Deformation Quantization of the Relational Symplectic Groupoid for Constant Poisson Structures

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Abstract

This thesis gives a first step for deformation quantization of the relational symplectic groupoid, by quantizing it with the Poisson σ-model for *constant* Poisson bivector fields. In Chapter 1 we introduce the notion of deformation quantization and the equivalence to a star product, describe the mathematical methods after M. Kontsevich as it is in [K] and work of A. S. Cattaneo, G. Felder, B. Keller, C. Torossian and A. Bruguières in [CKTB] and [CF1]. Chapter 2 is about the relational symplectic groupoid (RSG), which gives an introduction to groupoids itself and the definition of the RSG including its topological construction of the axioms. Most of this Chapter is based on the work of I. Contreras and A. S. Cattaneo in [IC], [CC1] and [CC2]. The main Chapter of the thesis is Chapter 3, which is an original part containing the quantization of the RSG. Based on the BV-BFV formalism and abelian *BF* theory as it is described in [CMR2] by A.S. Cattaneo, P. Mnev and N. Reshetikhin, we show that the states satisfy the modified quantum master equation and derive the Moyal-product by considering a topological gluing of the axioms for the RSG and the usage of an induced Poisson σ-model for the background fields. There are also appendices covering the notion of path integrals and Feynman diagrams, mostly from [MP1], [PS] and [ZJ], and aspects of supergeometry mostly of [L] and [DF].

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To my parents,
who have always supported me.

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First of all I want to thank my supervisor Prof. Dr. Alberto S. Cattaneo for the remarkable opportunity of doing such a thesis on this great subject and for giving me the chance of a comming PhD journey with him. Moreover, I want to thank him also for his great support during my studies and for the amazing offer of a lecture in Poisson geometry and deformation quantization, which helped me a lot. I want to thank Konstantin Wernli for his kind support and for all the time he has spend with me to discuss my questions. I also want to thank Michele Schiavina for helpfull explanations and for reading through partial parts of the thesis and giving me some important advices. I want to thank Ivan Contreras for letting me use his pictures of the RSG. My special thanks goes to Prof. Dr. Ashkan Nikeghbali, who gave me a lot of mental support during my studies, taught me important things for life and who I hope to label as a good friend by this time. I want to thank my parents for their financial support and for their guidance in my life, without whom I would not be writing this at all. Especially, I want to thank my mother who always gave everything such that I would have the best opportunities in my life and my father who always told me about many different things like philosophy and history, which inspired me to reach for knowledge at any time. Last but not least I want to thank all my friends who went with me through my time during university. I want to thank them for the great discussions at the coffee table and also for all the great time we had going out together.

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The question you raise, "How can such a formulation lead to computations?" doesn't bother me in the least! Throughout my whole life as a mathematician, the possibility of making explicit, elegant computations has always come out by itself, as a byproduct of a thorough conceptual understanding of what was going on. Thus I never bothered about whether what would come out would be suitable for this or that, but just tried to understand - and it always turned out that understanding was all that mattered.

— Alexander Grothendieck

1

Deformation Quantization and the Kontsevich Formula

1.1 Introduction

In classical mechanics we are used to work with sets of possible states, which is in particular given by a Poisson manifold $(\mathscr{P}, \alpha)$, which is a manifold $\mathscr{P}$ endowed with a Poisson bivector field α. In order to describe physical quantities (called observables), we can express them as smooth functions on $\mathscr{P}$, i.e. consider the space $C^\infty(\mathscr{P})$, which forms a commutative algebra and observe that α induces a Poisson bracket $\{\cdot,\cdot\}_\alpha$ on this algebra of functions and vice versa. In quantum mechanics, in order to describe quantum phenomena, we consider the space of all possible states, described by a Hilbert space $\mathcal{H}$, where the observables are given by self-adjoint (usually unbounded) operators, which form a non-commutative C^*-algebra.

A very important question in mathematical physics is the process of changing from a Poisson manifold (classical setting), to a Hilbert space (quantum setting). It actually turned out to be a non-trivial question, that gives rise to many interesting mathematical insights. There are different ways of mathematical theories, describing such a process. A common way is called deformation quantization, which only uses the data encoded in the given algebra, since a quantum system can be totally understood in terms of its C^*-algebra. As the name already points out, the commutative algebra of the classical setting is going to be *deformed* into a non-commutative algebra of the quantum setting, in the sense that there will be formulae, emerging by infinitesimal changes in terms of formal power series for the product operation between two smooth functions $f, g \in C^\infty(\mathscr{P})$ (star products). In terms of algebraic geometry, there is a whole theory behind such deformations, usually called *deformation theory*. The results for deformation quantization in [K] are described in terms of pure algebraic geometry in [AY]. More on deformation quantization can be found in [CKTB], [CI] and [MP2].

In this chapter we want to give a short overview to the notion of Poisson algebras and Poisson manifolds, corresponding to the classical setting of mechanics, discuss the notion of a formal deformation of an associative algebra and the properties of a star product. Moreover, we will actually see that there always exists a formal deformation for any finite dimensional Poisson manifold, which was proved by Maxim Kontsevich in [K]. Moreover, the resulting star product of Kontsevich gives us an *explicit* formula for such a star product on the underlying Poisson structure, which can be actually stated as a *path integral* formula, giving rise to a quantum field theory by using the notion of the *Poisson σ-model* (see [CF1] and [CF2]).

1.2 Poisson algebras and Poisson manifolds

1.2.1 Definitions and Properties

Definition 1.2.1 (Poisson Algebra). *A* Poisson algebra *is an associative algebra $\mathscr{A}$ equipped with a bracket $\{\cdot,\cdot\}:\mathscr{A}\times\mathscr{A}\to\mathscr{A}$ satisfying the following properties*

(*i*) *(Lie bracket). $\{\cdot,\cdot\}$ is skew symmetric and it satisfies for all $f,g,h\in\mathscr{A}$ the Jacobi identity*

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0.$$

(*ii*) *(Leibniz rule). $\{\cdot,\cdot\}$ acts as a derivation on a product of $\mathscr{A}$. That is*

$$\{f,gh\}=\{f,g\}h+g\{f,h\}.$$

A bracket $\{\cdot,\cdot\}$ satisfying these two conditions is called a Poisson bracket.

Definition 1.2.2 (Poisson manifold). *A* Poisson manifold *is a pair $(\mathscr{P},\alpha)$, where $\mathscr{P}$ is a smooth manifold and $\alpha\in\Gamma\left(\bigwedge^2 T\mathscr{P}\right)$ such that $(C^\infty(\mathscr{P}),\{\cdot,\cdot\}_\alpha)$ is a Poisson algebra, where*

$$\{f,g\}_\alpha:=\alpha(df,dg)$$

for all $f,g\in C^\infty(\mathscr{P})$, where we denote by $\Gamma(S)$ the set of smooth sections on some space S.

Remark 1.2.1. The bivector field $\alpha\in\Gamma\left(\bigwedge^2 T\mathscr{P}\right)$ is called a Poisson bivector field or a Poisson structure on $\mathscr{P}$. Moreover, any Poisson bracket induces a Poisson bivector field by the formula above. Thus, we can write α in local coordinates as

$$\alpha(x)=\sum_{1\leq i<j\leq \dim\mathscr{P}}\alpha^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}.$$

If we want to emphasize the condition of α being Poisson, we require in local coordinates the condition

$$\sum_{1\leq r\leq \dim \mathscr{P}} \alpha^{sr}(x)\frac{\partial}{\partial x^r}\alpha^{\ell k}(x) + \alpha^{kr}(x)\frac{\partial}{\partial x^r}\alpha^{s\ell}(x) + \alpha^{\ell r}(x)\frac{\partial}{\partial x^r}\alpha^{ks}(x) = 0.$$

Example 1.2.1 (Constant Poisson structure). Consider an open subset of $\mathbb{R}^d$. Then the constant Poisson structure is given by

$$\alpha^{ij}(x) \equiv c^{ij}$$

for some real constants $c^{ij} \in \mathbb{R}$ with the property $c^{ij} = -c^{ji}$. One can immediately see that this bivector field actually satisfies the conditions for a Poisson structure.

Example 1.2.2 (Linear Poisson structure). Let $\mathfrak{g}$ be a finite dimensional Lie algebra with structure constants $\{c_k^{ij}\}$ with respect to the basis $e_1, e_2, \ldots, e_n$. consider its dual space $\mathfrak{g}^*$; then, if x_l denotes the linear functions on $\mathfrak{g}^*$ which corresponds by duality to e_l, the bivector field α written down in local cordinates as

$$\alpha(x) = \sum_{1\leq i<j\leq \dim \mathfrak{g}} c_k^{ij}x^k\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$$

satisfies the conditions for being Poisson.

Example 1.2.3 (symplectic manifolds). If (M, ω) is a finite dimensional symplectic manifold, then with the bracket

$$\{f, g\}_\omega := \omega(X_f, X_g),$$

induced by the symplectic form, where X_f and X_g are Hamiltonian vector fields associated to f and g respectively, $(C^\infty(\mathscr{P}), \{\cdot,\cdot\}_\omega)$ is a Poisson algebra.

Definition 1.2.3 (Poisson morphisms)**.** *Let* $(M, \{\cdot,\cdot\}_M)$ *and* $(N, \{\cdot,\cdot\}_N)$ *be two Poisson manifolds. A map* $\varphi : M \to N$ *is called a* Poisson morphism *if the pull-back map*

$$\varphi^* : C^\infty(N) \to C^\infty(M)$$

is a Lie algebra homomorphism with respect to corresponding Poisson brackets.

Definition 1.2.4 (Coisotropic submanifolds)**.** *Let* $(\mathscr{P}, \alpha)$ *be a Poisson manifold and* $\mathscr{S}$ *be a submanifold of* $\mathscr{P}$. $\mathscr{S}$ *is called* coisotropic *if*

$$\alpha^\#(N^*\mathscr{S}) \subset T\mathscr{S},$$

where $N^*\mathscr{S}$ *is the conormal bundle of* $\mathscr{S}$*, defined by*

$$N_x^*\mathscr{S} := \{\beta \in T_x^*\mathscr{P} \mid \langle\beta, v\rangle = 0, \forall v \in T_x\mathscr{S}\},$$

where $\langle\cdot,\cdot\rangle$ *denotes the natural pairing between* $T_x^*\mathscr{P}$ *and* $T_x\mathscr{P}$ *and* $\alpha^\#$ *is the map given by*

$$\begin{aligned} \alpha^\# : T_x^*\mathscr{P} &\longrightarrow T_x\mathscr{P} \\ \beta &\longmapsto \alpha(x)(\beta,\cdot) \end{aligned}$$

1.3 Formal Deformation of associative Algebras

1.3.1 Basic Introduction

Let us now give the notion of a formal deformation as in the language of algebraic geometry and develop the structure of a star product in terms of its power series. Let $\mathscr{A}$ be an associative algebra over a ring k (i.e. a k-module) and let $\hbar$ denote here just a parameter. Denote by $k[[\hbar]]$ the ring of formal power series in $\hbar$ and by $\mathscr{A}[[\hbar]]$ the $k[[\hbar]]$-module of formal power series

$$\sum_{n\geq 0} a_n \hbar^n,$$

where the coefficients are in $\mathscr{A}$.

Definition 1.3.1 (Formal deformation / star product)**.** *Let $\mathscr{A}$ be an associative algebra over a ring k. A formal deformation (or star product) of $\mathscr{A}$ is a $k[[\hbar]]$-bilinear map*

$$\star : \mathscr{A}[[\hbar]] \times \mathscr{A}[[\hbar]] \to \mathscr{A}[[\hbar]],$$

such that $\psi \star \varphi \equiv \psi\varphi \mod \hbar\mathscr{A}[[\hbar]]$ and $(\xi \star \psi) \star \psi = \xi \star (\psi \star \varphi)$ for all formal power series $\xi, \psi, \varphi \in \mathscr{A}[[\hbar]]$.

Remark 1.3.1. The product of two elements $\mu, \nu \in \mathscr{A}$ is then of the form

$$\mu \star \nu = \mu\nu + B_1(\mu,\nu)\hbar + \cdots + B_n(\mu,\nu)\hbar^n + \cdots \in \mathscr{A}[[\hbar]]$$

where the B_i's are k-bilinear maps for all $i \geq 1$. Moreover, we assume $\mu \star 1 = 1 \star \mu = \mu$. The coefficients B_i actually determine the star product, since it is $k[[\hbar]]$-bilinear. Therefore, it is convenient to express the star product map as the power series including the bilinear maps. That is we can write

$$\star = \sum_{n\geq 0} B_n \hbar^n.$$

Remark 1.3.2. The product of two arbitrary elements $\left(\sum_{n\geq 0} \mu_n \hbar^n\right), \left(\sum_{n\geq 0} \nu_n \hbar^n\right) \in \mathscr{A}[[\hbar]]$ is defined by

$$\left(\sum_{n\geq 0} \mu_n \hbar^n\right) \star \left(\sum_{n\geq 0} \nu_n \hbar^n\right) := \sum_{k,\ell\geq 0} \mu_k \nu_\ell \hbar^{k+\ell} + \sum_{\substack{k,\ell\geq 0\\ m\geq 1}} B_m(\mu_k,\nu_\ell)\hbar^{k+\ell+m}.$$

For further discussion, let $\mathcal{G}$ be the group of $k[[\hbar]]$-module automorphisms g of $\mathscr{A}[[\hbar]]$ such that for all $\psi, \varphi \in \mathscr{A}[[\hbar]]$ we have

$$g(\psi) \equiv \psi \mod \hbar\mathscr{A}[[\hbar]]$$

Definition 1.3.2 (Equivalence of star products)**.** *We say that two formal deformation $\star$ and $\star'$ are* *equivalent if there is an element $g \in \mathcal{G}$ such that for all $\psi, \varphi \in \mathscr{A}[[\hbar]]$ we have*

$$g(\psi \star \varphi) = g(\psi) \star' g(\varphi).$$

We want to make sure, that the star product, defined as its power series in terms of the bilinear operators, is actually related to a Poisson structure on the underlying algebra $\mathscr{A}$ by the following lemma. Let therefore $\mathscr{A}$ be a commutative algebra.

Lemma 1.3.1 (Relation to Poisson structures)**.** *Let $\star$ be an associative formal deformation of the multiplication of $\mathscr{A}$. For $\mu, \nu \in \mathscr{A}$, set $\{\mu, \nu\}^- := B_1(\mu, \nu) - B_1(\nu, \mu)$.*

(*i*) *The map $\{\cdot, \cdot\}^-$ is a Poisson bracket on $\mathscr{A}$.*

(*ii*) *The bracket $\{\cdot, \cdot\}^-$ only depends on the equivalence class of $\star$.*

Proof. For (i), consider the map

$$[\cdot, \cdot] : (\psi, \varphi) \mapsto \frac{\psi \star \varphi - \varphi \star \psi}{\hbar}.$$

This map defines a Lie bracket on $\mathscr{A}[[\hbar]]$. The Bracket $\{\cdot, \cdot\}^-$ equals the reduction modulo $\hbar$ of $[\cdot, \cdot]$. Therefore, it is still a Lie bracket. The second equality follows from

$$[\psi, \varphi\eta] = [\psi, \varphi]\eta + \psi[\varphi, \eta],$$

which holds for all $\psi, \varphi, \eta \in \mathscr{A}[[\hbar]]$. For (ii), if $g \in \mathcal{G}$ yields the equivalence of $\star$ with $\star'$, then for all $\mu, \nu \in \mathscr{A}$ we have

$$B_1(\mu, \nu) + g(\mu\nu) = B_1'(\mu, \nu) + g(\mu)\nu + \mu g(\nu).$$

Thus the difference $B_1(\mu, \nu) - B_1'(\mu, \nu)$ is symmetric in μ, ν and does not contribute to $\{\cdot, \cdot\}^-$. □

Remark 1.3.3. Note that, by using the star product, we can also write

$$\{\mu, \nu\}^- = \lim_{\hbar \to 0} \frac{\mu \star \nu - \nu \star \mu}{\hbar}$$

Theorem 1.3.2 (Kontsevich)**.** *If $\mathscr{A}$ is the algebra of smooth functions on a differentiable manifold M, then each Poisson bracket on $\mathscr{A}$ lifts to an associative formal deformation.*

Remark 1.3.4. In particular, Konsevitch's theorem (theorem 1.3.2) states that there is a surjection from the set of equivalence classes of formal deformations of $\mathscr{A}$ onto the set of Poisson brackets on $\mathscr{A}$.

1.3.2 The Moyal product

There is a well known example of a formal deformation for the Poisson structure on $\mathscr{P} = \mathbb{R}^d$ with *constant* coefficients, which is called the *Moyal product*. Let therefore

$$\alpha = \sum_{1\leq i<j\leq d} c^{ij}\partial_i \wedge \partial_j \in \Gamma\left(\bigwedge^2 \mathbb{R}^d\right)$$

be the bivector field induced by the Poisson structure on $\mathbb{R}^d$, where $\partial_i = \frac{\partial}{\partial x^i}$ is the partial derivative in the direction of the coordinate x^i for all $1 \leq i \leq d$. The fact that α describes a Poisson structure on $\mathbb{R}^d$ means in particular that $c^{ij} = -c^{ji} \in \mathbb{R}$ for all $0 \leq i < j \leq d$. The *Moyal product* for two smooth functions $f, g \in C^\infty(\mathbb{R}^d)$ is then given by

$$\begin{aligned} f \star_M g &= fg + \hbar \sum_{i,j} c^{ij}\partial_i(f)\partial_j(g) + \frac{\hbar^2}{2}\sum_{i,j,k,\ell} c^{ij}c^{k\ell}\partial_i\partial_k(f)\partial_j\partial_\ell(g) + O(\hbar^3) \\ &= \sum_{n\geq 0}\frac{\hbar^n}{n!}\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}\prod_{k=1}^n c^{i_k,j_k}\left(\prod_{k=1}^n \partial_{i_k}\right)(f) \times \left(\prod_{k=1}^n \partial_{j_k}\right)(g), \end{aligned}$$

where $\times$ denotes the usual product. We can also alternatively write $\star_M$ in coordinates as

$$f \star_M g(x) = \exp\left\{\hbar c^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right\} f(x)g(y)\Big|_{x=y}.$$

Indeed, associativity is given as

$$\begin{aligned} ((f\star_M g)\star_M h)(x) &= \exp\left\{\hbar c^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}\right\}(f\star_M g)(x)h(z)\Big|_{x=z} \\ &= \exp\left\{\hbar c^{ij}\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}\right\}\exp\left\{\hbar c^{k\ell}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^\ell}\right\}f(x)g(y)h(z)\Big|_{x=y=z} \\ &= \exp\left\{\hbar\left(c^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}+c^{k\ell}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^\ell}+c^{mn}\frac{\partial}{\partial x^m}\frac{\partial}{\partial y^n}\right)\right\}f(x)g(y)h(z)\Big|_{x=y=z} \\ &= \exp\left\{\hbar c^{ij}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial y^j}+\frac{\partial}{\partial z^j}\right)\right\}\exp\left\{\hbar c^{k\ell}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^\ell}\right\}f(x)g(y)h(z)\Big|_{x=y=z} \\ &= (f\star_M (g\star_M h))(x) \end{aligned}$$

1.4 The result of Kontsevich

Theorem 1.3.2 tells us only about the existence of a lift for each Poisson bracket to an associative formal deformation. Nevertheless, Kontsevich was able to give a formula for a formal deformation for any Poisson structure on an open subset of $\mathbb{R}^d$ in [K]. Let now $U \subset \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$ and let $\mathscr{A} := C^\infty(U)$ denote the algebra of smooth functions on U. Recall that we have already seen that we need to describe the coefficients B_n for all $n \geq 0$, in order to determine the star product completely. The elements in $\mathscr{A}[[\hbar]]$, which correspond to Kontsevich's star product, are then expressed as

$$\star_K = \sum_{n\geq 0} B_n \hbar^n.$$

1.4.1 Graphical Setting

In order to give a well-defined description of the B_n's, we will make use of graphical structures, in particular we use the notion of an *oriented graph.*

Definition 1.4.1 (Oriented Graphs)**.** *An oriented graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $V_\Gamma \times V_\Gamma$.*

The elements of the set V_Γ are called the vertices of Γ and the elements of E_Γ are called the edges of Γ. We can write every edge $e \in E_\Gamma$ as a tuple of two vertices $(v_1, v_2) \in V_\Gamma \times V_\Gamma$. If $e = (v_1, v_2)$ is an edge, then we say e starts at the vertex v_1 and ends at the vertex v_2. We want to define a special class $\mathcal{G}_n$ of labeled graphs Γ. We say that a labeled graph Γ belongs to $\mathcal{G}_n$ if

(i) Γ has $n+2$ vertices and $2n$ edges,

(ii) the vertex set V_Γ is $\{1, \dots, n\} \sqcup \{L, R\}$, where L and R are just two symbols (meaning left and right),

(iii) edges of Γ are labeled by symbols $e_1^1, e_1^2, e_2^1, e_2^2, \dots, e_n^1, e_n^2$,

(iv) for every $k \in \{1, \dots, n\}$, the edges labeled by e_k^1 and e_k^2 start at the vertex k,

(v) for any $v \in V_\Gamma$, the ordered pair (v, v) is not an edge of Γ.

Remark 1.4.1. We have $|\mathcal{G}_n| = (n(n+1))^n$ for all $n \geq 1$ and $|\mathcal{G}_n| = 1$ for $n = 0$.

Example 1.4.1. An example of such a graph for $n = 3$ would be

$$(e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2) = ((1, L), (1, R), (2, R), (2, 3), (3, L), (3, R)). \tag{1.1}$$

For $n = 2$, we can have

$$(e_1^1, e_1^2, e_2^1, e_2^2) = ((1, 2), (1, L), (2, 1), (2, R)), \tag{1.2}$$

or we can also have

$$(e_1^1, e_1^2, e_2^1, e_2^2) = ((1, L), (1, R), (2, L), (2, R)). \tag{1.3}$$

1.4.2 The Kontsevich weights and main Theorem

Again, let $\mathscr{A} := C^\infty(U)$ be the algebra of smooth functions on an open subset $U \subset \mathbb{R}^d$. Now to each labeled graph $\Gamma \in \mathcal{G}_n$ we can associate a bidifferential operator

$$B_{\Gamma,a} : \mathscr{A} \times \mathscr{A} \to \mathscr{A},$$

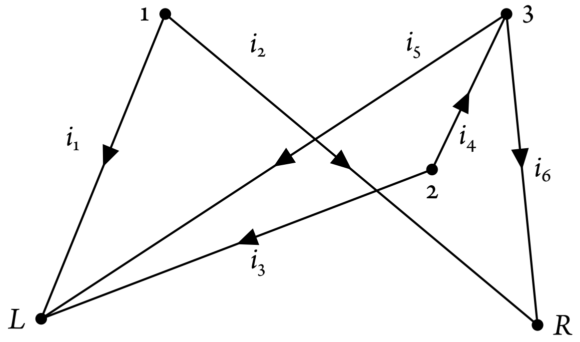


Figure 1.4.1: The illustration of the graph defined in (1.1). We can represent the edges by indices instead of $e^{\bullet}_{\bullet}$.

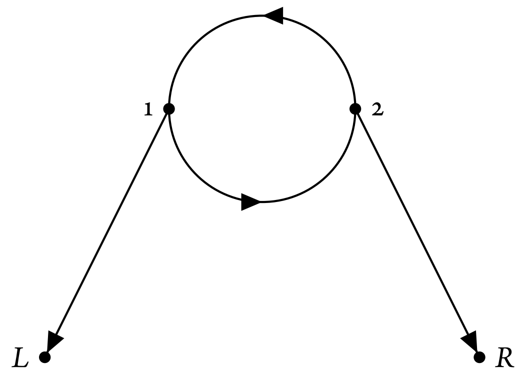


Figure 1.4.2: The illustration of the graph defined in (1.2).

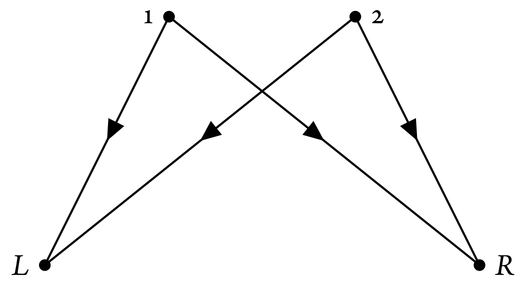


Figure 1.4.3: The illustration of the graph defined in (1.3).

where $\alpha = \sum_{1\leq i,j\leq d} \alpha^{ij}\partial_i \otimes \partial_j \in \Gamma((\mathbb{R}^d)^{\otimes 2})$ is any bivector field on U (not necessarily a Poisson bivector field). We are now able to give a general formula for every $B_{\Gamma,\alpha}$. For $f, g \in \mathscr{A}$, we set

$$B_{\Gamma,\alpha}(f,g) := \sum_I \left[\prod_{k=1}^{n} \left(\prod_{\substack{e\in E_\Gamma \\ e=(\bullet,k)}} \partial_{I(e)}\right) \left(\alpha^{I(e_k^1)I(e_k^2)}\right)\right] \times \left(\prod_{\substack{e\in E_\Gamma \\ e=(\bullet,L)}} \partial_{I(e)}\right)(f) \times \left(\prod_{\substack{e\in E_\Gamma \\ e=(\bullet,R)}} \partial_{I(e)}\right)(g),$$

where the sum is running over all maps $I : E_\Gamma \to \{1, \ldots, d\}$ and $e = (\bullet, v)$ stands for the arrow with target v and $\times$ denotes again the usual product. Moreover, we define the coefficients of the star product to be

$$B_n := \sum_{\Gamma\in\mathcal{G}_n} w_K(\Gamma) B_{\Gamma,\alpha},$$

where the $w_K(\Gamma)$'s are real constants depending on the graph Γ and which can be constructed in a proper way. They are called Kontsevich *weights*. They are constructed in such a way that associativity of $\star_K$ is guaranteed. In order to construct them, let $\mathscr{H}$ be the upper half plane in the complex plane, i.e. $\mathscr{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$ endowed with the *Lobachevsky* metric and let $z \neq w$ be two points on $\mathscr{H}$. Denote by $\varphi(z, w) \in \mathbb{R}/2\pi\mathbb{Z}$ the angle between the geodesic $\gamma(z, w)$, going from z to w with respect to the hyperbolic metric and the line $\gamma(z, i\infty)$, going from z to $i\infty$. Thus we get

$$\varphi(z,w) = \arg \frac{w-z}{w-\bar{z}} = \frac{1}{2i} \log \frac{(w-z)(\bar{w}-z)}{(w-\bar{z})(\bar{w}-\bar{z})}.$$

For the case of $z, w \in \mathscr{H} \sqcup \mathbb{R}$, we first let $\mathscr{H}_n$ to be the space of n numbered and pairwise distinct points

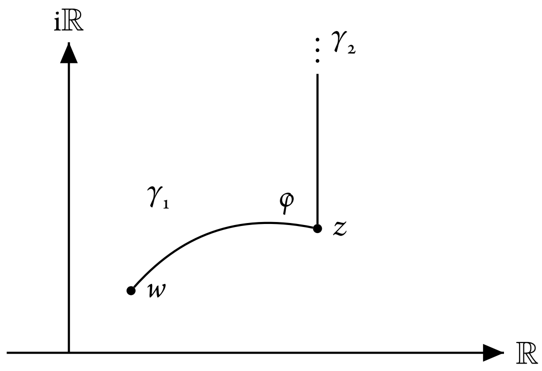


Figure 1.4.4: The illustration of $z \neq w$ on $\mathscr{H}$, the geodesics $\gamma_1 = \gamma(z, w)$, $\gamma_2 = \gamma(z, i\infty)$ and the angle $\varphi = \varphi(z, w)$ between them.

on $\mathscr{H}$, i.e. $\mathscr{H}_n$ is a set of the form

$$\mathscr{H}_n = \{(z_1, \ldots, z_n) \mid z_k \in \mathscr{H} \text{ for all } k \in \{1, \ldots, n\} \text{ and } z_j \neq z_\ell \text{ for } j \neq \ell \in \{1, \ldots, n\}\}.$$

One can see that every edge of a graph Γ defines an ordered pair (z, w) on $\mathscr{H} \sqcup \mathbb{R}$, and therefore also an angle $\varphi_e = \varphi(z, w)$. The differential of such an angle is denoted by $d\varphi_e$ and we can define the Kontsevich *weights* to be

$$w_K(\Gamma) := \frac{1}{(2\pi)^{2n}} \int_{\mathscr{H}_n} \bigwedge_{i=1}^{n} \left(d\varphi_{e_i^1} \wedge d\varphi_{e_i^2} \right) \in \mathbb{R}, \tag{1.4}$$

where $d = d_z + d_w$ with $d_z = dz\frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}}$.

Lemma 1.4.1 (Existence). *The integral in equation (1.4) converges absolutely.*

Finally, we are able to formulate the explicit formula of Kontsevich's star product for an arbitrary given Poisson structure on $\mathbb{R}^d$.

Theorem 1.4.2 (Kontsevich). *Let $\alpha = \sum_{1 \leq i < j \leq d} \alpha^{ij} \partial_i \wedge \partial_j \in \Gamma\left(\bigwedge^2 \mathbb{R}^d\right)$ be a Poisson bivector field on an open subset $U \subset \mathbb{R}^d$. Then for $f, g \in C^\infty(U)$ we have an explicit star product, denoted by $\star_K$, given by*

$$f \star_K g = \sum_{n \geq 0} \frac{\hbar^n}{n!} \sum_{\Gamma \in \mathcal{G}_n} w_K(\Gamma) B_{\Gamma,\alpha}(f, g),$$

which defines an associative formal deformation of the given Poisson structure. Moreover, its equivalence class is independent of the choice of coordinates in U.

Remark 1.4.2. Kontsevich got this result by an application of Stokes' theorem to compactifications of the spaces $\mathscr{H}_n$. This formula is in particular a special case of the *formality theorem*, also proved by Kontsevich, which states the existence of a so called L_∞*-quasi-isomorphism* between the space of multivector fields and the space of multidifferential operators, i.e. we obtain theorem 1.4.2 by restricting formality to bivector fields and bidifferential operators. Moreover, we see that $\star_M$, the Moyal product, is actually obtained from Kontsevich's star product restricted to constant Poisson structures, i.e. $\star_K |_{\{\text{const. } \alpha\}} = \star_M$. After all, the question was whether this abstract setting of a formal deformation could actually be related to a physical interpretation. Indeed, there is a way of formulating Kontsevich's star product in terms of a quantum field theory by using the *Poisson σ-model*, which was the result in [CF1] by Cattaneo and Felder.

