)}\sqrt{\sum_{i=1}^{n}[V^i(t)]^2 dt}.$$

Including the above reasoning, we then get

$$\int_{t_0}^{t} \sqrt{C}\sqrt{\sum_{i=1}^{n}[V^i(t)]^2 dt} \leq \mu(\tilde{t}) \leq \int_{t_0}^{t} \frac{1}{\sqrt{C}}\sqrt{\sum_{i=1}^{n}[V^i(t)]^2 dt}$$

$$\Longleftrightarrow \sqrt{C}\mu(t) \leq \tilde{\mu}(t) \leq \frac{1}{\sqrt{C}}\mu(t).$$

$\square$

B Conformal transformation

B.1 Geodesics

The metric $\bar{g}$ is defined by

$$ds^2 = \text{sgn}(f(t))[-(dt)^2 + \frac{1}{f(t)}(dx^1)^2 + \cdots + \frac{1}{f(t)}(dx^{n-1})^2],$$

and the corresponding Lagrangian and Euler-Lagrange equations are given by

$$\mathcal{L} = \frac{1}{2}\left(-\text{sgn}(f(t))\dot{t}^2 + \frac{\text{sgn}(f(t))}{f(t)}\sum_{i=1}^{n-1}(\dot{x}^i)^2\right) = \frac{-\text{sgn}(f(t))}{2}\left(\dot{t}^2 - \frac{1}{f(t)}\sum_{i=1}^{n-1}(\dot{x}^i)^2\right),$$

$$0 = \frac{\partial \mathcal{L}}{\partial t} - \frac{d}{ds}(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{t}}}_{-\text{sgn}(f(t))\dot{t}}) = -\frac{1}{2}\text{sgn}(f(t)) \cdot \frac{f'(t)}{f(t)^2}\sum_{i=1}^{n-1}(\dot{x^i})^2 + \text{sgn}(f(t))\ddot{t},$$

$$0 = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{ds}(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}^i}}_{\frac{\text{sgn}(f(t))}{f(t)}\dot{x}^i}) = -\frac{d}{ds}\frac{\text{sgn}(f(t))}{f(t)}\dot{x}^i$$

$\Longrightarrow \frac{\text{sgn}(f(t))}{f(t)}\dot{x}^i = c_{x^i}$ is a constant.

From this follows

$$\frac{1}{2}\text{sgn}(f(t)) \cdot \frac{f'(t)}{f(t)^2}\sum_{i=1}^{n-1}(\dot{x}^i)^2 = \text{sgn}(f(t))\ddot{t}$$

$$\Longleftrightarrow \ddot{t} = \frac{1}{2}\frac{f'(t)}{f(t)^2}\sum_{i=1}^{n-1}(\dot{x}^i)^2,$$

and $f(t) = \dot{x}^i \cdot \frac{\text{sgn}(f(t))}{c_{x^i}} \Longleftrightarrow \dot{x}^i = \frac{c_{x^i}}{\text{sgn}(f(t))} \cdot f(t)$.

Note that for the latter we have for any $i, j = 1, \ldots, (n-1)$ the following relationship at hand:

$$f(t) = \dot{x}^i \cdot \frac{\mathrm{sgn}(f(t))}{c_{x^i}} = \dot{x}^i \cdot \frac{\mathrm{sgn}(f(t))}{c_{x^j}},$$

therefore $\dot{x}^i = \frac{c_{x^i}}{c_{x^j}} \cdot \dot{x}^j$ and $\dot{x}^j = \frac{c_{x^j}}{c_{x^i}} \cdot \dot{x}^i$.

