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Globalization of effective actions in the BV formalism

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Abstract

The Batalin–Vilkovisky (BV) formalism is a powerful framework for handling quantum gauge theories, also providing a systematic way to compute an effective action S_{eff}. This thesis studies two techniques for globalizing a locally constructed S_{eff}. Both rest on embedding the global space into a resolution in which globality is encoded in a differential; the construction of the effective action is then realized by a chain map, and a strong deformation retract (or a full L_∞ transfer) back to the global functions, built via the homological perturbation lemma, yields globalization formulae and hence a global effective action.

The first technique uses formal geometry: a formal exponential map lifts the theory to a bundle carrying a flat Grothendieck connection D, whose degree-zero cohomology recovers the global functions; we carry this out for the one-dimensional AKSZ theory of the spinning particle. The second is Čech-theoretic: the effective action is computed patchwise, yielding a solution of the quantum master equation on the Thom–Whitney totalization of the Čech nerve of a cover, and finally globalized.

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Introduction

Quantum field theory frequently requires making sense of oscillatory integrals of the form $\int \mu e^{\mathrm{i}S/\hbar}$; when S carries a gauge symmetry the stationary phase formula breaks down, since the Hessian degenerates along the gauge orbits. The *Batalin–Vilkovisky* (BV) formalism [3, 4] is a very general framework that deals with this difficulty: it enlarges the field content by ghosts and antifields and replaces the naive integral by the BV integral of a solution of the (quantum) master equation over a gauge-fixing Lagrangian submanifold. Among other things, it lets one integrate out part of the fields (the fluctuations) and produce an *effective action* S_{eff} for the rest, the central object of this thesis.

The computation of S_{eff} is, however, typically local. What one would like instead is a single, globally defined effective action, and gluing the local data coherently into such an object is what we call *globalization*, the question this thesis is concerned with. We study two techniques that solve it.

The first technique makes use of *formal geometry* [7]. Its central device is a *formal exponential map*, which covariantly pulls back functions on the manifold to Taylor series along the tangent fibres and thereby organizes the perturbative expansion without breaking covariance. These Taylor series assemble into a bundle carrying a flat *Grothendieck connection* D, whose cohomology is concentrated in degree zero and consists precisely of the global functions. Splitting the fields via Hodge theory, one computes the effective BV action perturbatively as an element of this larger complex, which is then globalized by transferring it, via the homological perturbation lemma, into the global complex.

We test this machinery on the *spinning particle*, a one-dimensional AKSZ theory [1] which, once coupled to worldline supergravity, describes a relativistic spinning point particle. We first treat the case of a two-dimensional, orientable and time-orientable Lorentzian spacetime (M, g): admitting a global null frame, it lets the AKSZ target untwist into a cotangent bundle, so that the resulting AKSZ theory is what we call *split*. Split AKSZ theories admit a direct globalization procedure [11], which we carry out for the spinning particle. We then turn to the spinning particle on a spacetime M of arbitrary dimension, where this splitting is no longer available, and develop a globalization procedure inspired by the methods of Mnev and Wernli for Chern–Simons theory [30]. Both cases rest on the same strong deformation retractions and homological perturbation lemma of the formal-geometric procedure explained above.

The second technique is Čech-theoretic. Here we use an open cover $\mathcal{U} = \{U_\alpha\}$ of the manifold underlying the space of fields and the relevant resolution is the Thom–Whitney totalization of the Čech nerve of $\mathcal{U}$ [19]: while the Čech nerve with the Čech differential and the Thom–Whitney totalization, built from the de Rham complex on the simplices Δ^k, are quasi-isomorphic, only the latter carries the algebraic structure needed to define the antibracket. Splitting the BV field space by a *BV hedgehog* [10], we show that computing the effective action patchwise yields a solution to the quantum master equation on the Thom–Whitney complex, and construct, again via the homological perturbation lemma, the globalization formulae for a global effective action. We also present a construction due to Getzler and Pohorence [20], which builds a global partition function directly by gluing Lagrangian submanifolds across patches.

The thesis is organized as follows. We first collect the necessary background: the BV formalism, the AKSZ construction, and the homological algebra of L_∞ algebras and homotopy transfer. We then develop formal geometry, for both the symplectic and odd-symplectic cases, together with the algebraic BV structure such as the antibracket and the Laplacian, and the strong deformation retractions needed later. We construct the spinning particle and its BV theory via AKSZ and globalize its effective action, first in the two-dimensional split case and then in general. The remaining part develops the Čech globalization machinery.

1 BV formalism

This section will follow very closely the introductory book [29]. Some constructions in supergeometry were taken from [13] (with different sign conventions however) or from [27].

1.1 Motivation

In quantum field theory one is often interested in evaluating *Fresnel integrals*. These are integrals of the form

$$I(k) = \int_X \mu e^{\mathring{\imath} k S(x)},$$

where, for the sake of the exposition, we take X to be an oriented, n-dimensional manifold, $\mu \in \Omega_c^n(X)$ and $S \in C^\infty(X)$. We call a point $x_i \in X$ a *critical point* if $\mathrm{d}S_{x_i} = 0$ and we call S *non-degenerate* at a critical point x_i if the bilinear map $\mathrm{Hess} : T_{x_i}X \times T_{x_i}X \to \mathbb{R}$ given by $\mathrm{Hess}_{x_i}(v, w)S := v(wS)|_{x_i}$ is non-degenerate.

Assume that the critical points $x_0, ..., x_m$ of S are all non-degenerate on $\mathrm{Supp}(\mu) \subset X$. Then the *stationary phase formula* yields that in the asymptotics of $k \to \infty$ we have

$$I(k) \sim \sum_{i=0}^m e^{\mathring{\imath} k S(x_i)} \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} \left|\det S''(x_i)\right|^{-\frac{1}{2}} \cdot e^{\frac{\pi \mathring{\imath}}{4} \operatorname{sign} S''(x_i)} \cdot \mu_{x_i} + \mathcal{O}\left(k^{-\frac{n}{2}-1}\right)$$

where we have chosen a local coordinate chart $y_1^{(i)}, ..., y_n^{(i)}$ near each critical point x_i to write down the Hessian $S''(x_i)_{\ell k} := \partial^2 S / \partial y_\ell^{(i)} \partial y_k^{(i)}|_{y=0}$ in coordinates.

The limitation of the stationary phase formula is that it applies only when S has non-degenerate critical points, which is often not the case. The typical obstruction is a gauge symmetry, as for the Yang–Mills action: a Lie group G acts on X via $\rho : G \times X \to X$ leaving both S and μ invariant. Since S is G-invariant it is constant along each orbit, so at a critical point x the whole orbit $G \cdot x$ consists of critical points and the Hessian of S at x degenerates along the orbit directions $T_x(G \cdot x)$. Faddeev and Popov [16] therefore pass to the quotient X/G, on which these flat directions are gone. When G is compact and acts freely the invariant data descend and the integral factorizes as

$$I = \mathrm{Vol}(G) \int_{X/G} \tilde{\mu}\, e^{\mathring{\imath} k \tilde{S}},$$

where $\tilde{S}$ is the function induced by S on X/G and $\tilde{\mu}$ the induced measure; the prefactor $\mathrm{Vol}(G)$ is precisely the redundancy of integrating over each entire orbit of gauge-equivalent points. On the quotient the critical points are isolated and generically non-degenerate, and the stationary phase formula applies.

While this handles some cases, there is a hierarchy of constructions of increasing generality. In the Faddeev–Popov method the symmetries are described by a group action on a manifold; in the *BRST method* they are instead given by an involutive distribution of the tangent space; and in the *BV formalism* [3, 4], of which we make great use in this thesis, one may even drop integrability away from the critical points.

All three methods give rise to so-called *ghost fields*: odd (i.e. anticommuting) fields encoding a gauge symmetry of the system. Treating them rigorously calls for *supergeometry*, which extends the notion of a manifold to incorporate such odd structures. The next section collects the definitions and constructions we will need.

1.2 Supermanifolds

1.2.1 Preliminary definitions

To handle gauge symmetries, all approaches mentioned before resort to having *odd* fields. These are fields ψ^i which anticommute, i.e. $\psi^i\psi^j = -\psi^j\psi^i$. This is the structure that a supermanifold provides.

Definition 1.1. An $(n|m)$-supermanifold $\mathcal{M}$ is a smooth n-manifold M (the *body*, written $|\mathcal{M}| = M$) equipped with a sheaf $\mathcal{O}_{\mathcal{M}}$ of supercommutative algebras that is locally modelled on algebras $C^\infty(U) \otimes \wedge^\bullet V^*$, with $U \subset M$ open and V a fixed m-dimensional vector space. Equivalently, M carries an atlas of opens $U_\alpha \subset M$ with charts $\phi_\alpha : U_\alpha \to W = \mathbb{R}^n$ and algebra isomorphisms $\Phi_\alpha : \mathcal{O}_{\mathcal{M}}(U_\alpha) \to C^\infty(\phi_\alpha(U)) \otimes \wedge^\bullet V^*$.

In quantum field theory we care more about the algebra of fields $\mathcal{O}_{\mathcal{M}}$ than about the points themselves, which makes this algebro-geometric viewpoint on supermanifolds natural. Locally on U_α an element of $\mathcal{O}_{\mathcal{M}}$ reads

$$f|_{U_\alpha} = \sum_k \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant m} f_{i_1 \cdots i_k}(x) \cdot \theta_{i_1} \cdots \theta_{i_k}$$

where $x_1, ..., x_n$ are the *even* (commuting) coordinates on M and $\theta_1, ..., \theta_m \in V^*$ the *odd* (anticommuting) coordinates on V.

Definition 1.2. The augmentation $\wedge^\bullet V^* \to \mathbb{R}$ (projection onto degree zero) induces a globally well-defined *augmentation map*

$$h : \mathcal{O}_{\mathcal{M}} \to C^\infty(M).$$

An example that will be of importance in this thesis is the following.

Example 1.3. For an m-dimensional vector bundle $E \to M$ we can define the $(n|m)$-supermanifold ΠE with $|\Pi E| = M$ and $\mathcal{O}_{\Pi E} = \Gamma(M, \wedge^\bullet E^*)$.

We will apply this construction to the tangent bundle $E = TM$, which basically shifts the *parity* of the fiber coordinates (we denote the parity of some element as $|a| \in \{0, 1\}$); for local coordinates $x^1, ..., x^n$ and induced local coordinates on the fiber T_xM given by $y^1, ..., y^n$ this construction declares $|y^i| = 1$ for all $i \in \{1, \ldots, n\}$.

Theorem 1.4 (Batchelor, [5])**.** Every smooth supermanifold $\mathcal{M}$ is (non-canonically) isomorphic to a split supermanifold ΠE, for some vector bundle $E \to |\mathcal{M}|$ over its body.

For the following constructions, let $\mathcal{M} = (|M|, \mathcal{O}_{\mathcal{M}})$ always be an $(n|m)$ supermanifold.

Definition 1.5. A morphism of supermanifolds $\phi : \mathcal{M} \to \mathcal{N}$ is a smooth map of bodies $f : |\mathcal{M}| \to |\mathcal{N}|$ together with an extension to a morphism of sheaves of supercommutative algebras $\phi^* : \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{M}}$; that is, for every open $U \subset |\mathcal{N}|$ a map $\phi^*_U : \mathcal{O}_{\mathcal{N}}(U) \to \mathcal{O}_{\mathcal{M}}(f^{-1}(U))$ making the diagram commute

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{N}}(U) \arrow[r, "\phi^*"] \arrow[d, "h"] & \mathcal{O}_{\mathcal{M}}(f^{-1}(U)) \arrow[d, "h"] \\
C^\infty(U) \arrow[r, "f^*"] & C^\infty(f^{-1}(U))
\end{tikzcd}$$

Supermanifolds carry analogues of the standard constructions on ordinary manifolds.

Definition 1.6 (Vector bundle over a supermanifold, [13]). A *super vector bundle of rank* $(p|q)$ *over* M is a sheaf $\mathcal{V}$ of $\mathbb{Z}/2$-graded $\mathcal{O}_M$-modules on $|M|$ that is *locally free of rank* $(p|q)$: every point $x \in |M|$ admits an open neighborhood $U \subseteq |M|$ together with an isomorphism of sheaves of $\mathbb{Z}/2$-graded $\mathcal{O}_M|_U$-modules

$$\mathcal{V}|_U \;\cong\; \mathcal{O}_M|_U^p \oplus \Pi\mathcal{O}_M|_U^q,$$

where Π denotes the parity-reversal functor, and the isomorphism is required to be even (parity-preserving). A choice of such an isomorphism is called a *local frame*; concretely, it provides even sections $e_1, \ldots, e_p \in \mathcal{V}(U)$ and odd sections $\epsilon_1, \ldots, \epsilon_q \in \mathcal{V}(U)$ such that every section $s \in \mathcal{V}(U')$ on any $U' \subseteq U$ writes uniquely as

$$s \;=\; \sum_{i=1}^{p} f_i\, e_i + \sum_{\alpha=1}^{q} g_\alpha\, \epsilon_\alpha, \qquad f_i, g_\alpha \in \mathcal{O}_M(U').$$

The *global sections* of $\mathcal{V}$ are the elements of $\Gamma(M, \mathcal{V})$.

Definition 1.7 (Tangent sheaf of a supermanifold, [13]). The *tangent sheaf* $T\mathcal{M}$ of $\mathcal{M}$ is the sheaf on $|\mathcal{M}|$ whose sections over an open set $U \subseteq |\mathcal{M}|$ are the $\mathbb{R}$-linear graded derivations of the structure sheaf:

$$T\mathcal{M}(U) \;=\; \mathrm{Der}_{\mathbb{R}}\big(\mathcal{O}_{\mathcal{M}}(U)\big).$$

Here a *graded derivation of parity* $|X| \in \mathbb{Z}/2$ is an $\mathbb{R}$-linear map $X : \mathcal{O}_{\mathcal{M}}(U) \to \mathcal{O}_{\mathcal{M}}(U)$ that shifts parity by $|X|$ and satisfies the graded Leibniz rule

$$X(fg) \;=\; X(f)\, g + (-1)^{|X|\,|f|} f\, X(g), \qquad f, g \in \mathcal{O}_{\mathcal{M}}(U) \text{ homogeneous.}$$

The set $T\mathcal{M}(U)$ is a $\mathbb{Z}/2$-graded $\mathcal{O}_{\mathcal{M}}(U)$-module via $(fX)(g) = f \cdot X(g)$. As a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules, $T\mathcal{M}$ is locally free of rank $(n|m)$: on any coordinate chart U with even coordinates $x^1, \ldots, x^n$ and odd coordinates $\theta^1, \ldots, \theta^m$, the sections $\partial/\partial x^1, \ldots, \partial/\partial x^n$ (even) and $\partial/\partial\theta^1, \ldots, \partial/\partial\theta^m$ (odd) form a local frame, so that every section $X \in T\mathcal{M}(U)$ writes uniquely as

$$X \;=\; \sum_{i=1}^{n} f_i\, \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} g_\alpha\, \frac{\partial}{\partial\theta^\alpha}, \qquad f_i, g_\alpha \in \mathcal{O}_{\mathcal{M}}(U).$$

Thus $T\mathcal{M}$ defines a super vector bundle over $\mathcal{M}$ of rank $(n|m)$, called the *tangent bundle* of $\mathcal{M}$. Its global sections $\Gamma(\mathcal{M}, T\mathcal{M}) =: \mathfrak{X}(\mathcal{M})$ are the *vector fields* on $\mathcal{M}$. In particular they form a Lie superalgebra with Lie bracket

$$[X, Y] := X \circ Y - (-1)^{|X|\cdot|Y|} Y \circ X.$$

Definition 1.8 (Cotangent sheaf of a supermanifold, [13]). The *cotangent sheaf* $T^*\mathcal{M}$ of $\mathcal{M}$ is the sheaf of $\mathcal{O}_{\mathcal{M}}$-linear duals of the tangent sheaf: for each open set $U \subseteq |\mathcal{M}|$,

$$T^*\mathcal{M}(U) \;=\; \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}(U)}\big(T\mathcal{M}(U), \mathcal{O}_{\mathcal{M}}(U)\big),$$

where Hom denotes the $\mathbb{Z}/2$-graded $\mathcal{O}_{\mathcal{M}}(U)$-module of grading-respecting $\mathcal{O}_{\mathcal{M}}(U)$-linear maps.

As a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules, $T^*\mathcal{M}$ is locally free of rank $(n|m)$: on any coordinate chart U with even coordinates $x^1, \ldots, x^n$ and odd coordinates $\theta^1, \ldots, \theta^m$, the differentials $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ and $\mathrm{d}\theta^1, \ldots, \mathrm{d}\theta^m$ form a local frame dual to $\{\partial/\partial x^i, \partial/\partial\theta^\alpha\}$ (as $\mathcal{O}_{\mathcal{M}}$-module generators the $\mathrm{d}x^i$ are even and the $\mathrm{d}\theta^\alpha$ odd; their total parity, including the de Rham degree, is the opposite), so that every section $\omega \in T^*\mathcal{M}(U)$ writes uniquely as

$$\omega \;=\; \sum_{i=1}^{n} f_i\, \mathrm{d}x^i + \sum_{\alpha=1}^{m} g_\alpha\, d\theta^\alpha, \qquad f_i, g_\alpha \in \mathcal{O}_{\mathcal{M}}(U).$$

Thus $T^*\mathcal{M}$ defines a super vector bundle over $\mathcal{M}$ of rank $(n|m)$, called the *cotangent bundle* of $\mathcal{M}$. Its global sections $\Gamma(\mathcal{M}, T^*\mathcal{M}) =: \Omega^1(\mathcal{M})$ are the 1-*forms* on $\mathcal{M}$.

Remark 1.9. We now have the internal parity coming from the superstructure and the de Rham degree. We use the convention that the signs are governed by their sum, the total parity. Hence for a p–form ω, $|\omega|$ is p plus its internal degree.

We denote the pairing $\langle -, - \rangle : T\mathcal{M} \otimes T^*\mathcal{M} \to \mathcal{O}_\mathcal{M}$ such that the signs are given by $\langle fX, g\omega \rangle = (-1)^{(|X|+1)\cdot|g|} fg\langle X, \omega \rangle$. In particular $\omega(X) = (-1)^{|X|\cdot|\omega|}\langle X, \omega \rangle$.

Taking $\mathcal{O}_\mathcal{M}$-module tensor products of the sheaves above defines the differential forms $\Omega^p(\mathcal{M})$ as usual. The de Rham differential $\mathrm{d} : \mathcal{O}_\mathcal{M} \to T^*\mathcal{M}$ is fixed by $\langle X, \mathrm{d}f \rangle = Xf$ and extended uniquely to $\Omega^*(\mathcal{M})$ by $\mathrm{d}^2 = 0$ and

$$\mathrm{d}(\alpha\beta) = \mathrm{d}\alpha \cdot \beta + (-1)^{|\alpha|}\alpha \cdot \mathrm{d}\beta \quad \text{ for } \alpha \text{ in } \Omega^p(\mathcal{M}).$$

For these sign rules, the differential of a function $f \in \mathcal{O}(\mathcal{M})$ is given by the coordinate differentials on the left of the coefficients, i.e.

$$\mathrm{d}f = \sum_A \mathrm{d}z^A \frac{\partial f}{\partial z^A}.$$

Definition 1.10 (Contraction, [13])**.** Let X be a vector field of $\mathcal{M}$ Then X defines the contraction operator $\iota_X : \Omega^p \to \Omega^{p-1}$ of parity $|X| + 1$ (it lowers the de Rham degree by one). It is characterized by

(1) $\iota_X f = 0$ for $f \in \Omega^0(\mathcal{M})$,

(2) $\iota_X \alpha = \langle X, \alpha \rangle$ for $\alpha \in \Omega^1(\mathcal{M})$,

(3) $\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{(|X|+1)|\alpha|}\alpha \wedge \iota_X(\beta)$, for $\alpha \in \Omega^p(\mathcal{M})$.

A contraction of a differential form $\alpha \in \Omega^p(\mathcal{M})$ with vector fields $X_1, ..., X_p$ is then to be understood as

$$\alpha(X_1, ..., X_p) := \iota_{X_p} ... \iota_{X_1}\alpha.$$

As in the classical case, the Lie derivative can be defined through flows; we will only state its characterization and Cartan's formula.

Definition 1.11. For vector fields X, Y on $\mathcal{M}$ and $f \in \mathcal{O}_\mathcal{M}$ the Lie derivative is characterized by

$$\mathcal{L}_X(f) = Xf, \quad \mathcal{L}_X(Y) = [X, Y].$$

Furthermore, we have the equality of $\mathrm{Der}_{\mathcal{O}_\mathcal{M}}(\Omega^*(\mathcal{M}))$ given by

$$\mathcal{L}_X = \iota_X \mathrm{d} + (-1)^{|X|}\mathrm{d}\iota_X.$$

Definition 1.12 ([27])**.** For a super vector bundle $\mathcal{V}$ over $\mathcal{M}$, a *connection* on $\mathcal{V}$ is an even $\mathbb{R}$-linear morphism of sheaves

$$\nabla : \mathcal{V} \longrightarrow T^*\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{V}$$

satisfying the Leibniz rule

$$\nabla(fs) \;=\; \mathrm{d}f \otimes s + f\nabla s$$

for all homogeneous $f \in \mathcal{O}_\mathcal{M}(U)$ and $s \in \mathcal{V}(U)$ on any open $U \subseteq |\mathcal{M}|$.

In particular a connection assigns to each homogeneous vector field $X \in T\mathcal{M}(U)$ an $\mathbb{R}$-linear endomorphism $\nabla_X : \mathcal{V}(U) \to \mathcal{V}(U)$ of parity $|X|$, the *covariant derivative along* X, such that for all homogeneous $f \in \mathcal{O}_{\mathcal{M}}(U)$, $X, Y \in T\mathcal{M}(U)$, and $s \in \mathcal{V}(U)$:

$$\begin{aligned} \nabla_{X+Y}\, s &= \nabla_X s + \nabla_Y s, \\ \nabla_{fX}\, s &= f\, \nabla_X s, \\ \nabla_X(fs) &= X(f)\, s + (-1)^{|X|\,|f|}\, f\, \nabla_X s. \end{aligned}$$

One can define the covariant differential d^∇ associated to ∇ as an odd map $\Omega^* \otimes \mathcal{V} \to \Omega^{*+1} \otimes \mathcal{V}$ with

$$\begin{aligned} \nabla_X s &= \iota_X(\mathrm{d}^\nabla s), \\ \mathrm{d}^\nabla(\alpha \wedge \beta) &= \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathrm{d}^\nabla \beta, \end{aligned}$$

where $\alpha \in \Omega(\mathcal{M}), \beta \in \Omega(\mathcal{M}) \otimes \mathcal{V}$. The *curvature* of ∇ is the two form $R^\nabla \in \Omega^2(\mathcal{M}) \otimes \mathrm{End}(\mathcal{V})$ given by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - (-1)^{|X||Y|} \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

The connection is *flat* if $R^\nabla = 0$.

If we take $\mathcal{V}$ to be $T\mathcal{M}$, we define the torsion of ∇ to be the $(1, 2)$-tensor T^∇ defined by

$$T^\nabla(X, Y) = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y].$$

While supermanifolds now incorporate sign rules in the algebra of fields, we will need one more labelling.

Definition 1.13. Let $\mathcal{M}$ be a supermanifold and suppose the odd fiber $V = \oplus_k V_k$ and the even model space $W = \oplus_k W_k$ are both $\mathbb{Z}$-graded vector spaces, with only finitely many V_k, W_k nonzero. This induces a grading on the polynomial subalgebra $\mathrm{Sym}\, W^* \otimes \wedge V^*$ of $C^\infty(\phi_\alpha(U)) \otimes \wedge^\bullet V^*$: a linear function x^i on W_k is assigned degree $\mathrm{gh}(x^i) = -k$, and a linear function θ^α on V_k degree $\mathrm{gh}(\theta^\alpha) = -k$. If the chart transitions $\Phi_\alpha \circ \Phi_\beta^{-1}$ respect this grading, we call it a (global) $\mathbb{Z}$-grading on $\mathcal{M}$ and say that $\mathcal{M}$ is a $\mathbb{Z}$-graded supermanifold.

Remark 1.14. We will use the $\mathbb{Z}$-grading only as a labelling mechanism; the sign rules for graded commutativity are exclusively dictated by the parity which we will always denote by $|a|$. In particular they do not have to be compatible, i.e. we do not enforce $|a| \equiv \mathrm{gh}(a) \bmod 2$.

Example 1.15. A distinguished vector field, which also exists for conventional manifolds but is not globally well-defined, is the *Euler vector field* given in local coordinates by

$$\mathbb{E} = \sum_i \mathrm{gh}(x^i) \cdot x^i \frac{\partial}{\partial x^i} + \sum_\alpha \mathrm{gh}(\theta^\alpha) \cdot \theta^\alpha \frac{\partial}{\partial \theta^\alpha}$$

On a $\mathbb{Z}$-graded manifold it indeed glues to a global vector field, since the transition maps respect the grading and are therefore polynomial-homogeneous.

Applying $\mathbb{E}$ on $f \in \mathcal{O}_{\mathcal{M}}$ of well-defined degree $\mathrm{gh}(f)$ yields $\mathbb{E}f = \mathrm{gh}(f) \cdot f$.

Remark 1.16. Since $\mathbb{E}$ is even, Cartan's formula (Definition 1.11) gives $\mathcal{L}_{\mathbb{E}} = \iota_{\mathbb{E}}\mathrm{d} + \mathrm{d}\iota_{\mathbb{E}}$, and on a form α of homogeneous internal $\mathbb{Z}$-degree $\mathrm{gh}(\alpha)$ one has $\mathcal{L}_{\mathbb{E}}\alpha = \mathrm{gh}(\alpha)\, \alpha$ (here gh is the internal degree alone; the de Rham d raises the form degree but preserves gh). Hence if α is closed with $\mathrm{gh}(\alpha) \neq 0$, then

$$\alpha = \mathrm{d}\Big(\tfrac{1}{\mathrm{gh}(\alpha)} \iota_{\mathbb{E}} \alpha \Big),$$

so closed forms of nonzero internal degree are exact, with explicit primitive $\frac{1}{\mathrm{gh}(\alpha)} \iota_{\mathbb{E}} \alpha$. In particular the de Rham cohomology of a $\mathbb{Z}$-graded manifold is concentrated in internal degree 0.

Our main examples of $\mathbb{Z}$-graded supermanifolds will be of the following form.

Example 1.17. Let $E \to M$ be a graded vector bundle where the fibres are graded by odd degrees. We can then define the $\mathbb{Z}$-graded supermanifold $\mathcal{M}$ with $|\mathcal{M}| = M$ and $\mathcal{O}_{\mathcal{M}} = \Gamma(M, \wedge^\bullet E^*)$. Two examples of this construction are the following.

- $\mathcal{M} = T[1]M$. Locally we have coordinates x^i of M with $\mathrm{gh}(x^i) = 0$ and we declare $\mathrm{gh}(y^i) = 1$ for the fiber coordinates y^i. We can globally identify $\mathcal{O}_{\mathcal{M}}$ with forms on M, i.e. $\mathcal{O}_{\mathcal{M}} \cong \Omega^\bullet(M)$ where we identify $y^i \rightsquigarrow \mathrm{d}x^i$.

- $\mathcal{M} = T^*[-1]M$. Likewise, but with $\mathrm{gh}(y_i) = -1$. The identification now reads $\mathcal{O}_{\mathcal{M}} \cong \Gamma(M, \wedge^k TM)$ with the degrees flipped (a function of degree $-k$ corresponds to a k-vector field on M), again via $y_i \rightsquigarrow \partial/\partial x^i$.

1.2.2 Integration on supermanifolds

For readability we henceforth write $C^\infty(\mathcal{M}) := \mathcal{O}_{\mathcal{M}}(|\mathcal{M}|)$ for the global functions (and $C^\infty_c(\mathcal{M})$ for the compactly supported ones), used interchangeably with $\mathcal{O}_{\mathcal{M}}$.

Definition 1.18. Let $\mathcal{M} = \Pi E$ be the $(n|m)$-supermanifold of a vector bundle $E \to M$ as constructed in Example 1.3. Define the real line bundle $\mathrm{Ber}(\mathcal{M}) = \wedge^n T^*M \otimes \wedge^m E$ over $|\mathcal{M}| = M$; its sections $\mu \in \Gamma(\mathrm{Ber}(\mathcal{M}))$ are called *Berezinians.* They play the role top-degree forms play on an ordinary manifold, and let us integrate compactly supported elements of the structure sheaf,

$$\int_{\mathcal{M}} \mu \cdot - : C^\infty_c(\mathcal{M}) \to \mathbb{R}, \quad f \mapsto \int_M \langle \mu, (f)_m \rangle.$$

where $(f)_m$ is the component of f with the top exterior power of E^* and $\langle -, - \rangle$ denotes the conventional pairing of E with E^*.

Since $\mathrm{Ber}(\mathcal{M})$ is a line bundle over M alone, its sections carry no dependence on the fiber directions. More generally we set $\widetilde{\mathrm{Ber}}(\mathcal{M}) := \mathrm{Ber}(\mathcal{M}) \otimes \wedge^\bullet E^*$, whose sections, the *general Berezinians* $\mathrm{BER}(\mathcal{M}) := \Gamma(M, \widetilde{\mathrm{Ber}}(\mathcal{M})) = \Gamma(M, \mathrm{Ber}(\mathcal{M})) \otimes_{C^\infty(M)} C^\infty(\mathcal{M})$, form a module over $C^\infty(\mathcal{M})$. Integration then becomes a map $\int_{\mathcal{M}} : \mathrm{BER}(\mathcal{M}) \to \mathbb{R}$.

Example 1.19. For $T[1]M$ as given in Example 1.17, there is a special Berezinian μ characterized by

$$\int_{T[1]M} \mu \cdot f = \int_M \tilde{f},$$

where $\tilde{f} \in \Omega^\bullet(M)$ is the differential form given by the identification of $f \in C^\infty(\mathcal{M}) \rightsquigarrow \Omega^\bullet(M)$ in Example 1.17. In local coordinates we can write $\mu = \prod_i (\mathrm{d}x^i D\theta^i) =: \mathrm{d}^n x D^n \theta$.

Definition 1.20. Given a vector field X on $\mathcal{M}$ and a Berezinian $\mu \in \mathrm{BER}(\mathcal{M})$, the *divergence* $\mathrm{div}_\mu(X) \in C^\infty(\mathcal{M})$ is the function characterized by

$$\int_{\mathcal{M}} \mu \, Xf = -\int_{\mathcal{M}} \mu \, \mathrm{div}_\mu(X) f, \quad \text{for } f \in C^\infty_c(\mathcal{M}).$$

This is in essence integration by parts: in local coordinates x^i, θ^a with the Berezinian $\mu = \mathrm{d}^n x D^m \theta$, a vector field

$$X = \sum_i X^i(x,\theta) \frac{\partial}{\partial x^i} + \sum_a X^a(x,\theta) \frac{\partial}{\partial \theta^a}$$

has divergence

$$\mathrm{div}_\mu(X) = \sum_i \frac{\partial}{\partial x^i} X^i - (-1)^{|X|} \sum_a \frac{\partial}{\partial \theta^a} X^a.$$

We will need the following lemma later.

Lemma 1.21. For two Berezinians μ, μ' with $\mu' = \rho\mu$ with a nonvanishing $\rho \in C^\infty(\mathcal{M})$, we have

$$\mathrm{div}'_\mu(X) = \mathrm{div}_\mu(X) + \frac{X(\rho)}{\rho}.$$

1.2.3 Symplectic structure

We will now endow the supermanifolds with extra structure which will be fundamental for BV formalism.

Definition 1.22 (Odd-symplectic structure)**.** An *odd-symplectic structure* on $\mathcal{M}$ is a closed, non-degenerate, odd 2-form $\omega \in \Omega^2(\mathcal{M})$. A supermanifold carrying such a structure is called *odd-symplectic.*

In particular, only $(n|n)$-dimensional supermanifolds can be odd-symplectic.

Definition 1.23 (Lagrangian)**.** A submanifold $\mathcal{L}$ of an odd-symplectic manifold is *Lagrangian* if it is isotropic, $\omega|_{\mathcal{L}} = 0$, and maximal among isotropic submanifolds.

Much of conventional symplectic geometry carries over: one still has Darboux coordinates (x^i, ξ_i) with $\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}\xi_i$ locally, Weinstein's tubular neighbourhood theorem, graph Lagrangians, and so on.

Definition 1.24 (Hamiltonian vector field)**.** For $f \in C^\infty(\mathcal{M})$ we define the vector field $X_f \in \mathfrak{X}(M)$ by the equation $\iota_{X_f}\omega = \mathrm{d}f$ and call it the *Hamiltonian vector field* generated by the Hamiltonian f.

Note that the operator ι_{X_f} has parity $|X_f| + 1$, $|\omega| = 1$ and $\mathrm{d}f$ has parity $|f| + 1$. Putting it all together it follows that $|X_f| = |f| + 1$.

Definition 1.25 (Antibracket)**.** The *antibracket* is the operator $(-,-) : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ given by $(f, g) = (-1)^{|f|} X_f(g)$.

Example 1.26. Let (x^i, ξ_i) be a Darboux chart of $\mathcal{M}$ with x^i even and ξ_i odd, and $f \in C^\infty(\mathcal{M})$. Write $X_f = \sum_i A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial \xi_i}$; from $|X_f| = |f| + 1$ we then have the parities $|A^i| = |f| + 1$ and $|B^i| = |f|$. From Definition 1.10 (3) it follows that

$$\iota_{X_f}\omega = \sum_i \mathrm{d}\xi_i A^i + (-1)^{|f|} \mathrm{d}x^i B^i$$

Matching the coefficients with those of $\mathrm{d}f$, we obtain

$$X_f = \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x^i} + (-1)^{|f|} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial \xi_i}$$

and the antibracket hence reads

$$(f, g) = (-1)^{|f|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x^i} + \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \xi_i}.$$

Due to the numerous signs involved, it is useful to establish a relation between the antibracket and the inverse of the symplectic form in arbitrary coordinates.

Definition 1.27 (Dressed inverse)**.** Let z^i be local coordinates on $\mathcal{M}$ with parities $p_i := |z^i|$ and let $\omega = \frac{1}{2}\omega_{ij}\mathrm{d}z^i\mathrm{d}z^j$ (equivalently, $\omega_{ij} = \iota_{\partial_j}\iota_{\partial_i}\omega$) be a symplectic form of parity $|\omega| = k$, so that $|X_f| = |f| + k$. The *dressed inverse* of ω_{ij} is the tensor $\Pi^{ij} := X_{z^i}(z^j)$, of parity $p_i + p_j + k$ and satisfies

$$\sum_j (-1)^{(p_k+1)(p_i+p_j+k)}\,\Pi^{ij}\omega_{jk} = \delta^i_k, \tag{1}$$

i.e. it inverts the components of ω up to the signs of the graded contraction.

Lemma 1.28. For ω odd, the antibracket reads in coordinates

$$(f,g) = \sum_{i,j}(-1)^{p_i|f|}\frac{\partial f}{\partial z^i}\Pi^{ij}\frac{\partial g}{\partial z^j}, \qquad f,g \in \mathcal{O}_{\mathcal{M}}. \tag{2}$$

In particular $(z^i, z^j) = (-1)^{p_i}\Pi^{ij}$.

Proof. Let $f \in \mathcal{O}_{\mathcal{M}}$ and write $X_f = X^j\partial_j$ in components, where $X^j = X_f(z^j)$ has parity $|f|+1+p_j$ due to $|X_f| = |f|+1$. Contracting the defining equation $\iota_{X_f}\omega = \mathrm{d}f$ of Definition 1.24 with ι_{∂_k} gives

$$\sum_j (-1)^{(p_k+1)(|f|+1+p_j)}X^j\omega_{jk} = \iota_{\partial_k}\mathrm{d}f = \frac{\partial f}{\partial z^k}, \tag{†}$$

where we used $\iota_{X^j\partial_j} = X^j\iota_{\partial_j}$ and moved ι_{∂_k}, an operator of parity p_k+1, past the coefficient X^j. Choosing $f = z^i$ we have $X^j = \Pi^{ij}$, which is precisely (1) with $k = 1$.

For general f the equations (†) determine X_f uniquely due to the non-degeneracy of ω. We verify that they are solved by

$$X^j = \sum_i (-1)^{(p_i+1)(p_i+|f|)}\frac{\partial f}{\partial z^i}\Pi^{ij}.$$

Inserting, we split the sign of (†) as

$$(p_k+1)(|f|+1+p_j) = (p_k+1)(p_i+p_j+1) + (p_k+1)(|f|+p_i),$$

where the second summand does not depend on j. The sum over j is then exactly (1) and we compute

$$\begin{aligned}\sum_{i,j}(-1)^{(p_k+1)(|f|+1+p_j)}(-1)^{(p_i+1)(p_i+|f|)}\frac{\partial f}{\partial z^i}\Pi^{ij}\omega_{jk} &= \sum_i (-1)^{(p_i+1)(p_i+|f|)+(p_k+1)(|f|+p_i)}\frac{\partial f}{\partial z^i}\delta^i_k \\ &= \frac{\partial f}{\partial z^k},\end{aligned}$$

since for $i = k$ the two exponents agree and the sign squares to one. The antibracket then reads

$$(f,g) = (-1)^{|f|}X_f(g) = (-1)^{|f|+(p_i+1)(p_i+|f|)}\frac{\partial f}{\partial z^i}\Pi^{ij}\frac{\partial g}{\partial z^j}$$

and the exponent collapses modulo 2, $|f| + (p_i+1)(p_i+|f|) = |f| + p_i + p_i|f| + p_i + |f| = p_i|f|$. $\square$

For an even symplectic form the dressed inverse acquires, at points of the body, symmetry properties which we record for later use.

Lemma 1.29. Let ω be even, $k = 0$, and let $x \in |\mathcal{M}|$ be a point of the body. Then:

(1) $\omega_{ij}(x) = \Pi^{ij}(x) = 0$ unless $p_i = p_j$, and on the surviving blocks the dressing of (1) is trivial, $\sum_j \Pi^{ij}(x)\omega_{jk}(x) = \delta^i_k$;

(2) $\omega_{ji}(x) = -(-1)^{p_ip_j}\omega_{ij}(x)$ and $\Pi^{ji}(x) = -(-1)^{p_ip_j}\Pi^{ij}(x)$: the even block is antisymmetric, the odd block symmetric.

Proof. For (1), ω_{ij} and Π^{ij} are functions of parity $p_i + p_j$; at x all odd coordinates vanish, and an odd function has no constant term, so the mixed parity components vanish, while on the remaining blocks $p_i + p_j = 0$ kills the dressing. For (2), the contractions graded commute, so $\omega_{ij} = \iota_{\partial_j}\iota_{\partial_i}\omega = (-1)^{(p_i+1)(p_j+1)}\omega_{ji}$, which on the blocks $p_i = p_j =: p$ reads $\omega_{ji} = (-1)^{p+1}\omega_{ij} = -(-1)^{p_ip_j}\omega_{ij}$; blockwise inversion of (1) transfers both symmetries to $\Pi(x)$. $\square$

A noteworthy result of Schwarz [31], however, has no analogue in conventional symplectic geometry.

Theorem 1.30. Every odd-symplectic supermanifold $(\mathcal{M}, \omega)$ is globally symplectomorphic to $(\Pi T^*|M|, \omega_{\text{stand}})$, the odd cotangent bundle of its body with its standard odd-symplectic structure.

From now on our odd-symplectic supermanifold $(\mathcal{M}, \omega)$ comes equipped with a Berezinian $\mu \in \mathrm{BER}(\mathcal{M})$.

Definition 1.31 (Compatibility)**.** A Berezinian μ is *compatible* with ω if there is an atlas of Darboux charts in which $\mu = \mathrm{d}^n x D^n \xi$.

Definition 1.32 (BV Laplacian)**.** Let $(\mathcal{M}, \omega, \mu)$ be an odd-symplectic manifold with an arbitrary, not necessarily compatible, Berezinian μ. We define the operator $\Delta_\mu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ by

$$\Delta_\mu(f) = \frac{1}{2}\mathrm{div}_\mu(X_f).$$

In a Darboux chart (x^i, ξ_i) on $\mathcal{M}$, under the assumption that we can locally write $\mu = \rho(x,\xi)\mathrm{d}^n x D^n \xi$ with ρ a local density function, Example 1.26 and Lemma 1.21 yields

$$\Delta_\mu(f) = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} f + \frac{1}{2}(\log\rho, f).$$

In the case that μ is compatible with ω (and therefore $\rho = 1$) we have the induced operator

$$\Delta := \Delta_\mu = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}, \quad \mu \text{ compatible},$$

which we call the *Batalin–Vilkovisky* Laplacian. Due to the contraction of symmetric and antisymmetric indices it squares to 0, $\Delta^2 = 0$.

We now turn to partition functions in the BV formalism. First note that for a supermanifold $\mathcal{N}$ one has the relation

$$\mathrm{Ber}(\Pi T^*\mathcal{N})|_{\mathcal{N}} \cong \mathrm{Ber}(\mathcal{N})^{\otimes 2}, \tag{3}$$

where the restriction to $\mathcal{N}$ should be understood as the pullback of the inclusion map (or equivalently, setting the fiber coordinates generated by ΠT^* to 0). Indeed, by Batchelor's theorem (Theorem 1.4) we may write $\mathcal{N} \cong \Pi E$ for a vector bundle $E \to |\mathcal{N}|$, the ΠT^* functor generates new base coordinates (conjugate to the odd fiber coordinates of E) and new fiber coordinates (conjugate to the base coordinates of $|\mathcal{N}|$). This gives a new base E^* and a new fiber $E \oplus T^*|\mathcal{N}|$. Applying now the definition of $\mathrm{Ber}(-)$ to these spaces and restricting to $\mathcal{N}$ yields the identity above.

Since $\mathrm{Ber}(\mathcal{N})$ is a line bundle, we may take the square root of the image of $\mu \in \mathrm{Ber}(\Pi T^*\mathcal{N})|_\mathcal{N}$ under (3), written $\sqrt{\mu|_\mathcal{N}}$. In local coordinates X^A on $\mathcal{N}$ (of either parity), with induced Darboux fiber coordinates Θ_A on $\Pi T^*\mathcal{N}$, this square-root map reads

$$\sqrt{-|_\mathcal{N}} : \mathrm{Ber}(\Pi T^*\mathcal{N}) \to \mathrm{Ber}(\mathcal{N}), \quad \rho(X,\Theta)DXD\Theta \mapsto \sqrt{\rho(X,0)}DX.$$

Definition 1.33 (BV integral)**.** For $(\mathcal{M},\omega,\mu)$ an odd-symplectic manifold with a compatible Berezinian, a *BV* integral is an integral of the form

$$\int_{\mathcal{L}\subset\mathcal{M}} f\sqrt{\mu|_\mathcal{L}}$$

with $\mathcal{L}$ a Lagrangian submanifold of $\mathcal{M}$ and $f \in C^\infty(\mathcal{M})$ a function satisfying $\Delta_\mu f = 0$.

The following theorem is central to BV theory.

Theorem 1.34 (BV–Stokes, [31])**.** *Let $(\mathcal{M},\omega,\mu)$ be an odd-symplectic manifold with compact body endowed with a compatible Berezinian.*

(1) For any $g \in C^\infty(\mathcal{M})$ and $\mathcal{L} \subset \mathcal{M}$ a Lagrangian submanifold, one has

$$\int_\mathcal{L} \Delta_\mu g\sqrt{\mu|_\mathcal{L}} = 0.$$

(2) Let $\mathcal{L}$ and $\mathcal{L}'$ be two Lagrangian submanifolds whose bodies are homologous cycles in the body of $\mathcal{M}$ and let $f \in C^\infty(\mathcal{M})$ be a function satisfying $\Delta_\mu f = 0$. Then the following holds:

$$\int_\mathcal{L} f\sqrt{\mu|_\mathcal{L}} = \int_{\mathcal{L}'} f\sqrt{\mu|_{\mathcal{L}'}}.$$

Later we will treat Lagrangians simplicially, via embeddings $\iota : \Delta^n \times \mathcal{L} \to \mathcal{M}$. Part (2) above then says that for $\iota : \Delta^1 \times \mathcal{L} \to \mathcal{M}$ with $\iota(0,\mathcal{L}) = \mathcal{L}_0$ and $\iota(1,\mathcal{L}) = \mathcal{L}_1$ we have

$$\int_{\partial\iota} f\sqrt{\mu|_\mathcal{L}} = 0, \quad \text{for } \Delta_\mu f = 0,$$

where $\partial\iota = \mathcal{L}_1 - \mathcal{L}_0$. This theorem will later be refined to give a BV analogue of the de Rham theorem.

1.2.4 Algebraic structure

We now collect the algebraic structures we will work with. As before, $|a|$ denotes the parity of a, which need *not* agree mod 2 with its $\mathbb{Z}$-grading.

Definition 1.35 (BV algebra)**.** A *BV algebra* is a unital commutative graded algebra $(\mathcal{A},\cdot)$ over $\mathbb{R}$ equipped with

- A degree $+1$ antibracket $(-,-) : \mathcal{A}^j \otimes \mathcal{A}^k \to \mathcal{A}^{j+k+1}$ satisfying
  - skew-symmetry: $(x,y) = -(-1)^{(|x|+1)(|y|+1)}(y,x)$,
  - Leibniz identities:
$$(x,yz) = (x,y)z + (-1)^{(|x|+1)|y|}y(x,z),$$
$$(xy,z) = x(y,z) + (-1)^{|y|(|z|+1)}(x,z)y.$$
  - Jacobi identity:
$$(x,(y,z)) = ((x,y),z) + (-1)^{(|x|+1)(|y|+1)}(y,(x,z)).$$

- In addition, $\mathcal{A}^\bullet$ should carry a BV Laplacian, an $\mathbb{R}$-linear map $\Delta : \mathcal{A}^j \to \mathcal{A}^{j+1}$ satisfying
  - $\Delta^2 = 0$,
  - $\Delta(1) = 0$ (with 1 the unit in $\mathcal{A}^\bullet$),
  - second order Leibniz identity
  $$\Delta(xyz) \pm \Delta(xy)z \pm \Delta(xz)y \pm \Delta(yz)x \pm \Delta(x)yz \pm \Delta(y)xz \pm \Delta(z)xy = 0, \quad (4)$$
  - the antibracket arises as the "defect" of the first order Leibniz identity for Δ:
  $$\Delta(xy) = \Delta x \cdot y + (-1)^{|x|} x \cdot \Delta y + (-1)^{|x|}(x, y). \quad (5)$$

This is precisely the structure that an odd-symplectic, $\mathbb{Z}$-graded supermanifold with compatible Berezinian $(\mathcal{M}, \omega, \mu)$ has. The algebra is given by $\mathcal{A} = C^\infty(\mathcal{M})$, the antibracket by Definition 1.25 and the BV Laplacian by Definition 1.32. The grading is given by gh.

We are mostly interested in the elements of a BV algebra solving the following equations.

Definition 1.36 (Classical master equation)**.** In a *BV* algebra $(\mathcal{A}, \cdot, (-,-), \Delta)$, the *classical master equation* is solved by those $S \in \mathcal{A}^0$ with
$$(S, S) = 0,$$
and we denote this solution set $\mathrm{MC}(\mathcal{A})$.

Definition 1.37 (Quantum master equation)**.** In a *BV* algebra $(\mathcal{A}, \cdot, (-,-), \Delta)$, the *quantum master equation* is solved by those $S \in \mathcal{A}^0[[\hbar]]$ with
$$\frac{1}{2}(S, S) - i\hbar\Delta S = 0,$$
and we denote this solution set $\mathrm{MC}_\hbar(\mathcal{A})$.

Definition 1.38 (Quantum canonical BV transformation)**.** We say that two solutions $S, S' \in \mathrm{MC}_\hbar(\mathcal{A})$ are *equivalent* if there is a smooth family $S_t \in \mathcal{A}^0[[\hbar]]$ with $S_0 = S$, $S_1 = S'$ and a generator $R_t \in \mathcal{A}^{-1}[[\hbar]]$ such that
$$\frac{\mathrm{d}}{\mathrm{d}t} S_t = (S_t, R_t) - i\hbar\Delta R_t.$$

1.2.5 Half-densities

Next we absorb the Berezinian into the fields themselves; this will let us drop the choice of μ from the constructions.

Definition 1.39 ((Half-)densities)**.** A *density of weight* $\varkappa \in \mathbb{R}$ on an $(n|m)$-supermanifold $\mathcal{M}$ is a collection of local functions $\rho_{(\alpha)}(x_{(\alpha)}, \theta_{(\alpha)})$, one per chart of an atlas of $\mathcal{M}$, transforming on overlaps as
$$\rho_{(\alpha)}(x_{(\alpha)}, \theta_{(\alpha)}) = \rho_{(\beta)}(x_{(\beta)}, \theta_{(\beta)}) \cdot \left| \mathrm{Sdet} \frac{\partial(x_{(\alpha)}, \theta_{(\alpha)})}{\partial(x_{(\beta)}, \theta_{(\beta)})} \right|^\varkappa.$$
Here Sdet is the *superdeterminant*: for a block supermatrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with even diagonal blocks A, D and odd off-diagonal blocks B, C, one has $\mathrm{Sdet} = \det(A - BD^{-1}C)\det(D)^{-1}$. We denote the space of $\varkappa$-densities by $\mathrm{Dens}^\varkappa(\mathcal{M})$; for $\varkappa = 1$ we recover Berezinians. A *half-density* on $(\mathcal{M}, \omega)$ is a $\frac{1}{2}$-density. We write it in a Darboux chart (x^i, ξ_i) as
$$\sigma = \sigma(x, \xi) \cdot d^{\frac{1}{2}}x\, \mathcal{D}^{\frac{1}{2}}\xi,$$
where the symbol $d^{\frac{1}{2}}x\, \mathcal{D}^{\frac{1}{2}}\xi$ transforms by the square root of the Berezinian Jacobian.

Globally, half-densities are sections of the square root of the Berezinian line bundle:

$$\mathrm{Dens}^{\frac{1}{2}}\mathcal{M} = \Gamma(\mathcal{M}, \mathrm{Ber}(\mathcal{M})^{\otimes\frac{1}{2}}).$$

Definition 1.40 (Canonical BV Laplacian, [26])**.** The *canonical BV Laplacian* $\Delta : \mathrm{Dens}^{\frac{1}{2}}\mathcal{M} \to \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}$ is given in a Darboux chart by

$$\Delta : \sigma(x,\xi)\, d^{\frac{1}{2}}x\, \mathcal{D}^{\frac{1}{2}}\xi \longmapsto \left(\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}\sigma(x,\xi)\right) d^{\frac{1}{2}}x\, \mathcal{D}^{\frac{1}{2}}\xi.$$

One can show that the expression is globally well-defined and squares to zero.

Unlike the Schwarz BV Laplacian Δ_μ of Definition 1.32, this operator requires no choice of Berezinian. A compatible Berezinian μ on $\mathcal{M}$ does however provide a reference half-density $\sqrt{\mu} \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}$, and multiplication by it intertwines the two operators:

$$\Delta(\sqrt{\mu}\cdot f) = \sqrt{\mu}\cdot \Delta_\mu(f), \quad f \in C^\infty(\mathcal{M}). \tag{6}$$

The other key feature of half-densities is that they restrict to Berezinians on Lagrangians. For $\mathcal{L} \subset \mathcal{M}$ Lagrangian, the identity (3) (applied to $\mathcal{N} = \mathcal{L}$ together with the Weinstein neighborhood $\mathcal{M} \cong \Pi T^*\mathcal{L}$) yields a canonical restriction map

$$\mathrm{Dens}^{\frac{1}{2}}\mathcal{M} \to \mathrm{BER}(\mathcal{L}), \quad \sigma \mapsto \sigma|_{\mathcal{L}}.$$

In a Darboux chart (X^α, Ξ_α) for $\mathcal{M}$ in which $\mathcal{L} = \{\Xi = 0\}$, this map reads $\sigma(X,\Xi)\,\mathcal{D}^{\frac{1}{2}}X\,\mathcal{D}^{\frac{1}{2}}\Xi \mapsto \sigma(X,0)\,\mathcal{D}X$.

Definition 1.41 (BV integral, half-density version)**.** For $\sigma \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}$ and $\mathcal{L} \subset \mathcal{M}$ a Lagrangian submanifold, the *BV integral* of σ over $\mathcal{L}$ is

$$\int_{\mathcal{L}} \sigma := \int_{\mathcal{L}} \sigma|_{\mathcal{L}}.$$

The BV Stokes' theorem carries over verbatim via (6): for any $\beta \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}$ one has $\int_{\mathcal{L}} \Delta\beta = 0$, and for Δ-closed σ the integral $\int_{\mathcal{L}}\sigma$ depends only on the homology class of the body of $\mathcal{L}$. Whenever we have a compatible Berezinian μ on $\mathcal{M}$, this language recovers the earlier $\int_{\mathcal{L}} f\sqrt{\mu|_{\mathcal{L}}}$ upon setting $\sigma = f\cdot\sqrt{\mu}$.

1.3 Fiber BV integrals

We now come to the central player of the thesis, the *effective action*. It amounts to integrating out some of the BV fields while preserving the BV structure on the rest.

Let (B, ω_B) and (F, ω_F) be two odd-symplectic supermanifolds and consider their product

$$\mathcal{F} = B \times F, \quad \omega = \omega_B \oplus \omega_F,$$

which is again odd-symplectic. Denote by $\pi : \mathcal{F} \to B$ the projection onto the first factor.

Definition 1.42 (BV pushforward)**.** For $\mathcal{L} \subset F$ a Lagrangian submanifold, the *BV pushforward* along π is the $\mathbb{R}$-linear map

$$\pi_*^{(\mathcal{L})} : \mathrm{Dens}^{\frac{1}{2}}\mathcal{F} \to \mathrm{Dens}^{\frac{1}{2}}B, \quad \sigma \mapsto \int_{\mathcal{L}\subset F}\sigma,$$

i.e. for fixed $x' \in B$ one integrates the half-density σ over the copy of $\mathcal{L}$ in the fiber $\pi^{-1}(x')$. Equivalently, under the isomorphism $\mathrm{Dens}^{\frac{1}{2}}\mathcal{F} \cong \mathrm{Dens}^{\frac{1}{2}}B \otimes \mathrm{Dens}^{\frac{1}{2}}F$, $\pi_*^{(\mathcal{L})}$ is the identity on the first factor tensored with the ordinary BV integral on F.

The fundamental property is that the BV pushforward commutes with the canonical BV Laplacians and is well-behaved under Lagrangian homotopies.

Definition 1.43 (Lagrangian homotopy)**.** Let $(\mathcal{M}, \omega)$ be an odd-symplectic supermanifold. Two Lagrangians $\mathcal{L}_0 \sim \mathcal{L}_1$ are homotopic if there is a smooth family of Lagrangian embeddings $\iota : [0,1] \times \mathcal{L} \to \mathcal{M}$ with $\iota_0 = \mathcal{L}_0$ and $\iota_1 = \mathcal{L}_1$ (writing $\iota_t := \iota(t, -)$). Its associated velocity vector field is defined as

$$X := \mathrm{d}\iota\left(\frac{\partial}{\partial t}\right) \in \Gamma(\iota_t^* T\mathcal{M})$$

and its associated time dependent one form

$$\beta := \iota_t^*(\iota_X \omega) \in \Omega^1(L) \otimes \Omega^1([0,1]).$$

The Lagrangian condition reads $\iota_t^* \omega = 0$ and hence $\iota_t^*(\mathcal{L}_X \omega) = \frac{\mathrm{d}}{\mathrm{d}t}(\iota_t^* \omega) = 0$. By Cartan's magic formula and the closedness of ω we immediately obtain that $\mathrm{d}\beta = 0$. Because β has ghost degree -1 and the de Rham cohomology vanishes in nonzero ghost degree, we can define the Hamiltonian η of ι_t to be

$$\mathrm{d}\eta := \beta \in C^\infty(L) \otimes \Omega^1([0,1]), \tag{7}$$

which determines η uniquely: two solutions of $\mathrm{d}\eta = \beta$ differ by a d-closed 0-form, i.e. a constant, and constants sit in ghost degree 0 whereas η has ghost degree -1.

Since $\mathcal{F} = B \times F$ with $\omega = \omega_B \oplus \omega_F$, a Darboux chart on $\mathcal{F}$ is a union of Darboux charts on B and on F, and under $\mathrm{Dens}^{1/2}\mathcal{F} \cong \mathrm{Dens}^{1/2}B \otimes \mathrm{Dens}^{1/2}F$ the canonical BV Laplacian splits as

$$\Delta = \Delta_B + \Delta_F,$$

with Δ_B, Δ_F the BV Laplacians of the two factors.

Theorem 1.44 (BV–Stokes for fiber BV integrals, [10])**.** We have the two identities.

(1) $\pi_*^{(\mathcal{L})}$ is a chain map: for any $\sigma \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$,

$$\Delta_B \pi_*^{(\mathcal{L})} \sigma = \pi_*^{(\mathcal{L})} \Delta\sigma.$$

(2) Let $\mathcal{L}_0 \sim \mathcal{L}_1$ be two homotopic Lagrangians in F and $\sigma \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$ with $\Delta\sigma = 0$. Then

$$\pi_*^{(\mathcal{L}_1)}\sigma - \pi_*^{(\mathcal{L}_0)}\sigma = \Delta_B \int_0^1 \mathrm{d}t \int_L \eta\, \iota_t^*(\sigma).$$

Now assume both B and F are equipped with compatible Berezinians μ_B and μ_F; then $\mu = \mu_B \cdot \mu_F$ is compatible with ω on $\mathcal{F}$. In this case, half-densities on $\mathcal{F}$ of the form $\sigma = f \cdot \sqrt{\mu}$ with $f \in C^\infty(\mathcal{F})$ correspond to BV functions and we can recast the pushforward in terms of an *effective action*.

Definition 1.45 (Effective BV action)**.** Let $S \in C^\infty(\mathcal{F})_0[[\hbar]]$ be a solution of the quantum master equation on $\mathcal{F}$. The *effective action* S^{eff} on B induced by S via the fiber BV integral over $\mathcal{L} \subset F$ is defined by

$$\sqrt{\mu_B} \cdot e^{\mathrm{i}S^{\mathrm{eff}}/\hbar} = \pi_*^{(\mathcal{L})}\left(\sqrt{\mu} \cdot e^{\mathrm{i}S/\hbar}\right). \tag{8}$$

We will often omit the choice of the Lagrangian (which should be clear from the context) and just write S^{eff}.