1.5 The Poisson σ-model

1.5.1 The structure of the model

Let $(\mathscr{P}, \{\cdot,\cdot\}_\alpha)$ be a Poisson manifold with $\dim \mathscr{P} = d$, where $\{\cdot,\cdot\}_\alpha$ is the Poisson bracket on $\mathscr{P}$ with induced Poisson bivector field α. Recall that we can represent the Poisson bracket for two smooth functions $f, g \in C^\infty(\mathscr{P})$ as

$$\{f, g\}_\alpha = \alpha^{ij}\partial_i f \partial_j g,$$

for some smooth coefficient functions α^{ij}. Let us now introduce the Einstein convention for summation, that means summation over repeated indices is understood. For a general σ-model we need a space M, called the *target space* and a manifold Σ, called the *worldsheet*. For the Poisson σ-model, we are considering a connected, oriented, smooth 2-dimensional manifold worldsheet Σ and a Poisson manifold $(\mathscr{P}, \alpha)$ as the target, where α represents the Poisson bivector field of the Poisson bracket. We consider moreover the fields given by a map $X : \Sigma \to \mathscr{P}$ and a 1-form η, where $\eta(u) \in T^*_{X(u)}\mathscr{P}$ for $u \in \Sigma$. we will go with the convention that Greek indices will represent coordinates of the worldsheet and Latin indices will represent coordinates for the target space.

Definition 1.5.1 (Action functional)**.** *The* action functional *for the Poisson σ-model is given by*

$$S(X, \eta) = \int_\Sigma \left[\eta_{\mu i}\partial_\nu X^i + \frac{1}{2}\alpha^{ij}(X)\eta_{\mu i}\eta_{\nu i}\right] du^\mu du^\nu. \tag{1.5}$$

Moreover, if Σ has boundary, we choose the boundary condition $v^\mu(u)\eta_{\mu i}(u) = 0$ for all $u \in \partial\Sigma$ and for all vectors v which are tangent to $\partial\Sigma$.

Let us consider the worldsheet Σ given by the unit disc in $\mathbb{R}^2$, i.e. $\Sigma = D = \{u \in \mathbb{R}^2 \mid |u| \leq 1\}$ and thus $X : D \to \mathscr{P}$. Moreover, let η be a 1-form on D with $\eta \in \Gamma(X^*(T^*\mathscr{P}) \otimes T^*D)$. If we write X and η in local coordinates as $X^i(u)$ and $\eta_j(u) = \eta_{i\mu}(u)du^\mu$ respectively, where $1 \leq i \leq d$, we can write the action functional in (3.2) as

$$S(X,\eta) = \int_D \eta_i(u) \wedge dX^i(u) + \frac{1}{2}\alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u).$$

Moreover, we formulate the boundary condition for η such that for $u \in \partial D$, we get that $\eta_i(u)$ vanishes on vectors tangent to ∂D.

1.5.2 Kontsevich's star product

The Poisson σ-model is the main component one has to use for writing Kontsevich's star product for a finite dimensional Poisson manifold $(\mathscr{P}, \alpha)$, with $\dim \mathscr{P} = d$, in terms of a path integral. Therefore we need to consider a map $X : D \to \mathscr{P}$ and a 1-form $\eta \in \Gamma(D, X^*(T^*\mathscr{P}) \otimes T^*D)$ which are described as above. Then the theorem is given as follows.

Theorem 1.5.1 (Cattaneo-Felder)**.** *Let everything be as above. Then Kontsevich's star product is given by the semiclassical expansion of the path integral*

$$f \star_K g(x) = \int_{X(\infty)=x} f(X(1))g(X(0)) \exp\left\{\frac{\mathrm{i}}{\hbar}S(X,\eta)\right\} \mathscr{D}(X,\eta),$$

where $0, 1, \infty$ *represent any three cyclically ordered points on the unit circle.*

Remark 1.5.1. Cyclically ordered means in this setting that if we start from 0 and move on the unit circle counterclockwise we first meet 1 and then ∞.

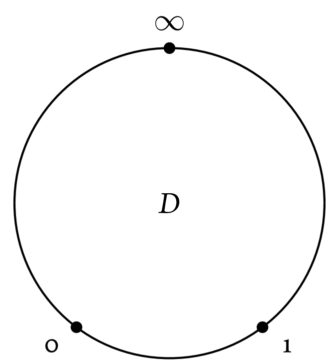


Figure 1.5.1: Cyclically ordered points on the disc

To evaluate this path integral, one needs to take gauge fixing and renormalization into account. We introduce an infinitesimal gauge parameter $\beta \in \Gamma(X^*(T^*\mathscr{P}))$, which vanishes on the boundary of the disc. The action is invariant under the following gauge transformation.

$$\begin{aligned}\delta_\beta X^i &= \alpha^{ij}(X)\beta_j \\ \delta_\beta \eta_i &= -d\beta_i - \partial_i \alpha^{jk}(X)\eta_j \beta_k\end{aligned}$$

In general, the commutator of two gauge transformations is only a gauge transformation modulo the equations of motion, since we get

$$\begin{aligned}[\delta_\beta, \delta_{\beta'}]X^i &= \delta_{\{\beta,\beta'\}}X^i \\ [\delta_\beta, \delta_{\beta'}]\eta_i &= \delta_{\{\beta,\beta'\}}\eta_i - \partial_i\partial_k\alpha^{rs}(X)\beta_r\beta'_s(dX^k + \alpha^{kj}(X)\eta_j),\end{aligned}$$

where $\{\beta, \beta'\}_i = -\partial_i\alpha^{kj}(X)\beta_j\beta'_k$ and $dX^k + \alpha^{kj}(X)\eta_j = 0$ is an Euler-Lagrange equation for the action. There is a special quantization formalism, called the BRST formalism, where β is referred to as an anticommuting *ghost field*. This is basically due to a gradation on the fields, called the *ghost number*. Moreover, there is a BRST operator δ_0, which is an odd derivative on the fields X, η, β, such that

$$\begin{aligned}\delta_0 X^i &= \alpha^{ij}(X)\beta_j \\ \delta_0\eta_i &= -d\beta_i - \partial_i\alpha^{kl}(X)\eta_k\beta_l \\ \delta_0\beta_i &= \frac{1}{2}\partial_i\alpha^{jk}(X)\beta_j\beta_k.\end{aligned}$$

Then the BRST operator is only a differential modulo the equations of motion, since we get

$$\begin{aligned}\delta_0^2 X^i &= \delta_0^2\beta_i = 0 \\ \delta_0^2\eta_i &= -\frac{1}{2}\partial_i\partial_k\alpha^{rs}(X)\beta_r\beta_s(dX^k + \alpha^{kj}(X)\eta_j)\end{aligned}$$

The BRST formalism actually works well for linear or constant Poisson structures, since then the operator is always a differential. In the general case, there is no well-defined cohomology to construct physical observables. Thus we need a more general procedure, which is called the *Batalin-Vilkovisky* formalism (BV formalism) (see chapter 3). We set the ghost number of our fields as $\mathrm{gh}(X^i) = \mathrm{gh}(\eta_i) = 0$ and $\mathrm{gh}(\beta_i) = 1$. Another gradation will be introduced, which measures the behaviour of our fields as differential forms on the disc. That is $\deg(X^i) = \deg(\beta_i) = 0$ and $\deg(\eta_i) = 1$. Now for the procedure of the BV formalism we will add some antifields $X^\dagger, \eta^\dagger, \beta^\dagger$ with ghost number and degree

$$\begin{aligned}\mathrm{gh}(\varphi_a^\dagger) &= -1 - \mathrm{gh}(\varphi^a) \\ \deg(\varphi_a^\dagger) &= 2 - \deg(\varphi_a).\end{aligned}$$

Moreover, we want to find an action $S_{BV}(\varphi, \varphi^\dagger)$, such that $S_{BV}(\varphi, 0) = S(\varphi)$, that means the BV action reduces to the calssical action if we set all antifields to zero. Here φ denotes all the fields $\varphi^1, \varphi^2, \ldots$ and $\varphi^\dagger$ denotes all the antifields $\varphi_1^\dagger, \varphi_2^\dagger, \ldots$ and we also want it to satisfy the *quantum master equation* (QME)

$$(S_{BV}, S_{BV}) - 2i\hbar\triangle S_{BV} = 0,$$

where $(\cdot,\cdot)$ denotes an odd Poisson bracket, called the *BV antibracket* and $\triangle$ is a differential operator, called the *BV Laplacian*. To describe these two objects, we want to make some small preparations first. Let us therefore introduce a Riemannian metric on D and denote by $\langle\cdot,\cdot\rangle_u$ the induced inner product on the exterior algebra of the cotangent space at u. Moreover, we have a volume form $\sqrt{g}du^1du^2 =: dv(u)$ on D. Then $\langle \alpha, \beta\rangle_u dv(u) = \alpha \wedge *\beta$, where $*$ is the Hodge star. The Hodge dual antifields[1] are given by $\varphi_a^* = *\varphi_a^\dagger$. Now we can define the BV Laplacian of a function of fields and antifields as

$$\triangle A = \sum_a (-1)^{\mathrm{gh}(\varphi_a)} \frac{\overrightarrow{\delta}^2 A}{\delta\varphi^a(u)\delta\varphi_a^*(u)}.$$

The functional derivatives of a function of fields and antifields, which we collectively denote by ψ^a, are the distributions defined by

$$\frac{d}{dt}A(\psi + t\rho)\bigg|_{t=0} = \int_D \left\langle \rho^a(u), \frac{\overrightarrow{\delta} A}{\delta\psi^a(u)}\right\rangle_u dv(u) = \int_D \left\langle \frac{A\overleftarrow{\delta}}{\delta\psi^a(u)}, \rho^a(u)\right\rangle_u dv(u),$$

for any test forms ρ^a with the same degree and ghost number as ψ^a. The Laplacian obeys the following product rule for two functions of fields and antifields A and B:

$$\triangle(AB) = \triangle(A)B + (-1)^{\mathrm{gh}(A)}(A, B) + (-1)^{\mathrm{gh}(A)}A\triangle(B),$$

where the BV antibracket is defined by

$$(A, B) = \sum_a \int_D \left(\left\langle \frac{A\overleftarrow{\delta}}{\delta\varphi^a(u)}, \frac{\overrightarrow{\delta} B}{\delta\varphi_a^*(u)}\right\rangle_u - \left\langle \frac{A\overleftarrow{\delta} A}{\delta\varphi_a^*(u)}, \frac{\overrightarrow{\delta} B}{\delta\varphi^a(u)}\right\rangle_u \right) dv(u).$$

Since this bracket is an odd Poisson bracket it clearly also satisfies the graded Jacobi identity and moreover we have the following rules.

(i)

$$(A, B) = -(-1)^{(\mathrm{gh}(A)-1)(\mathrm{gh}(B)-1)}(B, A)$$

(ii)

$$(A, BC) = (A, B)C + (-1)^{(\mathrm{gh}(A)-1)\,\mathrm{gh}(B)}B(A, C)$$

The Batalin-Vilkovisky theorem shows that an integral $\int_{\mathscr{L}}$ does not depend on the Lagrangian submanifold $\mathscr{L}$, which is an important fact for the expectation of observables. If we drop the second term of the quantum master equation we get the *classical master equation* (CME) $(S_{BV}, S_{BV}) = 0$. The BV action actually satisfies the CME and formally $\triangle S_{BV} = 0$ (BV harmonic). We can now define the BRST operator in the BV setting as

$$\delta A = (S_{BV}, A).$$

The CME now implies that δ is a differential. We have

(i)

$$\delta(AB) = \delta(A)B + (-1)^{\mathrm{gh}(A)}A\delta(B)$$

[1]One can also do the perturbative expansion without the Hodge dual antifields, as it is described in [CF2]

(ii)

$$\delta\varphi^a = (-1)^{\mathrm{gh}(\varphi^a)}\frac{\overrightarrow{\partial} S_{BV}}{\partial\varphi_a^\dagger},$$
$$\delta\varphi_a^\dagger = (-1)^{\mathrm{gh}(\varphi^a)+\deg(\varphi^a)}\frac{\overrightarrow{\partial} S_{BV}}{\partial\varphi^a}.$$

Now we start with the action

$$S_{BV}^0 = S + \int_D X_i^\dagger \delta_0 X^i + \eta^{\dagger i}\wedge\delta_0\eta_i - \beta^{\dagger i}\delta_0\beta_i,$$

which has BRST operator $\delta = \delta_0$. We require that the BV operator δ reduces to the BRST operator for the case that all antfields are set to zero. So if we set the antifields to zero, the action S_{BV}^0 reduces to the action S as wanted. Now we need to add some suitable terms to S_{BV}^0 such that $\delta^2 = 0$. It turns out that the BV action is given by

$$S_{BV} = S_{BV}^0 - \frac{1}{4}\int_D \eta^{\dagger i}\wedge\eta^{\dagger j}\partial_i\partial_j\alpha^{kl}(X)\beta_k\beta_l.$$

We do not discuss the case that $\triangle S_{BV} = 0$, which is more or less a fact of an appropriate renormalization ([CF1]). The computations simplify if one uses the *superfield formalism*, describing our fields and antifields as superfields. These are functions of even coordinates u^1, u^2 on D and odd (anticommuting) coordinates θ^1, θ^2. We can write a superfield in the form

$$\varphi(u,\theta) = \varphi^{(0)}(u) + \theta^\mu\varphi_\mu^{(1)}(u) + \theta^\mu\theta^\nu\frac{1}{2}\varphi_{\mu\nu}^{(2)},$$

where the component fields are given by a scalar function $\varphi^{(0)}$, a 1-form $\varphi^{(1)} = \varphi_\mu^{(1)}du^\mu$ and a two form $\varphi^{(2)} = \frac{1}{2}\varphi_{\mu\nu}^{(2)}du^\mu\wedge du^\nu$. We combine the fields of total degree[2] zero into even superfields $\tilde{X}^i$, called the *supercoordinates*:

$$\tilde{X}^i = X^i + \theta^\mu\eta_\mu^{\dagger i} - \frac{1}{2}\theta^\mu\theta^\nu\beta_{\mu\nu}^{\dagger i},$$

and the fields of total degree one into odd superfields $\tilde{\eta}_i$, called the *super 1-forms*:

$$\tilde{\eta}_i = \beta_i + \theta^\mu\eta_{i,\mu} + \frac{1}{2}\theta^\mu\theta^\nu X_{i,\mu\nu}^\dagger.$$

The BV action in this setting is then given by

$$S_{BV} = \int_D L^{(2)},$$

where $L^{(2)} = \int L d^2\theta$ with

$$L = \tilde{\eta}_i D\tilde{X}^i + \frac{1}{2}\alpha^{ij}(\tilde{X})\tilde{\eta}_i\tilde{\eta}_j$$

and $D = \theta^\mu\frac{\partial}{\partial u^\mu}$. One can then check that it actually satisfies the CME. The path integral is computed in

[2] ghost number + degree

the Lorentz-type gauge $d * \eta_i = 0$. On the upper half plane we may use the Euclidean metric in which we have

$$
\begin{aligned}
*du^1 &= du^2 \\
*du^2 &= -du^1
\end{aligned}
$$

In the general BV formalism one considers the integral $\int_{\mathscr{L}} \mathscr{O} \exp\left\{\frac{\mathrm{i}}{\hbar} S_{BV}\right\}$, for any *gauge fixing fermion* Ψ, which is a function of the fields of ghost number -1 and $\mathscr{O}$ is an observable, which means that it is a function of fields and antifields that is closed with respect to the *quantum* BRST operator $\Omega := -\mathrm{i}\hbar\triangle + \delta$, i.e.

$$
\Omega\mathscr{O} = -\mathrm{i}\hbar\triangle\mathscr{O} + (S_{BV}, \mathscr{O}) = 0.
$$

Moreover, the integral is taken over the *Lagrangian* submanifold $\mathscr{L}$ defined by the equations

$$
\varphi_a^\dagger = \overrightarrow{\partial}_{\varphi^a} \Psi.
$$

One can show that these integrals are invariant under variations of Ψ and thus *equal* to the original ill-defined path integral, i.e. $\int \mathscr{O} \exp\left\{\frac{\mathrm{i}}{\hbar} S(\varphi)\right\} \mathscr{D}(\varphi)$ (see A), with the action $S(\varphi) = S_{BV}(\varphi, 0)$, which is what one gets if $\Psi = 0$. To find such a Ψ, we introduce new anticommuting scalar fields (called *antighosts*) γ^i of ghost number -1 on D and scalar Lagrange multiplier fields λ^i of ghost number zero and also the corresponding antifields $\gamma_i^\dagger, \lambda_i^\dagger$. Moreover, $\lambda^i(u) = 0$ for $u \in \partial D$ and γ^i is constant on the boundary. The action for these fields and antifields is given by $-\int_D \lambda^i \gamma_i^\dagger$ and we can simply add it to the BV action. The gauge fixing condition $d\eta_i = 0$ is then encoded in the gauge fixing fermion $\Psi = -\int_D d\gamma^i * \eta_i$. With the condition of the fields on the Lagrangian submanifold, the fact that $\eta^{\dagger i} = *d\gamma^i$ and the boundary condition of γ^i beeing the same as that of $\eta^\dagger$ (vanishing on normal vectors), we get the *gauge fixed* action

$$
\begin{aligned}
S_{\mathrm{gf}} = \int_D \eta_i \wedge dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i \wedge \eta_j - *d\gamma^i \wedge (d\beta_i + \partial_i \alpha^{kl}(X)\eta_k\beta_l) \\
- \frac{1}{4} * d\gamma^i \wedge *d\gamma^j \partial_i\partial_j \alpha^{kl}(X)\beta_k\beta_l - \lambda^i d * \eta_i
\end{aligned}
$$

Now we can compute the *Feynman perturbation expansion* (see A.3) in powers of $\hbar$ around the classical solution $X(u) = x$ and $\eta(u) = 0$. We write $X(u) = x + \xi(u)$, where $\xi(u)$ is a fluctuation field with $\xi(\infty) = 0$. Then we can deduce the *Feynman propagator* from the *kinetic* part

$$
S_{\mathrm{gf}} = \int_D \eta_i d\xi^i - *d\gamma^i \wedge d\beta_i - \lambda^i d * \eta_i = \int_D \eta_i \wedge (d\xi^i + *d\lambda^i) + \beta_i d * d\gamma^i.
$$

of the gauge fixed action, where the other terms of S_{gf} are considered as perturbations. Now we need to invert the operators

$$d \oplus *d : \Omega^0(D) \oplus \Omega^0_0(D) \longrightarrow \Omega^1(D)$$
$$d * d : \Omega^0(D) \longrightarrow \Omega^2(D),$$

where $\Omega^0_0(D)$ is the space of functions with Dirichlet boundary condition on ∂D. In order to describe these integral kernels, we map the disc confromally onto the upper half plane $\mathscr{H}$ and use the standard complex coordinate of $\mathscr{H}$. The Green function (see A.4) for $(d * d)^{-1}$ is then given by $\frac{1}{2\pi}\psi(z, w)$, where

$$\psi(z, w) = \log \left| \frac{z - w}{z - \bar{w}} \right| .$$

The Green function for $(d \oplus *d)^{-1}$ is given by $G(z, w) = \frac{1}{2\pi}(*d_z\psi(z, w) \oplus d_z\varphi(z, w))$, where $d_z = dz\frac{\partial}{\partial z} + d\bar{z}\frac{\partial}{\partial \bar{z}}$ is the differential with respect to z and

$$\varphi(z, w) = \frac{1}{2\mathrm{i}} \log \frac{(z - w)(z - \bar{w})}{(\bar{z} - \bar{w})(\bar{z} - w)}.$$

Therefore, we get that $d_w * d_w\psi(z, w) = d_w * d_w\varphi(z, w) = 2\pi\delta_z(w)$, where $\delta_z(w)$ is the usual Dirac distribution and the boundary conditions for $w \in \partial\mathscr{H}$ are Dirichlet for ψ and Neumann for φ. The propagators (also called 2-point functions, see A.2) are then

$$\langle \gamma^k(w)\beta_j(z)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j\psi(z, w),$$
$$\langle \xi^k(w)\eta_j(z)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j d_z\varphi(z, w),$$
$$\langle \lambda^k(w)\eta_j(z)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j * d_z\psi(z, w)$$

Since $*d_w\psi(z, w) = d_w\varphi(z, w)$, we get $\langle *d\gamma^k(w)\beta_j(z)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j d_w\varphi(z, w)$. It follows that we can combine them into a *superpropagator*

$$\langle \xi^k(w)\eta_j(z)\rangle + \langle *d\gamma^k(w)\beta_j(z)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j d\varphi(z, w),$$

where $d = d_z + d_w$. If we consider the superfields $\tilde{\eta}_j(z, \theta) = \beta_j(z) + \theta^\mu\eta_{j,\mu}(w)$ and $\tilde{\xi}^k(w, \zeta) = \xi^k(w) + \zeta^\mu\eta^{\dagger j}_\mu(w)$, with $\eta^{\dagger j} = *d\gamma^j$, the superpropagator is

$$\langle \tilde{\xi}^k(w, \zeta)\tilde{\eta}_j(z, \theta)\rangle = \frac{\mathrm{i}\hbar}{2\pi}\delta^k_j D\varphi(z, w),$$

where $D = \theta^\mu\frac{\partial}{\partial z^\mu} + \zeta^\mu\frac{\partial}{\partial w^\mu}$. Then we can obtain the perturbation expansion by writing $S_{\mathrm{gf}} = S^0_{\mathrm{gf}} + S^1_{\mathrm{gf}}$ and expanding the integral

$$\int \exp\left\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{gf}}\right\}\mathscr{O} = \sum_{n\geq 0}\frac{\mathrm{i}^n}{\hbar^n n!}\int \exp\left\{\frac{\mathrm{i}}{\hbar}S^0_{\mathrm{gf}}\right\}(S^1_{\mathrm{gf}})^n\mathscr{O}.$$

We can compute this expression by using Wick's theorem (see A.2 (finite) and A.3 (infinite)) for Gaussian integrals

$$\int \exp\left\{\frac{\mathrm{i}}{\hbar}S^{0}_{\mathrm{gf}}\right\}\tilde{\xi}^{k_1}(w_1,\zeta_1)\cdots\tilde{\xi}^{k_N}(\omega_N,\zeta_N)\tilde{\eta}_{j_1}(z_1,\theta_1)\cdots\tilde{\eta}_{j_N}(z_N,\theta_N)\delta_x(X(\infty))$$
$$=\sum_{\sigma\in S_N}\langle\tilde{\xi}^{k_{\sigma(1)}}(w_{\sigma(1)},\zeta_{\sigma(1)})\tilde{\eta}_{j_1}(z_1,\theta_1)\rangle\cdots\langle\tilde{\xi}^{k_{\sigma(N)}}(w_{\sigma(N)},\zeta_{\sigma(N)})\tilde{\eta}_{j_N}(z_N,\theta_N)\rangle$$

We set the normalization of the integral such that $\int \exp\left\{\frac{\mathrm{i}}{\hbar}S^{0}_{\mathrm{gf}}\right\}\delta_x(X(\infty)) = 1$, so that for $\alpha = 0$ the star product coincides with the ordinary product. Here $\delta_x(X(t)) = \prod_{i=1}^{d}\delta(X^i - x^i)\gamma^i(t)$ fixes the value of the zero modes[3] of X and the γ's are needed since the integral is otherwise zero, owing to the presence of zero modes in the integration over γ. The Feynman perturbation expansion is then obtained by expanding the interaction term S^{1}_{gf} and the observable in powers of $\tilde{\xi}$, $\tilde{\eta}$. This gives the vertices

$$S^{1}_{\mathrm{gf}} = \frac{1}{2}\int_D\int d^2\theta\sum_{k\geq 0}\frac{1}{k!}\partial_{j_1}\cdots\partial_{j_k}\alpha^{ij}(x)\tilde{\xi}^{j_1}\cdots\tilde{\xi}^{j_k}\tilde{\eta}_i\tilde{\eta}_j. \tag{1.6}$$

Moreover, we can consider the observable of correct ghost number

$$\mathscr{O} = f(\tilde{X}(1))g(\tilde{X}(0))\delta_x(X(\infty)),$$

where $f, g \in C^\infty(\mathscr{P})$. Then expanding f and g in powers of $\tilde{\xi}$ we get an expansion in *Feynman diagrams*[4] (see A.2 and A.4). We can now label the term with n vertices (1.6) by the Kontsevich diagrams $\Gamma \in \mathcal{G}_n$. Let us also indicate the lines of a Kontsevich diagram with $(j, v_1(j))$, $(j, v_2(j))$ for $j = 1, \ldots, n$ (see figure 1.5.2)

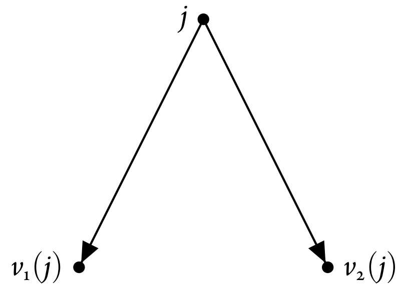


Figure 1.5.2: The principle of the lines going from a vertex j to the vertices $v_1(j)$ and $v_2(j)$

It is such that there is no line going from the vertex j to itself. The term which is now labeled by such a diagram Γ is then $B_{\Gamma,\alpha}(f,g)$ times

$$\frac{1}{n!}\left(\frac{\mathrm{i}}{\hbar}\right)^n\frac{1}{2^n}\left(\frac{\mathrm{i}\hbar}{2\pi}\right)^{2n}\int\bigwedge_{j=1}^{n}d\varphi(z_j,z_{v_1(j)})\wedge d\varphi(z_j,z_{v_2(j)}) = (-1)^n\left(\frac{\mathrm{i}\hbar}{2}\right)^n w_K(\Gamma).$$

[3]These are the constant functions.

[4]The arguments of f and g of our original integral are $X(1)$, $X(0)$, rather than the superfields. These additional terms do not contribute to the integral since they are of negative ghost number.

The factor $\frac{1}{\prod_j k_j!}$, where k_j is the number of lines pointing to j, is compensated by the fact that there are as many terms in the Wick theorem which give the same contribution beacause k_j arguments of $\tilde{\xi}$ are equal to each other. Now by the fact that $(-1)^n w_K(\Gamma) B_{\Gamma,\alpha}(g,f) = w_K(\overline{\Gamma}) B_{\overline{\Gamma},\alpha}(f,g)$, where $\overline{\Gamma}$ is Γ with R and L interchanged. Thus the product obtained here coincides with Kontsevich's except that it also involves tadpole diagrams[5]. One can handle them separately by considering an appropriate renormalization as it is described in [CF1].

Remark 1.5.2. This theorem is an application of BV theory which is, in some modified sense (BV-BFV), the important formalism which we will use in chapter 3. The Poisson σ-model is thus an important classical field theory in connection with deformation quantization, which we will use in chapter 3 to construct a deformation quantization of the *relational symplectic groupoid* for constant Poisson bivector fields, i.e. for the case of the Moyal product, by only considering its topological structure as a 2-dimensional manifold.

[5]These are diagrams with short loops

2

The Relational Symplectic Groupoid (RSG)

2.1 Introduction

Symplectic groupoids appear in Poisson and symplectic geometry and different aspects of topological field theories. A groupoid is basically a small category endowed with invertible morphisms. Moreover, if $G \rightrightarrows M$ denotes a groupoid over M, there is a symplectic form $\omega \in \Omega^2(G)$ such that the multiplication graph is a Lagrangian submanifold of $(G, \omega) \times (G, \omega) \times (G, -\omega)$. Actually, the Poisson σ-model as a reduced phase space of a 2-dimensional topological field theory has the structure of a symplectic Groupoid. In fact symplectic groupoids do not always exist, e.g. for Poisson σ-model of a non reduced phase space, whereas the more general construction, called the *relation symplectic groupoid*, does always exist (see [IC] and [CC1]). This chapter is mostly based on the work of I. Contreras in [IC]. In this chapter we want to give the mathematical notion, construction and descritpion of the axioms of the RSG and give a pictorial point of view of the axioms as topological manifolds. In order to do this, we first need to discuss some aspects of symplectic geometry and give a small introduction to general and symplectic groupoids. This chapter gives a rigorous definition of the RSG, whereas for our purpose of the quantization, we will only need the topological version of the axioms.

2.2 Symplectic Structures and Relations

2.2.1 Definitions and Properties

Definition 2.2.1 (non degeneracy)**.** *Let V be a vector space over $\mathbb{R}$. A skew symmetric form $\omega \in \bigwedge^2 V^*$ is called* non degenerate *if the induced linear map*

$$\begin{aligned} \omega^{\#} : V &\longrightarrow V^* \\ w &\longmapsto \omega^{\#}(v)(w) := \omega(v, w) \end{aligned}$$

is an isomorphism.

Definition 2.2.2 (weak non degeneracy)**.** *We call ω* weakly non degenerate *if $\omega^{\#}$ is injective.*

Remark 2.2.1. If V is finite dimensional, a *weakly non degenrate* form is also non degenerate.

Definition 2.2.3 (weak symplectic)**.** *A bilinear symmetric form ω on V is called* weak symplectic *if it is skew symmetric and weakly non degenerate.*

Definition 2.2.4 (weak symplectic vector space)**.** *A vector space V endowed with a weak symplectic form ω is called a* weak symplectic vector space.

Definition 2.2.5 (symplectomorphism)**.** *A bijective linear map $f : (V, \omega_V) \to (W, \omega_W)$ is called a* symplectomorphism *if $f^* \omega_W = \omega_V$.*

Definition 2.2.6 (symplectic orthogonal space)**.** *Let V be a symplectic space and W a linear subspace of V. We define its symplectic orthogonal space, by*

$$W^{\perp} := \{v \in V \mid \omega(v, w) = 0, \forall w \in W\}.$$

Proposition 2.2.1 (properties of the orthogonal symplectic space)**.** *Let V be a weak symplectic space and $W, Z \subset V$ subspaces of V. Then*

(*i*) $W \subset Z \Longrightarrow Z^{\perp} \subset W^{\perp}$.

(*ii*) $(W + Z)^{\perp} = W^{\perp} \cap Z^{\perp}$.

(*iii*) $W^{\perp} + Z^{\perp} \subset (W \cap Z)^{\perp}$.

(*iv*) $W \subset (W^{\perp})^{\perp}$.

(*v*) $W^{\perp} = ((W^{\perp})^{\perp})^{\perp}$.