From

$$\dot{x}^i = \frac{c_{x^i}}{\mathrm{sgn}(f(t))} \cdot f(t)$$

we get by integration

$$x^i(s) = \frac{c_{x^i}}{\mathrm{sgn}(f(t))} \int f(t) ds.$$

And by the substitutions $\dot{x}^i \longleftrightarrow \frac{f(t)}{\mathrm{sgn}(f(t))} \cdot c_{x^i}$ in the Lagrangian $\mathcal{L}$ we obtain

$$\begin{aligned} 2\mathcal{L} &= \left(-\mathrm{sgn}(f(t))\dot{t}^2 + \frac{\mathrm{sgn}(f(t))}{f(t)} \sum_{i=1}^{n-1} (\dot{x}^i)^2 \right) \\ &= -\mathrm{sgn}(f(t))\dot{t}^2 + \frac{\mathrm{sgn}(f(t))}{f(t)} \sum_{i=1}^{n-1} \left(\frac{f(t)}{\mathrm{sgn}(f(t))} \cdot c_{x^i} \right)^2 \\ &= -\mathrm{sgn}(f(t))\dot{t}^2 + \sum_{i=1}^{n-1} \frac{f(t)}{\mathrm{sgn}(f(t))} (c_{x^i})^2 \Longleftrightarrow \end{aligned}$$

$$\boxed{\dot{t}^2 = \frac{1}{\mathrm{sgn}(f(t))} \left(\sum_{i=1}^{n-1} \left(\frac{f(t)}{\mathrm{sgn}(f(t))} (c_{x^i})^2 \right) - 2\mathcal{L} \right) = \sum_{i=1}^{n-1} (f(t)(c_{x^i})^2) - \frac{2\mathcal{L}}{\mathrm{sgn}(f(t))}.}$$

By the substitutions

$$\dot{x}^i \longleftrightarrow \frac{f(t)}{\mathrm{sgn}(f(t))} \cdot c_{x^i}$$

in $\ddot{t} = \frac{1}{2} \frac{f'(t)}{f(t)^2} \sum_{i=1}^{n-1} (\dot{x}^i)^2$ we obtain

$$\ddot{t} = \frac{1}{2}\frac{f'(t)}{f(t)^2}\sum_{i=1}^{n-1}\left(\frac{f(t)}{\mathrm{sgn}(f(t))}\cdot c_{x^i}\right)^2 = \frac{1}{2}\frac{f'(t)}{f(t)^2}\sum_{i=1}^{n-1}\left(f(t)^2\cdot(c_{x^i})^2\right)$$
$$= \frac{1}{2}f'(t)\sum_{i=1}^{n-1}(c_{x^i})^2,$$

from which we get by integration

$$\boxed{\dot{t}(s) = \frac{1}{2}\sum_{i=1}^{n-1}(c_{x^i})^2\int\underbrace{\frac{df(t)}{dt}}_{f'(t)}ds.}$$

Expressed through the above partial results, we have

i) Geodesic equation for $x^i(s)$:

$$2\mathcal{L} = -\mathrm{sgn}(f(t))\dot{t}^2 + \sum_{i=1}^{n-1}\left(\frac{(c_{x^i})^2}{\mathrm{sgn}(f(t))}f(t)\right)$$

$$= -\mathrm{sgn}(f(t))(\underbrace{\frac{1}{2}\sum_{i=1}^{n-1}(c_{x^i})^2\int\frac{df(t)}{dt}ds}_{\dot{t}})^2 + \sum_{i=1}^{n-1}\left(\frac{(c_{x^i})^2}{\mathrm{sgn}(f(t))}\cdot\underbrace{\frac{\mathrm{sgn}(f(t))}{c_{x^j}}\dot{x}^j}_{f(t)}\right)$$

$$= -\frac{1}{4}\mathrm{sgn}(f(t))\left(\sum_{i=1}^{n-1}(c_{x^i})^2\right)^2\left(\int\frac{df(t)}{dt}ds\right)^2 + \sum_{i=1}^{n-1}\frac{(c_{x^i})^2}{c_{x^j}}\dot{x}^j.$$

This is equivalent to

$$\dot{x}^i = \frac{c_{x^j}}{\sum\limits_{i=1}^{n-1}(c_{x^i})^2}\left(2\mathcal{L} + \frac{1}{4}\mathrm{sgn}(f(t))\left(\sum_{i=1}^{n-1}(c_{x^i})^2\right)^2\left(\int\frac{df(t)}{dt}ds\right)^2\right),$$

which implies

$$x^j(s) = \frac{2\mathcal{L}c_{x^j}}{\sum\limits_{i=1}^{n-1}(c_{x^i})^2}s + \frac{\mathrm{sgn}(f(t))c_{x^j}}{4}\left(\sum_{i=1}^{n-1}(c_{x^i})^2\right)\int\left(\int\frac{df(t)}{dt}ds\right)^2 ds.$$

ii) Geodesic equation for $t(s)$:

