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Computation of Kontsevich Weights of Connection and Curvature Graphs for Symplectic Poisson Structures

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Abstract

The globalization of Kontsevich's local formula, as done in [11], presents us with a certain connection 1-form and its curvature 2-form, both of which given in terms of Kontsevich's formality map. In the main part of this thesis we use elementary methods, inspired by [31], to compute the Kontsevich weights of two families of graphs which appear in the above mentioned connection 1-form and its curvature 2-form in the case of a symplectic Poisson structure. We call the two families of graphs *connection* and *curvature graphs* accordingly.

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Contents

Introduction

In his famous paper [24], Kontsevich solved the problem of deformation quantization of Poisson manifolds. More precisely, he provided a formula for a star product in the local case and showed how this product can be globalized, thereby not only proving the existence of a deformation quantization for an arbitrary Poisson manifold, but at the same time giving a complete classification of deformation quantizations in terms of (equivalence classes) of formal Poisson structures. In [9] Cattaneo and Felder gave a quantum field theory interpretation of Kontsevich's quantization formula. More precisely, they showed that Kontsevich's local formula is given as a perturbative expansion of a path integral in the Poisson sigma model on a disk with three points marked on the boundary. It's well-known that the symmetries of the Poisson sigma model do not close off-shell. That's why the BRST formalism fails and one has to resort to a more general gauge fixing formalism to quantize the Poisson sigma model, the so-called BV-BFV formalism.

In the first chapter of this thesis we introduce the above mentioned BV-BFV formalism, which is a gauge fixing formalism for gauge theories on manifolds with boundary. We first treat the classical BV-BFV formalism as developed in [16], which, roughly speaking, connects the BV (Batalin-Vilkovisky) construction in the bulk with the BFV (Batalin-Fradkin-Vilkovisky) construction on the boundary. In a second step we then introduce the quantum BV-BFV formalism as presented in [17], which produces a quantum theory out of a classical BV-BFV theory. As an application we will then provide the perturbative quantization of so-called BF-like theories.
In a second chapter we start by introducing the AKSZ (Alexandrov-Kontsevich-Schwarz-Zaboronsky) construction as first presented in [1] and we will see that the resulting field theories are examples of BV-BFV theories on manifolds with boundary. We will then present the Poisson sigma model, which is a 2-dimensional example of an AKSZ theory. Following [14] we will then see how we can perturbatively quantize the Poisson sigma model globally by using methods of formal geometry.
In the third chapter we will introduce the concept of deformation quantization. After a short review of Poisson geometry we will define the notion of a star product on the deformed algebra of smooth functions on a manifold. Subsequently we will treat Kontsevich's results presented in [24]. In particular, we will discuss his formality map and see how it yields a star product in the local case. Following [11], we will then see how Kontsevich's local formula can be globalized. In this context one introduces a connection 1-form A and its curvature 2-form F, both given in terms of Kontsevich's formality map. Finally, we will also shortly mention how Kontsevich's quantization formula can be computed as the perturbative expansion of a correlation function in the Poisson sigma model on the disk.
In the fourth and main chapter of this thesis we will calculate the Kontsevich weights of three families of graphs using mainly Stokes' theorem as inspired by [31]. The three families of graphs are of course not randomly chosen, they are associated with Kontsevich's star product, the above mentioned connection 1-form and its curvature 2-form in the case of a symplectic Poisson manifold. For this reason we will call them *product graphs*, *connection graphs* and *curvature graphs* accordingly.
In the fifth and final chapter we will then use the Kontsevich weights computed in chapter 4 to find explicit expressions for a global (and in general nonassociative) bullet product, the connection 1-form A and its curvature 2-form F in the case of a symplectic Poisson structure. Finally, we will discuss the case of a cotangent bundle as a special case of a symplectic Poisson manifold and we will see how in that case the whole computation can be simplified by lifting a formal exponential map to the cotangent bundle.

1 The BV-BFV Formalism

In this chapter we will present the so-called BV-BFV formalism, which is a gauge fixing formalism for gauge theories on manifolds with boundary. After a short motivation we will first treat the BV-BFV formalism on a classical level before we introduce the quantum BV-BFV formalism. The whole section is mainly based on [17], but also on [14], [27], [16] and [13].

1.1 Motivation

A classical d-dimensional *Lagrangian field theory* associates to a d-manifold M a *space of fields* F_M, which is typically a space of sections on M, and an action functional $S_M : F_M \to \mathbb{R}$. Inspired by the Atiyah–Segal axioms, we want a (topological) *quantum field theory* to assign a so-called *partition function* ψ_M to the manifold M.
Now suppose that M is a closed manifold. In the path integral approach the partition function is constructed as

$$\psi_M = \int\limits_{F_M} e^{\frac{i}{\hbar}S_M(\phi)}\mathcal{D}\phi. \tag{1.1}$$

For this to work we must first make sense of the right-hand side of (1.1). Typically F_M is infinite-dimensional, so one cannot define $\mathcal{D}\phi$ measure theoretically. Instead, one defines the path integral as an asymptotic series in $\hbar \to 0$ by applying the stationary phase formula (see e.g. [27]). The main idea is that the fast oscillations cancel out except in the neighbourhood of critical points of the action functional S_M. This way one gets a series in $\hbar$ with coefficients given by sums of Feynman diagrams.
But in order to be able to apply the stationary phase formula, the critical points of the action functional must be isolated. For *gauge theories*, whose Lagrangians are invariant under certain local transformations, this is not the case (as described in [27], for gauge theories there is typically a tangential distribution ϵ on F_M which preserves the action, so the critical points come in ϵ-orbits and hence are not isolated).
The BV formalism offers a way out: The classical data (F_M, S_M) is replaced by a triple $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$, where $\mathcal{F}_M$ is a $\mathbb{Z}$-graded supermanifold, ω_M an odd symplectic form of degree -1 and $\mathcal{S}_M$ an even function of degree 0 on $\mathcal{F}_M$. One then replaces the path integral (1.1) by

$$\widehat{\psi}_M = \int\limits_{\mathcal{L}\subset\mathcal{F}_M} e^{\frac{i}{\hbar}\mathcal{S}_M}, \tag{1.2}$$

where $\mathcal{L}$ is a Lagrangian submanifold of $\mathcal{F}_M$. For an appropriate choice of $\mathcal{L}$ the right-hand side of (1.2) is then well-defined (as an asymptotic series using the stationary phase formula) and is invariant under deformations of $\mathcal{L}$. A choice of Lagrangian submanifold amounts to *gauge fixing* in the BV formalism.
For gauge theories on manifolds with boundary, the BV formalism is extended to the *BV-BFV formalism* developed in [16].

Remark 1.1. A short introduction to *supergeometry* is given in Appendix B. Furthermore, [28] provides an excellent and mathematically rigorous treatment of the perturbative expansion of path integrals near critical points of the action in terms of Feynman diagrams.

1.2 The Classical BV-BFV Formalism

In this section we give an outline of the classical BV-BFV formalism as presented in [17].

Definition 1.1. A *BV manifold* is a triple $(\mathcal{F}, \omega, \mathcal{S})$ where $\mathcal{F}$ is a supermanifold with an additional $\mathbb{Z}$-grading, ω an odd symplectic form of degree -1 and $\mathcal{S}$ an even function of degree 0 on $\mathcal{F}$ satisfying the *Classical Master Equation* (CME)

$$(\mathcal{S}, \mathcal{S}) = 0, \tag{1.3}$$

where $(\, , \,)$ denotes the odd Poisson bracket induced by ω.

Note that equivalently one can describe a BV manifold as a quadruple $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S})$ where $\mathcal{Q}$ is the Hamiltonian vector field of $\mathcal{S}$, given by

$$\iota_{\mathcal{Q}}\omega = \mathrm{d}\mathcal{S}, \tag{1.4}$$

and require it to satisfy $[\mathcal{Q}, \mathcal{Q}] = 2(\mathcal{Q})^2 = 0$.

Remark 1.2. In our case the $\mathbb{Z}_2$-grading from the supermanifold structure will always concide with the reduction of the $\mathbb{Z}$-grading modulo 2. We will refer to the $\mathbb{Z}$-grading as *ghost number* and denote it as $|\,.\,|$. And from now on we will call $\mathbb{Z}$-graded manifolds simply graded manifolds.

1.2.1 The BV Integral

Here we assume that the manifolds involved are finite-dimensional, orientable and equipped with a Berezinian measure μ.

Let $\mathcal{Y}$ be a graded supermanifold equipped with an odd symplectic form of ghost number -1. In [23] it was shown that the space of half-densities $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y})$ on $\mathcal{Y}$ can be equipped with an operator Δ of degree 1, given in local Darboux coordinates (x^i, y_i) on $\mathcal{Y}$ as $\sum_i \frac{\partial^2}{\partial x^i \partial y_i}$. This operator is called the *BV Laplacian*. In fact, one can show that $\Delta^2 = 0$.

Remark 1.3. If the Berezinian is compatible with the odd symplectic structure on $\mathcal{Y}$, i.e. if $\Delta\mu^{\frac{1}{2}} = 0$, where $\mu^{\frac{1}{2}} \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y})$ is the half-density corresponding to the Berezinian measure μ, then one can construct a μ-dependent BV-Laplacian Δ_μ on the space of smooth functions $\mathcal{C}^\infty(\mathcal{Y})$ on $\mathcal{Y}$ by setting $\mu^{\frac{1}{2}}\Delta_\mu f := \Delta(\mu^{\frac{1}{2}} f)$.

Now if we restrict a half-density on $\mathcal{Y}$ to a Lagrangian submanifold on $\mathcal{Y}$, we get a density which can be integrated. So let $\mathcal{L} \subset \mathcal{Y}$ be a Lagrangian submanifold. The BV integral is now defined as follows

$$\int_{\mathcal{L}} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}) \to \mathbb{C}, \quad \xi \mapsto \int_{\mathcal{L}} \xi := \int_{\mathcal{L}} \xi|_{\mathcal{L}}. \tag{1.5}$$

One of the main results is now the following (cf. Theorem 2.9, [17]):

Theorem 1.1 (Batalin-Vilkovisky-Schwarz; [4],[30])**.**

(i) Let ξ be a half-density on $\mathcal{Y}$ and $\mathcal{L} \subset \mathcal{Y}$ a Lagrangian submanifold. Assuming convergence of the integral we have that

$$\int_{\mathcal{L}} \Delta\xi = 0.$$

(ii) Let ξ be a half-density on $\mathcal{Y}$ satisfying $\Delta\xi = 0$ and let $(\mathcal{L}_t)$ be a smoothly varying family of Lagrangian submanifolds of $\mathcal{Y}$ parametrized by the real parameter t. Assuming that the integral $\int_{\mathcal{L}_t} \xi$ converges for each t, one has that

$$\frac{d}{dt}\int_{\mathcal{L}_t} \xi = 0.$$

As was already mentioned in section 1.1, a choice of Lagrangian submanifold corresponds to gauge fixing in the BV formalism. Part (ii) of the theorem above then tells us that for a half-density ξ satisfying $\Delta\xi = 0$ the BV integral is independent of the choice of gauge fixing.
Now suppose that ξ is a half-density satisfying $\Delta\xi = 0$ and $(\mathcal{L}_t)$ is a smoothly varying family of Lagrangian submanifolds for t in a neighbourhood of 0. Furthermore, suppose that the BV integral of ξ over $\mathcal{L}_0$ is ill-defined, whereas $\int_{\mathcal{L}_t} \xi$ is well-defined for all $t \neq 0$. Then part (ii) of the theorem above allows us to define

$$\int_{\mathcal{L}_0} \xi := \int_{\mathcal{L}_t} \xi$$

for some $t \neq 0$ in this neighbourhood of 0.
A case of special interest is the following: Consider the half-density $\xi = e^{\frac{i}{\hbar}\mathcal{S}}\mu^{\frac{1}{2}}$, where $\mathcal{S}$ is some even function on $\mathcal{Y}$ and μ is a compatible Berezinian. Then $\Delta\xi = 0$ if and only if

$$\frac{1}{2}(\mathcal{S},\mathcal{S}) - i\hbar\Delta_\mu\mathcal{S} = 0. \tag{1.6}$$

Equation (1.6) is called the *Quantum Master Equation* (QME). Assuming $\mathcal{S}$ can be expanded in $\hbar$, i.e. $\mathcal{S} = \mathcal{S}_0 + \hbar\mathcal{S}_1 + \hbar^2\mathcal{S}_2 + ...$, we immidiately see that the lowest order term of the QME is given by the CME $(\mathcal{S}_0,\mathcal{S}_0) = 0$, which motivates the definition of BV manifolds (cf. Definition 1.1).

Remark 1.4. In the setting of finite-dimensional integrals, which are defined measure theoretically, and for half-densities of the form $\xi = e^{\frac{i}{\hbar}\mathcal{S}}\mu^{\frac{1}{2}}$, one can consider the formal asymptotics for $\hbar \to 0$ of the BV integral given by the stationary phase formula (assuming $\mathcal{S}$ has only isolated critical points on $\mathcal{L}$) and get a formal power series in $\hbar$ with coefficients given by sums of Feynman diagrams. This approach translates to the infinite dimensonal case: In infinte dimensions one defines the BV integral of a half-density $\xi = e^{\frac{i}{\hbar}\mathcal{S}}\mu^{\frac{1}{2}}$ perturbatively, i.e. as a power series in $\hbar$ with coefficients given by sums of Feynman diagrams. Then, of course, theorems which hold in the measure theoretic setting have to be checked independently at the level of Feynman diagrams in the perturbative setting.

1.2.2 The BV Pushforward

Again, we assume that the manifolds involved are finite-dimensional, orientable and equipped with a Berezinian measure μ.

Now suppose that $(\mathcal{Y},\omega)$ is a direct sum of two graded supermanifolds $(\mathcal{Y}',\omega')$ and $(\mathcal{Y}'',\omega'')$ with odd symplectic structures ω' and ω'' of ghost number -1, i.e. $\mathcal{Y} = \mathcal{Y}' \oplus \mathcal{Y}''$ and $\omega = \omega' + \omega''$. Then the space of half-densities on $\mathcal{Y}$ is given as $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}')\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}'')$.

Remark 1.5. $\widehat{\otimes}$ denotes the topological tensor product, i.e. the unique tensor product such that $\mathcal{C}^\infty(\mathbb{R}^n)\widehat{\otimes}\mathcal{C}^\infty(\mathbb{R}^m) \cong \mathcal{C}^\infty(\mathbb{R}^{n+m})$.

Now let $\mathcal{L}'' \subset \mathcal{Y}''$ be a Lagrangian submanifold. BV integration over the second factor then defines a pushforward map, called the *BV pushforward*, as follows

$$\int_{\mathcal{L}''} : \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}) \to \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}'), \quad \xi = \xi' \otimes \xi'' \mapsto \int_{\mathcal{L}''} \xi := \xi' \otimes \int_{\mathcal{L}''} \xi''. \tag{1.7}$$

Similar to Theorem 1.1 we then have the following

Theorem 1.2. Assuming convergence of all the integrals we have:

(i) Let ξ be a half-density on $\mathcal{Y}$ and $\mathcal{L}'' \subset \mathcal{Y}''$ a Lagrangian submanifold. Then

$$\int_{\mathcal{L}''} \Delta\xi = \Delta' \int_{\mathcal{L}''} \xi,$$

where Δ' is the BV Laplacian on $\mathcal{Y}'$.

(ii) Let ξ be a half-density on $\mathcal{Y}$ satisfying $\Delta\xi = 0$ and let $(\mathcal{L}''_t)$ be a smoothly varying family of Lagrangian submanifolds of $\mathcal{Y}''$ parametrized by the real parameter t. Then

$$\frac{d}{dt}\int_{\mathcal{L}''_t} \xi = \Delta'\zeta_t,$$

where ζ_t is some half-density on $\mathcal{Y}'$ depending on the parameter t.

For a proof of the above theorem we refer to Theorem 2.9, [17].

Remark 1.6. Theorem 1.2 tells us that the BV pushforward is actually a chain map and that it defines a pushforward from the cohomology of Δ to the cohomology of Δ'.

1.2.3 BV-BFV Theories

Definition 1.2. A d-dimensional *BV theory* is the association of a BV manifold $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ to each closed d-manifold M.

Remark 1.7. In the field theoretic setting the graded manifold $\mathcal{F}$ is usually called the *space of fields* and is typically infinte-dimensional. In particular one then has to define the BV integral of exponential half-densities perturbatively, as already mentioned in Remark 1.4.

Now if we want to extend the gauge fixing formalism to manifolds with boundary we have to introduce the so-called *BFV formalism*.

Definition 1.3. A *BFV manifold* is a triple $(\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial)$ where $\mathcal{F}^\partial$ is a graded supermanifold, ω^∂ an even symplectic form of degree 0 and $\mathcal{S}^\partial$ an odd function of degree 1 on $\mathcal{F}^\partial$ satisfying the CME $(\mathcal{S}^\partial, \mathcal{S}^\partial) = 0$, where $(\ ,\ )$ is the Poisson bracket induced by ω^∂. The BFV manifold is called *exact* if there is a 1-form α^∂ such that $\omega^\partial = \delta\alpha^\partial$.

In the definiton above δ denotes the de Rham differential on the (usually infinite-dimensional) space of fields $\mathcal{F}^\partial$.
In [16] the classical BV framework was extended to manifolds with boundary. This was done by connecting the BV construction in the bulk with the BFV construction on the boundary. This leads us to the following:

Definition 1.4. A *BV-BFV manifold* over a given exact BFV manifold $(\mathcal{F}^\partial, \omega^\partial = \delta\alpha^\partial, \mathcal{S}^\partial)$ is a quintuple $(\mathcal{F}, \omega, \mathcal{S}, \mathcal{Q}, \pi)$ where $\mathcal{F}$ is a graded supermanifold, ω is an odd symplectic form of ghost number -1 on $\mathcal{F}$, $\mathcal{S}$ is an even function of ghost number 0 on $\mathcal{F}$, $\mathcal{Q}$ is a smooth vector field on $\mathcal{F}$ of degree 1 satisfying $[\mathcal{Q}, \mathcal{Q}] = 2(\mathcal{Q})^2 = 0$ and $\pi : \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion such that

$$\iota_{\mathcal{Q}}\omega = \delta S + \pi^*\alpha^\partial \tag{1.8}$$

and

$$\delta\pi\mathcal{Q} = \mathcal{Q}^\partial, \tag{1.9}$$

where $\mathcal{Q}^\partial$ is the Hamiltonian vector field of $\mathcal{S}^\partial$ and $\delta\pi$ denotes the differential of π.

It immidiately follows from the definition above that $(\mathcal{F}, \omega, \mathcal{S})$ is a BV manifold if $\mathcal{F}^\partial$ is a point. Furthermore, the conditions in Definition 1.4 above imply the following equation

$$\mathcal{Q}(\mathcal{S}) = \pi^*(2\mathcal{S}^\partial - \iota_{\mathcal{Q}^\partial}\alpha^\partial), \tag{1.10}$$

which reduces to the CME $(\mathcal{S}, \mathcal{S}) = 0$ if $\mathcal{F}^\partial$ is a point. The above equation can equivalently be written as

$$\iota_{\mathcal{Q}}\iota_{\mathcal{Q}}\omega = 2\pi^*\mathcal{S}^\partial, \tag{1.11}$$

which we call the *modified Classical Master Equation* (mCME).

Finally, let us generalize the notion of a BV theory:

Definition 1.5. A d-dimensional exact *BV-BFV theory* is the association of an exact BFV manifold $(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma = \delta\alpha^\partial_\Sigma, \mathcal{S}^\partial_\Sigma)$ to every closed $(d-1)$-manifold Σ and the association of a BV-BFV manifold $(\mathcal{F}_M, \omega_M, \mathcal{S}_M, \mathcal{Q}_M, \pi_M)$ over the exact BFV manifold $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M} = \delta\alpha^\partial_{\partial M}, \mathcal{S}^\partial_{\partial M})$ to every compact d-manifold M with boundary ∂M.

1.3 The Quantum BV-BFV Formalism

1.3.1 Geometric Quantization

Geometric quantization is a mathematical procedure, which, using the structure of symplectic manifolds, produces a quantum theory from a given classical theory. Here we only summarize the most important steps, for a more detailed treatment see e.g. [8], [5] or [19].

Prequantization: We start with a classical phase space, i.e. a symplectic manifold (M,ω). Under the condition that the integral of $\frac{\omega}{2\pi\hbar}$ over any closed surface is an integer (so-called *Weil integrality condition*), there exists a Hermitian line bundle $B \to M$, called the *prequantum line bundle*, equipped with a connection ∇ whose curvature 2-form is $\frac{\omega}{\hbar}$. The Hilbert space H_0 of square-integrable sections of B is called the *prequantum Hilber space*. It's not yet the Hilbert space of our quantized theory, it's still too big. But we do have a prequantization map $\mathcal{Q}: f \mapsto A_f := -i\hbar\nabla_{X_f} + f$ sending a classical observable $f \in \mathcal{C}^\infty(M)$ to an operator on H_0 satisfying $[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f,g\})$.

Polarization: In general, a choice of polarization is (locally) a choice of decomposition of the coordinates on M into "canonical coordinates" and "canonical momenta". Formally, a polarization $\mathcal{P} \subset TM \otimes \mathbb{C}$ is a complex distribution such that

- $\mathcal{P}_x \subset T_xM \otimes \mathbb{C}$ is a (complex) Lagrangian subspace for each $x \in M$,
- $\mathcal{P}$ is integrable,
- $\mathcal{P} \cap \overline{\mathcal{P}}$ has constant dimension throughout M, where $\overline{\mathcal{P}}$ is the complex conjugation of $\mathcal{P}$.

The *quantum space of states* $H_{\mathcal{P}} \subset H_0$ associated to $\mathcal{P}$ is then defined to be the space of all square-integrable sections s of B which are covariantly constant in the direction of the polarization $\mathcal{P}$, i.e. such that $\nabla_X(s) = 0 \; \forall X \in \mathcal{P}$.

Half-form correction: In many cases there are still several problems to resolve before we have a suitable quantum Hilbert space. One approach is the so-called *half-form* or *metaplectic correction*. For simplicity let us assume that we have a real polarization $\mathcal{P}$, i.e. $\mathcal{P} = \overline{\mathcal{P}}$. If M admits a so-called *metaplectic structure*, one can construct a line-bundle of half-forms $\bigwedge^{1/2}\mathcal{P} \to M$ associated to $\mathcal{P}$, wich, loosely speaking, correspond to the square-root of volume forms on $\mathcal{P}$ (for a detailed construction of this bundle see [5]). One can then replace the Hermitian line bundle B with the tensor bundle $B \otimes \bigwedge^{1/2}\mathcal{P}$ and define the *quantum space of states* $\mathcal{H}_{\mathcal{P}}$ associated to $\mathcal{P}$ as the space of square-integrable sections of this bundle which are covariantly constant in the direction of $\mathcal{P}$. This space is then equipped with an inner product and hence is a pre-Hilbert space. One can then define the *quantum Hibert space* as the completion of this pre-Hilbert space. Finally, one defines the quantum map $\widehat{\mathcal{Q}}: f \mapsto \widehat{A}_f$, sending classical observables to operators on $\mathcal{H}_{\mathcal{P}}$, as follows: $\widehat{A}_f(s \otimes \nu) = A_f s \otimes \nu - i\hbar s \otimes L_{X_f}\nu$ for sections $s \otimes \nu$ in $\mathcal{H}_{\mathcal{P}}$, where s is a section of B, ν is a half-form and A_f is the image of f under the prequantum map $\mathcal{Q}$ defined above.

1.3.2 Perturbative Quantization of BV-BFV Theories

Let us start with the definition of a perturbative quantum BV-BFV theory as proposed in [17]:

Definition 1.6. Given a classical d-dimensional exact BV-BFV theory (cf. Definition 1.5), the corresponding d-dimensional *quantum BV-BFV theory* is the association of

- A graded vector space $\mathcal{H}^{\mathcal{P}}_\Sigma$, called the *space of boundary states*, to every closed $(d-1)$-manifold Σ. It's constructed as a geometric quantization of the symplectic manifold $\mathcal{F}^\partial_\Sigma$ with a choice of polarization $\mathcal{P}$ on $\mathcal{F}^\partial_\Sigma$.
- A coboundary operator $\Omega^{\mathcal{P}}_\Sigma$ on $\mathcal{H}^{\mathcal{P}}_\Sigma$, called the *quantum BFV operator*, to each closed $(d-1)$-manifold Σ. This coboundary operator is a quantization of $\mathcal{S}^\partial_\Sigma$.
- A finite-dimensional graded supermanifold $\mathcal{V}^{\mathcal{P}}_M$ equipped with an odd symplectic form of degree -1, called the *space of residual fields*, to each compact d-manifold M with (possibly empty) boundary ∂M. We then define the graded vector space $\widehat{\mathcal{H}}^{\mathcal{P}}_M := \mathcal{H}^{\mathcal{P}}_{\partial M} \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}^{\mathcal{P}}_M)$, called the *space of states*, equipped with the two commuting coboundary operators $\widehat{\Omega}^{\mathcal{P}}_M := \Omega^{\mathcal{P}}_{\partial M} \otimes \mathrm{Id}$ and $\widehat{\Delta}^{\mathcal{P}}_M := \mathrm{Id} \otimes \Delta_{\mathcal{V}^{\mathcal{P}}_M}$, where $\Delta_{\mathcal{V}^{\mathcal{P}}_M}$ is the canonical BV Laplacian on half-densities on $\mathcal{V}^{\mathcal{P}}_M$.
- A state $\widehat{\psi}_M \in \widehat{\mathcal{H}}^{\mathcal{P}}_M$ to each compact d-manifold M which satisfies the so-called *modified Quantum Master Equation* (mQME)

$$(\hbar^2\widehat{\Delta}^{\mathcal{P}}_M + \widehat{\Omega}^{\mathcal{P}}_M)\widehat{\psi}_M = 0. \tag{1.12}$$

The goal now is to give a rough outline of the perturbative quantization scheme described in [17], producing a quantum BV-BFV theory out of the data of a classical BV-BFV theory. It should be noted that in infinite dimensions the reasoning is formal and needs to be checked in each concrete case.
So we start with a classical d-dimensional exact BV-BFV theory and a compact d-manifold M. Let us assume that we have a (real) polarization $\mathcal{P}$ on $\mathcal{F}^\partial_{\partial M}$ induced by a Lagrangian foliation with smooth leaf space $\mathcal{B}^\mathcal{P}_{\partial M}$, and with the property that $\alpha^\partial_{\partial M}$ restricted to the fibers of $\mathcal{P}$ vanishes. This then allows us to identify $\mathcal{H}^\mathcal{P}_{\partial M}$ with the space of half-densities $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^\mathcal{P}_{\partial M})$ (cf. geometric quantization in section 1.3.1). Furthermore, let us assume that we have a splitting of $\mathcal{F}_M \to \mathcal{B}^\mathcal{P}_{\partial M}$ given by $\mathcal{F}_M = \mathcal{B}^\mathcal{P}_{\partial M} \times \mathcal{Y}$ and that ω_M is a (weakly) nondegenerate 2-form on $\mathcal{Y}$ extended to the product $\mathcal{B}^\mathcal{P}_{\partial M} \times \mathcal{Y}$ (i.e. ω_M is constant over the base $\mathcal{B}^\mathcal{P}_{\partial M}$). Formally we may think of $\mathcal{B}^\mathcal{P}_{\partial M}$ as the space of *boundary fields* and of $\mathcal{Y}$ as the space of *bulk fields*. Using this assumption and the fact that the BV-BFV manifold satisfies equation (1.10), we find that

$$\frac{1}{2}(\mathcal{S}_M, \mathcal{S}_M) = \pi^*_M \mathcal{S}^\partial_{\partial M}, \tag{1.13}$$

where (,) is the Poisson bracket induced by the odd symplectic structure on $\mathcal{Y}$. This last equation is the fiberwise version of the mCME. Now let us quantize the boundary action $\mathcal{S}^\partial_{\partial M}$ as follows: For boundary fields $(b, p) \in \mathcal{F}^\partial_{\partial M}$, where $b \in \mathcal{B}^\mathcal{P}_{\partial M}$, let us replace p by $-i\hbar\frac{\delta}{\delta b}$ in $\mathcal{S}^\partial_{\partial M}$, i.e. set

$$\Omega^\mathcal{P}_{\partial M} := \mathcal{S}^\partial_{\partial M}\left(b, -i\hbar\frac{\delta}{\delta b}\right). \tag{1.14}$$

This is the so-called *standard quantization* of $\mathcal{S}^\partial_{\partial M}$. One then finds that

$$\Omega^\mathcal{P}_{\partial M} e^{\frac{i}{\hbar}\mathcal{S}_M} = \pi^*_M \mathcal{S}^\partial_{\partial M} \cdot e^{\frac{i}{\hbar}\mathcal{S}_M}.$$

Now if we assume that we have a Berezinian measure on $\mathcal{Y}$ which is compatible with the odd symplectic structure on $\mathcal{Y}$ we can construct a BV Laplacian Δ on functions on $\mathcal{Y}$ as explained in Remark 1.3. Then if $\Delta\mathcal{S}_M = 0$ we get the mQME for the exponential of the action:

$$(\hbar^2\Delta + \Omega^\mathcal{P}_{\partial M})e^{\frac{i}{\hbar}\mathcal{S}_M} = 0.$$

Remark 1.8. If $\Delta\mathcal{S}_M \neq 0$ one can define a new boundary action $\widetilde{\mathcal{S}}^\partial_{\partial M} = \mathcal{S}^\partial_{\partial M} + \mathcal{O}(\hbar)$ via $\pi^*_M\widetilde{\mathcal{S}}^\partial_{\partial M} = \frac{1}{2}(\mathcal{S}_M, \mathcal{S}_M) - i\hbar\Delta\mathcal{S}_M$. Then, after setting $\Omega^\mathcal{P}_{\partial M}$ to be the standard quantization of $\widetilde{\mathcal{S}}^\partial_{\partial M}$, the mQME is again satisifed. As a consequence $\Omega^\mathcal{P}_{\partial M}$ is a quantization of $\mathcal{S}^\partial_{\partial M}$, but not necessarily the standard quantization.

Remark 1.9. For local field theories the mQME is formal and needs a regularization. But then $\Delta\mathcal{S}_M = 0$ may not be compatible with the regularization, and hence one needs to fall back on Remark 1.8.

Remark 1.10. One says that a theory is *anomaly free* if $(\Omega^\mathcal{P}_{\partial M})^2 = 0$. This then allows us to interpret the physical states as the cohomology in degree zero of the coboundary operator $\hbar^2\Delta + \Omega^\mathcal{P}_{\partial M}$.

Now let us go on with the state: Assume that we have a splitting $\mathcal{Y} = \mathcal{V}^\mathcal{P}_M \times \mathcal{Y}'$, where $\mathcal{V}^\mathcal{P}_M$ is the so-called *space of residual fields*, which we assume to be finite-dimensional and which corresponds to the low energy fields, and $\mathcal{Y}'$ is the so-called *space of fluctuation fields*. In that case we have $\mathcal{F}_M = \mathcal{B}^\mathcal{P}_{\partial M} \times \mathcal{V}^\mathcal{P}_M \times \mathcal{Y}'$. Now define the bundle of residual fields $Z_M = \mathcal{B}^\mathcal{P}_{\partial M} \times \mathcal{V}^\mathcal{P}_M$ over $\mathcal{B}^\mathcal{P}_{\partial M}$, set $\widehat{\mathcal{H}}^\mathcal{P}_M = \mathrm{Dens}^{\frac{1}{2}}(Z_M) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^\mathcal{P}_{\partial M})\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}^\mathcal{P}_M)$, and define the BV Laplacian $\widehat{\Delta}^\mathcal{P}_M = \mathrm{Id} \otimes \Delta_{\mathcal{V}^\mathcal{P}_M}$. Finally, choose a Lagrangian submanifold $\mathcal{L}' \subset \mathcal{Y}'$ (as explained before, choosing such a Lagrangian submanifold corresponds to gauge fixing, cf. section 1.2.1). Now we define the state (or partition function) as the formal BV pushforward

$$\widehat{\psi}_M = \int_{\mathcal{L}'} e^{\frac{i}{\hbar}\mathcal{S}_M} \in \widehat{\mathcal{H}}^\mathcal{P}_M. \tag{1.15}$$

In the infinite-dimensional case $\widehat{\psi}_M$ is computed perturbatively using Feynman diagrams, i.e. $\widehat{\psi}_M$ is understood as a power series in $\hbar$ with coefficients given by sums of Feynman diagrams as already

mentioned in Remark 1.4 (here we assume that for each $\phi \in Z_M$, the restriction of $\mathcal{S}_M$ to $\{\phi\} \times \mathcal{L}'$ has isolated critical points on $\{\phi\} \times \mathcal{L}'$ in order to be able to apply the stationary phase formula).
By the preceding discussion we see that we formally get the mQME

$$(\hbar^2 \widehat{\Delta}^{\mathcal{P}}_M + \widehat{\Omega}^{\mathcal{P}}_M)\widehat{\psi}_M = 0,$$

where $\widehat{\Omega}^{\mathcal{P}}_M = \Omega^{\mathcal{P}}_{\partial M} \otimes \mathrm{Id}$. But since we replace integration by computations with Feynman diagrams in infinite dimensions, this equation needs to be verified separately (contrary to the finite-dimensional case, where we know the equation to hold).

Remark 1.11. In many cases, the action functional is of the form $\mathcal{S}_M = \mathcal{S}_0 + \mathcal{S}_{\text{pert}}$ with free or quadratic part $\mathcal{S}_0$ and a (small) perturbation $\mathcal{S}_{\text{pert}}$. In that case $\mathcal{V}_M$ is the space of critical points of $\mathcal{S}_0$ relative to the boundary polarization $\mathcal{P}$ modulo symmetries.

Remark 1.12. In the construction above we assumed that we can introduce a global gauge fixing. More generally one can consider a family of local gauge fixings, parametrized by a choice x_0 of solution to the Euler-Lagrange equations, in a formal neighbourhood of x_0. This produces a family of "local states". More precisely we get a horizontal section of the vector bundle of local states over the base space of allowed x_0's with respect to a flat Grothendieck connection on the base. The global state is then this family of local states. We will come back to this later when we consider the quantization of the so-called Poisson sigma model.