Remark 2.2.2. We can define $\omega^{\#W} : V \to W^*$ as the restriction of $\omega^{\#}(V)$ to W, namely,

$$\omega^{\#}(v)(w) := \omega(v, w)$$

for all $v \in V$ and $w \in W$. In this way, we get

$$W^{\perp} = \ker \omega^{\#W}$$

and therefore, we have the induced map $\omega^{\#W} : V/W^{\perp} \hookrightarrow W^{*}$. In the finite dimensional setting, we get

$$V/W^{\perp} \cong W^{*}$$

implying that $\dim W^{\perp} = \dim V - \dim W$. Let us denote by $\omega \mid_W$ the restriction of ω to W, to see that this induces a bilinear form on W and

$$(\omega \mid_W)^{\#} = (\omega^{\#W}) \mid_W .$$

Therefore

$$\ker(\omega \mid_W)^{\#} = W \cap W^{\perp}.$$

Definition 2.2.7 (symplectic, isotropic, coisotropic and Lagrangian)**.** *A subspace W of V is called*

(i) symplectic *if $\omega \mid_W$ is a symplectic form or equivalently, $W \cap W^{\perp} = \{0\}$.*

(ii) isotropic *if $W \subset W^{\perp}$.*

(iii) coisotropic *if $W^{\perp} \subset W$.*

(iv) Lagrangian *if $W^{\perp} = W$.*

2.2.2 Symplectic reduction

Definition 2.2.8 (reduction)**.** *Let W be a linear subspace of a symplectic vector space V. We define*

$$\underline{W} := W/W \cap W^{\perp},$$

called the reduction *of V. The form ω induces a symplectic form $\underline{\omega}$ on $\underline{W}$ given by*

$$\underline{\omega}([w_1], [w_2]) := \omega(w_1, w_2).$$

In particular, one can see that $\underline{\omega}$ is indeed skew-symmetric and non degenerate. Moreover, we can obtain the following.

(i) In the finite dimensional case, if W is a Lagrangian subspace, then $\dim W = \frac{1}{2} \dim V$.

(ii) If W is coisotropic, then $\underline{W} = W/W^{\perp}$.

(iii) W is coisotropic if and only if $\underline{W} = \{0\}$.

(iv) If $Z \subset L$, where L is Lagrangian, then Z is isotropic.

(v) If $L \subset Z$, where L is Lagrangian, then Z is coisotropic.

(vi) If $L \subset Z$, where L is Lagrangian and Z is isotropic, then $Z = L$.

2.2.3 Canonical relations

Definition 2.2.9 (relation)**.** *A relation R between two sets M and N is a subset of the cartesian product $M \times N$ and we will use the notation $R : M \nrightarrow N$. If $S \nrightarrow P$ is another relation, its composition is given by*

$$S \circ R := \{(m, p) \in M \times P \mid \exists n \in N, (m, n) \in R, (n, p) \in S\} : M \nrightarrow P.$$

Remark 2.2.3. In the case where the sets M and N are vector spaces, a relation R is called *linear* if it corresponds to a linear subspace of $M \oplus N$.

Definition 2.2.10 (canonical relation)**.** *Let (M, ω_M) and (N, ω_N) be symplectic spaces. A linear relation $L : M \nrightarrow N$ is called* canonical*, if R is a Lagrangian subspace of $\overline{M} \oplus N$, where $\overline{M}$ denotes the vector space M with its negative symplectic form $-\omega_M$.*

Remark 2.2.4. If $L : V \nrightarrow W$ is a canonical relation, then L is a Lagrangian subspace of $\overline{W} \oplus V$, so we have the canonical relation

$$L^\dagger : W \nrightarrow V$$

called the transpose of L.

2.3 Groupoids and Algebroids

2.3.1 Lie groupoids and Symplectic groupoids

Definition 2.3.1 (Groupoid)**.** *A* groupoid *in a small category* **C** *which has fiber products, corresponds to two objects G, M and a set of morphisms in* **C** *described in the following diagram*

$$G \times_{(s,t)} G \xrightarrow{\ \mu\ } G \xrightarrow{\ \ i\ \ } G \;\underset{s}{\overset{\overset{\varepsilon}{\longleftarrow}}{\underset{t}{\rightrightarrows}}}\; M$$

where $G \times_{(s,t)} G$ is the fiber product for the maps $s, t : G \to M$, such that the following axioms hold.

(G1) $s \circ \varepsilon = t \circ \varepsilon = id_M$

(G2) *If $g \in G_{(x,y)}$ and $h \in G_{(y,z)}$, then $\mu(g, h) \in G_{(x,z)}$*

(G3) $\mu(\varepsilon \circ s \times id_G) = \mu(id_G \times \varepsilon \circ t) = id_G$

(G4) $\mu(id_G \times i) = \varepsilon \circ t$

(G5) $\mu(i \times id_G) = \varepsilon \circ s$

(G6) $\mu(\mu \times id_G) = \mu(id_G \times \mu)$

where we denote $G_{(x,y)} := s^{-1}(x) \cap t^{-1}(y)$. We call the maps $s : G \to M$ and $t : G \to M$ the *source* and *target* map respectively. The map $\mu : G \times G \to G$ corresponds to a multiplication map, $\varepsilon : M \to G$ to the unit map and $i : G \to G$ to the inverse mapping.

Definition 2.3.2 (Lie groupoid). *if the category* **C** *in the above definition of a groupoid is one where the objects are smooth manifolds and the morphisms are smooth maps, we speak about a* Lie groupoid*.*

Remark 2.3.1. Since the category of manifolds does not have fiber product, we need to put an additional axiom for the definition of a Lie groupoid, namely that the map s has to be a surjective submersion.

Example 2.3.1 (Pair groupoid). In this case $G = M \times M$, the source and target corresponds to first and second projection respectively. The multiplication is given by

$$\mu((x, y), (y, z)) = (x, z),$$

the inverse is transposition and the unit map is the diagonal map.

Example 2.3.2 (Bundle groupoid). Let $E \to M$ be a vector bundle. The *vector bundle groupoid* has as objects the poins of the manifold M and the morphisms are the vectors of the fibers of E. The groupoid multiplication is fiber addition. The source and target coincide and correspond to the bundle projection, the inverse is the fiber inverse and the unit is the zero section.

Definition 2.3.3 (Symplectic groupoid). *A Lie groupoid such that the space of morphisms G is equipped with a nondegenerate closed 2-form $\omega \in \Omega^2(G)$ satisfying the following condition*

$$\mu^*(\omega) = \pi_1^*(\omega) + \pi_2^*(\omega),$$

where π_1 and π_2 denote the first and second projections of $G \times G$ onto G, is called a symplectic groupoid *and denoted by $(G, \omega) \rightrightarrows M$.*

Remark 2.3.2. We call an ω as in definition 2.3.3 *multiplicative*.

Lemma 2.3.1. *ω is multiplicative if and only if the graph of the multiplication map is a Lagrangian submanifold of $G \times G \times \bar{G}$, where $\bar{G}$ denotes the manifold G equipped with the opposite symplectic form structure $-\omega$.*

Proposition 2.3.2. *Let $(G, \omega) \rightrightarrows M$ be a finite dimensional symplectic groupoid. Then the following holds.*

(*i*) *The image of the unit map, $\varepsilon(M)$ is a Lagrangian submanifold of G.*

(*ii*) *The inverse map i is an antisymplectomorphism.*

(*iii*) *There is a unique Poisson structure on M such that s is a Poisson map.*

Example 2.3.3. The pair groupoid $M \times M \rightrightarrows M$, where (M, ω) is a symplectic manifold, is a trivial example of a symplectic groupoid, where the symplectic structure on $M \times M$ is given by $\omega \oplus -\omega$.

2.3.2 Lie algebroids

Definition 2.3.4 (Lie algebroid)**.** *A pair (A, ρ), where A is a vector bundle over M and ρ is a vector bundle morphism from A to TM is called a* Lie algebroid *if the following holds.*

(L1) *There is a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ such that the induced map $\rho_* : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.*

(L2) *The Leibniz rule holds:*

$$[X, fY]_A = f[X, Y]_A + \rho_*(X)(f)Y$$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Remark 2.3.3. The map ρ is called the *anchor map*.

2.4 Relational symplectic groupoids

Remark 2.4.1. The axioms for the relational symplectic groupoid are pictorially given in figure 2.4.1.

2.4.1 The axioms of the RSG

Definition 2.4.1 (Relational symplectic groupoid)**.** *A* relational symplectic groupoid *(RSG) is a triple $(\mathcal{G}, L, I)$ where*

(*i*) *$\mathcal{G}$ is a* weak *symplectic manifold*

(*ii*) *L is an immersed Lagrangian submanifold of $\mathcal{G}^3$.*

(*iii*) *I is an antisymplectomorphism of $\mathcal{G}$ called the* inversion,

satisfying the following axioms:

(*RSG1*) *L is* cyclically symmetric, *i.e. if $(x, y, z) \in L$, then $(y, z, x) \in L$.*

(*RSG2*) *I is an* involution, *i.e. $I^2 = id$.*

Remark 2.4.2. L is an immersed canonical relation $\mathcal{G} \times \mathcal{G} \nrightarrow \bar{\mathcal{G}}$ and will be denoted by L_{rel}. since the graph of I is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G}$, I is an immersed canonical relation $\bar{\mathcal{G}} \nrightarrow \mathcal{G}$ and will be denoted by I_{rel}. L and I can be regarded as well as immersed canonical relations

$$\bar{\mathcal{G}} \times \bar{\mathcal{G}} \nrightarrow \mathcal{G}$$

and

$$\mathcal{G} \nrightarrow \bar{\mathcal{G}}$$

respectively, which will be denoted by $\overline{L_{rel}}$ and $\overline{I_{rel}}$. The transposition

$$\begin{aligned} T : \mathcal{G} \times \mathcal{G} &\longrightarrow \mathcal{G} \times \mathcal{G} \\ (x, y) &\longmapsto (y, x) \end{aligned}$$

induces canonical relations

$$T_{rel} : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G} \times \mathcal{G}$$

and

$$\overline{T_{rel}} : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \nrightarrow \bar{\mathcal{G}} \times \bar{\mathcal{G}}.$$

The identity map $id : \mathcal{G} \to \mathcal{G}$ as a relation will be denoted by $id_{rel} : \mathcal{G} \nrightarrow \mathcal{G}$ and by $\overline{id_{rel}} : \bar{\mathcal{G}} \nrightarrow \bar{\mathcal{G}}$. Since I and T are diffeomorphisms, it follows that $I_{rel} \circ L_{rel}$ and $\overline{L_{rel}} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}})$ are immersed submanifolds. For a relational symplectic groupoid we want these two compositions to be morphisms $\mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G}$, and moreover we want them to coincide.

(*RSG3*) (1) *The composition $I_{rel} \circ L_{rel}$ and $\overline{L_{rel}} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}})$ are immersed submanifolds of $\mathcal{G}^3$.*

(2) *$I_{rel} \circ L_{rel}$ and $\overline{L_{rel}} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}})$ are Lagrangian submanifolds of $\overline{\mathcal{G}^2} \times \mathcal{G}$.*

(3)

$$I_{rel} \circ L_{rel} = \overline{L_{rel}} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}}).$$

Define $L_3 := I_{rel} \circ L_{rel} : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G}$. Then we can formulate a corollary of the previous axioms.

Corollary 2.4.1. $\overline{I_{rel}} \circ L_3 = \overline{L_3} \circ \overline{T_{rel}} \circ (\overline{I_{rel}} \times \overline{I_{rel}})$.

(*RSG4*) (1) *The compositions $L_3 \circ (L_3 \times id)$ and $L_3 \circ (id \times L_3)$ are immersed submanifolds of $\mathcal{G}^4$.*

(2) *$L_3 \circ (L_3 \times id)$ and $L_3 \circ (id \times L_3)$ are Lagrangian submanifolds of $\overline{G^3} \times \mathcal{G}$.*

(3)

$$L_3 \circ (L_3 \times id) = L_3 \circ (id \times L_3) \tag{2.1}$$

Remark 2.4.3. The graph of the map I, as a relation $\bullet \nrightarrow \mathcal{G} \times \mathcal{G}$ will be denoted by L_I.

(*RSG5*) (1) *The compositions $L_3 \circ L_I$ and $L_3 \circ (L - 3 \circ L_1 \times L_3 \circ L_I)$ are immersed submanifolds of $\mathcal{G}$.*

(2) *$L_3 \circ L_I$ and $L_3 \circ (L_3 \circ L_I \times L_3 \times L_I)$ are Lagrangian submanifolds of $\mathcal{G}$.*

(3) *Denoting by L_1 the morphism $L_1 := L_3 \circ L_I : \bullet \nrightarrow \mathcal{G}$, then*

$$L_3 \circ (L_1 \times L_1) = L_1. \tag{2.2}$$

Again, we can extract the following corollary.

Corollary 2.4.2. *$\overline{I_{rel}} \circ L_1 = \overline{L_1}$, which is also equivalent to $I \circ L_1 = \overline{L_1}$, where L_1 is regarded as an immersed Lagrangian submanifold of $\mathcal{G}$.*

(*RSG6*) (1) *$L_3 \circ (L_1 \times id)$ and $L_3 \circ (id \times L_1)$ are immersed submanifolds of $\mathcal{G} \times \mathcal{G}$.*

(2) *$L_3 \circ (L_1 \times id)$ and $L_3 \circ (id \times L_1)$ are Lagrangian submanifolds of $\bar{\mathcal{G}} \times \mathcal{G}$.*

(3) *If we define the morphism*

$$L_2 := L_3 \circ (L_1 \times id) : \mathcal{G} \nrightarrow \mathcal{G},$$

then the following equations hold.

(a) $L_2 = L_3 \circ (id \times L_1)$

(b) L_2 *leaves* L_1 *and* L_3 *invariant, i.e.*

$$
\begin{aligned}
L_2 \circ L_1 &= L_1 \\
L_2 \circ L_3 &= L_3 \circ (L_2 \times L_2) = L_3
\end{aligned}
$$

(c) $\overline{I_{rel}} \circ L_2 = \overline{L_2} \circ \overline{I_{rel}}$ *and* $L_2^\dagger = L_2$.

From the definition of L_2 *and equations (2.1) and (2.2), we get the following corollary.*

Corollary 2.4.3. L_2 *is* idempotent, *i.e.* $L_2 \circ L_2 = L_2$.

Definition 2.4.2 (Regular relational symplectic groupoid)**.** *A relational symplectic groupoid* $(\mathcal{G}, L, I)$ *is called* regular *if the following axioms are satisfied, where we consider* $\mathcal{G}$ *as a relation* $\bullet \not\rightarrow \mathcal{G}$ *and denote it by* $\mathcal{G}_{rel}$.

(rRSG1) $C := L_2 \circ \mathcal{G}$ *is an immersed submanifold of* $\mathcal{G}$.

Corollary 2.4.4. *C is an immersed coisotropic submanifold of* $\mathcal{G}$. *Moreover,* L_2 *is an equivalence relation in C.*

(rRSG2) *The partial reduction* $\underline{L_1} = L_1/(L_2 \cap L_1 \times L_1)$ *is a finite dimensional smooth manifold.*

(rRSG3) *The set* $S := \{(c, [\ell]) \in C \times \underline{L_1} \mid \exists \ell \in [\ell], g \in G : (\ell, c, g) \in L_3\}$ *is an immersed submanifold of* $\mathcal{G} \times \underline{L_1}$ *satisfying*

(1) $(S \times S) \circ L_2^{rel} = \triangle(\underline{L_1})$, *where* L_2^{rel} : **point** $\not\rightarrow C \times C$ *is the induced relation from* L_2 *with* **point** *denoting the connected zero dimensional manifold and* $\triangle(\underline{L_1})$ *denotes the diagonal submanifold* $\underline{L_1} \times \underline{L_1}$.

(2) *The induced relation* $dS := TS : T\mathcal{G} \to T\underline{L_1}$ *is surjective.*

Moreover, we can obtain another corollary.

Corollary 2.4.5. *The following holds.*

(i) *The relation*

$$T := \{(c, [\ell]) \in C \times \underline{L_1} \mid \exists \ell \in [\ell], g \in G : (c, \ell, g) \in L_3\} = I \circ S$$

is an immersed submanifold of $\mathcal{G} \times \underline{L_1}$.

(ii) *S and T regarded as relations from C to* $\underline{L_1}$ *are surjective submersions.*

2.4.2 Examples

Example 2.4.1 (Symplectic groupoid). Let G be a symplectic groupoid $G \rightrightarrows M$. Then we can endow it with a relational symplectic structure $(\mathcal{G}, L, I)$ by setting

$$\begin{aligned}
\mathcal{G} &= G \\
L &= \{(g_1, g_2, g_3) \in G^3 \mid (g_1, g_2) \in G \times_{(s,t)} G; g_3^{-1} = \mu(g_1, g_2)\} \\
I &= G \ni g \mapsto i(g)
\end{aligned}$$

The immersed canonical relations are then given by

$$\begin{aligned}
L_1 &= \varepsilon(M) \\
L_2 &= \triangle(G) \\
L_3 &= Gr(\mu)
\end{aligned}$$

Example 2.4.2 (Symplectic manifolds with a Lagrangian submanifold). Let (G, ω) be a symplectic manifold, φ an antisymplectomorphism and $\mathscr{L}$ an immersed Lagrangian submanifold of G such that $\varphi(\mathscr{L}) = \mathscr{L}$. We can then define a relational symplectic groupoid $(\mathcal{G}, L, I)$ by

$$\begin{aligned}
\mathcal{G} &= G \\
L &= \mathscr{L} \times \mathscr{L} \times \mathscr{L} \\
I &= \varphi
\end{aligned}$$

Moreover, the canonical relations are given by

$$\begin{aligned}
L_1 &= \mathscr{L} \\
L_2 &= \mathscr{L} \times \mathscr{L} \\
L_3 &= \mathscr{L} \times \mathscr{L} \times \mathscr{L}
\end{aligned}$$

2.4.3 Topological axioms of the RSG

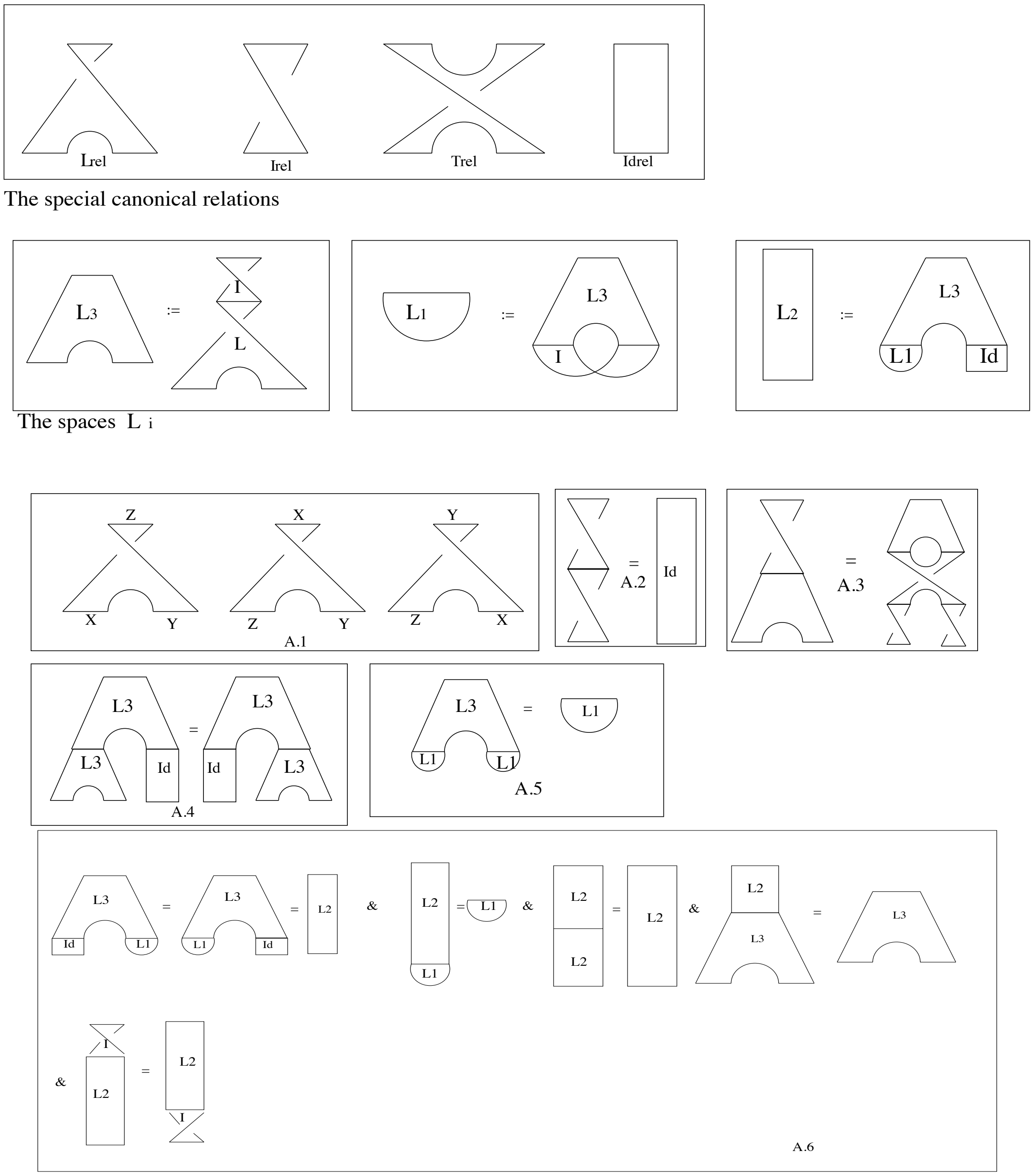


Figure 2.4.1: The relational symplectic groupoid as topological 2-manifolds. The morphisms I_{rel} and L_{rel} are represented by a twisted stripe and pair of *paper doll pants* respectively, and the induced immersed canonical relations L_1, L_2 and L_3 are constructed as compositions of L and I. As it is shown in the figure, they should satisfy the previously defined compatibility axioms. The horizontal segments in the boundary of the surfaces represent the weak symplectic manifold $\mathcal{G}$. The non horizontal segments have no meaning. The labels A1-A5 represent the axioms ($RSG1$)-($RSG5$).

3

Quantization of the RSG

3.1 Introduction

The BV-BFV formalism (*Batalin-Vilkovisky* and *Batalin-Fradkin-Vilkovisky*) is a formalism allowing the description of a perturbative quantization of gauge theories on manifolds with boundary. In [CMR2], Cattaneo, Mnev and Reshetikhin gave a general procedure of such a perturbative quantization, which is the main reference of this chapter. Therefore we may encourage the reader to first get familiar with the general procedure, since we will assume the terminology and procedure as in [CMR2]. Another good reference is [CWM], which shows a similar application of the BV-BFV formalism for Chern-Simons theory. For our particular process we need abelian *BF* theory which occurs as the unperturbed part in many AKSZ theories (see [AKSZ]) described as in [CMR2]. The main description of the classical BV-BFV formalism can be found in [CMR3] and more on general BV theory can be found in [DF],[DA1] and [BV]. In [CMR2], the procedure of the deformation quanitzation of the relational symplectic groupoid was discussed as a short application of the Poisson σ-model. We want to give an explicit computation for this process by considering constant Poisson structures. We need to compute the quantization of L_3 and L_1 by using their topological aspects and computing the Feynman diagrams (see A) appearing for constant Poisson structures. We can now use the BV-BFV quantization process, following to [CMR2], by considering the fields X and η as in the Poisson σ-model. Now the source is given by $M \in \{L_1, L_2, L_3\}$ and we consider as target the Poisson manifold $\mathscr{P} = \mathbb{R}^d$ with constant Poisson structure α. Following to [CMR2], we will write mQME for *modified quantum master equation*.

3.2 General procedure

Let D_n denote the disc whose boundary S^1 splits into $2n$ intervals I intersecting only at the end points and with the boundary condition $\eta = 0$ on alternating intervals. The remaining n intervals are free, so the space of boundary fields is $\mathcal{F}^\partial_{D_n} = (\mathcal{F}^\partial_I)^n$ with

$$\mathcal{F}^\partial_I = \Omega^\bullet(I) \otimes \mathbb{R}^n \oplus \Omega^\bullet_0(I) \otimes (\mathbb{R}^n)^*[1],$$

with $\Omega^\bullet_0(I)$ denoting the subcomplex of forms whose restriction to the end points is zero. Let us denote by $\mathcal{H}$ the vector space which quantizes $\mathcal{F}^\partial_I$ in one of the two usual polarizations. We can then view the state m_x associated to D_3 perturbing around a constant solution $X = x$ as a linear map $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. There are two inequivalent ways to cut D_4 into gluings of two D_3s. From this we see that m_x defines an associative structure in the $(\hbar^2\triangle + \Omega)$-cohomology for D_4. This provides a way of defining the deformation quantiziation of the relational symplectic groupoid. To compare the result with the deformation quantization of the Poisson manifold $\mathscr{P}$, we have to consider also D_1. We view the state σ_x associated to it as a linear map $\mathcal{H} \to \mathbb{C}[[\varepsilon]]$, with $\varepsilon = \frac{i\hbar}{2}$. If f is a function on $\mathscr{P}$, we may also take the expectation value of $f(X(u_0))$ where u_0 is a point in the interior of the interval with the boundary condition. We denote the result by $\tau_x f$. We may view τ_x as a linear map $C^\infty(\mathscr{P}) \otimes \mathbb{C}[[\varepsilon]] \to \mathcal{H}$. Kontsevich's star product is then obtained by composition:

$$f \star_K g(x) = \sigma_x(m_x(\tau_x f \otimes \tau_x g)).$$

3.3 Computations of the states

3.3.1 The setting for L_3

Let us start with the most complicated manifold $M = L_3$. Moreover, consider the polarization to be given as in figure 3.3.1 such that we have the boundary field $\mathsf{X}\mid_{\partial_1 M} = \mathbb{X}$ on $\partial_1 M$ and the boundary fields $\eta\mid_{\partial_2^{(1)}M} = \mathbb{E}^{(1)}$ and $\eta\mid_{\partial_2^{(2)}M} = \mathbb{E}^{(2)}$ on $\partial_2^{(1)}M$ and $\partial_2^{(2)}M$ respectively, where $\partial_2 M := \partial_2^{(1)}M \sqcup \partial_2^{(2)}M$. Therefore we choose the $\frac{\delta}{\delta\mathbb{X}}$-polarization on $\partial_2 M$ and the $\frac{\delta}{\delta\mathbb{E}}$-polarization on $\partial_1 M$. The boundary condition for η is such that η is zero on the black boundary components. The fields η and X can be decomposed into the sum of several different fields as

$$\eta = \mathbb{E} + \mathsf{e} + \mathscr{E}$$
$$\mathsf{X} = x + \mathbb{X} + \mathsf{x} + \mathscr{X},$$

where $\mathbb{E}, \mathbb{X}$ are the boundary fields as described above, e, x are the residual fields and $\mathscr{E}, \mathscr{X}$ are the fluctuation fields. The constant field $x : M \to \mathscr{P}$ is called the *background* field. It will play an essential role for the construction of the Moyal-product.

