TQFTs in the BV-BFV Formalism and Deformation Quantization

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Abstract

This thesis consists of the collection of the papers [Mos21a; MS22; CMW21; Mos21b; Mos20; MM22]. The results mainly consider the construction of (global) perturbative *topological quantum field theories (TQFTs)* with symmetry (gauge theories) in the *BV-BFV formalism*, i.e. on manifolds with boundary, and also higher codimension methods. Some parts of the results consider the field of *deformation quantization* and its relation to some algebraic properties derived from field-theoretic concepts.

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Contents

Part I

Preface

Chapter 1

Introduction

1.1 Topological Quantum Field Theories (TQFTs)

This thesis mainly deals with quantum field theories of *topological* type. The notion of a quantum field theory can be understood in many different settings. Two of the most important notions are *perturbative* and *functorial* (see [Ati88] for a detailed discussion). In this thesis we are mainly interested in the perturbative approach.

1.1.1 Perturbative TQFTs

In the perturbative setting, we consider functional integrals of the form

$$Z = \int_{\phi \in F_M} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\phi)} \mathscr{D}[\phi], \tag{1.1.1}$$

where F_M denotes some "space of fields" depending on some "space–time" manifold M, S some function on F_M (usually called the *action functional*), i the imaginary unit, $\hbar$ some small parameter and $\mathscr{D}$ some (possibly ill-defined) measure[1] on F_M. If the space of fields F_M is e.g. infinite-dimensional (which is the typical case in the setting of quantum field theory), the measure $\mathscr{D}$ is not defined and hence it is not clear how such an object can be defined. The way out of this problem is usually to consider a *perturbative expansion* in terms of *Feynman diagrams* (see e.g. [FH65; Pol05]), i.e. to write

$$Z \approx \sum_{n \geq 0} \sum_{\Gamma} w_\Gamma \hbar^n,$$

where Γ denotes a Feynman graph, $w_\Gamma \in \mathbb{R}$ denotes the *weight* associated to Γ and $\approx$ means the perturbative expansion when $\hbar \to 0$. In this approach, one considers the method of *stationary phase expansion* to expand perturbatively around classical solutions of S (i.e. solutions to the Euler–Lagrange equations $\delta S = 0$) to obtain a formal power series in $\hbar$ with coefficients given by the *weights* of the Feynman diagrams. These weights can be computed in terms of integrals involving a *propagator* (a.k.a. *Green's function*,

[1] We will sometimes drop the measure $\mathscr{D}$ in the notation of such an integral whenever it is clear.

a.k.a. *integral kernel*) for the corresponding system. The action functional S is typically assumed to be local, i.e. of the form

$$S(\phi) = \int_M \mathscr{L}(\phi, \partial\phi, \dots, \partial^N \phi) \tag{1.1.2}$$

for some integer $N \in \mathbb{Z}_{>0}$, where $\mathscr{L}$ denotes a density on M called the *Lagrangian*. This means that $\mathscr{L}$ depends on the field ϕ and its higher derivatives. Let us give the definition of a topological field theory in this setting:

Definition 1.1.1 (Topological field theory)**.** A classical field theory, described by an action functional S of the form (1.1.2), is said to be *topological* if it is independent of reparametrization and has no local degrees of freedom. In particular, S is independent of any metric.

Definition 1.1.2 (Topological quantum field theory)**.** A *topological quantum field theory* is a functional integral Z of the form (1.1.1), where the actional functional S describes a classical topological field theory.

1.1.2 Gauge theories

If the classical system carries any symmetries, i.e. if the Lagrangian $\mathscr{L}$ is invariant under the action of a Lie group G, we speak of a *gauge theory*. In this case, the formula for the stationary phase expansion fails,[2] and one has to come up with more sophisticated methods. One of the first developed methods is the *Faddeev–Popov ghost method* [FP67]. There, one introduces new type of fields, the *ghost fields*, and considers a graded configuration space of fields in order to show that the integral Z reduces to an integral which is indeed well-defined. However, this method is only restricted to the case of certain type of theories (those where the gauge group acts linearly on the space of fields) and hence needs to be enhanced. Another important approach was given in the construction of BRS(T)[3] [BRS74; BRS75; BRS76; Tyu76]. There, one considers the methods of Faddeev–Popov in a *cohomological formalism* by formulating the information of the symmetry into a cohomological vector field Q. The gauge invariance then reduce to the fact that the action functional S is closed with respect to Q and the cohomology can be considered since $Q^2 = 0$. This leads to reducing Z to an integral of the form

$$\int e^{\frac{i}{\hbar}(S+Q(\Psi))},$$

where $Q(\Psi)$ is some Q-exact term and Ψ denotes an odd function on the space of fields F_M called the *gauge-fixing fermion*. Actually, it can then happen that even though the

[2] The reason is that in the stationary phase formula we have a term of the form $\frac{1}{\sqrt{\det(\partial^2 S)}}$. This means that the critical points of S around which we want to expand, all have to be isolated. Unfortunately, this is never the case for gauge theories since all critical points appear in G-orbits.

[3] BRST stands for Becchi, Rouet, Stora and Tyutin. The papers by Becchi–Rouet–Stora are independent from the work of Tyutin who developed the same formalism by himself. Thus, sometimes it is also written BRS(T) instead of BRST in order to emphasize this independence.

critical points of S are all non-isolated, the ones of $S+Q(\Psi)$ are all isolated and this is a good choice of gauge-fixing.
Nevertheless, this method is again only applicable if the theory is of linear nature. As soon as the theory gets more general both methods fail and need to be adapted. The formalism that is suitable to use for the general case was provided by Batalin and Vilkovisky and is called the the *Batalin–Vilkovisky (BV) formalism* [BV77; BV81; BV83]. There, one considers a different space of fields $\mathcal{F}_M$, which is now a $\mathbb{Z}$-graded supermanifold, instead of F_M, endowed with an odd symplectic form of degree -1. One also introduces additional *anti-fields* which can be seen as the momentum variables for the fields[4]. In combination with the cohomological methods of the BRST formalism and with the additional methods of symplectic geometry, this formalism has proven to be the most powerful formalism in order to treat gauge theories in the quantum setting. In particular, one can show that the integral

$$\int_{F_M} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$$

can be replaced with an integral

$$\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}},$$

where $\mathcal{S}$ denotes an even function on $\mathcal{F}_M$ of degree 0. The gauge-fixing consists in choosing a *Lagrangian submanifold* $\mathcal{L}\subset\mathcal{F}_M$. The main theorem of Batalin–Vilkovisky (see also the work of Schwarz [Sch93] for a more mathematical approach) states that under certain circumstances, the integral $\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$ is independent of the choice of the Lagrangian submanifold $\mathcal{L}$. In particular, if certain assumptions hold, we can replace

$$\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}} \longrightarrow \int_{\mathcal{L}'} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$$

whenever one can continuously deform the Lagrangian $\mathcal{L}$ to the Lagrangian $\mathcal{L}'\subset\mathcal{F}_M$. The assumption for this is called the *quantum master equation (QME)* and describes an algebraic closedness property for the function $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$ with respect to an operator Δ, called the *BV Laplacian*. This means that if the integral $\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$ is not well-defined for some Lagrangian submanifold $\mathcal{L}\subset\mathcal{F}_M$ (e.g. F_M), but the QME holds, we can continuously deform $\mathcal{L}$ to some Lagrangian $\mathcal{L}'$ for which the integral $\int_{\mathcal{L}'} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$ is well-defined without changing the value of the original integral. The BV formalism is restricted to work on closed space–time manifolds M. Nevertheless, it is important to also consider manifolds with boundary in order to simplify the gauge formalism on complicated manifolds M by using cutting–gluing techniques. Such a formalism was provided by Cattaneo, Mnev and Reshetikhin [CMR14; CMR17]. They used the construction of Batalin–Vilkovisky in the bulk and combined it with the Hamiltonian approach developed by Batalin, Fradkin and Vilkovisky (BFV formalism) on the boundary [BF83; BF86; FV75; FV77; FF78]. They formulated everything in a coherent setting in the classical as well as in the quantum case for the symplectic cohomological formalism. This perturbative gauge formalism on manifolds with boundary is known today as the *BV-BFV formalism* [CM20]. The BV-BFV formalism is the main formalism considered in this thesis.

[4]The BV formalism is sometimes also called *anti-field formalism*.

1.2 The Batalin–Vilkovisky (BV) formalism

1.2.1 Classical BV formalism

Definition 1.2.1 (Lagrangian field theory)**.** A d-dimensional *Lagrangian field theory* assigns to a d-manifold M a pair (F_M, S_M), where F_M is some *space of fields*[5] and $S_M \in C^\infty(F_M)$ is a function on F_M, called the *action (functional).* We say that a Lagrangian field theory is *local* if we can express S_M as

$$S_M(\phi) = \int_M \mathscr{L}(j^k\phi), \quad \phi \in F_M.$$

where $\mathscr{L}$ denotes a *Lagrangian density* on M and $j^k\phi$ denotes the k-th jet prolongation of the field $\phi \in F_M$ as a section of the jet bundle of F_M.

Definition 1.2.2 (BV manifold)**.** A *BV manifold* consists of a triple $(\mathcal{F}, \mathcal{S}, \omega)$, where $\mathcal{F}$ is a $\mathbb{Z}$-graded[6] supermanifold[7], $\mathcal{S}$ an even function on $\mathcal{F}$ of degree 0 and ω and odd symplectic form on $\mathcal{F}$ of degree -1, such that the *classical master equation (CME)* holds:

$$(\mathcal{S}, \mathcal{S}) = 0. \tag{1.2.1}$$

Here, $(\ ,\ )$ denotes the odd Poisson bracket of degree $+1$ induced by ω.

Remark 1.2.3. We call $\mathcal{F}$ the *BV space of fields*, $\mathcal{S}$ the *BV action (functional)* and ω the *BV symplectic form*. The odd Poisson bracket $(\ ,\ )$ is usually called the *BV bracket* (or *anti-bracket*)[8]. If we consider a vector bundle over M and a section σ, we will denote by $\mathrm{gh}(\sigma) \in \mathbb{Z}$ the $\mathbb{Z}$-grading and by $|\sigma| \in \mathbb{Z}_2$ the parity. If σ is a differential form, we will denote by $\deg \sigma$ its form degree.

We also consider the Hamiltonian vector field Q of $\mathcal{S}$, i.e. the unique vector field of degree $+1$ satisfying the equation

$$\iota_Q \omega = \delta \mathcal{S},$$

where δ here denotes the de Rham differential on $\mathcal{F}$. It is easy to see that the vector field Q is actually *cohomological*, i.e. we have $Q^2 = 0$ and moreover, by definition, it is *symplectic*, i.e. we have $L_Q\omega = 0$. Here L denotes the Lie derivative. Note that, by definition, we have that $Q = (\mathcal{S},\ )$ and hence by the CME (1.2.1) we get $Q\mathcal{S} = 0$. We will call Q the *BV charge*[9].

Definition 1.2.4 (BV theory)**.** A d-dimensional *BV theory* is the assignment of a closed compact, connected d-manifold M to a BV manifold $(\mathcal{F}_M, \mathcal{S}_M, \omega_M)$.

[5] An important class of examples of F_M is given by the space of sections of a vector bundle over M, e.g. vector fields on M, differential forms on M. However, it can be much more complicated such as the space of connections or metrics on M.

[6] The $\mathbb{Z}$-grading is called the *ghost number* in the physics literature according to the construction by Faddeev–Popov [FP67].

[7] The supermanifold structure induces an additional $\mathbb{Z}_2$-grading which is referred to as *parity*, often called *odd* and *even*.

[8] The BV bracket is usually denoted by round brackets $(\ ,\ )$ instead of $\{\ ,\ \}$ in order to emphasize the fact that it is an *odd* Poisson bracket. This notation is due to the original notation by Batalin and Vilkovisky [BV81].

[9] The cohomological vector field Q is the same as the one in the BRST formalism [BRS74; BRS75; Tyu76] measuring the symmetries of the theory, thus often also called the *BRST charge*.

1.2.2 Quantum BV formalism

In the quantum setting we consider an operator Δ on functions on the BV space of fields $\mathcal{F}$, denoted by $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$, with the property that $\Delta^2 = 0$. To be more precise, let $\sigma \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ be some nowhere-vanishing reference half-density on $\mathcal{F}$. We can then define $\Delta_\sigma f := \frac{1}{\sigma}\Delta^{\frac{1}{2}}(f\sigma)$, where $\Delta^{\frac{1}{2}}$ denotes the canonical operator on $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ that squares to zero. Moreover, we have that

$$\Delta_\sigma(fg) = \Delta_\sigma f g \pm f\Delta_\sigma g \pm (f,g), \quad f,g \in C^\infty(\mathcal{F}).$$

This operator is canonical in the finite-dimensional case [Khu04] and needs to be properly regularized in the infinite-dimensional setting. We will just write Δ instead of Δ_σ whenever the reference half-density is understood and sometimes we will also write Δ instead of $\Delta^{\frac{1}{2}}$ whenever it is understood. The operator Δ is often called the *BV Laplacian* or *BV operator*. On an odd-symplectic supermanifold $(\mathcal{M},\omega)$ with local coordinates (x_i,θ^i), where x_i denote the even coordinates and θ^i denote the odd coordinates, it is given by

$$\Delta = \sum_i \frac{\partial^2}{\partial x_i \partial \theta^i}.$$

In particular, we have

$$\Delta f = \frac{1}{2}\mathrm{div}(f,\ \ ), \quad f \in C^\infty(\mathcal{F}). \tag{1.2.2}$$

Theorem 1.2.5 (Batalin–Vilkovisky–Schwarz[BV81; Sch93])**.** *Consider two half-densities f,g on an odd-symplectic supermanifold $(\mathcal{M},\omega)$. Then*

*(1) if $f = \Delta g$ (*BV exact*), we get that*

$$\int_{\mathcal{L}\subset\mathcal{M}} f = \int_{\mathcal{L}\subset\mathcal{M}} \Delta g = 0,$$

for any Lagrangian submanifold $\mathcal{L}\subset\mathcal{M}$.

*(2) if $\Delta f = 0$ (*BV closed*), we get that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t\subset\mathcal{M}} f = 0,$$

for any continuous family $(\mathcal{L}_t)$ of Lagrangian submanifolds of $\mathcal{M}$. In particular, when two Lagrangian submanifolds $\mathcal{L}_1\subset\mathcal{M}$ and $\mathcal{L}_2\subset\mathcal{M}$ can be continuously deformed into each other, we have

$$\int_{\mathcal{L}_1\subset\mathcal{M}} f = \int_{\mathcal{L}_2\subset\mathcal{M}} f.$$

Remark 1.2.6. Note that in the finite-dimensional case, when f is a half-density, then its restriction to a Lagrangian submanifold $\mathcal{L}$ makes it a density $f|_{\mathcal{L}}$ on $\mathcal{L}$.

For the setting of quantum gauge field theories, we are mainly interested in the case where our odd-symplectic supermanifold is given by a tuple $(\mathcal{F}, \omega)$ and the half-density f is of the form $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\rho \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$, where ρ denotes a reference Δ-closed, nowhere-vanishing half-density on $\mathcal{F}$. Then, when considering an integral of the form $\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\rho$, the choice of Lagrangian submanifold corresponds to choosing a gauge[10]. Note that the second point of Theorem 1.2.5 is then a condition for gauge-independence of the classical theory described by the action S. In the case of interest, we want thus

$$\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} = 0 \Longleftrightarrow \frac{1}{2}(S, S) - \mathrm{i}\hbar \Delta S = 0. \tag{1.2.3}$$

Equation (1.2.3) is called the *quantum master equation (QME)*. Note that in the semi-classical limit $\hbar \to 0$ we get the CME. When S depends on $\hbar$ as a formal power series $S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$, we can try to solve the QME order by order in $\hbar$. When we consider the BV action $\mathcal{S}$, which gives a solution to the CME (1.2.1), we need to make sure that $\Delta \mathcal{S} = 0$. This is true for many different theories of interest, but is not immediate in general. In general, we need to find a suitable regularization in order to make the QME hold. Note also that by Equation (1.2.2), we have

$$\Delta \mathcal{S} = \frac{1}{2} \mathrm{div}\, Q.$$

1.2.3 Example: 2D (abelian) BF theory

Let M be a connected closed 2-manifold and let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $P \to M$ be a principal G-bundle over M. Consider the space of connection 1-forms $\mathcal{A} := \Omega^1(M, \mathrm{ad}P)$ with values in the adjoint bundle of P. We define the space of fields for 2-dimensional BF theory to be

$$F_M := \mathcal{A} \oplus \Omega^0(M, \mathrm{ad}^* P),$$

where ad^*P denotes the coadjoint bundle of P. The BF action is given by

$$S_M := \int_M \langle B, F_A \rangle = \int_M \left(\langle B, \mathrm{d}A \rangle + \frac{1}{2} \langle B, [A, A] \rangle \right), \tag{1.2.4}$$

where $(A, B) \in F_M$ and $F_A := \mathrm{d}A + \frac{1}{2}[A, A]$ denotes the curvature 2-form of the connection 1-form A. We have denoted by $\langle \ , \ \rangle$ the pairing between the forms of adjoint and coadjoint type as an extension of the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. It is easy to see that the critical points of S_M are given by pairs $(A, B) \in F_M$ where A is a flat connection and $\mathrm{d}_A B = 0$, where d_A denotes the covariant derivative with respect to the connection A. Denote by $\mathcal{G}$ the group of gauge transformations of the space of connection 1-forms on P and let A^g denote the gauge transformed connection for a gauge transformation $g \in \mathcal{G}$. Then we can consider the extension to the space of fields F_M by the semi-direct

[10] When choosing a gauge-fixing fermion Ψ, the case where we can reduce to the cohomological setting of the BRST formalism is given by taking the Lagrangian submanifold to be the graph of the differential of the gauge-fixing fermion, i.e. we take $\mathcal{L} = \mathrm{graph}(\mathrm{d}\Psi)$.

product $\widetilde{\mathcal{G}} := \mathcal{G} \rtimes \Omega^0(M, \mathrm{ad}^*P)$, where $\mathcal{G}$ acts on $\Omega^0(M, \mathrm{ad}^*P)$ by coadjoint action. It is then not hard to see that the BF action (1.2.4) is invariant with respect to the gauge transformation

$$A \mapsto A^g, \tag{1.2.5}$$
$$B \mapsto B^{(g,f)} := \mathrm{Ad}^*_{g^{-1}} B + \mathrm{d}_{A^g} f, \tag{1.2.6}$$

i.e. we have

$$\begin{aligned} S_M(A^g, B^{(g,f)}) &= \int_M \left(\langle B^{(g,f)}, \mathrm{d}A^g \rangle + \frac{1}{2} \langle B^{(g,f)}, [A^g, A^g] \rangle \right) \\ &= \int_M \left(\left\langle \mathrm{Ad}^*_{g^{-1}} B + \mathrm{d}_{A^g} f, \mathrm{d}A^g \right\rangle + \frac{1}{2} \left\langle \mathrm{Ad}^*_{g^{-1}} B + \mathrm{d}_{A^g} f, [A^g, A^g] \right\rangle \right) \\ &= \int_M \left(\langle B, \mathrm{d}A \rangle + \frac{1}{2} \langle B, [A, A] \rangle \right) = S(A, B). \end{aligned}$$

If $\mathfrak{g} = \mathbb{R}$, we speak of *abelian* BF theory. In this case we have

$$F_M = \Omega^1(M) \oplus \Omega^0(M),$$

and

$$S_M = \int_M B \wedge \mathrm{d}A.$$

The critical points here are pairs $(A, B) \in F_M$ of the form $\mathrm{d}A = 0$ and $\mathrm{d}B = 0$. An example of 2-dimensional abelian BF theory is the *trivial Poisson sigma model*. There, we consider a 2-dimensional source manifold[11] Σ and some target manifold M. The space of fields F_M is given by vector bundle maps between the tangent bundle $T\Sigma$ of the source and the cotangent bundle T^*M of the target. The fields are thus tuples (X, η) where $X\colon \Sigma \to M$ is a map and $\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*M)$ is a 1-form with values in the pullback bundle X^*T^*M. The action is then of the form

$$S_\Sigma := \int_\Sigma \eta \wedge \mathrm{d}X.$$

The BV formulation of 2-dimensional BF theory is not hard to construct. The BV space of fields associated to the 2-manifold M is given by

$$\mathcal{F}_M := \Omega^\bullet(M, \mathrm{ad}P)[1] \oplus \Omega^\bullet(M, \mathrm{ad}^*P),$$

where $\Omega^\bullet = \bigoplus_{j=0}^2 \Omega^j$. The *superfields* are tuples $(\mathbf{A}, \mathbf{B}) \in \mathcal{F}_M$ of the form

$$\mathbf{A} := c + A + B^+, \tag{1.2.7}$$
$$\mathbf{B} := B + A^+ + c^+, \tag{1.2.8}$$