Corollary 1.46. As a corollary of Theorem 1.44 we have the following.

(1) If S solves QME on $\mathcal{F}$, then S^{eff} solves QME on B.

(2) If $\mathcal{L}_0 \sim \mathcal{L}_1$ are homotopic Lagrangians in F, then $S_0^{\mathrm{eff}} \sim S_1^{\mathrm{eff}}$ under a quantum canonical BV transformation on B.

(3) If $S_0 \sim S_1$ under a quantum canonical BV transformation on $\mathcal{F}$, then $S_0^{\mathrm{eff}} \sim S_1^{\mathrm{eff}}$ under a quantum canonical BV transformation on B.

1.4 BV formalism

Definition 1.47 (Classical BV theory)**.** A *classical BV theory* consists of the following data:

- an odd symplectic structure $\omega \in \Omega^2(\mathcal{F})_{-1}$ (BV form),
- a function $S \in C^\infty(\mathcal{F})_0$ satisfying the CME (BV action).

Definition 1.48 (Differential graded manifolds)**.** A vector field Q on a $\mathbb{Z}$-graded supermanifold is called a *cohomological vector field* if Q has degree 1 and $Q^2 = 0$. Then the pair $(\mathcal{M}, Q)$ is called a *differential graded manifold* (dg manifold).

Remark 1.49. In a BV theory the Hamiltonian vector field of S is such a cohomological vector field: $Q := X_S = (S, -)$ squares to zero by the CME.

A more general structure that we will need later on is the following.

Definition 1.50 (Hamiltonian dg manifold)**.** A *Hamiltonian dg manifold* of degree k consists of the following data:

- a dg manifold $(\mathcal{M}, Q)$,
- a symplectic form $\omega \in \Omega^2(\mathcal{M})_k$ of internal degree k,
- a Hamiltonian $\Theta \in C^\infty(\mathcal{M})_{k+1}$ with $(\Theta, \Theta) = 0$ where $(-,-)$ is the induced bracket of degree $-k$ of ω.

We note that the case $k = -1$ is exactly that of a classical BV theory. The reason to introduce Hamiltonian dg manifolds is that the starting point of the construction of a so-called n-dimensional *AKSZ theory* [1] is a Hamiltonian dg manifold of degree $k = n - 1$, which will be extremely useful to construct a BV theory from scratch.

Definition 1.51 (Quantum BV theory)**.** A *quantum BV theory* consists of the following data:

- a $\mathbb{Z}$-graded supermanifold $\mathcal{F}$,
- an odd symplectic structure $\omega \in \Omega^2(\mathcal{F})_{-1}$,
- a Berezinian $\mu \in \mathrm{BER}(\mathcal{F})$ compatible with ω,
- a function $S \in C^\infty(\mathcal{F})_0[[\hbar]]$ satisfying the QME.

For a quantum BV theory one then defines the partition function as the BV integral over a Lagrangian $\mathcal{L} \subset \mathcal{F}$ as

$$Z = \int_{\mathcal{L} \subset \mathcal{F}} \sqrt{\mu|_{\mathcal{L}}}\, e^{iS/\hbar}.$$

If the critical points of S are degenerate on that particular $\mathcal{L}$, Theorem 1.34 allows us to deform the Lagrangian $\mathcal{L} \rightsquigarrow \mathcal{L}'$ on which they are non-degenerate. The difference is then Δ-exact, which is a pure gauge term. The final partition function can then be computed with the stationary phase formula.

The strength of the BV formalism is that it still applies when the gauge symmetry is *open*: the infinitesimal gauge transformations may span a distribution that is not integrable away from the critical locus. We encode them by a vector bundle $E \to X$ over the space of classical fields, with an anchor $\rho : E \to TX$; a local frame $\{e_a\}$ gives the gauge vector fields $v_a = \rho(e_a) = \sum_i v_a^i(x)\, \partial_{x^i}$, and the classical action $S_{\mathrm{cl}} \in C^\infty(X)$ is gauge invariant, $v_a(S_{\mathrm{cl}}) = 0$. We only require the distribution $\rho(E)$ to be integrable along the critical locus X_{crit}, to which it is automatically tangent since $\mathrm{d}S_{\mathrm{cl}}(v) = v(S_{\mathrm{cl}}) = 0$.

We then build a classical BV theory on $\mathcal{F} = T^*[-1]E[1]$, the shifted cotangent bundle of $E[1]$, and look for a BV action $S \in C^\infty(\mathcal{F})_0$ with $(S, S) = 0$ fixed by

(1) $S|_{E[1]} = S_{\mathrm{cl}}$, and

(2) the term of S linear in the ghosts c^a (the fiber coordinates of $E[1]$) being $\sum_{a,i} c^a v_a^i(x)\, x_i^+$, where the x_i^+ are the fiber coordinates on $T^*[-1]$ dual to the base coordinates x^i.

These fix S up to higher order,

$$S = S_{\mathrm{cl}}(x) + \sum_{a,i} c^a v_a^i(x)\, x_i^+ + \cdots ,$$

the remaining terms being at least quadratic in the c^a and determined by imposing the CME.

Constructed this way, the higher-order terms of S must be guessed by hand. The AKSZ construction, to which we turn next, instead yields a solution S of the CME directly from the geometric data of a target.

1.5 AKSZ construction

In the AKSZ construction [1] the field space $\mathcal{F}$ is given by the mapping space between a source and a target. These are given by the following data.

Source. This is a closed, oriented n-dimensional manifold Σ. We denote the de Rham differential on the source with d, and for convenience will always use the local coordinates t^i on Σ.
Target. This is a Hamiltonian dg manifold $\mathcal{N}$ of degree $n-1$ with an exact symplectic form. To recall, it comes equipped with a cohomological vector field $Q_{\mathcal{N}}$, an exact symplectic form $\omega_{\mathcal{N}} \in \Omega^2(\mathcal{N})_{n-1}$ ($\delta\alpha_{\mathcal{N}} = \omega_{\mathcal{N}}$, where δ is the de Rham differential on $\mathcal{N}$) and $\Theta_{\mathcal{N}} \in C^\infty(\mathcal{N})_n$ such that $(\Theta_{\mathcal{N}}, \Theta_{\mathcal{N}}) = 0$.

The field space is then given by

$$\mathcal{F} = \mathrm{Map}(T[1]\Sigma, \mathcal{N}).$$

This is an infinite-dimensional graded supermanifold, which we describe through its *functor of points*: by Yoneda a supermanifold X is determined by the presheaf $S \mapsto X(S) := \mathrm{Mor}(S, X)$ of its S-points, with S ranging over all supermanifolds. We *define* $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ (here $\mathcal{M} = T[1]\Sigma$) to be the supermanifold representing

$$S \longmapsto \mathrm{Mor}(S \times \mathcal{M}, \mathcal{N}) \qquad \text{(contravariant in } S\text{)},$$

equivalently characterized via the adjunction $\mathrm{Mor}(S, \mathrm{Map}(\mathcal{M}, \mathcal{N})) = \mathrm{Mor}(S \times \mathcal{M}, \mathcal{N})$, natural in S: an S-point of $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ is an S-family of maps $\mathcal{M} \to \mathcal{N}$. Yoneda makes the representing object unique; its existence as an infinite-dimensional supermanifold is supplied locally by the model below. For $\mathcal{N}$ a super vector space one has $\mathrm{Map}(\mathcal{M}, \mathcal{N}) = C^\infty(\mathcal{M}) \hat{\otimes} \mathcal{N}$, and for general $\mathcal{N}$ one uses this in coordinate charts of $\mathcal{N}$. The body consists of the pt-points, i.e. the degree-preserving maps $\mathrm{Mor}(\mathcal{M}, \mathcal{N})$.

The mapping space admits the usual constructions of differential geometry. Tangent vectors, for example, are velocity vectors of curves in the space; in the language of the functor of points this amounts to considering curves $\gamma(t) \in \mathrm{Map}(\mathcal{M}, \mathcal{N})(\mathbb{R}^{1|0}) = \mathrm{Mor}(\mathbb{R}^{1|0} \times \mathcal{M}, \mathcal{N})$ for even vectors and $\mathbb{R}^{0|1}$ for odd vectors respectively. The tangent space at some point of the body, i.e. at $\phi \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$, is given by

$$T_\phi \, \mathrm{Map}(\mathcal{M}, \mathcal{N}) \cong \Gamma\left(\mathcal{M}, \phi^* T\mathcal{N}\right) \ .$$

We can lift the structure of the target $\mathcal{N}$ to the space of fields. This is done via the evaluation and projection maps. Consider the diagram below.

$$\begin{tikzcd} T[1]\Sigma \times \mathcal{F} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & \mathcal{N} \\ \mathcal{F} & \end{tikzcd}$$

To lift an object such as $\phi \in \Omega^p(\mathcal{N})$, we pull it back via the evaluation map ev^*, pick its $(0, p)$ degree form (on $\mathcal{N}$) and integrate it against $T[1]\Sigma$, effectively taking only the top degree form of Σ. We then have a form $\Phi = \pi_* \mathrm{ev}^* \phi$ of de Rham degree p but with internal degree decreased by n. We denote $\mathbb{T} = \pi_* \mathrm{ev}^* : \Omega^p(\mathcal{N})_j \to \Omega^p(\mathcal{F})_{j-n}$. Conventionally, we write $\int_\Sigma$ for the integration over $T[1]\Sigma$, i.e. $\int_\Sigma = \int \mathrm{d}^n t \mathrm{d}^n \theta$ with Berezin integration in the odd fibers; honest integrals of forms over the source, once we specialize to the circle, are instead denoted $\int_{S^1}$.

This construction defines us the *superfields* by applying the idea to the coordinate functions x^A of $\mathcal{N}$:

$$\boldsymbol{x}^a = \mathrm{ev}^* x^a \quad \in \quad C^\infty(T[1]\Sigma) \hat{\otimes} C^\infty(\mathcal{F}) \cong \Omega^\bullet(\Sigma) \hat{\otimes} C^\infty(\mathcal{F})$$

hence we can expand the superfield $\boldsymbol{x}^a$ in terms of odd fiber coordinates $\theta^i = \mathrm{d}t^i$ of $T[1]\Sigma$ as

$$\boldsymbol{x}^a = \sum_{k=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x^a_{i_1 \cdots i_k}(t) \theta^{i_1} \cdots \theta^{i_k}$$

and we use the functions $x^a_{i_1 \ldots i_k}(t) \in C^\infty(\Sigma)$ as coordinate functions on $\mathcal{F}$. Note that for each part of the sum to preserve the degree, we have that $|x^a_{i_1 \ldots i_k}| = |x^a| - k$.

Applied to a form $\phi \in \Omega^p(\mathcal{N})$, the procedure in local coordinates just amounts to replacing every coordinate by its superfield:

$$\phi = \sum_{a_1, \ldots, a_p} \phi_{a_1 \cdots a_p}(x) \delta x^{a_1} \wedge \cdots \wedge \delta x^{a_p} \quad \rightsquigarrow \quad \int_\Sigma \sum_{a_1, \ldots, a_p} \phi_{a_1 \cdots a_p}(\boldsymbol{x}) \delta \boldsymbol{x}^{a_1} \wedge \cdots \wedge \delta \boldsymbol{x}^{a_p} .$$

For a symplectic form ω we therefore obtain $\omega = \frac{1}{2}\omega_{ij}(\boldsymbol{x})\delta\boldsymbol{x}^i \wedge \delta\boldsymbol{x}^j$. For reasons to be seen later we include an extra sign, defining $\omega = (-1)^n\mathbb{T}\omega_{\mathcal{N}}$; in superfields,

$$\omega = (-1)^n \frac{1}{2}\sum_{a,b}\int_\Sigma \omega_{ab}(\boldsymbol{x})\cdot\delta\boldsymbol{x}^a \wedge \delta\boldsymbol{x}^b. \tag{9}$$

Next we construct a cohomological vector field Q on $\mathcal{F}$. On the target we have $Q_{\mathcal{N}}$ and on the source $T[1]\Sigma$ we have the de Rham differential $\mathrm{d} = \theta^i\partial/\partial t^i$. Both of these vector fields integrate to a flow on the respective manifolds with an odd flow parameter ϵ. More precisely, they generate two actions Φ^{d}, $\Phi^{Q_{\mathcal{N}}}$ of the supergroup $\mathbb{R}^{0|1}$ on Σ and $\mathcal{N}$ respectively, which can be lifted to left and right actions of $\mathbb{R}^{0|1}$ on the mapping space by

$$\phi \mapsto \phi \circ \Phi^{\mathrm{d}}, \quad \phi \mapsto \Phi^{Q_{\mathcal{N}}} \circ \phi.$$

Infinitesimally we have

$$\Phi_\epsilon^{\mathrm{d}*}(t^i) = t^i + \epsilon\theta^i,\ \Phi_\epsilon^{\mathrm{d}*}(\theta^i) = \theta^i \quad \text{and} \quad \Phi_\epsilon^{Q_{\mathcal{N}}*}(x^a) = x^a + \epsilon Q_{\mathcal{N}}^a(x)$$

of the two group actions. We can now consider the infinitesimal transformations of the compositions above by reading off the ϵ coefficient; we call the two resulting vector fields $\mathrm{d}_\Sigma^{\mathrm{lifted}}$ and $Q_{\mathcal{N}}^{\mathrm{lifted}}$, and set

$$Q = \mathrm{d}_\Sigma^{\mathrm{lifted}} + Q_{\mathcal{N}}^{\mathrm{lifted}} \in \mathfrak{X}(\mathcal{F})_1.$$

Note that because left and right group actions commute, the two vector fields commute. Together with the fact that they're both cohomological, it follows that Q is too, i.e. $Q^2 = 0$.

To read off Q in coordinates we apply the two lifted actions to the superfield $\boldsymbol{x}^a = \mathrm{ev}^*x^a$, viewed as a coordinate function on $\mathcal{F}$. Precomposition with $\Phi_\epsilon^{\mathrm{d}}$ substitutes the source flow into the argument,

$$\boldsymbol{x}^a(t,\theta) \mapsto \boldsymbol{x}^a(t+\epsilon\theta,\theta) = \boldsymbol{x}^a + \epsilon\,\theta^i\partial_{t^i}\boldsymbol{x}^a = \boldsymbol{x}^a + \epsilon\,\mathrm{d}\boldsymbol{x}^a,$$

so $\mathrm{d}_\Sigma^{\mathrm{lifted}}$ shifts $\boldsymbol{x}^a$ by $\mathrm{d}\boldsymbol{x}^a$, i.e. in components $x^a_{i_1\cdots i_k} \mapsto (\mathrm{d}\boldsymbol{x}^a)_{i_1\cdots i_k}$. Postcomposition with $\Phi_\epsilon^{Q_{\mathcal{N}}}$ instead substitutes the superfield into $Q_{\mathcal{N}}^a$,

$$\boldsymbol{x}^a \mapsto \boldsymbol{x}^a + \epsilon\, Q_{\mathcal{N}}^a(\boldsymbol{x}),$$

so $Q_{\mathcal{N}}^{\mathrm{lifted}}$ shifts $\boldsymbol{x}^a$ by $Q_{\mathcal{N}}^a(\boldsymbol{x})$. Writing each generator as a variational vector field then yields

$$Q = \int_\Sigma (\mathrm{d}\boldsymbol{x}^a)_{i_1\cdots i_k}(t)\frac{\delta}{\delta x^a_{i_1\cdots i_k}(t)} + Q_{\mathcal{N}}(\boldsymbol{x}(t))^a\frac{\delta}{\delta\boldsymbol{x}^a(t)} \tag{10}$$

Finally, we define the action S as

$$S = \iota_{\mathrm{d}_\Sigma^{\mathrm{lifted}}}\mathbb{T}\alpha_{\mathcal{N}} + \mathbb{T}\Theta_{\mathcal{N}}.$$

Note that $\iota_{\mathrm{d}_\Sigma^{\mathrm{lifted}}}$ replaces $\delta\boldsymbol{x}^a \rightsquigarrow \mathrm{d}\boldsymbol{x}^a$ and we obtain in coordinates

$$S = \int_\Sigma \sum_a \alpha_a(\boldsymbol{x})\mathrm{d}\boldsymbol{x}^a + \Theta_{\mathcal{N}}(\boldsymbol{x}) \tag{11}$$

Theorem 1.52. The data $(\mathcal{F}, \omega, Q, S)$ is a classical BV theory.

Proof. We must show that ω is odd-symplectic and that S satisfies the CME. For the first, observe that $\mathbb{T}\delta\phi = (-1)^n\delta\mathbb{T}\phi$ for any $\phi \in \Omega^*(\mathcal{N})$; this is clear in coordinates, since $\mathbb{T}$ pulls the coordinates back to superfields and commutes with δ (up to signs). In particular, it follows that if $\omega_{\mathcal{N}}$ is closed, then ω is closed. Non-degeneracy is checked by expanding (9) in superfields, and ω has internal degree -1 since $\mathbb{T}$ lowers the internal degree by n while $\omega_{\mathcal{N}}$ has degree $n-1$; hence ω is odd-symplectic.

To verify that $(S,S)=0$, we need to show that the Hamiltonian vector field of S squares to zero. We claim that Q is its Hamiltonian vector field (i.e. $\iota_Q\omega=\delta S$), and as we've seen above, $Q^2=0$ so we're done. We check

$$\iota_Q\omega=(-1)^n\left(\iota_{d_\Sigma^{\text{lifted}}}\mathbb{T}\omega_{\mathcal{N}}+\iota_{Q_{\mathcal{N}}^{\text{lifted}}}\mathbb{T}\omega_{\mathcal{N}}\right)=\iota_{d_\Sigma^{\text{lifted}}}\delta\mathbb{T}\alpha_{\mathcal{N}}+(-1)^n\mathbb{T}\iota_{Q_{\mathcal{N}}}\omega_{\mathcal{N}}$$

where for the first term we've used that $\omega_{\mathcal{N}}$ is exact and δ commutes with $\mathbb{T}$ and for the second, we can see by observing (10) that $\iota_{Q_{\mathcal{N}}^{\text{lifted}}}\mathbb{T}=\mathbb{T}\iota_{Q_{\mathcal{N}}}$. Now Definition 1.11 applied to the first term gives

$$\iota_{d_\Sigma^{\text{lifted}}}\delta\mathbb{T}\alpha_{\mathcal{N}}+(-1)^n\mathbb{T}\iota_{Q_{\mathcal{N}}}\omega_{\mathcal{N}}=L_{d_\Sigma^{\text{lifted}}}\mathbb{T}\alpha_{\mathcal{N}}+\delta\iota_{d_\Sigma^{\text{lifted}}}\mathbb{T}\alpha_{\mathcal{N}}+\delta\mathbb{T}\Theta_{\mathcal{N}}=\delta S,$$

where we have used that $L_{d_\Sigma^{\text{lifted}}}\mathbb{T}\alpha_{\mathcal{N}}=0$. Indeed, using the coordinate formula for $\mathrm{d}_\Sigma^{\text{lifted}}=V^a\partial/\partial\boldsymbol{x}^a$ with $V^a=\mathrm{d}\boldsymbol{x}^a$ from (10) and the formula for a Lie derivative of a one-form

$$\mathcal{L}_{\mathrm{d}_\Sigma^{\text{lifted}}}\alpha=(\mathrm{d}_\Sigma^{\text{lifted}}\alpha_a)\delta\boldsymbol{x}^a+(-1)^{|\alpha_a|}\alpha_b(\partial_aV^b)\delta\boldsymbol{x}^a,$$

which by the chain rule is exactly

$$\mathcal{L}_{\mathrm{d}_\Sigma^{\text{lifted}}}\alpha=\mathrm{d}(\alpha_a(\boldsymbol{x}))\delta\boldsymbol{x}^a+(-1)^{|\alpha_a|}\alpha_a(\boldsymbol{x})\mathrm{d}(\delta\boldsymbol{x}^a)=\mathrm{d}(\alpha_a(\boldsymbol{x})\delta\boldsymbol{x}^a),$$

which integrates to zero over Σ by Stokes' theorem. Here the sign $(-1)^{|\alpha_a|}$ is the Koszul sign produced when the odd source differential d passes the coefficient α_a; with it the two terms reassemble into the total derivative $\mathrm{d}(\alpha_a(\boldsymbol{x})\delta\boldsymbol{x}^a)$. $\square$

2 Algebraic structures

2.1 L_∞ algebras

This chapter collects numerous algebraic definitions and theorems, of which we will make extensive use later; they form the backbone of our globalization techniques. Recall again that the Koszul signs $|x|$ need not be congruent to the $\mathbb{Z}$-grading of the algebraic structures.

Definition 2.1 (dg space)**.** A *differential graded space* (dg space) is a graded space[1] V with a linear map $\mathrm{d} : V^\bullet \to V^{\bullet+1}$ such that $\mathrm{d}^2 = 0$.

Definition 2.2 (DGLA, [19])**.** A *1-shifted Lie superalgebra* (which we call DGLA for *differential graded Lie algebra*) is a dg space L equipped with a bilinear antibracket $(-,-) : L^k \times L^\ell \to L^{k+\ell+1}$ such that

(1) $\mathrm{d}(x,y) = (\mathrm{d}x, y) + (-1)^{|x|+1}(x, \mathrm{d}y)$ (Leibniz)

(2) $(y,x) = -(-1)^{(|x|+1)(|y|+1)}(x,y)$,

(3) $(x,(y,z)) = ((x,y),z) + (-1)^{(|x|+1)(|y|+1)}(y,(x,z))$ (Jacobi).

Definition 2.3 (Maurer–Cartan, [19])**.** The set of solutions to the *Maurer–Cartan* equation of a DGLA is given by even $x \in L^0$ such that

$$\mathrm{d}x + \frac{1}{2}(x,x) = 0,$$

and is denoted as $\mathrm{MC}(L)$.

Definition 2.4 (Complete graded space, [2] Def. 1.5)**.** A *complete graded space* is a graded space V equipped with a descending filtration

$$V = F^0V \supset \ldots \supset F^pV \supset \ldots$$

such that V is complete in the induced topology, i.e. $V \to \varprojlim V/F^pV$ is an isomorphism of graded spaces. A map $f : W \to V$ is *continuous* if $f(F^pW) \subset F^p(V)$ for all p. A *complete dg space* is a dg space with a filtration F^pV and continuous differential d; a *complete DGLA* additionally satisfies $(F^{p_1}V, F^{p_2}V) \subset F^{p_1+p_2}V$.

The natural filtration on a space $V[[\hbar]]$ is given by the powers $\hbar^p$.

Remark 2.5. Because we are interested in the classical part ($\hbar^0$) as well, we start our filtration at F^0V, so convergence is not guaranteed even for continuous expressions. We will therefore not mention this again: all definitions and expressions below are to be understood formally, and convergence has to be checked in each particular case by $\hbar$ degree.

Definition 2.6 (Gauge action, [19])**.** Let $y \in L^k$ and denote the map $\mathrm{ad}(y) : L^\bullet \to L^{\bullet+k+1}$, $x \mapsto (y,x)$. Then we define the gauge action of odd $y \in L^{-1}$ on an element $x \in L^0$ by

$$x \bullet y = x + \sum_{n=0}^{\infty} \frac{(-\,\mathrm{ad}(y))^n(\mathrm{d}y + (x,y))}{(n+1)!}.$$

[1] A space is always a vector space for us

One can easily check that $x \bullet y \in \mathrm{MC}(L)$. Furthermore it is the solution to the ordinary differential equation

$$\frac{\mathrm{d}(x \bullet ty)}{\mathrm{d}t} = \mathrm{d}y + (x \bullet ty, y),$$

with initial condition $x \bullet ty = x$ at $t = 0$.

One can equip a DGLA with more structure, which we will need in later chapters; for this we follow the definitions and notation of [2]. For a graded space V, denote $V^{\odot n}$ its n-th symmetric product (the signs are again given by Koszul rules on the parity). We consider the coalgebra $\bar{S}(V) = \oplus_{n \geqslant 1} V^{\odot n}$ with the unshuffle coproduct

$$\Delta\left(v_1 \odot \cdots \odot v_n\right) = \sum_{i=1}^{n-1} \sum_{\sigma \in S(i,n-i)} \varepsilon(\sigma) \left(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}\right) \otimes \left(v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}\right),$$

where $\varepsilon(\sigma)$ is the Koszul sign from permuting the v's, $S(i, n-i) \in S_n$ are all permutations which split into two blocks $\{\sigma(1), ..., \sigma(i)\}$ and $\{\sigma(i+1), ..., \sigma(n)\}$ and are strictly increasing in each of the blocks.

Given a map $F : \bar{S}(W) \to \bar{S}(V)$ between the symmetric coalgebras, denote by $f_i : W^{\odot i} \to V$ the composition $W^{\odot i} \hookrightarrow \bar{S}(W) \xrightarrow{F} \bar{S}(V) \twoheadrightarrow V$, the last arrow being the canonical projection. In particular, a coderivation $Q : \bar{S}(V) \to \bar{S}(V)$ (i.e. $\Delta Q = (Q \otimes \mathrm{Id} + \mathrm{Id} \otimes Q)\Delta$) and a morphism $F : \bar{S}(W) \to \bar{S}(V)$ (i.e. $\Delta F = (F \otimes F)\Delta$) are determined by their Taylor coefficients q_i and f_i, defined by the same composition, via

$$Q\left(v_1 \odot \cdots \odot v_n\right) = \sum_{i=1}^{n} \sum_{\sigma \in S(i,n-i)} \varepsilon(\sigma) q_i \left(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}\right) \odot v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}, \tag{12}$$

and

$$F\left(w_1 \odot \cdots \odot w_n\right) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{i_1,\ldots,i_k \geqslant 1 \\ i_1+\cdots+i_k=n}} \sum_{\sigma \in S(i_1,\ldots,i_k)} \varepsilon(\sigma) f_{i_1}\left(w_{\sigma(1)} \odot \cdots\right) \odot \cdots \odot f_{i_k}\left(\cdots \odot w_{\sigma(n)}\right). \tag{13}$$

where $S(i_1, \ldots, i_k)$ denotes the unshuffles into k blocks of sizes $i_1, \ldots, i_k$

Definition 2.7 (L_∞ algebra, [2] Def. 1.1)**.** An L_∞ algebra (V, Q) is a graded space V and a $+1$ coderivation Q on the space $\bar{S}(V)$ such that $Q^2 = 0$. Furthermore we say that it is a complete L_∞ algebra if V comes equipped with a filtration and the q_i are continuous, i.e. $q_i\left(F^{p_1}V \odot \cdots \odot F^{p_i}V\right) \subset F^{p_1+\cdots+p_i}V$, for all $i, p_1, \ldots, p_i \geqslant 1$.

Definition 2.8 (L_∞ morphism, [2] Def. 1.1)**.** An L_∞ morphism $F : (W, R) \to (V, Q)$ is a family of degree 0 Taylor coefficients $f_i : W^{\odot i} \to V$ for $i \geqslant 1$ such that the corresponding $F : \bar{S}(W) \to \bar{S}(V)$ intertwines R and Q.

A continuous L_∞ morphism $F : (W, R) \to (V, Q)$ between complete L_∞ algebras is an L_∞ morphism $F : (W, R) \to (V, Q)$ such that its Taylor coefficients are continuous, that is, $f_i\left(F^{p_1}W \odot \cdots \odot F^{p_i}W\right) \subset F^{p_1+\cdots+p_i}V$, for all $i, p_1, \ldots, p_i \geqslant 1$.

In particular, a DGLA $(L, \mathrm{d}, (-,-))$ is an L_∞ algebra by setting $q_1(x) = -\mathrm{d}x, q_2(x \odot y) = (-1)^{|x|+1}(x, y)$ and $q_i = 0$ $\forall i \geqslant 3$. The condition $Q^2 = 0$ then reproduces the Leibniz (1) and Jacobi (3) identities of Definition 2.2; antisymmetry (2) is automatic as q_2 lives on $L^{\odot 2}$, and $\mathrm{d}^2 = 0$ is the dg-space axiom.

We can also consider the MC set for an L_∞ algebra: define

$$\mathrm{MC}(V) := \left\{ x \in V^0 \,\middle|\, \sum_{i \geqslant 1} \frac{1}{i!} q_i(x^{\odot i}) = 0 \right\}. \tag{14}$$

For the q_1, q_2 of the DGLA case above, this reduces to Definition 2.3.

Note that a BV algebra $(\mathcal{A}, \cdot, \Delta, (-,-))$ is an example of a DGLA $(L, \mathrm{d}, (-,-))$ with an additional commutative product and compatibility conditions. It remains only to verify that

$$\Delta(x, y) = (\Delta x, y) + (-1)^{|x|+1}(x, \Delta y), \tag{15}$$

which follows by applying Δ to (5) and using that identity again together with $\Delta^2 = 0$.

Also note that the classical master equation (Def. 1.36) is just the $\mathrm{MC}(\mathcal{A})$ set with $q_1 \sim 0$ and $q_2 \sim (-,-)$ the BV bracket, while the quantum master equation (Def. 1.37) is $\mathrm{MC}_\hbar(\mathcal{A}) = \mathrm{MC}(\mathcal{A}[[\hbar]])$ with $q_1 \sim \mathring{\imath}\hbar\Delta$, $q_2 \sim (-,-)$. The BV canonical transformation (Def. 1.38) is then the statement that there is an $R \in \mathcal{A}^{-1}$ such that $S \bullet R = S'$ (taking R time-independent; see later).

2.2 Homological algebra

Definition 2.9 (Quasi-isomorphism)**.** A chain map $f : (V, \mathrm{d}_V) \to (W, \mathrm{d}_W)$ is a *quasi-isomorphism* if the induced map $H(f) : H(V) \to H(W)$ on cohomology is an isomorphism.

Probably the most important definition of the thesis is the following.

Definition 2.10 (Strong deformation retractions, [30], Def. A.1)**.** A *(complete) strong deformation retraction* is the data

$$W \underset{\pi}{\overset{\iota}{\rightleftharpoons}} V \circlearrowleft h$$

where $(W, \mathrm{d}_W), (V, \mathrm{d}_V)$ are (complete), differential graded spaces, $h \in \mathrm{Hom}^{-1}(V, V)$ and ι, π are cochain maps such that

$$\pi\iota = \mathrm{Id}_W, \quad \mathrm{d}_V h + h\mathrm{d}_V = \mathrm{Id} - \iota\pi.$$

with the side conditions

$$h\iota = 0, \qquad \pi h = 0, \qquad h^2 = 0.$$

Additionally, in the complete case the maps ι, π, h are required to be continuous.

Remark 2.11. In particular the maps ι and π are quasi-isomorphisms.

We will construct such SDRs in order to work in a resolution V of W, and then transport the relevant results back to W via π.

To construct new SDRs from old ones, we "perturb" the old one in the following sense.

Definition 2.12 (Perturbation)**.** For a differential graded space V, a *perturbation* of the differential d_V is a linear map $\partial \in \mathrm{Hom}^1(V, V)$ such that $(\mathrm{d}_V + \partial)^2 = 0$.

Lemma 2.13 (Homological perturbation lemma, [30] Lemma A.2)**.** Let (W, d_W) and (V, d_V) be a pair of complexes with an SDR given by (ι, π, h). Let ∂ be a perturbation of d_V. Then we can construct a new SDR between $(W, \mathrm{d}_W + \mathrm{d}_\partial)$ and $(V, \mathrm{d}_V + \partial)$ with the data

$$\begin{aligned}
\mathrm{d}_\partial &= \sum_{n \geqslant 0} \pi\partial(-h\partial)^n\iota \\
\iota_\partial &= \sum_{n \geqslant 0} (-h\partial)^n\iota, \\
\pi_\partial &= \sum_{n \geqslant 0} \pi(-\partial h)^n, \\
h_\partial &= \sum_{n \geqslant 0} h(-\partial h)^n.
\end{aligned}$$

Note that we can also write $\mathrm{d}_\partial = \pi\partial\iota_\partial$.

Later on we will need to construct a new SDR from the tensor product of a given SDR. The following is the symmetric-algebra form of the tensor trick of [22, 23], employed on the tensor coalgebra.

Lemma 2.14. Let
$$W \underset{\pi}{\overset{\iota}{\rightleftharpoons}} V \circlearrowleft h$$
be an SDR. For $n \geqslant 0$ let $\iota^{\odot n} : W^{\odot n} \to V^{\odot n}$ and $\pi^{\odot n} : V^{\odot n} \to W^{\odot n}$ be the induced maps on symmetric powers, and let $h_n : V^{\odot n} \to V^{\odot n}$ be the degree -1 map
$$h_n(v_1 \odot \cdots \odot v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{a=1}^{n} \pm\, \iota\pi(v_{\sigma(1)}) \odot \cdots \odot \iota\pi(v_{\sigma(a-1)}) \odot h(v_{\sigma(a)}) \odot v_{\sigma(a+1)} \odot \cdots \odot v_{\sigma(n)},$$
where $\pm$ is the Koszul sign of the permutation and of moving h into place ($|h| = -1$). Assembling
$$\widehat{\iota} = \prod_{n \geqslant 0} \iota^{\odot n}, \qquad \widehat{\pi} = \prod_{n \geqslant 0} \pi^{\odot n}, \qquad \widehat{h} = \prod_{n \geqslant 0} h_n$$
yields an SDR of the completed symmetric algebras, with their derivation differentials,
$$\widehat{S}W \underset{\widehat{\pi}}{\overset{\widehat{\iota}}{\rightleftharpoons}} \widehat{S}V \circlearrowleft \widehat{h}.$$
If the original SDR is complete, so is this one.

Proof. Every operator keeps the symmetric degree n, so the construction is block diagonal and it suffices to build an SDR $W^{\odot n} \rightleftharpoons V^{\odot n}$ for each n; assembling the blocks then gives the claim, continuity inherited degreewise. Fix n and write $A = \iota\pi$ which commutes with d_V.

We first lift the retraction to $V^{\otimes n}$, with differential $\mathrm{d}^{\otimes} = \sum_{a=1}^{n} \mathrm{Id}^{\otimes(a-1)} \otimes \mathrm{d}_V \otimes \mathrm{Id}^{\otimes(n-a)}$, maps $\iota^{\otimes n}, \pi^{\otimes n}$ and homotopy
$$H = \sum_{a=1}^{n} B_a, \qquad B_a = A^{\otimes(a-1)} \otimes h \otimes \mathrm{Id}^{\otimes(n-a)}.$$
Clearly $\pi^{\otimes n}\iota^{\otimes n} = (\pi\iota)^{\otimes n} = \mathrm{Id}$. For the retract relation we evaluate $\mathrm{d}^{\otimes} B_a + B_a \mathrm{d}^{\otimes}$ slot by slot: in slot a we get $\mathrm{d}_V h + h \mathrm{d}_V = \mathrm{Id} - A$, dressed with A's to its left and identities to its right; in a slot $b < a$ the two orders assemble into the commutator $[\mathrm{d}_V, A] = 0$; and in a slot $b > a$ the differential must cross the odd h in slot a, and the sign it picks up makes the two orders cancel (using that d_V, h are odd). What survives telescopes,
$$\mathrm{d}^{\otimes} H + H \mathrm{d}^{\otimes} = \sum_{a=1}^{n} (A^{\otimes(a-1)} \otimes \mathrm{Id}^{\otimes(n-a+1)} - A^{\otimes a} \otimes \mathrm{Id}^{\otimes(n-a)}) = \mathrm{Id}^{\otimes n} - A^{\otimes n} = \mathrm{Id} - \iota^{\otimes n}\pi^{\otimes n}.$$
The side conditions come for free from those of (ι, π, h): each term of $H\iota^{\otimes n}$ carries $h\iota = 0$, each term of $\pi^{\otimes n} H$ carries $\pi h = 0$, and in $B_a B_{a'}$ the two distinguished slots carry $h^2 = 0$ for $a = a'$, and $\iota\pi\, h = 0$ or $h\, \iota\pi = 0$ for $a \neq a'$; so $H^2 = 0$. Thus $(\iota^{\otimes n}, \pi^{\otimes n}, H)$ is an SDR.

For the symmetric product note that $V^{\odot n}$ sits inside $V^{\otimes n}$ as the image of the symmetrizer $t = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ (with Koszul signs). The maps $\mathrm{d}^{\otimes}, \iota^{\otimes n}, \pi^{\otimes n}, A^{\otimes n}$ are S_n-equivariant, hence commute with t and restrict on $V^{\odot n}$ to $\mathrm{d}_V, \iota^{\odot n}, \pi^{\odot n}$ and $\iota^{\odot n}\pi^{\odot n}$. Set $h_n = tHt$; spelling it out on $t(v_1 \otimes \cdots \otimes v_n)$ reproduces the formula in the statement. Since t commutes with $\mathrm{d}^{\otimes}$ and $A^{\otimes n}$, the retract relation descends, on $V^{\odot n}$ where $t = \mathrm{Id}$, to
$$\mathrm{d}_V h_n + h_n \mathrm{d}_V = t(\mathrm{Id} - A^{\otimes n})t = \mathrm{Id} - \iota^{\odot n}\pi^{\odot n},$$
and pulling t through $\iota^{\otimes n}, \pi^{\otimes n}$ gives $h_n \iota^{\odot n} = t(H\iota^{\otimes n})t = 0$ and $\pi^{\odot n} h_n = t(\pi^{\otimes n} H)t = 0$.

Only $(h_n)^2 = tHtHt = 0$ remains. Using $\sigma t = t$ to swallow the middle symmetrizer leaves a sum of terms tDt, each applying h in two slots and A or Id elsewhere. If the two slots agree the term dies by $h^2 = 0$. If they are distinct slots $a \neq b$, then D carries $h \otimes h$ there; two odd h's anticommute with the transposition τ_{ab}, which leaves the remaining even factors alone, so with $t\tau_{ab} = \tau_{ab}t = t$

$$tDt = tD\tau_{ab}t = -\,t\tau_{ab}Dt = -\,tDt$$

and the term vanishes. Hence $(h_n)^2 = 0$, and $(\iota^{\odot n}, \pi^{\odot n}, h_n)$ is an SDR. $\square$

The next theorem is in some sense a generalization of the homological perturbation lemma; it allows us to transfer entire L_∞ structure from one space to another and even more importantly for us, yields concrete formulae of morphisms between them.

Theorem 2.15 (Transfer of L_∞ structure, [2] Thm. 1.9)**.** Given an SDR

$$W \underset{\pi}{\overset{\iota}{\rightleftarrows}} V \circlearrowleft h$$

and an L_∞ algebra structure Q on V with linear part $q_1 = -\mathrm{d}_V$, there is an induced L_∞ algebra structure R on W with linear part $r_1 = -\mathrm{d}_W$, together with L_∞ morphisms $F : (W, R) \to (V, Q)$, $G : (V, Q) \to (W, R)$ with linear parts $f_1 = \iota$, $g_1 = \pi$ respectively. Denoting by F_i^k the composition $W^{\odot i} \hookrightarrow \bar{S}(W) \xrightarrow{F} \bar{S}(V) \twoheadrightarrow V^{\odot k}$, F and R are determined recursively by

$$f_i = \sum_{k=2}^{i} hq_k F_i^k \qquad \text{for } i \geqslant 2,$$

$$r_i = \sum_{k=2}^{i} \pi q_k F_i^k \qquad \text{for } i \geqslant 2.$$

Denote by $h_i : V^{\odot i} \to V^{\odot i}$ the degree minus one map defined by

$$h_i(v_1 \odot \cdots \odot v_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \sum_{j=1}^{i} \pm \iota\pi(v_{\sigma(1)}) \odot \cdots \odot \iota\pi(v_{\sigma(j-1)}) \odot h(v_{\sigma(j)}) \odot v_{\sigma(j+1)} \odot \cdots \odot v_{\sigma(i)},$$

where $\pm$ is the appropriate Koszul sign (taking into account that $|h| = -1$). Denoting by Q_i^k the composition $V^{\odot i} \hookrightarrow \bar{S}(V) \xrightarrow{Q} \bar{S}(V) \twoheadrightarrow V^{\odot k}$, the L_∞ morphism G is determined recursively by

$$g_i = \sum_{k=1}^{i-1} g_k Q_i^k h_i \qquad \text{for } i \geqslant 2.$$

We are mainly interested in the MC sets of L_∞ algebras, since these are exactly the solutions of the master equations. The following Proposition shows how to transport solutions between MC sets along L_∞ morphisms, which we construct via the transfer theorem above.

Proposition 2.16. Given an L_∞ morphism $F : (W, R) \to (V, Q)$, define the *push-forward* F_* as

$$F_*(x) = \sum_{i \geqslant 1} \frac{1}{i!} f_i(x^{\odot i}).$$

Then $x \in \mathrm{MC}(W) \implies F_*(x) \in \mathrm{MC}(V)$.

Proof. Maurer–Cartan elements are even of degree 0, so all Koszul signs among copies of x are trivial. Define $e^x := \sum_{n \geqslant 0} \frac{1}{n!} x^{\odot n}$ (including the $n = 0$ unit, so e^x lies in $\mathbb{R} \oplus \bar{S}(V)$).

With $w_1 = \cdots = w_n = x$ every $(i, n-i)$-unshuffle in (12) contributes the same summand, and there are $\binom{n}{i}$ of them, so $R(x^{\odot n}) = \sum_{i=1}^{n} \binom{n}{i} r_i(x^{\odot i}) \odot x^{\odot(n-i)}$. Dividing by $n!$, summing over n and reindexing $n = i + m$,

$$R(e^x) = \Big(\sum_{i \geqslant 1} \frac{1}{i!} r_i(x^{\odot i}) \Big) \odot e^x, \tag{$\dagger$}$$

hence $R(e^x) = 0 \iff \sum_{i \geqslant 1} \frac{1}{i!} r_i(x^{\odot i}) = 0 \iff x \in \mathrm{MC}(W)$. Likewise a fixed composition $i_1 + \cdots + i_k = n$ occurs $\frac{n!}{i_1! \cdots i_k!}$ times in (13), giving

$$F(x^{\odot n}) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \frac{n!}{i_1! \cdots i_k!} f_{i_1}(x^{\odot i_1}) \odot \cdots \odot f_{i_k}(x^{\odot i_k}).$$

Dividing by $n!$ and summing over n frees the constraint and factorizes the sum,

$$F(e^x) = \sum_{k \geqslant 0} \frac{1}{k!} \Big(\sum_{i \geqslant 1} \frac{1}{i!} f_i(x^{\odot i}) \Big)^{\odot k} = e^{F_*(x)}.$$

Since F is an L_∞ morphism, $QF = FR$. Computing $Q(e^{F_*(x)})$ with ($\dagger$) and then using these relations,

$$\Big(\sum_{i \geqslant 1} \frac{1}{i!} q_i\big(F_*(x)^{\odot i}\big) \Big) \odot e^{F_*(x)} = Q\big(e^{F_*(x)}\big) = Q\big(F(e^x)\big) = F\big(R(e^x)\big).$$

If $x \in \mathrm{MC}(W)$ then $R(e^x) = 0$ and the right-hand side vanishes. Projecting onto $V = V^{\odot 1}$ leaves $\sum_{i \geqslant 1} \frac{1}{i!} q_i(F_*(x)^{\odot i}) = 0$, that is $F_*(x) \in \mathrm{MC}(V)$. □

The push-forward is in fact one half of a bijection: a Maurer–Cartan element of V is pinned down by its image in $\mathrm{MC}(W)$ together with its homotopy datum. This is a formal incarnation of Kuranishi's theorem.

Theorem 2.17 (Formal Kuranishi theorem, [2] Thm. 1.13)**.** Keep the hypotheses of Theorem 2.15. The map

$$\rho : \mathrm{MC}(V) \longrightarrow \mathrm{MC}(W) \times h(V^0), \qquad x \longmapsto \big(G_*(x),\, hx\big),$$

is a bijection. Its inverse is obtained recursively: to a pair $(y, hv) \in \mathrm{MC}(W) \times h(V^0)$ one assigns the limit of the sequence $x_0 = 0$,

$$x_{n+1} = \iota(y) - q_1(hv) + \sum_{i \geqslant 2} \frac{1}{i!} \big(hq_i - \iota g_i\big)(x_n^{\odot i}),$$

which is the unique $x \in \mathrm{MC}(V)$ with $\rho(x) = (y, hv)$. For $hv = 0$ this recovers the push-forward of the previous proposition: $F_* = \rho^{-1}(-, 0)$ restricts to a bijection between $\mathrm{MC}(W)$ and the h-closed solutions $\{x \in \mathrm{MC}(V) \mid hx = 0\}$, with inverse π.

3 Formal geometry

Central to formal geometry is the following definition.

Definition 3.1 (Formal exponential map, [7]). A *formal exponential map* (FEM) is a smooth map $\phi : U \to M$ from a neighborhood $U \subset TM$ of the zero section such that (writing $\phi_x(y)$ for $x \in M, y \in T_xM \cap U$)

(1) $\phi_x(0) = x$,

(2) $\mathrm{d}\phi_x(0) = \mathrm{Id}$.

To write down the Feynman diagrams for actions, one usually has to take the Taylor expansion of the perturbation term to obtain expressions polynomial in the fields to give the explicit Feynman rules. Formal geometry does this covariantly, by first pulling back the action to the linear tangent space by a formal exponential map ϕ_x and then taking the Taylor expansion, denoted $\mathrm{T}\phi_x^* f$. If the formal exponential map is the geodesic exponential map of a torsion-free connection ∇ for example, this boils down to taking

$$\mathrm{T}\phi_x^* f = f(x) + \nabla_i f(x) y^i + \frac{1}{2!}\nabla_i \nabla_j f(x) y^i y^j + ...$$

There are multiple approaches to this, but all are essentially the same; we will show three ways of obtaining the connection and the formal exponential map, which however have a slightly different meaning in each approach.

The upshot of formal geometry is that one can define a flat connection D on the bundle where these Taylor expansions live. This connection has the property that it provides a resolution of the global functions; the cohomology $H_D^\bullet$ lives only in zeroth degree and consists of all global functions. This will be essential in the globalization procedure.

3.1 First construction

In the first construction we will first build a connection on the space of the formal power series. With this connection it will then be possible to give a characterization of the formal exponential map. We will closely follow [15].

3.1.1 Connection

We start with a manifold M (with local coordinates x^i) and we want to study the bundle $\widehat{S}T^*M$, i.e. the formal completion of the symmetric algebra of the cotangent bundle. This is naturally where the formal Taylor expansions will live by considering an exponential map $\phi_x : T_xM \to M$ and pulling back a $f \in C^\infty(M)$ to a function on the tangent fiber T_xM and then considering its Taylor expansion, creating a formal power series in the generators y^i of T^*M, i.e. a section in $\Gamma(\widehat{S}T^*M)$.

Let $a = \omega \otimes f \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$ and define the two operators

$$\delta : \Gamma(\Lambda^p T^*M \otimes \widehat{S}^q T^*M) \to \Gamma(\Lambda^{p+1} T^*M \otimes \widehat{S}^{q-1} T^*M)$$
$$\delta^{-1} : \Gamma(\Lambda^p T^*M \otimes \widehat{S}^q T^*M) \to \Gamma(\Lambda^{p-1} T^*M \otimes \widehat{S}^{q+1} T^*M)$$

by

$$\delta a = \left(\mathrm{d}x^i \wedge \frac{\partial}{\partial y^i}\right) a = \mathrm{d}x^i \wedge \omega \otimes \frac{\partial f}{\partial y^i}$$

and

$$\delta^{-1}a = \left(\frac{1}{p+q}y^i\iota_{\frac{\partial}{\partial x^i}}\right)a = \frac{1}{p+q}\iota_{\frac{\partial}{\partial x^i}}\omega\otimes y^i f, \quad \text{for } p+q\neq 0$$

and $\delta^{-1}a = 0$ else.

We define the *formal vector fields* to be the $C^\infty(M)$-linear derivations of $\Gamma(\widehat{S}T^*M)$, see [11]. In particular derivations are specified by their action on the generators y^i (extended by Leibniz); we have an isomorphism

$$\Gamma(\widehat{S}T^*M\otimes TM)\to \mathrm{Der}(\Gamma(\widehat{S}T^*M))$$

with the map

$$f\otimes v\mapsto (\Gamma(T^*M)\ni\alpha\mapsto\alpha(v)\,f)$$

and inverse

$$\mathcal{D}\mapsto\mathcal{D}(y^i)\otimes\frac{\partial}{\partial x^i}.$$

By taking $\frac{\partial}{\partial x^i}\rightsquigarrow\frac{\partial}{\partial y^i}$ we can write a formal vector field $Y\in\Gamma(\widehat{S}T^*M\otimes TM)$ in coordinates as

$$Y = Y^j(x,y)\frac{\partial}{\partial y^j}.$$

Consider now this bundle tensored with differential forms, i.e. $\Gamma(\Lambda T^*M\otimes\widehat{S}T^*M\otimes TM)$, in which an element has the form

$$Y = \mathrm{d}x^i Y_i^j(x,y)\frac{\partial}{\partial y^j}.$$

Letting δ,δ^{-1} act as defined above on the form part and trivially on the formal vector field part, we have the Hodge decomposition

$$a = a_{00} + (\delta^{-1}\delta+\delta\delta^{-1})a, \quad a\in\Gamma(\Lambda T^*M\otimes\widehat{S}T^*M\otimes TM), \tag{16}$$

where $\deg_y(a_{00}) = \deg_{\mathrm{d}x}(a_{00}) = 0$. This, and many other unproved statements here, will instead be later shown in Section 3.7 in a more general setting.

We now construct from a generic torsion-free connection ∇ on TM a new, flat *Grothendieck connection* D on the bundle $\widehat{S}T^*M$. Such a connection D is given locally by

$$Df = \mathrm{d}_x f + Y(f), \tag{17}$$

where $Y\in\Gamma(\Lambda T^*M\otimes\widehat{S}T^*M\otimes TM)$. Since D raises the form degree by one, we treat it as an odd operator on $\Gamma(\Lambda T^*M\otimes\widehat{S}T^*M)$; flatness is then the operator equation $\frac{1}{2}[D,D] = D^2 = 0$, where $[-,-]$ denotes the commutator of derivations, graded by the form degree. Expanding $D = \mathrm{d}_x + Y$ it reads

$$\mathrm{d}_x Y + \frac{1}{2}[Y,Y] = 0. \tag{18}$$

The power series Ansatz for Y is given by

$$Y = -\mathrm{d}x^i\frac{\partial}{\partial y^i} - \mathrm{d}x^\ell y^k\Gamma^i_{k\ell}(x)\frac{\partial}{\partial y^i} + X \tag{19}$$

where $\deg_y(X)\geqslant 2$, $\deg_{\mathrm{d}x}(X) = 1$. We read the leading term as the operator $-\delta$ and denote the Christoffel term by

$$\Gamma := -\mathrm{d}x^\ell y^k\Gamma^i_{k\ell}(x)\frac{\partial}{\partial y^i},$$

so that $\nabla = \mathrm{d}_x + \Gamma$ is the covariant derivative induced on $\widehat{S}T^*M$.

We would now like to obtain a condition on X such that (18) holds. This consists of computing every bracket $[-,-]$; given that Y is a derivation in y, we also have $\mathrm{d}_x Y = [\mathrm{d}_x, Y]$. Plugging in the decomposition of Y into (18) yields

$$0 = -[\mathrm{d}_x, \delta] + [\mathrm{d}_x, \Gamma] + [\mathrm{d}_x, X] + \frac{1}{2}([\delta, \delta] - [\delta, \Gamma] - [\Gamma, \delta] - [\delta, X] - [X, \delta] + [\Gamma, \Gamma] + [\Gamma, X] + [X, \Gamma] + [X, X]).$$

All parts of Y are of form degree one and hence odd, so the commutator is symmetric on them and the mixed terms pair up, e.g. $[\delta, \Gamma] = [\Gamma, \delta]$. We now group the terms. First, we define the curvature operator

$$R := [\mathrm{d}_x, \Gamma] + \frac{1}{2}[\Gamma, \Gamma]. \tag{20}$$

Second, we abbreviate the covariant derivative $\nabla X := [\nabla, X] = [\mathrm{d}_x, X] + [\Gamma, X]$ and similarly $\delta X := [\delta, X]$, so that the flatness condition takes the form

$$0 = R + \nabla X - \delta X + \frac{1}{2}[X, X] - [\mathrm{d}_x, \delta] - [\delta, \Gamma] + \frac{1}{2}[\delta, \delta].$$

The three remaining terms vanish: $\frac{1}{2}[\delta, \delta] = \delta^2 = 0$, since the antisymmetric $\mathrm{d}x^i \wedge \mathrm{d}x^j$ is contracted with the symmetric second derivative in y, and the mixed terms vanish by the following lemma.

Lemma 3.2. We have $[\mathrm{d}_x, \delta] = 0$, and $[\delta, \Gamma] = 0$ if and only if ∇ is torsion-free.

Proof. Both brackets are derivations of $\Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$ and annihilate the generators $f(x)$ and $\mathrm{d}x^i$, so it suffices to evaluate them on the generators y^i. For the first, $[\mathrm{d}_x, \delta]y^i = \mathrm{d}_x(\mathrm{d}x^i) = 0$. For the second, $\Gamma(\delta y^i) = \Gamma(\mathrm{d}x^i) = 0$ and

$$\delta(\Gamma y^i) = -\mathrm{d}x^m \frac{\partial}{\partial y^m}(\mathrm{d}x^\ell y^k \Gamma^i_{k\ell}) = -\mathrm{d}x^k \wedge \mathrm{d}x^\ell \Gamma^i_{k\ell},$$

which vanishes if and only if $\Gamma^i_{k\ell}$ is symmetric in its lower indices, i.e. iff ∇ is torsion-free. $\square$

Altogether, we obtain that the flatness condition on Y transfers to the condition on X that

$$R - \delta X + \nabla X + \frac{1}{2}[X, X] = 0. \tag{21}$$

The curvature R is a formal vector field, linear in y. To recover its components we evaluate on the generators: we compute

$$\begin{aligned} R(y^i) &= \mathrm{d}_x(\Gamma y^i) + \Gamma(\Gamma y^i) = \mathrm{d}x^\ell \wedge \mathrm{d}x^m y^k \frac{\partial \Gamma^i_{k\ell}}{\partial x^m} - \mathrm{d}x^\ell \wedge \mathrm{d}x^m y^k \Gamma^j_{km}\Gamma^i_{j\ell} \\ &= \frac{1}{2}\mathrm{d}x^\ell \wedge \mathrm{d}x^m y^k (\partial_m \Gamma^i_{k\ell} - \partial_\ell \Gamma^i_{km} + \Gamma^j_{k\ell}\Gamma^i_{jm} - \Gamma^j_{km}\Gamma^i_{j\ell}) = \frac{1}{2}\mathrm{d}x^\ell \wedge \mathrm{d}x^m y^k R^i_{km\ell}, \end{aligned}$$

where we antisymmetrized in ℓ, m against the wedge and recognized the components of the curvature in the convention $R(\partial_k, \partial_j)\partial_\ell = R^i_{\ell kj}\partial_i$. The factor $\frac{1}{2}$ is absorbed by the first Bianchi identity $R^i_{km\ell} + R^i_{m\ell k} + R^i_{\ell km} = 0$ (valid since ∇ is torsion-free): flipping the last two indices of the middle term gives $R^i_{km\ell} = R^i_{mk\ell} - R^i_{\ell km}$, and under the contraction with $\mathrm{d}x^\ell \wedge \mathrm{d}x^m$ we immediately obtain $R^i_{km\ell}\mathrm{d}x^\ell \wedge \mathrm{d}x^m = 2R^i_{mk\ell}\mathrm{d}x^\ell \wedge \mathrm{d}x^m$, arriving at

$$R = \mathrm{d}x^j \wedge \mathrm{d}x^\ell y^k R^i_{\ell kj}(x) \frac{\partial}{\partial y^i}.$$

The task is now to find power series solutions of X satisfying the equation above. A solution is not unique, but they are in one to one correspondence with $a \in \Gamma(\widehat{S}T^*M \otimes TM)$ with

$\deg_y(a) \geqslant 3$, such that $\delta^{-1}X = a$. Applying now the Hodge decomposition on X, where we use that $X_{00} = 0$ due to $\deg_{\mathrm{d}x}(X) = 1$, we obtain

$$X = \delta a + \delta^{-1}\left(R + \nabla X + \frac{1}{2}[X, X]\right). \tag{22}$$

We note that the degree-n part in y of the right hand side depends on X only through degrees lower than n due to δ^{-1}; we can solve this equation iteratively order by order. That every solution to (22) indeed satisfies (21) will be proven in Theorem 3.28. The resulting connection D_a is dependent on the choice of a which we will mostly omit in the notation.

3.1.2 Formal exponential map

Denote by $\pi : \Gamma(\widehat{S}T^*M) \to C^\infty(M)$ the evaluation on the zero section, i.e. $\sigma \mapsto \sigma|_{y=0}$. The definition/characterization of the formal exponential map is based on the following proposition.

Proposition 3.3. (Prop. 5.2, [27]) Let D be the connection defined in Section 3.1. For every $f \in C^\infty(M)$, there is a unique $\tilde{f} \in \Gamma(\widehat{S}T^*M)$ such that $\pi(\tilde{f}) = f$ and $D\tilde{f} = 0$.

This follows from the fact that one can solve $D\tilde{f} = 0$ with "initial condition" f iteratively order by order uniquely; the zeroth order thus determines all the other coefficients for a global function f.

Definition 3.4. Let D be the connection defined in Section 3.1. The formal exponential map $\mathrm{EXP}^* : C^\infty(M) \to \Gamma(\widehat{S}T^*M)$ is then defined as $\mathrm{EXP}^* f = \tilde{f}$, where $\tilde{f} \in \Gamma(\widehat{S}T^*M)$ is the unique section given by Proposition 3.3.

Note that in contrast with the first definition of a formal exponential map, this gives us directly the corresponding element in the bundle $\Gamma(\widehat{S}T^*M)$.

3.2 Second construction

3.2.1 Formal exponential map

Another view to formal geometry is given in [8] or [7]. The analogous action of EXP^* with the formal exponential map given in Definition 3.1 is the following.

Definition 3.5 (Taylor expansion)**.** Let ϕ_x be a formal exponential map as in Definition 3.1. The Taylor expansion of a function is defined as a map $\mathrm{T}\phi_x^* : C^\infty(M) \to \Gamma(\widehat{S}T^*M)$ by

$$\mathrm{T}\phi_x^* f = \sum_{I \in \mathbb{N}_0^n} \frac{1}{I!}\frac{\partial}{\partial y^I}(f \circ \phi_x)|_{y=0}\, y^I$$

with the multi index $I = (i_1, ..., i_n)$.