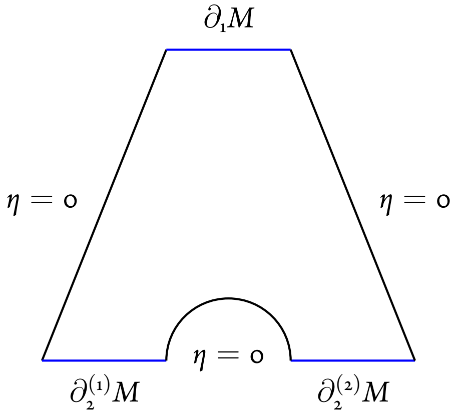


Figure 3.3.1: Setting of the boundary structure for L_3

Let us also first emphasize some important tools of formal geometry. We need to find a *generalized exponential* map

$$\begin{aligned}\phi : T\mathscr{P} &\longrightarrow \mathscr{P}\\ \mathscr{P} \oplus T_x\mathscr{P} \ni (x, y) &\longmapsto \phi_x(y)\end{aligned}$$

satisfying that $\phi_x(0) = x$ and $d_y\phi_x \mid_{y=0} = id$. For the case of $\mathscr{P}$ being some $\mathbb{R}^d$ we can easily find such a map by setting

$$\phi_x(y) = x + y. \tag{3.1}$$

Such a ϕ induces then a family (X_i) of vector fields in the y-direction on x. Such a vector field is given by

$$X_i = \sum_{j=1}^{d} X_i^j \frac{\partial}{\partial y^j}.$$

Moreover, we can define a connection D by

$$D = d + \sum_{i=1}^{d} dx^i X_i$$

on $\Gamma(\mathscr{P}, \hat{S}(T^*\mathscr{P}))$, where d is the usual de Rham differential and $\hat{S}$ is the completed symmetric algebra. For ϕ given as in (3.1) we get the vector fields $X_i = -\frac{\partial}{\partial y^i}$ and thus the connection is given by

$$D = d - \sum_{j=1}^{d} dx^j \frac{\partial}{\partial y^j} = \sum_{j=1}^{d} dx^j \left(\frac{\partial}{\partial x^j} - \frac{\partial}{\partial y^j} \right).$$

Thus for a function $f \in C^\infty(\mathscr{P})$ we get that

$$Df(x + y) = 0.$$

Using the Poisson σ-model action, we get an induced action

$$\mathcal{S}_M^{\phi} = \int_M \left(\eta_i \wedge d\hat{\mathsf{X}}^i + \frac{1}{2}\alpha^{ij}(x+\hat{\mathsf{X}})\eta_i \wedge \eta_j + \eta_i \wedge dx^i\right), \tag{3.2}$$

where $\hat{\mathsf{X}} = \mathbb{X} + \mathsf{x} + \mathscr{X}$. Since we got a slightly different action for this setting, we also need a more *special* mQME, which we will call *smQME*, that is for a state $\hat{\psi}$ given as

$$\left(\hbar^2\triangle + \Omega + \left(\frac{\hbar}{\mathrm{i}}\right) d\right)\hat{\psi} = 0. \tag{3.3}$$

Now for the case without cohomology, $\triangle = 0$ and the extra term in the action doesn't matter, since then the states are automatically killed by d, and the smQME is just the mQME anyway. We will still talk about the smQME for all the following settings. One can easily show that with the polarization as in figure 3.3.1 there is no cohomology and therefore no residual fields. Let now α be constant. This means that the only Feynman diagrams, that we have to compute are given as in figure 3.3.2. Let us denote by ζ the propagator on the disc and let us go with the convention that indices ij on the propagator ζ stands for the points u_i and u_j on the disc on which ζ is evaluated as a two point function. Moreover, let the index 1 always represent a point in the bulk and 2, 3 the points on the respective boundary component for $\partial_2 M$. Moreover, let the index 0 always represent a point on $\partial_1 M$. We start with the diagram (a). The perturbation term is obtained by the integral

$$\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M} = \frac{1}{2}\alpha^{ij}\int_{M\times C_2(\partial_2^{(1)}M)} \zeta_{12}\wedge\zeta_{13}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j,$$

where $\pi_{2,1}$ is the projection onto the first component of the configuration space $C_2(\partial_2^{(1)}M)$, $\pi_{2,2}$ the projection onto the second component of $C_2(\partial_2^{(1)}M)$ and ζ is the propagator of the disc for the points x^1 on the bulk and the points x^2, x^3 on the boundary component $\partial_2^{(1)}M$. The integration over the bulk vertex gives then

$$\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M} = \frac{1}{2}\alpha^{ij}\int_{C_2(\partial_2^{(1)}M)} \pi_1^*\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^*\mathbb{E}_j$$

where $\Theta_{L_3}\in\Omega^0(\partial_2 M)$ is the boundary propagator for L_3. Thus for (b) we get the perturbation term

$$\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(2)}M} = \frac{1}{2}\alpha^{ij}\int_{C_2(\partial_2^{(2)}M)} \pi_1^*\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^*\mathbb{E}_j.$$

The effective action for abelian *BF* theory of the free part, that means for the diagrams (c) and (d), is then given by

$$\mathcal{S}^{\text{eff}}_{\partial^{(k)}M} = -\int_{\partial_2^{(k)}M\times\partial_1 M} \pi_1^*\mathbb{E}_i\wedge\zeta\wedge\pi_2^*\mathbb{X}_i,$$

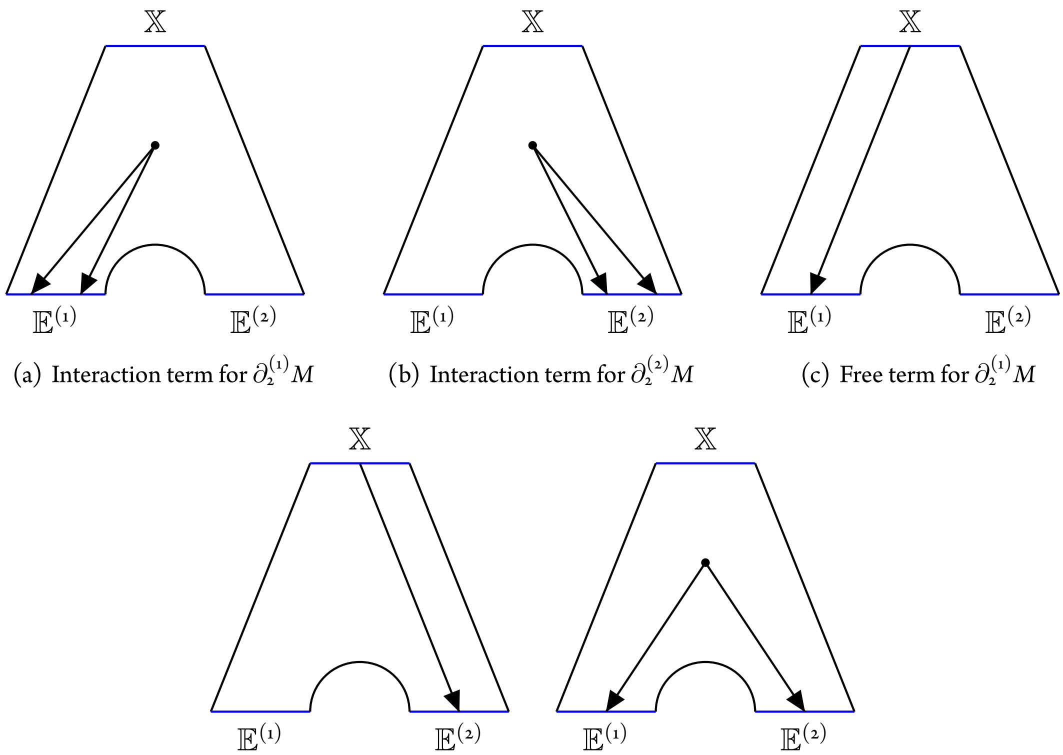
for $k\in\{1,2\}$ respectively, since there is no cohomology. The perturbation term for (e) is given by

$$\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} = \frac{1}{2}\alpha^{ij}\int_{\partial_2^{(1)}M\times\partial_2^{(2)}M} \pi_1^*\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^*\mathbb{E}_j.$$

The state for L_3 is then given by

$$\hat{\psi}_{L_3} = T_{L_3} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}^{\text{eff}}_{\partial^{(1)}M} + \mathcal{S}^{\text{eff}}_{\partial^{(2)}M} + \mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M} + \mathcal{S}^{\text{pert,eff}}_{\partial_2^{(2)}M} + \mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M \sqcup \partial_2^{(2)}M} \right) \right\} = T_{L_3} \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\}.$$

Remark 3.3.1. One can also show that the amplitudes T_{L_j}, for $j \in \{1, 2, 3\}$ are always equal to one.



(a) Interaction term for $\partial_2^{(1)}M$ (b) Interaction term for $\partial_2^{(2)}M$ (c) Free term for $\partial_2^{(1)}M$

(d) Free term for $\partial_2^{(2)}M$ (e) Interaction term for $\partial_1^{(1)}M$ and $\partial_2^{(2)}M$

Figure 3.3.2: The diagrams for L_3 which need to be computed. The diagrams (c) and (d) are the free terms of the action and (a), (b) and (e) are interaction terms.

3.3.2 The setting for L_1

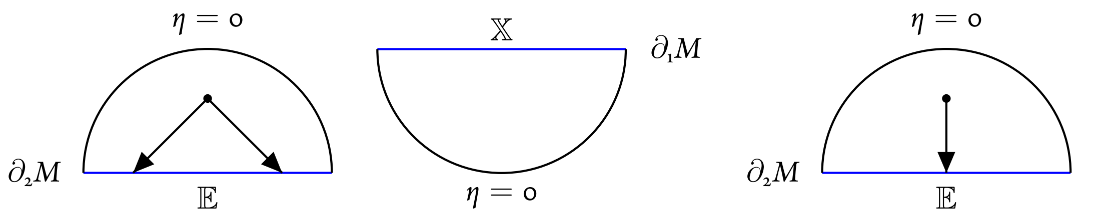
Now we set $M = L_1$ and consider the polarization as in figure 3.3.3 and the only Feynman diagram we have to consider is the one in figure 3.3.3 for the different polarizations.

For (b) we get the trivial state $\hat{\psi}^{(b)}_{L_1} = 1$, since there is no diagram to compute. The perturbation term for (a) is given by

$$\mathcal{S}^{\text{pert,eff}}_{\partial_2 M} = \frac{1}{2} a^{ij} \int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge \Theta_{L_1} \wedge \pi_2^* \mathbb{E}_j.$$

Now the free part consist also of residual fields, since with the chosen polarization we get a non vanishing cohomology, which basically means that we also need to consider a term as in (c).

Now we get an additional free term $\mathcal{S}^{\text{eff}}_{\partial_2 M} = \int_{\partial_2 M} z^i \mathbb{E}_i + z_i^\dagger dx^i$, since the volumeform on the bulk integrates to 1, and for the diagram (c) we get an additional perturbation term $\tilde{\mathcal{S}}^{\text{pert,eff}}_{\partial_2 M} = a^{ij} \int_{\partial_2 M} z_i^\dagger \mathbb{E}_j \wedge \tau$,



(a) Usual interaction term with the $\frac{\delta}{\delta \mathbb{X}}$-polarization.

(b) No diagram for the $\frac{\delta}{\delta \mathbb{E}}$-polarization.

(c) Interaction term appearing due to cohomology with the $\frac{\delta}{\delta \mathbb{X}}$-polarization.

Figure 3.3.3: The diagrams for L_1.

where $\{z^i, z_i^\dagger\}$ are some chosen coordinates on the cohomology and $\tau \in \Omega^1(\partial_2 M)$ is the result of the integral over the bulk vertex of the graph with one bulk vertex connected to one boundary vertex with the property that

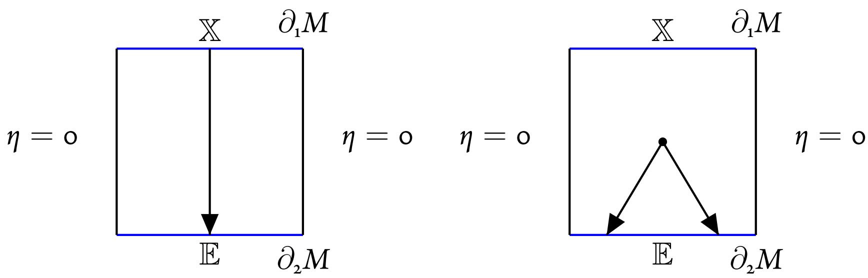
$$d\Theta_{L_1} = \pi_1^* \tau - \pi_2^* \tau. \tag{3.4}$$

Therefore the state for (a) is given by

$$\hat{\psi}_{L_1}^{(a)} = T_{L_1} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}_{\partial_2 M}^{\text{pert,eff}} + \mathcal{S}_{\partial_2 M}^{\text{eff}} + \tilde{\mathcal{S}}_{\partial_2 M}^{\text{pert,eff}} \right) \right\} = T_{L_1} \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}_{\partial M}^{\text{eff}} \right\}$$

3.3.3 The setting for L_2

Let us also take a look at $M = L_2$. We choose the polarization as in figure 3.3.4 such that also there will be no cohomology and thus no residual fields.



(a) Free term

(b) Interaction term

Figure 3.3.4: The diagrams for L_2

For the left diagram we get the only the free part of the effective action as

$$\mathcal{S}_{\partial M}^{\text{eff}} = -\int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{E}_i \wedge \zeta_{02} \wedge \pi_2^* \mathbb{X}_i,$$

and for the right diagram we have the perturbation term as

$$\mathcal{S}_{\partial_2 M}^{\text{pert,eff}} = \frac{1}{2} a^{ij} \int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge \Theta_{L_2} \wedge \pi_2^* \mathbb{E}_j.$$

Thus the state for L_2 is then

$$\hat{\psi}_{L_2} = T_{L_2} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}^{\mathrm{eff}}_{\partial M} + \mathcal{S}^{\mathrm{pert,eff}}_{\partial_2 M} \right) \right\}.$$

3.4 The (special) modified Quantum Master Equation

We need to make sure that the states $\hat{\psi}_{L_j}$, as computed before, satisfy the smQME, i.e.

$$\left(\hbar^2 \triangle + \Omega + \left(\frac{\hbar}{\mathrm{i}} \right) d \right) \hat{\psi}_{L_j} = 0$$

for all $j \in \{1, 2, 3\}$, where the quantum BRST operator $\Omega = \Omega_0 + \Omega_{\mathrm{pert}}$ splits into a free part Ω_0 and a perturbation part Ω_{pert}. The smQME will look different for different states, depending on the cohomology and the effective action.

3.4.1 The smQME for L_3

In the case of $M = L_3$, we can see that Ω is given by the sum of

$$\begin{aligned} \Omega_{\mathrm{pert}} &= -\frac{1}{2} \alpha^{ij} \left(\hbar^2 \int_{\partial_1 M} \frac{\delta}{\delta \mathbb{X}_i} \frac{\delta}{\delta \mathbb{X}_j} - \int_{\partial_2^{(1)} M \sqcup \partial_2^{(2)} M} \mathbb{E}_i \wedge \mathbb{E}_j \right) \\ \Omega_0 &= \mathrm{i}\hbar \left(\int_{\partial_1 M} d\mathbb{X}_i \frac{\delta}{\delta \mathbb{X}_i} + \int_{\partial_2^{(1)} M \sqcup \partial_2^{(2)} M} d\mathbb{E}_i \frac{\delta}{\delta \mathbb{E}_i} + \int_{\partial_1 M} \frac{\delta}{\delta \mathbb{X}_i} dx^i + \int_{\partial_2 M} \mathbb{E}_i dx^i \right), \end{aligned}$$

since $\eta = -\mathrm{i}\hbar \frac{\delta}{\delta \mathbb{X}}$ is the conjugated momentum for the η field. Let us also define each operator term of Ω with a derivative by

$$\begin{aligned} \Omega^{\mathrm{pert}}_{\mathbb{X}} &:= -\frac{\hbar^2}{2} \alpha^{ij} \int_{\partial_1 M} \frac{\delta}{\delta \mathbb{X}_i} \frac{\delta}{\delta \mathbb{X}_j} \\ \Omega_{\mathbb{X}} &:= \mathrm{i}\hbar \int_{\partial_1 M} d\mathbb{X}_i \frac{\delta}{\delta \mathbb{X}_i} \\ \Omega_{\mathbb{E}} &:= \mathrm{i}\hbar \int_{\partial_2^{(1)} M \sqcup \partial_2^{(2)} M} d\mathbb{E}_i \frac{\delta}{\delta \mathbb{E}_i}. \end{aligned}$$

Applying $\tilde{\Omega} = \Omega_{\mathrm{pert}} + \Omega_{\mathbb{X}} + \Omega_{\mathbb{E}}$ to the state for L_3, we get

$$\tilde{\Omega} \hat{\psi}_{L_3} = T_{L_3} \tilde{\Omega} \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\mathrm{eff}}_{\partial M} \right\} = T_{L_3} (\Omega_{\mathrm{pert}} + \Omega_{\mathbb{X}} + \Omega_{\mathbb{E}}) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\mathrm{eff}}_{\partial M} \right\}.$$

We want to compute each contribution of the different parts of $\tilde{\Omega}$. Let us therefore first apply Ω_{pert} and observe

$$\Omega_{\text{pert}} \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} = -\frac{\hbar^2}{2}a^{ij}\int_{\partial_1 M}\frac{\delta}{\delta\mathbb{X}_i}\frac{\delta}{\delta\mathbb{X}_j}\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.5}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.6}$$

$$= -\frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)\int_{\partial_1 M}\frac{\delta}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_j}\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.7}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.8}$$

$$= -\frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)^2\int_{\partial_1 M}\left(\frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j}\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_j}\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}\right) \tag{3.9}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.10}$$

$$= -\frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)^2\int_{\partial_1 M}\left(\frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_j}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.11}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.12}$$

Now we need to express the functional derivatives in (3.11) in terms of the propagator and the fields. Each term is given by a sum and the only terms which contribute to the derivative are

$$\frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j} = \frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j}+\frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j} \tag{3.13}$$

$$\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i} = \frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_i}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_i} \tag{3.14}$$

$$\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_j} = \frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_j}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_j}, \tag{3.15}$$

since the other terms of the effective action do not depend on the $\mathbb{X}$ field. Now we get

$$\frac{\delta^2\mathcal{S}^{\text{eff}}_{\partial^{(k)}M}}{\delta\mathbb{X}_i\delta\mathbb{X}_j} = 0 \tag{3.16}$$

$$\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(k)}M}}{\delta\mathbb{X}_i} = \int_{\partial_1 M\times\partial_2^{(k)}M}\zeta_{02}\wedge\pi_2^*\mathbb{E}_i \tag{3.17}$$

$$\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(k)}M}}{\delta\mathbb{X}_j} = \int_{\partial_1 M\times\partial_2^{(k)}M}\zeta_{03}\wedge\pi_2^*\mathbb{E}_j, \tag{3.18}$$

and hence

$$\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial M}}{\delta\mathbb{X}_j} = \frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_j}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_j}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_j}+\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_i}\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_j}. \tag{3.19}$$

This shows that the application of Ω_{pert} to the state is given by

$$\Omega_{\text{pert}} \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} = \frac{1}{2}a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2^{(1)}M)} \zeta_{02}\wedge\zeta_{03}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.20}$$

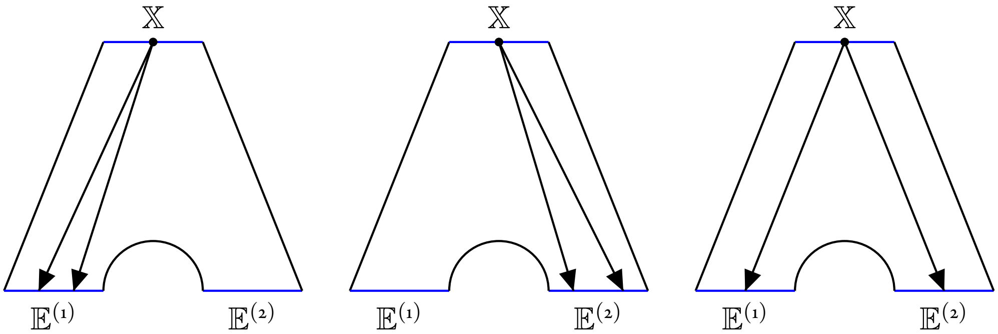
$$+\frac{1}{2}a^{ij}\left(\int_{\partial_1 M\times[\partial_2^{(1)}M\times\partial_2^{(2)}M]} \zeta_{02}\wedge\zeta_{03}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.21}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_1 M\times[\partial_2^{(2)}M\times\partial_2^{(1)}M]} \zeta_{02}\wedge\zeta_{03}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.22}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2^{(2)}M)} \zeta_{02}\wedge\zeta_{03}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.23}$$

$$+\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} \mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}. \tag{3.24}$$

For these terms we get the diagrams as in figure 3.4.1, where we have set the x^0 coordinate on $\partial_1 M$ and the x^2 and x^3 coordinates on $\partial_2^{(k)}M$ for $k\in\{1,2\}$.



(a) For $\partial_2^{(1)}M$ (b) For $\partial_2^{(2)}M$ (c) For $\partial_2^{(1)}M\times\partial_2^{(2)}M$

Figure 3.4.1: The diagrams for the terms of Ω_{pert}.

Now we want to compute the terms for $\Omega_{\mathbb{E}}$. Therefore we have

$$\Omega_{\mathbb{E}} \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} = \mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.25}$$

$$+\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i\frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.26}$$

$$+\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i\frac{\delta\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.27}$$

$$+\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i\frac{\delta\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.28}$$

$$+\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i\frac{\delta\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \tag{3.29}$$

We want to compute each term individually. Let us start with (3.25) and observe

$$\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i \frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(1)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}$$
$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(-\int_{\partial_2^{(1)}M\times\partial_1 M}\pi_1^* d\mathbb{E}_i\wedge\zeta\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}. \quad (3.30)$$

For the term (3.26) we get

$$\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i \frac{\delta\mathcal{S}^{\text{eff}}_{\partial^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}$$
$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(-\int_{\partial_2^{(2)}M\times\partial_1 M}\pi_1^* d\mathbb{E}_i\wedge\zeta\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}. \quad (3.31)$$

For the term (3.27) we get

$$\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M} d\mathbb{E}_i \frac{\delta\mathcal{S}^{\text{pert,eff}}_{\partial_2^{(1)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \quad (3.32)$$
$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}a^{ij}\left(\int_{C_2(\partial_2^{(1)}M)}\pi_1^* d\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^*\mathbb{E}_j+\int_{C_2(\partial_2^{(1)}M)}\pi_1^*\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^* d\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}. \quad (3.33)$$

Now using integration by parts we get

$$\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}a^{ij}\left(\int_{C_2(\partial_2^{(1)}M)}\pi_1^* d\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^*\mathbb{E}_j+\int_{C_2(\partial_2^{(1)}M)}\pi_1^*\mathbb{E}_i\wedge\Theta_{L_3}\wedge\pi_2^* d\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \quad (3.34)$$
$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}a^{ij}\left(\int_{C_2(\partial_2^{(1)}M)} d(\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{E}_j)\wedge\Theta_{L_3}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \quad (3.35)$$
$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}a^{ij}\left(\int_{\partial_2^{(1)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\} \quad (3.36)$$
$$+ \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}a^{ij}\left(\int_{C_2(\partial_2^{(1)}M)}\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{E}_j\wedge d\Theta_{L_3}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{eff}}_{\partial M}\right\}. \quad (3.37)$$

Since we have no cohomology, we can use Stoke's theorem to compute $d\Theta_{L_3}$ and because of the fact that

$$\Theta_{L_3} = \int_M \zeta_{12}\wedge\zeta_{13},$$

we get

$$d\Theta_{L_3} = \int_{\partial_1 M}\zeta_{02}\wedge\zeta_{03}.$$

Therefore we have that (3.37) is given as

$$\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}\alpha^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2^{(1)}M)}\zeta_{02}\wedge\zeta_{03}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}. \tag{3.38}$$

The same procedure holds for (3.28) and thus

$$\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}d\mathbb{E}_i\frac{\delta\mathcal{S}^{\mathrm{pert,eff}}_{\partial_2^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.39}$$

$$=\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}\alpha^{ij}\left(\int_{\partial_2^{(2)}M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.40}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}\alpha^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2^{(2)}M)}\zeta_{02}\wedge\zeta_{03}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.41}$$

The boundary propagator Θ_{L_3} is no longer in (3.36) and (3.37) since we have to integrate over the fiber of the configuration space and thus by the property of Θ_{L_3} its value is constant 1 on the fiber. Moreover, since the diagonal is a copy of the manifold itself, we get that integration over $\partial C_2(\partial_2^{(k)}M)$ is then actually given by integration over $\partial_2^{(k)}M$ with the remaining form $\mathbb{E}_i\wedge\mathbb{E}_j$, i.e. evaluated at the same point for $k\in\{1,2\}$. For the term (3.29) we have the same principle and thus

$$\mathrm{i}\hbar\left(\int_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}d\mathbb{E}_i\frac{\delta\mathcal{S}^{\mathrm{pert,eff}}_{\partial_2^{(1)}M\sqcup\partial_2^{(2)}M}}{\delta\mathbb{E}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.42}$$

$$=\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}\alpha^{ij}\left(\int_{\partial[\partial_2^{(1)}M\times\partial_2^{(2)}M]}\mathbb{E}_i\wedge\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.43}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\frac{1}{2}\alpha^{ij}\left(\int_{\partial_1 M\times[\partial_2^{(1)}M\times\partial_2^{(2)}M]}\zeta_{02}\wedge\zeta_{03}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}. \tag{3.44}$$

Moreover, we can observe that (3.43) vanishes, since we get integration over the double boundary and the fields vanish on the endpoints of the boundary. Now we need to compute the terms for $\Omega_{\mathbb{X}}$. Therefore we get

$$\Omega_{\mathbb{X}}\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}=\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\int_{\partial_1 M}d\mathbb{X}_i\frac{\delta\mathcal{S}^{\mathrm{eff}}_{\partial^{(1)}M}}{\delta\mathbb{X}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.45}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\int_{\partial_1 M}d\mathbb{X}_i\frac{\delta\mathcal{S}^{\mathrm{eff}}_{\partial^{(2)}M}}{\delta\mathbb{X}_i}\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}. \tag{3.46}$$

The term in (3.45) is then

$$\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(-\int_{\partial_2^{(1)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta\wedge\pi_2^*d\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}. \tag{3.47}$$

The term in (3.46) is then

$$\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(-\int_{\partial_2^{(2)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta\wedge\pi_2^*d\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\}. \tag{3.48}$$

Now if we combine (3.47) with (3.30) and (3.48) with (3.31) and using again integration by parts we get

$$\left(\int_{\partial_2^{(1)}M\times\partial_1 M}\pi_1^*d\mathbb{E}_i\wedge\zeta\wedge\pi_2^*\mathbb{X}_i+\int_{\partial_2^{(1)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta\wedge\pi_2^*d\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.49}$$

$$=\left(\int_{\partial_2^{(1)}M\times\partial_1 M}d(\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{X}_i)\wedge\zeta\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.50}$$

$$=\left(\int_{\partial[\partial_2^{(1)}M\times\partial_1 M]}\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.51}$$

$$+\left(\int_{\partial_2^{(1)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge d\zeta\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.52}$$

and

$$\left(\int_{\partial_2^{(2)}M\times\partial_1 M}\pi_1^*d\mathbb{E}_i\wedge\zeta\wedge\pi_2^*\mathbb{X}_i+\int_{\partial_2^{(2)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta\wedge\pi_2^*d\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.53}$$

$$=\left(\int_{\partial_2^{(2)}M\times\partial_1 M}d(\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{X}_i)\wedge\zeta\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.54}$$

$$=\left(\int_{\partial[\partial_2^{(2)}M\times\partial_1 M]}\pi_1^*\mathbb{E}_i\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.55}$$

$$+\left(\int_{\partial_2^{(2)}M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge d\zeta\wedge\pi_2^*\mathbb{X}_i\right)\exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\partial M}^{\mathrm{eff}}\right\} \tag{3.56}$$

respectively. Now again we can use that there is no cohomology and therefore $d\zeta = 0$ and thus the terms (3.52) and (3.56) vanish. Moreover, the terms (3.51) and (3.55) also vanish because of the principle we already had before. Now the term (3.20) cancels with (3.38), the term (3.21) cancels with (3.44) the term (3.23) cancels with (3.41) and finally the term in (3.24) cancels with the sum of the terms (3.36) and (3.40). Finally, for $\Omega_{0,1}=\int_{\partial_1 M}\frac{\delta}{\delta\mathbb{X}_i}dx^i$ we get a term $-\int_{\partial_2 M\times M}dx^i\mathbb{E}_i\wedge\zeta_{02}=-\int_{\partial_2 M}\mathbb{E}_i dx^i$ which cancels the multiplicative term in Ω_0. Now since $d\hat{\psi}_{L_3}=0$, the smQME for $\hat{\psi}_{L_3}$ is satisfied, because $\triangle=0$ without cohomology.

3.4.2 The smQME for L_1

Now we need to do the same computations for $M=L_1$ but with the difference that we have cohomology which means that $\triangle\neq 0$. Therefore we need to show that

$$\left(\hbar^2\triangle+\Omega_{(a)}+\left(\frac{\hbar}{\mathrm{i}}\right)d\right)\hat{\psi}_{L_1}^{(a)}=0,$$

where

$$\Omega_{(a)} = \int_{\partial_2 M} \left(\mathrm{i}\hbar d\mathbb{E}_i \frac{\delta}{\delta \mathbb{E}_i} - \hbar^2 \mathbb{E}_i dx^i + \frac{1}{2} a^{ij} \mathbb{E}_i \wedge \mathbb{E}_j \right) \tag{3.57}$$

$$\triangle = \sum_{i=1}^{n} (-1)^{1+\deg z_i} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_i^\dagger} \tag{3.58}$$

We can use the formula $\deg z_i = 1 - \deg \chi_i$, where $\{[\chi_i]\}$ is a basis for the cohomology (see [CMR2]), and since we have the cohomology of the disc, we get that $\deg \chi_i = 0$ and hence $\deg z_i = 1$. Therefore we have an even exponent and only the coefficients $+1$. Now let again $\Omega_{\mathbb{E}} := \mathrm{i}\hbar \int_{\partial_2 M} d\mathbb{E}_i \frac{\delta}{\delta \mathbb{E}_i}$. Then we get

$$\Omega_{\mathbb{E}} \hat{\psi}_{L_1}^{(a)} = \Omega_{\mathbb{E}} T_{L_1} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}^{\text{eff}}_{\partial M} \right) \right\} = T_{L_1} \Omega_{\mathbb{E}} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}^{\text{pert,eff}}_{\partial_2 M} + \mathcal{S}^{\text{eff}}_{\partial_2 M} + \tilde{\mathcal{S}}^{\text{pert,eff}}_{\partial_2 M} \right) \right\} \tag{3.59}$$

$$= T_{L_1} \left(\mathrm{i}\hbar \int_{\partial_2 M} d\mathbb{E}_i \frac{\delta}{\delta \mathbb{E}_i} \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(\mathcal{S}^{\text{pert,eff}}_{\partial_2 M} + \mathcal{S}^{\text{eff}}_{\partial_2 M} + \tilde{\mathcal{S}}^{\text{pert,eff}}_{\partial_2 M} \right) \right\} \right) \tag{3.60}$$

$$= T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \int_{\partial_2 M} \left(d\mathbb{E}_i \frac{\delta \mathcal{S}^{\text{pert,eff}}_{\partial_2 M}}{\delta \mathbb{E}_i} + d\mathbb{E}_i \frac{\delta \mathcal{S}^{\text{eff}}_{\partial_2 M}}{\delta \mathbb{E}_i} + d\mathbb{E}_i \frac{\delta \mathcal{S}^{\text{pert,eff}}_{\partial_2 M}}{\delta \mathbb{E}_i} \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.61}$$

$$= T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \frac{1}{2} a^{ij} \left(\int_{C_2(\partial_2 M)} \pi_1^* d\mathbb{E}_i \wedge \Theta_{L_1} \wedge \pi_2^* \mathbb{E}_j \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.62}$$

$$+ T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \frac{1}{2} a^{ij} \left(\int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge \Theta_{L_1} \wedge \pi_2^* d\mathbb{E}_j \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.63}$$

$$+ T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \left(\int_{\partial_2 M} z^i d\mathbb{E}_i \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.64}$$

$$+ T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) a^{ij} \left(\int_{\partial_2 M} z_i^\dagger d\mathbb{E}_j \wedge \tau \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.65}$$

Again, with integration by parts, we get that (3.62) together with (3.63) gives

$$T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \frac{1}{2} a^{ij} \left(\int_{\partial_2 M} \mathbb{E}_i \wedge \mathbb{E}_j \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\} \tag{3.66}$$

$$+ T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \frac{1}{2} a^{ij} \left(\int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge d\Theta_{L_1} \wedge \pi_2^* \mathbb{E}_j \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\}. \tag{3.67}$$

Now we can use (3.4) and we get that (3.67) is given by

$$T_{L_1} \mathrm{i}\hbar \left(\frac{\mathrm{i}}{\hbar} \right) \frac{1}{2} a^{ij} \left(\int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge \pi_1^* \tau \wedge \pi_2^* \mathbb{E}_j - \int_{C_2(\partial_2 M)} \pi_1^* \mathbb{E}_i \wedge \pi_2^* \tau \wedge \pi_2^* \mathbb{E}_j \right) \exp\left\{ \frac{\mathrm{i}}{\hbar} \mathcal{S}^{\text{eff}}_{\partial M} \right\}. \tag{3.68}$$

Therefore the term which arises from $\triangle$ cancels with (3.68). Moreover, we also get a term $\hbar^2 \int_{\partial_2 M} \mathbb{E}_i dx^i$, which cancels with the first multiplicative term of $\Omega_{(a)}$, and since (3.64) and (3.65) vanish, and clearly $d\hat{\psi}_{L_1}^{(a)} = 0$, we get that the smQME holds for $\hat{\psi}_{L_1}^{(a)}$. The smQME for $\hat{\psi}_{L_1}^{(b)}$ is trivially satisfied.

