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La costruzione Cattaneo-Mnev-Reshetikhin per teorie di campo su varietà con bordo

The Cattaneo-Mnev-Reshetikhin approach to field theories on manifolds with boundary

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sì sulle teste si può camminare.

I.Schiavone, *Strutture*

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Chapter 1

Introduction

Classical and quantum field theories on manifolds with boundary are extremely interesting from a physical point of view. In fact, the presence of a boundary influences the properties of the physical system and in many cases the physics of the boundary is fundamental.

Recently, a new approach to field theories on manifolds with boundary has been developed by Cattaneo, Mnev and Reshetikhin (CMR) [11]. The idea is to study the boundary terms emerging from the variation of the action S of the field theory, without requiring any boundary condition on bulk fields. The analysis in [11] shows that these terms induce a field theory on the boundary.

When the field theory in the bulk is a gauge theory, it is convenient to use the so called Batalin-Vilkovisky (BV) formalism ([5, 6]). This formalism, known also as the antifield formalism, has two great advantages. First of all, it can be used to deal with the perturbative expansion of the partition function for theories with gauge symmetries, not necessarily given by a group action on the space of classical fields (for which it is enough to use the Faddeev-Popov method [19] or the BRST quantization [8]). Second, it gives an useful and elegant interpretation of gauge theories, which are rewritten using the language of the graded symplectic geometry. The geometric intuition appears to be fundamental when we study a gauge theory on a manifold with boundary ([12]), in order to understand what kind of field theory lives on the boundary. Indeed, in ([12]) it is shown that the graded geometrical structures of a gauge theory in the bulk induce analogous structures (but with shifted degrees) on the boundary, which define a hamiltonian theory (that is, the space of boundary fields is a phase space).

One of the most interesting aspects of the CMR construction is that it is completely general and it can be applied to any local field theory defined by an action functional, depending on fields and on a finite number of their derivatives. Moreover, as pointed in [11, 12], the procedure can be iterated to lower dimensions. We can for instance think the boundary as the union of more elementary components. For example, a connected boundary of a two-dimensional manifold is a circle S^1, which can be thought as the union of two intervals, as in Fig. (1.1). Now every single boundary component has its boundary, that we call **corner**. In the case $d = 2$ (Fig. (1.1)), the corners are the extremal points of the intervals. Since the $(d-1)$-dimensional theory is local, *i.e.* expressed by a local action functional, we can apply the CMR procedure and get a $(d-2)$-dimensional theory. In general, this induction stops before arriving to $d = 0$ (points) for regularity reasons, but for some particular field theories (for example, of

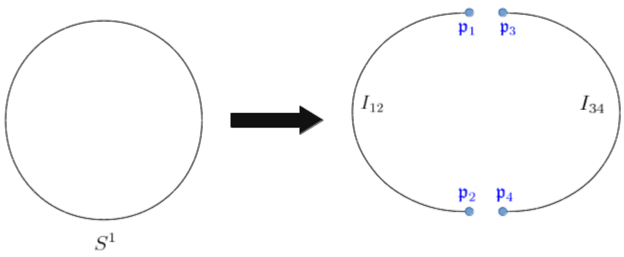

Figure 1.1: A circle S^1 can be decomposed as the union of two intervals, I_{12} and I_{34}. The intervals have two boundaries, called corners, given by their extremal points $\mathfrak{p}_i$.

the AKSZ type [1]), the procedure can be iterated until the points.

The goal behind this construction is to understand how to compute the partition function on a generic d-dimensional spacetime by first computing it on more elementary spaces and then gluing them by means of combinatorial procedures. For example a two dimensional surface M can be decomposed as in Fig. (1.2) in terms of discs, cylinders and pants. The procedure should tell how to calculate the partition function on M by first computing it on discs, cylinders and pants and then gluing the results. This idea was axiomatized by Atiyah ([3]) for topological quantum field theories, using the language of categories, and the treatment of Chern-Simons theory in three dimensions in [35] is the canonical example. The quantum version of the CMR construction, presented in [13], is therefore a proposal to lift Atiyah's axioms to the level of gauge theories.

It is natural to ask if it is possible to apply the same idea to a more refined partition of the spacetime, like for instance that given by a triangulation as in Fig. (1.3): the elementary pieces emerging from this decomposition are points, line segments, triangles and their higher dimensional counterparts. This is the idea underlying the formulation of the so called extended topological quantum field theory (ETQFT) (see for example [10, 21, 4]). This is a far reaching extension of Atiyah's framework, but at the present stage is very abstract and far from a clear application to physically interesting theories. Nevertheless, we can say that the basic idea is that a fully extended topological QFT can be defined by assigning something to points, *i.e.* in maximal codimension, and then computing quantities in lower codimension by universal gluing procedures. The CMR construction appears to be a good candidate to realize these ideas in an explicit and computable way. In this sense, an encouraging result is [23], in which the CMR construction is used to recover the partition function of a two-dimensional Yang-Mills theory on surfaces with boundary and corners. This is the first explicit example of the perturbative quantization of a gauge theory on a manifold with boundary and corners, using such techniques.

This thesis is a first step in the direction of understanding the role of the corners in the quantization. The idea we want to explore, already present in [23] and reflecting some aspects of the abstract analysis in the ETQFT approach (see [21]), is that the extended topological quantum field theory should assign to the corners $\partial\Sigma$ an associative algebra of operators and to the boundary Σ the Hilbert space of states, which should carry a representation of the algebra. The semiclassical version of this picture means that the theory associates a Poisson structure to $\partial\Sigma$ and to Σ a symplectic manifold (a

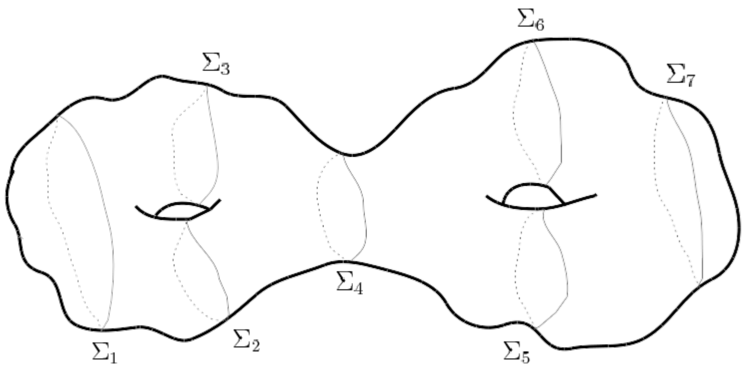

Figure 1.2: An example of cutting-gluing of a two-dimensional surface, cut along one-dimenional boundaries Σ_i. The partition function is calculated in the more elementary pieces, which are then glued together.

phase space), whose quantization, respectively, will give the operator algebra and the space of states. Moreover, a Poisson morphism from the observables on the corners to observables on the boundary should be defined in such a way that after quantization we get a quantized algebra acting on the Hilbert space of states.

A further complication we have to deal with is that we work with gauge theories and hence all these structures should encode gauge transformations. In the BV setting, this means that we have to replace vector spaces and Lie algebras with their homotopical generalizations, complexes and L_∞ algebras. We do not discuss L_∞ algebras here, but some basic notions are given in Appendix (C).

The need of such mathematical structures is a direct consequence of gauge invariance and represents a technical difficulty that should not make the full picture disappear. In this thesis, we discuss the semiclassical picture and we try to understand how these structures should be defined. In particular, we find that

1. After the choice of a polarization (a maximal set of commuting functionals, called "polarized functionals") on the space of corner fields, we can associate a L_∞ algebra to the corners.

2. The polarization induces corner conditions on boundary fields, in such a way that we obtain a differential graded Lie algebra on the boundary.

We are led to conjecture that there exist a L_∞ morphism, a collection of maps satisfying certain coherence relations (see Appendix (C)), between these structures. The first component of such a morphism is calculated in general and the full problem is solved in two concrete examples:

- The **two-dimensional BF theory**, which is a topological field theory with action

$$S = \int_M \langle B, F_A \rangle ,$$

where M is an oriented two-dimensional manifold (with boundary and corners); $F_A = dA + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$ is the field strength of the connection one form A

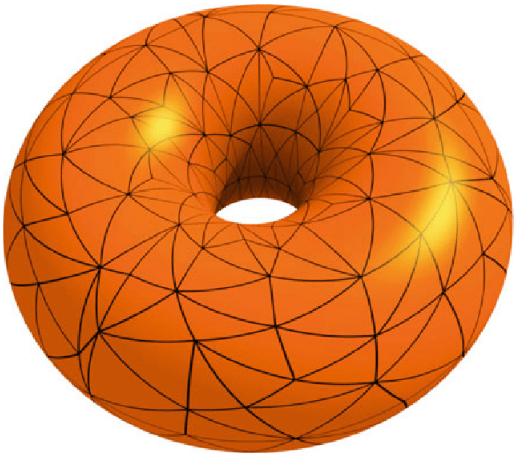

Figure 1.3: An example of triangulation of a torus. From this decomposition, faces, edges and corners emerge. This figure has been taken from [24].

(the gauge field) and $B \in \Omega^0(M, \mathfrak{g}^*)$. Here $\mathfrak{g}$ is a finite dimensional Lie algebra and $\langle , \rangle$ denotes the wedge product of differential forms and the standard pairing between the Lie algebra and its dual.

- The **three-dimensional Chern-Simons theory**, a topological field theory with action
$$S = \int_M \frac{1}{2} \langle A, A \rangle_g + \frac{1}{6} \langle A, [A, A] \rangle_g ,$$
where M is a three-dimensional manifold (with boundary and corners); $A \in \Omega^1(M, \mathfrak{g})$ is the connection one form; $\mathfrak{g}$ is a Lie algebra with a symmetric pairing and $\langle , \rangle_g$ denotes the wedge product of differential forms and the symmetric pairing between elements of $\mathfrak{g}$.

In particular, the guiding example is the three-dimensional Chern-Simons theory: in this case it is known that the quantization brings a representation of the Kac-Moody algebra, an example of infinite dimensional Lie algebra. Our construction shows that on the corners it is possible to associate directly to the corners the Kac-Moody algebra. Therefore, the quantization of the image of the Kac-Moody algebra by the L_∞ morphism is expected to act on the space of states, associated to the boundary, and to reproduce the known results for the quantization of Chern-Simons theory. This could be a first step in the direction of realizing Chern-Simons theory as an extended topological quantum field theory, according to [10, 21, 4].

This work is organized in the following way. After a brief study of graded geometry, we move to the **Chapter 2**, in which we recall the problems in the perturbative expansion of the partition function in gauge theories and the Faddeev-Popov and the BRST (a.k.a. cohomological) methods to deal with them. Next, we introduce the Batalin-Vilkovisky (BV) formalism and we describe in details the geometric interpretation it gives to gauge theories. Moreover, we explain how it can be used to perform the gauge-fixing.

In **Chapter 3**, we start our discussion on (classical) field theories on manifolds with boundary and we present the CMR construction. We focus on the case when the field theory in the bulk is a gauge theory and we extend the construction to the case of manifolds with boundary and corners. As detailed examples, we study Chern-Simons theory in three dimensions and BF theory in two dimensions.

Finally, **Chapter 4** is devoted to the study of the algebraic structures of polarized functionals on the space of corner fields and on the space of boundary fields. We focus on AKSZ theories. Our discussion leads to the conjecture that there exist a L_∞ morphism linking these algebraic structures. We build explicitly the first component of this morphism in general and we study the others in two concrete examples: the two-dimensional BF theory and the three-dimensional Chern-Simons theory. In this last example, for a particular choice of the polarization on the space of corner fields, we reproduce the Kac-Moody algebra on the corners.

Graded Geometry

Since we will make an extensive use of the language of graded geometry, we prefer to introduce at this point some basic notions on this subject. The spaces we will consider in this introductory discussion are finite-dimensional. However, the discussion can be extended in the infinite-dimensional case. We refer to [28] and [15] for basic notions of graded geometry.

The main idea behind graded geometry is to extend classical geometry by allowing coordinates with an integer **degree** different from zero. Classical geometry is recovered in degree zero. Supergeometry corresponds to having coordinates with degrees zero and one only.

Definition 1. *A graded (or $\mathbb{Z}$-graded) vector space is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) with the decomposition labelled by integers*

$$V = \oplus_{i\in\mathbb{Z}} V_i.$$

A vector $v_i \in V_i$ is called **homogeneous** and its **degree** is the quantity $|v_i| = i$. A non homogeneous element $v \in V$ can always be written as the sum of homogeneous components.

Many basic concepts from linear algebra and operations between vector spaces (tensor product, direct sum and so on) can be extended to the graded case. In particular, the **dual** vector space $(V_i)^*$ is defined as V^*_{-i}.

A relevant operation we can define on graded vector spaces is the **shift by k**, with $k \in \mathbb{Z}$.

Definition 2. *Let V be a $\mathbb{Z}$-graded vector space. The $\mathbb{Z}$-graded vector space $V[k]$ shifted by degree k is defined as*

$$V[k] = \oplus_{i\in\mathbb{Z}} V_{i+k}.$$

A crucial concept in graded geometry is that of **graded commutative algebra**.

Definition 3. *Let V be a graded vector space. If it is defined on V an associative product $\cdot$ which respects the gradings:*

$$V_i \cdot V_j \subset V_{i+j}$$

and it is ***commutative****:*

$$v_i v_j = (-1)^{|v_i||v_j|} v_j v_i,$$

then the couple $(V, \cdot)$ is called a ***commutative graded algebra****.*

A **derivation** D of degree $|D|$ of the product $\cdot$ is a map $D : V \to V$, such that

$$D(v_i v_j) = (Dv_i)v_j + (-1)^{|D||v_i|} v_i(Dv_j), \quad v_i \in V_i,\, v_j \in V_j.$$

Example 1. If V is a graded vector space over a $\mathbb{R}$ (or $\mathbb{C}$), we define the **graded symmetric algebra** $S(V)$ as the set of polynomial functions on V:

$$S(V) \ni f = \sum_k \sum_{a_1,..,a_k} f_{a_1,a_2....,a_k} v^{a_1} v^{a_2} ....v^{a_k} \tag{1.1}$$

where $f_{a_1,...,a_k}$ are real (complex) numbers. Homogeneous components in $S(V)$ are homogeneous polynomials. The associative product in $S(V)$ is simply the product of functions, which preserves the gradings. Therefore $(S(V), \cdot)$ is a graded commutative algebra.

Definition 4. *(Differential graded Lie algebra) A* ***differential graded Lie algebra*** *$(V, D, [,])$ is a graded vector space V together with a bilinear map $[\bullet, \bullet] : V_i \otimes V_j \to V_{i+j}$ and a differential $D : V_i \to V_{i+1}$, with $D^2 = 0$, such that:*

- *For $v_i \in V_i$ and $v_j \in V_j$, we have $[v_i, v_j] = (-1)^{ij+1}[v_j, v_i]$.*
- *For $v_i \in V_i$, $v_j \in V_j$ and $v_k \in V_k$, we have*

  $$[v_i, [v_j, v_k]] = [[v_i, v_j], v_k] + (-1)^{ij}[v_j, [v_i, v_k]].$$

  This is the graded Jacobi identity.
- *The differential D is a derivation for $[,]$.*

Remark 1. Let $(V, D, [,])$ be a differential graded Lie algebra. Assume that V is an associative commutative algebra (3) with an associative product $\cdot$ in such a way that:

- For $v_i \in V_i$, $v_j \in V_j$ and $v_k \in V_k$, $[,]$ satisfies the graded Leibniz property

  $$[v_i \cdot v_j, v_k] = v_i \cdot [v_j, v_k] + (-1)^{jk}[v_i, v_k]v_j;$$

- For $v_i \in V_i$ and $v_j \in V_j$, D is a derivation for the associative product.

Then, $(V, \cdot, D, [,])$ is a **differential graded Poisson algebra**.

Graded Manifolds

Recall that an "ordinary" non-graded manifold M is a topological space which is locally homeomorphic to $\mathbb{R}^n$, that is, we can associate local coordinates to open sets on M. The global structure of the manifold M is recovered by gluing smoothly the different coordinate systems in overlapping charts.

A graded manifold $\mathcal{M}$ is defined in a similar way. It is a topological space with local assignments of graded coordinates $x_j \in \mathbb{R}^n$, $v_i \in V_i$, where $V = \oplus_i V_i$ is a graded vector space. These coordinate systems are glued together from a local patch to an other with smooth degree-preserving maps.

The set of functions over $\mathcal{M}$, $C^\infty(\mathcal{M})$, is locally isomorphic to $C^\infty(U) \otimes S(V)$, where U is homeomorphic to $\mathbb{R}^n$ and $S(V)$ is the graded symmetric algebra of V. This means that a function $f \in C^\infty(\mathcal{M})$ can be locally written as

$$f = \sum_k \sum_{a_1,..,a_k} f_{a_1,..,a_k}(x) v^{a_1} v^{a_2} ...v^{a_k},$$

where $f_{a_1,...,a_k}(x) \in C^\infty(U)$.

Example 2. (Shifted tangent bundle) Consider an ordinary n-dimensional manifold M and its **shifted tangent bundle** $T[1]M$. Local coordinates on M, $\{x_1, .., x_n\}$, have degree zero, while the corresponding coordinates on the fibers, $(\theta_1, ..., \theta_n)$ have degree $|\theta_i| = |x_i| + 1 = 1$. The gluing rule is defined as:

$$\hat{x}^i = \hat{x}^i(x), \quad \hat{\theta}^i = \frac{\partial \hat{x}^i}{\partial x^j}\theta^j. \tag{1.2}$$

Functions on $T[1]M$ have the local expansion

$$f(x,\theta) = \sum_{k=0}^{n} \sum_{a_1,..,a_k} f_{a_1,...,a_k}(x)\theta^{a_1}....\theta^{a_k},$$

which are naturally identified with differential forms (θ^i tranforms like dx^i):

$$C^\infty(T[1]M) = \Omega^\bullet(M). \tag{1.3}$$

Remark 2. Given the identification $C^\infty(T[1]M) = \Omega^\bullet(M)$, we can interpret the integration of a top form ω on M as

$$\int_M \omega = \int_{T[1]M} \tilde{\omega} \tag{1.4}$$

where $\tilde{\omega}$ is the element in $C^\infty(T[1]M)$ corresponding to $\omega \in \Omega^{top}(M)$. In local coordinates, the RHS of Eq. (1.4) is

$$\int dx^1...dx^n d\theta^1...d\theta^n\, \omega_{1,...,n}(x)\theta^1...\theta^n, \tag{1.5}$$

where the integration in the θ's is intended as a Grassmann integration

The measure $dx^1...dx^n d\theta^1...d\theta^n$ is also called a **Berezinian** and it is denoted with μ. It is independent on the choice of coordinates. Notice that $|\mu| = n$.

Example 3. (Shifted cotangent bundle) Consider again an ordinary n-dimensional manifold M. We can associate to it an other graded manifold, the **shifted cotangent bundle** $T^*[-1]M$. Local coordinates are $(x_1, ..., x_n; \theta_1, ..., \theta_n)$, where the x_i's are local coordinates on M with degree zero and θ_i's are local coordinates on the fibers with degree $|\theta_i| = -|x_i| - 1 = -1$. The gluing rule is the following

$$\hat{x}^i = \hat{x}^i(x), \quad \hat{\theta}_i = \frac{\partial x^i}{\partial \hat{x}^j}\theta_j. \tag{1.6}$$

Functions on $T^*[-1]M$ have the local expansion

$$f(x,\theta) = \sum_{k=0}^{n} \sum_{a_1,..,a_k} f_{a_1,...,a_k}(x)\theta^{a_1}...\theta^{a_k}, \tag{1.7}$$

which, since θ_i transforms like ∂_i, are naturally identified with multivector fields

$$C^\infty(T^*[-1]M = \Gamma(\Lambda^\bullet TM)).$$

Given a graded manifold $\mathcal{M}$, graded vector fields are defined as graded derivations of $C^\infty(\mathcal{M})$. The set of graded vector fields on $\mathcal{M}$ is a graded Lie algebra with the graded bracket

$$[X,Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X,$$

and we denote it with $Vect(\mathcal{M})$.

Example 4. Every graded manifold is equipped with the so called **Euler vector field**, which is locally written as

$$E = \sum_k |v^k| v^k \frac{\partial}{\partial v^k}, \quad |E| = 0, \tag{1.8}$$

and acts on a homogeneous function f as

$$E(f) = |f| f. \tag{1.9}$$

Definition 5. *A* ***cohomological vector field*** *is a graded vector field $Q \in Vect(\mathcal{M})$ such that:*

- $|Q| = 1$*;*
- $[Q, Q] = 0$*.*

The couple $(\mathcal{M}, Q)$ is called ***differential graded manifold****.*

Example 5. Let M be an ordinary manifold and consider the shifted tangent bundle $T[1]M$. Define the vector field (in local coordinates):

$$Q = \sum_k \theta^k \frac{\partial}{\partial x^k}. \tag{1.10}$$

Since $|Q| = 1$ and $[Q, Q] = 0$, $(T[1]M, Q)$ is a differential graded manifold. The action of Q on $C^\infty(T[1]M)$ corresponds to the de Rham differential on $\Omega^\bullet(M)$.

Example 6. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, with dimension n, and consider the shifted $\mathfrak{g}[1]$, whose coordinates are $\{c_i\}_{i=1}^n$ of degree 1. Functions on $\mathfrak{g}[1]$ have the form

$$f = \sum_k \sum_{a_1,..,a_k} f_{a_1,..,a_k} c^{a_1} ... c^{a_k}, \quad f_{a_1,..,a_k} \in \mathbb{R}\,(\mathbb{C}), \tag{1.11}$$

and can be identified with elements of the exterior algebra of the dual Lie algebra, $\Lambda^\bullet \mathfrak{g}^*$.

On $\mathfrak{g}[1]$ we can define a cohomological vector field d_{CE} given by

$$d_{CE} = \frac{1}{2} \sum_{i,j,k} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k}, \tag{1.12}$$

where f_{ij}^k are the structure constant of $\mathfrak{g}$ in a base $\{e_i\}_{i=1}^n$, $[e_i, e_j] = \sum_k f_{ij}^k e_k$.

Notice that d_{CE} has degree 1. The fact that it squares to zero follows from the Jacobi identity for the commutator in $\mathfrak{g}$. Therefore, $(\mathfrak{g}[1], d_{CE})$ is a differential graded manifold. The complex $(\Lambda^\bullet \mathfrak{g}^*, d_{CE})$ is known as the **Chevalley-Eilenberg** complex

Chapter 2

Batalin-Vilkovisky Formalism

In order to study a quantum field theory, deifned by the action S, we have to give a meaning to the partition function $\mathcal{Z}$:

$$\mathcal{Z} = \int_{\mathcal{F}} e^{\frac{i}{\hbar}S} D\phi, \tag{2.1}$$

with $\mathcal{F}$ the space of fields. One way to study the integral above is by performing a saddle-point approximation at $\hbar \to 0$, which is an expansion around the critical points of the action, namely the field configurations which extremize the action S. In order for the stationary phase method to work, the critical points of S must be non-degenerate (isolated). If the field theory has a gauge symmetry, this is not the case, since critical field configurations are defined up to gauge transformations, and one fails to make sense of the asymptotic expansion of the partition function. One way to solve this problem is to consider field configurations linked by a gauge tranformation as equivalent, so that one is led to the problem of defining (2.1) as an integral over the orbits of the gauge action, or rather to choose one representative for each orbit. The choice of the representative and the suppression of the infinite equivalent field configurations is known as **gauge fixing**. In order to do this, Faddeev and Popov (FP) ([19]) introduced unphysical fermionic fields, called ghost fields, and lagrange multipliers in the action. Their method is explained in more details in the next subsection.

As in [8, 7] Becchi, Rouet, Stora (and indipendently Tyutin, [32]) observed for the first time, the Faddeev-Popov gauge fixed action, S_{FP}, has a fermionic symmetry, called in their honour "BRST symmetry". This symmetry is defined as a gauge transformation with the ghost field as gauge parameter. Since it associates a fermionic field to a bosonic field, this symmetry is fermionic. Moreover, it squares to zero. The gauge fixed action S_{FP} can be rearranged as the sum of the old non gauge-fixed action S and a BRST exact term. This result was then generalized for more general actions including also matter fields. Since then the BRST method is the modern way to think to the FP construction and to perform the gauge fixing.

The BRST method can be applied only in the case of field theories with gauge symmetries given by the action of a Lie group. However, one can be interested in actions with a symmetry which, for instance, closes only when the field equations of motion are satisfied or which are not irreducible. Examples are supergravity (without auxiliary fields) and the BF theory (the "topological limit" of Yang-Mills-type theories) in dimension $d \geq 4$. The **Batalin-Vilkovisky formalism** (BV) ([5, 6]) was introduced in order to perform the gauge fixing for these kind of theories too. Moreover, it reduces to the BRST method when the symmetry is given by a group action. An other relevant aspect of the BV formalism is that it provides an useful and elegant geometrical picture of gauge theories, reformulated using the language of graded symplectic geometry.

As we will see in the next chapter, this last aspect becomes fundamental when one studies gauge theories on manifolds with boundary.

In this chapter we first recall the main features of the Faddeev-Popov construction and of the BRST approach, and then we move to the BV formalism. The last section is devoted to illustrating a relevant kind of BV sigma models, the so called AKSZ theories.

We mainly follow the lectures [27]. Other references used are [14, 20, 31, 1]. For other insightful discussions on the BV, see [34].

2.0.1 Faddeev-Popov method

Suppose that the space of fields $\mathcal{F}$ is acted on by a Lie group $\mathcal{G}$, whose Lie algebra we denote with $\mathfrak{g}$, and that the action S is invariant under the action of $\mathcal{G}$. This means that $S[\Phi] = S[\Phi^g]$, where $\Phi \in \mathcal{F}$ and Φ^g denotes the action by $g \in \mathcal{G}$. The gauge invariance of the action tells that S essentially depends on the gauge orbits on the space of fields and therefore we can look at S as a function on the quotient $\mathcal{F}/\mathcal{G}$.
Following these ideas, we understand the integral (2.1) as an integral over the coset

$$\mathcal{Z} = Vol\mathcal{G} \int_{\mathcal{F}/\mathcal{G}} e^{\frac{i}{\hbar}\tilde{S}} D[\phi], \tag{2.2}$$

where $D[\phi]$ is a measure on the coset and $\tilde{S} \in C^\infty(\mathcal{F}/\mathcal{G})$ is such that

$$S = p^*\tilde{S},$$

with $p : \mathcal{F} \to \mathcal{F}/\mathcal{G}$ the quotient map. The factor $Vol(\mathcal{G})$ is the volume of the gauge orbit, which is the source of the degeneracy problem of critical points of the action.

The gauge-fixing of the partition function is performed in the following way. Choose a smooth function $F : \mathcal{F} \to \mathfrak{g}$ with the property that $\sigma = F^{-1}(0) \subset \mathcal{F}$ intersects transversally each $\mathcal{G}$-orbit once[1]. We call F the gauge fixing function. The transversality means that the tangent space of $\mathcal{F}$ at a certain field configuration ϕ can be decomposed in the direct sum

$$T_\phi\mathcal{F} = T_\phi\sigma \oplus T_\phi Orb(\phi) = T_\phi\sigma \oplus \mathfrak{g},$$

since $T_\phi Orb(\phi)$ is isomorphic to $\mathfrak{g}$[2].

The integral (2.2) is then rewritten as

$$\mathcal{Z} = Vol\mathcal{G} \int_{\mathcal{F}} \delta_0(F(\phi)) \cdot detFP(\phi) \cdot e^{\frac{i}{\hbar}S} D\phi, \tag{2.3}$$

where:

- $\delta_0(F(\phi))$ is a functional delta at $0 \in \mathfrak{g}$;
- $detFP(\phi)$ is the so called Faddeev-Popov determinant. The matrix $FP(\phi)$ is the differential of the map $F(\phi)$, restricted to the tangent space of the gauge orbit.

[1] Or a finite number of times in general.

[2] The isomorphism is given by the association to the generators T_a of $\mathfrak{g}$ of the fundamental vector fields v_a, which span $T_\phi Orb(\phi)$.

The presence of the determinant ensures invariance under deformations of the gauge fixing function F.
It is useful to rewrite[3] (2.3) by introducing the Faddeev-Popov ghosts $c \in \Omega^0(M, \mathfrak{g}[1])$, $\bar{c} \in \Omega^d(M, \mathfrak{g}^*[-1])$ and the Lagrange multiplier $\lambda \in \Omega^d(M, \mathfrak{g}^*)$. We use an integral representation of the delta function (the Fourier transform on the unity) and the rules of gaussian integration of fermionic fields to write

$$\delta_0(F(\phi)) = \int D\lambda\, e^{\frac{i}{\hbar}\int_M \langle \lambda, F(\phi)\rangle},$$
$$detFP(\phi) = \int DcD\bar{c}\, e^{\frac{i}{\hbar}\int_M \langle \bar{c}, FP(\phi)c\rangle},$$

where $\langle\rangle$ is the canonical pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

Therefore, integral 2.3 becomes

$$Vol\mathcal{G} \int_{\mathcal{F}_{FP}} e^{\frac{i}{\hbar}S_{FP}} D\phi DcD\bar{c}D\lambda, \tag{2.4}$$

where the **Faddeev-Popov action** is

$$S_{FP}[\phi, c, \bar{c}, \lambda] = S[\phi] + \int_M \langle \lambda, F(\phi)\rangle + \int_M \langle \bar{c}, FP(\phi)c\rangle \tag{2.5}$$

and the space of Faddeev-Popov fields is

$$\mathcal{F}_{FP} = \underset{\phi}{\mathcal{F}} \oplus \Omega^0(\underset{c}{M}, \mathfrak{g}[1]) \oplus \Omega^d(M, \underset{\bar{c}}{\mathfrak{g}}^*[-1]) \oplus \Omega^d(\underset{\lambda}{M}, \mathfrak{g}^*).$$

Therefore, at the price of introducing new non-physical fields, the action we obtain is non degenerate and allows for a well-defined asymptotic expansion of the partition function.

Example 7. Let (M, g_M) be a Riemannian manifold and $\mathcal{F} = \Omega^1(M) \otimes \mathfrak{g}$ the space of connection one-forms of a trivial principal G-bundle over M (gauge fields A), with $\mathfrak{g}$ the Lie algebra of the structure group G, which is a compact Lie group. The Yang-Mills action is given by

$$S_{YM}[A] = \frac{1}{2}\int_M tr(F_A \wedge *F_A) \in C^\infty(\mathcal{F}),$$

where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature two-form (the field strength) and $*$ is the Hodge operation associated to the metric g_M.

The action is invariant under the transformation

$$A \to A^g = gAg^{-1} + g^{-1}dg,$$

where $g \in C^\infty(M, G) \equiv \mathcal{G}$ (namely, a function with values in the structure group G), which is the group of gauge transformations. The infinitesimal version of the tranformation above can be found by expanding g around the identity $g \simeq 1 + \alpha$, with $\alpha \in \Omega^0(M, \mathfrak{g}) \equiv Lie(\mathcal{G})$ (which is the Lie algebra of the group of gauge transformations):

$$A \to d_A\alpha = d\alpha + [A, \alpha].$$

[3] We use the notation $\Omega^p(M, \mathfrak{g}) \equiv \Omega^p(M) \otimes \mathfrak{g}$ to indicate the space of differential p-forms with values in $\mathfrak{g}$.