Remark 3.6. Note that for an exponential map ϕ_x we have the conditions $\phi_x(0) = x$ and $(\mathrm{d}\phi_x)|_{y=0} = \mathrm{Id}$. In particular, this means that in coordinates we can always write

$$\mathrm{T}\phi_x^* f = f(x) + \partial_{x^i} f(x) y^i + \mathcal{O}(y^2).$$

Furthermore, the Jacobian of ϕ_x given by

$$J = J^i_j(x, y) = \frac{\partial \phi^i_x}{\partial y^j}(y)$$

and is invertible at the zero section due to $J(x, 0) = \mathrm{Id}$.

3.2.2 Connection

Proposition 3.7. Let ϕ be a FEM in the sense of Definition 3.1. Define

$$Y_\phi := -\mathrm{d}x^a \mathrm{T}\Big[(J^{-1})^k_j \frac{\partial \phi^j_x}{\partial x^a}\Big]\frac{\partial}{\partial y^k} \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM),$$

with J the Jacobian of Remark 3.6, set $D_\phi := \mathrm{d}_x + Y_\phi$ and denote T the Taylor expansion with respect to y. Then:

(1) $D_\phi(\mathrm{T}\phi^* f) = 0$ for all $f \in C^\infty(M)$;

(2) D_ϕ is flat;

(3) D_ϕ is the unique connection of the form $\mathrm{d}_x + Y$ with $Y \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM)$ that annihilates $\mathrm{Im}(\mathrm{T}\phi^*)$.

The key step is the following pointwise determination lemma, which we will reuse repeatedly.

Lemma 3.8. Let $A \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM)$ and ϕ a formal exponential map. If $A(\mathrm{T}\phi^* f) = 0$ for all $f \in C^\infty(M)$, then $A = 0$.

Proof. Since A is $C^\infty(M)$-linear and acts fiberwise, $(A\sigma)(x)$ depends only on $\sigma(x) \in \widehat{S}T^*_x M$; in local coordinates $A = \alpha^j \frac{\partial}{\partial y^j}$ with form-valued coefficients $\alpha^j = \alpha^j(x,y) \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$, so $A = 0$ if and only if each α^j vanishes. As $A(y^m) = \alpha^m$ it suffices to show that A annihilates each generator y^m fiberwise. Indeed, fix x and define with a bump function χ

$$u^m := \chi \cdot (y^m \circ \phi_x^{-1}) \in C^\infty(M).$$

Then $u^m \circ \phi_x = y^m$ near $0 \in T_xM$, so its Taylor expansion at the zero section is $(\mathrm{T}\phi^* u^m)(x) = y^m$. By hypothesis $A(\mathrm{T}\phi^* u^m) = 0$, and evaluating at x gives $\alpha^m(x,\cdot) = 0$. As x and m were arbitrary, $A = 0$. □

Proof of Proposition 3.7. For (1) we use the chain rule: for $f \in C^\infty(M)$,

$$\frac{\partial}{\partial x^a}(f \circ \phi_x)(y) = (\partial_j f)(\phi_x(y))\frac{\partial \phi^j_x}{\partial x^a}, \qquad \frac{\partial}{\partial y^k}(f \circ \phi_x)(y) = (\partial_j f)(\phi_x(y)) J^j_k.$$

Inverting the second relation for $(\partial_j f) \circ \phi_x$ and inserting it into the first gives

$$\frac{\partial}{\partial x^a}(f \circ \phi_x) = \frac{\partial (f \circ \phi_x)}{\partial y^k}(J^{-1})^k_j \frac{\partial \phi^j_x}{\partial x^a},$$

i.e. $(\mathrm{d}_x + Y_\phi)(f \circ \phi_x) = 0$; applying the Taylor expansion, which commutes with ∂_x, ∂_y and products of series, yields (1). For (2), the curvature $F := \mathrm{d}_x Y_\phi + \frac{1}{2}[Y_\phi, Y_\phi]$ is an element of $\Gamma(\Lambda^2 T^*M \otimes \widehat{S}T^*M \otimes TM)$ and D_ϕ^2 acts by F; by (1) we have $F(\mathrm{T}\phi^* f) = D^2_\phi(\mathrm{T}\phi^* f) = 0$ for all f, so $F = 0$ by Lemma 3.8. For (3), if $D' = \mathrm{d}_x + Y'$ also annihilates $\mathrm{Im}(\mathrm{T}\phi^*)$, then $A := Y_\phi - Y' = D_\phi - D'$ satisfies the hypothesis of Lemma 3.8, so $A = 0$. □

Proposition 3.9. Let ∇ be a torsion-free connection on M and ϕ its geodesic exponential map. Then the two constructions agree for the choice $a = 0$ in (22), i.e. $\mathrm{T}\phi^*_x = \mathrm{EXP}^*$ and $D_\phi = D_{a=0}$.

This is the consequence of Proposition 2.3, Lemma 5.4, Theorem 5.6 and Corollary 5.8 of [27], (where τ is the notation used for EXP^*).

3.3 Third variant

This variant was introduced in [27].

Definition 3.10. Denote by $\mathcal{D}(M)$ the algebra of differential operators on a graded manifold M. The formal exponential map associated to a connection ∇ on TM is the morphism of left $C^\infty(M)$-modules $\mathrm{pbw} : \Gamma(STM) \to \mathcal{D}(M)$ given by the initial conditions

$$\mathrm{pbw}(f) = f, \forall f \in C^\infty(M)$$
$$\mathrm{pbw}(X) = X, \forall X \in \Gamma(TM)$$

and inductively, for homogeneous $X_0, \dots, X_n \in \Gamma(TM)$,

$$\mathrm{pbw}\left(X_0 \odot \cdots \odot X_n\right) = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \big[X_k \cdot \mathrm{pbw}(X^{\hat{k}}) - \mathrm{pbw}(\nabla_{X_k}(X^{\hat{k}}))\big], \tag{23}$$

where $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$ is the Koszul sign and $X^{\hat{k}} = X_0 \odot ... \odot X_{k-1} \odot X_{k+1} \odot ... \odot X_n$.

The "real formal exponential map" in the sense of the two previous definitions, i.e. the Taylor expansion in the formal bundle, is then given by constructing a flat connection from pbw (Lemma 5.4 of [27]) and defining $\mathrm{EXP}^* : C^\infty \to \Gamma(\widehat{S}T^*M)$ with Proposition 3.3. Corollary 5.7 of [27] then assures that EXP^* has the form

$$\mathrm{EXP}^*(f) = \sum_I \frac{1}{I!} y^I \otimes \mathrm{pbw}(\overleftarrow{\partial_x^I})(f),$$

where, for a multi-index $I = (i_1, \dots, i_n)$, the ordering of the derivatives is fixed by

$$\overleftarrow{\partial_x^I} = \underbrace{\partial_{x^n} \odot \cdots \odot \partial_{x^n}}_{i_n} \odot \cdots \odot \underbrace{\partial_{x^1} \odot \cdots \odot \partial_{x^1}}_{i_1}.$$

For a classical manifold this definition matches with the others; this is the content of Proposition 2.3 in [27], which states that

$$\mathrm{pbw}\left(X_0 \odot \cdots \odot X_k\right)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t_0}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}t_1}\right|_0 \cdots \left.\frac{\mathrm{d}}{\mathrm{d}t_k}\right|_0 f\left(\exp\left(t_0 X_0 + t_1 X_1 + \cdots + t_k X_k\right)\right),$$

where exp is the exponential map associated to ∇.

3.4 Extension to other bundles

So far everything was phrased for the bundle $\widehat{S}T^*M$ of formal functions; the same constructions extend verbatim to formal tensor fields, as mentioned in [7]. Consider the sections $\Gamma(\Lambda TM \otimes \widehat{S}T^*M)$, the *formal multivector fields*, and $\Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$, the *formal differential forms*, by again replacing $\frac{\partial}{\partial x^i} \rightsquigarrow \frac{\partial}{\partial y^i}$ and $\mathrm{d}x^i \rightsquigarrow \mathrm{d}y^i$. We can then let D act on these sections by Lie derivative, i.e. $D\sigma = \mathrm{d}_x\sigma + \mathcal{L}_Y\sigma$. In coordinates, writing $Y = \mathrm{d}x^a Y_a^j(x,y)\frac{\partial}{\partial y^j}$, a formal multivector field $X = \frac{1}{k!}X^{i_1\cdots i_k}(x,y)\frac{\partial}{\partial y^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{i_k}}$ and a formal differential form $\omega = \frac{1}{k!}\omega_{i_1\cdots i_k}(x,y)\mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}$, the Lie derivative acts componentwise as

$$(\mathcal{L}_Y X)^{i_1\cdots i_k} = \mathrm{d}x^a \left(Y_a^j \frac{\partial X^{i_1\cdots i_k}}{\partial y^j} - \sum_{m=1}^{k} \frac{\partial Y_a^{i_m}}{\partial y^j} X^{i_1\cdots j \cdots i_k} \right),$$

$$(\mathcal{L}_Y \omega)_{i_1\cdots i_k} = \mathrm{d}x^a \left(Y_a^j \frac{\partial \omega_{i_1\cdots i_k}}{\partial y^j} + \sum_{m=1}^{k} \frac{\partial Y_a^j}{\partial y^{i_m}} \omega_{i_1\cdots j\cdots i_k} \right),$$

where in the sums the index j replaces i_m. We can also allow symmetric powers, i.e. the bundles $S^kTM \otimes \widehat{S}T^*M$ and $S^kT^*M \otimes \widehat{S}T^*M$, with the same formulas where the wedge is replaced by the symmetric product.

The Taylor expansion extends to these bundles as well. Since ϕ_x is formally a diffeomorphism onto its image, it transports any tensor: we push multivector fields forward by $(\phi_x^{-1})_*$, pull differential forms back by ϕ_x^* and then Taylor expand the resulting coefficients. Recall J the Jacobian of Remark 3.6 and write T for the Taylor expansion in y. Each tensor index then receives one factor of the Jacobian, J^{-1} for vector indices and J for form indices: for a multivector field $X = \frac{1}{k!}X^{i_1\cdots i_k}(x)\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}$ and a differential form $\omega = \frac{1}{k!}\omega_{i_1\cdots i_k}(x)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ we set

$$\begin{aligned}
\mathrm{T}\phi_x^*X &= \frac{1}{k!}\mathrm{T}[(J^{-1})^{j_1}_{i_1}\cdots(J^{-1})^{j_k}_{i_k}X^{i_1\cdots i_k}(\phi_x(y))]\frac{\partial}{\partial y^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}} \in \Gamma(\Lambda^kTM \otimes \widehat{S}T^*M),\\
\mathrm{T}\phi_x^*\omega &= \frac{1}{k!}\mathrm{T}[\omega_{i_1\cdots i_k}(\phi_x(y))J^{i_1}_{j_1}\cdots J^{i_k}_{j_k}]\mathrm{d}y^{j_1} \wedge \cdots \wedge \mathrm{d}y^{j_k} \in \Gamma(\Lambda^kT^*M \otimes \widehat{S}T^*M).
\end{aligned} \tag{24}$$

On the symmetric powers S^kTM and S^kT^*M we use the same formulas with $\wedge$ replaced by $\odot$ and for $k = 0$ both reduce to the Taylor expansion of functions.

The extension is compatible with the tensor algebra structure: it is multiplicative, i.e. $\mathrm{T}\phi_x^*(fs) = \mathrm{T}\phi_x^*f \cdot \mathrm{T}\phi_x^*s$, where $\cdot$ denotes the module structure of the formal tensor fields over $\Gamma(\widehat{S}T^*M)$ and s is any section of the bundles above.

Moreover $\mathrm{T}\phi_x^*$ commutes with contractions; for example, for a one-form α and a vector field v we have

$$\langle \mathrm{T}\phi_x^*\alpha, \mathrm{T}\phi_x^*v\rangle = \mathrm{T}[\alpha_i(\phi_x(y))J^i_j(J^{-1})^j_l v^l(\phi_x(y))] = \mathrm{T}\phi_x^*\langle\alpha, v\rangle,$$

since the Jacobians cancel pairwise between upper and lower indices.

Finally, the projection π extends to a projection from these formal bundles to the tensor bundles on M by setting $y = 0$ and strictly replacing $\partial/\partial y^i, \mathrm{d}y^i \rightsquigarrow \partial/\partial x^i, \mathrm{d}x^i$. This is admissible because at the zero section $y = 0$ we have that $J(x, 0) = \mathrm{Id}$, and therefore the Taylor corrections vanish.

3.5 Globality

Because of flatness, one can also treat the connection D as a differential on the complex $\Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$ which raises the form degree (on x) by one.

We package the globalization as a strong deformation retract, recovering the full connection D from its $-\delta$ part by homological perturbation (Def. 2.12 and Lemma 2.13).

Lemma 3.11 (Grothendieck SDR)**.** Write $D = -\delta + \partial$ with $\partial = \mathrm{d}_x + Y + \delta$, and let (ι, Π_0, h) be the SDR data of $-\delta$, i.e. the inclusion ι of $C^\infty(M)$ in $\mathrm{d}x$- and y-degree zero, the projection $\Pi_0 : a \mapsto a_{00}$, and the homotopy $h = -\delta^{-1}$. Perturbing along ∂ yields a strong deformation retract

$$(C^\infty(M), 0) \underset{\pi_D}{\overset{\iota_D}{\rightleftarrows}} (\Gamma(\Lambda T^*M \otimes \widehat{S}T^*M), D) \circlearrowleft h_D$$

of the Grothendieck complex onto the global functions, with data

$$\iota_D = \sum_{n\geqslant 0}(\delta^{-1}\partial)^n\iota = \mathrm{T}\phi_x^*, \qquad \pi_D = \Pi_0, \qquad h_D = \sum_{n\geqslant 0}h(-\partial h)^n.$$

Proof. The triple (ι, Π_0, h) is an SDR for $-\delta$ (Def. 2.10): the retract relation $(-\delta)h + h(-\delta) = \mathrm{Id} - \iota\Pi_0$ is the Hodge decomposition (16), and the side conditions $h^2 = 0$, $h\iota = 0$,

$\Pi_0 h = 0$ hold since δ^{-1} squares to zero, vanishes in dx-degree zero, and raises the y-degree. We treat ∂ as a perturbation of $-\delta$ (Def. 2.12), which is admissible since $(-\delta+\partial)^2 = D^2 = 0$ by flatness. As ∂ raises the total degree while h preserves it, the operators $(-h\partial)^n$ increase it without bound, so the series of the homological perturbation lemma (Lemma 2.13) converge in each total degree and produce the retract above. The transferred differential is $\mathrm{d}_\partial = \sum_{n\geqslant 0} \Pi_0 \partial(-h\partial)^n \iota$; since ∂ raises $\deg_{\mathrm{d}x}$ we have $\Pi_0\partial = 0$, whence $\mathrm{d}_\partial = 0$ and $\pi_D = \Pi_0$. Finally $\iota_D = \sum_{n\geqslant 0}(\delta^{-1}\partial)^n \iota$ is the unique D-flat section (as a chain map) restricting to the identity on the zero section, so $\iota_D = \mathrm{T}\phi_x^*$ (Proposition 3.3). ☐

The retract makes the cohomology of the Grothendieck complex immediate.

Corollary 3.12. The cohomology of D sits in degree 0 and corresponds to global functions, i.e. $H_D^i = 0$ for all $i \geqslant 1$ and $H_D^0 = \mathrm{Im}(\mathrm{T}\phi_x^*) \cong C^\infty(M)$.

Remark 3.13. By the exact same arguments we also obtain (in the expressions below, the first ΛT^*M factor is the form part in dx, while the middle factors are the formal tensor parts in ∂_y resp. dy)

$$H_D^0(\Gamma(\Lambda T^*M \otimes \Lambda TM \otimes \widehat{S}T^*M)) = \Gamma(\Lambda TM)$$

$$H_D^0(\Gamma(\Lambda T^*M \otimes \Lambda T^*M \otimes \widehat{S}T^*M)) = \Gamma(\Lambda T^*M)$$

$$H_D^0(\Gamma(\Lambda T^*M \otimes S^jTM \otimes \widehat{S}T^*M)) = \Gamma(S^jTM)$$

$$H_D^0(\Gamma(\Lambda T^*M \otimes S^jT^*M \otimes \widehat{S}T^*M)) = \Gamma(S^jT^*M)$$

and $H_D^{\geqslant 1} = 0$; the isomorphisms in degree zero are realized by $\mathrm{T}\phi_x^*$ as in (24).

3.6 Symplectic structure

In Section 5 we will have special interest in the cotangent bundle T^*M of a manifold with its canonical symplectic structure. Let again x^i be a coordinate atlas on M and denote its induced fiber coordinates by p_i. Denote the subalgebra of functions which are polynomial in the fiber coordinates $C^\infty_{\mathrm{poly}}(T^*M) \subset C^\infty(T^*M)$. Note that this is well-defined: the induced fiber coordinates transform linearly in p, with the inverse Jacobian of the coordinate change as (x-dependent) coefficient matrix, so the polynomial degree in the fibers is chart-independent. We then have the canonical identification of $C^\infty_{\mathrm{poly}}(T^*M) \cong \Gamma(STM)$ via

$$f(x,p) = \frac{1}{k!} X^{i_1\cdots i_k}(x) p_{i_1}\cdots p_{i_k} \mapsto X_f(x) = \frac{1}{k!} X^{i_1\cdots i_k}(x) \frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_k}}.$$

As this is a part of the bundle $\Gamma(S^kTM)$ which we have considered in Section 3.4 and Remark 3.13, the entire machinery applies by defining a formal exponential map $\phi_x : T_xM \to M$ and using the identification above to identify $C^\infty_{\mathrm{poly}}(T^*M)$ with the tensor bundle above; this allows us for example to define $\mathrm{T}\phi_x^* f := \mathrm{T}\phi_x^* X_f$ with the formula (24).

An equivalent approach is to take the cotangent lift of ϕ_x; define the map

$$\Phi_x := (\phi_x, (\mathrm{d}\phi_x^*)^{-1}) : T^*(T_xM) \to T^*M.$$

In particular, this pulls back the fiber coordinates p_i to $\Phi_x^*(p_i) = \mathrm{T}((J^{-1})_i^j) z_j$, where z_j are the coordinates dual to y^j. If we now apply $\Phi_x^* f$ for the $f(x,p)$ defined above, we note that we obtain precisely the same result as $\mathrm{T}\phi_x^* X_f$ when we use the same identification trick and replace $z_i \rightsquigarrow \partial/\partial y^i$. Hence we can use the cotangent lift Φ_x as the formal exponential map to pullback functions and Taylor expand on the subalgebra $C^\infty_{\mathrm{poly}}(T^*M)$.

We can now consider the symplectic structure on T^*M. Denote by $\alpha = p_i \mathrm{d}x^i$ the canonical one-form and by $\omega = \mathrm{d}\alpha = \mathrm{d}p_i \wedge \mathrm{d}x^i$ the symplectic form, which induces the Poisson bracket $(-,-)$ with $(x^i, p_j) = \delta^i_j$. The symplectic structure under the isomorphism of $C^\infty_{\mathrm{poly}}(T^*M) \cong \Gamma(STM)$ is translated to the symmetric Schouten bracket; it is a map

$$[-,-]_{\mathrm{SN}} : \Gamma(S^kTM) \times \Gamma(S^\ell TM) \to \Gamma(S^{k+\ell-1}TM)$$

characterized algebraically by

$$[g,h]_{\mathrm{SN}} = 0, \quad [V,g]_{\mathrm{SN}} = -V(g), \quad [V,W]_{\mathrm{SN}} = -[V,W]$$

where $g,h \in C^\infty(M)$, $V,W \in \Gamma(TM)$ and the rightmost bracket is the regular Lie bracket. The minus signs relative to the usual conventions for the Schouten bracket are due to our sign convention $(x^i, p_j) = \delta^i_j$ for the canonical bracket. This is then extended via

$$[V, W \odot Z]_{\mathrm{SN}} = [V,W]_{\mathrm{SN}} \odot Z + W \odot [V,Z]_{\mathrm{SN}},$$

and as a biderivation in each argument to all of $\Gamma(STM)$. We then have

$$(f,g) = [X_f, X_g]_{\mathrm{SN}}.$$

We can now define the exact same structure on $\Gamma(STM \otimes \widehat{S}T^*M)$ by taking $[-,-]_{\mathrm{SN,fib}}$ to satisfy the same relations as mentioned but with y^i as the main variable instead of x^i, which is now simply a spectator under all operations. We can then easily show that for any $f,g \in C^\infty_{\mathrm{poly}}(T^*M)$

$$\mathrm{T}\phi^*_x([X_f, X_g]_{\mathrm{SN}}) = [\mathrm{T}\phi^*_x X_f, \mathrm{T}\phi^*_x X_g]_{\mathrm{SN,\,fib}}.$$

In the second approach we can define the symplectic structure on $T^*(T_xM)$ as $\alpha_{\mathrm{fib}} = z_i \mathrm{d}y^i$. Due to the construction of the cotangent lift we then immediately have $\Phi^*_x(\alpha) = \alpha_{\mathrm{fib}}$, i.e. a local symplectomorphism. Note that $[-,-]_{\mathrm{SN,\,fib}}$ is again the induced symplectic structure by α_{fib} under the identification in the formal space.

The two constructions above rest on the fact that the cotangent bundle carries this simple canonical symplectic structure; for a general symplectic manifold a different approach is needed.

For a general symplectic manifold (M,ω) we proceed as in [15]. The goal is to construct a flat, "symplectic connection" D on $\widehat{S}T^*M$. Let $(-,-)$ be the corresponding Poisson bracket and Π^{ij} the Poisson tensor, so that $\Pi^{ik}\omega_{kl} = \delta^i_l$, and consider again $\widehat{S}T^*M$ equipped with the Poisson structure $(-,-)_{\mathrm{fib}}$ given by

$$(a,b)_{\mathrm{fib}} = \Pi^{ij}(x) \frac{\partial a}{\partial y^i} \frac{\partial b}{\partial y^j}$$

i.e. the corresponding Poisson tensor does not depend on the fiber coordinates. The symplecticity condition on D then reads

$$D(a,b)_{\mathrm{fib}} = (Da,b)_{\mathrm{fib}} + (a,Db)_{\mathrm{fib}}. \tag{25}$$

Writing this out with the Ansatz (17) and (19): using that Π^{ij} does not depend on y, the d_x part fails the Leibniz rule by

$$\mathrm{d}_x(a,b)_{\mathrm{fib}} - (\mathrm{d}_x a, b)_{\mathrm{fib}} - (a, \mathrm{d}_x b)_{\mathrm{fib}} = \mathrm{d}x^l \partial_l \Pi^{ij} \frac{\partial a}{\partial y^i} \frac{\partial b}{\partial y^j},$$

while for the formal vector field (one-form) $Y = Y^k(x,y)\frac{\partial}{\partial y^k}$ we compute

$$Y(a,b)_{\mathrm{fib}} - (Ya,b)_{\mathrm{fib}} - (a,Yb)_{\mathrm{fib}} = -\left(\Pi^{kj}\frac{\partial Y^i}{\partial y^k} + \Pi^{ik}\frac{\partial Y^j}{\partial y^k}\right)\frac{\partial a}{\partial y^i}\frac{\partial b}{\partial y^j} = (\mathcal{L}_Y \Pi_{\mathrm{fib}})^{ij} \frac{\partial a}{\partial y^i}\frac{\partial b}{\partial y^j},$$

i.e. the failure is measured by the Lie derivative of the fiberwise Poisson tensor $\Pi_{\text{fib}} = \frac{1}{2}\Pi^{ij}(x)\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$. We apply this to the three parts of Y. The part $-\mathrm{d}x^i\frac{\partial}{\partial y^i}$ has constant coefficients in y and is therefore automatically compatible. The Christoffel part Γ combines with the d_x term to the condition

$$\mathrm{d}x^l\left[\partial_l\Pi^{ij} + \Gamma^i_{lk}\Pi^{kj} + \Gamma^j_{lk}\Pi^{ik}\right] = \mathrm{d}x^l(\nabla_l\Pi^{ij}) = 0,$$

i.e. ∇ must be a symplectic connection (equivalently $\nabla\omega = 0$). The remaining condition is $\mathcal{L}_X\Pi_{\text{fib}} = 0$, i.e. X must be a fiberwise symplectic formal vector field (one-form valued). Because the fibres are vector spaces, a fiberwise symplectic formal vector field is automatically fiberwise Hamiltonian, i.e. there is $r \in \Gamma(T^*M \otimes \widehat{S}T^*M)$ with $X = (r, -)_{\text{fib}}$.

Proposition 3.14 ([15] Thm. 7)**.** The map $\mathrm{T}\phi_x^*$ is a morphism of Poisson algebras

$$\mathrm{T}\phi_x^* : (C^\infty(M), (-,-)) \to (\Gamma(\widehat{S}T^*M), (-,-)_{\text{fib}})$$

if and only if X is a Hamiltonian vector field.

Proof. We first assume that X is Hamiltonian. We note that D can then be split up in Hamiltonian parts: X is Hamiltonian by assumption and the part $-\mathrm{d}x^i\frac{\partial}{\partial y^i}$ has Hamiltonian $\omega_{ij}y^i\mathrm{d}x^j$. This allows us to write

$$Da = \nabla a + (\omega_{ij}y^i\mathrm{d}x^j, a)_{\text{fib}} + (r, a)_{\text{fib}}.$$

In particular D satisfies the compatibility (25) by the computation above. Now for any $f \in C^\infty(M)$, we have that by Remark 3.6 we can write $\mathrm{T}\phi_x^* f = f(x) + \partial_{x^i}f(x)y^i + \mathcal{O}(y^2)$ and it follows immediately that

$$(\mathrm{T}\phi_x^* f, \mathrm{T}\phi_x^* g)_{\text{fib}}|_{y=0} = (f, g).$$

By (25) and $D\mathrm{T}\phi_x^* f = D\mathrm{T}\phi_x^* g = 0$ we have that $D(\mathrm{T}\phi_x^* f, \mathrm{T}\phi_x^* g)_{\text{fib}} = 0$. By Proposition 3.3 it follows that

$$(\mathrm{T}\phi_x^* f, \mathrm{T}\phi_x^* g)_{\text{fib}} = \mathrm{T}\phi_x^*(f, g)$$

and this direction follows.

Conversely, assume that $\mathrm{T}\phi_x^*$ intertwines the brackets. Then for any $f, g \in C^\infty(M)$ all three terms of (25) applied to $\mathrm{T}\phi_x^* f, \mathrm{T}\phi_x^* g$ vanish: $D\mathrm{T}\phi_x^* f = D\mathrm{T}\phi_x^* g = 0$ and also $D(\mathrm{T}\phi_x^* f, \mathrm{T}\phi_x^* g)_{\text{fib}} = D\mathrm{T}\phi_x^*(f, g) = 0$. By the computation above (using that ∇ is symplectic), the failure of the Leibniz rule is tensorial and we obtain

$$0 = (\mathcal{L}_X\Pi_{\text{fib}})^{ij}\,\frac{\partial \mathrm{T}\phi_x^* f}{\partial y^i}\frac{\partial \mathrm{T}\phi_x^* g}{\partial y^j}.$$

Choosing for f, g the local coordinate functions x^k, x^l, the chain rule gives $\partial_{y^j}\mathrm{T}\phi_x^* x^k = \mathrm{T}[J^k_j]$, which is an invertible formal power series. We conclude $\mathcal{L}_X\Pi_{\text{fib}} = 0$, i.e. X is symplectic and therefore Hamiltonian. □

Remark 3.15. In the proof we used that the parts $-\mathrm{d}x^i\frac{\partial}{\partial y^i}$ and X are Hamiltonian, while the term $\nabla = \mathrm{d}_x + \Gamma$ was kept together as a covariant derivative. Indeed, the Christoffel part Γ alone is in general not even symplectic: by the computation above,

$$(\mathcal{L}_\Gamma\Pi_{\text{fib}})^{ij} = \mathrm{d}x^l\left(\Gamma^i_{lk}\Pi^{kj} + \Gamma^j_{lk}\Pi^{ik}\right) = -\mathrm{d}x^l\partial_l\Pi^{ij},$$

where we used $\nabla_l\Pi^{ij} = 0$ in the second equality. Hence Γ is symplectic if and only if $\partial_l\Pi^{ij} = 0$ (equivalently $\partial_l\omega_{ij} = 0$), i.e. if and only if ω has constant coefficients in the chosen chart, as in Darboux coordinates. We can therefore in general not write $\Gamma = (h_\Gamma, -)_{\text{fib}}$ for some Hamiltonian h_Γ; in the splitting of D into Hamiltonian parts, the term $\nabla = \mathrm{d}_x + \Gamma$ has to be kept together as a covariant derivative.

As before, one can iteratively solve the flatness condition to obtain r. Here the r's are in one-to-one correspondence to $f \in \Gamma(\widehat{S}T^*M)$ with $\deg_y(f) \geqslant 4$, via the normalization condition $\delta^{-1}r = f$ (see Theorem 3.28). Indeed, since $\deg_y(X) \geqslant 2$, the Hamiltonian r is at least of degree 3 in y, and hence $\delta^{-1}r$ at least of degree 4. The condition analogous to (21) reads

$$0 = \hat{R} - \delta r + \nabla r + \frac{1}{2}(r, r),$$

where

$$\hat{R} = \frac{1}{4}\omega_{ij} R^j_{k\ell m} y^i y^k \mathrm{d}x^\ell \wedge \mathrm{d}x^m.$$

The two-form $\hat{R}$ is a Hamiltonian for the curvature term of (21); this uses that $\omega_{ij} R^j_{k\ell m}$ is symmetric in i, k for a symplectic connection. Using the condition $\delta^{-1}r = f$ then gives us the final equation

$$r = \delta f + \delta^{-1}(\hat{R} + \nabla r + \frac{1}{2}(r, r)),$$

which again can be computed iteratively for r.

3.7 Odd–symplectic structure

We now begin to extend the constructions of this chapter to a $\mathbb{Z}$-graded supermanifold $\mathcal{M}$ with an odd symplectic form ω; the goal is again a flat, symplectic Grothendieck connection as in Section 3.6.

3.7.1 Connection and FEM

As in Section 3.1 we study the bundle $\widehat{S}T^*\mathcal{M}$, now the formal completion of the *graded* symmetric algebra of the cotangent bundle: for local coordinates z^i with parities $p_i := |z^i|$ the fiber generators y^i carry the parities $|y^i| = p_i$, while the one-forms $\mathrm{d}z^i$ have total parity $p_i + 1$. On the complex $\Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M})$, bigraded by $(p, q) := (\deg_{\mathrm{d}z}, \deg_y)$ as before, we define by the same formulas the operators

$$\delta := \mathrm{d}z^i \frac{\partial}{\partial y^i}, \qquad \delta^* := y^i \iota_{\frac{\partial}{\partial z^i}}, \qquad \delta^{-1} := \frac{1}{p+q}\,\delta^* \text{ for } (p, q) \neq (0, 0)$$

and $\delta^{-1} := 0$ else.

Lemma 3.16 (Graded Hodge decomposition)**.** The operators satisfy $\delta^2 = (\delta^{-1})^2 = 0$ and $\delta\delta^{-1} + \delta^{-1}\delta = \mathrm{Id}$ on bidegrees $(p, q) \neq (0, 0)$. In particular every $a \in \Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M})$ decomposes as

$$a = a_{00} + (\delta^{-1}\delta + \delta\delta^{-1})a, \qquad \deg_{\mathrm{d}z}(a_{00}) = \deg_y(a_{00}) = 0. \tag{26}$$

Proof. Both δ and δ^* are odd derivations preserving the total degree $p + q$. The squares δ^2, $(\delta^*)^2$ and the commutator $[\delta, \delta^*]$ are even derivations, hence determined by their action on the generators $f(z)$, y^j, $\mathrm{d}z^j$. From

$$\delta y^j = \mathrm{d}z^j, \qquad \delta\, \mathrm{d}z^j = 0, \qquad \delta^* y^j = 0, \qquad \delta^*\, \mathrm{d}z^j = y^j,$$

and $\delta f = \delta^* f = 0$, we read off

$$\delta^2 = 0, \qquad (\delta^*)^2 = 0, \qquad [\delta, \delta^*] = (\deg_{\mathrm{d}z} + \deg_y)\,\mathrm{Id}\,.$$

Since δ and δ^* preserve $p + q$, the normalization $\frac{1}{p+q}$ commutes with them, and dividing the last two identities by $p + q$ yields $(\delta^{-1})^2 = 0$ and $\delta\delta^{-1} + \delta^{-1}\delta = \mathrm{Id}$ on $(p, q) \neq (0, 0)$; the decomposition (26) follows. □

We now construct the Grothendieck connection. The formal vector fields are again the $\mathcal{O}_{\mathcal{M}}$-linear derivations of $\Gamma(\widehat{S}T^*\mathcal{M})$, written in coordinates as $Y = Y^j(z,y)\frac{\partial}{\partial y^j}$, and we consider their one-form valued analogues

$$Y = \mathrm{d}z^i Y_i^j(z,y)\frac{\partial}{\partial y^j} \in \Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M} \otimes T\mathcal{M}),$$

which we require to be odd with respect to the total parity, i.e. $|Y_i^j| = p_i + p_j$. The connection is again given by $D = \mathrm{d}_z + Y$ as in (17), an odd operator raising $\deg_{\mathrm{d}z}$ by one, with flatness condition $\frac{1}{2}[D,D] = 0$; the only difference to Section 3.1 is that the commutator is now graded by the total parity instead of the form degree.

Let ∇ be a torsion-free connection on $T\mathcal{M}$ (Definition 1.12) with Christoffel symbols $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k(z)\partial_k$ of parity $|\Gamma_{ij}^k| = p_i + p_j + p_k$. The torsion of Definition 1.12 evaluates on coordinate vector fields to

$$T^\nabla(\partial_i, \partial_j) = (\Gamma_{ij}^k - (-1)^{p_i p_j}\Gamma_{ji}^k)\partial_k,$$

so torsion-freeness now reads $\Gamma_{ij}^k = (-1)^{p_i p_j}\Gamma_{ji}^k$. The power series Ansatz for Y is the same as in (19),

$$Y = -\mathrm{d}z^i\frac{\partial}{\partial y^i} - \mathrm{d}z^\ell y^k \Gamma_{k\ell}^i(z)\frac{\partial}{\partial y^i} + X \tag{27}$$

with X odd, $\deg_y(X) \geqslant 2$ and $\deg_{\mathrm{d}z}(X) = 1$. Note that all three terms are indeed odd as required.

Lemma 3.17. As in the classical case, $[\mathrm{d}_z, \delta] = 0$, and $[\delta, \Gamma] = 0$ if and only if ∇ is torsion-free.

Proof. The first holds for the exact same reason. For the second, again $\Gamma(\delta y^i) = \Gamma(\mathrm{d}z^i) = 0$, while now

$$\delta(\Gamma y^i) = -\mathrm{d}z^m\frac{\partial}{\partial y^m}(\mathrm{d}z^\ell y^k \Gamma_{k\ell}^i) = -(-1)^{p_k(p_\ell+1)}\mathrm{d}z^k\mathrm{d}z^\ell\Gamma_{k\ell}^i =: -S^i.$$

The sum S^i vanishes by the graded symmetry of Γ: relabelling $k \leftrightarrow \ell$ and using torsion-freeness gives

$$S^i = (-1)^{p_\ell(p_k+1)}(-1)^{p_k p_\ell}\mathrm{d}z^\ell\mathrm{d}z^k\Gamma_{k\ell}^i = (-1)^{p_\ell}\mathrm{d}z^\ell\mathrm{d}z^k\Gamma_{k\ell}^i,$$

while instead commuting the two one-forms gives

$$S^i = (-1)^{p_k(p_\ell+1)+(p_k+1)(p_\ell+1)}\mathrm{d}z^\ell\mathrm{d}z^k\Gamma_{k\ell}^i = -(-1)^{p_\ell}\mathrm{d}z^\ell\mathrm{d}z^k\Gamma_{k\ell}^i,$$

hence $S^i = -S^i = 0$. □

The curvature is defined exactly as in (20), $R := [\mathrm{d}_z, \Gamma] + \frac{1}{2}[\Gamma, \Gamma]$, and is again a formal vector field, linear in y; evaluating on the generators as before, now keeping track of the Koszul signs, yields

$$R(y^i) = (-1)^{p_\ell+p_k}\,\mathrm{d}z^\ell y^k\mathrm{d}z^m\frac{\partial\Gamma_{k\ell}^i}{\partial z^m} + (-1)^{p_\ell+1}\,\mathrm{d}z^\ell\mathrm{d}z^m y^n\Gamma_{nm}^k\Gamma_{k\ell}^i.$$

We refrain from rewriting this in terms of a graded curvature tensor; the explicit component form will not be needed.

The computation of the flatness condition is now verbatim the one of Section 3.1: expanding $\frac{1}{2}[D,D]$ produces the nine brackets as before, the mixed terms pair up since all parts of Y are odd, the terms $[\mathrm{d}_z,\delta]$, $[\delta,\Gamma]$ and $\frac{1}{2}[\delta,\delta] = \delta^2$ vanish by Lemma 3.17 and Lemma 3.16, and the flatness condition transfers to

$$R - \delta X + \nabla X + \frac{1}{2}[X,X] = 0, \tag{28}$$

where again $\delta X := [\delta, X]$ and $\nabla X := [\nabla, X]$. Applying the graded Hodge decomposition (26) to X turns this into the recursion

$$X = \delta a + \delta^{-1}(R + \nabla X + \frac{1}{2}[X, X]),$$

which is solved iteratively in $\deg_y$ exactly as in (22), with solutions in bijection with the normalizations $a = \delta^{-1}X$ of $\deg_y(a) \geqslant 3$. That the iteration in the graded case indeed solves (28) is Proposition 5.1 of [27], where the Fedosov resolution is carried out for graded manifolds (stated there for the normalization $a = 0$); the argument for general a is the same, and we will spell it out in the Hamiltonian setting in the proof of Theorem 3.28.

Note also that Proposition 3.3, which we cited from [27], is proven there for graded manifolds in the first place, so the formal exponential map EXP* of Definition 3.4 is defined in the graded setting as well. The explicit formula was given in Section 3.3

$$\mathrm{EXP}^* f = \sum_I \frac{1}{I!} y^I \mathrm{pbw}(\overleftarrow{\partial_z^I})(f). \tag{29}$$

In particular

$$\mathrm{EXP}^* f = f + y^i \frac{\partial f}{\partial z^i} + \mathcal{O}(y^2), \qquad |\mathrm{EXP}^* f| = |f|,$$

the latter since $\mathrm{pbw}(\overleftarrow{\partial_z^I})$ has the same parity as y^I.

Also the globalization statement of Section 3.5 carries over.

Proposition 3.18. Write $D = -\delta + \partial$ with $\partial = \mathrm{d}_z + Y + \delta$ as in Lemma 3.11. Perturbing the SDR data $(\iota, \Pi_0, h = -\delta^{-1})$ of $-\delta$ along ∂ yields a strong deformation retract

$$(\mathcal{O}_{\mathcal{M}}, 0) \underset{\pi_D}{\overset{\iota_D}{\rightleftarrows}} (\Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M}), D) \circlearrowleft h_D$$

with

$$\iota_D = \sum_{n \geqslant 0} (\delta^{-1}\partial)^n \iota = \mathrm{EXP}^*, \qquad \pi_D = \Pi_0, \qquad h_D = \sum_{n \geqslant 0} h(-\partial h)^n.$$

In particular $H_D^0 = \mathrm{Im}(\mathrm{EXP}^*) \cong \mathcal{O}_{\mathcal{M}}$ and $H_D^{\geqslant 1} = 0$.

Proof. The proof of Lemma 3.11 applies verbatim, with the graded Hodge decomposition (Lemma 3.16) providing the SDR data of $-\delta$. This is Theorem 7.1 of [27], proven there by the same homological perturbation argument. □

3.7.2 Antibracket

We now bring in the symplectic structure. Freezing the base point, the formal fiber carries the constant-coefficient odd symplectic form $\omega_{\text{fib}} := \frac{1}{2}\omega_{ij}(z)\mathrm{d}y^i\mathrm{d}y^j$, and Lemma 1.28 applies fiberwise.

Definition 3.19 (Fiberwise antibracket)**.** The *fiberwise antibracket* on $\Gamma(\widehat{S}T^*\mathcal{M})$ is

$$(a, b)_{\text{fib}} := \sum_{i,j} (-1)^{p_i|a|} \frac{\partial a}{\partial y^i} \Pi^{ij}(z) \frac{\partial b}{\partial y^j}, \tag{30}$$

where $\Pi^{ij}(z)$ is the dressed inverse of ω_{ij} from Definition 1.27, $k = 1$. The bracket is extended to $\Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M})$ by the same formula, with $|a|$ the total parity.

In particular $(y^i, y^j)_{\text{fib}} = (-1)^{p_i}\Pi^{ij}(z)$, skew symmetry and the Jacobi identity hold fiberwise. The symplecticity condition on the Grothendieck connection now reads

$$D(a, b)_{\text{fib}} = (Da, b)_{\text{fib}} + (-1)^{|a|+1}(a, Db)_{\text{fib}}, \tag{31}$$

replacing (25). This is precisely the Leibniz rule of Definition 2.2 for the odd operator D.

Definition 3.20. Let ∇ be a torsion-free connection on $T\mathcal{M}$, extended to one-forms as the dual connection, characterized by

$$X(\iota_Y\alpha) = \iota_{\nabla_X Y}\alpha + (-1)^{|X|(|Y|+1)}\iota_Y(\nabla_X\alpha), \qquad \alpha \in \Omega^1(\mathcal{M}).$$

We call ∇ *symplectic* if

$$\nabla_X(\iota_Y\omega) = \iota_{\nabla_X Y}\omega \qquad \text{for all } X, Y \in \mathfrak{X}(\mathcal{M}).$$

This is the condition $\nabla\omega = 0$. In coordinates, choosing $X = \partial_l$, $Y = \partial_j$ and contracting with ι_{∂_k} via the characterization above, using $\iota_{fX} = f\iota_X$, the condition on ω_{jk} reads

$$\frac{\partial\omega_{jk}}{\partial z^l} = (-1)^{(p_j+p_m)(p_k+1)}\Gamma^m_{lj}\omega_{mk} + \Gamma^m_{lk}\omega_{jm}, \tag{32}$$

which for vanishing parities reduces to the familiar $\partial_l\omega_{jk} = \Gamma^m_{lj}\omega_{mk} + \Gamma^m_{lk}\omega_{jm}$ of Section 3.6.

Remark 3.21. The definition above holds for both even and odd ω's. The formula (32) is the same for the even case as well.

Lemma 3.22. The connection ∇ is symplectic if and only if

$$\nabla_l\Pi^{ij} := \frac{\partial\Pi^{ij}}{\partial z^l} + (-1)^{p_i+p_m(p_i+p_l)}\Gamma^i_{ml}\Pi^{mj} + (-1)^{p_l(p_i+1)}\Pi^{im}\Gamma^j_{ml} = 0. \tag{33}$$

Proof. Abbreviate $\sigma_{ijk} := (-1)^{(p_k+1)(p_i+p_j+1)}$ and $\lambda_j := (-1)^{p_l(p_i+p_j+1)}$, with i, k, l fixed throughout and only j summed. We show

$$\sum_j \sigma_{ijk}\left[(\nabla_l\Pi^{ij})\,\omega_{jk} + \lambda_j\,\Pi^{ij}\,\nabla_l\omega_{jk}\right] = 0. \tag{†}$$

Since λ_j is the Koszul sign of ∂_l passing Π^{ij}, the derivative terms of (33) and (32) assemble to $\partial_l(\sum_j \sigma_{ijk}\Pi^{ij}\omega_{jk}) = \partial_l\delta^i_k = 0$, and it remains to show that the four Christoffel terms, denoted T_1, T_2 for (33) and T_3, T_4 for (32), cancel. In T_1 the dressing σ_{ijk} differs from σ_{mjk} by $(-1)^{(p_k+1)(p_i+p_m)}$, independently of j, so the sum over j collapses as

$$T_1 = \sum_{j,m}\sigma_{ijk}(-1)^{p_i+p_m(p_i+p_l)}\,\Gamma^i_{ml}\Pi^{mj}\omega_{jk} = (-1)^{p_i+p_k(p_i+p_l)+(p_k+1)(p_i+p_k)}\,\Gamma^i_{kl} = (-1)^{p_kp_l}\,\Gamma^i_{kl}.$$

In T_4 we move Γ^m_{lk}, of parity $p_l + p_k + p_m$, to the left of Π^{ij}; the Koszul cost converts the dressing $\sigma_{ijk}\lambda_j = (-1)^{(p_k+p_l+1)(p_i+p_j+1)}$ precisely into σ_{ijm}, and the sum collapses again:

$$T_4 = -\sum_{j,m}\sigma_{ijk}\lambda_j\,\Pi^{ij}\Gamma^m_{lk}\omega_{jm} = -\sum_{j,m}\sigma_{ijm}\,\Gamma^m_{lk}\Pi^{ij}\omega_{jm} = -\Gamma^i_{lk} = -(-1)^{p_kp_l}\,\Gamma^i_{kl},$$

so $T_1 + T_4 = 0$ by torsion-freeness. In T_3 we relabel $j \leftrightarrow m$ and swap the lower slots of Γ; the resulting sign agrees identically with that of T_2,

$$(p_k+1)(p_i+p_m+1)+p_l(p_i+p_m+1)+(p_m+p_j)(p_k+1)+p_lp_m = (p_k+1)(p_i+p_j+1)+p_l(p_i+1),$$

so $T_2 + T_3 = 0$, both terms carrying the factors $\Pi^{im}\Gamma^j_{ml}\omega_{jk}$ but opposite overall signs. This proves (†). $\square$

We go back to the fiber now.

Definition 3.23 (Defect)**.** For an operator V of parity $|V|$ on $\Gamma(\Lambda T^*\mathcal{M}\otimes\widehat{S}T^*\mathcal{M})$, its *defect* with respect to the fiberwise antibracket is

$$\mathrm{def}_V(a,b) := V(a,b)_{\mathrm{fib}} - (Va,b)_{\mathrm{fib}} - (-1)^{|V|(|a|+1)}(a,Vb)_{\mathrm{fib}}.$$

The symplecticity condition (31) thus reads $\mathrm{def}_D = 0$. We first check this for the covariant part of the Ansatz.

Lemma 3.24. For $\nabla = \mathrm{d}_z + \Gamma$ we have on the generators

$$\mathrm{def}_\nabla(y^i, y^j) = (-1)^{p_i} \mathrm{d}z^l \nabla_l \Pi^{ij},$$

with $\nabla_l \Pi$ as in (33). In particular $\mathrm{def}_\nabla = 0$ if and only if ∇ is symplectic.

Proof. Since ∇ is a derivation and $(-,-)_{\mathrm{fib}}$ is a derivation in each argument, the mixed second order terms cancel and the defect is again a derivation in each argument; moreover all three terms vanish whenever one argument does not contain y. The defect is therefore determined by its values on the pairs of generators (y^i, y^j). There, using $(y^i, y^j)_{\mathrm{fib}} = (-1)^{p_i} \Pi^{ij}$ and $\nabla y^i = -\mathrm{d}z^l y^k \Gamma^i_{kl}$, we compute

$$\begin{aligned}
\nabla(y^i, y^j)_{\mathrm{fib}} &= (-1)^{p_i} \mathrm{d}z^l \frac{\partial \Pi^{ij}}{\partial z^l}, \\
(\nabla y^i, y^j)_{\mathrm{fib}} &= -\sum_m (-1)^{p_m(p_i+p_l)} \mathrm{d}z^l \Gamma^i_{ml} \Pi^{mj}, \\
(y^i, \nabla y^j)_{\mathrm{fib}} &= -(-1)^{p_i} \sum_m (-1)^{p_m(p_l+1)} \Pi^{im} \mathrm{d}z^l \Gamma^j_{ml}.
\end{aligned}$$

Collecting the three terms with the signs of Definition 3.23 and commuting $\mathrm{d}z^l$ to the left, the exponents collapse to the dressings of (33) and we obtain

$$\mathrm{def}_\nabla(y^i, y^j) = (-1)^{p_i} \mathrm{d}z^l \nabla_l \Pi^{ij}. \qquad \square$$

The leading term $-\delta$ and the correction term X are treated through their Hamiltonians.

Definition 3.25. Let $\mathrm{d}_y := \mathrm{d}y^i \frac{\partial}{\partial y^i}$ be the vertical de Rham differential. We call a formal vector field X *fiberwise symplectic* if $\mathcal{L}_X \omega_{\mathrm{fib}} = (-1)^{|X|} \mathrm{d}_y(\iota_X \omega_{\mathrm{fib}}) = 0$.

Since we're in the fiber, every fiberwise symplectic formal vector field is fiberwise Hamiltonian. We use the fiberwise Euler vector field $\mathbb{E}_y := y^i \frac{\partial}{\partial y^i}$.

Lemma 3.26 (Fiberwise Hamiltonians)**.** Let W be a formal vector field of $\deg_y = n$ (with possibly $\mathrm{d}z$-valued coefficients). If W is fiberwise symplectic then it is fiberwise Hamiltonian,

$$\iota_W \omega_{\mathrm{fib}} = \mathrm{d}_y r, \qquad r = \frac{1}{n+1} \iota_{\mathbb{E}_y} \iota_W \omega_{\mathrm{fib}},$$

and consequently $W = (-1)^{|r|}(r, -)_{\mathrm{fib}}$ with $|r| = |W| + 1$ and $\deg_y(r) = n + 1$.

Proof. The fiberwise Euler vector field is even, so Cartan's formula gives $\mathcal{L}_{\mathbb{E}_y} = \iota_{\mathbb{E}_y} \mathrm{d}_y + \mathrm{d}_y \iota_{\mathbb{E}_y}$, and $\mathcal{L}_{\mathbb{E}_y}$ is the even derivation counting the total weight $\deg_y + \deg_{\mathrm{d}y}$, as one checks on the generators y^i and $\mathrm{d}y^i$ (compare Remark 1.16). The one-form $\alpha := \iota_W \omega_{\mathrm{fib}}$ has weight $n+1 \geqslant 1$ and is closed, since W is fiberwise symplectic: $\mathrm{d}_y(\iota_W \omega_{\mathrm{fib}}) = 0$ by Definition 3.25. Hence

$$(n+1)\alpha = \mathcal{L}_{\mathbb{E}_y} \alpha = \mathrm{d}_y(\iota_{\mathbb{E}_y} \alpha),$$

i.e. $\iota_W \omega_{\mathrm{fib}} = \mathrm{d}_y r$ with r as above, which says $W = X_r$ for the fiberwise symplectic structure. The relation to the bracket is Definition 1.25 applied fiberwise, $(r, b)_{\mathrm{fib}} = (-1)^{|r|} X_r(b)$, together with $|X_r| = |r| + 1$. $\square$

The defect of fiberwise Hamiltonian vector fields vanish.

Lemma 3.27. For $W = (-1)^{|r|}(r, -)_{\mathrm{fib}}$ we have $\mathrm{def}_W = 0$.

Proof. This is the Jacobi identity: with $w = |r| + 1$ the parity of W,

$$\begin{aligned}
W(a,b)_{\mathrm{fib}} = (-1)^{|r|}(r, (a,b)_{\mathrm{fib}})_{\mathrm{fib}} &= (-1)^{|r|}((r,a)_{\mathrm{fib}}, b)_{\mathrm{fib}} + (-1)^{|r| + (|r|+1)(|a|+1)}(a, (r,b)_{\mathrm{fib}})_{\mathrm{fib}} \\
&= (Wa, b)_{\mathrm{fib}} + (-1)^{w(|a|+1)}(a, Wb)_{\mathrm{fib}}. \qquad \square
\end{aligned}$$

We can now assemble the symplectic Grothendieck connection. Applying Lemma 3.26 to the leading term $-\delta$, which is fiberwise symplectic, gives us

$$h_\delta := \iota_{\mathbb{E}_y}\iota_{-\delta}\omega_{\mathrm{fib}} = -\sum_{k,m}(-1)^{(p_k+1)(p_m+1)}y^m \mathrm{d}z^k\omega_{km}(z), \qquad -\delta = (h_\delta, -)_{\mathrm{fib}},$$

even and of $\deg_y = 1$. Second, write $\mathrm{ad}_r := (r,-)_{\mathrm{fib}}$, an operator of parity $|r|+1$. The Jacobi identity gives $[\mathrm{ad}_r, \mathrm{ad}_s] = \mathrm{ad}_{(r,s)_{\mathrm{fib}}}$ as in the proof of Lemma 3.27, while the vanishing defects of ∇ and $-\delta$ give

$$[\nabla, \mathrm{ad}_r] = \mathrm{ad}_{\nabla r}, \qquad [\delta, \mathrm{ad}_r] = \mathrm{ad}_{\delta r}\,. \tag{34}$$

Third, the curvature $R = \nabla^2 = [\mathrm{d}_z, \Gamma] + \frac{1}{2}[\Gamma,\Gamma]$ of (20) is a formal vector field, even and linear in y: evaluated on the generators the second-order part of ∇^2 cancels, leaving the graded Riemann tensor $R(y^i)$. It is fiberwise symplectic: since ∇ is symplectic, $\nabla\omega_{\mathrm{fib}} = 0$ by (32), and as ∇ acts on formal forms by Lie derivative (Section 3.4) its square acts as $\mathcal{L}_R$, so that $\mathcal{L}_R\omega_{\mathrm{fib}} = \nabla^2\omega_{\mathrm{fib}} = 0$. Since R is even, the Hamiltonian of Lemma 3.26 generates $-R$, and we absorb the sign into

$$\hat{R} := -\frac{1}{2}\iota_{\mathbb{E}_y}\iota_R\omega_{\mathrm{fib}}, \qquad R = (\hat{R}, -)_{\mathrm{fib}},$$

odd and of $\deg_y = 2$.

Theorem 3.28. Let ∇ be a torsion-free symplectic connection on $T\mathcal{M}$ and take the Ansatz (27) with $X = (r,-)_{\mathrm{fib}}$, where $r \in \Gamma(\Lambda^1 T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M})$ is even with $\deg_y(r) \geqslant 3$. Then D is symplectic, and it is flat if and only if

$$\hat{R} - \delta r + \nabla r + \frac{1}{2}(r,r)_{\mathrm{fib}} = 0. \tag{35}$$

Solutions exist and are obtained iteratively in $\deg_y$ from

$$r = \delta h + \delta^{-1}(\hat{R} + \nabla r + \frac{1}{2}(r,r)_{\mathrm{fib}}),$$

in bijection with the normalizations $h = \delta^{-1}r$ of $\deg_y(h) \geqslant 4$.

Proof. The parts ∇, $-\delta$ and X have vanishing defect by Lemma 3.24 for ∇ and by Lemma 3.27 for $-\delta = (h_\delta, -)_{\mathrm{fib}}$ and $X = (r,-)_{\mathrm{fib}}$; the defect is additive in the operator, so D is symplectic. For flatness, insert $X = \mathrm{ad}_r$ into (28) and use the three commutator identities above:

$$R - \delta X + \nabla X + \frac{1}{2}[X,X] = \mathrm{ad}\left(\hat{R} - \delta r + \nabla r + \frac{1}{2}(r,r)_{\mathrm{fib}}\right).$$

The argument of ad has all components of $\deg_y \geqslant 2$, namely 2 for $\hat{R}$, $\geqslant 2$ for δr, $\geqslant 3$ for ∇r and $\geqslant 4$ for $(r,r)_{\mathrm{fib}}$, while the kernel of ad is y-independent; flatness is therefore equivalent to (35).

It remains to establish the correspondence between solutions and normalizations. If r solves (35), the Hodge decomposition (26) gives $r = \delta\delta^{-1}r + \delta^{-1}\delta r$ (r carries form degree one, so $r_{00} = 0$), and inserting $\delta r = \hat{R} + \nabla r + \frac{1}{2}(r,r)_{\mathrm{fib}}$ shows that r satisfies the recursion with $h = \delta^{-1}r$, odd and of $\deg_y \geqslant 4$. Conversely, fix such an h. Since δ^{-1} raises $\deg_y$, the degree-q part of the right hand side of the recursion depends on r only through degrees $< q$, so the recursion has a unique solution r, obtained iteratively, which is even with $\deg_y(r) \geqslant 3$. Moreover $\delta^{-1}r = \delta^{-1}\delta h = h$, using $(\delta^{-1})^2 = 0$ and then (26) on h, which has form degree zero.

We are left to show that the solution of the recursion solves (35); we adapt the classical argument of Fedosov (proof of Theorem 3.2 in [17]). Denote by $F := \hat{R} - \delta r + \nabla r + \frac{1}{2}(r,r)_{\mathrm{fib}}$

the left hand side, so that the identity above reads $D^2 = \mathrm{ad}_F$, and $\deg_y(F) \geqslant 2$. First, F obeys the Bianchi identity

$$\delta F = \nabla F + (r, F)_{\mathrm{fib}}.$$

Indeed, the graded Jacobi identity for the commutator gives $[D, D^2] = \frac{1}{2}[D, [D, D]] = 0$, while the identities (34) applied to F and the Jacobi identity $[\mathrm{ad}_r, \mathrm{ad}_F] = \mathrm{ad}_{(r,F)_{\mathrm{fib}}}$ yield

$$0 = [D, \mathrm{ad}_F] = \mathrm{ad}(\nabla F - \delta F + (r, F)_{\mathrm{fib}});$$

the argument of ad has $\deg_y \geqslant 1$ and therefore vanishes. Second, F is normalized: the recursion says $\delta^{-1}(\hat{R} + \nabla r + \frac{1}{2}(r, r)_{\mathrm{fib}}) = r - \delta h$, while (26) together with $\delta^{-1} r = h$ gives $\delta^{-1}\delta r = r - \delta h$ as well; subtracting, $\delta^{-1} F = 0$. Now apply (26) to F itself (F carries form degree two, so $F_{00} = 0$) and insert the Bianchi identity:

$$F = \delta^{-1}\delta F = \delta^{-1}(\nabla F + (r, F)_{\mathrm{fib}}).$$

The right hand side raises $\deg_y$, by one in the first term and by at least three in the second, so the degree-q component of F is determined by the components of lower degree. As F has no components below $\deg_y = 2$, induction on q gives $F = 0$. □

Finally, the formal exponential map of such a connection respects the antibrackets; this is the odd analogue of Proposition 3.14.

Proposition 3.29. Let D be a flat connection of the Ansatz (27) with ∇ symplectic and X fiberwise symplectic. Then

$$\mathrm{EXP}^* : (\mathcal{O}_{\mathcal{M}}, (-,-)) \to (\Gamma(\widehat{S}T^*\mathcal{M}), (-,-)_{\mathrm{fib}})$$

is a morphism of odd Poisson algebras.