Combining the above two results for $\dot{t}$ we have

$$\begin{cases} \dot{t}^2 = \sum_{i=1}^{n-1} \left(f(t)(c_{x^i})^2\right) - \frac{2\mathcal{L}}{\mathrm{sgn}(f(t))} \\ \dot{t} = \frac{1}{2} \sum_{i=1}^{n-1} (c_{x^i})^2 \int \frac{df(t)}{dt} ds \end{cases},$$

which yields

$$\left(\int \frac{df(t)}{dt} ds\right)^2 = \frac{4}{\mathrm{sgn}(f(t)) \left(\sum_{i=1}^{n-1} (c_{x^i})^2\right)^2} \left(\sum_{i=1}^{n-1} \left(\frac{f(t)}{\mathrm{sgn}(f(t))} (c_{x^i})^2\right) - 2\mathcal{L}\right)$$

$$\Longleftrightarrow \left(\int \frac{df(t)}{dt} ds\right)^2 = \frac{4f(t)}{\sum_{i=1}^{n-1} (c_{x^i})^2} - \frac{8\mathcal{L}}{\mathrm{sgn}(f(t)) \left(\sum_{i=1}^{n-1} (c_{x^i})^2\right)^2}.$$

B.2 Turning Point

A pseudo-timelike geodesic has its turning point (in the Riemannian region) at $\dot{t}(s) = 0$. This means

$$\dot{t}(s) = \frac{1}{2} \sum_{i=1}^{n-1} (c_{x^i})^2 \int \frac{df(t)}{dt} ds = 0$$

$$\Longleftrightarrow \int \frac{df(t)}{dt} ds = 0$$

$$\Longleftrightarrow \frac{4f(t)}{\sum_{i=1}^{n-1} (c_{x^i})^2} - \frac{8\mathcal{L}}{\mathrm{sgn}(f(t)) \left(\sum_{i=1}^{n-1} (c_{x^i})^2\right)^2} = 0$$

$$\Longleftrightarrow f(t) = \frac{2\mathcal{L}}{\mathrm{sgn}(f(t)) \sum_{i=1}^{n-1} (c_{x^i})^2} = \frac{1}{\mathrm{sgn}(f(t)) \sum_{i=1}^{n-1} (c_{x^i})^2}.$$

Calculation via the Lagrangian when $\dot{t}(s) = 0$:

$$
\begin{aligned}
2\mathcal{L} &= \frac{\operatorname{sgn}(f(t))}{f(t)} \sum_{i=1}^{n-1} (\dot{x}^i)^2 \\
\iff f(t) &= \frac{\operatorname{sgn}(f(t))}{2\mathcal{L}} \sum_{i=1}^{n-1} (\dot{x}^i)^2 \\
\iff \frac{1}{f(t)} &= \frac{\operatorname{sgn}(f(t))}{2\mathcal{L}} \sum_{i=1}^{n-1} \left(\frac{c_{x^i}}{\operatorname{sgn}(f(t))}\right)^2 \\
\iff f(t) &= \frac{2\mathcal{L}}{\operatorname{sgn}(f(t))} \frac{1}{\sum_{i=1}^{n-1} (c_{x^i})^2} = \frac{1}{\operatorname{sgn}(f(t))} \frac{1}{\sum_{i=1}^{n-1} (c_{x^i})^2}.
\end{aligned}
$$

B.3 Intersection Point

A pseudo-timelike geodesic intersects itself (in the Lorentzian region) at

$$
-2f(t) = \frac{-4\mathcal{L}}{\operatorname{sgn}(f(t)) \left(\sum_{i=1}^{n-1} (c_{x^i})^2\right)^2} \cdot \sum_{i=1}^{n-1} (c_{x^i})^2 = \frac{2}{\operatorname{sgn}(f(t)) \left(\sum_{i=1}^{n-1} (c_{x^i})^2\right)^2} \cdot \sum_{i=1}^{n-1} (c_{x^i})^2.
$$

C Local transformation theorem

An alternative proof for the " $\Longrightarrow$ " direction of the local Transformation Theorem, using radical-adapted Gauss-like coordinates:

Proof. " $\Longrightarrow$ " Recall from Section 9 that $(M, \tilde{g})$ is a transverse, signature-type changing manifold with a transverse radical if and only if for any $q \in \mathcal{H}$ there exist a neighborhood $U(q)$ and smooth radical-adapted Gauss-like coordinates such that $\tilde{g} = -t(dt)^2 + g_{ij}dx^i dx^j$, for $i, j \in \{1, \ldots, n-1\}$.