Finally, let us have a look at the gluing of manfiolds: Suppose we have two compact d-manifolds M_1 and M_2 with a common boundary Σ, which is a $(d-1)$-manifold. Now let us glue those manifolds along their common boundary to produce a compact d-manifold M. We write this as

$$M = M_1 \cup_\Sigma M_2.$$

The state $\widehat{\psi}_M$ can then be obtained from the states $\widehat{\psi}_{M_1}$ and $\widehat{\psi}_{M_2}$ by the *gluing formula*

$$\widehat{\psi}_M = P_*(\widehat{\psi}_{M_1} *_\Sigma \widehat{\psi}_{M_2}), \tag{1.16}$$

where $*_\Sigma$ denotes the pairing of states in $\mathcal{H}^{\mathcal{P}}_\Sigma$ and P_* is the BV pushforward with respect to the odd symplectic fibration of residual fields $P : \mathcal{V}^{\mathcal{P}}_{M_1} \times \mathcal{V}^{\mathcal{P}}_{M_2} \to \mathcal{V}^{\mathcal{P}}_M$.

Remark 1.13. The gluing formula allows us to cut a manifold into simple pieces, then to compute the partition functions on those pieces and finally to assemble them to get the partition function on the entire manifold.

1.3.3 Quantization of BF-like Theories

Now we want to apply the quantization scheme outlined in section 1.3.2 to a class of topological field theories called *BF-like theories*. Again we are mainly following [17], but also [14].

Let us start with the necessary definitions:

Definition 1.7. A d-dimensional *abelian BF theory* with shift k associates to a compact d-manifold M the triple $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ with space of fields $\mathcal{F}_M = \Omega^\bullet(M)[k] \oplus \Omega^\bullet(M)[d-k-1]$ with elements the fields $(X, \eta) \in \mathcal{F}_M$, with odd symplectic form

$$\omega_M = \int_M \delta\eta \wedge \delta X \tag{1.17}$$

of degree or ghost number -1, and with even action functional

$$\mathcal{S}_M = \int_M \eta \wedge \mathrm{d}X \tag{1.18}$$

of ghost number 0. As before, δ is the de Rham differential on $\mathcal{F}_M$ and d the de Rham differential on M. If M is closed then $\mathcal{S}_M$ satisfies the CME (1.3) and hence $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ is a BV manifold (cf. Definition 1.1).

Remark 1.14. If M has a nonempty boundary we additionally get the space of boundary fields with corresponding even symplectic form and odd boundary action by simply restricting everything to the boundary (in practice just replace ∂M with M in all the expressions above). The whole data then defines a BV-BFV theory (cf. Definition 1.5) as was shown in [16].

Definition 1.8. A d-dimensional *BF-like theory* associates to a compact d-manifold M the triple $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ with space of fields $\mathcal{F}_M = \big(\Omega^\bullet(M) \otimes V[1]\big) \oplus \big(\Omega^\bullet(M) \otimes V^*[d-2]\big)$, where V is an n-dimensional graded vector space with dual space V^*, with odd symplectic form

$$\omega_M = \int_M \langle \delta\eta \wedge \delta X \rangle \tag{1.19}$$

of degree -1 and with even action functional $\mathcal{S}_M = \mathcal{S}_{M,0} + \mathcal{S}_{M,\mathrm{pert}}$ of degree 0 with free or unperturbed part $\mathcal{S}_{M,0}$ given as a sum of copies of an abelian BF theory

$$\mathcal{S}_{M,0} = \int_M \langle \eta \wedge \mathrm{d}X \rangle, \tag{1.20}$$

where $\langle\,,\,\rangle$ is the canonical pairing between V and V^*, and with interaction part $\mathcal{S}_{M,\mathrm{pert}}$ given as the integral of a (density-valued) function $\mathcal{V}$ of the fields X and η, i.e.

$$\mathcal{S}_{M,\mathrm{pert}} = \int_M \mathcal{V}(X, \eta), \tag{1.21}$$

with the additional requirement that $\mathcal{V}$ depends on the fields but not on their derivatives. Finally, we demand that the action functional $\mathcal{S}_M$ satisfies the CME for M without boundary.

Remark 1.15. The notation $\langle \eta \wedge X \rangle$ is to be understood as follows: We canonically pair the vectors of V and its dual V^* and wedge the forms on M.

Remark 1.16. Suppose that M has nonempty boundary. Then the fact that $\mathcal{V}$ does not depend on the derivatives of the fields guarantees that the space of boundary fields is simply given by $\mathcal{F}^\partial_{\partial M} = (\Omega^\bullet(\partial M) \otimes V[1]) \oplus (\Omega^\bullet(\partial M) \otimes V^*[d-2])$ with symplectic structure $\omega^\partial_{\partial M} = \delta\alpha^\partial_{\partial M}$, where $\alpha^\partial_{\partial M} = (-1)^d \int_{\partial M} \langle \eta \wedge \delta X \rangle$.

The perturbative quantization of BF-like theories is done in great detail in [17]. Here we will only give a rough outline and summarize the most important results: Let us start with a BF-like theory and a compact d-manifold M. Remember that we assume a splitting

$$\mathcal{F}_M = \mathcal{B}^{\mathcal{P}}_{\partial M} \oplus \mathcal{V}^{\mathcal{P}}_M \oplus \mathcal{Y}' \tag{1.22}$$

of the space of fields.

Polarizations: Let us write the boundary ∂M as a disjoint union of two (possibly empty) parts $\partial_1 M$ and $\partial_2 M$, so that $\mathcal{F}^\partial_{\partial M} = \mathcal{F}^\partial_{\partial_1 M} \times \mathcal{F}^\partial_{\partial_2 M}$. On $\partial_1 M$ we choose the $\frac{\delta}{\delta\eta}$-polarization and identify the space of leaves of the associated foliation with $\mathcal{B}_1 := \Omega^\bullet(\partial_1 M) \otimes V[1]$, whose elements are the X-fields. On $\partial_2 M$ we choose the $\frac{\delta}{\delta X}$-polarization and identify the space of leaves of the associated foliation with $\mathcal{B}_2 := \Omega^\bullet(\partial_2 M) \otimes V^*[d-2]$, whose elements are the η-fields. The polarization $\mathcal{P}$ on $\mathcal{F}^\partial_{\partial M}$ is then given as the direct product of the two polarizations. This way we get $\mathcal{B}^{\mathcal{P}}_{\partial M} = \mathcal{B}_1 \times \mathcal{B}_2$. Finally, we will denote the X-fields by $\mathbb{X} \in \mathcal{B}_1$ and the η-fields by $\mathbb{E} \in \mathcal{B}_2$.

Remark 1.17. From now on we will denote the $\frac{\delta}{\delta\eta}$-polarization as the $\mathbb{X}$-representation and the $\frac{\delta}{\delta X}$-polarization as the $\mathbb{E}$-representation.

Residual fields: For BF-like theories the space of residual fields is given in terms of relative cohomology. More precisely, we have

$$\mathcal{V}^{\mathcal{P}}_M = \big(H^\bullet(M, \partial_1 M) \otimes V[1]\big) \oplus \big(H^\bullet(M, \partial_2 M) \otimes V^*[d-2]\big), \tag{1.23}$$

which is a finite-dimensional BV manifold.

The propagator: Using the splitting of the space of fields (1.22), let us write an element $(X, \eta) \in \mathcal{F}_M$ as

$$\begin{aligned} X &= \mathbb{X} + \mathtt{x} + \mathcal{X} \\ \eta &= \mathbb{E} + \mathtt{e} + \mathcal{E} \end{aligned} \tag{1.24}$$

with $\mathbb{X}$, $\mathbb{E} \in \mathcal{B}^{\mathcal{P}}_{\partial M}$ the boundary fields, $\mathtt{x}$, $\mathtt{e} \in \mathcal{V}^{\mathcal{P}}_M$ the residual fields and $\mathcal{X}$, $\mathcal{E} \in \mathcal{Y}'$ the fluctuation fields. The propagator, denoted as ζ, is determined by the free part $\mathcal{S}_{M,0}$ of the action, or, more precisely, by $\widehat{\mathcal{S}}_{M,0} = \int_M \langle \mathcal{E} \wedge d\mathcal{X} \rangle$. One way of constructing the propagator is by using Hodge theory on manifolds with boundary as is done in [17]. This Hodge theoretic propagator is the integral kernel of an integral operator, which is related to the gauge fixing Lagrangian $\mathcal{L}' \subset \mathcal{Y}'$. So in particular the propagator depends on the choice of gauge fixing.
Now the propagator ζ constructed this way restricts to a smooth $(d-1)$-form away from the diagonal of $M \times M$. So if we define the open *configuaration space*

$$\mathrm{Conf}_2(M) := \{(x_1, x_2) \in M^2 | \, x_1 \neq x_2\},$$

then the propagator restricts to a smooth form on $\mathrm{Conf}_2(M)$. Now actually we will work with compactified versions of configuration spaces. More precisely with the so-called *FMAS compactification* of configuration spaces due to Fulton/MacPherson [21] and Axelrod/Singer [2]. Let us denote the FMAS compactification of $\mathrm{Conf}_2(M)$ by $\overline{C}_2(M)$. Then $\overline{C}_2(M)$ is a compact smooth manifold with corners. Moreover, the propagator ζ extends to $\overline{C}_2(M)$ as a smooth $(d-1)$-form.

The quantum state: Recall that we started with a BF-like theory and a compact d-manifold M, we assumed a splitting (1.22) of the space of fields $\mathcal{F}_M$ and we already have the polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M}$.

Definition 1.9. Given a gauge fixing Lagrangian $\mathcal{L}' \subset \mathcal{Y}'$, the *principal part of the state* is defined as the formal perturbative expansion of the BV pushforward

$$\widehat{\psi}_M = \int\limits_{\mathcal{L}' \subset \mathcal{Y}'} e^{\frac{i}{\hbar}\mathcal{S}_M} \in \widehat{\mathcal{H}}^{\mathcal{P}}_M \tag{1.25}$$

using the Feynman rules and diagrams of the given theory.

Hence the state takes the form

$$\widehat{\psi}_M(\mathbb{X}, \mathbb{E}; \mathtt{x}, \mathtt{e}) = \prod_{j=1}^{n} T_M^{(k_j)} \cdot \exp\left(\frac{i}{\hbar} \sum_{\Gamma} \frac{(-i\hbar)^{l(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \int\limits_{\overline{C}_\Gamma(M)} \omega_\Gamma(\mathbb{X}, \mathbb{E}; \mathtt{x}, \mathtt{e}) \right). \tag{1.26}$$

In the above expression, the sum is taken over all *connected* Feynman diagrams, i.e. over all connected oriented graphs Γ. $\mathrm{Aut}(\Gamma)$ is the set of all automorphisms of Γ, $l(\Gamma)$ denotes the number of loops of Γ and the coefficients $T_M^{(k_i)}$ are constants which are constructed using the so-called *Reidemeister torsion* (see [17] for more details on this). If Γ is a graph with r points in the bulk and $s = s_1 + s_2$ points on the boundary $\partial M = \partial_1 M \sqcup \partial_2 M$, then $\overline{C}_\Gamma(M)$ denotes the FMAS compactified configuration space of r points in the bulk and s points on the boundary. Finally, ω_Γ is a smooth form on $\overline{C}_\Gamma(M)$ which is constructed using the Feynman rules below.

The Feynman rules for BF-like theories, which are summarized in figure 1 below, are the following:

- We have $r \geq 0$ bulk vertices in M decorated by "vertex tensors"

$$\mathcal{V}^{j_1 \cdots j_k}_{i_1 \cdots i_l} := \frac{\partial^{k+l}}{\partial X^{i_1} \cdots \partial X^{i_l} \partial \eta_{j_1} \cdots \partial \eta_{j_k}} \mathcal{V}(X, \eta),$$

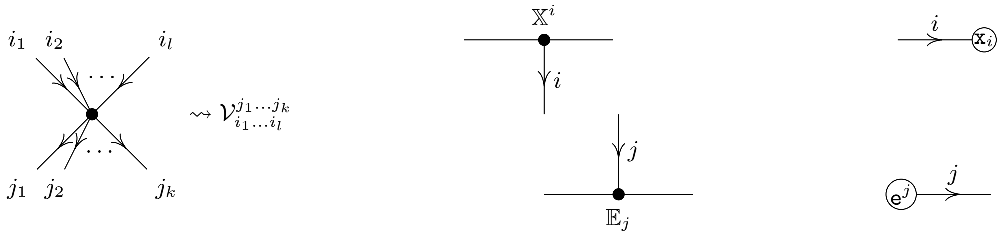
  where l is the in- and k the out-valency of the vertex.

- We have $s_1 \geq 0$ boundary vertices on $\partial_1 M$ with no outgoing half-edges and a single incoming half-edge decorated by $\mathbb{X}^i$ evaluated at the vertex location on $\partial_1 M$.

- We have $s_2 \geq 0$ boundary vertices on $\partial_2 M$ with no incoming half-edges and a single outgoing half-edge decorated by $\mathbb{E}_i$ evaluated at the vertex location on $\partial_2 M$.

- Edges are decorated with $\zeta \cdot \delta_i^j$, where ζ is the propagator induced by the chosen gauge fixing Lagrangian $\mathcal{L}' \subset \mathcal{Y}'$.
- Loose half-edges, sometimes called leaves, are allowed and are decorated with the residual fields $\mathtt{x}_i$ for out-orientation and $\mathtt{e}^i$ for in-orientation.

The differential form ω_Γ on the compactified configuration space $\overline{C}_\Gamma(M)$ is then the wedge product of the decorations above. Observe that ω_Γ is a polynomial in boundary and residual fields whose order is determined by the number of boundary vertices and leaves in Γ.



(a) Interaction vertex (b) Boundary vertices (c) Residual fields

Figure 1: Feynman rules and diagrams of a BF-like theory.

Now let us define the *effective action* $\mathcal{S}_M^{\text{eff}}$ as the sum

$$\mathcal{S}_M^{\text{eff}}(\mathbb{X},\mathbb{E};\mathtt{x},\mathtt{e}) := \sum_{\Gamma\text{ connected}} \frac{(-i\hbar)^{l(\Gamma)}}{|\text{Aut}(\Gamma)|} \int\limits_{\overline{C}_\Gamma(M)} \omega_\Gamma(\mathbb{X},\mathbb{E};\mathtt{x},\mathtt{e}) \tag{1.27}$$

over all *connected* Feynman graphs. Furthermore, let us define

$$T_M := \prod_{j=1}^{n} T_M^{(k_j)}. \tag{1.28}$$

Then we can write the state $\widehat{\psi}_M$ as

$$\widehat{\psi}_M(\mathbb{X},\mathbb{E};\mathtt{x},\mathtt{e}) = T_M \cdot e^{\frac{i}{\hbar}\mathcal{S}_M^{\text{eff}}(\mathbb{X},\mathbb{E};\mathtt{x},\mathtt{e})}. \tag{1.29}$$

There is also a gluing procedure for the principal part of the state, which allows to assemble the states on compact manifolds with a common boundary to a state on the glued manifold (cf. the gluing formula (1.16) in section 1.3.2). So the principal part of the state is enough for gluing purposes, but it may not satisfy the mQME (1.12). The reason is that the quantum BFV operator $\Omega_{\partial M}^{\mathcal{P}}$ generally contains higher functional derivatives, which need to be regularized. One way to do this is to introduce *composite fields* as higher powers of a boundary field $\mathbb{A}$ ($\mathbb{A} = \mathbb{X}$ or $\mathbb{E}$), written as $[\mathbb{A}^{i_1}\cdots\mathbb{A}^{i_p}]$ or simply as $[\mathbb{A}^I]$ for a multi-index $I = (i_1,...,i_p)$. We can then regard a higher functional derivative $\frac{\delta^p}{\delta\mathbb{A}^{i_1}\cdots\delta\mathbb{A}^{i_p}}$ as a first order functional derivative $\frac{\delta}{\delta[\mathbb{A}^{i_1}\cdots\mathbb{A}^{i_p}]}$ acting on the algebra generated by expressions of the form

$$\int\limits_{\overline{C}_{s_1}(\partial_1 M)\times\overline{C}_{s_2}(\partial_2 M)} L^{J_1\cdots J_{s_2}}_{I_1\cdots I_{s_1}} \wedge \pi_{1,1}^*[\mathbb{X}^{I_1}]\wedge\cdots\wedge\pi_{1,s_1}^*[\mathbb{X}^{I_{s_1}}]\wedge\pi_{2,1}^*[\mathbb{E}_{J_1}]\wedge\cdots\wedge\pi_{2,s_2}^*[\mathbb{E}_{J_{s_2}}], \tag{1.30}$$

where I_i, J_j are multi-indices, $L^{J_1\cdots J_{s_2}}_{I_1\cdots I_{s_1}}$ is a smooth differential form on $\overline{C}_{s_1}(\partial_1 M)\times\overline{C}_{s_2}(\partial_2 M)$ depending on the residual fields and $\pi_{i,l}$ denotes the projection onto the l-th component of the compactified configuration space $\overline{C}_{s_i}(\partial_i M)$ for $i = 1, 2$. The product, denoted by $\bullet$, of two expressions of the form (1.30) is obtained by adding all possible ways of restricting to a diagonal in the product of the configuration spaces and by combining the corresponding composite fields in a single bracket (for more details and examples of this product see [17]).

In terms of Feynman diagrams, the construction of composite fields and the corresponding first order functional derivatives amounts to introducing additional boundary vertices of higher valency as shown in figure 2 below.

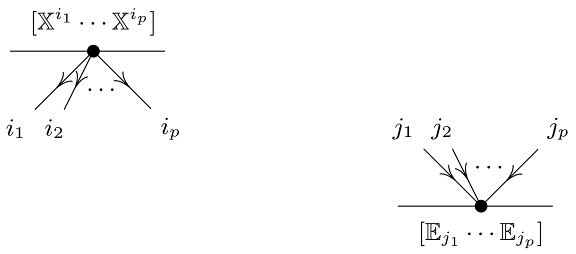


Figure 2: Composite field boundary vertices.

This finally leads us to the following

Definition 1.10. Given a gauge fixing Lagrangian $\mathcal{L}'\subset\mathcal{Y}'$, the *full quantum state* is defined as the formal perturbative expansion of the BV pushforward

$$\widehat{\boldsymbol{\psi}}_M=\int\limits_{\mathcal{L}'\subset\mathcal{Y}'} e^{\frac{i}{\hbar}\mathcal{S}_M}\in\widehat{\mathcal{H}}^{\mathcal{P}}_M \tag{1.31}$$

using the Feynman rules and diagrams of the given BF-like theory summarized in figure 1 and additionally with the boundary vertices of higher valency for composite fields as in figure 2 .

The boundary state space: We need a slight generalization of expressions of the form (1.30), where we also allow products of composite fields. So let us define *regular functionals* on the space of boundary fields as linear combinations of expressions of the form

$$\begin{aligned}\int\limits_{\overline{C}_{s_1}(\partial_1 M)\times\overline{C}_{s_2}(\partial_2 M)} & L^{J_1^1\cdots J_1^{n_1}\cdots J_{s_2}^1\cdots J_{s_2}^{n_{s_2}}}_{I_1^1\cdots I_1^{m_1}\cdots I_{s_1}^1\cdots I_{s_1}^{m_{s_1}}}\wedge\pi_{1,1}^*\prod_{i=1}^{m_1}[\mathbb{X}^{I_1^i}]\wedge\cdots\\ &\wedge\pi_{1,s_1}^*\prod_{i=1}^{m_{s_1}}[\mathbb{X}^{I_{s_1}^i}]\wedge\pi_{2,1}^*\prod_{j=1}^{n_1}[\mathbb{E}_{J_1^j}]\wedge\cdots\wedge\pi_{2,s_2}^*\prod_{j=1}^{n_{s_2}}[\mathbb{E}_{J_{s_2}^j}],\end{aligned} \tag{1.32}$$

where I_i^j, J_i^j are multi-indices and $L^{J_1^1\cdots J_1^{n_1}\cdots J_{s_2}^1\cdots J_{s_2}^{n_{s_2}}}_{I_1^1\cdots I_1^{m_1}\cdots I_{s_1}^1\cdots I_{s_1}^{m_{s_1}}}$ is a smooth differential form on $\overline{C}_{s_1}(\partial_1 M)\times\overline{C}_{s_2}(\partial_2 M)$ depending on the residual fields. Note that the bullet product $\bullet$ introduced above may be extended to regular functionals.

Definition 1.11. The *space of boundary states* $\mathcal{H}^{\mathcal{P}}_{\partial M}$ is the linear span of expressions of the form (1.32), i.e. it's the space of regular functionals.

The quantum BFV operator: The quantum BFV operator can be written as a sum

$$\boldsymbol{\Omega}_{\partial M}=\Omega_0+\boldsymbol{\Omega}_{\text{pert}}, \tag{1.33}$$

where Ω_0 is the standard quantization of the free part of the boundary action

$$\mathcal{S}^{\partial}_{\partial M,0}=\int\limits_{\partial M}\langle\mathbb{E}\,\overset{\wedge}{,}\,\mathrm{d}\mathbb{X}\rangle \tag{1.34}$$

and $\boldsymbol{\Omega}_{\text{pert}}$ is determined by boundary configuration integrals. The BFV operator is an element of the algebra of differential operators acting on the space of boundary states, which is generated by products of Ω_0 and so-called simple operators. For more details on Ω_0 and simple operators and how they act on regular functionals and, in particular, for more details on the BFV operator and how it is constructed I refer to [17].

Remark 1.18. The operator in (1.33) is constructed using composite fields introduced above and, in analogy to the state, it may be called the *full* quantum BFV operator. Then there is also the *principal part* of the BFV operator which can be defined without the notion of composite fields.

Finally, let us formulate the main result for BF-like theories:

Theorem 1.3 (Cattaneo-Mnev-Reshetikhin; [17])**.** Let M be a compact d-manifold. Then we have the following:

(i) The (full) quantum BFV operator squares to zero

$$(\boldsymbol{\Omega}_{\partial M})^2 = 0.$$

(ii) The full quantum state $\widehat{\boldsymbol{\psi}}_M$ satisfies the mQME

$$(\hbar^2 \Delta_{\mathcal{V}_M^{\mathcal{P}}} + \boldsymbol{\Omega}_{\partial M})\widehat{\boldsymbol{\psi}}_M = 0.$$

(iii) Under a change of gauge the state changes by an exact term:

$$\widehat{\boldsymbol{\psi}}_M \mapsto \widehat{\boldsymbol{\psi}}_M + (\hbar^2 \Delta_{\mathcal{V}_M^{\mathcal{P}}} + \boldsymbol{\Omega}_{\partial M})\phi,$$

where ϕ is an element of the space of states $\widehat{\mathcal{H}}_M^{\mathcal{P}}$.

2 Perturbative Quantization of the Poisson Sigma Model

In the first part of this chapter, which is mainly based on [16] and [27], we will introduce the AKSZ construction of topological field theories for compact manifolds (possibly with boundary) and we will see that the resulting theories, called AKSZ sigma models, are examples of BV-BFV theories. We will then introduce the so-called Poisson sigma model, which is a 2-dimensional example of an AKSZ theory.
In a second part we will then discuss the perturbative quantization of the Poisson sigma model as presented in [14] in a more general setting. In particular, we will have to linearize our theory around critical points of the action.

2.1 The Poisson Sigma Model as an AKSZ Theory

First introduced in [1], the AKSZ construction for closed manifolds - named after Alexandrov, Kontsevich, Schwarz and Zaboronsky - produces a solution of the CME (1.3) on the mapping space between two supermanifolds, called source and target manifold. In [16] the construction was extended to compact manifolds with boundary, producing a BV-BFV theory as in Definition 1.5.

2.1.1 The AKSZ Construction

Let us start with the following definiton:

Definition 2.1. Let $\mathcal{N}$ be a $\mathbb{Z}$-graded manifold. A smooth vector field $\mathcal{Q}$ of degree or ghost number 1 on $\mathcal{N}$ satisfying $[\mathcal{Q}, \mathcal{Q}] = 0$, where $[\,,\,]$ is the graded Lie bracket, is called a *cohomological vector field.* A graded manifold equipped with a cohomological vector field is called a *differential graded (dg) manifold.* A dg manifold $(\mathcal{N}, \mathcal{Q})$ equipped with a symplectic form ω of degree or ghost number k is called a *dg symplectic manifold* of degree k if ω is $\mathcal{Q}$-invariant, i.e. if $L_{\mathcal{Q}}\omega = 0$. Finally, a dg symplectic manifold $(\mathcal{N}, \omega, \mathcal{Q})$ of degree k is called a *Hamiltonian dg manifold* of degree k if there is a function H on $\mathcal{N}$ of ghost number $k + 1$ satisfying

$$\{H, H\} = 0, \tag{2.1}$$

where $\{\,,\,\}$ is the graded Poisson bracket induced by ω, and such that $\mathcal{Q}$ is the Hamiltonian vector field of H, i.e. $\iota_{\mathcal{Q}}\omega = \mathrm{d}H$. A Hamiltonian dg manifold $(\mathcal{N}, \omega, H)$ is called *exact* if the symplectic form ω is exact, i.e. if $\omega = \mathrm{d}\alpha$ for a primitive 1-form α.

The AKSZ construction requires source and target data:

- The *source manifold* is a shifted tangent bundle $T[1]M$ where M is a compact oriented d-manifold, possibly with boundary.
- The *target manifold* is an exact Hamiltonian dg manifold $(\mathcal{N}, \omega = \mathrm{d}\alpha, H)$ of degree $d - 1$.

The *space of fields* is then given by the mapping space

$$\mathcal{F}_M = \mathrm{Map}(T[1]M, \mathcal{N}) \tag{2.2}$$

between graded supermanifolds, which is itself a (usually infinite-dimensional) graded supermanifold.

Remark 2.1. Given graded supermanifolds $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ the mapping space $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ is characterized by the property

$$\mathrm{Mor}(\mathcal{L}, \mathrm{Map}(\mathcal{M}, \mathcal{N})) \cong \mathrm{Mor}(\mathcal{L} \times \mathcal{M}, \mathcal{N}), \tag{2.3}$$

where $\mathrm{Mor}(\,.\,,\,.\,)$ denotes the space of morphisms between graded supermanifolds. In the case where $\mathcal{N}$ is a graded vector space the mapping space is simply given by $\mathrm{Map}(\mathcal{M}, \mathcal{N}) = \mathcal{C}^\infty(\mathcal{M}) \otimes \mathcal{N}$.

The goal now is to lift geometric structures from source and target to the mapping space. Let us start with the cohomological structure: The target manifold $\mathcal{N}$ is a dg manifold with cohomological vector field $\mathcal{Q}$; the source manifold $\mathcal{M} := T[1]M$ is a dg manifold as well with cohomological vector field given by the de Rham differential d on M. As explained in [18], the groups of diffeomorphisms Diff($\mathcal{M}$) and Diff($\mathcal{N}$) act on the mapping space $\mathcal{F}_M$ by composition (from the right and left respectively). We can use the corresponding infinitesimal actions $L : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{F}_M)$ and $R : \mathfrak{X}(\mathcal{N}) \to \mathfrak{X}(\mathcal{F}_M)$ between the Lie algebras of vector fields to lift d and $\mathcal{Q}$ to the mapping space, with images denoted by d^L and $\mathcal{Q}^R$ respectively. Finally, we define the cohomological vector field on $\mathcal{F}_M$ (of ghost number 1) to be the sum of the two lifted vector fields, i.e. we set

$$\mathcal{Q}_M := \mathrm{d}^L + \mathcal{Q}^R. \tag{2.4}$$

Now let us continue with the symplectic structure: Consider the evaluation map $\mathrm{ev} : \mathcal{M} \times \mathcal{F}_M \to \mathcal{N}$ and let τ be an r-form on $\mathcal{N}$. Then the pullback $\mathrm{ev}^*\tau$ is an r-form on $\mathcal{M} \times \mathcal{F}_M$. Choose the bidegree $(0, r)$ component of $\mathrm{ev}^*\tau$, i.e. the component in $\Omega^0(\mathcal{M})\widehat{\otimes}\Omega^r(\mathcal{F}_M)$, and integrate the result with respect to the canonical Berezinian on $\mathcal{M}$. Let us denote this procedure as p_* (since it can be regarded as the pushforward of the projection map $p : \mathcal{M} \times \mathcal{F}_M \to \mathcal{F}_M$).

Remark 2.2. The *canonical Berezinian measure* on $\mathcal{M} = T[1]M$ is defined as

$$\int_{T[1]M} f := \int_M j(f), \tag{2.5}$$

where $j : \mathcal{C}^\infty(T[1]M) \to \Omega^\bullet(M)$ denotes the obvious isomorphism.

We now have a map

$$T := p_*\mathrm{ev}^* : \Omega^r(\mathcal{N}) \to \Omega^r(\mathcal{F}_M) \tag{2.6}$$

called the *transgression map*. This map leaves the form degree invariant, but reduces the ghost number of the forms by d (the dimension of the manifold M).
This transgression map can now be used to equip the mapping space $\mathcal{F}_M$ with a symplectic structure. We simply set

$$\omega_M := T(\omega), \tag{2.7}$$

where ω is the symplectic form of ghost number $d-1$ on the target. The symplectic form $\omega_M \in \Omega^2(\mathcal{F}_M)$ is then of ghost number $(d-1) - d = -1$.

Now that we have equipped the space of fields with a cohomological vector field and a symplectic structure, let us introduce the action functional: In the AKSZ construction the action functional is defined as

$$\mathcal{S}_M := \iota_{\mathrm{d}^L} T(\alpha) + T(H), \tag{2.8}$$

which is an element of $\mathcal{C}^\infty(\mathcal{F}_M)$ of ghost number 0. In the above expression α is the primitive 1-form on $\mathcal{N}$ satisfying $\mathrm{d}\alpha = \omega$.

Let us introduce local coordinates (x^a) on the target manifold $\mathcal{N}$ and local coordinates (u^i, θ^i) on the source manifold $\mathcal{M} = T[1]M$, where (u^i) are local coordinates on M and (θ^i) the corresponding degree 1 fiber coordinates. Now denote by X^a the composition of an element $X \in \mathcal{F}_M$, called a *superfield*, and the coordinate function x^a. This can then be written as

$$X^a(u, \theta) = \sum_{k=0}^{d} \sum_{1 \le i_1 < \cdots < i_k \le d} X^a_{i_1 \cdots i_k}(u)\theta^{i_1} \cdots \theta^{i_k}. \tag{2.9}$$

Remark 2.3. The coefficient functions $X^a_{i_1 \cdots i_k} \in \mathcal{C}^\infty(M)$ can be regarded as the coordinates on the space of fields $\mathcal{F}_M$ of degree or ghost number $|x^a| - k$, where $|x^a|$ is the degree of the corresponding coordinate on the target $\mathcal{N}$.

In local coordinates we can now write the symplectic form ω_M as

$$\omega_M = \frac{1}{2} \int_M \sum_{a,b} \omega_{ab}(X) \wedge \delta X^a \wedge \delta X^b, \tag{2.10}$$

where δ is the de Rham differential on the space of fields and ω_{ab} are the coefficient functions of the target symplectic form ω in these local coordinates (i.e. $\omega = \frac{1}{2} \sum_{a,b} \omega_{ab}(x) \mathrm{d}x^a \wedge \mathrm{d}x^b$).

Remark 2.4. We regard X^a as a smooth function on $\mathcal{F}_M$ with values in forms on M. Thus the integrand of (2.10) is a 2-form on the space of fields with values in forms on M. More precisely, we have that $X^a = \mathrm{ev}^* x^a \in \Omega^\bullet(M)\widehat{\otimes}\mathcal{C}^\infty(\mathcal{F}_M)$.

Similarly, we can write the action functional in local coordinates as

$$\mathcal{S}_M[X] = \int_M \left(\sum_a \alpha_a(X) \wedge \mathrm{d}X^a + H(X) \right), \tag{2.11}$$

where $\mathrm{d} = \sum_i \theta^i \frac{\partial}{\partial u^i}$ is the de Rham differential on M and α_a are the coefficient functions of the target primitive 1-form α in these coordinates. The action functional can be written as the sum of a "kinetic" and an "interaction" term: $\mathcal{S}_M = \mathcal{S}_M^{\mathrm{kin}} + \mathcal{S}_M^{\mathrm{int}}$ with $\mathcal{S}_M^{\mathrm{kin}}[X] = \int_M \sum_a \alpha_a(X) \wedge \mathrm{d}X^a$ and $\mathcal{S}_M^{\mathrm{int}}[X] = \int_M H(X)$. For completeness sake let us also write down the cohomological vector field on $\mathcal{F}_M$ in local coordinates:

$$\mathcal{Q}_M = \int_M \left(\mathrm{d}X^a + \omega^{ab}(X) \wedge \frac{\partial H}{\partial x^b}(X) \right) \wedge \frac{\delta}{\delta X^a}. \tag{2.12}$$

For the case without boundary we then have the following result (cf. Theorem 4.108, [13]):

Proposition 2.1. The AKSZ construction given above defines a d-dimensional BV theory, i.e. the quadruple $(\mathcal{F}_M, \omega_M, \mathcal{Q}_M, \mathcal{S}_M)$ is a BV manifold for each closed oriented d-manifold M.