[11] Note that here Σ takes the place of M before. We have chosen this notation here because of historical reasons.

where $\mathrm{gh}(c) = 1$, $\mathrm{gh}(A) = \mathrm{gh}(B) = 0$, $\mathrm{gh}(A^+) = \mathrm{gh}(B^+) = -1$, $\mathrm{gh}(c^+) = -2$. Moreover, $\deg(c) = \deg(B) = 0$, $\deg(A) = \deg(A^+) = 1$, $\deg(B^+) = \deg(c^+) = 2$. For a classical field ϕ, we have denoted by ϕ^+ its *anti-field* with the property $\mathrm{gh}(\phi) + \mathrm{gh}(\phi^+) = -1$ and $\deg(\phi) + \deg(\phi^+) = 2$. Note that, we have denoted by c the *ghost field*. We can define the curvature of the superconnection $\mathbf{A}$ by $\mathbf{F_A}$ defined by

$$\mathbf{F_A} := F_{A_0} + \mathrm{d}_{A_0}\mathbf{a} + \frac{1}{2}[\mathbf{a}, \mathbf{a}],$$

where here $[\ ,\ ]$ denotes now the induced bracket on the super Lie algebra, A_0 is some reference connection 1-form and $\mathbf{a} := \mathbf{A} - A_0 \in \Omega^\bullet(M, \mathrm{ad}P)[1]$. The BV action is then given by

$$\mathcal{S}_M(\mathbf{A}, \mathbf{B}) := \int_M \langle \mathbf{B}, \mathbf{F_A} \rangle, \tag{1.2.9}$$

where $\langle\ ,\ \rangle$ here denotes the pairing between the forms of adjoint and coadjoint type as an extension of the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ with shifted degree, i.e. with additional sign coming from the $\mathbb{Z}$-grading. It is then not hard to see that $\mathcal{S}_M$ satisfies the CME $(\mathcal{S}_M, \mathcal{S}_M) = 0$. Note also that $Q_M\mathbf{A} = (\mathcal{S}_M, \mathbf{A}) = \mathbf{F_A}$ and $Q_M\mathbf{B} = (\mathcal{S}_M, \mathbf{B}) = \mathrm{d}_\mathbf{A}\mathbf{B}$. In particular, the cohomological vector field of degree $+1$ is given by

$$\begin{aligned} Q_M = (\mathcal{S}_M,\ ) &= \int_M \left(\mathrm{d}\mathbf{A}\frac{\delta}{\delta\mathbf{A}} + \mathrm{d}\mathbf{B}\frac{\delta}{\delta\mathbf{B}}\right) \\ &= \int_M \left(\mathrm{d}c\frac{\delta}{\delta c} + \mathrm{d}A\frac{\delta}{\delta A} + \mathrm{d}B^+\frac{\delta}{\delta B^+} + \mathrm{d}B\frac{\delta}{\delta B} + \mathrm{d}A^+\frac{\delta}{\delta A^+} + \mathrm{d}c^+\frac{\delta}{\delta c^+}\right). \end{aligned} \tag{1.2.10}$$

The BV symplectic form of degree -1 is given by

$$\omega_M = \int_M \delta\mathbf{A} \wedge \delta\mathbf{B} = \int_M \left(\delta c \wedge \delta c^+ + \delta A \wedge \delta A^+ + \delta B \wedge \delta B^+\right).$$

Remark 1.2.7. The general structure of BF theory can be easily generalized to arbitrary dimension.

1.2.4 Example: AKSZ theories

An important class of BV theories were developed by Alexandrov, Kontsevich, Schwarz and Zaboronsky in [Ale+97] known today as AKSZ theories. They formulated a method how to obtain a solution of the CME by considering a special form for the space of fields, namely, mapping spaces between supermanifolds. Let us first describe the ingredients needed to formulate these type of theories. Let M be a closed, connected, oriented d-manifold and consider a differential graded symplectic manifold $(\mathcal{M}, \omega)$ where the symplectic form is exact, i.e. $\omega = \mathrm{d}\alpha$, and of degree $d-1$. Moreover, let Θ be a function on $\mathcal{M}$ of degree d such that $\{\Theta, \Theta\}_\omega = 0$, where $\{\ ,\ \}_\omega$ denotes the Poisson bracket of degree $1+d$ induced by ω. Moreover, consider the Hamiltonian vector field Q for the Hamiltonian function Θ which is cohomological by definition, i.e. $Q^2 = 0$. Define the space of fields as

$$\mathcal{F}_M^{\mathrm{AKSZ}} := \mathrm{Map}(T[1]M, \mathcal{M}).$$

Using the symplectic structure on $\mathcal{M}$, we can construct a symplectic structure on $\mathcal{F}_M^{\text{AKSZ}}$ by *transgression*. Namely, we have a diagram

$$\begin{tikzcd} \mathrm{Map}(T[1]M,\mathcal{M})\times T[1]M \arrow[r, "\mathsf{ev}"] \arrow[d, "\mathsf{p}"'] & \mathcal{M} \arrow[dl, dashed] \\ \mathrm{Map}(T[1]M,\mathcal{M}) & \end{tikzcd}$$

where ev denotes the evaluation map and p the projection onto the first factor. Thus, on the level of forms, we can define a *transgression map* $\mathbb{T}\colon \Omega^\bullet(\mathcal{M})\to\Omega^\bullet(\mathrm{Map}(T[1]M,\mathcal{M}))$ by

$$\mathbb{T}(\eta):=\mathsf{p}_*\mathsf{ev}^*\eta=\int_{T[1]M}\mu\mathsf{ev}^*\eta,\quad \eta\in\Omega^\bullet(\mathcal{M})$$

where μ is the standard Berezinian on $T[1]M$. Hence, we define a symplectic form on $\mathcal{F}_M^{\text{AKSZ}}$ by

$$\omega_M:=\mathbb{T}(\omega)=\int_{T[1]M}\mu\mathsf{ev}^*\omega,$$

where ω is the symplectic form on $\mathcal{M}$. Note that since ω was of degree $d-1$, we get that ω_M is of degree $(d-1)-d=-1$ which is the correct degree for a BV symplectic form. We can define a cohomological vector field on the mapping space as the sum of the lift of the de Rham differential on M and the lift of the cohomological vector field on $\mathcal{M}$:

$$Q_M:=\widehat{\mathrm{d}_M}+\widehat{Q},$$

where the hats denote the lift to the mapping space. The BV action is then constructed by using the primitive α of ω and the Hamiltonian function Θ. We define

$$\mathcal{S}_M^{\text{AKSZ}}:=\iota_{\widehat{\mathrm{d}_M}}\mathbb{T}(\alpha)+\mathbb{T}(\Theta).$$

It is then not difficult to see that $\mathcal{S}_M^{\text{AKSZ}}$ is indeed of degree 0 and satisfies the CME.

Remark 1.2.8. Many interesting theories are of AKSZ type, such as e.g. Chern–Simons theory [CS74; Wit89; AS91; AS94], the Poisson sigma model [Ike94; SS94; CF01a], or Witten's A- and B-twisted sigma models [Wit88b; Ale+97]. The most important one for the purposes of this thesis are so-called BF-like theories, which are *deformations* of abelian BF theory.

1.3 Extension to Manifolds with Boundary: The BV-BFV Formalism

1.3.1 Classical BV-BFV formalism

In the BV formalism the source manifold M was always assumed to have empty boundary (i.e. $\partial M=\varnothing$). In order to overcome this and still make sense of a gauge formalism for manifolds with boundary, one couples the Lagrangian approach of the BV theory in the bulk to the Hamiltonian approach of the *BFV theory*[12] on the boundary such that everything is coherent at the end. The coupling of bulk and boundary was considered in the classical setting first in [CMR14].

Definition 1.3.1 (BFV manifold)**.** A *BFV manifold* is a quadruple

$$(\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial)$$

such that $\mathcal{F}^\partial$ is a $\mathbb{Z}$-graded supermanifold, $\mathcal{S}^\partial$ is an odd function on $\mathcal{F}^\partial$ of degree $+1$, ω is an even symplectic form on $\mathcal{F}^\partial$ of degree 0 and Q^∂ is a cohomological vector field on $\mathcal{F}^\partial$ of degree $+1$ such that it $\iota_{Q^\partial}\omega^\partial = \delta\mathcal{S}^\partial$. Note that we than get that $\{\mathcal{S}^\partial, \mathcal{S}^\partial\}_{\omega^\partial} = 0$, where $\{\ ,\ \}_{\omega^\partial}$ denotes the even Poisson bracket of degree 0 induced by the symplectic form ω^∂. We say that a BFV manifold is *exact*, if the symplectic form is exact, i.e. $\omega^\partial = \delta\alpha^\partial$.

Remark 1.3.2. We call $\mathcal{F}^\partial$ the *BFV space of fields*, $\mathcal{S}^\partial$ the *BFV action (functional)*, ω^∂ the *BFV symplectic form* and Q^∂ the *BFV charge*. If we consider the exact case, we will call α^∂ the *BFV 1-form.*

Definition 1.3.3 (BV-BFV manifold over exact BFV manifold)**.** A *BV-BFV manifold over an exact BFV manifold* $(\mathcal{F}^\partial, \omega^\partial, \mathcal{S}^\partial, Q^\partial)$ is a quintuple

$$(\mathcal{F}, \omega, \mathcal{S}, Q, \mathsf{p}),$$

where $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold, $\mathcal{S}$ is an even function on $\mathcal{F}$ of degree 0, ω is an odd symplectic form on $\mathcal{F}$ of degree -1, Q is the cohomological vector field of degree $+1$ and $\mathsf{p}\colon \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion, such that

(1) $\iota_Q\omega = \delta\mathcal{S} + \mathsf{p}^*\alpha^\partial$,

(2) $\delta\mathsf{p}Q = Q^\partial$.

Remark 1.3.4 (modified CME)**.** It is easy to see that the conditions of Definition 1.3.3 implies the *modified classical master equation (mCME)*

$$Q\mathcal{S} = \mathsf{p}^*(2\mathcal{S}^\partial - \iota_{Q^\partial}\alpha^\partial). \tag{1.3.1}$$

Note that in the closed setting, we have required the CME $Q\mathcal{S} = 0$, whereas in the setting with boundary it is not zero anymore, but given entirely in terms of boundary data. We call condition (1) the *modified* CME [CMR14]. Condition (2) tells us that the BV charge is projectable onto the BFV charge Q^∂. For examples and more insights on the classical BV-BFV formalism we refer to [CMR14; CM20]. Note also that if $\mathcal{F}^\partial$ is a point, Definition 1.3.3 reduces to the one of a BV manifold.

Definition 1.3.5 (BV-BFV theory)**.** A *BV-BFV theory* assigns to each d-manifold M with boundary $(d-1)$-manifold ∂M a BV-BFV manifold

$$(\mathcal{F}_M, \omega_M, \mathcal{S}_M, Q_M, \mathsf{p}_M),$$

with $\mathsf{p}_M\colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$, over an exact BV-BFV manifold $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M} = \delta\alpha^\partial_{\partial M}, \mathcal{S}^\partial_{\partial M}, Q^\partial_{\partial M})$.

[12] The letters BFV stand for Batalin–Fradkin–Vilkovisky due to their work [BF83; BF86; FV75; FV77] where they developed the Hamiltonian setting.

1.3.2 Quantum BV-BFV formalism

The quantum BV-BFV formalism, for a given BV-BFV theory, is given by the following data:

(i) The *state space* $\mathcal{H}^{\mathcal{P}}_{\Sigma}$, a graded vector space associated to some $(d-1)$-manifold Σ with a choice of a *polarization*[13] $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\Sigma}$. The state space is constructed through the methods of *geometric quantization* (see e.g. [Kir85; Woo97; BW12]) for the symplectic manifold $(\mathcal{F}^{\partial}_{\Sigma}, \omega^{\partial}_{\Sigma})$.

(ii) The *quantum BFV operator* $\Omega^{\mathcal{P}}_{\Sigma}$, a coboundary operator on the state space $\mathcal{H}^{\mathcal{P}}_{\Sigma}$ which is determined as a quantization of the BFV action $\mathcal{S}^{\partial}_{\Sigma}$.

(iii) The *space of residual fields* $\mathcal{V}_M$, a finite-dimensional graded manifold endowed with a symplectic form of degree -1 associated to a d-manifold M and a polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M}$. Moreover, we can define the graded vector space

$$\widehat{\mathcal{H}}^{\mathcal{P}}_M := \mathcal{H}^{\mathcal{P}}_{\partial M} \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_M),$$

where $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_M)$ denotes the space of half-densities on $\mathcal{V}_M$. This graded vector space is endowed with two commuting coboundary operators

$$\begin{aligned} \widehat{\Omega}^{\mathcal{P}}_M &:= \Omega^{\mathcal{P}}_{\partial M} \otimes \mathrm{id}, \\ \widehat{\Delta}^{\mathcal{P}}_M &:= \mathrm{id} \otimes \Delta_{\mathcal{V}_M}, \end{aligned}$$

where $\Delta_{\mathcal{V}_M}$ denotes the canonical BV Laplacian on half-densities on residual fields.

(iv) A *state* $\widehat{\psi}_M \in \widehat{\mathcal{H}}^{\mathcal{P}}_M$ that satisfies the *modified quantum master equation (mQME)*

$$\left(\hbar^2 \widehat{\Delta}^{\mathcal{P}}_M + \widehat{\Omega}^{\mathcal{P}}_M\right) \widehat{\psi}_M = 0.$$

This equation is the quantum version of Equation (1.3.1).

1.4 Deformation Quantization

The theory of *deformation quantization* was proposed by Dirac [Dir30] and Weyl [Wey31] in order capture the mathematical transition form the commutative algebra of classical observables to the non-commutative algebra of quantum observables. These ideas were carried further by the *operator quantization* and the *exact quantization question*. It was Groenewold [Gro46] who showed that an exact quantization is actually not possible in general and thus the question for *deformations* came into play. These mathematical questions have been made popular by Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer [Bay+78a; Bay+78b]. In particular, one would like to map the classical commutative

[13] A *polarization* is the choice of an involutive Lagrangian subbundle of the tangent bundle of a manifold. In our case, $\mathcal{P}$ is an involutive Lagrangian subbundle of $T\mathcal{F}^{\partial}_{\Sigma}$.

product on smooth functions to a *deformation* of it by a small deformation parameter. Usually, the deformation parameter is $\hbar$. Thus, for two smooth functions f and g we have

$$fg \mapsto f \star g := fg + \sum_{k\geq 0} B_k(f,g)\hbar^k,$$

where the B_k denote some bidifferential operators with certain additional properties. The power series on the right has to be understood in a *formal* way, i.e. we are not interested in convergence issues. The phase space arising in the classical setting is usually given by a symplectic manifold (M,ω). In particular, when considering the space of classical observables $C^\infty(M)$, it can be endowed with a Poisson bracket $\{\ ,\ \}$ induced by the symplectic structure ω. A requirement in the definition of a star product is that the B_1 are given in terms of the Poisson bracket $\{\ ,\ \}$. We say that $\star$ is a deformation of the pointwise product on $C^\infty(M)$ in *direction* of $\{\ ,\ \}$. The local case of $(\mathbb{R}^{2n},\omega_0)$, where $\omega_0 = \sum_{1\leq i<j\leq n} \mathrm{d}q^i \wedge \mathrm{d}p^j$ is the standard symplectic form with (q_i,p_i) coordinates in $\mathbb{R}^{2n}$, there is a construction of a star product proposed by Moyal [Moy49]. The globalization to any symplectic manifold was constructed later independently by De Wilde–Lecomte [DL83] and Fedosov [Fed94]. The general construction, i.e. the case when the Poisson bracket is coming from a general Poisson structure π, was given by Kontsevich [Kon03]. He gave an explicit formula for a star product which was formulated for the local case when starting with a Poisson manifold $(\mathbb{R}^d,\pi)$. The underlying objects in Kontsevich's formula are *graphs* (also called *Kontsevich graphs*), which in fact are related to Feynman diagrams for a certain perturbative quantum field theory, in particular the *Poisson sigma model* [Ike94; SS94; CF01c] on the 2-dimensional disk, as it was shown by Cattaneo–Felder [CF00]. Each graph contributes a weight (a real number), which is given in terms of *configuration space integrals*, i.e. integrals over configuration spaces of points on the upper half-plane. Actually, Kontsevich gave a much more general result in [Kon03], which is known as *formality*. He showed that the differential graded Lie algebra of multidifferential operators endowed with the Gerstenhaber bracket and the Hochschild differential is L_∞-quasi-isomorphic to the differential graded Lie algebra of multivector fields endowed with the Schouten–Nijenhuis bracket and the zero differential. This result implies the result for deformation quantization for the case of bidifferential operators and bivector fields. Another approach to prove Kontsevich's formality theorem was proposed by Kontsevich and Tamarkin in [Kon99] and [Tam03], respectively, by using the notion of *operads*. This method has several advantages and showed to be useful in order to prove (higher versions of) *Deligne's conjecture* [Del95]. The gobalization of Kontsevich's star product to any Poisson manifold (M,π) was given by Cattaneo–Felder–Tomassini [CFT02]. In their approach, they used similar techniques as Fedosov in [Fed94] for the Poisson case mixed with methods of *formal geometry* developed by Gelfand–Fuks [GF69; GF70], Gelfand–Kazhdan [GK71] and Bott [Bot10]. In particular, they used Kontsevich's L_∞-morphism to construct a *connection* and *curvature* term and showed that these satisfy a similar globalization equation as in [Fed94].

Symplectic groupoids and reduction

The field-theoretic formulation of Kontsevich's star product leads to important insights in the field of symplectic and Poisson geometry. In particular, the *Poisson sigma model* turns out to carry interesting geometric structure when considering its phase space, as it was shown by Cattaneo–Felder [CF01d]. They show that the phase space, given by the space of leaves for a Hamiltonian foliation, has a natural groupoid structure and is in fact given by a *symplectic groupoid* if it is a manifold. This is considered after gauge-reduction. However, the case before reduction yields also an interesting structure as it was shown by Cattaneo–Contreras [CC13; CC15]. They have formulated an axiomatic approach to this construction and called the resulting mathematical object a *relational symplectic groupoid* (RSG). An RSG is given by some (infinite-dimensional) Banach manifold $\mathcal{G}$ together with some Lagrangian relations $L_1 \subset \mathcal{G}, L_2 \subset \mathcal{G} \times \mathcal{G}, L_3 \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ and an involution operation $\mathcal{I}$. Interpreting the Lagrangian relations as 2-dimensional disks with a particular way of choosing polarization and boundary conditions on the boundary, it was shown in [CMW17] that the quantization of the Moyal star product, following [CF00], can be constructed by using the BV-BFV formalism. It is expected that the general case of Kontsevich's star product can also be constructed with techniques of cutting–gluing. In [Haw08], Hawkins showed how to quantize a symplectic groupoid using methods of geometric quantization together with concepts of C^*-algebras. A first step towards the quantization for the *relational* case has been done in [CMW21] which is part of this thesis.

Chapter 2

Paper Abstracts and Main Results

2.1 4-Manifold Topology, Donaldson–Witten Theory, Floer Homology and Higher Gauge Theory Methods in the BV-BFV Formalism

2.1.1 Abstract

We study the behavior of Donaldson's invariants of 4-manifolds [Don87] based on the moduli space of anti self-dual connections (instantons) in the perturbative field theory setting where the underlying source manifold has boundary. It is well-known that these invariants take values in the instanton Floer homology groups of the boundary 3-manifold [Flo89; Don02]. Gluing formulae for these constructions lead to a functorial topological field theory description according to a system of axioms developed by Atiyah [Ati87], which can be also regarded in the setting of perturbative quantum field theory, as it was shown by Witten [Wit88a], using a version of supersymmetric Yang–Mills theory, known today as Donaldson–Witten theory. One can actually formulate an AKSZ model [Ale+97] which recovers this theory for a certain gauge-fixing. We consider these constructions in a perturbative quantum gauge formalism for manifolds with boundary that is compatible with cutting and gluing, called the BV-BFV formalism [CMR14; CMR17], which was recently developed by Cattaneo, Mnev and Reshetikhin. We prove that this theory satisfies a modified Quantum Master Equation and extend the result to a global picture when perturbing around constant background fields. These methods are expected to extend to higher codimensions and thus might help getting a better understanding for fully extendable n-dimensional field theories (in the sense of Baez–Dolan [BD95] and Lurie [Lur09]) in the perturbative setting, especially when $n \leq 4$. Additionally, we relate these constructions to Nekrasov's partition function [Nek03] by treating an equivariant version of Donaldson–Witten theory in the BV formalism [Bon+20]. Moreover, we discuss the extension, as well as the relation, to higher gauge theory and enumerative geometry methods, such as Gromov–Witten [Gro85; Wit91] and Donaldson–Thomas theory [DT98] and recall their correspondence conjecture for general Calabi–Yau 3-folds. In particular, we discuss the corresponding (relative) partition functions, defined as the generating function for the given invariants, and gluing phenomena.