Let $\tilde{g}$ be a transverse, signature-type changing metric with a transverse radical. Consequently for any $q \in \mathcal{H}$ we have $d(\det(\tilde{g}_{\mu\nu})) \neq 0$ and there exists a neighbourhood $U(q)$ and smooth radical-adapted Gauss-like coordinates such that the metric takes the form $\tilde{g} = -t(dt)^2 + \tilde{g}_{ij}dx^i dx^j$, for $i, j \in \{1, \ldots, n-1\}$. Then in the neighborhood $U := \bigcup_{q \in \mathcal{H}} U(q)$ of $\mathcal{H}$ the associated matrix representation is given by

$$[\tilde{g}_{\mu\nu}] = \left(\begin{array}{c|ccc} -t & 0 & \cdots & 0 \\ \hline 0 & \tilde{g}_{11} & \cdots & \tilde{g}_{1n-1} \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{g}_{n-11} & \cdots & \tilde{g}_{n-1n-1} \end{array}\right) = \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{G} & \\ 0 & & & \end{array}\right) + \left(\begin{array}{c|ccc} 1-t & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array}\right),$$

which can be written as $\tilde{g} = g + f \cdot E_{11}$, with the matrix unit E_{11} and $f(t, \hat{x}) = 1 - h(t, \hat{x}) = 1 - t$.[102]Notice that because of the properties of $\tilde{g}$, the metric g is non-degenerate, hence Lorentzian, and properly defined on U.

In these coordinates the locus of signature change is at $t = 0$, the Lorentzian sector M_L is determined by $t > 0$ and the Riemannian sector M_R is set by $t < 0$. Furthermore in the neighborhood U of $\mathcal{H}$ there is a unique absolute time function defined by $h(t, \hat{x}) := t$,[103] and in the Lorentzian sector we have $h \mid_{M_L} (t, \hat{x}) =$

[102]The E_{11} matrix unit is an n-by-n matrix with only one nonzero entry with value 1 in the 1st row and 1st column. With respect to radical-adapted Gauss-like coordinates is E_{11} is the matrix representation of the $(0,2)$-tensor field $dt \otimes dt = \delta^0_\mu \delta^0_\nu dx^\mu \otimes dx^\nu$ for $\mu, \nu \in \{0, \ldots, n-1\}$. This is because E_{11} is the matrix with the components $\delta^0_\mu \delta^0_\nu$.

[103]For this particular class of metrics this is predicated on the fact that on the hypersurface, the coordinate basis vector ∂_t is the only eigenvector for the eigenvalue $g_{00} \mid_{\mathcal{H}} = \lambda_{00} \mid_{\mathcal{H}} = 0$. From this then follows that $Rad_q = \text{span}(\{\partial_t\})$ for $q \in \mathcal{H}$ and $g(\partial_t, \boldsymbol{.})(q) = 0 \Longrightarrow \partial_t$ is lightlike.

$-\tilde{g}\mid_{M_L}(\partial_t, \partial_t) = t > 0$. According to Lemma 13.11 we can thus on M_L define a smooth non-vanishing timelike line element field $\{W, -W\}$ corresponding to $\frac{\partial}{\partial t} =: W$ in $M_L \cap U$. Hence, locally in $M_L \cap U$ this will be a smooth timelike vector field W (i.e. a timelike line element field with determined sign). Since the absolute time function allows to introduce a time concept in $M_R \cap U$, this implies also that $\frac{\partial}{\partial t}$ is future pointing there. For this reason, $W = \frac{\partial}{\partial t}$ on U, and moreover $\{W, -W\}$ on U is a smooth non-vanishing timelike line element field with respect to g.