For the gauge fixing function $F : \mathcal{F} \to Lie(\mathcal{G})$ we can choose

$$F(A) = d^\dagger A,$$

where $d^\dagger$ is the codifferential. In coordinates, this is just the divergence $\partial_\mu A^\mu$, which is a function on M with values in $\mathfrak{g}$. In this case, the Faddeev-Popov matrix is

$$FP(A) = d^\dagger d_A : Lie(\mathcal{G}) \to Lie(\mathcal{G}),$$

that is, we compose the gauge fixing function with the infinitesimal gauge action d_A. The gauge fixed action (2.5) is therefore

$$S_{FP}(A, c, \bar{c}, \lambda) = S_{YM}[A] + \int_M \langle \lambda, d^\dagger A \rangle + \int_M \langle \bar{c}, d^\dagger d_A c \rangle . \tag{2.6}$$

Now the kinetic term in (2.6) is non degenerate and we can define the perturbative expansion of the partition function. Therefore, we can infer the Feynman rules for the propagators of the gauge field and of the ghosts fields and for the interactions between them.

2.0.2 BRST approach

The main idea of the Faddeev-Popov method is to take the quotient of the space of fields with the gauge group of the theory, in such a way to "count only one time" all those field configurations linked by a gauge transformation. Now we move to the discussion of the **BRST** (or **cohomological**) **approach**, which was born as a reformulation of the Faddeev-Popov method. We won't follow the original approach of the autors, but a more abstract one ([27]).

First of all, a **quantum BRST theory** is defined as the set of the following data:

- A **space of BRST fields**, $\mathcal{F}_{BRST}$, endowed with a $\mathbb{Z}$-grading, called **ghost** grading, such that its 0-th degree part is the space of physical fields $\mathcal{F}$.
- A vector field Q, acting on $\mathcal{F}_{BRST}$, with $gh(Q) = 1$ and satifying the nilpotency property $Q^2 = [Q, Q] = 0$. Q is called **BRST operator** and it encodes the gauge transformations on fields. This vector field is called cohomological (see definition 5).
- An **action** $S \in C^\infty(\mathcal{F}_{BRST})$, with $gh(S) = 0$, such that $Q(S) = 0$ (namely it is gauge invariant.)
- A **measure** μ on $\mathcal{F}_{BRST}$ such that

$$div_\mu Q = 0. \tag{2.7}$$

An immediate consequence of property (2.7) is that the integral of Q-exact functions of BRST fields is zero:

$$\int_{\mathcal{F}_{BRST}} Q(f)\, \mu = 0, \quad \forall f \in C^\infty(\mathcal{F}_{BRST}). \tag{2.8}$$

In fact, by definition of the divergence, we have that

$$\int_{\mathcal{F}_{BRST}} Q(f)\mu = -\int_{\mathcal{F}_{BRST}} (div_\mu Q \cdot f)\,\mu = 0, \quad \forall f \in C^\infty(\mathcal{F}_{BRST}).$$

We define a BRST integral as an integral of a Q-closed (gauge invariant) function $f \in C^\infty(\mathcal{F}_{BRST})$:

$$\int_{\mathcal{F}_{BRST}} f\,\mu, \quad Q(f) = 0. \tag{2.9}$$

Since the BRST integral of Q-exact functions is zero, the BRST integral is a map which assigns real (or complex) numbers to the cohomology classes of Q.

This result allows to perform the gauge-fixing in the BRST setting. In fact, we are interested in integrals of the kind

$$\mathcal{Z} = \int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}S}\mu$$

Since S is Q-closed, we know that an asymptotic expansion around its critical points can not be performed. However, the integrand is Q-closed too. The idea is to deform the integrand by a Q-exact term in such a way to find a perturbatively well-defined integral:

$$\mathcal{Z} = \int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}S}\mu = \int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}(S+Q(\Psi))}\mu, \tag{2.10}$$

where the function $\Psi \in C^\infty(\mathcal{F}_{BRST})$ is known as the **gauge fixing fermion**. Notice that $gh(\Psi) = -1$, in order to add a $gh = 0$ term to the classical action.

Remark 3. Correlation functions are calculated in the same way as the partition function. If $\{\mathcal{O}_i\}_{i=1}^n$ is a set of observables, $\mathcal{O}_i \in C^\infty(\mathcal{F}_{BRST})$ and $Q(\mathcal{O}_i) = 0$, the expectation value of their product is

$$\langle \mathcal{O}_1....\mathcal{O}_n\rangle = \frac{1}{Z}\int_{\mathcal{F}_{BRST}} \mathcal{O}_1....\mathcal{O}_n e^{\frac{i}{\hbar}S}\mu = \frac{1}{Z}\int_{\mathcal{F}_{BRST}} \mathcal{O}_1....\mathcal{O}_n e^{\frac{i}{\hbar}(S+Q(\Psi))}\mu, \tag{2.11}$$

since the integrand is Q-closed.

Example 8. (Yang-Mills: Faddeev-Popov via BRST). Consider a pure Yang-Mills theory, like in example (7). Let's try to build a quantum BRST package in the following way:

- The space of BRST fields is $\mathcal{F}_{BRST} = \underset{A}{\Omega^1(M,\mathfrak{g})} \oplus \underset{c}{\Omega^0(M,\mathfrak{g}[1])}$. By definition, $gh(A) = 0$ and $gh(c) = 1$.
- The BRST operator is

$$Q = \int_M \left\langle d_A c, \frac{\delta}{\delta A}\right\rangle + \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c}\right\rangle. \tag{2.12}$$

  Its action on BRST fields is $QA = d_A c$, namely the infinitesimal gauge transformation with gauge parameter the field c, and $Qc = \frac{1}{2}[c,c]$, which is the action of the Chevalley-Eilenberg differential. In particular, the condition that Q squares to zero leads to the Jacobi identity for the structure constant of $\mathfrak{g}$ and to the property that the infinitesimal action $Lie(\mathcal{G}) \to Vect(\mathcal{F})$ preserves the bracket.

- The action is S_{YM}, which has zero ghost degree. The equation $Q(S_{YM}) = 0$ is equivalent to the gauge-invariance of the classical action.

Assuming that a measure μ on $\Omega^1(M,\mathfrak{g})\oplus\Omega^0(M,\mathfrak{g})$ is defined, $\mu = DADc$, the partition function is

$$\mathcal{Z} = \int_{\mathcal{F}_{BRST}} e^{\frac{i}{\hbar}S_{YM}} DADc.$$

Notice that it is not possible to construct a non zero gauge fixing fermion with $gh = -1$ with the fields at disposal. Moreover, since the Yang-Mills action does not depend on ghosts, the integral over ghost fields is identically zero. Therefore, the package we have built is incomplete.

This suggest that we have to enlarge the space of fields. If we recall $\mathcal{F}_{min} = \Omega^1(M,\mathfrak{g})\oplus\Omega^0(M,\mathfrak{g}[1])$, $Q_{min} \equiv Q$, $\mu_{min} \equiv DADc$, we construct the true BRST package in the following way:

- The new space of BRST fields is

$$\mathcal{F}_{new} = \mathcal{F}_{min} \oplus \Omega^d(M, \underset{\bar{c}}{\mathfrak{g}^*}[-1]) \oplus \Omega^d(\underset{\lambda}{M}, \mathfrak{g}^*), \tag{2.13}$$

  where $gh(\bar{c}) = -1$ and $gh(\lambda) = 0$.

- The new BRST operator

$$Q = Q_{min} + (-1)^d \int_M \left\langle \lambda, \frac{\delta}{\delta \bar{c}} \right\rangle, \tag{2.14}$$

  acting on the new fields as $Q\bar{c} = (-1)^d\lambda$ and $Q\lambda = 0$. The sign $(-1)^d$ is conventional.

- We assume that on $\mathcal{F}_{new}$ it is defined the measure $\mu_{new} = \mu_{min} D\bar{c}D\lambda$.

Remark 4. The set of data $(\mathcal{F}_{min}, Q_{min}, \mu_{min})$ is sometimes called a **minimal** BRST theory, while $(\mathcal{F}_{new}, Q_{new}, \mu_{new})$ a **non minimal** BRST theory.

Now we are able to define a gauge fixing fermion Ψ with ghost degree -1. For example, we could choose the class of functionals:

$$\Psi_F = \int_M \langle \bar{c}, F(A)\rangle \in C^\infty(\mathcal{F}_{BRST}), \quad gh(\Psi_F) = -1, \tag{2.15}$$

where $F : A_M \to \Omega^0(M,\mathfrak{g})$. If we choose $F(A) = d^\dagger A$:

$$\begin{aligned} Q(\Psi_F) &= Q\int_M \langle \bar{c}, F(A)\rangle = \\ &= (-1)^d \int_M \langle Q\bar{c}, F(A)\rangle + (-1)^d(-1)^{d-1}\int_M \left\langle \bar{c}, Qd^\dagger A\right\rangle = \\ &= \int_M \left\langle \lambda, d^\dagger A\right\rangle + \int_M \left\langle \bar{c}, d^\dagger d_A c\right\rangle. \end{aligned} \tag{2.16}$$

The gauge fixed action is precisely the Faddeev-Popov action (2.6) and the partition function is now perturbatively well defined.

Remark 5. Notice that when we commute the action of Q with that of the integral over M, there is a sign $(-1)^d$. In general, a derivation $\mathcal{D}$ of degree $|\mathcal{D}|$, acting on functions of the fields, commutes with the integral:

$$\mathcal{D}\int_M f = \mathcal{D}\int_{T[1]M} \mu\,\tilde{f} = (-1)^{d|\mathcal{D}|}\int_{T[1]M} \mu\,\mathcal{D}(\tilde{f}) = (-1)^{d|\mathcal{D}|}\int_M \mathcal{D}(f), \qquad (2.17)$$

where f is a function of the fields times a top form on M; $\tilde{f}$ is the same function of the fields times an element of $C^\infty(T[1]M)$; $\mu = d^n x\, d^n\theta$ is the integration measure (the Berezinian, see remark (1.4)).

Remark 6. The BRST formalism can be applied when the symmetry group acting on the space of fields is reducible. Roughly speaking, this means that gauge transformations are defined up to other gauge transformations and so on, so that the introduction of "higher" ghosts associated to these "higher" symmetries is required.

The typical example is the p-form electrodynamics, where the gauge field $A \equiv A^{(p)}$ is a p-form, instead of a one-form connection. The action is $S = \frac{1}{2}\int_M dA^{(p)} \wedge *dA^{(p)}$, which is invariant under the transformation

$$A^{(p-1)} \to A^{(p)} + dB^{(p-1)}.$$

The gauge group is therefore the set of $p-1$ differential forms, $\Omega^{p-1}(M)$. These gauge transformations are defined up to gauge transformations of the kind

$$B^{(p-1)} \to B^{(p-1)} + dC^{(p-2)},$$

that is the first stabilizer of the gauge group is the set of $p-2$ forms, $\Omega^{p-2}(M)$, and so on. The "tower" of infinitesimal symmetries is

$$\Omega^0(M) \to .... \to \Omega^{p-2}(M) \to \Omega^{p-1}(M) \to \Omega^p(M) = \mathcal{F}, \qquad (2.18)$$

where the arrows represent the action of the LHS term on the RHS. In this case, the space of classical BRST fields contains "higher" ghosts, that is fields with higher ghost number (up to p), $\mathcal{F}_{BRST} = \oplus_{i=0}^p \Omega^{p-i}(M)[i]$.

2.1 BV-formalism

BRST formalism is a powerful tool, mostly used in the quantization of field theories. However, there could be cases in which the symmetry is more involved than that given by the action of a Lie group. For example, we could have a field theory with gauge symmetries whose algebra closes only when fields are on-shell (such supergravity without auxiliary fields). In these cases, we say that the gauge symmetry is given by a non integrable distribution on the space of fields. The power of Batalin-Vilkovisky formalism lies in providing a construction which allows to deal also with these symmetries. A BRST system can be pictured in BV formalism.

An other useful and interesting aspect is the geometric interpretation the BV gives to gauge theory, which are rewritten in the language of graded symplectic geometry. As we have already stated, this picture becomes fundamental when we study gauge theories on manifolds with boundary (see chapter (3).

2.1.1 Graded symplectic geometry

In order to discuss BV formalism properly, we need to introduce some basic notions on **graded symplectic geometry**, which is symplectic geometry in the graded case. In the following, we will focus on the finite-dimensional case. The generalization to the infinite-dimensional one (*i.e.* to field theory) will be considered as straightforward.

For an introduction on standard (non graded) symplectic geometry, see Appendix (A). References followed for this brief summary are [28, 15].

Graded symplectic geometry

Let $\mathcal{F}$ be a $\mathbb{Z}$-graded manifold and x^i local coordinates of degree $|x^i|$. Similarly to the case of example (2), differential forms on $\mathcal{F}$ are by definition functions on the degree-shifted tangent bundle $T[1]\mathcal{F}$. Fiber coordinates θ^i have degree $|\theta^i| = |x^i| + 1$. A differential form $\omega \in C^\infty(T[1]\mathcal{F})$ can be written as

$$\omega = \sum_p \sum_{i_1 \cdots i_p} f_{i_1,\dots,i_p}(x^i)\theta^{i_1} \cdots \theta^{i_p} = \sum_p \omega^{(p)}, \tag{2.19}$$

where $f_{i_1,\dots,i_p}(x^i)$ are local functions on $\mathcal{F}$. For a fixed p and $f_{i_1,\dots,i_p}(x^i)$ homogeneous, the degree of $\omega^{(p)}$ is

$$|\omega^{(p)}| = |f_{i_1,\dots,i_p}(x^i)| + \sum_{l=0}^{p} |x^l| + p.$$

It is useful to think of the gradings of $\omega^{(p)} \in \Omega^p(\mathcal{F})$ in this way:

- $p \equiv dR(\omega^{(k)})$ as the de Rham degree of the differential form;
- $|f_{i_1,..,i_p}(x^i)| + \sum_{l=0}^{p} |x^l| \equiv gh(\omega^{(p)})$ as the internal degree (or ghost degree) of $\omega^{(p)}$, coming from the $\mathbb{Z}$-grading of coordinates on $\mathcal{F}$.

The sum $dR(\omega^{(p)}) + gh(\omega^{(p)}) = |\omega^{(p)}|$ is called total degree of the differential form $\omega^{(p)}$.

Remark 7. In this notation, notice that $gh(\theta^i) = |x^i|$.

Remark 8. The de Rham differential d is a cohomological vector field on $C^\infty(T[1]\mathcal{F})$. In local coordinates:

$$d = \sum_i \theta^i \frac{\partial}{\partial x^i}, \quad |d| = 1.$$

Moreover, if i_X is the contraction with the vector field $X \in Vect(\mathcal{F})$, $|i_X| = |X| - 1$, the Lie derivative along X is the graded commutator

$$L_X = [d, i_X] = di_X + (-1)^{|X|} i_X d,$$

Therefore, the Cartan calculus can be extended to the graded case.

Example 9. Let $\mathcal{F} = \mathbb{R}[1]$ the shifted real line. Its shifted tangent bundle $T[-1]\mathbb{R}[1] = \mathbb{R}[1] \oplus \mathbb{R}$. If θ is the degree-1 coordinate on $\mathbb{R}[1]$ and x the degree-0 coordinate on $\mathbb{R}$ a differential form is a function $\omega(x, \theta)$ of the kind

$$\omega(x, \theta) = \sum_p x^p(a + b\theta), \quad a, b \in \mathbb{R}. \tag{2.20}$$

Definition 6. *Let $\mathcal{F}$ be a $\mathbb{Z}$-graded manifold. A graded symplectic structure of (ghost) degree k on $\mathcal{F}$ is a differential two-form $\omega^{(2)} \equiv \omega \in C^\infty(T[1]\mathcal{F})$ such that*

- *Its total degree is $|\omega| = k+2$;*
- *ω is closed, $d\omega = 0$;*
- *ω is non-degenerate, that is the map*

$$\omega : T\mathcal{F} \to T^*[k]\mathcal{F}$$

is an isomorphism.

The couple $(\mathcal{F}, \omega)$ is called graded symplectic manifold of degree k.

Remark 9. As in the non graded case, a vector field $X \in Vect(\mathcal{F})$ such that

$$L_X\omega = 0$$

is called symplectic. If in addition

$$i_X\omega = df, \quad f \in C^\infty(\mathcal{F}),$$

$X \equiv X_f$ is called hamiltonian with hamiltonian function f. The set of hamiltonian functions $\mathbf{H}_\omega$ with the bracket $\{,\}$ ($|\{,\}| = -k$), induced by ω, is a graded Poisson algebra. Indeed, for $f, g, h \in \mathbf{H}_\omega$

- $\{f,g\} = (-1)^{(|f|-k)(|g|-k)+1}\{g,f\}$, (graded skew-symmetry)
- $\{f,gh\} = \{f,g\}h + (-1)^{|g|(|f|-k)}g\{f,h\}$, (graded Leibniz)
- $\{f,\{g,h\}\} = \{\{f,g\},h\} + (-1)^{(|f|-k)(|g|-k)}\{g,\{f,h\}\}$. (graded Jacobi).

If $k = -1$, we have the so called **Gertenhaber algebra**.

Remark 10. Any graded simplectic form of degree $k \neq 0$ is exact. The primitive is $\frac{i_E\omega}{k}$, where E is the Euler vector field (see example (4)). In fact

$$\delta i_E\omega = L_E\omega = k\omega \;\Rightarrow\; \omega = \delta\left(\frac{1}{k}i_E\omega\right). \tag{2.21}$$

Remark 11. In the finite dimensional case it is enough to require ω to be inective in order to guarantee that it is not degenerate. In the infinite dimensional case, this is not true anymore, and we speak of weakly non-degenerate forms (injective but not surjective).

Example 10. Consider example (9). The two-form $\omega = ax^2 = ad\theta \wedge d\theta$ is a symplectic structure of ghost degree 2 on $\mathbb{R}[1]$.

Example 11. Given a graded manifold $\mathcal{M}$, consider its shifted cotangent bundle $T^*[k]\mathcal{M}$. Let x^i be graded coordinates on the base and ξ_i the corresponding graded coordinates on the fiber. On $T^*[k]\mathcal{F}$ it is defined a symplectic two form of degree k:

$$\omega_{st} = \sum_{ij} \delta_i^j dx^i \wedge d\xi_j.$$

This form is also called **canonical symplectic form**.

In standard symplectic geometry, symplectic manifolds have locally the structure of a cotangent bundle. This is the Darboux theorem (Appendix (A)). In the graded case, a similar result holds.

Theorem 1. *(Schwarz) Let $(\mathcal{F}, \omega)$ be a graded symplectic manifold of degree k. There exists a degree-0 diffeomorphism $\phi : (\mathcal{F}, \omega) \to (T^*[k]F, \omega_{st})$, such that $\omega = \phi^*\omega_{st}$ globally. Moreover, F can be chosen as a smooth ungraded manifold.*

Notice that, on the contrary of the non-graded case, this graded version of Darboux theorem is true globally: in other words, all graded symplectic manifolds are shifted cotangent bundles. Locally, this means that we can always choose coordinates on $\mathcal{F}$ in such a way that

$$\omega = \sum_{ij} \delta_i^j dx^i \wedge d\xi_j.$$

This local coordinates are called Darboux coordinates.

As in the non graded case (see Appendix (A)), a submanifold $\mathcal{L}$ of a graded symplectic manifold $(\mathcal{F}, \omega)$ is called **isotropic** if the graded symplectic form, restricted to $\mathcal{L}$, is zero. If $\mathcal{L} \subset \mathcal{F}$ is isotropic and it is not a proper submanifold of an other isotropic submanifold, $\mathcal{L}$ is called **lagrangian**.

Example 12. Consider the shifted cotangent bundle $T^*[k]\mathcal{F}$ of a graded manifold $\mathcal{F}$. Given a function $\Psi \in C^\infty(\mathcal{F})$ of degree k, we may associate to it the lagrangian submanifold

$$\mathcal{L}_\psi \equiv graph(d\Psi) \subset T^*[k]\mathcal{F}.$$

If (x^i, ξ_i) are local coordinates on $T^*[k]\mathcal{F}$, the graph lagrangian is given by $(x^i, \xi_i = \frac{\partial \Psi}{\partial x^i})$.

BV laplacian and Stokes integral

Definition 7. *A **BV manifold** is a triple $(\mathcal{F}, \omega, \mu)$, where $(\mathcal{F}, \omega)$ is a graded symplectic manifold of degree $k = -1$ and μ is a measure of integration such that there exist an atlas $\{U_\alpha,\, (x^i_{(\alpha)}, \xi_{(\alpha)_i})\}$ of Darboux charts such that locally:*

$$\mu = \Pi_i \, dx^i_{(\alpha)} d\xi_{(\alpha)_i}.$$

*A measure μ with these properties is called **compatible** with ω.*

A measure μ on $(\mathcal{F}, \omega)$ allows to define the notion of divergence of a vector field $X \in Vect(\mathcal{F})$:

$$\int_{\mathcal{F}} \mu X(g) = -\int_{\mathcal{F}} \mu \, div_\mu(X) g, \tag{2.22}$$

where $f, g \in C^\infty(\mathcal{F})$.

Given a compatible measure μ, we define the **BV operator** $\Delta_\mu : C^\infty(\mathcal{F}) \to C^\infty(\mathcal{F})$ as

$$\Delta_\mu(f) = \frac{1}{2} div_\mu(X_f), \quad f \in C^\infty(\mathcal{F}), \tag{2.23}$$

where X_f is the hamiltonian vector field of f. Notice that $|\Delta_\mu| = 1$.

In local Darboux coordinates (x^i, ξ_i) on $\mathcal{M}$, the BV operator can be expressed as

$$\Delta_\mu = \sum_i (-1)^{|x^i|} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i}. \tag{2.24}$$

Notation 1. We call $\partial_i \equiv \frac{\partial}{\partial x^i}$ and $\partial^i \equiv \frac{\partial}{\partial \xi_i}$. The BV operator is $\Delta_\mu = \sum_i (-1)^{|x^i|} \partial_i \partial^i$.

Lemma 1. The BV operator Δ_μ has the following properties:

- $\Delta_\mu^2 = 0$;
- •• $\Delta_\mu(fg) = \Delta_\mu(f)g + (-1)^{|f|} f \Delta_\mu(g) + (-1)^{|f|}\{f, g\}, \quad f, g \in C^\infty(\mathcal{F})$;
- ••• $\Delta_\mu\{f, g\} = \{\Delta_\mu(f), g\} + (-1)^{|f|+1}\{f, \Delta_\mu(g)\}, \quad f, g \in C^\infty(\mathcal{F})$.

Remark 12. The set of functions $C^\infty(\mathcal{F})$ with the degree-1 Poisson bracket $\{,\}$ and the BV operator with the properties above is called **BV algebra**.

The main reason why we are interested in the BV operator is that it plays a key role in the generalization of Stokes theorems in the graded symplectic case.
First of all, we introduce the notion of BV integral.

Definition 8. *Let $(\mathcal{F}, \omega, \mu)$ be a BV manifold and $\mathcal{L}$ a lagrangian submanifold of $\mathcal{F}$. A **BV integral** is an integral of the form*

$$\int_{\mathcal{L}} \sqrt{\mu|_{\mathcal{L}}}\, f, \tag{2.25}$$

where $f \in C^\infty(\mathcal{F})$ is such that $\Delta_\mu f = 0$ and $\sqrt{\mu|_{\mathcal{L}}}$ is the measure induced on $\mathcal{L}$.

Remark 13. It can be shown that every lagrangian submanifold of $\mathcal{F}$ has an induced measure. Indeed, let (x^i, ξ_i) Darboux coordinats on $\mathcal{F}$ such that the (compatible) measure on $\mathcal{F}$ is $\mu = \Pi_i dx^i d\xi_i$. If we consider graph lagrangians $\mathcal{L}_\psi$ (see example (12)), the induced measure is given by $\sqrt{\mu|_{\mathcal{L}}} = \sqrt{\mu|_{\xi_i = \frac{\partial \Psi}{\partial x^i}}}$. In particular:

$$\int_{\mathcal{L}_\Psi} f \sqrt{\mu|_{\mathcal{L}}} = \int f|_{\xi_i = \partial_i \Psi}\, dx^1 .... dx^n, \tag{2.26}$$

where the integral is intended as a Grassmann integral along the odd variables and a Lebesgue integral along the even ones.

From now on, we will consider graph lagrangians only.

The following theorem generalizes the Stokes theorem to the graded case:

Theorem 2. *Let $(\mathcal{F}, \omega, \mu)$ be a BV manifold. The following results hold:*

- *For any $f \in C^\infty(\mathcal{F})$ and $\mathcal{L}_\Psi \subset \mathcal{F}$ be a lagrangian submanifold, it holds that*

$$\int_{\mathcal{L}_\Psi} \Delta_\mu(f) \sqrt{\mu|_{\mathcal{L}_\Psi}} = 0; \tag{2.27}$$

- •• *For any $f \in C^\infty(\mathcal{F})$ such that $\Delta_\mu(f) = 0$ the BV integral*

$$\int_{\mathcal{L}_\Psi} f \sqrt{\mu|_{\mathcal{L}_\Psi}} \tag{2.28}$$

is invariant under deformation of $\mathcal{L}_\Psi$.

Proof. Consider $f \in C^\infty(\mathcal{F})$. Let (x^i, ξ_i) be a local Darboux chart and let $\mu = \Pi_i dx^i d\xi_i$ be the integration measure on $T^*[-1]\mathcal{F}$.

- In order to calculate the BV integral (2.27), we have to restrict the function $\Delta_\mu(f)$ to the lagrangian submanifold $\mathcal{L}_\Psi$. This is done by putting $\xi_i = \partial_i \Psi$. However, notice that, since Ψ does depend on x^i:

$$\begin{aligned}\sum_i (-1)^{|x|^i} \partial_i \left(\partial^i f|_{\xi_i=\partial_i\Psi}\right) = &\sum_i (-1)^{|x^i|} \left(\partial_i \partial^i f\right)|_{\xi_i=\partial_i\Psi} + \\ &+ \sum_{i,j} (-1)^{|x|^i} \partial_i \partial_j \Psi \partial^j \partial^i f|_{\xi_i=\partial_i\Psi}.\end{aligned} \tag{2.29}$$

  The second term on the RHS vanishes by symmetry reason, while the term on the LHS is a total derivative. Therefore, if there are no contributes at infinity, we have:

$$0 = \int \sum_i (-1)^{|x^i|} \left(\partial_i \partial^i f\right)|_{\xi_i=\partial_i\Psi} dx^1 ... dx^n,$$

  which is exactly integral (2.27).

•• Consider a family of odd functions Ψ_t and the BV integrals

$$I_t = \int_{\mathcal{L}_{\Psi_t}} f \sqrt{\mu|_{\mathcal{L}_{\Psi_t}}}. \tag{2.30}$$

  By taking the derivative along the parameter t, we find:

$$\begin{aligned}\frac{dI_t}{dt} &= \int \frac{d}{dt} f|_{\xi_i=\partial_i\Psi}\, dx^1 ... dx^n = \\ &= \int \left(\partial_i \dot{\Psi}_t \partial^i f\right)_{\xi_i=\partial_i\Psi_t} = \\ &= \pm \int \Delta_\mu(\dot{\Psi}_t f) dx^1 .... dx^n = 0,\end{aligned} \tag{2.31}$$

  where we used the Leibniz rule of Δ_μ and the fact that Ψ is a function of x^i only. The last integral is zero since the integrand is Δ_μ-exact.

□

As we will see, these theorems are the keys to understand the gauge-fixing in the BV setting

2.1.2 Gauge-Fixing in BV formalism

Let $(\mathcal{F}, \omega, \mu)$ be an infinite-dimensional BV manifold and $\mathcal{S} \in C^\infty(\mathcal{F})$ a function with $gh(\mathcal{S}) = 0$. Now ω is a weakly-non degenerate symplectic form, namely $\mathbf{H}_\omega \subset \mathcal{F}$. We will use δ to indicate the de Rham differential on $(\mathcal{F}, \omega)$.

The partition function is defined as

$$\mathcal{Z} = \int_{\mathcal{L}_\Psi \subset \mathcal{F}} \sqrt{\mu|_\mathcal{L}}\, e^{\frac{i}{\hbar}\mathcal{S}}, \tag{2.32}$$

where $\Psi \in C^\infty(\mathcal{F})$, with $gh(\Psi) = -1$.
From Stokes theorem in BV formalism, the integral above is invariant under deformations of the graph lagrangian if the integrand is closed under the BV laplacian Δ_μ, which in a Darboux chart (Φ^a, Φ_a^+) on $\mathcal{F}$ is

$$\Delta_\mu = \sum_a (-1)^{|\Phi^a|} \int_M \frac{\delta}{\delta \Phi^a} \frac{\delta}{\delta \Phi_a^+}. \tag{2.33}$$

The Φ^as are the fields and the Φ_a^+s the corresponding antifields. This expression extends definition (2.24) to the infinite-dimensional case.
The idea of gauge fixing in BV formalism is that, if the integrand in (2.32) is Δ_μ-closed and if $\mathcal{S}$ has degenerate critical points on a lagrangian $\mathcal{L}_\Psi$, we use the freedom to deform $\mathcal{L}_\Psi$ to an other lagrangian $\mathcal{L}_{\Psi'}$ such that $\mathcal{S}$ has non-degenerate critical points on it. One can apply the stationary phase formula to calculate the integral on the new lagrangian $\mathcal{L}'_\Psi$.
Before seeing this procedure at work, we have to check that the integrand in (2.32) is Δ_μ-closed.

Lemma 2. Let $(\mathcal{F}, \omega, \mu)$ an infinite-dimensional BV manifold and $\mathcal{S} \in C^\infty(\mathcal{F})$ a $gh = 0$-function. Then $\Delta_\mu e^{\frac{i}{\hbar}\mathcal{S}} = 0$ if and only if

$$\frac{1}{2}\{\mathcal{S}, \mathcal{S}\} - i\hbar \Delta_\mu \mathcal{S} = 0. \tag{2.34}$$

This equation is known as the **quantum master equation** (QME).

Proof. First of all, notice that

$$\Delta_\mu (\mathcal{S})^n = n(\mathcal{S})^{n-1} \Delta_\mu \mathcal{S} + \frac{n(n-1)}{2} (\mathcal{S})^{n-2} \{\mathcal{S}, \mathcal{S}\},$$

which follows from the application of property (••) of lemma (1). Therefore:

$$\begin{aligned} \Delta_\mu e^{\frac{i}{\hbar}\mathcal{S}} &= \Delta_\mu \sum_n \frac{1}{n!} \left(\frac{i}{\hbar}\right)^n (\mathcal{S})^n = \\ &= (-i\hbar)^{-2} (\frac{1}{2}\{\mathcal{S}, \mathcal{S}\} - i\hbar \Delta_\mu \mathcal{S}) e^{\frac{i}{\hbar}\mathcal{S}}, \end{aligned}$$

and we infer that

$$\Delta_\mu e^{\frac{i}{\hbar}\mathcal{S}} = 0 \Rightarrow \frac{1}{2}\{\mathcal{S}, \mathcal{S}\} - i\hbar \Delta_\mu \mathcal{S} = 0.$$

□

Remark 14. A solution of the QME can be searched as a series in powers of $\hbar$:

$$\mathcal{S} = \mathcal{S}^{(0)} + (-i\hbar)\mathcal{S}^{(1)} + (-i\hbar)^2 \mathcal{S}^{(2)} + ...$$

The QME can be solved perturbatively in $\hbar$. At order zero, we find:

$$\{\mathcal{S}^{(0)}, \mathcal{S}^{(0)}\} = 0. \tag{2.35}$$

Eq. (2.35) is called **Classical Master Equation** (CME). As for higher orders, we have:

$$\begin{aligned} &\{S^{(0)}, S^{(1)}\} + \Delta_\mu \mathcal{S}^{(0)} = 0; \\ &\{S^{(0)}, S^{(2)}\} + \frac{1}{2}\{\mathcal{S}^{(1)}, \mathcal{S}^{(1)}\} + \Delta_\mu \mathcal{S}^{(1)} = 0 \\ &\dots \end{aligned} \tag{2.36}$$

Let $\{\mathcal{S}^{(0)}, \bullet\} = \mathcal{Q}$ be the hamiltonian vector field of $\mathcal{S}^{(0)}$. Since the CME for $\mathcal{S}^{(0)}$ holds and $|\mathcal{Q}| = 1$, this is a cohomological vector field. The first-order correction in $\hbar$ to the classical action $\mathcal{S}^{(0)}$ can be found if the term $\Delta_\mu \mathcal{S}^{(0)}$ is $\mathcal{Q}$-exact. Notice that this is automatically $\mathcal{Q}$-closed (this follows from the CME for $\mathcal{S}^{(0)}$ and lemma (1)). Therefore, a primitive $\mathcal{S}^{(1)}$ does exist only if the cohomology class of $\Delta_\mu \mathcal{S}^{(0)}$ in $H_1(\mathcal{Q})$ vanishes.