Proof. Since X is fiberwise symplectic, $\mathrm{def}_D = 0$ as in Theorem 3.28. For $f, g \in \mathcal{O}_{\mathcal{M}}$ the section $(\mathrm{EXP}^* f, \mathrm{EXP}^* g)_{\mathrm{fib}}$ is then D-flat, and its value on the zero section is

$$(\mathrm{EXP}^* f, \mathrm{EXP}^* g)_{\mathrm{fib}}\big|_{y=0} = \sum_{i,j} (-1)^{p_i |f|} \frac{\partial f}{\partial z^i} \Pi^{ij} \frac{\partial g}{\partial z^j} = (f, g),$$

by $\mathrm{EXP}^* f = f + y^i \frac{\partial f}{\partial z^i} + \mathcal{O}(y^2)$, $|\mathrm{EXP}^* f| = |f|$ and Lemma 1.28. Proposition 3.3 therefore gives $(\mathrm{EXP}^* f, \mathrm{EXP}^* g)_{\mathrm{fib}} = \mathrm{EXP}^*(f, g)$. □

3.7.3 BV Laplacian

Next we want to understand how the differentials (the Grothendieck connection D and the BV Laplacians) and the antibracket interplay between formal geometry and the underlying manifold. Define the *fiberwise BV Laplacian*

$$\Delta_{\mathrm{fib}} := \frac{1}{2} \sum_{i,j} (-1)^{p_i p_j} \Pi^{ij}(z) \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} \quad \text{on } \Gamma(\widehat{S}T^*\mathcal{M}). \tag{36}$$

The Koszul factor is fixed by the requirement that Δ_{fib} generates the fiberwise antibracket (30). Note that the skew-symmetry of the antibracket together with $(z^i, z^j) = (-1)^{p_i}\Pi^{ij}$ yields the graded symmetry $\Pi^{ji} = (-1)^{p_i p_j}\Pi^{ij}$, which we use below.

Lemma 3.30. The odd operator Δ_{fib} generates the fiber bracket in the sense of (5),

$$\Delta_{\mathrm{fib}}(ab) = \Delta_{\mathrm{fib}} a\, b + (-1)^{|a|} a \Delta_{\mathrm{fib}} b + (-1)^{|a|}(a, b)_{\mathrm{fib}},$$

and squares to zero, $\Delta_{\mathrm{fib}}^2 = 0$.

Proof. Expanding $\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}(ab)$ by the Leibniz rule, the terms with both derivatives on one factor produce $\Delta_{\mathrm{fib}}a\,b + (-1)^{|a|}a\Delta_{\mathrm{fib}}b$. The cross terms give

$$\frac{1}{2}\sum_{i,j}(-1)^{p_ip_j}\Pi^{ij}((-1)^{p_i(|a|+p_j)}\frac{\partial a}{\partial y^j}\frac{\partial b}{\partial y^i} + (-1)^{p_j|a|}\frac{\partial a}{\partial y^i}\frac{\partial b}{\partial y^j});$$

commuting Π^{ij} past the first derivative factor and using the graded symmetry of Π in the first summand, both summands collapse to $\frac{1}{2}(-1)^{|a|+p_i|a|}\frac{\partial a}{\partial y^i}\Pi^{ij}\frac{\partial b}{\partial y^j}$, and the sum is $(-1)^{|a|}(a,b)_{\mathrm{fib}}$. For nilpotency, passing the derivatives through $\Pi^{kl}(z)$ gives

$$\Delta_{\mathrm{fib}}^2 = \frac{1}{4}\sum(-1)^{p_ip_j+p_kp_l+(p_i+p_j)(p_k+p_l+1)}\,\Pi^{ij}\Pi^{kl}\,\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^k}\frac{\partial}{\partial y^l}.$$

The summand with the pairs (i,j) and (k,l) exchanged equals minus the original summand: restoring the original order of the Π's and of the derivatives and collecting all Koszul signs, the relative factor is, with $u := p_i + p_j$ and $v := p_k + p_l$,

$$(-1)^{u(v+1)+v(u+1)+(u+1)(v+1)+uv} = -1.$$

The sum cancels in pairs and $\Delta_{\mathrm{fib}}^2 = 0$. □

Compatibility of Δ_{fib} with the Grothendieck connection is more delicate and imposes a condition on the correction term of D.

Lemma 3.31. Let D be as in Theorem 3.28, with ∇ symplectic and $X = (r,-)_{\mathrm{fib}}$, $\deg_y r \geqslant 3$. Then

$$[\Delta_{\mathrm{fib}}, D] = (\Delta_{\mathrm{fib}}r, -)_{\mathrm{fib}}.$$

In particular $[\Delta_{\mathrm{fib}}, D] = 0$ if and only if $\Delta_{\mathrm{fib}}r = 0$.

Proof. Since Δ_{fib} is a derivation of the fiberwise antibracket, i.e. $\mathrm{def}_{\Delta_{\mathrm{fib}}} = 0$ (this is (15)), the mechanism of (34) gives

$$[\Delta_{\mathrm{fib}}, (h,-)_{\mathrm{fib}}] = (\Delta_{\mathrm{fib}}h, -)_{\mathrm{fib}}. \tag{†}$$

Split $D = \nabla + (h_\delta,-)_{\mathrm{fib}} + (r,-)_{\mathrm{fib}}$ as in Theorem 3.28. For the covariant part we claim $[\Delta_{\mathrm{fib}}, \nabla] = 0$. Expanding $[\Delta_{\mathrm{fib}}, \nabla](ab)$ with the ∇-Leibniz rule, the generating identity of Lemma 3.30 and the symplecticity of ∇, the four non-derivation terms cancel pairwise:

$$\nabla a\,\Delta_{\mathrm{fib}}b:\ (-1)^{|a|+1} + (-1)^{|a|} = 0, \qquad \Delta_{\mathrm{fib}}a\,\nabla b:\ (-1)^{|a|} + (-1)^{|a|+1} = 0,$$
$$(\nabla a, b)_{\mathrm{fib}}:\ (-1)^{|a|+1} + (-1)^{|a|} = 0, \qquad (a, \nabla b)_{\mathrm{fib}}:\ 1 + (-1) = 0,$$

so $[\Delta_{\mathrm{fib}}, \nabla]$ is an even derivation. It annihilates $f(z)$ and $\mathrm{d}z^i$ (as Δ_{fib} kills $\deg_y \leqslant 1$) and y^i (as ∇y^i is linear in y), hence vanishes.

By (†) the part $(h_\delta,-)_{\mathrm{fib}}$ commutes with Δ_{fib} as well, its Hamiltonian being linear in y, and the remaining part gives $[\Delta_{\mathrm{fib}}, D] = (\Delta_{\mathrm{fib}}r, -)_{\mathrm{fib}}$. As the kernel of $s \mapsto (s,-)_{\mathrm{fib}}$ consists of the y-independent elements and $\deg_y(\Delta_{\mathrm{fib}}r) \geqslant 1$, this vanishes if and only if $\Delta_{\mathrm{fib}}r = 0$. □

Remark 3.32. The fiberwise BV Laplacian defined above is induced by the constant Berezinian $\mu_0 = D^n y$ on the fiber $T_z\mathcal{M}$.

Lemma 3.33 (BV-perturbed SDR)**.** Assuming $\Delta_{\mathrm{fib}}r = 0$, perturbing the retract of Proposition 3.18 along $-\mathring{\imath}\hbar\Delta_{\mathrm{fib}}$ yields a strong deformation retract

$$(\mathcal{O}_\mathcal{M}, -\mathring{\imath}\hbar\Delta) \underset{\pi_\Delta}{\overset{\iota_\Delta}{\rightleftarrows}} (\Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M}), D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}) \circlearrowleft h_\Delta$$

with

$$\iota_\Delta = \iota_D = \mathrm{EXP}^*, \qquad \pi_\Delta = \sum_{n\geqslant 0} \Pi_0(\mathring{\imath}\hbar\,\Delta_{\mathrm{fib}} h_D)^n, \qquad h_\Delta = \sum_{n\geqslant 0} h_D(\mathring{\imath}\hbar\,\Delta_{\mathrm{fib}} h_D)^n,$$

and transferred differential $-\mathring{\imath}\hbar\Delta$ on $\mathcal{O}_\mathcal{M}$, where the induced BV Laplacian is given by $\Delta = \Pi_0\,\Delta_{\mathrm{fib}}\,\mathrm{EXP}^*$.

Proof. As $D^2 = \Delta_{\mathrm{fib}}^2 = 0$ and $[D, \Delta_{\mathrm{fib}}] = 0$ we have $(D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}})^2 = 0$, so $-\mathring{\imath}\hbar\Delta_{\mathrm{fib}}$ is a perturbation of D (Def. 2.12). Each step of the perturbation series carries a power of $\hbar$, so the series of Lemma 2.13 converge $\hbar$-adically and produce the data above.

For ι_Δ observe that h_D decreases the dz-degree by one; since $\mathrm{EXP}^* f$ has dz-degree 0 and Δ_{fib} preserves it, $h_D\Delta_{\mathrm{fib}}\mathrm{EXP}^* = 0$, so

$$\iota_\Delta = \sum_{n\geqslant 0}(\mathring{\imath}\hbar\, h_D\Delta_{\mathrm{fib}})^n\iota_D = \iota_D = \mathrm{EXP}^*.$$

The transferred differential $-\mathring{\imath}\hbar\Delta$ is thus given by

$$-\mathring{\imath}\hbar\Delta = \pi_D(-\mathring{\imath}\hbar\Delta_{\mathrm{fib}})\iota_\Delta = -\mathring{\imath}\hbar\,\Pi_0\Delta_{\mathrm{fib}}\mathrm{EXP}^*$$

The other formulae are simply restating the HPL formulae for our setup. $\square$

The fiber bracket makes the formal complex into a DGLA, and the retract transports this structure to the global functions.

Proposition 3.34. The triple $\big(\Gamma(\Lambda T^*\mathcal{M} \otimes \widehat{S}T^*\mathcal{M}),\ D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}},\ (-,-)_{\mathrm{fib}}\big)$ is a DGLA. Transferring it along the SDR of Lemma 3.33 via Theorem 2.15 gives back a DGLA: the induced brackets on $(\mathcal{O}_\mathcal{M}, -\mathring{\imath}\hbar\Delta)$ are (up to signs) $r_1 = -\mathring{\imath}\hbar\Delta$, $r_2 = (-,-)$ (the antibracket of $(\mathcal{M},\omega)$) and $r_i = 0$ for $i \geqslant 3$, together with L_∞ quasi-isomorphisms

$$(\mathcal{O}_\mathcal{M}, -\mathring{\imath}\hbar\Delta, (-,-)) \underset{G}{\overset{F}{\rightleftarrows}} (\Gamma(\widehat{S}T^*\mathcal{M}), D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}, (-,-)_{\mathrm{fib}})$$

with linear parts $f_1 = \mathrm{EXP}^*$ and $g_1 = \pi_\Delta$, the first of which is strict.

Proof. The operator $D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}$ squares to zero (Lemma 3.33) and is a derivation of $(-,-)_{\mathrm{fib}}$: for D this is (31), established in Theorem 3.28, and for Δ_{fib} it follows from (15). With $q_1(a) = -(D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}})a$, $q_2(a \odot b) = (-1)^{|a|+1}(a,b)_{\mathrm{fib}}$ and $q_{\geqslant 3} = 0$ this is a DGLA.

We compute the r_i: by Proposition 3.29 we have $q_2(\mathrm{EXP}^* f, \mathrm{EXP}^* g) = (\mathrm{EXP}^* f, \mathrm{EXP}^* g)_{\mathrm{fib}} = \mathrm{EXP}^*(f,g)$, which lands in $\mathrm{Im}\,\mathrm{EXP}^* = \mathrm{Im}\,\iota_\Delta$; the homotopy annihilates this ($h_\Delta\iota_\Delta = 0$), so

$$f_2 = h_\Delta\, q_2(\mathrm{EXP}^* \odot \mathrm{EXP}^*) = h_\Delta\,\mathrm{EXP}^*(-,-) = 0.$$

Inductively, for $i \geqslant 3$ every monomial of F_i^2 is $f_{j_1} \odot f_{j_2}$ with $j_1 + j_2 = i$ and $j_1, j_2 \geqslant 1$, so some $j_\ell \geqslant 2$ and $f_{j_\ell} = 0$; hence $F_i^2 = 0$, giving $f_i = h_\Delta q_2 F_i^2 = 0$ and $r_i = \pi_\Delta q_2 F_i^2 = 0$. Thus $F = \mathrm{EXP}^*$ is strict and $r_i = 0$ for $i \geqslant 3$. The surviving bracket is $r_2 = \pi_\Delta\, q_2(\mathrm{EXP}^* \odot \mathrm{EXP}^*) = \pi_\Delta\,\mathrm{EXP}^*(-,-) = (-,-)$, using $\pi_\Delta\iota_\Delta = \mathrm{Id}$. $\square$

The induced BV Laplacian can be computed explicitly.

Corollary 3.35. The BV Laplacian $\Delta = \Pi_0\Delta_{\mathrm{fib}}\mathrm{EXP}^*$ induced on $\mathcal{O}_\mathcal{M}$ reads

$$\Delta f = \frac{1}{2}\sum_{i,j}(-1)^{p_i p_j}\Pi^{ij}(z)\Big(\frac{\partial^2 f}{\partial z^i \partial z^j} - \Gamma_{ij}^k\frac{\partial f}{\partial z^k}\Big).$$

In particular Δ depends on ∇ but not on the choice of r, it squares to zero and it generates the antibracket of $(\mathcal{M},\omega)$ in the sense of (5).

Proof. Since Δ_{fib} lowers the y-degree by two and Π_0 evaluates at $y = 0$, only the quadratic part of EXP^*f contributes. Solving $D(\text{EXP}^*f) = 0$ to second order in y gives $\text{EXP}^*f = f + y^i \frac{\partial f}{\partial z^i} + y^m y^k S_{mk} + \mathcal{O}(y^3)$ with

$$S_{mk} = \frac{1}{2}(-1)^{p_k p_m} \frac{\partial^2 f}{\partial z^m \partial z^k} - \frac{1}{2}\Gamma^i_{km} \frac{\partial f}{\partial z^i},$$

obtained by one step $\delta^{-1}\nabla$ of the recursion $\iota_D = \sum_n (\delta^{-1}\partial)^n \iota$ applied to the linear term $y^i \frac{\partial f}{\partial z^i}$; the correction term X does not enter, since it raises the y-degree of the linear term at least to three. Differentiating twice and relabelling with the graded symmetry of Π,

$$\Delta f = \Pi_0 \Delta_{\text{fib}}(y^m y^k S_{mk}) = \sum_{m,k} \Pi^{mk} S_{mk},$$

which is the claimed formula after applying $\Gamma^i_{km} = (-1)^{p_k p_m}\Gamma^i_{mk}$. $\Delta^2 = 0$ holds since $-\mathring{\imath}\hbar\Delta$ is the transferred differential of the retract. Finally, EXP^* is a morphism of algebras (Corollary 5.3 of [27]) and of antibrackets (Proposition 3.29) and Π_0 is multiplicative, so applying Π_0 to the generating identity of Lemma 3.30 for $\text{EXP}^*f \cdot \text{EXP}^*g$ yields

$$\Delta(fg) = \Delta f\, g + (-1)^{|f|} f \Delta g + (-1)^{|f|}(f, g). \qquad \square$$

3.8 Choice of the formal exponential map

The constructions of this chapter all depend on the choice of the formal exponential map ϕ. In this section we make this dependence precise, following [7] to some extent.

Since all constructions until now only involved the vertical jets of ϕ along the zero section, we identify from now on two formal exponential maps whose vertical derivatives all agree. Denote the set of formal exponential maps (Def. 3.1) as $\mathcal{E}(M)$ and we consider two formal exponential maps to be equal, if their jets agree, i.e. if they share the same collection of formal power series

$$\phi^i_x(y) = x^i + y^i + \frac{1}{2}\phi^i_{jk}(x) y^j y^k + \dots$$

Definition 3.36 (Fiberwise formal diffeomorphisms)**.** Denote by $\mathcal{G}$ the set of collections $g = (g_x)_{x \in M}$ of formal power series

$$g^i_x(y) = y^i + \sum_{k \geqslant 2} \frac{1}{k!} g^i_{j_1 \cdots j_k}(x) y^{j_1} \cdots y^{j_k}, \quad g^i_{j_1 \cdots j_k} \in \Gamma(S^k T^*M \otimes TM),$$

together with the multiplication $(gh)_x := g_x \circ h_x$ given by composition of formal power series.

Lemma 3.37. $\mathcal{G}$ is a group.

Proof. The composition of two such series is again of this form, since the 1-jet of a composition is the composition of the 1-jets. For the inverse, write $g = \text{Id} + g_{\geqslant 2}$ and solve $g_x(h_x(y)) = y$ order by order in y: in degree n the equation reads $h_n + P_n = 0$, where h_n is the degree-n part of h and P_n is a polynomial expression in $g_2, \dots, g_n$ and $h_2, \dots, h_{n-1}$ only. This determines h_n inductively and uniquely. $\square$

Remark 3.38. The group $\mathcal{G}$ is the exponential of the Lie algebra of formal vector fields of y-degree at least two: for $C \in \Gamma(\widehat{S}T^*M \otimes TM)$ with $\deg_y(C) \geqslant 2$, the series $\exp(C)_x(y) := \sum_n \frac{1}{n!}(C^n_x\, y)$ is well-defined degree by degree, since C raises the polynomial degree by at least one, and lies in $\mathcal{G}$; the same induction as in Lemma 3.37 shows that exp is a bijection: in y-degree k the part of $\exp(C)$ is $C_k + P_k(C_2, \dots, C_{k-1})$ with P_k depending only on the lower parts, so $\exp(C) = g$ solves recursively and uniquely for C, giving the inverse log.

Definition 3.39 (Torsor)**.** Let G be a group. A set S with a right G-action is called a G*-torsor* if S is nonempty and for any two $s, s' \in S$ there is a unique $g \in G$ with $s' = s \cdot g$, i.e. the action is free and transitive.

Lemma 3.40. The formula $(\phi \cdot g)_x := \phi_x \circ g_x$ defines a right action of $\mathcal{G}$ on $\mathcal{E}(M)$, and $\mathcal{E}(M)$ is a $\mathcal{G}$-torsor.

Proof. We have $\phi_x(g_x(0)) = \phi_x(0) = x$ and $\mathrm{d}(\phi_x \circ g_x)(0) = \mathrm{d}\phi_x(0)\,\mathrm{d}g_x(0) = \mathrm{Id}$, so $\phi \cdot g \in \mathcal{E}(M)$; associativity of the action is associativity of composition. The set $\mathcal{E}(M)$ is nonempty by the geodesic construction of Section 3.1. Since $\mathrm{d}\phi_x(0) = \mathrm{Id}$, the series ϕ_x admits a formal inverse ϕ_x^{-1} (same induction as in Lemma 3.37). Given $\phi^0, \phi^1 \in \mathcal{E}(M)$, set $g_x := (\phi_x^0)^{-1} \circ \phi_x^1$; then $g_x(0) = 0$ and $\mathrm{d}g_x(0) = \mathrm{Id}$, so $g \in \mathcal{G}$ and $\phi^1 = \phi^0 \cdot g$. Freeness follows by composing $\phi^0 \cdot g = \phi^0 \cdot g'$ with $(\phi_x^0)^{-1}$. □

Remark 3.41. The set $\mathcal{E}(M)$ is moreover path-connected: Let g be the unique element in $\mathcal{G}$ such that $\phi_1 = \phi_0 \cdot g$. Then we have a path ϕ_t in $\mathcal{E}(M)$ given by $\phi_0 \cdot \exp(t \log(g))$ from ϕ_0 to ϕ_1.

We now describe how the induced objects transform. For $g \in \mathcal{G}$ define the automorphism $\hat{g}$ of $\Gamma(\widehat{S}T^*M)$ by

$$(\hat{g}\sigma)_x := \sigma_x \circ g_x.$$

It is $C^\infty(M)$-linear, multiplicative, and satisfies $\pi \circ \hat{g} = \pi$, since $g_x(0) = 0$. By the same Jacobian pattern as in Section 3.4 it extends to all formal tensor bundles. To compare the Grothendieck connections we first record an explicit formula for D in terms of ϕ alone, valid for an arbitrary (not necessarily geodesic) formal exponential map.

Theorem 3.42 (Change of the formal exponential map)**.** Let $\phi \in \mathcal{E}(M)$ and $g \in \mathcal{G}$. Then

$$\mathrm{T}(\phi \cdot g)^* = \hat{g} \circ \mathrm{T}\phi^*, \qquad D_{\phi \cdot g} = \hat{g} \circ D_\phi \circ \hat{g}^{-1}, \qquad \pi \circ \hat{g} = \pi.$$

In particular, by Lemma 3.40, the data $(\mathrm{T}\phi^*, D_\phi)$ of any two formal exponential maps are conjugate by the automorphism $\hat{g}$ of the unique group element relating them.

Proof. The first identity holds since the Taylor expansion of a composition is the composition of the Taylor expansions:

$$\mathrm{T}(\phi \cdot g)^* f = \mathrm{T}[f \circ \phi_x \circ g_x] = \mathrm{T}[f \circ \phi_x] \circ g_x = \hat{g}(\mathrm{T}\phi^* f).$$

For the second, set $D' := \hat{g} \circ D_\phi \circ \hat{g}^{-1}$. We claim D' is again of the form $\mathrm{d}_x + Y'$: the difference $D' - \mathrm{d}_x$ is a derivation of $\Gamma(\widehat{S}T^*M)$ in each form degree: we can see that it is $C^\infty(M)$-linear by

$$D'(h\sigma) = \hat{g}\big(\mathrm{d}_x h \cdot \hat{g}^{-1}\sigma + h\, D_\phi \hat{g}^{-1}\sigma\big) = \mathrm{d}_x h \cdot \sigma + h\, D'\sigma,$$

so that $D' - \mathrm{d}_x$ obeys the $C^\infty(M)$-linear Leibniz rule and is therefore a formal vector field valued one-form by the identification of Section 3.1. Moreover D' is flat and annihilates $\hat{g}(\mathrm{Im}\,\mathrm{T}\phi^*) = \mathrm{Im}(\mathrm{T}(\phi \cdot g)^*)$. By Proposition 3.7 (3) applied to $\phi \cdot g$ we conclude $D' = D_{\phi \cdot g}$. The last identity was noted above. □

Corollary 3.43. The cohomology $H^\bullet_{D_\phi}$, together with the identification $H^0_{D_\phi} \cong C^\infty(M)$ of Section 3.5, is independent of the choice of ϕ: the automorphism $\hat{g}$ maps flat sections to flat sections and commutes with π. The same holds for the formal tensor bundles of Remark 3.13.

Proof. Let $\phi, \phi' \in \mathcal{E}(M)$ and let $g \in \mathcal{G}$ be the unique element with $\phi' = \phi \cdot g$. By Theorem 3.42 we have $D_{\phi'} \circ \hat{g} = \hat{g} \circ D_\phi$, so $\hat{g}$ is an isomorphism of complexes and induces an isomorphism $H^\bullet_{D_\phi} \cong H^\bullet_{D_{\phi'}}$ in every degree. □

Infinitesimally, the conjugation of Theorem 3.42 becomes a flow equation for the connection, which is the form in which the statement appears in [7].

Proposition 3.44 (Gauge flow)**.** Let $(\phi_t)_{t\in I}$ be a smooth path in $\mathcal{E}(M)$ and write $Y_t := Y_{\phi_t}$, J_t for the Jacobian of ϕ_t. Define the time dependent formal vector field

$$C_t := -\mathrm{T}\Big[(J_t^{-1})^k_j\, \partial_t \phi_t^j\Big] \frac{\partial}{\partial y^k} \in \Gamma(\widehat{S}T^*M \otimes TM).$$

Then $\deg_y(C_t) \geqslant 2$ and

$$\partial_t Y_t = \mathrm{d}_x C_t + [Y_t, C_t].$$

Proof. The degree bound holds since the 0- and 1-jets of ϕ_t are fixed by the definition of $\mathcal{E}(M)$, so that $\partial_t \phi_t = \mathcal{O}(y^2)$. For the flow equation we use the trick of [7] and assemble the path into a single formal exponential map on $M \times I$: define

$$\psi_{(x,t)}(y, \tau) := (\phi_{t,x}(y),\, t + \tau),$$

which satisfies $\psi_{(x,t)}(0,0) = (x,t)$ and $\mathrm{d}\psi(0) = \mathrm{Id}$, i.e. $\psi \in \mathcal{E}(M \times I)$. Its Jacobian in the fiber variables (y,τ) is block diagonal, $\tilde{J} = \mathrm{diag}(J_t, 1)$, and the base derivatives are $\partial_x \psi = (\partial_x \phi_t, 0)$ and $\partial_t \psi = (\partial_t \phi_t, 1)$. The formula of Proposition 3.7 therefore gives

$$Y_\psi = Y_t + \Big(C_t - \frac{\partial}{\partial \tau}\Big)\mathrm{d}t.$$

By Proposition 3.7 (2) applied on $M \times I$, Y_ψ satisfies the flatness equation with respect to $\tilde{\mathrm{d}} = \mathrm{d}_x + \mathrm{d}t\,\partial_t$. We decompose it by form type. The $\mathrm{d}x \wedge \mathrm{d}x$ part is the flatness of Y_t for every fixed t. For the mixed part, write $E := C_t - \partial_\tau$ and compute

$$\tilde{\mathrm{d}}\, Y_\psi + \frac{1}{2}[Y_\psi, Y_\psi] \;\supset\; \mathrm{d}t \wedge \partial_t Y_t - \mathrm{d}t \wedge \mathrm{d}_x E - \mathrm{d}t \wedge [Y_t, E],$$

where we used $\tilde{\mathrm{d}}(E\mathrm{d}t) = (\mathrm{d}_x E) \wedge \mathrm{d}t = -\mathrm{d}t \wedge \mathrm{d}_x E$ and $[Y_t, E\,\mathrm{d}t] = -\mathrm{d}t \wedge [Y_t, E]$, and the term quadratic in $\mathrm{d}t$ vanishes. Now $\mathrm{d}_x \partial_\tau = 0$ and $[Y_t, \partial_\tau] = 0$, since the coefficients of Y_t do not depend on τ. Setting the mixed component to zero therefore yields $\partial_t Y_t = \mathrm{d}_x C_t + [Y_t, C_t]$. □

Proposition 3.44 describes how the connection changes along a path; the companion statement for the Taylor pullbacks follows from the same chain rule that produced Y_ϕ in Proposition 3.7.

Lemma 3.45 (Flow of the pullbacks)**.** Let $(\phi_t)_{t\in I}$ be a smooth path in $\mathcal{E}(M)$ with generators C_t as in Proposition 3.44. Then for every $f \in C^\infty(M)$

$$\partial_t\, \mathrm{T}\phi_t^* f = -C_t(\mathrm{T}\phi_t^* f),$$

where C_t acts as a fiberwise derivation of $\Gamma(\widehat{S}T^*M)$.

Proof. Write $\sigma_t := \mathrm{T}\phi_t^* f = \mathrm{T}[f \circ \phi_{t,x}]$ and recall that T commutes with ∂_t, ∂_y and with products of series. The chain rule gives

$$\partial_t \sigma_t = \mathrm{T}[(\partial_j f)(\phi_t)\, \partial_t \phi_t^j], \qquad \frac{\partial \sigma_t}{\partial y^k} = \mathrm{T}[(\partial_j f)(\phi_t)\, J^j_k],$$

with $J^j_k = \partial \phi_t^j / \partial y^k$ the Jacobian of Remark 3.6. Inverting the second relation for $\mathrm{T}[(\partial_j f)(\phi_t)] = \mathrm{T}[(J^{-1})^k_j]\, \partial_{y^k} \sigma_t$ and inserting it into the first gives

$$\partial_t \sigma_t = \mathrm{T}[(J_t^{-1})^k_j\, \partial_t \phi_t^j]\, \partial_{y^k} \sigma_t = -C_t(\sigma_t),$$

by the definition of C_t in Proposition 3.44. □

Remark 3.46 (Symplectic reduction of $\mathcal{G}$)**.** When M is symplectic and D is built from a symplectic connection as in Section 3.6, the group $\mathcal{G}$ reduces accordingly. Call $g \in \mathcal{G}$ *symplectic* if every g_x is a formal symplectomorphism of the fiber $(T_xM, \omega_{\text{fib}}(x))$ fixing the origin, and write $\mathcal{G}_\omega \subset \mathcal{G}$ for the subgroup they form. Theorem 8 of [15] then likewise shows that under a different choice of a symplectic connection we obtain a change of the connection and formal exponential map under conjugation as in Theorem 3.42 by $\mathcal{G}_\omega$.

4 The Spinning Particle I: AKSZ construction

We will now construct a one-dimensional AKSZ theory. This entails a one-dimensional source manifold Σ, which we will take (later on) to be $\Sigma = S^1$. To construct the target and its symplectic structure, we mainly follow [12] and [19]. We will then lift this to an AKSZ theory.

4.1 Target manifold

4.1.1 Setting

For a one-dimensional AKSZ theory, we need a Hamiltonian dg manifold of degree 0 with an exact symplectic form. The symplectic form hence has internal degree 0.

Let (M, g) be a Lorentzian manifold, which we shall call the *spacetime*. We start with the cotangent bundle $T^*(\Pi TM)$ with fibre coordinates (P_i, λ_i) conjugate to the base coordinates (x^i, θ^i) respectively. The full set of coordinates is

$$(x^i,\, \theta^i,\, P_i,\, \lambda_i) \;\text{ with parities } (0,\, 1,\, 0,\, 1).$$

Definition 4.1 (Canonical form)**.** The canonical 1-form on $T^*(\Pi TM)$ is

$$\alpha_{\mathrm{can}} = P_i \mathrm{d}x^i + \lambda_i \mathrm{d}\theta^i \tag{37}$$

and the canonical symplectic form is

$$\omega_{\mathrm{can}} = \mathrm{d}\alpha_{\mathrm{can}} = \mathrm{d}P_i \wedge \mathrm{d}x^i + \mathrm{d}\lambda_i \wedge \mathrm{d}\theta^i. \tag{38}$$

This is a closed, non-degenerate 2-form of (internal) degree 0. The Poisson structure reads

$$(x^i, P_j) = \delta^i_j, \quad (\theta^i, \lambda_j) = \delta^i_j.$$

The coordinate P_j is however not a covector as the notation would suggest.

Lemma 4.2. Under a coordinate change $\bar{x}^{\bar{\imath}} = \phi^{\bar{\imath}}(x)$ on M we have

$$\lambda_j = \bar{\lambda}_{\bar{\imath}} \frac{\partial \bar{x}^{\bar{\imath}}}{\partial x^j}, \tag{39}$$

$$P_j = \frac{\partial \bar{x}^{\bar{\imath}}}{\partial x^j} \bar{P}_{\bar{\imath}} - \bar{\lambda}_{\bar{\imath}} \frac{\partial^2 \bar{x}^{\bar{\imath}}}{\partial x^j \, \partial x^k} \theta^k. \tag{40}$$

Proof. For the differentials we have the following transformations:

$$\mathrm{d}\bar{x}^{\bar{i}} = \mathrm{d}x^j \frac{\partial \bar{x}^{\bar{i}}}{\partial x^j}, \quad \mathrm{d}\bar{\theta}^{\bar{i}} = -\frac{\partial^2 \bar{x}^{\bar{i}}}{\partial x^k \partial x^j} \theta^k \mathrm{d}x^j + \mathrm{d}\theta^j \frac{\partial \bar{x}^{\bar{i}}}{\partial x^j}$$

where we used that $\mathrm{d}x^j$ and θ^k are both odd. Since we know that α_{can} is invariant under coordinate transformations, we can substitute the differentials above into the expression and group the terms to obtain

$$P_j \mathrm{d}x^j + \lambda_j \mathrm{d}\theta^j = \left(\bar{P}_{\bar{i}} \frac{\partial \bar{x}^{\bar{i}}}{\partial x^j} - \bar{\lambda}_{\bar{i}} \frac{\partial^2 \bar{x}^{\bar{i}}}{\partial x^k \partial x^j} \theta^k \right) \mathrm{d}x^j + \left(\bar{\lambda}_{\bar{i}} \frac{\partial \bar{x}^{\bar{i}}}{\partial x^j} \right) \mathrm{d}\theta^j. \tag{41}$$

Comparison of both sides proves the Lemma. □

The last term in (40) spoils covariance; this motivates the following redefinition.

Definition 4.3 (Covariant momentum)**.** Given a connection Γ on TM, define

$$p_i := P_i + \Gamma^\ell_{ik} \lambda_\ell \theta^k. \tag{42}$$

Proposition 4.4. The quantity p_i transforms as a covector.

Proof. Under $\bar{x}^{\bar{\imath}} = \phi^{\bar{\imath}}(x)$:

$$p_j = P_j + \Gamma^{\ell}_{jk}\,\lambda_\ell\,\theta^k = \frac{\partial \bar{x}^{\bar{\imath}}}{\partial x^j}\,\bar{P}_{\bar{\imath}} - \bar{\lambda}_{\bar{\imath}}\,\frac{\partial^2 \bar{x}^{\bar{\imath}}}{\partial x^j\,\partial x^k}\,\theta^k + \Gamma^{\ell}_{jk}\,\bar{\lambda}_{\bar{m}}\,\frac{\partial \bar{x}^{\bar{m}}}{\partial x^\ell}\,\theta^k.$$

The standard transformation law of Christoffel symbols,

$$\Gamma^{\ell}_{jk}\,\frac{\partial \bar{x}^{\bar{m}}}{\partial x^\ell} = \frac{\partial \bar{x}^{\bar{\imath}}}{\partial x^j}\,\bar{\Gamma}^{\bar{m}}_{\bar{\imath}\bar{n}}\,\frac{\partial \bar{x}^{\bar{n}}}{\partial x^k} + \frac{\partial^2 \bar{x}^{\bar{m}}}{\partial x^j\,\partial x^k},$$

shows that the two inhomogeneous terms cancel, yielding $p_j = \frac{\partial \bar{x}^{\bar{\imath}}}{\partial x^j}\,\bar{p}_{\bar{\imath}}$. □

We obtain three fields $(\theta^i, \lambda_i, p_i)$ each transforming covariantly (i.e. as the index in the notation suggests) under a change of coordinates of the base M. We have effectively, through the choice of a connection Γ, untwisted the space of fields $T^*(\Pi TM)$ to

$$T^*(\Pi TM) = \Pi TM \oplus_M \Pi T^*M \oplus_M T^*M. \tag{43}$$

Any connection on TM may be used for this splitting; since we work with the metric g, we take the Levi-Civita connection. In the resulting coordinates the canonical 1-form reads

$$\alpha_{\mathrm{can}} = p_i \mathrm{d}x^i - \Gamma^{\ell}_{ik}\lambda_\ell\theta^k \mathrm{d}x^i + \lambda_i \mathrm{d}\theta^i$$

and hence

$$\omega_{\mathrm{can}} = \mathrm{d}p_i \wedge \mathrm{d}x^i - \frac{\partial \Gamma^{\ell}_{ik}}{\partial x^j}\lambda_\ell\theta^k \mathrm{d}x^j \wedge \mathrm{d}x^i - \Gamma^{\ell}_{ik}\theta^k \mathrm{d}\lambda_\ell \wedge \mathrm{d}x^i + \Gamma^{\ell}_{ik}\lambda_\ell \mathrm{d}\theta^k \wedge \mathrm{d}x^i + \mathrm{d}\lambda_i \wedge \mathrm{d}\theta^i. \tag{44}$$

4.1.2 Constraint submanifold

We will now use the Lorentzian metric g as a vector bundle morphism to constrain the target manifold to a symplectic submanifold.

Definition 4.5 (Constraint surface)**.** The constraint surface is the sub-supermanifold N defined by the embedding $\iota_N : \Pi TM \oplus_M T^*M \hookrightarrow \Pi TM \oplus_M \Pi T^*M \oplus_M T^*M$ given in coordinates by

$$\iota_N(x^i, \theta^i, p_i) = \big(x^i,\ \theta^i,\ p_i,\ \tfrac{1}{2}\,g_{ij}(x)\,\theta^j\big).$$

Note that this is globally well defined because all quantities transform covariantly.

Proposition 4.6 (Second-class constraints)**.** Define $\phi_i := \lambda_i - \frac{1}{2}\,g_{ij}(x)\,\theta^j \in C^\infty(T^*(\Pi TM))$. Their Poisson bracket matrix with respect to ω_{can} is

$$C_{ij} := (\phi_i,\, \phi_j) = -g_{ij}.$$

Since g is non-degenerate, the restriction is symplectic.

Proof. With the Poisson structure (44) one can easily check that $(\theta^i, \lambda_j) = \delta^i_j, (\lambda_i, x^j) = 0$ and $(\theta^i, x^j) = 0$. Hence we have

$$C_{ij} = (\phi_i,\, \phi_j) = -\tfrac{1}{2}\,g_{j\nu}\,(\lambda_i,\, \theta^\nu) - \tfrac{1}{2}\,g_{i\mu}\,(\theta^\mu,\, \lambda_j) = -\tfrac{1}{2}\,g_{ji} - \tfrac{1}{2}\,g_{ij} = -g_{ij}.$$

Now $\omega_N := \iota_N^*\omega_{\mathrm{can}}$ is non-degenerate whenever the constraint matrix $C_{ij} = (\phi_i, \phi_j)$ is non-degenerate. Hence (N, ω_N) is a symplectic supermanifold. □

Plugging in the constraint in all formulae above yields

$$\alpha_N = p_i \mathrm{d}x^i - \frac{1}{2}\partial_k g_{rj}\theta^r\theta^k \mathrm{d}x^j + \frac{1}{2}g_{ij}\theta^i \mathrm{d}\theta^j,$$

with symplectic form

$$\omega_N = \mathrm{d}p_i \wedge \mathrm{d}x^i - \frac{1}{2}\theta^r\theta^k\partial_\ell\partial_k g_{rj}\mathrm{d}x^\ell \wedge \mathrm{d}x^j - g_{kj}\Gamma^k_{i\ell}\theta^\ell \mathrm{d}x^i \wedge \mathrm{d}\theta^j + \frac{1}{2}g_{ij}\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j.$$

Lemma 4.7 (Poisson brackets)**.** The Poisson brackets are given by

$$(x^i, p_j) = \delta^i_j, \quad (\theta^i, \theta^j) = -g^{ij}, \quad (\theta^i, p_j) = -\Gamma^i_{jk}\theta^k, \quad (p_i, p_j) = -\frac{1}{2}R_{\ell kij}\theta^\ell\theta^k.$$

Proof. Every bracket is $(z^A, z^B) = (-1)^{|z^A|}X_{z^A}(z^B)$ by Definition 1.25, so it suffices to read the Hamiltonian vector fields (Definition 1.24) off the defining equation $\iota_{X_{z^A}}\omega_N = \mathrm{d}z^A$.

For x^i this is solved by $X_{x^i} = \partial_{p_i}$, since $\iota_{\partial_{p_i}}$ annihilates every term of ω_N but $\mathrm{d}p_m \wedge \mathrm{d}x^m$, on which it returns $\mathrm{d}x^i$. Hence $(x^i, p_j) = \delta^i_j$ and $(x^i, \theta^j) = (x^i, x^j) = 0$.

For θ^i write $X_{\theta^i} = A^k\partial_{x^k} + B_k\partial_{p_k} + C^k\partial_{\theta^k}$. The $\mathrm{d}p$-part of $\iota_{X_{\theta^i}}\omega_N$ comes only from $\mathrm{d}p_m \wedge \mathrm{d}x^m$ and reads $-A^k\mathrm{d}p_k$, so $A^k = 0$; the $\mathrm{d}\theta$-part then comes only from $\frac{1}{2}g_{mn}\mathrm{d}\theta^m \wedge \mathrm{d}\theta^n$ and equals $C^k g_{kn}\mathrm{d}\theta^n$, whence $C^k = g^{ik}$. As θ^i is odd,

$$(\theta^i, \theta^j) = -X_{\theta^i}(\theta^j) = -g^{ij}.$$

The remaining component is fixed by the vanishing of the $\mathrm{d}x$-part, to which $\mathrm{d}p_m \wedge \mathrm{d}x^m$ contributes $B_k\mathrm{d}x^k$ and the cross term $-g_{sj}\Gamma^s_{k\ell}\theta^\ell\,\mathrm{d}x^k \wedge \mathrm{d}\theta^j$ contributes, through $C^m = g^{im}$, the term $-\Gamma^i_{k\ell}\theta^\ell\,\mathrm{d}x^k$. Their sum vanishes, so $B_k = \Gamma^i_{k\ell}\theta^\ell$ and

$$(\theta^i, p_j) = -X_{\theta^i}(p_j) = -\Gamma^i_{j\ell}\theta^\ell.$$

Next, for p_i set $X_{p_i} = A^k\partial_{x^k} + B_k\partial_{p_k} + C^k\partial_{\theta^k}$. Now the $\mathrm{d}p$-part forces $A^k = -\delta^k_i$, and the vanishing $\mathrm{d}\theta$-part, to which the cross term now also contributes, gives $C^k = \Gamma^k_{i\ell}\theta^\ell$. The bracket itself sits in the $\mathrm{d}x$-part, which receives B_j from $\mathrm{d}p_m \wedge \mathrm{d}x^m$, a contribution from $-\frac{1}{2}\theta^r\theta^s\partial_\ell\partial_s g_{rj}\,\mathrm{d}x^\ell \wedge \mathrm{d}x^j$ through $A^k = -\delta^k_i$, and one from the cross term through $C^k = \Gamma^k_{i\ell}\theta^\ell$. Writing ω_{ij} for the coefficient of $\mathrm{d}x^i \wedge \mathrm{d}x^j$ in ω_N, its vanishing yields

$$(p_i, p_j) = \omega_{ij} + g_{k\ell}\,\Gamma^\ell_{im}\,\Gamma^k_{jn}\,\theta^m\theta^n. \tag{45}$$

We anti-symmetrize the terms in $\theta^m\theta^n$. First, observe that

$$\omega_{ij} = \frac{1}{2}\left[\partial_i\partial_m g_{nj} - \partial_j\partial_m g_{ni}\right]\theta^m\theta^n,$$

which after anti-symmetrization gives

$$\omega_{ij} = \frac{1}{4}\left[\partial_i\partial_m g_{nj} - \partial_i\partial_n g_{mj} - \partial_j\partial_m g_{ni} + \partial_j\partial_n g_{mi}\right]\theta^m\theta^n.$$

The same for the other term:

$$g_{k\ell}\,\Gamma^\ell_{im}\,\Gamma^k_{jn}\,\theta^m\theta^n = \frac{1}{2}\,g_{k\ell}\left(\Gamma^\ell_{im}\Gamma^k_{jn} - \Gamma^\ell_{in}\Gamma^k_{jm}\right)\theta^m\theta^n.$$

Putting everything together yields

$$(p_i, p_j) = -\frac{1}{2}R_{\ell kij}\,\theta^\ell\theta^k. \qquad \square$$

4.1.3 Coupling to supergravity

The last step consists in coupling the fields to supergravity and writing down the Hamiltonian of the system. Because the supergravity fields live in bundles of $\mathbb{R}[1]$ and hence are not affected by any curvature, all the results from the previous subsections can be kept and amended trivially for the supergravity fields.

We replace $M \to M \times \mathbb{R}[1]$ in the construction. This gives us fields in the space $T^*(\Pi T\mathbb{R}[1]) \cong \mathbb{R}[1] \times \mathbb{R}[-1] \times \Pi\mathbb{R}[1] \times \Pi\mathbb{R}[-1]$ as in Table 1.

The canonical symplectic structure is

$$\alpha = b\mathrm{d}c + \beta\mathrm{d}\gamma, \quad \omega = \mathrm{d}b \wedge \mathrm{d}c + \mathrm{d}\beta \wedge \mathrm{d}\gamma, \quad (c, b) = -1,\ (\gamma, \beta) = 1.$$

Space	Field	Ghost number	Parity
$\mathbb{R}[1]$	c	1	1
$\mathbb{R}[-1]$	b	-1	1
$\Pi\mathbb{R}[1]$	γ	1	0
$\Pi\mathbb{R}[-1]$	β	-1	0

Table 1: Supergravity fields of the spinning particle, from $T^*(\Pi T\mathbb{R}[1])$.

Denote the new space of fields as

$$\mathcal{N} = N \times T^*(\Pi T\mathbb{R}[1]) = (\Pi TM \oplus_M T^*M) \times T^*\big(\Pi T\mathbb{R}[1]\big)$$

The full symplectic structure is then

$$\alpha_{\mathcal{N}} = p_i \mathrm{d}x^i - \frac{1}{2}\partial_k g_{rj}\theta^r\theta^k \mathrm{d}x^j + \frac{1}{2}g_{ij}\theta^i \mathrm{d}\theta^j + b\mathrm{d}c + \beta\mathrm{d}\gamma \tag{46}$$

and

$$\begin{aligned}\omega_{\mathcal{N}} = {}& \mathrm{d}p_i \wedge \mathrm{d}x^i - \frac{1}{2}\theta^r\theta^k\partial_\ell\partial_k g_{rj}\mathrm{d}x^\ell \wedge \mathrm{d}x^j - g_{kj}\Gamma^k_{i\ell}\theta^\ell \mathrm{d}x^i \wedge \mathrm{d}\theta^j + \frac{1}{2}g_{ij}\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j \\ & + \mathrm{d}b \wedge \mathrm{d}c + \mathrm{d}\beta \wedge \mathrm{d}\gamma.\end{aligned} \tag{47}$$

4.2 Hamiltonian

Since p_i transforms as a covector and θ^i as a vector, the quantity $Q = p_i\theta^i$ is well defined globally. We define the Hamiltonian

$$\Theta_{\mathcal{N}} = \frac{1}{2}(Q, Q)c + Q\gamma + \gamma^2 b. \tag{48}$$

Proposition 4.8. $(\Theta_{\mathcal{N}}, \Theta_{\mathcal{N}}) = 0$.

Proof. Given the symplectic pairing of the fields under the Poisson brackets (Lemma 4.7) , there are only four contributions which do not vanish a priori. The first is $\frac{1}{4}((Q,Q)c, (Q,Q)c) = 0$, because the product rule gives a $\sim c^2 = 0$ on the only possible non-vanishing term. The second is $(Q\gamma, Q\gamma) = (Q,Q)\gamma^2$. The third (double) is $\frac{1}{2}((Q,Q)c, Q\gamma) = 0$ which vanishes due to the Jacobi identity. Finally the fourth is the double contribution $\frac{1}{2}((Q,Q)c, \gamma^2 b) = -\frac{1}{2}(Q,Q)\gamma^2$. Summing all contributions up yields 0, as desired. □

Proposition 4.9 (Hamiltonian)**.** The Hamiltonian is given by

$$\Theta_{\mathcal{N}} = -\frac{1}{2}p_i p_j g^{ij} c + p_i\theta^i\gamma + \gamma^2 b. \tag{49}$$

Proof. We compute

$$(Q, Q) = (p_i\theta^i, p_j\theta^j) = p_i(\theta^i, p_j)\theta^j + p_i p_j(\theta^i, \theta^j) + (p_i, p_j)\theta^i\theta^j - p_j(p_i, \theta^j)\theta^i.$$

Using Lemma 4.7, we obtain

$$= p_i(-\Gamma^i_{j\ell}\theta^\ell)\theta^j - p_i p_j g^{ij} - \frac{1}{2}R_{\ell kij}\theta^\ell\theta^k\theta^i\theta^j - p_j\Gamma^j_{i\ell}\theta^\ell\theta^i.$$

The first and the fourth term vanish by contracting the torsion-free Levi-Civita connection with the odd fields. The third term vanishes due to $R_{[\ell kij]} = 0$ which follows from the Bianchi identities. The claim follows. □

4.3 With magnetic field

The spinning particle is sometimes endowed with a background magnetic field. Let P be the principal $U(1)$ bundle over the manifold M. A magnetic field is described by a connection one-form A, such that for an open $U \subset M$ after choosing a local gauge $s : U \to \mathsf{P}$, we can write $s^*A = A_i(x)\mathrm{d}x^i$ as a local gauge field. The theory then changes by amending the symplectic structure

$$\alpha_{\mathcal{N}} \ni p_i \mathrm{d}x^i \rightsquigarrow (p_i + A_i)\mathrm{d}x^i,$$

which results in the change

$$\omega_{\mathcal{N}} \ni \mathrm{d}p_i \wedge \mathrm{d}x^i \rightsquigarrow \mathrm{d}p_i \wedge \mathrm{d}x^i + \frac{1}{2}F_{ij}(x)\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

where F_{ij} is the usual electromagnetic field tensor given by $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x)$. Only the $\omega_{x^i x^j}$ part changes, and looking back at the calculation of Lemma 4.7, we can see that the only bracket where this component of the symplectic form entered our calculations was for the last one. We can resume from (45) and add $+\frac{1}{2}F_{ij}(x)$, to obtain

$$(p_i, p_j) = -\frac{1}{2}R_{\ell k i j}\theta^\ell \theta^k + F_{ij}, \tag{50}$$

while the other brackets are unchanged.

For the Hamiltonian, going back to the proof of Proposition 4.8, we observe that the only relevant change with the introduction of $A_i(x)$ is the bracket (Q, Q). However this was never of importance for the vanishing of any terms. Therefore we can keep (48). Correcting the (p_i, p_j) bracket in the calculation of Prop. 4.9, we obtain

$$\Theta^A_{\mathcal{N}} := -\frac{1}{2}p_i p_j g^{ij} c + \frac{1}{2}F_{ij}\theta^i\theta^j c + p_i\theta^i\gamma + \gamma^2 b\ . \tag{51}$$

which solves $(\Theta^A_{\mathcal{N}}, \Theta^A_{\mathcal{N}}) = 0$. Note that these quantities are only defined locally due to the choice of the local gauge. In the following sections we will however not use any magnetic field, $A = 0$.

4.4 Two-dimensional case

In this section we consider the special case where the spacetime (M, g) is two-dimensional, orientable and time-orientable. In this case the theory simplifies, as the target untwists back into a cotangent bundle (which it was before we imposed the constraint).

In the following we will show that there exists a supermanifold $P = M \times \mathbb{R}^{0|1}$ and a symplectomorphism

$$(N, \omega_N) \cong (T^*P, \omega_{\mathrm{P}}).$$

where ω_P is the canonical symplectic form on T^*P. We can ignore all the supergravity fields for now, as they are split from the rest anyway.

Lemma 4.10 (Null splitting)**.** Let (M, g) be an oriented and time-oriented two-dimensional Lorentzian manifold. Then TM admits a global splitting

$$TM \ = \ L_+ \oplus L_-,$$

where L_+ and L_- are trivial real line bundles whose fibres are null with respect to g. In particular, TM is trivializable.

Proof. Time-orientability provides a global timelike vector field on M. Normalizing it, we obtain $T \in \Gamma(TM)$ with $g(T, T) = -1$. Let ϵ denote the Lorentzian volume form determined by the orientation and the metric. Define the 1-form

$$\sigma := \iota_T \epsilon,$$

and let $S := \sigma^\sharp$ be its metric dual, $g(S,V) = \sigma(V) = \epsilon(T,V)$ for all V.

We verify the properties of S. First, S is everywhere nonvanishing given that ϵ is non-degenerate and the condition $g(T,T) = -1$.

Second, $S \perp T$:

$$g(S,T) = \sigma(T) = \epsilon(T,T) = 0,$$

by antisymmetry of ϵ.

Third, $g(S,S) = +1$, as

$$g(S,S) = g^{ij}\sigma_i\sigma_j = T^aT^b\epsilon_{ai}\epsilon_{bj}g^{ij} = -T^aT^bg_{ab} = -g(T,T) = 1.$$

Thus (T,S) is a global orthonormal frame: $g(T,T) = -1$, $g(S,S) = +1$, $g(T,S) = 0$.

Now define

$$e_+ \;:=\; \frac{1}{\sqrt{2}}(T+S), \qquad e_- \;:=\; \frac{1}{\sqrt{2}}(T-S). \tag{52}$$

such that the null properties hold

$$\begin{aligned} g(e_+,e_+) &= 0,\\ g(e_-,e_-) &= 0,\\ g(e_+,e_-) &= -1. \end{aligned}$$

Since $e_\pm$ are global, nowhere-vanishing sections of the null line bundles $L_\pm := \mathrm{span}(e_\pm)$, both L_+ and L_- are trivial, and $TM = L_+ \oplus L_-$. □

Let (e^+, e^-) denote the dual coframe, characterized by $\langle e_\tau, e^\sigma\rangle = \delta^\sigma_\tau$ for $\sigma,\tau \in \{+,-\}$. The completeness relation is

$$\delta^i_j = e^i_+e^+_j + e^i_-e^-_j. \tag{53}$$

The metric and its inverse decompose as

$$g_{ij} = -\big(e^+_ie^-_j + e^-_ie^+_j\big), \qquad g^{ij} = -\big(e^i_+e^j_- + e^i_-e^j_+\big). \tag{54}$$

Definition 4.11. Define the globally well-defined odd functions on ΠTM

$$\psi := e^+_i(x)\,\theta^i, \qquad \chi := -e^-_i(x)\,\theta^i. \tag{55}$$

These are scalars on M: under a coordinate change $x^i \mapsto \bar{x}^i(x)$, the covector e^+_i and the vector θ^i transform inversely, so ψ is invariant (and the same for χ).

Lemma 4.12. The odd fibre coordinates decompose as

$$\theta^i \;=\; e^i_+\,\psi \;-\; e^i_-\,\chi. \tag{56}$$

Proof. Completeness relation (53): $\theta^i = \delta^i_j\,\theta^j = e^i_+(e^+_j\theta^j) + e^i_-(e^-_j\theta^j) = e^i_+\psi - e^i_-\chi$. □

This automatically gives us the correspondence $\Pi TM \cong M \times \mathbb{R}^{0|2}$, where the isomorphism is defined as

$$\Phi^*(\psi) = e^+_i(x)\theta^i, \quad \Phi^*(\chi) = -e^-_i(x)\theta^i,$$

and trivially on the base M. We now consider the cotangent bundle of both sides and consider their canonical one-forms to obtain a cotangent lift of Φ. Denote by $\bar\psi, \bar\chi$ the conjugate coordinates to ψ and χ respectively, and $\tilde{p}_i$ the conjugate momentum to x^i. Comparing (37) with $\alpha = \tilde{p}_i\mathrm{d}x^i + \bar\psi\mathrm{d}\psi + \bar\chi\mathrm{d}\chi$, we can read off

$$\Phi^*(\bar\psi) = e^i_+\lambda_i, \quad \Phi^*(\bar\chi) = -e^i_-\lambda_i, \quad \Phi^*(\tilde{p}_i) = P_i - \lambda_j\partial_i\theta^j,$$

obtained by expanding

$$\lambda_i\mathrm{d}\theta^i = -\lambda_i(\partial_ke^i_+\psi\mathrm{d}x^k - \partial_ke^i_-\chi\mathrm{d}x^k - e^i_+\mathrm{d}\psi + e^i_-\ \mathrm{d}\chi).$$

This gives a symplectomorphism $(T^*\Pi TM, \omega_{\text{can}}) \cong (T^*(M \times \mathbb{R}^{0|2}), \omega)$ with the symplectic form

$$\omega = \mathrm{d}\tilde{p}_i \wedge dx^i + \mathrm{d}\bar{\psi} \wedge \mathrm{d}\psi + \mathrm{d}\bar{\chi} \wedge \mathrm{d}\chi.$$

The symplectic constraint now reads

$$\bar{\psi} = \frac{1}{2} e^i_+(x) g_{i\ell}[e^\ell_+(x)\psi - e^\ell_-(x)\chi], \quad \bar{\chi} = -\frac{1}{2} e^i_-(x) g_{i\ell}[e^\ell_+(x)\psi - e^\ell_-(x)\chi].$$

Using the duality relations of the null frame, we immediately obtain that $\bar{\psi} = \chi/2, \bar{\chi} = \psi/2$. Plugging this into the symplectic form we obtain

$$\omega = \mathrm{d}\tilde{p}_i \wedge \mathrm{d}x^i + \mathrm{d}\psi \wedge \mathrm{d}\chi.$$

This is precisely (T^*P, ω) for $P = M \times \mathbb{R}^{0|1}$. Amending P with the supergravity fields $\mathcal{P} := P \times \Pi T\mathbb{R}[1]$ we obtain

$$\omega_{\mathcal{P}} = \mathrm{d}\tilde{p}_i \wedge \mathrm{d}x^i + \mathrm{d}\psi \wedge \mathrm{d}\chi + \mathrm{d}b \wedge \mathrm{d}c + \mathrm{d}\beta \wedge \mathrm{d}\gamma.$$

Next we give the Hamiltonian in these coordinates. We have that the covariant momentum p_i from the general spinning particle is related to $\tilde{p}_i$ by

$$p_i = \tilde{p}_i + \Gamma^\ell_{ik} \lambda_\ell \theta^k + \lambda_j \partial_i \theta^j.$$

On the constraint surface $\lambda_i = \frac{1}{2} g_{ij}\theta^j$; substituting this together with $\theta^i = e^i_+\psi - e^i_-\chi$, the frame derivatives combine with the Christoffel symbols into covariant derivatives and collapse to

$$p_i = \tilde{p}_i + g_{k\ell}\, e^k_-\, \nabla_i e^\ell_+\, \psi\chi = \tilde{p}_i - e^+_\ell\, \nabla_i e^\ell_+\, \psi\chi,$$

using $g_{k\ell}\, e^k_- = -e^+_\ell$. We substitute this into $\Theta_{\mathcal{N}} = -\frac{1}{2} p_i p_j g^{ij} c + p_i \theta^i \gamma + \gamma^2 b$. The correction drops out of $p_i\theta^i$, since $\psi\chi\,\theta^i = 0$ by $\psi^2 = \chi^2 = 0$, so that $p_i\theta^i = \tilde{p}_i(e^i_+\psi - e^i_-\chi)$. In the quadratic term $(\psi\chi)^2 = 0$ keeps only the cross term,

$$-\frac{1}{2} p_i p_j g^{ij} c = -\frac{1}{2}\tilde{p}_i\tilde{p}_j g^{ij} c + g^{ij}\tilde{p}_i\, e^+_\ell\, \nabla_j e^\ell_+\, \psi\chi\, c.$$

Hence

$$\Theta_{\mathcal{P}} = -\frac{1}{2}\tilde{p}_i\tilde{p}_j g^{ij} c + g^{ij}\tilde{p}_i\, e^+_\ell\, \nabla_j e^\ell_+\, \psi\chi\, c + \tilde{p}_i e^i_+ \psi\gamma - \tilde{p}_i e^i_- \chi\gamma + \gamma^2 b.$$

4.5 Summary of the target data

We collect the results of the preceding subsections. The full target is a Hamiltonian dg manifold $(\mathcal{N}, \omega_{\mathcal{N}}, \Theta_{\mathcal{N}})$ of degree 0.