Then in U we have

$$-t = \tilde{g}(\partial_t, \partial_t) = \underbrace{g(\partial_t, \partial_t)}_{-1} + \underbrace{f}_{1-t} \cdot \underbrace{E_{11}(\partial_t, \partial_t)}_{1}.$$

From this follows directly (notice that $\flat$ is with respect to the metric g, not $\tilde{g}$)

$$1 = g(\partial_t, \partial_t) \cdot g(\partial_t, \partial_t) = \underbrace{g(W, W)}_{W^\flat(W)} \otimes \underbrace{g(W, W)}_{W^\flat(W)} = (W^\flat \otimes W^\flat)(W, W)$$

$$\Longleftrightarrow 1 = (W^\flat \otimes W^\flat)(\partial_t, \partial_t),$$

$$0 = g(\partial_t, \partial_t) \cdot \underbrace{g(\partial_t, \partial_i)}_{0} = g(W, \partial_t) \cdot g(W, \partial_i)$$

$$= (g(W, \centerdot) \otimes g(W, \centerdot))(\partial_t, \partial_i) = W^\flat \otimes W^\flat(\partial_t, \partial_i)$$

$$\Longleftrightarrow 0 = (W^\flat \otimes W^\flat)(\partial_t, \partial_i)$$

and

$$0 = \underbrace{g(\partial_t, \partial_i)}_{0} \cdot \underbrace{g(\partial_t, \partial_j)}_{0} = g(W, \partial_i) \cdot g(W, \partial_j)$$

$$= (g(W, \centerdot) \otimes g(W, \centerdot))(\partial_i, \partial_j) = W^\flat \otimes W^\flat(\partial_i, \partial_j)$$

$$\Longleftrightarrow 0 = (W^\flat \otimes W^\flat)(\partial_i, \partial_j),$$

for $i, j \in \{1, \dots, n-1\}$.

Hence, $E_{11} \coloneqq W^\flat \otimes W^\flat$ is a $(0,2)$-tensor field on U where $W^\flat = -dt$, and we are able to write the metric $\tilde{g} = -t(dt)^2 + \tilde{g}_{ij}dx^i dx^j$ in the following way:

$$\tilde{g} = g + f \cdot E_{11} = \left(-(dt)^2 + g_{ij}dx^i dx^j\right) + (1-t)W^\flat \otimes W^\flat,$$

with $f(t, \hat{x}) \coloneqq 1 - h(t, \hat{x}) = 1 - t$ and $g_{ij} = \tilde{g}_{ij}$ for $i, j \in \{1, \dots, n-1\}$ on U. $\square$

D Global pseudo-timelike loops

Coordinates adapted to the foliation $(S_t)_{t\in\mathbb{R}}$ in the Lorentzian sector M_L of the signature-type changing, n-dimensional manifold $(M, \tilde{g})$ are introduced [35] on each hypersurface S_t as a spatial coordinate system $\zeta = (x^1, \ldots, x^{n-1})$ that varies smoothly between adjacent hypersurfaces. Then $\zeta_t = (t, x^1, \ldots, x^{n-1})$ is a well-behaved coordinate system of M_L. On each T_pM the basis associated to ζ_t is given by the time vector $\partial_t =: m + \beta = Nn + \beta$ and $\partial_i := \frac{\partial}{\partial_i}$, $i \in \{1, \ldots, n-1\}$, where $m = Nn$ is the normal evolution vector (adapted to the temporal function $\mathcal{T}$), N the lapse function,[104] and β is the shift vector that measures the difference between ∂_t and m.[105]

It is noteworthy that ∂_t is tangent to the curves of constant spatial coordinates, and $\partial_i \in T_pS_t \subset T_pM$. Moreover, the vector $\beta =: \beta^i\partial_i$ is tangent to the hypersurfaces S_t, and the vector n is normal to S_t. As an immediate consequence $\beta = \gamma(\partial_t)$can be regarded as the orthogonal projection

$$\gamma : T_pM_L \longrightarrow T_pS_t$$

$$\partial_t \mapsto \partial_t + (n \cdot \partial_t)n =: \beta$$

of ∂_t onto S_t. Thus we can decompose ∂_t into its normal and tangential parts with respect to the surfaces S_t. Based on this reasoning we can easily deduce the metric components with respect to coordinates $\zeta_t = (t, x^1, \ldots, x^{n-1})$ adapted to the foliation $(S_t)_{t\in\mathbb{R}}$:

$$g_{00} = g(\partial_t, \partial_t) = -N^2 + \beta_i\beta^i,$$

and

$$g(\partial_t, \partial_t) < 0 \Longleftrightarrow \beta_i\beta^i < N^2,$$

[104]The lapse function, or specifically $N(t) = g_{00}$ being a function of coordinate time, can be normalized away, ensuring proper time equals coordinate time.