Remark 3.4.1. We skip the computation of the smQME for the state of L_2 since we won't need it for the following computations, although one can easily see that also here the smQME holds and the computations are exactly the same as for the other states.

3.5 Associativity and gluing

We want to show that the associativity of the gluing for $M = L_3$ is indeed satisfied, which is illustrated in figure 3.5.1.

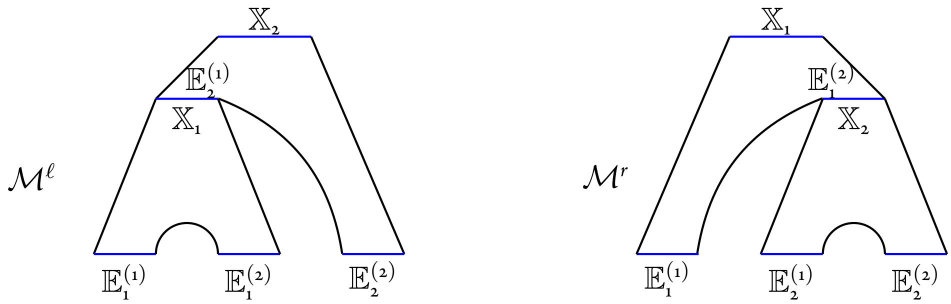


Figure 3.5.1: The associativity of L_3

We can show that the *gluing* of the states as in the figure, i.e. the left and the right one, with the chosen polarization gives us the same result up to some Ω-exact term. Let us compute the state of the glued manifold as in the left figure. Thus let Σ^ℓ be the identification of the boundary $\partial_2^{(1)} M$ for M the upper glued L_3, which we call L_3^1, and $\partial_1 M$ for M the lower glued L_3, which we call L_3^2. Let us denote by $\mathcal{M}^\ell$ the glued manifold, i.e. $\mathcal{M}^\ell := L_3^1 \cup_{\Sigma^\ell} L_3^2$. Formally the state for $\mathcal{M}^\ell$ is then given by

$$\begin{aligned}\tilde{\psi}_{\mathcal{M}^\ell} &= \int_{\mathbb{X}_1^{\Sigma^\ell}, \mathbb{E}_2^{\Sigma^\ell}} \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{\Sigma^\ell} \mathbb{X}_1^{\Sigma^\ell} \wedge \mathbb{E}_2^{\Sigma^\ell}\right\} \hat{\psi}_{L_3^1}\hat{\psi}_{L_3^2} \\ &= \int_{\mathbb{X}_1^{\Sigma^\ell}, \mathbb{E}_2^{\Sigma^\ell}} \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{\Sigma^\ell} \mathbb{X}_1^{\Sigma^\ell} \wedge \mathbb{E}_2^{\Sigma^\ell}\right\} \exp\left\{\frac{\mathrm{i}}{\hbar}\left(\mathcal{S}^{\text{eff}}_{\partial L_3^1} + \mathcal{S}^{\text{eff}}_{\partial L_3^2}\right)\right\}. \end{aligned} \tag{3.69}$$

The correpsonding state for $\mathcal{M}^r := L_3^1 \cup_{\Sigma^r} L_3^2$, where Σ^r is the identification of the boundary component $\partial_2^{(2)} M$ for M the upper glued L_3, denoted L_3^1 and $\partial_1 M$ for M the lower L_3, denoted L_3^2, is given by

$$\begin{aligned}\tilde{\psi}_{\mathcal{M}^r} &= \int_{\mathbb{X}_2^{\Sigma^r}, \mathbb{E}_1^{\Sigma^r}} \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{\Sigma^r} \mathbb{X}_2^{\Sigma^r} \wedge \mathbb{E}_1^{\Sigma^r}\right\} \hat{\psi}_{L_3^1}\hat{\psi}_{L_3^2} \\ &= \int_{\mathbb{X}_2^{\Sigma^r}, \mathbb{E}_1^{\Sigma^r}} \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{\Sigma^r} \mathbb{X}_1^{\Sigma^r} \wedge \mathbb{E}_2^{\Sigma^r}\right\} \exp\left\{\frac{\mathrm{i}}{\hbar}\left(\mathcal{S}^{\text{eff}}_{\partial L_3^1} + \mathcal{S}^{\text{eff}}_{\partial L_3^2}\right)\right\}. \end{aligned} \tag{3.70}$$

We want to compute these states explicitly by considering a general construction of a manifold, which gives us the associativity as a special case. Therefore we first need to construct the manifold and the corresponding states.

3.5.1 General associativity construction

Let us consider the Manifold $M = \mathcal{M}^n$ given as in figure 3.5.2. Then we can write

$$\partial_2 M = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_{n-1} \sqcup I_n$$

as the disjoint union of n Intervals.

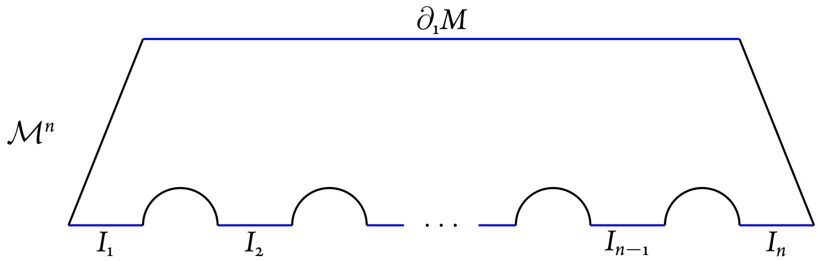


Figure 3.5.2: The more general manifold $\mathcal{M}^n$.

Then the state for $M = \mathcal{M}^n$ is easily computed by considering the free and the interaction terms for the effective action similar as for L_3. Therefore we have the effective action

$$\mathcal{S}^{\text{eff}}_{\mathcal{M}^n} = -\int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{E}_i \wedge \zeta^n \wedge \pi_2^* \mathbb{X}_i + \frac{1}{2}\alpha^{ij} \int_{M \times C_2(\partial_2 M)} \zeta^n_{12} \wedge \zeta^n_{13} \wedge \pi^*_{2,1}\mathbb{E}_i \wedge \pi^*_{2,2}\mathbb{E}_j,$$

where $C_2(\partial_1 M) = C_2(I_1) \sqcup C_2(I_2) \sqcup \cdots \sqcup I_1 \times I_2 \sqcup I_1 \times I_3 \sqcup \cdots \sqcup I_{n-1} \times I_n$ and ζ^n is the bulk propagator for $\mathcal{M}^n$. Since ζ^n is a propagator, we can decompose $\zeta^n = \zeta + d\kappa^n$, where ζ is the usual bulk propagator of the disc and κ^n some zero form. We consider a family $\zeta^{n,t}$ of these propagators given by $\zeta^{n,t} = \zeta + td\kappa^n$ and consider the action of this family as

$$\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}(t) = -\int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{E}_i \wedge \zeta^{n,t} \wedge \pi_2^* \mathbb{X}_i + \frac{1}{2}\alpha^{ij} \int_{M \times C_2(\partial_2 M)} \zeta^{n,t}_{12} \wedge \zeta^{n,t}_{13} \wedge \pi^*_{2,1}\mathbb{E}_i \wedge \pi^*_{2,2}\mathbb{E}_j. \tag{3.71}$$

Then we have the state

$$\hat{\psi}_{\mathcal{M}^n}(t) = \exp\left\{\frac{i}{\hbar}\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}(t)\right\}. \tag{3.72}$$

We claim that

$$\partial_t \hat{\psi}_{\mathcal{M}^n}(t) = \Omega\left(\hat{\psi}_{\mathcal{M}^n}(t) \int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{E}_i \wedge \kappa^n \wedge \pi_2^* \mathbb{X}_i\right) \tag{3.73}$$

$$+ \Omega\left(\hat{\psi}_{\mathcal{M}^n}(t)\frac{1}{2}\alpha^{ij} \int_{M \times C_2(\partial_2 M)} \zeta^{n,t}_{12} \wedge \kappa^n_{13} \wedge \pi^*_{2,1}\mathbb{E}_i \wedge \pi^*_{2,2}\mathbb{E}_j\right) \tag{3.74}$$

$$+ \Omega\left(\hat{\psi}_{\mathcal{M}^n}(t)\frac{1}{2}\alpha^{ij} \int_{M \times C_2(\partial_2 M)} \kappa^n_{12} \wedge \zeta^{n,t}_{13} \wedge \pi^*_{2,1}\mathbb{E}_i \wedge \pi^*_{2,2}\mathbb{E}_j\right) \tag{3.75}$$

$$= \Omega\left(\hat{\psi}_{\mathcal{M}^n}(t) \cdot \varphi\right) \tag{3.76}$$

with

$$\varphi = \int_{\partial_2 M\times\partial_1 M} \pi_1^*\mathbb{E}_i \wedge \kappa^n \wedge \pi_2^*\mathbb{X}_i + \frac{1}{2}\alpha^{ij}\int_{M\times C_2(\partial_2 M)} \zeta_{12}^{n,t}\wedge\kappa_{13}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j \\ + \frac{1}{2}\alpha^{ij}\int_{M\times C_2(\partial_2 M)} \kappa_{12}^n\wedge\zeta_{13}^{n,t}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j,$$

where Ω is given as in the smQME of L_3. Indeed, we can first observe that

$$\partial_t\hat{\psi}_{\mathcal{M}^n}(t) = \left(\frac{\mathrm{i}}{\hbar}\right)\hat{\psi}_{\mathcal{M}^n}(t)(\partial_t\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}(t)), \tag{3.77}$$

which means that we only have to compute $\partial_t\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}(t)$. Let us first consider the free term $\mathcal{S}^{\text{free,eff}}_{\mathcal{M}^n}(t)$ of the action. Then we get that

$$\partial_t\mathcal{S}^{\text{free,eff}}_{\mathcal{M}^n}(t) = \partial_t\left(-\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge(\zeta + td\kappa^n)\wedge\pi_2^*\mathbb{X}_i\right) = -\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge d\kappa^n\wedge\pi_2^*\mathbb{X}_i. \tag{3.78}$$

The derivative of the perturabation part $\mathcal{S}^{\text{pert,eff}}_{\mathcal{M}^n}(t)$ is then given by

$$\partial_t\mathcal{S}^{\text{pert,eff}}_{\mathcal{M}^n}(t) = 2\frac{1}{2}\alpha^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge\partial_t\zeta_{13}^{n,t}\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j = \alpha^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge d\kappa_{13}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j. \tag{3.79}$$

Hence we get that

$$\partial_t\hat{\psi}_{\mathcal{M}^n}(t) = \left(\frac{\mathrm{i}}{\hbar}\right)\hat{\psi}_{\mathcal{M}^n}(t)\left(-\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge d\kappa^n\wedge\pi_2^*\mathbb{X}_i + \alpha^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge d\kappa_{13}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right). \tag{3.80}$$

Now we compute $\Omega(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi)$. We get

$$\Omega(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) = (\Omega_0 + \Omega_{\text{pert}})(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) = \Omega_0(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) + \Omega_{\text{pert}}(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) \tag{3.81}$$

$$= \underbrace{(\Omega_0\hat{\psi}_{\mathcal{M}^n}(t))\varphi}_{(1)} + \underbrace{(\Omega_0\varphi)\hat{\psi}_{\mathcal{M}^n}(t)}_{(2)} + \underbrace{\Omega_{\text{pert}}(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi)}_{(3)}. \tag{3.82}$$

Let us first compute term (3). Then we get

$$\Omega_{\text{pert}}(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) = -\frac{\hbar^2}{2}a^{ij}\int_{\partial_1 M}\frac{\delta}{\delta\mathbb{X}_i}\frac{\delta}{\delta\mathbb{X}_j}(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) + \frac{1}{2}a^{ij}\int_{\partial_2 M}\mathbb{E}_i\wedge\mathbb{E}_j(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) \tag{3.83}$$

$$= -\frac{\hbar^2}{2}a^{ij}\int_{\partial_1 M}\frac{\delta}{\delta\mathbb{X}_i}\left[\left(\frac{\mathrm{i}}{\hbar}\right)\frac{\delta\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}}{\delta\mathbb{X}_j}\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi + \frac{\delta\varphi}{\delta\mathbb{X}_j}\hat{\psi}_{\mathcal{M}^n}(t)\right] \tag{3.84}$$

$$+ \frac{1}{2}a^{ij}\int_{\partial_2 M}\mathbb{E}_i\wedge\mathbb{E}_j(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) \tag{3.85}$$

$$= -\frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)^2\int_{\partial_1 M}\frac{\delta\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}}{\delta\mathbb{X}_i}\cdot\frac{\delta\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}}{\delta\mathbb{X}_j}\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi \tag{3.86}$$

$$+ \frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)\int_{\partial_1 M}\frac{\delta\varphi}{\delta\mathbb{X}_i}\cdot\frac{\delta\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}}{\delta\mathbb{X}_j}\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.87}$$

$$+ \frac{\hbar^2}{2}a^{ij}\left(\frac{\mathrm{i}}{\hbar}\right)\int_{\partial_1 M}\frac{\delta\varphi}{\delta\mathbb{X}_j}\cdot\frac{\delta\mathcal{S}^{\text{eff}}_{\mathcal{M}^n}}{\delta\mathbb{X}_i}\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.88}$$

$$+ \frac{1}{2}a^{ij}\int_{\partial_2 M}\mathbb{E}_i\wedge\mathbb{E}_j(\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi) \tag{3.89}$$

Analyzing the terms, we get that (3.86) is given by

$$-\frac{\hbar^2}{2}a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2 M)}\zeta^{n,t}_{12}\wedge\zeta^{n,t}_{13}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\varphi\cdot\hat{\psi}_{\mathcal{M}^n}(t), \tag{3.90}$$

the term (3.87) is given by

$$\frac{\hbar^2}{2}a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2 M)}\zeta^{n,t}_{12}\wedge\kappa^n_{13}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t), \tag{3.91}$$

and the term (3.88) is given by

$$\frac{\hbar^2}{2}a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2 M)}\zeta^{n,t}_{13}\wedge\kappa^n_{12}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t), \tag{3.92}$$

We can take the sum of (3.91) and (3.92) to obtain

$$\hbar^2 a^{ij}\left(\int_{\partial_1 M\times C_2(\partial_2 M)}\zeta^{n,t}_{12}\wedge\kappa^n_{13}\wedge\pi^*_{2,1}\mathbb{E}_i\wedge\pi^*_{2,2}\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t), \tag{3.93}$$

Next we need to compute (1). Thus we get

$$(\Omega_0\hat{\psi}_{\mathcal{M}^n}(t))\varphi = \left(\frac{\mathrm{i}}{\hbar}\right)(\Omega_0\mathcal{S}^{\text{eff}}_{\mathcal{M}^n})\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi \tag{3.94}$$

$$= -\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\overbrace{\int_{\partial_2 M\times\partial_1 M}\pi_1^*d\mathbb{E}_i\wedge\zeta_{12}^{n,t}\wedge\pi_2^*\mathbb{X}_i+\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta_{12}^{n,t}\wedge\pi_2^*d\mathbb{X}_i}^{\text{integration by parts}=\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\zeta_{02}^{n,t}\wedge\pi_2^*\mathbb{X}_i=0}\right)\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi \tag{3.95}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\frac{1}{2}a^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge\zeta_{13}^{n,t}\wedge\pi_1^*d\mathbb{E}_i\wedge\pi^*\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi \tag{3.96}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\frac{1}{2}a^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge\zeta_{13}^{n,t}\wedge\pi_1^*\mathbb{E}_i\wedge\pi^*d\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi \tag{3.97}$$

$$= \mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\underbrace{\left(\int_{\partial_1 M\times C_2(\partial_2 M)}d(\zeta_{02}^{n,t}\wedge\zeta_{03})\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)}_{=0}\cdot\varphi \tag{3.98}$$

$$+\,\mathrm{i}\hbar\left(\frac{\mathrm{i}}{\hbar}\right)\left(\frac{1}{2}a^{ij}\int_{\partial_2 M}\mathbb{E}_i\wedge\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t)\cdot\varphi. \tag{3.99}$$

The term (2) gives us

$$(\Omega_0\varphi)\hat{\psi}_{\mathcal{M}^n}(t) = \mathrm{i}\hbar\left(\int_{\partial_2 M\times\partial_1 M}\pi_1^*d\mathbb{E}_i\wedge\kappa_{01}^n\wedge\pi_2^*\mathbb{X}_i+\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge\kappa_{01}^n\wedge\pi_2^*d\mathbb{X}_i\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.100}$$

$$+\,\mathrm{i}\hbar\left(a^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge\kappa_{13}^n\wedge\pi_{2,1}^*d\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.101}$$

$$+\,\mathrm{i}\hbar\left(a^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge\kappa_{13}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*d\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.102}$$

$$= \mathrm{i}\hbar\left(\int_{\partial_2 M\times\partial_1 M}\pi_1^*\mathbb{E}_i\wedge d\kappa_{12}^n\wedge\pi_2^*\mathbb{X}_i\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.103}$$

$$+\,\mathrm{i}\hbar\left(a^{ij}\int_{\partial_1 M\times C_2(\partial_2 M)}\zeta_{02}^{n,t}\wedge\kappa_{03}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.104}$$

$$+\,\mathrm{i}\hbar\left(a^{ij}\int_{M\times C_2(\partial_2 M)}\zeta_{12}^{n,t}\wedge d\kappa_{13}^n\wedge\pi_{2,1}^*\mathbb{E}_i\wedge\pi_{2,2}^*\mathbb{E}_j\right)\hat{\psi}_{\mathcal{M}^n}(t) \tag{3.105}$$

Rearranging the terms, and by the fact that $\hat{\psi}_{\mathcal{M}^n}(t)$ satisfies the smQME, we see that the claim holds. This provides a general way of showing that associativity is indeed satisfied, namely, since $\hat{\psi}_{\mathcal{M}^n}$ changes by an Ω-exact term, we can say that $\hat{\psi}^\ell - \hat{\psi}_{\mathcal{M}^3}$ is given by some Ω-exact term, say $\Omega(B)$ and $\hat{\psi}^r - \hat{\psi}_{\mathcal{M}^3}$ is also given by some other Ω-exact term, say $\Omega(A)$. Hence we can say that $\hat{\psi}^\ell$ and $\hat{\psi}^r$ also differ by an Ω-exact term since $\hat{\psi}^\ell - \hat{\psi}_{\mathcal{M}^3} + \hat{\psi}_{\mathcal{M}^3} - \hat{\psi}^r = \Omega(A+B)$. This result is essential for the associativity of the star product, using a particular gluing, as we will see. As another point of view, one can also compute the effective action to all the diagrams which appear in $\hat{\psi}^\ell$ and $\hat{\psi}^r$ by identifying the fields on the glued boundary and compare them to eachother. Let us give the example for the free part of the effective action, i.e. with the diagram as in figure 3.5.3.

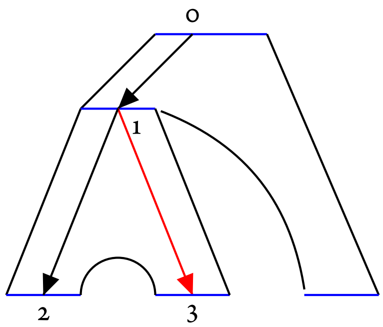


Figure 3.5.3: A diagram for the free part of the effective action on $\mathcal{M}^\ell$. The red arrow indicates the other possible diagram, meaning the diagram going from 0 to 1 and then to 3 instead of 2.

The corresponding effective action of the upper arrow is given by

$$\mathcal{S}_1^\ell = -\int_{\Sigma \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \zeta_{01} \wedge \pi_2^* \mathbb{X}_i$$

and the one for the lower arrow

$$\mathcal{S}_2^\ell = -\int_{\partial_2^{(1)} M_2 \times \Sigma} \pi_1^* \mathbb{E}_i \wedge \zeta_{12} \wedge \pi_2^* \mathbb{X}_i,$$

where we set $M_1 := L_3^1$ and $M_2 := L_3^2$ and Σ to be the identification of $\partial_2^{(1)} M_1$ with $\partial_1 M_2$. Therefore the glued action is given by identfying the fields $\mathbb{E}_2^{(1)}$ and $\mathbb{X}_1$ and integratig over Σ. Thus we get

$$\mathcal{S}_{g1}^\ell = -\int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \zeta_{02}^\ell \wedge \pi_2^* \mathbb{X}_i, \tag{3.106}$$

where ζ^ℓ is the bulk propagator of $\mathcal{M}^\ell$. The same holds for the free part of $\mathcal{M}^r$ as in figure 3.5.4, by renaming the boundary components of $\mathcal{M}^r$, i.e.

$$\mathcal{S}_{g2}^r = -\int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \zeta_{02}^r \wedge \pi_2^* \mathbb{X}_i,$$

where ζ^r is the bulk propagator for $\mathcal{M}^r$.

Consider now the states $\hat{\psi}^{g1}_{\mathcal{M}^\ell} = \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^\ell_{g1}\right\}$ and $\hat{\psi}^{g1}_{\mathcal{M}^r} = \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^r_{g1}\right\}$ Now since ζ^ℓ and ζ^r are both propagators, we can again write them as $\zeta^\ell = \zeta + d\kappa^\ell$ and $\zeta^r = \zeta + d\kappa^r$, with some zero forms κ^ℓ and κ^r. Moreover we can again parametrize the states by a parameter t by setting $\zeta_t^\ell = \zeta + td\kappa^\ell$ and $\zeta^r = \zeta + td\kappa^r$.

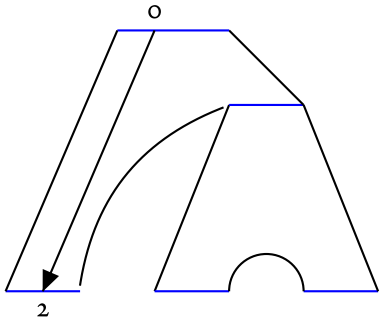


Figure 3.5.4: The equivalent diagram for the free part of the effective action as in figure 3.5.3 on $\mathcal{M}^r$

Thus we get then

$$\partial_t \hat{\psi}^{g_1}_{\mathcal{M}^\ell}(t) = -\left(\frac{\mathrm{i}}{\hbar}\right) \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^\ell_{g_1}\right\} (\partial_t \mathcal{S}^\ell_{g_1}) d\kappa^\ell_{02} \tag{3.107}$$

$$\partial_t \hat{\psi}^{g_1}_{\mathcal{M}^r}(t) = -\left(\frac{\mathrm{i}}{\hbar}\right) \exp\left\{\frac{\mathrm{i}}{\hbar}\mathcal{S}^r_{g_1}\right\} (\partial_t \mathcal{S}^r_{g_1}) d\kappa^r_{02}, \tag{3.108}$$

where

$$\partial_t \mathcal{S}^\ell_{g_1} = \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge d\kappa^\ell_{02} \wedge \pi_2^* \mathbb{X}_i \tag{3.109}$$

$$\partial_t \mathcal{S}^r_{g_1} = \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge d\kappa^r_{02} \wedge \pi_2^* \mathbb{X}_i. \tag{3.110}$$

Now, using integration by parts, we get

$$\partial_t \mathcal{S}^\ell_{g_1} = \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* d\mathbb{E}_i \wedge \kappa^\ell_{02} \wedge \pi_2^* \mathbb{X}_i + \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \kappa^\ell_{02} \wedge \pi_2^* d\mathbb{X}_i \tag{3.111}$$

$$\partial_t \mathcal{S}^r_{g_1} = \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* d\mathbb{E}_i \wedge \kappa^r_{02} \wedge \pi_2^* \mathbb{X}_i + \int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \kappa^r_{02} \wedge \pi_2^* d\mathbb{X}_i. \tag{3.112}$$

Therefore one can observe that

$$\partial_t \hat{\psi}^{g_1}_{\mathcal{M}^\ell}(t) = \hat{\psi}^{g_1}_{\mathcal{M}^\ell}(t)\Omega_0 \left(\int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \kappa^\ell_{02} \wedge \pi_2^* \mathbb{X}_i\right) \tag{3.113}$$

$$\partial_t \hat{\psi}^{g_1}_{\mathcal{M}^r}(t) = \hat{\psi}^{g_1}_{\mathcal{M}^r}(t)\Omega_0 \left(\int_{\partial_2^{(1)} M_2 \times \partial_1 M_1} \pi_1^* \mathbb{E}_i \wedge \kappa^r_{02} \wedge \pi_2^* \mathbb{X}_i\right). \tag{3.114}$$

Since the propagator ζ^r for the diagram on $\mathcal{M}^r$ is the bulk propagator ζ, we get that $\hat{\psi}^{g_1}_{\mathcal{M}^\ell}$ is the same as $\hat{\psi}^{g_1}_{\mathcal{M}^r}$ up to an Ω_0-exact term. For the interaction part of the effective action we will end up with the result that $\hat{\psi}_{\mathcal{M}^\ell}$ is the same as $\hat{\psi}_{\mathcal{M}^r}$ up to an Ω-exact term. The same procedure works for the interaction diagrams, i.e. one compares the computation for a possible interaction diagram on $\mathcal{M}^\ell$ with the same on

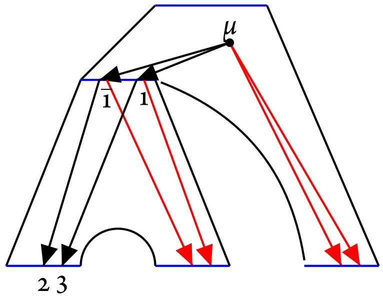


Figure 3.5.5: All possible combinations for the interaction term of the effective action for $\mathcal{M}^\ell$

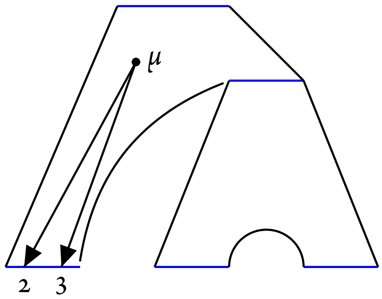


Figure 3.5.6: The equivalent interaction term as in figure 3.5.5 for $\mathcal{M}^r$

$\mathcal{M}^r$ as in figure 3.5.5 and 3.5.6.

3.6 Main gluing and the Moyal product

Let us first take the L_1 with the $\frac{\delta}{\delta \mathbb{X}}$-polarization with the difference that we put a delta function on the $\eta = 0$ boundary (see figure 3.6.1).

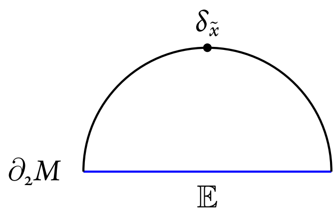


Figure 3.6.1: L_1 with $\frac{\delta}{\delta \mathbb{X}}$-polarization endowed with a delta function

Here $\delta_{\tilde{x}}(x) = \delta(x - \tilde{x})$, where $\tilde{x}$ and x are points in $\mathscr{P} = \mathbb{R}^d$. Therefore the state for this setting is given by

$$\hat{\psi}_{L_1^\delta}(\mathbb{E}, z, z^\dagger, x, dx) = \delta(x + z - \tilde{x}) \exp\left\{ \frac{\mathrm{i}}{\hbar} \left(z^i \int_{\partial_2 M} \mathbb{E}_i + z_i^\dagger dx^i \right) \right\}.$$

Now let us consider the gluing of an L_1 manifold with the $\frac{\delta}{\delta \mathbb{E}}$-polarization with an L_1 manifold with the $\frac{\delta}{\delta \mathbb{X}}$-polarization, where the first one is endowed with a smooth function f and the second one with the delta function as before as it is given in figure 3.6.2 whereas we only consider the case for $a = 0$.

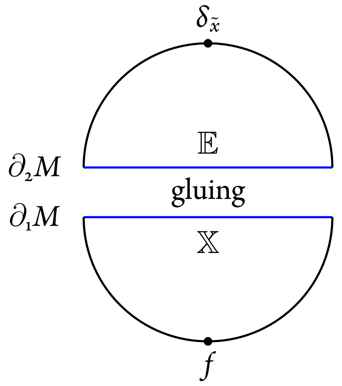


Figure 3.6.2: The gluing of the different L_1

We should expect to end up with the observable f using this particular gluing. Indeed, first we notice that the diagrams we get in the L_1 manifold with the $\frac{\delta}{\delta \mathbb{E}}$-polarization are all possible arrows going from $\partial_1 M$ to the observable f and hence, by Wick's theorem (see A), is given by

$$\sum_{n\geq 0} \frac{(-i\hbar)^n}{n!} \int_{(\partial_1 M)^n} \mathbb{X}(u_1)\cdots\mathbb{X}(u_n)\zeta_{u_1 i_1}\cdots\zeta_{u_n i_n}\partial_{i_1}\cdots\partial_{i_n} f(x),$$

where $u_1, \ldots, u_n$ are points on $\partial_1 M$ and we set $\zeta_{u_j i_j} := \zeta(u_j, u_0)_{i_j}$ with u_0 a point on the lower boundary where f is attached. Using the fact that $\int_{\partial_1 M} \zeta_{u_j i_j} = 1$ holds, we can use the gluing principle of identifying the $\mathbb{X}$ with the $\mathbb{E}$ fields to obtain

$$\hat{\psi}_{f,\tilde{x}} = \delta(x + z - \tilde{x}) \sum_{n\geq 0} \frac{1}{n!} z^{i_1}\cdots z^{i_n}\partial_{i_1}\cdots\partial_{i_n} f(x) \exp\left\{\frac{i}{\hbar} z_j^{\dagger} dx^j\right\},$$

which, by considering the sum as a Taylor expansion of f, can be written as

$$\hat{\psi}_{f,\tilde{x}} = \delta(x + z - \tilde{x}) f(x + z) \exp\left\{\frac{i}{\hbar} z_j^{\dagger} dx^j\right\},$$

Let us now write $\rho(x) := \delta(x - \tilde{x}) f(x)$. Then

$$\hat{\psi}_{f,\tilde{x}} = \hat{\psi}_{\rho} = \rho(x + z) \exp\left\{\frac{i}{\hbar} z_j^{\dagger} dx^j\right\}.$$

Thus we get

$$\triangle \hat{\psi}_{\rho} = \frac{i}{\hbar} \sum_{k=1}^{d} dx^k \partial_k \rho(x + z) \exp\left\{\frac{i}{\hbar} z_j^{\dagger} dx^j\right\}$$

and

$$d\hat{\psi}_{\rho} = \sum_{k=0}^{d} dx^k \partial_k \rho(x + z) \exp\left\{\frac{i}{\hbar} z_j^{\dagger} dx^j\right\}$$

and therefore $\triangle \hat{\psi}_{\rho} = \frac{i}{\hbar} d\hat{\psi}_{\rho}$ and since $\Omega \hat{\psi}_{\rho} = 0$ we get that the smQME holds. Let us now consider the

Lagrangian submanifold $\mathscr{L} = \{z = 0\}$. Then we get the BV integral

$$\int_{\mathscr{L}} \hat{\psi}_\rho \,|_{z=0}\, dz^\dagger \cdots dz^\dagger = \rho(x)\left(\frac{i}{\hbar}\right)^n d^n x = \left(\frac{i}{\hbar}\right)^n f(x)\delta(x - \tilde{x})d^n x.$$

Now integrating this term over the whole Poisson manifold, we get

$$\int_{\mathscr{P}} \left(\int_{\mathscr{L}} \hat{\psi}_\rho \,|_{z=0}\, dz^\dagger \cdots dz^\dagger\right) = \left(\frac{i}{\hbar}\right)^n f(\tilde{x}).$$

Therefore we end up with the observable which was the claim. Now let us consider the case where α is constant different from zero. Then the integration over $\mathscr{P}$ is actually given by the integration of the star product between f and δ, since we get the usual disc as the glued manifold and by the fact of the appearing Feynman diagrams. This means we get

$$\int_{\mathscr{P}} f \star \delta_{\tilde{x}} d^n x = \underbrace{\int_{\mathscr{P}} f(x)\delta(x - \tilde{x})d^n x}_{=f(\tilde{x})} + \underbrace{\alpha^{ij}\int_{\mathscr{P}} \partial_i f(x)\partial_j \delta(x - \tilde{x})d^n x}_{=\alpha^{ij}\underbrace{\int_{\mathscr{P}} \partial_i\partial_j f(x)\delta(x-\tilde{x})d^n x}_{=0}+\cdots} + \cdots$$

up to some constants dependig on $\hbar$, where we have used the fact that α is an antisymmetric tensor to obtain zero for the other terms. Therefore we get the same result as before. Now using this construction, we can observe the Moyal-product by the gluing as in figure 3.6.3.