Supermanifold. The target supermanifold is the direct product

$$\mathcal{N} = (\Pi TM \oplus_M T^*M) \times T^*(\Pi T\mathbb{R}[1])$$

with local coordinates, ghost numbers, and parities as in Table 2.

Symplectic structure. The primitive 1-form and symplectic 2-form of internal degree 0 decompose as sums over the two factors:

$$\alpha_{\mathcal{N}} = p_i \mathrm{d}x^i - \tfrac{1}{2}\,\partial_k g_{rj}\,\theta^r\theta^k \mathrm{d}x^j + \tfrac{1}{2} g_{ij}\,\theta^i \mathrm{d}\theta^j + b\mathrm{d}c + \beta\mathrm{d}\gamma$$

$$\begin{aligned}\omega_{\mathcal{N}} = {}& \mathrm{d}p_i \wedge \mathrm{d}x^i - \frac{1}{2}\theta^r\theta^k \partial_\ell\partial_k g_{rj} \mathrm{d}x^\ell \wedge \mathrm{d}x^j - g_{kj}\Gamma^k_{i\ell}\theta^\ell \mathrm{d}x^i \wedge \mathrm{d}\theta^j + \frac{1}{2} g_{ij}\mathrm{d}\theta^i \wedge \mathrm{d}\theta^j \\ & + \mathrm{d}b \wedge \mathrm{d}c + \mathrm{d}\beta \wedge \mathrm{d}\gamma.\end{aligned}$$

Space	Field	Ghost Number	Parity
M	x^i	0	0
T^*M	p_i	0	0
ΠTM	θ^i	0	1
$\mathbb{R}[1]$	c	1	1
$\mathbb{R}[-1]$	b	-1	1
$\Pi\mathbb{R}[1]$	γ	1	0
$\Pi\mathbb{R}[-1]$	β	-1	0

Table 2: Fields of the spinning particle for an n-dimensional spacetime.

Poisson brackets. The nonvanishing Poisson brackets are

$$(x^i, p_j) = \delta^i_j, \quad (\theta^i, \theta^j) = -g^{ij}, \quad (\theta^i, p_j) = -\Gamma^i_{jk}\theta^k, \quad (p_i, p_j) = -\frac{1}{2}R_{\ell kij}\theta^\ell\theta^k,$$
$$(c, b) = -1, \quad (\gamma, \beta) = 1.$$

Hamiltonian and classical master equation. The Hamiltonian $\Theta_\mathcal{N} \in C^\infty(\mathcal{N})_1$ is

$$\Theta_\mathcal{N} = -\frac{1}{2}p_i\, p_j\, g^{ij} c + p_i\theta^i\gamma + \gamma^2\, b\,, \tag{57}$$

and satisfies

$$(\Theta_\mathcal{N},\, \Theta_\mathcal{N}) = 0.$$

The associated cohomological vector field $Q_\mathcal{N} := (\Theta_\mathcal{N}, -)$ makes $(\mathcal{N}, Q_\mathcal{N})$ into a dg manifold. Together, the data $(\mathcal{N},\, \alpha_\mathcal{N},\, \omega_\mathcal{N},\, \Theta_\mathcal{N})$ constitute the target of the AKSZ construction.

Two-dimensional version In this case (M, g) is a Lorentzian, 2-dimensional, orientable and time-orientable manifold, and we have that $\mathcal{N} = T^*(M \times \mathbb{R}^{0|1} \times \Pi T\mathbb{R}[1]) = T^*\mathcal{P}$ with fields

Space	Field	Ghost Number	Parity
M	x^i	0	0
T^*M	$\tilde{p}_i$	0	0
$\mathbb{R}^{0\|1}$	ψ	0	1
$T^*(\mathbb{R}^{0\|1})$	χ	0	1
$\mathbb{R}[1]$	c	1	1
$\mathbb{R}[-1]$	b	-1	1
$\Pi\mathbb{R}[1]$	γ	1	0
$\Pi\mathbb{R}[-1]$	β	-1	0

Table 3: Fields of the spinning particle for a two-dimensional spacetime.

symplectic structure

$$\omega_\mathcal{P} = \mathrm{d}\tilde{p}_i \wedge \mathrm{d}x^i + \mathrm{d}\psi \wedge \mathrm{d}\chi + \mathrm{d}b \wedge \mathrm{d}c + \mathrm{d}\beta \wedge \mathrm{d}\gamma$$

and Hamiltonian

$$\Theta_{\mathcal{P}} = -\frac{1}{2}\tilde{p}_i\tilde{p}_j g^{ij}(x)c + g^{ij}\tilde{p}_i\, e^+_\ell\, \nabla_j e^\ell_+\, \psi\chi\, c + \tilde{p}_i e^i_+(x)\psi\gamma - \tilde{p}_i e^i_-(x)\chi\gamma + \gamma^2 b.$$

4.6 AKSZ construction

We now follow the AKSZ construction procedure of Section 1.5 to obtain a BV theory from the target given in the last section.

4.6.1 General case

We start with the target data for the general case and lift this now to the mapping space $\mathcal{F}(T[1]\Sigma, \mathcal{N})$ (where we will take $\Sigma = S^1$). First we lift the coordinates/fields of the target, which give us the new field space generated by the superfields listed below.

superfield	field	gh	par	field [dt]	gh	par
$\boldsymbol{x^i}$	x^i	0	0	p^{+i}	-1	1
$\boldsymbol{p_i}$	p_i	0	0	x^+_i	-1	1
$\boldsymbol{\theta^i}$	θ^i	0	1	θ^{+i}	-1	0
$\boldsymbol{c}$	c	1	1	e	0	0
$\boldsymbol{b}$	e^+	-1	1	c^+	-2	0
$\boldsymbol{\gamma}$	γ	1	0	ξ	0	1
$\boldsymbol{\beta}$	ξ^+	-1	0	γ^+	-2	1

Table 4: Superfields of the AKSZ theory for an n-dimensional spacetime.

The AKSZ action (11) reads

$$S_{\text{AKSZ}} = \int_\Sigma \boldsymbol{p}_i d\boldsymbol{x}^i - \frac{1}{2}\partial_k g_{rj}(\boldsymbol{x})\boldsymbol{\theta}^r\boldsymbol{\theta}^k \wedge \mathrm{d}\boldsymbol{x}^j + \frac{1}{2}g_{ij}(\boldsymbol{x})\boldsymbol{\theta}^i \wedge \mathrm{d}\boldsymbol{\theta}^j + \boldsymbol{b}\mathrm{d}\boldsymbol{c} + \boldsymbol{\beta}\mathrm{d}\boldsymbol{\gamma}$$
$$- \frac{1}{2}\boldsymbol{p}_i\boldsymbol{p}_j g^{ij}(\boldsymbol{x})\boldsymbol{c} + \boldsymbol{p}_i\boldsymbol{\theta}^i\boldsymbol{\gamma} + \boldsymbol{\gamma}^2\boldsymbol{b}$$

and the symplectic form (9) on $\mathcal{F}$:

$$\omega_{\text{AKSZ}} = -\int_\Sigma \delta\boldsymbol{p}_i \wedge \delta\boldsymbol{x}^i - \frac{1}{2}\boldsymbol{\theta}^r\boldsymbol{\theta}^k\partial_\ell\partial_k g_{rj}(\boldsymbol{x})\delta\boldsymbol{x}^\ell \wedge \delta\boldsymbol{x}^j - g_{kj}(\boldsymbol{x})\Gamma^k_{i\ell}(\boldsymbol{x})\boldsymbol{\theta}^\ell\delta\boldsymbol{x}^i \wedge \delta\boldsymbol{\theta}^j$$
$$+ \frac{1}{2}g_{ij}(\boldsymbol{x})\delta\boldsymbol{\theta}^i \wedge \delta\boldsymbol{\theta}^j + \delta\boldsymbol{b} \wedge \delta\boldsymbol{c} + \delta\boldsymbol{\beta} \wedge \delta\boldsymbol{\gamma}.$$

Plugging in the superfield expansion in local coordinates and only picking the dt part we obtain the full AKSZ action

$$\begin{aligned} S_{\mathrm{AKSZ}} = \int_\Sigma p_i \mathrm{d}x^i &+ \frac{1}{2} g_{ij}(x)\theta^i \mathrm{d}\theta^j \\ &+ e^+ \mathrm{d}c + \xi^+ \mathrm{d}\gamma \\ &- \frac{1}{2}\partial_k g_{rj}(x)\theta^r\theta^k \mathrm{d}x^j \\ &+ \left(x_i^+ p_j g^{ij}(x) + \frac{1}{2} p_i p_j \partial_k g^{ij}(x) p^{+k} \right) c \\ &- \frac{1}{2} p_i p_j g^{ij}(x) e - x_i^+ \theta^i \gamma + p_i \theta^{+i}\gamma + p_i \theta^i \xi \\ &- 2\gamma\xi e^+ + \gamma^2 c^+. \end{aligned}$$

Theorem 1.52 now guarantees that S_{AKSZ} solves the CME, $(S_{\mathrm{AKSZ}}, S_{\mathrm{AKSZ}}) = 0$.

4.6.2 Two-dimensional case

The field space is the one of the general case, with the superfield $\boldsymbol{\theta^i}$ replaced by the two superfields $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$:

superfield	field	gh	par	field [dt]	gh	par
$\boldsymbol{x^i}$	x^i	0	0	p^{+i}	-1	1
$\boldsymbol{p_i}$	p_i	0	0	x_i^+	-1	1
$\boldsymbol{\psi}$	ψ	0	1	χ^+	-1	0
$\boldsymbol{\chi}$	χ	0	1	ψ^+	-1	0
$\boldsymbol{c}$	c	1	1	e	0	0
$\boldsymbol{b}$	e^+	-1	1	c^+	-2	0
$\boldsymbol{\gamma}$	γ	1	0	ξ	0	1
$\boldsymbol{\beta}$	ξ^+	-1	0	γ^+	-2	1

Table 5: Superfields of the AKSZ theory for a two-dimensional spacetime.

The AKSZ action reads

$$\begin{aligned} S_{\mathrm{AKSZ}} &= \int_\Sigma \boldsymbol{p}_i \mathrm{d}\boldsymbol{x}^i + \boldsymbol{\chi}\mathrm{d}\boldsymbol{\psi} + \boldsymbol{b}\mathrm{d}\boldsymbol{c} + \boldsymbol{\beta}\mathrm{d}\boldsymbol{\gamma} - \frac{1}{2}\boldsymbol{p}_i\boldsymbol{p}_j g^{ij}(\boldsymbol{x})\boldsymbol{c} + g^{ij}\boldsymbol{p}_i\, e_\ell^+\, \nabla_j e_+^\ell\, \boldsymbol{\psi}\boldsymbol{\chi}\,\boldsymbol{c} \\ &\qquad + \boldsymbol{p}_i e_+^i(\boldsymbol{x})\boldsymbol{\psi}\boldsymbol{\gamma} - \boldsymbol{p}_i e_-^i(\boldsymbol{x})\boldsymbol{\chi}\boldsymbol{\gamma} + \boldsymbol{\gamma}^2\boldsymbol{b}. \end{aligned} \tag{58}$$

with symplectic form on $\mathcal{F}$

$$\omega_{\mathrm{AKSZ}} = -\int_\Sigma \delta\boldsymbol{p}_i \wedge \delta\boldsymbol{x}^i + \delta\boldsymbol{\psi}\wedge\delta\boldsymbol{\chi} + \delta\boldsymbol{b}\wedge\delta\boldsymbol{c} + \delta\boldsymbol{\beta}\wedge\delta\boldsymbol{\gamma}.$$

5 The Spinning Particle II: 2-dim globalization

In this chapter we compute the effective action from the action constructed via AKSZ of the two-dimensional version of the spinning particle and globalize it with the general procedure given in [7] and [11]. This procedure requires the target of the AKSZ theory to be *split*: it must be a cotangent bundle with the canonical symplectic form. This is precisely what we have at hand for the two dimensional case.

Admittedly, some constructions (in particular concerning the operators in formal space) are a bit handwavy in the following development; we will however make them precise in Section 6.

5.1 In formal geometry

Recall the AKSZ action of the spinning particle in two dimensions

$$S_{\text{AKSZ}} = \int_\Sigma \boldsymbol{p}_i \mathrm{d}\boldsymbol{x}^i + \boldsymbol{\chi}\mathrm{d}\boldsymbol{\psi} + \boldsymbol{b}\mathrm{d}\boldsymbol{c} + \boldsymbol{\beta}\mathrm{d}\boldsymbol{\gamma} - \frac{1}{2}\boldsymbol{p}_i\boldsymbol{p}_j g^{ij}(\boldsymbol{x})\boldsymbol{c} + g^{ij}\boldsymbol{p}_i\, e_\ell^+\, \nabla_j e_+^\ell\, \boldsymbol{\psi}\boldsymbol{\chi}\,\boldsymbol{c} \\ + \boldsymbol{p}_i e_+^i(\boldsymbol{x})\boldsymbol{\psi}\boldsymbol{\gamma} - \boldsymbol{p}_i e_-^i(\boldsymbol{x})\boldsymbol{\chi}\boldsymbol{\gamma} + \boldsymbol{\gamma}^2\boldsymbol{b}. \tag{59}$$

To globalize the effective action we will first lift the theory into formal geometry. First, we choose a connection on $\mathcal{P}$. Given that $\mathcal{P} = M\times\mathbb{R}^{0|1}\times\Pi T\mathbb{R}[1]$ where M comes equipped with a Lorentzian metric, it is natural to take the Levi Civita connection on M and take the trivial connection on the rest. This gives us an exponential map $\phi_x = \exp_x : T_x\mathcal{P} \to \mathcal{P}$. Consider the exponential map on a fiber with base point $x \in M$. For the perturbative expansion we need the coordinate formula for the exponential map. Use barred field notation for the respective tangent fields in $T_x\mathcal{P}$. Then the pullbacks are given by

$$\mathrm{T}\phi_x^*(x^i) = x^i + \bar{x}^i - \frac{1}{2}\Gamma^i_{jk}(x)\,\bar{x}^j\bar{x}^k - \frac{1}{6}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]\bar{x}^j\bar{x}^k\bar{x}^l + O(|\bar{x}|^4),$$
$$\mathrm{T}\phi_x^*(\psi) = \bar{\psi}, \quad \mathrm{T}\phi_x^*(c) = \bar{c}, \quad \mathrm{T}\phi_x^*(\gamma) = \bar{\gamma}.$$

Now our target is $T^*\mathcal{P}$. We have seen in Section 3.6 how we can use ϕ_x to pull back the subalgebra $C^\infty_{\text{poly}}(T^*\mathcal{P})$ with the symplectic structure into formal space (which is admissible for our case, c.f. (59)). Construct the cotangent lift

$$\Phi_x := (\phi_x, (\mathrm{d}\phi_x^*)^{-1}) : T^*T_x\mathcal{P} \to T^*\mathcal{P}$$

The pullback formulae for Φ_x of the base coordinates remain the same. For the pullback formulae of the fiber coordinates let J_x be the Jacobian of ϕ_x. It has the structure $J_x = \mathrm{diag}(\partial\mathrm{T}\phi_x^*(x^i)/\partial\bar{x}^j, 1, 1, 1)$, where

$$\frac{\partial\mathrm{T}\phi_x^*(x^i)}{\partial\bar{x}^j} = \delta^i_j - \Gamma^i_{jk}(x)\bar{x}^k - \frac{1}{12}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]_{(jkl)}\bar{x}^k\bar{x}^l + \mathcal{O}(|\bar{x}|^3),$$

where (jkl) denotes the sum over all six permutations of the indices j, k, l inside the brackets. To compute the cotangent lift in coordinates, we need the inverse of the Jacobian, which we can compute by using $(\mathrm{Id} - X)^{-1} = I + X + X^2 + ...$, yielding

$$J_x^{-1} = \mathrm{diag}\left(\delta^i_j + \Gamma^i_{jk}(x)\bar{x}^k + Q^i_{jkl}(x)\bar{x}^k\bar{x}^l + \mathcal{O}\left(|\bar{x}|^3\right), 1, 1, 1\right),$$

where we abbreviated the (in k, l symmetric) coefficient

$$Q^i_{jkl} := \frac{1}{6}\left[\partial_k\Gamma^i_{jl} + \partial_l\Gamma^i_{jk} + \partial_j\Gamma^i_{kl} - 2\Gamma^i_{jm}\Gamma^m_{kl} + \Gamma^i_{km}\Gamma^m_{jl} + \Gamma^i_{lm}\Gamma^m_{jk}\right].$$

The cotangent lift maps the fiber coordinates by the inverse Jacobian,

$$\mathrm{T}\Phi_x^*(p_i) = (J_x^{-1})^j{}_i\,\bar{p}_j = \bar{p}_i + \Gamma^j_{ik}(x)\bar{x}^k\bar{p}_j + Q^j_{ikl}(x)\bar{x}^k\bar{x}^l\bar{p}_j + \mathcal{O}(|\bar{x}|^3),$$

$$\mathrm{T}\Phi_x^*(\chi) = \bar{\chi}, \quad \mathrm{T}\Phi_x^*(b) = \bar{b}, \quad \mathrm{T}\Phi_x^*(\beta) = \bar{\beta}.$$

Next we want to extend the exponential map into the mapping space. Fix a constant field $\varphi \in \mathrm{Map}(T[1]\Sigma, \mathcal{P})$ given by $\varphi = x \in M$. Consider the field space $\mathrm{Map}(T[1]\Sigma, T^*T_x\mathcal{P})$ and denote the respective superfields in bold, barred notation, i.e. $\bar{\boldsymbol{x}}^i = \bar{x}^i(t) + \bar{p}_+^i(t)\mathrm{d}t$ etc. We can then define the new exponential map

$$\Phi_x : \mathrm{Map}(T[1]\Sigma, T^*T_x\mathcal{P}) \to \mathrm{Map}(T[1]\Sigma, T^*\mathcal{P})$$

by taking the pullback via Φ_x and plugging in the superfields for the fields, i.e. replacing $f \mapsto \mathbf{f}$ for all fields. We obtain for the x superfields

$$\begin{aligned}\mathrm{T}\Phi_x^*(\boldsymbol{x}^i) = {}& x^i + \bar{x}^i + \bar{p}_+^i \mathrm{d}t - \frac{1}{2}\Gamma^i_{jk}(x)[\bar{x}^j\bar{x}^k + 2\bar{x}^j\bar{p}_+^k\mathrm{d}t] + \\ & - \frac{1}{6}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]\bar{x}^j\bar{x}^k\bar{x}^l - \frac{1}{12}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]_{(jkl)}\bar{p}_+^j\bar{x}^k\bar{x}^l\mathrm{d}t + O(|\bar{x}|^4)\end{aligned}$$

In field components this then reads

$$\begin{aligned}\mathrm{T}\Phi_x^*(x^i) &= x^i + \bar{x}^i - \frac{1}{2}\Gamma^i_{jk}(x)\bar{x}^j\bar{x}^k - \frac{1}{6}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]\bar{x}^j\bar{x}^k\bar{x}^l + O(|\bar{x}|^3),\\ \mathrm{T}\Phi_x^*(p_+^i) &= \bar{p}_+^i - \Gamma^i_{jk}(x)\bar{x}^j\bar{p}_+^k - \frac{1}{12}\left[\partial_l\Gamma^i_{jk} - 2\Gamma^i_{mk}\Gamma^m_{jl}\right]_{(jkl)}\bar{p}_+^j\bar{x}^k\bar{x}^l + O(|\bar{x}|^3).\end{aligned}$$

Similarly we obtain for the momenta superfield

$$\begin{aligned}\mathrm{T}\Phi_x^*(p_i) &= \bar{p}_i + \Gamma^j_{ik}(x)\bar{x}^k\bar{p}_j + Q^j_{ikl}(x)\bar{x}^k\bar{x}^l\bar{p}_j + \mathcal{O}(|\bar{x}|^3),\\ \mathrm{T}\Phi_x^*(x_i^+) &= \bar{x}_i^+ + \Gamma^j_{ik}(x)\left[\bar{x}^k\bar{x}_j^+ + \bar{p}_+^k\bar{p}_j\right] + Q^j_{ikl}(x)\left[\bar{x}^k\bar{x}^l\bar{x}_j^+ + 2\bar{p}_+^k\bar{x}^l\bar{p}_j\right] + \mathcal{O}(|\bar{x}|^3),\end{aligned}$$

and for all the other fields trivially

$$\begin{array}{llll}\mathrm{T}\Phi_x^*(\psi) = \bar{\psi}, & \mathrm{T}\Phi_x^*(c) = \bar{c}, & \mathrm{T}\Phi_x^*(\gamma) = \bar{\gamma}, & \mathrm{T}\Phi_x^*(\chi) = \bar{\chi},\\ \mathrm{T}\Phi_x^*(e^+) = \bar{e}^+, & \mathrm{T}\Phi_x^*(\xi^+) = \bar{\xi}^+, & \mathrm{T}\Phi_x^*(\chi^+) = \bar{\chi}^+, & \mathrm{T}\Phi_x^*(e) = \bar{e},\\ \mathrm{T}\Phi_x^*(\xi) = \bar{\xi}, & \mathrm{T}\Phi_x^*(\psi^+) = \bar{\psi}^+, & \mathrm{T}\Phi_x^*(c^+) = \bar{c}^+, & \mathrm{T}\Phi_x^*(\gamma^+) = \bar{\gamma}^+.\end{array}$$

To perform perturbation theory, we split the AKSZ action into a kinetic part S^0 and a perturbative part S^P. For S^0 we take the part inherited from the canonical one form. Given that we used the cotangent lift, we have quadratic in fields

$$S_x^0 := \mathrm{T}\Phi_{\mathrm{x}}^*\, S_0 = \int_\Sigma \bar{\boldsymbol{p}}_i\mathrm{d}\bar{\boldsymbol{x}}^i + \bar{\boldsymbol{\chi}}\mathrm{d}\bar{\boldsymbol{\psi}} + \bar{\boldsymbol{b}}\mathrm{d}\bar{\boldsymbol{c}} + \bar{\boldsymbol{\beta}}\mathrm{d}\bar{\boldsymbol{\gamma}}.$$

To obtain the Taylor expansion of the perturbative part $S_x^P := \mathrm{T}\Phi_{\mathrm{x}}^*\, S^P$ one then simply pulls back the fields via Φ_x^*, writes out the result in superfield expansion and then Taylor expands the tensors g^{ij}, e_+^i, e_-^i around $x \in M$. The result is a power series in the barred field coordinates. Note that in particular due to the form of the formal exponential map that S_x^P is always at least cubic in the fibered field coordinates and hence does not contribute to the kinetic part S_x^0.

Now consider the vector field $Y \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM)$ given from the Grothendieck connection (c.f. (19)):

$$\mathrm{d}x^iY_i = \mathrm{d}x^iY_i^k(x,\bar{x})\frac{\partial}{\partial\bar{x}^k}$$

where we denote $\bar{x}$ for the fiber coordinates (in Section 3.1 they were called y^i). Taking the fiberwise bracket $(-,-)_{\mathrm{fib}}$ on the tangent bundle, we can easily construct the Hamiltonian one-form of Y simply by

$$\mathrm{d}x^iY_i^k(x,\bar{x})\frac{\partial}{\partial\bar{x}^k} \rightsquigarrow \mathrm{d}x^iY_i^k(x,\bar{x})\bar{p}_k := H_Y \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM),$$

where $\bar{p}_k$ is the fiber coordinate conjugate to $\bar{x}^k$. In other words, for any $X \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes TM)$, this construction satisfies $H_{\mathcal{L}_Y X} = (H_X, H_Y)_{\text{fib}}$, where we denote H to be the Hamiltonian of X, Y respectively. This is precisely the construction of Section 3.6, but the other way around.

This construction can be lifted to the mapping space as usual by replacing the fields above with the superfields. Lifting H_Y we get

$$S_Y := \int_\Sigma \mathrm{d}x^i Y_i^k(x, \bar{\boldsymbol{x}}) \bar{\boldsymbol{p}}_k$$

which by construction satisfies the lift rule

$$S_{\mathcal{L}_Y X} = (-1)^n (S_X, S_Y)_{\text{fib}},$$

the sign being that of the AKSZ symplectic form $\omega = (-1)^n \mathbb{T}\omega_{\mathcal{N}}$ ($n = \dim \Sigma = 1$ here).

We are now ready to define the modified effective action. First observe that our space of fields is now

$$\mathsf{F}_x = \mathrm{Map}(T[1]\Sigma, T^*T_x\mathcal{P}) \cong \Omega^\bullet(\Sigma) \otimes (T_x\mathcal{P} \oplus T_x^*\mathcal{P})$$

After taking the Taylor expansion of the fields at some $x \in M$, we obtain

$$S_x := S_x^0 + S_x^P \in \Gamma(\widehat{S}\mathsf{F}^*)$$

which satisfies $\mathrm{d}_x S_x + (S_Y, S_x)_{\text{fib}} = 0$ by construction.

Consider now $\widetilde{S}_x := S_x + S_Y \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{F}^*)$, where the S part is of form degree 0 and S_Y of form degree 1. Note that formally $\Delta_{\text{fib}} S_0 = 0$, since S_0 is bilinear in the fields with a single worldline derivative, so that $\Delta_{\text{fib}} S_0 \sim \partial_t \delta(t - t')|_{t'=t} = 0$ as principal value. Assume for now formally that $\Delta_{\text{fib}} S_x = \Delta_{\text{fib}} S_Y = 0$ (we will expand on this in the next section). From the equation above and the flatness condition on Y we get that it formally satisfies the modified QME

$$\mathrm{d}_x \widetilde{S}_x + \frac{1}{2}(\widetilde{S}_x, \widetilde{S}_x)_{\text{fib}} - i\hbar \Delta_{\text{fib}} \widetilde{S}_x = 0.$$

We will later construct a Hodge decomposition of $\Omega^\bullet(\Sigma)$ and choose a Lagrangian $\mathcal{L}$ in the complement of the harmonic forms. This leaves, after integrating out this part, the zero modes in the space $\mathsf{H}_x := H^\bullet(\Sigma) \otimes T_x\mathcal{P} \oplus H^\bullet(\Sigma) \oplus T_x^*\mathcal{P}$. We denote the induced symplectic structure as $(-,-)_{\text{fib}}^{\mathsf{H}}$ and $\Delta_{\text{fib}}^{\mathsf{H}}$ (we will construct these more precisely in Section 6.2). Defining

$$\widetilde{Z}_x := \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x},$$

we can compute

$$\mathrm{d}_x \widetilde{Z}_x = \int_{\mathcal{L}} \mathrm{d}_x e^{\frac{i}{\hbar}\widetilde{S}_x} = \frac{i}{\hbar}\int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \mathrm{d}_x \widetilde{S}_x = -\frac{i}{2\hbar}\int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x}(\widetilde{S}_x, \widetilde{S}_x)_{\text{fib}} = -\frac{i}{\hbar}\Delta_{\text{fib}}^{\mathsf{H}} \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x},$$

i.e. we have

$$\mathrm{d}_x \widetilde{Z} - i\hbar \Delta_{\text{fib}}^{\mathsf{H}} \widetilde{Z} = 0.$$

Defining the effective action as $e^{\frac{i}{\hbar}S_{\text{eff}}} := \widetilde{Z}_x$ (omit the x for notational simplicity), understood perturbatively by taking the connected Feynman diagrams, we formally obtain the modified QME

$$\mathrm{d}_x S_{\text{eff}} + \frac{1}{2}(S_{\text{eff}}, S_{\text{eff}})_{\text{fib}}^{\mathsf{H}} - i\hbar \Delta_{\text{fib}}^{\mathsf{H}} S_{\text{eff}} = 0.$$

Note that $S_{\text{eff}} \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*[[\hbar]])$. The QME can then be split up in form degrees on ΛT^*M.

Remark 5.1. From the coordinate formulae of $\mathrm{T}\Phi_x^*$ above we see that S_x is polynomial in all fields except the $\bar{x}^i$, in which it is an honest power series, the Taylor tails of the tensors. The same holds for the tree level part of S_{eff}, while at loop order the cycle graphs produce infinite series also in the other zero modes (Section 5.2), so that altogether

$$S_{\text{eff}} \in \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M \otimes \widehat{S}\widetilde{\mathsf{H}}^*)[[\hbar]],$$

with $\widetilde{\mathsf{H}}_x \subset \mathsf{H}_x$ the complement of the $H^0(S^1) \otimes T_xM$ part and the $\bar{x}$-series singled out as the $\widehat{S}T^*M$ factor. This reflects the formal geometry, compare also [7]: the modes $\bar{x}^i$ are counted not among the residual fields but as the fiber coordinates of the formal bundle. Since we globalize over $x \in M$ only, the Grothendieck projection collapses exactly this power series to an honest global function on M, while the residual fields in $\widetilde{\mathsf{H}}$ remain fibered; as a bundle over M, $\widetilde{\mathsf{H}}$ is identified with its global counterpart built from the target bundles via a connection, such as the one contained in the choice of ϕ.

Modified quantum canonical transformations (without time dependence) are given by $T \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*[[\hbar]])$ of total degree -1. The infinitesimal transformation reads

$$\delta S_{\text{eff}} = \mathrm{d}_x T + (S_{\text{eff}}, T)^{\mathsf{H}}_{\text{fib}} - i\hbar\Delta^{\mathsf{H}}_{\text{fib}} T \tag{60}$$

which can be integrated $\int_0^1 \mathrm{d}t$ to obtain finite transformations. As before, we can let T carry time dependence and substitute $\delta \rightsquigarrow \mathrm{d}/\mathrm{d}t$ in that case.

We now want to describe how the theory changes under a choice of formal exponential map. By Remark 3.41 any two formal exponential maps are connected by a path ϕ_t, with velocity encoded by the generators $C_t \in \Gamma(\widehat{S}T^*M \otimes TM)$, $\deg_{\bar{x}}(C_t) \geqslant 2$, of Proposition 3.44. The two infinitesimal flow equations we need are

$$\partial_t Y_t = \mathrm{d}_x C_t + [Y_t, C_t], \qquad \partial_t\, \mathrm{T}\phi_t^* f = -C_t(\mathrm{T}\phi_t^* f),$$

the first from Proposition 3.44, the second from Lemma 3.45.

Next we lift the generator to the space of fields, in complete analogy to the lift of Y: we set

$$C^k(x,\bar{x})\frac{\partial}{\partial \bar{x}^k} \rightsquigarrow H_C := C^k(x,\bar{x})\,\bar{p}_k, \qquad S_C := \int_\Sigma C^k(x,\bar{\boldsymbol{x}})\,\bar{\boldsymbol{p}}_k \in \Gamma(\widehat{S}\mathsf{F}^*),$$

and the lifted flow acts on functionals by $(-, S_C)_{\text{fib}}$, with $S_{\mathcal{L}_C X} = (S_X, S_C)_{\text{fib}}$ as before. In contrast to S_Y, the generator C carries no form degree on M, so S_C has total degree -1; this is precisely the degree of a generator of a modified quantum canonical transformation.

Lemma 5.2. Let (ϕ_t) be a path of formal exponential maps and $S_{x,t} = \mathrm{T}\phi_t^* S_x$. Define $\widetilde{S}_t := S_{x,t} + S_{Y_t}$ the associated formal global actions. Then

$$\partial_t \widetilde{S}_t = \mathrm{d}_x S_{C_t} + (\widetilde{S}_t, S_{C_t})_{\text{fib}}.$$

If moreover $\Delta_{\text{fib}} S_{C_t} = 0$[2] , this is an infinitesimal modified quantum canonical transformation with generator $T = S_{C_t}$.

Proof. We compute the two parts separately. For the Grothendieck part, S_Y is linear in Y, so by the flow equation for Y_t

$$\partial_t S_{Y_t} = S_{\partial_t Y_t} = S_{\mathrm{d}_x C_t} + S_{[Y_t, C_t]} = \mathrm{d}_x S_{C_t} + (S_{Y_t}, S_{C_t})_{\text{fib}},$$

where d_x passes through the Hamiltonian lift since the lift acts fiberwise. For the part $S_{x,t}$ we use $\partial_t \mathrm{T}\phi_t^* S = -\mathcal{L}_{C_t}(\mathrm{T}\phi_t^* S)$ which again lifts to $\partial_t S_{x,t} = (S_{C_t}, S_{x,t})_{\text{fib}}$ as before. □

[2] By the same reason as before, c.f. Section 5.3

Lemma 5.3. Let $T \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{F}^*[[\hbar]])$ be of total degree -1. Then

$$(\mathrm{d}_x - i\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}) \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x}\, \frac{i}{\hbar} T = \frac{i}{\hbar} \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \left[\mathrm{d}_x T + (\widetilde{S}_x, T)_{\mathrm{fib}} - i\hbar\Delta_{\mathrm{fib}} T\right].$$

Proof. This is the analogous result of Corollary 1.46 (3) in the modified complex. Exponentiating the modified QME gives

$$\Delta_{\mathrm{fib}} e^{\frac{i}{\hbar}\widetilde{S}_x} = e^{\frac{i}{\hbar}\widetilde{S}_x} \left(\frac{i}{\hbar}\Delta_{\mathrm{fib}}\widetilde{S}_x - \frac{1}{2\hbar^2}(\widetilde{S}_x, \widetilde{S}_x)_{\mathrm{fib}}\right) = \frac{1}{\hbar^2}\, e^{\frac{i}{\hbar}\widetilde{S}_x}\, \mathrm{d}_x \widetilde{S}_x.$$

Using $\Delta_{\mathrm{fib}}(e^{\frac{i}{\hbar}\widetilde{S}_x} T) = (\Delta_{\mathrm{fib}} e^{\frac{i}{\hbar}\widetilde{S}_x}) T + e^{\frac{i}{\hbar}\widetilde{S}_x} \Delta_{\mathrm{fib}} T + \frac{i}{\hbar} e^{\frac{i}{\hbar}\widetilde{S}_x} (\widetilde{S}_x, T)_{\mathrm{fib}}$ and that the fiber BV integral intertwines Δ_{fib} with $\Delta^{\mathsf{H}}_{\mathrm{fib}}$ (Thm. 1.44 (1)), we compute

$$\mathrm{d}_x \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \frac{i}{\hbar} T = \frac{i}{\hbar} \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \left(\frac{i}{\hbar}\, \mathrm{d}_x \widetilde{S}_x\, T + \mathrm{d}_x T\right),$$

$$-i\hbar\, \Delta^{\mathsf{H}}_{\mathrm{fib}} \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \frac{i}{\hbar} T = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_x} \left(\frac{1}{\hbar^2}\, \mathrm{d}_x \widetilde{S}_x\, T + \Delta_{\mathrm{fib}} T + \frac{i}{\hbar}(\widetilde{S}_x, T)_{\mathrm{fib}}\right).$$

The terms containing $\mathrm{d}_x \widetilde{S}_x\, T$ cancel and the claim follows. □

With the Lemmas we immediately obtain the following result.

Corollary 5.4. We have

$$\partial_t \widetilde{Z} = (\mathrm{d}_x - i\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}) \int_{\mathcal{L}} e^{\frac{i}{\hbar}\widetilde{S}_t}\, \frac{i}{\hbar} S_{C_t},$$

and consequently the class of $\widetilde{Z}$ in the $(\mathrm{d}_x - i\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}})$-cohomology does not depend on the choice of the formal exponential map.

5.2 Feynman graphs

We can now describe S_{eff} in terms of Feynman diagrams. First, we will construct a Lagrangian $\mathcal{L}$ inside the space of fluctuations of F_x, where

$$\mathsf{F}_x = \mathrm{Map}(T[1]\Sigma, T^*T_x\mathcal{P}) \cong \Omega^\bullet(\Sigma) \otimes (T_x\mathcal{P} \oplus T^*_x\mathcal{P}),$$

to fix the gauge. For this, we use a Hodge decomposition of $\Omega^\bullet(\Sigma)$. This entails the inclusion

$$\iota\colon H^\bullet(\Sigma) \to \Omega^\bullet(\Sigma)$$

of the harmonic representatives, the projection

$$p\colon \Omega^\bullet(\Sigma) \to H^\bullet(\Sigma)$$

onto the de Rham cohomology of Σ, and

$$K\colon \Omega^\bullet(\Sigma) \to \Omega^{\bullet-1}(\Sigma)$$

a linear operator, such that with $P := \iota p$ the triple (ι, p, K) satisfies

(I) $\mathrm{d}K + K\mathrm{d} = \mathrm{Id} - P$, (II) $PK = KP = 0$, (III) $K^T = -K$, (IV) $K^2 = 0$.

The transpose is w.r.t. the Poincaré pairing on forms $\int_\Sigma \bullet \wedge \bullet$. The operator K is an integral operator with a distributional integral kernel $\omega \in \Omega^1(\Sigma \times \Sigma)$, which is our propagator.

The spinning particle is a one dimensional field theory. We will take $\Sigma = S^1$. We use the Hodge decomposition

$$\Omega^\bullet(S^1) = \Omega^\bullet_{\mathrm{Harm}}(S^1) \oplus \bar{\Omega}^0(S^1) \oplus \bar{\Omega}^1(S^1),$$

where the harmonic forms are spanned by $\{1, \mathrm{d}t\}$, and $\iota_{S^1} : H^\bullet(S^1) \to \Omega^\bullet(S^1)$ is the tautological inclusion of these representatives. The associated chain homotopy operator is

$$K_{S^1} : g(t)\mathrm{d}t \mapsto \int_{S^1} \omega_{S^1}(t, t')g(t')\mathrm{d}t'$$

with the integral kernel

$$\omega_{S^1}(t, t') = \theta(t - t') - t + t' - \frac{1}{2}$$

and the projection

$$p_{S^1} : f(t) + g(t)\mathrm{d}t \mapsto \int_{S^1} (\mathrm{d}t' - \mathrm{d}t) \wedge (f(t') + g(t')\mathrm{d}t').$$

The gauge fixing below relies on a factorization of the fiberwise symplectic form ω_{fib} (Section 3.7). The split target $T^*\mathcal{P}$ is a cotangent bundle, so in the barred fiber superfields $(\bar{\boldsymbol{q}}^a, \bar{\boldsymbol{p}}_a)$ its coefficients are the canonical δ^a_b,

$$\omega_{\mathrm{fib}} = -\int_\Sigma \delta\bar{\boldsymbol{p}}_a \wedge \delta\bar{\boldsymbol{q}}^a,$$

writing as $\alpha \otimes u \in \Omega^\bullet(\Sigma) \otimes (T_x\mathcal{P} \oplus T^*_x\mathcal{P}) = \mathsf{F}_x$, one reads off

$$\omega_{\mathrm{fib}}(\alpha \otimes u, \beta \otimes v) = -(-1)^{|u||\beta|}\langle u, v\rangle \int_\Sigma \alpha \wedge \beta, \qquad \alpha, \beta \in \Omega^\bullet(\Sigma), \quad u, v \in T_x\mathcal{P} \oplus T^*_x\mathcal{P}, \tag{61}$$

with $\langle -, -\rangle$ the canonical pairing of the cotangent fibre $T_x\mathcal{P} \oplus T^*_x\mathcal{P}$. The leftover $\int_\Sigma \alpha \wedge \beta$ is the source Poincaré pairing, so ω_{fib} is the tensor product of the two.
Now define the Lagrangian $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_K \otimes (T_x\mathcal{P} \oplus T^*_x\mathcal{P}), \qquad \text{with } \mathcal{L}_K = \operatorname{Ker} P \cap \operatorname{Ker} K \tag{62}$$

This is precisely a Lagrangian inside the complement of the harmonic forms.

Proposition 5.5. $\mathcal{L}_K$ is Lagrangian.

Proof. Considering the splitting of (61) and the fact that we are tensoring with the entire target factor $T_x\mathcal{P} \oplus T^*_x\mathcal{P}$ that is symplectic, it suffices to show that $\mathcal{L}_K = \operatorname{Ker} P \cap \operatorname{Ker} K$ is Lagrangian inside the fluctuation space $\operatorname{Ker} P$, on which the pairing is nondegenerate.

We claim that $\alpha \in \mathcal{L}_K \iff \alpha = K\mathrm{d}\alpha$. If $\alpha \in \mathcal{L}_K$, then $K\alpha = 0$ and $P\alpha = 0$, so (I) gives $\alpha = \mathrm{d}K\alpha + K\mathrm{d}\alpha + P\alpha = K\mathrm{d}\alpha$. Conversely, if $\alpha = K\mathrm{d}\alpha$ then $K\alpha = K^2\mathrm{d}\alpha = 0$ by (IV) and $P\alpha = PK\mathrm{d}\alpha = 0$ by (II), so $\alpha \in \operatorname{Ker} P \cap \operatorname{Ker} K$.

For isotropy, we write $\alpha, \beta \in \mathcal{L}_K$ as $\alpha = K\mathrm{d}\alpha$, $\beta = K\mathrm{d}\beta$ and move one K across the pairing via (III):

$$\int_\Sigma \alpha \wedge \beta = \int_\Sigma K\mathrm{d}\alpha \wedge K\mathrm{d}\beta \overset{\text{(III)}}{=} -\int_\Sigma \mathrm{d}\alpha \wedge K^2\mathrm{d}\beta \overset{\text{(IV)}}{=} 0.$$

For half-dimensionality, note that $\mathcal{L}_K = \operatorname{Ker}(K|_{\operatorname{Ker} P})$ and we claim that $\operatorname{Ker}(K|_{\operatorname{Ker} P}) = \operatorname{Im}(K|_{\operatorname{Ker} P})$ from which the result follows. First note that applying d once on the left and once on the right on (I) yields $P\mathrm{d} = \mathrm{d} - \mathrm{d}K\mathrm{d} = \mathrm{d}P$. Now if $\alpha \in \operatorname{Ker}(K|_{\operatorname{Ker} P})$, then by the above $\alpha = K\mathrm{d}\alpha = K(\mathrm{d}\alpha)$. Because $P(\mathrm{d}\alpha) = \mathrm{d}(P\alpha) = 0$, we obtain $\alpha \in \operatorname{Im}(K|_{\operatorname{Ker} P})$. Now if $\alpha \in \operatorname{Im}(K|_{\operatorname{Ker} P})$, then $K(\alpha) = K(K\beta) = 0$ by (IV). $\square$

The perturbative expansion of the effective action is the usual perturbative expansion via Feynman diagrams of the BV pushforward of Definition 1.45. In our field space F_x with Hodge decomposition, in which we can split every field $\bar{\phi} \in \mathsf{F}_x$ to write $\bar{\phi} = \bar{\phi}^{\mathsf{H}} + \bar{\phi}^{\mathrm{fl}}$, this amounts to computing the perturbative expansion of

$$S_{\mathrm{eff}}(\bar{\phi}^{\mathsf{H}}; \hbar) = \int_{\mathcal{L}} e^{iS(\bar{\phi}^{\mathsf{H}} + \bar{\phi}^{\mathrm{fl}})/\hbar}\mathrm{d}\phi^{\mathrm{fl}}$$

which is a sum over connected oriented graphs Γ giving the total form of the effective action

$$S_{\text{eff}}(\bar{\phi}^{\mathsf{H}};\hbar) = \sum_{\Gamma} \frac{(i\hbar)^{l(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \; W_\Gamma^{\text{target}} \cdot W_\Gamma^{\text{source}} \tag{63}$$

where $l(\Gamma)$ stands for the number of loops, $|\mathrm{Aut}(\Gamma)|$ is the number of graph automorphisms, W_Γ^{target} is called the target part and W_Γ^{source} the source part.

Remark 5.6. The factorization into source- and target contributions for Feynman diagrams in the expansion (63) is due to the factorization of the space of fields (5.1) and to the fact that our ansatz for the gauge fixing (62) is compatible with this factorization.

Remark 5.7. We orient every edge from its momentum half-edge $(\bar{\boldsymbol{p}}_i, \bar{\boldsymbol{\chi}}, \bar{\boldsymbol{b}}, \bar{\boldsymbol{\beta}})$ to its position half-edge $(\bar{\boldsymbol{x}}^i, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{\gamma}})$.

The target part W_Γ^{target} is a polynomial in the fields H_x of degree equal to the number of leaves and computed with the following rules.

(1) to an incoming leaf of Γ decorated by $\{1, \mathrm{d}t\}$ one associates the corresponding field $\bar{\boldsymbol{x}}^{i\mathsf{H}}, \bar{\boldsymbol{\psi}}^{\mathsf{H}}, \bar{\boldsymbol{c}}^{\mathsf{H}}, \bar{\boldsymbol{\gamma}}^{\mathsf{H}}$,

(2) to an outgoing leaf decorated by $\{1, \mathrm{d}t\}$ one associates the corresponding field $\bar{\boldsymbol{p}}_i^{\mathsf{H}}, \bar{\boldsymbol{\chi}}^{\mathsf{H}}$, $\bar{\boldsymbol{b}}^{\mathsf{H}}, \bar{\boldsymbol{\beta}}^{H}$,

(3) to a vertex with m inputs and n outputs one associates the expression

$$\partial_{i_1} \cdots \partial_{i_m} T^{j_1 \cdots j_n}$$

which is the m-th derivative of n-vector contribution to the action S. These are the coefficients of the Taylor expansions of g^{ij}, e_+^i, e_-^i and $\mathrm{d}x^\ell Y_\ell^i$ at $x \in M$ respectively,

(4) for every edge contract the dummy Latin indices for the two constituent half-edges.

The result of the contraction is a polynomial function on H_x.

The source part W_Γ^{source} is given by

$$W_\Gamma^{\text{source}} = \int_{\Sigma^{\times V(\Gamma)}} \left(\prod_{\text{edges } (h_{\text{in}}, h_{\text{out}})} \pi^*_{v(h_{\text{in}}), v(h_{\text{out}})} \omega \right) \cdot \left(\prod_{\text{leaves } l} \pi^*_{v(l)} \chi_{\alpha_l} \right) \tag{64}$$

Here $V(\Gamma)$ denotes the set of vertices, $\pi_v : \Sigma^{\times V(\Gamma)} \to \Sigma$ picks out the copy of Σ labelled by v, and $\pi_{u,v} : \Sigma^{\times V(\Gamma)} \to \Sigma \times \Sigma$ retains the copies labelled by u and v; the symbol $v(h)$ stands for the vertex at which the half-edge h is attached. Notice that all these integrals converge.

There are some statements about Feynman diagrams we can make solely with the source part.

Proposition 5.8. Every vertex has exactly one leaf decorated by the de Rham degree 1 harmonic form $\mathrm{d}t$, all other leaves, if any, are decorated by constants.

Proof. This follows solely by degree counting on (64). Because the propagator is of de Rham degree 0, for every vertex exactly one leaf has to carry a $\mathrm{d}t$ contribution. □

Proposition 5.9. Every vertex is either a corolla or contains a loop. It follows that all tree graphs vanish.

Proof. Given that every vertex has exactly one leaf decorated by $\mathrm{d}t$ and the rest are constants, the product of the leaf part of (64) at every vertex is $\sim \mathrm{d}t \sim P$. If there is a single edge connecting the vertex, we have an integral proportional to KP which vanishes by (II). If the graph is not a corolla, then it follows that $E \geqslant V$ and hence $F - 2 = E - V \geqslant 0$, i.e. the graphs must contain at least one loop. □

Example 5.10. One can consider the connected graph given by two vertices, each of degree 3, connected by two edges with each one leaf $\mathrm{d}t_1$ and $\mathrm{d}t_2$ respectively. Assume that the edges have a clockwise orientation.

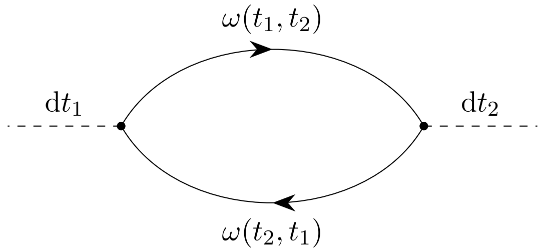

Figure 1: A one-loop graph C_2: two trivalent vertices joined by two propagators, each vertex carrying a single $\mathrm{d}t$ leaf.

This gives a contribution of

$$W_\Gamma^{\text{source}} = -\int_{S^1 \times S^1} \omega_{S^1}\,(t_1, t_2)^2\,\mathrm{d}t_1 \mathrm{d}t_2 = -\frac{1}{12}.$$

We also have some restrictions on the target side. We first observe from S_0 that $\bar{\boldsymbol{\gamma}}$ only has a propagator with $\bar{\boldsymbol{\beta}}$, which is not present in the interaction part. Therefore the $\bar{\boldsymbol{\gamma}}$ field in a Feynman graph has to be a leaf. Due to Proposition 5.8, the vertex $\bar{\boldsymbol{\gamma}}^2\bar{\boldsymbol{b}}$ hence does not contribute to any Feynman graph (except as a corolla). Since the interaction vertex with $\bar{\boldsymbol{b}}$ cannot contribute, the $\bar{\boldsymbol{c}}$ field can also not interact with any of the leftover interaction vertices; for the same reason, the field $\bar{\boldsymbol{c}}$ is also simply a leaf.

Apart from these restrictions, another obvious one is the fact that e_+, e_- constitutes a null frame. As such, any contraction of such a section with one of the same kind via the metric vanishes.

Decompose $S_{\text{eff}} \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*[[\hbar]])$ as $S_{\text{eff}} = \sum_{i=0}^n S^{(i)}$, where $n = \dim M$ (in our case $n = 2$) and $S_{\text{eff}}^{(i)}$ is the i-form component. From Proposition 5.9 we know that at $\hbar^0$ there are only the corollas; at de Rham degree 0, these are simply all the vertices of $\mathrm{T}\Phi_{\mathrm{x}}^*\, S_P$ and at de Rham degree 1 S_Y, all restricted to H (i.e. if after replacing every field $\bar{\phi} = \bar{\phi}^{\mathsf{H}} + \bar{\phi}^{\mathrm{fl}}$ and only taking fully H-dressed contributions).

For the $\hbar^1$ part there is already an enormous amount of terms. First of all we note that due to $V - E + 2 = 2$ and the fact that every vertex needs to have at least 2 adjacent edges (see Prop. 5.9), the graphs are all just cycle graphs C_n, with n the amount of vertices. Every possible graph C_n gives a power series in the zero mode fields.

5.3 BV Laplacian

To justify $\Delta_{\text{fib}} S_x = 0$, $\Delta_{\text{fib}} S_Y = 0$ and $\Delta_{\text{fib}} S_C = 0$, we present the arguments of [9]. Take a manifold T^*M and its mapping space $\mathrm{Map}(T[1]\Sigma, T^*M)$ with conjugate superfields $\bar{b}^I(t, \theta)$,

$\bar{a}_J(t,\theta)$, the capital indices running over all coordinates. The BV Laplacian is formally given by

$$\Delta_{\text{fib}} \sim \int_\Sigma \frac{\delta^2}{\delta \bar{b}^I(t,\theta)\,\delta \bar{a}_I(t,\theta)}.$$

We now have

$$K(t,\theta;t',\theta') = \Delta_{\text{fib}}\big(\bar{b}^I(t,\theta)\,\bar{a}_J(t',\theta')\big) = \delta^I_J\,\delta(t-t')(\theta-\theta').$$

Acting on a functional $S_\gamma = \int_\Sigma \gamma(\bar{b})\bar{a}$ produces a term $\sim \delta(0)$, requiring regularization of Δ_{fib} to obtain sensible expressions.

The rule is provided by the propagator. By the Ward identity[3] the ω of the previous section satisfies the distributional equation

$$\mathrm{d}\omega = \phi - K,$$

where ϕ is the smooth representative of the Poincaré dual of the diagonal $D_\Sigma \subset \Sigma \times \Sigma$ given by the harmonic kernel; for $\Sigma = S^1$ of unit length, $\phi = \mathrm{d}t' - \mathrm{d}t$. Moreover $\omega|_{D_\Sigma} = 0$ as principal value. Solving for $K = \phi - \mathrm{d}\omega$ at the diagonal, the second term dropping by $\omega|_{D_\Sigma} = 0$, yields the regularization

$$\delta(t-t')(\theta-\theta')\big|_{D_\Sigma} := \psi(t)$$

with $\psi := \phi|_{D_\Sigma}$, and we define Δ_{reg} to be Δ_{fib} with this assignment (when we write Δ_{fib} we will always mean the regularized one). On the generators this yields

$$\Delta_{\text{reg}}(\bar{b}^I\bar{a}_J)(t) = \delta^I_J\,\psi(t).$$

The one-form ψ represents the Euler class of Σ [9]; for $\Sigma = S^1$ we have $\psi = (\mathrm{d}t' - \mathrm{d}t)|_{D_\Sigma} = 0$ pointwise, consistent with $\chi(S^1) = 0$. Now consider a functional of the form

$$S_\gamma = \int_\Sigma \gamma^{i_1,\dots,i_k}(\bar{b}(t))\bar{a}_{i_1}(t)\dots\bar{a}_{i_k}(t)$$

with $\gamma \in \Gamma(S^kTM)$. Applying Δ_{reg} and using the symmetry of γ and the commutativity of the $\bar{a}$'s we obtain

$$\Delta_{\text{reg}}S_\gamma = -k\int_\Sigma \psi(t)\partial_i\gamma^{ii_1\dots i_{k-1}}(\bar{b}(t))\bar{a}_{i_1}\dots\bar{a}_{i_{k-1}} = 0.$$

For our case the $\bar{b}^I$ comprise all coordinates of $\mathcal{P}$, among them the base fields $\bar{\boldsymbol{x}}^i$, and the $\bar{a}_I$ their conjugate momenta, among them $\bar{\boldsymbol{p}}_i$. Taking $\gamma = g^{ij}, e^i_+, e^i_-, Y^k, C^k$ then gives the results $\Delta_{\text{reg}}S_x = 0, \Delta_{\text{reg}}S_Y = 0$ and $\Delta_{\text{reg}}S_C = 0$.

5.4 Globalization of the effective action

5.4.1 Via canonical transformation

In the following we want to apply (modified) quantum canonical transformations such that we collapse

$$S_{\text{eff}} \rightsquigarrow \widetilde{S_{\text{eff}}}^{(0)} + S_Y|_{\mathsf{H}}, \quad D\widetilde{S_{\text{eff}}}^{(0)} = 0$$

where D is the induced Grothendieck connection on the subbundle $\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*[[\hbar]])$. This allows us to find the globalized (in x) effective action $\widetilde{S_{\text{eff}}}^{(0)} = \mathrm{T}\Phi_{\mathrm{x}}^*\,S_{\text{eff}}^{\text{glob}}$. This is the general globalization procedure presented in [7].

[3] Without going into too much detail, by standard BV properties we have $\Delta_{\text{fib}}(e^{\frac{i}{\hbar}S}\mathcal{O}) = e^{\frac{i}{\hbar}S}(\Delta_{\text{fib}}\mathcal{O} - \frac{i}{\hbar}(S,\mathcal{O})_{\text{fib}})$. Integrating over some $\mathcal{L}$ in the fluctuation space, pulling out the BV differential from the fibered integral and forming the expectation value w.r.t. S_0 we obtain $\Delta^{\mathsf{H}}_{\text{fib}}\langle\mathcal{O}\rangle_0 = \langle\Delta_{\text{fib}}\mathcal{O} - \frac{i}{\hbar}(S,\mathcal{O})_{\text{fib}}\rangle_0$. Setting $\mathcal{O} = \bar{b}\otimes\bar{a}$ eventually yields the equation $\mathrm{d}\omega = \phi - K$, see [9].

The globalization procedure works as follows. First denote $S^{(i)}_{\text{eff}} = \sum_{k=0}^{\infty} \hbar^k S^{(i)}_k$. We first observe that due to Proposition 5.9 we have that $S^{(i)}_0 = 0$ for all $i > 1$. The $S^{(0)}_0$ and $S^{(1)}_0$ are simply S and S_Y restricted to H_x. Denoting $\bar{S}^{(1)} = S^{(1)}_{\text{eff}} - S^{(1)}_0$ and $\bar{S}^{(i)} = S^{(i)}_{\text{eff}}$ for $i > 1$, we then claim that there is a quantum canonical transformation starting at $\hbar^1$ which results in $\bar{S}^{(i)} = 0$ for $i \geqslant 1$. We prove this by induction on the order $\hbar^k$.

$\hbar^{k+1}$	$\bar{S}^{(0)}_{k+1}$	$\bar{S}^{(1)}_{k+1}$		$\bar{S}^{(m-1)}_{k+1}$	$\bar{S}^{(m)}_{k+1}$
$\hbar^{k}$	$\bar{S}^{(0)}_{k}$	$\bar{S}^{(1)}_{k}$		$\bar{S}^{(m-1)}_{k}$	$\bar{S}^{(m)}_{k}$
$\hbar^{k-1}$	$\bar{S}^{(0)}_{k-1}$				
$\vdots$					
$\hbar^{1}$	$\bar{S}^{(0)}_{1}$				
$\hbar^{0}$	$\bar{S}^{(0)}_{0}$				
	Ω^0	Ω^1	$\cdots$	Ω^{m-1}	Ω^m

Table 6: Bookkeeping of the induction at fixed $\hbar^k$: the shaded cells are already cleared, and clearing the top de Rham degree Ω^m propagates downward until only the Ω^0 column survives.

Indeed, assume that $\bar{S}^{(i)}_r = 0\ \forall i \geqslant 1$ and $\forall r < k$ (see figure). Denote $D = \mathrm{d}_x + (S^{(1)}_0, -)^{\mathsf{H}}_{\text{fib}}$ and $\Omega = (S^{(0)}_0, -)^{\mathsf{H}}_{\text{fib}}$. Because of the flatness condition and CME, we know that $D^2 = \Omega^2 = 0$ and that D and Ω commute. The QME now on the degrees $\hbar^k, \Omega^{i+1}$ reads

$$D\bar{S}^{(i)}_k + \Omega S^{(i+1)}_k = 0,$$

for $i \geqslant 0$. In particular, due to dimensions, we have that $D\bar{S}^{(m)} = 0$. Because $H^{\bullet}_D = H^0$, $\exists \tau \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*)$ such that $D\tau = \bar{S}^{(m)}$. By defining $T = -\hbar^k \tau$ and integrating the transformation

$$\delta \bar{S}^{(i)}_k = DT_k(i-1) + \Omega T^{(i)}_k$$

for $i = m$ we obtain $\bar{S}^{(m)}_k \rightsquigarrow 0$ and for $i = m-1$ we obtain $\bar{S}^{(m-1)}_k \rightsquigarrow \bar{S}^{(m-1)}_k - \Omega\tau$, such that now $\bar{S}^{(m-1)}_k$ satisfies $D\bar{S}^{(m-1)}_k = 0$ and hence we can apply the same procedure to make it vanish, all up to $i = 1$. The claim follows.