Proof. First note that $\mathcal{Q}_M$ is clearly cohomological since it's the sum of the liftings of two cohomological vector fields (the liftings are automatically cohomological and anticommute with each other).
Now observe that the transgression map T is a chain map, i.e. $T\mathrm{d}\phi = (-1)^d \delta T\phi$ for each form ϕ on the target $\mathcal{N}$. Using this, it then immidiately follows that $\omega_M = T\omega$ is a closed 2-form on $\mathcal{F}_M$, i.e. $\delta\omega_M = \pm T\mathrm{d}\omega = 0$. The fact that ω_M is weakly nondegenerate follows from the fact that ω is nondegenerate and expression (2.10). So we can conclude that ω_M is an odd symplectic form on $\mathcal{F}_M$.
Now let us show that $\mathcal{Q}_M$ is the Hamiltonian vector field of $\mathcal{S}_M$:

$$\begin{aligned}\iota_{\mathcal{Q}_M}\omega_M &= \iota_{\mathcal{Q}_M} T\omega = \iota_{\mathrm{d}^L} T\omega + \iota_{\mathcal{Q}^R} T\omega = (-1)^d \iota_{\mathrm{d}^L}\delta T\alpha + T\iota_Q\omega \\ &= (-1)^d \underbrace{L_{\mathrm{d}^L} T\alpha}_{=0} + (-1)^d \delta\iota_{\mathrm{d}^L} T\alpha + (-1)^d \delta T H = (-1)^d \delta\mathcal{S}_M,\end{aligned}$$

where in the fourth step we used Cartan calculus and in the last step we used the fact that $L_{\mathrm{d}^L} T\alpha$ is the integral over M of a total derivative and thus vanishes (since by assumption M is closed).
Finally, since $\mathcal{Q}_M$ squares to zero and is the Hamiltonian vector field of $\mathcal{S}_M$ it follows that $(\mathcal{S}_M, \mathcal{S}_M) = 0$ □

Now suppose that M has nonempty boundary. The boundary ∂M is then in particular a closed oriented $(d-1)$-manifold an we can do the same construction as above but with M replaced by ∂M. So we get the space of boundary fields

$$\mathcal{F}^\partial_{\partial M} = \mathrm{Map}(T[1]\partial M, \mathcal{N}) \tag{2.13}$$

together with a surjective submersion $\pi_M : \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ which is simply given by the restriction of maps. Let us also denote the boundary fields by $X \in \mathcal{F}^\partial_{\partial M}$. In local coordinates the boundary cohomological vector field on $\mathcal{F}^\partial_{\partial M}$ is then given by

$$\mathcal{Q}^\partial_{\partial M} = \int_{\partial M} \left(\mathrm{d}X^a + \omega^{ab}(X) \wedge \frac{\partial H}{\partial x^b}(X) \right) \wedge \frac{\delta}{\delta X^a}. \tag{2.14}$$

Similarly, the boundary symplectic form is

$$\omega^\partial_{\partial M} = \frac{1}{2} \int_{\partial M} \sum_{a,b} \omega_{ab}(X) \wedge \delta X^a \wedge \delta X^b. \tag{2.15}$$

And since ∂M is now a $(d-1)$-manifold the ghost number of $\omega^{\partial}_{\partial M}$ is $(d-1)-(d-1)=0$.
And finally, the boundary action is locally given as

$$\mathcal{S}^{\partial}_{\partial M}[X] = \int\limits_{\partial M} \left(\sum_a \alpha_a(X) \wedge \mathrm{d}X^a + H(X) \right), \tag{2.16}$$

which is a smooth function on $\mathcal{F}^{\partial}_{\partial M}$ of ghost number 1.
Using Theorem 2.1 above (with M replaced by the closed manifold ∂M) and taking the ghost numbers of the boundary symplectic form and the boundary action into account, it immidiately follows that the quadruple $(\mathcal{F}^{\partial}_{\partial M}, \omega^{\partial}_{\partial M}, \mathcal{Q}^{\partial}_{\partial M}, \mathcal{S}^{\partial}_{\partial M})$ is a BFV manifold as in Definition 1.3. Moreover, we have that $\omega^{\partial}_{\partial M} = \delta\alpha^{\partial}_{\partial M}$ with

$$\alpha^{\partial}_{\partial M} = \int\limits_{\partial M} \sum_a \alpha_a(X) \wedge \delta X^a. \tag{2.17}$$

So the BFV manifold is exact. And actually it's not hard to see that this result holds more generally: The quadruple $(\mathcal{F}^{\partial}_{\Sigma}, \omega^{\partial}_{\Sigma}, \mathcal{Q}^{\partial}_{\Sigma}, \mathcal{S}^{\partial}_{\Sigma})$, which is obtained by simply replacing ∂M by Σ in the above expressions, is an exact BFV manifold for each closed $(d-1)$-manifold Σ.
Now we have the following result (cf. Proposition 6.3, [16]):

Lemma 2.1. The cohomological vector field $\mathcal{Q}_M$ is Hamiltonian up to a boundary term:

$$\iota_{\mathcal{Q}_M}\omega_M = (-1)^d \delta\mathcal{S}_M + \pi^*_M \alpha^{\partial}_{\partial M}. \tag{2.18}$$

Proof. Using Stokes' theorem we find that $L_{\mathrm{d}^L} T\alpha = (-1)^d \pi^*_M \alpha^{\partial}_{\partial M}$. Now similar to the proof of Proposition 2.1 above we have

$$\iota_{\mathcal{Q}_M}\omega_M = (-1)^d L_{\mathrm{d}^L} T\alpha + (-1)^d \delta\iota_{\mathrm{d}^L} T\alpha + (-1)^d \delta TH = \pi^*_M \alpha^{\partial}_{\partial M} + (-1)^d \delta\mathcal{S}_M$$

□

Lemma 2.1 and Proposition 2.1 finally allow us to conclude the following result:

Proposition 2.2. The AKSZ construction given above defines a d-dimensional exact BV-BFV theory, i.e. the quadruple $(\mathcal{F}^{\partial}_{\Sigma}, \omega^{\partial}_{\Sigma}, \mathcal{Q}^{\partial}_{\Sigma}, \mathcal{S}^{\partial}_{\Sigma})$ is an exact BFV manifold for each closed oriented $(d-1)$-manifold Σ and the quintuple $(\mathcal{F}_M, \omega_M, \mathcal{Q}_M, \mathcal{S}_M, \pi_M)$ is a BV-BFV manifold over the exact BFV manifold $(\mathcal{F}^{\partial}_{\partial M}, \omega^{\partial}_{\partial M}, \mathcal{Q}^{\partial}_{\partial M}, \mathcal{S}^{\partial}_{\partial M})$ for each compact oriented d-manifold M with boundary ∂M.

Now befor we go on and describe the Poisson sigma model, let us define a special type of AKSZ sigma models:

Definition 2.2. Given the source manifold $T[1]M$, where M is a compact oriented d-manifold, we call an AKSZ sigma model *split* if the target manifold is of the form

$$\mathcal{N} = T^*[d-1]N \tag{2.19}$$

with canonical symplectic structure, where N is a graded manifold.

2.1.2 The Poisson Sigma Model

Let M be a compact oriented 2-manifold (possibly with boundary) and (N, π) a Poisson manifold with Poisson bivector field $\pi \in \Gamma(\bigwedge^2 TN)$. In the classical picture the space of fields F_M of the Poisson sigma model is given by vector bundle maps $TM \to T^*N$ with elements $(X, \eta) \in F_M$, where $X : M \to N$ and $\eta \in \Omega^1(M, X^*T^*N)$, and the action functional is

$$S_M = \int\limits_M \left(\langle \eta \overset{\wedge}{,} \mathrm{d}X \rangle + \frac{1}{2} \langle \pi(X), \eta \wedge \eta \rangle \right), \tag{2.20}$$

where $\langle\,,\,\rangle$ is the canonical pairing of the tangent and cotangent space of N and d is the de Rham differential on M.

Remark 2.5. We have that $\eta \in \Omega^1(M, X^*T^*N)$ and $\mathrm{d}X \in \Omega^1(M, X^*TN)$. So the notation $\langle \eta \overset{\wedge}{,} \mathrm{d}X \rangle$ is understood as the canonical pairing of the tangent and cotangent fibers of N and the wedging of 1-forms on M. This then produces a number-valued 2-form on M which can then be integrated. The second term is simply understood as the canonical pairing of $\pi(X) \in \Gamma(M, \bigwedge^2 X^*TN)$ and $\eta \wedge \eta \in \Omega^2(M, \bigwedge^2 X^*T^*N)$.

In local coordinates on N introduced above, the action may be written as

$$S_M = \int\limits_M \sum_i \left(\eta_i \wedge \mathrm{d}X^i + \frac{1}{2} \sum_j \pi^{ij}(X) \eta_i \wedge \eta_j \right) \tag{2.21}$$

with the η_i's being 1-forms on M.
The Euler-Lagrange equations corresponding to the above action functional are given by

$$\begin{aligned} \mathrm{d}X + \langle \pi(X), \eta \rangle &= 0, \\ \mathrm{d}\eta + \frac{1}{2} \langle \partial \pi(X), \eta \wedge \eta \rangle &= 0 \end{aligned} \tag{2.22}$$

and the action functional is invariant under the following infinitesimal gauge transformations:

$$\begin{aligned} X &\mapsto X + \epsilon \langle \pi(X), b \rangle, \\ \eta &\mapsto \eta + \epsilon \big(\mathrm{d}b + \langle \partial \pi(X), \eta \wedge b \rangle \big), \end{aligned} \tag{2.23}$$

where $b \in \Gamma(M, X^*T^*N)$ is the generator of the gauge transformation and ϵ is an infinitesimal parameter.

Remark 2.6. The symmetries of the Poisson sigma model do not close off-shell. More precisely, the commutator of two transformations of the form (2.23) is in general not a transformation of the same type (it is, however, again a transformation of the same type on-shell, i.e. modulo the Euler-Lagrange equations). That's the reason why the BRST formalism fails and one has to resort to the BV formalism to quantize the Poisson sigma model.

Remark 2.7. The Poisson sigma model is connected to deformation quantization and Kontsevich's star product. More precisely, it was shown in [9] that the perturbative expansion of a path integral given in terms of the Poisson sigma model reproduces Kontsevich's formula. We will come back to this in section 3 below.

As was done in [10], this classical picture can be extended and the Poisson sigma model can be constructed as an AKSZ theory. To do this let M and (N, π) be as above. As the source manifold we have $\mathcal{M} := T[1]M$, as the target manifold $\mathcal{N} := T^*[1]N$. Let us introduce local coordinates (q^i, p_i) on $\mathcal{N}$ with coordinates (q^i) on N and (p_i) the corresponding degree 1 fiber coordinates. In these coordinates the Poisson bivector field can be written as $\pi = \frac{1}{2} \sum_{i,j} \pi^{ij}(q) \partial_i \wedge \partial_j$. Now $(\mathcal{N}, \omega = \mathrm{d}\alpha, H)$ is an exact Hamiltonian dg manifold of degree 1 with canonical symplectic form ω of ghost number 1 locally given as

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i, \tag{2.24}$$

with Liouville 1-form α given by

$$\alpha = \sum_i p_i \mathrm{d}q^i \tag{2.25}$$

and with Hamiltonian H locally described as

$$H = \frac{1}{2} \sum_{i,j} \pi^{ij}(q) p_i p_j. \tag{2.26}$$

The space of fields then is

$$\mathcal{F}_M = \mathrm{Map}(\mathcal{M}, \mathcal{N}) = \mathrm{Map}(T[1]M, T^*[1]N) \tag{2.27}$$

with elements $(\widetilde{X}, \widetilde{\eta}) \in \mathcal{F}_M$ with $\widetilde{X}$ a map $T[1]M \to N$ and $\widetilde{\eta}$ a section of $X^*T^*[1]N$.

Remark 2.8. Note that the Poisson sigma model is an example of a split AKSZ sigma model as in Definition 2.2.

The superfields $\widetilde{X}$ and $\widetilde{\eta}$ can be written in terms of homogeneous field components (fields and antifields, cf. Remark 2.9 below) as

$$\begin{aligned} \widetilde{X} &= X + \eta^+ + \beta^+, \\ \widetilde{\eta} &= \beta + \eta + X^+, \end{aligned} \tag{2.28}$$

with base map $X : M \to N$ of ghost number $\mathrm{gh}(X) = 0$ and with

$$\begin{aligned} \eta^+ &\in \Omega^1(M, X^*TN),\ \mathrm{gh}(\eta^+) = -1, \\ \beta^+ &\in \Omega^2(M, X^*TN),\ \mathrm{gh}(\beta^+) = -2, \\ \beta &\in \Omega^0(M, X^*T^*N),\ \mathrm{gh}(\beta) = 1, \\ \eta &\in \Omega^1(M, X^*T^*N),\ \mathrm{gh}(\eta) = 0, \\ X^+ &\in \Omega^2(M, X^*T^*N),\ \mathrm{gh}(X^+) = -1. \end{aligned} \tag{2.29}$$

Note that the sum of the form degree and the ghost number is 0 for all the components of $\widetilde{X}$ and 1 for all the components of $\widetilde{\eta}$.

Remark 2.9. Given a field ϕ, its *antifield* is the conjugate field with respect to the odd symplectic form and is denoted as ϕ^+. For the form degree (denoted as deg) of field and antifield we have that $\deg(\phi^+) = \dim(M) - \deg(\phi) = 2 - \deg(\phi)$, and for the ghost number we have that $\mathrm{gh}(\phi) + \mathrm{gh}(\phi^+) = -1$.

In local coordinates on $\mathcal{N}$ the odd symplectic form ω_M of ghost number -1 on $\mathcal{F}_M$ is given by

$$\omega_M = \int\limits_M \sum_i \delta\widetilde{\eta}_i \wedge \delta\widetilde{X}^i, \tag{2.30}$$

the cohomological vector field is

$$\mathcal{Q}_M = \int\limits_M \sum_i \left(\left(\mathrm{d}\widetilde{X}^i + \sum_j \pi^{ij}(\widetilde{X}) \wedge \widetilde{\eta}_j \right) \wedge \frac{\delta}{\delta \widetilde{X}^i} + \left(\mathrm{d}\widetilde{\eta}_i + \frac{1}{2} \sum_{j,k} \partial_i \pi^{jk}(\widetilde{X}) \wedge \widetilde{\eta}_j \wedge \widetilde{\eta}_k \right) \wedge \frac{\delta}{\delta \widetilde{\eta}_i} \right) \tag{2.31}$$

and the action functional is given as

$$\mathcal{S}_M[(\widetilde{X}, \widetilde{\eta})] = \int\limits_M \sum_i \left(\widetilde{\eta}_i \wedge \mathrm{d}\widetilde{X}^i + \frac{1}{2} \sum_j \pi^{ij}(\widetilde{X}) \wedge \widetilde{\eta}_i \wedge \widetilde{\eta}_j \right). \tag{2.32}$$

This may also be written in a coordinate independent way as

$$\mathcal{S}_M[(\widetilde{X}, \widetilde{\eta})] = \int\limits_M \left(\langle \widetilde{\eta} \overset{\wedge}{,} \mathrm{d}\widetilde{X} \rangle + \frac{1}{2} \langle \pi(\widetilde{X}), \widetilde{\eta} \wedge \widetilde{\eta} \rangle \right), \tag{2.33}$$

where $\langle\ ,\ \rangle$ is the canonical pairing of the tangent and cotangent space of N as before.

Remark 2.10. We will denote the kinetic term of the action of the Poisson sigma model as $\mathcal{S}_{0,M}$ and the interaction term as $\mathcal{S}_{\pi,M}$, where $\mathcal{S}_{0,M} = \int_M \langle \widetilde{\eta} \overset{\wedge}{,} \mathrm{d}\widetilde{X} \rangle$ and $\mathcal{S}_{\pi,M} = \int_M \frac{1}{2} \langle \pi(\widetilde{X}), \widetilde{\eta} \wedge \widetilde{\eta} \rangle$, so that we may write $\mathcal{S}_M = \mathcal{S}_{0,M} + \mathcal{S}_{\pi,M}$.

Now suppose that M has nonempty boundary ∂M. Let us also write down the boundary data: The space of boundary fields is simply

$$\mathcal{F}^\partial_{\partial M} = \mathrm{Map}(T[1]\partial M, T^*[1]N) \tag{2.34}$$

with surjective submersion $\pi_M : \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ given by the restriction of maps. Let us also denote the boundary fields by $\widetilde{X}$ and $\widetilde{\eta}$. In local coordinates on $\mathcal{N}$ we then have

$$\begin{aligned}
\alpha^\partial_{\partial M} &= \int_{\partial M} \sum_i \widetilde{\eta}_i \wedge \delta \widetilde{X}^i, \\
\omega^\partial_{\partial M} &= \int_{\partial M} \sum_i \delta\widetilde{\eta}_i \wedge \delta \widetilde{X}^i, \\
\mathcal{Q}^\partial_{\partial M} &= \int_{\partial M} \sum_i \left(\left(\mathrm{d}\widetilde{X}^i + \sum_j \pi^{ij}(\widetilde{X}) \wedge \widetilde{\eta}_j \right) \wedge \frac{\delta}{\delta \widetilde{X}^i} + \left(\mathrm{d}\widetilde{\eta}_i + \frac{1}{2} \sum_{j,k} \partial_i \pi^{jk}(\widetilde{X}) \wedge \widetilde{\eta}_j \wedge \widetilde{\eta}_k \right) \wedge \frac{\delta}{\delta \widetilde{\eta}_i} \right), \\
\mathcal{S}^\partial_{\partial M} &= \int_{\partial M} \sum_i \left(\widetilde{\eta}_i \wedge \mathrm{d}\widetilde{X}^i + \frac{1}{2} \sum_j \pi^{ij}(\widetilde{X}) \wedge \widetilde{\eta}_i \wedge \widetilde{\eta}_j \right).
\end{aligned} \tag{2.35}$$

Since the Poisson sigma model is an example of an AKSZ sigma model we then immidiately know that it defines an exact BV-BFV theory (cf. Proposition 2.2).

2.2 Perturbative Quantization of the Poisson Sigma Model

As is done in [14] for split AKSZ theories, we will now quantize the Poisson sigma model, which is an example of a split AKSZ sigma model. In order to do so we first need to linearize the theory. In particular this means replacing the space of fields with the formal neighbourhood of a constant field. This will then produce a BF-like theory, which allows us to apply the perturbative quantization scheme introduced in section 1.3.

2.2.1 Formal Geometry

Let us first introduce some notions of formal geometry which we will need for the linearization. For a more detailed treatment we refer to [6] and [11]. To keep the notation as short and clean as possible, we will make use of the Einstein summation convention in this section.

Formal Exponential Map: Let N be a smooth manifold and $U \subset TN$ an open neighbourhood of the zero section. A *generalized exponential map* is a smooth map $\phi : U \to N$, $(x, y) \mapsto \phi_x(y)$, satisfying $\phi_x(0) = x$ and $\mathrm{d}_y\phi_x(0) = Id_{T_xN}$ for all $x \in N$.
Choosing local coordinates (x^i) on N and (y^i) on the fiber of the tangent bundle, one can write explicitly

$$\phi^i_x(y) = x^i + y^i + \frac{1}{2}\phi^i_{x,jk} y^j y^k + \frac{1}{3!}\phi^i_{x,jkl} y^j y^k y^l + \dots. \tag{2.36}$$

Based on the above expansion we define an equivalence relation as follows: Two generalized exponential maps are equivalent if all their partial derivatives with respect to the y^i's evaluated at 0 coincide for all points x on the base N. A *formal exponential map* is an equivalence class of generalized exponential maps, which is completely characterized by the coefficients $\phi_{x,\bullet}$. It's worth noting that the coefficients $\phi^i_{x,jk}$ transform like the coefficients of a torsion-free connection. As explained in [11], this then allows one to construct a formal exponential map ϕ from a torsion-free connection Γ via the formal geodesic flow. In local coordinates this can be written as

$$\phi^i_x(y) = x^i + y^i - \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + \frac{1}{3!}\left(2\Gamma^i_{rj}(x)\Gamma^r_{kl}(x) - \partial_j \Gamma^i_{kl}(x)\right) y^j y^k y^l + \dots. \tag{2.37}$$

This construction in particular shows that formal exponential maps exist.
Now let ϕ be some representative of a formal exponential map and consider a function $f \in \mathcal{C}^\infty(N)$. Then $\phi^*_x f \in \mathcal{C}^\infty(U_x)$ and we can consider its Taylor expansion $T\phi^*_x f \in \widehat{S}T^*_x N$ in the y variables around $y = 0$. Here $\widehat{S}T^*_x N$ denotes the formal completion of the symmetric algebra $S^\bullet T^*_x N$ (i.e. the algebra of formal

power series). In local coordinates we can write

$$T\phi_x^* f(y) = f(x) + y^i \partial_i f(x) + \frac{1}{2} y^j y^k (\partial_j \partial_k f(x) + \phi^i_{x,jk} \partial_i f(x)) + \dots . \tag{2.38}$$

Note that actually $T\phi_x^* f$ does not depend on the choice of representative and hence is well defined for any formal exponential map. Finally, by varying x, we get a section $T\phi^* f \in \Gamma(\widehat{S}T^*N)$. So given a formal exponential map ϕ, we can associate a smooth section $T\phi^* f$ to any function $f \in \mathcal{C}^\infty(N)$.

Grothendieck Connection: For every section σ of $\widehat{S}T^*N$ we can define a section $R(\sigma)$ of $T^*N \otimes \widehat{S}T^*N$ by taking the Taylor expansion with respect to the y variables of $-\mathrm{d}_y \sigma \circ (\mathrm{d}_y \phi)^{-1} \circ \mathrm{d}_x \phi$. This gives us a $\mathcal{C}^\infty(N)$-linear map $R : \Gamma(\widehat{S}T^*N) \to \Gamma(T^*N \otimes \widehat{S}T^*N)$ and hence we get a connection $D_G = \mathrm{d}_x + R$ on $\widehat{S}T^*N$, the so-called *Grothendieck connection*. Note that we can regard R as a 1-form with values in $TN \otimes \widehat{S}T^*N$, i.e. $R \in \Gamma(T^*N \otimes TN \otimes \widehat{S}T^*N)$.
In local coordinates (x^i) on the base and (y^i) on the fiber we have $R = R_i dx^i$ and

$$R_i(x;y) = -\left(\left(\frac{\partial \phi_x}{\partial y}\right)^{-1}\right)^k_j \frac{\partial \phi_x^j}{\partial x^i} \frac{\partial}{\partial y^k} =: Y_i^k(x;y) \frac{\partial}{\partial y^k}. \tag{2.39}$$

Here $Y_i^k \in \Gamma(\widehat{S}T^*N)$ is a formal power series in the second argument and thus can be written as

$$Y_i^k(x;y) = \sum_{l=0}^{\infty} \sum_{i_1,\dots,i_l} Y^k_{i,i_1\cdots i_l}(x) y^{i_1} \cdots y^{i_l}. \tag{2.40}$$

One can show that the Grothendieck connection is flat or, equivalently, that the corresponding differential on the complex of $\widehat{S}T^*N$-valued differential forms, which we will also denote as D_G, squares to zero. This flatness condition can be expressed as the *Maurer-Cartan equation*

$$\mathrm{d}_x R + \frac{1}{2}[R,R] = 0, \tag{2.41}$$

where $[\ ,\ ]$ is the Lie bracket of vector fields.
An important result which characterizes the D_G-closed sections of $\widehat{S}T^*N$ is the following:

Proposition 2.3. A section σ of $\widehat{S}T^*N$ lies in the image of $T\phi^*$, i.e. is of the form $\sigma = T\phi^* f$ for some function $f \in \mathcal{C}^\infty(N)$, if and only if $\mathrm{d}_x \sigma + R(\sigma) = D_G \sigma = 0$.

Proof. We denote by ϕ a representative of the formal exponential map. First observe that $\mathrm{d}_x(\phi^* f) = \mathrm{d}f \circ \mathrm{d}_x \phi$ and $\mathrm{d}_y(\phi^* f) = \mathrm{d}f \circ \mathrm{d}_y \phi$. Now since ϕ is a generalized exponential map there exists an open neighbourhood $V \subset TN$ of the zero section on which $\mathrm{d}_y \phi$ is invertible. On V we then have

$$\mathrm{d}_x(\phi^* f) = \mathrm{d}_y(\phi^* f) \circ (\mathrm{d}_y \phi)^{-1} \circ \mathrm{d}_x \phi.$$

Now by taking the Taylor expansion with respect to the y variables on both sides we immidiately see that $\mathrm{d}_x \sigma + R(\sigma) = 0$ if $\sigma = T\phi^* f$.
Conversely, suppose that $\mathrm{d}_x \sigma + R(\sigma) = 0$. Define a function f by $f(x) := \sigma_x(0)$. Then it follows from (2.38) and the fact that $\mathrm{d}_x \sigma = -R(\sigma)$ that $\sigma = T\phi^* f$ □

It's not hard to see that the sections of the form $T\phi^* f$ for $f \in \mathcal{C}^\infty(N)$, i.e. the D_G-closed sections, form a subalgebra of the algebra $\Gamma(\widehat{S}T^*N)$. Furthermore, we note that D_G is a derivation, i.e. that $D_G(\sigma\tau) = (D_G\sigma)\tau + \sigma(D_G\tau)$. So it follows that the subalgebra of D_G-closed sections of $\widehat{S}T^*N$ can be identified with the algebra $\mathcal{C}^\infty(N)$.
A further result concerning the Grothendieck connection, a proof of which can be found in [6], is the following:

Proposition 2.4. The cohomology of the differential D_G is concentrated in degree 0 and $H^0_{D_G} = T\phi^* \mathcal{C}^\infty(N)$.

Formal Vertical Tensor Fields: We can now extend everything to so-called tensor bundles: A tensor bundle $E \to N$ is a bundle which is a tensor product or a symmetric or antisymmetric product of the tangent or cotangent bundle, or a direct sum thereof. Given a tensor bundle E, we define the associated *formal vertical bundle* $\widehat{E} := E \otimes \widehat{S}T^*N$. Then a formal exponential map ϕ induces an injective map

$$T\phi^* : \Gamma(E) \to \Gamma(\widehat{E}). \tag{2.42}$$

As an example, let us consider a bivector field π, i.e. let $E = \bigwedge^2 TN$. We then have that

$$(T\phi_x^* \pi)^{ij}(y) = \pi^{rs}(\phi_x(y)) \left(\left(\frac{\partial \phi_x}{\partial y} \right)^{-1} \right)_r^i \left(\left(\frac{\partial \phi_x}{\partial y} \right)^{-1} \right)_s^j. \tag{2.43}$$

If ϕ is determined by a connection as in (2.37), then we can write this as

$$(T\phi_x^* \pi)^{ij}(y) = \pi^{ij}(x) + \nabla_r \pi^{ij}(x) y^r + \left(\frac{1}{2} \nabla_r \nabla_s \pi^{ij}(x) - \frac{1}{6} R^{[i}_{rks}(x) \pi^{j]k}(x) \right) y^r y^s + \dots, \tag{2.44}$$

where the bracket [ ] denotes the antisymmetric part of the enclosed indices.

Remark 2.11. As observed in [11], if π is Poisson then so is $T\phi^*\pi$ since the pushforward preserves the Lie bracket.

Now we can let R act on $\Gamma(\widehat{E})$ by the Lie derivative. This way we again get a Grothendieck connection $D_G = \mathrm{d}_x + R$ on $\widehat{E}$ and hence a differential, also denoted as D_G, on the corresponding complex of differential forms. As before, we have that D_G is a flat connection (this can again be expressed as the Maurer-Cartan equation). Furthermore, we again have that a section $\sigma \in \Gamma(\widehat{E})$ lies in the image of $T\phi^*$ if and only if $\mathrm{d}_x \sigma + R(\sigma) = 0$. And finally, we also have that the cohomology of D_G is concentrated in degree zero and that $H^0_{D_G} = T\phi^*(\Gamma(E))$.

2.2.2 Linearization

Let us proceed with the linearization of the Poisson sigma model. In this section we mainly follow [11] and [6], but also [14]. So suppose now that we have a Poisson sigma model as introduced in section 2.1.2 with space of fields $\mathcal{F}_M = \mathrm{Map}(T[1]M, T^*[1]N)$, where M is a compact oriented 2-manifold, (N, π) is a Poisson manifold and $T^*[1]N$ is equipped with the canonical symplectic structure ω. Recall that the elements of the space of fields are given as $(\widetilde{X}, \widetilde{\eta}) \in \mathcal{F}_M$ with superfields $\widetilde{X}$ and $\widetilde{\eta}$, which can be written in terms of homogeneous field components as in (2.29), and the action functional is

$$\mathcal{S}_M[(\widetilde{X}, \widetilde{\eta})] = \mathcal{S}_{0,M}[(\widetilde{X}, \widetilde{\eta})] + \mathcal{S}_{\pi,M}[(\widetilde{X}, \widetilde{\eta})] = \int_M \left(\langle \widetilde{\eta} \overset{\wedge}{,} \mathrm{d}\widetilde{X} \rangle + \frac{1}{2} \langle \pi(\widetilde{X}), \widetilde{\eta} \wedge \widetilde{\eta} \rangle \right). \tag{2.45}$$

Now let us pick a critical point of the action $\mathcal{S}_M$ (i.e. a solution of the Euler-Lagrange equations) of the form $(\widetilde{X}, \widetilde{\eta}) \equiv (x, 0) \in \mathcal{F}_M$ where $x \in N$ (here x is the constant map with image $x \in N$, so we denote both the constant map as well as the point in N by x). The perturbative expansion around such a critical point only depends on a formal neighbourhood of it. For $x \in N$ let us define $\mathcal{F}_{M,x} := \mathrm{Map}(T[1]M, T^*[1]T_xN)$. The argument that now follows is from [11]: Let us denote the space of supermaps $T[1]M \to N$ as $\mathcal{X}$ and the space of supermaps $T[1]M \to T_xN$ as $\mathcal{Y}$. Note that we can regard the spaces of fields as the shifted cotangent bundles of the spaces of supermaps, i.e. $\mathcal{F}_M = T^*[1]\mathcal{X}$ and $\mathcal{F}_{M,x} = T^*[1]\mathcal{Y}$, with respective canonical symplectic structure. Now let us fix some neighbourhood $\mathcal{X}_x$ of the constant map $\widetilde{X} \equiv x$ in the space $\mathcal{X}$. For the perturbative expansion around a critical point $(\widetilde{X}, \widetilde{\eta}) \equiv (x, 0)$ it is enough to restrict the space of fields to $T^*[1]\mathcal{X}_x \subset \mathcal{F}_M$. Now let us also fix a neighbourhood $\mathcal{Y}^0_x$ of the zero map in the space $\mathcal{Y}$. Finally, choose some formal exponential map ϕ for N. Now we can define a map $\widetilde{\phi}_x : \mathcal{Y}^0_x \to \mathcal{X}_x$, $A \mapsto \widetilde{X}$, by setting

$$\widetilde{X} = \phi_x(A), \tag{2.46}$$

where the equation is understood pointwise. Note that $\widetilde{\phi}_x$ is a diffeomorphism for appropriately chosen neighbourhoods $\mathcal{X}_x$ and $\mathcal{Y}^0_x$ (since the formal exponential map ϕ is a local diffeomorphism). Hence we can extend this map canonically to the shifted cotangent bundles via pullback. More precisely, we get a map $\widetilde{\phi}_A : T^*[1]_A\mathcal{Y}_x^0 \to T^*[1]_{\widetilde{\phi}_x(A)}\mathcal{X}_x$, $B \mapsto \widetilde{\eta}$, by setting

$$\widetilde{\eta} = \big(\mathrm{d}\phi_x(A)\big)^{*,-1}B, \tag{2.47}$$

where again the equation is understood pointwise.
Finally, observe that the map $\widetilde{\phi} : T^*[1]\mathcal{Y}_x^0 \to T^*[1]\mathcal{X}_x$ is a symplectomorphism and (formally) unimodular. Hence the perturbative expansion of our Poisson sigma model around $(\widetilde{X},\widetilde{\eta}) \equiv (x,0)$ corresponds to the perturbative expansion around $(A,B) \equiv (0,0)$ in the new superfields $(A,B) \in T^*[1]\mathcal{Y}_x^0 \subset \mathcal{F}_{M,x}$.
Following the notation of the old superfields, let us write the new fields as $(\widehat{X},\widehat{\eta}) := (A,B)$. Then the action in the new fields is given by

$$\mathcal{S}_{M,x}[(\widehat{X},\widehat{\eta})] := (T\widetilde{\phi}_x^*\mathcal{S}_M)[(\widehat{X},\widehat{\eta})] = \int_M \left(\langle \widehat{\eta} \overset{\wedge}{,} \mathrm{d}\widehat{X}\rangle + \frac{1}{2}\langle T\phi_x^*\pi(\widehat{X}), \widehat{\eta}\wedge\widehat{\eta}\rangle\right), \tag{2.48}$$

which can again be written as the sum of a kinetic and an interaction term $\mathcal{S}_{M,x} = \mathcal{S}_{0,M,x} + \mathcal{S}_{\pi,M,x}$.
In local coordinates (q^i,p_i) on $T^*[1]T_xN$, with (q^i) the coordinates on T_xN and (p_i) the corresponding degree 1 fiber coordinates, the action functional can be written as

$$\mathcal{S}_{M,x}[(\widehat{X},\widehat{\eta})] = \int_M \sum_i \left(\widehat{\eta}_i \wedge \mathrm{d}\widehat{X}^i + \frac{1}{2}\sum_j (T\phi_x^*\pi)^{ij}(\widehat{X}) \wedge \widehat{\eta}_i \wedge \widehat{\eta}_j\right). \tag{2.49}$$

Now consider the triple $(\mathcal{F}_{M,x},\omega_{M,x},\mathcal{S}_{M,x})$ with odd symplectic form $\omega_{M,x} = \int_M \sum_i \delta\widehat{\eta}_i \wedge \delta\widehat{X}^i$. Note that it can be obtained from the AKSZ construction with target $T^*[1]T_xN$ and Hamiltonian function $H_x := \frac{1}{2}\sum_{i,j}(T\phi_x^*\pi)^{ij}(q)p_ip_j$. Proposition 2.1 then shows that $(\mathcal{F}_{M,x},\omega_{M,x},\mathcal{S}_{M,x})$ is a BV manifold for a closed manifold M. In particular, we have that $\mathcal{S}_{M,x}$ satisfies the CME (1.3). It's then not hard to see that the triple $(\mathcal{F}_{M,x},\omega_{M,x},\mathcal{S}_{M,x})$ defines a BF-like theory as in Definition 1.8. This will allow us to use the quantization scheme for BF-like theories described in section 1.3.3.
In the case where M has nonempty boundary we can use Lemma 2.1 to find that

$$\iota_{\mathcal{Q}_{M,x}}\omega_{M,x} = \delta\mathcal{S}_{M,x} + \pi_{M,x}^*\alpha_{\partial M,x}^{\partial}, \tag{2.50}$$

where as usual $\pi_{M,x} : \mathcal{F}_{M,x} \to \mathcal{F}_{\partial M,x}^{\partial}$ is the surjective submersion given by restriction to the boundary, $\mathcal{F}_{\partial M,x}^{\partial} = \mathrm{Map}(T[1]\partial M, T^*[1]T_xN)$ is the space of boundary fields, $\alpha_{\partial M,x}^{\partial} = \int_{\partial M}\sum_i \widehat{\eta}_i \wedge \delta\widehat{X}^i$ is the tautological 1-form and $\mathcal{Q}_{M,x}$ is the Hamiltonian vector field of $\mathcal{S}_{M,x}$. Moreover, by Proposition 2.2 we have that $(\mathcal{F}_{M,x},\omega_{M,x},\mathcal{S}_{M,x},\mathcal{Q}_{M,x},\pi_{M,x})$ is a BV-BFV manifold over the exact BFV manifold $(\mathcal{F}_{\partial M,x}^{\partial},\omega_{\partial M,x}^{\partial} = \delta\alpha_{\partial M,x}^{\partial},\mathcal{S}_{\partial M,x}^{\partial})$, where $\mathcal{S}_{\partial M,x}^{\partial}$ is the boundary action with Hamiltonian vector field $\mathcal{Q}_{\partial M,x}^{\partial}$.