2.1.2 Main results

The purpose of this paper is to provide the reader with:

(i) a concise overview of the field of 4-manifold topology, both from the pure mathematical and field-theoretic point of view. Especially, to explain some of the relations of the literature as well as to new developements such as the BV-BFV techniques developed recently.

(ii) a concise overview of (instanton and Lagrangian [Flo88]) Floer (co)homology and how it fits into the 4D-3D bulk-boundary correspondence through Chern–Simons theory [CS74; AS91; AS94].

(iii) a detailed formulation of a (global) perturbative quantization of the AKSZ formulation of Donaldson–Witten theory on source manifolds with boundary which fits into the gauge theory setting of the BV-BFV formalism by using methods of configuration space integrals.

(iv) a short discussion of the quantization of higher defect theories through the recently developed methods of shifted (symplectic or Poisson) structures.

(v) a concise overview of the relation to Seiberg–Witten theory [SW94b; SW94a] through Nekrasov's construction by using equivariant methods of localization and an equivariant version of the BV formalism in combination with Donaldson–Witten theory and a description of the relation with Donaldson's invariants through an equivariant version of instanton Floer (co)homology [AB96].

(vi) a description of the relation to constructions of Donaldson–Thomas theory.

(vii) a concise overview of the conjectured correspondence of Donaldson–Thomas theory and Gromov–Witten theory and connections to other fields, such as e.g. topological recursion or supergeometry [BL75; Lei80; CS11; CM20].

2.2 Formal Global Perturbative Quantization of the Rozansky–Witten Model in the BV-BFV Formalism (Joint with D. Saccardo)

2.2.1 Abstract

We describe a globalization construction for the Rozansky–Witten model in the BV-BFV formalism for a source manifold with and without boundary in the classical and quantum case. After having introduced the necessary background, we define an AKSZ sigma model, which, upon globalization through notions of formal geometry extended appropriately to our case, is shown to reduce to the Rozansky–Witten model. The relations with other relevant constructions in the literature are discussed. Moreover, we split the model as a BF-like theory and we construct a perturbative quantization of the model in the quantum BV-BFV framework. In this context, we are able to prove the modified differential

Quantum Master Equation and the flatness of the quantum Grothendieck BFV operator. Additionally, we provide a construction of the BFV boundary operator in some cases.

2.2.2 Main results

In this paper, we continue the effort in analyzing TQFTs within the quantum BV-BFV formalism by studying the *Rozansky–Witten (RW) theory*. The RW model is a topological sigma model with a source 3-dimensional manifold Σ_3, which was introduced by Rozansky and Witten in [RW97] through a *topological twist* of a 6-dimensional supersymmetric sigma model with target a hyperKähler manifold M. Of particular interest is the perturbative expansion of the RW partition function. Rozansky and Witten obtained this expansion as a combinatorial sum in terms of Feynman diagrams Γ, which are shown to be trivalent graphs:

$$Z_M(\Sigma_3) = \sum_{\Gamma} b_\Gamma(M) I_\Gamma(\Sigma_3), \tag{2.2.1}$$

the $b_\Gamma(M)$ are complex valued functions on trivalent graphs constructed from the target manifold, while $I_\Gamma(\Sigma_3)$ contains the integral over the propagators of the theory and depends on the source manifold. There are evidences which suggest that $I_\Gamma(\Sigma_3)$ are the *LMO invariants* of Le, Murakami and Ohtsuki [LMO98]. On the other hand, Rozansky and Witten showed that $b_\Gamma(M)$ satisfy the famous AS (which is reflected in the absence of tadpoles diagrams) and IHX relations. As a result, $b_\Gamma(M)$ constitute the *Rozansky–Witten weight system* for the *graph homology*, the space of linear combinations of equivalence classes of trivalent graphs (modulo the AS and IHX relations). This means that the RW weights can be used to construct new finite type topological invariants for 3-dimensional manifolds [Bar95].

The main contribution of this paper is to add the RW theory to the list of TQFTs which have been studied successfully within the globalized version of the quantum BV-BFV framework [CMW19]. This will be a step towards the higher codimension quantization of RW theory, which will possibly lead to new insights towards the 3-dimensional correspondence between CS theory [Wit89] and the Reshetikhin–Turaev construction [RT91] from the point of view of (perturbative) extended field theories described by Baez–Dolan [BD95] and Lurie [Lur09]. Moreover this could also help in understanding (generalizations of a globalized version of the) Berezin–Toeplitz quantization (star product) [Sch10] through field-theoretic methods using cutting and gluing similarly as it was done for Kontsevich's star product [Kon03] in the case of the Poisson sigma model in [CMW20].

We construct the BV-BFV extension of an AKSZ model having a 3-dimensional manifold Σ_3 (possibly with boundary) as source and a holomorphic symplectic manifold M as target with holomorphic symplectic form Ω. Following [Kap99], we define a formal holomorphic exponential map φ. This is used to linearize the space of fields of our model obtaining

$$\widetilde{\mathcal{F}}_{\Sigma_3,x} = \Omega^\bullet(\Sigma_3) \otimes T_x^{1,0}M, \tag{2.2.2}$$

where $\Omega^\bullet(\Sigma_3)$ denotes the complex of de Rham forms on the source manifold and $T_x^{1,0}M$ is the holomorphic tangent space on the target. In order to vary the constant solution around which we perturb, we define a classical Grothendieck connection which can be seen as a complex extension of the Grothendieck connection used in [CMW19; CMW20]. In this way, we construct a *formal global action* for our model, i.e.

$$\widetilde{\mathcal{S}}_{\Sigma_3,x} := \int_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i d\hat{\mathbf{X}}^j + \left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^j + \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}\right) \tag{2.2.3}$$

with $\hat{\mathbf{X}}^i$ the coordinates of the spaces of fields $\widetilde{\mathcal{F}}_{\Sigma_3,x}$ organized as superfields, x is the constant map over which we expand, $\left(\hat{R}^i_{\Sigma_3}\right)_j$ and $\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}$ the components of the Grothendieck connection given by

$$\begin{aligned} R^i_j(x;y)dx^j &:= -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial\varphi^p}{\partial x^j}dx^j, \\ R^i_{\bar{j}}(x;y)dx^{\bar{j}} &:= -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial\varphi^p}{\partial x^{\bar{j}}}dx^{\bar{j}}, \end{aligned} \tag{2.2.4}$$

where $\{y^i\}$ are the generators of the fiber of $\widehat{\mathrm{Sym}}^\bullet(T^{\vee 1,0}M)$. The formal action is such that the *differential Classical Master Equation* (dCME) is satisfied, namely

$$d_M\widetilde{\mathcal{S}}_{\Sigma_3,x} + \frac{1}{2}(\widetilde{\mathcal{S}}_{\Sigma_3,x},\widetilde{\mathcal{S}}_{\Sigma_3,x}) = 0, \tag{2.2.5}$$

with $d_M = d_x + d_{\bar{x}}$ the sum of holomorphic and antiholomorphic Dolbeault differentials on M. The dCME presented here is different from the one presented in e.g. [BCM12; CMW19; CMW20] since there d_M was the de Rham differential on the body of the target manifold.
The globalized model is then shown to be a globalization of the RW model [RW97], which reduces to the RW model itself in the appropriate limits. Our globalization of the RW model is compared with other globalization constructions as the one developed in [CLL17] for a closed source manifold by using Costello's approach [Cos11b; Cos11a] to *derived geometry* [Toë06; Toë14; Pan+13], the procedure in [Ste17] which extends the work of [CLL17] to manifolds with boundary and the procedure in [QZ10; KQZ13]. In general, our model is compatible with all these apparently different views. In particular, we give a detailed account of the similarities between our method and the one in [CLL17], thus confirming the claim in Remark 3.6 in [CMW19] about the equivalence between Costello's approach and ours.
In order to quantize the theory according to the quantum BV-BFV formalism, we formulate a split version of our globalized RW model. Since the globalization is controlled by an L_∞-algebra, following [Ste17] and inspired by the work of Cattaneo, Mnev and Wernli for Chern–Simons theory [CMW17], we assume that we can split the L_∞-algebra in two isotropic subspaces. The action of the *globalized split RW model* is then

$$\widetilde{\mathcal{S}}^{\mathrm{S}}_{\Sigma_3,x} = \left\langle\hat{\mathbf{B}}, D\hat{\mathbf{A}}\right\rangle + \left\langle\left(\hat{R}_{\Sigma_3}\right)_j(x;\hat{\mathbf{A}}+\hat{\mathbf{B}})dx^j, \hat{\mathbf{A}}+\hat{\mathbf{B}}\right\rangle + \left\langle\left(\hat{R}_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{A}}+\hat{\mathbf{B}})dx^{\bar{j}}, \hat{\mathbf{A}}+\hat{\mathbf{B}}\right\rangle, \tag{2.2.6}$$

where $\langle -,-\rangle$ denotes the BV symplectic form on the space of fields $\widetilde{\mathcal{F}}^{\mathrm{S}}_{\Sigma_3,x}$ with values in the Dolbeault complex of M, $\hat{\mathbf{A}}^i$ and $\hat{\mathbf{B}}_i$ are the fields found from the splitting of the field

$\hat{\mathbf{X}}^i$, and D denotes the superdifferential. Note that d is the de Rham differential on the target, not on the source.
Finally, we quantize the globalized split RW model within the quantum BV-BFV formalism framework. Here, we obtained the following two theorems.

Theorem 2.2.1 (Flatness of the qGBFV operator)**.** *The* quantum Grothendieck BFV (qGBFV) operator ∇_{G} *for the anomaly-free globalized split RW model squares to zero, i.e.*

$$(\nabla_{\mathrm{G}})^2 \equiv 0, \tag{2.2.7}$$

where

$$\nabla_{\mathrm{G}} = d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3} = d_x + d_{\overline{x}} - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}, \tag{2.2.8}$$

with d_M the sum of the holomorphic and antiholomorphic Dolbeault differentials on the target M, $\Delta_{\mathcal{V}_{\Sigma_3,x}}$ the BV Laplacian and $\mathbf{\Omega}_{\partial\Sigma_3}$ the full BFV boundary operator.

Theorem 2.2.2 (mdQME for anomaly-free globalized split RW model)**.** *Consider the full covariant perturbative state $\hat{\boldsymbol{\psi}}_{\Sigma_3,x}$ as a quantization of the anomaly-free globalized split RW model. Then*

$$\left(d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}\right)\hat{\boldsymbol{\psi}}_{\Sigma_3,x} = 0. \tag{2.2.9}$$

We also provide an explicit expression for the BFV boundary operator up to one bulk vertices in the $\mathbb{B}$-representation by adapting to our case the degree counting techniques of [CMW19]. Unfortunately, due to some complications related to the number of Feynman rules, we are not able to provide an explicit expression of the BFV boundary operator in the $\mathbb{B}$-representation in the case of a higher number of bulk vertices.

2.3 Convolution Algebras for Relational Symplectic Groupoids and Reduction (Joint with I. Contreras and K. Wernli)

2.3.1 Abstract

We introduce the notions of relational groupoids and relational convolution algebras. We provide various examples arising from the group algebra of a group G and a given normal subgroup H. We also give conditions for the existence of a Haar system of measures on a relational groupoid compatible with the convolution, and we prove a reduction theorem that recovers the usual convolution of a Lie groupoid.

2.3.2 Main results

We study various examples of relational convolution algebras that arise from extending Haar systems of measures to relational groupoids, and we prove the main result: a *reduction theorem for relational convolution algebras*, which recovers the usual groupoid

convolution algebra. This is also the first step towards proving the "quantization commutes with reduction" conjecture by Guillemin and Sternberg [GS82] in the setting of groupoid quantization.
In particular, this result serves as the first step towards reduction of its quantization (convolution algebras for relational groupoids). The next step is to construct the polarized algebra for relational symplectic groupoids. In addition to this, we hope to use relational convolution algebras to recover the C^*-algebra quantization of Poisson pencils via reduction, recovering the results obtained in [Bon+14] regarding the Bohr-Sommerfeld groupoid.
The second main idea behind this paper is that the relational symplectic groupoids could be used to study the relation between groupoid quantization and deformation quantization in a field-theoretic way, as follows. Relational symplectic groupoids were introduced in [CC15; CC13] in order to describe the groupoid structure of the phase space of a 2-dimensional topological field theory, the *Poisson Sigma Model* (PSM) [Ike94; SS94; CF01b], *before* gauge reduction [CF01d]. In [CF00], Cattaneo and Felder have shown that the perturbative quantization of the PSM using the Batalin–Vilkovisky (BV) formalism [BV77; BV81; BV83] yields Kontsevich's star product [Kon03], a deformation quantization associated to any Poisson manifold.
Recently [CMW17], this formalism has been applied to the relational symplectic groupoid for constant Poisson structures, linking the BV-BFV perturbative quantization of the relational symplectic groupoid and Kontsevich's star product in this case by methods of cutting and gluing for Lagrangian evolution relations. Another motivation to this paper is the connection between groupoids and Frobenius objects in a dagger monoidal category. For instance, a representative example of a relational convolution algebra is the relational group algebra, a version *up to equivalence*, of the group algebra of a group G. Group algebras are particular cases of Frobenius algebras, so relational convolution algebras provide a new class of examples of Frobenius objects in the category of sets and relations, which are also in correspondence with groupoids [HCC13; MZ20].

2.4 On Quantum Obstruction Spaces and Higher Codimension Gauge Theories

2.4.1 Abstract

Using the quantum construction of the BV-BFV method for perturbative gauge theories, we show that the obstruction for quantizing a codimension 1 theory is given by the second cohomology group with respect to the boundary BRST charge. Moreover, we give an idea for the algebraic construction of codimension k quantizations in terms of $\mathbb{E}_k$-algebras and higher shifted Poisson structures by formulating a higher version of the quantum master equation.

2.4.2 Main results

We show that the obstruction for the quantization of manifolds with boundary is controlled by the second cohomology group with respect to the cohomological vector field on the boundary fields. Moreover, we formulate a classical extension of higher codimension k theories as in [CMR14] which we call BF^{k}V theories. The coupling for each stratum, in fact, is easily extended in the classical setting (BV-BF^{k}V theories), whereas for the quantum setting it might be rather involved. In order to formulate a fully extended topological quantum field theory in the sense of Baez–Dolan [BD95] or Lurie [Lur09], the coupling is indeed necessary. Since one layer of the quantum picture, namely the quantum master equation, is described in terms of deformation quantization, we can formulate an algebraic approach for the higher codimension extension in terms of $\mathbb{E}_k$- and $\mathbb{P}_k$-algebras [Lur17; Saf18]. Here $\mathbb{E}_k$ denotes the ∞-operad of little k-dimensional disks [Lur17; Kon99; FW20]. Moving to one codimension higher corresponds to the shift of the Poisson structure by -1 since the symplectic form is shifted by $+1$ (see [Pan+13] for the shifted symplectic setting). This is controlled by the operad $\mathbb{P}_k$ on codimension k which corresponds to $(1-k)$-shifted Poisson structures [Cal+17; Saf17]. Using this notion, we give some ideas for the quantization in higher codimension. Moreover, if one uses the notion of Beilinson–Drinfeld ($\mathbb{BD}$) algebras [BD04; CG16], in particular $\mathbb{BD}_0$- and $\mathbb{BD}_1$-algebras, one can try to consider the action of $\mathbb{P}_0 \cong \mathbb{BD}_0/\hbar$ (for $\hbar \to 0$) on $\mathbb{P}_1 \cong \mathbb{BD}_1/\hbar$ (for $\hbar \to 0$) in order to capture the algebraic structure of the classical bulk-boundary coupling (see also [Saf17, Section 5]). Here $\cong$ denotes an isomorphism of operads. In general, one can define the $\mathbb{BD}_k$ operads to provide a certain interpolation between the $\mathbb{P}_k$ and $\mathbb{E}_k$ operads in the sense that they are graded Hopf [LP08] differential graded (dg) operads over $\mathbf{K}[\![\hbar]\!]$, where $\hbar$ is of weight $+1$ and $\mathbf{K}$ a field of characteristic zero, together with the equivalences

$$\mathbb{BD}_k/\hbar \cong \mathbb{P}_k, \qquad \mathbb{BD}_k[\![\hbar^{-1}]\!] \cong \mathbb{E}_k(\!(\hbar)\!).$$

The formality of the $\mathbb{E}_k$ operad [Tam03; Kon99; FW20] implies the equivalence $\mathbb{BD}_k \cong \mathbb{P}_k[\![\hbar]\!]$. There is a formulation of a $\mathbb{BD}_2$-algebra in terms of brace algebras [CW15; Saf18] and one can show that there is in fact a quasi-isomorphism $\mathbb{P}_2 \cong \mathbb{BD}_2/\hbar$ (for $\hbar \to 0$). However, the notion of a $\mathbb{BD}_k$-algebra for $k \geq 3$ in terms of braces is currently not defined, but there should not be any obstruction to do this. Using these operads, one can define a deformation quantization of a $\mathbb{P}_{k+1}$-algebra A to be a $\mathbb{BD}_{k+1}$-algebra $A_\hbar$ together with an equivalence of $\mathbb{P}_{k+1}$-algebras $A_\hbar/\hbar \cong A$ (see [Cal+17; MS18] for a detailed discussion).

2.5 Formal Global AKSZ Gauge Observables and Generalized Wilson Surfaces

2.5.1 Abstract

We consider a construction of observables by using methods of supersymmetric field theories. In particular, we give an extension of AKSZ-type observables constructed in [Mne15] using the Batalin–Vilkovisky structure of AKSZ theories to a formal global version with methods of formal geometry. We will consider the case where the AKSZ theory is "split" which will give an explicit construction for formal vector fields on base and fiber within the formal global action. Moreover, we consider the example of formal global generalized Wilson surface observables whose expectation values are invariants of higher-dimensional knots by using BF field theory. These constructions give rise to interesting global gauge conditions such as the differential Quantum Master Equation and further extensions.

2.5.2 Main results

The aim of this paper is to extend the constructions of [Mne15] to a formal global construction. In fact we will construct formal global observables by using the notion of a Hamiltonian Q-bundle [KS15] together with notions of formal geometry, and we will study the formal global extension of Wilson loop type observables for the Poisson sigma model. Additionally, we discuss the formal global extension of Wilson surface observables which have been studied in [CR05] by using the AKSZ formulation of BF theories. We will show that these constructions lead to interesting gauge conditions such as the *differential Quantum Master Equation* (and further extensions).
These constructions are expected to extend to manifolds with boundary by using the BV-BFV formalism as the globalization constructions have been studied for nonlinear split AKSZ theories on manifolds with boundary [CMW19].

2.6 Computation of Kontsevich Weights of Connection and Curvature Graphs for Symplectic Poisson Structures (Joint with F. Musio)

2.6.1 Abstract

We give a detailed explicit computation of weights of Kontsevich graphs which arise from connection and curvature terms within the globalization picture as in [CMW20] for the special case of symplectic manifolds. We will show how the weights for the curvature graphs can be explicitly expressed in terms of the hypergeometric function as well as by a much simpler formula combining it with the explicit expression for the weights of its underlined connection graphs. Moreover, we consider the case of a cotangent bundle, which will simplify the curvature expression significantly.

2.6.2 Main results

Let M be a smooth manifold and let $\phi\colon TM \to M$ be a formal exponential map and consider the lift $\overline{\phi}\colon TN \to N$ to the cotangent bundle $N = T^*M$. We set $x = (q, p) \in N$ and $y = (\bar{q}, \bar{p}) \in T_xN$. Note that this is a particular case of a canonical symplectic manifold. We will consider the lifted vector fields $\overline{R}$ to the cotangent case, which induce lifted interaction vertices within the Feynman graphs which appear in the computation of the connection 1-form and its curvature 2-form and see how these terms simplify. First we note that $A(\overline{R}, \mathsf{T}\overline{\phi}^*\pi)$ is still given by

$$A(\overline{R}_x, \mathsf{T}\overline{\phi}_x^*\pi)(\sigma_x) = \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \frac{1+(-1)^n}{2^{n+1}(n+1)} (\mathsf{T}\overline{\phi}_x^*\pi)^{i_1 j_1} \cdots (\mathsf{T}\overline{\phi}_x^*\pi)^{i_n j_n} (\overline{R}_x)^k_{i,i_1\cdots i_n}(\sigma_x)_{,kj_1\cdots j_n}, \tag{2.6.1}$$

where T denotes the Taylor expansion around $y = 0$. The simplification in this case is a small one: All summands containing a term $(\overline{R}_x)^k_{i,i_1\cdots i_n}$ with more than one derivative with respect to $\bar{p}$ will vanish [Mos19]. For the case of the curvature 2-form F^N the simplification is more interesting. Since for each non-vanishing coefficient $(\mathsf{T}\overline{\phi}_x^*\pi)^{ij}$ one of the two outgoing edges is always representing a $\bar{q}$-derivative and the other corresponding edge representing a $\bar{p}$-derivative (since we work with Darboux coordinates around $x \in N$), we see that the sum of the Weyl curvature terminates at $n = 2$. Or put differently, we only have to consider the graphs Γ_n up to $n = 2$, i.e. with at most two wedges attached to the wheel consisting of two $\overline{R}$-vertices. Moreover, since the Kontsevich weights[1] w_{Γ_n} are, up to $n = 2$, given by $w_{\Gamma_0} = 0$, $w_{\Gamma_1} = \frac{1}{24}$ and $w_{\Gamma_2} = 0$, we get

$$F_x^N = F(\overline{R}_x, \overline{R}_x, \mathsf{T}\overline{\phi}_x^*\pi) = \frac{\hbar}{48} (\mathsf{T}\overline{\phi}_x^*\pi)^{rs} (\overline{R}_x)^k_{i,lr} (\overline{R}_x)^l_{j,ks} \mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{2.6.2}$$

where we sum over the indices i, j, r, s, k, l and where again summands containing a term $(\overline{R}_x)^k_{i,lr}$ with more than one derivative with respect to $\bar{p}$ vanish. So in the case of a cotangent bundle we get a much simpler expression for the Weyl curvature F^N.