Analogous considerations hold for the second-order correction: if the term $\frac{1}{2}\{\mathcal{S}^{(1)}, \mathcal{S}^{(1)}\} + \Delta_\mu \mathcal{S}^{(1)}$ (which is $\mathcal{Q}$-closed) is also $\mathcal{Q}$-exact, we can find a primitive $\mathcal{S}^{(2)}$ (the second-order correction) and so on. In general, if we search for the correction of order $n+1$, we must verify that there are no obstructions in $H_1(\mathcal{Q})$. In this case, we can find the $n+1$-th correction to the solution of the CME, otherwise the procedure stops.

Notice that in this approach the fundamental ingredient needed to solve the QME is a solution of the CME $\mathcal{S}^{(0)}$.

Definition 9. *(Classical BV theory) A* ***classical BV theory*** *the set of following data:*

- *A $\mathbb{Z}$-graded manifold $\mathcal{F}$ (the set of BV fields);*
- *A graded symplectic differential form $\omega \in \Omega^2(\mathcal{F})$, $gh(\omega) = -1$.*
- *A function $\mathcal{S} \in C^\infty(\mathcal{F})$, $gh(\mathcal{S}) = 0$, satisfying the CME:*

$$\{\mathcal{S}, \mathcal{S}\} = 0. \tag{2.37}$$

This function is called classical BV action.

Remark 15. If the form ω above has ghost degree k, we speak of BF^{k+1}V (Batalin-Fradkin-Vilkovisky) theories. In particular, the case $k = 0$ emerges in the BFV formalism, the hamiltonian counterpart of the BV formalism. Moreover, case $k \geq 0$ arises as the target structure for $(k+1)$-dimensional AKSZ sigma models [1], see section (2.2).

Definition 10. *(Quantum BV theory) A* ***quantum BV theory*** *the set of the following data:*

- *A BV manifold $(\mathcal{F}, \omega, \mu)$;*
- *A function $\mathcal{S} \in C^\infty(\mathcal{F})$, with $gh(\mathcal{S}) = 0$, satisfying the QME*

$$\frac{1}{2}\{\mathcal{S}, \mathcal{S}\} - i\hbar \Delta_\mu \mathcal{S} = 0. \tag{2.38}$$

This function is called quantum BV action.

Given a quantum BV theory, the partition function (2.32) is invariant under deformations of the lagrangian submanifold. Therefore, if $\mathcal{S}$ has degenerate critical points on $\mathcal{L}$, we can deform it to a lagrangian $\mathcal{L}_\Psi$ in such a way that the critical points of $\mathcal{S}$ are non degenerate on it. This is what we call gauge-fixing in BV formalism

We can study quantum observables in BV formalism. Given a function $\mathcal{O} \in C^\infty(\mathcal{F})$, written as a series in power of $\hbar$, its expectation value is

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{L}\subset\mathcal{F}} \sqrt{\mu|_{\mathcal{L}}} e^{\frac{i}{\hbar}\mathcal{S}}\mathcal{O}. \tag{2.39}$$

The integral is invariant under deformations of $\mathcal{L}$ iff the integrand is Δ_μ-closed. This condition is satisfied if $\{\mathcal{S},\mathcal{O}\} - i\hbar\Delta_\mu O = 0$, which is the statement of gauge invariance of the function $\mathcal{O}$ in BV formalism. Such a function $\mathcal{O}$ is called **quantum observable**.

Remark 16. (BRST in BV formalism) A quantum BRST theory (see subsection (2.0.2)) can be extended to a quantum BV theory. Let M be a closed oriented d-dimensional manifold and $\mathcal{F}_{BRST}$ the space of BRST fields on M. Moreover, let Q_{BRST} and S_{BRST} be the BRST operator and the BRST action respectively.

The space of BV fields is defined as

$$\mathcal{F}_{BV} = T^*[-1]\mathcal{F}_{BRST}, \tag{2.40}$$

with graded symplectic structure given by the standard symplectic form on the shifted cotangent:

$$\omega_{BV} = \int_M \delta\Phi^\alpha \wedge \delta\Phi^+_\alpha, \quad gh(\omega_{BV}) = -1 \tag{2.41}$$

where Φ^α are the BRST fields and Φ^+_α are the corresponding antifields (the coordinates on the fiber). As for the measure on the space of BV fields, we can take $\mu_{BV} = \mu\otimes\mu^+$, where μ is the integration measure over BRST fields and μ^+ is that over the antifields. Locally, $\mu_{BV} = \Pi_{\alpha,\beta} D\Phi^\alpha D\Phi^+_\beta$.

Consider now the following action $\mathcal{S}_{BV} \in C^\infty(\mathcal{F}_{BV})$

$$\mathcal{S}_{BV}(\Phi,\Phi^+) = S_{BRST}(\Phi) + \int_M Q^\alpha_{BRST}(\Phi)\Phi^+_\alpha, \tag{2.42}$$

where $Q^\alpha_{BRST}(\Phi)$ are the components of the BRST operator.

This action satisfies the QME (2.38). In fact, since $\mathcal{S}_{BV}$ does not depend on $\hbar$, the QME reduces to two separated equations, $\{\mathcal{S}_{BV},\mathcal{S}_{BV}\} = 0$, which is the CME and follows from direct calculations, and $\Delta_\mu\mathcal{S}_{BV} = 0$, which holds since $div_\mu(Q_{BRST}) = 0$. The set $(\mathcal{F}_{BV},\omega_{BV},\mu_{BV},\mathcal{S}_{BV})$ is a quantum BV theory.

Let's move to the gauge-fixing of the partition function of such a theory. Denote with $\mathcal{L}_0$ the zero section of $\mathcal{F}_{BV} = T^*[-1]\mathcal{F}_{BRST}$ (namely, we immerse $\mathcal{F}_{BRST}$ into $\mathcal{F}_{BV}$ as a submanifold with the antifields equals to zero) and $\mathcal{L}_\Psi \subset T^*[-1]\mathcal{F}_{BRST}$ the graph lagrangian submanifold associated to a functional $\Psi \in C^\infty(\mathcal{F}_{BRST})$, with $gh(\Psi) = -1$. The gauge fixing consists in the replacement:

$$\int_{\mathcal{L}_0\subset\mathcal{F}_{BV}} \sqrt{\mu_{BV}} e^{\frac{i}{\hbar}\mathcal{S}_{BV}(\Phi,\Phi^+)} \to \int_{\mathcal{L}_\Psi\subset\mathcal{F}_{BV}} \sqrt{\mu_{BV}} e^{\frac{i}{\hbar}\mathcal{S}_{BV}(\Phi,\Phi^+)}.$$

The first integral reduces to

$$\int_{\mathcal{F}_{BRST}} \Pi_\alpha D\Phi^\alpha \, e^{\frac{i}{\hbar}S_{BRST}(\Phi)},$$

since $\mathcal{S}_{BV}$ reduces to the BRST action on the zero section. As for the second, notice that on the graph lagrangian, the BV action reduces to $\mathcal{S}_{BV}(\Phi^\alpha,\Phi^+_\alpha = \frac{\delta}{\delta\Phi^\alpha}\Psi) = S_{BRST}(\Phi) + Q_{BRST}\Psi$. Therefore, the second integral can be rewritten as

$$\int_{\mathcal{F}_{BRST}} \Pi_\alpha D\Phi^\alpha \, e^{\frac{i}{\hbar}(S_{BRST}(\Phi)+Q_{BRST}\Psi)}.$$

Thus, gauge fixing in BRST formalism, performed by shifting the BRST action by a Q_{BRST}-exact function, is obtained in the BV setting, as a deformation of the lagrangian submanifold on which we integrate the partition function.

Example 13. (Yang-Mills) Consider the BRST formulation of Yang-Mills theory, as in example (8). The space of BV fields is

$$\begin{aligned}\mathcal{F}_{BV} &= T^*[-1]\mathcal{F}_{BRST} = \\ &= T^*[-1]\mathcal{F} \oplus T^*[-1]\left(\Omega^0(M,\mathfrak{g}[1])\right) = \\ &= \underset{A}{\Omega^1(M,\mathfrak{g})} \oplus \underset{A^+}{\Omega^{d-1}(M,\mathfrak{g}^*[-1])} \oplus \underset{c}{\Omega^0(M,\mathfrak{g}[1])} \oplus \underset{c^+}{\Omega^d(M,\mathfrak{g}^*[-2])},\end{aligned} \tag{2.43}$$

where A^+ and c^+ are the antifields associated to A and c. The BV two form is

$$\omega_{BV} = \int_M \langle \delta A, \delta A^+ \rangle + \langle \delta c, \delta c^+ \rangle . \tag{2.44}$$

As a consequence of Eq. (2.42) and the form (2.12) of the Yang-Mills BRST operator, the BV action is:

$$\mathcal{S}_{BV} = S_{YM}[A] + \int_M \langle d_A c, A^+ \rangle + \int_M \left\langle \frac{1}{2}[c,c], c^+ \right\rangle , \tag{2.45}$$

where $S_{YM}[A] = \frac{1}{2}\int_M tr\,(F_A \wedge *F_A)$.

The cohomological vector field $\mathcal{Q}_{BV}$ is found by variating the BV action:

$$\begin{aligned}\delta\mathcal{S}_{BV} &= (-1)^d \int_M \langle d_A(*F_A) + [c,A^+], \delta A\rangle + \langle d_A A^+ + [c,c^+], \delta c\rangle + \\ &\quad + (-1)^d \int_M \langle d_A c, \delta A^+\rangle + \left\langle \frac{1}{2}[c,c], \delta c^+ \right\rangle = \\ &= i_{\mathcal{Q}_{BV}}\omega_{BV}.\end{aligned} \tag{2.46}$$

Therefore:

$$\begin{aligned}\mathcal{Q}_{BV} &= (-1)^d \int_M \left\langle d_A c, \frac{\delta}{\delta A}\right\rangle + \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c}\right\rangle + \\ &\quad + (-1)^d \int_M \left\langle d_A(*F_A) + [c,A^+], \frac{\delta}{\delta A^+}\right\rangle + \left\langle d_A A^+ + [c,c^+], \frac{\delta}{\delta c^+}\right\rangle .\end{aligned} \tag{2.47}$$

In order to perform the gauge fixing, we need to define the BV theory corresponding to the non-minimal BRST formulation of Yang-Mills theory we introduced in example (8). This is given by the following set of data.

- The space of non-minimal BV fields is

$$\begin{aligned}\mathcal{F}_{BV}^{nm} &= \mathcal{F}_{BV} \oplus T^*[-1]\left(\Omega^d(M,\mathfrak{g}^*[-1]) \oplus \Omega^d(M,\mathfrak{g}^*)\right) = \\ &= \mathcal{F}_{BV} \oplus \underset{\bar{c}}{\Omega^d(M,\mathfrak{g}^*[-1])} \oplus \underset{\lambda}{\Omega^d(M,\mathfrak{g}^*)} \oplus \underset{\bar{c}^+}{\Omega^0(M,\mathfrak{g})} \oplus \underset{\lambda^+}{\Omega^0(M,\mathfrak{g}[-1])};\end{aligned}$$

- The BV two form is

$$\omega_{BV}^{nm} = \omega_{BV} + \int_M \langle \delta\lambda, \delta\lambda^+ \rangle + \int_M \langle \delta\bar{c}, \delta\bar{c}^+ \rangle ;$$

- The BV action is

$$\mathcal{S}_{BV}^{nm} = \mathcal{S}_{BV} + \int_M \langle \lambda, \bar{c}^+ \rangle$$

- The integration measure is

$$\mu_{BV}^{nm} = DADA^+ DcDc^+ D\bar{c}D\bar{c}^+ D\lambda D\lambda^+$$

The starting partition function is

$$\mathcal{Z} = \int_{\mathcal{L}_0 \subset \mathcal{F}_{BV}^{nm}} e^{\frac{i}{\hbar}\mathcal{S}_{BV}^{nm}}$$

The Faddeev-Popov gauge fixing can be obtained by choosing a graph Lagrangian $\mathcal{L}_\Psi \subset \mathcal{F}_{BV}^{nm}$, with

$$\Psi = \int_M \langle \bar{c}, F(A) \rangle \in C^\infty(\mathcal{F}_{BRST}), \quad gh(\Psi) = -1,$$

where $F : \mathcal{F} \to \Omega^0(M, \mathfrak{g})$. For example, we can choose $F(A) = d^\dagger A$.

Locally, $\mathcal{L}_\Psi$ is given by:

$$\begin{cases} A, c, \bar{c}, \lambda; \\ A^+ = \frac{\delta\Psi}{\delta A} = -d^\dagger \bar{c}; \\ c^+ = \frac{\delta\Psi}{\delta c} = 0; \\ \bar{c}^+ = \frac{\delta\Psi}{\delta \bar{c}} = (-1)^{d(d-1)} d^\dagger A = d^\dagger A; \\ \lambda^+ = \frac{\delta\Psi}{\delta\lambda} = 0. \end{cases} \tag{2.48}$$

Therefore:

$$\begin{aligned} \mathcal{S}_{BV}^{nm}|_{\mathcal{L}_\Psi} &= S_{YM} + \int_M \langle d_A c, -d^\dagger \bar{c} \rangle + \int_M \langle \lambda, d^\dagger A \rangle = \\ &= S_{YM} + \int_M \langle -d^\dagger \bar{c}, d_A c \rangle + \int_M \langle \lambda, d^\dagger A \rangle = \\ &= S_{YM} + \int_M \langle \bar{c}, d^\dagger d_A c \rangle + \int_M \langle \lambda, d^\dagger A \rangle, \end{aligned}$$

which is exactly the Faddeev-Popov action (2.6).

Remark 17. (Non integrable symmetries) We have said that the power of the Batalin-Vilkovisky formalism is that it can be used to treat gauge symmetry not given by a group action on the space of classical fields, but rather given by a non integrable distribution on the space of classical fields. An example is the case when the algebra of infinitesimal gauge transformations closes only on shell, that is modulo Euler-Lagrange equations. The main result ([5]) is that the most general solution of the classical master equation is a power series in antifields

$$\mathcal{S}_{BV}(\Phi, \Phi^+) = \sum_{n=0}^{\infty} \sum_{a_1,..,a_n} \mathcal{S}^{a_1,...,a_n}(\Phi) \Phi^+_{a_1} \Phi^+_{a_2} ... \Phi^+_{a_n}, \tag{2.49}$$

where Φ is the set of BRST fields and Φ^+ the corresponding antifields.

When the gauge symmetry is the action of a group on the space of classical fields, Eq. (2.49) reduces to (2.42) (namely, it is linear in the antifields). Higher polynomial terms appear when the gauge symmetry is given by a non integrable distribution on the space of classical fields.

2.2 AKSZ theories

In the previous section we have seen that a classical field theory in the BV formalism is specified by means of an action $\mathcal{S}$ on the space of BV fields, which resolves the classical master equation $\{\mathcal{S},\mathcal{S}\} = 0$, where $\{,\}$ is the graded Poisson bracket, $|\{,\}| = 1$, induced by a BV symplectic form ω_{BV} (see definition (9)). From a geometric point of view, the CME means that the hamiltonian vector field $\mathcal{Q} = \{\mathcal{S}, \bullet\}$ squares to zero. Moreover, a solution of the CME is a fundamental ingredient in order to construct a solution of the quantum master equation (see remark (14)).

In this chapter, we are going to study the so called **AKSZ theories**, which were first introduced in [1]. For these particular BV field theories, there exists a sistematic procedure which allows to construct a solution of the classical master equation, given some geometrical data in input. Rather than constructing a solution of the CME starting from a classical action, the AKSZ construction gives the full set of fields and antifields and the solution of the CME from the beginning. As we will see, AKSZ theories are sigma models. For appropriate choices of the source and the target, the AKSZ construction produces the BV action functionals for many known topological field theories, like the Chern-Simons theory or the BF theory.

2.2.1 AKSZ construction

The AKSZ construction requires the following geometrical data in input:

- A closed oriented d-dimensional manifold M, the **source**;
- A differential graded symplectic manifold $(\mathcal{N}, \omega_{\mathcal{N}}, D)$, with $gh(\omega_{\mathcal{N}}) = d-1$ and hamiltonian function $\Theta_{\mathcal{N}}$ for D. This space is the **target space**.

In addition, we require that $\omega_{\mathcal{N}}$ is globally exact, that is $\omega_{\mathcal{N}} = \delta\alpha_{\mathcal{N}}$. If $d \neq 1$ (and so $gh(\omega) \neq 0$), since $\mathcal{N}$ has a $\mathbb{Z}$-grading, there always exist a distinguished primitive $\alpha_{\mathcal{N}}$, coming from the contraction with the Euler vector field on $\mathcal{N}$.

Notation 2. We call d the de Rham differential acting on differential forms on the source and δ the de Rham differential on forms on the target space and on the space of fields.

From now on, we assume that $\mathcal{N}$ is a graded vector space. The space of fields of the model is defined as

$$\mathcal{F}_{\mathcal{M}} = C^{\infty}(T[1]M) \otimes \mathcal{N} \simeq \Omega^{\bullet}(M) \otimes \mathcal{N}, \tag{2.50}$$

since $C^{\infty}(T[1]M) \simeq \Omega^{\bullet}(M)$.

Let $X \in \mathcal{F}_M$ and $x^a \in C^{\infty}(\mathcal{N})$ be local coordinates on the target. The composition $\mathbb{X}^a \equiv x^a \circ X$ is a function on $T[1]M$. In local coordinates $(u_1, ..., u_n, \theta^1, ..., \theta^n) \equiv (u, \theta)$ on $T[1]M$, we have

$$\mathbb{X}^a(u,\theta) = \sum_{k=0}^{n} \sum_{1 \leq i_1 \leq ... \leq i_k \leq k} X^a_{i_1,...,i_k}(u)\theta^{i_1} \cdots \theta^{i_k}. \tag{2.51}$$

The elements $\mathbb{X}^a(u,\theta)$ are called **superfields**. The differential forms on the RHS of definition (2.51) are the field content of the model of ghost degree $gh(X^a_{i_1,...,i_k}) = |x^a| - k$.

We are going to show now that on $\mathcal{F}_M$ it is possible to define a BV theory (see definition (9)), namely that on $(F_M$ there are a graded symplectic form ω_M with $gh(\omega_M) = -1$ and a cohomological vector field $\mathcal{Q}_M$, which is hamiltonian and with hamiltonian function the classical BV action satisfying the CME.
Assume that the target symplectic form is written as

$$\omega_{\mathcal{N}} = \frac{1}{2} \sum_{a,b} \omega_{ab}(x) \delta x^a \wedge \delta x^b = \delta \left(\sum_a \alpha_a(x) \delta x^a \right). \tag{2.52}$$

Define the following 2-form on $\mathcal{F}_M$:

$$\omega_M = \frac{1}{2} \sum_{ab} \int_M \omega_{ab}(\mathbb{X}) \delta \mathbb{X}^a \wedge \delta \mathbb{X}^b. \tag{2.53}$$

The previous expression has to be intended in the following way: we "substitute" the coordinates on the target with the superfields; we select the top form in the explicit expression of the integrand and then we integrate over M. The result is a two-form on $\mathcal{F}_M$.

Lemma 3. ω_M is a BV form on $\mathcal{F}_M$.

Proof. First of all, ω_M is closed. This follows from $\delta \int_M = (-1)^d \int_M \delta$ (see remark (5)). As for the non-degeneracy, ω_M is weakly non-degenerate. The ghost degree of ω_M is

$$\begin{aligned} gh(\omega_M) &= |\omega_M| - 2 = \\ &= |\omega_{\mathcal{N}}| - d - 2 = \\ &= (d - 1 + 2) - d - 2 = -1 \end{aligned}$$

Therefore, ω_M is a BV symplectic form. □

As cohomological vector field on $\mathcal{F}_M$ we consider

$$\mathcal{Q}_M = \int_M d\mathbb{X}^a \frac{\delta}{\delta \mathbb{X}^a} + D(\mathbb{X})^a \frac{\delta}{\delta \mathbb{X}^a} \equiv \hat{d}_M + \hat{D}. \tag{2.54}$$

The terms on the RHS are simply the composition of the de Rham differential on the source and the cohomological vector field on the target with the superfields.

The vector $\mathcal{Q}_M$ has $gh(\mathcal{Q}_M) = 1$ and squares to zero: in fact, d_M and D square to zero (they are cohomological) and anticommute (they act on different spaces) and so do their composition with the superfields (since the only difference is that we have the superfields in place of the coordinates on the source/target).

Thus, we arrive at the following Lemma:

Lemma 4. If M has no boundary, $\mathcal{Q}_M$ is ω_M-hamiltonian and its hamiltonian function is

$$\mathcal{S}_M = (-1)^d \int_M \alpha_a(\mathbb{X}) d\mathbb{X}^a + \Theta_{\mathcal{N}}(\mathbb{X}),$$

where $\Theta_{\mathcal{N}}$ is the hamiltonian function of D. Moreover, $\mathcal{S}_M$ satisfies the CME:

$$\{\mathcal{S}_M, \mathcal{S}_M\}.$$

Proof. Let's study $i_{\mathcal{Q}_M}\omega_M = i_{\hat{d}_M}\omega_M + i_{\hat{D}}\omega_M$. The second term is simply (recall that $|i_{\mathcal{Q}_M}| = 0$):

$$\begin{aligned} i_{\hat{D}}\omega_M &= i_{\hat{D}} \int_M \omega_{\mathcal{N}}(\mathbb{X}) = \\ &= \int_M i_{\hat{D}}\omega_{\mathcal{N}}(\mathbb{X}) = \\ &= \int_M \delta\Theta_{\mathcal{N}}(\mathbb{X}) = \\ &= (-1)^d \delta \int_M \Theta_{\mathcal{N}}(\mathbb{X}). \end{aligned}$$

In order to study the first term, we perform the following calculation:

$$\begin{aligned} \delta i_{\hat{d}_M} \int_M \alpha_a(\mathbb{X})\delta\mathbb{X}^a &= \delta \int_M \alpha_a(\mathbb{X}) d\mathbb{X}^a = \\ &= (-1)^d \int_M \left(\frac{\delta\alpha_a(\mathbb{X})}{\delta\mathbb{X}^b} \delta\mathbb{X}^b d\mathbb{X}^a \right) + \int_M \alpha_a(\mathbb{X})\delta d\mathbb{X}^a = \\ &= (-1)^d \int_M \left(\frac{\delta\alpha_a(\mathbb{X})}{\delta\mathbb{X}^b} \delta\mathbb{X}^b d\mathbb{X}^a \right) - \int_M \alpha_a(\mathbb{X}) d\delta\mathbb{X}^a. \end{aligned}$$

Integrating by parts the last term on the RHS:

$$\int_M \alpha_a(\mathbb{X}) d\delta\mathbb{X}^a = -(-1)^d \int_M \left(\frac{\delta\alpha_a}{\delta\mathbb{X}^b} d\mathbb{X}^b \delta\mathbb{X}^a \right) + \cancel{(-1)^d \int_M d\left(\alpha_a(\mathbb{X})\delta\mathbb{X}^a\right)}.$$

Therefore, we obtain

$$\begin{aligned} \delta i_{\hat{d}_M} \int_M \alpha_a(\mathbb{X})\delta\mathbb{X}^a &= (-1)^d \int_M \left(\frac{\delta\alpha_i}{\delta\mathbb{X}^j} \delta\mathbb{X}^j d\mathbb{X}^i + \frac{\delta\alpha_i}{\delta\mathbb{X}^j} d\mathbb{X}^j \delta\mathbb{X}^i \right) = \\ &= (-1)^d i_{\hat{d}_M} \int_M \delta\left(\alpha_a(\mathbb{X})\delta\mathbb{X}^a\right) = \\ &= (-1)^d i_{\hat{d}_M} \int_M \omega_{\mathcal{N}}(\mathbb{X}) \end{aligned}$$

Therefore:

$$\begin{aligned} i_{\mathcal{Q}_M}\omega_M &= (-1)^d \delta i_{\hat{d}_M} \int_M \alpha_a(\mathbb{X})\delta\mathbb{X}^a + (-1)^d \delta \int_M \Theta_{\mathcal{N}}(\mathbb{X}) = \\ &= (-1)^d \delta \left(\int_M \alpha_a(\mathbb{X}) d\mathbb{X}^a + \Theta_{\mathcal{N}}(\mathbb{X}) \right) \equiv \\ &\equiv \delta\mathcal{S}_M, \end{aligned}$$

where $\mathcal{S}_M = (-1)^d \int_M \alpha_a(\mathbb{X}) d\mathbb{X}^a + \Theta_{\mathcal{N}}(\mathbb{X})$. Since $\mathcal{Q}_M$ squares to zero, $\mathcal{S}_M$ satisfies the CME. □

Remark 18. If M has a boundary, $\mathcal{S}_M$ is not the hamiltonian function of $\mathcal{Q}_M$ anymore, since there is also the boundary term. We will study this boundary term in the next chapter.

Corollary 1. *The data $(\mathcal{F}_M, \omega_M, \mathcal{Q}_M, \mathcal{S}_M)$ defines a classical BV theory.*

Remark 19. Starting from some geometric data, the AKSZ construction provides a solution of the classical master equation on the space of AKSZ fields (2.50). The BV theory obtained can be regarded as a BV extension of a certain classical gauge system, whose data can be read off from the AKSZ theory by expanding the BV action $\mathcal{S}_M$ in the homogeneous components of fields. For example, since classical fields have ghost number zero, the classical action S_{cl} is the BV action $\mathcal{S}_M$ restricted to classical fields $S_{cl} = \mathcal{S}_M|_{\mathbb{X}^a \to \mathbb{X}^a_{cl}}$, where $\mathbb{X}^a_{cl}$ is the $k = |x^a|$ term in the sum (2.51).

For appropriate choices of the target manifold, we obtain several BV extensions of known field theories. We examine the cases of Chern-Simons and of the BF theory.

Example 14. (Chern-Simons Theory). The BV extension of Chern-Simons theory can be obtained from the AKSZ construction. Let M be a closed oriented three-dimensional manifold and $\mathcal{N} = \mathfrak{g}[1]$, with $\mathfrak{g}$ a Lie algebra with a symmetric bilinear form g, invariant under the adjoint representation (that is, $\mathfrak{g}$ is a **quadratic** Lie algebra). Let $\{T_a\}$ be the generators of $\mathfrak{g}$. The biliner form g can be used to raise and lower indices and induces an isomorphism between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, $T^a = g^{ab}T_b$. The inner product defined in $\mathfrak{g}$ is the following:

$$\begin{aligned} \langle \bullet, \bullet \rangle_g &: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \\ \langle v, w \rangle_g &= \sum_{a,b} g_{ab} v^a w^b = \sum_a v_a w^a, \quad \forall v, w \in \mathfrak{g}. \end{aligned} \tag{2.55}$$

Let x^a be coordinates on $\mathcal{N}$, $|x^a| = 1$. On the target space $\mathfrak{g}$, we can define a symplectic structure:

$$\begin{aligned} \omega_{\mathcal{N}} &= \frac{1}{2} \sum_{ab} g_{ab} \delta x^a \wedge \delta x^b = \\ &= \delta \left(\frac{1}{2} \sum_{ab} g_{ab} x^a \delta x^b \right) \end{aligned} \tag{2.56}$$

Notice that $gh(\omega_{\mathcal{N}}) = 2$. The canonical bracket between coordinate functions is $\{x^a, x^b\} = g^{ab}$ $(\sum_b g^{ab} g_{bc} = \delta^a_c)$.

The set of smooth functions on $\mathcal{N}$ is $C^\infty(\mathfrak{g}[1]) = \Lambda^\bullet \mathfrak{g}^*$. From example (6), we know that it is a complex with cohomological vector field

$$D = d_{CE} = \sum_{a,b,c} \frac{1}{2} f^a_{bc} x^b x^c \frac{\partial}{\partial x^a}. \tag{2.57}$$

Since g is invariant, D is hamiltonian with hamiltonian function given by

$$\Theta_{\mathcal{N}} = \sum_{a,b,c,d} \frac{1}{6} g_{ab} f^b_{cd} x^a x^c x^d, \quad |\Theta_{\mathcal{N}}| = 3. \tag{2.58}$$

Now let's move to the corresponding AKSZ theory. The space of fields is

$$\mathcal{F}_M = \Omega^\bullet(M) \otimes \mathfrak{g}[1]. \tag{2.59}$$

The BV field content is encoded in one superfield $\mathbb{X}$, since we have only the degree-1 coordinate function $x = x^a T_a$:

$$\mathbb{X} = c + A + A^+ + c^+, \tag{2.60}$$

where:

- c is a zero-form with values in $\mathfrak{g}$ with ghost number 1. It is identified with the generators of infinitesimal gauge transformations;
- A is a one-form with values in $\mathfrak{g}$ with ghost number zero. It is the classical gauge field;
- A^+ is a two-form with values in $\mathfrak{g}$ with ghost number -1. It is the antifield corresponding to A;
- c^+ is a three-form with values in $\mathfrak{g}$ with ghost number -2. It is the antifield correpsonding to c.

The BV symplectic form (2.53) on $\mathcal{F}_M$ is (by picking up only the top-degree content of the integrand):

$$\begin{aligned}\omega_M &= \frac{1}{2}\sum_{ab}\int_M g_{ab}\delta\mathbb{X}^a \wedge \delta\mathbb{X}^b = \\ &= \sum_{ab} g_{ab}\int_M \delta A^a \wedge \delta A^{+b} + \delta c^a \wedge \delta c^{+b} = \\ &= \int_M \left\langle \delta A, \delta A^+\right\rangle_g + \left\langle \delta c, \delta c^+\right\rangle_g ,\end{aligned} \tag{2.61}$$

where $\langle \delta X, \delta X^+\rangle_g \equiv \sum_{ab} g_{ab}\delta X^a \wedge \delta X^{+b}$.