Now we want to see what happens if we vary the point $x \in N$. To this purpose let us define the map $\widehat{\mathcal{S}}_M : x \mapsto \mathcal{S}_{M,x}$. We now want to compute $\mathrm{d}_x\widehat{\mathcal{S}}_M$. To do this let us first define

$$\mathcal{S}_{R,M,x} := \int_M \sum_{i,j} Y_j^i(x;\widehat{X})\widehat{\eta}_i \wedge dx^j, \tag{2.51}$$

where Y_j^i is defined in (2.39), and set $\mathcal{S}_{R,M} : x \mapsto \mathcal{S}_{R,M,x}$. For M closed we then have that $(\widehat{\mathcal{S}}_{0,M},\mathcal{S}_{M,R}) = 0$. Furthermore, we know from section 2.2.1 that $\mathrm{d}_x(T\phi^*\pi) = -R(T\phi^*\pi)$. So it follows that

$$\mathrm{d}_x\widehat{\mathcal{S}}_M = (\mathcal{S}_{R,M},\widehat{\mathcal{S}}_M) \tag{2.52}$$

for a closed manifold M. Now set $\widetilde{\mathcal{S}}_M := \widehat{\mathcal{S}}_M + \mathcal{S}_{R,M}$. Using the flatness of the Grothendieck connection or, more precisely, the Maurer-Cartan equation (2.41), the fact that $(\mathcal{S}_{R,M},\mathcal{S}_{R,M}) = \mathcal{S}_{[R,R],M}$ as well as equation (2.52) above, we can conclude that

$$\mathrm{d}_x\widetilde{\mathcal{S}}_M + \frac{1}{2}(\widetilde{\mathcal{S}}_M,\widetilde{\mathcal{S}}_M) = 0 \tag{2.53}$$

for M closed, which is called the *differential Classical Master Equation* (dCME).
Now consider the case where M has a nonempty boundary. Define the map $\widehat{\mathcal{Q}}_M : x \mapsto \mathcal{Q}_{M,x}$. Now note that for each $x \in N$ we can lift R_x to a vector field $R_{M,x}$ on $\mathcal{F}_{M,x}$. Define the map $R_M : x \mapsto R_{M,x}$ and set $\widetilde{\mathcal{Q}}_M := \widehat{\mathcal{Q}}_M + R_M$. Then it follows from equation (2.50) and the fact that $\iota_{R_{M,x}}\omega_{M,x} = \delta S_{R,M,x}$ that

$$\iota_{\widetilde{\mathcal{Q}}_{M,x}}\omega_{M,x} = \delta\widetilde{\mathcal{S}}_{M,x} + \pi^*_{M,x}\alpha^{\partial}_{\partial M,x}, \tag{2.54}$$

which we call the *modified differential Classical Master Equation* (mdCME).

2.2.3 Quantization

Following [14], let us now discuss the quantization of our linearized theory. So far we have a bundle $\bigsqcup_x(\mathcal{F}_{M,x}, \omega_{M,x}, \mathcal{S}_{M,x})$ of BF-like theories over the Poisson manifold N and hence we can apply the quantization scheme described in section 1.3.3 fiberwise: For each $x \in N$ we have a splitting of the space of fields

$$\mathcal{F}_{M,x} = \mathcal{B}^{\mathcal{P}}_{\partial M,x} \oplus \mathcal{V}^{\mathcal{P}}_{M,x} \oplus \mathcal{Y}'_x \tag{2.55}$$

with the base space of boundary fields $\mathcal{B}^{\mathcal{P}}_{\partial M,x}$, the space of residual fields

$$\mathcal{V}^{\mathcal{P}}_{M,x} = \big(H^\bullet(M, \partial_1 M) \otimes T_x N[1]\big) \oplus \big(H^\bullet(M, \partial_2 M) \otimes T^*_x N\big), \tag{2.56}$$

where we write the boundary ∂M of the compact oriented 2-manifold M as a disjoint union $\partial M = \partial_1 M \sqcup \partial_2 M$, and with the space of fluctuation fields $\mathcal{Y}'_x$, which is the symplectic complement of $\mathcal{B}^{\mathcal{P}}_{\partial M,x} \oplus \mathcal{V}^{\mathcal{P}}_{M,x}$. The polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M,x}$ is the direct product of the $\mathbb{X}$-representation (i.e. the $\frac{\delta}{\delta\widehat{\eta}}$-polarization) on $\partial_1 M$ and the $\mathbb{E}$-representaion (i.e. the $\frac{\delta}{\delta\widehat{X}}$-polarization) on $\partial_2 M$. As before, we write the fields $(\widehat{X}, \widehat{\eta}) \in \mathcal{F}_{M,x}$ as

$$\begin{aligned} \widehat{X} &= \mathbb{X} + \mathsf{x} + \mathcal{X}, \\ \widehat{\eta} &= \mathbb{E} + \mathsf{e} + \mathcal{E}. \end{aligned} \tag{2.57}$$

Furthermore, we denote the boundary state space (cf. Definition 1.11) as $\mathcal{H}^{\mathcal{P}}_{\partial M,x}$ and set $\widehat{\mathcal{H}}^{\mathcal{P}}_{M,x} = \mathcal{H}^{\mathcal{P}}_{\partial M,x}\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}^{\mathcal{P}}_{M,x})$, which is the space of states. Finally, let us fix a Lagrangian submanifold $\mathcal{L}'_x \subset \mathcal{Y}'_x$. Now as before, we can first define the principal part of the state:

Definition 2.3. The *prinicpal part of the state* is defined as the formal perturbative expansion of the BV pushforward

$$\widehat{\psi}_{M,x}(\mathbb{X}, \mathbb{E}; \mathsf{x}, \mathsf{e}) = \int_{\mathcal{L}'_x \subset \mathcal{Y}'_x} e^{\frac{i}{\hbar}\mathcal{S}_{M,x}[(\widehat{X},\widehat{\eta})]} \in \widehat{\mathcal{H}}^{\mathcal{P}}_{M,x} \tag{2.58}$$

using the Feynman rules and diagrams of the given theory (cf. Feynman rules and graphs in section 1.3.3).

Now consider the bundle $\widehat{\mathcal{H}}^{\mathcal{P}}_{M,\mathrm{tot}} \to N$ given by the union of the spaces of states $\widehat{\mathcal{H}}^{\mathcal{P}}_{M,x}$. The principal states $\widehat{\psi}_{M,x}$ then assemble into a section of this bundle and one may ask oneself whether they actually assemble into a covariantly constant section of this bundle. The answer is no, but luckily this can be remedied.

Definition 2.4. Given a formal exponential map ϕ on N and a linearized Poisson sigma model (i.e. linearized around a critical point $(x, 0) \in \mathcal{F}_M$) with action functional $\mathcal{S}_{M,x} = \mathcal{S}_{0,M,x} + \mathcal{S}_{\pi,M,x}$ for $x \in N$, we define the *formal globalized action* to be

$$\widetilde{S}_{M,x} := \mathcal{S}_{M,x} + \mathcal{S}_{R,M,x} = \mathcal{S}_{0,M,x} + \mathcal{S}_{\pi,M,x} + \mathcal{S}_{R,M,x}. \tag{2.59}$$

In local coordinates on the target the formal globalized action is given by

$$\widetilde{\mathcal{S}}_{M,x}[(\widehat{X}, \widehat{\eta})] = \int_M \sum_i \left(\widehat{\eta}_i \wedge \mathrm{d}\widehat{X}^i + \frac{1}{2}\sum_j (T\phi^*_x\pi)^{ij}(\widehat{X}) \wedge \widehat{\eta}_i \wedge \widehat{\eta}_j + \sum_j Y^j_i(x; \widehat{X}) \wedge \widehat{\eta}_j \wedge \mathrm{d}x^i \right). \tag{2.60}$$

When working with the formal globalized action the Feynman rules are the same as before (summarized in figure 1) but with additional vertices in the bulk stemming from the term $\mathcal{S}_{R,M,x}$: To a vertex labeled by R with l incoming half-edges labeled by $i_1, \dots, i_l$ and one outgoing half-edge labeled by j we associate the "vertex tensor" $Y^j_{i;i_1\cdots i_l}(x)\mathrm{d}x^i$ as given in (2.40). Furthermore, note that $(T\phi^*\pi)^{ij} \in \Gamma(\widehat{S}T^*N)$ and hence $(T\phi^*_x\pi)^{ij}$ is a formal power series in y which can be written as

$$(T\phi^*_x\pi)^{ij}(y) = \sum_{l=0}^{\infty} \sum_{i_1,\dots,i_l} (T\phi^*_x\pi)^{ij}{}_{,i_1\cdots i_l} y^{i_1} \cdots y^{i_l}. \tag{2.61}$$

So to an interaction vertex (stemming from the usual interaction part $\mathcal{S}_{\pi,M,x}$ of the action functional) with l incoming half-edges labeled by $i_1, \dots, i_l$ and two outgoing half-edges labeled by i and j we associate the "vertex tensor" $\frac{1}{2}(T\phi^*_x\pi)^{ij}{}_{;i_1\cdots i_l}$ from above. The Feynman rules and diagrams are summarized in figure 3 below.

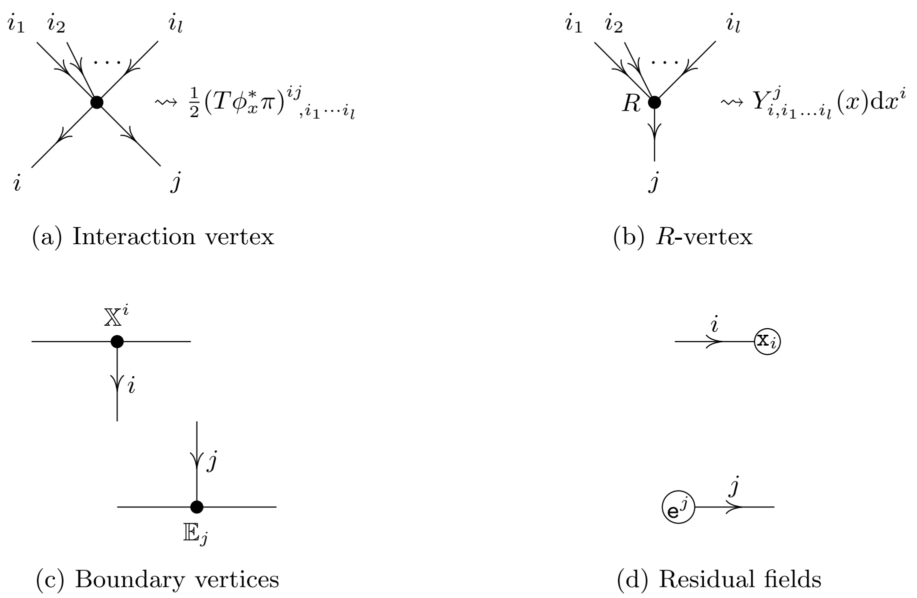


Figure 3: Feynman rules and diagrams for a linearized and globalized Poisson sigma model.

Using the formal globalized action and the Feynman rules introduced above let us define the covariant principle state:

Definition 2.5. The *covariant principle state* is given by the formal perturbative expansion of the BV pushforward

$$\widetilde{\psi}_{M,x}(\mathbb{X}, \mathbb{E}; \mathtt{x}, \mathtt{e}) = \int_{\mathcal{L}'_x \subset \mathcal{Y}'_x} e^{\frac{i}{\hbar}\widetilde{\mathcal{S}}_{M,x}[(\widehat{X},\widehat{\eta})]} \in \Omega^\bullet(N, \widehat{\mathcal{H}}^{\mathcal{P}}_{M,x}) \tag{2.62}$$

using the Feynman rules and diagrams summarized in figure 3 above.

In order for a global version of the mQME (1.12) to hold we must regularize higher functional derivatives using the notion of composite fields as before. This then leads to the definition of the covariant version of the full quantum state:

Definition 2.6. The *full covariant quantum state* is given as the formal perturbative expansion of the BV pushforward

$$\widetilde{\boldsymbol{\psi}}_{M,x}(\mathbb{X}, \mathbb{E}; \mathtt{x}, \mathtt{e}) = \int_{\mathcal{L}'_x \subset \mathcal{Y}'_x} e^{\frac{i}{\hbar}\widetilde{\mathcal{S}}_{M,x}[(\widehat{X},\widehat{\eta})]} \in \Omega^\bullet(N, \widehat{\mathcal{H}}^{\mathcal{P}}_{M,x}) \tag{2.63}$$

using the Feynman rules and diagrams summarized in figure 3 and additionally with the rules for the boundary vertices of higher valency for composite fields shown in figure 2.

As mentioned above, for the globalized version of the Poisson sigma model (or, more generally, for globalized split AKSZ theories) the mQME needs to be adjusted. What we then get is a more general condition called the *modified differential Quantum Master Equation* (mdQME), which is given by (2.64) in Theorem 2.1 below.
The mdQME may be spoiled by short loops (i.e. arrows starting and ending at the same vertex). For the $T\phi_x^*\pi$-vertex (i.e. the interaction vertex stemming from the term $\mathcal{S}_{\pi,M,x}$) short loops are absent if we assume that the target Poisson structure is *unimodular* (a Poisson structure is unimodular if its divergence is a Hamiltonian vector field; cf. [12], section 4.2), which is in particular the case when the Poisson structure is symplectic. Basically, if the Poisson structure is unimodular one may assume that $\Delta\mathcal{S}_{\pi,M,x} = 0$, which in turn implies that short loops are absent for the interaction vertex. More precisely and as mentioned in [6], if π is unimodular one may pick a volume form v on N such that $\mathrm{div}_v\pi = 0$ and then define Δ accordingly. Furthermore, for the R-vertex we can always find a formal exponential map such that $\Delta\mathcal{S}_{R,M,x} = 0$, as is explained in [6]. So we may always assume that short loops are absent for the R-vertex.
Now let us formulate the main result of this section:

Theorem 2.1 (Cattaneo-Moshayedi-Wernli; [14])**.** Consider the full covariant quantum state $\widetilde{\boldsymbol{\psi}}_{M,x}$ as a perturbative quantization of a unimodular Poisson sigma model. Then

$$\left(\mathrm{d}_x - i\hbar\Delta_{\mathcal{V}^{\mathcal{P}}_{M,x}} + \frac{i}{\hbar}\boldsymbol{\Omega}_{\partial M}\right)\widetilde{\boldsymbol{\psi}}_{M,x} = 0, \tag{2.64}$$

where d_x is the de Rham differential on the Poisson manifold N.

Remark 2.12. In [14] the theorem is shown more generally for split AKSZ theories. In that case one needs an additional assumption, namely that the theory has no hidden faces anomalies. But it can be shown that any 2-dimensional theory is hidden faces anomaly free, hence we don't need to formulate this assumption for the Poisson sigma model.

Remark 2.13. In [6] it is shown how the (full) boundary BFV operator $\boldsymbol{\Omega}_{\partial M}$ is construced for the formal globalized action of the Poisson sigma model (cf. Definition 2.59) in the case of a single connected boundary component in $\mathbb{E}$- or in $\mathbb{X}$-representation.

Now let us define the operator

$$\nabla_G := \mathrm{d}_x - i\hbar\Delta_{\mathcal{V}^{\mathcal{P}}_{M,x}} + \frac{i}{\hbar}\boldsymbol{\Omega}_{\partial M}, \tag{2.65}$$

which is called the *quantum Grothendieck BFV operator*. It's an operator of total degree 1 on forms valued in sections of the total bundle of states. The previous theorem then tells us that the full covariant quantum state defines a closed section with respect to the quantum Grothendieck BFV operator.
Finally we also have the following result:

Theorem 2.2 (Cattaneo-Moshayedi-Wernli; [14])**.** The quantum Grothendieck BFV operator ∇_G is a coboundary operator, i.e.

$$(\nabla_G)^2 = 0.$$

Remark 2.14. As is done in [14], one can additionally show that the cohomology class of $\widetilde{\boldsymbol{\psi}}_{M,x}$ is independent of the choices made. Altogether one then sees that the state gives a well-defined ∇_G-cohomology class.

3 Deformation Quantization

In this section we will introduce the notion of deformation quantization. It's a quantization procedure which focuses on the algebra of observables, contrary to e.g. geometric quantization (cf. section 1.3.1), which focuses on the space of states. More precisely, deformation quantization is concerned with the deformation of commutative algebras of classical observables to noncommutative algebras of quantum observables. The classical input is usually a Poisson manifold, so in a first step we will give a short introduction to Poisson geometry, mainly following [29]. In a second step, still mainly following [29], we will present the main notions of deformation quantization of Poisson manifolds, above all the notion of a star product on the algebra of smooth functions. In a third and last part we will then discuss Kontsevich's famous results from [24]. In particular, we will introduce his local quantization formula and discuss its globalization.

3.1 Poisson Geometry

Let M be a smooth manifold and consider the graded commutative algebra $\Gamma(M, \bigwedge^\bullet TM)$ of *polyvector fields* on M. The *degree* of a polyvector field $a \in \Gamma(M, \bigwedge^p TM)$ is $|a| = p$. It is well known that the usual Lie bracket of vector fields can be uniquely extended to a graded Lie bracket $[\ ,\ ]$ of degree -1 on $\Gamma(M, \bigwedge^\bullet TM)$, called the *Schouten-Nijenhuis bracket* or simply the *Schouten bracket*, satisfying

(i) $[a, b] = -(-1)^{(|a|+1)(|b|+1)}[b, a]$ (Antisymmetry),

(ii) $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1)(|b|+1)}[b, [a, c]]$ (Jacobi identity),

(iii) $[a, b \wedge c] = [a, b] \wedge c + (-1)^{|b||c|}[a, c] \wedge b$ (Poisson identity),

(iv) $[f, g] = 0$ for all $f, g \in \mathcal{C}^\infty(M) \subset \Gamma(M, \bigwedge^\bullet TM)$,

(v) $[X, f] = Xf$ for all $X \in \Gamma(M, TM) \subset \Gamma(M, \bigwedge^\bullet TM)$ and $f \in \mathcal{C}^\infty(M)$.

Definition 3.1. A *Poisson manifold* is a smooth manifold M equipped with a bivector field $\pi \in \Gamma(M, \bigwedge^2 TM)$, called a *Poisson bivector field*, satisfying $[\pi, \pi] = 0$.

Remark 3.1. Every manifold M is a Poisson manifold with the zero bivector field $\pi = 0$.

As in symplectic geometry there is also the notion of a Hamiltonian vector field:

Definition 3.2. Let (M, π) be a Poisson manifold and $f \in \mathcal{C}^\infty(M)$. The *Hamiltonian vector field* associated to f is the vector field

$$X_f = [\pi, f]. \tag{3.1}$$

Remark 3.2. The vector field $X_f = [\pi, f]$ can be defined for any bivector field $\pi \in \Gamma(M, \bigwedge^2 TM)$, not necessarily satisfying $[\pi, \pi] = 0$.

Definition 3.3. Let (M, π) be a Poisson manifold and $f, g \in \mathcal{C}^\infty(M)$. The *Poisson bracket* on $\mathcal{C}^\infty(M)$ is defined as

$$\{f, g\} = X_f g. \tag{3.2}$$

Proposition 3.1. Let (M, π) be a Poisson manifold. The Poisson bracket is a bilinear map $\{\ ,\ \} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ satisfying

(i) $\{f, g\} = -\{g, f\}$ (Antisymmetry),

(ii) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity),

(iii) $\{f, gh\} = g\{f, h\} + \{f, g\}h$ (Leibniz rule).

Proof. Bilinearity of the Poisson bracket follows immidiately from (3.2) and the bilinearity of the Schouten bracket. Now $\{f, g\} = [[\pi, f], g]$ and, using properties (i), (ii) and (iv) of the Schouten bracket, we find that

$$\{f, g\} = [[\pi, f], g] = [f, [\pi, g]] = -[[\pi, g], f] = -\{g, f\},$$

which proves (i). Furthermore, using the Jacobi identity of the Schouten bracket and $[\pi,\pi]=0$, it follows that $[\pi, X_f]=0$ for any $f\in\mathcal{C}^\infty(M)$. For $f,g\in\mathcal{C}^\infty(M)$ we then find that

$$[X_f,X_g]=[X_f,[\pi,g]]=[[X_f,\pi],g]+[\pi,[X_f,g]]=\underbrace{[[X_f,\pi],g]}_{=0}+X_{\{f,g\}}=X_{\{f,g\}}.$$

But then it follows that

$$\{f,\{g,h\}\}=X_fX_gh=[X_f,X_g]h+X_gX_fh=X_{\{f,g\}}h+X_gX_fh=\{\{f,g\},h\}+\{g,\{f,h\}\}$$

which proves (ii). Finally, (iii) follows directly from the Leibniz rule for vector fields. □

Definition 3.4. A *Poisson algebra* is a commutative algebra A together with a Lie bracket $\{\ ,\ \}: A\times A\to A$ satisfying the Leibniz rule

$$\{f,gh\}=g\{f,h\}+\{f,g\}h. \tag{3.3}$$

Now let A be a commutative algebra. We will denote the vector space of *derivations* (i.e. linear maps from A to A satisfying the Leibniz rule) by $\mathrm{Der}(A)$. Similarly, we will denote by $\mathrm{Der}^k(A)$ the subspace of $\mathrm{Hom}(A^{\otimes k},A)$ of completely antisymmetric maps which are derivations in each argument. It is well-known that the map $\Gamma(M,TM)\to\mathrm{Der}(\mathcal{C}^\infty(M))$, $X\mapsto(f\mapsto[X,f])$ is an isomorphism. More generally, we have the following:

Proposition 3.2. Let M be a smooth manifold. The map

$$\Gamma(M,\bigwedge^k TM)\to\mathrm{Der}^k(\mathcal{C}^\infty(M)),\ \alpha\mapsto((f_1,\dots,f_k)\mapsto[\dots[\alpha,f_1],f_2],\dots,f_k]$$

is an isomorphism of vector spaces.

Theorem 3.1. Let M be a smooth manifold. The structure of a Poisson manifold on M (i.e. a Poisson bivector field π on M) is equivalent to the structure of a Poisson algebra on $\mathcal{C}^\infty(M)$.

Proof. Given a Poisson bivector field π on M, it follows from Proposition 3.1 that the bracket (3.2) defines a Poisson algebra structure on $\mathcal{C}^\infty(M)$. Conversely, suppose that we have a Lie bracket $\{\ ,\ \}$ on $\mathcal{C}^\infty(M)$ satisfying the Leibniz rule. By Proposition 3.2 the bracket corresponds to a bivector field π on M. More precisely, we have that

$$\{f,g\}=[[\pi,f],g]=X_fg.$$

Now $\{\ ,\ \}$ satisfies the Jacobi identity. So we get that

$$X_fX_gh=\{f,\{g,h\}\}=\{g,\{f,h\}\}+\{\{f,g\},h\}=X_gX_fh+X_{\{f,g\}}h,$$

which implies that $[X_f,X_g]=X_{\{f,g\}}$. But we also have that $[X_f,X_g]=[[X_f,\pi],g]+X_{\{f,g\}}$. So we must have $[X_f,\pi]=0$ for every $f\in\mathcal{C}^\infty(M)$. But using the Jacobi identity of the Schouten bracket we find that $[[\pi,\pi],f]=2[\pi,[\pi,f]]=2[\pi,X_f]$ for every $f\in\mathcal{C}^\infty(M)$, which in turn implies that $[\pi,\pi]=0$. Hence π is a Poisson bivector field. □

Definition 3.5. Let (M,π_M) and (N,π_N) be Poisson manifolds. A *Poisson map* is a smooth map $f:M\to N$ such that the induced map $f^*:\mathcal{C}^\infty(N)\to\mathcal{C}^\infty(M)$ preserves brackets, i.e. such that

$$\{f^*a,f^*b\}_M=f^*\{a,b\}_N \tag{3.4}$$

for $a,b\in\mathcal{C}^\infty(N)$.

One can then show (cf. Proposition 6.26, [29]) that a smooth map $f:M\to N$ is Poisson if and only if $(df)_x(\pi_{M,x})=\pi_{N,f(x)}$.

To conclude this section, let us give an example of an important family of Poisson manifolds (cf. Theorem 6.17, [29]):

Proposition 3.3. Let M be a smooth manifold and let $\pi \in \Gamma(M, \bigwedge^2 TM)$. Then if the induced map $\pi^\# : T^*M \to TM$ is an isomorphism, its inverse $\omega^\# : TM \to T^*M$ is induced by a nondegenerate 2-form $\omega \in \Gamma(M, \bigwedge^2 T^*M)$. Furthermore, $d\omega = 0$ if and only if $[\pi, \pi] = 0$.

So a Poisson manifold (M, π) is also a symplectic manifold if $\pi^\#$ is an isomorphism. Hence symplectic manifolds form a subset of Poisson manifolds. It's worth noting that in this case the Hamiltonian vector field X_f defined by

$$\iota_{X_f}\omega = df \tag{3.5}$$

coincides with the Hamiltonian vector field given by (3.1).

3.2 Deformation Quantization of Poisson Manifolds

As already mentioned at the beginning, in this part we will give a general introduction to deformation quantization. We still mainly follow [29], but also [22], [25] and [24].

3.2.1 Motivation

Roughly speaking, quantization is the process of forming a quantum system starting from a classical one, whereas the deformation of a mathematical object yields a family of "similar" objects depending on some parameter. The main motivation for deformation quantization comes, not surprisingly, from physics: In classical mechanics the phase space is a Poisson manifold M, the observables are smooth real-valued functions on M which form a Poisson algebra (commutative!) and the evolution of an observable $f \in \mathcal{C}^\infty(M)$ is governed by the Hamilton equation

$$\frac{df}{dt} = \{H, f\}, \tag{3.6}$$

where H is the Hamiltonian function of the system.
In quantum mechanics on the other hand the space of states is a Hilbert space $\mathcal{H}$, observables are self-adjoint operators on $\mathcal{H}$ which form the associative algebra $\mathrm{End}(\mathcal{H})$ (not commutative!) and the evolution of an observable $A \in \mathrm{End}(\mathcal{H})$ is governed by the Heisenberg equation

$$-i\hbar\frac{dA}{dt} = [\widehat{H}, A], \tag{3.7}$$

where $\widehat{H}$ is the Hamiltonian operator of the system.
The *correspondence principle* now tells us that for $\hbar \to 0$ a quantum mechanical system should be described by a classical system. So in this limit the Heisenberg equation (3.7) should go over to the Hamilton equation (3.6). In particular, one might expect that the associatve algebra $\mathrm{End}(\mathcal{H})$ becomes the commutative algebra $\mathcal{C}^\infty(M)$ and the quantum Hamiltonian $\widehat{H}$ becomes the classical Hamiltonian H in the limit $\hbar \to 0$.
The main idea of deformation quantization, motivated by the considerations above, now is to understand the quantum mechanical observables, which in general do not commute, as a deformation in the direction of the Poisson bracket of the Poisson algebra $\mathcal{C}^\infty(M)$ of classical observables, which do commute. Or to put it differently, we want to get a family of associative algebras over $\hbar$ becoming commutative at $\hbar = 0$.

3.2.2 Star Products

Definition 3.6. Let $\mathcal{A}$ be a commutative algebra over $\mathbb{R}$. A *formal deformation* of $\mathcal{A}$ is an associative unital algebra structure on $\mathcal{A}[[\hbar]]$ such that

(i) the product is $\mathbb{R}[[\hbar]]$-linear in each argument,

(ii) the map $\mathcal{A}[[\hbar]]/\hbar\mathcal{A}[[\hbar]] \to \mathcal{A}$ is an isomorphism of algebras.

The product on $\mathcal{A}[[\hbar]]$ is usually denoted by $\star$ and is called a star product. Equivalently, we then have the following

Definition 3.7. Let $\mathcal{A}$ be a commutative algebra over $\mathbb{R}$. A *star product* on $\mathcal{A}$ is an associative $\mathbb{R}[[\hbar]]$-linear product $\star : \mathcal{A}[[\hbar]] \otimes_{\mathbb{R}[[\hbar]]} \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ on $\mathcal{A}[[\hbar]]$ which reduces to the product on $\mathcal{A}$ modulo $\hbar$.

A star product on $\mathcal{A}[[\hbar]]$ is completely determined by what it is on $\mathcal{A}$, and on $\mathcal{A}$ it is given by

$$a \star b = ab + \sum_{n=1}^{\infty} B_n(a,b)\hbar^n, \qquad a, b \in \mathcal{A}, \tag{3.8}$$

with $\mathbb{R}$-bilinear maps $B_n : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. From now on we will actually additionally assume that each map B_n, $n \geq 1$, is a *bidifferential operator* of degree at most n.
Now note that for each associative algebra the commutator $[a, b] = ab - ba$ satisfies

$$[a, bc] = b[a, c] + [a, b]c \tag{3.9}$$

and

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]. \tag{3.10}$$

By applying these formulas to the associative algebra $(\mathcal{A}[[\hbar]], \star)$ and by taking the appropriate term of the expansion in $\hbar$ on both sides we then find that

$$\{a, b\} := B_1(a, b) - B_1(b, a), \qquad a, b \in \mathcal{A}, \tag{3.11}$$

defines a Poisson bracket on $\mathcal{A}[[\hbar]]$.

Definition 3.8. Two star products $\star$, $\star'$ on $\mathcal{A}$ are *gauge equivalent* if there exists a map $g : \mathcal{A} \to \mathcal{A}[[\hbar]]$ such that

(i) g is the identity modulo $\hbar$

(ii) $g(a \star b) = g(a) \star' g(b)$

Remark 3.3. To be more precise, we have a gauge group acting on star prodcucts on the algebra $\mathcal{A}$. The group consists of automorphisms of $\mathcal{A}[[\hbar]]$ (regarded as a $\mathbb{R}[[\hbar]]$-module) of the form

$$a \mapsto a + \sum_{k=1}^{\infty} \hbar^k D_k(a), \qquad a \in \mathcal{A} \subset \mathcal{A}[[\hbar]], \tag{3.12}$$

where $D_i : \mathcal{A} \to \mathcal{A}$ are differential operators. If $D = 1 + \sum_{k=1}^{\infty} \hbar^k D_k$ is such an automorphism, than it acts on the set of star products on $\mathcal{A}$ as

$$\star \mapsto \star', \; a(\hbar) \star' b(\hbar) := D(D^{-1}(a(\hbar)) \star D^{-1}(b(\hbar))) \tag{3.13}$$

for $a(\hbar), b(\hbar) \in \mathcal{A}[[\hbar]]$.

In the case where $\mathcal{A}$ is the algebra of smooth functions on a manifold we then have the following result (cf. Proposition 8.8, [29]):

Proposition 3.4. Let M be a smooth manifold and set $\mathcal{A} := \mathcal{C}^\infty(M)$. Then any star product on $\mathcal{A}$ is gauge equivalent to one where B_1 is antisymmetric.

So up to gauge equivalence a star product on $\mathcal{C}^\infty(M)$ may always be taken of the form

$$f \star g = fg + \frac{\hbar}{2}\{f, g\} + \dots. \tag{3.14}$$

In particular we see that gauge equivalence classes of star products modulo $\hbar^2$ are classified by Poisson structures on M. But note that at this point it is not clear whether there exists a star product with the term of order $\hbar$ equal to a given Poisson structure.

Definition 3.9. Let (M, π) be a Poisson manifold and $(\mathcal{C}^\infty(M), \{\ ,\ \})$ the corresponding Poisson algebra of smooth functions. A *deformation quantization* of M (or of $\mathcal{C}^\infty(M)$) is given by a star product $\star$ on $\mathcal{C}^\infty(M)[[\hbar]]$ (or, equivalently, by a formal deformation of $\mathcal{C}^\infty(M)$) with the following properties:

(i) The maps B_n appearing in the star product are bidifferential operators of degree at most n

(ii) $B_1(f, g) - B_1(g, f) = \{f, g\}$ for all $f, g \in \mathcal{C}^\infty(M)$, where $\{\ ,\ \}$ is the Poisson bracket on $\mathcal{C}^\infty(M)$

(iii) $1 \star f = f \star 1 = f$ for all $f \in \mathcal{C}^\infty(M)$

So a deformation quantization of a Poisson manifold M is an associative algebra $(\mathcal{C}^\infty(M)[[\hbar]], \star)$ with the star product satisfying (i)-(iii) above. In particular we then have that

$$[f, g] = f \star g - g \star f = \hbar\{f, g\} + \mathcal{O}(\hbar^2). \tag{3.15}$$

Now a question that naturally arises is the following: Do all Poisson manifolds allow a deformation quantization? This question was acutally answered in the affirmative by Kontsevich in [24]. And what is more, he gave a concrete formula for a star product. Next section we will come back to this, but first let us conclude this section with a nontrivial example of a star product:

Example. Let $M = \mathbb{R}^{2n}$ with a constant Poisson structure π given by

$$\pi = \sum_{i,j} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \tag{3.16}$$

where $\pi^{ij} = -\pi^{ji} \in \mathbb{R}$. In that case the Poisson bracket is given by

$$\{f, g\} = \sum_{i,j} \pi^{ij} (\partial_i f)(\partial_j g). \tag{3.17}$$

The so-called *Moyal product* on $\mathbb{R}^{2n}$ is the product

$$f \star g = fg + \frac{\hbar}{2} \sum_{i,j} \pi^{ij} (\partial_i f)(\partial_j g) + \frac{\hbar^2}{8} \sum_{i,j,k,l} \pi^{ij} \pi^{kl} (\partial_i \partial_k f)(\partial_j \partial_l g) + \dots \tag{3.18}$$

for $f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n})$. One can show that the Moyal product is associative. Furthermore, we clearly have that $f \star g - g \star f = \hbar\{f, g\} + \dots$ and $f \star 1 = 1 \star f = f$. So the Moyal product defines a deformation quantization of the Poisson manifold $(\mathbb{R}^{2n}, \pi)$.

3.3 Kontsevich Quantization Formula

In this section we will present some important results due to Kontsevich, mainly following [24], but in certain parts also [12]. In particular, we will present a canonical construction of an equivalence class of star products for Poisson manifolds, known as the *Kontsevich quantization formula*.

Let M be a smooth manifold and consider a formal power series

$$\pi = \pi_1 \hbar + \pi_2 \hbar^2 + \dots \in \Gamma(M, \bigwedge^2 TM)[[\hbar]] \tag{3.19}$$

of bivector fields on M. If π is Poisson, i.e. if $[\pi, \pi] = 0$, then we call such a series a *formal family of Poisson structures*. We then have the following important result:

Theorem 3.2 (Kontsevich; [24])**.** The set of gauge equivalence classes of star products on a smooth manifold M can be naturally identified with the set of equivalence classes of formal Poisson structures $\pi = \sum_{k \geq 1} \pi_k \hbar^k$ modulo the action of the group of formal paths in the diffeomorphism group of M, starting at the identity diffeomorphism.

This theorem not only proves the existence of a deformation quantization of an arbitrary Poisson manifold, but it also gives a complete classification of deformation quantizations. And acutally it is an immidiate Corollary of Kontsevich's formaility theorem, which he stated and proved in [24] and which we are going to present below.