[1]The weights w_{Γ_n} are the weights of graphs of a wheel consisting of two points, i.e. two edges between two points, and n wedge-graphs attached to it.

Bibliography

[AB96] D. M. Austin and P. J. Braam. “Equivariant Floer Theory and Gluing Donaldson Polynomials”. In: *Topology* 35.1 (1996), pp. 167–200.

[Ale+97] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky. “The geometry of the master equation and topological quantum field theory”. In: *Internat. J. Modern Phys. A* 12.7 (1997), pp. 1405–1429.

[AS91] S. Axelrod and I. M. Singer. “Chern-Simons perturbation theory”. In: *Differential geometric methods in theoretical physics, Proceedings, New York.* Vol. 1. 1991, pp. 3–45.

[AS94] S. Axelrod and I. M. Singer. “Chern-Simons perturbation theory. II”. In: *J. Differ. Geom.* 39.1 (1994), pp. 173–213.

[Ati87] M. F. Atiyah. “New invariants of three-dimensional and four-dimensional manifolds”. In: *The Mathematical Heritage of Hermann Weyl, Proceedings.* 1987, pp. 285–299.

[Ati88] M. F. Atiyah. “Topological quantum field theories”. In: *Publ.math. IHÉS* 68.1 (1988), pp. 175–186.

[Bar95] D. Bar-Natan. “On the Vassiliev knot invariants”. In: *Topology* 34.2 (1995), pp. 423–472.

[Bay+78a] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. “Deformation theory and quantization. I. Deformations of symplectic structures”. In: *Ann. Phys.* 111.1 (1978), pp. 61–110.

[Bay+78b] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. “Deformation theory and quantization. II. Physical applications”. In: *Ann. Phys.* 111.1 (1978), pp. 111–151.

[BCM12] F. Bonechi, A. S. Cattaneo, and P. Mnev. “The Poisson sigma model on closed surfaces”. In: *J. High Energy Phys.* 1 (2012), pp. 099, 26.

[BD04] A. Beilinson and V. Drinfeld. *Chiral Algebras.* Vol. 51. Amer. Math. Society, Colloquium publications, 2004, p. 375.

[BD95] J. C. Baez and J. Dolan. “Higher dimensional algebra and topological quantum field theory”. In: *J. Math. Phys.* 36 (1995), pp. 6073–6105.

[BF83] I. A. Batalin and E. S. Fradkin. "A generalized canonical formalism and quantization of reducible gauge theories". In: *Phys. Lett. B* 122.2 (1983), pp. 157–164.

[BF86] I. A. Batalin and E. S. Fradkin. "Operator quantization and abelization of dynamical systems subject to first-class constraints". In: *La Rivista Del Nuovo Cimento Series 3* 9.10 (1986), pp. 1–48.

[BL75] F. A. Berezin and D. A. Leites. "Supermanifolds". In: *Dokl. Akad. Nauk SSSR* 224.3 (1975), pp. 505–508.

[Bon+14] F. Bonechi, N. Ciccoli, J. Qui, and M. Tarlini. "Quantization of Poisson manifolds from the integrability of the modular function". In: *Commun. Math. Phys.* 331 (2014), pp. 851–885.

[Bon+20] F. Bonechi, A. S. Cattaneo, J. Qiu, and M. Zabzine. "Equivariant Batalin-Vilkovisky formalism". In: *J. Geom. Phys.* 154 (2020).

[Bot10] R. Bott. "Some Aspects of Invariant Theory in Differential Geometry". In: *Differential Operators on Manifolds.* Springer Berlin Heidelberg, 2010, pp. 49–145.

[BRS74] C. Becchi, A. Rouet, and R. Stora. "The abelian Higgs Kibble model, unitarity of the S-operator". In: *Phys. Lett. B* 52.3 (1974), pp. 344–346.

[BRS75] C. Becchi, A. Rouet, and R. Stora. "Renormalization of the abelian Higgs-Kibble model". In: *Commun. Math. Phys.* 42.2 (1975), pp. 127–162.

[BRS76] C. Becchi, A. Rouet, and R. Stora. "Renormalization of gauge theories". In: *Annals of Physics* 98.2 (1976), pp. 287–321.

[BV77] I. A. Batalin and G. A. Vilkovisky. "Relativistic S-matrix of dynamical systems with boson and fermion constraints". In: *Phys. Lett. B* 69.3 (1977), pp. 309–312.

[BV81] I. A. Batalin and G. A. Vilkovisky. "Gauge algebra and quantization". In: *Phys. Lett. B* 102.1 (1981), pp. 27–31.

[BV83] I. A. Batalin and G. A. Vilkovisky. "Quantization of gauge theories with linearly dependent generators". In: *Phys. Rev. D* 28.10 (1983), pp. 2567–2582.

[BW12] S. Bates and A. Weinstein. *Lectures on the Geometry of Quantization.* University Reprints, 2012.

[Cal+17] D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. "Shifted Poisson structures and deformation quantization". In: *J. Topol.* 10.2 (2017), pp. 483–584.

[CC13] A. S. Cattaneo and I. Contreras. "Groupoids and Poisson sigma models with boundary". In: *Geometric, Algebraic and Topological Methods for Quantum Field Theory.* World Scientific, 2013.

[CC15] A. S. Cattaneo and I. Contreras. "Relational Symplectic Groupoids". In: *Lett. Math. Phys.* 105.5 (2015), pp. 723–767.

[CF00] A. S. Cattaneo and G. Felder. “A path integral approach to the Kontsevich quantization formula”. In: *Commun. Math. Phys.* 212 (2000), pp. 591–611.

[CF01a] A. S. Cattaneo and G. Felder. “On the AKSZ formulation of the Poisson sigma model”. In: *Lett. Math. Phys.* 56.2 (2001), pp. 163–179.

[CF01b] A. S. Cattaneo and G. Felder. “On the Globalization of Kontsevich’s Star Product and the Perturbative Poisson Sigma Model”. In: *Progress of Theoretical Physics Supplement* 144 (2001), pp. 38–53.

[CF01c] A. S. Cattaneo and G. Felder. “Poisson sigma models and deformation quantization”. In: *Mod. Phys. Lett. A* 16 (2001), pp. 179–190.

[CF01d] A. S. Cattaneo and G. Felder. “Poisson sigma models and symplectic groupoids”. In: *Quantization of Singular Symplectic Quotients.* Birkhäuser Basel, 2001, pp. 61–93.

[CFT02] A. S. Cattaneo, G. Felder, and L. Tomassini. “Fedosov connections on jet bundles and deformation quantization”. In: *Halbout G. Deformation quantization* (2002), pp. 191–202.

[CG16] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory, Volume 2.* 2016.

[CLL17] K. Chan, N. C. Leung, and Q. Li. “BV Quantization of the Rozansky–Witten Model”. In: *Commun. Math. Phys.* 355 (2017), pp. 97–144.

[CM20] A. S. Cattaneo and N. Moshayedi. “Introduction to the BV-BFV formalism”. In: *Rev. Math. Phys.* 32 (2020), p. 67.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Classical BV Theories on Manifolds with Boundary”. In: *Commun. Math. Phys.* 332.2 (2014), pp. 535–603.

[CMR17] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Perturbative Quantum Gauge Theories on Manifolds with Boundary”. In: *Commun. Math. Phys.* 357.2 (2017), pp. 631–730.

[CMW17] A. S. Cattaneo, P. Mnev, and K. Wernli. “Split Chern–Simons Theory in the BV-BFV Formalism”. In: Quantization, Geometry, Noncommutative Structures in Mathematics, and Physics–Springer International Publishing, 2017, pp. 293–324.

[CMW19] A. S. Cattaneo, N. Moshayedi, and K. Wernli. “Globalization for Perturbative Quantization of Nonlinear Split AKSZ Sigma Models on Manifolds with Boundary”. In: *Commun. Math. Phys.* 372.1 (2019), pp. 213–260.

[CMW20] A. S. Cattaneo, N. Moshayedi, and K. Wernli. “On the Globalization of the Poisson Sigma Model in the BV-BFV Formalism”. In: *Commun. Math. Phys.* 375.1 (2020), pp. 41–103.

[CMW21] I. Contreras, N. Moshayedi, and K. Wernli. “Convolution algebras for Relational Groupoids and Reduction”. In: *Pac. J. Math.* 313.1 (2021), pp. 75–102.

[Cos11a] K. Costello. *A geometric construction of the Witten genus, II.* 2011. arXiv: 1112.0816v2.

[Cos11b] K. Costello. *Renormalization and Effective Field Theory.* Vol. 170. Mathematical Surveys and Monographs. American Mathematical Society (AMS), 2011.

[CR05] A. S. Cattaneo and C. Rossi. "Wilson Surfaces and Higher Dimensional Knot Invariants". In: *Commun. Math. Phys.* 256 (2005), pp. 513–537.

[CS11] A. S. Cattaneo and F. Schätz. "Introduction to Supergeometry". In: *Reviews in Mathematical Physics* 23.6 (2011), pp. 669–690.

[CS74] S.-S. Chern and J. Simons. "Characteristic Forms and Geometric Invariants". In: *Ann. of Math. (2)* 99.1 (1974), pp. 48–69.

[CW15] D. Calaque and T. Willwacher. "Triviality of the higher formality theorem". In: *Proc. Amer. Math. Soc.* 143 (2015), pp. 5181–5193.

[Del95] P. Deligne. "Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte". In: *Sel. Math. New Ser.* 1.4 (1995), pp. 667–697.

[Dir30] P. A. M. Dirac. *The Principles of Quantum Mechanics.* Oxford University Press, 1930.

[DL83] M. DeWilde and P. B. A. Lecomte. "Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds". In: *Lett. Math. Phys.* 7.6 (1983), pp. 487–496.

[Don02] S. K. Donaldson. *Floer Homology Groups in Yang-Mills Theory.* Cambridge Tracts in Mathematics. Cambridge University Press, 2002.

[Don87] S. K. Donaldson. "The orientation of Yang-Mills moduli spaces and 4-manifold topology". In: *J. Differ. Geom.* 26.3 (1987), pp. 397–428.

[DT98] S. K. Donaldson and R. P. Thomas. "Gauge theory in higher dimensions". In: *The geometric universe (Oxford, 1996), Oxford Univ. Press* (1998), pp. 31–47.

[Fed94] B. V. Fedosov. "A simple geometrical construction of deformation quantization". In: *J. Differ. Geom.* 40.2 (1994), pp. 213–238.

[FF78] E. S. Fradkin and T. E. Fradkina. "Quantization of relativistic systems with boson and fermion first- and second-class constraints". In: *Phys. Lett. B* 72.3 (1978), pp. 343–348.

[FH65] R. P. Feynman and A. R. Hibbs. *Quantum mechanics and path integrals.* International series in pure and applied physics. New York : McGraw-Hill, 1965.

[Flo88] A. Floer. "Morse theory for Lagrangian intersections". In: *J. Differ. Geom.* 28 (1988), pp. 513–547.

[Flo89] A. Floer. "An instanton-invariant for 3-manifolds". In: *Commun. Math. Phys.* 118.2 (1989), pp. 215–240.

[FP67] L. D. Faddeev and V. N. Popov. "Feynman diagrams for the Yang-Mills field". In: *Phys. Lett. B* 25.1 (1967), pp. 29–30.

[FV75] E. S. Fradkin and G. A. Vilkovisky. "Quantization of relativistic systems with constraints". In: *Phys. Lett. B* 55.2 (1975), pp. 224–226.

[FV77] E. S. Fradkin and G. A. Vilkovisky. "Quantization of Relativistic Systems with Constraints: Equivalence of Canonical and Covariant Formalisms in Quantum Theory of Gravitational Field". In: *CERN Preprint* CERN-TH-2332 (1977).

[FW20] B. Fresse and T. Willwacher. "The intrinsic formality of E_n-operads". In: *J. Eur. Math. Soc.* 22.7 (2020), pp. 2047–2133.

[GF69] I. M. Gelfand and D. B. Fuks. "The cohomology of the Lie algebra of vector fields on a smooth manifold". In: *J. Funct. Analysis* 33 (1969), pp. 194–210.

[GF70] I. M. Gelfand and D. B. Fuks. "The cohomology of the Lie algebra of formal vector fields". In: *Izv. AN SSR* 34 (1970), pp. 110–116.

[GK71] I. M. Gelfand and D. A. Kazhdan. "Some problems of the differential geometry and the calculation of cohomologies of Lie algebras of vector fields". In: *Dokl. Akad. Nauk Ser. Fiz.* 200 (1971), pp. 269–272.

[Gro46] H. J. Groenewold. "On the principles of elementary quantum mechanics". In: *Physics* 12 (1946), pp. 405–460.

[Gro85] M. Gromov. "Pseudo holomorphic curves in symplectic geometry". In: *Invent. Math.* 82 (1985), pp. 307–347.

[GS82] V. Guillemin and S. Sternberg. "Geometric quantization and multiplicities of group representations". In: *Invent. Math.* 67.3 (1982), pp. 515–538.

[Haw08] E. Hawkins. "A groupoid approach to quantization". In: *J. Symplectic Geom.* 6.1 (2008), pp. 61–125.

[HCC13] C. Heunen, A. S. Cattaneo, and I. Contreras. "Relative Frobenius algebras are groupoids". In: *J. Pure Appl. Algebra* 217.1 (2013), pp. 114–124.

[Ike94] N. Ikeda. "Two-Dimensional Gravity and Nonlinear Gauge Theory". In: *Ann. Phys.* 235.2 (1994), pp. 435–464.

[Kap99] M. Kapranov. "Rozansky–Witten invariants via Atiyah classes". In: *Compos. Math.* 115.1 (1999), pp. 71–113.

[Khu04] H. M. Khudaverdian. "Semidensities on Odd Symplectic Supermanifolds". In: *Commun. Math. Phys.* 247.2 (2004), pp. 353–390.

[Kir85] A. Kirillov. "Geometric quantization". In: *Dynamical systems - 4* 4 (1985), pp. 141–176.

[Kon03] M. Kontsevich. "Deformation quantization of Poisson manifolds". In: *Lett. Math. Phys.* 66.3 (2003), pp. 157–216.

[Kon99] M. Kontsevich. "Operads and Motives in Deformation Quantization". In: *Lett. Math. Phys.* 48 (1999), pp. 35–72.

[KQZ13] J. Källén, J. Qiu, and M. Zabzine. "Equivariant Rozansky–Witten classes and TFTs". In: *J. Geom. Phys.* 64 (2013), pp. 222–242.

[KS15] A. Kotov and T. Strobl. "Characteristic classes associated to Q-bundles". In: *International Journal of Geometric Methods in Modern Physics* 12.01 (2015), p. 23.

[Lei80] D. A. Leites. "Introduction to the theory of supermanifolds". In: *Uspekhi Mat. Nauk* 35.1(211) (1980), pp. 3–57.

[LMO98] T. T. Q. Le, J. Murakami, and T. Ohtsuki. "On a universal perturbative invariant of 3-manifolds". In: *Topology* 37.3 (1998), pp. 539–574.

[LP08] M. Livernet and F. Patras. "Lie theory for Hopf operads". In: *J. Algebra* 319.12 (2008), pp. 4899–4920.

[Lur09] J. Lurie. "On the classification of topological field theories". In: *Current Developments in Mathematics* 2008 (2009), pp. 129–280.

[Lur17] J. Lurie. *Higher Algebra*. available at https://www.math.ias.edu/~lurie/. 2017.

[MM22] N. Moshayedi and F. Musio. *Computation of Kontsevich Weights of Connection and Curvature Graphs for Symplectic Poisson Structures*. 2019, to appear in Adv. Theor. Math. Phys. 25.5 (2022). arXiv: 1912.08742.

[Mne15] P. Mnev. "A Construction of Observables for AKSZ Sigma Models". In: *Lett. Math. Phys.* 105 (2015), pp. 1735–1783.

[Mos19] N. Moshayedi. *On Globalized Traces for the Poisson Sigma Model*. 2019. arXiv: 1912.02435.

[Mos20] N. Moshayedi. "Formal Global AKSZ Gauge Observables and Generalized Wilson Surfaces". In: *Ann. Henri Poincaré* 21 (2020), pp. 2951–2995.

[Mos21a] N. Moshayedi. *4-Manifold Topology, Donaldson-Witten Theory, Floer Homology and Higher Gauge Theory Methods in the BV-BFV Formalism*. 2021. arXiv: 2107.00304.

[Mos21b] N. Moshayedi. "On Quantum Obstruction Spaces and Higher Codimension Gauge Theories". In: *Phys. Lett. B* 815 (2021).

[Moy49] J. E. Moyal. "Quantum mechanics as a statistical theory". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 45.01 (1949), p. 99.

[MS18] V. Melani and P. Safronov. "Derived coisotropic structures II: stacks and quantization". In: *Sel. Math.* 24 (2018), pp. 3119–3173.

[MS22] N. Moshayedi and D. Saccardo. "Formal Global Perturbative Quantization of the Rozansky-Witten Model in the BV-BFV Formalism". In: *J. Geom. Phys.* 174 (2022).

[MZ20] R. Mehta and R. Zhang. "Frobenius objects in the category of relations". In: *Lett. Math. Phys.* 110.7 (2020), pp. 1941–1959.

[Nek03] N. Nekrasov. "Seiberg-Witten prepotential from instanton counting". In: *Adv. Theor. Math. Phys.* 7.5 (2003), pp. 831–864.

[Pan+13] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. "Shifted symplectic structures". In: *Publications mathématiques de l'IHÉS* 117 (2013), pp. 271–328.

[Pol05] M. Polyak. "Feynman diagrams for pedestrians and mathematicians". In: *Proc. Symp. Pure Math.* 73 (2005), pp. 15–42.

[QZ10] J. Qiu and M. Zabzine. "Odd Chern-Simons Theory, Lie Algebra Cohomology and Characteristic Classes". In: *Commun. Math. Phys.* 300.3 (2010), pp. 789–833.

[RT91] N. Y. Reshetikhin and V. G. Turaev. "Invariants of 3-manifolds via link polynomials and quantum groups". In: *Invent. Math.* 103.1 (1991), pp. 547–597.

[RW97] L. Rozansky and E. Witten. "Hyper-Kähler geometry and invariants of three-manifolds". In: *Selecta Math.* 3 (1997), pp. 401–458.

[Saf17] P. Safronov. "Poisson reduction as a coisotropic intersection". In: *Higher Structures* 1.1 (2017), pp. 87–121.

[Saf18] P. Safronov. "Braces and Poisson additivity". In: *Compos. Math.* 154.8 (2018), pp. 1698–1745.

[Sch10] M. Schlichenmaier. "Berezin-Toeplitz Quantization for compact Kähler Manifolds. A Review of Results". In: *Adv. Math. Phys.* 2010 (2010).

[Sch93] A. Schwarz. "Geometry of Batalin-Vilkovisky quantization". In: *Commun. Math. Phys.* 155.2 (1993), pp. 249–260.

[SS94] P. Schaller and T. Strobl. "Poisson structure induced (topological) field theories". In: *Mod. Phys. Lett. A* 09.33 (1994), pp. 3129–3136.

[Ste17] P. Steffens. "Functorial Field Theory from Perturbative BV-BFV Quantization". Master Thesis. University of Amsterdam. 2017.

[SW94a] N. Seiberg and E. Witten. "Monopole Condensation, And Confinement In $N = 2$ Supersymmetric Yang-Mills Theory". In: *Nucl. Phys. B* 426 (1994), pp. 19–52.

[SW94b] N. Seiberg and E. Witten. "Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD". In: *Nucl. Phys. B* 431 (1994), pp. 484–550.