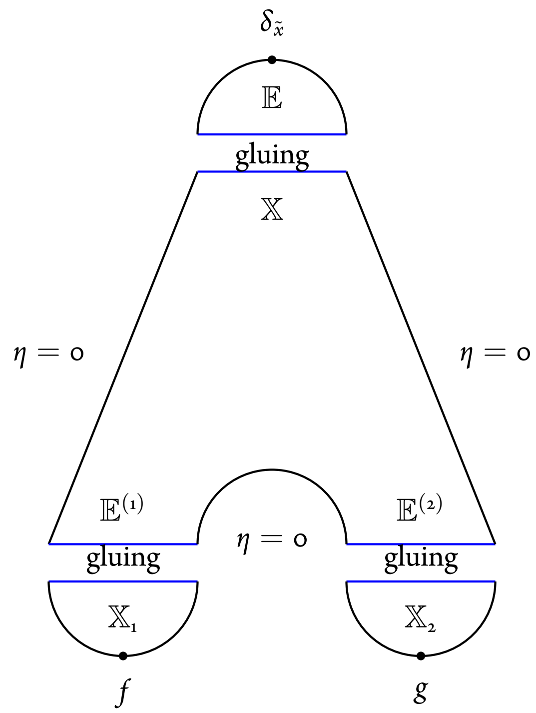


Figure 3.6.3: The gluing for the Moyal product

This can be thought of as using the process before by using the gluing as in figure 3.6.2 but with the difference that we attach $f \star g$ on the lower manifold insted of f. It is not really that case, since the propagators will and derivatives will split to two different L_1's where f and g are attached on it. Therefore one needs to observe that the lower manifold, with $f \star g$ attached to it, as in figure 3.6.2 does indeed have the same state as the gluing in figure 3.6.3 but without the cap glued at the top. The diagrams which do appear in the exponential for this gluing are given in figure 3.6.4.

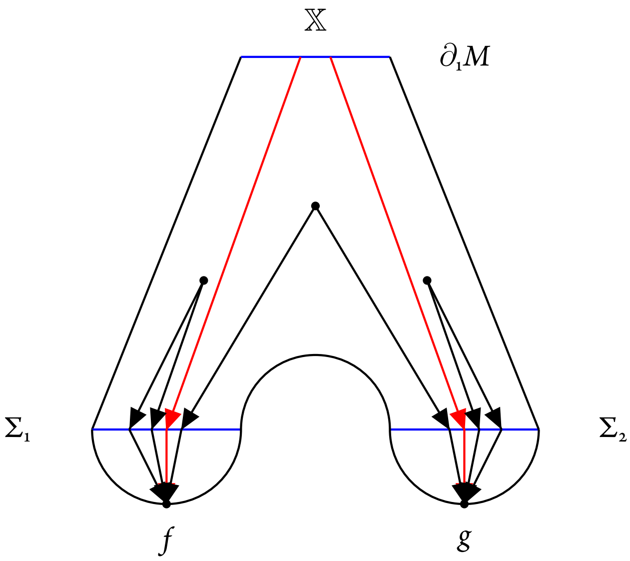


Figure 3.6.4: Appearing diagrams, where Σ_1 and Σ_2 are corresponding identificated boundaries for the gluing. The red arrows represent the free part terms. We can also observe that by antisymmetry the two diagrams where two interaction term arrows go to the same glued boundary component vanish. Thus the Moyal-product will appear as the glued result of the interaction diagram going in both directions.

Using Wick's theorem, the effective action terms of L_3 and the gluing properties, we can observe the state of the glued manifold by the same arguments as before. Thus we get

$$\sum_{\substack{n\geq 0\\ m\geq 0\\ \ell\geq 0}} \frac{(-i\hbar)^{n+m+\ell}}{n!m!\ell!} \int_{C_{n+m}(\partial_1 M)} \prod_{k_1=1}^{n}\prod_{k_2=1}^{m} \mathbb{X}^{i_{k_1}}(u_{k_1})\zeta_{u_{k_1}v_0}\mathbb{X}^{j_{k_2}}(\tilde{u}_{k_2})\zeta_{\tilde{u}_{k_2}v_1} \prod_{k_3=1}^{\ell} \alpha^{i_{k_3}j_{k_3}}\partial_{i_{k_1}}\partial_{i_{k_3}}(f(x))\partial_{j_{k_1}}\partial_{j_{k_3}}(g(x)), \tag{3.115}$$

where $\partial_1 M$ is the top boundary, $u_1, \ldots, u_n, \tilde{u}_1, \ldots, \tilde{u}_m$ are distinct points in $\partial_1 M$, the $\mathbb{X}^{i_k}$ are the $\mathbb{X}$-fields on $\partial_1 M$ correpsonding to the Σ_1 gluing and $\mathbb{X}^{j_k}$ are the $\mathbb{X}$-fields on $\partial_1 M$ corresponding to the Σ_2 gluing. Here the propagators ar given as before with the difference that v_0 represents a point on the lower boundary where f is attached and v_1 represents a point on the lower boundary where g is attached.

Now by the argument as before we can consider the gluing as in figure 3.6.6, which is then basically the same as the one in figure 3.6.3.

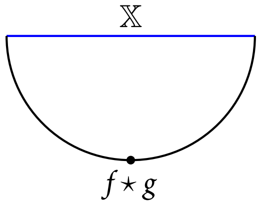


Figure 3.6.5: The L_1 manifold with the $\frac{\delta}{\delta \mathbb{E}}$-polarization with the star product on it.

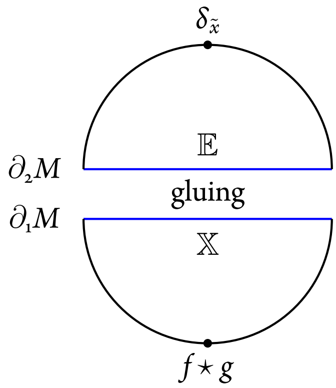


Figure 3.6.6: The cap gluing with the star product.

Finally, using the procedure for the free part term, we end up with the integral

$$\int_{\mathscr{P}} [(f \star g) \star \delta_{\tilde{x}}](x) d^n x,$$

which is given by $f \star g(\tilde{x})$. Therefore we obtain the Moyal-product using this particular gluing. We have already shown that associativity is satisfied, since the states of the L_3 gluings only differ by an Ω-exact term, i.e. the smQME is directly satisfied and therefore associativity holds for this construction.

A

Mathematical Methods of Quantum Field Theory: Path Integrals and Feynman Diagrams

A.1 Introduction

The notion of a *path integral* is a very interesting and elegant way of describing a quantum system (see [ZJ]). In this part, we want to give the mathematical notion of a path integral as in [MP1], the way of getting the graphical interpretation of it in terms of the mathematical setting of graphs, called *Feynman diagrams*, out of it and describe the relation to deformation quantization. The path integral formalism of quantum mechanics goes back to Richard P. Feynman (therefore also called a *Feynman integral*), who formulated the path integral in [F] and showed its equivalence to the Schrödinger equation, which is the equation of motion for a quantum system. Similar as in classical mechanics, where we have the least action principle, we can consider the same for the quantum setting, by formulating a corresponding functional integral. To do so, one has to consider all possible paths between two points. The mathematical problem appearing in this approach is the computation of an integral of the form

$$\int_{\mathcal{F}(M)} f(S(\varphi))\mathscr{D}(\varphi),$$

where M is a space-time manifold and $\varphi : M \to \mathbb{R}$ is called a field, $\mathscr{D}(\varphi)$ is a (mathematical ill defined) measure on an infinite dimensional space, in particular a measure on the space of all possible paths (fields), $\mathcal{F}(M)$ the space of fields and $f(S(\varphi))$ a function of the classical action $S(\varphi)$, actually defined as $f(t) = \exp\left\{\frac{i}{\hbar}t\right\}$. Formulating a path integral is often associated to formulating a *quantum field theory* (see [ZJ]). Problems appear with the treatment of the measure $\mathscr{D}$ when one wants to make computations with a path integral. Despite these problems, there is still a whole theory behind it, which leads to interesting mathematical objects that are related to many other theories. A nice way of graphical illustrations for path integrals are called *Feynman diagrams* (see [ZJ]). They are usually related to operator elements appearing in path integrals and also give rise to an impressive amount of mathematical (for example graph theory) and physical theories (illustration of quantum field theory, Feynman rules (see [ZJ])). We want to start with some preliminary definitions right away. As we have already mentioned, we consider $\mathcal{F}(M)$ as the space of fields. For a Lagrangian $L : \mathcal{F}(M) \to \mathbb{R}$, the action $S : \mathcal{F}(M) \to \mathbb{R}$ is given by

$$S(\varphi) = \int_M L(\varphi)dx,$$

In quantum field theory, we consider the partition function given as a path integral

$$Z = \int_{\mathcal{F}(M)} \exp\left\{\frac{\mathrm{i}}{\hbar}S(\varphi)\right\} \mathscr{D}(\varphi).$$

In order to compute probability amplitudes, we also define the *expectation* of an observable $\mathscr{O} : \mathcal{F}(M) \to \mathbb{R}$, to be

$$\langle \mathscr{O} \rangle = \frac{1}{Z}\int_{\mathcal{F}(M)} \mathscr{O}(\varphi) \exp\left\{\frac{\mathrm{i}}{\hbar}S(\varphi)\right\} \mathscr{D}(\varphi),$$

and for a collection $\mathscr{O}_1, ..., \mathscr{O}_m$ of observables we have the m-point correlation function, given by

$$\langle \mathscr{O}_1, ..., \mathscr{O}_m \rangle = \frac{1}{Z}\int_{\mathcal{F}(M)} \prod_{j=1}^{m} \mathscr{O}_j(\varphi) \exp\left\{\frac{\mathrm{i}}{\hbar}S(\varphi)\right\} \mathscr{D}(\varphi).$$

For simplicity, we want to deal with euclidean spaces. Therefore we can apply something called a *Wick rotation* to the path integral as it is explained in [ZJ]. Thus one can do all computations with

$$Z = \int_{\mathcal{F}(M)} \exp\left\{-\frac{1}{\hbar}S(\varphi)\right\} \mathscr{D}(\varphi).$$

A.2 Finite dimensional quantum field theory

The first focus should be on the finite dimensional case, i.e. the case where the space of fields is some $\mathbb{R}^d$ for $d \geq 1$, in particular $\mathcal{F}(M) = \mathbb{R}^d$. The theory for finite dimensional quantum field theory is important for the further understanding of the general infinite dimensional case. As soon as we have developed the method for the finite dimensional setting, we are able to manipulate formulae in a way, where we can make reasonable assertions for the general case. For an action $S : \mathbb{R}^d \to \mathbb{R}$, we want to consider the partition function given by

$$Z = \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{\hbar}S(x)\right\} d^dx,$$

where d^dx is the Lebesgue measure on $\mathbb{R}^d$. The interesting situation appears in the limit for $\hbar \to 0$, which is called the *quasi-classical limit*. For our further discussion, it suffices to consider the Taylor expansion of S. Since the linear terms of this expansion vanishes, we need to make sense of the integral given by

$$\int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \hbar U(x)\right\} d^dx.$$

The start of analyzing the path integral in finite dimensions, is to develop several formulae and relations, which we can rewrite for the infinite dimensional case. One of the most central formulae, which are used for those computations are the Gaussian integral formulae and are therefore our first point of discussion. Let us start with the first relation. Let us denote the space of all $(d \times d)$-matrices with entries in a ring k by $\mathcal{M}at(d \times d, k)$.

Proposition A.2.1 (Gaussian relation I). *Let $A = (A_{ij})$ be a positive-definite matrix in $\mathcal{M}at(d \times d, \mathbb{R})$ and let $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the euclidean inner product on $\mathbb{R}^d$. Moreover, let $x = (x_1, ..., x_d) \in \mathbb{R}^d$*

and let $a > 0$. Then

$$\int_{\mathbb{R}^d} \exp\left\{-\frac{a}{2}\langle Ax, x\rangle\right\} d^d x = \left(\frac{2\pi}{a}\right)^{\frac{d}{2}} \frac{1}{\sqrt{\det A}}, \tag{A.1}$$

where $d^d x$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Proof. We have to use the well known relation

$$\int_{\mathbb{R}} \exp\left\{-\frac{a}{2}x^2\right\} dx = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}}.$$

Consider now the given integral as

$$\int_{\mathbb{R}^d} \exp\left\{-\frac{a}{2}\langle Ax, x\rangle\right\} d^d x = \int_{\mathbb{R}} \times \cdots \times \int_{\mathbb{R}} \exp\left\{-\frac{a}{2}\sum_{k,l=1}^{d} A_{kl}x_k x_l\right\} dx^1 \cdots dx^d.$$

We can use the fact that one can consider the matrix A to be symmetric. In general, we can always consider A as a symmetric $(d \times d)$-matrix. Indeed, we can write

$$\sum_{k,\ell=1}^{d} A_{k\ell}x_k x_\ell = \frac{1}{2}\sum_{k,\ell=1}^{d}(A_{k\ell} + A_{\ell k})x_k x_\ell,$$

which gives us a new symmetric matrix $(A_{k\ell} - A_{\ell k})$. Considering now A to be symmetric, we can always find an orthogonal matrix $P \in O(d \times d, \mathbb{R})$ such that

$$A = P^T \Lambda P,$$

where $\Lambda_k = \sum_{\ell=1}^{d} \lambda_\ell \delta_{k\ell}$ with $\{\lambda_\ell\}_{1\leq \ell \leq d} \subset \mathbb{R}$ being the eigenvalues of A. Thus, we finally get

$$\begin{aligned}
&\int_{\mathbb{R}} \times \cdots \times \int_{\mathbb{R}} \exp\left\{-\frac{a}{2}\sum_{k,\ell=1}^{d} A_{k\ell}x_k x_\ell\right\} dx^1 \cdots dx^d \\
&\qquad = \int_{\mathbb{R}} \times \cdots \times \int_{\mathbb{R}} \exp\left\{-\frac{a}{2}\sum_{k=1}^{d} y_k^2 \lambda_k\right\} dy^1 \cdots dy^d = \prod_{k=1}^{d}\sqrt{\frac{2\pi}{a\lambda_k}} = \left(\frac{2\pi}{a}\right)^{\frac{d}{2}} \frac{1}{\sqrt{\det A}}.
\end{aligned}$$

□

Remark A.2.1. We can also make sense of this integral relation for a general d-dimensional vector space V, by looking at a positive-definite symmetric operator $A : V \to V^*$, where V^* denotes the dual of V, and thus we consider the pairing given by $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$. A induces a map $\det A : \bigwedge^d V \to \bigwedge^d V^*$, meaning $\det A \in (\bigwedge^d V^*)^{\otimes 2} = (\bigwedge^d V^*) \otimes (\bigwedge^d V^*)$.

Remark A.2.2. A more general setting would be to consider an integrand given by

$$\exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \langle J, x\rangle\right\},$$

where $J \in \mathbb{R}^d$, as we will see later on. Let us therefore describe another relation, which is similar to proposition A.2.1.

Proposition A.2.2 (Gaussian relation II). *Let everything be as in proposition A.2.1 and let $J \in \mathbb{R}^d$. Then*

$$Z_1(J) = \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \langle J, x\rangle\right\} d^d x = (2\pi)^{\frac{d}{2}} \frac{\exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}}{\sqrt{\det A}}.$$

Remark A.2.3. By the definition of $Z_1(J)$, we note that the integral in equation $(A.1)$ can be written as $Z_a(0)$, for $a \geq 1$, and thus we can write

$$Z_1(J) = Z_1(0) \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}.$$

Proof of proposition A.2.2. We can do the change of coordinate $x \mapsto x + A^{-1}J$. The Jacobian of this change of coordinate is then given by (δ_{ij}). Hence we get

$$\begin{aligned} Z_1(J) &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \langle J, A^{-1}J\rangle\right\} d^d x \\ &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle - \frac{1}{2}\langle Ax, A^{-1}J\rangle - \frac{1}{2}\langle J, x\rangle - \frac{1}{2}\langle J, A^{-1}J\rangle + \langle J, x\rangle + \langle J, A^{-1}J\rangle\right\} d^d x. \end{aligned}$$

Note that $\langle Ax, A^{-1}J\rangle = \langle x, J\rangle$, since

$$\langle Ax, A^{-1}J\rangle = \sum_{i=1}^d (Ax)_i (AJ)_j = \sum_{i,j=1}^d A_{ij}A^{ij}x_i J_j = \sum_{i,j=1}^d \delta_{ij}x_i J_j = \sum_{i=1}^d x_i J_i,$$

where $A^{-1} = (A^{ij})$. Therefore we get

$$\begin{aligned} Z_1(J) &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle - \frac{1}{2}\langle Ax, A^{-1}J\rangle - \frac{1}{2}\langle J, x\rangle - \frac{1}{2}\langle J, A^{-1}J\rangle + \langle J, x\rangle + \langle J, A^{-1}J\rangle\right\} d^d x \\ &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle - \frac{1}{2}\langle x, J\rangle - \frac{1}{2}\langle J, x\rangle - \frac{1}{2}\langle J, A^{-1}J\rangle + \langle J, x\rangle + \langle J, A^{-1}J\rangle\right\} d^d x \\ &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \frac{1}{2}\langle J, A^{-1}J\rangle\right\} d^d x = Z_1(0) \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}. \end{aligned}$$

□

Now we are able to describe the correlation function for observables. The correlation function of observables is an important object, which is basically the correlation function of statistical mechanics, measuring the correlation between random variables, which are in our case the observables of a quantum system (for example a self-adjoint operator) as measurable operators. As the expectation of an observable, we are able to write the correlation function in a similar way using the partition function. Let us start with the definition of the correlation function in finite dimensions.

Definition A.2.1 (m-point correlation function). *Let $\mathcal{O}_1, \ldots, \mathcal{O}_m : \mathbb{R}^d \to \mathbb{R}$ be observables, that are measurable functions (or measurable operators). Then we set the m-point correlation function of those to be*

$$\langle \mathcal{O}_1, \ldots, \mathcal{O}_m\rangle = \frac{1}{Z_1(0)} \int_{\mathbb{R}^d} \prod_{j=1}^d \mathcal{O}_j(x) \exp\left\{-\frac{1}{2}\langle Ax, x\rangle\right\} d^d x.$$

An important observation is the special case, where the observables are given as coordinate functions. For those, we can express the correlation function with a formula, which is essential for the formulation of the main theorem giving the connection between path integrals and the graphs, appearing out of those (see theorem A.2.5). Now by the definition of the correlation function, we get the following corollary.

Corollary A.2.3. *The m-point correlation function of coordinate functions $x^{i_1}, \dots, x^{i_m}$ is given by*

$$\langle x^{i_1}, \dots, x^{i_m} \rangle = \frac{1}{Z_1(0)} \prod_{k=1}^{m} \partial_{i_k} Z_1(J)\Big|_{J=0} = \prod_{k=1}^{m} \partial_{i_k} \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Big|_{J=0}, \tag{A.2}$$

where $\partial_i = \frac{\partial}{\partial J_i}$.

Proof. First of all we have to note that

$$\begin{aligned}\partial_i Z_1(J) = \frac{\partial}{\partial J_i} Z_1(J) &= \frac{\partial}{\partial J_i} \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \langle J, x\rangle\right\} d^d x \\ &= \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \langle J, x\rangle\right\} x^i d^d x.\end{aligned}$$

Now it is easy to see that for one coordinate function x^i we have

$$\langle x^i \rangle = \frac{1}{Z_1(0)} \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle\right\} x^i d^d x = \frac{1}{Z_1(0)} \partial_i Z_1(J)\Big|_{J=0},$$

and hence for coordinate functions $x^{i_1}, \dots, x^{i_m}$, we get

$$\langle x^{i_1}, \dots, x^{i_m} \rangle = \frac{1}{Z_1(0)} \partial_{i_1} \cdots \partial_{i_m} Z_1(J)\Big|_{J=0} = \frac{1}{Z_1(0)} \prod_{k=1}^{m} \partial_{i_k} Z_1(J)\Big|_{J=0}.$$

Replacing $Z_1(J)$ by $Z_1(0) \exp\left\{-\frac{1}{2}\langle J, A^{-1}J\rangle\right\}$, we get the result. □

According to the formulae we have derived for the partition function and by using the definition of the correlation function, we can obtain a special formula, where the observables are expressed as formal power series for the coordinate functions x^i. Using equation (A.2), we get the following proposition.

Proposition A.2.4. *Let $\mathcal{O}_1, \dots, \mathcal{O}_m$ be formal power series. Then we get*

$$\langle \mathcal{O}_1, \dots, \mathcal{O}_m \rangle = \prod_{k=1}^{m} \mathcal{O}_k(\partial_J) \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Big|_{J=0}, \tag{A.3}$$

where $\partial_J = \frac{\partial}{\partial J}$.

Proof. Let us start with one observable $\mathcal{O}_i = \sum_{k_i=1}^{l_i} a^i_{k_i} (x^i)^{k_i}$ given as a formal power series for the coordinate functions x^i. Then, by linearity of the integral, the definition of the expectation of a single observable and equation (A.2), we get

$$\frac{1}{Z_1(0)}\int_{\mathbb{R}^d}\mathcal{O}_i \exp\left\{-\frac{1}{2}\langle Ax,x\rangle\right\}d^dx = \left\langle \sum_{k_i=1}^{\ell_i} a_k^i (x^i)^{k_i}\right\rangle$$
$$= \sum_{k_i=1}^{\ell_i} a_{k_i}^i \frac{1}{Z_1(0)}\int_{\mathbb{R}^d}\underbrace{x^i\cdots x^i}_{k_i}\exp\left\{-\frac{1}{2}\langle Ax,x\rangle\right\}d^dx$$
$$= \sum_{k_i=1}^{\ell_i} a_{k_i}^i\langle x^i,\dots,x^i\rangle = \sum_{k_i=1}^{\ell_i} a_{k_i}^i \partial_i^{k_i}\exp\left\{\frac{1}{2}\langle J,A^{-1}J\rangle\right\}\Bigg|_{J=0}.$$

Considering now the observables, given as formal power series for the coordinate functions x^i by

$$\mathcal{O}_1 = \sum_{k_1=1}^{\ell_1} a_{k_1}^1 (x^{i_1})^{k_1},\dots,\mathcal{O}_m = \sum_{k_m=1}^{\ell_m} a_{k_m}^m (x^{i_m})^{k_m},$$

we can deduce the the formula for the general correlation function by the computation

$$\frac{1}{Z_1(0)}\int_{\mathbb{R}^d}\prod_{j=1}^{d}\mathcal{O}_j\exp\left\{-\frac{1}{2}\langle Ax,x\rangle\right\}d^dx = \langle\mathcal{O}_1,\dots,\mathcal{O}_m\rangle = \prod_{j=1}^{m}\sum_{k_j=1}^{\ell_j} a_{k_j}^j(\partial_{i_j})^{k_j}\exp\left\{\frac{1}{2}\langle J,A^{-1}J\rangle\right\}\Bigg|_{J=0}$$
$$= \sum_{k_1=1}^{\ell_1} a_{k_1}^1(\partial_{i_1})^{k_1}\sum_{k_2=1}^{\ell_2} a_{k_2}^2(\partial_{i_2})^{k_2}\cdots\sum_{k_m=1}^{\ell_m} a_{k_m}^m(\partial_{i_m})^{k_m}\exp\left\{\frac{1}{2}\langle J,A^{-1}J\rangle\right\}\Bigg|_{J=0}.$$

The main thing is to replace x^{i_j} in the formal power series with ∂_{i_j}, since the coordinate functions appear out of derivation of the exponential function with respect to J. Writing the series again as an observable we get the result

$$\langle\mathcal{O}_1,\dots,\mathcal{O}_m\rangle = \prod_{k=1}^{m}\mathcal{O}_k(\partial_J)\exp\left\{\frac{1}{2}\langle J,A^{-1}J\rangle\right\}.$$

□

In order to describe the theory, which relates the path integrals to some special graphs, called the Feynman diagrams, we need to formulate a theorem where the heart of this mathematical relation of quantum field theory lies. This theorem allows us to interpret the operator elements as a structure of edges and vertices for the graphical structure and gives an important tool to deduce the physical amplitude from a given diagram. We will now formulate this theorem and describe the way of interpreting the diagram structure out of it. After formulating this for the finite dimensional case, we can make sense to this theorem for the general infinite dimensional case. Moreover, we will look at the physical structure behind these diagrams later on when we talk about the Feynman rules in more detail.

Theorem A.2.5 (Wick)**.** *Let $A \in \mathcal{M}at(d\times d,\mathbb{R})$ and $J\in\mathbb{R}^d$. Then*

$$\prod_{k=1}^{m}\partial_{i_k}\exp\left\{\frac{1}{2}\langle J,A^{-1}J\rangle\right\}\Bigg|_{J=0} = \begin{cases}\sum_W\prod_{l=1}^{m-1}A^{j_l j_{l+1}}, & \text{for } m \text{ even}\\ 0, & \text{for } m \text{ odd}\end{cases}$$

where the sum runs over the index set $W = \{(j_l, j_{l+1}) \mid 1\le l\le m-1\}$.

Proof. The proof of Wick's theorem can be found for example in [MP1]. □

A more general way of stating Wick's theorem is to consider arbitrary linear functions, which is more convenient to work with.

Theorem A.2.6 (Wick)**.** *let $\mathcal{O}_1(x), \ldots, \mathcal{O}_m(x)$ be arbitrary linear functions of the coordinates x^i. Then*

$$\langle \mathcal{O}_1, \ldots, \mathcal{O}_m \rangle = \begin{cases} \sum_{\{(i_k, i_{k+1}) \mid 1 \leq k \leq m-1\}} \prod_{j=1}^{m-1} \langle \mathcal{O}_{i_j}, \mathcal{O}_{i_{j+1}} \rangle, & \text{for } m \text{ even} \\ 0, & \text{for } m \text{ odd} \end{cases}$$

Remark A.2.4. It turns out that the convention of interpreting the edges of a graph as correlation functions, is a better way for quantum field aspects instead of labeling them with the according operator elements by Wick's theorem. Thus, we want to represent a term $\prod_{j=1}^{m-1} \langle \mathcal{O}_{i_j}, \mathcal{O}_{i_{j+1}} \rangle$ by a graph. Therefore, let us first give some more specific mathematical definitions of a graph.

Definition A.2.2 (valency)**.** *The valency (or degree) of a vertex is defined as the number of edges incident to the vertex, where loops at a vertex are counted as two edges.*

Definition A.2.3 (n-valent graph)**.** *A graph Γ is called an n-valent graph if every vertex $v \in V_\Gamma$ is of degree n.*

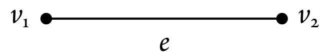


Figure A.2.1: An example of a univalent graph with two vertices v_1 and v_2 and one edge e

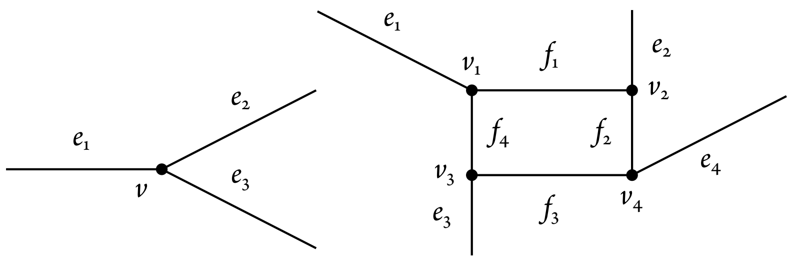


Figure A.2.2: Two examples of a 3-valent graph. The graph on the right has internal edges f_i and external edges e_i, whereas the graph on the left only has external edges e_i, but still they both represent a 3-valent graph, since every vertex of the graph has valence three.