3.3.1 L_∞-algebras

Definition 3.10. A (counital coassociative) *coalgebra* over a field k is a k-vector space C together with k-linear maps $\Delta : C \to C \otimes C$, called the *comultiplication*, and $\epsilon : C \to k$, called the *counit*, such that

(i) $(\mathrm{Id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}_C) \circ \Delta$,

(ii) $(\mathrm{Id}_C \otimes \epsilon) \circ \Delta = \mathrm{Id}_C = (\epsilon \otimes \mathrm{Id}_C) \circ \Delta$.

A coalgebra is called *cocommutative* if $\sigma \circ \Delta = \Delta$, where $\sigma : C \otimes C \to C \otimes C$, $a \otimes b \mapsto \sigma(a \otimes b) = b \otimes a$ for all $a, b \in C$.

Definition 3.11. Let C and C' be two coalgebras. A *coalgebra morphism* is a linear map $f : C \to C'$ such that $(f \otimes f) \circ \Delta = \Delta' \circ f$.

Now let V be a graded vector space and consider the *symmetric algebra* $S(V) = \bigoplus_{n\geq 0} S^n V$, which is the free graded unital commutative algebra of V. Now actually $S(V)$ is also a coalgebra. More precisely, it's the cofree cocommutative counital coalgebra of V with comultiplication

$$\Delta(v_1 \cdots v_n) = \sum_{k=0}^{n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \chi(\sigma; v_1, \ldots, v_n)(v_{\sigma(1)} \cdots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \cdots v_{\sigma(n)}), \tag{3.20}$$

where we sum over *unshuffle* permutations

$$\mathrm{Sh}(k, n-k) = \{\sigma \in S_n | \; \sigma(1) < \cdots < \sigma(k), \; \sigma(k+1) < \cdots < \sigma(n)\} \tag{3.21}$$

and where $\chi(\sigma; v_1, \ldots, v_n)$ is the Koszul sign, which is obtained as follows: Writing the permutation σ as a compostition of m transpositions which exchange only neighbouring elements, the Koszul sign $\chi(\sigma; v_1, \ldots, v_n)$ is the product of the usual sign $\mathrm{sgn}(\sigma) = (-1)^m$ and factors of $(-1)^{|v_i||v_i+1|}$ for each involved interchange $\cdots v_i v_{i+1} \cdots \mapsto \cdots v_{i+1} v_i \cdots$.
Finally, let us write $S^+(V) := \bigoplus_{n\geq 1} S^n V$ for the cofree cocommutative coalgebra of V without counit.

Definition 3.12. A *coderivation* of a coalgebra C is a linear map $D : C \to C$ satisfying the *co-Leibniz rule*

$$\Delta \circ D = (D \otimes \mathrm{Id}_C + \mathrm{Id}_C \otimes D) \circ \Delta. \tag{3.22}$$

Definition 3.13. An L_∞-*algebra* is a pair (V, D) where V is a graded vector space and D is a coderivation of degree $+1$ on the graded cocommutative coalgebra $S^+(V[1])$ satisfying $D^2 = 0$.

Remark 3.4. For an L_∞-algebra (V, D) the coderivation is given by its Taylor coefficients

$$D_i : S^i(V[1]) \to V[2], \qquad i \geq 1, \tag{3.23}$$

which satisfy certain relations. In particular we have that D_1 is a differential, and hence we can regard $(V[1], D_1)$ as a cochain complex.

Definition 3.14. An L_∞-morhpism between two L_∞-algebras (V, D) and (V', D') is a morphism of graded cocommutative coalgebras $U : S^+(V[1]) \to S^+(V'[1])$ satisfying $U \circ D = D' \circ U$.

Remark 3.5. An L_∞-morphism $U : S^+(V[1]) \to S^+(V'[1])$ is uniquely determined by its Taylor coefficients

$$U_i : S^i(V[1]) \to V'[1], \qquad i \geq 1, \tag{3.24}$$

(or $\bigwedge^i V \to V'[1-i]$), where U_i is the restriction to $S^i(V[1])$ of $p' \circ U$ with projection $p' : S^+(V'[1]) \to V'[1]$. The first Taylor coefficient is then a chain map

$$U_1 : (V[1], D_1) \to (V'[1], D'_1). \tag{3.25}$$

Definition 3.15. A *quasi-isomorphism* between L_∞-algebras (V, D) and (V', D') is an L_∞-morphism $U : S^+(V[1]) \to S^+(V'[1])$ such that the first Taylor component $U_1 : V[1] \to V'[1]$ induces an isomorphism between cohomology groups of the complexes $(V[1], D_1)$ and $(V'[1], D'_1)$.

3.3.2 The Kontsevich Formality Theorem

Definition 3.16. A *differential graded Lie algebra* is a graded vector space $L = \bigoplus_i L_i$ over a field k (of characteristic zero) together with a linear map $[\ ,\ ] : L_i \otimes L_j \to L_{i+j}$ and a differential $\mathrm{d} : L_i \to L_{i+1}$ satisfying

(i) $[a, b] = -(-1)^{|a||b|}[b, a]$

(ii) $(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0$ (graded Jacobi identity)

(iii) $\mathrm{d}[a, b] = [\mathrm{d}a, b] + (-1)^{|a|}[a, \mathrm{d}b]$ (graded Leibniz rule)

for all homogeneous elements $a, b, c \in L$.

Remark 3.6. Differential graded Lie algebras are L_∞-algebras (L, D) with $D_k = 0$ for all $k \geq 3$.

Now let M be a smooth manifold and set $\mathcal{A} := \mathcal{C}^\infty(M)$. We will introduce two differential graded Lie algebras over $\mathbb{R}$ associated with M:
The first one is the differential graded Lie algebra of *polydifferential operators* $D^\bullet_{\mathrm{poly}}(M)$. For each $n \geq -1$ we have that $D^n_{\mathrm{poly}}(M) \subset \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathcal{A})$ and in local coordinates (x^i) any element of $D^n_{\mathrm{poly}}(M)$ can be written as

$$f_0 \otimes \cdots \otimes f_n \mapsto \sum_{I_0,\ldots,I_n} A^{I_0,\ldots,I_n}(x)\partial_{I_0}(f_0)\cdots\partial_{I_n}(f_n), \tag{3.26}$$

where the I_k are multi-indices and the f_k and $A^{I_0,\ldots,I_n}$ are functions in (x^i). Actually $D^\bullet_{\mathrm{poly}}(M)$ is a subalgebra of the (shifted) Hochschild cochain complex of the algebra of functions $\mathcal{A}$. So the differential on $D^\bullet_{\mathrm{poly}}(M)$ is the usual differential of the Hochschild complex (shifted by one) and the Lie bracket is the so-called Gerstenhaber bracket. For more details on the Hochschild complex, the Gerstenhaber bracket and the differential we refer to [24], section 3.4.2.
The second one is the differential graded Lie algebra of *polyvector fields* $T^\bullet_{\mathrm{poly}}(M) := \Gamma(\bigwedge^{\bullet+1} TM)$ (shifted by one), which we have already introduced in section 3.1 above. The Lie bracket is the Schouten-Nijenhuis bracket and the differential is $\mathrm{d} = 0$.

Now there is an evident map $\mathcal{U}_1^{(0)} : T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$ given by

$$\xi_0 \wedge \cdots \wedge \xi_n \mapsto \Big(f_0 \otimes \cdots \otimes f_n \mapsto \frac{1}{(n+1)!}\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma)\prod_{i=0}^{n} \xi_{\sigma(i)}(f_i)\Big) \tag{3.27}$$

for vector fields $\xi_i \in \Gamma(M, TM) \subset T_{\mathrm{poly}}(M)$ and by

$$h \mapsto (1 \mapsto h) \tag{3.28}$$

for functions $h \in \mathcal{A} \subset T_{\mathrm{poly}}(M)$. It turns out that this map is actually a quasi-isomorphism of complexes (cf. Theorem 4.10, [24]). With this we are finally able to state the main result of this section, called Kontsevich's formality theorem:

Theorem 3.3 (Kontsevich; [24])**.** There exists an L_∞-morphism

$$\mathcal{U} : T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M) \tag{3.29}$$

such that $\mathcal{U}_1 = \mathcal{U}_1^{(0)}$.

So in other words $T_{\mathrm{poly}}(M)$ and $D_{\mathrm{poly}}(M)$ are quasi-isomorphic differential graded Lie algebras.
Kontsevich constructed an explicit L_∞-morphism for $M = \mathbb{R}^d$ and then extended the construction to general manifolds. We will sketch this construction in the sections to come, but it's worth noting that there are other possible quasi-isomorphisms between $T_{\mathrm{poly}}(M)$ and $D_{\mathrm{poly}}(M)$.

Remark 3.7. As already mentioned before, Theorem 3.2, which classifies all deformation quantizations of an arbitrary Poisson manifold, is a direct consequence of the formality theorem above. The idea is the following: Let L and K be two differential graded Lie algebras and F an L_∞-morphism from L to K.

Then any solution $\gamma \in L_1 \otimes \mathfrak{m}$ of the Maurer-Cartan equation, where $\mathfrak{m}$ is a nilpotent nonunital algebra, produces a solution of the Maurer-Cartan equation in $K_1 \otimes \mathfrak{m}$, i.e.

$$\mathrm{d}\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \implies \mathrm{d}\widetilde{\gamma} + \frac{1}{2}[\widetilde{\gamma}, \widetilde{\gamma}] = 0,$$

where $\widetilde{\gamma} = \sum_{n\geq 1} \frac{1}{n!} F_n(\gamma \wedge \cdots \wedge \gamma) \in K_1 \otimes \mathfrak{m}$. The same formula is also applicable for solutions of the Maurer-Cartan equation which formally depend on $\hbar$, i.e. for solutions of the form $\gamma(\hbar) = \gamma_1 \hbar + \gamma_2 \hbar^2 + \ldots \in L_1[[\hbar]]$. Now one has to observe that solutions of the Maurer-Cartan equation in $T_{\mathrm{poly}}(M)$ are exactely Poisson structures on M, i.e. elements $\pi \in T^1_{\mathrm{poly}}(M) = \Gamma(\bigwedge^2 TM)$ satisfying $[\pi, \pi] = 0$. Any such π then defines a solution $\gamma(\hbar) := \pi \cdot \hbar \in T^1_{\mathrm{poly}}(M)[[\hbar]]$ formally depending on $\hbar$. On the other side, solutions of the Maurer-Cartan equation in $D_{\mathrm{poly}}(M)$ formally depending on $\hbar$ are precisely the star products on $\mathcal{C}^\infty(M)[[\hbar]]$. From this one obtains Theorem 3.2 as a Corollary of the formality theorem above.

3.3.3 Configuration Spaces

Let $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ be the upper half plane and let $n, m \in \mathbb{N}$ such that $2n + m \geq 2$. Define

$$\mathrm{Conf}_{n,m} := \{(x_1, \ldots, x_n, q_1, \ldots, q_m) \in \mathbb{H}^n \times \mathbb{R}^m | \ x_i \neq x_j \text{ for } i \neq j, \ q_1 < \ldots < q_m\}. \tag{3.30}$$

The 2-dimensional real Lie group of orientation preserving affine transformations of the real line

$$G^{(1)} = \{z \mapsto az + b | \ a, b \in \mathbb{R}, \ a > 0\}$$

acts freely on $\mathrm{Conf}_{n,m}$. The quotient space

$$C_{n,m} := \mathrm{Conf}_{n,m}/G^{(1)} \tag{3.31}$$

is then a smooth manifold of dimension $2n + m - 2$.
Analogously, let $n \geq 2$ and let us define

$$\mathrm{Conf}_n := \{(x_1, \ldots, x_n) \in \mathbb{C}^n | \ x_i \neq x_j \text{ for } i \neq j\}. \tag{3.32}$$

The 3-dimensional real Lie group

$$G^{(2)} = \{z \mapsto az + b | \ a \in \mathbb{R}, \ a > 0, \ b \in \mathbb{C}\} \tag{3.33}$$

acts freely on Conf_n. The quotient space

$$C_n := \mathrm{Conf}_n/G^{(2)} \tag{3.34}$$

is then a smooth manifold of dimension $2n - 3$.
As is done in [24], one can construct compactifications $\overline{C}_n$ of C_n and $\overline{C}_{n,m}$ of $C_{n,m}$, which are smooth manifolds with corners of dimension $2n - 3$ and $2n + m - 2$ respectively. We will leave out the details of the construction here. What is important for us is the following: Stokes' theorem applies to smooth differential top forms on manifolds with corners. For this one only needs the strata of codimension 1, which correspond to limiting configurations in which a group of points collapses to a point (possibly on the boundary) in such a way that within the group the relative position after rescaling remains fixed.
Let us quickly describe the codimension 1 strata of $\overline{C}_{n,m}$:

- *Strata of type I:* A subset S of $n' \geq 2$ out of n points in the upper half plane collapse at a point in the upper half plane. The relative position of the collapsing points is described by a configuration on the plane and the remaining points and the point of collapse are given by a configuration on $\mathbb{H} \cup \mathbb{R}$:

$$\partial_S \overline{C}_{n,m} = \overline{C}_{n'} \times \overline{C}_{n-n'+1,m}. \tag{3.35}$$

- *Strata of type II:* A subset S of n' out of n points in the upper half plane and a subset T of m' out of m points on the real line with $2n' + m' \geq 2$ collapse at a point on the real line. Both the relative position of the collapsing points as well as the remaining points together with the point of collapse are described by a configuration on $\mathbb{H} \cup \mathbb{R}$:

$$\partial_{S,T} \overline{C}_{n,m} = \overline{C}_{n',m'} \times \overline{C}_{n-n',m-m'+1}. \tag{3.36}$$

3.3.4 Universal Formula

The goal is now to give an explicit formula for an L_∞-morphism $\mathcal{U} : T_{\text{poly}}(M) \to D_{\text{poly}}(M)$ in the case where $M = \mathbb{R}^d$. In order to construct the morphism we will work with special sets of graphs. The graphs we work with will all be finite and oriented, and they will have no multiple edges between two vertices and no short loops (i.e. no edges starting and ending at the same vertex).

Definition 3.17. A graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that $E_\Gamma \subset (V_\Gamma \times V_\Gamma)\backslash\Delta$, where Δ is the diagonal.

Elements of V_Γ are called the *vertices* of Γ, elements of E_Γ are the *edges*. If $e = (v_1, v_2) \in E_\Gamma$ is an edge, then we say that e starts at v_1 and ends at v_2.

Definition 3.18. An *admissible graph* is a graph Γ such that

- The set of vertices V_Γ is $\{1, \dots, n\} \sqcup \{\bar{1}, \dots, \bar{m}\}$, where $n, m \in \mathbb{N}$ and $2n + m - 2 \geq 0$. Vertices from the set $\{1, \dots, n\}$ are called vertices of the first type, vertices from $\{\bar{1}, \dots, \bar{m}\}$ are called vertices of the second type.
- Every edge $(v_1, v_2) \in E_\Gamma$ starts at a vertex of first type, $v_1 \in \{1, \dots, n\}$.
- For every vertex $k \in \{1, \dots, n\}$ of the first type, the set of edges

$$\text{Star}(k) := \{(v_1, v_2) \in E_\Gamma |\ v_1 = k\} \tag{3.37}$$

  starting from k is labeled by symbols $\{e_k^1, \dots, e_k^{\#\text{Star}(k)}\}$.

Now consider the upper half plane $\mathbb{H} = \{z \in \mathbb{C} |\ \text{Im}(z) > 0\}$ endowed with the Poincaré metric and let $u, v \in \mathbb{H}$, $u \neq v$. For our construction we will use the harmonic angle function

$$\varphi(u, v) := \arg\left(\frac{v - u}{v - \bar{u}}\right) = \frac{1}{2i}\log\left(\frac{(v - u)(\bar{v} - u)}{(v - \bar{u})(\bar{v} - \bar{u})}\right) \in \mathbb{R}/2\pi\mathbb{Z}, \tag{3.38}$$

which measures the angle at u formed by the two geodesic lines $l(u, v)$ passing through u and v and $l(u, \infty)$ through u and the point ∞. The direction of the measurement of the angle is counterclockwise from $l(u, \infty)$ to $l(u, v)$. By continuity $\varphi(u, v)$ can also be defined for $u, v \in \mathbb{H} \cup \mathbb{R}$, $u \neq v$. The angle function is depicted in figure 4 below.

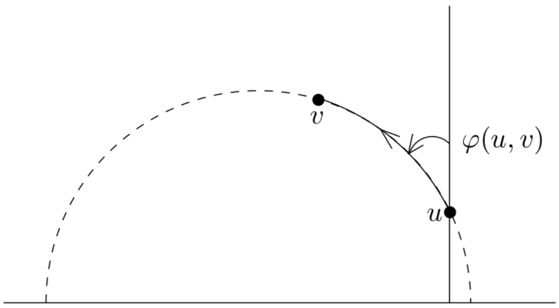


Figure 4: The harmonic angle function on $\mathbb{H} \cup \mathbb{R}$.

As introduced above, the harmonic angle function is a smooth map $\varphi : \text{Conf}_{2,0} \to \mathbb{R}/2\pi\mathbb{Z}$, which then induces a smooth map $C_{2,0} \to \mathbb{R}/2\pi\mathbb{Z}$. This map in turn can then be extended to the compactification $\overline{C}_{2,0}$. So all in all we get a smooth map

$$\varphi : \overline{C}_{2,0} \to \mathbb{R}/2\pi\mathbb{Z} \tag{3.39}$$

and we will denote $\varphi([(x, y)])$ simply by $\varphi(x, y)$ for $x, y \in \mathbb{H} \cup \mathbb{R}$, $x \neq y$.
It's not hard to see that the restriction of φ to the boundary component $\overline{C}_2 \cong S^1$ of $\overline{C}_{2,0}$ is the angle measured in anti-clockwise direction from the vertical line, and that $\mathrm{d}\varphi(x, y) = 0$ for $x \in \mathbb{R}$.

Let us define the set $G_{n,m}$ of admissible graphs Γ with n vertices of the first type, m vertices of the second type and exactely $2n+m-2$ edges (recall that the dimension of $\overline{C}_{n,m}$ is precisely $2n+m-2$).
Now to a graph $\Gamma \in G_{n,m}$ we associate the weight

$$w_\Gamma = \prod_{k=1}^{n} \frac{1}{(\#\mathrm{Star}(k))!} \frac{1}{(2\pi)^{2n+m-2}} \int_{\overline{C}_{n,m}} \bigwedge_{e \in E_\Gamma} \mathrm{d}\varphi_e. \tag{3.40}$$

At this point a few remarks concerning the above formula are in order:

Remark 3.8. The orientation of $\mathrm{Conf}_{n,m}$ is the product of the standard orientation on the coordinate space $\mathbb{R}^m \supset \{(q_1, \dots, q_m)|\, q_1 < \dots < q_m\}$ with the product of standard orientations on the plane $\mathbb{R}^2$ (for points $x_i \in \mathbb{H} \subset \mathbb{R}^2$). The group $G^{(1)}$ is even dimensional and naturally oriented because it acts freely and transitively on the complex manifold $\mathbb{H}$. Thus, the quotient space $C_{n,m} = \mathrm{Conf}_{n,m}/G^{(1)}$ carries again a natural orientation.

Remark 3.9. Every edge e of Γ defines a map from $\overline{C}_{n,m}$ to $\overline{C}_{2,0}$. The pullback of the function φ by the map $\overline{C}_{n,m} \to \overline{C}_{2,0}$ corresponding to an edge e is denoted by φ_e.

Remark 3.10. The ordering in the wedge product of 1-forms $\mathrm{d}\varphi_e$ is fixed by enumeration of the set of sources of edges and by the enumeration of the set of edges with a given source.

Finally, we observe that the integral in the definition of the weight w_Γ is absolutely convergent because it is an integral of a smooth differential top form over a compact manifold with corners.

Now for a graph $\Gamma \in G_{n,m}$ we will define a linear map

$$\mathcal{U}_\Gamma : \bigwedge^{n} T_{\mathrm{poly}}(\mathbb{R}^d) \to D_{\mathrm{poly}}(\mathbb{R}^d)[1-n]. \tag{3.41}$$

Set $k_i := \#\mathrm{Star}(i)$ for the vertices $i \in \{1, \dots, n\}$ of first type of the graph Γ. Then $\mathcal{U}_\Gamma$ has only one nonzero graded component, namely $(\mathcal{U}_\Gamma)_{(k_1-1,\dots,k_n-1)}$. For polyvector fields $\gamma_1, \dots, \gamma_n \in \Gamma(M, \bigwedge^\bullet TM)$ of degrees $k_1, \dots, k_n$ and functions $f_1, \dots, f_m$ on $\mathbb{R}^d$ we get a function

$$F := \big(\mathcal{U}_\Gamma(\gamma_1 \wedge \dots \wedge \gamma_n)\big)(f_1 \otimes \dots \otimes f_m) \tag{3.42}$$

on $\mathbb{R}^d$. It is a sum over all configurations of indices running from 1 to d, labeled by the edges E_Γ:

$$F = \sum_{I: E_\Gamma \to \{1,\dots,d\}} F_I. \tag{3.43}$$

The functions F_I are defined as follows: To a vertex $1 \le i \le n$ of first type we associate a function g_i on $\mathbb{R}^d$ which is a coefficient of the polyvector field γ_i, namely

$$g_i = \gamma_i^{I(e_i^1)\cdots I(e_i^{k_i})}. \tag{3.44}$$

Remark 3.11. Here we use the following convention: A polyvector field $\gamma \in \Gamma(M, \bigwedge^k T\mathbb{R}^d)$ is locally written as

$$\gamma = \sum_{i_1,\dots,i_k} \frac{1}{k!} \gamma^{i_1\cdots i_k} \frac{\partial}{\partial_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial_{i_k}} \tag{3.45}$$

with coefficient functions $\gamma^{i_1,\dots,i_k}$.

To a vertex $\bar{j}$ of second type we associate the function $g_{\bar{j}} = f_j$.
Finally, the function F_I is the product of the partial derivatives

$$\Big(\prod_{e \in E_\Gamma, e=(*,\nu)} \partial_{I(e)}\Big) g_\nu \tag{3.46}$$

over all $n+m$ vertices ν of Γ.

Example. Consider the graph $\Gamma \in G_{1,2}$ in figure 5 below, with a bivector field $\pi = \sum_{i,j} \frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$ and functions f, g on $\mathbb{R}^d$.

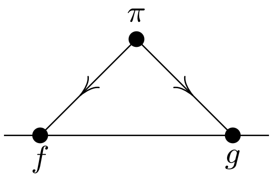


Figure 5: The graph $\Gamma \in G_{1,2}$

Then according to the description above we get that

$$\big(\mathcal{U}_\Gamma(\pi)\big)(f \otimes g) = \sum_{i,j=1}^{d} \pi^{ij}\partial_i(f)\partial_j(g). \tag{3.47}$$

Remark 3.12. Note that the construction of the function F from the graph Γ, the polyvector fields γ_i and the functions f_j is invariant under the action of the group of affine transformations of $\mathbb{R}^d$ because we contract upper and lower indices.

Now let us define a map

$$\mathcal{U} : T_{\text{poly}}(M) \to D_{\text{poly}}(M) \tag{3.48}$$

via its nth Taylor coefficients $\mathcal{U}_n$ for $n \geq 1$ (cf. Remark 3.5), regarded as linear maps from $\bigwedge^n T_{\text{poly}}(M)$ to $D_{\text{poly}}(M)[1-n]$ given by

$$\mathcal{U}_n := \sum_{m\geq 0} \sum_{\Gamma \in G_{n,m}} w_\Gamma \times \mathcal{U}_\Gamma. \tag{3.49}$$

We then have the following:

Theorem 3.4 (Kontsevich; [24])**.** The map $\mathcal{U}$ is an L_∞-morphism and also a quasi-isomorphism.

3.3.5 Quantization Formula

Now let again $M = \mathbb{R}^d$ and let π be a Poisson bivector field on $\mathbb{R}^d$. As explained in [24], using the formality map $\mathcal{U}$ we obtain a star product $\star$ on $\mathbb{R}^d$ from the Poisson bivector field π as follows:

$$f \star g := \sum_{n\geq 0} \sum_{\Gamma \in G_{n,2}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\pi \wedge \cdots \wedge \pi)(f \otimes g) \tag{3.50}$$

for $f, g \in \mathcal{C}^\infty(\mathbb{R}^d)$.
Let us have a closer look at this formula: First of all we note that for the definition of the star product we use all linear maps $\mathcal{U}_n$, including the case $n = 0$, i.e. we work with $\mathcal{U} + \mathcal{U}_0$. The term $\mathcal{U}_0$ comes from the unique graph $\Gamma_0 \in G_{0,2}$ with two vertices of second type, no vertices of first type and no edges at all. It's not hard to see that $w_{\Gamma_0} = 1$, $\mathcal{U}_0 = \mathcal{U}_{\Gamma_0}$ and $\mathcal{U}_{\Gamma_0}(f \otimes g) = fg$.
Furthermore, since $\mathcal{U}_\Gamma$ has only one nonzero graded component and since we are only working with a bivector field π, the sum in (3.50) could be taken over only the set $\widetilde{G}_{n,2} \subset G_{n,2}$ consisting of admissible graphs where each vertex of first type has exactely two outgoing edges.
Finally we have the following:

Theorem 3.5 (Kontsevich; [24])**.** Let π be a Poisson bivector field on an open subset U of $\mathbb{R}^d$. The *Kontsevich quantization formula*

$$f \star g = fg + \sum_{n\geq 1} \sum_{\Gamma \in \widetilde{G}_{n,2}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\pi \wedge \cdots \wedge \pi)(f \otimes g) \tag{3.51}$$

defines a star product on U. Furthermore, under a change of coordinates one obtains a gauge equivalent star product.

Example. Let us consider a constant Poisson structure $\pi = \sum_{i,j} \pi^{ij}\partial_i \wedge \partial_j$ on $\mathbb{R}^d$. In that case all the partial derivatives of the functions π^{ij} vanish and we only have to consider graphs where all the vertices of first type have no incoming and exactely two outgoing edges. The weights of the remaining graphs will be explicitly calculated in section 4.3 below. For now we will just state the result: In the case of a constant Poisson structure the Kontsevich quantization formula reads

$$f \star g = fg + \frac{\hbar}{2}\sum_{i,j} \pi^{ij}(\partial_i f)(\partial_j g) + \frac{\hbar^2}{8}\sum_{i,j,k,l} \pi^{ij}\pi^{kl}(\partial_i\partial_k f)(\partial_j\partial_l g) + \dots, \tag{3.52}$$

which is precisely the Moyal product (3.18).

3.3.6 Globalization

In this section we will describe the globalization of Kontsevich's local formula (3.50), mainly following [11] and Appendix B of [15] . We will do so by using Kontsevich's formula fiberwise on the tangent bundle of a Poisson manifold M.

We again start in the local case, i.e. let π be a Poisson bivector field on $\mathbb{R}^d$ and let ξ, ζ be any two vector fields on $\mathbb{R}^d$. Let us define

$$P(\pi) := \sum_{n\geq 0}\sum_{\Gamma \in G_{n,2}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\pi \wedge \dots \wedge \pi), \tag{3.53}$$

$$A(\xi, \pi) := \sum_{n\geq 0}\sum_{\Gamma \in G_{n+1,1}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi \wedge \pi \wedge \dots \wedge \pi), \tag{3.54}$$

$$F(\xi, \zeta, \pi) := \sum_{n\geq 0}\sum_{\Gamma \in G_{n+2,0}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi \wedge \zeta \wedge \pi \wedge \dots \wedge \pi). \tag{3.55}$$

By definition of the maps $\mathcal{U}_\Gamma$ it follows immidiately that $P(\pi)$ is a bidifferential operator, $A(\xi,\pi)$ a differential operator and $F(\xi,\zeta,\pi)$ a smooth function. Furthermore, observe that $P(\pi)(f \otimes g) = f \star g$ is precisely Kontsevich's star product (3.50). So in particular, it's a deformation of the commutative pointwise product fg along the direction of the Poisson structure π. Similarly, $A(\xi, \pi) = \xi + \mathcal{O}(\hbar)$ is a deformation of the Lie derivative compatible with the star product. This deformed Lie derivative is however not a Lie algebra homomorphism and F actually measures its failure to be one.

Remark 3.13. Later on we will interpret A as a connection 1-form and F as its curvature 2-form.

Now let (M, π) be a Poisson manifold. We will use the notions of formal geometry introduced in section 2.2.1 above. In particular, we will choose a formal exponential map $\phi : U \to M$, with $U \subset TM$ an open neighbourhood of the zero section, and we will consider the Grothendieck connection $D_G = \mathrm{d} + R$ on $\widehat{S}T^*M$, which, in local coordinates (x^i) on the base and (y^i) on the fiber, is given by (2.39). As explained in [11], D_G is a derivation, D_G is a flat connection and the subalgebra of D_G-closed sections is isomorphic to the algebra of smooth functions on M.
Now as is mentioned in Remark 2.11 above, the Taylor expansion $T\phi_x^*\pi$ of π at x is still Poisson with values in $\widehat{S}T_x^*M$. In particular $P(T\phi^*\pi)$ defines pointwise (with respect to $x \in M$) Kontsevich's star product on sections of $\widehat{S}T^*M$:

$$\sigma \star \tau := P(T\phi^*\pi)(\sigma \otimes \tau), \qquad \sigma, \tau \in \Gamma(\widehat{S}T^*M), \tag{3.56}$$

which is a deformation of the pointwise product in the direction of $T\phi^*\pi$.
Now let us consider the bundle $\mathcal{E} := \widehat{S}T^*M[[\hbar]]$. Our goal now is to find a subalgebra $\mathcal{A} \subset \Gamma(\mathcal{E})$ that is a deformation quantization of the subalgebra $\mathcal{C}^\infty(M) \subset \Gamma(\mathcal{E})$. The subalgebra $\mathcal{A}$ will be given by closed sections with respect to a deformation of the Grothendieck connection. For this to work, the deformed Grothendieck connection must in particular be flat and it must be a derivation. We will perform this

deformation in two steps:
For a tangent vector $\xi \in T_xM$ let us define $\widehat{\xi}$ by

$$\widehat{\xi}(x;y) = \iota_\xi R(x;y) = \xi^i Y_i^k(x;y)\frac{\partial}{\partial y^k}. \tag{3.57}$$

Then we can define

$$\mathcal{D}_G^\xi := \xi + A(\widehat{\xi}, T\phi_x^*\pi). \tag{3.58}$$

This can be written as

$$\mathcal{D}_G = \mathrm{d} + A(R, T\phi^*\pi), \tag{3.59}$$

where we interpret $A(R, T\phi^*\pi)$ as a 1-form with values in differential operators on $\mathcal{E}$. In local coordinates (x^i) around some point $x \in M$ we can write

$$A(R_x, T\phi_x^*\pi) = \mathrm{d}x^i A(R_i(x), T\phi_x^*\pi). \tag{3.60}$$

As explained in [11], $\mathcal{D}_G$, which we call the *deformed Grothendieck connection*, is a globally well-defined connection on $\Gamma(\mathcal{E})$ and it is a derivation. In general, however, $\mathcal{D}_G$ is not flat. But at least $(\mathcal{D}_G)^2$ is a so-called *inner derivation*, i.e.

$$(\mathcal{D}_G)^2\sigma = [F^M, \sigma]_\star := F^M \star \sigma - \sigma \star F^M \tag{3.61}$$

for any section $\sigma \in \Gamma(\mathcal{E})$. Here F^M is the so-called *Weyl curvature* tensor of $\mathcal{D}_G$ and it is given by $F^M(\xi,\zeta) := F(\widehat{\xi}, \widehat{\zeta}, T\phi_x^*\pi)$ for two tangent vectors $\xi, \zeta \in T_xM$. F^M is a 2-form with values in sections of $\mathcal{E}$ and in local coordinates around a point $x \in M$ it can be written as

$$F_x^M = F(R_x, R_x, T\phi_x^*\pi) = \mathrm{d}x^i \wedge \mathrm{d}x^j F(R_i(x), R_j(x), T\phi_x^*\pi). \tag{3.62}$$

The formality theorem then implies the *Bianchi identity*

$$\mathcal{D}_G F^M = 0. \tag{3.63}$$

Now since the deformed Grothendieck connection $\mathcal{D}_G$ is not yet flat, we must modify it slightly such that it becomes flat but still remains a derivation. The way to do this is to define the so-called *modified deformed Grothendieck connection*

$$\overline{\mathcal{D}}_G := \mathcal{D}_G + [\gamma,\ .\ ]_\star \tag{3.64}$$

for some 1-form $\gamma \in \Omega^1(M,\mathcal{E})$. The connection defined this way is a derivation and it's Weyl curvature is given by

$$\overline{F}^M = F^M + \mathcal{D}_G\gamma + \gamma \star \gamma. \tag{3.65}$$

Remark 3.14. For any two 1-forms $\gamma, \sigma \in \Omega(M,\mathcal{E})$ one defines their star product as

$$\gamma \star \sigma := \sum_{i,j} (\gamma_i \star \sigma_j) dx^i \wedge dx^j. \tag{3.66}$$

It was shown in [11] that there exists a $\gamma \in \Omega^1(M,\mathcal{E})$ such that $\overline{F}^M = F^M + \mathcal{D}_G\gamma + \gamma\star\gamma = 0$. This in turn implies that $(\overline{\mathcal{D}}_G)^2 = 0$, i.e. that the modified deformed Grothendieck connection is flat (for this choice of γ). This then guarantees that $\overline{\mathcal{D}}_G$-closed sections form a nontrivial subalgebra $\mathcal{A} \subset \Gamma(\mathcal{E})$.

It now remains to see that the subalgebra $\mathcal{A}$ of $\overline{\mathcal{D}}_G$-closed sections provides a deformation quantization of $\mathcal{C}^\infty(M)$. We have mentioned above that $\mathcal{C}^\infty(M)$ is isomorphic to the subalgebra $A \subset \Gamma(\widehat{S}T^*M)$ of D_G-closed sections. Now $\mathcal{A}$ provides a deformation quantization of M if there exists a module isomorphism between $A[[\hbar]]$ and $\mathcal{A}$ such that the product on $\mathcal{A}$ is a deformation of the product on the image of $A[[\hbar]]$ in the direction of the Poisson bracket (plus the usual conditions, cf. Definition 3.9). More precisely, one can construct a map $\rho : \Gamma(\widehat{S}T^*M)[[\hbar]] \to \Gamma(\mathcal{E})$ that deforms the identity and which satisfies $\overline{\mathcal{D}}_G\rho(\sigma) = \rho(D_G\sigma)$ for all $\sigma \in \Gamma(\widehat{S}T^*M)[[\hbar]]$. Such a map is called a *quantization map*. As explained in [11], such a map always exists and it is possible to find one of the form $\rho = \mathrm{Id} + \sum_{k\geq 1} \hbar^k \rho_k$, where the ρ_k's are differential operators with respect to y of order $\leq k$, vanishing on constants and depending smoothly on $x \in M$. Moreover, for a given connection $\overline{\mathcal{D}}_G$ there is a unique ρ satisfying $\rho\big|_{y=0} = \mathrm{Id}$. Finally, for $f, g \in \mathcal{C}^\infty(M)$ one can define a global star product on M as follows:

$$f \star_M g := \Big(\rho^{-1}\big(\rho(T\phi^* f) \star \rho(T\phi^* g)\big)\Big)\Big|_{y=0}. \tag{3.67}$$

So $\mathcal{A}$ indeed provides a deformation quantization of $\mathcal{C}^\infty(M)$.