[Tam03] D. E. Tamarkin. "Formality of Chain Operad of Little Discs". In: *Lett. Math. Phys.* 66 (2003), pp. 65–72.

[Toë06] B. Toën. "Higher and derived stack: A global overview". In: *Algebraic Geometry-Seattle 2005, Proc. Sympos. Pure Math* 80 (2006).

[Toë14] B. Toën. *Derived Algebraic Geometry.* 2014.

[Tyu76] I. V. Tyutin. "Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism". In: *Preprints of P.N. Lebedev Physical Institute, No. 39* (1976).

[Wey31] H. Weyl. "The theory of groups and quantum mechanics". In: *Dover, New York, translated from* Quantenmechanik und Gruppentheorie*, Z. Physik (1927)* 46 (1931), pp. 1–46.

[Wit88a] E. Witten. "Topological quantum field theory". In: *Commun. Math. Phys.* 117.3 (1988), pp. 353–386.

[Wit88b] E. Witten. "Topological sigma models". In: *Commun. Math. Phys.* 118.3 (1988), pp. 411–449.

[Wit89] E. Witten. "Quantum field theory and the Jones polynomial". In: *Commun. Math. Phys.* 121.3 (1989), pp. 351–399.

[Wit91] E. Witten. "Two-dimensional gravity and intersection theory on moduli space". In: *Surveys in differential geometry* (1991), pp. 243–310.

[Woo97] N. Woodhouse. *Geometric Quantization.* Oxford University Press (OUP), 1997.

Part II

Collected Papers

Chapter 1

4-Manifold Topology, Donaldson–Witten Theory, Floer Homology and Higher Gauge Theory Methods in the BV-BFV Formalism

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4-MANIFOLD TOPOLOGY, DONALDSON–WITTEN THEORY, FLOER HOMOLOGY AND HIGHER GAUGE THEORY METHODS IN THE BV-BFV FORMALISM

NIMA MOSHAYEDI

Dedicated to Jürg Fröhlich on the occasion of his 75th birthday

Abstract. We study the behavior of Donaldson's invariants of 4-manifolds based on the moduli space of anti self-dual connections (instantons) in the perturbative field theory setting where the underlying source manifold has boundary. It is well-known that these invariants take values in the instanton Floer homology groups of the boundary 3-manifold. Gluing formulae for these constructions lead to a functorial topological field theory description according to a system of axioms developed by Atiyah, which can be also regarded in the setting of perturbative quantum field theory, as it was shown by Witten, using a version of supersymmetric Yang–Mills theory, known today as Donaldson–Witten theory. One can actually formulate an AKSZ model which recovers this theory for a certain gauge-fixing. We consider these constructions in a perturbative quantum gauge formalism for manifolds with boundary that is compatible with cutting and gluing, called the BV-BFV formalism, which was recently developed by Cattaneo, Mnev and Reshetikhin. We prove that this theory satisfies a modified Quantum Master Equation and extend the result to a global picture when perturbing around constant background fields. These methods are expected to extend to higher codimensions and thus might help getting a better understanding for fully extendable n-dimensional field theories (in the sense of Baez–Dolan and Lurie) in the perturbative setting, especially when $n \leq 4$. Additionally, we relate these constructions to Nekrasov's partition function by treating an equivariant version of Donaldson–Witten theory in the BV formalism. Moreover, we discuss the extension, as well as the relation, to higher gauge theory and enumerative geometry methods, such as Gromov–Witten and Donaldson–Thomas theory and recall their correspondence conjecture for general Calabi–Yau 3-folds. In particular, we discuss the corresponding (relative) partition functions, defined as the generating function for the given invariants, and gluing phenomena.

Contents

1. Introduction

1.1. **Overview and motivation.** There is no doubt that the study of gauge theories had a big influence on modern mathematics and physics. A particularly influential and important one is the study of the topology of 4-manifolds by using Yang–Mills theory and the notion of anti self-dual connections, also called *instantons*, which can be considered as a class of critical points of the Yang–Mills action functional. An early attempt, maybe even one of the starting points, was the construction of instantons given in [Ati+78]. Shortly thereafter, Donaldson introduced in [Don83; Don84; Don90] his famous polynomial invariants which are a type of topological invariants based on the theory of characteristic classes on vector bundles (topological K-theory) and the construction of the moduli space of anti self-dual connections. In particular, these invariants are described as integrals of a product of certain cohomology classes over the moduli space. Thus, defining these invariants relies very much on the behaviour of the moduli space as a sufficiently "nice" manifold. Many different people, including Uhlenbeck [Uhl82a; Uhl82b], Freed [FU84], Freedman [Fre82] and Taubes [Tau82], provided results which contributed to the fact that these moduli spaces are indeed "nice" enough. However, at first, these invariants have been only defined for the case when the 4-manifold is closed.

A new drive came into play when Floer introduced in [Flo89a; Flo89b] a type of topological invariant of closed 3-manifolds by using methods of gauge theories based on ideas of Witten regarding the constructions of Morse theory in the setting of perturbative quantum field theory [Wit82]. In particular, he constructed an infinite-dimensional version of Morse theory based on the moduli space of anti self-dual connections together with the Chern–Simons action functional playing the role of the Morse function. The main invariants are then given by the homology groups, called *instanton Floer homology groups*, similarly as the Morse homology groups in the finite setting. Braam and Donaldson then realized in [BD95b; Don02] that the polynomial invariants defined by Donaldson can be extended to 4-manifolds with boundary by imposing that the invariants are then exactly valued in these Floer groups. Moreover, they proved a gluing formula which endows the invariant with the structure of a functorial TQFT according to Atiyah's axioms [Ati88].

Another type of a similar approach to a homology theory for Floer's construction was considered in [Flo88] by using the symplectic manifolds and corresponding transversal Lagrangian submanifolds thereof. Besides important insights regarding the Arnold conjecture (see e.g. [HZ94]), it was soon realized that this type of homology theory plays a fundamental role in order to formulate and understand the mirror symmetry appearing in string theory [Yau92; Hor+03] from a homological point of view. In particular, it was Fukaya who constructed an A_∞-category in [Fuk93] (see also [Fuk+09a; Fuk+09b]) from out of this notion and Kontsevich who used these categories to formulate a conjecture which, for two mirror Calabi–Yau manifolds, relates this category of one side of the mirror to the derived category of coherent sheaves of the other side of the mirror through an equivalence of triangulated categories. This is known famously today as the *homological mirror symmetry conjecture* [Kon94a].

On the other hand, based on ideas of Atiyah [Ati87], Witten gave a way of obtaining Donaldson's polynomials by considering the perturbative expansion of the expectation value (path integral quantization) for a certain observable with respect to a local action functional. He also gave an argument involving the case of manifolds with boundary by interpreting the boundary states as the Floer groups in agreement with Donaldson's observation. The perturbative methods of treating quantum gauge theories have developed through time by different approaches.

The *Batalin–Vilkovisky (BV) formalism* [BV77; BV81; BV83a] provides a nice way of dealing with quantum gauge theories in a cohomological symplectic formalism [KT79] by using methods of functional integrals. In fact, the gauge-fixing there is equivalent to the choice of a Lagrangian submanifold which, by similar methods as the *BRST formalism* [BRS74; BRS75; Tyu76] and *Faddeev–Popov ghosts* [FP67], gives a way of computing the partition function by the perturbative expansion into Feynman graphs. The methods described by Batalin and Vilkovisky are considered from a Lagrangian point of view, whereas the Hamiltonian counterpart was described in the work of Batalin, Fradkin, Fradkina and Vilkovisky [BF83; BF86; FV75; FF78], usually called the *BFV formalism*. Note that these constructions can be considered separately for closed spacetime manifolds. However, one is often interested in the quantization picture for manifolds with boundary in order to use the concept of locality for the simplification through cutting and gluing properties. Additionally, everything should be consistent with Atiyah's TQFT axioms and Segal's axioms regarding conformal field theories [Seg88]. Such an extension has been recently provided by Cattaneo, Mnev and Reshetikhin to deal with the classical and quantum formalism of local gauge theories on manifolds with boundary in the cohomological symplectic setting by coupling the BV construction in the bulk to the BFV construction on the boundary [CMR14; CMR17]. These constructions can be easily extended to higher codimensions in the classical setting, but need more sophisticated techniques in the quantum setting. Nevertheless, this formalism is expected to give a reasonable candidate for a perturbative formulation of (fully) extended TQFTs (in the sense of Baez–Dolan [BD95a] or Lurie [Lur09]) for the case of interest.

As it was shown in [Ike11], it turns out that the field theory constructed by Witten can be naturally formulated by using a type of BV theory developed by Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) in [Ale+97] for a special gauge-fixing. This formulation extends in a nice way to the BV-BFV formalism since AKSZ theories often appear as suitable deformations of abelian BF theory [Mne19]. This is also the case for Donaldson–Witten theory, formulated as a 4-dimensional AKSZ theory with target $\mathfrak{h}[1]\oplus\mathfrak{h}[2]$ for some Lie algebra $\mathfrak{h}$. An approach to globalize a special type of AKSZ theories has been given in [CMW20]. There one starts with an AKSZ theory of any dimension which is of *split-type*. Choosing a background field as an element of the moduli space of classical solutions, one can vary the local theories over the target manifold by considering the *Grothendieck connection*

as in the setting of formal geometry developed by Gelfand–Fuks [GF69; GF70], Gelfand–Kazhdan [GK71] and Bott [Bot10]. Within the BV-BFV formalism, this leads to more general gauge conditions, such as the modified *differential* Quantum Master Equation.
Another approach, which is related to Donaldson's construction, to obtain topological invariants of 4-manifolds is due to Seiberg and Witten [SW94b; SW94a] (see also [Nic00] for a more mathematical introduction). They considered $\mathcal{N} = 2$ supersymmetric Yang–Mills theory[1] and formulated a set of equations (*Seiberg–Witten equations*) which contains the same information of the 4-manifold as the Yang–Mills equations but has the advantage that it is much easier to deal with. Solutions of these equations are usually called *monopoles*[2].
The field-theoretic approach of Seiberg–Witten theory is in its nature given by an A-model as in [Wit88b]. The low energy effective action can actually be written in terms of a holomorphic function F, called the *Seiberg–Witten prepotential*. In [Nek03], Nekrasov showed how one can compute this prepotential F as a certain limit by using techniques of equivariant localization for a given torus action to define a partition function proportional to the volume of the moduli space of instantons. In particular, this partition function is given as a generating function with coefficients given by the volumes of the connected components of the moduli space of instantons controlled through the *instanton numbers*. This is achieved by formulating a 6-dimensional theory (i.e. a given 4-manifold, or locally $\mathbb{R}^4$, times a torus $\mathbb{T}$) and then perform dimensional reduction. This is also called the $\mathcal{N} = 2$ *supersymmetric Yang–Mills theory in the* Ω*-background*, where Ω is a 4×4 matrix with entries given by one of the two generators of the torus $\mathbb{T}$ with a sign or zero in a certain way. This background in fact allows one to integrate out all the *fluctuations (high energy modes)* appearing in the functional integral quantization, and hence get rid of any divergencies.
Since the underlying $\mathcal{N} = 2$ supersymmetric Yang–Mills theory in this setting is based on the techniques of equivariant localization, one can study an equivariant version of Donaldson–Witten theory in the AKSZ-BV setting and consider a (regularized) perturbative quantization in order to obtain the mentioned partition function. In order to deal with with the quantization of such an equivariant field theory, it is natural to formulate an equivariant version of the BV formalism as it was developed in [Bon+20]. However, the extension to manifolds with boundary is still open but following the constructions of Atiyah, Donaldson and Witten for the non-equivariant setting, the boundary states, hence the geometric quantization procedure for the state space in the BV-BFV formalism, should recover an equivariant version of instanton Floer homology as discussed in [AB96].
An approach to the construction of Donaldson for higher dimensional gauge theories was initiated through the work of Donaldson and Thomas in [DT98] and developed further by Thomas in [Tho00]. In particular, using a holomorphic version of Chern–Simons theory [CS74], Thomas constructed topological invariants for Calabi–Yau 3-folds by constructing a holomorphic version of the Casson invariant [Sav99] which counts the bundles over the given Calabi–Yau 3-fold and extends to the counting of curves in algebraic 3-folds. The formulation of these invariants in terms of a weighted Euler characteristic was given by Behrend in [Beh09]. One can consider the Donaldson–Thomas partition function, similarly as for Gromov–Witten theory [Wit91; Beh97], as a generating function for the corresponding invariants on any Calabi–Yau 3-fold X. Moreover, one can prove gluing formulae for these partition functions, first considered in [LW15] by considering an associated divisor and the properties of the Hilbert scheme of 1-dimensional subschemes of X. By the nature

[1] Recall that $\mathcal{N}$ denotes the number of irreducible real spin representations in a supersymmetry (SUSY) algebra.
[2] According to the equations for magnetic monopoles in the theory of electromagnetism.

of its construction, Donaldson–Thomas theory is related to the construction of Nekrasov theory through its way of instanton counting (the modified moduli space of instantons in Nekrasov's theory is replaced by the Hilbert scheme of 1-dimensional subschemes of X in Donaldson–Thomas theory). Moreover, it seems plausible that the Donaldson–Thomas partition function is related to the Gromov–Witten partition function by a certain correspondence. Such a correspondence for general Calabi–Yau 3-folds was conjectures by Maulik, Nekrasov, Okounkov and Pandharipande in [Mau+06a; Mau+06b] and proven for toric 3-folds in [Mau+11] by Maulik, Oblomkov, Okounkov and Pandharipande. The correspondence seems to formulate a version of the homological mirror symmetry conjecture, which was already known to be true for toric 3-folds.

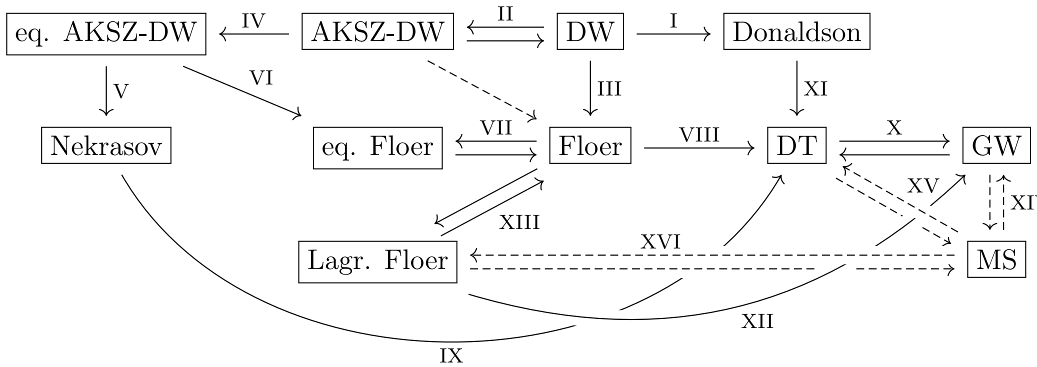

FIGURE 1.1.1. Diagrammatic illustration of the relations discussed in this paper.

Legend for Figure 1.1.1:

I Witten's construction to obtain Donaldson polynomials via field theory,
II AKSZ-BV formulation of Donaldson–Witten theory,
III When the source manifold has boundary. Boundary states are given by Floer groups,
IV Equivariant formulation of the AKSZ-BV formulation of Donaldson–Witten theory through an equivariant BV construction,
V BV partition function of equivariant Donaldson–Witten theory cooresponds to Nekrasov's partition function,
VI BV-BFV quantization of equivariant Donaldson–Witten theory produces equivariant Floer groups on the boundary,
VII Equivariant formulation of Floer (co)homology,
VIII Through Taubes' construction: the Euler characteristic with respect to Floer homology for any homology 3-sphere coincides with its Casson invariant. In particular, Donaldson–Thomas invariants are a complex version of such Euler characteristic when using the Floer groups with respect to the holomorphic Chern–Simons action functional for certain Calabi–Yau manifolds,
IX Through enumerative counting methods and the nature of its formulation,
X Conjectured correspondence by Maulik, Nekrasov, Okounkov and Pandharipande,
XI Higher gauge theory approach when passing to complex geometry,
XII Formulation of Gromov–Witten invariants through the moduli space of pseudo-holomorphic curves as an A-model path integral,
XIII The Atiyah–Floer conjecture,
XIV Ingredients for mirror symmetry,

XV Ingredients for mirror symmetry,
XVI Ingredients for mirror symmetry.

1.2. **Main purpose of the paper.** The purpose of this paper is to provide the reader with:

(*i*) a concise overview of the field of 4-manifold topology, both from the pure mathematical and field-theoretic point of view. Especially, to explain some of the relations of the literature as well as to new developements such as the BV-BFV techniques developed recently.

(*ii*) a concise overview of (instanton and Lagrangian) Floer (co)homology and how it fits into the 4D-3D bulk-boundary correspondence through Chern–Simons theory.

(*iii*) a detailed formulation of a (global) perturbative quantization of the AKSZ formulation of Donaldson–Witten theory on source manifolds with boundary which fits into the gauge theory setting of the BV-BFV formalism by using methods of configuration space integrals.

(*iv*) a short discussion of the quantization of higher defect theories through the recently developed methods of shifted (symplectic or Poisson) structures.

(*v*) a concise overview of the relation to Seiberg–Witten theory through Nekrasov's construction by using equivariant methods of localization and an equivariant version of the BV formalism in combination with Donaldson–Witten theory and a description of the relation with Donaldson's invariants through an equivariant version of instanton Floer (co)homology.

(*vi*) a description of the relation to constructions of Donaldson–Thomas theory.

(*vii*) a concise overview of the conjectured correspondence of Donaldson–Thomas theory and Gromov–Witten theory and connections to other fields, such as e.g. topological recursion or supergeometry.

1.3. **Structure of the paper.** The paper is structured as follows:

- In Section 2 we recall the definition of the moduli space of anti self-dual connections (instantons) together with some properties and define Donaldson's invariants. Moreover, we briefly discuss Witten's construction by using methods of field theory in order to obtain the aforementioned invariants.
- In Section 3 we recall the construction of Morse theory and Morse homology in order to construct instanton Floer homology by using the Chern–Simons action functional as a Morse function. We describe the relation of Donaldson's invariants with this homology theory through the setting of 4-manifolds with boundary. Moreover, we recall the construction of Lagrangian Floer homology, the Atiyah–Floer conjecture and give a remark on an approach for proving it.
- In Section 4 we recall the classical and quantum BV-BFV formalism, the notion of BV algebras and give the important examples of AKSZ theories and abelian BF theories. Moreover, we briefly discuss higher codimension (defects, branes) extensions and give some ideas for the quantization approach.
- In Section 5 we give an AKSZ construction of Donaldson–Witten theory and show how it fits into the classical BV-BFV setting.
- In Section 6 we consider the quantization of the classical setting obtained in Section 5 by describing the Feynman rules and the partition function. In particular, we prove that the Donaldson–Witten partition function satisfies a modified version of the Quantum Master Equation by using configuration space integrals and show that we get a well-defined cohomology theory on the boundary state space. Moreover, we construct a perturbative globalization approach by using methods of formal geometry

and argue that the partition function constructed through the globalization approach lies in the kernel of a certain operator that squares to zero which also leads to a cohomology theory in this setting, similarly as for the nonglobal case.

- In Section 7 we introduce Nekrasov's partition function in terms of polynomial counts through the Hilbert scheme of monomial ideals in $\mathbb{C}[u_1, u_2]$ and the Ext^1-groups by using the modular interpretation of the tangent bundle. We also consider an equivariant version of the BV gauge formalism and hence consider an equivariant version of Donaldson–Witten theory. On closed 4-manifolds the quantum picture corresponds to Nekrasov's partition function, whereas for 4-manifolds with boundary it induces an equivariant version of instanton Floer (co)homology as the boundary state. We describe the notion of equivariant Floer (co)homology following the construction of Austin–Braam in the Cartan model. Moreover, we discuss its relation to Donaldson's invariants. In particular, we consider the fact that these equivariant (co)homology groups appear as the (dual) image of the invariants.
- In Section 8 we recall Donaldson–Thomas invariants through their description as the relative topological Euler characteristic of the Hilbert scheme of 1-dimensional subschemes of a Calabi–Yau 3-fold by using Behrend's construction. We also recall Taubes' construction for their relation to Floer homology by considering them as a holomorphic version of the Casson invariant. Moreover, we construct the Donaldson–Thomas partition function as a generating function with respect to the Donaldson–Thomas invariants, extend it to a relative version when considering appropriate divisors of the underlying Calabi–Yau 3-fold and describe their gluing properties.
- In Section 9 we recall Kontsevich's moduli space of stable maps for a Calabi–Yau 3-fold and define the corresponding Gromov–Witten invariants. We also recall the partition function for Gromov–Witten invariants, similarly as before, as a generating series for these invariants and extend also this to a relative version for appropriate divisors. We briefly explain some ideas of topological recursion and its connection to Gromov–Witten invariants. To close the circle, we recall the Gromov–Witten/Donaldson–Thomas correspondence conjecture for Calabi–Yau 3-folds. Finally, we give some ideas for Gromov–Witten invariants on (graded) supermanifolds by considering ideas of Keßler–Sheshmani–Yau in the case of super Riemann surfaces.