[105]These relation and properties mean that the any adjacent hypersurface $S_{t+\delta t}$ can be reached from the hypersurface S_t by the infinitesimal displacement δtm of each point of S_t.

$$g_{0i} = g(\partial_t, \partial_i) = (m + \beta) \cdot \partial_i = \underbrace{m\partial_i}_{0} + \beta\partial_i = \beta_i,$$

where β_i are the components of the metric dual form of $\beta =: \beta^i \partial_i$ with respect to the spatial coordinates $\zeta = (x^1, \ldots, x^{n-1})$. As ∂_i and ∂_j are both tangent to S_t,[106]

$$g_{ij} = g(\partial_i, \partial_j) = \gamma(\partial_i, \partial_j) = \gamma_{ij},$$

where γ_{ij} are the components of the $(n-1)$-metric with respect to the coordinates $\zeta = (x^1, \ldots, x^{n-1})$.

Taking into consideration that $\beta_i = \gamma_{ij}\beta^j$, this yields the following line element

$$\begin{aligned} g &= -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (-N^2 + \underbrace{\gamma_{ij}\beta^j\beta^i}_{\beta_i\beta^i})dt^2 + \underbrace{\gamma_{ij}\beta^j}_{\beta_i} dt dx^i + \underbrace{\gamma_{ij}\beta^i}_{\beta_j} dt dx^j + \underbrace{\gamma_{ij}}_{g_{ij}} dx^i dx^j, i, j \in \{1, \ldots, n-1\}, \end{aligned}$$

and the expression for the associated matrix representation

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k\beta^k & \beta_1 & \cdots & \beta_{n-1} \\ \beta_1 & \gamma_{11} & \cdots & \gamma_{1n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \gamma_{n-11} & \cdots & \gamma_{n-1n-1} \end{pmatrix} = \left(\begin{array}{c|c} -N^2 + \beta_k\beta^k & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right),$$

Thus we constructed a globally defined pointwise splitting of the metric $g = -N(t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where $N(t)$ is a smooth function on M and γ is a Riemannian metric on each level S_t. Note that the determinant of g is given by $\det g = -N^2 \cdot \det \gamma$, where $\gamma = \gamma_{ij} dx^i dx^j$, $i, j \in \{1, \ldots, n-1\}$.

Since in our case, the hypersurface of signature change $\mathcal{H}$, radical-adapted Gauss-like coordinates and the absolute time function are given, it makes sense to initially define on $\mathcal{H}$ a scalar field $N(t)$ as well as a vector field β in some neighborhood of $\mathcal{H}$,[107] such that for $x^0 = t = 0$ is $S_0 := \mathcal{H}$. Furthermore, in the neighborhood

[106] In the neighborhood of the hypersurface of signature change $\mathcal{H}$ we have established the condition that ∂_t is timelike, and this consequently means that $g(\partial_t, \partial_t) < 0 \Longleftrightarrow \beta_i\beta^i < N^2$.

[107] Clearly, within neighborhood patches equipped with radical-adapted Gauss-like coordinates we have $N(t) = \sqrt{t}$ and $\beta = 0$.

of $\mathcal{H}$ we have established the condition that ∂_t is time-like, and this consequently means that $g(\partial_t, \partial_t) < 0 \iff \beta_i \beta^i < N^2$. Then the lapse function at each point of $\mathcal{H}$ determines a unique vector $m = Nn$ as described above, and accordingly the location of the "ensuing" hypersurface $S_{\delta t}$. Then the shift vector prescribes how to propagate the coordinate system from $\mathcal{H} = S_0$ to $S_{\delta t}$.

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