Remark 3.15. One may be tempted to define a product

$$(f \bullet g)(x) := \big(P(T\phi^*\pi)(T\phi^* f \otimes T\phi^* g)\big)(x; 0). \tag{3.68}$$

This is indeed a well-defined global product on $\mathcal{C}^\infty(M)[[\hbar]]$, but it is in general not associative. To make this product associative one has to introduce a quantization map ρ as above, which then again leads to the product (3.67).

3.3.7 Connection to the Poisson Sigma Model

As described in [9], Kontsevich's star product on $\mathbb{R}^d$ can be computed as the perturbative expansion of a certain correlation function in the Poisson sigma model on the closed disk with three points marked on the boundary and with values in $\mathbb{R}^d$. The corresponding path integral may be computed in terms of Feynman diagrams which precisely coincide with the graphs that appear in Kontsevich's quantization formula introduced above.
More precisely, consider the Poisson sigma model on a closed disk D with classical action functional

$$S_D = \int_D \Big(\langle \eta \stackrel{\wedge}{,} \mathrm{d}X\rangle + \frac{1}{2}\langle \pi(X), \eta \wedge \eta\rangle\Big) \tag{3.69}$$

as introduced in section 2.1.2. Given two smooth maps f and g on $\mathbb{R}^d$, their star product is given as the perturbative expansion (cf. section 2.2) of the following path integral:

$$f \star g(x) = \int_{X(\infty)=x} f(X(0))g(X(1))e^{\frac{i}{\hbar}S_D[(X,\eta)]}. \tag{3.70}$$

Here 0, 1 and ∞ are three cyclically ordered points on the boundary ∂D of the disk D (i.e. if we start at 0 and move counterclockwise on the circle we first meet 1 and then ∞). The path integral is taken over all $X : D \to \mathbb{R}^d$, $\eta \in \Omega^1(D, X^*T^*\mathbb{R}^d)$ satisfying the boundary conditions $X(\infty) = x$ and $\eta(u)(\xi) = 0$ for $u \in \partial D$ and ξ tangent to ∂D. Finally, the expansion is taken around the classical solution $X(u) = x$, $\eta(u) = 0$. For a more detailed discussion of the computation of the perturbative expansion of the above expectation value we refer to [9].
Now let (M, π) be a general Poisson manifold and consider a Poisson sigma model on a closed disk D with three marked points 0, 1 and ∞ on the boundary and with values in M. Given two smooth functions $f, g \in \mathcal{C}^\infty(M)$ we can again compute the expectation value of $f(X(0))g(X(1))$ (here we assume the same boundary conditions as above). More precisely, introducing a formal exponential map $\phi : TM \supset U \to M$ and linearizing our model as explained in section 2.2.2, we are actually dealing with the expectation value of $T\phi^* f(x; \widehat{X}(0))T\phi^* g(x; \widehat{X}(1))$. As explained in [11], computing the perturbative expansion of this expectation value yields the product $f \bullet g(x)$ introduced in Remark 3.15 above. This approach yields a well-defined global formula, but, as we have already mentioned before, the bullet product is in general not associative. As brought up in Remark 3.15, the remedy is to introduce a quantization map ρ, whose path integral interpretation is, as far as we know, not clear. A more in depth discussion of this problem and its connection to the presence of a boundary can be found in [11], section 6.

4 Computation of Kontsevich weights

Let (M, π) be a *symplectic* Poisson manifold (i.e. the induced map $\pi^{\#}$ is an isomorphism with an inverse induced by a symplectic form ω, so M is in particular also a symplectic manifold; cf. Proposition 3.3) and $\phi : TM \supset U \to M$ some formal exponential map. Anticipating the computation of the star product $P(T\phi^*\pi)$, the connection 1-form $A(R, T\phi^*\pi)$ and its curvature 2-form $F(R, R, T\phi^*\pi)$ as introduced in section 3.3.6, we will explicitly compute the Kontsevich weights of three families of graphs in this section, which we call product, connection and curvature graphs respectively.

As explained in detail in section 3.3, we are working on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ and we use the harmonic angle function

$$\varphi(u, v) = \arg\left(\frac{v - u}{v - \bar{u}}\right) = \frac{1}{2i}\log\left(\frac{(v - u)(\bar{v} - u)}{(v - \bar{u})(\bar{v} - \bar{u})}\right), \tag{4.1}$$

which measures the angle on $\mathbb{H} \cup \mathbb{R}$ as depicted in figure 4. The propagator used in the computation of the Kontsevich weights is then simply given by $\mathrm{d}\varphi(u, v)$ and is usually called the *Kontsevich propagator*. Now let $\Gamma \in G_{n,m}$ be an admissible graph (cf. Definition 3.18) with n vertices of first type, m vertices of second type and with $2n + m - 2$ edges. As in section 3.3, we then use this propagator to compute the Kontsevich weight w_Γ of Γ as follows:

$$w_\Gamma = \int\limits_{C_{n,m}} c_\Gamma, \tag{4.2}$$

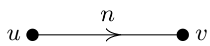
where $C_{n,m}$ is the $(2n + m - 2)$-dimensional configuration space introduced in section 3.3.3 and with smooth differential form

$$c_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \bigwedge_{\text{edges } e} d\varphi_e \tag{4.3}$$

on $C_{n,m}$ with the wedge product being over all $2n + m - 2$ edges e of the graph Γ.

Remark 4.1. In section 3.3 we worked over the compactified configuration spaces $\overline{C}_{n,m}$. But we will see below that all the integrals over the non-compactified configuration spaces $C_{n,m}$ of the graphs we are considering converge and are thus finite. So it's not necessary to introduce the compactifications.

To simplify the notation we will use graphical language where



corresponds to a factor of $d(\varphi(u, v)^n)/(2\pi)^n$ in w_Γ. If there is no n above the arrow this then simply means that $n = 1$.

Now we know that the dimension of the configuration space $C_{n,m}$ is $2n + m - 2$. And since we work on a symplectic Poisson manifold M (with Darboux coordinates around each point $x \in M$), a vertex of first type is either a $T\phi_x^*\pi$-vertex with precisely two outgoing and no incoming edges, or a R-vertex with precisely one outgoing edge and arbitrarily many incoming edges (this will be discussed in more detail in section 5 below). So we may write $n = p + r$, where p is the number of $T\phi_x^*\pi$-vertices and r is the number of R-vertices. We then have that $\deg(c_\Gamma) = 2p + r$, and in order for the integral (4.2) not to vanish, we must have that c_Γ is a top form, i.e. that $2n + m - 2 = 2p + r$. This then implies that

$$r + m = 2. \tag{4.4}$$

So we have to distinguish three different cases, namely $(r, m) = (2, 0)$, $(r, m) = (1, 1)$ and $(r, m) = (0, 2)$, which we will treat separately in what follows.

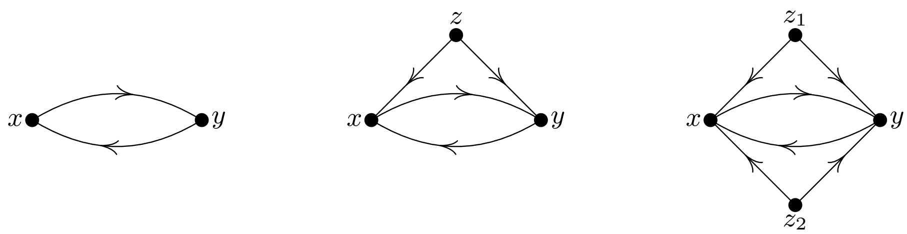
4.1 Kontsevich Weights of Curvature Graphs

We will first treat the case $(r, m) = (2, 0)$, i.e. the case where we have no boundary vertices and exactely two R-vertices. In that case we get a family of graphs $(\Gamma_n)_{n\geq 0}$, called the *curvature graphs*, where Γ_n is the graph with n wedges as in figure 6(a) (stemming from n $T\phi_x^*\pi$-vertices) attached to the wheel as in figure 6(b) (stemming from the two R-vertices).

(a) wedge (b) wheel

Figure 6: Curvature graphs consist of: (a) wedges stemming from $T\phi_x^*\pi$-vertices attached to (b) a wheel stemming from the two R-vertices.

Examples of the graphs Γ_n are given in figure 7 below for $n = 0, 1, 2$.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 7: Graphs Γ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the wheel.

The Kontsevich weight of the graph Γ_n for $n \geq 0$ is given by

$$w_{\Gamma_n} = \frac{1}{(2\pi)^{2n+2}} \int\limits_{C_{n+2,0}} \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x)\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,y)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,y). \tag{4.5}$$

Remark 4.2. For simplicity and to keep the expressions as short as possible we will drop the wedge product between the Kontsevich propagators from now on.

Remark 4.3. The sign of the weight c_Γ depends on the ordering of the edges of the graph Γ (i.e. the ordering of the propagator 1-forms in the integrand), and thus the ordering must always be specified. Throughout this whole section we will stick to the ordering given in (4.5) above.

The goal now is to compute (4.5) explicitly. We will do this in several steps, mainly using Stokes' theorem as in [31].

4.1.1 Step 1

In a first step we want to integrate out the wedges. More precisely, for a wedge as in figure 6(a) we want to compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y), \tag{4.6}$$

i.e. we want to integrate out z (with $x, y \in \mathbb{H}$ fixed). To do this we make a branch cut such that $\varphi(z,x) \in (0, 2\pi)$ and use Stokes' theorem

$$\int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y) = \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,y),$$

where ∂ is the boundary of the integration domain depicted in figure 8 below.

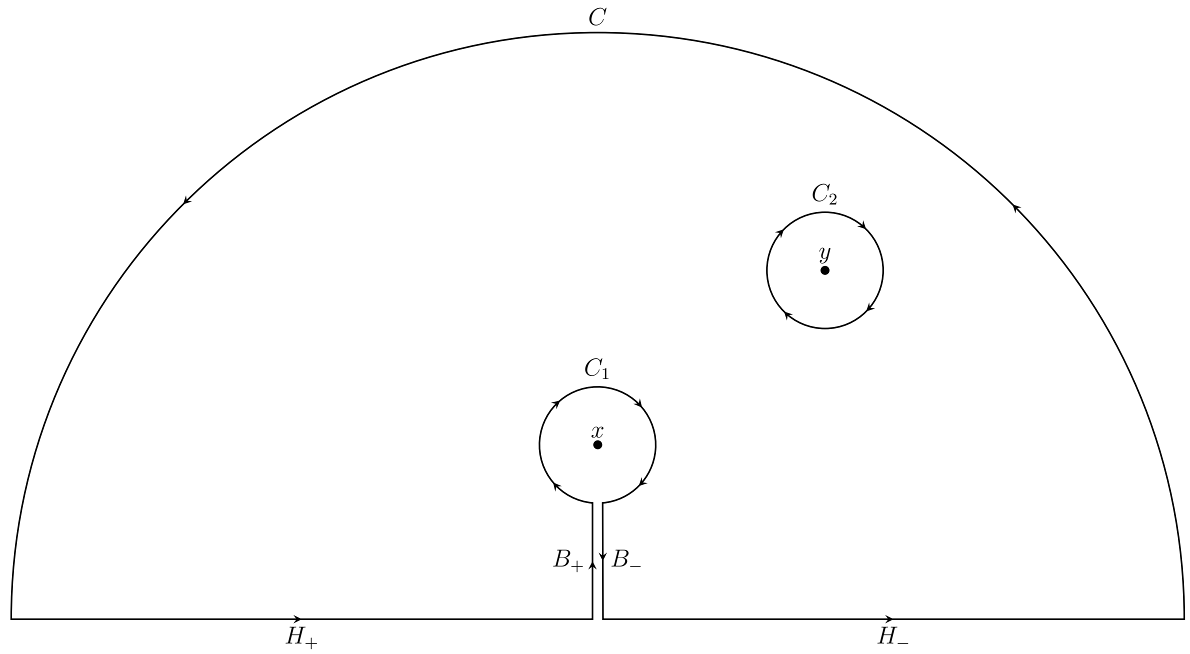


Figure 8: Boundary ∂ of the integration domain: C is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the circles C_1 and C_2 have infinitesimal radius and $H_+ \cup H_-$ is the real line.

Now using (3.38) we can discuss the different boundary components:

- On $H_+ \cup H_-$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z,y) = \mathrm{darg}(1) = 0$
- On B_+: $\varphi(z,x) = 2\pi$
- On B_-: $\varphi(z,x) = 0$
- On C_1: $z = x + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies \mathrm{d}\varphi(z,y) = \mathrm{darg}(\frac{y-x}{y-\bar{x}}) = 0$
- On C_2: $z = y + \epsilon e^{-i\theta}$ for $\epsilon \to 0$ and $\theta \in [0, 2\pi) \implies \varphi(z,x) \to \varphi(y,x)$, $\mathrm{d}\varphi(z,y) = -\mathrm{d}\theta$
- On C: $z = Re^{i\theta}$ for $R \to \infty$ and $\theta \in [0,\pi] \implies \varphi(z,x) = \varphi(z,y) = 2\theta$, $\mathrm{d}\varphi(z,y) = 2\mathrm{d}\theta$

We then finally get

$$\begin{aligned}\int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y) &= \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,y) = 2\pi \int\limits_{B_+} \mathrm{d}\varphi(z,y) + \int\limits_0^\pi 4\theta \mathrm{d}\theta - \varphi(y,x)\int\limits_0^{2\pi} \mathrm{d}\theta \\ &= 2\pi\big(\varphi(x,y) - \varphi(y,x) + [x;y]\pi\big),\end{aligned} \tag{4.7}$$

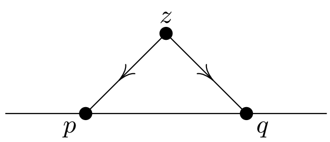
where

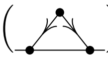
$$[x;y] = \begin{cases} +1, & \text{if } Re(x) > Re(y) \\ -1, & \text{if } Re(x) < Re(y). \end{cases} \tag{4.8}$$

Dividing the result (4.7) by $(2\pi)^2$ we then get

$$\frac{1}{2\pi}\big(\varphi(x,y)-\varphi(y,x)\big) \pm \frac{1}{2}, \tag{4.9}$$

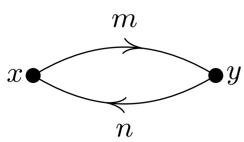
which agrees with the result given in [3], Lemma 5.3.
Finally, consider the limit $(x,y)\to(p,q)$ for $p,q\in\mathbb{R}$ with $p<q$. Using (4.9) and the fact that $\varphi(p,q)=2\pi$ and $\varphi(q,p)=0$, we can compute the Kontsevich weight of the graph



We get that $\int\limits_{C_{1,2}} \Big($  $\Big) = \frac{1}{2}$, which agrees with the result in [24], section 6.4.3.

4.1.2 Step 2

In a second step we want to compute the weight of the graph



for $n,m\geq 1$, i.e. we want to explicitly compute the integral

$$\frac{1}{(2\pi)^{n+m}}\int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n. \tag{4.10}$$

As before we make the branch cut such that $\varphi(x,y)\in(0,2\pi)$ and use Stokes' theorem

$$\int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n = \int\limits_{\partial} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n$$

with boundary ∂ of the integration domain depicted in figure 9 below.

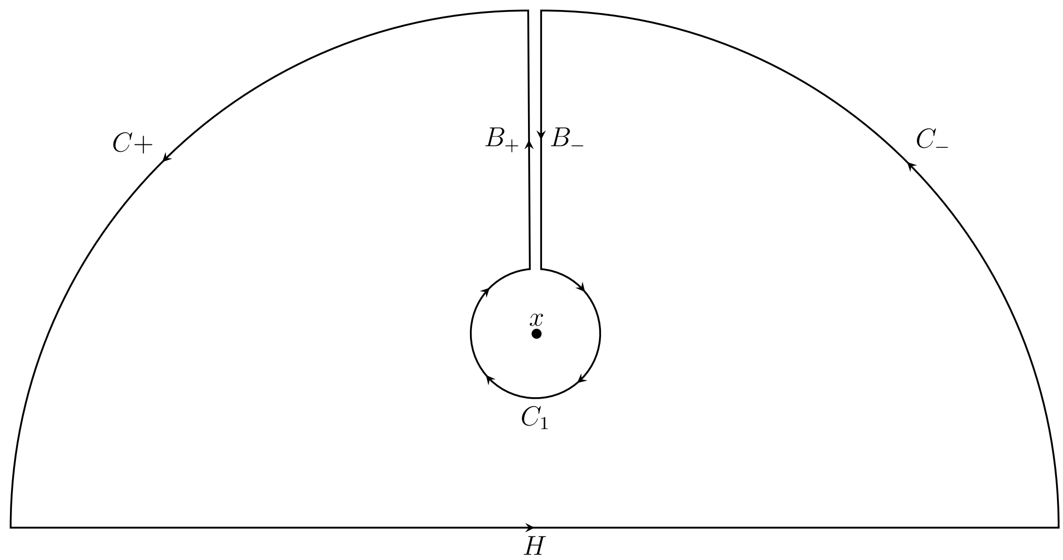


Figure 9: Boundary ∂ of the integration domain: $C_- \cup C_+$ is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the circle C_1 has infinitesimal radius and H is the real line.

Again we discuss the different boundary components:

- On H: $y \in \mathbb{R} \implies \mathrm{d}\varphi(y,x) = \mathrm{darg}(1) = 0$
- On $B_- \cup B_+$: $\mathrm{d}\varphi(y,x) = 0$
- On C_-: $y = Re^{i\theta}$ for $R \to \infty \implies \varphi(x,y) = 2\pi$
- On C_+: $y = Re^{i\theta}$ for $R \to \infty \implies \varphi(x,y) = 0$
- On C_1: $y = x + \epsilon e^{-i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$ and $\varphi(y,x) = \begin{cases} \frac{\pi}{2} - \theta, & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{5\pi}{2} - \theta, & \text{for } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$

With this, let us compute the integral

$$\int\limits_{y \in \mathbb{H} \setminus \{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n = \int\limits_{\partial} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n = (2\pi)^m \int\limits_{C_-} \mathrm{d}\varphi(y,x)^n + \int\limits_{C_1} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n$$
$$= (2\pi)^m \pi^n - n \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta - n \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta.$$

Now we use the substitution $a = \frac{\pi}{2} - \theta$ to compute

$$\begin{aligned} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta &= \int\limits_0^{\pi} (\pi + a)^m a^{n-1} \mathrm{d}a = \sum_{k=0}^{m} \binom{m}{k} \pi^k \int\limits_0^{\pi} a^{m+n-k-1} \mathrm{d}a \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{\pi^{m+n}}{m+n-k}. \end{aligned} \tag{4.11}$$

Similarly, we use the substitution $a = \frac{3\pi}{2} - \theta$ to compute

$$\begin{aligned}\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta = \int\limits_0^\pi a^m (\pi + a)^{n-1} \mathrm{d}a = \sum_{k=0}^{n-1} \binom{n-1}{k} \pi^k \int\limits_0^\pi a^{m+n-k-1} \mathrm{d}a \\ = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\pi^{m+n}}{m+n-k}.\end{aligned} \tag{4.12}$$

Putting everything together we then get

$$\int\limits_{y\in\mathbb{H}\setminus\{x\}} d\varphi(x,y)^m d\varphi(y,x)^n = \left(2^m - \sum_{k=0}^{m} \binom{m}{k} \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{n}{m+n-l}\right) \pi^{m+n}. \tag{4.13}$$

It's not hard to see that for $n = 1$ the above formula simplifies to

$$\int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x) = 2^m \left(1 - \frac{2}{m+1}\right) \pi^{m+1},$$

which agrees with the result in [31], section 4.

4.1.3 Step 3

In a third step we want to compute an integral similar to (4.10), but with an additional factor $[x;y]$ as defined in (4.8). So we want to compute the integral

$$\frac{1}{(2\pi)^{n+m}} \int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y] \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n. \tag{4.14}$$

As usual, we use Stokes' theorem:

$$\begin{aligned}\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y] \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n &= \int\limits_{\substack{y\in\mathbb{H}\setminus\{x\} \\ Re(y)<Re(x)}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int\limits_{\substack{y\in\mathbb{H}\setminus\{x\} \\ Re(y)>Re(x)}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ &= \int\limits_{\partial_+} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int\limits_{\partial_-} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n\end{aligned}$$

with boundaries ∂_+ and ∂_- of the integration domain depicted in figure 10 below.

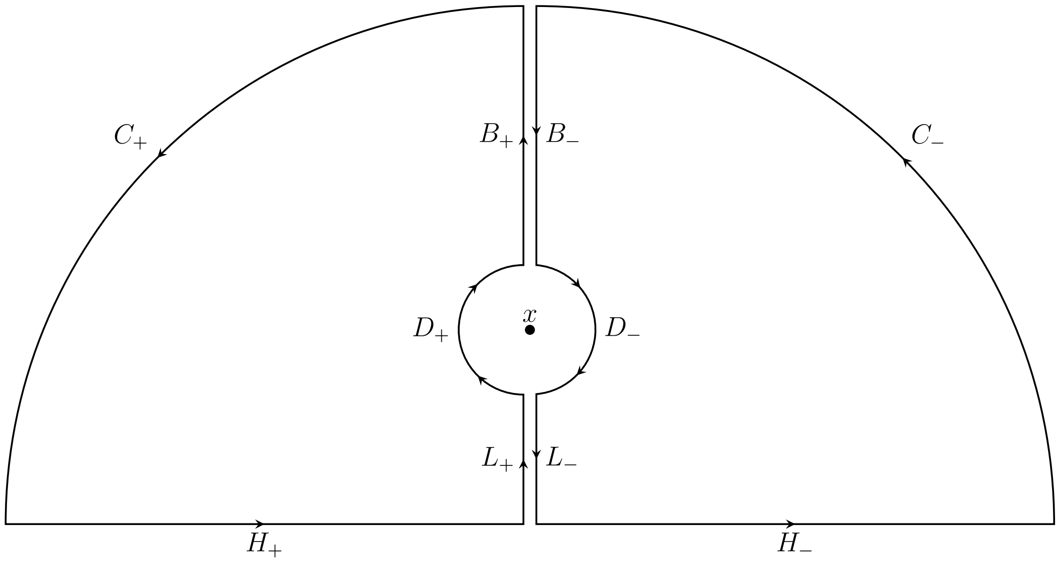


Figure 10: Boundaries ∂_+ (on the left) and ∂_- (on the right) of the integration domain: $C_- \cup C_+$ is the half-circle at infinity, B_+ and B_- as well as L_+ and L_- are infinitesimally close together, the circle $D_+ \cup D_-$ has infinitesimal radius and $H_+ \cup H_-$ is the real line.

As before, we discuss the different boundary components:

- On $H_+ \cup H_-$: $\mathrm{d}\varphi(y,x) = 0$
- On $B_\pm \cup L_\pm$: $\mathrm{d}\varphi(y,x) = 0$
- On C_-: $\varphi(x,y) = 2\pi$
- On C_+: $\varphi(x,y) = 0$
- On D_-: $y = x + \epsilon e^{-i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$, $\varphi(y,x) = \frac{\pi}{2} - \theta$
- On D_+: $y = x + \epsilon e^{-i\theta}$ for $\epsilon \to 0$ and $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$, $\varphi(y,x) = \frac{5\pi}{2} - \theta$

With this we compute the integral

$$\begin{aligned}
&\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]\mathrm{d}\varphi(x,y)^m d\varphi(y,x)^n = \int\limits_{\partial_+} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int\limits_{\partial_-} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\
&= -n \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta - (2\pi)^m \pi^n + n \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta \qquad (4.15) \\
&= \left(-2^m + \sum_{k=0}^{m} \binom{m}{k} \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{n}{m+n-l}\right) \pi^{m+n},
\end{aligned}$$

where we used (4.11) and (4.12) in the last step.

4.1.4 Putting Everything Together

Finally, we are able to compute the Kontsevich weight (4.5) of the graphs Γ_n described at the beginning of section 4.1. Integrating over z_i, $i = 1, \dots, n$, and applying the result (4.7) obtained in the first step we get

$$\begin{aligned}
w_{\Gamma_n} &= \frac{1}{(2\pi)^{2n+2}} \int\limits_{C_{n+2,0}} \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x)\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,y)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,y) \\
&= \frac{1}{(2\pi)^{n+2}} \int\limits_{y\in\mathbb{H}\setminus\{x\}} \big(\varphi(x,y)-\varphi(y,x)+[x;y]\pi\big)^n \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x) \\
&= \frac{1}{(2\pi)^{n+2}} \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}(-1)^l \int\limits_{y\in\mathbb{H}\setminus\{x\}} \varphi(x,y)^{n-k-l}\varphi(y,x)^l([x;y]\pi)^k \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x) \\
&= \frac{1}{2^{n+2}} \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^l}{\pi^{n-k+2}(n-k-l+1)(l+1)} \int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1}.
\end{aligned} \tag{4.16}$$

Now we note that for even k we have

$$\begin{aligned}
&\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} = \int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&= \left(2^{n-k-l+1} - \sum_{r=0}^{n-k-l+1}\binom{n-k-l+1}{r}\frac{l+1}{n-k-r+2} - \sum_{s=0}^{l}\binom{l}{s}\frac{l+1}{n-k-s+2}\right)\pi^{n-k+2},
\end{aligned}$$

where we have used (4.13). Similarly, for odd k we get

$$\begin{aligned}
&\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} = \int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]\mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&= \left(-2^{n-k-l+1} + \sum_{r=0}^{n-k-l+1}\binom{n-k-l+1}{r}\frac{l+1}{n-k-r+2} - \sum_{s=0}^{l}\binom{l}{s}\frac{l+1}{n-k-s+2}\right)\pi^{n-k+2},
\end{aligned}$$

where we have used (4.15).
We will now try to simplify the expressions we got. We start by observing a few things:
First of all, we clearly have that

$$\mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} = -\mathrm{d}\varphi(y,x)^{l+1}\mathrm{d}\varphi(x,y)^{n-k-l+1}. \tag{4.17}$$

Similarly, we also have that

$$[x;y] = -[y;x]. \tag{4.18}$$

Furthermore, we can obviously swap x and y in the integral and get the same result, i.e.

$$\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^m \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} = \int\limits_{x\in\mathbb{H}\setminus\{y\}} [y;x]^m \mathrm{d}\varphi(y,x)^{n-k-l+1}\mathrm{d}\varphi(x,y)^{l+1}. \tag{4.19}$$

Now assume that n is even. Applying (4.17), (4.18) and (4.19) to the last line of (4.16), it then follows that

$$w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}\binom{n-k}{\frac{n-k}{2}} \frac{(-1)^{\frac{n-k}{2}}}{\pi^{n-k+2}(\frac{n-k}{2}+1)^2} \int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^{\frac{n-k}{2}+1}\mathrm{d}\varphi(y,x)^{\frac{n-k}{2}+1}. \tag{4.20}$$

So most of the terms cancel for n even. Now using (4.13) we observe that

$$\frac{1}{\pi^{2m}} \int_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^m = 2^m - \sum_{k=0}^{m} \binom{m}{k} \frac{m}{2m-k} - \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{m}{2m-l}$$

$$= 2^m - \sum_{k=0}^{m-1} \left(\binom{m}{k} + \binom{m-1}{k} \right) \frac{m}{2m-k} - 1 = 2^m - \sum_{k=0}^{m-1} \binom{m}{k} - 1 = 2^m - \sum_{k=0}^{m} \binom{m}{k} = 0.$$

Plugging this result into (4.20) with $m = \frac{n-k}{2}$ we finally find that

$$w_{\Gamma_n} = 0$$

for $n \geq 0$, n even.

For n odd the different terms in the last line of (4.16) do not cancel anymore. Instead, we will try to write (4.13) and (4.15) more compactly. To do this, let us introduce the so-called *hypergeometric function* ${}_2F_1(a,b;c;z)$. It is defined by the series

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{4.21}$$

for $z \in \mathbb{C}$, $|z| < 1$, where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k = \begin{cases} 1, & \text{if } k = 0 \\ a(a+1)\cdots(a+k-1), & \text{if } k > 0. \end{cases} \tag{4.22}$$

It's not hard to see that the series terminates if either a or b is a non-positive integer. In that case the hypergeometric function reduces to a polynomial and can therefore also be defined for $|z| \geq 1$.

We can now use the hypergeometric function to write

$$\sum_{k=0}^{m} \binom{m}{k} \frac{1}{m+n-k} = \frac{{}_2F_1(-m,-m-n;1-m-n;-1)}{m+n}$$

and

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{m+n-k} = \frac{{}_2F_1(1-n,-m-n;1-m-n;-1)}{m+n}.$$

This then allows us to write (4.13) as

$$\int_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n$$
$$= \left(2^m - \frac{n}{m+n} \left({}_2F_1(-m,-m-n;1-m-n;-1) + {}_2F_1(1-n,-m-n;1-m-n;-1) \right) \right) \pi^{m+n}$$

and (4.15) as

$$\int_{y\in\mathbb{H}\setminus\{x\}} [x;y] \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n$$
$$= \left(-2^m + \frac{n}{m+n} \left({}_2F_1(-m,-m-n;1-m-n;-1) - {}_2F_1(1-n,-m-n;1-m-n;-1) \right) \right) \pi^{m+n}.$$

Plugging those results into (4.16) we finally get for all $n \geq 0$

$$\begin{aligned} w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \frac{(-1)^l}{(n-k-l+1)(l+1)} \Big((-1)^k 2^{n-k-l+1} \\ - \frac{l+1}{n-k+2} \big({}_2F_1(-l,-n+k-2;-n+k-1;-1) \\ + (-1)^k {}_2F_1(-n+k+l-1,-n+k-2;-n+k-1;-1) \big) \Big). \end{aligned} \tag{4.23}$$

The Kontsevich weights of the first few graphs are given in table 1 below.

n	0	1	2	3	4	5	6	7	8	9
w_{Γ_n}	0	$\frac{1}{24}$	0	$\frac{1}{320}$	0	$\frac{1}{2688}$	0	$\frac{1}{18432}$	0	$\frac{1}{112640}$

Table 1: Kontsevich weights of the curvature graphs Γ_n for $n = 0, 1, \ldots, 9$.

As a sanity check we have the following: For $n = 0$ the graph Γ_0 is just a wheel with two vertices (see figure 7(a)) and its weight is zero according to [24], Lemma 7.3. For $n = 1$ the graph Γ_1 is just a wheel with two spokes pointing outward (see figure 7(b)) and its weight is $1/24$ according to [31], Proposition 1.1. So at least for $n = 0, 1$ our formula (4.23) for the Kontsevich weights w_{Γ_n} produces the correct values.

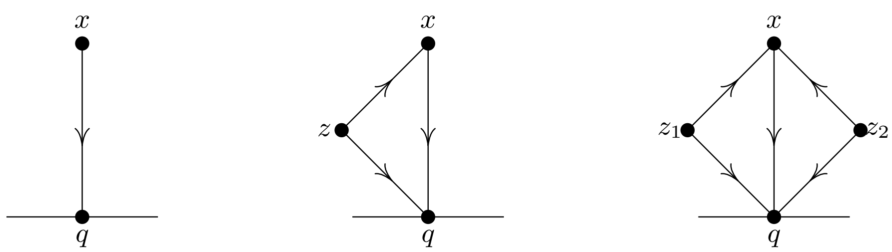
4.2 Kontsevich Weights of Connection Graphs

We will now treat the case $(r, m) = (1, 1)$, i.e. the case where we have one boundary vertex and one R-vertex. In that case we get a family of graphs $(\Upsilon_n)_{n\geq 0}$, called the *connection graphs*, where Υ_n is the graph with n wedges as in figure 11(a) (stemming from n $T\phi_x^*\pi$-vertices) attached to the graph containing a single edge from the R-vertex to the the boundary vertex as in figure 11(b) below.

(a) wedge (b) single edge

Figure 11: Connection graphs consist of: (a) wedges stemming from $T\phi_x^*\pi$-vertices attached to (b) a single edge from the R-vertex to the boundary vertex.

Examples of the graphs Υ_n are given in figure 12 below for $n = 0, 1, 2$.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 12: Graphs Υ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the single edge from the R-vertex to the boundary vertex.

The Kontsevich weight of the graph Υ_n for $n \geq 0$ is given by

$$w_{\Upsilon_n} = \frac{1}{(2\pi)^{2n+1}} \int\limits_{C_{n+1,1}} \mathrm{d}\varphi(x, q)\mathrm{d}\varphi(z_1, x)\mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, x)\mathrm{d}\varphi(z_n, q). \tag{4.24}$$

Remark 4.4. As before, the ordering of the edges of the graph Υ_n specified in (4.24) above determines the sign of w_{Υ_n}. Throughout this whole section we will stick to this ordering.

The goal now is to compute (4.24) explicitly. And as before, we will do this in several steps.

4.2.1 Step 1

For a wedge as in figure 11(a) we want to compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int_{z\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,q), \tag{4.25}$$

i.e. we want to integrate out z (with $x, q \in \mathbb{H}\cup\mathbb{R}$ fixed). The computation is almost the same as the one we have already done in section 4.1.1 above: Again we make a branch cut such that $\varphi(z,x) \in (0, 2\pi)$ and use Stokes' theorem:

$$\int_{z\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,q) = \int_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,q),$$

where ∂ is the boundary of the integration domain depicted in figure 13 below.