Notation. Throughout the paper, Σ will denote a 4-manifold and N a 3-manifold. It might happen that N appears as the boundary 3-manifold of a 4-manifold Σ, otherwise the boundary of Σ is denoted by $\partial\Sigma$. Riemann surfaces of genus g will be denoted by Σ_g. Moduli spaces, of different flavor however, will be denoted by the calligraphic letter $\mathcal{M}$ and specified through different decorations. We will denote real numbers by $\mathbb{R}$, complex numbers by $\mathbb{C}$ and integers by $\mathbb{Z}$. If a (co)homology group takes values in $\mathbb{R}$, we will not further emphasize it, i.e. we write $H^\bullet(\Sigma)$ instead of $H^\bullet(\Sigma, \mathbb{R})$, whereas for integer-valued groups we will try to indicate it, i.e. we write $H^\bullet(\Sigma, \mathbb{Z})$. The exterior product between differential forms is sometimes explicitly written and sometimes left out to avoid any cumbersome notation. The Einstein summation convention is assumed, i.e. we sum over repeating indices, whenever the summation sign is not written explicitly. Algebro-geometrical objects (e.g. (projective) algebraic varieties, schemes, etc.) are usually denoted by X, which should be distinguished from the classical base field in the AKSZ formulation of Donaldson–Witten theory, which will be also denoted by X. Classical action functionals are usually denoted by S (with additional decorations, depending on the given theory), whereas the BV action is denoted by the calligraphic version $\mathcal{S}$. The same holds for the space of fields, usually denoted by F, and its BV counterpart $\mathcal{F}$. The space of connections on some principal bundle P is usually denoted by $\mathcal{A}$ (sometimes also $\mathcal{A}(P)$ to emphasize the bundle). Smooth maps on a manifold M will be denoted either by $C^\infty(M)$ or by $\mathcal{O}_M$ (as the structure sheaf) depending on the context. Sections of a bundle E are denoted by $\Gamma(E)$. The space of smooth differential forms on a manifold M will be denoted by $\Omega^\bullet(M)$. Differential forms on M with values in some bundle E will be denoted by $\Omega^\bullet(M,E)$. Finally, we will denote by $\mathrm{i}:=\sqrt{-1}$ the imaginary unit.

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2. Donaldson–Witten theory

2.1. **Moduli space of anti self-dual connections.** Let (Σ,g) be a 4-dimensional Riemannian manifold and let $\mathfrak{g}=\mathrm{Lie}(G)$ be the Lie algebra of a Lie group G. Moreover, let $P\to\Sigma$ be a principal G-bundle and consider its adjoint bundle $\mathrm{ad}\,P:=P\times_G\mathfrak{g}$. Using the Hodge star operator $*$, we can define for a 2-form $\chi\in\Omega^2(\Sigma)$ its *self-dual* and *anti self-dual* parts by

$$\chi^\pm=\frac{1}{2}(\chi\pm *\chi). \tag{2.1.1}$$

Note that in four dimensions we have $*\colon H^2(\Sigma)\to H^2(\Sigma)$. We can extend the splitting of (2.1.1) naturally to differential forms with values in $\mathrm{ad}\,P$. In particular we can extend this to the curvature 2-form F_A of a connection 1-form $A\in\Omega^1(\Sigma,\mathrm{ad}\,P)$. A connection $A\in\Omega^1(\Sigma,\mathrm{ad}\,P)$ is called *anti self-dual* or *instanton* if $F_A^+=0$. We define the *instanton number* of A to be

$$k_A:=\frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(F_A\wedge F_A). \tag{2.1.2}$$

This number is an integer only if $G=\mathrm{SU}(2)$. Moreover, if the corresponding SU(2)-bundle lifts to a 2-dimensional complex vector bundle E, we have $k_A=\int_\Sigma c_2(E)\in\mathbb{Z}$, i.e. the instanton number is determined by the second Chern class since $c_1(E)=0$ (because in this case $\mathrm{Tr}(F_A\wedge F_A)=0$). Note that if A is an anti self-dual connection, the instanton number is positive. Consider the *Yang–Mills action functional*[3]

$$S_\Sigma^{\mathrm{YM}}(A)=\int_\Sigma\|F_A\|^2, \tag{2.1.3}$$

where we have considered the curvature locally as $F_A=\frac{1}{2}F_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j$. Hence, we get that

$$S_\Sigma^{\mathrm{YM}}(A)=\int_\Sigma\|F_A\|^2=\int_\Sigma\|F_A^-\|^2+\int_\Sigma\|F_A^+\|^2\geq 8\pi^2k_A. \tag{2.1.4}$$

This shows that the critical points of S_Σ^{YM} are bounded by $8\pi^2k_A$, and in fact the minimum is attained if A is anti self-dual, i.e. $F_A^+=0$. Note that $\mathrm{Tr}(F_A\wedge F_A)=-\|F_A\|^2$ and thus $\mathrm{Tr}(F_A\wedge F_A)=-\left(\|F_A^+\|^2-\|F_A^-\|^2\right)$. Let us denote by $\mathcal{A}:=\Omega^1(\Sigma,\mathrm{ad}\,P)$ the space of connections on P. One can check that this is an infinite-dimensional space. Define the map

$$\begin{aligned} s\colon \mathcal{A}&\to\Omega^2_{\mathrm{ASD}}(\Sigma,\mathrm{ad}\,P),\\ A&\mapsto F_A^+, \end{aligned} \tag{2.1.5}$$

[3] We will not always indicate the measure whenever it is clear. When we write $\int\|F_A\|^2$, we actually mean $\int\|F_A\|^2\mathrm{d}\mu$.

where we have denoted by $\Omega^2_{\mathrm{ASD}}(\Sigma, \operatorname{ad} P)$ the anti self-dual 2-forms on Σ with values in $\operatorname{ad} P$. Moreover, denote by $\mathcal{G} := \Gamma(\mathrm{Aut}(\operatorname{ad} P))$ the infinite-dimensional Lie group of gauge transformations of $\operatorname{ad} P$. Then we can define the *moduli space of anti self-dual connections*[4] by

$$\mathcal{M}^k_{\mathrm{ASD}}(\Sigma, G) := s^{-1}(0) = \{[A] \in \mathcal{A}/\mathcal{G} \mid s(A) = 0,\, k_A = k\}. \tag{2.1.6}$$

2.2. **Donaldson polynomials.** In [Don83], Donaldson defined topological invariants for 4-manifolds by using methods of gauge theory. If Σ is an oriented, compact and simply connected 4-manifold, we can split $H^2(\Sigma)$ into eigenspaces $H^2_+(\Sigma) \oplus H^2_-(\Sigma)$ by using the Hodge star (dual and anti self-dual part). Let $b^2_+ := \dim H^2_+(\Sigma)$, $P \to \Sigma$ a principal SU(2)-bundle with second Chern class $c_2(P)$ and $k := \int_\Sigma c_2(P) \in \mathbb{Z}$, the instanton number (2.1.2). The moduli space of anti self-dual connections $\mathcal{M}_{\mathrm{ASD}}$ in this setting[5], as defined in (2.1.6), is of dimension[6]

$$\dim \mathcal{M}_{\mathrm{ASD}} = 8k - 3(1 + b^2_+), \tag{2.2.1}$$

which is even if b^2_+ is odd. Let us assume that b^2_+ is indeed odd and let $\dim \mathcal{M}_{\mathrm{ASD}} = 2d$. We also need the notion of an *irreducible* connection. A connection $A \in \mathcal{A}$ is called *irreducible*, if the image of the holonomy representation is not contained[7] in a one parameter subgroup of SU(2). Denote the space of irreducible connections by $\mathcal{A}^* \subset \mathcal{A}$. Consider the quotient $\mathcal{A}^*/\mathcal{G}$ by the gauge transformations and note that $\mathcal{M}_{\mathrm{ASD}} \subset \mathcal{A}^*/\mathcal{G}$ is a subspace of the quotient. Moreover, it is quite compact and almost defines an element of $H_{2d}(\mathcal{A}^*/\mathcal{G}, \mathbb{Z})$.
Let $\mathcal{R}$ denote the space of all Riemannian metrics on Σ of class C^r for some $r \geq 0$. Freed and Uhlenbeck have shown in [FU84] that for all $k > 0$, the moduli space

$$\mathcal{M}_{\mathcal{R}} := (\mathcal{A}^*/\mathcal{G}) \times \mathcal{R}$$

is in fact a manifold. Moreover, for $k > 0$, there exists a subset of $\mathcal{R}' \subset \mathcal{R}$ (elements of $\mathcal{R}'$ are called *generic metrics*) such that $\mathcal{M}_{\mathrm{ASD}}$ is a smooth submanifold of $\mathcal{A}^*/\mathcal{G}$ for all $g \in \mathcal{R}'$ and with virtual dimension equal to the expected dimension.

2.2.1. *Uhlenbeck compactification.* By a combination of two theorems in [Uhl82a; Uhl82b], Uhlenbeck showed that there exists a natural compactification $\overline{\mathcal{M}}_{\mathrm{ASD}}$ of the moduli space $\mathcal{M}_{\mathrm{ASD}}$. If we denote the moduli space of anti self-dual connections by $\mathcal{M}^k_{\mathrm{ASD}}$ to indicate its

[4]Sometimes it is also called *moduli space of instantons*. Moreover, we drop the dependence on the instanton number k, the 4-manifold Σ and the group G from the notation and just write $\mathcal{M}_{\mathrm{ASD}}$ whenever it is clear.

[5]For $k > 0$, there is a more complicated notion of dimension called the *virtual dimension* (see Footnote 44 for more details on the *virtual fundamental class*) for the moduli space $\mathcal{M}_{\mathrm{ASD}}$. The virtual dimension and the dimension might be different for certain metrics. However, it was argued in [FU84] that for the subspace of Riemannian metrics on Σ consisting of *generic metrics*, the virtual dimension and the actual dimension of $\mathcal{M}_{\mathrm{ASD}}$ coincide, i.e.

$$\operatorname{vdim} \mathcal{M}_{\mathrm{ASD}} = \dim \mathcal{M}_{\mathrm{ASD}} = 8k - 3(1 + b^2_+).$$

[6]A general formula for the dimension is given by $\dim \mathcal{M}_{\mathrm{ASD}} = 4a(G)k - \dim G(1 + b^2_+)$, where $a(G) \in \mathbb{Z}$ is an integer depending on the group G (in our case $a(\mathrm{SU}(2)) = 2$). Note that we have $\dim \mathrm{SU}(2) = 3$. A first approach to the dimension was considered in [AHS77; AHS78] for S^4 which is given by $8k - 3$ since then $b_+ = 0$.

[7]This is equivalent to saying that the holonomy group of the connection is precisely SU(2) and not a proper subgroup.

dependence of the instanton number k, we get that

$$\overline{\mathcal{M}}_{\mathrm{ASD}} \subset \bigcup_{j=0}^{k} \mathcal{M}_{\mathrm{ASD}}^{k-j} \cup \mathrm{Sym}^j(\Sigma). \tag{2.2.2}$$

This holds for any generic metric $g \in \mathcal{R}'$. The following theorem due to Uhlenbeck will be very important also for later constructions:

Theorem 2.2.1 (Uhlenbeck[Uhl82b]). *Let A be an anti self-dual connection in a principal bundle P over the punctured 4-ball $B^4 \setminus \{0\}$. If the L^2-norm of the curvature F_A of A is finite, i.e. if*

$$\int_{B^4\setminus\{0\}} \|F_A\|^2 < \infty,$$

then there exists a gauge in which the bundle P extends to a smooth bundle $\widetilde{P}$ over B^4 and the connection A extends to a smooth anti self-dual connection $\widetilde{A}$ in B^4.

Note that Theorem 2.2.1 implies in particular that every anti self-dual connection over $\mathbb{R}^4$ with bounded Yang–Mills action functional with respect to the L^2-norm can be obtained from an anti self-dual connection over $S^4 = \mathbb{R}^4 \cup \{\infty\}$.

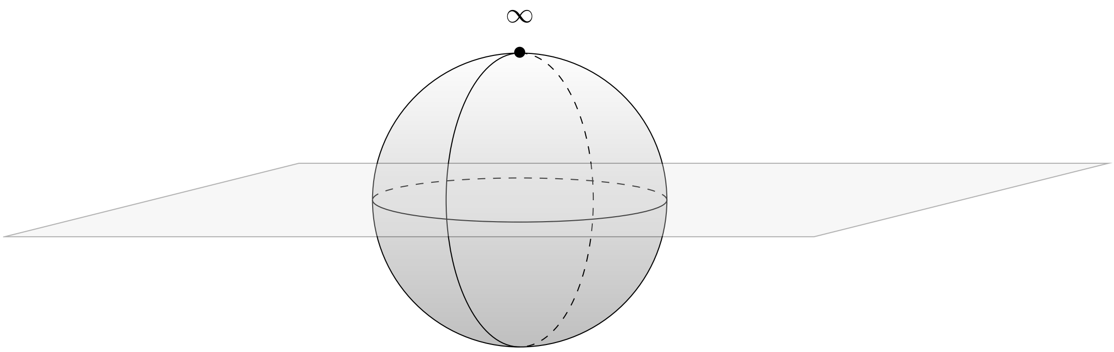

FIGURE 2.2.1. Compactification of $\mathbb{R}^4$ to S^4.

2.2.2. *Construction of invariants.* On $(\mathcal{A}^*/\mathcal{G}) \times \Sigma$ one can define a principal SO(3)-bundle $\widetilde{P}$ as $(\pi^*P \to \mathcal{A}^* \times \Sigma)/\mathcal{G}$ with $\pi\colon \mathcal{A} \times \Sigma \to \Sigma$ being the projection onto the second factor. Note that an SO(3)-bundle $P \to \Sigma$ is classified by the second Stiefel–Whitney class $w_2(P) \in H^2(\Sigma, \mathbb{Z}/2)$ and the first Pontryagin class $p_1(P) \in H^4(\Sigma, \mathbb{Z})$. When $w_2(P) = 0$, then the bundle lifts to an SU(2)-bundle with second Chern class $c_2(P) = -\frac{1}{4}p_1(P)$. However, also if $w_2(P) \neq 0$, everything can be done more or less similarly as before (although the orientability of the moduli space will be more complicated[8]). Using the first Pontryagin class $p_1(\widetilde{P}) \in H^4((\mathcal{A}^*/\mathcal{G}) \times \Sigma, \mathbb{Z})$, we can define a map

$$\begin{aligned} \mu\colon H_2(\Sigma, \mathbb{Z}) &\to H^2(\mathcal{A}^*/\mathcal{G}, \mathbb{Z}), \\ [C] &\mapsto \int_C p_1(\widetilde{P}). \end{aligned} \tag{2.2.3}$$

[8]In fact, if w is a class in $H^2(\Sigma, \mathbb{Z}/2)$ and devide the integral lifts c of w into equivalence classes through the relation $c \sim c$ if and only if $\frac{1}{2}(c - c')$ is even. Note that when Σ is spin, then there is only one equivalence class and there are two otherwise. The orientation on the moduli space is then induced by the orientation of H^2_+ and a choice of equivalence class of the integral lifts of the second Stiefel–Whitney class w_2 of the given SO(3)-bundle.

Thus, one can define polynomials $\mathcal{D}(\Sigma)$ of degree d on $H_2(\Sigma,\mathbb{Z})$ as a map

$$\begin{aligned}&\mathcal{D}(\Sigma)\colon H_2(\Sigma,\mathbb{Z})\times\cdots\times H_2(\Sigma,\mathbb{Z})\to\mathbb{Z},\\ &([C_1],\ldots,[C_d])\mapsto\mathcal{D}(\Sigma)([C_1],\ldots,[C_d]):=\int_{\overline{\mathcal{M}}_{\mathrm{ASD}}}\prod_{i=1}^d\mu([C_i]).\end{aligned}\tag{2.2.4}$$

Theorem 2.2.2 (Donaldson[Don83]). *Suppose that $b_+^2>1$ and $k>\frac{3}{2}\left(\frac{b_+^2+1}{2}\right)$. Then the polynomials $\mathcal{D}(\Sigma)$ are independent of the metric and indeed only depend on the homology classes of $C_1,\ldots,C_d$, hence $\mathcal{D}(\Sigma)$ define topological invariants of Σ.*

Theorem 2.2.3 (Donaldson[Don90]). *Suppose Σ is a simply connected, oriented 4-manifold with b_+^2 odd and there is an orientation preserving diffeomorphism between Σ and an oriented connected sum of manifolds Σ_1, Σ_2 both having $b_+^2>0$. Then $\mathcal{D}(\Sigma)=0$ for all k.*

2.3. **Field theory formulation.** In [Wit88a], Witten provided a way of obtaining the Donaldson polynomials by formulating a topological quantum field theory and using methods of functional integrals. In particular, he computed the expectation value of a certain observable by a perturbative expansion for a suitable Lagrangian density. The action that was used is given in components by

$$\begin{aligned}S_\Sigma^{\mathrm{DW}}=\int_\Sigma \mathrm{d}^4u\sqrt{g}\,\mathrm{Tr}\bigg(&\frac{3}{8}F_{ij}F^{ij}+\frac{1}{2}\phi D_iD^i\lambda-\mathrm{i}\eta D_i\psi^i+\mathrm{i}D_i\psi_j\chi^{ij}-\\ &-\frac{\mathrm{i}}{8}\phi[\chi_{ij},\chi^{ij}]-\frac{\mathrm{i}}{2}\lambda[\psi_i,\psi^i]-\frac{\mathrm{i}}{2}\phi[\eta,\eta]-\frac{1}{8}[\phi,\lambda]^2\bigg),\end{aligned}\tag{2.3.1}$$

where the fields are defined as in [Wit88a]. We will refer to this action as the *classical Donaldson–Witten (DW) action functional.* Denote by Φ the collection of all fields of the theory. Then, Donaldson's polynomials (2.2.4) can be obtained by using the correlation function with respect to (2.3.1) as

$$\langle O_{a_1}\cdots O_{a_d}\rangle:=\int\exp(\mathrm{i}S_\Sigma^{\mathrm{DW}}(\Phi)/\hbar)\prod_{j=1}^d O_{a_j}(\Phi)\mathscr{D}[\Phi],$$

for some observables $O_{a_1},\ldots,O_{a_d}$. The observables of interest are given by

$$O^{(\gamma)}:=\int_\gamma W_{k_\gamma},\tag{2.3.2}$$

where $\gamma\in H^{k_\gamma}(\Sigma,\mathbb{Z})$ and, for $k_\gamma=0,\ldots,4$, we have $W_0:=\frac{1}{2}\mathrm{Tr}(\phi\wedge\phi)$, $W_1:=\mathrm{Tr}(\phi\wedge\psi)$, $W_2:=\mathrm{Tr}(\frac{1}{2}\psi\wedge\psi+\mathrm{i}\phi\wedge F)$, $W_3:=\mathrm{i}\,\mathrm{Tr}(\psi\wedge F)$ and $W_4:=-\frac{1}{2}\mathrm{Tr}(F\wedge F)$. To each $O^{(\gamma)}$, one actually associates a $(4-k_\gamma)$-form $\mu(\gamma)$ on $\overline{\mathcal{M}}_{\mathrm{ASD}}$ as in Section 2.2. Explicitly, we get

$$\begin{aligned}\left\langle O^{(\gamma_1)}\cdots O^{(\gamma_d)}\right\rangle&=\int\exp(\mathrm{i}S_\Sigma^{\mathrm{DW}}(\Phi)/\hbar)\prod_{j=1}^d O^{(\gamma_j)}(\Phi)\mathscr{D}[\Phi]\\ &=\int\exp(\mathrm{i}S_\Sigma^{\mathrm{DW}}(\Phi)/\hbar)\prod_{j=1}^d\int_{\gamma_j}W_{k_j}(\Phi)\mathscr{D}[\Phi]\\ &=\int_{\overline{\mathcal{M}}_{\mathrm{ASD}}}\prod_{j=1}^d\mu(\gamma_j)=\mathcal{D}(\Sigma)(\gamma_1,\ldots,\gamma_d).\end{aligned}\tag{2.3.3}$$

Remark 2.3.1. The integral in (2.3.3) is rigorously defined since the differential forms $\mu(\gamma)$ can be obtained from each observable $O^{(\gamma)}$ by integrating out the high energy modes.