Lemma A.2.7. *The edge of a graph Γ can be represented by the a 2-point correlation function.*

Corollary A.2.8. *Let everything be as in theorem A.2.6 and let $A \in \mathcal{M}at(d \times d, \mathbb{R})$. Then the 2-point correlation function of two observables $\mathcal{O}_i, \mathcal{O}_j : \mathbb{R}^d \to \mathbb{R}$, for $1 \leq i, j \leq d$, is given by*

$$\langle \mathcal{O}_i, \mathcal{O}_j \rangle = \langle \mathcal{O}_j, A^{-1} \mathcal{O}_j \rangle.$$

Proof. It is enough to prove this for the coordinate functions x^i and x^j, since one can deduce the general case then immediately from Wick's theorem (theorem A.2.5). Therefore, we want to show that $\langle x^i, x^j \rangle = A^{ij}$. Remember that we have already seen that $\langle x^i, x^j \rangle = \partial_i \partial_j \exp \left\{ \frac{1}{2} \langle J, A^{-1} J \rangle \right\} \Big|_{J=0}$. Computing this expression explicitly, we get

$$\partial_i \partial_j \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\} = \partial_j\left(\frac{1}{2}\left(\sum_{i=1}^{d} 2A^{ij}J_i\right)\exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\right)$$
$$= A^{ij} + \left(\sum_{i=1}^{d} A^{ij}J_i\right)\left(\sum_{j=1}^{d} A^{ij}J_j\right)\exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}.$$

Now setting $J = 0$, we get the claim. □

Finally, we are able to represent an edge of a graph Γ by $e = (j, k)$, which denotes the 2-point correlation function

$$\langle \mathcal{O}_i, \mathcal{O}_j\rangle = A_e^{-1} = \langle \mathcal{O}_i, A^{-1}\mathcal{O}_j\rangle.$$

Hence with Wick's theorem (theorem A.2.6) and only considering univalent graphs Γ, we can write

$$\langle \mathcal{O}_1, \dots, \mathcal{O}_m\rangle = \sum_{\Gamma}\prod_{e\in E_\Gamma} A_e^{-1}.$$

Example A.2.1. Consider the coordinate functions x^1, x^2, x^3, x^4. Using theorem A.2.5, we get

$$\langle x^1, x^2, x^3, x^4\rangle = A^{12}A^{34} + A^{13}A^{24} + A^{14}A^{23} \quad (A.4)$$

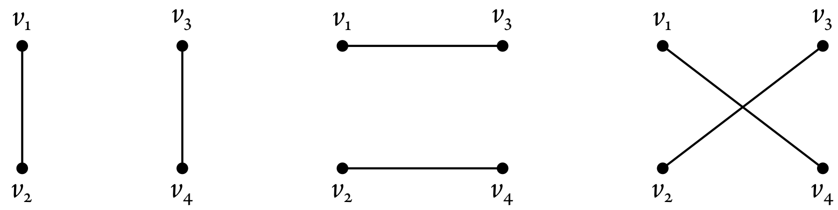


Figure A.2.3: The corresponding graphs for (A.4).

Remark A.2.5. To summarize things, the main point of Wick's theorem states that the operator elements do exactly describe the edges of the univalent graphs. In particular, the correlation function can be interpreted as a representation of an edge.

To get a general physical notion of the partition function, we want to consider the partition function with a potential $U(x) : \mathbb{R}^d \to \mathbb{R}$. That is, we look at partition functions of the form

$$Z_1(U) = \int_{\mathbb{R}^d} \exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \hbar U(x)\right\} d^d x.$$

Remark A.2.6. With proposition A.2.4, using that $\mathcal{O} = \exp\{\hbar U(x)\}$ and the fact that the correlation function of observables $\mathcal{O}_1, \dots, \mathcal{O}_m$ is given by

$$\langle \mathcal{O}_1, \dots, \mathcal{O}_m\rangle_U = \frac{1}{Z_1(U)}\int_{\mathbb{R}^d}\prod_{j=1}^{m}\mathcal{O}_j(x)\exp\left\{-\frac{1}{2}\langle Ax, x\rangle + \hbar U(x)\right\} d^d x,$$

we get that

$$Z_1(U) = Z_1(0)\exp\{\hbar U(\partial J)\}\exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Big|_{J=0} \quad (A.5)$$

and hence

$$\langle \mathcal{O}_1, \dots, \mathcal{O}_m \rangle_U = \frac{Z_1(0)}{Z_1(U)} \exp\left\{\hbar U(\partial_J)\right\} \prod_{j=1}^{m} \mathcal{O}_j(\partial_J) \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Bigg|_{J=0}.$$

Example A.2.2 (Cubic potential). Let us consider an example of a certain potential $U(x)$. Let therefore $U(x) = \sum_{ijk} U_{ijk}x^i x^j x^k$ and consider the expansion of the partition function $Z_1(U)$, defined as in (A.5), for $\hbar$ in power series. Then the coefficients are given by

$$\frac{Z_1(0)}{n!}\left(\sum_{i,j,k} U_{ijk}\partial_i\partial_j\partial_k\right)^n \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Bigg|_{J=0}.$$

The coefficients of the linear terms vanish by Wick's theorem and the coefficients of the square term is given by

$$\frac{Z_1(0)}{2}\sum_{i,j,k}\sum_{\mu,\nu,\rho} U_{ijk}U_{\mu\nu\rho}\partial_i\partial_j\partial_k\partial_\mu\partial_\nu\partial_\rho \exp\left\{\frac{1}{2}\langle J, A^{-1}J\rangle\right\}\Bigg|_{J=0} = \frac{Z_1(0)}{2}\sum_{i,j,k}\sum_{\mu,\nu,\rho} U_{ijk}U_{\mu\nu\rho}\sum_{\tilde{W}}\prod_{j=1}^{5} A^{i_j i_{j+1}}, \tag{A.6}$$

where the index set $\tilde{W}$ is the set of all pairings $(i_1, i_2), \dots, (i_5, i_6)$ of i, j, k, μ, ν, ρ. We can represent the coefficients U_{ijk} in terms of diagrams if we consider the triple (i, j, k) as trivalent vertices. To preserve the labels, we can write them on the ends of the edges meeting in this new vertex (i.e. on the star of the vertex). Similarly, we can represent $U_{\mu\nu\rho}$ by connecting the triple (μ, ν, ρ) to a second trivalent vertex. Note, that each of these labelled graphs is considered up to its automorphisms, i.e. maps of a graph onto itself, mapping edges to edges and vertices to vertices and preserving the incidence relation. Indeed, while the application of an automorphism changes the labels, it preserves their pairing and the way they are united in triples, thus corresponds to the same term in the right hand side of (A.6). Instead of summing over the automorphism classes of graphs, we may sum over all labelled graphs, but divide the term corresponding to a graph Γ by the number $|Aut(\Gamma)|$ of its automorphisms. also, when summing the resulting expressions over all indices, we observe that the terms which corresponds to the indices i, j, k, μ, ν, ρ and to the indices μ, ν, ρ, i, j, k are the same. Hence, we can write the coefficients of the square term as

$$Z_1(0)\sum_{\Gamma}\frac{1}{|Aut(\Gamma)|}\sum_{\mathscr{L}}\prod_{v} U_v \prod_{e} A_e^{-1}, \tag{A.7}$$

where $\mathscr{L}$ is, here, the set of all trivalent graphs with two vertices and labelling of their edges $U_v = U_{ijk}$ for a vertex v with the labels i, j, k of the adjacent edges and $A_e^{-1} = A^{ij}$ for an edge e with labels i, j.

We can now consider a general potential and deduce the general setting from the example above. Consider therefore a potential of the form $U(x) = U_{i_1 \dots i_k}x^{i_1}\cdots x^{i_k}$, where the k'th degree term will lead to the representation of $U_{i_1 \dots i_k}$ in terms of k-valent vertices. We call such a vertex an *internal vertex*. Moreover, let us assume that all internal vertices of any Feynman graph Γ are of valence $n \geq 3$ and we denote this number by $|\Gamma|$. We will also denote by $\mathcal{G}^\circ$ the set of all graphs Γ with no legs and we denote, for $\gamma \geq 1$, by $\mathcal{G}^\gamma$ the set of all *non-vacuum*[1] graphs Γ with γ ordered legs.

[1]non-vacuum graphs are graphs such that each connected component has at least one leg.

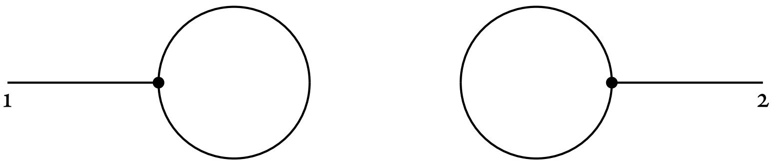


Figure A.2.4: An example of a degree two graph with two legs.

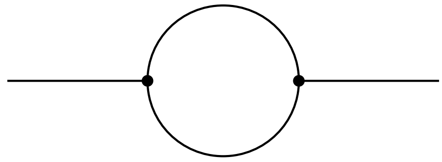


Figure A.2.5: An example of a degree two graph with two legs.

Proposition A.2.9. *Let Γ° and $|\Gamma|$ be as above and let everything else be as in (A.7). Then*

$$Z_1(U) = Z_1(0) \sum_{\Gamma \in \mathcal{G}^{\circ}} \frac{\hbar^{|\Gamma|}}{|Aut(\Gamma)|} \sum_{\mathscr{L}} \prod_{v} U_v \prod_{e} A_e^{-1}.$$

Proposition A.2.10. *Let $\mathcal{G}^{\gamma}$ and $|\Gamma|$ be as above and let everything be as in (A.7). Then, for coordinate functions $x^{i_1}, \ldots, x^{i_\gamma}$, we get*

$$\langle x^{i_1}, \ldots, x^{i_\gamma} \rangle_U = \sum_{\Gamma \in \mathcal{G}^{\gamma}} \frac{\hbar^{|\Gamma|}}{|Aut(\Gamma)|} \sum_{\mathscr{L}} \prod_{v} U_v \prod_{e} A_e^{-1}.$$

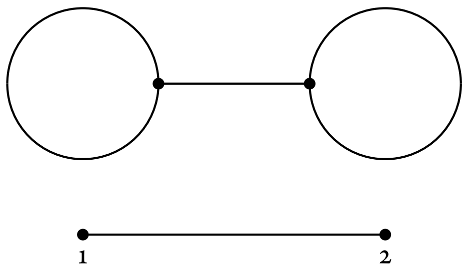


Figure A.2.6: An example of a degree two graph with two legs.

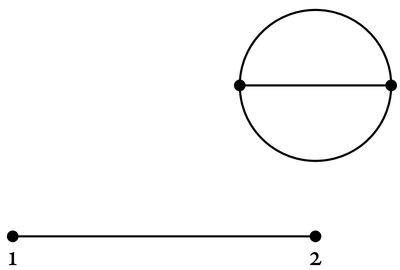


Figure A.2.7: An example of a degree two graph with two legs.

We want to reformulate the above in terms of a general notion of weights for graphs. To do so, we need the notion of a weight system, and derived from that, the notion of a general weight for a graph $\Gamma \in \mathcal{G}^\gamma$.

Definition A.2.4 (Weight system). *Let V be a vector space. A weight system is a collection $(a, \{u_k\}_{k=3}^{\infty})$ of $a \in S^2(V)$ and $u_k \in S^k(V^*)$.*

Definition A.2.5 (Weight). *Let w be a weight system. A weight w_Γ of a graph $\Gamma \in \mathcal{G}^\gamma$ is a map $w_\Gamma : (V^*)^{\otimes\gamma} \to \mathbb{R}$, which is defined as follows: Assign to each internal vertex v of valence k an element $u_k \in S^k(V^*)$ and associate each copy of V^* with an edge. Moreover, for $i \in \{1, ..., \gamma\}$ assign some $f_i \in V^*$ to the i'th leg of Γ. Finally, for each edge contract two copies of V^* associated to its ends using $a \in S^2(V)$. After all copies of V^* get contracted, we obtain a number $w_\Gamma(f_1, ..., f_\gamma) \in \mathbb{R}$.*

Remark A.2.7. Note that in our case we can observe that A^{-1} and $\hbar U(x)$ determine a weight system. Take $a = A^{-1}$ and let u_v be the degree k part of $\hbar U(x)$. In terms of a physical theory, these rules of computing the weight system are known as *Feynman rules*.

We can now reformulate the above formulae as

$$Z_1(U) = Z_1(0) \sum_{\Gamma \in \mathcal{G}^0} \frac{1}{|Aut(\Gamma)|} w_\Gamma,$$

$$\langle f_1, .., f_\gamma \rangle_U = \sum_{\Gamma \in \mathcal{G}^\gamma} \frac{1}{|Aut(\Gamma)|} w_\Gamma(f_1, ..., f_\gamma).$$

A.3 Infinite dimensional quantum field theory

After we have developed the theory of path integrals in finite dimensions, we are ready to look at the infinite dimensional case. As already mentioned, path integrals are mathematically not very well defined, because of the measure $\mathscr{D}$ on the space of fields. However, we can use the finite dimensional setting to deduce reasonable formulae for the infinite dimensional case. Let us now start with the dictionary. Instead of the discrete set $i \in \{1, ..., d\}$ of indices we consider a continuous variable $x \in M$, where M is a n-dimensional manifold, say for example $\mathbb{R}^n$. Thus the sum over i becomes an integral over x. Moreover, vectors $x = (x^1, ..., x^d)$ become fields $\varphi(x)$ and $J = (J^1, ..., J^d)$ also becomes a field $J(x)$. The quadratic form $A = (A_{ij})$ becomes an integral kernel $K(x, y)$. Moreover, the pairings in terms of the inner product[2] $\langle Ax, x \rangle = \sum_{ij} A_{ij} x^i x^j$ and $\langle J, x \rangle = \sum_i J^i x^i$ become

$$\langle K\varphi, \varphi \rangle = \int K(x, y)\varphi(x)\varphi(y)dxdy$$

and

$$\langle J, \varphi \rangle = \int J(x)\varphi(x)dx$$

respectively. The partition function $Z_1(J)$ becomes a path integral $Z_1(J)$ of the space of fields $\mathcal{F}(M)$. In particular,

$$Z_1(J) = \int_{\mathcal{F}(M)} \exp\left\{-\frac{1}{2}\langle K\varphi, \varphi\rangle + \langle J, \varphi\rangle\right\} \mathscr{D}(\varphi).$$

We also have a nice correspondence for the inverse of the matrix A, which is now a function $G = K^{-1}$ defined by the relation

$$\int K(x, z)G(z, y)dz = \delta(x - y),$$

[2] The inner product is actually given by the L^2 Hilbert space structure on $\mathbb{R}^n$ if the manifold M is $\mathbb{R}^n$, otherwise we consider a Riemannian manifold and work with a L^2 structure of the tangent space.

where δ is the usual delta distribution. Using the function G and rewrite the finite dimensional formula for $Z_1(J)$, we can obtain

$$Z_1(J) = Z_1(0) \exp\left\{\frac{1}{2}\langle J, GJ\rangle\right\}.$$

Finally, the correlation function $\langle x^{i_1}, \dots, x^{i_m}\rangle$ is then given by

$$\langle\varphi(x_1), \dots, \varphi(x_m)\rangle = \frac{1}{Z_1(0)} \int_{\mathcal{F}(M)} \prod_{j=1}^{m} \varphi(x_j) \exp\left\{-\frac{1}{2}\langle K\varphi, \varphi\rangle\right\} \mathscr{D}(\varphi).$$

We need to make sure what the meaning of the derivative $\frac{\partial}{\partial x^i}$ is for the infinite dimensional setting. The right choice will be the notion of a *functional derivative* $\frac{\delta}{\delta\varphi(x)}$. In order to describe the functional derivative, let $F(\varphi)$ be a functional. Consider the limit, given by

$$DF(\varphi)(r) = \lim_{\varepsilon\to 0} \frac{F(\varphi + \varepsilon r) - F(\varphi)}{\varepsilon}.$$

If this limit can be written as an integral of the form

$$\int r(x)h(x)dx,$$

for some $h(x)$, then one can define the functional derivative to be exactly this function, i.e. $\frac{\delta F}{\delta\varphi(x)} = h(x)$. There are two essential properties, which make the functional derivative actually similar to usual derivatives. These two relation are

(i) *(Distribution relation)*

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta(x - y), \tag{A.8}$$

(ii) *(Leibniz rule)*

$$\frac{\delta}{\delta\varphi(x)}(F(\varphi)H(\varphi)) = \frac{\delta F(\varphi)}{\delta\varphi(x)}H(\varphi) + F(\varphi)\frac{\delta H(\varphi)}{\delta\varphi(x)}, \tag{A.9}$$

where δ on the right hand side of the equation (A.8) is the usual delta distribution.

Example A.3.1 (Symmetric potential). To see how the functional derivative with respect to certain variables actually acts on functionals depending on different variables, we want to consider a symmetric potential given by

$$U(\varphi) = \sum_n \frac{1}{n!} \int U_n(x_1, \dots, x_n) \prod_{j=1}^{n} \varphi(x_j)dx_j.$$

Now, using the relation (A.8) and (A.9), we can easily obtain that

$$\frac{\delta U(\varphi)}{\delta\varphi(y)} = \sum_n \frac{1}{n!} \int U_{n+1}(y, x_1, \dots, x_n) \prod_{j=1}^{n} \varphi(x_j)dx_j.$$

Corollary A.3.1 (Inverse of the integral kernel)**.** *According to the finite dimensional case, $G(x, y)$ can be written as*

$$G(x, y) = \frac{1}{Z_1(0)} \frac{\delta^2}{\delta J(x)\delta J(y)} Z_1(J)\bigg|_{J=0}$$

Corollary A.3.2 (m-point correlation function)**.** *According to the finite dimensional case, we can define the* *m-point correlation function to be given by*

$$\langle \varphi(x_1), \ldots, \varphi(x_m) \rangle = \frac{1}{Z_1(0)} \frac{\delta^m}{\prod_{J=1}^m \delta J(x_j)} Z_1(J) \Bigg|_{J=0}.$$

As already pointed out for the finite dimensional case, Wick's theorem is the main result for getting a mathematical notion of Feynman graphs out of the path integral. After formulating theorem A.2.5, we can, according to it, formulate a more general theorem, by using the derived formulae of the infinite dimensional case. We get the following formulation of Wick's theorem.

Theorem A.3.3 (Wick)**.** *Reformulating theorem A.2.5 in terms of the infinite dimensional case, we get*

$$\frac{\delta^m}{\prod_{j=1}^m \delta J(x_j)} \exp\left\{ \frac{1}{2} \langle J, GJ \rangle \right\} \Bigg|_{J=0} = \sum_W \prod_{j=1}^{m-1} G(x_{i_j}, x_{i_{j+1}}),$$

where the sum is over the index set $W = \{(i_j, i_{j_1}) \mid 1 \leq j \leq m-1\}$.

By the adding of a potential, we can also make sense of the formulae of the finite dimensional case, using the more general partition function, for the infinite dimensional setting, meaning that we consider the path integral

$$Z_1(U) = \int_{\mathcal{F}(M)} \exp\left\{ -\frac{1}{2} \langle K\varphi, \varphi \rangle + \hbar U(\varphi) \right\} \mathscr{D}(\varphi)$$

and the corresponding formula for (A.5) is given by

$$Z_1(U) = Z_1(0) \exp\left\{ \hbar U\left(\frac{\delta}{\delta J} \right) \right\} \exp\left\{ \frac{1}{2} \langle J, GJ \rangle \right\} \Bigg|_{J=0}.$$

Using theorem A.3.3, we can formulate $Z_1(U)$ in terms of Feynman graphs as

$$Z_1(U) = \sum_{\Gamma \in \mathcal{G}^0} \frac{\hbar^{|\Gamma|}}{|Aut(\Gamma)|} \int_{\mathscr{L}} \prod_v U_v \prod_e G_e,$$

where $\mathscr{L}$ is the set of all labelings of the ends of edges, $U_v = U(x_1, \ldots, x_k)$ for a k-valent vertex with the labels $x_1, \ldots, x_k$ of the adjacent edges and $G_e = G(x_i, x_j)$ for an edge with labels x_i and x_j.

A.4 The Feynman propagator and Feynman rules

After developing the mathematical theory of path integrals, we want to see how the Feynman rules arise as physical properties, such as conservation principles for the momentum space, and look at Feynman graphs from another point of view. Moreover, we want to use from now on the Einstein summation convention, meaning that summation over repeated indices is understood. Most of this section is from [PS], [ZJ] and [HO]. In this section we set $\hbar = 1$ and we consider the space-time manifold $M = \mathbb{R}^4$. Let us also denote simply by dx the 4-dimensional Lebesgue measure for the coordinate space of M. Moreover, we will use the Minkowski metric $\eta = diag(-1, 1, \ldots, 1)$. Let us now consider the Lagrangian of a free scalar field theory, which is given by

$$L_0(\varphi) = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2.$$

The corresponding action functional is then given by

$$S_0(\varphi) = \int_M L_0(\varphi)dx = -\int_M \frac{1}{2}\varphi(x)\Delta\varphi(x)dx,$$

where we have set $\Delta = -\Box + m^2 = \partial_t^2 - \nabla^2 + m^2$. Let us work out the details for the path integral

$$Z^0(J) = \int_{\mathcal{F}(M)} \exp\left\{ iS_0(\varphi) + i\int_M J(x)\varphi(x)dx \right\} \mathscr{D}(\varphi).$$

We can obtain, by the rules of the functional derivative, that

$$\frac{\delta S_0(\varphi)}{\delta\varphi(x)} = -\Delta\varphi(x).$$

Using $\tilde{\varphi}(x) = \varphi(x) - \Delta^{-1}J(x)$, we obtain that

$$S_0(\varphi) + \int_M J(x)\varphi(x)dx = S_0(\tilde{\varphi}) + \frac{1}{2}\int_M J(x)\Delta^{-1}J(x)dx.$$

Moreover, we can compute

$$\begin{aligned} S_0(\tilde{\varphi}) = -\frac{1}{2}\int_M \big(\varphi(x)\Delta\varphi(x) - \varphi(x)J(x) + J(x)\Delta^{-1}J(x) - \Delta^{-1}J(x)\Delta\varphi(x)\big)\, dx \\ = -\frac{1}{2}\int_M \big(\varphi(x)\Delta\varphi(x) - 2\varphi(x)J(x) + J(x)\Delta^{-1}J(x)\big)\, dx. \end{aligned}$$

Therefore, choosing the normalization $Z_1(0) = 1$, we get

$$Z^0(J) = \exp\left\{ \frac{i}{2}\int_M J(x)\Delta^{-1}J(x)dx \right\}.$$

In order to describe Δ^{-1}, we need to note that Δ is a differential operator and thus its inverse is a Green's function. The problem can be therefore reformulated by solving the equation

$$-\Delta_x\mathscr{F}(x-y) = \delta(x-y). \tag{A.10}$$

To solve this equation, we need to use Fourier transforms. The Fourier transform into the momentum space of the Green's function is then given by

$$\hat{\mathscr{F}}(p) = \int_M \exp\{-i\langle p, x\rangle\}\mathscr{F}(x)dx = \int_M \exp\{-i\langle p, (x-y)\rangle\}\mathscr{F}(x-y)dx,$$

where we use the standard inner product $\langle p, x\rangle = p_i x^i$ on M. If we multiply the above equation with $\exp\{-i\langle p, (x-y)\rangle\}$ and integrate over all x, we get

$$-\int_M \exp\{-i\langle p, (x-y)\rangle\}\Delta_x\mathscr{F}(x-y)dx = \int_M \delta(x-y)\exp\{-i\langle p, (x-y)\rangle\}\, dx = 1.$$

By replacing the operator Δ_x by its definition, we integrate the left hand side twice and drop the surface terms to obtain

$$-\int_M \Delta_x \left(\exp\{-\langle p,(x-y)\rangle\}\right)\mathscr{F}(x-y)dx = -\int_M (p^2+m^2)\exp\{-i\langle p,(x-y)\rangle\}\mathscr{F}(x-y)dx = 1.$$

Hence we obtain now the equation

$$-(p^2+m^2)\hat{\mathscr{F}}(p) = 1.$$

This differential equation is not uniquely determined, if we don't require some boundary conditions. However, we can define the *Feynman propagator*, out of this equation, to be

$$\hat{\mathscr{F}}(p) = -\frac{1}{p^2+m^2-\mathrm{i}\varepsilon}.$$

More precisely, we consider the distribution obtained in the limit $\varepsilon \to 0$. Let us now consider the more general situation of a interacting scalar field theory. Therefore let the Lagrangian be given by

$$L(\varphi) = L_0(\varphi) - U(\varphi),$$

with a corresponding action

$$S(\varphi) = \int_M L(\varphi)dx.$$

Now considering the same procedure as above, we are able to describe the path integral

$$Z(J) = \int_{\mathcal{F}(M)} \exp\left\{\mathrm{i}S(\varphi) + \mathrm{i}\int_M J(x)\varphi(x)dx\right\}\mathscr{D}(\varphi)$$

by the computation

$$\begin{aligned}
Z(J) &= \exp\left\{-\mathrm{i}\int_M U\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)dxZ^0(J)\right\}\\
&= \exp\left\{-\mathrm{i}\int_M U\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)dx\right\}\exp\left\{-\frac{\mathrm{i}}{2}\int_M J(x)\mathscr{F}(x-y)J(y)dxdy\right\}\\
&= \exp\left\{\frac{\mathrm{i}}{2}\frac{\delta}{\delta\varphi(x)}\mathscr{F}(x-y)\frac{\delta}{\delta\varphi(y)}dxdy\right\}\exp\left\{-\mathrm{i}\int_M (U(\varphi(x)) - J(x)\varphi(x))dx\right\}\Bigg|_{\varphi=0},
\end{aligned}$$

where we have used the relation

$$-\mathrm{i}\frac{\delta}{\delta J(x)}\exp\left\{\mathrm{i}\int_M J(x)\varphi(x)dx\right\} = \varphi(x)\exp\left\{\mathrm{i}\int_M J(x)\varphi(x)dx\right\}$$

of the functional derivative and the following lemma.

Lemma A.4.1. *Let G and F be two functionals. Then we have the equation*

$$G\left(-\mathrm{i}\frac{\delta}{\delta J}\right)F(\mathrm{i}J) = F\left(\frac{\delta}{\delta\varphi}\right)G(\varphi)\exp\left\{\mathrm{i}\int_M xJ(x)\varphi(x)dx\right\}\Bigg|_{\varphi=0}.$$

To obtain the Feynman rules for the space-time coordinates, one has to do a perturbative expansion of this path integral. Therefore, we can state the Feynman rules for space-time coordinates as follows.

Theorem A.4.2 (Feynman rules for space-time)**.** *The Feynman rules for space-time coordinates are* *described as follows.*

(i) *A line between two points x and y (which is an edge* $e = (x, y)$*) represents a propagator* $\mathrm{i}\mathscr{F}(x-y)$ *(see figure (A.4.1)).*

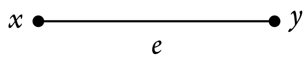


Figure A.4.1: The edge between x and y

(ii) *An n-valent vertex represents a factor* $-\mathrm{i}U^{(n)}(0)$*, where* $U^{(n)}(\varphi) = \frac{\delta^n U(\varphi)}{\delta \varphi^n}$ *(see figure (A.4.2)).*

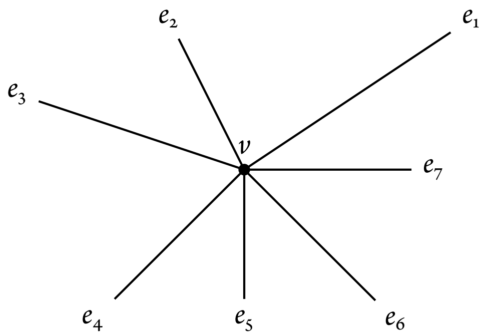


Figure A.4.2: Example with $n = 7$ edges

(iii) *A univalent vertex is represented by* $\mathrm{i}J(x)$ *(see figure (A.4.3)).*

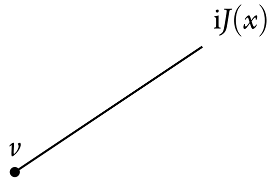


Figure A.4.3: Univalent vertex with one outgoing line

(iv) *To consider all vertices of the given graph, one has to integrate over space coordiantes.*

The use of the Feynman rules for space-time coordinates is sometimes a bit harder, since the appearing integrals can be hard to compute. Therefore, one can use the corresponding Feynman rules for the *momentum space coordinates*, while using physical aspect, such as *momentum conservation*. For the description of this slightly different rules, we want to consider the Fourier transform of the m-point correlation function

$$F(p_1, \ldots, p_n) := \int_M \langle \varphi(x_1), \ldots, \varphi(x_m) \rangle \exp\{\mathrm{i}\langle p, x \rangle\}\, dx,$$

where we obtain a delta function of the sum of all momentum coordinates $\delta\left(\sum_i p_i\right)$. Moreover, formulating the Feynman propagator with its Fourier inverse transform, we can write

$$\mathrm{i}\mathscr{F}(x-y) = \int_M \exp\{\mathrm{i}\langle p, (x-y)\rangle\} \left(-\frac{\mathrm{i}}{p^2 + m^2 - \mathrm{i}\varepsilon}\right) \frac{dp}{(2\pi)^{\dim M}}.$$

If we consider the rule, telling us that we need to integrate over each space coordinate for all vertices, we can again obtain a delta function. Now using the properties of the functional derivative, we are able to reformulate the Feynman rules in terms of momentum space coordinates. Moreover, let us use the convention that internal momenta are described by the letter ϱ and the outgoing momenta by the usual letter p.

Theorem A.4.3 (Feynman rules for the momentum space I)**.** *The Feynman rules for the momentum space coordinates are described as follows.*

(*i*) *An internal line with associated momentum ϱ represents a propagator* $-\frac{\mathrm{i}}{\varrho^2+m^2-\mathrm{i}\varepsilon}$.

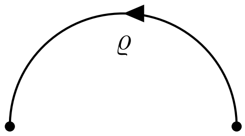


Figure A.4.4: An internal edge with associated momentum ϱ

(*ii*) *An external line with associated momentum p represents a propagator* $-\frac{\mathrm{i}}{p^2+m^2}$.

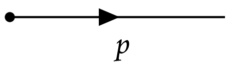


Figure A.4.5: An external edge with associated momentum p

(*iii*) *An n-valent vertex represents a factor*

$$-\mathrm{i}U^{(n)}(0)(2\pi)^{\dim M}\delta\left(\sum_i p_i\right).$$

(*iv*) *To consider all internal lines, we need to integrate over the momenta with respect to the measure*

$$\frac{d\varrho_i}{(2\pi)^{\dim M}}.$$

Example A.4.1 (Loop). Let us look at the principle of momentum conservation in this context with a simple example of an internal loop diagram. Consider therefore the Feynman diagram given by figure (A.4.6).