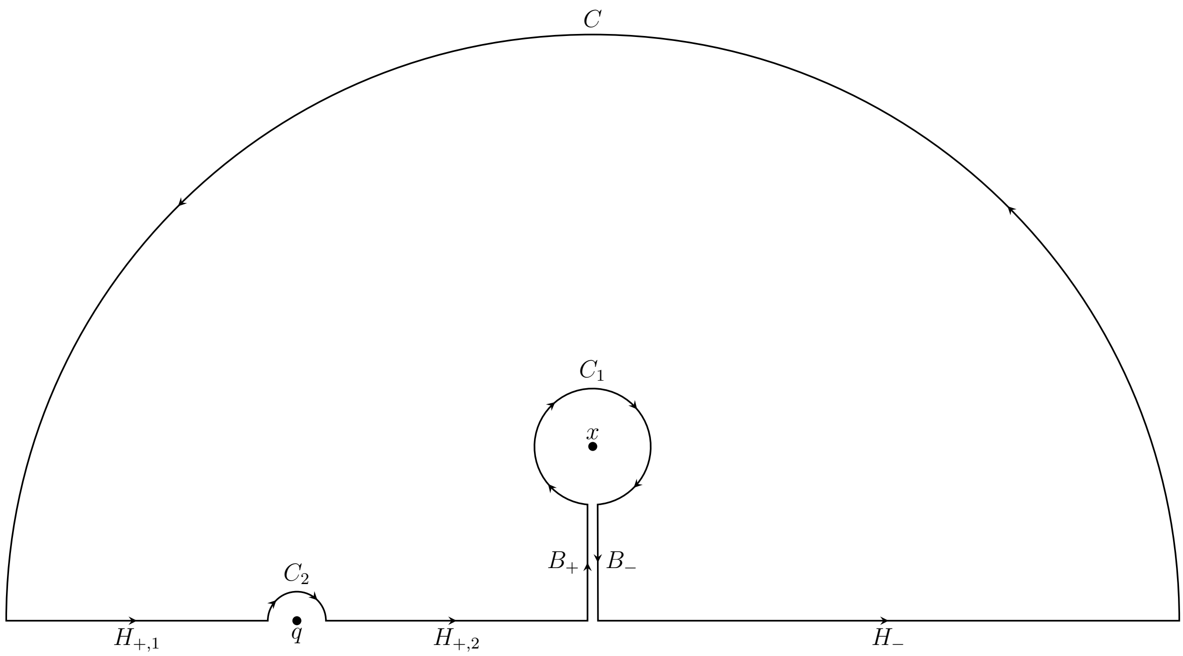


Figure 13: Boundary ∂ of the integration domain: C is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the (half) circles C_1 and C_2 have infinitesimal radius and $H_{+,1} \cup H_{+,2} \cup H_-$ is the real line.

Let us have a look at the different boundary components:

- On $H_{+,1} \cup H_{+,2} \cup H_-$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z,q) = 0$
- On B_+: $\varphi(z,x) = 2\pi$
- On B_-: $\varphi(z,x) = 0$
- On C_1: $z = x + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies \mathrm{d}\varphi(z,q) = \mathrm{d}\arg(\frac{q-x}{q-\bar{x}}) = 0$
- On C_2: $z = q + \epsilon e^{-i\theta}$ for $\epsilon \to 0$ and $\theta \in [-\pi, 0] \implies \varphi(z,x) \to \varphi(q,x)$, $\varphi(z,q) = -2\theta$

- On C: $z = Re^{i\theta}$ for $R \to \infty$ and $\theta \in [0, \pi] \implies \varphi(z, x) = \varphi(z, q) = 2\theta$

We can then compute the integral

$$\begin{aligned}\int\limits_{z\in\mathbb{H}\setminus\{x,q\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,q) &= \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,q) = 2\pi \int\limits_{B_+} \mathrm{d}\varphi(z,q) + \int\limits_0^\pi 4\theta \mathrm{d}\theta - 2\varphi(q,x)\int\limits_{-\pi}^0 \mathrm{d}\theta \\ &= 2\pi\big(\varphi(x,q) - \varphi(q,x) + [x;q]\pi\big),\end{aligned} \tag{4.26}$$

where

$$[x;q] = \begin{cases} +1, & \text{if } Re(x) > q \\ -1, & \text{if } Re(x) < q. \end{cases} \tag{4.27}$$

Dividing the result (4.26) by $(2\pi)^2$ we get

$$\frac{1}{2\pi}\big(\varphi(x,q) - \varphi(q,x)\big) \pm \frac{1}{2}, \tag{4.28}$$

which agrees with the result in [31], Lemma 5.3.
Finally, observe that one obtains (4.26) by simply taking the limit $y \to q \in \mathbb{R}$ in (4.7).

4.2.2 Step 2

In a second step we now want to compute the integral

$$\frac{1}{(2\pi)^{m+n}} \int\limits_{C_{1,1}} \varphi(q,x)^m \mathrm{d}\varphi(x,q)^n \tag{4.29}$$

for $n \geq 1$ and $m \geq 0$.
First note that $C_{1,1}$, shown in figure 14 below, is a smooth manifold of dimension 1 which is homeomorphic to an open interval.

Remark 4.5. We work with the standard orientation on $C_{1,1}$, which is induced by the standard orientation on the plane $\mathbb{R}^2$.

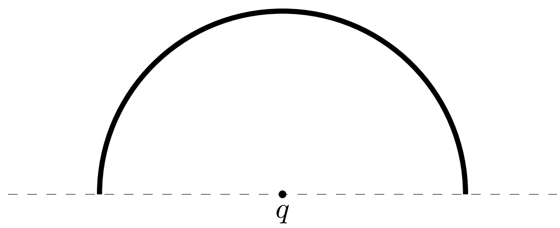


Figure 14: The manifold $C_{1,1}$ is the product of a (fixed) single point q on the real line and an open half circle.

Now it's not hard to see that the boundary $\partial C_{1,1}$ is just a two-element set. More precisely $\partial C_{1,1} = \{(q,s),(q,t)\}$ with $s < q$ and $t > q$ (for a more detailed treatment see [24]).
But now we have to make a branch cut such that $\varphi(q,x) \in (0, 2\pi)$. Then the boundary ∂ of the integration domain, depicted in figure 15 below, contains four points, namely

$$\partial = \{(q,s),(q,t),(q,y_+),(q,y_-)\}, \tag{4.30}$$

where y is the point on the half circle directly above q, i.e. with $\mathrm{Re}(y) = q$, and y_+ and y_- are the limits $x \to y$ on the half circle from the left (i.e. from the region $\mathrm{Re}(x) < q$ of the half circle) and from the right (i.e. from the region $\mathrm{Re}(x) > q$) respectively.

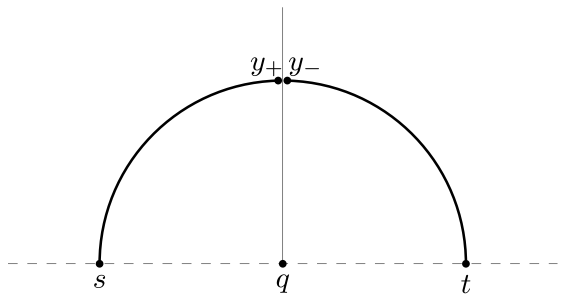


Figure 15: $C_{1,1}$ with branch cut and its boundary ∂ consisting of four points.

Finally, using Stokes' theorem and the fact that $\mathrm{d}\varphi(q,x) = 0$ for $q \in \mathbb{R}$, we find that

$$\begin{aligned}
&\int\limits_{C_{1,1}} \varphi(q,x)^m \mathrm{d}\varphi(x,q)^n = \int\limits_{\partial} \varphi(q,x)^m \varphi(x,q)^n \\
&= \varphi(q,s)^m\varphi(s,q)^n - \varphi(q,y_+)^m\varphi(y_+,q)^n + \varphi(q,y_-)^m\varphi(y_-,q)^n - \varphi(q,t)^m\varphi(t,q)^n \\
&= \begin{cases} (2\pi)^n, & \text{if } m = 0 \\ 2^m\pi^{m+n}, & \text{if } m > 0. \end{cases}
\end{aligned} \tag{4.31}$$

4.2.3 Step 3

Let us start with writing (4.26) as follows:

$$2\pi\big(\varphi(x,q) - \varphi(q,x) + \pi[x;q]\big) = 2\pi\big(\varphi(x,q) - \varphi(q,x) - \pi + 2\pi(x;q)\big), \tag{4.32}$$

where

$$(x;q) = \begin{cases} +1, & \text{if } Re(x) > q \\ 0, & \text{if } Re(x) < q. \end{cases} \tag{4.33}$$

In this step we then want to compute an integral similar to (4.29), but with an additional factor $(x;y)$ as defined above. So we want to compute

$$\frac{1}{(2\pi)^{m+n}} \int\limits_{C_{1,1}} (x;q)\varphi(q,x)^m \mathrm{d}\varphi(x,q)^n \tag{4.34}$$

for $n \geq 1$ and $m \geq 0$.
As before we use Stokes' theorem and find that

$$\begin{aligned}
\int\limits_{C_{1,1}} (x;q)\varphi(q,x)^m \mathrm{d}\varphi(x,q)^n &= \int\limits_{\substack{C_{1,1} \\ \mathrm{Re}(x)>q}} \varphi(q,x)^m \mathrm{d}\varphi(x,q)^n \\
&= \varphi(q,y_-)^m\varphi(y_-,q)^n - \varphi(q,t)^m\varphi(t,q)^n = 2^m\pi^{m+n}
\end{aligned} \tag{4.35}$$

for all $m \geq 0$ and all $n \geq 1$.

4.2.4 Putting Everything Together

Now we can use the results from steps 1-3 to compute the Kontsevich weight (4.24) of the graphs Υ_n described at the beginning of section 4.2 for $n \geq 0$. Integrating over z_i, $i = 1, \ldots, n$, and applying (4.32) we get

$$\begin{aligned}
w_{\Upsilon_n} &= \frac{1}{(2\pi)^{2n+1}} \int\limits_{C_{n+1,1}} \mathrm{d}\varphi(x,q)\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,q)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,q) \\
&= \frac{1}{(2\pi)^{n+1}} \int\limits_{C_{1,1}} \big(\varphi(x,q)-\varphi(q,x)-\pi+2\pi(x;q)\big)^n \mathrm{d}\varphi(x,q) \\
&= \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l} \binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}(-1)^{l+s} \\
&\int\limits_{C_{1,1}} \varphi(x,q)^{n-k-l-s}\varphi(q,x)^s\pi^l(2\pi(x;q))^k\mathrm{d}\varphi(x,q) \\
&= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l} \binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}\frac{(-1)^{l+s}}{2^{n-k+1}\pi^{n-k-l+1}(n-k-l-s+1)} \\
&\int\limits_{C_{1,1}} (x;q)^k\varphi(q,x)^s\mathrm{d}\varphi(x,q)^{n-k-l-s+1}.
\end{aligned} \tag{4.36}$$

We note that for $k=0$ we have

$$\int\limits_{C_{1,1}} \varphi(q,x)^s\mathrm{d}\varphi(x,q)^{n-l-s+1} = \begin{cases} (2\pi)^{n-l+1}, & \text{if } s=0 \\ 2^s\pi^{n-l+1}, & \text{if } s>0, \end{cases}$$

where we have used (4.31). Similarly, for $k \geq 1$ we have

$$\int\limits_{C_{1,1}} (x;q)^k\varphi(q,x)^s\mathrm{d}\varphi(x,q)^{n-k-l-s+1} = \int\limits_{C_{1,1}} (x;q)\varphi(q,x)^s\mathrm{d}\varphi(x,q)^{n-k-l-s+1} = 2^s\pi^{n-k-l+1},$$

where we have used (4.35).
Plugging the above two results into the last line of (4.36) we get

$$\begin{aligned}
w_{\Upsilon_n} = &\underbrace{\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l} \binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}\frac{(-1)^{l+s}}{2^{n-k-s+1}(n-k-l-s+1)}}_{=:A(n)} \\
&-\underbrace{\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^{n+1}(n-l+1)}}_{=:B(n)} + \underbrace{\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^l(n-l+1)}}_{=:C(n)}.
\end{aligned} \tag{4.37}$$

As shown in Appendix A, we have that

$$\begin{aligned}
A(n) &= \frac{(-1)^n}{2^{n+1}(n+1)}, \\
B(n) &= \frac{(-1)^n}{2^{n+1}(n+1)}, \\
C(n) &= \frac{1+(-1)^n}{2^{n+1}(n+1)}.
\end{aligned} \tag{4.38}$$

And hence we finally find that

$$w_{\Upsilon_n} = \frac{1+(-1)^n}{2^{n+1}(n+1)} \tag{4.39}$$

for $n \geq 0$. In particular we see that

$$w_{\Upsilon_n} = 0$$

for all $n \geq 1$, n odd.
The Kontsevich weights of the first few graphs are given in table 2 below.

n	0	1	2	3	4	5	6	7	8	9
w_{Υ_n}	1	0	$\frac{1}{12}$	0	$\frac{1}{80}$	0	$\frac{1}{448}$	0	$\frac{1}{2304}$	0

Table 2: Kontsevich weights of the graphs Υ_n for $n = 0, 1, \ldots, 9$.

As a sanity check we have the following: For $n = 0$ the graph Υ_0 is just a single edge as in figure 11(b) and its weight is 1 according to [24], section 6.4.3. For $n = 1$ the graph Υ_1 is just a single edge with one wedge attached as in figure 12(b) and its weight is 0 according to [20], Appendix B. For $n = 2$ the graph Υ_2 is a single edge with two wedges attached as in figure 12(c) and its weight is $\frac{1}{12}$ according to [26], Appendix A. So at least for $n = 0, 1, 2$ our formula (4.39) for the Kontsevich weights w_{Υ_n} produces the correct values.

4.3 Kontsevich Weights of Product Graphs

At last we will treat the case $(r, m) = (0, 2)$, i.e. the case where we have exactely two boundary vertices and no R-vertex. In that case we get a family of graphs $(\Lambda_n)_{n \geq 0}$, called the *product graphs*, where Λ_n is the graph with n wedges as in figure 16 (stemming from n $T\phi_x^*\pi$-vertices) attached to the two boundary vertices.

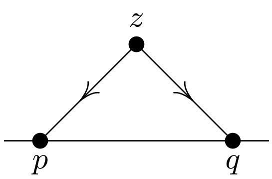


Figure 16: Product graphs consist of wedges attached to the two boundary vertices.

Examples of the graphs Λ_n are given in figure 17 below for $n = 0, 1, 2$.

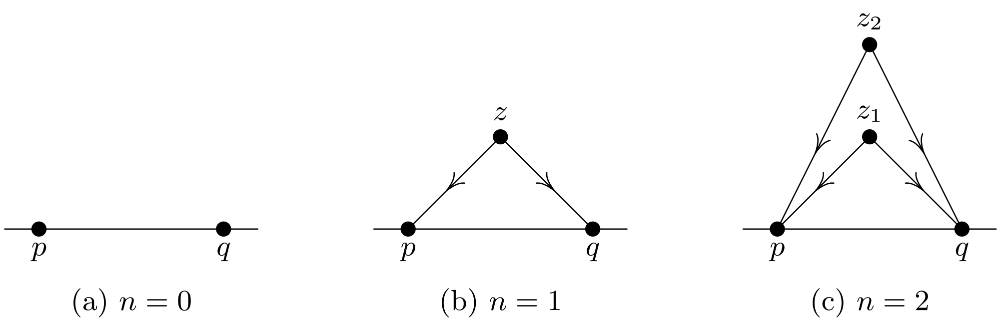


Figure 17: Graphs Λ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the two boundary vertices.

The Kontsevich weight of the graph Λ_n for $n \geq 0$ is given by

$$w_{\Lambda_n} = \frac{1}{(2\pi)^{2n}} \int\limits_{C_{n,2}} \mathrm{d}\varphi(z_1, p)\mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, p)\mathrm{d}\varphi(z_n, q). \tag{4.40}$$

Remark 4.6. For $n = 0$ we simply set $\mathrm{d}\varphi(z_1, p)\mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, p)\mathrm{d}\varphi(z_n, q) = 1$ in the integral above.

Remark 4.7. As before, the ordering of the edges of the graph Λ_n specified in (4.40) above determines the sign of w_{Λ_n}. Throughout this whole section we will stick to this ordering.

Now also for this third and final family of graphs our goal is to compute (4.24) explicitly. But this time the computation is much easier and shorter than before.
For the boundary vertices $p, q, \in \mathbb{R}$ with $p < q$ we have already computed the Kontsevich weight of a wedge as in figure 16 at the end of section 4.1.1. Our result was

$$\frac{1}{(2\pi)^2} \int\limits_{C_{1,2}} \mathrm{d}\varphi(z, p)\mathrm{d}\varphi(z, q) = \frac{1}{2}. \tag{4.41}$$

For completeness sake and to make sure that we get the same result let us nonetheless do a direct computation. So for a wedge as in figure 16 let us compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int\limits_{z \in \mathbb{H}} \mathrm{d}\varphi(z, p)\mathrm{d}\varphi(z, q) \tag{4.42}$$

with $p, q \in \mathbb{R}$, $p < q$ fixed. As before, we use Stokes' theorem:

$$\int\limits_{z \in \mathbb{H}} \mathrm{d}\varphi(z, p)\mathrm{d}\varphi(z, q) = \int\limits_{\partial} \varphi(z, p)\mathrm{d}\varphi(z, q),$$

where ∂ is the boundary of the integration domain depicted in figure 18 below.

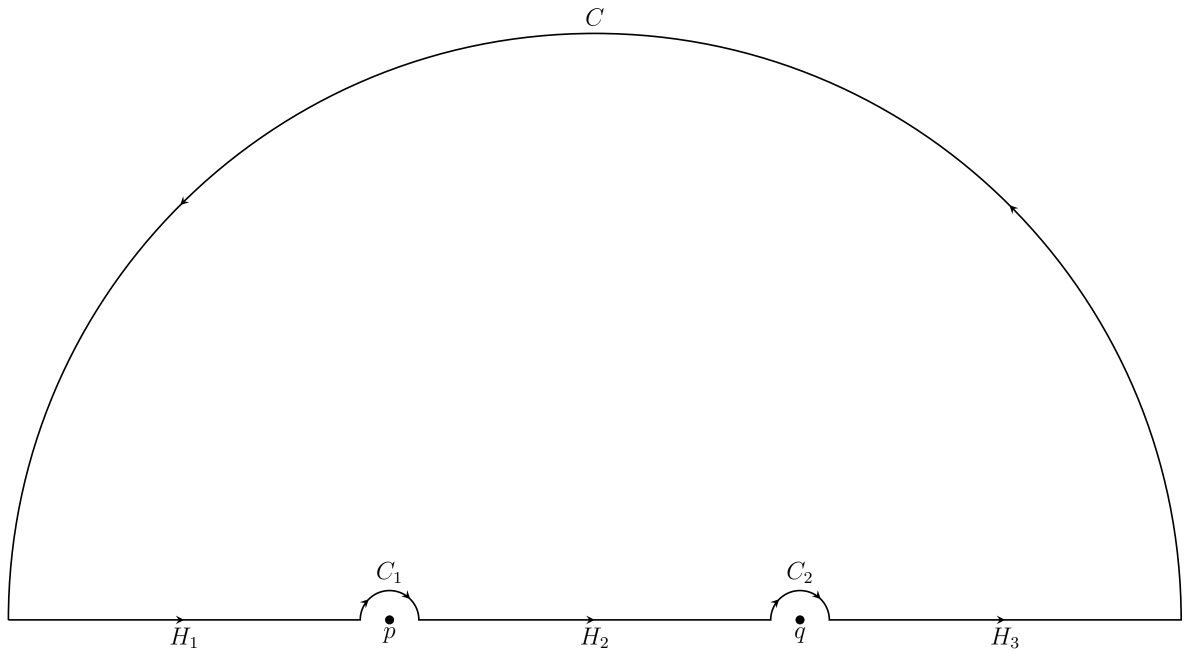


Figure 18: Boundary ∂ of the integration domain: C is the half-circle at infinity, the half circles C_1 and C_2 have infinitesimal radius and $H_1 \cup H_2 \cup H_3$ is the real line.

As usual, let us have a look at the different boundary components:

- On $H_1 \cup H_2 \cup H_3$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z, q) = 0$
- On C_1: $z = p + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies \mathrm{d}\varphi(z, q) = \mathrm{d}\arg(1) = 0$
- On C_2: $z = q + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies \varphi(z, p) \to \varphi(q, p) = 0$

- On C: $z = Re^{i\theta}$ for $R \to \infty$ and $\theta \in [0, \pi] \implies \varphi(z,p) = \varphi(z,q) = 2\theta$

We can then compute the integral

$$\int_{z\in\mathbb{H}} \mathrm{d}\varphi(z,p)\mathrm{d}\varphi(z,q) = \int_{\partial} \varphi(z,p)\mathrm{d}\varphi(z,q) = \int_0^{\pi} 4\theta\mathrm{d}\theta = 2\pi^2,$$

which indeed agrees with (4.41) after dividing by $(2\pi)^2$.
With this result at hand it is now easy to compute the Kontsevich weight of the graph Λ_n for $n \geq 1$:

$$w_{\Lambda_n} = \frac{1}{(2\pi)^{2n}} \int_{C_{n,2}} \mathrm{d}\varphi(z_1,p)\mathrm{d}\varphi(z_1,q)\cdots\mathrm{d}\varphi(z_n,p)\mathrm{d}\varphi(z_n,q) = \frac{1}{(2\pi)^{2n}}(2\pi^2)^n = \frac{1}{2^n}.$$

For $n = 0$ it's not hard to see that

$$w_{\Lambda_0} = \int_{C_{0,2}} 1 = 1$$

since $C_{0,2}$ is a single point. So all in all we finally find that

$$w_{\Lambda_n} = \frac{1}{2^n} \tag{4.43}$$

for all $n \geq 0$.

5 Deformation Quantization of Symplectic Poisson Manifolds

Let (M,π) be a Poisson manifold and $\phi : TM \supset U \to M$ a formal exponential map. In section 3.3.6 we have seen that the subalgebra $\mathcal{A}$ of $\overline{\mathcal{D}}_G$-closed sections, where $\overline{\mathcal{D}}_G$ is the modified deformed Grothendieck connection (3.64), provides a deformation quantization of $\mathcal{C}^\infty(M)$. In this section we will treat the case where M is a *symplectic* Poisson manifold and we are going to investigate the modified deformed Grothendieck connection more closely. To do this we will in particular have a closer look at the connection 1-form $A(R, T\phi^*\pi)$ and its Weyl curvature $F(R,R,T\phi^*\pi)$ introduced at the very beginning of section 3.3.6.

5.1 The Symplectic Case

So let (M,π) be a d-dimensional symplectic Poisson manifold and ϕ a formal exponential map as above. We are now going to have a look at the possible vertices of first type of the graphs which appear in the computation of $P(T\phi^*\pi)$, $A(R,T\phi^*\pi)$ and $F(R,R,T\phi^*\pi)$:
Fix $x \in M$, choose Darboux coordinates (x^i) around x and let (y^i) be the corresponding fiber coordinates of the tangent bundle. As remarked in [6] and [11], if the Poisson structure on M is symplectic, then it's always possible to find a formal exponential map $\phi : TM \to M$ such that $T\phi_x^*\pi(y) = T\phi^*\pi(x;y)$ is constant in y (and smooth in x). As a consequence, the $T\phi_x^*\pi$-vertex is bivalent, with precisely two outgoing edges labeled by i and j and no incoming edges. Now as for the Poisson sigma model (cf. section 2.2.3), we can decorate the $T\phi_x^*\pi$-vertex with the "vertex tensor" $\frac{1}{2}(T\phi_x^*\pi)^{ij}$, which will appear in the map $\mathcal{U}_\Gamma$ (3.41) defined in section 3.3.4.
Besides $T\phi_x^*\pi$-vertices stemming from the Poisson tensor $T\phi_x^*\pi$, we also have R-vertices stemming from the vector field $R(x;y) = R_i(x;y)\mathrm{d}x^i$ with

$$R_i(x;y) = Y_i^k(x;y)\frac{\partial}{\partial y^k} \tag{5.1}$$

defined in (2.39). Those R-vertices have one outgoing edge and, in general, arbitrarily many incoming edges. In a concrete case where an R-vertex has, let's say, l incoming edges labeled by $i_1,\dots,i_l$ and an outgoing edge labeled by k, we may decorate it with the "vertex tensor" $Y^k_{i,i_1\dots i_l}(x)\mathrm{d}x^i$ which will also appear in the map $\mathcal{U}_\Gamma$. The two possible vertices of first type are depicted in figure 19 below.

(a) $T\phi_x^*\pi$-vertex (b) R-vertex

Figure 19: The two possible vertices of first type.

Now that we know the possible vertices of first type, we can do a dimensional analysis as was already done at the beginning of section 4: Consider an admissible graph $\Gamma \in G_{n,m}$ with n vertices of first type, m vertices of second type and with $2n+m-2$ edges. Writing

$$w_\Gamma = \int_{C_{n,m}} c_\Gamma$$

as we have done in (4.2), we immidiately see that the smooth differential form c_Γ is of top degree $\deg(c_\Gamma) = 2n+m-2$ since the graph Γ has $2n+m-2$ edges. Now let us write $n = p+r$, where p is the number of $T\phi_x^*\pi$-vertices and r is the number of R-vertices of Γ. Then we also have that $\deg(c_\Gamma) = 2p + r$, and hence that $2n + m - 2 = 2p + r$. This then implies that

$$r + m = 2.$$

So one has to distinguish three different cases, namely $(r, m) = (2, 0)$, $(r, m) = (1, 1)$ and $(r, m) = (0, 2)$, as we have done in section 4.

Now we note that the graphs that appear in the computation of $P(T\phi_x^*\pi)$ contain precisely two vertices of second type, as follows from (3.53) in section 3.3.6. So we are in the case $(r, m) = (0, 2)$ and it's then not hard to see that the contributing graphs are precisely given by the family $(\Lambda_n)_{n\geq 0}$ treated in section 4.3, and hence the name product graphs.
Similarly, we note that the graphs that appear in the computation of $A(R, T\phi_x^*\pi)$ contain precisely one vertex of second type, as follows from (3.54). So we are in the case $(r, m) = (1, 1)$ and it's then immidiately clear that the contributing graphs are precisely given by the family $(\Upsilon_n)_{n\geq 0}$ treated in section 4.2, and hence the name connection graphs.
And finally, we note that the graphs that appear in the computation of $F(R, R, T\phi_x^*\pi)$ contain no vertices of second type, as follows from (3.55) in section 3.3.6. So we are in the case $(r, m) = (2, 0)$ and it then follows that the contributing graphs are precisely given by the family $(\Gamma_n)_{n\geq 0}$ treated in section 4.1, and hence the name curvature graphs.
So we see that we can now use the Kontsevich weights computed in section 4 to give explicit expressions for $P(T\phi_x^*\pi)$, $A(R, T\phi_x^*\pi)$ and $F(R, R, T\phi_x^*\pi)$.

5.1.1 The Product $P(T\phi^*\pi)$

In section 4.3 we obtained the Kontsevich weights

$$w_{\Lambda_n} = \frac{1}{2^n}, \qquad n \geq 0, \tag{5.2}$$

of the product graphs $(\Lambda_n)_{n\geq 0}$. Now let $\sigma, \tau \in \Gamma(\mathcal{E})$ with $\mathcal{E} = \widehat{S}T^*M[[\hbar]]$ as before and let $x \in M$. Using the Kontsevich weights above we get

$$P(T\phi_x^*\pi)(\sigma_x \otimes \tau_x) = \sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n} n!} (T\phi_x^*\pi)^{i_1 j_1} \cdots (T\phi_x^*\pi)^{i_n j_n} (\sigma_x)_{,i_1 \cdots i_n} (\tau_x)_{,j_1 \cdots j_n}, \tag{5.3}$$

where we sum over all the indices $i_1, \ldots, i_n, j_1, \ldots, j_n$ from 1 to $d = \dim(M)$.
This then also yields an explicit expression of the bullet product (3.68) introduced in Remark 3.15:

$$\begin{aligned}(f \bullet g)(x) &= \big(P(T\phi^*\pi)(T\phi^* f \otimes T\phi^* g)\big)(x; 0) \\ &= \left(\sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n} n!} (T\phi_x^*\pi)^{i_1 j_1} \cdots (T\phi_x^*\pi)^{i_n j_n} (T\phi_x^* f)_{,i_1 \cdots i_n} (T\phi_x^* g)_{,j_1 \cdots j_n}\right)(0).\end{aligned} \tag{5.4}$$

5.1.2 The Connection 1-form $A(R, T\phi^*\pi)$

In section 4.2 we obtained the Kontsevich weights

$$w_{\Upsilon_n} = \frac{1 + (-1)^n}{2^{n+1}(n+1)}, \qquad n \geq 0, \tag{5.5}$$

of the connection graphs $(\Upsilon_n)_{n\geq 0}$. As before, let $\sigma \in \Gamma(\mathcal{E})$ and fix $x \in M$. For R and Y as in (5.1) we set $R_x(y) := R(x; y)$ and $(Y_x)_i^k(y) := Y_i^k(x; y)$. Using the Kontsevich weights above we then get

$$\begin{aligned}A(R_x, T\phi_x^*\pi)(\sigma_x) &= \mathrm{d}x^i A\left((Y_x)_i^k \frac{\partial}{\partial y^k}, T\phi_x^*\pi\right)(\sigma_x) \\ &= \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \frac{1 + (-1)^n}{2^{n+1}(n+1)} (T\phi_x^*\pi)^{i_1 j_1} \cdots (T\phi_x^*\pi)^{i_n j_n} (Y_x)^k_{i, i_1 \cdots i_n} (\sigma_x)_{,k j_1 \cdots j_n},\end{aligned} \tag{5.6}$$

where we again sum over all indices $i, k, i_1, \dots, i_n, j_1, \dots, j_n$.
This then allows us to write down an explicit expression for the deformed Grothendieck connection (3.59), namely

$$\begin{aligned}(\mathcal{D}_G)_x &= \mathrm{d}_x + A(R_x, T\phi_x^*\pi) = \left(\frac{\partial}{\partial x^i} + A\big((R_x)_i, T\phi_x^*\pi\big)\right)\mathrm{d}x^i \\ &= \left(\frac{\partial}{\partial x^i} + \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}\frac{1+(-1)^n}{2^{n+1}(n+1)}(T\phi_x^*\pi)^{i_1 j_1}\cdots(T\phi_x^*\pi)^{i_n j_n}(Y_x)^k_{i,i_1\cdots i_n}\frac{\partial^{n+1}}{\partial y^{j_n}\cdots\partial y^{j_1}\partial y^k}\right)\mathrm{d}x^i.\end{aligned} \tag{5.7}$$

5.1.3 The Curvature 2-form $F(R, R, T\phi^*\pi)$

In section 4.1 we obtained the Kontsevich weights

$$\begin{aligned}w_{\Gamma_n} = \frac{1}{2^{n+2}}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^l}{(n-k-l+1)(l+1)}\Big((-1)^k 2^{n-k-l+1} \\ -\frac{l+1}{n-k+2}\big({}_2F_1(-l,-n+k-2;-n+k-1;-1) \\ +(-1)^k{}_2F_1(-n+k+l-1,-n+k-2;-n+k-1;-1)\big)\Big)\end{aligned} \tag{5.8}$$

of the curvature graphs $(\Gamma_n)_{n\geq 0}$ for $n \geq 0$. Using the Kontsevich weights above we then get for $x \in M$:

$$\begin{aligned}F(R_x, R_x, T\phi_x^*\pi) &= \mathrm{d}x^i \wedge \mathrm{d}x^j F\big((R_x)_i, (R_x)_j, T\phi_x^*\pi\big) \\ &= \mathrm{d}x^i \wedge \mathrm{d}x^j \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}w_{\Gamma_n}(T\phi_x^*\pi)^{i_1 j_1}\cdots(T\phi_x^*\pi)^{i_n j_n}(Y_x)^k_{i,li_1\cdots i_n}(Y_x)^l_{j,kj_1\cdots j_n},\end{aligned} \tag{5.9}$$

where, as usual, we sum over the indices $i, j, k, l, i_1, \dots, i_n, j_1, \dots, j_n$.
We can then write down the modified deformed Grothendieck connection as defined in (3.64), namely

$$\overline{\mathcal{D}}_G = \mathcal{D}_G + [\gamma,\ .\ ]_\star, \tag{5.10}$$

where the deformed Grothendieck connection $\mathcal{D}_G$ is explicitly given by (5.7), the star product is explicitly given by (5.3) and where $\gamma \in \Omega^1(M, \mathcal{E})$ is such that

$$F^M + \mathcal{D}_G\gamma + \gamma \star \gamma = 0 \tag{5.11}$$

with Weyl curvature $F^M = F(R, R, T\phi^*\pi)$ explicitly given by (5.9).

5.1.4 The 1-form γ

We have already mentioned that there always exists a 1-form γ satisfying (5.11). Following [11], we will now explain this in a little more detail: Since γ takes values in $\mathcal{E}$ we may write

$$\gamma = \gamma_0 + \hbar\gamma_1 + \hbar^2\gamma_2 + \dots. \tag{5.12}$$

Similarly, we may write

$$\mathcal{D}_G = D_G + \hbar^2\mathcal{D}_2 + \hbar^4\mathcal{D}_4 + \dots, \tag{5.13}$$

where $D_G = \mathrm{d} + R$ is the Grothendieck connection and where we have used that the Kontsevich weights (5.5) satisfy $w_{\Upsilon_0} = 1$ and $w_{\Upsilon_n} = 0$ for all odd $n \geq 1$.
Finally, we can also write

$$F^M = \hbar F_1 + \hbar^3 F_3 + \hbar^5 F_5 + \dots, \tag{5.14}$$

where we have used that that the Kontsevich weights (5.8) satisfy $w_{\Gamma_n} = 0$ for all even $n \geq 0$.
This now allows us to decompose equation (5.11) into a system of equations depending on the order of $\hbar$. In order $\hbar^0$ we get the equation

$$D_G \gamma_0 = 0, \tag{5.15}$$

which, according to Proposition 2.3, can be solved by $\gamma_0 = T\phi^* f$ for some smooth function $f \in \mathcal{C}^\infty(M)$. In order $\hbar^1$ we get the equation

$$F_1 + D_G \gamma_1 + (\gamma_0 \star \gamma_0)_1 = 0. \tag{5.16}$$

By the Bianchi identity (3.63) we see that $D_G F_1 = 0$ and by equation (5.15) above it also immidiately follows that $D_G(\gamma_0 \star \gamma_0)_1 = 0$. So we see that $D_G \gamma_1$ is equal to a D_G-closed form. But according to Proposition 2.4 the corresponding cohomology group is trivial, i.e. a D_G-closed form is also D_G-exact. So it follows that $D_G \gamma_1$ is equal to a D_G-exact form, and hence it is possible to find a γ_1 that solves equation (5.16) above.
By induction one can then show that in each order $\hbar^k$, $k \geq 1$, $D_G \gamma_k$ is equal to a D_G-closed and hence D_G-exact form $\alpha\big(F_{\leq k}, \gamma_{<k}\big)$ depending on the lower order coefficients of F^M and γ. In particular, it then follows that there exists a γ_k solving the equation $D_G \gamma_k = \alpha\big(F_{\leq k}, \gamma_{<k}\big)$.