Let us carry this out explicitly at order $\hbar^1$ for our two-dimensional spacetime, where the de Rham degree runs only up to $m = 2$. Take the homotopy gauge $\tau = h_D \bar{S}^{(m)}$, with h_D the Grothendieck homotopy of Lemma 3.11, so that $D\tau = \bar{S}^{(m)}$. For the $\hbar^1$ computation, the only $\hbar^0$ datum is $S^{(0)}_0$, hence $\Omega = (S^{(0)}_0, -)^{\mathsf{H}}_{\text{fib}}$ is the only bracket that enters. Clearing the top degree shifts $\bar{S}^{(1)}_1 \rightsquigarrow \bar{S}^{(1)}_1 - \Omega\, h_D\, \bar{S}^{(2)}_1$, and clearing the resulting degree-one term shifts $\bar{S}^{(0)}_1$ by $-\Omega\, h_D\, (\bar{S}^{(1)}_1 - \Omega\, h_D\, \bar{S}^{(2)}_1)$. Collecting, the first-order globalized action is

$$\widetilde{S}^{(0)}_1 = \bar{S}^{(0)}_1 - \big(S^{(0)}_0, h_D\, \bar{S}^{(1)}_1\big)^{\mathsf{H}}_{\text{fib}} + \big(S^{(0)}_0, h_D\, (S^{(0)}_0, h_D\, \bar{S}^{(2)}_1)^{\mathsf{H}}_{\text{fib}}\big)^{\mathsf{H}}_{\text{fib}}.$$

The global representative is then $\widetilde{S}^{(0)}_1$ evaluated at the zero section.

We note that the higher $\hbar^p$ orders will become increasingly tedious to compute. The canonical transformations we apply at lower $\hbar$ order affect each of the terms at higher $\hbar$ degree which we would have to track, before reaching that row to collapse it to 0 de Rham degree.

5.4.2 Via L_∞ pushforward

The equation

$$\mathrm{d}_x S_{\text{eff}} + \frac{1}{2}(S_{\text{eff}}, S_{\text{eff}})^{\mathsf{H}}_{\text{fib}} - i\hbar\Delta^{\mathsf{H}}_{\text{fib}} S_{\text{eff}} = 0$$

can be rewritten for $S'_{\text{eff}} = S_{\text{eff}} - S_Y|_{\mathsf{H}}$, as

$$(\mathrm{d}_x + (S_Y|_{\mathsf{H}}, -)^{\mathsf{H}}_{\text{fib}})S'_{\text{eff}} + \frac{1}{2}(S'_{\text{eff}}, S'_{\text{eff}})^{\mathsf{H}}_{\text{fib}} - i\hbar\Delta^{\mathsf{H}}_{\text{fib}} S'_{\text{eff}} = 0,$$

which in other words means $S'_{\text{eff}} \in \mathrm{MC}(\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*))$ for the DGLA with the induced Grothendieck differential D, BV Laplacian on the zero modes and induced BV bracket. The globalization procedure above consists in applying gauge transformations (which in this case are modified BV canonical transformations (60)) consecutively until, at a given order of $\hbar$, the element is only in $\deg_{\mathrm{d}x} = 0$ and thus in $\operatorname{Ker} h_\Delta$. One can then apply ι_Δ, which on $\operatorname{Ker} h_\Delta$ collapses to Π_0, to obtain the global effective action.

An equivalent way of globalizing the solution is to use the pushforward G_* on S'_{eff} from the transfer of Proposition 3.34, which by Proposition 2.16 guarantees to give us a solution in the global BV space.

Lemma 5.11. The Taylor components of G read $g_1 = P$ and $g_i = g_{i-1} Q_i^{i-1} K_i$ for $i \geqslant 2$.

Proof. Recall (12) that the coderivation Q is given by

$$Q\left(v_1 \odot \cdots \odot v_i\right) = \sum_{\ell=1}^{n} \sum_{\sigma \in S(\ell, i-\ell)} \varepsilon(\sigma) q_\ell \left(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(\ell)}\right) \odot v_{\sigma(\ell+1)} \odot \cdots \odot v_{\sigma(i)},$$

hence Q_i^k is nonzero when q_ℓ contributes with $i - \ell + 1 = k$. Because $q_\ell = 0$ for $\ell \geqslant 3$, we have that $i - k < 2$. The sum for g_i runs until $k \leqslant i - 1$, such that only the $k = i - 1$ term survives. $\square$

Remark 5.12 ($\hbar$-convergence)**.** Suppose $S_0^{(0)}$ is the only datum at order $\hbar^0$, every other component $\bar{S}^{(n)}$ being $\mathcal{O}(\hbar)$, as in the present case; set $d = \dim M$, the dimension of the manifold we are globalizing over. Track the de Rham degree on M. The bracket q_2 wedges the $\mathrm{d}x$-forms and hence adds degrees, while the homotopy and projection are the towers

$$h_\Delta = \sum_{m \geqslant 0} h_D (i\hbar\Delta^{\mathsf{H}}_{\text{fib}} h_D)^m, \qquad \pi_\Delta = \sum_{m \geqslant 0} \Pi_0 (i\hbar\Delta^{\mathsf{H}}_{\text{fib}} h_D)^m,$$

in which each block $\Delta^{\mathsf{H}}_{\text{fib}} h_D$ lowers the degree by one and carries a factor of $\hbar$. A monomial $g_i(\bar{S}^{(n_1)} \odot \cdots \odot \bar{S}^{(n_i)})$ thus uses $i-1$ base homotopies together with some number M of these $\Delta^{\mathsf{H}}_{\text{fib}} h_D$ blocks; since it must land in degree zero, the leg degrees obey

$$\text{(I)} \quad \sum_{j=1}^{i} n_j = (i-1) + M, \qquad \text{(II)} \quad \operatorname{ord}_\hbar \left(g_i(\cdots)\right) = \sum_{j=1}^{i} \operatorname{ord}_\hbar \bar{S}^{(n_j)} + M.$$

Each leg of degree $\geqslant 1$ and each block costs at least one $\hbar$, and no leg carries degree above d. Writing a_n for the number of legs of degree n,

$$\operatorname{ord}_\hbar g_i \overset{\text{(II)}}{\geqslant} \sum_{n \geqslant 1} a_n + M \geqslant \frac{1}{d} \sum_{n \geqslant 1} n\, a_n + M \overset{\text{(I)}}{=} \frac{(i-1)+M}{d} + M \geqslant \frac{i-1}{d},$$

the second step using $n \leqslant d$. As orders are integers, $\operatorname{ord}_\hbar g_i \geqslant \lceil (i-1)/d \rceil$, so at order $\hbar^k$ only the finitely many g_i with $i \leqslant dk+1$ contribute and the pushforward converges $\hbar$-adically. For our two-dimensional spacetime $d = 2$, whence $\hbar^0$ sees g_1 alone and $\hbar^1$ sees g_1, g_2, g_3.

We carry out the first order explicitly. By the remark there are no insertions at $\hbar^1$, so $\pi_\Delta \rightsquigarrow \Pi_0$ and $h_\Delta \rightsquigarrow h$, and the transfer maps (Theorem 2.15) read

$$\begin{aligned} g_1(x) &= \Pi_0(x), \\ g_2(x^{\odot 2}) &= \Pi_0\, q_2\big(\underbrace{h(x)\odot x + \mathrm{T}\phi_x^*\Pi_0(x)\odot h(x)}_{K_2(x\odot x)}\big), \\ g_3(x^{\odot 3}) &= \Pi_0\, q_2\big(h\, q_2(h(x)\odot x)\odot x\big) + \cdots , \end{aligned}$$

only the tree of g_3 above surviving the inputs below. Feed in $x = S'_{\mathrm{eff}} = \bar{S}^{(0)} + \bar{S}^{(1)} + \bar{S}^{(2)}$ and keep the $\hbar^1$, $\deg_{\mathrm{d}x} = 0$ part. Since h annihilates every degree-zero leg $S_0^{(0)}$, the homotopy must fall on the single quantum leg. In g_2 this leaves $h\bar{S}_1^{(1)}$ bracketed with $S_0^{(0)}$; as $S_0^{(0)}$ is already flat ($\mathrm{T}\phi_x^*\Pi_0 S_0^{(0)} = S_0^{(0)}$, the procedure starting only at $\hbar^1$) the two summands of K_2 coincide and, with the $\frac{1}{2!}$,

$$\tfrac{1}{2!}\, g_2(S'^{\odot 2}_{\mathrm{eff}})\big|_{\hbar^1} = -\,\Pi_0\big(S_0^{(0)},\, h\,\bar{S}_1^{(1)}\big)^{\mathsf{H}}_{\mathrm{fib}}.$$

The lone g_3-tree threads the degree-two leg through two homotopies,

$$\tfrac{1}{3!}\, g_3(S'^{\odot 3}_{\mathrm{eff}})\big|_{\hbar^1} = +\,\Pi_0\big(S_0^{(0)},\, h\,(S_0^{(0)},\, h\,\bar{S}_1^{(2)})^{\mathsf{H}}_{\mathrm{fib}}\big)^{\mathsf{H}}_{\mathrm{fib}},$$

while g_1 gives $\Pi_0\bar{S}_1^{(0)}$. Summing,

$$G_*(S'_{\mathrm{eff}})\big|_{\hbar^1} = \Pi_0\bar{S}_1^{(0)} - \Pi_0\big(S_0^{(0)}, h\,\bar{S}_1^{(1)}\big)^{\mathsf{H}}_{\mathrm{fib}} + \Pi_0\big(S_0^{(0)}, h\,(S_0^{(0)}, h\,\bar{S}_1^{(2)})^{\mathsf{H}}_{\mathrm{fib}}\big)^{\mathsf{H}}_{\mathrm{fib}} = \Pi_0\widetilde{S}_1^{(0)},$$

in agreement with the result from before.

Remark 5.13. Remember that we are actually only globalizing over $x \in M$, as we have seen in Remark 5.1. The respective Grothendieck SDR hence does not land in the entire global complex, but only in the sections global in x. We will be more precise in the next section, see also Proposition 6.13.

6 The Spinning Particle III: n-dim globalization

We now globalize the effective action of the general n-dimensional spinning particle following the methods of [30] to some extent. We will first lift the theory with its symplectic structure to formal geometry with the constructions developed in Section 3.7. We will justify the operations done in the last section via the homological perturbation lemma, in particular when passing down to cohomology.

6.1 In formal geometry

Recall the AKSZ construction of the spinning particle for a spacetime of any dimension: it has the mapping space $\mathcal{F} = \mathrm{Map}(T[1]\Sigma, \mathcal{N})$, where $\Sigma = S^1$ and $\mathcal{N} = (\Pi TM \oplus_M T^*M) \times T^*(\Pi T\mathbb{R}[1])$ is the target of Section 4.5, generated by the superfields collected below.

superfield	field	gh	par	field $[\mathrm{d}t]$	gh	par
$\boldsymbol{x^i}$	x^i	0	0	p^{+i}	-1	1
$\boldsymbol{p_i}$	p_i	0	0	x_i^+	-1	1
$\boldsymbol{\theta^i}$	θ^i	0	1	θ^{+i}	-1	0
$\boldsymbol{c}$	c	1	1	e	0	0
$\boldsymbol{b}$	e^+	-1	1	c^+	-2	0
$\boldsymbol{\gamma}$	γ	1	0	ξ	0	1
$\boldsymbol{\beta}$	ξ^+	-1	0	γ^+	-2	1

The AKSZ action reads

$$S_{\mathrm{AKSZ}} = \int_\Sigma \boldsymbol{p}_i \mathrm{d}\boldsymbol{x}^i - \frac{1}{2}\partial_k g_{rj}(\boldsymbol{x})\, \boldsymbol{\theta}^r \boldsymbol{\theta}^k \wedge \mathrm{d}\boldsymbol{x}^j + \frac{1}{2} g_{ij}(\boldsymbol{x})\, \boldsymbol{\theta}^i \wedge \mathrm{d}\boldsymbol{\theta}^j + \boldsymbol{b}\,\mathrm{d}\boldsymbol{c} + \boldsymbol{\beta}\,\mathrm{d}\boldsymbol{\gamma}$$
$$- \frac{1}{2}\boldsymbol{p}_i \boldsymbol{p}_j\, g^{ij}(\boldsymbol{x})\, \boldsymbol{c} + \boldsymbol{p}_i \boldsymbol{\theta}^i\, \boldsymbol{\gamma} + \boldsymbol{\gamma}^2\, \boldsymbol{b}, \tag{65}$$

and solves the classical master equation $(S_{\mathrm{AKSZ}}, S_{\mathrm{AKSZ}}) = 0$ by Theorem 1.52. It has symplectic structure

$$\omega_{\mathrm{AKSZ}} = -\int_\Sigma \delta\boldsymbol{p}_i \wedge \delta\boldsymbol{x}^i - \frac{1}{2}\boldsymbol{\theta}^r \boldsymbol{\theta}^k \partial_\ell \partial_k g_{rj}(\boldsymbol{x}) \delta\boldsymbol{x}^\ell \wedge \delta\boldsymbol{x}^j - g_{kj}(\boldsymbol{x})\Gamma^k_{i\ell}(\boldsymbol{x})\boldsymbol{\theta}^\ell \delta\boldsymbol{x}^i \wedge \delta\boldsymbol{\theta}^j$$
$$+ \frac{1}{2} g_{ij}(\boldsymbol{x})\delta\boldsymbol{\theta}^i \wedge \delta\boldsymbol{\theta}^j + \delta\boldsymbol{b} \wedge \delta\boldsymbol{c} + \delta\boldsymbol{\beta} \wedge \delta\boldsymbol{\gamma}.$$

which gives rise to an antibracket.

Now assume we have constructed a symplectic connection (this can be done for any symplectic and odd-symplectic manifold) on the target $\mathcal{N}$ and denote its Christoffel symbols as $\Gamma^k_{ij}(z)$ in local coordinates. It satisfies (32), see also Remark 3.21. This connection can be lifted to the mapping space with transgressed Christoffel symbols $\Gamma^k_{ij}(\boldsymbol{z})$ and, together with ω_{AKSZ}, it also satisfies (32).

Fix again a constant map $\varphi = x \in M$. With a symplectic connection constructed, we can now construct the Grothendieck connection on the formal bundle $\widehat{S}\mathsf{F}_x^*$ with

$$\mathsf{F}_x = \mathrm{Map}(T[1]\Sigma, T_x\mathcal{N}) \cong \Omega^\bullet(\Sigma) \otimes T_x\mathcal{N}.$$

It is given by $D = \mathrm{d}_x + Y$ with

$$\mathrm{d}_x = \mathrm{d}x^m \frac{\partial}{\partial x^m}, \qquad Y = \int_\Sigma \mathrm{d}x^m Y^j_m(x, \bar{\boldsymbol{z}}) \frac{\partial}{\partial \bar{\boldsymbol{z}}^j},$$

where $\bar{\boldsymbol{z}}$ denotes the fiber superfields of F_x, the analogues of the barred fields of Section 5. We can split again $Y = -\delta + \Gamma + X$ with

$$\delta = \int_\Sigma \mathrm{d}x^m \frac{\partial}{\partial \bar{\boldsymbol{z}}^m}, \quad \Gamma = -\int_\Sigma \mathrm{d}x^\ell \bar{\boldsymbol{z}}^k \Gamma^i_{k\ell}(x) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}, \quad X = \int_\Sigma \mathrm{d}x^m X^j_m(x, \bar{\boldsymbol{z}}) \frac{\partial}{\partial \bar{\boldsymbol{z}}^j}$$

As we have seen, X is fiberwise symplectic and hence we can write it as ad_{S_X}, for its Hamiltonian S_X.

At the basepoint $x \in M$ we have components of $\omega_{\text{fib}} = \int_\Sigma \frac{1}{2}\omega_{ij}\delta\bar{\boldsymbol{z}}^i\delta\bar{\boldsymbol{z}}^j$ given by

$$\omega_{\bar{\boldsymbol{p}}_i\bar{\boldsymbol{x}}^j}(x) = -\delta^i_j, \quad \omega_{\bar{\boldsymbol{\theta}}^i\bar{\boldsymbol{\theta}}^j}(x) = -g_{ij}(x), \quad \omega_{\bar{\boldsymbol{b}}\bar{\boldsymbol{c}}} = -1, \quad \omega_{\bar{\boldsymbol{\beta}}\bar{\boldsymbol{\gamma}}} = -1.$$

with the dressed inverse Π^{ij} (Definition 1.27 with $k = 0$) given by

$$\Pi^{\bar{\boldsymbol{x}}^i\bar{\boldsymbol{p}}_j} = +\delta^i_j, \quad \Pi^{\bar{\boldsymbol{\theta}}^i\bar{\boldsymbol{\theta}}^j} = +g^{ij}(x), \quad \Pi^{\bar{\boldsymbol{c}}\bar{\boldsymbol{b}}} = +1, \quad \Pi^{\bar{\boldsymbol{\gamma}}\bar{\boldsymbol{\beta}}} = +1, \tag{66}$$

The connection induces a formal exponential map $\mathrm{T}\phi^*_x$ (we use this notation instead of EXP*), with which we can lift S_{AKSZ}; we obtain $S_x = \mathrm{T}\phi^*_x S_{\text{AKSZ}} \in \widehat{S}\mathsf{F}^*_x$. By the symplectic condition, it satisfies $(S_x, S_x)_{\text{fib}} = 0$ and given its globality, $DS_x = 0$.

The fiberwise BV Laplacian is given as in (36) with (66). Taking the source to be $\Sigma = S^1$ again the arguments regarding Δ_{fib} presented in Section 5.3 go through as well (with the change that we have the tensor $\Pi^{ij}(x)$ inside the integral, constant along Σ and in the fields since the base point is frozen, which still gets eliminated by the vanishing Euler class of S^1). In particular we obtain $\Delta_{\text{fib}} S_x = 0$ and, very importantly, $\Delta_{\text{fib}} S_X = 0$, which has the consequence due to Lemma 3.31 that $[\Delta_{\text{fib}}, D] = 0$. In particular S_x is a solution of the modified quantum master equation

$$(D - \mathring{\imath}\hbar\Delta_{\text{fib}})S_x + \frac{1}{2}(S_x, S_x)_{\text{fib}} = 0.$$

Split $S_x = S^0_x + S^P_x$ with kinetic part

$$S^0_x = \int_\Sigma \Big(\bar{\boldsymbol{p}}_i \, \mathrm{d}\bar{\boldsymbol{x}}^i + \tfrac{1}{2} g_{ij}(x) \, \bar{\boldsymbol{\theta}}^i \, \mathrm{d}\bar{\boldsymbol{\theta}}^j + \bar{\boldsymbol{b}} \, \mathrm{d}\bar{\boldsymbol{c}} + \bar{\boldsymbol{\beta}} \, \mathrm{d}\bar{\boldsymbol{\gamma}} \Big),$$

and S^P_x the rest. This is indeed the full quadratic part of S_x: by the properties of the formal exponential map, $\mathrm{T}\phi^*_x(z^i) = z^i(x) + \bar{\boldsymbol{z}}^i + \mathcal{O}(\bar{\boldsymbol{z}}^2)$, and at the base point $x \in M$ every coordinate except the x^i vanishes, so constant terms enter S_x only through the tensors evaluated at x. The quadratic part of $\mathrm{T}\phi^*_x S_{\text{AKSZ}}$ therefore consists of the kinetic terms with frozen coefficients, while all vertices of the perturbation part are at least cubic in the fluctuations.

6.2 Effective action

Consider the worldline de Rham differential d_Σ. It acts on $\mathsf{F}_x = \Omega^\bullet(S^1) \otimes T_x\mathcal{N}$ as $\mathrm{d}_\Sigma \otimes \mathrm{Id}_{T_x\mathcal{N}}$ and induces a cohomology $\mathsf{H}_x = H^\bullet(S^1) \otimes T_x\mathcal{N}$. The worldline Hodge decomposition (Section 5.2) gives a strong deformation retraction

$$(\mathsf{H}_x, 0) \underset{p}{\overset{\iota}{\rightleftharpoons}} (\mathsf{F}_x, \mathrm{d}_\Sigma) \circlearrowleft K, \qquad K = K_{S^1} \otimes \mathrm{Id}_{T_x\mathcal{N}}, \tag{67}$$

with K_{S^1} the worldline homotopy (kernel $\omega_{S^1}(t, t') = \theta(t - t') - t + t' - \frac{1}{2}$, Section 5.2). Note that the triple (ι, p, K) is x-independent. For later use we recall the defining relations of the Hodge data from Section 5.2,

(I) $\mathrm{d}K + K\mathrm{d} = \mathrm{Id} - P$, (II) $PK = KP = 0$, (III) $K^T = -K$, (IV) $K^2 = 0$.

As in Section 5.2, the gauge fixing rests on a factorization of the symplectic form, which works just as well in the present case. Splitting a field $\bar{\boldsymbol{z}}(t) \in \Omega^\bullet(\Sigma) \otimes T_x\mathcal{N}$ as $\bar{\boldsymbol{z}}(t) = \alpha \otimes u$, with $\alpha \in \Omega^\bullet(\Sigma), u \in T_x\mathcal{N}$ we can write, given that ω_{fib} freezes the base point dependence of the symplectic form given by ω_{AKSZ},

$$\omega_{\mathrm{fib}}(\alpha \otimes u,\ \beta \otimes v) = -(-1)^{|u||\beta|}\omega_{\mathcal{N}}|_x(u,v)\int_\Sigma \alpha \wedge \beta, \qquad \alpha,\beta \in \Omega^\bullet(\Sigma),\ u,v \in T_x\mathcal{N}, \quad (68)$$

so ω is the source Poincaré pairing tensored with the frozen target form $\omega_{\mathcal{N}}|_x$.

This is all the gauge fixing needs. The subspace

$$\mathcal{L} = \operatorname{Im} K = \operatorname{Im} K_{S^1} \otimes T_x\mathcal{N} \subset \mathsf{F}_x \qquad (69)$$

is Lagrangian by the same arguments as in Section 5.2. The proof of Proposition 5.5 works exactly the same using only that $\omega_{\mathcal{N}}$ is non-degenerate.

Our next step is to construct the SDR for the completed symmetric tensor algebra $\Gamma(\widehat{S}\mathsf{F}_x^*)$ from the SDR (67) and perturb it with the BV Laplacian Δ_{fib} and the constructed Grothendieck connection D. We first record that the perturbation is compatible with d_Σ. We naturally extend d_Σ from F_x to $\widehat{S}\mathsf{F}_x^*$ via Leibniz.

Lemma 6.1. As operators on $\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{F}^*)$, the worldline differential graded-commutes with both perturbing operators,

$$[\mathrm{d}_\Sigma, D] = 0, \qquad [\mathrm{d}_\Sigma, \Delta_{\mathrm{fib}}] = 0.$$

Proof. The three operators read

$$\mathrm{d}_\Sigma = \int_\Sigma \theta\, \partial_t \bar{\boldsymbol{z}}^k \frac{\partial}{\partial \bar{\boldsymbol{z}}^k}, \qquad D = \mathrm{d}_x + \int_\Sigma \mathrm{d}x^\ell\, Y_\ell^i(x,\bar{\boldsymbol{z}}) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}, \qquad \Delta_{\mathrm{fib}} = \tfrac{1}{2}\int_\Sigma (-1)^{p_i p_j}\Pi^{ij}(x) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}\frac{\partial}{\partial \bar{\boldsymbol{z}}^j}.$$

The base part d_x acts only on the x-dependent coefficients and hence commutes with d_Σ; it suffices to treat the vector field part of D.

Connection: The operator d_Σ acts on D as

$$\int_\Sigma \theta\, \partial_t \bar{\boldsymbol{z}}^k \frac{\partial}{\partial \bar{\boldsymbol{z}}^k}\big(\mathrm{d}x^\ell Y_\ell^i\big) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i} = (-1)^{p_\ell+1}\int_\Sigma \theta\, \mathrm{d}x^\ell\, (\partial_t Y_\ell^i) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}.$$

where we have used the chain rule. In the second, D's derivative strikes the coefficient $\partial_t \bar{\boldsymbol{z}}^k$ of d_Σ, producing the super delta $\frac{\partial \bar{\boldsymbol{z}}^k(t,\theta)}{\partial \bar{\boldsymbol{z}}^i(t',\theta')} = \delta_i^k\, \delta(t-t')(\theta-\theta')$ of the expansion in Lemma 6.2:

$$\int_\Sigma\!\int_\Sigma \mathrm{d}x^\ell Y_\ell^i(\bar{\boldsymbol{z}}(t')) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i(t')}\big(\theta\, \partial_t \bar{\boldsymbol{z}}^k(t)\big) \frac{\partial}{\partial \bar{\boldsymbol{z}}^k(t)} = (-1)^{p_i}\int_\Sigma\!\int_\Sigma \mathrm{d}x^\ell Y_\ell^i(\bar{\boldsymbol{z}}(t'))\, \theta(\theta-\theta')\, \partial_t\delta(t-t') \frac{\partial}{\partial \bar{\boldsymbol{z}}^i(t)}.$$

Integrating by parts in t on the closed S^1 moves ∂_t onto Y:

$$= (-1)^{p_i + |Y_\ell^i|}\int_\Sigma \theta\, \mathrm{d}x^\ell\, (\partial_t Y_\ell^i) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}.$$

The two contributions are the same object, and by the odd parity of the D, $(p_\ell+1) + |Y_\ell^i| + p_i = 1$, they carry opposite signs and cancel: $[\mathrm{d}_\Sigma, D] = 0$.

BV Laplacian: The coefficient $\Pi^{ij}(x)$ only depends on the base point so d_Σ's derivative has nothing to differentiate and a single term survives, from Δ_{fib}'s derivatives striking $\partial_t \bar{\boldsymbol{z}}$:

$$[\mathrm{d}_\Sigma, \Delta_{\mathrm{fib}}] = \int_\Sigma\!\int_\Sigma (-1)^{p_i p_j}\Pi^{ij}(x)\, \theta(\theta-\theta')\, \partial_t\delta(t-t') \frac{\partial}{\partial \bar{\boldsymbol{z}}^j(t')}\frac{\partial}{\partial \bar{\boldsymbol{z}}^i(t)}.$$

Write $\bar{\boldsymbol{z}}^j(t) = \bar{z}_0^j(t) + \bar{z}_1^j(t)\theta$. In components both derivatives land on the $\bar{z}_1$'s ($\theta(\theta-\theta') = -\theta\theta'$ saturating the two odd integrations) and the kernel reduces, up to an overall sign, to $\Pi^{ij}(x)\,\partial_t\delta(t-t')$ against the pair $\frac{\delta}{\delta\bar{z}_1^j(t')}\frac{\delta}{\delta\bar{z}_1^i(t)}$. Only the part of a kernel carrying the graded symmetry of the derivative pair can contribute, and the two symmetries are opposite. Indeed, writing $p := p_i = p_j$, the mixed blocks of Π vanishing by Lemma 1.29, the exchange $(i,t)\leftrightarrow(j,t')$ multiplies the kernel by $(-1)^p$: a factor $-(-1)^p$ from $\Pi^{ji} = -(-1)^{p_ip_j}\Pi^{ij}$ and -1 from the antisymmetry of $\partial_t\delta$, while the derivative pair, of parities $p+1$, swaps with $(-1)^{p+1}$. The contraction therefore formally vanishes. □

Lemma 6.2. As operators on $\Lambda T^*M\otimes \mathsf{F}_x^*$

$$[\delta, K] = 0, \qquad [\Gamma, K] = 0.$$

Proof. In essence these statements are true because they act on different parts of the superfields (source or target). We will still verify them by hand. With $K\bar{\boldsymbol{z}}^j(t) = (K_{S^1}\bar{z}_1^j)(t)$ and $\frac{\partial}{\partial\bar{\boldsymbol{z}}^i(t')}(K_{S^1}\bar{z}_1^j)(t) = \delta_i^j\,\omega_{S^1}(t,t')$, we get

$$\begin{aligned}\Gamma K\bar{\boldsymbol{z}}^j(t) &= -\int_\Sigma \mathrm{d}x^\ell\,\bar{\boldsymbol{z}}^k(t')\,\Gamma^i_{k\ell}(x)\,\frac{\partial}{\partial\bar{\boldsymbol{z}}^i(t')}\,(K_{S^1}\bar{z}_1^j)(t)\\ &= -\int_\Sigma \mathrm{d}x^\ell\,\bar{\boldsymbol{z}}^k(t')\,\Gamma^j_{k\ell}(x)\,\omega_{S^1}(t,t') = (-1)^{p_k+1}\,\mathrm{d}x^\ell\,\Gamma^j_{k\ell}(x)\,(K_{S^1}\bar{z}_1^k)(t),\end{aligned}$$

the sign from θ' crossing $\mathrm{d}x^\ell$ and $\bar{z}_1^k$. For the reverse order,

$$\Gamma\bar{\boldsymbol{z}}^j(t,\theta) = -\int_\Sigma \mathrm{d}x^\ell\,\bar{\boldsymbol{z}}^k(t')\,\Gamma^j_{k\ell}(x)\,\delta(t-t')(\theta-\theta') = (-1)^{p_k+1}\,\mathrm{d}x^\ell\,\Gamma^j_{k\ell}(x)\,\bar{\boldsymbol{z}}^k(t,\theta),$$

and hence

$$K\Gamma\bar{\boldsymbol{z}}^j(t) = (-1)^{p_k}\,\mathrm{d}x^\ell\,\Gamma^j_{k\ell}(x)\,(K_{S^1}\bar{z}_1^k)(t).$$

Likewise

$$\delta K\bar{\boldsymbol{z}}^j(t) = \int_\Sigma \mathrm{d}x^m\,\frac{\partial}{\partial\bar{\boldsymbol{z}}^m(t')}\,(K_{S^1}\bar{z}_1^j)(t) = \int_\Sigma \mathrm{d}x^j\,\omega_{S^1}(t,t') = 0,$$

the integrand being independent of θ' and hence killed by the Berezin integration, while $\delta\bar{\boldsymbol{z}}^j(t) = \mathrm{d}x^j$ is field-independent and K vanishes on constants,

$$K\delta\bar{\boldsymbol{z}}^j(t) = K\,\mathrm{d}x^j = 0.$$

□

We first have to elevate the SDR (67) to the completed algebra to start the perturbation.

Corollary 6.3. Lemma 2.14 applies to (67) and lifts it to an SDR on the completed symmetric algebra,

$$(\widehat{S}\mathsf{H}_x^*, 0) \underset{\widehat{p}}{\overset{\widehat{\iota}}{\rightleftharpoons}} (\widehat{S}\mathsf{F}_x^*,\ \mathrm{d}_\Sigma) \circlearrowleft \widehat{K},$$

where $\widehat{\iota},\widehat{p}$ extend ι, p to each factor, and $\widehat{K}$ acts on a monomial of $\deg_{\bar{\boldsymbol{z}}} = n$ (its x-dependent coefficient being a spectator) by

$$\widehat{K}(\bar{\boldsymbol{z}}^{i_1}\ldots\bar{\boldsymbol{z}}^{i_n}) = \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{a=1}^{n}\pm\,\iota p(\bar{\boldsymbol{z}}^{i_{\sigma(1)}})\cdots\iota p(\bar{\boldsymbol{z}}^{i_{\sigma(a-1)}})K(\bar{\boldsymbol{z}}^{i_{\sigma(a)}})\bar{\boldsymbol{z}}^{i_{\sigma(a+1)}}\cdots\bar{\boldsymbol{z}}^{i_{\sigma(n)}}, \tag{70}$$

with $\pm$ the Koszul sign considering $|K| = -1$.

Remark 6.4. The maps of the Hodge decomposition on the smooth forms were defined as maps on the tangent space. Without mentioning it, we naturally let them act on the functionals (i.e. on the cotangent space) via precomposition.

Proposition 6.5. Set $\partial = D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}$. Perturbing Corollary 6.3 along ∂ yields a strong deformation retract

$$(\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*_x),\, \mathrm{d}_\partial) \underset{p_\partial}{\overset{\iota_\partial}{\rightleftarrows}} \big(\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{F}^*_x),\, \mathrm{d}_\Sigma + D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}\big) \circlearrowleft K_\partial$$

with data

$$\iota_\partial = \sum_{n\geqslant 0}(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota}, \qquad p_\partial = \sum_{n\geqslant 0}\widehat{p}\,(-\partial_{\mathrm{red}}\widehat{K})^n, \qquad K_\partial = \sum_{n\geqslant 0}\widehat{K}\,(-\partial_{\mathrm{red}}\widehat{K})^n,$$

where we introduced the reduced perturbation $\partial_{\mathrm{red}} = \mathrm{ad}_{S_X} - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}$. The transferred differential reads $\mathrm{d}_\partial = \sum_{n\geqslant 0}\widehat{p}\,\partial\,(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota} = \widehat{p}\,\partial\,\iota_\partial$ on $\widehat{S}\mathsf{H}^*_x$.

Proof. The triple $(\widehat{\iota}, \widehat{p}, \widehat{K})$ is an SDR by the Corollary above, extended ΛT^*M-linearly to the form valued sections since the perturbation carries form degree, which we perturb with $\partial = D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}$. The perturbation is admissible: $\mathrm{d}_\Sigma^2 = D^2 = \Delta_{\mathrm{fib}}^2 = 0$, $[\mathrm{d}_\Sigma, D] = [\mathrm{d}_\Sigma, \Delta_{\mathrm{fib}}] = 0$ by Lemma 6.1, and $[D, \Delta_{\mathrm{fib}}] = 0$ by Lemma 3.31 (since $\Delta_{\mathrm{fib}}S_X = 0$); hence $(\mathrm{d}_\Sigma + D - \mathring{\imath}\hbar\Delta_{\mathrm{fib}})^2 = 0$.

The two summands of ∂ control convergence of the series of Lemma 2.13. The connection D raises the form degree $\deg_{\mathrm{d}x}$ on the base, bounded by $\dim M$ as M is an ordinary manifold, while Δ_{fib} preserves it and carries a power of $\hbar$; thus in each $\mathrm{d}x$-degree the series is a power series in $\hbar$ and converges $\hbar$-adically.

The last thing we need to check is that the perturbation reduces to $\partial_{\mathrm{red}} = \mathrm{ad}_{S_X} - \mathring{\imath}\hbar\Delta_{\mathrm{fib}}$ in the series. This follows from Lemma 6.2, as δ and Γ both act as derivations on the completed algebra and commute with K, ιp and Id. □

Up to here the perturbation was handled purely algebraically. From now on we evaluate the transferred operators on elements, e.g. the differential d_∂ on functions of the zero modes, the projection p_∂ on $e^{\mathring{\imath}S^P_x/\hbar}$, and every term of the perturbative series acquires a diagrammatic meaning: each $\widehat{K}$ propagates a leg, each Δ_{fib} contracts two legs, each ad_{S_X} inserts a vertex, and the SDR conditions (I)–(IV) become vanishing statements for diagrams. We argue in this language below.

We will take a closer look at the induced differential d_∂. We split it as

$$\mathrm{d}_\partial = \widehat{p}D\widehat{\iota} - \mathring{\imath}\hbar\widehat{p}\Delta_{\mathrm{fib}}\widehat{\iota} + \widehat{p}\,\mathrm{ad}_{S_X}\sum_{n\geqslant 1}(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota} - \mathring{\imath}\hbar\widehat{p}\Delta_{\mathrm{fib}}\sum_{n\geqslant 1}(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota}.$$

Our first claim is that the third term vanishes.

Proposition 6.6. In the splitting of d_∂ above we have that

$$\widehat{p}\,\mathrm{ad}_{S_X}\sum_{n\geqslant 1}(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota} = 0.$$

Proof. Each $\widehat{K}$ creates a propagated leg $\bar{\boldsymbol{z}}(t) \rightsquigarrow \bar{\boldsymbol{z}}'(t')$. Due to $\widehat{p}$ at the end, which kills off any free K-leg at the end due to $pK = 0$, for the term to contribute, these have to be consumed by the operators in ∂_{red}; either by ad_{S_X} which bites it off and inserts S_X on that leg or Δ_{fib} which connects it to another free leg. Note that Δ_{fib} cannot eat two K-legs: their source pairing is $\int_\Sigma K\alpha \wedge K\beta = -\int_\Sigma \alpha \wedge K^2\beta = 0$ by (III) and (IV), i.e. $\operatorname{Im} K$ is isotropic, exactly as in Proposition 5.5. Since a term of order n in the series carries n operators $\widehat{K}$ and $n-1$ operators from ∂_{red} (on the left) and one ad_{S_X}, each of the latter must therefore consume exactly one K-leg.

Now in the sum above, the leading ad_{S_X} (i.e. the ad_{S_X} left of the sum) must in particular eat a K-leg. Its vertex is hence connected to the diagram by a single ω edge, while all its remaining legs are projected to the harmonics by $\widehat{p}$. The integration at that vertex therefore meets the edge kernel alone and yields $\int_{S^1}\omega_{S^1}(s,t)\mathrm{d}t = 0$, which is the SDR condition $KP = PK = 0$, the same mechanism by which the tree diagrams vanished in Proposition 5.9. $\square$

Proposition 6.7. The operator $D^{\mathsf{H}} := \widehat{p}D\widehat{\iota}$ is the Grothendieck connection on $\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*)$.

Proof. All parts of D act on the target index and on the x-dependent coefficients only, the worldline dependence being a spectator, so they pass through the mode decomposition: $\widehat{p}\,\mathrm{d}_x\widehat{\iota} = \mathrm{d}_x$, $\widehat{p}\,\Gamma\widehat{\iota} = \Gamma|_{\mathsf{H}}$ and $\widehat{p}\,\delta\widehat{\iota} = \delta|_{\mathsf{H}}$. In $\widehat{p}\,\mathrm{ad}_{S_X}\,\widehat{\iota}$ the bracket contracts one leg of S_X with a zero mode and $\widehat{p}$ projects the remaining legs, which is the zero-mode bracket with $S_X|_{\mathsf{H}} := \widehat{p}S_X$. Together

$$\widehat{p}D\widehat{\iota} = \mathrm{d}_x - \delta|_{\mathsf{H}} + \Gamma|_{\mathsf{H}} + \mathrm{ad}_{S_X|_{\mathsf{H}}}.$$

For flatness take the perturbation $\partial = D$ alone in Proposition 6.5: the transferred differential is $\widehat{p}D\widehat{\iota} + \widehat{p}\,\mathrm{ad}_{S_X}\sum_{n\geqslant 1}(-\widehat{K}\,\mathrm{ad}_{S_X})^n\widehat{\iota}$, whose higher terms vanish term by term by Proposition 6.6; hence D^{H} is itself a transferred differential and squares to zero by the HPL. Now note that D^{H} has the correct Ansatz for the bundle $\widehat{S}\mathsf{H}^*$ together with the fact that the last vector field is Hamiltonian and makes the connection flat. $\square$

Similarly, we define the induced fiberwise BV Laplacian $\Delta^{\mathsf{H}}_{\mathrm{fib}} := \widehat{p}\Delta_{\mathrm{fib}}\widehat{\iota}$; its properties (and justification to call it the BV Laplacian) are collected in Lemma 6.12 below.

Theorem 6.8. The perturbed projection computes the effective action of Section 5.2: acting on $e^{\mathring{\imath}S_x^P/\hbar}$ it is the perturbative expansion of the BV pushforward of $\widetilde{S}_x = S_x^0 + S_x^P + S_X$,

$$p_\partial(e^{\mathring{\imath}S_x^P/\hbar}) = e^{\mathring{\imath}S_{\mathrm{eff}}/\hbar} = \int_{\mathcal{L}} e^{\mathring{\imath}\widetilde{S}_x(\bar{\phi}^{\mathsf{H}}+\bar{\phi}^{\mathrm{fl}})/\hbar}\mathrm{d}\phi^{\mathrm{fl}},$$

with the kinetic part S_x^0 providing the propagator and the vertices given by $S_x^P + S_X$.

The relationship between the HPL and the effective action is known, see [24] or [14].

Lemma 6.9. The kinetic part generates the worldline differential,

$$(S_x^0, -)_{\mathrm{fib}} = \mathrm{d}_\Sigma, \tag{71}$$

and the split $S_x = S_x^0 + S_x^P$ is compatible with the remaining structure:

$$DS_x^0 = DS_x^P = 0, \qquad (S_x^P, S_x^P)_{\mathrm{fib}} = 0.$$

Proof. The equality (71) can be quickly verified in coordinates.

For DS_x^0 we use that D is symplectic, $\mathrm{def}_D = 0$; the mechanism of (34) then gives $[D, \mathrm{ad}_h] = \mathrm{ad}_{Dh}$, so with (71) and Lemma 6.1

$$\mathrm{ad}_{DS_x^0} = [D, \mathrm{ad}_{S_x^0}] = [D, \mathrm{d}_\Sigma] = 0.$$

The kernel of $s \mapsto (s, -)_{\mathrm{fib}}$ consists of the field-independent elements, while every part of DS_x^0 is at least linear in the fields; hence $DS_x^0 = 0$, and $DS_x^P = DS_x - DS_x^0 = 0$ by globality.

Finally the CME splits: expanding with (71),

$$0 = (S_x, S_x)_{\mathrm{fib}} = \mathrm{d}_\Sigma S_x^0 + 2\,\mathrm{d}_\Sigma S_x^P + (S_x^P, S_x^P)_{\mathrm{fib}} = (S_x^P, S_x^P)_{\mathrm{fib}},$$

again by closedness of the source. $\square$

Corollary 6.10. The effective action $e^{\mathring{\imath}S_{\mathrm{eff}}/\hbar} = p_\partial(e^{\mathring{\imath}S_x^P/\hbar})$ solves

$$(D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}} + R)e^{\mathring{\imath}S_{\mathrm{eff}}/\hbar} = 0, \qquad R := -\mathring{\imath}\hbar\,\widehat{p}\Delta_{\mathrm{fib}}\sum_{n\geqslant 1}(-\widehat{K}\partial_{\mathrm{red}})^n\widehat{\iota}.$$

Proof. Exponentiating the generating property of Lemma 3.30 gives

$$\Delta_{\mathrm{fib}}\, e^{\mathring{i}S_x^P/\hbar} = e^{\mathring{i}S_x^P/\hbar}\Big(\frac{\mathring{i}}{\hbar}\Delta_{\mathrm{fib}}S_x^P - \frac{1}{2\hbar^2}(S_x^P, S_x^P)_{\mathrm{fib}}\Big),$$

hence

$$(\mathrm{d}_\Sigma + D - \mathring{i}\hbar\Delta_{\mathrm{fib}})\, e^{\mathring{i}S_x^P/\hbar} = \frac{\mathring{i}}{\hbar}e^{\mathring{i}S_x^P/\hbar}(\mathrm{d}_\Sigma S_x^P + DS_x^P + \frac{1}{2}(S_x^P, S_x^P)_{\mathrm{fib}} - \mathring{i}\hbar\Delta_{\mathrm{fib}}S_x^P) = 0,$$

term by term: $\mathrm{d}_\Sigma S_x^P = 0$ since the integrand becomes d_Σ-exact and the source is closed, $DS_x^P = 0$ and $(S_x^P, S_x^P)_{\mathrm{fib}} = 0$ by Lemma 6.9, and $\Delta_{\mathrm{fib}}S_x^P = 0$ by Section 5.3. The perturbed projection is a chain map by Lemma 2.13, $p_\partial(\mathrm{d}_\Sigma + \partial) = \mathrm{d}_\partial\, p_\partial$, so $\mathrm{d}_\partial\, e^{\mathring{i}S_{\mathrm{eff}}/\hbar} = 0$; inserting the splitting of d_∂ together with Proposition 6.6 yields the claim. $\square$

Proof of Theorem 6.8. (Sketch) Expand the series $p_\partial = \sum_{n\geqslant 0}\widehat{p}(-\partial_{\mathrm{red}}\widehat{K})^n$ and the exponential $e^{\mathring{i}S_x^P/\hbar} = \sum_k \frac{1}{k!}(\frac{\mathring{i}}{\hbar})^k(S_x^P)^k$, and compare with the diagrammatic expansion (63) in four steps.

1. Blocks. The operator $\widehat{K}$ selects by (70) one leg $\bar{\boldsymbol{z}}^i(t)$, bites off its de Rham degree one component and propagates it, $K\bar{\boldsymbol{z}}^i(t) = \int_{S^1}\omega_{S^1}(t,t')\bar{z}_1^i(t')\mathrm{d}t'$, projecting the legs to its left by ιp. The operator Δ_{fib} contracts two legs at coincident points with the kernel $\Pi^{ij}(x)$ of (66); contractions within a single vertex vanish by $\psi(t) = 0$ as in Section 5.3, and by the counting of Proposition 6.6 one of the two legs must be a K-leg. The two operators compose to an edge: on a pair of legs, for instance,

$$-\partial_{\mathrm{red}}\widehat{K}\,(\bar{\boldsymbol{p}}_i(t)\,\bar{\boldsymbol{x}}^j(t')) = \mathring{i}\hbar\,\Delta_{\mathrm{fib}}\widehat{K}\,(\bar{\boldsymbol{p}}_i(t)\,\bar{\boldsymbol{x}}^j(t')) + \cdots = \mathring{i}\hbar\,\delta_i^j\,\omega_{S^1}(t',t) + \ldots,$$

the inverse of the kinetic operator of $\bar{\boldsymbol{p}}_i\mathrm{d}\bar{\boldsymbol{x}}^i \subset S_x^0$, and similarly in the other sectors of (66): an edge of (64), weighted by $\mathring{i}\hbar$. The operator ad_{S_X} contracts one leg of its argument with one leg of S_X through the contact kernel $\Pi^{ij}(x)\delta(t-t')$ and multiplies with the remaining vertex $\partial S_X/\partial\bar{\boldsymbol{z}}$; since by the counting it must consume a K-leg, it inserts the vertex S_X on an edge, reproducing the connection vertex of rule (3) in Section 5.2. Finally $\widehat{p}$ projects all remaining legs to the harmonics: the leaves.

2. Powers of $\hbar$. Every vertex carries $\frac{\mathring{i}}{\hbar}$ from the exponential, every edge carries $\mathring{i}\hbar$ from its Δ_{fib}; a connected diagram with V vertices and E edges therefore carries $(\mathring{i}\hbar)^{E-V} = (\mathring{i}\hbar)^{l(\Gamma)}(\mathring{i}\hbar)^{-1}$, matching (63) after extracting the overall $\frac{\mathring{i}}{\hbar}$ of $e^{\mathring{i}S_{\mathrm{eff}}/\hbar}$. Disconnected diagrams exponentiate, since $\widehat{p}$ and ιp are algebra morphisms and the contractions act by the Leibniz rule; taking the logarithm leaves the connected diagrams.

3. Symmetry factors. At order n the series performs n contractions in every possible order, each exactly once; the symmetrization $\frac{1}{n!}\sum_\sigma$ of (70) together with the $\frac{1}{k!}$ of the exponential assigns weight one to every unordered configuration of edges and vertices, leaving $|\mathrm{Aut}(\Gamma)|^{-1}$ per isomorphism class of diagrams.

$\square$

We note that just as we have observed in the last chapter that the tree graphs vanish under Proposition 5.9, here, for the exact same reason, they do as well. Note also that S_X cannot contribute a corolla: it enters only through the operators ad_{S_X} of p_∂, which always contract one of its legs. Therefore the $\hbar^0$ contribution to S_{eff} is simply given by $S_x^P|_{\mathsf{H}}$.

Remark 6.11. The density of Remark 3.32 reads here formally $\mu_0 = D^n\bar{\boldsymbol{z}}(t)$. The transfer data $\widehat{p}, \widehat{\iota}, \widehat{K}$ do not touch the Berezinian, however D does. Vector field acts on half-densities by the Lie derivative,

$$\mathcal{L}_V(f\sqrt{\mu_0}) = (Vf)\sqrt{\mu_0} + \tfrac{1}{2}(\mathrm{div}_{\mu_0}V)f\sqrt{\mu_0},$$

and we claim that formally $\operatorname{div}_{\mu_0} V = 0$.

The shift δ has constant coefficients, so its divergence vanishes. The Hamiltonian part satisfies $\operatorname{div}_{\mu_0} X = \pm 2\Delta_{\mathrm{fib}} S_X = 0$. For the linear part Γ, note that for a formal vector field of the usual form $V = \int_\Sigma V^i \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}$ we have that the divergence reads

$$\operatorname{div}_{\mu_0} V = \int_\Sigma (-1)^{p_i} \frac{\partial V^i(t,\theta)}{\partial \bar{\boldsymbol{z}}^i(t',\theta')}\Big|_{(t',\theta')=(t,\theta)}.$$

Hence for $\Gamma = -\int_\Sigma \mathrm{d}x^\ell\, \bar{\boldsymbol{z}}^k\, \Gamma^i_{k\ell}(x) \frac{\partial}{\partial \bar{\boldsymbol{z}}^i}$ we obtain

$$\operatorname{div}_{\mu_0} \Gamma = -\mathrm{d}x^\ell \sum_i (-1)^{p_i}\, \Gamma^i_{i\ell}(x) \int_\Sigma \delta(t-t')(\theta-\theta')\Big|_{(t',\theta')=(t,\theta)} = \int_{S^1} \psi = 0$$

Hence D preserves μ_0.

When passing to cohomology we obtain

$$\mu_0^{\mathsf{H}} = D(\bar{z}_0, \bar{z}_1) = \prod_i D\bar{z}_0^i\, D\bar{z}_1^i,$$

the density behind $\Delta^{\mathsf{H}}_{\mathrm{fib}}$ of Lemma 6.12. Every target direction appears twice, as $\bar{z}_0^i$ of parity p_i and as $\bar{z}_1^i$ of parity p_i+1, so the lift $V_A = \bar{z}_0^k A^i_k \frac{\partial}{\partial \bar{z}_0^i} + \bar{z}_1^k A^i_k \frac{\partial}{\partial \bar{z}_1^i}$ of any linear operator A satisfies

$$\operatorname{div}_{\mu_0^{\mathsf{H}}} V_A = \sum_i (-1)^{p_i} A^i_i + \sum_i (-1)^{p_i+1} A^i_i = 0,$$

exactly and without regularization. In particular all vector field parts of D^{H} are μ_0^{H}-divergence free: the shift and Hamiltonian parts as before, and $\operatorname{div}_{\mu_0^{\mathsf{H}}}(\Gamma|_{\mathsf{H}}) = 0$ now exactly.

6.3 Globalization

The rest term R is the only deviation from the modified QME of Section 5.4. In this section we remove it along the acyclic directions of D^{H} and transfer the corrected equation to the global complex. Write the harmonic superfields in components,

$$\bar{\boldsymbol{z}}^i|_{\mathsf{H}} = \bar{z}_0^i + \bar{z}_1^i \mathrm{d}t,$$

so that $\widehat{S}\mathsf{H}^*_x$ is generated by the zero modes $\bar{z}_0^i$ of parity p_i and their conjugate modes $\bar{z}_1^i$ of parity p_i+1. We first record that $\Delta^{\mathsf{H}}_{\mathrm{fib}}$ is an honest BV Laplacian on these generators.

Lemma 6.12. On $\widehat{S}\mathsf{H}^*$ we have $\Delta_{\mathrm{fib}}\widehat{\iota} = \widehat{\iota}\,\Delta^{\mathsf{H}}_{\mathrm{fib}}$ with

$$\Delta^{\mathsf{H}}_{\mathrm{fib}} = \sum_{i,j} (-1)^{p_i(p_j+1)}\, \Pi^{ij}(x) \frac{\partial}{\partial \bar{z}_0^i} \frac{\partial}{\partial \bar{z}_1^j}.$$

It squares to zero, generates the zero-mode antibracket $(-,-)^{\mathsf{H}}_{\mathrm{fib}}$, with pairings $(\bar{z}_0^i, \bar{z}_1^j)^{\mathsf{H}}_{\mathrm{fib}} = (-1)^{p_i}\Pi^{ij}(x)$ on the generators, and it commutes with the connection,

$$[D^{\mathsf{H}}, \Delta^{\mathsf{H}}_{\mathrm{fib}}] = 0.$$

Proof. With $\frac{\partial}{\partial \bar{\boldsymbol{z}}^k(t')} = \frac{\delta}{\delta \bar{z}_1^k(t')} - \theta' \frac{\delta}{\delta \bar{z}_0^k(t')}$ as in Lemma 6.2, the mode functionals have the derivative kernels

$$\frac{\partial}{\partial \bar{\boldsymbol{z}}^k(t')}\, \widehat{\iota}(\bar{z}_0^i) = -\delta^i_k\, \theta', \qquad \frac{\partial}{\partial \bar{\boldsymbol{z}}^k(t')}\, \widehat{\iota}(\bar{z}_1^i) = \delta^i_k.$$

Inserting these into (30), the Berezin integration leaves

$$(\widehat{\iota}\bar{z}_0^i, \widehat{\iota}\bar{z}_1^j)_{\mathrm{fib}} = (-1)^{p_i}\Pi^{ij}(x), \qquad (\widehat{\iota}\bar{z}_0^i, \widehat{\iota}\bar{z}_0^j)_{\mathrm{fib}} = (\widehat{\iota}\bar{z}_1^i, \widehat{\iota}\bar{z}_1^j)_{\mathrm{fib}} = 0,$$

the latter two by $\theta'^2 = 0$ and by the absence of θ' respectively. Since Δ_{fib} is second order, kills linear functionals, and $\widehat{\iota}$ is an algebra morphism, $\Delta_{\mathrm{fib}}\widehat{\iota}$ is determined by these pairings through the generating property of Lemma 3.30 and agrees with $\widehat{\iota}\Delta^{\mathsf{H}}_{\mathrm{fib}}$ for the displayed operator, whose Koszul factor is fixed by the same requirement as in (36). In particular $\widehat{p}\Delta_{\mathrm{fib}}\widehat{\iota} = \Delta^{\mathsf{H}}_{\mathrm{fib}}$, and $(\Delta^{\mathsf{H}}_{\mathrm{fib}})^2 = 0$ as in Lemma 3.30.

For the commutator take $D^{\mathsf{H}} = \mathrm{d}_x - \delta|_{\mathsf{H}} + \Gamma|_{\mathsf{H}} + \mathrm{ad}_{S_X|_{\mathsf{H}}}$ from Proposition 6.7. The part $\delta|_{\mathsf{H}} = \mathrm{d}x^m\partial/\partial\bar{x}^m_0$ commutes, since all coefficients in the computation are constant on the fiber: the commutator reduces to that of the bare partial derivatives, which graded commute. For $\nabla|_{\mathsf{H}} = \mathrm{d}_x + \Gamma|_{\mathsf{H}}$ the computation of Lemma 3.31 applies similarly and gives $[\Delta^{\mathsf{H}}_{\mathrm{fib}}, \nabla|_{\mathsf{H}}] = 0$ by (33). For the Hamiltonian part, the identity (†) in the proof of Lemma 3.31 uses only that the Laplacian generates the bracket, so $[\Delta^{\mathsf{H}}_{\mathrm{fib}}, \mathrm{ad}_{S_X|_{\mathsf{H}}}] = \mathrm{ad}_{\Delta^{\mathsf{H}}_{\mathrm{fib}}(S_X|_{\mathsf{H}})}$. The Hamiltonian S_X has the form

$$S_X = \int_\Sigma \mathrm{d}x^\ell\, r_\ell(x, \bar{\boldsymbol{z}}).$$

Its restriction to the zero modes carries exactly one $\bar{z}_1$-leg due to the Berezin integral; splitting this from the rest of the power series yields $\mathrm{d}x^\ell \bar{z}^k_1 \mathrm{d}t \frac{\partial r_\ell}{\partial \bar{z}^k_0}(x, \bar{z}_0)$, and swapping the order of the $\mathrm{d}t$ to the left, graded commuting with the field $\bar{z}^k_1$ and $\mathrm{d}x^\ell$, to integrate, we obtain

$$S_X|_{\mathsf{H}} = (-1)^{(p_\ell+1)+(p_k+1)}\mathrm{d}x^\ell\, \bar{z}^k_1\, \frac{\partial r_\ell}{\partial \bar{z}^k_0}(x, \bar{z}_0).$$

Acting with $\Delta^{\mathsf{H}}_{\mathrm{fib}}$ we get

$$\begin{aligned}\Delta^{\mathsf{H}}_{\mathrm{fib}}(S_X|_{\mathsf{H}}) = \sum_{k,j} &(-1)^{p_j(p_k+1)}(-1)^{(p_k+1)(p_\ell+1)}(-1)^{p_j(p_\ell+1)}(-1)^{(p_\ell+1)+(p_k+1)}\times \\ &\times \mathrm{d}x^\ell\, \Pi^{jk}(x)\, \frac{\partial^2 r_\ell}{\partial \bar{z}^j_0 \partial \bar{z}^k_0}(x, \bar{z}_0).\end{aligned}$$

Now because $\omega_{\mathcal{N}}$ is even, $\Pi^{ij}(x)$ satisfies the conditions of Lemma 1.29. In particular, from (1) we have $p_k = p_j$ and hence we obtain, ignoring the p_ℓ contributions which do not matter, the total factor of $(-1)^{(p_k+1)(p_j+1)}$, symmetric in k and j. Using now $\Pi^{ji}(x) = -(-1)^{p_ip_j}\Pi^{ij}(x)$ by Lemma 1.29 (2), while the Hessian of r_ℓ is graded symmetric, the contraction cancels in pairs, the odd diagonal terms dying by $\partial^2 = 0$. □

The connection D^{H} carries its own Grothendieck SDR. Note that its leading term contracts only part of the fiber: $\delta|_{\mathsf{H}} = \mathrm{d}x^m\partial/\partial\bar{x}^m_0$ with m running over the directions of M. Hence when we speak of *global* we mean *global in x* as the following Proposition makes precise.

Proposition 6.13. In analogy to Remark 5.1, let $\widetilde{\mathsf{H}}_x \subset \mathsf{H}_x$ be the complement of the $H^0(S^1) \otimes T_xM$ part, i.e. spanned by the zero modes of all superfields but $\bar{\boldsymbol{x}}$ together with all the $\bar{z}^i_1$; the sections of $\widehat{S}\widetilde{\mathsf{H}}^*$ are then exactly those not involving the $\bar{x}^m_0$. With the Hodge pair $(\mathrm{d}x^m, \bar{x}^m_0)$ in place of $(\mathrm{d}z^i, y^i)$, Proposition 3.18 applies verbatim and yields the strong deformation retract

$$(\Gamma(\widehat{S}\widetilde{\mathsf{H}}^*), 0) \underset{\pi_G}{\overset{\iota_G}{\rightleftarrows}} (\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*), D^{\mathsf{H}}) \circlearrowleft h_G,$$

where π_G evaluates at $\mathrm{d}x = \bar{x}^m_0 = 0$ and ι_G is the formal exponential map of D^{H}. In particular $H^0_{D^{\mathsf{H}}} = \mathrm{Im}\,\iota_G \cong \Gamma(\widehat{S}\widetilde{\mathsf{H}}^*)$ and $H^{\geqslant 1}_{D^{\mathsf{H}}} = 0$.