3. Instanton and Lagrangian Floer homology

3.1. **Morse homology.** Let us recall the construction of *Morse homology.* Fix a compact, closed manifold Σ. A function $f\colon \Sigma \to \mathbb{R}$ is called *Morse function*, if all its critical points are non-degenerate, i.e. for all critical points, the Hessian of f is invertible. In other words, the section $\mathrm{d}f$ is transverse to the zero section of $T^*\Sigma$. Since the Hessian is self-adjoint, it has real spectrum and hence we define $\mathrm{ind}_p f$ to be the dimension of the sum of all negative eigenspaces. Moreover, we assume that for any two critical points p, q of f we have that $\mathrm{ind}_p f > \mathrm{ind}_q f$ implies $f(p) > f(q)$. The function f is then said to be *self-indexing.* If we choose a Riemannian metric on Σ, we can look at the gradient ∇f and the corresponding flow equations

$$\dot{\gamma}(t) = -\nabla_{\gamma(t)} f. \tag{3.1.1}$$

For two critical points p, q of f, we define the moduli space $\mathcal{M}(p, q)$ of solutions of (3.1.1) such that

$$\lim_{t\to-\infty} \gamma(t) = p, \qquad \lim_{t\to+\infty} \gamma(t) = q. \tag{3.1.2}$$

Note that $\mathcal{M}(p, q)$ is empty unless $f(p) > f(q)$. In fact, we have the following theorem:

Theorem 3.1.1 (Morse–Smale)**.** *Let p, q be two distinct critical points of a Morse function f. Then $\mathcal{M}(p, q)$ is a manifold of dimension* $\mathrm{ind}_p f - \mathrm{ind}_q f$.

There is a free and proper $\mathbb{R}$-action on $\mathcal{M}(p, q)$ given by reparametrization $\gamma(t) \mapsto \gamma(t - a)$ for some $a \in \mathbb{R}$. We consider then the quotient

$$\overline{\mathcal{M}}(p, q) := \mathcal{M}(p, q)/\mathbb{R}.$$

One can check that if $\mathrm{ind}_p f = \mathrm{ind}_q f + 1$, then $\overline{\mathcal{M}}(p, q)$ is a compact, oriented 0-dimensional manifold. Hence, by counting its points by signs, we can deduce

$$\#\overline{\mathcal{M}}(p, q) \in \mathbb{Z}. \tag{3.1.3}$$

We can now construct a chain complex as follows. Define the chain groups to be given by

$$CM_k(\Sigma, f) := \bigoplus_{\substack{p \text{ critical point of } f\\ \mathrm{ind}_p f = k}} \mathbb{Z}\langle p\rangle. \tag{3.1.4}$$

The boundary operator ∂ will be constructed by counting flow lines. Namely, for critical points p of f with $\mathrm{ind}_p f = k$ we define the boundary operator by

$$\partial p = \sum_{\substack{q \text{ critical point of } f\\ \mathrm{ind}_q f = k-1}} \#\overline{\mathcal{M}}(p, q) \cdot q \tag{3.1.5}$$

One can show (although it is non-trivial) that $\partial^2 = 0$ and thus indeed defines a differential. Finally, *Morse homology* is given by the homology of this complex, and we denote it by

$$HM_\bullet(\Sigma, f) := \ker \partial/\mathrm{im}\, \partial.$$

3.2. **The (holomorphic) Chern–Simons action functional.** An important action functional for further discussions is given by Chern–Simons theory [CS74], which is a topological field theory on a 3-manifold with many connections to other mathematical theories [Wit89; RT91; CM08]. Let us briefly recall its construction and extension to a holomorphic version which will be important later (see Section 8). Let N be a real 3-manifold and consider a vector bundle $E \to N$ with structure group G. The curvature F_A of a connection $A \in \mathcal{A}$ defines a closed 1-form

$$a \mapsto \frac{1}{4\pi^2}\int_N \mathrm{Tr}(a \wedge F_A), \qquad a \in \Omega^1(N, \mathrm{ad}\, E) \tag{3.2.1}$$

on $\mathcal{A}$. In fact, we can extend (3.2.1) to gauge equivalence classes since this expression is gauge invariant. Fix a base point A_0 in the space of gauge equivalence classes. One can show that (3.2.1) actually appears as the exterior derivative of a local action functional given by

$$S_N^{\mathrm{CS}}(A) := \int_N \mathrm{Tr}\left(\mathrm{d}_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a\right), \qquad A = A_0 + a. \tag{3.2.2}$$

This is the Chern–Simons action functional. One can check that this is gauge invariant under transformations connected to the identity. Moreover, on the gauge equivalence classes it is well-defined modulo $\mathbb{Z}$. Note also that critical points are given by flat connections. One can extend this picture to Calabi–Yau 3-folds, i.e. smooth compact Kähler 3-folds X with trivial canonical bundle $K_X \cong \mathcal{O}_X$. Consider now the space $\mathcal{A}_{\mathrm{hol}} := \{\bar\partial\text{-operators on a fixed smooth bundle } E \to X\}$ and the closed 1-form

$$a \mapsto \frac{1}{4\pi^2}\int_X \mathrm{Tr}\left(a \wedge F_A^{0,2}\right) \wedge \mathrm{dvol}_X, \qquad a \in \Omega^{0,1}(X, \mathrm{ad}\, E), \tag{3.2.3}$$

where $F_A^{0,2}$ denotes the antiholomorphic curvature (i.e. with respect to $\bar\partial$) of a holomorphic connection $A \in \mathcal{A}_{\mathrm{hol}}$ and dvol_X denotes the complex volume form on X. similarly as before, the 1-form (3.2.3) is gauge invariant and thus descends to the space of gauge equivalence classes. Again, fixing a basepoint $A_0 \in \mathcal{A}_{\mathrm{hol}}$, (3.2.3) appears as the exterior derivative of a local holomorphic action functional given by

$$S_X^{\mathrm{hol,CS}}(A) := \frac{1}{4\pi^2}\int_X \mathrm{Tr}\left(\bar\partial_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a\right) \wedge \mathrm{dvol}_X, \qquad A = A_0 + a. \tag{3.2.4}$$

This the *holomorphic* Chern–Simons action functional. Again, one can check that (3.2.4) is gauge invariant with respect to transformations connected to the identity. The critical points of (3.2.4) are given by integrable holomorphic structures on the bundle E.

3.3. **A 4D-3D bulk-boundary correspondence on (infinite) cylinders.** Let Σ be a 4-manifold. Moreover, let $G = \mathrm{SU}(2)$ and consider a principal G-bundle $P \to \Sigma$. Then P has one characteristic class, the second *Chern class* $c_2(P) \in H^2(\Sigma, \mathbb{Z})$. Using the *Chern–Weil formalism* [Wei49; Che52], we can identify it with its de Rham cohomology representative in the image of $H^2(\Sigma, \mathbb{R})$, which is given by

$$c_2(P) = \frac{1}{8\pi^2}\mathrm{Tr}(F_A \wedge F_A), \tag{3.3.1}$$

for some connection 1-form $A \in \mathcal{A}$ on P and its corresponding curvature 2-form $F_A = \mathrm{d}A + \frac{1}{2}[A, A]$. Indeed, this cohomology class is independent of the choice of connection and it is a closed 2-form. Note also that its integral over any 4-manifold Σ is an integer, since

$c_2(P)$ is an integral class. If $G = \mathrm{SO}(3)$, we get that the corresponding class is given by the first Pontryagin class of the associated bundle $P^{\mathrm{SO}(3)} := P \times_{\mathrm{SO}(3)} \mathbb{R}^3$, which is given by

$$p_1(P^{\mathrm{SO}(3)}) = -\frac{1}{2\pi^2}\,\mathrm{Tr}(F_A \wedge F_A). \tag{3.3.2}$$

Note that these classes vanish on 3-manifolds. However, there is a way how we can construct invariants on 3-manifolds by using Chern–Simons theory (see Section 3.2). One can actually check that

$$\left(\int_\Sigma \mathrm{Tr}(F_A \wedge F_A)\right) \Big/ 8\pi^2\mathbb{Z} \tag{3.3.3}$$

depends only on the gauge equivalence class of $A\big|_{\partial\Sigma}$ and not on Σ nor on A in the bulk. If A extends[9] to a connection A' on an extended bundle $P' \to \Sigma'$, then we can glue Σ to Σ' along their common boundary N and obtain a new bundle $P'' \to \Sigma'' := \Sigma \cup_N \Sigma'$. Hence

$$\int_\Sigma \mathrm{Tr}(F_A \wedge F_A) - \int_{\Sigma'} \mathrm{Tr}(F_{A'} \wedge F_{A'}) = \int_{\Sigma''} \mathrm{Tr}(F_{A''} \wedge F_{A''}) \in 8\pi^2\mathbb{Z}. \tag{3.3.4}$$

Let B be a connection on $P \to N$. Then the Chern–Simons action functional $S_N^{\mathrm{CS}}(B)$ at B is given by (3.3.3) where F_A is the curvature of a connection A that is given as an extension of B to some bundle over Σ. Let B_0 be a fixed connection on P and consider a family of connections $(B_t)_{t\in[0,1]}$ such that $B_1 = B$, which we can regard as a connection A on $I \times P \to I \times N$. Then we have

$$S^{\mathrm{YM}}_{I\times N, B_0}(B) = \int_{I\times N} \|F_A\|^2 = -\int_{I\times N} \mathrm{Tr}(F_A \wedge F_A). \tag{3.3.5}$$

Consider the path $B_t = B_0 + tb$ for some b such that $B = B_0 + b$, and assume that B_0 is a trivial connection by choosing a trivialization for $G = \mathrm{SU}(2)$. Then

$$F_A = \mathrm{d}(tb) + \frac{t^2}{2}[b \wedge b] = \mathrm{d}t \wedge b + t\mathrm{d}b + \frac{t^2}{2}[b\wedge b], \tag{3.3.6}$$

which gives

$$\mathrm{Tr}(F_A \wedge F_A) = \mathrm{d}t \wedge \mathrm{Tr}(b \wedge (2t\mathrm{d}b + t^2[b \wedge b])). \tag{3.3.7}$$

Hence, we get the following chain of equality:

$$\begin{aligned} S^{\mathrm{YM}}_{I\times N}(B) &= -\int_{I\times N} \mathrm{Tr}(F_A \wedge F_A) \\ &= -\int_{I\times N} \mathrm{d}t \wedge \mathrm{Tr}(b \wedge (2t\mathrm{d}b + t^2[b\wedge b])) \\ &= -\int_N \mathrm{Tr}\left(b \wedge \mathrm{d}b + \frac{2}{3} b\wedge b \wedge b\right) = -S^{\mathrm{CS}}_{N,B_0}(b). \end{aligned} \tag{3.3.8}$$

Moreover, if B_0 does not arise from a trivialization, we get

$$S^{\mathrm{YM}}_{I\times N}(B) = -\int_N \mathrm{Tr}\left(2b \wedge F_{B_0} + b \wedge \mathrm{d}b + \frac{2}{3} b \wedge b \wedge b\right), \tag{3.3.9}$$

where F_{B_0} is the curvature of the connection B_0.

[9]In fact, one can always extend an SU(2)- or SO(3)-bundle on a closed oriented 3-manifold over some compact oriented 4-manifold. In particular, this follows from the fact that for these groups the cobordism group is given by the third homology group of their classifying spaces.

3.4. **Instanton Floer homology.** We will restrict ourselves to the case where $G = \mathrm{SU}(2)$ and N is an oriented integral homology 3-sphere[10] endowed with a Riemannian metric. Denote by $\mathcal{A}$ the set of all connections on $P \to N$, which is an affine space over $\Omega^1(N, \mathrm{ad}\, P)$. Recall that a gauge transformation g is a bundle automorphism of P covering the identity map of Σ. For such a transformation $g\colon P \to P$, we can construct a new map $\hat{g}\colon P \to G$ by $g(p) = p\hat{g}(p)$, with the property $\hat{g}(ph) = h^{-1}\hat{g}(p)h$ for all $p \in P$ and $h \in G$. We call the set of all gauge transformations $\mathcal{G}$. The elements of $\mathcal{G}$ are identified with sections of $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} G$. We would like to construct a Morse chain complex (as in Section 3.1) over $\mathcal{A}/\mathcal{G}$. Indeed, this can be done since by completing $\mathcal{A}$ and $\mathcal{G}$ with respect to the *Sobolev norm topology*[11], the spaces $\mathcal{A}$ and $\mathcal{G}$ have the structure of an infinite-dimensional manifold. Passing to the quotient will produce a manifold by a local slice theorem for the natural action of $\mathcal{G}$ on $\mathcal{A}$, which is given by pulling back 1-forms on P, i.e. $A \mapsto g^*A$. Note that the curvature F_A of A would transform to the curvature of g^*A by $F_{g^*A} = \mathrm{Ad}_{\hat{g}} F_A$. Denote by $\mathcal{A}^* \subset \mathcal{A}$ the subspace of irreducible connections and consider the tangent space of $\mathcal{A}^*$ at a reference connection B, which is given by

$$T_B\mathcal{A}^* \cong \Omega^1(N, \mathrm{ad}\, P). \tag{3.4.1}$$

Next, we would like to consider the tangent space at some equivalence class $[B] \in \mathcal{A}^*/\mathcal{G}$. For this purpose, we use the Hodge star $*\colon \Omega^j(N) \to \Omega^{3-j}(N)$ induced by the metric on N. Moreover, recall that on j-forms we have $*^2 = (-1)^{j(3-j)}$. Denoting by $\mathrm{d}_B\colon \Omega^j(N, \mathrm{ad}\, P) \to \Omega^{j+1}(N, \mathrm{ad}\, P)$ the covariant derivative with respect to B, we can define its formal adjoint by $\mathrm{d}_B^* := - * \mathrm{d}_B *$. Then we can obtain

$$T_{[B]}(\mathcal{A}^*/\mathcal{G}) \cong \ker(\mathrm{d}_B^*). \tag{3.4.2}$$

The key point for our Morse complex is that we want to consider the Chern–Simons action functional

$$S_N^{\mathrm{CS}}\colon \mathcal{A}^*/\mathcal{G} \to \mathbb{R}/8\pi^2\mathbb{Z}$$

to play the role of a Morse function[12]. For convenience, we actually want to consider $-S_N^{\mathrm{CS}}$. Recall from Section 3.2 that critical points of the Chern–Simons action functional on $\mathcal{A}^*$ are given by flat connections and on $\mathcal{A}^*/\mathcal{G}$ by gauge equivalence classes of flat connections.

The Floer chains $CF_\bullet(N)$ are given by the $\mathbb{Z}$-module with generators $[B]$ being gauge equivalence classes of flat connections of the trivial $\mathrm{SU}(2)$-bundle over N. For two flat connections B_0, B_1 consider a path $t \mapsto B_t$ connecting them. The corresponding path of operators has a spectral flow which is defined by pursuing the net number of eigenvalues crossing zero. Denote by $\mathrm{Eig}^{\mp}(B_t)$ the set of eigenvalues that pass along the path of operators B_t from negative to positive and by $\mathrm{Eig}^{\pm}(B_t)$ the set of eigenvalues that pass along the path of

[10] This is a 3-manifold N whose homology groups are the same as for S^3. Namely, $H_0(N,\mathbb{Z}) = H_3(N,\mathbb{Z}) \cong \mathbb{Z}$ and $H_1(N,\mathbb{Z}) = H_2(N,\mathbb{Z}) = 0$.

[11] For the analytic sides of the construction, one needs the notion of a Sobolev space L^p_ℓ of sections of bundles associated to P, i.e. sections locally represented by functions with first ℓ derivatives in L^p. There one can define a norm $\|\ \|_{L^p_\ell(A)}$ for any smooth connection A, for example, if $p = 2$ and $\ell = 1$, we have

$$\|\sigma\|^2_{L^2_1(A)} = \|\mathrm{d}_A\sigma\|^2_{L^2} + \|\sigma\|^2_{L^2} = \int_\Sigma \mathrm{d}\mu \left(\|\mathrm{d}_A\sigma\|^2 + \|\sigma\|^2\right).$$

[12] Of course, this requires the Chern–Simons action functional to actually be Morse, i.e. with critical points being nondegenerate. In particular, one can consider small perturbations ε (*holonomy perturbations*) such that $S^{\mathrm{CS}} + \varepsilon$ is indeed sufficiently nice. Here, $\varepsilon\colon \mathcal{A}^*/\mathcal{G} \to \mathbb{R}/8\pi^2\mathbb{Z}$ is some admissible perturbation function. In particular, the existence of regular values for the perturbed action is connected to the properties of being smooth and its differential to be a *Fredholm operator*.

operators B_t from positive to negative. Following [APS76], we define the *spectral flow* to be the number

$$\mathrm{sf}(B_0, B_1) := \#\mathrm{Eig}^{\mp}(B_t) - \#\mathrm{Eig}^{\pm}(B_t). \tag{3.4.3}$$

Note that in finite dimensions, this corresponds to the difference of the index of critical points. This will replace the grading of the Floer chains for the infinite-dimensional setting. Consider the space of connections A on $\mathbb{R} \times P$, which is a bundle over $\mathbb{R} \times N$, satisfying:

(1) A is anti-self dual, i.e. $F_A = - * F_A$,
(2) $\lim_{t\to-\infty}[A|_{\{t\}\times N}] =: A_-$,
(3) $\lim_{t\to+\infty}[A|_{\{t\}\times N}] =: A_+$,
(4) A has finite energy, i.e. the curvature F_A has finite L_2-norm:

$$\|F_A\|_2^2 = \int_{\mathbb{R}\times N} \|F_A\|^2 < \infty.$$

If we denote the space of such connections by $\widehat{\mathcal{M}}(A_-, A_+)$, we can define a moduli space

$$\mathcal{M}(A_-, A_+) := \widehat{\mathcal{M}}(A_-, A_+)/\mathcal{G},$$

where $\mathcal{G} := \mathrm{Aut}(\mathbb{R} \times P) = \{f : \mathbb{R} \times N \to \mathrm{SU}(2)\}$ denotes the gauge transformations. We can define an $\mathbb{R}$-action on $\mathcal{M}(A_-, A_+)$ by shifting the t variable and define

$$\overline{\mathcal{M}}(A_-, A_+) := \mathcal{M}(A_-, A_+)/\mathbb{R}.$$

In fact, there exists (see [Flo89a; Don02; Sav01]) a small (holonomy) perturbation $\varepsilon > 0$ of S_N^{CS} which leads to a moduli space $\mathcal{M}_\varepsilon(A_-, A_+)$ such that $\overline{\mathcal{M}}_\varepsilon(A_-, A_+)$ is a smooth oriented manifold with

$$\dim \overline{\mathcal{M}}_\varepsilon(A_-, A_+) = \mathrm{sf}(A_-, A_+) - 1.$$

Moreover, if $\dim \overline{\mathcal{M}}_\varepsilon(A_-, A_+) = 0$, then $\overline{\mathcal{M}}_\varepsilon(A_-, A_+)$ is compact. Finally, the boundary operator is defined through the counting of instantons as

$$\partial A_- := \sum_{\substack{A_+\in\mathcal{A}_{\mathrm{flat}} \\ \mathrm{sf}(A_-,A_+)=1}} \#\overline{\mathcal{M}}_\varepsilon(A_-, A_+) \cdot A_+,$$

where $\mathcal{A}_{\mathrm{flat}}$ denotes the space of flat connections. One can show that the resulting complex $CF_\bullet(N, \partial)$ is actually independent of the metric and the perturbation and that $\partial^2 = 0$. Therefore, we get a well-defined homology theory [Flo89a; Don02]. The corresponding homology, denoted by $HF_\bullet(N) := \ker\partial/\mathrm{im}\,\partial$, is called *instanton Floer homology*[13].