5.2 The Case of a Cotangent Bundle

We will now consider a special case of a symplectic Poisson manifold, namely the case of a cotangent bundle T^*M with canonical Poisson structure (i.e. the Poisson structure corresponding to the canonical symplectic structure). We will see that we can lift a formal exponential map $\phi : TM \to M$ to T^*M, which will then lead to a simplification of the R-vertex shown in figure 19 above. This in turn will then reduce the number of possible graphs in the computation of the curvature 2-form F^M considerably.

5.2.1 Lifting Formal Exponential Maps to the Cotangent Bundle

Let M be a smooth manifold and $\phi : TM \to M$ a formal exponential map. First note that $TT^*M \cong \pi^*TM \oplus \pi^*T^*M$, where $\pi : T^*M \to M$ is the projection map of the bundle. So for $(q,p) \in T^*M$ we have that $T_{(q,p)}T^*M \cong T_qM \oplus T_q^*M$. Now we want to lift the formal exponential map ϕ to the cotangent bundle T^*M, i.e. to a map $\overline{\phi} : TT^*M \to T^*M$. This can be done via pullback: Consider

$$\overline{\phi}_{(q,p)} : T_{(q,p)}T^*M \cong T_qM \oplus T_q^*M \to T^*M$$

and let $(\bar{q}, \bar{p}) \in T_{(q,p)}T^*M \cong T_qM \oplus T_q^*M$. Note that $\mathrm{d}_{\bar{q}}\phi$ is invertible on an appropriately chosen neighbourhood $U \subset TM$ of the zero section. Then since $\phi_q : T_qM \to M$ and $T_{\bar{q}}T_qM \cong T_qM$, we get that

$$\big(\mathrm{d}_{\bar{q}}\phi_q\big)^{*,-1} : T_q^*M \to T^*_{\phi_q(\bar{q})}M.$$

So the lift can be defined as

$$\overline{\phi}_{(q,p)}(\bar{q}, \bar{p}) := \Big(\phi_q(\bar{q}), \big(\mathrm{d}_{\bar{q}}\phi_q\big)^{*,-1}\bar{p}\Big) \in T^*M. \tag{5.17}$$

Now let $x := (q,p) \in T^*M$ and $y := (\bar{q}, \bar{p}) \in T_xT^*M$ and set $\overline{\phi}^{\bar{q}} : (q,p;\bar{q},\bar{p}) \mapsto \phi_q(\bar{q})$ and $\overline{\phi}^{\bar{p}} : (q,p;\bar{q},\bar{p}) \mapsto \big(\mathrm{d}_{\bar{q}}\phi_q\big)^{*,-1}\bar{p}$. Then we have that

$$\mathrm{d}_y\overline{\phi} = \begin{pmatrix} \mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{q}} & 0 \\ \mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{p}} & \mathrm{d}_{\bar{p}}\overline{\phi}^{\bar{p}} \end{pmatrix},$$
$$\mathrm{d}_x\overline{\phi} = \begin{pmatrix} \mathrm{d}_q\overline{\phi}^{\bar{q}} & 0 \\ \mathrm{d}_q\overline{\phi}^{\bar{p}} & 0 \end{pmatrix}.$$

Note that (on an appropriately chosen neighbourhood $V \subset TT^*M$ of the zero section) $\mathrm{d}_y\overline{\phi}$ is invertible with

$$\left(\mathrm{d}_y\overline{\phi}\right)^{-1} = \begin{pmatrix} \left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{q}}\right)^{-1} & 0 \\ -\left(\mathrm{d}_{\bar{p}}\overline{\phi}^{\bar{p}}\right)^{-1} \circ \left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{p}}\right) \circ \left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{q}}\right)^{-1} & \left(\mathrm{d}_{\bar{p}}\overline{\phi}^{\bar{p}}\right)^{-1} \end{pmatrix}.$$

Now we can use $\overline{\phi}$ to define $\overline{R}$ as in section 2.2.1: Let $N := T^*M$ and $\sigma \in \Gamma(\widehat{S}T^*N)$. Then we define $\overline{R}(\sigma)$ as the Taylor expansion with respect to the y variables of $-\mathrm{d}_y\sigma \circ \left(\mathrm{d}_y\overline{\phi}\right)^{-1} \circ \mathrm{d}_x\overline{\phi}$. In local coordinates (x^i) on the base and (y^i) on the fiber we then have $\overline{R} = \overline{R}_i \mathrm{d}x^i$ and

$$\overline{R}_i(x;y) = -\left(\left(\frac{\partial\overline{\phi}_x}{\partial y}\right)^{-1}\right)^k_j \frac{\partial\overline{\phi}^j_x}{\partial x^i}\frac{\partial}{\partial y^k} =: \overline{Y}^k_i(x;y)\frac{\partial}{\partial y^k}. \tag{5.18}$$

As a matrix we may write $\overline{Y}$ as

$$\overline{Y} = -\left(\mathrm{d}_y\overline{\phi}\right)^{-1} \circ \mathrm{d}_x\overline{\phi} = \begin{pmatrix} -\left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{q}}\right)^{-1} \circ \mathrm{d}_q\overline{\phi}^{\bar{q}} & 0 \\ \left(\mathrm{d}_{\bar{p}}\overline{\phi}^{\bar{p}}\right)^{-1} \circ \left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{p}}\right) \circ \left(\mathrm{d}_{\bar{q}}\overline{\phi}^{\bar{q}}\right)^{-1} \circ \mathrm{d}_q\overline{\phi}^{\bar{q}} - \left(\mathrm{d}_{\bar{p}}\overline{\phi}^{\bar{p}}\right)^{-1} \circ \mathrm{d}_q\overline{\phi}^{\bar{p}} & 0 \end{pmatrix}. \tag{5.19}$$

The key observation now is that $\overline{Y}$ is linear in $\bar{p}$. As a consequence, expressions of the form $\overline{Y}^j_{i,i_1\cdots i_l}(x)$ can contain at most one derivative with respect to a $\bar{p}$ variable, else they will vanish.

5.2.2 Simplified Vertices

Let $\phi : TM \to M$ be a formal exponential map and consider the lift $\overline{\phi} : TN \to N$ to the cotangent bundle $N = T^*M$ as constructed in the last section. As before, let $x = (q,p) \in N$ and $y = (\bar{q},\bar{p}) \in T_xN$. Then, as observed above, we have that $\overline{Y} = -\left(\mathrm{d}_y\overline{\phi}\right)^{-1} \circ \mathrm{d}_x\overline{\phi}$ is linear in $\bar{p}$.
Now let us consider an $\overline{R}$-vertex as in figure 19. To such a vertex we associate a "vertex tensor" of the form $\overline{Y}^j_{i,i_1\cdots i_l}(x)\mathrm{d}x^i$. And since $\overline{Y}^j_i$ is linear in $\bar{p}$, it immidiately follows that this $\overline{R}$-vertex can have at most one incoming $\bar{p}$ edge.
Now as already remarked in section 5.1, since N is symplectic we may choose a formal exponential map $\phi : TN \to N$ such that $T\phi^*\pi(x;y)$ is constant in y (and smooth in x) and as a consequence the $T\phi^*_x\pi$-vertex is bivalent, with precisely two outgoing and no incoming edges. But now we would like to work with the lift $\overline{\phi}$ to the cotangent bundle of a formal exponential map $\phi : TM \to M$, so that we still have the simplified $\overline{R}$-vertices. That's why we will make the following

Assumption: Let M be a smooth manifold and consider the cotangent bundle $N = T^*M$ equipped with the canonical symplectic Poisson structure π. We then assume that there exists a formal exponential map $\phi : TM \to M$ such that $T\overline{\phi}^*_x\pi(y) = T\overline{\phi}^*\pi(x;y)$ is constant in y and smooth in x, where $\overline{\phi}$ is the lift of ϕ to the cotangent bundle N as described in the last section.

Remark 5.1. It's not hard to see that the assumption above holds for $M = \mathbb{R}^n$ with the formal exponential map $\phi : T\mathbb{R}^n \to \mathbb{R}^n$, $(q,p) \mapsto q + p$. So the assumption is true at least for some special cases. Of course it would now be interesting in its own right to investigate whether the assumption holds more generally.

So under the above assumption we now have that the $T\overline{\phi}^*_x\pi$-vertex has only two outgoing and no incoming edges and is decorated with the "vertex tensor" $\frac{1}{2}(T\overline{\phi}^*_x\pi)^{ij}$. Furthermore, since $(T\overline{\phi}^*_x\pi)^{ij} \in \widehat{S}T^*_xN$ is constant in y we may identify it with its image in $\mathbb{R}$, i.e. we may think of it as an element $(T\overline{\phi}^*_x\pi)^{ij} \in \mathbb{R}$.
Now from (2.43) we know that

$$(T\overline{\phi}^*_x\pi)^{ij}(y) = \pi^{rs}(\overline{\phi}_x(y))\left(\left(\frac{\partial\overline{\phi}_x}{\partial y}\right)^{-1}\right)^i_r \left(\left(\frac{\partial\overline{\phi}_x}{\partial y}\right)^{-1}\right)^j_s. \tag{5.20}$$

From this expression it then immidiately follows that the matrix of the coefficients $(T\overline{\phi}^*_x\pi)^{ij}$ is invertible. Furthermore, $T\overline{\phi}^*\pi$ is still Poisson, as noted in Remark 2.11, and by the considerations above it's nondegenerate (i.e. symplectic). So by choosing Darboux coordinates around each point $x \in N$, we may always assume that the matrix of coefficients $(T\overline{\phi}_x^*\pi)^{ij}$ is in standard form. In particular, we see that for each non-vanishing "vertex tensor" $\frac{1}{2}(T\overline{\phi}_x^*\pi)^{ij}$ one of the two outgoing edges is always a $\bar{q}$ edge and the other the corresponding $\bar{p}$ edge. In combination with the simplified $\overline{R}$-vertex (at most one incoming $\bar{p}$ edge), this reduces the number of possible graphs with a non-vanishing contribution to $A(\overline{R}, T\overline{\phi}^*\pi)$ and $F(\overline{R}, \overline{R}, T\overline{\phi}^*\pi)$ considerably.

Finally, the simplified vertices of first type in the case of a cotangent bundle are summarized in figure 20 below.

(a) $T\overline{\phi}_x^*\pi$-vertex (b) $\overline{R}$-vertex

Figure 20: The two possible vertices of first type: (a) The $T\overline{\phi}^*\pi$-vertex with two outgoing edges; (b) The $\overline{R}$-vertex with at most one incoming $\bar{p}$ edge represented by the dashed line.

Remark 5.2. In the case of a Poisson sigma model with a cotangent bundle as a target manifold one can also lift a formal exponential map and then use the same arguments as above to simplify the Feynman rules of the model (cf. figure 3) in a similar fashion.

5.2.3 The curvature 2-form $F(\overline{R}, \overline{R}, T\overline{\phi}^*\pi)$

As before, let M be a smooth manifold, let $\phi : TM \to M$ be a formal exponential map and consider the lift $\overline{\phi} : TN \to N$ to the cotangent bundle $N = T^*M$. As usual we set $x = (q, p) \in N$ and $y = (\bar{q}, \bar{p}) \in T_xN$. We will now use the simplified $\overline{R}$-vertex in the graphs which appear in the computation of the connection 1-form and its curvature 2-form and see how the terms simplify: First we note that $A(\overline{R}, T\overline{\phi}^*\pi)$ is still given by

$$A(\overline{R}_x, T\overline{\phi}_x^*\pi)(\sigma_x) = \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \frac{1 + (-1)^n}{2^{n+1}(n+1)} (T\overline{\phi}_x^*\pi)^{i_1 j_1} \cdots (T\overline{\phi}_x^*\pi)^{i_n j_n} (\overline{Y}_x)^k_{i,i_1\cdots i_n} (\sigma_x)_{,kj_1\cdots j_n}. \quad (5.21)$$

The simplification in this case is a small one: All summands with a term $(\overline{Y}_x)^k_{i,i_1\cdots i_n}$ with more than one derivative with respect to a $\bar{p}$ variable will vanish.
In the case of the curvature 2-form F^N the simplification is much more interesting: Since for each non-vanishing coefficient $(T\overline{\phi}_x^*\pi)^{ij}$ one of the two outgoing edges is always a $\bar{q}$ edge and the other the corresponding $\bar{p}$ edge (since we work with Darboux coordinates around $x \in N$), we see that the sum in (5.9) terminates at $n = 2$. Or put differently, we only have to consider the curvature graphs Γ_n up to $n = 2$, i.e. with at most two wedges attached to the wheel consisting of two $\overline{R}$-vertices (cf. figure 7). And since up to $n = 2$ the Kontsevich weights (5.8) are given by $w_{\Gamma_0} = 0$, $w_{\Gamma_1} = \frac{1}{24}$ and $w_{\Gamma_2} = 0$, we find that

$$F_x^N = F(\overline{R}_x, \overline{R}_x, T\overline{\phi}_x^*\pi) = \frac{\hbar}{48} (T\overline{\phi}_x^*\pi)^{rs} (\overline{Y}_x)^k_{i,lr} (\overline{Y}_x)^l_{j,ks} \mathrm{d}x^i \wedge \mathrm{d}x^j, \quad (5.22)$$

where we sum over the indices i, j, r, s, k, l and where again summands with a term $(\overline{Y}_x)^k_{i,lr}$ with more than one derivative with respect to a $\bar{p}$ variable will vanish. So in the case of a cotangent bundle we get a much simpler expression for the Weyl curvature F^N.

Appendix A Binomial Sums

Here we will treat the binomial sums appearing in the expression (4.37) and show that we indeed get the results (4.38). To do this we will use the well-known identities

$$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k} \tag{A.1}$$

and

$$\sum_{k=0}^{n}(-1)^l\binom{n+1}{k} = (-1)^n \tag{A.2}$$

as well as the Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{l}a^k b^{n-k}. \tag{A.3}$$

A.1 $B(n)$

Let us start with

$$B(n) = \sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^{n+1}(n-l+1)}.$$

Using the identities (A.1) and (A.2) we find that

$$B(n) = \frac{1}{2^{n+1}(n+1)}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l = \frac{(-1)^n}{2^{n+1}(n+1)}. \tag{A.4}$$

A.2 $C(n)$

Let us continue with

$$C(n) = \sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^l(n-l+1)}.$$

Using the identity (A.1) we can write

$$(n+1)C(n) = \sum_{l=0}^{n}\binom{n+1}{l}\Big(-\frac{1}{2}\Big)^l.$$

Using the Binomial theorem we then find

$$\sum_{l=0}^{n}\binom{n+1}{l}\Big(-\frac{1}{2}\Big)^l = \Big(\frac{1}{2}\Big)^{n+1} - \Big(-\frac{1}{2}\Big)^{n+1}$$

and hence

$$C(n) = \frac{1+(-1)^n}{2^{n+1}(n+1)}. \tag{A.5}$$

A.3 $A(n)$

Finally, let us treat the case

$$A(n) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l}\binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}\frac{(-1)^{l+s}}{2^{n-k-s+1}(n-k-l-s+1)}.$$

Write

$$A(n) = \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^{l}}{2^{n-k+1}}\sum_{s=0}^{n-k-l}\binom{n-k-l}{s}\frac{(-2)^{s}}{(n-k-l-s+1)}. \tag{A.6}$$

We first treat the innermost sum: Set $m = n-k-l$. Then

$$\sum_{s=0}^{n-k-l}\binom{n-k-l}{s}\frac{(-2)^{s}}{(n-k-l-s+1)} = \sum_{s=0}^{m}\binom{m}{s}\frac{(-2)^{s}}{(m-s+1)}.$$

Using the identity (A.1) we find that

$$\sum_{s=0}^{m}\binom{m}{s}\frac{(-2)^{s}}{(m-s+1)} = \frac{1}{m+1}\sum_{s=0}^{m}\binom{m+1}{s}(-2)^{s}.$$

Applying the Binomial theorem we then get that

$$\sum_{s=0}^{m}\binom{m+1}{s}(-2)^{s} = (-1)^{m+1}\big(1-2^{m+1}\big).$$

Plugging all of this into (A.6) we find

$$\begin{aligned} A(n) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^{l}}{2^{n-k+1}}\frac{(-1)^{n-k-l+1}}{n-k-l+1}\big(1-2^{n-k-l+1}\big) \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k+1}}{2^{n-k+1}}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{1}{n-k-l+1}\big(1-2^{n-k-l+1}\big). \end{aligned} \tag{A.7}$$

Using identity (A.1) and the Binomial theorem we obtain the following:

$$\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{1}{n-k-l+1} = \frac{1}{n-k+1}\sum_{l=0}^{n-k}\binom{n-k+1}{l} = \frac{1}{n-k+1}\big(2^{n-k+1}-1\big),$$

and

$$\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{2^{n-k-l+1}}{n-k-l+1} = \frac{1}{n-k+1}\sum_{l=0}^{n-k}\binom{n-k+1}{l}2^{n-k-l+1} = \frac{1}{n-k+1}\big(3^{n-k+1}-1\big).$$

Now plugging those results into (A.7) we find that

$$\begin{aligned} A(n) &= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k+1}}{2^{n-k+1}(n-k+1)}\big(2^{n-k+1}-3^{n-k+1}\big) \\ &= (-1)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{n-k+1}\Big(1-\Big(\frac{3}{2}\Big)^{n-k+1}\Big). \end{aligned} \tag{A.8}$$

Using (A.1), (A.2) and the Binomial theorem a final time we see that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{n-k+1} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}(-1)^{k} = \frac{(-1)^{n}}{n+1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n-k+1}\Big(\frac{3}{2}\Big)^{n-k+1} = \frac{(-1)^{n+1}}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\Big(-\frac{3}{2}\Big)^{n-k+1}$$
$$= \frac{1}{2^{n+1}(n+1)} + \frac{(-1)^n}{n+1}.$$

Plugging everything into (A.8) we finally find that

$$A(n) = (-1)^{n+1}\Big(\frac{(-1)^n}{n+1} - \frac{1}{2^{n+1}(n+1)} - \frac{(-1)^n}{n+1}\Big) = \frac{(-1)^n}{2^{n+1}(n+1)}. \tag{A.9}$$

Appendix B Supergeometry

In this chapter we will briefly introduce the main notions of supergeometry. For a more detailed treatment we refer to [18] and [13].

B.1 Locally Ringed Spaces

Let us start with some algebraic preliminaries which will be needed later on.

Definition B.1. Let X be a topological space. A *presheaf F of rings* on X consists of:

(i) A ring $F(U)$ for every open subset $U \subset X$.

(ii) A ring homomorphism $\mathrm{res}_{V,U} : F(U) \to F(V)$, called the *restriction map*, for every pair $V \subset U$ of open subsets of X satisfying $\mathrm{res}_{U,U} = \mathrm{Id}_{F(U)}$ and $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ whenever $W \subset V \subset U$.

Definition B.2. Let F and G be two presheaves of rings on X. A *morphism of presheaves* $\varphi : F \to G$ is a family of ring homomorphisms $\varphi_U : F(U) \to G(U)$ such that for every pair $V \subset U$ of open subsets of X the following diagram commutes:

$$\begin{array}{ccc} F(U) & \xrightarrow{\ \varphi_U\ } & G(U) \\ {\scriptstyle \mathrm{res}_{V,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ F(V) & \xrightarrow[\ \varphi_V\ ]{} & G(V) \end{array}$$

Remark B.1. If F is a presheaf of rings on X, $V \subset U$ are open subsets of X and $s \in F(U)$, then one usually abbreviates

$$s\big|_V := \mathrm{res}_{V,U}(s). \tag{B.1}$$

Definition B.3. Let X be a topological space. A *sheaf F of rings* on X is a presheaf of rings satisfying the following two conditions:

(i) For any open subset $U \subset X$ and any open cover $\{U_i |\ i \in I\}$ of U, if $s, t \in F(U)$ such that $s\big|_{U_i} = t\big|_{U_i}$ for all $i \in I$, then $s = t$.

(ii) For any open subset $U \subset X$ and any open cover $\{U_i |\ i \in I\}$ of U, if we are given a collection $s_i \in F(U_i)$ satisfying $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$ for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, then there exists a unique $s \in F(U)$ such that $s\big|_{U_i} = s_i$ for all $i \in I$.

Remark B.2. A morphism of sheaves $\varphi : F \to G$ is simply a morphism of the underlying presheaves.

Definition B.4. Let F be a sheaf of rings on a topological space X and let $x \in X$. The *stalk* of F at x, denoted as F_x, is given as the direct limit

$$F_x := \varinjlim_{U \ni x} F(U) \tag{B.2}$$

indexed over all open subsets containing x.

Definition B.5. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X togther with a sheaf of rings $\mathcal{O}_X$ on X, called the *structure sheaf* of X. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings (i.e. they contain a unique maximal ideal).

Definition B.6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, φ), where $f : X \to Y$ is a continuous map and $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism (of sheaves) from the structure sheaf of Y to the direct image of the structure sheaf of X.
If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces, then (f, φ) is a *morphism of locally ringed spaces* if in addition the ring homomorphisms induced by φ between the stalks of $\mathcal{O}_Y$ and the stalks of $f_*\mathcal{O}_X$ are *local homomorphisms* (i.e. they map unique maximal ideal to unique maximal ideal).

B.2 Supermanifolds

Before we introduce the concept of a supermanifold, let us start with the linear case:

Definition B.7. A *superspace* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, $0, 1 \in \mathbb{Z}_2$. The *parity* of a (nonzero) homogeneous element $v \in V_i$ is $|v| = i$, $i = 0, 1$. Elements of parity 0 are called *even* and those of parity 1 are called *odd*. Accordingly, V_0 is called the space of even vectors and V_1 the space of odd vectors.

Definition B.8. Given a superspace V, the *parity reversed* space ΠV is defined to be the superspace with even and odd subspaces interchanged, i.e. with $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$. Note that the change of parity Π is actually an endofunctor on the category of superspaces.

Definition B.9. Let V and W be two superspaces. A *morphism of superspaces* $f : V \to W$ is a linear map which preserves the grading, i.e. which satisfies $f(V_i) \subset W_i$, $i = 0, 1$.

Definition B.10. Let $V = V_0 \oplus V_1$ be a superspace. The *dual space* of V is the superspace $V^* = V_0^* \oplus V_1^*$, where the space of even functionals V_0^* is given by the functionals on V that vanish on V_1 and, similarly, the space of odd functionals V_1^* is given by those functionals that vanish on V_0.

Finally, let us introduce the notion of a supermanifold:

Definition B.11. A *supermanifold* $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to

$$(U, \mathcal{C}^\infty(U) \otimes \bigwedge V^*), \tag{B.3}$$

where U is an open subset of $\mathbb{R}^n$, V is a finite dimensional real vector space and $\bigwedge V^* := \bigoplus_k \bigwedge^k V^*$ is the exterior algebra of the dual space V^*. The above isomorphism is in the category of $\mathbb{Z}_2$-graded algebras. This means that the parity of a homogeneous element, given by

$$\mathcal{C}^\infty(U) \otimes \bigwedge^k V^* \to \mathbb{Z}_2,\ f \otimes x \mapsto |f \otimes x| := |x| = k \text{ mod } 2, \tag{B.4}$$

must be preserved.

Locally a supermanifold looks like an open subset U of $\mathbb{R}^n$ with even coordinates (x^i), together with some odd coordinates (θ^μ) corresponding to a basis of V^*. Algebraically, we have the relations

$$\begin{aligned} x^i x^j &= x^j x^i, \\ x^i \theta^\mu &= \theta^\mu x^i, \\ \theta^\mu \theta^\nu &= -\theta^\nu \theta^\mu. \end{aligned} \tag{B.5}$$

The supermanifold described by such a local piece is denoted by $U \times \Pi V$ and, accordingly, we write

$$\mathcal{C}^\infty(U \times \Pi V) := \mathcal{C}^\infty(U) \otimes \bigwedge V^*. \tag{B.6}$$

In the global picture, the algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ is defined as the algebra of global sections of the sheaf $\mathcal{O}_M$ associated to $\mathcal{M}$. It's worth noting that the parity extends to $\mathcal{C}^\infty(\mathcal{M})$ and that this way the algebra of smooth functions becomes a graded commutative algebra, i.e. for homogeneous elements $f, g \in \mathcal{C}^\infty(\mathcal{M})$ we have that $fg = (-1)^{|f||g|} gf$.

Definition B.12. Let $\mathcal{M}$ and $\mathcal{N}$ be two supermanifolds. A *morphism of supermanifolds* from $\mathcal{M}$ to $\mathcal{N}$ is simply a morphism $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ of the corresponding locally ringed spaces.

Example. Let M be a smooth manifold. The algebra of differential forms $\Omega(M) := \bigoplus_k \Omega^k(M)$ is locally given by $\mathcal{C}^\infty(U) \otimes T_x^* M$, where U is some chart domain on M containing x. So (M, Ω_M), where Ω_M is the sheaf of differential forms on M, is a supermanifold.

Example. Let M be a smooth manifold. Then we can consider the *odd tangent bundle* ΠTM and the *odd cotangent bundle* ΠT^*M. Both are supermanifolds and the corresponding algebras of smooth functions are given by

$$\mathcal{C}^\infty(\Pi TM) = \Gamma\left(\bigwedge T^*M\right) = \Omega(M),$$
$$\mathcal{C}^\infty(\Pi T^*M) = \Gamma\left(\bigwedge TM\right) = T_{\text{poly}}(M).$$

Now let us also have a look at integration on supermanifolds: Consider the supermanifold $U \times \Pi V$, where U is an open subset of $\mathbb{R}^n$ with even coordinates $(x^i)_{i=1}^n$ and ΠV is an m-dimensional vector space with odd coordinates $(\theta^j)_{j=1}^m$.

Definition B.13. The *Berezinian integral* on ΠV is the unique linear functional $\int_{\Pi V} \mathrm{d}\theta$ satisfying

- $\int\limits_{\Pi V} \theta^m \cdots \theta^1 \mathrm{d}\theta = 1$,
- $\int\limits_{\Pi V} \frac{\partial f}{\partial \theta^i} \mathrm{d}\theta = 0$, $i = 1, \dots, m$, for all $f \in \bigwedge V^*$.

Remark B.3. We call $\mathrm{d}\theta$ a *Berezinian measure.*

Given a change of variables $\theta^i = \theta^i(\xi^1, \dots, \xi^m)$ and the corresponding Jacobian

$$J = \left(\frac{\partial \theta^i}{\partial \xi^j}\right),$$

where $\partial/\partial\xi^j$ corresponds to the so-called *right derivative* (i.e. $\partial(\xi^1\xi^2)/\partial\xi^2 = \xi^1$ and $\partial(\xi^1\xi^2)/\partial\xi^1 = -\xi^2$), it's not hard to see that

$$\int f(\theta)\mathrm{d}\theta = \int f(\theta(\xi))(\det J)^{-1}\mathrm{d}\xi. \tag{B.7}$$

Now given a function $f \in \mathcal{C}^\infty(U \times \Pi V) = \mathcal{C}^\infty(U) \otimes \bigwedge V^*$, the Berezinian integral of f is defined as the number

$$\int\limits_{U\times\Pi V} f(x,\theta)\mathrm{d}\theta\mathrm{d}x = \int\limits_{U}\int\limits_{\Pi V} f(x,\theta)\mathrm{d}\theta\mathrm{d}x. \tag{B.8}$$

Again we can consider a change of coordinates $x^i = x^i(y, \xi)$ and $\theta^i = \theta^i(y, \xi)$ with corresponding Jacobian

$$J = \frac{\partial(x,\theta)}{\partial(y,\xi)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = \partial x/\partial y$, $B = \partial x/\partial \xi$, $C = \partial\theta/\partial x$ and $D = \partial\theta/\partial\xi$. Note that the diagonal blocks A and D are even and the off-diagonal blocks B and C are odd.

Definition B.14. The *Berezinian* or *superdeterminant* of the supermatrix J is defined as

$$\mathrm{Ber}J := \frac{\det(A - BD^{-1}C)}{\det D}. \tag{B.9}$$

Finally, the formula for the above change of coordinates is given by

$$\int f(x,\theta)\mathrm{d}\theta\mathrm{d}x = \int f\big(x(y,\xi),\theta(y,\xi)\big)\epsilon\mathrm{Ber}J\mathrm{d}\xi\mathrm{d}y, \tag{B.10}$$

where ϵ is the sign of the orientaion of the map $y \mapsto x(y, 0)$.
Finally, to perform integration on a general supermanifold $\mathcal{M}$, which locally looks like $U \times \Pi V$, one needs to introduce the notion of a *density*: A *density* on the supermanifold $\mathcal{M}$ is a section of the *Berezinian bundle* tensor the orientation bundle of the underlying manifold (so they have the desired transformation behaviour). One can then define the integral of densities as for ordinary manifolds.

B.3 Graded Manifolds

Let us again start with the linear case:

Definition B.15. A *$\mathbb{Z}$-graded vecor space* V, often simply denoted as a *graded vector space*, is a collection of vector spaces $(V_i)_{i\in\mathbb{Z}}$. The *degree* of a (nonzero) homogeneous element $v \in V_i$ is $|v| = i$, $i \in \mathbb{Z}$.

Remark B.4. Consider a graded vector space $V = (V_i)_{i\in\mathbb{Z}}$ where only finitely many of the spaces V_i are nontrivial. Then we may regard V as a superspace by setting $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with

$$V_{\bar{0}} := \bigoplus_{i\in 2\mathbb{Z}} V_i,$$
$$V_{\bar{1}} := \bigoplus_{i\in 2\mathbb{Z}+1} V_i.$$

Definition B.16. Given a graded vector space V and an integer k, the *k-shift* $V[k]$ of V is given by $(V[k])_i := V_{k+i}$, $i \in \mathbb{Z}$.

Definition B.17. Let V and W be two graded vector spaces. A *morphism of graded vector spaces* $V \to W$, also called a *graded linear map*, is a collection of linear maps $(f_i : V_i \to W_i)_{i\in\mathbb{Z}}$. A graded linear map of *degree* k is a morphism of graded vector spaces $V \to W[k]$.

Definition B.18. Let V be a graded vector space. The dual space of V is the graded vector space $V^* = (V^*_{-i})_{i\in\mathbb{Z}}$.

Definition B.19. Let V be a graded vector space. The *symmetric algebra* $S(V)$ of V is the quotient of the tensor algebra $T(V)$ by the ideal generated by elements of the form

$$v \otimes w - (-1)^{|v||w|} w \otimes v$$

for any homogeneous elements $v, w \in V$.

Finally, let us introduce the notion of a graded manifold:

Definition B.20. Let M be a smooth manifold. A *graded manifold* $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to

$$(U, \mathcal{C}^\infty(U) \otimes S(V^*)), \tag{B.11}$$

where U is an open subset of $\mathbb{R}^n$ and V is a graded vector space. The manifold M is called the *body* of $\mathcal{M}$ and $\mathcal{O}_M$ is the structure sheaf of M. The above isomorphism is in the category of $\mathbb{Z}$-graded algebras.

The algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ is defined as the algebra of global sections of the sheaf $\mathcal{O}_M$ and it automatically inherits a $\mathbb{Z}$-grading. And as for supermanifolds, *morphisms of graded manifolds* are simply morphisms of the corresponding locally ringed spaces.

Example. A *graded vector bundle* E over a manifold M is a collection of vector bundles $(E_i)_{i\in\mathbb{Z}}$ over M. The sheaf

$$U \mapsto \Gamma\big(U, S(E\big|_U{}^*)\big)$$

then corresponds to a graded manifold. One can actually show that every graded manifold is isomorphic to a graded manifold associated to a graded vector bundle.

Definition B.21. Let $\mathcal{M}$ be a graded manifold. A *graded vector field of degree k* on $\mathcal{M}$ is a graded linear map

$$X : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[k]$$

which satisfies the *graded Leibniz rule*

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g) \tag{B.12}$$

for all homogeneous functions f and g.

Locally a graded vector field can be described as follows: Let V be a graded vector space with homogeneous coordinates (x^i), which correspond to a basis of V^*. A graded vector field on V is a linear combination

$$X = \sum_i X^i \frac{\partial}{\partial x^i},$$

where the X^i's are elements of $S(V^*)$ and $\left(\frac{\partial}{\partial x^i}\right)$ is a basis of V dual to (x^i). The vector field X then acts on the algebra of functions according to the rules $\frac{\partial}{\partial x^i}(x^j) = \delta_i^j$ and $\frac{\partial}{\partial x^i}(fg) = \left(\frac{\partial}{\partial x^i}(f)\right)g + (-1)^{|x^i||f|} f\left(\frac{\partial}{\partial x^i}(g)\right)$.

Now let us denote the space of graded vector fields on the graded manifold $\mathcal{M}$ by $\mathfrak{X}(\mathcal{M})$. Then we can define the *graded Lie bracket* $[\ ,\ ] : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ by

$$[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X. \tag{B.13}$$

The space $\mathfrak{X}(\mathcal{M})$ endowed with the graded Lie bracket then becomes a graded Lie algebra.

Definition B.22. Let $\mathcal{M}$ be a graded manifold. A *cohomological vector field* is a graded vector field X of degree 1 on $\mathcal{M}$ such that $[X, X] = 0$. A graded manifold endowed with a cohomological vector field is called a *differential graded manifold*, or dg manifold for short.

Now let us also introduce the notion of differential forms on a graded manifold $\mathcal{M}$: Locally the algebra of differential forms on $\mathcal{M}$ is constructed by adding new coordinates $(\mathrm{d}x^i)$ of degree $|\mathrm{d}x^i| = |x^i| + 1$ to a system of homogeneous coordinates (x^i) on $\mathcal{M}$. The global picture is as follows: The shifted tangent bundle $T[1]\mathcal{M}$ carries the structure of a dg manifold with cohomological vector field Q, locally given by

$$Q = \sum_i \mathrm{d}x^i \frac{\partial}{\partial x^i}.$$

The de Rham complex $(\Omega(\mathcal{M}), \mathrm{d})$ of $\mathcal{M}$ then corresponds to $(\mathcal{C}^\infty(T[1]\mathcal{M}), Q)$.

Remark B.5. A differential form on a graded manifold $\mathcal{M}$ has a *total degree*, i.e. the degree in $\mathcal{C}^\infty(T[1]\mathcal{M})$, and a *form degree*. We then define the *degree* of a differential form as the difference of the total degree minus the form degree.

Definition B.23. Let $\mathcal{M}$ be a graded manifold. A *graded symplectic form of degree k* on $\mathcal{M}$ is a 2-form ω which is homogeneous of degree k, closed with respect to the de Rham differential on $\mathcal{M}$ and nondegenerate, i.e. the induced morphism of graded vector bundles

$$\omega^\# : T\mathcal{M} \to T^*[k]\mathcal{M}$$

is an isomorphism. One then calls the tuple $(\mathcal{M}, \omega)$ a *graded symplectic manifold* of degree k.

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