The important thing is that the incoming momentum p is the sum of the two outgoing momenta ϱ_1 and ϱ_2 and further since they need to fulfill the momentum conservation for v_1, that is we have no loss of momentum $(p_1 - \varrho_1 - \varrho_2 = 0)$, we get a delta function of the form $\delta(p_1 - \varrho_1 - \varrho_2)$. Therefore, by the same argument, we get a delta function on v_2 as $\delta(\varrho_1 + \varrho_2 - p_2)$. Integrating over all internal momenta, we get the final delta function $\delta(p_1 - p_2)$. Let us emphasize a formula due to Euler, which basically gives a relation between the number of loops, the number of internal edges and the number of vertices of a connected diagram. The formula is given by

$$|L_\Gamma| = |\tilde{E}_\Gamma| - |V_\Gamma| + 1,$$

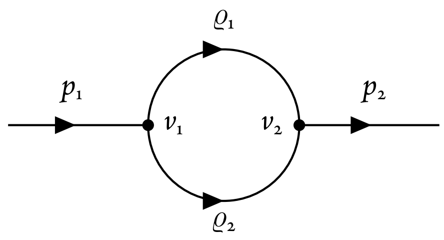


Figure A.4.6: Internal loop diagram. Note the importance of the orientation for momentum conservation.

where L_Γ is the set of all loops of the graph Γ and $\tilde{E}_\Gamma$ the set of all internal edges and V_Γ the set of all vertices of the graph Γ. By the Feynman rules, we get that the corresponding integral is given by

$$\int_M \prod_{v=1}^{|V_\Gamma|} (2\pi)^{\dim M} \prod_{e=1}^{|\tilde{E}_\Gamma|} \frac{d\varrho_i}{(2\pi)^{\dim M}} \delta\left(\sum_i p_{i,v}\right). \tag{A.11}$$

The momentum conservation is imposed inside this formula for all edges incident at each vertex v. Hence, considering only connected graphs, we can use Euler's relation to obtain by the delta function a modification of formula (A.11), which is given as

$$\int_M (2\pi)^{\dim M} \delta\left(\sum_i p_i\right) \prod_{\ell=1}^{|L|} \frac{d\varrho_\ell}{(2\pi)^{\dim M}}.$$

One can notice that there are some factors i according to the Feynman rules, that is that for each internal edge there is a factor of $(-\mathrm{i})$ and also for each vertex, which means that, by Euler's relation, we have a total

$$(-\mathrm{i})^{|E_\Gamma|}(-\mathrm{i})^{|V_\Gamma|} = (-1)^{|V_\Gamma|}\mathrm{i}^{1-|L|}.$$

With this condition, we can easily simplify the Feynman rules for momentum coordinates.

Theorem A.4.4 (Feynman rules for the momentum space II)**.** *The Feynman rules can now be described as follows.*

(i) *An internal line with associated momentum ϱ represents a propagator $\frac{1}{\varrho^2+m^2-\mathrm{i}\varepsilon}$.*

(ii) *An external line with associated momentum p represents a propagator $-\frac{\mathrm{i}}{p^2+m^2}$.*

(iii) *An n-valent vertex represents a factor $-U^{(n)}(0)$.*

(iv) *Add a factor $-\mathrm{i}$ for each loop and integrate over undetermined loop momenta*

$$\int_M \frac{d\varrho}{(2\pi)^{\dim M}}.$$

(v) *There is always an factor of $\mathrm{i}(2\pi)^{\dim M}$ times a delta function, imposing the momentum conservation.*

We want to look at some straightforward examples of the usage of those rules for a given diagram, i.e. we will compute the amplitudes by using the Feynman rules. Moreover, we will also look at two examples of diagrams appearing in quantum electrodynamics (QED) and in quantum chromodynamics (QCD) (see example A.4.5 and A.4.6 respectively), which are two important subfields of quantum field theory. We will start with a simple example.

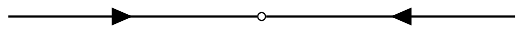


Figure A.4.7: straight momentum conservation. The white circle indicates that there is no vertex.

Example A.4.2 (straight line momenta). Let us consider the diagram given by figure (A.4.7).
Then the corresponding amplitude is, by the Feynman rules, given by

$$\frac{1}{p_1^2 + m^2}(2\pi)^{\dim M}\delta(p_1 + p_2).$$

Remark A.4.1. For diagrams with loops there is the possibility that we end up with divergent integrals.

Example A.4.3 (Internal loop diagram I). Let us consider the diagram given as in figure (A.4.8).

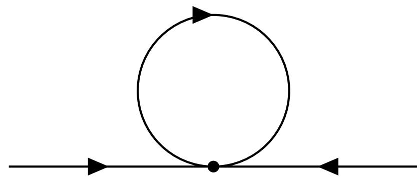


Figure A.4.8: Internal loop diagram I

The corresponding amplitude is then, by the Feynman rules, given by

$$\frac{(-\mathrm{i})^2\mathrm{i}(2\pi)^{\dim M}\delta(p_1 + p_2)}{2(p_1^2 + m^2)(p_2^2 + m^2)}\left\{-U^{(4)}(0)\right\}(-\mathrm{i})\int_M \frac{1}{\varrho^2 + m^2 - \mathrm{i}\varepsilon}\frac{d\varrho}{(2\pi)^{\dim M}}.$$

Note that we can add a factor of 2, because of the symmetry of the diagram. Thus we get

$$\frac{U^{(4)}(0)\delta(p_1 + p_2)}{2(p_1^2 + m^2)(p_2^2 + m^2)}\int_M \frac{d\varrho}{\varrho^2 + m^2 - \mathrm{i}\varepsilon}.$$

Example A.4.4 (Internal loop diagram II). Let us consider the diagram given as in figure (A.4.6). Then the correpsonding amplitude is, by the Feynman rules, given by

$$\frac{(-\mathrm{i})^2\mathrm{i}(2\pi)^{\dim M}\delta(p_1 + p_2)}{2(p_1^2 + m^2)(p_2^2 + m^2)}\left\{-U^{(3)}(0)\right\}^2(-\mathrm{i})\int_M \frac{1}{(\varrho^2 + m^2 - \mathrm{i}\varepsilon)((\varrho - p_1)^2 + m^2 - \mathrm{i}\varepsilon)}\frac{d\varrho}{(2\pi)^{\dim M}}.$$

Again we can use the symmetry factor 2. Thus we get

$$-\frac{\left\{U^{(3)}(0)\right\}^2\delta(p_1 + p_2)}{2(p_1^2 + m^2)(p_2^2 + m^2)}\int_M \frac{d\varrho}{(\varrho^2 + m^2 - \mathrm{i}\varepsilon)((\varrho - p_1)^2 + m^2 - \mathrm{i}\varepsilon)}.$$

Example A.4.5 (QED). An example of a Feynman diagram in physics, in particular in QED, is given in figure (A.4.9), where we consider an electron e^- and a positron e^+ interacting to a muon μ^- and an antimuon μ^+. The waved line is often used to represent a photon γ, which is emitted or absorbed by the particles.

Again, one can use the Feynman rules for the momentum space, by considering the external momenta of all four particles representing the external lines and the internal momentum of the photon representing the internal line, to compute the amplitude of this diagram. Even if the mathematical properties are the

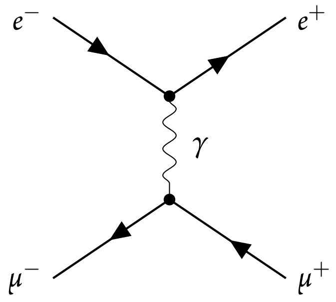


Figure A.4.9: Example of a diagram in QED: $e^+e^- \to \mu^+\mu^-$

same as before, there are slightly different rules considered in QED (see [ZJ]) but we will not go into more detail on that.

Example A.4.6 (QCD). Another example of a Feynman diagram in physics, in particular QCD, is given in figure (A.4.10), where we represent a particle interacting by the emission or absorption of gluons, wich are represented by the curly lines and again, the waved lines are representing photons. This is also a nice example of an internal loop diagram, where ϱ_1 and ϱ_2 are the internal momenta of the particle. Again, one can compute the amplitude with the Feynman rules for the momentum space, but with slight differences used for QCD (see [ZJ]).

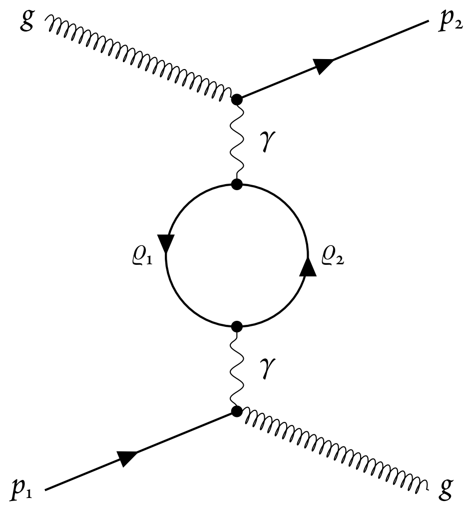


Figure A.4.10: Example of a diagram in QCD

Feynman diagrams are an important tool for physicists to describe the amplitude of certain quantum systems, which leads to a probability of the system and the Feynman rules are used to get them. The mathematical interest lies in the relation between those diagrams and the corresponding path integral for a given Lagrangian density mostly because of the aim of a rigorous formulation for infinite dimensional structures, which is an obstacle for mathematicians, speaking about the fact that the path integral is defined over a mathematically ill defined measure $\mathscr{D}$, which we have pointed out several times. As we have seen, it is still possible to get a mathematical formalism, considering the finite dimensional structure and deal with well known mathematical objects to derive formulae, which don't appeal to the measure $\mathscr{D}$ if we transfer them to the infinite dimensional case. Moreover, many mathematical physicists use the notion of path integrals and Feynman diagrams in order to relate to quantum field theory if the mathematics is going to get to formal, where one can easily lose the ground of physics, such that having a path integral or equivalently a Feynman diagram included, it is definitely related to physics. More can be found in [JD], [SW] and [FR].

B

Elements of Supergeometry

B.1 Linear superspaces and properties

We want to extend some of the usual structures to this special setting (see [L] for more details). The first structure to reformulate in this setting is that of a vector space.

Definition B.1.1 (Linear superspace). *Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denote the field of residues modulo 2. A linear space $\mathscr{M}$ is called a superspace if it admits a decomposition*

$$\mathscr{M} = \mathscr{M}_{\bar{0}} \oplus \mathscr{M}_{\bar{1}}.$$

The elements of $\mathscr{M}_{\bar{0}}$ and $\mathscr{M}_{\bar{1}}$ are called homogeneous even and odd elements of $\mathscr{M}$ respectively.

Equivalently, we can think of a superspace as an ordinary vector space V endowed with an automorphism $\alpha : V \to V$ such that $\alpha^2 = id_V$. We say that, in this case, V_0 is the 1-eigenspace and V_1 is the (-1)-eigenspace. The *dimension* of a linear superspace is given as a pair $(\dim \mathscr{M}_{\bar{0}}, \dim \mathscr{M}_{\bar{1}})$. If we have that $\dim \mathscr{M}_{\bar{i}} = d_i$ for $i \in \{\bar{0}, \bar{1}\}$, then we say that $\mathscr{M}$ is a $(d_{\bar{0}}|d_{\bar{1}})$-dimensional linear superspace.

Example B.1.1 (Linear graded superspace). Let $\mathscr{M} = \bigoplus_{k\in\mathbb{Z}} \mathscr{M}_k$ be a $\mathbb{Z}$-graded vector space. Then one can consider $\mathscr{M}$ as a linear superspace with

$$\mathscr{M}_{\bar{0}} = \bigoplus_{k\in 2\mathbb{Z}} \mathscr{M}_k, \quad \mathscr{M}_{\bar{1}} = \bigoplus_{k\in 2\mathbb{Z}+1} \mathscr{M}_k$$

If $(\mathscr{M}_\bullet, \partial)$ is a chain complex with a degree 1 differential, then ∂ can be seen as a map

$$\begin{cases} \partial : \mathscr{M}_{\bar{0}} \to \mathscr{M}_{\bar{1}} \\ \partial : \mathscr{M}_{\bar{1}} \to \mathscr{M}_{\bar{0}} \end{cases}$$

Let now $\mathscr{M}$ and $\mathscr{N}$ be two linear superspaces. A morphism $\phi : \mathscr{M} \to \mathscr{N}$ between two superspaces is a linear map, which preserves the grading, i.e. if $\alpha_{\mathscr{M}}$ and $\alpha_{\mathscr{N}}$ are the automorphisms of the linear superspaces respectively as before, then the diagram

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\alpha_{\mathscr{M}}} & \mathscr{M} \\ \phi \big\downarrow & & \big\downarrow \phi \\ \mathscr{N} & \xrightarrow[\alpha_{\mathscr{N}}]{} & \mathscr{N} \end{array}$$

commutes. Moreover, let us denote the category of vector spaces over a field K by $\mathbf{Vect}_K$ and the category of linear superspaces by $\mathbf{SuperVect}_K$. Let us define a map Π on the linear superspace $(\mathscr{M}, \alpha_{\mathscr{M}}) \in \mathbf{SuperVect}_K$, where the tupel $(\mathscr{M}, \alpha_{\mathscr{M}})$ represents a linear superspace with its automorphism map $\alpha_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$, satisfying $\alpha^2_{\mathscr{M}} \equiv 0$, such that

$$\Pi : (\mathscr{M}, \alpha_{\mathscr{M}}) \mapsto (\mathscr{M}, -\alpha_{\mathscr{M}}),$$

meaning that $(\Pi\mathscr{M})_{\bar{0}} = \mathscr{M}_{\bar{1}}$ and $(\Pi\mathscr{M})_{\bar{1}} = \mathscr{M}_{\bar{0}}$. Therefore Π is an endofunctor on the category **SuperVect**. It is important to look at the case of a vector space, that is if V is a vector space, we can write $V[1] = 0 \oplus V[1] = (V, -id_V) = \Pi V$, where $V[1]$ represents the odd shift of V. Let $\mathscr{M}$ and $\mathscr{N}$ be two linear superspaces. Then we have a natural tensor product on the category **SuperVect**, given as

$$\mathscr{M} \otimes \mathscr{N} = (\mathscr{M} \otimes \mathscr{N})_{\bar{0}} \oplus (\mathscr{M} \otimes \mathscr{N})_{\bar{1}},$$

where $(\mathscr{M} \otimes \mathscr{N})_{\bar{0}} = (\mathscr{M}_{\bar{0}} \otimes \mathscr{N}_{\bar{0}}) \oplus (\mathscr{M}_{\bar{1}} \otimes \mathscr{N}_{\bar{1}})$ and $(\mathscr{M} \otimes \mathscr{N})_{\bar{1}} = (\mathscr{M}_{\bar{0}} \otimes \mathscr{N}_{\bar{1}}) \oplus (\mathscr{M}_{\bar{1}} \otimes \mathscr{N}_{\bar{0}})$, which can be equivalently reduced to $\alpha_{\mathscr{M}\otimes\mathscr{N}} = \alpha_{\mathscr{M}} \otimes \alpha_{\mathscr{N}}$. Moreover, we can define a map, called the *braiding* as

$$\begin{aligned} \sigma_{\mathscr{M},\mathscr{N}} : \mathscr{M} \otimes \mathscr{N} &\longrightarrow \mathscr{N} \otimes \mathscr{M} \\ x \otimes y &\longmapsto (-1)^{deg^{\mathscr{M}}(x) deg^{\mathscr{N}}(y)} (y \otimes x), \end{aligned}$$

where $deg^{\mathscr{M}} : \mathscr{M} \to \{\bar{0}, \bar{1}\}$ is the degree map for an element in $\mathscr{M}$. That is, for $x \in \mathscr{M}_k$ we get $deg(x) = k$, where $k \in \{\bar{0}, \bar{1}\}$ since $\mathscr{M} = \mathscr{M}_{\bar{0}} \oplus \mathscr{M}_{\bar{1}}$. In fact we have a natural embedding

$$\begin{aligned} \mathbf{Vect} &\longrightarrow \mathbf{SuperVect} \\ V &\longmapsto (V, id_V) \end{aligned}$$

This implies that the n-th symmetric power of V is given by

$$S^n(V) = (V^{\otimes n})_{S_n} = V^{\otimes n} \Big/ \{ \textstyle\bigotimes_{k=1}^n x_k - \sigma\left(\bigotimes_{k=1}^n x_k\right) \mid \sigma \in S_n \} \cdot$$

Therefore we can note that

$$S^n(\mathscr{M}_{\bar{0}} \oplus 0) = S^n(\mathscr{M}_{\bar{0}}), \qquad S^n(0 \oplus \mathscr{M}_{\bar{1}}) = \bigwedge^n \mathscr{M}_{\bar{1}},$$

and for a general linear superspace $\mathscr{M} = \mathscr{M}_{\bar{0}} \oplus \mathscr{M}_{\bar{1}}$ we can obtain

$$S^n(\mathscr{M}) = S^n(\mathscr{M}_{\bar{0}} \oplus \mathscr{M}_{\bar{1}}) = \bigoplus_{k=0}^n \left(S^k(\mathscr{M}_{\bar{0}}) \otimes \bigwedge^{n-k} \mathscr{M}_{\bar{1}} \right).$$

Let again $\mathscr{M}$ be a linear superspace. Then we denote by $\mathscr{M}^*$ its dual and we actually get that $a_{\mathscr{M}^*} = (a_{\mathscr{M}})^*$. This implies that the even linear functionals on $\mathscr{M}$ are those functionals that are zero on odd vectors and odd linear functionals on $\mathscr{M}$ are those functionals that are zero on even vectors. Let us define the space of formal series (or regular functions) on $\mathscr{M}$ by

$$\mathcal{R}(\mathscr{M}) = \varinjlim_n S^n(\mathscr{M}^*).$$

Let now $\mathscr{M}$ be a $(d_{\bar{0}}|d_{\bar{1}})$-dimensional linear superspace and let $\{e_1, \ldots, e_{d_{\bar{0}}}\}$ be a basis of $\mathscr{M}_{\bar{0}}$ and $\{f_1, \ldots, f_{d_{\bar{1}}}\}$ be a basis of $\mathscr{M}_{\bar{1}}$. Then $\{x^1, \ldots, x^{d_{\bar{0}}}, \theta^1, \ldots, \theta^{d_{\bar{1}}}\}$ is a basis of $\mathscr{M}^*$, where the x^i are even and the θ^k are odd. The linear functionals x^i are called the *even coordinates* on $\mathscr{M}$ and the functionals θ^k are called the *odd coordinates*. If we consider, for example, the space $S^2(\mathscr{M}^*)$, we get that

$$\begin{aligned} x^i x^j &= x^j x^i \\ x^i \theta^k &= \theta^k x^i \\ \theta^k \theta^l &= -\theta^l \theta^k \end{aligned}$$

where we can see that the x^i are commutating and the θ^k are anticommuting variables. Thus, if moreover $\mathscr{M} \in \mathbf{SuperVect}_K$, where K is a field, then

$$\mathcal{R}(\mathscr{M}) = K\left[\left[x^1, \ldots, x^{d_{\bar{0}}}\right]\right] \wedge \theta^1 \wedge \cdots \wedge \theta^{d_{\bar{1}}}$$

meaning that the regular functions are formal power series for the dual variables with coefficients in K.

Recall that a usual Lie algebra is actually a vector space $\mathfrak{g}$ endowed with a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. Let us consider the Lie bracket as a map

$$[\cdot,\cdot] : S^2(\Pi\mathfrak{g}) \to \Pi\mathfrak{g},$$

with Π being the endofunctor on **SuperVect** as defined before and let $[\cdot,\cdot]^* : \Pi\mathfrak{g}^* \to S^2(\Pi\mathfrak{g}^*)$ be its dual map. Since the space $S^2(\Pi\mathfrak{g}^*)$ has an embedding into the space $\mathcal{R}(\Pi\mathfrak{g})$, we can think of $[\cdot,\cdot]^*$ as a map $\Pi\mathfrak{g}^* \to \mathcal{R}(\Pi\mathfrak{g})$. If we now define a map $\delta : \mathcal{R}(\Pi\mathfrak{g}) \to \mathcal{R}(\mathfrak{g})$ by requiring it to satisfy the Leibniz rule

$$\delta(\phi_1\phi_2) = \delta(\phi_1)\phi_2 + (-1)^{deg(\phi)}\phi_1\delta(\phi_2),$$

where *deg* is again the degree map, we get an extension of $[\cdot,\cdot]^*$ to δ, which is then a degree 1 derivative and moreover δ is a differential, i.e. $\delta^2 = 0$. Indeed, let $\phi_1, \ldots, \phi_n \in \Pi\mathfrak{g}^*$. Then

$$\delta\left(\prod_{j=1}^n \phi_j\right) = \sum_{0\leq k\leq n} (-1)^{k-1} \prod_{\substack{0\leq j\leq n \\ \delta(\phi_j)=\phi_j \text{ for } k\neq j \\ \delta(\phi_j)=\delta(\phi_j) \text{ for } j=k}} \delta(\phi_j).$$

Thus we get

$$\begin{aligned} \delta^2\left(\prod_{j=1}^n \phi_j\right) &= \delta^2(\phi_1)\phi_2\cdots\phi_n - \delta(\phi_1)\delta(\phi_2)\cdots\phi_n + \cdots + (-1)^{n-1}\delta(\phi_1)\phi_2\cdots\phi_2\cdots\delta(\phi_n) \\ &\quad + \delta(\phi_1)\delta(\phi_2)\cdots\phi_n + \phi_1\delta^2(\phi_2)\cdots\phi_n + \cdots + \phi_1\phi_2\cdots\delta(\phi_n) \\ &= \delta^2(\phi_1)\phi_2\cdots\phi_n + \phi_1\delta^2(\phi_2)\cdots\phi_n + \cdots + \phi_1\phi_2\cdots\delta^2(\phi_n), \end{aligned}$$

therefore we only need to show that $\delta^2(\phi) = 0$ for any $\phi \in \Pi\mathfrak{g}^*$. We know that $\delta \mid_{\Pi\mathfrak{g}^*} = [\cdot, \cdot]^*$ and thus

$$\langle \delta(\phi), g_1 \wedge g_2 \rangle = \langle \phi, [g_1, g_2] \rangle,$$

and hence

$$\langle \delta^2(\phi), g_1 \wedge g_2 \wedge g_3 \rangle = \left\langle \phi, \sum_{(i,j,k)=(1,2,3)} [[g_i, g_j], g_k] \right\rangle,$$

where $\sum_{(i,j,k)=(1,2,3)}$ represents the cyclic sum for the indices i, j, k. This means that the condition $\delta^2 = 0$ is identical to the Jacobi identity of $[\cdot, \cdot]$, which is clearly satisfied.

B.2 Superdomains and Supermanifolds

We want to describe the notion of superdomains and supermanifolds with the language of algebraic geometry, meaning that we will use the notion of structure sheaves and ringed spaces. Let us start with the usual definitions right away. Let X be a topological space. Suppose that to each subset $U \subset X$ there is a set $\mathscr{F}(U)$ assigned to it and that for any open subsets $U \subset V$ there is a restriction mapping $r^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$ such that $r^V_U \circ r^W_V = r^W_U$ for any chain of open subsets $U \subset V \subset W$. Moreover, $\mathscr{F}(\varnothing)$ is a singleton and r^U_U is the identity mapping.

Definition B.2.1 (Sheaf). *A family $\mathscr{F}$ of sets $\mathscr{F}(U)$ and mappings r^V_U is called a* sheaf *if for any collection $\{U_i\}_{i\in I}$, for some index set I, of open sets in X with $U = \bigcup_{i\in I} U_i$ the following are satisfied.*

(*i*) *If $\xi, \eta \in \mathscr{F}(U)$ and $r^V_{U_i}(\xi) = r^V_{U_i}(\eta)$, then $\xi = \eta$ for all $i \in I$, where $U \subset V$,*

(*ii*) *If $\xi_i \in \mathscr{F}(U_i)$ and $r^V_{U_i\cup U_j}(\xi_i) = r^V_{U_i\cup U_j}(\xi_j)$ for all $i, j \in I$, then there is a $\xi \in \mathscr{F}(U)$ such that $r^V_{U_i}(\xi) = \xi_i$ for all $i \in I$, where $U \subset V$.*

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over X. Then a morphism $h : \mathscr{F} \to \mathscr{G}$ is a collection of mappings $h_U : \mathscr{F}(U) \to \mathscr{G}(U)$, one for each open $U \subset X$, such that $r^V_U h_U = h_U r^V_U$ for any $U \subset V$ of X. If all the h_U are inclusions, then $\mathscr{F}$ is called a subsheaf of $\mathscr{G}$. If all the $\mathscr{F}$ are groups or modules (or superalgebras or...) and the r^V_U are homeomorphisms of these structures, then $\mathscr{F}$ is called a sheaf of groups or modules (or superalgebras or...). The restriction of a sheaf $\mathscr{F}$ over X to a sheaf over an open subset $U \subset X$ is defined in the obvious way and denoted by $\mathscr{F} \mid_U$.

Definition B.2.2 (Ringed spaces and morphisms). *A* ringed space *is a pair $(X, \mathscr{F})$, where X is a topological space and $\mathscr{F}$ is a sheaf of rings over X. If $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ are ringed spaces, a* morphism *$\phi : (X, \mathscr{F}) \to (Y, \mathscr{G})$ is a collection $(\tilde{\phi}, \phi^*_U)$, where $\tilde{\phi} : X \to Y$ is a continuous mapping and $\phi^*_U : \mathscr{G}(U) \to \mathscr{F}(\tilde{\phi}^{-1}(U))$ a homeomorphism of rings, one for each open subset U of Y, compatible with the restriction mappings, that is*

$$r^{\tilde{\phi}^{-1}(V)}_{\tilde{\phi}^{-1}(U)} \circ \phi^*_V = \phi^*_U \circ r^V_U.$$

Example B.2.1 (Structure sheaf on a manifold). Let $k = \mathbb{R}$ and let $\mathscr{M}$ be a Hausdorff space with a countable basis of its topology and locally homeomorphic to an open subset k^m. suppose that every point $p \in \mathscr{M}$ has an open subset of k^m such that for any open set $U \subset V$ the function $f : U \to k$ is smooth on $h_V(U)$. Let $U = \bigcup_{i\in I} U_i$, where the U_i are open subsets of $\mathscr{M}$. A mapping $f : U \to k$ is called a *smooth function* if $f\mid_{U_i}$ is smooth for all $i \in I$. A ringed space $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ is called a *smooth manifold of dimension m* if $\mathscr{O}_{\mathscr{M}}(U)$ is an algebra of smooth functions on $\mathscr{M}$.

Definition B.2.3 (Superspace and Superdomain). *Let $(k^p, \mathscr{O}_{k^p})$ be a ringed space. A ringed space*

$$\mathscr{K}^{p,q} = (k^p, \mathscr{O}_{\mathscr{K}^{p,q}}),$$

where $\mathcal{O}_{\mathcal{K}^{p,q}} = \mathcal{O}_{k^p} \otimes \bigwedge(q)$*, is called a* smooth superspace. *Here* $\bigwedge(n)$ *denotes the Grassmann algebra in n variables. We say functions on* $\mathcal{K}^{p,q}$ *are functions on* k^p *with values in* $\bigwedge(q)$*. Here* k^p *is called the* underlying space *of* $\mathcal{K}^{p,q}$*. Let U be a domain in* k^p*. A ringed space*

$$\mathcal{U}^{p,q} = (U, \mathcal{O}_{\mathcal{K}^{p,q}|_U})$$

is called a superdomain of dimension (p, q).

Finally, we are able to give the definition of a supermanifold.

Definition B.2.4 (Supermanifold I)**.** *A supermanifold* $\mathcal{M}$ *is a ringed space* $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$*, where* $\mathcal{O}_{\mathcal{M}}$ *is a sheaf of commutative superalgebras on* $\mathcal{M}$*, such that*

(*i*) $\mathcal{M}$ *is a Hausdorff space with a countable basis of its topology,*

(*ii*) *every point* $m \in \mathcal{M}$ *has a neighborhood U such that the ringed space* $(U, \mathcal{O}_{\mathcal{M}}(U))$ *is isomorphic to a superdomain* $\mathcal{U}$*.*

There is also a definition of a supermanifold, where one doesn't appeal to the notion of a sheaf. To state that definition, we need some other notions. Let $\mathcal{M}$ be a Hausdorff space with a countable basis of its topology. A chart on $\mathcal{M}$ is a pair $(\mathcal{U}, c)$, where $\mathcal{U}$ is a superdomain and $c : U \to \mathcal{M}$, usually identified with U. Let $(\mathcal{U}_1, c_1)$ and $(\mathcal{U}_2, c_2)$ be two charts with $c_1(U_1), c_2(U_2) \subset \mathcal{M}$ and $W = c_1(U_1) \cap c_2(U_2)$. We put $U_1' = c_1^{-1}(W) \subset U_1$, $U_2' = c_2^{-1}(W) \subset U_2$ and let $\gamma'_{U_1U_2} : U_1' \to U_2'$ denote the composite $c_2^{-1} \circ c_1$, which is clearly a homeomorphism. A *compatibility* between two charts $(\mathcal{U}_1, c_1)$ and $(\mathcal{U}_2, c_2)$ is an isomorphism of superdomains $\gamma'_{\mathcal{U}_1\mathcal{U}_2} : \mathcal{U}_2' \to \mathcal{U}_2'$ such that the underlying mapping $\tilde{\gamma}_{\mathcal{U}_1\mathcal{U}_2}$ coincides with $\gamma'_{U_1U_2}$.

Definition B.2.5 (Atlas)**.** *An* atlas *is a collection of charts* $\{(\mathcal{U}_\alpha, c_\alpha)\}$*, where α ranges over some indexing sets, and a set of compatibilities* $\gamma_{\alpha\beta} = \gamma_{\mathcal{U}_\alpha\mathcal{U}_\beta}$ *between* $(\mathcal{U}_\alpha, c_\alpha)$ *and* $(\mathcal{U}_\beta, c_\beta)$ *for all index pair* α, β *such that*

(*i*) *the set of* $c_\alpha(U_\alpha)$ *covers* $\mathcal{M}$*,*

(*ii*) *for all* α, β, δ *the composite* $\gamma_{\alpha\beta} \circ \gamma_{\beta\alpha} \circ \gamma_{\delta\alpha}$ *is the identity on the open superdomain in* $\mathcal{U}_\alpha$ *on which it is defined,*

(*iii*) $\gamma_{\alpha\alpha} : \mathcal{U}_\alpha \to \mathcal{U}_\alpha$ *is the identity for all α. In particular,* $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}^{-1}$.

Definition B.2.6 (Supermanifold II)**.** *A* super manifold *is a space together with an atlas on it.*

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