The cohomological vector field on $\mathcal{F}_M$ is obtained as in definition (2.54). From the expression (2.60) of the superfield and the differential (2.57), we obtain:

$$\begin{aligned}\mathcal{Q}_M &= \int_M \left\langle d\mathbb{X} + \frac{1}{2}[\mathbb{X},\mathbb{X}], \frac{\delta}{\delta\mathbb{X}}\right\rangle_g = \\ &= \int_M \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c}\right\rangle_g + \left\langle d_A c, \frac{\delta}{\delta A}\right\rangle_g + \left\langle F_A + [c, A^+], \frac{\delta}{\delta A^+}\right\rangle_g + \\ &+ \left\langle d_A A^+ + [c, c^+], \frac{\delta}{\delta c^+}\right\rangle_g ,\end{aligned} \tag{2.62}$$

where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature two-form and $d_A = d + [A, \cdot]$. Notice that the first two terms on the RHS give the BRST operator.
Finally, the BV action is (see lemma (4))

$$\begin{aligned}\mathcal{S}_M &= -\int_M \frac{1}{2}\left\langle \mathbb{X}, d\mathbb{X}\right\rangle_g + \frac{1}{6}\left\langle \mathbb{X}, [\mathbb{X},\mathbb{X}]\right\rangle_g = \\ &= -\int_M \frac{1}{2}\left\langle A, dA\right\rangle_g + \frac{1}{6}\left\langle A, [A,A]\right\rangle_g + \left\langle A^+, dc + [A,c]\right\rangle_g + \frac{1}{2}\left\langle c^+, [c,c]\right\rangle_g .\end{aligned} \tag{2.63}$$

The first two terms of the BV action are the classical action of Chern-Simons theory. The terms linear in antifields carries informations on the infinitesimal gauge transformations acting on BRST fields. Notice that there are no quadratic terms in the BV action, confirming that the symmetries of the theory are given by the action of a Lie group.

Example 15. (BF theory). First of all, we present the main features of a BF theory. Let M be a d-dimensional closed manifold and let $\mathfrak{g}$ be a finite-dimensional Lie algebra.

We assume that $\mathfrak{g}$ is quadratic and we call g the symmetric bilinear form, invariant under the adjoint representation, defined on $\mathfrak{g}$. In BF theory this assumption is not necessary (on the contrary to Chern-Simons theory), but it simplifies the discussion. We denote with $\langle\rangle_g$ the inner product defined by g in $\mathfrak{g}$. This pairing provides an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. Nevertheless, we will keep using distinguished notations for the Lie algebra $\mathfrak{g}$ and the dual $\mathfrak{g}^*$.

The classical action of the BF theory is

$$S_{cl} = \int_M \langle B, F_A \rangle_g \,, \tag{2.64}$$

where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature (the field strength) of the connection one-form (gauge field) $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^{d-2}(M, \mathfrak{g}^*)$. The variation of the action gives the equations of motion:

$$\begin{aligned} &F_A = 0 \\ &dB + [A, B] = 0. \end{aligned} \tag{2.65}$$

The first equation says that A is a flat connection, while the second is the vanishing of the covariant derivative of B, $d_A B$.
It is interesting to study the symmetries of the action. The infinitesimal transformations that leave the action unchanged are the following:

$$\begin{aligned} &A \to A + d_A\gamma; \\ &B \to B + d_A\lambda_1 - [B, \gamma], \end{aligned} \tag{2.66}$$

where $\gamma \in \Omega^0(M, \mathfrak{g})$ and $\lambda_1 \in \Omega^{d-3}(M, \mathfrak{g}^*)$. Indeed, at the first order in γ and λ:

$$\begin{aligned} S_{cl} \to & S_{cl} + \int_M \langle B, [F_A, \gamma] \rangle_g - \langle [B, \gamma], F_A \rangle_g + \langle d_A\lambda_1, F_A \rangle_g = \\ & S_{cl} + \int_M \cancel{\langle B, [F_A, \gamma] \rangle_g} - \cancel{\langle B, [F_A, \gamma] \rangle_g} + \cancel{\langle \lambda_1, d_A F_A \rangle_g (-1)^{d-3}}, \end{aligned} \tag{2.67}$$

where the last term is zero as a consequence of the second Bianchi identity.

It is important to notice that the gauge transformations on B are defined up to other gauge transformations, but only when the field A is on shell. Indeed, consider the transformation $\lambda_1 \to \lambda_1 + d_A\lambda_2$, with $\lambda_2 \in \Omega^{d-4}(M, \mathfrak{g}^*)$. The gauge transformation on B transforms as $B \to B + d_A\lambda_1 + d_A^2\lambda_2$. The action changes as

$$S_{cl} \to S_{cl} + \int_M \cancel{\langle d_A\lambda_1, F_A \rangle_g} + \left\langle d_A^2\lambda_2, F_A \right\rangle_g .$$

Since $d_A^2\lambda_2 = [F_A, \lambda_2]$, the third term is

$$\int_M \langle [F_A, \lambda_2], F_A \rangle_g = \int_M \langle \lambda_2, [F_A, F_A] \rangle_g \,,$$

which is zero only if the connection A is flat.

If A is flat, we could change $\lambda_2 \to \lambda_2 + d_A\lambda_3$, with $\lambda_3 \in \Omega^{d-5}(M, \mathfrak{g}^*)$, without affecting the gauge transformation on the gauge transformation and so on. At the step n, the classical action changes as

$$\begin{aligned} S_{cl} \to & S_{cl} + \int_M \cancel{\langle d_A\lambda_1, F_A \rangle_g} + \left\langle d_A^2\lambda_2, F_A \right\rangle_g + ... + \langle d_A\lambda_n, F_A \rangle_g = \\ & = S_{cl} - \int_M \langle \lambda_2, [F_A, F_A] \rangle_g - \langle d_A\lambda_3, [F_A, F_A] \rangle_g - .... - \left\langle d_A^{n-2}\lambda_n, [F_A, F_A] \right\rangle_g \end{aligned}$$

and each term is zero only if the connection is flat.

Therefore, we have "higher gauge transformations". On the contrary of the case of the p-form electrodynamics, now the stabilizers emerge only when the field A is on shell. Notice that if $d < 4$, the stabilizers are trivial.

Let's move now to the BV formulation of the BF theory and show that it can be obtained from the AKSZ construction. For simplicity, we consider the case $d = 4$. Let us set the target $\mathcal{N} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[2]$ and let ψ^a and ξ_a be coordinates on $\mathfrak{g}[1]$ and $\mathfrak{g}^*[2]$ respectively. The target can be endowed with

- A graded symplectic two-form

$$\omega_{\mathcal{N}} = \sum_a \delta\xi_a \wedge \delta\psi^a = \delta\left(\sum_a \xi_a \wedge \delta\psi^a\right), \quad gh(\omega_{\mathcal{N}}) = 3;$$

- A cohomological vector field

$$Q_{\mathcal{N}} = \sum_{a,b,c} \frac{1}{2} f^a_{bc}\psi^a\psi^b \frac{\partial}{\partial\psi^c} - f^a_{bc}\psi^b\xi_a \frac{\partial}{\partial\xi_c},$$

  whose hamiltonian function is

$$\Theta_{\mathcal{N}} = \sum_{a,b,c} \frac{1}{2} f^a_{bc}\xi_a\psi^b\psi^c, \quad gh(\Theta_{\mathcal{N}}) = 4.$$

The space of AKSZ fields is

$$\mathcal{F}_M = \Omega^\bullet(M, \mathfrak{g}[1]) \oplus \Omega^\bullet(M, \mathfrak{g}^*[2]). \tag{2.68}$$

mapping Now there are two kinds of superfield. The superfield associated to $\psi = \sum_a \psi^a T_a$:

$$\mathbb{A} = c + A + B^+ + \lambda_1^+ + \lambda_2^+, \quad |\mathbb{A}| = 1, \tag{2.69}$$

and that one associated to $\xi = \sum_a \xi_a T^a$

$$\mathbb{B} = \lambda_2 + \lambda_1 + B + A^+ + c^+, \quad |\mathbb{B}| = 2. \tag{2.70}$$

Let's comment the field content of $\mathbb{A}$ and $\mathbb{B}$.

- The $gh = 0$ part of the superfields, namely A and B, are the classical fields;
- The fields with $gh = 1$, c and λ_1 are associated to the gauge transformations acting on A and B. The field with $gh = 2$, λ_2 encodes the gauge transformations acting on λ_1 (the non trivial stabilizer);
- The other fields are the antifields and their ghost number is lower than zero.

The BV two-form on $\mathcal{F}_M$ is

$$\begin{aligned}\omega_M &= \int_M \langle \delta\mathbb{B}, \delta\mathbb{A}\rangle_g = \\ &= \int_M \langle \delta c, \delta c^+\rangle_g + \langle \delta A, \delta A^+\rangle_g + \langle \delta B, \delta B^+\rangle_g + \langle \delta\lambda_1, \delta\lambda_1^+\rangle_g + \langle \delta\lambda_2, \delta\lambda_2^+\rangle_g ,\end{aligned} \tag{2.71}$$

while the BV action is

$$\begin{aligned}\mathcal{S}_M &= \int_M \left\langle \mathbb{B}, d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \right\rangle_g = \\ &= \int_M \langle B, F_A \rangle_g + \langle A^+, d_A c \rangle_g + \langle B^+, [c, B] + d_A \lambda_1 \rangle_g + \langle \lambda_1^+, d_A \lambda_2 \rangle_g + \\ &\qquad + \langle \lambda_2^+, [c, \lambda_2] \rangle_g + \left\langle c^+, \frac{1}{2}[c, c] \right\rangle_g + \frac{1}{2} \langle \lambda_2, [B^+, B^+] \rangle_g .\end{aligned} \tag{2.72}$$

The first term is the classical action and the linear terms in the antifields encode the gauge transformation on fields. The appearance of the quadratic term in the antifields is syntomatic of the non-integrability of the gauge symmetry.
Finally, the cohomological vector field is

$$\begin{aligned}\mathcal{Q}_M &= \int_M \left\langle d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}], \frac{\delta}{\delta \mathbb{A}} \right\rangle_g + \int_M \left\langle d_{\mathbb{A}} \mathbb{B}, \frac{\delta}{\delta \mathbb{B}} \right\rangle_g = \\ &= \int_M \left\langle \frac{1}{2}[c, c], \frac{\delta}{\delta c} \right\rangle_g + \left\langle d_A c, \frac{\delta}{\delta A} \right\rangle_g + \left\langle F_A, \frac{\delta}{\delta B^+} \right\rangle_g + \left\langle d_A B^+, \frac{\delta}{\delta \lambda_1^+} \right\rangle_g + \\ &\quad + \left\langle d_A \lambda_1^+, \frac{\delta}{\delta \lambda_2^+} \right\rangle_g + \int_M \left\langle [c, \lambda_2], \frac{\delta}{\delta \lambda_2} \right\rangle_g + \left\langle d_A \lambda_2 + [c, \lambda_1], \frac{\delta}{\delta \lambda_1} \right\rangle_g + \\ &\quad + \left\langle d_A \lambda_1 + [c, B] + [B^+, \lambda_2], \frac{\delta}{\delta B} \right\rangle_g + \\ &\quad + \left\langle d_A B + [c, A^+] + [B^+, \lambda_1] + [\lambda_1^+, \lambda_2], \frac{\delta}{\delta A^+} \right\rangle_g + \\ &\quad + \left\langle d_A A^+ + [c, c^+] + [B^+, B] + [\lambda_1^+, \lambda_1] + [\lambda_2^+, \lambda_2] \frac{\delta}{\delta c^+} \right\rangle_g .\end{aligned} \tag{2.73}$$

Remark 20. The construction above generalizes to all dimensions. In two and three dimensions, the gauge symmetry is easier to deal with, since there are no non trivial stabilizers and higher ghosts are not present. While in two and three dimensions it is enough to employ the Faddeev-Popov technique to perform a gauge fixing, in four dimension the BV formalism becomes essential.

Chapter 3

The CMR approach

In this chapter we introduce the approach to field theories on manifolds with boundary, developed in ([11], [12]) by Cattaneo, Mnev and Reshetikhin (CMR).

The main idea is the following. When one introduces the Euler-Lagrange equations by computing the variation of the action, there are boundary terms that are usually set to zero thanks to a suitable choice of boundary conditions on the set of fields. In the CMR approach, the boundary conditions are not fixed and the additional boundary term is interpreted as a one form on the space of boundary fields. In particular, the construction shows that this term induces a field theory on the boundary.

When the field theory in the bulk possesses a gauge symmetry, it is convenient to use the BV formalism we have introduced in chapter (2). We have seen that this formalism allows to describe gauge theories using the language of graded symplectic geometry. The CMR construction is used to induce analogous geometric structures, but with shifted degree, on the boundary. These structures define a hamiltonian (BFV) theory (namely, the space of boundary fields is a phase space). Therefore, the main result is that a BV theory in the bulk induces a BFV theory on the boundary. The collection of the two theories is called BV-BFV theory.

One of the most prominent features of this procedure is that is that it is general and it applies to any local field theory, defined by an action functional which depends on fields and a finite number of their derivatives. Furthermore, it is sistematic and it can be iterated to induce a field theory in lower dimension from a field theory in greater dimension. In general, this procedure stops at some point, since, as we will see, it involves a presymplectic reduction (see Appendix (A)), which can be singular. However, for some particular kind of field theories (for example, AKSZ theories) the procedure goes on until $d = 0$.

The basic references we followed in this chapter are [11, 12]. For a more recent review, see [14].

3.1 Classical Field Theory with boundary

First of all, recall that a classical lagrangian field theory is defined by the collection of the following data:

- A source manifold M, with possible extra structures such as a metric;

- The space of fields $\mathcal{F}_M$ on M, which can include functions on M (e.g. scalar fields), sections of vector bundles, connections on a principal bundle over M (gauge fields) and so on;
- A density $\mathcal{L}$ of the fields and finitely many derivatives of the fields (the locality condition), called lagrangian density.

With these ingredients, we can construct the main object of a classical lagrangian field theory, the functional **action** S:

$$S = \int_M \mathcal{L}(\phi_i, \partial\phi_i, \partial\partial\phi_i, ...). \tag{3.1}$$

From a classical point of view, the physical dinamical content of a lagrangian field theory is totally encoded in the variational problem of S. Suppose that M has no boundary. The variational problem gives the Euler-Lagrange equations for the fields in the bulk:

$$\delta S = \int_M \left(EL^i_M\right) \delta\phi_i. \tag{3.2}$$

The critical locus of S, or the zero locus of δS, is the set of the on-shell fields, namely the solutions of the equations of motion.

What happens if we consider a manifold M with a boundary ∂M? Besides the term above, the variational problem gives rise also to a boundary term BT:

$$\delta S = \int_M \left(EL^i_M\right) \delta\phi_i + BT \tag{3.3}$$

Usually, what one does at this point is fixing some boundary conditions on the bulk fields in order to cancel the boundary term on the RHS of (3.3).
The approach we will follow is based on avoiding this last step and considering all the possible boundary values the bulk fields can assume. The question that naturally arises is what is the right interpretation for the boundary term BT in 3.3.

Before studying the general case, let's study the following example:

Example 16. (First-order Electrodynamics). Consider the case of electrodynamics, reformulated in the so called first-order formalism. Let (M, g_M) be a d-dim Riemannian manifold with boundary ∂M. The space of fields is

$$\mathcal{F}_M = Conn_M \times \Omega^{d-2}(M),$$

where $Conn_M$ is the set of connections one-form for a principal $U(1)$-bundle over M (gauge fields). The action is

$$S_M[A, B] = \int_M B \wedge dA + \frac{1}{2} B \wedge^* B, \quad (A, B) \in \mathcal{F}_M$$

where $*$ is the Hodge operation. Let now $\mathcal{F}_{\partial M}$ be the set of all the boundary configurations of bulk fields and $p : \mathcal{F_M} \to \mathcal{F}_{\partial M}$ the surjection which sends the bulk fields to the corresponding boundary value. The variational problem leads to

$$\delta S_M = \int_M dB\delta A + (dA + *B)\,\delta B + \int_M d\,(B \wedge \delta A)\,.$$

The first terms on the RHS are the EL equations for the fields A and B. Notice that, if we put the field B on shell, the action $S_M[A,B]$ reduces to the usual electrodynamics' action. The last term is a total derivative, which we can interpret as a one-form $\breve{\alpha} \in \Omega^1(\mathcal{F}_M)$ on the space of bulk fields (δ being the de Rham differential on $\Omega^\bullet(\mathcal{F}_M)$). If $B_{\partial M}$ and $A_{\partial M}$ are two boundary configurations of B and A, we can write

$$\breve{\alpha} = \int_M d\left(p^*(B_{\partial M} \wedge \delta A_{\partial M})\right) \equiv p^* \alpha_{\partial M},$$

where $\alpha_{\partial M} = \int_{\partial M} B_{\partial M} \wedge \delta A_{\partial M} \in \Omega^1(\mathcal{F}_{\partial M})$. By taking its variation, we obtain a weakly non-degenerate two-form $\omega_{\partial M} = \delta\alpha_{\partial M}$. This two form is interpreted as a symplectic form on $\mathcal{F}_{\partial M}$, the space of boundary configurations of bulk fields, which is seen as the space of boundary fields.

This particular example shows a rather important fact: the field theory on the bulk induces a symplectic structure on the space of boundary fields, given by the boundary configurations of the bulk fields.

In general, we can write the variation (3.3) as

$$\delta S_M = \mathcal{EL}_M + \breve{\alpha}, \tag{3.4}$$

where $\mathcal{EL}_M$ is the integral over M of the Euler-Lagrange equations of bulk fields and $\breve{\alpha} \in \Omega^1(\mathcal{F}_M)$. We then introduce the form $\breve{\omega} = \delta\breve{\alpha} \in \Omega^2(\mathcal{F}_M)$. In general this two-form will have a non-trivial kernel, but it is assumed to be presymplectic (see Appendix (A)). Therefore, $Ker(\breve{\omega})$ is an integrable distribution in $Vect(\mathcal{F}_M)$ and we can define the quotient space $\mathcal{F}_M/Ker(\breve{\omega}) \equiv \mathcal{F}_{\partial M}$. Let $p : \mathcal{F}_M \to \mathcal{F}_{\partial M}$ be the natural surjection on the leaf space. If the leaf space is smooth, the form $\breve{\omega}$ reduces to a weakly non degenerate form $\omega_{\partial M}$ on the leaf space such that $\breve{\omega} = p^*\omega_{\partial M}$. The presymplectic reduction $(\mathcal{F}_{\partial M}, \omega_{\partial M})$ is therefore a symplectic manifold, with $\omega_{\partial M}$ weakly non-degenerate.
Since $\omega_{\partial M}$ is closed, it can always be locally written as

$$\omega_{\partial M} = \delta\alpha_{\partial M}, \tag{3.5}$$

where $\alpha_{\partial M} \in \Omega^1(\mathcal{F}_{\partial M})$ is the symplectic potential. It is immediate to see that $\breve{\alpha} = p^*\alpha_{\partial M}$.
In the following, we suppose that (3.5) is globally true.

Remark 21. Notice that we may change $\alpha_{\partial M}$ to

$$\alpha^f_{\partial M} = \alpha_{\partial M} + \delta f,$$

where $f \in C^\infty(\mathcal{F}_{\partial M})$, without changing the symplectic form $\omega_{\partial M}$. The variational problem (3.4) is preserved if we change S_M to

$$S^f_M = S_M - p^* f.$$

These transformations do not alter the equations of motion and can be considered as gauge transformations (which correspond to change the action by a boundary term).

3.2 BV-BFV construction

The main point of the previous section is that, given a classical field theory on a manifold M with boundary ∂M, the boundary term in the variation of the action give rise to a symplectic structure on the space of boundary fields, obtained after a presymplectic reduction.

Let assume that the field theory possesses a gauge symmetry. The question that naturally arises is how the local symmetries of the theory project to those of the field theory on the boundary.

In order to answer to this question, consider a BV theory $(\mathcal{F}_M, \mathcal{S}_M, \mathcal{Q}_M, \omega_M)$ in the bulk. Recall that $(\mathcal{F}_M, \omega_M)$ is a $\mathbb{Z}$-graded symplectic manifold, with $gh(\omega_M) = -1$, and that, if M has no boundary, $\mathcal{Q}_M$ and $\mathcal{S}_M$ are related by

$$i_{\mathcal{Q}_M}\omega_M = \delta\mathcal{S}_M.$$

From a geometrical point of view, this means that $\mathcal{Q}_M$ is the hamiltonian vector field for the BV action. In more physical terms, the variation of the BV action leads to the EoM, written in a similar way to hamiltonian mechanics (the degrees are obviously different). As a consequence, the CME $\{\mathcal{S}_M, \mathcal{S}_M\} = 0$ holds.

As we have seen in the previous section, if M has a boundary, there will be a non zero boundary term in the variation of $\mathcal{S}_M$, which we can interpret as a one-form on $\mathcal{F}_M$:

$$i_{\mathcal{Q}_M}\omega_M = \delta\mathcal{S}_M + \breve{\alpha}, \tag{3.6}$$

Eq. (3.6) tells that $\mathcal{Q}_M$ is not the hamiltonian vector field for $\mathcal{S}_M$ anymore. In this sense, $\breve{\alpha}$ measures the failure of the CME. The set $(\mathcal{F}_M, \omega_M, \mathcal{Q}_M, \mathcal{S}_M)$, with $\mathcal{Q}_M$ and $\mathcal{S}_M$ such that relation (3.6) holds, is called a **relaxed BV theory**.
Indeed, $\mathcal{Q}_M$ is not even symplectic. This can be easily seen by taking the exterior derivative δ on both sides of relation (3.6) (and recalling that ω_M is closed):

$$L_{\mathcal{Q}_M}\omega_M = \delta\breve{\alpha} \equiv \breve{\omega} \in \Omega^2(\mathcal{F}_M). \tag{3.7}$$

Assuming $\breve{\omega}$ presymplectic, we interpret the presymplectic reduction $(\mathcal{F}_{\partial M}, \omega_{\partial M})$ as the space of boundary fields. Notice that $gh(\breve{\omega}) = gh(\omega_{\partial M}) = 0$.

Suppose that $\omega_{\partial M} = \delta\alpha_{\partial M}$ globally, with $\alpha_{\partial M} \in \Omega^1(\mathcal{F}_{\partial M})$. If $p : \mathcal{F}_M \to \mathcal{F}_{\partial M}$ is the restriction map, then $\alpha = p^*\alpha_{\partial M}$. Therefore, we arrive at the fundamental equation of the BV theory for manifolds with boundary:

$$\boxed{i_{\mathcal{Q}_M}\omega_M = \delta\mathcal{S}_{\mathcal{M}} + p^*\alpha_\partial}. \tag{3.8}$$

We still have to study how gauge symmetries on the bulk are related to those on the boundary.

Lemma 5. The cohomological vector field $\mathcal{Q}_M$ is projectable to the reduction $\mathcal{F}_{\partial M}$.

Proof. In order to show this Lemma, it is enough to see that $\mathcal{Q}_M$ preserves $Ker(\breve{\omega})$. First of all, because of Eq. (3.7):

$$L_{\mathcal{Q}_M}\breve{\omega} = 0,$$

namely $\mathcal{Q}_M$ preserves $\breve{\omega}$. Let now $X \in Ker(\breve{\omega})$. If $\mathcal{Q}_M$ preserves $Ker(\breve{\omega})$, it must hold that $L_{\mathcal{Q}_M}X = [\mathcal{Q}_M, X] \in Ker(\breve{\omega})$. However, this follows from the identites

$$i_{[\mathcal{Q}_M, X]}\breve{\omega} = [L_{\mathcal{Q}_M}, i_X]\,\breve{\omega} = 0.$$

We conclude that there is a uniquely well-defined $Q_{\partial M}$ on $\mathcal{F}_{\partial M}$, to which $\mathcal{Q}_M$ projects:

$$p_*\mathcal{Q}_M = \mathcal{Q}_{\partial M}.$$

The vector field $\mathcal{Q}_{\partial M}$ is automatically cohomological and symplectic:

$$1)\ \mathcal{Q}^2_{\partial M} = [\mathcal{Q}_{\partial M}, \mathcal{Q}_{\partial M}] = [p_*\mathcal{Q}_M, p_*\mathcal{Q}_M] = p_*[\mathcal{Q}_M, \mathcal{Q}_M] = 0;$$
$$2)\ 0 = L_{\mathcal{Q}_M}\breve{\omega} = L_{\mathcal{Q}_M}(p^*\omega_{\partial M}) = p^*\left(L_{p_*\mathcal{Q}_M}\omega_{\partial M}\right) = p^*\left(L_{\mathcal{Q}_{\partial M}}\omega_{\partial M}\right).$$

□

Moreover, the cohomological vector field $\mathcal{Q}_{\partial M}$ is $\omega_{\partial M}$-hamiltonian with hamiltonian function $i_E i_{\mathcal{Q}_{\partial M}}\omega_{\partial M} \equiv \mathcal{S}_{\partial M}$, where E is the Euler vector field (see example (4)). In fact:

$$\begin{aligned}\delta i_E i_{\mathcal{Q}_{\partial M}}\omega_{\partial M} &= \left([L_E, i_{\mathcal{Q}_{\partial M}}] + i_{\mathcal{Q}_{\partial M}} L_E\right)\omega_{\partial M} = \\ &= [L_E, i_{\mathcal{Q}_{\partial M}}]\,\omega_{\partial M} = \\ &= i_{[E,\mathcal{Q}_{\partial M}]}\omega_{\partial M} = \\ &= i_{\mathcal{Q}_{\partial M}}\omega_{\partial M},\end{aligned} \tag{3.9}$$

where we used that $L_E\omega_{\partial M} = 0$ (since $gh(\omega_{\partial M}) = 0$) and $L_E(\mathcal{Q}_{\partial M}) = \mathcal{Q}_{\partial M}$ (since $\mathcal{Q}_{\partial M}$ has ghost degree 1).

Let's summarize what we have found:

- In the bulk M we have the collection of data $(\mathcal{F}_M, \omega_M, \mathcal{Q}_M, \mathcal{S}_M)$ defining a relaxed BV theory;
- In the boundary we have the data $(\mathcal{F}_{\partial M}, \omega_{\partial M}, \mathcal{Q}_{\partial M}, \mathcal{S}_{\partial M})$, where:
  1. $(\mathcal{F}_{\partial M}, \omega_{\partial M})$ is the presymplectic reduction of $(\mathcal{F}_M, \breve{\omega})$, with $|\omega_{\partial M}| = 0$;
  2. $p_*\mathcal{Q}_M = \mathcal{Q}_{\partial M}$;
  3. $i_{\mathcal{Q}_{\partial M}}\omega_{\partial M} = \delta\mathcal{S}_{\partial M}$, with $|\mathcal{S}_{\partial M}| = -1$.

The set of data on the boundary is a **BFV theory** (see remark (15)) and the collection of the two theories in the bulk and in the boundary is what we call an **exact BV-BFV theory**. The adjective "exact" is due to the exactness of the symplectic form $\omega_{\partial M}$.

Notation 3. In the following we will use the same notation for bulk fields and boundary fields. It will be clear from the context if we refer to bulk or boundary fields.

Remark 22. Notice that in principle it is not mandatory to require $\omega_{\partial M}$ to be exact, but the formulation of the theory becomes more involved.

Example 17. (First-order Electrodynamics). We return to the first-order electrodynamics, seen in example (16). We specialize to the case $d = 4$. We need to extend the theory to a BV theory. First of all, since we want to implement gauge transformations for A, we add the ghost field $c \in \Omega^0(M)[1]$ and we define the space of BRST fields $E_M = \Omega^0(M)[1] \oplus \mathcal{F}_M$. The space of BV fields is simply

$$\mathcal{F}_M = T^*[-1]E_M = E_M \oplus \underset{B^+}{\Omega^2(M)[-1]} \oplus \underset{A^+}{\Omega^3(M)[-1]} \oplus \underset{c^+}{\Omega^4(M)[-2]}, \tag{3.10}$$

where A^+, B^+, c^+ are the antifields associated to the BRST fields A, B, c. The BV symplectic form on the space of BV fields is the canonical symplectic form on the shifted cotangent $T^*[-1]E_M$,

$$\omega_M = \int_M \delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+.$$

The BV action $\mathcal{S}_M$ is

$$\mathcal{S}_M = \int_M \left(B \wedge dA + \frac{1}{2} B \wedge *B + A^+ \wedge dc \right)$$

and the cohomological vector field $\mathcal{Q}_M$ is

$$\mathcal{Q}_M = \int_M \left(dc \wedge \frac{\delta}{\delta A} + dB \wedge \frac{\delta}{\delta A^+} + (*B + dA) \wedge \frac{\delta}{\delta B^+} + dA^+ \wedge \frac{\delta}{\delta c^+} \right).$$

Let's turn now to the induced theory on the boundary. The variation of the BV action leads to the EoM plus a boundary term of the form:

$$\breve{\alpha} = \int_M d\left(B \wedge \delta A + A^+ \wedge \delta c\right) \Rightarrow \breve{\omega} = -\int_M d\left(\delta B \wedge \delta A + \delta A^+ \wedge \delta c\right). \tag{3.11}$$

We immediately see that the kernel of $\breve{\omega}$ consists of the B^+-direction and the c^+-direction. Therefore, the space of boundary fields is simply

$$\mathcal{F}_{\partial M} = E_{\partial M} \oplus \underset{A^+}{\Omega^3(\partial M)[-1]},$$

where $E_{\partial M} = \Omega^0(\partial M)[1] \oplus \mathcal{F}_{\partial M}$. Notice that, since in this case the BV fields are differential forms on the bulk, the projection $p : \mathcal{F}_M \to \mathcal{F}_{\partial M}$ is taken as the pullback of the natural immersion $\mathfrak{i} : \partial M \to M$. The symplectic form on the space of BFV fields is

$$\omega_{\partial M} = -\int_{\partial M} \left(\delta B \wedge \delta A + \delta A^+ \wedge \delta c\right), \quad \breve{\omega} = p^* \omega_{\partial M}$$

which is clearly exact. As for the symmetries on the boundary, the vector field $\mathcal{Q}_M$ projects to

$$\mathcal{Q}_{\partial M} = \int_M \left(dB \wedge \frac{\delta}{\delta A^+} + dc \wedge \frac{\delta}{\delta A} \right),$$

which is $\omega_{\partial M}$-hamiltonian with hamiltonian function

$$\mathcal{S}_{\partial M} = \int_{\partial M} c\, dB.$$

Theories in lower dimension

Up to now, we have described a sistematic procedure which allows to induce a BFV theory on the boundary starting from a BV theory in the bulk. However, this construction can be applied iteratively to go to lower dimension. In particular, we could consider **manifolds with boundary and corners**.

A detailed discussion of manifolds with corners is beyond the scope of this thesis. Roughly speaking, a manifold with boundary and corners is a manifold which is modelled on open subsets of quadrants in $\mathbb{R}^d$, which are subsets of $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n : x_1 \geq 0, ..., x_k \geq 0\}$, where $0 \leq k \leq d$, and which is supposed to be of class C^r, $r \geq 1$. A subset of the quadrants of the form $\{x \in \mathbb{R}^n : x_1 = 0, ..., x_l = 0, x_{l+1} \geq 0, ..., x_k \geq 0\}$ is called a **corner** of dimension $d - l$. Notice that if $k = 1$ we retrieve the basic notion of manifold with boundary.