Proof. Lemma 3.16 holds verbatim for $\delta|_{\mathsf{H}}$ with $\delta^*_{\mathsf{H}} = \bar{x}^m_0\iota_{\partial_{x^m}}$, the counting operator now being $\deg_{\mathrm{d}x} + \deg_{\bar{x}^m_0}$; the perturbative argument of Proposition 3.18 then goes through unchanged, the recursion raising the degree of $\bar{\boldsymbol{z}}$ at every step. □

Corollary 6.14. Since $[D^{\mathsf{H}}, \Delta^{\mathsf{H}}_{\mathrm{fib}}] = 0$, perturbing along $-\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}$ as in Lemma 3.33 yields the strong deformation retract

$$(\Gamma(\widehat{S}\widetilde{\mathsf{H}}^*)[[\hbar]], -\mathring{\imath}\hbar\Delta_{\widetilde{\mathsf{H}}}) \underset{\pi_\Delta}{\overset{\iota_G}{\rightleftarrows}} (\Gamma(\Lambda T^*M \otimes \widehat{S}\mathsf{H}^*)[[\hbar]],\ D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}) \circlearrowleft h_\Delta$$

with unchanged inclusion, $\pi_\Delta = \sum_{n\geqslant 0} \pi_G(\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}} h_G)^n$, $h_\Delta = \sum_{n\geqslant 0} h_G(\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}} h_G)^n$, and the induced global BV Laplacian $\Delta_{\widetilde{\mathsf{H}}} = \pi_G \Delta^{\mathsf{H}}_{\mathrm{fib}} \iota_G$. In particular

$$\Delta^{\mathsf{H}}_{\mathrm{fib}}\, \iota_G = \iota_G\, \Delta_{\widetilde{\mathsf{H}}}.$$

Proof. The proof of Lemma 3.33 applies here as well: h_G lowers the form degree while $\Delta^{\mathsf{H}}_{\mathrm{fib}}$ preserves it, so $h_G\Delta^{\mathsf{H}}_{\mathrm{fib}}\iota_G = 0$ and the inclusion does not deform. The last identity is the form degree zero part of the chain map property $(D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}})\iota_G = \iota_G(-\mathring{\imath}\hbar\Delta_{\widetilde{\mathsf{H}}})$. □

Abbreviate for the rest of the section

$$A := D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}, \qquad E := e^{\mathring{\imath}S_{\mathrm{eff}}/\hbar}, \qquad F := R\,E,$$

so that $A^2 = 0$ by Lemma 6.12, Corollary 6.10 reads $AE = -F$, and consequently

$$AF = -A^2E = 0.$$

The components of F sit in form degrees $2 \leqslant p \leqslant m := \dim M$. Indeed, a term of R without any ad_{S_X} ends in $\Delta_{\mathrm{fib}}\widehat{\imath} = \widehat{\imath}\Delta^{\mathsf{H}}_{\mathrm{fib}}$ and dies by $\widehat{K}\widehat{\imath} = 0$. In a term with a single ad_{S_X}, first note that ad_{S_X} has to sit at the rightmost position, otherwise the argument with $\Delta_{\mathrm{fib}}\widehat{\imath} = \widehat{\imath}\Delta^{\mathsf{H}}_{\mathrm{fib}}$ applies here as well and kills the contribution. Hence contract that ad_{S_X} with the fields being fed in. All legs except those of S_X are harmonic due to the beginning $\widehat{\imath}$ and as such all the coming K contractions must be made on S_X, as they die on harmonic fields. Now the only surviving Δ_{fib} contractions to be made are between harmonic and harmonic fields; these however do not consume a K-leg, and as such there will be a free K leg once $\widehat{p}$ is applied and hence the contribution dies. This is not the case if we instead have two ad_{S_X} as in that case the Δ_{fib} can contract these two together.

The defect F can now be pushed down the form degrees via a Zig-Zag argument.

Proposition 6.15. Set $c^{(p)} := 0$ for $p \geqslant m$ and define, descending in the form degree,

$$c^{(p-1)} := h_G(F^{(p)} + \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}} c^{(p)}), \qquad p = m, m-1, \ldots, 1.$$

Then $\sigma := E + c$ with $c := \sum_{p=0}^{m-1} c^{(p)} = \mathcal{O}(\hbar)$ satisfies

$$A\sigma = -\mathring{\imath}\hbar\,\iota_G(w), \qquad w := \pi_G(\Delta^{\mathsf{H}}_{\mathrm{fib}} c^{(0)}) \in \Gamma(\widehat{S}\widetilde{\mathsf{H}}^*)[[\hbar]], \qquad \Delta_{\widetilde{\mathsf{H}}}\, w = 0.$$

Proof. In form degree $p \geqslant 1$ we need $(A\sigma)^{(p)} = D^{\mathsf{H}}c^{(p-1)} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(p)} - F^{(p)} = 0$, i.e. that h_G in the recursion is a genuine inverse of D^{H}. The SDR gives

$$D^{\mathsf{H}}c^{(p-1)} = (\mathrm{Id} - \iota_G\pi_G - h_G D^{\mathsf{H}})(F^{(p)} + \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(p)}),$$

π_G annihilates positive form degrees, and the last term vanishes by descending induction:

$$D^{\mathsf{H}}(F^{(p)} + \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(p)}) = D^{\mathsf{H}}F^{(p)} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}D^{\mathsf{H}}c^{(p)} = D^{\mathsf{H}}F^{(p)} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}F^{(p+1)} = (AF)^{(p+1)} = 0,$$

using $[D^{\mathsf{H}}, \Delta^{\mathsf{H}}_{\mathrm{fib}}] = 0$, $(\Delta^{\mathsf{H}}_{\mathrm{fib}})^2 = 0$, the recursion at $p+1$ and $AF = 0$.

In degree zero only $(A\sigma)^{(0)} = -\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(0)}$ survives. It is D^{H}-closed,

$$D^{\mathsf{H}}\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(0)} = -\Delta^{\mathsf{H}}_{\mathrm{fib}}D^{\mathsf{H}}c^{(0)} = -\Delta^{\mathsf{H}}_{\mathrm{fib}}(F^{(1)} + \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\mathrm{fib}}c^{(1)}) = -\Delta^{\mathsf{H}}_{\mathrm{fib}}F^{(1)} = 0,$$

the last equality being the degree one component of $AF = 0$ together with $F^{(0)} = 0$. A D^{H}-closed element of form degree zero lies in $\mathrm{Im}\,\iota_G$: apply the retract relation $y = \iota_G\pi_G y +$

$D^{\mathsf{H}}h_G y + h_G D^{\mathsf{H}} y$ and note that h_G vanishes on form degree zero. Hence $\Delta^{\mathsf{H}}_{\text{fib}} c^{(0)} = \iota_G(w)$ with $w = \pi_G(\Delta^{\mathsf{H}}_{\text{fib}} c^{(0)})$, and

$$\iota_G(\Delta_{\widetilde{\mathsf{H}}} w) = \Delta^{\mathsf{H}}_{\text{fib}}\, \iota_G(w) = (\Delta^{\mathsf{H}}_{\text{fib}})^2 c^{(0)} = 0$$

gives $\Delta_{\widetilde{\mathsf{H}}} w = 0$, ι_G being injective. □

Corollary 6.16. Assume the class of w in the $\Delta_{\widetilde{\mathsf{H}}}$-cohomology vanishes[4], $w = \Delta_{\widetilde{\mathsf{H}}} u$. Then $\widetilde{\sigma} := \sigma - \iota_G(u)$ solves the modified QME on the zero modes exactly,

$$(D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\text{fib}})\, \widetilde{\sigma} = 0.$$

Proof. By Corollary 6.14 and $D^{\mathsf{H}}\iota_G = 0$ we have $A\iota_G(u) = -\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\text{fib}}\iota_G(u) = -\mathring{\imath}\hbar\iota_G(\Delta_{\widetilde{\mathsf{H}}} u) = -\mathring{\imath}\hbar\iota_G(w)$, so $A\widetilde{\sigma} = A\sigma + \mathring{\imath}\hbar\iota_G(w) = 0$ by Proposition 6.15. □

The $\widetilde{\sigma} := e^{\mathring{\imath}\widetilde{S_{\text{eff}}}/\hbar}$ we have constructed is the analogous $e^{\mathring{\imath}S'_{\text{eff}}/\hbar}$, with $S'_{\text{eff}} = S_{\text{eff}} - S_Y|_{\mathsf{H}}$ that we have obtained in the last chapter for free. To globalize it to a global $S^{\text{glob}}_{\text{eff}}$ we can proceed similarly, by first writing out the modified QME on the level of $\widetilde{S_{\text{eff}}}$, given by

$$(D^{\mathsf{H}} - \mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\text{fib}})\widetilde{S_{\text{eff}}} + \frac{1}{2}(\widetilde{S_{\text{eff}}}, \widetilde{S_{\text{eff}}})^{\mathsf{H}}_{\text{fib}} = 0,$$

and then using BV canonical transformations or L_∞ transfer to obtain a global (in M) effective action. These apply unchanged because $\widetilde{S_{\text{eff}}}$ carries the same $\hbar$-structure as in Section 5.4. Indeed, both corrections are of order $\hbar$ relative to E. In Proposition 6.15 we have seen that c is $\mathcal{O}(\hbar)$, while $\iota_G(u)$ is even of order $\hbar^2$: $c^{(1)}$ is of order $\hbar$, hence by the recursive formula $c^{(0)} = h_G\mathring{\imath}\hbar\Delta^{\mathsf{H}}_{\text{fib}}c^{(1)}$ (using $F^{(1)} = 0$) is $\mathcal{O}(\hbar^2)$, and the same follows for w and u. Together with the observation after Theorem 6.8, that at order $\hbar^0$ only the corollas $S^P_x|_{\mathsf{H}}$ survive, we obtain

$$\widetilde{S_{\text{eff}}}^{(0)} = S^P_x|_{\mathsf{H}} + \mathcal{O}(\hbar), \qquad \widetilde{S_{\text{eff}}}^{(p)} = \mathcal{O}(\hbar), \quad p \geqslant 1 :$$

all classical data sits in form degree zero. This is precisely the situation of Section 5.4 (compare Remark 5.12, with the degree one slot now even empty at order $\hbar^0$), and the inductive collapse as well as the L_∞ pushforward start at order $\hbar$ as before.

[4]The author believes that this is indeed the case for the spinning particle; in this case the proof would be shown in a future publication.

7 Čech globalization

In the following we shall see another technique to globalize effective actions. Before, we used formal geometry to bring the effective action into a space where we have control over the global objects via the Grothendieck connection, whose closed sections in degree zero were precisely the global functions. Here we do the analogous thing with Čech machinery.

Concretely, we work with a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of a (super)manifold $\mathcal{M}$ and place the data (such as solutions of the master equation) on the Thom–Whitney totalization of its Čech nerve, which carries a DGLA structure. We then again make use of the HPL and transfer theorems of Section 2 to globalize the data back to the global complex, as we have done for formal geometry.

7.1 Totalizations

The following totalization constructions are mainly taken from [19].

7.1.1 The Čech totalization

We will make extensive use of simplicial sets in the following sections, motivating a simplicial approach to the Čech machinery.

Definition 7.1. We denote by Δ the category with

- objects: totally ordered sets $[k] = (0 < ... < k)$ for $k \in \mathbb{N}$,
- morphisms: $f : [k] \to [\ell]$ such that $i \leqslant j \Longrightarrow f(i) \leqslant f(j)$, for $i, j \in [k]$,

In particular the morphisms are generated by the usual face maps $d_i : [k] \to [k+1]$ which skip i and the degeneracy maps $s_i : [k] \to [k-1]$ which send both $s_i(i) = s_i(i+1) = i \in [k-1]$.

Definition 7.2 (Simplicial set, [21])**.** A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$. Concretely, it consists of sets X_k for $k \geqslant 0$ together with face maps $d_i : X_k \to X_{k-1}$ and degeneracy maps $s_i : X_k \to X_{k+1}$, for $0 \leqslant i \leqslant k$, satisfying the *simplicial identities*

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i, && i < j,\\
s_i s_j &= s_{j+1} s_i, && i \leqslant j,\\
d_i s_j &= s_{j-1} d_i, && i < j,\\
d_i s_j &= \mathrm{Id}, && i = j \text{ or } i = j+1,\\
d_i s_j &= s_j d_{i-1}, && i > j+1.
\end{aligned}$$

The category of simplicial sets is denoted **sSet**.

Definition 7.3 (Simplicial supermanifold)**.** A *simplicial supermanifold* is a functor from Δ^{op} to the category of graded supermanifolds. Denote the category of simplicial supermanifolds as **sSmfd**.

The k-th space $N_k\mathcal{U}$ of the nerve of a cover $\mathcal{U}$ is given by the disjoint union

$$N_k\mathcal{U} = \bigsqcup_{(\alpha_0\ldots\alpha_k)\in I^{k+1}_{\leqslant}} U_{\alpha_0\ldots\alpha_k}, \quad \text{where} \quad U_{\alpha_0\ldots\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$$

and $I^{k+1}_{\leqslant} = \{(\alpha_0, \ldots, \alpha_k) \in I^{k+1} \mid \alpha_0 \leqslant \cdots \leqslant \alpha_k\}$ denotes the non-decreasing tuples (repeated indices allowed). The Čech nerve $N_\bullet\mathcal{U}$ is precisely such a functor corresponding to a simplicial supermanifold, since deleting an index (d_i) and repeating one (s_i) both preserve the non-decreasing order. The face maps d_i and s_i correspond to the inclusion $d_i : N_{k+1}\mathcal{U} \to N_k\mathcal{U}$

$$N_{k+1}\mathcal{U} \supset U_{\alpha_0\ldots\alpha_{k+1}} \mapsto U_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{k+1}} \subset N_k\mathcal{U}$$

and the identification $s_i : N_{k-1}\mathcal{U} \to N_k\mathcal{U}$

$$N_{k-1}\mathcal{U} \supset U_{\alpha_0\ldots\alpha_{k-1}} \mapsto U_{\alpha_0\ldots\alpha_i\alpha_i\ldots\alpha_k} \subset N_k\mathcal{U},$$

respectively.

Definition 7.4. A *cosimplicial object* of a category $\mathcal{C}$ is a covariant functor $X^\bullet : \Delta \to \mathcal{C}$.

The field space functor $\mathcal{F}$ can be seen as a contravariant functor from the category of graded supermanifolds to the category of graded Lie superalgebras (with the antibracket). Given that $N_\bullet\mathcal{U}$ is also a contravariant functor, its composition is a covariant functor and we therefore obtain the cosimplicial object

$$V^k := \mathcal{F}(N_k\mathcal{U}) = \prod_{(\alpha_0,\ldots,\alpha_k)\in I_{\leqslant}^{k+1}} \mathcal{F}\left(U_{\alpha_0\ldots\alpha_k}\right) \tag{72}$$

of the category of graded Lie superalgebras. An element is a tuple ν consisting of data $\nu_{\alpha_0\ldots\alpha_k}$ for $(\alpha_0,\ldots,\alpha_k) \in I_{\leqslant}^{k+1}$. It has the induced coface maps $d^i : V^{k-1} \to V^k$

$$(d^i\nu)_{\alpha_0\ldots\alpha_k} = \nu_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_k}|_{U_{\alpha_0\ldots\alpha_k}},$$

where the hatted index is omitted, and codegeneracy maps $s^i : V^k \to V^{k-1}$

$$\left(s^i\nu\right)_{\alpha_0\ldots\alpha_{k-1}} = \nu_{\alpha_0\ldots\alpha_i\alpha_i\ldots\alpha_{k-1}}.$$

To eliminate the double indices, we normalize the cosimplicial object by taking

$$N^k(V) = \bigcap_{i=0}^{k-1} \operatorname{Ker}\left(s^i : V^k \to V^{k-1}\right).$$

In our setting this just amounts to imposing that $\nu_{\alpha_0\ldots\alpha_k} = 0$ whenever $\alpha_i = \alpha_{i+1}$ for some i.

We can now construct the usual Čech complex with differential

$$\check{\delta} = \sum_{i=0}^{k+1}(-1)^i d^i : N^k(V) \to N^{k+1}(V)$$

If V^k is itself a complex with differential $\mathrm{d} : V^{j,k} \to V^{j+1,k}$ descending to a differential $\mathrm{d} : N^k\left(V^j\right) \to N^k\left(V^{j+1}\right)$ we define the totalization of the double complex $V^{\bullet\bullet}$ as the graded superspace

$$|V|^n = \prod_{k=0}^{\infty} N^k\left(V^{n-k}\right)$$

with differential $\mathrm{d}_{\mathrm{tot}} = \check{\delta} + (-1)^k\mathrm{d}$.

While this complex is itself a genuine cochain complex (with the differential $\check{\delta} + (-1)^k\mathrm{d}$ above), it is not a DGLA: the levelwise antibrackets do not assemble into a single strict bracket on the totalization, since combining them across Čech degrees needs a graded-commutative product on cochains and the natural (Alexander–Whitney cup) product is not graded-commutative. The Thom–Whitney totalization supplies such a product (via $\Omega(\Delta^\bullet)$), assembling the levelwise antibrackets into a single antibracket (75) on the totalization and making $\|V\|$ a genuine DGLA.

7.1.2 The Thom–Whitney totalization

Define the polynomial, free commutative algebra of differential forms on a simplex, denoted by $\Omega_k = \Omega(\Delta^k)$. It has generators $\{t_i\}_{0\leqslant i\leqslant k}$ of degree 0 satisfying $\sum_{i=0}^k t_i = 1$ and the generators $\{\mathrm{d}t_i\}_{0\leqslant i\leqslant k}$ of degree 1 satisfying $\sum_{i=0}^k \mathrm{d}t_i = 0$. $\Omega^\bullet$ is a functor from Δ^{op} to the category of differential graded commutative algebras and the morphisms $f : [k] \to [\ell]$ act on Ω_k as

$$f^* t_i = \sum_{f(j)=i} t_j, \quad 0 \leqslant i \leqslant \ell.$$

For the Thom–Whitney totalization we take the simplicial $\Omega_\bullet$ with maps $f^* : \Omega_\ell \to \Omega_k$ and a cosimplicial superspace $V^\bullet$ with maps $f_* : V^k \to V^\ell$. From the big product

$$\prod_{k\geqslant 0} \Omega_k \otimes V^k$$

with elements given by a tuple $\omega = (\omega_0, \omega_1, ...)$ where $\omega_k \in \Omega_k \otimes V^k$, we take the subspace

$$N_{\mathrm{TW}}(V) = \left\{\omega \mid (f^* \otimes 1)\, \omega_\ell = (1 \otimes f_*)\, \omega_k \text{ in } \Omega_k \otimes V^\ell, \text{ for every } f : [k] \to [\ell] \text{ in } \Delta\right\}.$$

This is nothing but imposing simplicial compatibility in a tuple, i.e. that the simplicial operations match between the layers.

Each of the two factors carries a grading: the differential forms Ω_k^p by the form degree p (with de Rham differential δ) and the graded superspace $V^{j,k}$ with differential d. Denote by $N_{\mathrm{TW}}^p(V)$ the same subspace as above with the additional condition that $\omega_i \in \Omega^p$ $\forall i$. We then define the Thom–Whitney totalization of $V^{\bullet\bullet}$

$$\|V\|^n = \prod_{p=0}^{\infty} N_{\mathrm{TW}}^p\left(V^{n-p}\right),$$

with total differential $d_{\mathrm{TW}} = \delta + \mathrm{d}$, the Koszul sign $(-1)^p$ being exhibited concretely in (74) below. For our purposes, we will always take the same V as in (72).

The advantage of this totalization is that $\|V\|$ can now be equipped with a differential and an antibracket to form a DGLA. On $\Omega_k \otimes V^k$ with $\beta, \beta_1, \beta_2 \in \Omega_k^p$ and $v, v_1, v_2 \in V^{jk}$ define the product

$$(\beta_1 \otimes v_1)\,(\beta_2 \otimes v_2) = (-1)^{p_2|v_1|}\beta_1\beta_2 v_1 v_2, \tag{73}$$

the differential

$$D(\beta \otimes v) = \delta\beta \otimes v + (-1)^p \beta \otimes \mathrm{d}v, \tag{74}$$

and the bracket

$$(\beta_1 \otimes v_1, \beta_2 \otimes v_2) = (-1)^{p_2(|v_1|+1)}\beta_1\beta_2\,(v_1, v_2)\,. \tag{75}$$

Given that this totalization has the structure of a DGLA, we can consider its MC set (Def. 2.3), given by the elements $S \in \|V\|$ whose components $S^p_{\alpha_0\ldots\alpha_k} \in \Omega^p_k \otimes \mathcal{F}^{-p}(U_{\alpha_0\ldots\alpha_k})$ satisfy

$$\delta S^{p-1}_{\alpha_0\ldots\alpha_k} + \mathrm{d}S^p_{\alpha_0\ldots\alpha_k} + \frac{1}{2}\sum_{i=0}^{p}(S^i_{\alpha_0\ldots\alpha_k}, S^{p-i}_{\alpha_0\ldots\alpha_k}) = 0$$

on all intersections for all k.

We will soon see that the two complexes are quasi-isomorphic through an SDR and hence contain the same information for our purposes.

Remark 7.5 (Functoriality under refinement)**.** The TW totalization is functorial under refinement of covers. If $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ is a refinement of $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ via a map $\phi : J \to I$ with

$V_\beta \subset U_{\phi(\beta)}$, then restriction of fields gives a morphism of cosimplicial DGLAs $\mathcal{F}(N_\bullet\mathcal{U}) \to \mathcal{F}(N_\bullet\mathcal{V})$, which in turn induces a DGLA quasi-isomorphism $\|\mathcal{F}(N_\bullet\mathcal{U})\| \to \|\mathcal{F}(N_\bullet\mathcal{V})\|$, see [19] for details.

Remark 7.6. From now on we take $\Omega_\bullet$ to consist of all smooth forms on the simplices, not just the polynomial ones. The constructions that follow are carried out in the literature for the polynomial (Sullivan) forms, but they work just as well in the general smooth case.

7.2 BV in simplicial language

We will give some results of the BV theory we have established in the first chapter in simplicial language. Before we do that, we must introduce some more simplicial machinery.

7.2.1 Simplicial homotopy theory

We give some important definitions and theorems in simplicial homotopy theory, mostly taken from [21], where one can also find the details and proofs. Central are the definitions of the simplicial category $\boldsymbol{\Delta}$ (Def. 7.1) and the definition of a simplicial set (Def. 7.2).

Example 7.7. Define Δ^n as the object of the category **sSet** by

$$\Delta^n = \mathrm{Hom}_\Delta(-,[n]),$$

i.e. it is the functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ represented by $[n]$. We then denote Δ^n_k to be the set $\mathrm{Hom}_\Delta([k],[n])$ and Id_n to be the natural identity of the set Δ^n_n.

Definition 7.8 (Simplicial maps)**.** Let X, Y be objects of **sSet**. A natural transformation $\varphi : X \to Y$ (between functors $\Delta^{\mathrm{op}} \to \mathbf{Set}$) is called a *simplicial map*. These are the morphisms of the category **sSet**. It consists of a family of maps $\varphi_k : X_k \to Y_k$ satisfying the usual commutative diagram.

Let Y be any simplicial set. Yoneda's Lemma gives a natural bijection between the simplicial maps $\Delta^n \to Y$ and the n-simplices of Y

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y) \cong Y_n$$

The bijection is given by identifying a simplicial map $\varphi : \Delta^n \to Y$ with the n-simplex $\varphi_n(\mathrm{Id}_n) = x \in Y_n$.

Definition 7.9 (Boundary)**.** The boundary of Δ^n is the simplicial subset $\partial\Delta^n$ given by

$$\partial\Delta^n_k = \{f \in \mathrm{Hom}_\Delta([k],[n]) |\ f \text{ is not surjective}\}.$$

The idea behind this definition is to keep all the elements of the set $\mathrm{Hom}_\Delta([k],[n])$ except $\mathrm{Id}_n \in \Delta^n_n$.

Definition 7.10 (Horn)**.** The k-th horn of Δ^n, denoted Λ^n_k, is the simplicial subset whose set of j-simplices is

$$(\Lambda^n_k)_j = \{f \in (\partial\Delta^n)_j \mid \mathrm{im}\, f \neq [n]\backslash\{k\}\}.$$

In other words, among all boundary simplices we ban those whose image is exactly everything except k.

Definition 7.11 (Fibration)**.** A simplicial map $p : X \to Y$ is called a *fibration* if for all $n \geqslant 1$ and $0 \leqslant k \leqslant n$, every commutative diagram of simplicial maps given by the left diagram below admits a simplicial map h extending it to the commutative diagram on the right:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "g"'] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

In very broad terms, what this diagram tells us, is that if we find a way to fill a shape downstairs in the base space Y, a fibration guarantees that we can always lift that "filling strategy" up to the total space X such that it glues onto whatever boundary information (given by f) is present.

Definition 7.12 (Kan complex)**.** A *Kan complex* is a simplicial set X such that the canonical map $p : X \to \Delta^0$ is a fibration. In other words, for each horn $f : \Lambda^n_k \to X$, one can find an extension $h : \Delta^n \to X$ such that $h \circ i = f$.

A Kan complex is therefore a space where one can draw any horn inside which is always fillable. We will see that these are the complexes that enable us to define homotopies.

Definition 7.13 (Simplicial homotopy)**.** We say that there is a *simplicial homotopy* between two simplicial maps $f, g : X \to Y$ if there is a simplicial map $h : X \times \Delta^1 \to Y$ such that the diagram

$$\begin{tikzcd} X \times \Delta^0 = X \arrow[d, "1\times d^1"'] \arrow[drr, "f"] & & \\ X \times \Delta^1 \arrow[rr, "h"] & & Y \\ X \times \Delta^0 = X \arrow[u, "1\times d^0"] \arrow[urr, "g"'] & & \end{tikzcd}$$

commutes.

We will first be interested in the homotopies of two vertices, i.e. the homotopies of two simplicial maps $\Delta^0 \to Y$. It turns out that this is an equivalence relation if Y is a Kan complex, as the following Lemma shows.

Lemma 7.14. If Y is a Kan complex, the simplicial homotopy of vertices $\Delta^0 \to Y$ is an equivalence relation.

Proof. Let d_i, s_i be the usual induced simplicial morphisms on the simplicial set Y. Let $x, y \in Y_0$. We need to show that the existence of $e \in Y_1$ with $d_1 e = y$ and $d_0 e = x$ is an equivalence relation.

Reflexivity: Define $e = s_0 x$. By the simplicial identity $d_j s_j = d_{j+1} s_j = \mathrm{Id}$ we have $d_1 e = d_0 e = x$.

Transitivity: Suppose that we have two edges e_2 and e_0 such that $d_0 e_2 = y, d_1 e_2 = x$ and $d_0 e_0 = z, d_1 e_0 = y$. Given that $d_0 e_2$ and $d_1 e_0$ match, we have a well-defined map $f : \Lambda^2_1 \to Y$. For this we can choose an extension $h : \Delta^2 \to Y$ and define $e_1 = d_1 h$; we get

$$d_0 e_1 = d_0 d_1 h = d_0 d_0 h = z, \quad d_1 e_1 = d_1 d_1 h = d_1 d_2 h = x.$$

Symmetry: Let e_2 be an edge such that $d_1e_2 = y, d_0e_2 = x$. Define the edge $e_1 = s_0x$. Due to $d_1e_1 = d_1e_2$, we have again a well-defined map $f : \Lambda_0^2 \to Y$, for which we can extend via $h : \Delta^2 \to Y$ and define $e_0 = d_0h$. Indeed,

$$d_0e_0 = d_0d_0h = d_0d_1h = x, \quad d_1e_0 = d_1d_0h = d_0d_2h = y. \qquad \square$$

We can finally define the homotopy sets of a simplicial set.

Definition 7.15. For a Kan complex X, one defines $\pi_0(X)$ to be the set of vertices of X modulo homotopy, i.e.

$$\pi_0(X) = X_0/\sim$$

One could also define the higher homotopy groups; these are not of direct relevance for our purposes and we will omit their definition. One last notion of importance is that of a *weak equivalence*: a map $f : X \to Y$ between Kan complexes that induces an isomorphism on the higher homotopy groups (c.f. [21], equation (7.9)) and, importantly for us, a bijection $f_* : \pi_0(X) \to \pi_0(Y)$.

7.2.2 BV in simplicial terms

We have seen above that the tensor product of a DGLA L with a differential graded commutative algebra such as $\Omega^\bullet$ again yields a DGLA, with product, differential and brackets given as in (73), (74) and (75).

Furthermore it is, due to the contravariance of Ω, a simplicial DGLA. Taking the MC functor, we obtain a simplicial set $\mathrm{Del}_\infty(L)_k = \mathrm{MC}(\Omega(\Delta^k) \otimes L)$, often called the *Deligne groupoid* [2]. The 0-vertices are then the MC solutions of L, the 1-edges are homotopies between them, and so on.

In the previous section we used the Kan property to define homotopy correctly. One can indeed prove that $\mathrm{Del}_\infty(L)$ is a Kan complex if L is nilpotent ([25] Prop. 2.2.3) or we work with a complete L (where the filtration starts at F^1 and not at F^0, as we have defined it; [2] Cor. 2.11), since in these cases the infinite sums encountered in Section 2 converge automatically.

We see from the proof of Lemma 7.14 that we need to be able to fill the two horns to define π_0. In our setting, this boils down to having a well-defined composition of gauge actions (c.f. 2.6) for transitivity, and inverse gauge actions for symmetry. Assuming their convergence, we can define the following.

Definition 7.16. Let $S_0, S_1 \in \mathrm{MC}(L)$. We say that S_0 and S_1 are equivalent, if they are in the same class of $\pi_0(\mathrm{Del}_\infty(L))$, that is, if $\exists A_t \in \mathrm{Del}_\infty(L)_1 = \mathrm{MC}(\Omega^*(\Delta^1; L))$ such that $d_1A_t = S_0$ and $d_0A_t = S_1$.

Unpacking the definition, we observe that an element A_t can be written as $A_t = S_t + R_t\mathrm{d}t$ (where we take $t := t_1$ and $t_0 = 1 - t$ respectively, so that the face d_1, i.e. $\{t_1 = 0\}$, is $t = 0$). The MC equations for S_0 and S_1 spell out

$$\mathrm{d}S_0 + \frac{1}{2}(S_0, S_0) = 0, \quad \mathrm{d}S_1 + \frac{1}{2}(S_1, S_1) = 0$$

and for A_t

$$\delta S_t - \mathrm{d}R_t = (S_t, R_t), \quad \mathrm{d}S_t + \frac{1}{2}(S_t, S_t) = 0,$$

with the condition $d_1A_t = A_t|_{t=0} = S_t|_{t=0} = S_0$ and $d_0A_t = S_1$. This is exactly the content of the BV canonical transformation (if we use $\mathrm{d} = -\mathring{\imath}\hbar\Delta$).

Definition 7.17. Let $\mathcal{M}$ be a manifold with odd-symplectic structure. Define the simplicial set of families of Lagrangian submanifolds of $\mathcal{M}$:

$$\Xi(\mathcal{M})_\bullet = \{\iota : \Delta^\bullet \times \mathcal{L} \to \mathcal{M} | \ \iota(t, -) \subset \mathcal{M} \ \text{ Lagrangian}\},$$

where $\mathcal{L}$ is a Lagrangian submanifold of $\mathcal{M}$. The face and degeneracy maps are induced by precomposition with the standard maps of $\Delta^\bullet$, and the simplicial identities of Definition 7.2 are satisfied since $\Delta^\bullet$ itself satisfies them.

We will integrate over these Lagrangian families and hence form the free abelian group $\mathbb{Z}\Xi(\mathcal{M})$ on $\Xi(\mathcal{M})$ to define the differential complex $(\mathbb{Z}\Xi(\mathcal{M}), \partial)$ where $\partial = \sum_i (-1)^i d_i$.

Consider now again the half-densities $\mathrm{Dens}^{1/2}(\mathcal{M})$. Denote the integration map

$$I_\sigma(\iota) = \int_{\mathcal{L}} \iota^* \sigma, \quad \iota \in \Xi(\mathcal{M}), \sigma \in \mathrm{Dens}^{1/2}(\mathcal{M}).$$

For a Δ-closed σ we have then seen that, for any $\iota \in \Xi(\mathcal{M})_1$ with $d_1\iota = \mathcal{L}_0$ and $d_0\iota = \mathcal{L}_1$, the BV integral of Definition 1.41 satisfies $I_\sigma(\partial\iota) = 0$.

7.2.3 Effective action

We have seen that to define the effective action one usually has a splitting of fields given by a cartesian product $\mathcal{F} = B \times F$, where one integrates the fluctuation fields in F away by choosing $\mathcal{L} \subset F$, see Definition 1.42. This setup can be generalized.

Definition 7.18 (BV hedgehog, [10, Rmk. 2.13])**.** A *BV hedgehog* is a submersion $\pi : \mathcal{F} \to B$ of odd symplectic supermanifolds, with typical fiber $F = \pi^{-1}(x)$, $x \in B$, also odd symplectic, such that there is an open cover $\{U_\alpha\}_{\alpha \in I}$ of B and symplectomorphisms

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F, \quad \mathrm{pr}_B \circ \phi_\alpha = \pi,$$

over U_α, pulling $\omega_B \oplus \omega_F$ back to $\omega_{\mathcal{F}}$, and whose fiber transition maps $R_{\alpha\beta}$ (see below) are connected to the identity.

Since the fibers are odd symplectic, we have a decomposition $T_p\mathcal{F} = V_p \oplus H_p$ with $V_p = \mathrm{Ker}\, \mathrm{d}\pi_p$ and $H_p = V_p^{\perp_\omega}$. For any $v \in H$ and $w \in V$,

$$\omega(\mathrm{d}\phi_{\alpha\beta}(v), w) = \omega(v, \mathrm{d}\phi_{\alpha\beta}^{-1}(w)),$$

by symplecticity. Since $\phi_{\alpha\beta}$ covers Id, we have $\mathrm{d}\phi_{\alpha\beta}^{-1}(w) \in V$, and $v \in V^{\perp_\omega}$ kills any vertical vector, so $\mathrm{d}\phi_{\alpha\beta}(v) \in H$. In coordinates:

$$\phi_{\alpha\beta}(x, y) = (x, R_{\alpha\beta}(y)), \quad R_{\alpha\beta} := (\phi_\alpha \circ \phi_\beta^{-1})|_F \in \mathrm{Symp}(F),$$

i.e. the transition maps act only in the fiber direction, and moreover independently of x, since preserving H forces $\partial_x R_{\alpha\beta} = 0$. The $R_{\alpha\beta}$ satisfy the usual cocycle conditions, $R_{\alpha\beta}R_{\beta\gamma} = R_{\alpha\gamma}$ with $R_{\alpha\alpha} = \mathrm{Id}$.

Let $\sigma \in \mathrm{Dens}^{1/2}(\mathcal{F})$ be a global solution to the QME. Choosing $\mathcal{L} \subset F$ and using each trivialization ϕ_α, one defines a local effective action

$$\sigma_\alpha^{\mathrm{eff}} := \int_{\mathcal{L}} (\phi_\alpha^{-1})^* \sigma \in \mathrm{Dens}^{1/2}(U_\alpha),$$

which solves the QME on U_α by Corollary 1.46. On an overlap $U_{\alpha\beta}$ however,

$$\sigma_\beta^{\mathrm{eff}} - \sigma_\alpha^{\mathrm{eff}} = \int_{R_{\alpha\beta}(\mathcal{L})} (\phi_\alpha^{-1})^* \sigma - \int_{\mathcal{L}} (\phi_\alpha^{-1})^* \sigma,$$

a discrepancy since $R_{\alpha\beta}$ need not preserve $\mathcal{L}$. By Theorem 1.44 the mismatch is controlled by a Lagrangian homotopy between $\mathcal{L}$ and $R_{\alpha\beta}(\mathcal{L})$ in F. Encoding such homotopies coherently over the full Čech nerve requires some further constructions.

Definition 7.19 (Simplicial group and action, [28] Def. 18.1)**.** A simplicial group is a simplicial set $G : \mathbf{\Delta} \to \mathbf{Grp}$. We write $e_k \in G_k$ for the unit. A simplicial group G *acts from the left* on a simplicial set E if there is a simplicial map $\phi : G \times E \to E$, written $\phi(g, x) = g \cdot x$, such that $e_k \cdot x = x$ and $g \cdot (g' \cdot x) = (gg') \cdot x$.

Definition 7.20 (Twisting function and twisted Cartesian product, [28] Def. 18.3)**.** Let F, B be simplicial sets and G a simplicial group acting on F from the left. A *twisting function* is a family of maps $\tau : B_k \to G_{k-1}$, $k \geqslant 1$, satisfying

$$\begin{aligned} d_0\tau(b) &= \tau(d_0 b)^{-1}\tau(d_1 b), && \text{for } b \in B_{k\geqslant 2},\\ d_i\tau(b) &= \tau(d_{i+1} b), && \text{for } i > 0,\\ s_i\tau(b) &= \tau(s_{i+1} b), && \text{for } i \geqslant 0,\\ \tau(s_0 b) &= e_k. \end{aligned}$$

The associated *twisted Cartesian product* $F \times_\tau B$ is the simplicial set with $(F \times_\tau B)_k = F_k \times B_k$ and

$$d_i(f, b) = (d_i f, d_i b), \quad i > 0, \quad d_0(f, b) = (\tau(b) \cdot d_0 f,\ d_0 b), \quad s_i(f, b) = (s_i f, s_i b),\ i \geqslant 0.$$

The identities on τ are precisely the requirement that $F \times_\tau B$ be a simplicial set.

We specialise to the hedgehog. The base is the Čech nerve $B = N_\bullet\mathcal{U}$ and the fiber is the simplicial set $\Xi(F)$ of Lagrangian families in F (Def. 7.17); the structure group is the constant group complex $G = \mathrm{Symp}(F)$, i.e. $G_k = \mathrm{Symp}(F)$ with all $d_i, s_i = \mathrm{Id}$. It acts on $\Xi(F)$ from the left by post-composition $g \cdot \iota = g \circ \iota$. This is a simplicial map: the operator induced on $\Xi(F)$ by a morphism θ of $\mathbf{\Delta}$ is precomposition on the source, $\theta^*\iota = \iota \circ (\theta \times \mathrm{Id}_{\mathcal{L}})$ so the square defining a simplicial map commutes; in particular this holds for every face d_i and degeneracy s_i.

For the twisting function we take the transition cocycle of the hedgehog read off the leading edge,

$$\tau : N_k\mathcal{U} \to \mathrm{Symp}(F), \qquad \tau(\alpha_0 \ldots \alpha_k) = R^{-1}_{\alpha_0\alpha_1},$$

and we set $\Xi_{\tau\mathcal{U}}(F) := \Xi(F) \times_\tau N_\bullet\mathcal{U}$.

Lemma 7.21. τ *is a twisting function, so* $\Xi_{\tau\mathcal{U}}(F)$ *is a simplicial set, with*

$$d_i(\iota, \underline{\alpha}) = (d_i\iota, d_i\underline{\alpha})\ (i > 0), \quad d_0(\iota, \underline{\alpha}) = (R^{-1}_{\alpha_0\alpha_1} \circ d_0\iota, d_0\underline{\alpha}), \quad s_i(\iota, \underline{\alpha}) = (s_i\iota, s_i\underline{\alpha}).$$

Proof. As G is constant, $d_i\tau = s_i\tau = \tau$, so the four identities reduce to relations on R. The first reads $R^{-1}_{\alpha_0\alpha_1} = R_{\alpha_1\alpha_2}R^{-1}_{\alpha_0\alpha_2}$, i.e. the cocycle relation $R_{\alpha_0\alpha_1}R_{\alpha_1\alpha_2} = R_{\alpha_0\alpha_2}$; the second and third hold because deleting or repeating a vertex of index $\geqslant 1$ fixes the leading edge; and $\tau(s_0\underline{\alpha}) = R^{-1}_{\alpha_0\alpha_0} = \mathrm{Id}$ since $R_{\alpha\alpha} = \mathrm{Id}$. ☐

Definition 7.22 (Simplicial section)**.** A *simplicial section* of a simplicial map $p : E \to B$ is a simplicial map $s : B \to E$ with $p \circ s = \mathrm{Id}_B$.

A *flexible Lagrangian* is a simplicial section $s : N_\bullet\mathcal{U} \to \Xi_{\tau\mathcal{U}}(F)$ of the projection, normalised by $\iota_\alpha = \mathrm{Id}_{\mathcal{L}}$. Concretely it assigns to each $\underline{\alpha} = (\alpha_0 \ldots \alpha_k)$ a Lagrangian family $\iota_{\alpha_0\ldots\alpha_k} : \Delta^k \times \mathcal{L} \to F$ with

$$\iota_{\alpha_0\ldots\alpha_k}|_{t_i=0} = \iota_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}\ (i \geqslant 1), \qquad \iota_{\alpha_0\ldots\alpha_k}|_{t_0=0} = R_{\alpha_0\alpha_1} \circ \iota_{\alpha_1\ldots\alpha_k},$$

together with $\iota_{s_i\underline{\alpha}} = s_i\iota_{\underline{\alpha}}$ on degeneracies.

Fix a flexible Lagrangian. We construct its Hamiltonian one-form as in (7). On each simplex, with $\underline{\alpha} = (\alpha_0 \ldots \alpha_k)$ as above, the family $\iota_{\underline{\alpha}}$ (written ι when the simplex is fixed) has velocity fields $X_{j,\underline{\alpha}} := \mathrm{d}\iota_{\underline{\alpha}}(\partial/\partial t_j)$, $j = 0, \ldots, k$. Since every $\iota_{\underline{\alpha}}(t, -)$ is Lagrangian, $\iota_{\underline{\alpha}}(t, -)^*\omega_F = 0$, and differentiating along t_j shows each

$$\beta_{j,\underline{\alpha}} := \iota_{\underline{\alpha}}^*(\iota_{X_{j,\underline{\alpha}}}\omega_F) \in \Omega^1(\mathcal{L})$$

is closed. Since $\mathrm{gh}(\beta_{j,\underline{\alpha}}) = -1$ and cohomology in nonzero ghost degree vanishes, we can define the following.

Definition 7.23 (Hamiltonian one-form)**.** The *Hamiltonian one-form* of the family is

$$\eta_{\underline{\alpha}} = \sum_{j=0}^{k} \eta_{j,\underline{\alpha}}\, \mathrm{d}t_j \;\in\; \Omega_k^1 \otimes C^\infty(\mathcal{L}), \qquad \mathrm{d}_{\mathcal{L}}\eta_{j,\underline{\alpha}} = \beta_{j,\underline{\alpha}},$$

Due to $\mathrm{gh}(\eta_{j,\underline{\alpha}}) = -1$ it is unique.

Lemma 7.24. The Hamiltonian one-forms satisfy

$$\eta_{\underline{\alpha}}\big|_{t_i=0} = \eta_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}\ (i \geqslant 1), \qquad \eta_{\underline{\alpha}}\big|_{t_0=0} = \eta_{\alpha_1\ldots\alpha_k}.$$

Proof. Restriction to $\{t_i = 0\}$ commutes with differentiation with respect to the simplicial coordinate t_j with $j \neq i$. Therefore the computation of $X_{j,\underline{\alpha}}|_{t_i=0}$ for $j \neq i$ is equivalent to first setting $t_i = 0$ on the family $\iota_{\underline{\alpha}}$, and then taking the simplicial differential. The term of $i = j$ does not contribute anyway, as it will get killed by $\mathrm{d}t_j = 0$.

Faces $i \geqslant 1$: The section condition gives $\iota_{\underline{\alpha}}|_{t_i=0} = \iota_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$, hence $X_{j,\underline{\alpha}}|_{t_i=0} = X_{j,\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$ and therefore $\beta_{j,\underline{\alpha}}|_{t_i=0} = \beta_{j,\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$.

Face $i = 0$: Set $R := R_{\alpha_0\alpha_1}$. Here $\iota_{\underline{\alpha}}|_{t_0=0} = R \circ \iota_{\alpha_1\ldots\alpha_k}$ with $R \in \mathrm{Symp}(F)$ constant in t, so the velocities push forward, $X_{j,\underline{\alpha}}|_{t_0=0} = R_* X_{j,\alpha_1\ldots\alpha_k}$. By naturality of the interior product, $R^*(\iota_{R_*V}\omega_F) = \iota_V(R^*\omega_F)$, together with $R^*\omega_F = \omega_F$,

$$\beta_{j,\underline{\alpha}}\big|_{t_0=0} = (R \circ \iota_{\alpha_1\ldots\alpha_k})^*\big(\iota_{R_*X_{j,\alpha_1\ldots\alpha_k}}\omega_F\big) = \iota_{\alpha_1\ldots\alpha_k}^*\big(\iota_{X_{j,\alpha_1\ldots\alpha_k}}R^*\omega_F\big) = \beta_{j,\alpha_1\ldots\alpha_k}.$$

From β to η: Pullback along a coface commutes with $\mathrm{d}_{\mathcal{L}}$, hence $\eta_{\underline{\alpha}}|_{t_i=0}$ satisfies the equation $\mathrm{d}_{\mathcal{L}}(\eta_{\underline{\alpha}}|_{t_i=0}) = \beta_{\underline{\alpha}}|_{t_i=0}$; as that primitive is unique, it equals $\eta_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$ for $i \geqslant 1$ and $\eta_{\alpha_1\ldots\alpha_k}$ for $i = 0$. □

With these, the patchwise effective action of a global $\sigma \in \mathrm{Dens}^{1/2}(\mathcal{F})$, for a flexible Lagrangian with base Lagrangian $\mathcal{L}$, is

$$\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k} := \pi_{*,\alpha_0\ldots\alpha_k}\sigma := \int_{\mathcal{L}} e^{-i\eta_{\alpha_0\ldots\alpha_k}/\hbar}\,(\phi_{\alpha_0}^{-1} \circ \iota_{\alpha_0\ldots\alpha_k})^*\sigma \;\in\; \Omega_k \otimes \mathrm{Dens}^{1/2}(U_{\alpha_0\ldots\alpha_k}).$$

Proposition 7.25. $\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}$ is an element of the Thom–Whitney complex $\|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\|$.

Proof. Each $\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}$ lies in $\Omega_k \otimes \mathrm{Dens}^{1/2}(U_{\alpha_0\ldots\alpha_k})$, so the collection is a point of $\prod_k \Omega_k \otimes \mathrm{Dens}^{1/2}(N_k\mathcal{U})$. What remains is the Thom–Whitney matching $(f^* \otimes 1)\omega_\ell = (1 \otimes f_*)\omega_k$ for every $f : [k] \to [\ell]$ in $\mathbf{\Delta}$; since these are generated by the cofaces and codegeneracies, it suffices to treat those.

Cofaces: For a coface $f = d_i$ the map f^* restricts a form to the i-th face $\{t_i = 0\}$ and f_* is the Čech coface, so the condition reads

$$\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}\big|_{t_i=0} = \sigma^{\mathrm{eff}}_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}\big|_{U_{\alpha_0\ldots\alpha_k}}.$$

For $i \geqslant 1$ the face relations of the flexible Lagrangian and of η give $\iota_{\alpha_0\ldots\alpha_k}|_{t_i=0} = \iota_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$ and $\eta_{\alpha_0\ldots\alpha_k}|_{t_i=0} = \eta_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$; since α_0 survives, the reference trivialization ϕ_{α_0} is unchanged and the whole integrand restricts to that of $\sigma^{\mathrm{eff}}_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_k}$. For $i = 0$ we instead have $\iota_{\alpha_0\ldots\alpha_k}|_{t_0=0} = R_{\alpha_0\alpha_1} \circ \iota_{\alpha_1\ldots\alpha_k}$ and $\eta|_{t_0=0} = \eta_{\alpha_1\ldots\alpha_k}$, so

$$\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}\big|_{t_0=0} = \int_{\mathcal{L}} e^{-i\eta_{\alpha_1\ldots\alpha_k}/\hbar}\,(\phi_{\alpha_0}^{-1} \circ R_{\alpha_0\alpha_1} \circ \iota_{\alpha_1\ldots\alpha_k})^*\sigma.$$

By definition of the transition map $\phi_{\alpha_0}^{-1} \circ R_{\alpha_0\alpha_1} = \phi_{\alpha_1}^{-1}$, so the right-hand side equals $\sigma^{\mathrm{eff}}_{\alpha_1\ldots\alpha_k}|_{U_{\alpha_0\ldots\alpha_k}}$.

Codegeneracies: Here we observe that a codegeneracy can never change the first index α_0 (it can double it but not replace it); the rest is tracking the simplicial maps s_i^* through the construction, which are all well-behaved, to show $s^{i*}\sigma^{\mathrm{eff}}_{\underline{\alpha}} = \sigma^{\mathrm{eff}}_{s_i\underline{\alpha}}$.

$\square$

Theorem 7.26. The fiberwise BV integration $\pi_{*,\underline{\alpha}}$ is a chain map between the complexes $(\mathrm{Dens}^{1/2}(\mathcal{M}), -i\hbar\Delta)$ and $(\|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\|, \delta - i\hbar\Delta_B)$, i.e.

$$(\delta - i\hbar\Delta_B)\pi_{*,\alpha_0\ldots\alpha_k}\sigma = \pi_{*,\alpha_0\ldots\alpha_k}\big(-i\hbar\Delta\sigma\big). \tag{76}$$

Proof. We adapt the proof of Theorem 4.1 of [20] for this case. Work in an adapted Darboux chart $(x^A, \zeta_A; y^a, \xi_a)$ around some point in $\mathcal{F}$, with (x^A, ζ_A) on B and (y^a, ξ_a) on the fiber F, $\omega_F = \mathrm{d}\xi_a \wedge \mathrm{d}y^a$; these split coordinates are the ones furnished by the trivialization ϕ_{α_0}. As ϕ_{α_0} is a (t-independent) symplectomorphism of F, the one-forms $\beta_{j,\underline{\alpha}}$ (and hence $\eta_{\underline{\alpha}}$) are read off $\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}}$ just as off $\iota_{\underline{\alpha}}$ (naturality, as in Lemma 7.24). After a fiber-direction reparametrization we may thus assume $\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}}$ is realized as a section of $\Pi T^*\mathcal{L}$ in F:

$$y^a = y^a, \qquad \xi_a = \xi_a(y,t), \qquad \xi_a(y,0) = 0.$$

In this chart $\beta_{j,\underline{\alpha}} = \partial_{t_j}\xi_a\,\mathrm{d}y^a$, so the defining relation $\mathrm{d}_{\mathcal{L}}\eta_{j,\underline{\alpha}} = \beta_{j,\underline{\alpha}}$ reads $\partial\eta_{j,\underline{\alpha}}/\partial y^a = \partial\xi_a/\partial t_j$. The components of $\delta\eta_{\underline{\alpha}}$ are then independent of y:

$$\frac{\partial}{\partial y^a}\Big(\frac{\partial\eta_{j,\underline{\alpha}}}{\partial t_i} - \frac{\partial\eta_{i,\underline{\alpha}}}{\partial t_j}\Big) = \frac{\partial^2\xi_a}{\partial t_i\partial t_j} - \frac{\partial^2\xi_a}{\partial t_j\partial t_i} = 0,$$

so they are constant along $\mathcal{L}$; being of ghost number -1 they vanish, i.e. $\delta\eta_{\underline{\alpha}} = 0$.

Write $\sigma = f(x,\zeta,y,\xi)\,\mathrm{d}y\,\mathrm{d}x$, so

$$(\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}})^*\sigma \;=\; f\big(x,\zeta,y,\xi(y,t)\big)\,\mathrm{d}y\,\mathrm{d}x.$$

Applying δ and using $\delta\eta_{\underline{\alpha}} = 0$,

$$\delta\big(e^{-i\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}})^*\sigma\big) \;=\; e^{-i\eta_{\underline{\alpha}}/\hbar}\sum_a \delta\xi_a(y,t)\,\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\mathrm{d}y\,\mathrm{d}x.$$

The defining relation gives $\delta\xi_a(y,t) = \partial\eta_{\underline{\alpha}}/\partial y^a$, hence

$$\delta\big(e^{-i\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}})^*\sigma\big) \;=\; e^{-i\eta_{\underline{\alpha}}/\hbar}\sum_a \frac{\partial\eta_{\underline{\alpha}}}{\partial y^a}\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\mathrm{d}y\,\mathrm{d}x. \tag{†}$$

To turn the integrand of (†) into a fiber Laplacian plus a total derivative, expand the latter by Leibniz

$$i\hbar\sum_a \frac{\partial}{\partial y^a}\Big[e^{-i\eta_{\underline{\alpha}}/\hbar}\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\Big] = e^{-i\eta_{\underline{\alpha}}/\hbar}\sum_a \frac{\partial\eta_{\underline{\alpha}}}{\partial y^a}\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)} + i\hbar\, e^{-i\eta_{\underline{\alpha}}/\hbar}\sum_a \frac{\partial}{\partial y^a}\Big(\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\Big).$$

The chain rule through $\xi(y,t)$ splits the last sum into the fiber Laplacian and a cross term,

$$\sum_a \frac{\partial}{\partial y^a}\Big(\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\Big) = \sum_a \frac{\partial^2 f}{\partial y^a\partial\xi_a}\Big|_{\xi(y,t)} + \sum_{a,b}\frac{\partial^2 f}{\partial\xi_b\partial\xi_a}\Big|_{\xi(y,t)}\frac{\partial\xi_b}{\partial y^a}.$$

The cross term vanishes: the Lagrangian condition $\iota_{\underline{\alpha}}(t,-)^*\omega_F = 0$ reads $\frac{\partial\xi_a}{\partial y^b}\,\mathrm{d}y^b \wedge \mathrm{d}y^a = 0$, i.e. $\partial\xi_b/\partial y^a$ is symmetric in a, b, whereas $\partial^2 f/\partial\xi_b\partial\xi_a$ is antisymmetric as the ξ_a are odd. Solving the Leibniz identity for the integrand of (†) and using $e^{-i\eta_{\underline{\alpha}}/\hbar}\Delta_F f|_{\xi(y,t)}\,\mathrm{d}y\,\mathrm{d}x = e^{-i\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1} \circ \iota_{\underline{\alpha}})^*\Delta_F\sigma$, equation (†) becomes

$$\delta\big(e^{-i\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*\sigma\big) \;=\; e^{-i\eta_{\underline{\alpha}}/\hbar}\,(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*(-i\hbar\Delta_F\sigma) \;+\; i\hbar\sum_a \frac{\partial}{\partial y^a}\Big[e^{-i\eta_{\underline{\alpha}}/\hbar}\frac{\partial f}{\partial\xi_a}\Big|_{\xi(y,t)}\Big]\,\mathrm{d}y\,\mathrm{d}x.$$

Integrating the equation above over $\mathcal{L}$ (i.e. over the y^a), the total y-derivative term vanishes by Stokes, leaving

$$\delta\int_{\mathcal{L}} e^{-\mathring{i}\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*\sigma \;=\; \int_{\mathcal{L}} e^{-\mathring{i}\eta_{\underline{\alpha}}/\hbar}(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*(-\mathring{i}\hbar\Delta_F\sigma). \tag{‡}$$

Since Δ_B involves only (x,ζ) it commutes with $\int_{\mathcal{L}}$ and $(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*$. We have

$$\int_{\mathcal{L}} e^{-\mathring{i}\eta_{\underline{\alpha}}/\hbar}\,(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*(-\mathring{i}\hbar\Delta_B\sigma) \;=\; (-1)^p(-\mathring{i}\hbar)\Delta_B\int_{\mathcal{L}} e^{-\mathring{i}\eta_{\underline{\alpha}}/\hbar}\,(\phi_{\alpha_0}^{-1}\circ\iota_{\underline{\alpha}})^*\sigma \;=\; -\mathring{i}\hbar\,\Delta_B\,\sigma_{\underline{\alpha}}^{\mathrm{eff}},$$

where on the right Δ_B acts on $\Omega_k\otimes\mathrm{Dens}^{1/2}$ with its Koszul sign, i.e. the $(-1)^p$ is absorbed into the differential of the totalization. Adding this to (‡) and using $\Delta = \Delta_B + \Delta_F$ yields the claim. □

If we now start with a global $\sigma = e^{\mathring{i}S/\hbar} \in \mathrm{Dens}^{1/2}(\mathcal{F})$ which solves the quantum master equation, the right-hand side of (76) vanishes and hence $\sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}$ is a cocycle of the $\delta - \mathring{i}\hbar\Delta_B$ complex. If we have a global Berezinian $\mu_B \in \mathrm{BER}(B)$ with $\Delta\sqrt{\mu_B} = 0$, we can write $\sqrt{\mu_B}e^{\mathring{i}S_{\alpha_0\ldots\alpha_k}/\hbar} = \sigma^{\mathrm{eff}}_{\alpha_0\ldots\alpha_k}$ and obtain

$$\delta S^{p-1}_{\alpha_0\ldots\alpha_k} - \mathring{i}\hbar\Delta_B S^p_{\alpha_0\ldots\alpha_k} + \frac{1}{2}\sum_{i=0}^{p}(S^i_{\alpha_0\ldots\alpha_k}, S^{p-i}_{\alpha_0\ldots\alpha_k}) = 0,$$

hence $S_{\underline{\alpha}} \in \mathrm{MC}_\hbar(\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|)$.

7.3 Strong deformation retractions

The first goal is to construct two SDRs

$$(\mathcal{F}(\mathcal{M})[[\hbar]], -\mathring{i}\hbar\Delta) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|, \check{\delta} - (-1)^k\mathring{i}\hbar\Delta), \underset{p}{\overset{w}{\rightleftarrows}} (\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, \delta - \mathring{i}\hbar\Delta),$$

to compose them into a single SDR

$$(\mathcal{F}(\mathcal{M})[[\hbar]], -\mathring{i}\hbar\Delta) \underset{P}{\overset{I}{\rightleftarrows}} (\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, \delta - \mathring{i}\hbar\Delta).$$

The space $\mathcal{F}$ can be replaced by $\mathrm{Dens}^{1/2}$ here; for $\mathcal{F}$ we assume throughout a global Berezinian μ with $\Delta\sqrt{\mu}$ to define $\Delta = \Delta_\mu$, for $\mathrm{Dens}^{1/2}$ we can keep using the canonical BV Laplacian.