Given a manifold M with boundary ∂M and $(d-2)$-dimensional corners, we know that a BV theory in the bulk induces a BFV theory on the boundary. However, our boundary ∂M now has a "boundary", given by corners $\partial\partial M$ of dimension $d-2$. Now, the theory on the boundary is a relaxed BFV theory, since the condition that $\mathcal{S}_{\partial M}$ is the hamiltonian function for $\mathcal{Q}_{\partial M}$ is not true. The error is given by a one-form $\check{\alpha}_{\partial M}$ on $\mathcal{F}_{\partial M}$. The presymplectic reduction of $\mathcal{F}_{\partial M}$ by the kernel of the two-form $\check{\omega}_{\partial M} = \delta\check{\alpha}_{\partial M}$ gives the space of corner fields $\mathcal{F}_{\partial\partial M}$, which is a graded symplectic manifold with graded symplectic form $\omega_{\partial\partial M}$ of degree $+1$, such that $\check{\omega}_{\partial M} = p_\partial^* \omega_{\partial\partial M}$ with $p_\partial : \mathcal{F}_{\partial M} \to \mathcal{F}_{\partial\partial M}$ the projection of boundary fields to corner fields. Again, the cohomological vector field on $\mathcal{F}_{\partial M}$ projects to a cohomological vector field on $\mathcal{F}_{\partial\partial M}$, which will be $\omega_{\partial\partial M}$-hamiltonian, with hamiltonian function a degree-2 action $\mathcal{S}_{\partial\partial M}$.

The data $(\mathcal{F}_{\partial\partial M}, \omega_{\partial\partial M}, \mathcal{Q}_{\partial\partial M}, \mathcal{S}_{\partial\partial M})$ defines a $\mathrm{BF}^2\mathrm{V}$ theory (see remark (15)), living on the $(d-2)$-dimensional corners.

If the $(d-2)$-dimensional corners have $(d-3)$-dimensional corners, the same construction can be iterated. In particular, we find that the $\mathrm{BF}^2\mathrm{V}$ theory on the $(d-2)$-corners induces a $\mathrm{BF}^2\mathrm{V}$ theory on the $(d-3)$-dimensional corners and so on.

Tipically, at some point the reduced action $\mathcal{S}_{\partial\partial.....\partial M} = 0$ and the procedure stops. Moreover, the presymplectic reduction could be singular. We will see in the next section that for some paricular kind of topological field theories (the AKSZ-like ones) this construction can be iterated until we arrive to $d = 0$.

3.3 AKSZ theory

In the previous section we have studied the BV-BFV construction and its extension to lower dimension. In general, the construction stops at some point, but, as we will see, for AKSZ theories we can proceed further and get something non trivial even in dimension zero. We say that AKSZ theories are **fully extendable theories**.

Consider an AKSZ theory on a manifold M. Let $(\mathcal{N}, \omega_{\mathcal{N}}, D)$ be the target manifold, with $\mathcal{N}$ a graded vector space, and $\mathcal{F}_M = \Omega^\bullet(M) \otimes \mathcal{N}$ the space of AKSZ fields. Recall that the graded symplectic form on the target induces a BV two-form on $\mathcal{F}_M$ (see definition (2.53)), which can also be endowed with a cohomological vector field $\mathcal{Q}_M$ (2.54).

Assume now that M has a boundary ∂M. The AKSZ construction allows to induce a BFV theory on ∂M by taking:

- The space of boundary AKSZ fields $\mathcal{F}_{\partial M} = \Omega^\bullet(\partial M) \otimes \mathcal{N}$, which is parametrized by the superfields

$$\mathbb{X}^a_{\partial M}(u', \theta') = \sum_{k=0}^{n-1} \sum_{1 \le i_1 \le \ldots \le i_k \le k} X^a_{i_1,\ldots,i_k}(u')\theta'^{i_1} \cdots \theta'^{i_k}, \tag{3.12}$$

  where (u', θ') are a list of local coordinate in $T[1]\partial M$;

- The cohomological vector field $Q_{\partial M}$ is the composition of $d_{\partial M}$ and D with the superfields $\mathbb{X}^a_{\partial M}$;

- The symplectic form $\omega_{\partial M} = \frac{1}{2}\sum_{ab}\int_M \omega_{ab}(\mathbb{X}_{\partial M})\delta\mathbb{X}^a_{\partial M} \wedge \delta\mathbb{X}^b_{\partial M}$;

- The action $\mathcal{S}_{\partial M} = (-1)^{d-1}\int_{\partial M} \alpha_a(\mathbb{X}_{\partial M})\, d\mathbb{X}^a_{\partial M} + \Theta_{\mathcal{N}}(\mathbb{X}_{\partial M})$. Its ghost degree is $gh(\mathcal{S}_{\partial M}) = 1$.

Roughly speaking, in order to obtain the theory on the boundary, it is enough to "substitute" M with ∂M and $\mathbb{X}_M$ with $\mathbb{X}_{\partial M}$.

Proposition 1. The BV data on the bulk and the BFV data on the boundary described above define a BV-BFV theory.

Proof. First of all, notice that it is defined a projection $p : \mathcal{F}_M \to \mathcal{F}_{\partial M}$, given by the pullback of the injection $\mathfrak{i} : \partial M \to M$, and that the cohomological vector field $\mathcal{Q}_M$ projects to $\mathcal{Q}_{\partial M}$.

We have to show that Eq. (3.8) holds. Recall that in the proof of lemma (4) we cancelled the boundary term, whom now we have to take into account. In particular, we find that

$$\begin{aligned} i_{\mathcal{Q}_M}\omega_M &= \delta\mathcal{S}_M + \int_M d\left(\alpha_a(\mathbb{X})\delta\mathbb{X}^a\right) = \\ &= \delta\mathcal{S}_M + p^* \int_{\partial M} \alpha_a(\mathbb{X}_{\partial M})\delta\mathbb{X}^a_{\partial M} = \\ &= \delta\mathcal{S}_M + p^*\alpha_{\partial M}, \end{aligned} \tag{3.13}$$

where $\alpha_{\partial M} = \int_{\partial M} \alpha_a(\mathbb{X}_{\partial M})\delta\mathbb{X}^a_{\partial M}$ is a one-form the space of boundary fields.

□

Therefore, Eq. (3.8) holds and the boundary term is $\alpha_{\partial M}$.

Remark 23. Notice that, with our conventions, the AKSZ symplectic form on the boundary is $\omega_{\partial M} = (-1)^{d-1}\delta\alpha_{\partial M}$. Therefore, $\check{\omega} = (-1)^{d-1}p^*\omega_{\partial M}$.

Remark 24. An AKSZ theory can be maximally extended. In fact, on a $(d-l)$-dimensional corner Σ, the space of AKSZ fields is $\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma) \otimes \mathcal{N}$ and the superfields are

$$\mathbb{X}^a_\Sigma(u'', \theta'') = \sum_{k=0}^{n-l} \sum_{1\le i_1 \le \ldots \le i_k \le k} X^a_{i_1,\ldots,i_k}(u'')\theta''^{i_1}\cdots\theta''^{i_k},$$

with (u'', θ'') a collection of local coordinates on $T[1]\Sigma$. As for the cohomological vector field, the graded symplectic two-form ω_Σ on $\mathcal{F}_\Sigma$ and the AKSZ action S_Σ, these are obtained as above by simply "substituting" ∂M with Σ and the superfields $\mathbb{X}_{\partial M}$ with $\mathbb{X}_\Sigma$. Obviously, the grading is different from the d, $d-1$-dimensional case:

$$gh(\omega_\Sigma) = l - 1, \quad gh(\mathcal{S}_\Sigma) = l$$

Example 18. (Chern-Simons). Consider Chern-Simons theory on a 3-dimensional manifold M with boundary ∂M and corners $\partial\partial M$. The theory in the bulk has been studied in example (14).
As for the boundary BFV theory, boundary fields are simply the pullback of the bulk fields to the boundary

$$\mathcal{F}_{\partial M} = \Omega^\bullet(\partial M) \otimes \mathfrak{g}[1],$$

where $\mathfrak{g}$ is a quadratic Lie algebra with symmetric bilinear form g. The field content of the theory on the boundary is encoded in the superfield

$$\mathbb{X} = c + A + A^+. \tag{3.14}$$

It is important to stress that, even if we are using the same notation for boundary fields as well as for the bulk fields, A and A^+ are not dual coordinates on the 2-dimensional boundary.

The exact BFV structure is the following:

- The boundary term from the variational principle gives the one-form

$$\alpha_{\partial M} = \frac{1}{2}\int_M \langle A, \delta A\rangle_g + \langle c, \delta A^+\rangle_g + \langle A^+, \delta c\rangle_g ,$$

where $\langle\rangle_g$ is the inner product in $\mathfrak{g}$ induced by g. Its exterior derivative is the BFV symplectic form

$$\omega_{\partial M} = \int_{\partial M} \frac{1}{2}\langle \delta A, \delta A\rangle_g + \langle \delta A^+, \delta c\rangle_g .$$

Therefore, we infer that on the boundary the antifield corresponding to c is A^+.

- The cohomological vector fields is

$$\mathcal{Q}_{\partial M} = \int_{\partial M} \left\langle d_A c, \frac{\delta}{\delta A}\right\rangle_g + \left\langle (F_A + [c, A^+]), \frac{\delta}{\delta A^+}\right\rangle_g + \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c}\right\rangle_g$$

- The action is

$$\mathcal{S}_{\partial M} = \int_{\partial M} \langle c, F_A\rangle_g + \left\langle \frac{1}{2}[c,c], A^+\right\rangle_g .$$

Studying the variational principle for the BFV theory, we can find out the theory living on the corners $\partial\partial M$, which is a BF^{2}V theory. The space of fields is $\mathcal{F}_{\partial\partial M} = \Omega^\bullet(\partial\partial M)\otimes\mathfrak{g}[1]$. Corner fields are therefore c and A. Furthermore:

- The graded symplectic two-form on $\mathcal{F}_{\partial\partial M}$ is

$$\omega_{\partial\partial M} = \int_{\partial\partial M} \langle \delta A, \delta c\rangle_g ;$$

- The cohomological vector field is

$$\mathcal{Q}_{\partial\partial M} = \int_{\partial\partial M} \left\langle d_A c, \frac{\delta}{\delta A}\right\rangle_g + \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c}\right\rangle_g ;$$

- The action is

$$\mathcal{S}_{\partial\partial M} = -\int_{\partial\partial M} \left\langle c, \frac{1}{2}dc + \frac{1}{3}[A,c]\right\rangle_g + \frac{1}{6}\langle A, [c,c]\rangle_g$$

Example 19. (BF theory). In example (15) we have studied the BV formulation of BF theory on closed oriented 4-dimensional manifold M. Recall that, even if we assume to have a pairing $\langle\rangle_g$ on $\mathfrak{g}$, with g the symmetric bilinear form, invariant under the adjoint representation, and therefore an isomorphism $\mathfrak{g}\simeq\mathfrak{g}^*$, we keep the distinguished notation $\mathfrak{g}$ for the Lie algebra and $\mathfrak{g}^*$ for its dual.

Suppose that M has boundary ∂M and corners $\partial\partial M$. The space of boundary fields is

$$\mathcal{F}_{\partial M} = \Omega^\bullet(\partial M, \mathfrak{g}[1]) \oplus \Omega^\bullet(\partial M, \mathfrak{g}^*)$$

and the projection form bulk fields to boundary fields is again the pullback of the inclusion map $\mathfrak{i} : \partial M \to M$. This space is parametrized by the superfields

$$\mathbb{A} = c + A + B^+ + \lambda_1^+$$
$$\mathbb{B} = \lambda_2 + \lambda_1 + B + A^+.$$

The boundary term emerging from the variational principle of the BV action $\mathcal{S}_M$ gives the one-form on the space of boundary fields

$$\alpha_{\partial M} = \int_{\partial M} \langle B, \delta A \rangle_g + \langle \lambda_2, \delta \lambda_1^+ \rangle_g + \langle \lambda_1, \delta B^+ \rangle_g + \langle A^+, \delta c \rangle_g .$$

With our conventions, the AKSZ symplectic form on the boundary is

$$\omega_{\partial M} = -\delta \alpha_{\partial M} = \int_{\partial M} \langle \delta B, \delta A \rangle_g + \langle \delta \lambda_2, \delta \lambda_1^+ \rangle_g + \langle \delta \lambda_1, \delta B^+ \rangle_g + \langle \delta A^+, \delta c \rangle_g .$$

The cohomological vector field acting on $\mathcal{F}_{\partial M}$ is

$$\mathcal{Q}_{\partial M} = \int_{\partial M} \left\langle d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}], \frac{\delta}{\delta \mathbb{A}} \right\rangle_g + \left\langle d_{\mathbb{A}} \mathbb{B}, \frac{\delta}{\delta \mathbb{B}} \right\rangle_g .$$

Finally, the BFV action is

$$\begin{aligned} S_{\partial M} &= -\int_{\partial M} \left\langle \mathbb{B}, d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \right\rangle_g = \\ &= -\int_{\partial M} \langle \lambda_2, d_A B^+ + [c, \lambda_1^+] \rangle_g + \langle \lambda_1, F_A \rangle_g + \langle B, d_A c \rangle_g + \left\langle A^+, \frac{1}{2}[c, c] \right\rangle_g . \end{aligned}$$

We can now repeat the same reasoning to induce the theory on the two-dimensional corners. The data defining the theory living on $\partial\partial M$ are the following:

- The superfields encoding fields living on the corners

$$\begin{aligned} \mathbb{A} &= c + A + B^+ \\ \mathbb{B} &= \lambda_2 + \lambda_1 + B. \end{aligned}$$

- The graded symplectic form

$$\omega_{\partial\partial M} = \int_{\partial\partial M} \langle \delta \lambda_2, \delta B^+ \rangle_g + \langle \delta \lambda_1, \delta A \rangle_g + \langle \delta B, \delta c \rangle_g , \quad gh(\omega_{\partial\partial M}) = 1;$$

- The cohomological vector field

$$\mathcal{Q}_{\partial\partial M} = \int_{\partial\partial M} \left\langle \mathbb{B}, d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \right\rangle_g ;$$

- The action

$$\mathcal{S}_{\partial\partial M} = \int_{\partial\partial M} \langle \lambda_2, F_A \rangle_g + \langle \lambda_1, d_A c \rangle_g , \quad gh(\mathcal{S}_{\partial\partial M}) = 2.$$

Consider now the case of BF in two dimensions. The boundary has dimension one and corners are points. An analogous discussion (starting from example (15)) in the case $d = 2$ leads to the following data on intervals and points:

•• On the boundary ∂M:

- The space of boundary fields is

$$\mathcal{F}_{\partial M} = \Omega^1(\underset{A}{\partial M}, \mathfrak{g}) \oplus \Omega^0(\underset{B}{\partial M}, \mathfrak{g}) \oplus \Omega^0(\underset{c}{\partial M}, \mathfrak{g}[1]) \oplus \Omega^2(\underset{A^+}{\partial M}, \mathfrak{g}[-1])$$

  and the superfields encoding the fields living on the boundary are:

$$\mathbb{A} = c + A;$$
$$\mathbb{B} = B + A^+.$$

- The symplectic two-form $\omega_{\partial M}$ on $\mathcal{F}_{\partial M}$ is

$$\omega_{\partial M} = \int_{\partial M} \langle \delta B, \delta A \rangle_g + \langle \delta A^+, \delta c \rangle_g , \quad gh(\omega_{\partial M}) = 0.$$

- The AKSZ action is

$$S_{\partial M} = \int_{\partial M} \langle B, d_A c \rangle_g + \left\langle A^+, \frac{1}{2}[c,c] \right\rangle_g .$$

- The cohomological vector field on $\mathcal{F}_{\partial M}$:

$$\mathcal{Q}_{\partial M} = \int_{\partial M} \left\langle d_A c, \frac{\delta}{\delta A} \right\rangle_g + \left\langle \frac{1}{2}[c,c], \frac{\delta}{\delta c} \right\rangle_g + \left\langle [c,B], \frac{\delta}{\delta B} \right\rangle_g + \\ + \left\langle d_A B + [c, A^+], \frac{\delta}{\delta A^+} \right\rangle_g ,$$

•• The space of corner fields is the target $\mathcal{F}_{\partial\partial M} = \mathcal{N} = \mathfrak{g}[1] \oplus \mathfrak{g}^*$. The hamiltonian structure is $(\omega_{\mathcal{N}},\ Q_{\mathcal{N}}, \Theta_{\mathcal{N}})$, where $\Theta_{\mathcal{N}}$ is the action on the target.

Chapter 4

Boundary-corners algebraic relations

From the previous chapter, we know that, given a manifold M with boundary Σ and corners $\partial\Sigma$ and under some regularity assumptions, a BV theory on the bulk induces a BFV theory on the boundary, which in turns induces a $\mathrm{BF}^2\mathrm{V}$ theory on the corners. We will focus on classical AKSZ theories, where this procedure is easier to describe. The goal of this chapter is to explore in more details the relations linking the field theory on the boundary and that one on the corners.

On $\partial\Sigma$ we have a $\mathrm{BF}^2\mathrm{V}$ theory and a symplectic form of degree 1, which determines an odd Poisson algebra. Nevertheless, combining it with the cohomological vector field $\mathcal{Q}_{\partial\Sigma}$, encoding the gauge symmetries on the space of corner fields, we can define an algebraic structure of degree zero, the *derived algebra of corner observables*. This structure satisfies the Jacoby identity and the Leibniz properties w.r.t. the associative product of functionals, but it is not skew-symmetric. In order to make it skew-symmetric, we need to choose a polarization (see Appendix (A)) on the space of corner fields, *i.e* a maximal set of commuting functionals (called "polarized functionals"). However, the introduction of a polarization complicates the picture and the set of polarized functionals acquires the structure of a L_∞ algebra.

On the other hand, on the space of boundary fields we have a degree zero symplectic form, but, due to the presence of the corners, the cohomological vector field $\mathcal{Q}_\Sigma$ is not symplectic nor hamiltonian. The polarization on the space of corner fields is used to impose corner conditions on boundary fields, in such a way to make $\mathcal{Q}_\Sigma$ symplectic and hamiltonian. If $\mathcal{Q}_\Sigma$ does preserve these corner conditions, we can define a differential graded Lie algebra (see definition (4)) on the space of boundary fields.

Once we have built these structures, we then look for a L_∞ morphism (a homotopical extension of a Lie algebra morphism) linking them. Such morphisms are given by a collection of maps related by certain coherence relations (see Appendix (C)). We are led to the conjecture that such a morphism does exist and we construct explicitly its first component in general. We solve the full problem of finding the other components in two different examples: the two-dimensional BF theory and the three-dimensional Chern-Simons theory.

We study the two-dimensional BF theory first, both in the horizontal and the vertical polarization (for the definition of such polarizations, see Appendix (A)). In the horizontal polarization we find that the L_∞ algebra on $\partial\Sigma$ is actually a Poisson algebra and that on Σ there is a differential graded Lie algebra. The L_∞ morphism is simply a morphism of Poisson algebras. On the other hand, in the vertical polarization we find that to $\partial\Sigma$ we can associate a chain complex and to Σ a differential graded Lie algebra. In particular, it emerges that this chain complex is just the Chevalley-Eilenberg complex related to the Poisson algebra we found in the horizontal polarization. In this case, the L_∞ morphism between these structures is curved, namely it has also a zero component (a representative element on the space of boundary fields).

As for three-dimensional Chern Simons, we discuss first the horizontal polarization, where the algebra of polarized corner observables is the Kac-Moody algebra and to Σ we can associate a differential graded Lie algebra. The L_∞ morphism has only two components. In the case of vertical polarization, some problems concerning the definition of the right algebraic structures on $\partial\Sigma$ and on Σ emerge. A more careful and precise discussion will be required in order to deal with this polarization.

4.1 The derived algebra of corner observables

Consider an AKSZ theory, living on a d-dimensional oriented manifold M, with $(d-1)$-dimensional boundary Σ and $(d-2)$-dimensional corners $\partial\Sigma$.

We know that:

- On Σ lives a relaxed BFV field theory, $(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{Q}_\Sigma, \mathcal{S}_\Sigma)$.
- On $\partial\Sigma$ lives a BF^{2}V field theory, $(\mathcal{F}_{\partial\Sigma}, \omega_{\partial\Sigma}, \mathcal{Q}_{\partial\Sigma}, \mathcal{S}_{\partial\Sigma})$.

Recall that "relaxed" means that $\mathcal{Q}_\Sigma$ is neither ω_Σ-hamiltonian nor symplectic, since $L_{\mathcal{Q}_\Sigma}\omega_\Sigma = \check{\omega}_\Sigma$ is a presymplectic two form on the space of boundary fields.

Definition 11. *For each $f_\mathcal{N} \in C^\infty(\mathcal{N})$ and $\gamma \in \Omega^p(\Sigma)$, $\phi \in \Omega^p(\partial\Sigma)$, we define the **AKSZ functionals** on the space of boundary or corner fields:*

$$\begin{aligned} f_{\gamma,\Sigma} &= \int_\Sigma \gamma\, f_\mathcal{N}(\mathbb{X}_\Sigma) \in C^\infty(\mathcal{F}_\Sigma), \\ f_{\phi,\partial\Sigma} &= \int_{\partial\Sigma} \phi\, f_\mathcal{N}(\mathbb{X}_{\partial\Sigma}) \in C^\infty(\mathcal{F}_{\partial\Sigma}). \end{aligned} \tag{4.1}$$

Remark 25. In the expressions (4.1), the function $f_\mathcal{N}$ composed with the superfields is expanded in differential forms of Σ and the only component that gives a contribution is the form of degree top$-p$.

From now on, we will use the notation $\mathbb{X}$ for both $\mathbb{X}_\Sigma$ and $\mathbb{X}_{\partial\Sigma}$. It will be clear from the context if we are referring to boundary or corners superfields.

Lemma 6. Let $X_{f_\mathcal{N}}$ be the hamiltonian vector field of $f_\mathcal{N}$. AKSZ functionals are hamiltonian and their hamiltonian vector field is

$$\begin{aligned} X_{f_{\gamma,\Sigma}} &= (-1)^{|\Sigma||f_{\gamma,\Sigma}|+|\gamma|} \int_\Sigma \gamma\, X^a_{f_\mathcal{N}}(\mathbb{X}) \frac{\delta}{\delta \mathbb{X}^a} \\ X_{f_{\phi,\partial\Sigma}} &= (-1)^{|\partial\Sigma||f_{\tilde{\phi},\Sigma}|+|\phi|} \int_{\partial\Sigma} \phi\, X^a_{f_\mathcal{N}}(\mathbb{X}) \frac{\delta}{\delta \mathbb{X}^a}, \end{aligned} \tag{4.2}$$

where $X^a_{f_\mathcal{N}}(\mathbb{X})$ are the components of $X_{f_\mathcal{N}}$ evaluated on the superfields.

Proof. This is a straightforward calculation:

$$\begin{aligned}
i_{X_{f_{\gamma,\Sigma}}}\omega_\Sigma &= i_{X_{f_{\gamma,\Sigma}}} \int_\Sigma \frac{1}{2}\sum_{a,b} \omega_{ab}(\mathbb{X})\delta\mathbb{X}^a \wedge \delta\mathbb{X}^b = \\
&= (-1)^{|\Sigma|[|f_{\gamma,\Sigma}|-1]} \sum_{a,b} \int_\Sigma i_{X_{f_{\gamma,\Sigma}}} \left(\frac{1}{2}\omega_{ab}(\mathbb{X})\delta\mathbb{X}^a \wedge \delta\mathbb{X}^b\right) = \\
&= (-1)^{|\Sigma|[|f_{\gamma,\Sigma}|-1]} (-1)^{|\Sigma||f_{\gamma,\Sigma}|+|\gamma|} \int_\Sigma \gamma\, \delta f_{\mathcal{N}}(\mathbb{X}) = \\
&= \delta \int_\Sigma \gamma\, f_{\mathcal{N}}(\mathbb{X}) = \\
&= \delta f_{\gamma,\Sigma}
\end{aligned} \tag{4.3}$$

and analogously for $f_{\phi,\partial\Sigma}$. Since ω_Σ and $\omega_{\partial\Sigma}$ are weakly non degenerate, these hamiltonian vector fields are unique. □

We call $\mathbf{H}_\Sigma \subset C^\infty(\mathcal{F}_\Sigma)$ and $\mathbf{H}_{\partial\Sigma} \subset C^\infty(\mathcal{F}_{\partial\Sigma})$ the sets of AKSZ functionals on the space of boundary and corner fields respectively.

Remark 26. Notice that, since $\mathcal{Q}_\Sigma$ is not symplectic, it is not a derivation of the even bracket $\{,\}$ induced by ω_Σ. Therefore, the set $(\mathbf{H}_\Sigma, \{,\})$ is not a differential graded Lie algebra (recall definition (4)) with $\mathcal{Q}_\Sigma$.

First of all, let's study the algebraic structures that it is possible to define on $\mathbf{H}_{\partial\Sigma}$. The symplectic form $\omega_{\partial\Sigma}$ induces an odd bracket $(,)$ in $\mathbf{H}_{\partial\Sigma}$:

$$(f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma}) = \omega_{\partial\Sigma}(X_{f_{\phi,\partial\Sigma}}, X_{g_{\psi,\partial\Sigma}}), \quad |(,)| = -1.$$

Lemma 7. The set $(\mathbf{H}_{\partial\Sigma}, \mathcal{Q}_{\partial\Sigma}, (,))$ is a differential graded Lie algebra, where $(,)$ is the Poisson bracket induced by $\omega_{\partial\Sigma}$.

Proof. We have to prove that $\mathcal{Q}_{\partial\Sigma}$ preserves $\mathbf{H}_{\partial\Sigma}$, namely that, if $f_{\phi,\partial\Sigma}$ is an AKSZ functional, $\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma})$ is still an AKSZ functional. Recall that $\mathcal{Q}_{\partial\Sigma} = \hat{d}_{\partial\Sigma} + \hat{D}_{\partial\Sigma}$ is the composition of the De Rham differential on $\partial\Sigma$ and the cohomological vector field in the target with the corners superfields (see expression (2.54)).

Let $f_{\phi,\partial\Sigma} \in \mathbf{H}_{\partial\Sigma}$. We have:

$$\begin{aligned}
\mathcal{Q}_{\partial\Sigma} f_{\phi,\partial\Sigma} &= \mathcal{Q}_{\partial\Sigma} \int_{\partial\Sigma} \phi\, f_{\mathcal{N}}(\mathbb{X}) = \\
&= \left(\hat{d}_{\partial\Sigma} + \hat{D}_{\partial\Sigma}\right) \int_{\partial\Sigma} \phi\, f_{\mathcal{N}}(\mathbb{X}) = \\
&= (-1)^{|\partial\Sigma|+|\phi|} \int_{\partial\Sigma} \phi\, \hat{d}_{\partial\Sigma} f_{\mathcal{N}}(\mathbb{X}) + (-1)^{|\partial\Sigma|+|\phi|} \int_{\partial\Sigma} \phi\, \hat{D}_{\partial\Sigma} f_{\mathcal{N}}(\mathbb{X}) = \\
&= (-1)^{|\partial\Sigma|} \int_{\partial\Sigma} \phi\, d f_{\mathcal{N}}(\mathbb{X}) = \\
&= (-1)^{|\Sigma|} \int_{\partial\Sigma} d\phi\, f_{\mathcal{N}}(\mathbb{X}) + (-1)^{|\partial\Sigma|+|\phi|} \int_{\partial\Sigma} \phi\, (D f_{\mathcal{N}})(\mathbb{X}) = \\
&= (-1)^{|\Sigma|} f_{d\phi,\partial\Sigma} + (-1)^{|\partial\Sigma|+|\phi|} (Df)_{\phi,\partial\Sigma},
\end{aligned}$$

which is the sum of two AKSZ functionals.

As for the bracket $(,)$, we have only to show that it preserves $\mathbf{H}_{\partial\Sigma}$. For any $f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma} \in \mathcal{H}_{\partial\Sigma}$, we have

$$\begin{aligned}(f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma}) = X_{f_{\phi,\partial\Sigma}}(g_{\psi,\partial\Sigma}) = \\ &= (-1)^{|\partial\Sigma|+|\phi|+|\psi||f_{\phi,\partial\Sigma}|+|\psi|}\int_{\partial\Sigma}\psi\,\phi\,\{f_{\mathcal{N}}, g_{\mathcal{N}}\}(\mathbb{X}) = \\ &= (-1)^{|\partial\Sigma|+|\phi|+|\psi||f_{\phi,\partial\Sigma}|+|\psi|}\{f,g\}_{\psi\phi,\partial\Sigma},\end{aligned} \tag{4.4}$$

which is an AKSZ functional. Here, $\{f_{\mathcal{N}}, g_{\mathcal{N}}\}$ is the bracket between $f_{\mathcal{N}}$ and $g_{\mathcal{N}}$ in the target and $\psi\phi \equiv \psi \wedge \phi$.

Finally, since $\mathcal{Q}_{\partial\Sigma}$ is symplectic (and hamiltonian), it is a derivation of $(,)$. In fact:

$$\begin{aligned}\mathcal{Q}_{\partial\Sigma}\,(f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma}) =& \cancel{(L_{\mathcal{Q}_{\partial\Sigma}}\omega)\,(X_{f_{\phi,\partial\Sigma}}, X_{g_{\psi,\partial\Sigma}})} + (\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), g_{\psi,\partial\Sigma}) + \\ &+ (-1)^{|f_{\phi,\partial\Sigma}|-1}(f_{\phi,\partial\Sigma}, \mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma})).\end{aligned} \tag{4.5}$$

Therefore, the lemma is proved.

□

Besides $(,)$, there is an other algebraic operation we can define in $\mathbf{H}_{\partial\Sigma}$. Starting from $\mathcal{Q}_{\Sigma}$ and $(,)$, it is possible to construct an even bracket in $\mathbf{H}_{\check{\omega}_{\partial\Sigma}}$, called the **derived bracket** (see [26] or [29] for further details):

$$f_{\phi,\partial\Sigma} \circ g_{\psi,\partial\Sigma} \equiv (-1)^{|f_{\phi,\partial\Sigma}|}(\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}),\, g_{\psi,\partial\Sigma}), \quad |\circ| = 0. \tag{4.6}$$

Remark 27. $\mathbf{H}_{\partial\Sigma}$ is closed under the derived bracket, since it is closed under $\mathcal{Q}_{\partial\Sigma}$ and $(,)$.

Let's study the properties of the derived bracket.

- **$\circ$ is not skew-symmetric**
  This follows from calculation (4.5):

$$\begin{aligned}f_{\phi,\partial\Sigma} \circ g_{\psi,\partial\Sigma} =& (-1)^{|f_{\phi,\partial\Sigma}||g_{\psi,\partial\Sigma}|+1} g_{\psi,\partial\Sigma} \circ f_{\phi,\partial\Sigma} + \\ &+ (-1)^{|f_{\phi,\partial\Sigma}|} Q(f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma}).\end{aligned} \tag{4.7}$$

  The failure of the skew-symmetry of $\circ$ is measured by the $\mathcal{Q}_{\partial\Sigma}$-exact term $\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}, g_{\psi,\partial\Sigma})$.

- **Left Jacobi identity**
  Let's write $(\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), (\mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma}), h_{\alpha,\partial\Sigma}))$ using Jacobi identity for $(,)$:

$$\begin{aligned}(\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), (\mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma}), h_{\alpha,\partial\Sigma})) &= ((\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), \mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma})), h_{\alpha,\partial\Sigma}) + \\ &\quad + (-1)^{|f_{\phi,\partial\Sigma}||g_{\psi,\partial\Sigma}|}(\mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma}), (\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), h_{\alpha,\partial\Sigma})) = \\ &= (-1)^{|f_{\phi,\partial\Sigma}|}(\,\mathcal{Q}_{\partial\Sigma}\,(\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), g_{\psi,\partial\Sigma}), h_{\alpha,\partial\Sigma}) + \\ &\quad + (-1)^{|f_{\phi,\partial\Sigma}||g_{\psi,\partial\Sigma}|}(\mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma}), (\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}), h_{\alpha,\partial\Sigma})).\end{aligned}$$

  We multiply both sides by $(-1)^{|f_{\phi,\partial\Sigma}|}(-1)^{|g_{\psi,\partial\Sigma}|}$ and we obtain

$$\begin{aligned}f_{\phi,\partial\Sigma} \circ (g_{\psi,\partial\Sigma} \circ h_{\alpha,\partial\Sigma}) =& (f_{\phi,\partial\Sigma} \circ g_{\psi,\partial\Sigma}) \circ h_{\alpha,\partial\Sigma} + \\ &+ (-1)^{|f_{\phi,\partial\Sigma}||g_{\psi,\partial\Sigma}|} g_{\psi,\partial\Sigma} \circ (f_{\phi,\partial\Sigma} \circ h_{\alpha,\partial\Sigma}).\end{aligned} \tag{4.8}$$

- **Q is a derivation of $\circ$.**
  This follows from the fact that $\mathcal{Q}_{\partial\Sigma}$ is a derivation for $(,)$:

$$\begin{aligned}\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma}\circ g_{\psi,\partial\Sigma}) =&\mathcal{Q}_{\partial\Sigma}(f_{\phi,\partial\Sigma})\circ g_{\psi,\partial\Sigma}+\\ &+(-1)^{|f_{\phi,\partial\Sigma}|}f_{\phi,\partial\Sigma}\circ\mathcal{Q}_{\partial\Sigma}(g_{\psi,\partial\Sigma})\end{aligned} \tag{4.9}$$

Since the derived bracket is not skew-symmetric, it is not a Lie bracket. On the other hand, $(\mathbf{H}_{\partial\Sigma},\circ)$ is a **graded left Leibniz algebra** (or a graded Loday algebra, see ([26])), namely a graded algebra whose binary operation satisfies Jacoby from the left. Since $\mathcal{Q}_{\partial\Sigma}$ is a derivation of $\circ$, the set $(\mathbf{H}_{\partial\Sigma},\mathcal{Q}_{\partial\Sigma},\circ)$ is a **differential graded left Leibniz algebra**.

Remark 28. Notice that the derived bracket satisfies the Leibniz rule for the associative product in $\mathbf{H}_{\partial\Sigma}$, which follows from the Leibniz rule for $(,)$:

$$\begin{aligned}f_{\phi,\partial\Sigma}\circ(g_{\psi,\partial\Sigma}h_{\alpha,\partial\Sigma}) =& (f_{\phi,\partial\Sigma}\circ g_{\psi,\partial\Sigma})h_{\alpha,\partial\Sigma}+\\ &+(-1)^{|f_{\phi,\partial\Sigma}||g_{\psi,\partial\Sigma}|}g_{\psi,\partial\Sigma}(f_{\phi,\partial\Sigma}\circ h_{\alpha,\partial\Sigma}).\end{aligned} \tag{4.10}$$

The set $(\mathbf{H}_{\partial\Sigma},\circ)$ with the associative product an example of what in [26] is called generalized Loday-Poisson algebra.

Remark 29. The reason why we are interested in the derived algebra is that the derived bracket has degree zero, like the bracket $\{,\}$ in $\mathbf{H}_{\Sigma}$.

4.2 From corner to boundary observables

Our next goal is to describe how we can send functionals in $\mathbf{H}_{\partial\Sigma}$ to functionals on $\mathbf{H}_{\Sigma}$.

Definition 12. *Let $\mathfrak{i}:\partial\Sigma\hookrightarrow\Sigma$ be the inclusion map of the corners into the boundary. An **extension map** of differential forms on $\partial\Sigma$ to Σ is a map $\mathfrak{e}:\Omega^p(\partial\Sigma)\to\Omega^p(\Sigma)$, such that*

$$\mathfrak{i}^*\circ\mathfrak{e}=Id.$$

Remark 30. There is not a unique way to extend forms on the corners to forms on the boundary. In fact, for $\phi\in\Omega^p(\partial\Sigma)$, given an extension $\mathfrak{e}$, we can always define an other extension $\mathfrak{e}_2$ such that:

$$\mathfrak{e}_2(\phi)=\mathfrak{e}(\phi)+\alpha,\quad \alpha\in\Omega^p(\partial\Sigma):\,\mathfrak{i}^*(\alpha)=0. \tag{4.11}$$

Remark 31. In general, the extension map does not preserve either the differential or the wedge product. For ϕ and ψ differential forms on $\partial\Sigma$:

$$\begin{aligned}&\mathfrak{e}\circ d\neq d\circ\mathfrak{e}\\ &\mathfrak{e}(\phi\psi)\neq\mathfrak{e}(\phi)\mathfrak{e}(\psi).\end{aligned} \tag{4.12}$$

Notation 4. In the following, we assume to have chosen an extension map $\mathfrak{e}$ and, if $\phi\in\Omega^p(\partial\Sigma)$ we use the notation $\mathfrak{e}(\phi)\equiv\tilde{\phi}$ to indicate the extended form on the boundary. Moreover, since $|\tilde{\phi}|=|\phi|$, we will use $|\phi|$ for both the degree of the extension and of the form on the corners.

Let now $p : \mathcal{F}_\Sigma \to \mathcal{F}_{\partial\Sigma}$ be the restriction map. The pullback p^* acts on $f_{\phi,\partial\Sigma} \in \mathbf{H}_{\partial\Sigma}$ as:

$$p^* f_{\phi,\partial\Sigma} = \int_\Sigma d\left(\tilde{\phi}\, f_{\mathcal{N}}(\mathbb{X})\right). \tag{4.13}$$

Notice that $p^* f_{\phi,\partial\Sigma}$ does not depend on the extension, but only on ϕ.

The following lemma holds:

Lemma 8. $p^* f_{\phi,\partial\Sigma}$ is $\check{\omega}_\Sigma$-hamiltonian and $\check{X}_{p^* f_{\phi,\partial\Sigma}} = (-1)^{|\partial\Sigma|+|f_{\tilde{\phi},\Sigma}|} X_{f_{\tilde{\phi},\Sigma}}$ is a $\check{\omega}_\Sigma$-hamiltonian vector field for $p^* f_{\phi,\partial\Sigma}$.

Proof. Recall that $\check{\omega}_\Sigma = L_{\mathcal{Q}_\Sigma}\omega_\Sigma$ is a presymplectic two-form on the space of boundary fields.

We have:

$$\begin{aligned} i_{X_{f_{\tilde{\phi},\Sigma}}}\check{\omega}_\Sigma &= (-1)^{|\partial\Sigma|} i_{X_{f_{\tilde{\phi},\Sigma}}} p^*\omega_{\partial\Sigma} = \\ &= (-1)^{|\partial\Sigma|} i_{X_{f_{\tilde{\phi},\Sigma}}} \int_\Sigma d\left(\frac{1}{2}\sum_{a,b}\omega_{ab}(\mathbb{X})\delta\mathbb{X}^a \wedge \delta\mathbb{X}^b\right) = \\ &= (-1)^{|\partial\Sigma|}(-1)^{|\Sigma||f_{\tilde{\phi},\Sigma}|+|\phi|}(-1)^{|\partial\Sigma||f_{\phi,\partial\Sigma}|}\int_\Sigma d\left(\tilde{\phi}\,\delta f_{\mathcal{N}}(\mathbb{X})\right) = \\ &= (-1)^{|\Sigma||f_{\tilde{\phi},\Sigma}|}(-1)^{|\partial\Sigma||f_{\phi,\partial\Sigma}|}\,\delta p^* f_{\phi,\partial\Sigma} = \\ &= (-1)^{|\partial\Sigma|+|f_{\tilde{\phi},\Sigma}|}\,\delta p^* f_{\phi,\partial\Sigma}. \end{aligned} \tag{4.14}$$

Therefore, $(-1)^{|\partial\Sigma|+|f_{\tilde{\phi},\Sigma}|} X_{f_{\tilde{\phi},\Sigma}} \equiv \check{X}_{p^* f_{\phi,\partial\Sigma}}$ is a $\check{\omega}_\Sigma$-hamiltonian vector field for $p^* f_{\phi,\partial\Sigma}$. □

Remark 32. Notice that $|f_{\phi,\partial\Sigma}| = |f_{\tilde{\phi},\Sigma}| + 1$.

The lemma above is fundamental. In fact, on the one hand it affirms that the function $p^* f_{\phi,\partial\Sigma}$ has a $\check{\omega}_\Sigma$-hamiltonian vector field. This property is not obvious, since in the presymplectic case (see Appendix (A)) such a vector field could not exist. On the other hand, this lemma tells that, amongst all the possible $\check{\omega}_\Sigma$-hamiltonian vector fields of $p^* f_{\phi,\partial\Sigma}$ (which can always be shifted by an element of $Ker(\check{\omega}_\Sigma)$), there exist a preferred one (once we have chosen the extension), given by $X_{f_{\tilde{\phi},\Sigma}}$ (modulo signs), which is the ω_Σ-hamiltonian vector field of $f_{\tilde{\phi},\Sigma}$.

Lemma (8) can be used to define a map $\mathbf{i} : \mathbf{H}_{\partial\Sigma} \longrightarrow \mathbf{H}_\Sigma$, such that

$$\mathbf{i}(f_{\phi,\partial\Sigma}) = (-1)^{|\partial\Sigma|+|f_{\tilde{\phi},\partial\Sigma}|} f_{\tilde{\phi},\Sigma}, \quad |\mathbf{i}| = -1 \tag{4.15}$$

In order to obtain a degree-0 map, we compose $\mathbf{i}$ with the differential $\mathcal{Q}_{\partial\Sigma}$. Define $\Phi = \mathbf{i} \circ \mathcal{Q}_{\partial\Sigma}$ such composition.

Lemma 9. Let $f_{\phi,\partial\Sigma} \in \mathbf{H}_{\partial\Sigma}$. The explicit action of the map Φ on $f_{\phi,\partial\Sigma}$ is

$$\boxed{\Phi(f_{\phi,\partial\Sigma}) = (-1)^{|f_{\tilde{\phi},\Sigma}|}\left\{f_{\widetilde{d\phi},\Sigma} + (-1)^{|\phi|+1}(Df)_{\tilde{\phi},\Sigma}\right\}}, \tag{4.16}$$

which can also be rewritten as:

$$\boxed{\Phi(f_{\phi,\partial\Sigma}) = (-1)^{|f_{\tilde{\phi},\Sigma}|}\left(p^* f_{\phi,\partial\Sigma} + (-1)^{|\partial\Sigma|}\,\mathcal{Q}_\Sigma(f_{\tilde{\phi},\Sigma})\right) + (-1)^{|f_{\tilde{\phi},\Sigma}|}\, f_{\widetilde{d\phi}-d\tilde{\phi},\,\Sigma}}. \tag{4.17}$$

Proof. Recall that

$$p^*\left(\mathcal{Q}_{\partial\Sigma} f_{\phi,\partial\Sigma}\right) = (-1)^{|\Sigma|} p^* f_{d\phi,\partial\Sigma} + (-1)^{|\partial\Sigma|+|\phi|} p^*(Df)_{\phi,\partial\Sigma}. \tag{4.18}$$

If we apply the map $\mathbf{i}$ to (4.18), we obtain:

$$\begin{aligned}\mathbf{i}\left(p^*\mathcal{Q}_{\partial\Sigma} f_{\phi,\partial\Sigma}\right) &= \mathbf{i}\left((-1)^{|\Sigma|} p^* f_{d\phi,\partial\Sigma} + (-1)^{|\partial\Sigma|+|\phi|} p^*(Df)_{\phi,\partial\Sigma}\right) = \\ &= (-1)^{|f_{\phi,\partial\Sigma}|+1} f_{\widetilde{d\phi},\Sigma} + (-1)^{|f_{\phi,\partial\Sigma}|+|\phi|}(Df)_{\tilde{\phi},\Sigma}.\end{aligned}$$

Therefore:

$$\Phi(f_{\phi,\partial\Sigma}) = (-1)^{|f_{\tilde{\phi},\Sigma}|}\left\{f_{\widetilde{d\phi},\Sigma} + (-1)^{|\phi|+1}(Df)_{\tilde{\phi},\Sigma}\right\}.$$

This expression can be rewritten as (4.17). In fact:

$$\begin{aligned}\mathcal{Q}_\Sigma(f_{\tilde{\phi},\Sigma}) &= \left(\hat{d}_\Sigma + \hat{D}_\Sigma\right)\int_\Sigma \tilde{\phi}\, f_{\mathcal{N}}(\mathbb{X}) = \\ &= (-1)^{|\Sigma|+|\phi|}\int_\Sigma \tilde{\phi}\,\hat{d}_\Sigma f_{\mathcal{N}}(\mathbb{X}) + (-1)^{|\Sigma|+|\phi|}\int_\Sigma \tilde{\phi}\,(Df)_{\mathcal{N}}(\mathbb{X}) = \\ &= (-1)^{|\partial\Sigma|} f_{d\tilde{\phi},\Sigma} + (-1)^{|\Sigma|+|\phi|}(Df)_{\tilde{\phi},\partial\Sigma} + (-1)^{|\Sigma|} p^* f_{\phi,\partial\Sigma},\end{aligned}$$

We can obtain from the above expression $(Df)_{\tilde{\phi},\Sigma}$ and substitute into Eq. (4.16). Thus:

$$\Phi(f_{\phi,\partial\Sigma}) = (-1)^{|f_{\tilde{\phi},\Sigma}|}\left(p^* f_{\phi,\partial\Sigma} + (-1)^{|\partial\Sigma|}\mathcal{Q}_\Sigma(f_{\tilde{\phi},\Sigma})\right) + (-1)^{|f_{\tilde{\phi},\Sigma}|} f_{\widetilde{d\phi}-d\tilde{\phi},\Sigma}.$$

□

4.3 A polarization on the space of corner fields

The construction we made in the previous subsection allows to define:

- A graded Lie algebra $(\mathbf{H}_\Sigma, \{,\})$ on the space of boundary fields;
- A differential graded left Leibniz algebra $(\mathbf{H}_{\partial\Sigma}, \mathcal{Q}_{\partial\Sigma}, \circ)$ on the space of corner fields.

Moreover, we built a map $\Phi : \mathbf{H}_{\partial\Sigma} \to \mathbf{H}_\Sigma$, whose algebraic properties have still to be investigated.

Our goal is to understand these structures as (homotopical version of) Lie algebras and use Φ as a morphism between them. In this moment we do not have these ingredients, since the algebraic structures on the boundary and on the corners we introduced above are not the same and they can not linked by a morphism. Therefore, this general scheme is not quite satisfactory.

Moreover, as we have noticed in remark (26), $\mathcal{Q}_\Sigma$ is not a derivation of $\{,\}$, since it is not symplectic:

$$L_{\mathcal{Q}_\Sigma}\omega_\Sigma = \check{\omega}_\Sigma.$$

Therefore, on the space of boundary fields we are not able to include $\mathcal{Q}_\Sigma$ in our discussion. Since $\mathcal{Q}_\Sigma$ plays a role in the definition of what is a classical observable (a $\mathcal{Q}_\Sigma$-exact functional) on the boundary (and therefore what kind of functionals can be quantized), we would like to include it in our construction.

In order to promote it to a differential for $(\mathbf{H}_\Sigma, \{,\})$, we could think to restrict the space of boundary fields $\mathcal{F}_\Sigma$ to a submanifold where $\tilde{\omega} = 0$. We want to show that this restriction can be induced by a **polarization** on the space of corner fields.

Let F be a polarization (see Appendix (A)) on $\mathcal{F}_{\partial\Sigma}$ and $P_F : \mathcal{F}_{\partial\Sigma} \to \mathcal{L}_F$ be the projection to the leaf space. Assume that, besides the natural projection P_F, an immersion $I_F : \mathcal{L}_F \to \mathcal{F}_{\partial\Sigma}$ of the leaf space into the space of corner fields is defined, such that we have $P_F \circ I_F = Id_{\mathcal{L}_F}$.

Starting from P and I, we can define a projection map π_F from functionals on $\mathcal{F}_{\partial\Sigma}$ to polarized functionals on $\mathcal{L}_F$ as

$$\pi_F = P_F^* \circ I_F^* : C^\infty(\mathcal{F}_{\partial\Sigma}) \to P^*(C^\infty(\mathcal{L}_F)) \subset C^\infty(\mathcal{F}_{\partial\Sigma}).$$

Remark 33. We assume that, if we apply π_F to an AKSZ functional, we obtain a polarized AKSZ functional. That is, $\pi_F(\mathbf{H}_{\partial\Sigma}) \subset \mathbf{H}_{\partial\Sigma}$. This is true in the examples we have studied. We call $\mathbf{H}^F_{\partial\Sigma} \equiv \pi_F(\mathbf{H}_{\partial\Sigma})$.

A natural question is what kind of algebraic structures the projection π does induce on $\mathbf{H}^F_{\partial\Sigma}$. The answer is provided by the following general theorem, due to Voronov (see [33]):

Theorem 3. *Let $(L, Q, (,))$ be a differential graded Lie algebra and $\pi : L \to V$ a projection into an abelian subalgebra V of L, such that it satisfies the distributive law (Voronov's law)*

$$\pi(a, b) = \pi(\pi(a), b) + \pi(a, \pi(b)), \quad a, b \in L \tag{4.19}$$

Then we can endow V with the structure of a L_∞-algebra, whose operations are

$$\begin{aligned}
&\{f\}_1 = (\pi Q)(f),\\
&\{f_1, f_2\}_2 = \pi(\, f_1 \circ f_2 \,),\\
&\{f_1, f_2, f_3\}_3 = \pi(\, (f_1 \circ f_2, f_3) \,),\\
&....\\
&\{f_1, f_2, ..., f_n\}_n = \pi(....(f_1 \circ f_2, f_3),\ f_4), ...,\ f_{n-1}),\ f_n)\\
&....,
\end{aligned} \tag{4.20}$$

with $f_i \in V$ and $\circ$ is the derived bracket.

Proof. See ([33]). □

Remark 34. We can apply this theorem to $L = \mathbf{H}_{\partial\Sigma}$, $V = \mathbf{H}^F_{\partial\Sigma}$ and $\pi = \pi_F$. Indeed, $\mathbf{H}^F_{\partial\Sigma}$ is an abelian subalgebra of $\mathbf{H}_{\partial\Sigma}$, since F is a polarization. In the case $\mathcal{F}_{\partial\Sigma}$ has the structure of a shifted cotangent bundle, we can check directly that π_F satisfies property (4.19) (see Appendix (D) for the check in the finite-dimensional case).

Therefore, $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet)$ is a L_∞-algebra. For a brief introduction on the subject, see Appendix (C).

Remark 35. The introduction of a L_∞-algebra on the space of corner fields, due to the choice of a polarization, is only a technical aspect we are obliged to take into account. This rather involved algebraic structure emerges since we are not working in cohomology (that is, modulo $\mathcal{Q}_{\partial\Sigma}$-exact functionals): as a consequence, the Lie algebras we would like to work with are actually defined up to homotopy and become L_∞-algebras. However, the introduction of L_∞-algebras does not change the physical substance, since classical observables are $\mathcal{Q}_{\partial\Sigma}$ closed and these L_∞-algebras restrict to Lie algebras in cohomology.

Remark 36. Notice that, if $\mathcal{Q}_{\partial\Sigma}$ preserves $\mathbf{H}^F_{\partial\Sigma}$, the first bracket $\{\}_1$ is exactly $\mathcal{Q}_{\partial\Sigma}$ (since π acts as the identity). In this case, the derived bracket and higher brackets are zero and $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet) = (\mathbf{H}^F_{\partial\Sigma},\, \mathcal{Q}_{\partial\Sigma})$ is a **chain complex**.

If $\mathcal{Q}_{\partial\Sigma}$ does not preserve the polarization, but the derived bracket does, brackets $\{\}_n$ with $n > 2$ are zero and $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet) = (\mathbf{H}^F_{\partial\Sigma},\, \pi\mathcal{Q}_{\partial\Sigma},\, \circ)$ is a differential graded Lie algebra. Moreover, if $\pi\mathcal{Q}_{\partial\Sigma} = 0$, the differential is trivial and $(\mathbf{H}^F_{\partial\Sigma}, \circ)$ is a **graded Lie algebra**.

The corner conditions

The choice of a polarization F on the space of corner fields and the definition of Voronov's projection π_F can be used to impose some corner conditions on the boundary fields. These are given by considering only those boundary fields whose restriction to the corners lies in $Im(I_F)$. Let $\mathcal{F}^\pi_\Sigma \subset \mathcal{F}_\Sigma$ be this submanifold and $\mathcal{I} : \mathcal{F}^\pi_\Sigma \hookrightarrow \mathcal{F}_\Sigma$ its immersion as a submanifold of the space of boundary fields.

We can prove the following lemma.

Lemma 10. If the immersion $I_F : \mathcal{L}_F \to \mathcal{F}_{\partial\Sigma}$ is lagrangian, that is $I_F^*\omega_{\partial\Sigma} = 0$, then

$$\check{\omega}_\Sigma|_{\mathcal{F}^\pi_\Sigma} = 0. \tag{4.21}$$

Proof. The presymplectic form, restricted to $\mathcal{F}^\pi_\Sigma$, is:

$$\begin{aligned}\check{\omega}_\Sigma|_{\mathcal{F}^\pi_\Sigma} &= \mathcal{I}^*\,\check{\omega}_\Sigma = \\ &= (\mathcal{I}^* \circ p^*)\omega_{\partial\Sigma}.\end{aligned} \tag{4.22}$$

The map $p \circ \mathcal{I}$ is just the restriction map of boundary fields in $\mathcal{F}^\pi_\Sigma$ to their values on the corners, which by definition lies in $Im(I_F)$. However, notice that we can insert an identity in the projection $p \circ \mathcal{I}$ in the following way:

$$(Id)|_{Im(I_F)} \circ (p \circ \mathcal{I}) = (I_F \circ P_F)|_{Im(I_F)} \circ (p \circ \mathcal{I})\,,$$

whose pullback is

$$(p \circ \mathcal{I})^* \circ Id_{\Omega^\bullet(Im(I_F))} = (p \circ \mathcal{I})^* \;\circ\; (I_F \circ P_F)\,|^*_{Im(I_F)}.$$

Inserting this identity into Eq. (4.22), we obtain

$$(p \circ \mathcal{I})^* \;\circ\; (I_F \circ P_F)\,|^*_{Im(I_F)}\omega_{\partial\Sigma},$$

which is zero, since $I_F^*\omega_{\partial\Sigma} = 0$. □

The fundamental point of this lemma is that for suitable choices of the immersion I, which must be lagrangian, we can impose corner conditions on the space of boundary fields (and therefore restrict $\mathcal{F}_\Sigma$ to $\mathcal{F}^\pi_\Sigma$ as described above) in such a way to cancel $\check{\omega}_\Sigma$.

These considerations lead to the following lemma:

Lemma 11. Assume that $\mathcal{Q}_\Sigma$ preserves the corner conditions on boundary fields. In this case, the set of data $(\mathcal{F}^\pi_{\partial\Sigma}, \omega_\Sigma, \mathcal{Q}_\Sigma, \mathcal{S}_\Sigma)$ is an exact BFV theory.

Corollary 2. *Let $\mathbf{H}^{\pi_F}_\Sigma$ be the AKSZ functionals in $\mathbf{H}_\Sigma$ restricted to $\mathcal{F}^\pi_\Sigma$. Then, if $\mathcal{Q}_\Sigma$ preserves the corner conditions, $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$ is a differential graded Lie algebra.*

Remark 37. We stress that the fact that $\mathcal{Q}_\Sigma$ preserves the corner conditions on boundary fields does depend on the lagrangian immersion I of the leaf space.

The "new" map Φ

Let's resume what we have built in the previous subsections:

- An L_∞ algebra $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet)$ on the space of corner fields, after the choice of a polarization F on $\mathcal{F}_{\partial\Sigma}$;
- A differential graded Lie algebra $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$ on the space of boundary fields, assuming that $\mathcal{Q}_\Sigma$ does preserve the corner conditions induced by the polarization on $\mathcal{F}_{\partial\Sigma}$.

This two sets can be linked by a slightly modifed version of the map Φ we defined in section (4.2). Define the map Φ_{π_F} as:

$$\mathbf{H}^F_{\partial\Sigma} \underset{\mathcal{Q}_{\partial\Sigma}}{\longrightarrow} \mathbf{H}_{\partial\Sigma} \underset{\mathbf{i}}{\longrightarrow} \mathbf{H}_\Sigma \underset{\mathcal{I}^*}{\longrightarrow} \mathbf{H}^{\pi_F}_\Sigma. \tag{4.23}$$

Notice that the pullback $\mathcal{I}^*$ acts on a hamiltonian functional $f_{\tilde{\phi},\Sigma} \in \mathbf{H}_\Sigma$ in such a way that:

$$\left(\mathcal{I}^*(f_{\tilde{\phi},\Sigma})\right)|_{corn.} = \pi(f_{\phi,\partial\Sigma}).$$

This map Φ_{π_F} has degree zero and has the same explicit form of the "old" map Φ. It links polarized functionals on the space of corner fields with the (restricted) hamiltonian functionals on the space of boundary fields.

Conjecture 1. *The map Φ_{π_F} is the first component of a L_∞-morphism $\Phi_\bullet$ between $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet)$ and $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$.*

Studying this problem in its generality is rather involved. We specialize to the following cases:

1- $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet) = (\mathbf{H}^F_{\partial\Sigma}, \mathcal{Q}_\Sigma)$ is a chain complex.

2- $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet) = (\mathbf{H}^F_{\partial\Sigma}, 0, \circ)$ is a graded Lie algebra.

For the definition of L_∞ morphism in this two cases, see Appendix (C).

4.4 BF in two dimensions

We have discussed the theories living on the one-dimensional boundary and on the zero-dimensional corners at the end of example (19). We know that the space of corner fields is the target $\mathcal{N} = \mathfrak{g}[1] \oplus \mathfrak{g}^* = T^*[-1]\mathfrak{g}[1]$. Recall that we made the (unnecessary) hypothesis that a paring $\langle\rangle_g$ was defined in $\mathfrak{g}$, induced by a symmetric *ad*-invariant bilinear form g on it. Even if this pairing provides an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$, we keep distinguishing the notations $\mathfrak{g}$ and $\mathfrak{g}^*$.

The space of boundary AKSZ fields is $\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma) \otimes \mathcal{N}$, which is parametrized by the superfields:

$$\begin{aligned} \mathbb{A} &= c + A, \quad |\mathbb{A}| = 1; \\ \mathbb{B} &= B + A^+, \quad |\mathbb{B}| = 0. \end{aligned} \tag{4.24}$$

Recall that on the boundary A^+ is not the antifield associated to A. We are using the same notation for bulk and boundary fields.

We study the cases of the horizontal and vertical polarizations (see Appendix (A)).

Remark 38. In the following we assume that the corners are given by a single point and therefore that the boundary is "half" of an infinite segment. This assumption allows to define an extension of constant functions S on the point to the same constant on the boundary Σ, namely

$$\mathfrak{e}(S) = \tilde{S} = S.$$

With this prescription:

$$\begin{aligned} \mathfrak{e}(SS') &= SS' = \mathfrak{e}(S)\mathfrak{e}(S') \\ \mathfrak{e}(dS) &= dS = d\mathfrak{e}(S) = 0. \end{aligned}$$

For this reason, we will use the same notation for constants on the point and their extension to the boundary.

Horizontal Polarization

This polarization is characterized by the following projection and immersion (as before, we take the zero section):

$$\begin{aligned} P_F &: \underset{(\xi,\psi)}{\mathcal{N}} \to \underset{\xi}{\mathfrak{g}^*}, \\ I_F &: \underset{\xi}{\mathfrak{g}^*} \to \underset{(\xi,0)}{\mathcal{N}}. \end{aligned} \tag{4.25}$$

The set of polarized functionals is $\mathbf{H}^F_{\partial\Sigma} = C^\infty(\mathfrak{g}^*) = Sym^\bullet(\mathfrak{g})$, the symmetric algebra of polynomials on $\mathfrak{g}$:

$$S_\xi = \sum_n \sum_{a_1,\dots,a_n} S_{a_1,\dots,a_n} \xi^{a_1}\xi^{a_2}\dots\xi^{a_n}, \tag{4.26}$$

where $S_{a_1,..,a_n}$ are real numbers.

Voronov's projection is $\pi_F = P_F^* \circ I_F^* : C^\infty(\mathcal{N}) \to \mathbf{H}^F_{\partial\Sigma}$.

The L_∞ structure induced by π on $\mathbf{H}^F_{\partial\Sigma}$ is described in the following lemma.

Lemma 12. $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet)$ is a Poisson algebra $(\mathbf{H}^F_{\partial\Sigma}, 0, \circ)$, where the derived bracket is the Kirillov-Kostant bracket.

Proof. Since polynomial functions are generated by monomials, we consider only the generators of $\mathbf{H}^F_{\partial\Sigma}$, that is elements of the form

$$\sum_a S_a \xi^a \equiv \langle S, \xi \rangle_g.$$

The cohomological vector field $Q_\mathcal{N}$ does not preserve the polarization. Indeed

$$Q_\mathcal{N}(\langle S, \xi\rangle) = -\langle S, [\psi, \xi]\rangle \Rightarrow \pi_F \circ Q_\mathcal{N} = 0.$$

On the other hand, the derived bracket is (we omit the sum symbol)

$$\begin{aligned} \Big(Q_\mathcal{N} \langle S, \xi\rangle_g, \langle U, \xi\rangle_g \Big) &= -S_a U_d f^a_{bc}(\psi^b \xi^c, \xi^d) = \\ &= -S_a U_d f^{ad}_c \xi^c = \\ &= -\langle [S, U], \xi\rangle_g. \end{aligned} \tag{4.27}$$

Therefore,

$$\langle S,\xi\rangle_g \circ \langle U,\xi\rangle_g = -\langle [S,U],\xi\rangle_g .$$

which is the Kirillov-Kostant bracket in $\mathfrak{g}^*$. Notice that the polarization is preserved and therefore higher brackets are zero.

Since the derived bracket satisfies the Leibniz rule w.r.t. the associative product in $\mathbf{H}^F_{\partial\Sigma}$, the set $(\mathbf{H}^F_{\partial\Sigma}, 0, \circ)$ is a Poisson algebra. □

As for the set $\mathcal{F}^\pi_\Sigma$ of restricted boundary fields, these are contrained by the corner conditions:

$$\begin{aligned}(p\circ\mathcal{I})(B) &= I_F(\xi) = \xi\\ (p\circ\mathcal{I})(c) &= I_F(\psi) = 0.\end{aligned} \tag{4.28}$$

Notice that $\mathcal{Q}_\Sigma$ preserves these conditions, since $(p\circ\mathcal{I})(\mathcal{Q}_\Sigma(c)) = 0$. Therefore, the set $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$ is a differential graded Lie algebra.

The following Lemma holds:

Lemma 13. There exists a L_∞ morphism between $(\mathbf{H}^F_{\partial\Sigma}, 0, \circ)$ and $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$.