7.3.1 First SDR: global → Čech

We begin with an SDR between the algebra of global functions and the Čech totalization.

$$\mathcal{F}(\mathcal{M})[[\hbar]] \underset{\pi}{\overset{\iota}{\rightleftarrows}} (|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|, \check{\delta}) \circlearrowleft h \tag{77}$$

This is a special case where $\mathcal{F}(\mathcal{M})[[\hbar]]$ is simply the cohomology of $(|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|, \check{\delta})$, since the global functions sit in cohomology degree 0 and the higher degrees vanish.

We define

- $\iota\colon \mathcal{F}(\mathcal{M})[[\hbar]] \to |\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|$ by sending $S \in \mathcal{F}(\mathcal{M})[[\hbar]]$ to the Čech 0-cochain
$$\iota(f) := \big(S|_{U_\alpha}\big)_{\alpha\in I},$$
extended by 0 in higher Čech degrees.

- $\pi\colon |\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]| \to \mathcal{F}(\mathcal{M})[[\hbar]]$ averaging the 0-cochain part of a Čech cochain ν with a partition of unity
$$\pi(\nu) := \sum_{\alpha\in I} \rho_\alpha \nu_\alpha.$$
- $h\colon |\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]| \to |\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|$ of degree -1 by
$$\big(h(\nu)\big)_{\alpha_0\cdots\alpha_{p-1}} := \sum_{\alpha\in I} \rho_\alpha \nu_{\alpha\alpha_0\cdots\alpha_{p-1}} \quad \text{for } p \geqslant 1,$$
and $h(\nu) := 0$ on 0-cochains.

Throughout, the components $\nu_{\alpha_0\ldots\alpha_k}$ are stored only on increasing strings $\alpha_0 < \cdots < \alpha_k$, and we read them off on an arbitrary string by total antisymmetry: for any permutation $\tau \in S_{k+1}$,
$$\nu_{\alpha_{\tau(0)}\ldots\alpha_{\tau(k)}} := \operatorname{sgn}(\tau)\, \nu_{\alpha_0\ldots\alpha_k},$$
and $\nu_{\alpha_0\ldots\alpha_k} = 0$ whenever two indices coincide. In words, to evaluate ν on an unordered string one sorts the indices increasingly and multiplies by the sign of the sorting permutation; for $k = 1$ this is just $\nu_{\beta\alpha} = -\nu_{\alpha\beta}$.

Lemma 7.27. The triple (ι, π, h) defines an SDR as in (77); that is,

(i) $\pi \circ \iota = \mathrm{Id}_{\mathcal{F}(\mathcal{M})[[\hbar]]}$;

(ii) $\check{\delta}h + h\check{\delta} = \mathrm{Id} - \iota\pi$;

(iii) $h\iota = 0, \quad \pi h = 0, \quad h^2 = 0.$

Proof. (i) For $f \in \mathcal{F}(\mathcal{M})[[\hbar]]$,
$$\pi(\iota(f)) = \sum_\alpha \rho_\alpha f|_{U_\alpha} = f \cdot \sum_\alpha \rho_\alpha = f.$$

(ii) From [6]: For a Čech p-cochain ν with $p \geqslant 1$ we use $(\check{\delta}\nu)_{\alpha\alpha_0\cdots\alpha_p} = \nu_{\alpha_0\cdots\alpha_p} - \sum_{i=0}^p (-1)^i \nu_{\alpha\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p}$ to compute
$$(\check{\delta}h\nu)_{\alpha_0\cdots\alpha_p} = \sum_{i=0}^p (-1)^i (h\nu)_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p} = \sum_\alpha \sum_{i=0}^p (-1)^i \rho_\alpha\, \nu_{\alpha\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p},$$
$$(h\check{\delta}\nu)_{\alpha_0\cdots\alpha_p} = \sum_\alpha \rho_\alpha (\check{\delta}\nu)_{\alpha\alpha_0\cdots\alpha_p} = \nu_{\alpha_0\cdots\alpha_p} - \sum_\alpha \sum_{i=0}^p (-1)^i \rho_\alpha\, \nu_{\alpha\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p},$$
where in the second line we used $\sum_\alpha \rho_\alpha = 1$. Adding the two, the double sums cancel and $(\check{\delta}h + h\check{\delta})\nu = \nu$; since $\iota\pi$ vanishes in positive Čech degree this is $(\mathrm{Id} - \iota\pi)\nu$.

For $p = 0$, $h(\nu) = 0$, so we only need to compute $h(\check{\delta}\nu)$. Using the sign convention $(\check{\delta}\nu)_{\alpha\beta} = \nu_\beta - \nu_\alpha$,
$$\big(h(\check{\delta}\nu)\big)_\beta = \sum_\alpha \rho_\alpha (\check{\delta}\nu)_{\alpha\beta} = \sum_\alpha \rho_\alpha (\nu_\beta - \nu_\alpha) = \nu_\beta - \pi(\nu) = \big(\nu - \iota\pi(\nu)\big)_\beta.$$
Hence $(\check{\delta}h + h\check{\delta})(\nu) = h(\check{\delta}\nu) = (\mathrm{Id} - \iota\pi)\nu$, as required.

(iii) The identity $h\iota = 0$ is immediate from $h(\nu) = 0$ on 0-cochains.

For $\pi h = 0$, let ν be a Čech p-cochain with $p \geqslant 1$:
$$\pi(h(\nu)) = \sum_\beta \rho_\beta (h\nu)_\beta = \sum_\beta \rho_\beta \sum_\alpha \rho_\alpha \nu_{\alpha\beta} = \sum_{\alpha,\beta} \rho_\alpha \rho_\beta \nu_{\alpha\beta} = 0,$$

where the last equality follows by antisymmetry of $\nu_{\alpha\beta}$ in (α,β) against the symmetry of $\rho_\alpha\rho_\beta$.

For $h^2=0$, applied to a p-cochain with $p \geqslant 2$,

$$(h^2\nu)_{\alpha_0\cdots\alpha_{p-2}} = \sum_\beta \rho_\beta (h\nu)_{\beta\alpha_0\cdots\alpha_{p-2}} = \sum_{\alpha,\beta} \rho_\alpha\rho_\beta\nu_{\alpha\beta\alpha_0\cdots\alpha_{p-2}} = 0$$

by the same antisymmetric/symmetric contraction argument. $\square$

Next, we will introduce the BV Laplacian in the Čech complex. We want to use it as a perturbation of the Čech differential and apply the homological perturbation lemma to construct the updated SDR. Let the perturbation be the differential $\partial = -(-1)^k \mathring{\imath}\hbar\Delta$, where k is the Čech degree. Care must be taken with the degree-dependent sign inside the geometric series of Lemma 2.13: every h lowers the Čech degree by one, so acting on a k-cochain the successive Δ's in $\pi(-\partial h)^n$ and $h(-\partial h)^n$ sit in degrees $k-1, k-2, \ldots, k-n$ and contribute

$$(-1)^{(k-1)+(k-2)+\cdots+(k-n)} = (-1)^{nk+\frac{n(n+1)}{2}}.$$

We then have that

$$\begin{aligned}
\mathrm{d}_{\mathcal{F}(\mathcal{M})[[\hbar]]} &\rightsquigarrow \mathrm{d}_\Delta = -\mathring{\imath}\hbar\,\pi\Delta\iota_\Delta,\\
\iota &\rightsquigarrow \iota_\Delta = \sum_{n\geqslant 0} (\mathring{\imath}\hbar)^n (h\Delta)^n \iota,\\
\pi &\rightsquigarrow \pi_\Delta = \sum_{n\geqslant 0} (-1)^{nk+\frac{n(n+1)}{2}} (\mathring{\imath}\hbar)^n \pi(\Delta h)^n,\\
h &\rightsquigarrow h_\Delta = \sum_{n\ \geqslant 0} (-1)^{nk+\frac{n(n+1)}{2}} (\mathring{\imath}\hbar)^n h(\Delta h)^n.
\end{aligned}$$

Now observe that in our case, we have that $\Delta\iota = \iota\Delta$, where the Δ on the right-hand side is to be understood as the regular BV Laplacian on the global functions. Due to $h\iota = 0$ we therefore have that in fact $\iota_\Delta = \iota$. Using this in the formula for d_Δ, we obtain immediately that $\mathrm{d}_\Delta = -\mathring{\imath}\hbar\Delta$ on $\mathcal{F}(\mathcal{M})[[\hbar]]$. Hence the only two changes are h_Δ and π_Δ.

All in all we upgraded the SDR to one which contains the BV Laplacian as differential

$$(\mathcal{F}(\mathcal{M})[[\hbar]], -\mathring{\imath}\hbar\Delta) \underset{\pi_\Delta}{\overset{\iota}{\rightleftarrows}} (|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|, \check{\delta} - (-1)^k \mathring{\imath}\hbar\Delta) \circlearrowleft h_\Delta \tag{78}$$

7.3.2 Second SDR: Čech → Thom–Whitney

We now construct an SDR

$$(|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|, \check{\delta} - (-1)^k \mathring{\imath}\hbar\Delta) \underset{p}{\overset{w}{\rightleftarrows}} (\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, \delta - \mathring{\imath}\hbar\Delta) \circlearrowleft s \tag{79}$$

where the $\check{\delta}$ is the Čech differential and δ the de Rham operator on forms on the simplices.

We have the following constructions from [18]:

- For a Čech k-cochain $\nu = (\nu_{\alpha_0\cdots\alpha_k})$ we define the Whitney embedding w by

$$w(\nu) = \sum_{(\alpha_0,\cdots,\alpha_k)\in I_{\leqslant}^{k+1}} \epsilon_{\alpha_0\cdots\alpha_k} \otimes \nu_{\alpha_0\cdots\alpha_k}, \tag{80}$$

where $\epsilon_{\alpha_0\cdots\alpha_k}$ is the *Whitney elementary form*

$$\epsilon_{\alpha_0\cdots\alpha_k} = k! \sum_{i=0}^{k} (-1)^i t_{\alpha_i} \mathrm{d}t_{\alpha_0} \cdots \widehat{\mathrm{d}t_{\alpha_i}} \cdots \mathrm{d}t_{\alpha_k}. \tag{81}$$

Equivalently, summing over arbitrary $(k+1)$-tuples (and using the antisymmetric convention),

$$w(\nu) = \frac{1}{k+1} \sum_{\alpha_0,\ldots,\alpha_k \in I^{k+1}} \sum_{i=0}^{k} (-1)^i t_{\alpha_i} \mathrm{d}t_{\alpha_0} \cdots \widehat{\mathrm{d}t_{\alpha_i}} \cdots \mathrm{d}t_{\alpha_k} \otimes \nu_{\alpha_0\cdots\alpha_k}.$$

One can easily show that w is a chain map. For the $\check{\delta}$ and δ we have that ([20]):

$$w(\check{\delta}\nu) = \frac{1}{k+2} \sum_{\alpha_0,\ldots,\alpha_{k+1} \in I^{k+1}} \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} (-1)^{i+j} t_{\alpha_i} \mathrm{d}t_{\alpha_0} \ldots \widehat{\mathrm{d}t}_{\alpha_i} \ldots \mathrm{d}t_{\alpha_{k+1}} \otimes \nu_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_{k+1}}.$$

Due to $\sum_i \mathrm{d}t_{\alpha_i} = 0$ only $i = j$ terms survive. Using $\sum_i t_{\alpha_i} = 1$ yields

$$w(\check{\delta}\nu) = \sum_{\alpha_1,\ldots,\alpha_{k+1} \in I^{k+1}} \mathrm{d}t_{\alpha_1} \ldots \mathrm{d}t_{\alpha_{k+1}} \nu_{\alpha_1\ldots\alpha_{k+1}}.$$

which is exactly the same as $\delta w(\nu)$. Similarly, we have

$$\begin{aligned} w(-(-1)^k \mathring{\imath}\hbar\Delta\nu) &= \frac{-(-1)^k \mathring{\imath}\hbar}{k+1} \sum_{\alpha_0,\ldots,\alpha_k \in I^{k+1}} \sum_{i=0}^{k} (-1)^i t_{\alpha_i} \mathrm{d}t_{\alpha_0} \cdots \widehat{\mathrm{d}t_{\alpha_i}} \cdots \mathrm{d}t_{\alpha_k} \otimes \Delta\nu_{\alpha_0\cdots\alpha_k} \\ &= \frac{-\mathring{\imath}\hbar\Delta}{k+1} \sum_{\alpha_0,\ldots,\alpha_k \in I^{k+1}} \sum_{i=0}^{k} (-1)^i t_{\alpha_i} \mathrm{d}t_{\alpha_0} \cdots \widehat{\mathrm{d}t_{\alpha_i}} \cdots \mathrm{d}t_{\alpha_k} \otimes \nu_{\alpha_0\cdots\alpha_k} \\ &= -\mathring{\imath}\hbar\Delta w(\nu) \end{aligned}$$

where we have used the Koszul sign rule to push the Δ past the differential forms.

- The map p is defined via integration of differential forms over the simplices,

$$(p\omega)_{\alpha_0\ldots\alpha_p} = \int_{\Delta^p} \omega_{\alpha_0\ldots\alpha_p} \tag{82}$$

where Δ^p is the corresponding simplex of $\alpha_0\ldots\alpha_p$. It leaves the $\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]$ part untouched, picks the top form and integrates it to a number, such that the output is in $|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|$. Note that:

  - For $p = 0$: p is evaluation at the vertex e_{α_0}, i.e. setting $t_{\alpha_0} = 1$ and all other equal to 0.
  - For $p \geqslant 1$: only p-forms with exactly p differentials $\mathrm{d}t_{\alpha_j}$ contribute. After eliminating t_{α_0} via the relation $t_{\alpha_0} + t_{\alpha_1} + \cdots + t_{\alpha_p} = 1$ on the simplex, the basic integral is (omitting the $\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]$ part)

$$\left[p\left(t_{\alpha_1}^{b_1} \cdots t_{\alpha_p}^{b_p} \mathrm{d}t_{\alpha_1} \cdots \mathrm{d}t_{\alpha_p}\right)\right]_{\alpha_0\ldots\alpha_p} = \frac{b_1! b_2! \cdots b_p!}{(b_1 + b_2 + \cdots + b_p + p)!}.$$

The map p is a chain map: due to Stokes we have

$$(p\delta\omega)_{\alpha_0\ldots\alpha_p} = \int_{\Delta^p} (\delta\omega)_{\alpha_0\ldots\alpha_p} = \int_{\partial\Delta^p} \omega_{\alpha_0\ldots\alpha_p} = \sum_{i=0}^{p} (-1)^i \int_{\partial_i\Delta^p} \omega_{\alpha_0\ldots\alpha_p}$$

where $\partial_i\Delta^p$ is the i-th face of Δ^p, i.e. $t_i = 0$. The TW simplicial compatibility conditions force $\omega_{\alpha_0\ldots\alpha_p}|_{\partial_i\Delta^p} = \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p}$ and hence

$$\begin{aligned} \sum_{i=0}^{p} (-1)^i \int_{\partial_i\Delta^p} \omega_{\alpha_0\ldots\alpha_p} &= \sum_{i=0}^{p} (-1)^i \int_{\Delta^{p-1}} \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p} \\ &= \sum_{i=0}^{p} (-1)^i (p\omega)_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p} = (\check{\delta}p\omega)_{\alpha_0\ldots.\alpha_p}. \end{aligned}$$

For the same reason as for w, we also obtain $-(-1)^k \mathring{\imath}\hbar\Delta p\omega = p(-\mathring{\imath}\hbar\Delta\omega)$.

- The homotopy s is the *Dupont contraction.* It is a degree -1 simplicial endomorphism $s_\bullet : \Omega^*_\bullet \to \Omega^{*-1}_\bullet$ which is defined as follows. Define for $0 \leqslant i \leqslant n$ the maps $\phi_i : [0,1] \times \Delta^n \to \Delta^n$ by
$$\phi_i(t,x) = tx + (1-t)e_i,$$
where e_i is the i'th vertex. Denote $\pi_* : \Omega^*([0,1] \times \Delta^n) \to \Omega^{*-1}(\Delta^n)$ the integration along t. Define the operator $h^i_n : \Omega^*_n \to \Omega^{*-1}_n$ by the formula
$$h^i_n(\omega) = \pi_*(\phi_i^*\omega).$$
The Dupont contraction s is then constructed as
$$s_n = \sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \epsilon_{i_0 \dots i_k} h^{i_k}_n \dots h^{i_0}_n, \quad n \geqslant 0.$$

The fact that this describes an SDR is stated in the Lemma below. The proofs are tedious and not overly insightful, which is the reason why we reference to literature instead.

Lemma 7.28. The triple (w,p,s) defines an SDR as in (79):

(i) $p \circ w = \mathrm{Id}_{|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]|}$;

(ii) $(\delta - \mathring{\imath}\hbar\Delta)s + s(\delta - \mathring{\imath}\hbar\Delta) = \mathrm{Id} - wp$;

(iii) $sw = 0, \quad ps = 0, \quad s^2 = 0.$

Proof. Item (i) follows from the duality $I_{\alpha_0\cdots\alpha_k}(\epsilon_{\beta_0\cdots\beta_k}) = \delta_{\underline{\alpha}\underline{\beta}}$ between elementary forms and simplices (on the diagonal the $k!$-normalisation $I_{\alpha_0\cdots\alpha_k}(\epsilon_{\alpha_0\cdots\alpha_k}) = 1$, off it vanishing). For (ii), note first that it reduces to the form-level statement $\delta s + s\delta = \mathrm{Id} - wp$: s acts only on the $\Omega_\bullet$ factor and lowers the form degree by one, so it anticommutes with the Koszul-signed Laplacian, $\Delta(s\beta \otimes v) = (-1)^{p-1} s\beta \otimes \Delta v = -s\,\Delta(\beta \otimes v)$, and the two Δ-terms cancel. The form-level statement can then be found in [18], Theorem 3.7. For (iii), $ps = 0$ is Lemma 3.4, i); s^2 is the content of Theorem 3.11 and finally $sw = 0$ follows from $s^2 = 0$ and Lemma 3.4 ii). $\square$

Remark 7.29. None of the three maps of this SDR really requires polynomiality of the forms. Indeed Dupont's contraction is originally defined on the full de Rham complex $\Omega^\bullet(\Delta^n)$ of smooth forms, and the polynomial model of [18] used here is a restriction of it. Lemma 7.28 therefore holds verbatim with $\Omega_\bullet$ taken to be the smooth de Rham complex $\Omega^\bullet(\Delta^\bullet)$ in place of the polynomial one, as anticipated in Remark 7.6, and every construction of this section carries over to smooth forms unchanged.

7.3.3 Composition of SDRs

We now compose the two SDRs to obtain a single SDR
$$(\mathcal{F}(\mathcal{M})[[\hbar]], -\mathring{\imath}\hbar\Delta) \underset{P}{\overset{I}{\rightleftharpoons}} (\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, \delta - \mathring{\imath}\hbar\Delta) \circlearrowleft K. \tag{83}$$
by defining
$$I := w \circ \iota, \qquad P := \pi_\Delta \circ p, \qquad K := s + wh_\Delta p.$$

Proposition 7.30. The triple (I,P,K) is an SDR as given in (83).

Proof. We first show the homotopy relation. For brevity and clarity we abbreviate for this proof $\check{\delta} := \check{\delta} - (-1)^k \mathring{\imath}\hbar\Delta$ in the Čech complex and $\delta := \delta - \mathring{\imath}\hbar\Delta$ in the TW complex. Using that w and p are chain maps with $w\check{\delta} = \delta w$ and $\check{\delta} p = p\delta$, we compute

$$\begin{aligned} IP &= w\iota\pi_\Delta p \\ &= w(\mathrm{Id} - (\check{\delta} h_\Delta + h_\Delta \check{\delta}))p \\ &= wp - w\check{\delta} h_\Delta p - w h_\Delta \check{\delta} p \\ &= wp - \delta(w h_\Delta p) - (w h_\Delta p)\delta. \end{aligned}$$

Now using $wp = \mathrm{Id} - (\delta s + s\delta)$ from the second SDR,

$$IP = \mathrm{Id} - \delta(s + w h_\Delta p) - (s + w h_\Delta p)\delta = \mathrm{Id} - (\delta K + K\delta).$$

Thus $\delta K + K\delta = \mathrm{Id} - IP$, as required.

The side conditions are then given by

$$\begin{aligned} PI &= \pi_\Delta p w \iota = \pi_\Delta (pw)\iota = \pi\iota = \mathrm{Id}\,. \\ KI &= (s + w h_\Delta p)(w\iota) = [sw]\iota + w h_\Delta p w \iota = w[h_\Delta \iota] = 0 \\ PK &= (\pi_\Delta p)(s + w h_\Delta p) = \pi_\Delta [ps] + \pi_\Delta p w h_\Delta p = [\pi_\Delta h_\Delta] p = 0. \\ K^2 &= (s + w h_\Delta p)(s + w h_\Delta p)[s^2] + [sw] h_\Delta p + w h_\Delta [ps] + w h_\Delta p w h_\Delta p \\ &= w[h_\Delta^2] p = 0, \end{aligned}$$

where bracketed terms vanish due to the side conditions on the individual SDRs. □

7.4 Globalization

7.4.1 Globalization on the level of densities

The developed SDR in the previous sections is sufficient to globalize densities. In this case we consider the complex $\|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\|$. Convergence of π_Δ and ι_Δ is automatic here, as we will always assume a finite nerve $N_\bullet\mathcal{U}$.

Take a cocycle $\sigma \in \|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\|$. The components satisfy

$$\delta\sigma^{p-1}_{\underline{\alpha}} - \mathring{\imath}\hbar\Delta\sigma^{p}_{\underline{\alpha}} = 0, \quad \forall \underline{\alpha} \in N_\bullet\mathcal{U}.$$

To apply $P = \pi_\Delta \circ p$ on σ, we first note that p performs the simplicial integration, retaining on the k-th Čech level the top form; thus $p\sigma$ is a Čech cochain, which π_Δ then maps to a global density. Now recall that $\pi_\Delta = \sum_{n \geq 0} (-1)^{nk + \frac{n(n+1)}{2}} (\mathring{\imath}\hbar)^n \pi(\Delta h)^n$; on a k-cochain only the term $n = k$ survives, so

$$\pi_\Delta \sigma = (-1)^{\frac{k(k-1)}{2}} (\mathring{\imath}\hbar)^k\, \pi(\Delta h)^k \sigma.$$

Applying the definitions of h and π from the right, each h introduces a fresh summed index $\beta_i \in I$ weighted by ρ_{β_i}:

$$\begin{aligned} (h\sigma)_{\gamma_0\cdots\gamma_{k-1}} &= \sum_{\beta_k \in I} \rho_{\beta_k}\, \sigma_{\beta_k \gamma_0 \cdots \gamma_{k-1}}, \\ (h\Delta h\sigma)_{\gamma_0\cdots\gamma_{k-2}} &= \sum_{\beta_{k-1} \in I} \rho_{\beta_{k-1}}\, \Delta \sum_{\beta_k \in I} \rho_{\beta_k}\, \sigma_{\beta_k \beta_{k-1} \gamma_0 \cdots \gamma_{k-2}}, \\ &\vdots \\ \pi(\Delta h)^k \sigma &= \sum_{\beta_0 \in I} \rho_{\beta_0}\, \Delta \sum_{\beta_1 \in I} \rho_{\beta_1}\, \Delta \cdots \Delta \sum_{\beta_k \in I} \rho_{\beta_k}\, \sigma_{\beta_k \cdots \beta_0}. \end{aligned}$$

Hence, summing over the full nerve,

$$\pi_\Delta\sigma = (-1)^{\frac{k(k-1)}{2}}(\mathring{\imath}\hbar)^k \sum_{(\beta_0,\ldots,\beta_k)\in I^{k+1}} \rho_{\beta_0}\,\Delta\,\rho_{\beta_1}\,\Delta\cdots\Delta\,\rho_{\beta_k}\;\sigma_{\beta_k\beta_{k-1}\cdots\beta_0}, \tag{84}$$

the sum running over *all* $(k+1)$-tuples in I^{k+1}, with σ read off antisymmetrically (so only distinct indices with $U_{\beta_0\cdots\beta_k} \neq \varnothing$ contribute). Grouping these by their underlying increasing tuple $\alpha_0 < \cdots < \alpha_k$, the coefficient of $\sigma_{\alpha_0\ldots\alpha_k}$ is the antisymmetrisation

$$\Psi_{\alpha_0\ldots\alpha_k} = \hbar^k\,\mathrm{Alt}\big(\rho_{\alpha_k}\Delta\rho_{\alpha_{k-1}}\Delta\cdots\Delta\rho_{\alpha_0}\big) := \hbar^k \sum_{\tau\in S_{k+1}} \mathrm{sgn}(\tau)\,\rho_{\alpha_{\tau(k)}}\Delta\rho_{\alpha_{\tau(k-1)}}\cdots\Delta\rho_{\alpha_{\tau(0)}}, \tag{85}$$

where ρ_α is multiplication by the partition function. Using this on the entire σ we obtain the global Δ-cocycle $P(\sigma) \in \mathrm{Dens}^{1/2}(\mathcal{M})$:

$$P(\sigma) = \sum_k c_k \sum_{(\alpha_0\ldots\alpha_k)\in I^{k+1}_{\leqslant}} \int_{\Delta^k} \Psi_{\alpha_0\ldots\alpha_k}\sigma_{\alpha_0\ldots\alpha_k}, \qquad c_k := (-1)^{\frac{k(k+1)}{2}}\,\mathring{\imath}^k = \begin{cases} 1, & k \text{ even},\\ -\mathring{\imath}, & k \text{ odd},\end{cases}$$

where, on top of the prefactor of π_Δ, another $(-1)^k$ factor is picked up by pushing the simplicial integration in front of Ψ (which contains k Laplacians and hence has parity k).

In particular, this allows us to globalize the $\delta - \mathring{\imath}\hbar\Delta_B$ cocycle $\sigma^{\mathrm{eff}}_{\underline{\alpha}}$ (c.f. (76)). The partition function is then simply given as the BV integral

$$Z'(\sigma) = \int_{\mathcal{L}} P(\sigma) = \sum_k c_k \sum_{(\alpha_0\ldots\alpha_k)\in I^{k+1}_{\leqslant}} \int_{\Delta^k}\int_{\mathcal{L}} \Psi_{\alpha_0\ldots\alpha_k}\sigma_{\alpha_0\ldots\alpha_k} \tag{86}$$

for some Lagrangian $\mathcal{L} \subset \mathcal{M}$.

Remark 7.31. For a cochain complex (V, d_V) of R-modules, one can consider its dual $V^* = \mathrm{Hom}_R(V, R)$ with differential $\mathrm{d}_{V^*}\varphi(v) = \varphi(\mathrm{d}_V v)$. A partition function of V can be seen as a d_{V^*}-closed $\varphi \in V^*$. The construction above is then simply applying the quasi-isomorphism π^* to the partition function Z of the global complex, obtaining a $\delta - \mathring{\imath}\hbar\Delta$-closed $\pi^* Z = Z'$.

We can even go further and globalize the cocycle over a *Lagrangian submanifold of a simplicial odd symplectic supermanifold* (LSS), which replaces a global Lagrangian by a family of Lagrangians which glue coherently. This idea is due to Getzler and Pohorence [20]. We present the idea here with two major changes: first, our definition of the LSS is different but equivalent, using simplicial machinery; second, we have chosen the Čech nerve to carry ordered indices $I^{k+1}_{\leqslant}$.

Definition 7.32 (Bisimplicial set, [21])**.** A *bisimplicial set* X is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$. Equivalently, it can be viewed as a functor $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. It consists of sets $X_{m,n}$ with the appropriate face and degeneracy maps. The *diagonal simplicial set* is the composite functor

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\Delta} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{Set},$$

which has the n-simplices given by $X_{n,n}$.

Let $\Xi(\mathcal{M})_\bullet$ be the simplicial set of families of Lagrangian submanifolds of $\mathcal{M}$ (see Definition 7.17). Then $\Xi(-)$ is a functor $\Xi(-) : \mathbf{sSmfd} \to \mathbf{sSet}$. Composing this with the functor $N_\bullet\mathcal{U} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSmfd}$, we obtain the bisimplicial set $\Xi(N_\bullet\mathcal{U})$. Denote by $\Xi_{\mathcal{U}}$ its diagonal simplicial set, with

$$(\Xi_{\mathcal{U}})_k = \bigsqcup_{\underline{\alpha}\in N_k\mathcal{U}} \Xi(U_{\underline{\alpha}})_k,$$

an element being a pair $(\underline{\alpha}, \iota)$ with $\iota \in \Xi(U_{\underline{\alpha}})_k$, and with

$$d_i(\underline{\alpha}, \iota) = (d_i\underline{\alpha},\, d_i\iota), \qquad s_i(\underline{\alpha}, \iota) = (s_i\underline{\alpha},\, s_i\iota).$$

This is well-defined: $U_{\underline{\alpha}} \subseteq U_{d_i\underline{\alpha}}$, so $d_i\iota \in \Xi(U_{\underline{\alpha}})_{k-1} \subseteq \Xi(U_{d_i\underline{\alpha}})_{k-1}$, while $U_{s_i\underline{\alpha}} = U_{\underline{\alpha}}$; the simplicial identities are inherited from $N_\bullet\mathcal{U}$ and from each $\Xi(U_{\underline{\alpha}})$. Forgetting the family gives a simplicial projection

$$p\colon \Xi_{\mathcal{U}} \longrightarrow N_\bullet\mathcal{U}, \qquad p(\underline{\alpha}, \iota) = \underline{\alpha}.$$

Definition 7.33 (LSS)**.** An LSS of $N_\bullet\mathcal{U}$ is a simplicial section (Definition 7.22) of p. Explicitly, it assigns to each simplex $\underline{\alpha} = (\alpha_0 \leqslant \cdots \leqslant \alpha_k)$ a fiberwise Lagrangian family

$$\iota_{\underline{\alpha}}\colon \Delta^k \times \mathcal{L} \to U_{\alpha_0\cdots\alpha_k}$$

satisfying

$$d_i\iota_{\underline{\alpha}} = \iota_{d_i\underline{\alpha}} \quad (0 \leqslant i \leqslant k), \qquad s_i\iota_{\underline{\alpha}} = \iota_{s_i\underline{\alpha}}.$$

Its Hamiltonian one-form is $\eta_{\underline{\alpha}} \in \Omega^1_k \otimes C^\infty(\mathcal{L})$ (Definition 7.23).

Theorem 7.34 ([20], Thm 4.1)**.** *Let $\iota \in \Xi(\mathcal{M})_k$, with Hamiltonian one-form η (Def. 7.23), and $\sigma \in \mathrm{Dens}^{1/2}_c(\mathcal{M}) \otimes \Omega_k$. Then*

$$\delta \int_{\mathcal{L}} e^{-\mathring{\imath}\eta/\hbar}\iota^*\sigma = \int_{\mathcal{L}} e^{-\mathring{\imath}\eta/\hbar}\iota^*(\delta - \mathring{\imath}\hbar\Delta)\sigma.$$

The proof is the simpler version of Theorem 7.26.

Getzler's construction now builds a partition function $Z : \|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\| \to \mathbb{C}$ with $Z(D\sigma) = 0$ for a gauge-fixing LSS ι of $N_\bullet\mathcal{U}$, where $D := \delta - \mathring{\imath}\hbar\Delta$.

We start with the elements on the patches. Naively we would define

$$Z_0(\sigma) = \sum_{\alpha_0 \in I^1_{\leqslant}} \int_{\Delta^0}\int_{\mathcal{L}_{\alpha_0}} \iota^*_{\alpha_0}\rho_{\alpha_0}\sigma_{\alpha_0}.$$

Writing $\rho_{\alpha_0}D\sigma = D(\rho_{\alpha_0}\sigma) - [D, \rho_{\alpha_0}]\sigma$ and applying Thm. 7.34, the first summand becomes δ-exact on Δ^0 and integrates to zero, leaving the defect

$$Z_0(D\sigma) = -\sum_{\alpha_0 \in I^1_{\leqslant}} \int_{\Delta^0}\int_{\mathcal{L}_{\alpha_0}} \iota^*_{\alpha_0}[D, \rho_{\alpha_0}]\sigma. \tag{†}$$

This must be cancelled one level up. Set

$$Z_1(\sigma) = -\mathring{\imath} \sum_{(\alpha_0\alpha_1) \in I^2_{\leqslant}} \int_{\Delta^1}\int_{\mathcal{L}_{\alpha_0\alpha_1}} e^{-\mathring{\imath}\eta_{\alpha_0\alpha_1}/\hbar}\iota^*_{\alpha_0\alpha_1}(\Phi_{\alpha_0\alpha_1}\sigma),$$

with the operator $\Phi_{\alpha_0\alpha_1}$ to be determined. It must have degree 1, $\Phi_{\alpha_0\alpha_1}D\sigma = [D, \Phi_{\alpha_0\alpha_1}]\sigma - D(\Phi_{\alpha_0\alpha_1}\sigma)$; applying again Thm. 7.34 to the second term and Stokes on Δ^1,

$$\begin{aligned} Z_1(D\sigma) = &-\mathring{\imath} \sum_{(\alpha_0\alpha_1)} \int_{\Delta^1}\int_{\mathcal{L}_{\alpha_0\alpha_1}} e^{-\mathring{\imath}\eta_{\alpha_0\alpha_1}/\hbar}\iota^*_{\alpha_0\alpha_1}([D, \Phi_{\alpha_0\alpha_1}]\sigma) \\ &+ \mathring{\imath} \sum_{(\alpha_0\alpha_1)} \int_{\partial\Delta^1}\int_{\mathcal{L}_{\alpha_0\alpha_1}} e^{-\mathring{\imath}\eta_{\alpha_0\alpha_1}/\hbar}\iota^*_{\alpha_0\alpha_1}(\Phi_{\alpha_0\alpha_1}\sigma). \end{aligned}$$

On $\partial\Delta^1$ the Hamiltonian $\eta_{\alpha_0\ldots\alpha_1}$ restricts to a vertex, which vanishes. Letting α_0 be the surviving index and using an antisymmetry notation convention for unordered indices on Φ, the Stokes term yields

$$-\mathring{\imath} \sum_{(\alpha_0) \in I^1_{\leqslant}} \int_{\mathcal{L}_{\alpha_0}} \iota^*_{\alpha_0}\Big(\sum_\beta \Phi_{\alpha_0\beta}\,\sigma\Big).$$

For this to cancel with (†), we need that $-\mathring{\imath}\sum_\beta \Phi_{\alpha_0\beta} = [D,\rho_{\alpha_0}]\ \forall\alpha_0 \in I^1_{\leqslant}$, i.e., as $[D,\rho_{\alpha_0}] = -\mathring{\imath}\hbar[\Delta,\rho_{\alpha_0}]$, when $\sum_\beta \Phi_{\alpha_0\beta} = \hbar[\Delta,\rho_{\alpha_0}]$. Iterating between Z_k and Z_{k+1} works identically: on a $(k+1)$-simplex Φ contains $k+1$ Laplacians and hence has parity $k+1$, so $\Phi D\sigma = (-1)^{k+1}\big(D(\Phi\sigma) - [D,\Phi]\sigma\big)$, while regrouping the boundary faces of Δ^{k+1} by their surviving k-simplex (moving the summed index to the last slot by antisymmetry) produces a compensating $(-1)^{k+1}$. The cancellation between the levels then reads

$$c_{k+1}\sum_\beta \Phi_{\alpha_0\dots\alpha_k\beta} = (-1)^k c_k\,[D,\Phi_{\alpha_0\dots\alpha_k}].$$

We fix the normalization such that the operators satisfy the recursion relation

$$\sum_\beta \Phi_{\alpha_0\dots\alpha_k\beta} = \hbar\,[\Delta,\Phi_{\alpha_0\dots\alpha_k}], \qquad \Phi_{\alpha_0} = \rho_{\alpha_0}, \tag{87}$$

with unordered indices reordered by antisymmetry. Using that $[D,\Phi] = -\mathring{\imath}\hbar[\Delta,\Phi]$ implies $c_{k+1} = (-1)^{k+1}\mathring{\imath}\,c_k$. Telescoping up to k with $c_0 = 1$ yields

$$c_k = (-1)^{1+2+\dots+k}\,\mathring{\imath}^k = (-1)^{\frac{k(k+1)}{2}}\,\mathring{\imath}^k = \begin{cases} 1, & k \text{ even},\\ -\mathring{\imath}, & k \text{ odd}.\end{cases}$$

We now claim that $\Psi_{\underline{\alpha}}$ of π_Δ from (85) solves the recursion relation. Set $H_\alpha := [\Delta,\rho_\alpha]$, which is $[\Delta,-]$-closed, $[\Delta,H_\alpha] = 0$. The descending string of Ψ then collapses onto the H's,

$$\Phi_{\alpha_0\dots\alpha_k} := \Psi_{\alpha_0\dots\alpha_k} = \hbar^k\,\mathrm{Alt}\big(\rho_{\alpha_k}\Delta\rho_{\alpha_{k-1}}\Delta\cdots\Delta\rho_{\alpha_0}\big) = \hbar^k\,\mathrm{Alt}\big(\rho_{\alpha_k}H_{\alpha_{k-1}}H_{\alpha_{k-2}}\cdots H_{\alpha_0}\big). \tag{88}$$

The second equality holds because under the antisymmetrisation any monomial with two adjacent ρ's vanishes; the ρ's commute, so $\rho_{\alpha_i}\rho_{\alpha_j}$ is symmetric against the alternating sign. The lowest cases read $\Phi_{\alpha_0} = \rho_{\alpha_0}$, $\Phi_{\alpha_0\alpha_1} = \hbar(\rho_{\alpha_1}H_{\alpha_0} - \rho_{\alpha_0}H_{\alpha_1})$ and

$$\Phi_{\alpha_0\alpha_1\alpha_2} = \hbar^2\big(\rho_{\alpha_2}[H_{\alpha_1},H_{\alpha_0}] - \rho_{\alpha_1}[H_{\alpha_2},H_{\alpha_0}] + \rho_{\alpha_0}[H_{\alpha_2},H_{\alpha_1}]\big).$$

Lemma 7.35. The operators (88) satisfy the recursion (87).

Proof. As $[\Delta,-]$ is a graded derivation with $[\Delta,H_{\alpha_i}] = 0$ and $[\Delta,\rho_{\alpha_k}] = H_{\alpha_k}$, we obtain

$$[\Delta,\Phi_{\alpha_0\dots\alpha_k}] = \hbar^k\,\mathrm{Alt}\big(H_{\alpha_k}H_{\alpha_{k-1}}\cdots H_{\alpha_0}\big),$$

Appending one index, $\Phi_{\alpha_0\dots\alpha_k\beta} = \hbar^{k+1}\mathrm{Alt}\big(\rho_\beta H_{\alpha_k}\cdots H_{\alpha_0}\big)$, the antisymmetrisation running over the $k+2$ labels $\beta,\alpha_k,\dots,\alpha_0$. Sum over $\beta\in I$ and group by the label carried by the single ρ: the term with ρ_β in that slot gives $\big(\sum_\beta\rho_\beta\big)\mathrm{Alt}(H_{\alpha_k}\cdots H_{\alpha_0}) = \mathrm{Alt}(H_{\alpha_k}\cdots H_{\alpha_0})$, while every term in which β sits on an H-slot is linear in H_β and drops out, since $\sum_\beta H_\beta = [\Delta,\sum_\beta\rho_\beta] = [\Delta,1] = 0$. Hence

$$\sum_\beta \Phi_{\alpha_0\dots\alpha_k\beta} = \hbar^{k+1}\,\mathrm{Alt}\big(H_{\alpha_k}\cdots H_{\alpha_0}\big) = \hbar\,[\Delta,\Phi_{\alpha_0\dots\alpha_k}],$$

which is (87). $\square$

Corollary 7.36. For $\sigma \in \|\mathrm{Dens}^{1/2}(N_\bullet\mathcal{U})\|$ the functional

$$Z(\sigma) := \sum_i Z_i(\sigma) = \sum_{k\geqslant 0} c_k \sum_{(\alpha_0\dots\alpha_k)\in I^{k+1}_{\leqslant}} \int_{\Delta^k}\int_{\mathcal{L}_{\alpha_0\dots\alpha_k}} e^{-\mathring{\imath}\eta_{\alpha_0\dots\alpha_k}/\hbar}\,\iota^*_{\alpha_0\dots\alpha_k}\big(\Phi_{\alpha_0\dots\alpha_k}\,\sigma_{\alpha_0\dots\alpha_k}\big),$$

with $\Phi_{\underline{\alpha}}$ as in (88) and $c_k = (-1)^{\frac{k(k+1)}{2}}\mathring{\imath}^k$ as before, satisfies $Z(D\sigma) = 0$.

Proof. By the constructions above the Z_k and Z_{k+1} terms cancel their contributions successively. The only term with no partner is the bracket of a top simplex $\underline{\alpha} = (\alpha_0 \dots \alpha_K)$ that meets no further patch, i.e. $U_{\beta\alpha_0\dots\alpha_K} = \varnothing$ for every $\beta \notin \underline{\alpha}$. Here the H-form (88) makes the vanishing immediate. Since $[\Delta, -]$ is a derivation with $[\Delta, H_{\alpha_i}] = 0$ and $[\Delta, \rho_{\alpha_K}] = H_{\alpha_K}$, the leading ρ simply turns into an H,

$$[D, \Phi_{\alpha_0\dots\alpha_K}] = -\mathring{\imath}\hbar\,[\Delta, \Phi_{\alpha_0\dots\alpha_K}] = -\mathring{\imath}\hbar^{K+1}\,\mathrm{Alt}\big(H_{\alpha_K} H_{\alpha_{K-1}} \cdots H_{\alpha_0}\big).$$

On $U_{\alpha_0\dots\alpha_K}$ only the patches $\alpha_0, \dots, \alpha_K$ are present, so the partition of unity restricts to $\sum_{i=0}^K \rho_{\alpha_i} = 1$ there, whence

$$\sum_{i=0}^{K} H_{\alpha_i} = \Big[\Delta, \sum_{i=0}^{K} \rho_{\alpha_i}\Big] = [\Delta, 1] = 0.$$

The $K+1$ operators $H_{\alpha_0}, \dots, H_{\alpha_K}$ are therefore linearly dependent; expanding $H_{\alpha_K} = -\sum_{i<K} H_{\alpha_i}$ in the last argument, multilinearity and the alternating property of Alt force $\mathrm{Alt}(H_{\alpha_K} \cdots H_{\alpha_0}) = 0$. Hence the top bracket term vanishes and $Z(D\sigma) = 0$. ☐

We note that the obtained partition function Z coincides, not surprisingly, with the one (86) in the case that our LSS is in fact a global Lagrangian, i.e. the patches and intersection parts etc. are simply the global Lagrangian restricted to their respective domains. In this case all $\eta_{\underline{\alpha}} = 0$ and we obtain equality.

Remark 7.37. In [20] Getzler works with the full, unordered Čech nerve, where the Φ operators are the symmetric

$$\Phi^{\mathrm{G}}_{\alpha_0\dots\alpha_k} = \frac{\hbar^k}{k+1} \sum_{i=0}^{k} (-1)^i\, H_{\alpha_0} \cdots H_{\alpha_{i-1}}\, \rho_{\alpha_i}\, H_{\alpha_{i+1}} \cdots H_{\alpha_k}.$$

The functional Z is moreover independent of the chosen partition of unity. To see this one lets it vary: introduce an auxiliary simplex Δ^n with de Rham differential d, allow the ρ_α to depend on Δ^n (and the $\iota_{\underline{\alpha}}$ automatically as well), and adjoin d to the differential, so that the $\Phi_{\underline{\alpha}}$ acquire coefficients in $\Omega^\bullet(\Delta^n)$. The recursion and Corollary 7.36 are unaffected. With no change in the proof, Corollary 7.36 becomes

$$Z\big((\mathrm{d} + \delta - \mathring{\imath}\hbar\Delta)\sigma\big) + \mathrm{d}Z(\sigma) = 0.$$

Corollary 7.38. An *observable* is a bosonic, ghost-number-0 cocycle of the total differential, $(\mathrm{d} + \delta - \mathring{\imath}\hbar\Delta)\sigma = 0$. For such σ the value $Z(\sigma)$ is unchanged by a change of the partition of unity and is the same for homologous LSSs.

7.4.2 Globalization on the level of actions

We now place the additional q_2 structure (i.e. the BV bracket) on the Thom–Whitney totalization. We therefore equip $\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|$ with the L_∞ structure $Q = (q_1, q_2, q_3, \dots)$ given by

$$q_1(\omega_1) = -(\delta - \mathring{\imath}\hbar\Delta)\omega_1, \qquad q_2(\omega_1 \odot \omega_2) = (-1)^{|\omega_1|+1}(\omega_1, \omega_2), \qquad q_k = 0 \quad \text{for all } k \geqslant 3.$$

Theorem 7.39. Applying the transfer of Theorem 2.15 to the contraction

$$(\mathcal{F}(\mathcal{M})[[\hbar]], -\mathring{\imath}\hbar\Delta) \underset{P}{\overset{I}{\rightleftharpoons}} (\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, -\mathring{\imath}\hbar\Delta + \delta) \circlearrowleft K$$

and additionally the L_∞ structure Q on $\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|$ given by the bracket $(-,-)$, the transferred L_∞ structure $R = (r_1, r_2, r_3, \dots)$ on $\mathcal{F}(\mathcal{M})[[\hbar]]$ has:

(i) $r_1(S_1) = -(-i\hbar\Delta)S_1$,

(ii) $r_2(S_1 \odot S_2) = (-1)^{|S_1|+1}(S_1, S_2)$, the BV bracket on $\mathcal{F}(\mathcal{M})[[\hbar]]$,

(iii) $r_k = 0$ for all $k \geqslant 3$.

Proof. Using the recursive formulae of Theorem 2.15 with the SDR (83), $f_1 = I$, $g_1 = P$, $q_2 = \pm(-,-)$ and $q_k = 0$ for $k \geqslant 3$, they read

$$f_i = Kq_2F_i^2, \qquad r_i = Pq_2F_i^2 \qquad (i \geqslant 2). \tag{89}$$

For (ii) we now have that from (89) and (13)

$$F_2^2(S_1 \odot S_2) = f_1(S_1) \odot f_1(S_2) = I(S_1) \odot I(S_2),$$

so

$$r_2(S_1 \odot S_2) = (-1)^{|S_1|+1}P\big((I(S_1), I(S_2))\big).$$

On each simplex of the nerve, $(I(S_1), I(S_2))$ restricts to $(S_1, S_2)|_{U_{\alpha_0\cdots\alpha_p}}$ by locality of the bracket. Applying $P = \pi_\Delta p$, the projection p keeps only the vertex evaluations (since the input has de Rham degree 0) and π_Δ averages with the partition of unity:

$$r_2(S_1, S_2) = (-1)^{|S_1|+1}\sum_\alpha \rho_\alpha(S_1, S_2)|_{U_\alpha} = (-1)^{|S_1|+1}(S_1, S_2).$$

Note that the sum of π_Δ contributes in this case only for $n = 0$ (i.e. $\pi_\Delta = \pi$) as the parts of Čech degree $\geqslant 1$ are all 0.

(iii) We first observe that $f_i = 0$ for $i \geqslant 2$. In fact, observe that

$$f_2(S_1, S_2) = (-1)^{|S_1|+1}K\big((I(S_1), I(S_2))\big)$$

Now the inclusion I of a global S_j into the TW complex yields $1 \otimes S_j$ with the appropriate restriction on every Čech level. The bracket therefore is of the from $(I(S_1), I(S_2)) = 1 \otimes (S_1, S_2) = I((S_1, S_2))$ and because $KI = 0$, we immediately obtain $f_2 = 0$.

For $i \geqslant 3$ we repeat the proof we have seen before. Assume $f_2 = \cdots = f_{i-1} = 0$. By formula (13), every monomial in F_i^2 is of the form $f_{j_1}(\cdots) \odot f_{j_2}(\cdots)$ with $j_1 + j_2 = i$ and $j_1, j_2 \geqslant 1$; since $i \geqslant 3$, at least one of the indices j_ℓ is $\geqslant 2$, so $f_{j_\ell} = 0$ by the inductive hypothesis. Therefore $F_i^2 = 0$, and in particular $f_i = Kq_2F_i^2 = 0$.

It now follows immediately from $r_i = Pq_2F_i^2 = 0$ and the fact that $F_i^2 = 0$ for $i \geqslant 3$ that $r_i = 0$ for $i \geqslant 3$. $\square$

Recall the MC set of a L_∞ algebra (14):

$$\mathrm{MC}(V) := \left\{ x \in V^0 \;\middle|\; \sum_{i\geqslant 1}\frac{1}{i!}q_i(x^{\odot i}) = 0 \right\},$$

which we can now consider on both $(\mathcal{F}(\mathcal{M})[[\hbar]], R)$ and $(\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|, Q)$. The MC equation is a finite sum, given that both $r_i = q_i = 0$ for $i \geqslant 3$. We can use the pushforward G_* to obtain solutions in the global complex, using again Lemma 5.11 (the proof goes through in this case as well), which states that $g_i = g_{i-1}Q_i^{i-1}K_i$ for $i \geqslant 2$.

Remark 7.40 ($\hbar$-convergence)**.** The pushforward $G_*(x) = \sum_{i\geqslant 1}\frac{1}{i!}g_i(x^{\odot i})$ is in general a formal sum. As we saw in Remark 5.12, it converges in the $\hbar$-adic topology once the higher data are quantum, here every Čech component $x^{(n)}$ of de Rham degree $n \geqslant 1$ being $\mathcal{O}(\hbar)$. The $\hbar$-counting is the very same; only the ingredients change. We now grade by the simplicial form degree, and the base homotopy is $K = s + wh_\Delta p$, whose block-free part $s + whp$ (Dupont contraction and partition of unity) has deg-weight -1 and whose Δh insertions are the $\hbar$-blocks: each lowers the Čech degree, but the trailing w reconverts it into a simplicial drop, so a block still carries deg-weight -1 and one $\hbar$. With P retaining only the vertex part and $d = \dim N_\bullet\mathcal{U}$, the budget and the estimate $\operatorname{ord}_\hbar g_i \geqslant \lceil (i-1)/d \rceil$ are exactly those of Remark 5.12, so at order $\hbar^k$ only the g_i with $i \leqslant dk+1$ contribute and the series converges $\hbar$-adically. For a cover with one dimensional nerve (no triple overlaps) $d = 1$, so $\hbar^0$ sees g_1 alone and $\hbar^1$ sees g_1, g_2, as in the example below.

Example 7.41 (Three patches with only double overlaps)**.** We carry out the pushforward to first order on a three patch cover with all double overlaps but no triple one. Let $\mathcal{U} = \{U_0, U_1, U_2\}$ with U_{01}, U_{02}, U_{12} nonempty but $U_{012} = \varnothing$, and a partition of unity $\rho_0 + \rho_1 + \rho_2 = 1$. The nerve is then still one dimensional (three vertices and three edges, no 2-cell), so $d = \dim N_\bullet\mathcal{U} = 1$. The Maurer–Cartan datum is

$$x = (S_0, S_1, S_2;\ S_{01}(t)+R_{01}(t)\mathrm{d}t,\ S_{02}(t)+R_{02}(t)\mathrm{d}t,\ S_{12}(t)+R_{12}(t)\mathrm{d}t) \in \mathrm{MC}\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|,$$

the S_α on the patches and the three edge entries on the double overlaps; the de Rham degree one parts $R_{ij}(t)\mathrm{d}t$ are taken to be $\mathcal{O}(\hbar)$.

Since the highest Čech degree is one, two simplifications occur: $h_\Delta = h$ exactly (every $h\Delta h$ string ends with h applied to a 0-cochain, which vanishes), and $\pi_\Delta = \pi$ up to its $n = 1$ term $\mathring{\imath}\hbar\pi\Delta h$, which acting on the $\mathcal{O}(\hbar)$ edge data is $\mathcal{O}(\hbar^2)$ and can be dropped here; by Remark 7.40 only g_1 and g_2 reach order $\hbar^1$. Integrating over the simplices,

$$p(x) = (S_0, S_1, S_2;\ \bar{R}_{01}, \bar{R}_{02}, \bar{R}_{12}), \qquad \bar{R}_{ij} = \int_0^1 R_{ij}(t)\mathrm{d}t,$$

the degree zero edge parts $S_{ij}(t)$ dropping out. The first term is the partition of unity average

$$g_1(x) = \pi p(x) = \rho_0 S_0 + \rho_1 S_1 + \rho_2 S_2.$$

For the second we use $K_2(x \odot x) = K(x) \odot x + IP(x) \odot K(x)$. On the edge s annihilates the de Rham degree zero part, and on a one form $g(t)\mathrm{d}t$ it reads

$$s(g(t)\mathrm{d}t) = \int_0^t g(t')\mathrm{d}t' - t\int_0^1 g(t')\mathrm{d}t',$$

which vanishes at the two vertices $t = 0$ and $t = 1$. Since the final $P = \pi_\Delta p$ keeps only the vertex evaluations, the s part of K drops out and $K(x) \rightsquigarrow whp(x)$, whose vertex values $(hp(x))_\beta = \sum_\alpha \rho_\alpha \bar{R}_{\alpha\beta}$ are, using $\bar{R}_{\beta\alpha} = -\bar{R}_{\alpha\beta}$,

$$(hp(x))_0 = -\rho_1\bar{R}_{01} - \rho_2\bar{R}_{02}, \quad (hp(x))_1 = \rho_0\bar{R}_{01} - \rho_2\bar{R}_{12}, \quad (hp(x))_2 = \rho_0\bar{R}_{02} + \rho_1\bar{R}_{12}.$$

The mixed term $IP(x) \odot K(x)$ dies under P. Here $IP(x) = \bar{S}$ with $\bar{S} = \rho_0 S_0 + \rho_1 S_1 + \rho_2 S_2$ is a global element, the same on every patch, so it factors out of the average and leaves $Pq_2(IP(x) \odot K(x)) = -(\bar{S}, \pi hp(x))$. This vanishes because $\pi h = 0$ (a side condition of the first SDR, Lemma 7.27); explicitly,

$$\pi h\nu = \sum_\beta \rho_\beta (h\nu)_\beta = \sum_{\alpha\beta} \rho_\alpha\rho_\beta \nu_{\alpha\beta} = 0,$$

the symmetric weight $\rho_\alpha\rho_\beta$ contracting the antisymmetric $\nu_{\alpha\beta} = -\nu_{\beta\alpha}$ to zero. The first term, in contrast, survives because its bare leg S_α is patch dependent; it equals $\sum_\alpha \rho_\alpha((hp(x))_\alpha, S_\alpha)$, that is

$$\rho_0(-\rho_1\bar{R}_{01} - \rho_2\bar{R}_{02}, S_0) + \rho_1(\rho_0\bar{R}_{01} - \rho_2\bar{R}_{12}, S_1) + \rho_2(\rho_0\bar{R}_{02} + \rho_1\bar{R}_{12}, S_2).$$

Collecting each $\bar{R}_{ij}$, which occurs once paired with S_i and once with S_j,

$$\begin{aligned} g_2(x \odot x) &= \rho_0\rho_1(\bar{R}_{01}, S_1 - S_0) + \rho_0\rho_2(\bar{R}_{02}, S_2 - S_0) + \rho_1\rho_2(\bar{R}_{12}, S_2 - S_1) \\ &= \sum_{i<j} \rho_i\rho_j(\bar{R}_{ij}, S_j - S_i). \end{aligned}$$

Every $g_{i \geqslant 3}$ carries at least two of the $\bar{R}_{ij}$ and is $\mathcal{O}(\hbar^2)$. Assembling $G_*(x) = \sum_{i\geqslant 1} \frac{1}{i!} g_i(x^{\odot i})$,

$$G_*(x) = \rho_0 S_0 + \rho_1 S_1 + \rho_2 S_2 + \frac{1}{2}\sum_{i<j} \rho_i\rho_j(\bar{R}_{ij}, S_j - S_i) + \mathcal{O}(\hbar^2).$$

Remark 7.42 (Moduli space)**.** Up to canonical transformations, the real object of interest is the set of MC solutions modulo BV equivalence (Definition 7.16), that is $\pi_0(\mathrm{Del}_\infty(L))$. Since the SDR provides two L_∞ weak equivalences F, G, applying the Del_∞ functor should (if Kan property is satisfied) give a weak equivalence of the corresponding Deligne–Getzler groupoids ([18], Prop. 4.9), hence a bijection $\pi_0(\mathrm{Del}_\infty(\mathcal{F}(\mathcal{M})[[\hbar]])) \cong \pi_0(\mathrm{Del}_\infty(\|\mathcal{F}(N_\bullet\mathcal{U})[[\hbar]]\|))$ under which $G_*(S)$ represents $[S]$.

Conclusion

We have seen two methods to obtain a globalized effective action in the BV formalism.

At its core the globalization procedure is the same for both. We lift the action, a solution of the quantum master equation (QME), into a resolution of the global functionals, in which globality is encoded by an additional differential D, and compute the effective action there. It is computed by a fiber BV integral, which is a chain map for the BV Laplacian, so the effective action again satisfies a modified master equation. This is in fact a Maurer–Cartan equation, whose bracket is the antibracket and whose differential is the BV Laplacian Δ of that space together with the globality differential D; the ordinary QME is the case $D = 0$. Since D is acyclic away from degree zero, where the global functionals sit, this solution can be moved there, either by modified BV canonical transformations or, what amounts to the same, by the L_∞ homotopy-transfer projection formulae. The result is a genuinely global effective action.

The first method is formal-geometric. We developed its odd-symplectic version (Section 3.7) and applied it to the AKSZ theory of the spinning particle: the action is lifted to the tangent fibre and Taylor expanded there, and the resulting resolution is governed by the Grothendieck differential D, acyclic except in degree zero. When the spacetime (M, g) is two-dimensional the target is split, so the effective action solves the modified QME directly (Section 5.4) and globalizes with no further work. For a spacetime of general dimension the target is no longer split, and the effective action solves the modified QME only up to a rest term; removing that term meets a potential cohomological obstruction, but once a genuine solution is at hand the globalization proceeds exactly as in the split case.

In the second method we used Čech cohomology. Here again the differential is the Thom–Whitney de Rham differential δ, which by construction acts as the Čech differential on the complex, and is again acyclic except in degree zero for the global functions. As the underlying splitting of the BV field space we used a BV hedgehog; constructing the effective action on it turned out to be a chain map into the Thom–Whitney complex. The globalization procedure is then again by L_∞ transfer. It would be interesting to find an example of a BV theory on which we could carry out this method explicitly.

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