Proof. As usual, consider the action of the map Φ_{π_F} on elements of $\mathbf{H}^F_{\partial\Sigma}$:

$$\Phi_{\pi_F}(\langle S,\xi\rangle_g) = +\int_\Sigma \left\langle S, [A,B] + [c,A^+]\right\rangle_g . \tag{4.29}$$

First of all, $\mathcal{Q}_\Sigma\Phi_{\pi_F}$ because of our boundary conditions, *i.e.* Φ_{π_F} preserves the differential. As a consequence, we can take the representative element $\Phi_0 = 0$.

As for the bracket, on the one hand we have

$$\Phi_{\pi_F}(\langle S,\xi\rangle_g \circ \langle U,\xi\rangle_g) = -\int_\Sigma \left\langle [S,U], [A,B] + [c,A^+]\right\rangle_g .$$

On the other:

$$\begin{aligned}\{\Phi_{\pi_F}\langle S,\xi\rangle_g, \Phi_{\pi_F}\langle U,\xi\rangle_g\} &= X_{\Phi_{\pi_F}\langle S,\xi\rangle_g}\left(\Phi\langle U,\xi\rangle_g\right) =\\ &= -\int_\Sigma \left\langle [S,U], [A,B] + [c,A^+]\right\rangle_g ,\end{aligned}$$

where the ω_Σ-hamiltonian vector field of $\Phi_{\pi_F}\langle S,\xi\rangle_g$ is

$$\begin{aligned}X_{\Phi_{\pi_F}\langle S,\xi\rangle_g} = -\int_\Sigma &\left\langle [S,A], \frac{\delta}{\delta A}\right\rangle_g + \left\langle [S,B], \frac{\delta}{\delta B}\right\rangle_g + \left\langle [S,A^+], \frac{\delta}{\delta A^+}\right\rangle_g +\\ &+ \left\langle [S,B^+], \frac{\delta}{\delta B^+}\right\rangle_g\end{aligned} \tag{4.30}$$

Therefore the map Φ_{π_F} provides a morphism of differential graded Lie algebras. Higher components Φ_n, with $n \geq 2$, are identically zero.

Finally, the L_∞-morphism is the following:

$$\Phi = \begin{cases} \Phi_0 = 0;\\ \Phi_1 = \Phi_{\pi_F};\\ \Phi_n = 0, \quad n\geq 2,\end{cases} \tag{4.31}$$

which is a morphism of differential graded Lie algebras. □

Remark 39. Since $\mathbf{H}^F_{\partial\Sigma}$ is generated by the linear elements $\langle S, \xi \rangle_g$, we can extend the action of Φ_{π_F} to the whole $\mathbf{H}^F_{\partial\Sigma}$ by requiring that it preserves the associative product:

$$\Phi_{\pi_F}\left(\Pi_a \langle S_a, \xi_a \rangle\right) \equiv \Pi_a \Phi_{\pi_F} \langle S_a, \xi_a \rangle_g . \tag{4.32}$$

In this way, Φ_{π_F} becomes a morphism of Poisson algebras.

Vertical Polarization

Consider now the vertical polarization. The projection P_F and the lagrangian immersion I_F (we take again the zero section) are

$$\begin{aligned} P_F &: \underset{(\xi,\psi)}{\mathcal{N}} \to \underset{\psi}{\mathfrak{g}[1]} \\ I_F &: \underset{\psi}{\mathfrak{g}[1]} \to \underset{(0,\psi)}{\mathcal{N}} . \end{aligned} \tag{4.33}$$

The set of polarized functionals is $\mathbf{H}^F_{\partial\Sigma} = C^\infty(\mathfrak{g}[1]) = \Lambda^\bullet \mathfrak{g}^*$, the exterior algebra of $\mathfrak{g}^*$, whose elements are of the form:

$$R_\psi = \sum_{a_1,\dots,a_n} R_{a_1,..,a_n} \psi^{a_1}\psi^{a_2}....\psi^{a_n}, \tag{4.34}$$

with the coefficients $R_{a_1,\dots,a_n}$ are skew-symmetric. Voronov's projection is $\pi_F = P_F^* \circ I_F^* : C^\infty(\mathcal{N}) \to \mathbf{H}^F_{\partial\Sigma}$.

As for the L_∞ structure on $\mathbf{H}^F_{\partial\Sigma}$ induced by π_F, we have that:

Lemma 14. $(\mathbf{H}^F_{\partial\Sigma}, \{\}_\bullet)$ is the Chevalley-Eilenberg complex $(\mathbf{H}^F_{\partial\Sigma}, Q_\mathcal{N})$.

Proof. The proof follows from the fact that $Q_\mathcal{N}$ acts on elements of $\mathbf{H}^F_{\partial\Sigma}$ as the Chevalley-Eilenberg differential. Therefore, higher brackets are zero and $(\mathbf{H}^F_{\partial\Sigma}, Q_\mathcal{N})$ is the Chevalley-Eilenberg complex. □

The corner conditions induced by the vertical polarization on $\mathcal{N}$ are

$$\begin{aligned} (p \circ \mathcal{I})(B) &= I_F(\xi) = 0; \\ (p \circ \mathcal{I})(c) &= I_F(\psi) = \psi. \end{aligned} \tag{4.35}$$

As in the case of the horizontal polarizarion, these conditions are preserved by $\mathcal{Q}_\Sigma$, since $(p \circ \mathcal{I})(\mathcal{Q}_\Sigma(B)) = 0$. Therefore, $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$ is a differential graded Lie algebra.

The following Lemma holds:

Lemma 15. There exist an L_∞ morphism between $(\mathbf{H}^F_{\partial\Sigma}, Q_\mathcal{N})$ and $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$.

Proof. The map Φ_{π_F} acts on an element $\langle R, \psi \rangle$ as

$$\Phi_{\pi_F}\left(\langle R, \psi \rangle_g\right) = -\int_\Sigma \langle R, [A, c] \rangle_g . \tag{4.36}$$

Now the differential does not annihilate $\Phi_{\pi_F}(\langle R, \psi \rangle_g)$. Therefore, we can take $\Phi_0 = \mathcal{S}_\Sigma$ as the representative element of the L_∞ morphism:

$$\mathcal{Q}_\Sigma(\Phi_{\pi_F}(\langle R, \psi \rangle_g)) + \cancel{\Phi_{\pi_F}(Q_\mathcal{N} \langle R, \psi \rangle_g)} = \{\Phi_0, \Phi_{\pi_F}(\langle R, \psi \rangle_g)\}. \tag{4.37}$$

Notice that the ω_Σ-hamiltonian vector field of $\Phi_{\pi_F}\langle R,\psi\rangle_g$ is

$$X_{\Phi_{\pi_F}\langle R,\psi\rangle_g} = \int_\Sigma \left\langle [R,c], \frac{\delta}{\delta B}\right\rangle_g + \left\langle [R,A], \frac{\delta}{\delta A^+}\right\rangle_g.$$

Therefore $\{\Phi_{\pi_F}(\bullet), \Phi_{\pi_F}(\bullet)\} = 0$ and Φ_{π_F} preserves the bracket. The map Φ_2 and higher components are automatically zero.

Finally, the L_∞ morphism is the following:

$$\Phi = \begin{cases} \Phi_0 = S_\Sigma; \\ \Phi_1 = \Phi_{\pi_F}; \\ \Phi_n = 0, \quad n \geq 2. \end{cases} \tag{4.38}$$

□

Remark 40. Similarly to the case of the horizontal polarization, we can extend the map Φ_{π_F} to the whole $\mathbf{H}^F_{\partial\Sigma}$ by requiring that

$$\Phi_{\pi_F}(\Pi_a \langle R_a, \psi_a\rangle_g) \equiv \Pi_a \Phi_{\pi_F}(\langle R_a, \psi_a\rangle_g). \tag{4.39}$$

4.5 Chern-Simons in three dimensions

In example (18) we studied the field theories living on the two-dimensional boundary and the one-dimensional corner, induced by a Chern-Simons theory on a three-dimensional manifold.

Recall that the space of boundary and corner fields are respectively

$$\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma) \otimes \mathfrak{g}[1]$$
$$\mathcal{F}_{\partial\Sigma} = \Omega^\bullet(\partial\Sigma) \otimes \mathfrak{g}[1],$$

where $\mathfrak{g}$ is a quadratic Lie algebra with ad-invariant symmetric bilinear form g (recall example 14) and that g provides an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. Notice that the space of corner fields has the structure of a shifted cotangent bundle, $\mathcal{F}_{\partial\Sigma} = T^*[-1]\Omega^1(S^1, \mathfrak{g}[1]) = \Omega^0(S^1, \mathfrak{g}[1]) \oplus \Omega^1(S^1, \mathfrak{g})$.

The superfield encoding the field content of the boundary theory is

$$X = c + A + A^+ \tag{4.40}$$

and that one parametrizing the space of corner fields is

$$\mathbb{X} = c + A. \tag{4.41}$$

Notice that we use the same notiation for boundary and corner fields.

Remark 41. From now on, we assume that $\partial\Sigma$ has only one component and that $\partial\Sigma = S^1$.

The set of AKSZ functionals $\mathbf{H}_{S^1}$ on the space of corner fields is made up by elements of the kind

$$\mathbf{X}_{\phi,S^1} = \sum_k \frac{1}{k!}\int_{S^1} \sum_{a_1,..,a_k} \mathbb{X}^{a_1}...\mathbb{X}^{a_k}\phi_{a_1,...,a_k}, \tag{4.42}$$

where $\phi \in \Omega^\bullet(S^1, \Lambda^\bullet\mathfrak{g})$ and $\phi_{a_1,..,a_k} \in \Omega^\bullet(S^1)$ are the components of ϕ.

Like in the case of two-dimensional BF theory, there are two main polarizations we can consider, the horizontal and vertical polarizations.

Horizontal Polarization

The horizontal polarization is specified by the following projection:

$$P_F : \underset{(A,c)}{T^*[-1]\Omega^1(S^1, \mathfrak{g}[1])} \to \underset{A}{\Omega^1(S^1, \mathfrak{g})}, \tag{4.43}$$

and as lagrangian immersion we take the zero section:

$$I_F : \underset{A}{\Omega^1(S^1, \mathfrak{g})} \to \underset{(A,0)}{T^*[-1]\Omega^1(S^1, \mathfrak{g}[1])}. \tag{4.44}$$

As for Voronov's projection, it is the composition $\pi_F = P_F^* \circ I_F^*$. It acts on functionals (4.42) as:

$$\pi_F \left(\mathbf{X}_{\phi,S^1}\right) = \mathbf{X}_{\phi^0,S^1} \tag{4.45}$$

where $\phi^0 \in \Omega^0(S^1, \mathfrak{g})$.

Therefore, the set of polarized AKSZ functionals $\mathbf{H}^F_{S^1}$ is generated by

$$\mathbf{X}_{\phi^0,S^1} = \sum_a \int_{S^1} \mathbb{X}^a \, \phi^0_a = \int_{S^1} \left\langle A, \phi^0 \right\rangle_g, \tag{4.46}$$

where $\langle\rangle_g$ is the pairing induced by g and we have used (4.41).

Remark 42. The vector space $\Omega^0(S^1, \mathfrak{g})$ is a Lie algebra. For any $\phi^0, \psi^0 \in \Omega^0(S^1, \mathfrak{g})$ the Lie bracket is given by:

$$[\phi^0, \psi^0] = \sum_{a,b} \phi^0_a \, \psi^0_b \, [T^a, T^b], \tag{4.47}$$

that is, by the multiplication of functions and the Lie bracket of $\mathfrak{g}$ between the generators. This algebra is known as **loop algebra** ([22]). Consider the following central extension of the loop algebra: the vector space is the direct sum $\Omega^0(S^1, \mathfrak{g}) \oplus \mathbb{R}$ and the Lie bracket is

$$[\phi^0, \psi^0]' = [\phi^0, \psi^0] + \int_{S^1} \left\langle \psi^0, d\phi^0 \right\rangle_g. \tag{4.48}$$

This central extension of the loop algebra is an example of **Kac-Moody algebra** (see [22]) and we denote it with $(\hat{\mathfrak{g}}, [,]')$.

Lemma 16. Let F be the horizontal polarization. The L_∞ algebra $(\mathbf{H}^F_{S^1}, \{\}_\bullet)$, defined in theorem 3, is isomorphic to the Kac-Moody algebra $(\hat{\mathfrak{g}}, [,]')$.

Proof. First of all, $\mathcal{Q}_{S^1}$ does not preserve $\mathbf{H}^F_{S^1}$:

$$\mathcal{Q}_{S^1}(\mathbf{X}_{\phi^0,S^1}) = \int_{S^1} \left\langle c, d\phi^0 \right\rangle_g - \int_{S^1} \left\langle [A, c], \phi^0 \right\rangle_g \notin \mathbf{H}^F_{S^1}. \tag{4.49}$$

Therefore, Voronov's differential, $\pi_F \circ \mathcal{Q}_{S^1} = 0$. On the other hand, the derived bracket between two elements of $\mathbf{H}^F_{\partial\Sigma}$ gives

$$\mathbf{X}_{\phi^0,S^1} \circ \mathbf{X}_{\psi^0,S^1} = \int_{S^1} \left\langle \psi^0, d\phi^0 \right\rangle_g + \mathbf{X}_{[\phi^0,\psi^0],S^1} \in \mathbf{H}^F_{S^1}. \tag{4.50}$$

Since the derived bracket preserves $\mathbf{H}^F_{S^1}$ and Voronov's projection acts as the identity on it, higher brackets are zero and the L_∞ algebra $(\mathbf{H}^F_{S^1}, \{\}_\bullet)$ is a Lie algebra $(\mathbf{H}^F_{S^1}, 0, \circ)$.

Let now $\mathfrak{r} : \hat{\mathfrak{g}} \to \mathbf{H}^F_{S^1}$ be the map

$$\begin{aligned} \mathfrak{r}(\phi^0) &= \mathbf{X}_{\phi^0, S^1} \\ \mathfrak{r}(1) &= 1. \end{aligned} \tag{4.51}$$

This map $\mathfrak{r}$ is a morphism of Lie algebras:

$$\mathfrak{r}([\phi^0, \psi^0]') = \mathfrak{r}(\phi^0) \circ \mathfrak{r}(\psi^0). \tag{4.52}$$

In fact:

$$\begin{aligned} \mathfrak{r}([\phi^0, \psi^0]') &= \mathfrak{r}\left([\phi^0, \psi^0] + \int_{S^1} \langle \psi^0, d\phi^0 \rangle_g \right) = \\ &= \mathbf{X}_{[\phi^0, \psi^0], S^1} + \int_{S^1} \langle \psi^0, d\phi^0 \rangle_g = \\ &= \mathbf{X}_{\phi^0, S^1} \circ \mathbf{X}_{\psi^0, S^1} = \\ &= \mathfrak{r}(\phi^0) \circ \mathfrak{r}(\psi^0). \end{aligned} \tag{4.53}$$

Therefore, $(\mathbf{H}^F_{S^1}, \{\}_\bullet) = (\mathbf{H}^F_{S^1}, 0, \circ)$ is isomorphic to $(\hat{\mathfrak{g}}, [,]')$ and the isomorphism is provided by the map $\mathfrak{r}$. $\square$

Let us discuss now the conditions that the boundary fields must satisfy on the corner. The set $\mathcal{F}^{\pi_F}_\Sigma$ is given by those boundary fields configurations such that A is not constrained and $c|_{S^1} = 0$.

From lemma (10), we know that $\mathcal{Q}_\Sigma$ becomes ω_Σ-hamiltonian. Moreover, it preserves these conditions, since $\mathcal{Q}_\Sigma(c)|_{S^1} = 0$. As a consequence, it restricts to $\mathbf{H}^{\pi_F}_\Sigma$ and its restriction is hamiltonian. The algebra $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$ is a differential graded Lie algebra.

Since the functionals $\mathbf{X}_{\phi^0, S^1} \in \mathbf{H}^F_{S^1}$ and $\mathbf{X}_{\widetilde{\phi^0}, S^1} \in \mathbf{H}^{\pi_F}_\Sigma$ depends only on ϕ^0 and $\widetilde{\phi^0}$, we will use the notation

$$\begin{aligned} \mathbf{X}_{\phi^0, \partial\Sigma} &\equiv \phi^0 \\ \mathbf{X}_{\widetilde{\phi^0}, \Sigma} &\equiv \widetilde{\phi^0}. \end{aligned}$$

The following Lemma holds:

Lemma 17. There exists a L_∞ morphism $\Phi_\bullet$ between $(\mathbf{H}^F_{S^1}, 0, \circ)$ and $(\mathbf{H}^{\pi_F}_\Sigma, \mathcal{Q}_\Sigma, \{,\})$, such that

$$\Phi_\bullet = \begin{cases} \Phi_0 = 0 \\ \Phi_1 = \begin{cases} \Phi_1(\phi^0) = \Phi_{\pi_F}(\phi^0) \\ \Phi_1(1) = 1 \end{cases} \\ \Phi_2 = \begin{cases} \Phi_2(\phi^0, \psi^0) = \widetilde{[\phi^0, \psi^0]} - [\widetilde{\phi^0}, \widetilde{\psi^0}] \\ \Phi_2(1, \bullet) = 0 \end{cases} \\ \Phi_n \equiv 0, \ \ n \geq 3 \end{cases}, \tag{4.54}$$

where Φ_{π_F} has been defined in (4.23).

Proof. Recall that the explicit action of Φ_{π_F} has been given in (4.16). We have:

$$\begin{aligned}\Phi_1(\int_{S^1} \langle \phi^0, A \rangle_g) = -\int_\Sigma d\left\langle A, \widetilde{\phi^0}\right\rangle_g + \mathcal{Q}_\Sigma \int_\Sigma \left\langle A^+, \widetilde{\phi}^0\right\rangle_g - \\ -\int_\Sigma \left\langle A, \widetilde{d\phi^0} - d\widetilde{\phi^0}\right\rangle_g,\end{aligned} \tag{4.55}$$

while, when it acts on constant functionals

$$\Phi_1(\int_{S^1} \langle \psi^0, d\phi^0 \rangle_g) = \int_{S^1} \langle \psi^0, d\phi^0\rangle_g \Phi_1(1) = \int_\Sigma d\langle \widetilde{\psi^0, d\phi^0}\rangle_g, \tag{4.56}$$

Notice that Φ_1 depends only on the extensions of functions ϕ^0 and exact one-forms $d\phi^0$. Therefore, the following extension for functions

$$\begin{aligned}\mathfrak{e}(\phi^0) &= \widetilde{\phi^0} \\ \mathfrak{e}(c) &= c,\end{aligned}$$

with c the constant function, can be used to choose the extension of exact one forms

$$\mathfrak{e}(d\phi^0) = d\mathfrak{e}(\phi^0).$$

With this choice, the last term on the RHS of Eq. (4.55) vanishes.

In order to study the other components of the L_∞-morphism $\Phi_\bullet$, we have first to study how Φ_1 does intertwine the differentials. We have (recall example (23)):

$$\mathcal{Q}_\Sigma \left(\Phi_1(\phi^0)\right) + 0 = \{\Phi_0, \Phi_1(\phi^0)\}. \tag{4.57}$$

However:

$$\begin{aligned}\mathcal{Q}_\Sigma \left(\Phi_{\pi_F}(\phi^0)\right) &= + \int_\Sigma d\left\langle \mathcal{Q}_\Sigma A, \phi^0\right\rangle_g = \\ &= \int_\Sigma d\left\langle d_A c, \phi^0\right\rangle_g = 0,\end{aligned} \tag{4.58}$$

as a consequence of the corner conditions $c|_{S^1} = 0$. The same holds when Φ_1 acts on constant functionals. Therefore, Φ_1 is a morphism of complexes and Φ_0 can be chosen as zero.

As for the failure of Φ_1 to preserve the bracket between, we have:

$$\begin{aligned}
&\Phi_1(\phi^0\circ\psi^0)-\left\{\Phi_1(\phi^0),\Phi_1(\psi^0)\right\}=\\
&=\int_\Sigma d\langle\widetilde{\psi^0,d\phi^0}\rangle_g-\int_\Sigma d\langle\widetilde{\psi^0,d\phi^0}\rangle_g-\\
&-\int_\Sigma\left\langle A,\,d[\widetilde{\phi^0,\psi^0}]-[\widetilde{d\phi^0},\tilde{\psi}^0]-[\tilde{\phi}^0,\widetilde{d\psi^0}]\right\rangle_g+\\
&+\int_\Sigma\left\langle\frac{1}{2}[A,A]+[A^+,c],\,[\widetilde{\phi^0,\psi^0}]-[\widetilde{\phi^0},\widetilde{\psi^0}]\right\rangle_g=\\
&=-\int_\Sigma\left\langle A,\,d[\widetilde{\phi^0,\psi^0}]-d[\widetilde{\phi^0},\widetilde{\psi^0}]\right\rangle_g+\\
&+\int_\Sigma\left\langle\hat{D}A^+,[\widetilde{\phi^0,\psi^0}]-[\widetilde{\phi^0},\widetilde{\psi^0}]\right\rangle_g=\\
&=\int_\Sigma\left\langle\mathcal{Q}_\Sigma A^+,[\widetilde{\phi^0,\psi^0}]-[\widetilde{\phi^0},\widetilde{\psi^0}]\right\rangle_g=\\
&=\mathcal{Q}_\Sigma\int_\Sigma\left\langle A^+,[\widetilde{\phi^0,\psi^0}]-[\widetilde{\phi^0},\widetilde{\psi^0}]\right\rangle_g=\\
&=\mathcal{Q}_\Sigma\Phi_2(\phi^0,\psi^0),
\end{aligned}\tag{4.59}$$

where the map Φ_2 is defined as

$$\Phi_2(\phi^0,\psi^0)=[\widetilde{\phi^0,\psi^0}]-[\widetilde{\phi^0},\widetilde{\psi^0}].\tag{4.60}$$

In the calculation, we used that the ω_Σ-hamiltonian vector field for $\Phi_1(\phi^0)$ is

$$X_{\Phi_1(\phi^0)}=\int_\Sigma\left\langle\widetilde{d\phi^0}+[A,\widetilde{\phi^0}],\frac{\delta}{\delta A}\right\rangle_g+\left\langle[c,\widetilde{\phi^0}],\frac{\delta}{\delta c}\right\rangle_g+\left\langle[A^+,\widetilde{\phi^0}],\frac{\delta}{\delta A^+}\right\rangle_g.\tag{4.61}$$

When of the entries is the constant functional, since $1\circ\alpha^0=\{\Phi_1(1),\Phi_1(\alpha^0)\}=0$, we can define

$$\Phi_2(1,\alpha^0)=0.\tag{4.62}$$

As it concerns the third component, this emerges from the following relation:

$$\begin{aligned}
&\mathcal{Q}_\Sigma\Phi_3(\phi^0,\psi^0,\alpha^0)+\Phi_2(\phi^0\circ\psi^0,\,\alpha^0)+\Phi_2(\psi^0\circ\alpha^0,\,\phi^0)+\Phi_2(\alpha^0\circ\phi^0,\,\psi^0)-\\
&-\left\{\Phi_1(\phi^0),\Phi_2(\psi^0,\alpha^0)\right\}-\left\{\Phi_1(\psi^0),\Phi_2(\alpha^0,\phi^0)\right\}-\left\{\Phi_1(\alpha^0),\Phi_2(\phi^0,\psi^0)\right\}=0.
\end{aligned}\tag{4.63}$$

Let's calculate at first the terms of the type $\Phi_2(\phi^0\circ\psi^0,\,\alpha^0)$. We have:

$$\begin{aligned}
\Phi_2(\phi^0\circ\psi^0,\,\alpha^0)&=\Phi_2(\int_{S^1}\langle\psi^0,d\phi^0\rangle_g+[\phi^0,\psi^0],\alpha^0)=\\
&=\Phi_2([\phi^0,\psi^0],\alpha^0)=\\
&=[[\widetilde{\phi^0,\psi^0],\alpha^0}]-[[\widetilde{\phi^0,\psi^0}],\widetilde{\alpha^0}].
\end{aligned}\tag{4.64}$$

Therefore:

$$\begin{aligned}
&\Phi_2(\phi^0\circ\psi^0,\,\alpha^0)+\Phi_2(\psi^0\circ\alpha^0,\,\phi^0)+\Phi_2(\alpha^0\circ\phi^0,\,\psi^0)=\\
&=\underbrace{[[\widetilde{\phi^0,\psi^0],\alpha^0}]+[[\widetilde{\psi^0,\alpha^0],\phi^0}]+[[\widetilde{\alpha^0,\phi^0],\psi^0}]}_{0}-\\
&-[[\widetilde{\phi^0,\psi^0}],\widetilde{\alpha^0}]-[[\widetilde{\psi^0,\alpha^0}],\widetilde{\phi^0}]-[[\widetilde{\alpha^0,\phi^0}],\widetilde{\psi^0}],
\end{aligned}\tag{4.65}$$

where the first terms are zero because of the Jacoby identity for the commutator. On the other hand, terms of the kind $\{\Phi_1(\phi^0), \Phi_2(\psi^0, \alpha^0)\}$ give

$$\begin{aligned}\{\Phi_1(\phi^0), \Phi_2(\psi^0, \alpha^0)\} &= X_{\Phi_{\pi_F}(\phi^0)}\left(\int_\Sigma \left\langle A^+, \widetilde{[\psi^0, \alpha^0]} - [\widetilde{\psi^0}, \widetilde{\alpha^0}]\right\rangle_g\right) = \\ &= \int_\Sigma \left\langle X_{\Phi_1(\phi^0)}(A^+), \widetilde{[\psi^0, \alpha^0]} - [\widetilde{\psi^0}, \widetilde{\alpha^0}]\right\rangle_g = \\ &= \int_\Sigma \left\langle [A^+, \widetilde{\phi^0}], \widetilde{[\psi^0, \alpha^0]} - [\widetilde{\psi^0}, \widetilde{\alpha^0}]\right\rangle_g = \\ &= \int_\Sigma \left\langle A^+, [\widetilde{\phi^0}, \widetilde{[\psi^0, \alpha^0]}] - [\widetilde{\phi^0}, [\widetilde{\psi^0}, \widetilde{\alpha^0}]]\right\rangle_g = \\ &= [\widetilde{\phi^0}, \widetilde{[\psi^0, \alpha^0]}] - [\widetilde{\phi^0}, [\widetilde{\psi^0}, \widetilde{\alpha^0}]].\end{aligned} \tag{4.66}$$

Therefore:

$$\begin{aligned}\{\Phi_1(\phi^0), \Phi_2(\psi^0, \alpha^0)\} + \{\Phi_1(\psi^0), \Phi_2(\alpha^0, \phi^0)\} + \{\Phi_1(\alpha^0), \Phi_2(\phi^0, \psi^0)\} = \\ = \underbrace{[\widetilde{\phi^0}, \widetilde{[\psi^0, \alpha^0]}] + [\widetilde{\psi^0}, \widetilde{[\alpha^0, \phi^0]}] + [\widetilde{\alpha^0}, \widetilde{[\phi^0, \psi^0]}] - \\ -[\widetilde{\phi^0}, [\widetilde{\psi^0}, \widetilde{\alpha^0}]] - [\widetilde{\psi^0}, [\widetilde{\alpha^0}, \widetilde{\phi^0}]] - [\widetilde{\alpha^0}, [\widetilde{\phi^0}, \widetilde{\psi^0}]]}_{0}\end{aligned} \tag{4.67}$$

where again we used the Jacoby identity. Finally, as a consequence of (4.65) and (4.67), Eq. (4.63) gives

$$\mathcal{Q}_\Sigma \Phi_3(\phi_0, \psi_0, \alpha_0) = 0,$$

which tells that $\Phi_3(\phi^0, \psi^0, \alpha^0)$ can be taken zero.

The same choice can be done when one of the entry is the constant functional $\int_{S^1} \langle \phi^0, d\psi^0 \rangle_g$. Therefore, the map Φ_3 can be taken identically zero.
As for higher brackets, since $\Phi_3 \equiv 0$, if $n = 4$ the only term we have to calculate is:

$$\mathcal{Q}_\Sigma \Phi_4(\phi^0, \psi^0, \alpha^0, \beta^0) = \{\Phi_2(\phi^0, \psi^0), \Phi_2(\alpha^0, \beta^0)\}, \tag{4.68}$$

which is zero, since the ω-hamiltonian vector field of $\Phi_2(\phi^0, \psi^0)$ is

$$X_{\Phi_2(\phi^0, \psi^0)} = \int_\Sigma \left\langle \widetilde{[\phi^0, \psi^0]} - [\widetilde{\phi^0}, \widetilde{\psi^0}], \frac{\delta}{\delta c}\right\rangle_g.$$

The same result holds when one or more of the entries are the constant functional $\int_{S^1} \langle \phi^0, d\psi^0 \rangle_g$.

It follows that higher brackets with $n \geq 4$ are identically zero.

□

Vertical Polarization

Consider now the vertical polarization. The projection into the leaf space and its lagrangian immersion, given by the zero section, are

$$\begin{aligned}P_F &: \underset{(A,c)}{T^*[-1]\Omega^1(S^1, \mathfrak{g}[1])} \to \underset{c}{\Omega^0(S^1, \mathfrak{g}[1])} \\ I_F &: \underset{c}{\Omega^0(S^1, \mathfrak{g}[1])} \to \underset{(0,c)}{T^*[-1]\Omega^1(S^1, \mathfrak{g}[1])}\end{aligned} \tag{4.69}$$

Again, Voronov's projection is the composition of the pullbakcs, $\pi_F = P_F^* \circ I_F^*$ and its action on (4.42) is given by

$$\pi_F(\mathbf{X}_{\phi,S^1}) = \mathbf{X}_{\phi^1,S^1}, \tag{4.70}$$

where $\phi^1 \in \Omega^1(S^1, \Lambda^\bullet \mathfrak{g})$. Thus, the set of polarized functionals $\mathbf{H}^F_{\partial\Sigma}$ is generated by elements of the kind

$$\mathbf{X}_{\phi^1,S^1} = \sum_k \frac{1}{k!} \sum_{a_1,..,a_k} \frac{1}{k} \int_{S^1} c^{a_1} ... c^{a_k} \phi^1_{a_1,..,a_k}, \tag{4.71}$$

with $\phi^1_{a_1,..,a_k} \in \Omega^1(S^1)$.

As for the L_∞ structure induced on $\mathbf{H}^F_{\partial\Sigma}$, it is easy to show the following lemma.

Lemma 18. Let F be the vertical polarization. The L_∞ algebra $(\mathbf{H}^F_{S^1}, \{\}_\bullet)$, defined in theorem (3) is isomorphic to the complex $(\Omega^1(S^1, \Lambda^\bullet \mathfrak{g}), d_{CE})$, where d_{CE} is the Chevalley-Eilenberg differential of $\mathfrak{g}$.

Proof. This follows from the fact that $\mathcal{Q}_{S^1}$ acts on the ghost fields c as

$$\mathcal{Q}_{S^1} c = \frac{1}{2}[c, c].$$

Thus, we can write that

$$\mathcal{Q}_{S^1}(\mathbf{X}_{\phi^1,S^1}) = \mathbf{X}_{d_{CE}\phi^1,S^1}, \tag{4.72}$$

where d_{CE} is the Chevalley-Eilenberg differential.

Since the differential $\mathcal{Q}_{S^1}$ preserves $\mathbf{H}^F_{S^1}$, the derived bracket and higher brackets are zero: the L_∞ algebra $(\mathbf{H}^F_{S^1}, \{\}_\bullet)$ is a chain complex $(\mathbf{H}^F_{S^1}, \mathcal{Q}_{S^1})$. Due to (4.72), this complex is isomorphic to $(\Omega^1(S^1, \Lambda^\bullet \mathfrak{g}), d_{CE})$, where the isomorphism is the map $\mathfrak{l}$, such that:

$$\begin{aligned} &\mathfrak{l} : \Omega^1(S^1, \Lambda^\bullet \mathfrak{g}) \to \mathbf{H}^F_{S^1}, \\ &\mathfrak{l}(\phi^1) = \mathbf{X}_{\phi^1,S^1}, \\ &\mathfrak{l} \circ d_{CE} = \mathcal{Q}_{S^1} \circ \mathfrak{l}. \end{aligned} \tag{4.73}$$

□

Remark 43. For simplicity, we assume to have chosen an extension map which commutes with the differential. If it is defined a projection of the boundary to the corners, $\mathbf{p} : \Sigma \to \partial\Sigma$, such an extension map is simply the pullback $\mathbf{p}^*$ of differential forms on the corners to differential forms on the boundary.

As it concerns the corner conditions on boundary fields, the set $\mathcal{F}^\pi_\Sigma$ is given by those boundary fields such that $A|_{S^1} = 0$.

Notice that, on the contrary to the horizontal polarization, now $\mathcal{Q}_\Sigma$ does not preserve the corner conditions, since $(\mathcal{Q}_\Sigma A)|_{S^1} = dc \neq 0$. As a consequence, lemma (11) does not hold and $\mathcal{Q}_\Sigma$ is not a differential for $(\mathbf{H}^{\pi_F}_\Sigma, \{,\})$. Moreover, the chain complex in lemma (18) is the Chevalley-Eilenberg complex associated to the loop algebra and we do not have any information of the central extension.

As a consequence, we should modify the algebraic structures living on the boundary and on the corner.

Chapter 5

Conclusions

This thesis is devoted to the study of classical gauge theories of the AKSZ type on manifolds with boundary and corners. In particular, the construction we have developed is the following:

- After the choice of a polarization on the space of corner fields, we have built a L_∞ algebra on the space of corner observables.
- The polarization has been used to induce corner conditions on boundary fields. Under the assumption that the cohomological vector field $\mathcal{Q}_\Sigma$, encoding the gauge symmetries acting on boundary fields, preserved these conditions, we have defined a differential graded Lie algebra on boundary observables.
- We have conjectured that there exists a L_∞ morphism between these algebraic structures, whose first component has been assigned.

The full problem of finding the other components has been solved in two examples: the two-dimensional BF theory both in the vertical and the horizontal polarization and the three-dimensional Chern-Simons theory. In Chern-Simons theory, we have shown that in the horizontal polarization the L_∞ algebra on the corner is the Kac-Moody algebra rand we have mapped it on the space of boundary fields. Furthermore, we have seen that our construction has to be extended in order to deal with the vertical polarization.

This classical discussion is preliminary to the study of the quantization, which is the real goal of the research program. Following the ideas already present in [23], we expect that to the corners we have to associate an associative algebra of classical observables and to the boundary a phase space: the quantization sends the classical algebra to an operator algebra and the phase space to a Hilbert space of states, which carries a representation of the operator algebra. This would constitute an explicit realization of the ideas of the extended approach, described for example in [10, 21, 4]. In particular, we hope that our work will contribute to realize Chern-Simons theory as an extended topological field theory.

However, the classical description has still to be totally understood. The next step is to extend our results by studying the three-dimensional Chern-Simons theory in the vertical polarization and the two-dimensional BF theory in more general polarizations and higher dimensions. An other interesting development could be the study of others three-dimensional AKSZ theories. Indeed, Chern-Simons theory is a particular class of a family classified by Courant algebroids [30]. In these cases, what should be the analogue of the Kac-Moody algebra is unknown and we expect that this approach is the right one to understand it.

On the other hand, the construction of the higher components of the morphism is quite *ad hoc* for our examples. It is natural to ask if it is possible to build them for a general AKSZ theory and therefore to complete the discussion in the AKSZ case. The analysis of non-AKSZ theories, such as General Relativity, whose BV-BFV formulation has been studied in [16, 17], could complete the classical analysis, which will furnish the tools to deal with the quantization.

Appendix A

Elements of symplectic geometry

We refer to [9] for basic discussions and [18] for more advanced discussions.

Definition 13. *Let M be a smooth m-dimensional manifold without boundary and ω a two-form on it. We say that ω is a symplectic form if it is non-degenerate and closed. The couple (M, ω) is called* ***symplectic manifold****.*

"Closed" means that $d\omega = 0$, where d is the exterior derivative:

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \qquad d^2 = 0. \tag{A.1}$$

Remark 44. Since ω is closed, it can be locally written as

$$\omega = d\alpha, \quad \alpha \in \Omega^1(M).$$

This one-form α is called **symplectic potential**. Notice that it is not unique, since it is defined up to closed one-forms.

"Non-degenerate" means that, for each point $p \in M$, the skew symmetric map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is non degenerate and, as a consequence, that the map $\tilde{\omega}_p : T_pM \to T_p^*M$, given by:

$$X \in T_pM \longrightarrow \tilde{\omega}_p(X) = \omega(X, .) \equiv i_X\omega, \tag{A.2}$$

is invertible. As a consequence,the dimension of the tangent space and so of the manifold must be even[1].

This local map induces a global isomorphism between the tangent bundle TM and the cotangent bundle T^*M.

Definition 14. *Let (M, ω) be a symplectic manifold. A submanifold Y of M is* ***isotropic*** *if, at each point $p \in Y$, $\omega|_{T_pY} \equiv 0$. An isotropic submanifold Y is* ***lagrangian*** *if $dimY = \frac{1}{2}dimM$.*

[1]Infact, if K in an antisimmetric matrix of dimension n, $K = -K^T$, we have:

$$detK = \frac{1}{2}(detK + detK) = \frac{1}{2}(detK + detK^T) = \frac{1}{2}(detK + (-1)^n detK) = 0 \tag{A.3}$$

if n is odd.

Example 20. The most important example of a symplectic manifold is the cotangent bundle $M = T^*Q$. Let $\{q^k\}$ be local coordinates on the base space. They can be extended to a coordinate system $\{q^k, p_k\}$ on T^*Q, with $\{p_k\}$ coordinates along the fiber. The symplectic two-form is

$$\omega_0 = \sum_{k=0}^{n} dq^k \wedge dp_k = d\alpha_0, \tag{A.4}$$

where the one form $\alpha = \sum_{k=0}^{n} q^k dp_k$ and $n = dim(Q)$. It can be shown that ω_0 is globally exact, that is, $\omega_0 = d\alpha_0$ is globally valid. The form ω_0 is called **canonical symplectic form** and the potential α_0 the **tautological form**.

Theorem 4. *Let (M, ω) be a $2n$-dimensional symplectic manifold and let p be any point in M. Then, there is a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that on U:*

$$\omega = \sum_{i=1}^{n} \delta_i^j dx^i \wedge dy_j \tag{A.5}$$

*This chart is called **Darboux chart**.*

An immediate consequence of this theorem is that symplectic manifolds can be distinguished only by their global properties, as they locally have the structure of $(\mathbb{R}^{2n}, \omega_0)$.

Symplectic and Hamiltonian vector fields

Let $Vect(M)$ be the set of smooth vector fields on (M, ω).

Definition 15. *A vector field $X \in Vect(M)$ is called a **symplectic vector field** if:*

$$\mathcal{L}_X \omega = \{d, i_X\}\omega = 0, \tag{A.6}$$

where $\{d, i_X\} = di_X + i_X d$.

Notice that, because of the closedness of ω, we have:

$$\mathcal{L}_X \omega = d(i_X \omega) = 0, \tag{A.7}$$

that is the one-form $i_X \omega$ is closed too.

Definition 16. *A symplectic vector field X is **hamiltonian** if the one-form $i_X \omega$ is exact, that is if it exists a function $f \in C^\infty(M)$ such that:*

$$i_X \omega = df. \tag{A.8}$$

*The primitive f of $i_X \omega$ is called **hamiltonian function** of X.*

Remark 45. Locally, on every open set, every symplectic vector field is hamiltonian. If the first cohomology group[2] $H^1_{deRham}(M) = 0$, then globally every symplectic vector field is hamiltonian. In general, $H^1_{deRham}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

[2]Recall that the k-de Rham cohomology group $H^k_{deRham}(M)$ is the set of closed k-forms on M modulo exact k-forms.

Remark 46. Notice that, in the finite dimensional case, since ω is symplectic, every function $f \in C^\infty(M)$ is hamiltonian. This is not true in the infinite dimensional case. We call X_f the hamiltonian vector field associated to the function f.

Definition 17. *Let $f, g \in C^\infty(M)$. Their **Poisson bracket** is*

$$\{f, g\} = \omega(X_f, X_g) = X_f(g). \tag{A.9}$$

The Poisson bracket satisfies the Jacoby identity:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{A.10}$$

and the Leibniz rule wrt the associative product of C^∞-functions:

$$\begin{aligned} \{f, gh\} &= \{f, g\}h + g\{f, h\} \\ \{fg, h\} &= f\{g, h\} + \{f, h\}g. \end{aligned} \tag{A.11}$$

Remark 47. The set $(C^\infty(M), \{,\})$ (namely $C^\infty(M)$ with the associative product of functions and the Poisson bracket) is a Lie algebra. With the associative product of functions, it is a Poisson algebra.

Corresponding to the Lie algebra of hamiltonian functions, we have the Lie algebra of hamiltonian vector fields, whose Lie product is given by the commutator:

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad X, Y \in \mathcal{X}_{ham}(M), \quad f \in C^\infty(M) \tag{A.12}$$

In fact, the commutator between two hamiltonian vector fields is an other hamiltonian vector field:

$$\begin{aligned} i_{[X_f, X_g]}\omega &= (\mathcal{L}_{X_f} i_{X_g} - i_{X_g}\mathcal{L}_{X_f})\omega = \mathcal{L}_{X_f} i_{X_g}\omega = \\ &= \{d, i_{X_f}\} i_{X_g}\omega = d i_{X_f} i_{X_g}\omega \\ &= d(\{f, g\}) \\ &= i_{X_{\{f,g\}}}\omega. \end{aligned} \tag{A.13}$$

Hence:

$$X_{\{f,g\}} = [X_f, X_g]. \tag{A.14}$$

Polarization

Basically, a polarization is a foliation of (M, ω) by lagrangian subspaces.

Definition 18. *Let (M, ω) be a symplectic manifold. A **polarization** F on (M, ω) is a distribution of the tangent bundle TM with the following properties:*

- (i) *It is **involutive**;*
- (ii) *It is **lagrangian**;*
- (iii) *$dim(F_p \cap T_pM)$ is constant for every $p \in M$.*

Property (i) means that, for any two vector fields X, Y directed along F, their commutator $[X, Y]$ is still directed along F. This implies that F is the tangent bundle of a foliation of M, whose leaf spaces must be lagrangian because of property (ii). Property (iii) ensures that we can take the quotient of M with the lagrangian foliation.

Definition 19. *A function $f \in C^\infty(M)$ is called* ***polarized*** *if it is constant along the directions spanning the polarization F:*

$$V(f) = 0, \quad \forall V \in \Gamma(F). \tag{A.15}$$

Let $\mathcal{L}_F$ be the quotient space of M with the polarization, with $P_F : M \to \mathcal{L}_F$ the associated natural projection.

Definition 20. *An* ***immersion*** *of the quotient $\mathcal{L}_F$ into M is an injection I_F such that*

$$P_F \circ I_F = Id|_{\mathcal{L}_F}.$$

Moreover, we call I_F ***lagrangian*** *if*

$$I_F^* \omega = 0.$$

Remark 48. Such an immersion does not always exist for generic polarizations. Moreover, if it exists, it is not unique.

If a lagrangian immersion exists, then $\mathcal{L}_F$ is a lagrangian submanifold of (M, ω). Therefore, (the pullback of) the set of functions on $\mathcal{L}_F$ is an abelian subalgebra of $(C^\infty(M), \{,\})$. In fact, given $f, h \in C^\infty(\mathcal{L}_F)$:

$$\begin{aligned} \{I_F^* f, I_F^* g\} &= I_F^* \{f, g\} = \\ &= (I_F^* \omega)(X_f, X_g) = \\ &= 0. \end{aligned} \tag{A.16}$$

Example 21. Consider a cotangent bundle (T^*Q, ω_0), with canonical coordinates $\{q^k, p_k\}$ and standard symplectic form $\omega_0 = \sum_k dq^k \wedge dp_k$. There are two main real polarizations we can define:

- The **vertical polarization**, locally spanned by the vectors $\{\frac{\delta}{\delta p_k}\}$ tangent to the fibers ($\simeq \mathbb{R}^n$), which foliate T^*Q. The quotient space is isomorphic to the base space Q. Polarized functions are functions on T^*Q constant along the fibers, namely they depend only on the q's.

  Since Q is the base space of the cotangent bundle, we can immerse it into T^*Q, for example as the graph of a function $g \in C^\infty(Q)$:

  $$I_F : \underset{q^k}{Q} \to \underset{(q^k, p_k = \frac{\partial g}{\partial q^k})}{T^*Q}. \tag{A.17}$$

  This immersion is lagrangian and it is not unique, since it depends on the choice of g. If g is the constant function, this immersion is called **zero section**:

  $$I_F^* f(q) = f(q, 0). \tag{A.18}$$

- The **horizontal polarization**, locally spanned by $\{\frac{\partial}{\partial q^k}\}$, tangent to the configuration space Q. The quotient space is isomorphic to the fiber and polarized functions depend only on the p's. As before, we can immerse the fiber into the cotangent bundle as the graph of a function $g(p_1, .., p_n)$ and this immersion is lagrangian.

Presymplectic Reduction

Consider a manifold M and a closed two form $\breve{\omega}$ on it. For p a point in M:

$$Ker(\breve{\omega})_p = \{X \in T_pM : i_X\breve{\omega}_p = 0\} \neq \emptyset. \tag{A.19}$$

Assume that $\breve{\omega}$ has constant rank, namely that $Ker(\breve{\omega}_p)$ has constant codimension

$$Codim(Ker(\breve{\omega}_p)) = dim(M) - dim(Ker(\breve{\omega}_p))$$

for all $p \in M$. We call such a $\breve{\omega}$ **presymplectic**.

we can still define the concept of hamiltonian function and hamiltonian vector field for $\breve{\omega}$. However, not every function in $C^\infty(M)$ is hamiltonian. Moreover, hamiltonian functions do not have a unique hamiltonian vector field, since it can always be shifted by an element of the kernel of $\breve{\omega}$.

If $\breve{\omega}$ is presymplectic, $Ker(\breve{\omega}) = \cup_{p\in M} Ker(\breve{\omega}_p)$ defines a distribution of TM, which is integrable, since, given any two $X, Y \in Ker(\breve{\omega})$, $[X, Y]$ is still in $Ker(\breve{\omega})$. Therefore, we can take the quotient $M/Ker(\breve{\omega})$. If the quotient manifold is smooth, the presymplectic form $\breve{\omega}$ on M induces a well-defined symplectic form ω on the quotient, such that

$$\breve{\omega} = p^*\omega, \tag{A.20}$$

where $p : M \to M/Ker(\breve{\omega})$ is the natural projection to the quotient.

The couple $(M/Ker(\breve{\omega}), \omega)$ is called the **presymplectic reduction** of $(M, \breve{\omega})$.

Appendix B

Atiyah's axioms of TQFT

Atiyah's axioms formalize topological quantum field theories (TQFT) [2, 3]. We first give the axioms and then we remark their physical interpretation. For a brief and simple presentation, see [27].

In Atiyah's interpretation, a topological quantum field theory (TQFT) in dimension d is a functor Z which assigns:

A) a finite dimensional vector space $Z(\Sigma)$ over $\mathbb{C}$ to each oriented closed d-dimensional smooth manifold Σ;

B) a vector $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(d+1)$-dimensional manifold M with boundary ∂M.

Furthermore, the functor Z is subject to the following axioms.

1) Z is **functorial** with respect to orientation preserving diffeomorphism of Σ and M.

This means that, if $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism, then $Z(f) : Z(\Sigma_1) \to Z(\Sigma_2)$ is an isomorphism of vector spaces. "Functoriality" menas that $Z(gf) = Z(g)Z(f)$, for $g : \Sigma_2 \to \Sigma_3$. Moreover, if $F : M_1 \to M_2$ is an orientation preserving diffeomorphism, with $\partial M_1 = \Sigma_1$ and $\partial M_2 = \Sigma_2$, then $Z(F)$ sends the vector $Z(M_1)$ to the vector $Z(M_2)$.

2) Z is **involutory**.

This means that, if $\bar{\Sigma}$ is Σ with reversed orientation, then $Z(\bar{\Sigma}) = Z(\Sigma)^*$, where $Z(\Sigma)^*$ is the dual vector space of $Z(\Sigma)$.

3) Z is **multiplicative**.

This means that

3.1) $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

3.2) If $\partial M_1 = \bar{\Sigma}_1 \sqcup \Sigma_2$, $\partial M_2 = \bar{\Sigma}_2 \sqcup \Sigma_3$ and $M = M_1 \cup_{\Sigma_2} M_2$ is the manifold obtained by gluing M_1 and M_2 over the common boundary Σ_2, then we require that

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \in Z(\bar{\Sigma}_1) \otimes Z(\Sigma_2),$$

where $\langle , \rangle$ denotes the natural pairing between the dual spaces $Z(\Sigma_2)$ and $Z(\Sigma_2)^*$. Axiom (**2**) tells also that $Z(M)$ is actually a homomorphism of vector spaces, $Z(M) : Z(\Sigma_1) \to Z(\Sigma_3)$, resulting from the composition $Z(M_1) \circ Z(M_2)$.

Remark 49. Such closed oriented smooth $(d+1)$-dimensional manifolds M, with $\partial M = \bar{\Sigma}_1 \sqcup \Sigma_2$ are called **cobordisms**.

Other consequences of these axioms are the following. First of all, the multiplicativity axiom is used also to state that, for $M = \Sigma \times I$ (a cylinder),

3.3) $Z(\Sigma \times I) = Id_\Sigma$,

where Id_Σ is the identity map on Σ.

Moreover, from axiom (**3.1**) we infer that, when $\Sigma = \emptyset$, we can set $Z(\emptyset)$ equal to zero or to $\mathbb{C}$. We set

3.4) $Z(\emptyset) = \mathbb{C}$.

As a consequence of (**3.4**), when M has $\partial M = \emptyset$, $Z(M) : \mathbb{C} \to \mathbb{C}$ is simply the multiplication by a complex number. An other way of state this result is that the TQFT assigns numerical invariants (complex numbers) to closed $(d+1)$-dimensional manifolds. If we cut M along a d-dimensional manifold Σ, we get two smaller $(d+1)$-dimensional manifolds, M_1 and M_2, with $\partial M_1 = \emptyset \sqcup \bar{\Sigma}$ and $\partial M_2 = \Sigma \sqcup \emptyset$. Then, $Z(M_1) \in Z(\Sigma)^*$ and $Z(M_2) \in Z(\Sigma)$ and, from property (**3.2**) we obtain

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \in \mathbb{C}.$$

Therefore, for M a closed $(d+1)$-dimensional manifold, the invariant $Z(M)$ can be computed from any decomposition $M = M_1 \cup_\Sigma M_2$.

We can summarize all these axioms in the followig definition.

Definition 21. *(TQFT) A TQFT is a functor of symmetric monoidal categories,* $Cob_{d+1} \to Vect_{\mathbb{C}}$.

The categorical structure of these categories is the following.

- Objects in Cob_n are closed oriented d-dimensional manifolds and morphisms are oriented cobordisms M, with $\partial M = \bar{\Sigma}_1 \sqcup \Sigma_2$. The composition of morphism is the gluing along a common boundary component and the identity morphism is the cylinder $M = \Sigma \times I$. The monoidal product is the disjoint union $\sqcup$ and the monoidal unit is the empty set $\emptyset$.

•• Objects in $Vect_{\mathbb{C}}$ are complex vector spaces and morphisms are homomorphisms between them. The composition of morphisms is the composition of homomorphisms and the identity morphism is the identity automorphism of vector spaces. The monoidal product is the tensor product $\otimes$ and the monoidal unit is the field of complex numbers $\mathbb{C}$.

From a physical point of view, Atiyah's axioms can be interpreted as it follows. For a closed oriented $(d+1)$-dimensional manifold M, the numerical invariant $Z(M)$ is the partition function of the theory. Property (**3.2**) implies that we can compute the partition function of the theory through cutting and gluing rules.

For Σ a d-dimensional closed oriented manifold, the vector space $Z(\Sigma)$ is the Hilbert space of the theory. Moreover, if M is such that $\partial M = \Sigma$, the vector $Z(M)$ is interpreted as the vacuum state defined by M of the theory. If $M = \Sigma \times I$, with I a time direction, property (**3.3**) means that the Hilbert space does not evolve in time and there is no dynamics. If we have a cobordism M, with $\partial M = \bar{\Sigma}_1 \sqcup \Sigma_2$, the initial Hilbert space $Z(\Sigma)_1$ evolves non trivially and the homomorphism $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$ is the evolution operator.

Appendix C

L_∞ algebras

We introduce here the basic definition of L_∞ algebras. We refer to [25] for a short and clear discussion on the subject.

Definition 22. *(L_∞ algebra) An L_∞ algebra is a graded vector space $V^\bullet$ endowed with multilinear, graded skew-symmetric operations $l_n : \Lambda^n V \to V$ such that*

- *l_n has degree $2-n$;*
- *The **strong homotopy Jacoby identity** holds, namely, for all $n > 0$:*

$$\sum_{n=i+j,\ i\geq 1, j\geq 1} \ \sum_{\sigma\in Sh(i,j)} \pm l_i\left(v_{\sigma_1}, .., v_{\sigma_{i-1}},\ l_j\left(v_{\sigma_i}, .., v_{\sigma_n}\right)\right) = 0, \tag{C.1}$$

where $(v_1, ..., v_n) \in V^\bullet$ is an n-tuple of vectors.

Remark 50. In the definition, $Sh(i,j)$ is the group of permutations of numbers $1, ... n = i+j$ such that $\sigma_1 < \sigma_2 < .... < \sigma_i$ and $\sigma_{i+1} < .... < \sigma_n$.

Remark 51. If the L_∞ algebra admits also an operation l_0, which is a l_1-closed element of V and enters in relations (C.1), we call it **curved L_∞ algebra**.

Example 22. Consider $n = 1$. Relation (C.1) takes the form

$$l_1(l_1(v_1)) = 0.$$

Assume now $n = 2$. Now we have also a binary operation l_2. Relation (C.1) tells that

$$l_1(l_2(v_1, v_2)) = l_2(l_1(v_1), v_2) + (-1)^{|v_1|} l_2(v_1, l_1(v_2)),$$

which states that l_1 is a derivation of l_2.

Notice that if $n = 3$, the map l_2 satisfies the Jacoby identity only up to homotopy. Indeed, from (C.1):

$$\begin{aligned} &l_2(v_1, l_2(v_2, v_3)) - l_2(l_2(v_1, v_2, v_3) + (-1)^{|v_1||v_2|} l_2(v_2, l_2(v_1, v_3)) = \\ &= \pm l_1(l_3(v_1, v_2, v_3)) \pm l_3(l_1(v_1), v_2, v_3) \pm l_3(v_1, l_1(v_2), v_3) \pm l_3(v_1, v_2, l_1(v_3)). \end{aligned}$$

The map l_3 encodes the failure of the l_2 to satisy the Jacoby identity. This is the reason why L_∞ algebra are also called homotopy Lie algebras.

Remark 52. A chain complex $(V^\bullet, l_1)$ is a L_∞ algebra with $l_n = 0$, $n > 1$.
A differential graded Lie algebra $(V^\bullet, l_1, l_2)$ is a L_∞ algebra with $l_n = 0$, $n > 2$.

Let $(V^\bullet, l_n)$ and $(W^\bullet, m_n)$ be two L_∞ algebras. We can define the concept of L_∞ morphism between them. Since we are interested only in chain complexes and differential graded Lie algebras, we introduce this notion only for this two cases.

Definition 23. *Let $(V^\bullet, l_1, l_2)$ and $(W^\bullet, m_1, m_2)$ be two differential graded Lie algebras. A* ***curved*** ***L_∞-morphism*** *between them is a collection of maps $\Phi_\bullet = \{\Phi_0, \Phi_1, ...\}$ from $(V^\bullet, l_1, l_2)$ to $(W^\bullet, m_1, m_2)$ such that:*

- *The component Φ_0 is a homogeneous element in $W^\bullet$;*
- *The component Φ_n is an n-ary operation of degree $1-n$.*
- *For any $n = 0, 1, 2, ..$ and homogeneous elements $v_i \in V$, the following relation holds:*

$$\begin{aligned} & m_1\left(\Phi_n(v_1, v_2, ..., v_n)\right) - \sum_{i_1}^{n} \pm \Phi_n(v_1, ..., l_1(v_i), ..., v_n) = \\ & = \frac{1}{2} \sum_{k,j\geq 0,\, k+j=n} \frac{1}{k!j!} \sum_{\sigma \in Sh(k,j)} \pm m_2\left(\Phi_k(v_{\sigma_1}, ..., v_{\sigma_k}), \Phi_j(v_{\sigma_{k+1}}, ..., v_{\sigma_n})\right) + \\ & + \sum_{i<j} \pm \Phi_{n-1}(l_2(v_i, v_j), v_1, ...v_n). \end{aligned} \tag{C.2}$$

Remark 53. A curved L_∞ morphism $\Phi_\bullet$ with $\Phi_0 = 0$ is simply called L_∞ **morphism**.

Example 23. Consider the following particular cases.

- For $n = 0$, we find:
$$m_1(\Phi_0) = m_2(\Phi_0, \Phi_0). \tag{C.3}$$
- For $n = 1$, we find:
$$\Phi_1(l_1(v_1)) - m_1(\Phi_1(v_1)) = m_2(\Phi_0, \Phi_1(v_1)), \tag{C.4}$$
that is, the map Φ_1 preserves the differential up to a term involving the representative Φ_0.
- For $n = 2$, we find:
$$\begin{aligned} \Phi_1(l_2(v_1, v_2)) - m_2(\Phi_1(v_1), \Phi_1(v_2)) = & \; m_1(\Phi_2(v_1, v_2)) - \\ & - \Phi_2(l_1(v_1), v_2) - \\ & - (-1)^{|v_1|}\Phi_2(v_1, l_1(v_2)) - \\ & - m_2(\Phi_0, \Phi_2(v_1, v_2), \end{aligned} \tag{C.5}$$
that is, the map Φ_1 preserves the bracket up to terms involving the second component Φ_2 and the representative Φ_0.

Relations between higher components can be found in a similar way.

Remark 54. A morphism of differential graded Lie algebra, as well as a morphism of chain complexes, can be seen as a L_∞-morphism $\Phi_\bullet$ with only the unary component Φ_1 non trivial.

Appendix D

Projection to Polarized Functions

Consider a graded symplectic manifold $(\mathcal{M}, \check{\omega})$. Let F be a polarization on it and $\mathcal{L}_F$ the leaf space. Define the following maps:

- The projection $P : \mathcal{M} \to \mathcal{L}_F$;
- An immersion $I : \mathcal{L}_F \to \mathcal{M}$, such that $P \circ I = Id_{\mathcal{L}_F}$.

Let $\pi_F = P_F^* \circ I_F^* : C^\infty(\mathcal{M}) \to P_F^*(C^\infty(\mathcal{L}_F)) \subset C^\infty(\mathcal{M})$.

Lemma 19. The map π_F is a projection.

Proof. Let $f \in P_F^*(C^\infty(\mathcal{L}_F))$, that is $f = P_F^* g$, for a $g \in C^\infty(\mathcal{L}_F)$. We have:

$$\pi_F f = (P_F^* \circ I_F^*) P_F^* g = P_F^*(I_F^* \circ P_F^*) g = P_F^* g = f. \tag{D.1}$$

□

We should study how π_F acts on $\{f, g\}$, for $f, g \in C^\infty(\mathcal{M})$ and verify that it satisfies the Voronov's distributive law:

$$\pi_F\{f, g\} = \pi_F\{\pi_F f, g\} + \pi_F\{f, \pi_F g\} \tag{D.2}$$

Example 24. Consider a trivial shifted cotangent bundle $\mathcal{M} = T^*[1]V$ and let $\omega = \delta c^a \wedge \delta x_a$, where x_a are degree-0 coordinates on V and c^a degree-1 coordinates on the odd fiber. If we choose the **vertical polarization**, we have:

$$\begin{aligned} P_F &: T^*[1]V \to V, \qquad & P_F(x, c) &= x \\ I_F &: V \to T^*[1]V, \qquad & I_F(x) &= (x, 0) \end{aligned} \tag{D.3}$$

Therefore $\pi_F f(x, c) = f(x, 0)$. Let's see if Voronov's law is verified. Recall that the ω-hamiltonian vector fields of f is $X_f = \frac{\partial f(x,c)}{\partial x_a} \frac{\partial}{\partial c^a} + \frac{\partial f(x,c)}{\partial c^a} \frac{\partial}{\partial x_a}$ and analogously for g. On the one hand we have:

$$\pi_F\{f, g\} = \pi_F(X_f(g)) = \frac{\partial f(x, 0)}{\partial x_a} \left[\frac{\partial g(x, c)}{\partial c^a}\right]_{c=0} + \left[\frac{\partial f(x, c)}{\partial c^a}\right]_{c=0} \frac{\partial g(x, 0)}{\partial x_a} \tag{D.4}$$

On the other hand:

$$\pi_F\{\pi_F f, g\} = \pi_F(X_{\pi_F f}(g)) = \frac{\partial f(x,0)}{\partial x^a}\left[\frac{\partial g(x,c)}{\partial c_a}\right]_{c=0}, \tag{D.5}$$

and

$$\pi_F\{f, \pi_F g\} = \pi_F(X_f(\pi_F g) = \left[\frac{\partial f(x,c)}{\partial c_a}\right]_{c=0}\frac{\partial g(x,0)}{\partial x^a}. \tag{D.6}$$

In this case, Voronov's distributive law is satisfied.
Consider now the **horizontal polarization**:

$$\begin{aligned} P_F : T^*[1]V \to V^*[1], \qquad & P_F(x,c) = c \\ I_F : V^*[1] \to T^*[1]V, \qquad & I_F(c) = (x_0, c), \end{aligned} \tag{D.7}$$

Now we have $\pi_F f(x,c) = f(x_0, c)$. In this case:

$$\begin{aligned} \pi_F\{f,g\} &= \left[\frac{\partial f(x,c)}{\partial x_a}\right]_{x=x_0}\frac{\partial g(x_0,c)}{\partial c^a} + \frac{\partial f(x_0,c)}{\partial c^a}\left[\frac{\partial g(x,c)}{\partial x_a}\right]_{x=x_0} \\ \pi_F\{\pi_F f, g\} &= \frac{\partial f(x_0,c)}{\partial c^a}\left[\frac{\partial g(x,c)}{\partial x_a}\right]_{x=x_0} \\ \pi_F\{f, \pi_F g\} &= \left[\frac{\partial f(x,c)}{\partial x_a}\right]_{x=x_0}\frac{\partial g(x_0,c)}{\partial c^a} \end{aligned} \tag{D.8}$$

Therefore, Voronov's law is satisfied in this polarization too.

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