3.5. **Relation to Donaldson polynomials.** After Floer defined his homology groups, Donaldson soon realized how they were related to the polynomials he has constructed. Good expositions can be also found in [Ati87; Bra91]. Assume that $\Sigma = \Sigma_1 \cup_N \Sigma_2$, where N is an oriented homology 3-sphere and Σ_1 (resp. Σ_2) is a simply connected 4-manifold with boundary N (resp. N^{opp})[14]. By the assumption that $b_+^2 > 0$ for both Σ_1 and Σ_2, Donaldson defined polynomials

$$\mathcal{D}(\Sigma_1) : H_2(\Sigma_1, \mathbb{Z}) \times \cdots \times H_2(\Sigma_1, \mathbb{Z}) \to (HF_\bullet(N))^*, \tag{3.5.1}$$

$$\mathcal{D}(\Sigma_2) : H_2(\Sigma_2, \mathbb{Z}) \times \cdots \times H_2(\Sigma_2, \mathbb{Z}) \to (HF_\bullet(N^{\mathrm{opp}}))^*, \tag{3.5.2}$$

[13] We will sometimes also just call it *Floer homology*, i.e. dropping the word *instanton*, whenever it is clear. We mainly consider this type of Floer homology in this paper. There are different types of Floer homology constructions such as e.g. Lagrangian Floer homology (see Section 3.7) which, e.g., plays an important role in the formulation of Kontsevich's homological mirror symmetry conjecture (Conjecture 3.8.6)

[14] Here we denote N^{opp} to be N with opposite orientation.

that is that the polynomials $\mathcal{D}$ are valued in the dual of the Floer homology on the boundary. In fact, one can define a pairing $\langle\ ,\ \rangle_{HF}$ between elements of $(HF(N))^*$ and $(HF(N^{\mathrm{opp}}))^*$

$$\langle\ ,\ \rangle_{HF}\colon (HF_j(N))^* \times (HF_{3-j}(N^{\mathrm{opp}}))^* \to \mathbb{Z} \tag{3.5.3}$$

by using the fact that $CF_j(N) = CF_{3-j}(N^{\mathrm{opp}})$ and for irreducible flat connections $A \in CF_j(N)$ and $B \in CF_{j-1}(N)$, that both, $\langle \mathrm{d}_N A, B\rangle$ and $\langle A, \mathrm{d}_{N^{\mathrm{opp}}} B\rangle$, are the number of flow lines from A to B counted with sign. Here we have denoted by d_N the de Rham differential on N and by $\mathrm{d}_{N^{\mathrm{opp}}}$ the de Rham differential on N^{opp}. In fact, the signs are the same and hence these numbers do agree. Note that by defining a cochain complex $CF^j := \mathrm{Hom}(CF_j, \mathbb{Z})$, we get

$$HF_j(N) \cong HF^{3-j}(N^{\mathrm{opp}}).$$

Theorem 3.5.1 (Braam–Donaldson[BD95b; Don02])**.** *We have*

$$\mathcal{D}(\Sigma_1 \cup_N \Sigma_2) = \langle \mathcal{D}(\Sigma_1), \mathcal{D}(\Sigma_2)\rangle_{HF},$$

where $\langle\ ,\ \rangle_{HF}$ *denotes the pairing as in* (3.5.3)*.*

Theorem 3.5.1 tells us how Donaldson polynomials glue along boundaries of 4-manifolds. This will be interesting in connection to the perturbative field-theoretic approach of quantum gauge theories on manifolds with boundary which in the cohomological symplectic setting is compatible with cutting and gluing [CMR17]. We will see how this result fits into this framework.

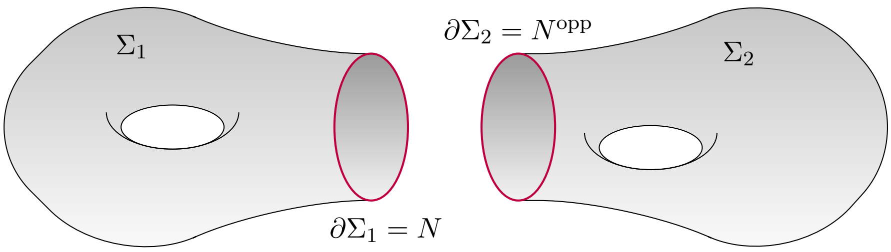

FIGURE 3.5.1. Gluing of two manifolds along a common boundary with opposite orientations. Think of Σ_1 and Σ_2 as two 4-manifolds with boundary 3-manifolds $\partial\Sigma_1$ and $\partial\Sigma_2$, respectively.

3.6. **Field theory approach to instanton Floer homology.** Based on ideas of Atiyah [Ati87], Witten gave an approach for a quantum field theoretic construction regarding the appearance of Floer homology in Donaldson theory for 4-manifolds with boundary in [Wit88b]. His construction uses a supersymmetric field theory approach to Morse theory developed in [Wit82]. Moreover, he gave a quantum field theoretic interpretation for the gluing of 4-manifolds along a common boundary 3-manifold with pairing ground states on the boundary contained in the instanton Floer homology groups which recovers the result of Theorem 3.5.1. Recall that, in general, we are considering expectation values to be given by an integral of the form

$$\langle O\rangle = \int \exp(\mathrm{i}S(\Phi)/\hbar)O(\Phi)\mathscr{D}[\Phi], \tag{3.6.1}$$

where S is the action of the theory and Φ denotes the collection of all integration variables. Let Σ be the underlying 4-manifold with boundary $\partial\Sigma$ and consider the field theory as in

Section 2.3. Choosing boundary conditions on $\partial\Sigma$ is usually required for the computation of a path integral as in (3.6.1). For the 3-manifold $\partial\Sigma$ (more precisely, the infinite cylinder $\partial\Sigma\times\mathbb{R}$), we can consider the associated state space $\mathcal{H}_{\partial\Sigma}$. Let $\Phi|_{\partial\Sigma}$ be the restriction of all integration variables to $\partial\Sigma$ and note that then $\mathcal{H}_{\partial\Sigma}$ denotes the space of functionals depending on $\Phi|_{\partial\Sigma}$ and a state corresponds to a functional $\Psi(\Phi|_{\partial\Sigma})$. If we consider the state $\Psi(\Phi|_{\partial\Sigma})$ to define a boundary condition, we can consider the path integral (3.6.1) in terms of this condition to be

$$\langle O\Psi(\Phi|_{\partial\Sigma})\rangle = \int \exp(\mathrm{i}S^{\mathrm{DW}}_\Sigma(\Phi)/\hbar)O(\Phi)\Psi(\Phi|_{\partial\Sigma})\mathscr{D}[\Phi], \tag{3.6.2}$$

where S^{DW}_Σ is defined as in (2.3.1) and

$$O := \prod_{j=1}^d\int_{\gamma_j} W_{k_j}, \tag{3.6.3}$$

with $\gamma_j\in H_{k_j}(\Sigma,\mathbb{Z})$ and W_{k_j} defined as in Section 2.3. In fact, (3.6.2) turns out to be a topological invariant if Ψ represents an instanton Floer cohomology class and it depends only on the cohomology class represented by Ψ. The observables are chosen similarly as for the Donaldson polynomials (see Section 2.3). Hence, one obtains the Donaldson polynomials with values in the (dual of the) instanton Floer cohomology groups as in Section 2.2.

3.7. **Lagrangian Floer homology.** Besides the instanton construction [Flo89b], there is another type of Floer homology theory which uses the data of a symplectic manifold [Flo88]. We will mainly use the excellent introductory paper [Aur14] for this section and we refer to it and the references within for more details and further constructions using Lagrangian Floer homology. We should remark that there the construction is dual to Floer's original construction and thus the grading convention will be reversed (so we should rather speak of cohomology instead of homology, but we decide to keep the original term). Let (Σ,ω) be a compact symplectic manifold and consider two compact Lagrangian submanifolds $\mathcal{L}_0,\mathcal{L}_1\subset\Sigma$. Then we can associate to the pair $(\mathcal{L}_0,\mathcal{L}_1)$ of Lagrangians a chain complex $CF(\mathcal{L}_0,\mathcal{L}_1)$ which is freely generated by the intersection points of $\mathcal{L}_0$ and $\mathcal{L}_1$ together with a differential $\partial\colon CF(\mathcal{L}_0,\mathcal{L}_1)\to CF(\mathcal{L}_0,\mathcal{L}_1)$ such that $\partial^2=0$ and thus we can consider its corresponding homology $HF(\mathcal{L}_0,\mathcal{L}_1):=\ker\partial/\mathrm{im}\,\partial$. Moreover, if there is a Hamiltonian isotopy between two compact Lagrangian submanifolds $\mathcal{L}_1$ and $\mathcal{L}_1'$, we get an isomorphism $HF(\mathcal{L}_0,\mathcal{L}_1)\cong HF(\mathcal{L}_0,\mathcal{L}_1')$ and if there is a Hamiltonian isotopy between $\mathcal{L}_0$ and $\mathcal{L}_1$, then $HF(\mathcal{L}_0,\mathcal{L}_1)\cong H^\bullet(\mathcal{L}_0)$. Similarly as before, Lagrangian Floer homology can be formally viewed as an infinite version of Morse homology with respect to the Morse function given by the functional defined on the universal cover of the path space $\mathrm{Path}(\mathcal{L}_0,\mathcal{L}_1):=\{\gamma\colon[0,1]\to\Sigma\mid\gamma(0)\in\mathcal{L}_0,\ \gamma(1)\in\mathcal{L}_1\}$ which is given by

$$S(\gamma,[\Gamma])=-\int_\Gamma\omega,$$

with $\gamma\in\mathrm{Path}(\mathcal{L}_0,\mathcal{L}_1)$ and $[\Gamma]$ the equivalence class of a homotopy $\Gamma\colon[0,1]\times[0,1]\to\Sigma$ between γ and a fixed base point in the connected component of $\mathrm{Path}(\mathcal{L}_0,\mathcal{L}_1)$ containing γ. Usually, we want that the two Lagrangian submanifolds intersect transversally, such that there is a finite set of intersection points. The Lagrangian Floer chain complex is then given by

$$CF(\mathcal{L}_0,\mathcal{L}_1):=\bigoplus_{p\in\mathcal{L}_0\cap\mathcal{L}_1}\Lambda\cdot p,$$

where $\Lambda := \{\sum_{i\geq 0} a_i T^{\lambda_i} \mid a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} = +\infty\}$ denotes the *Novikov field* for some field $\mathbb{K}$. Let J be a ω-compatible almost-complex structure on Σ. We can define the differential ∂ by counting *pseudo-holomorphic strips* in Σ with boundary contained in $\mathcal{L}_0$ and $\mathcal{L}_1$. This is parametrized through the following construction: Let $p, q \in \mathcal{L}_0 \cap \mathcal{L}_1$. Then the coefficient of q in ∂p is given by the moduli space of maps $u\colon \mathbb{R}\times[0,1]\to\Sigma$ such that:

(1) it solves the Cauchy–Riemann equation

$$\bar{\partial}_J u := \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0,$$

with respect to the boundary conditions

$$\begin{cases} u(s,0)\in\mathcal{L}_0 \text{ and } u(s,1)\in\mathcal{L}_1,\ \forall s\in\mathbb{R},\\ \lim_{s\to+\infty} u(s,t) = p,\ \lim_{s\to-\infty} u(s,t) = q \end{cases}$$

(2) it has finite energy (symplectic area of the strip):

$$\int u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 \mathrm{d}s\mathrm{d}t < \infty.$$

Let $\widehat{\mathcal{M}}(p,q;[u],J)$ denote the moduli space defined through the conditions (1) and (2) above and $\mathcal{M}(p,q;[u],J)$ the moduli space after taking the quotient by the $\mathbb{R}$-action given by reparametrization (i.e. $u(s,t)\mapsto u(s-a,t)$ for some $a\in\mathbb{R}$). We have denoted by $[u]$ the homotopy class of u in $\pi_2(\Sigma,\mathcal{L}_0\cap\mathcal{L}_1)$. Moreover, define the *Maslov index* of the homotopy class $[u]$ as

$$\mathrm{ind}([u]) := \mathrm{ind}_{\mathbb{R}}(D_{\bar{\partial}_J,u}) = \dim\ker D_{\bar{\partial}_J,u} - \dim\operatorname{coker} D_{\bar{\partial}_J,u},$$

where $D_{\bar{\partial}_J,u}$ denotes the linearization of $\bar{\partial}_J$ at a given solution u. One can show that $D_{\bar{\partial}_J,u}$ is indeed a Fredholm operator and thus one can compute its Fredholm index. In particular, it can be shown that $\widehat{\mathcal{M}}(p,q;[u],J)$ is a smooth manifold of dimension $\mathrm{ind}([u])$ if all solutions of (1) and (2) are regular, i.e. the operator $D_{\bar{\partial}_J,u}$ is surjective at each point $u\in\widehat{\mathcal{M}}(p,q;[u],J)$. Thus, $\mathcal{M}(p,q;[u],J)$ is an oriented 0-dimensional manifold whenever $\mathrm{ind}([u])=1$. Compactness of the moduli space is given through *Gromov's compactness theorem* [Gro85]. Indeed, Gromov showed that any sequence of J-holomorphic curves with uniformly bounded energy admits a subsequence which converges, up to reparametrization, to a nodal tree of J-holomorphic curves. Denote by $\mathrm{LGr}(n)$ the Grassmannian of Lagrangian n-planes in the standard symlectic space $(\mathbb{R}^{2n},\omega_0)$ and by $\mathrm{LGr}(T\Sigma)$ the Grassmannian of Lagrangian planes in $T\Sigma$ as an $\mathrm{LGr}(n)$-bundle over Σ. The $\mathbb{Z}$-grading of the complex is obtained by ensuring that the difference of the index of a strip depends only on the difference between the degrees of the two generators that it connects and not on its homotopy class[15]. Then we can construct the grading as follows: For all $p\in\mathcal{L}_0\cap\mathcal{L}_1$ we can obtain a homotopy class of a path connecting $T_p\mathcal{L}_0$ and $T_p\mathcal{L}_1$ in $\mathrm{LGr}(T_p\Sigma)$ by connecting the chosen graded lifts of the tangent spaces at p through a path in $\widetilde{\mathrm{LGr}}(T_p\Sigma)$. Composing this path with

[15]This is actually obstructed by the following two conditions: The first Chern class of Σ has to be 2-torsion and the *Maslov class* $\mu_{\mathcal{L}}\in\mathrm{Hom}(\pi_1(\mathcal{L}),\mathbb{Z}) = H^1(\mathcal{L},\mathbb{Z})$ of $\mathcal{L}$ vanishes, i.e. $2c_1(T\Sigma)=0$, which allows to lift the Grassmannian $\mathrm{LGr}(T\Sigma)$ to a fiberwise universal cover $\widetilde{\mathrm{LGr}}(T\Sigma)$ given as the Grassmannian of *graded* Lagrangian planes in $T\Sigma$. In particular, if we have a nowhere vanishing section $\sigma\in\Gamma(\bigwedge^n_{\mathbb{C}} T^*\Sigma\otimes\bigwedge^n_{\mathbb{C}} T^*\Sigma)$, the argument of σ assigns to each Lagrangian plane ℓ a phase $\phi(\ell) := \arg(\sigma|_\ell)\in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Thus, one defines a graded lift of ℓ to be the choice of a real lift $\tilde{\varphi}(\ell)\in\mathbb{R}$ of $\varphi(\ell)$. The Maslov class is defined as the obstruction for choosing graded lifts of the tangent spaces to $\mathcal{L}$, i.e. lifting the section of $\mathrm{LGr}(T\Sigma)$ over $\mathcal{L}$ given by $p\mapsto T_p\mathcal{L}$ to a section of the infinite cyclic cover $\widetilde{\mathrm{LGr}}(T\Sigma)$. The Lagrangian $\mathcal{L}$ together with such a choice of lift is called *graded Lagrangian submanifold* of Σ.

the canonical short path from $T_p\mathcal{L}_0$ and $T_p\mathcal{L}_1$, denoted by $-\lambda_p$, we get a closed loop in $\mathrm{LGr}(T_p\Sigma)$. We can then define the degree of p to be the Maslov index of this loop. One can check that for any strip u connecting p and q, we get

$$\mathrm{ind}(u) = \deg(q) - \deg(p),$$

The differential can be then defined as

$$\partial p = \sum_{\substack{q\in\mathcal{L}_0\cap\mathcal{L}_1\\ \mathrm{ind}([u])=1}} \#\mathcal{M}(p,q;[u],J)T^{\omega([u])}q,$$

where $\#\mathcal{M}(p,q;[u],J)\in\mathbb{Z}$ (or $\mathbb{Z}_2$) is the signed (or unsigned) count of pseudo-holomorphic strips connecting p to q in the class $[u]$, and $\omega([u]) = \int u^*\omega < \infty$ is the symplectic area of these strips. One can observe that ∂ is indeed of degree $+1$. The fact that it squares to zero is a non-trivial observation which requires some assumptions. Let us go back to the compactness argument. In our case, we consider J-holomorphic strips $u\colon \mathbb{R}\times[0,1]\to\Sigma$ with boundary on Lagrangian submanifolds $\mathcal{L}_0$ and $\mathcal{L}_1$. Then we have three situations that can appear: the first situation is called *strip breaking* (see Figure 3.7.1) and occurs if energy concentrates at either end $s\to\pm\infty$, i.e. there exists a sequence (a_n) with $a_n\to\pm\infty$ such that the sequence of strips $u_n(s-a_n,t)$ converges to a non-constant limit strip. The second situation is called *disk bubbling* (see Figure 3.7.2) and occurs if energy concentrates at a point on the boundary of the strip, i.e. when $t\in\{0,1\}$, where suitable rescalings of u_n converge to a J-holomorphic disk in Σ with boundary completely contained in either $\mathcal{L}_0$ or $\mathcal{L}_1$. the third situation occurs if energy concentrates at an interior point of the strip, where suitable rescalings of u_n converge to a J-holomorphic sphere in Σ.

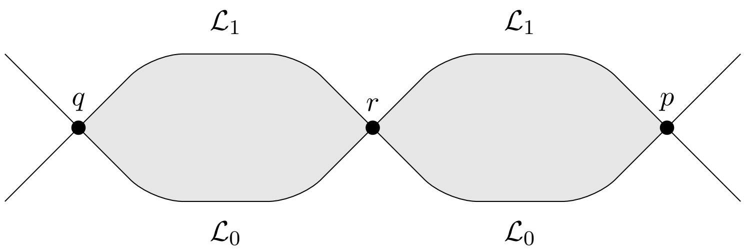

FIGURE 3.7.1. Example of a broken strip situation.

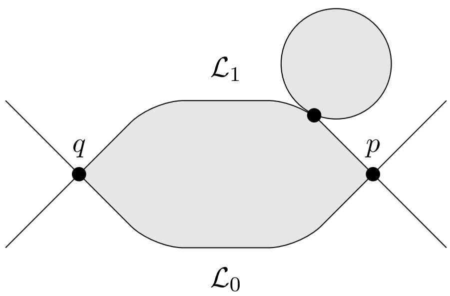

FIGURE 3.7.2. Example of a disk bubbling situation.

Strip breaking is in fact the ingredient needed for the differential ∂ to square to zero whenever disk bubbling can be excluded. We can make sure that disk and sphere bubbles do not appear by imposing the condition $[\omega]\cdot\pi_2(\Sigma,\mathcal{L}_j)=0$ for $j\in\{0,1\}$. There are also other more general ways to avoid disk and sphere bubbling, e.g. to impose a lower bound on the Maslov index by considering the case when the symplectic area of disks and their Maslov index are proportional to each other. One then speaks of *monotone* Lagrangian submanifolds in monotone symplectic manifolds.

3.8. **The Atiyah–Floer conjecture.** In [Ati87], Atiyah conjectured that instanton Floer homology should be related to Lagrangian Floer homology in the following way (see also [Sal95; Weh05b; Weh05a; SW08] for a slightly different approach to the same conjecture using Lagrangian boundary conditions):

Conjecture 3.8.1 (Atiyah–Floer[Ati87])**.** Let Σ_g be a Riemann surface of genus $g\geq 1$. Then the space of flat SU(3)-connections on Σ_g, denoted by $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(\Sigma_g)$, up to isomorphism has a symplectic structure. Suppose N is an integral homology sphere with Heegaard splitting along the surface Σ_g, given by

$$N=H^1_g\cup_{\Sigma_g}H^2_g,$$

where H^i_g denotes a handle-body of genus g. Then the space $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^i_g)$ of flat connections on Σ_g which extend to flat connections on H^i_g determines a Lagrangian subspace of $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(\Sigma_g)$. In particular, we have

$$H_{\mathrm{inst}}F_\bullet(N)\cong H_{\mathrm{Lagr}}F_\bullet(\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^1_g),\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^2_g)),$$

where $H_{\mathrm{inst}}F_\bullet$ denotes the instanton Floer homology and $H_{\mathrm{Lagr}}F_\bullet$ denotes the Lagrangian Floer homology. Here, $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^1_g)$ and $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^2_g)$ are considered as Lagrangian submanifolds of $\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(\Sigma_g)$. Usually, $H_{\mathrm{Lagr}}F_\bullet(\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^1_g),\mathcal{A}^{\mathrm{SU}(2)}_{\mathsf{flat}}(H^2_g))$ is called the *symplectic instanton Floer homology of N.*

Remark 3.8.2. The Atiyah–Floer conjecture leads to a better understanding of the instanton Floer homology for 3-manifolds with boundary. In particular, for a 4-manifold together with a principal $\mathbb{P}$U(2)-bundle, where $\mathbb{P}$U denotes the *projective unitary group*, one can define numerical invariants as in (2.2.4) due to Donaldson. For a 3-manifold together with a principal $\mathbb{P}$U(2)-bundle, one can define the instanton Floer homology group as in Section 3.4. For a 2-manifold together with a principal $\mathbb{P}$U(2)-bundle, we can construct an A_∞-category[16]. Moreover, as we have seen in Section 3.5, the Donaldson invariants of a 4-manifold Σ with boundary is valued in the instanton Floer homology of $\partial\Sigma$. Finally, it is expected that the Floer homology of a 3-manifold N with boundary gives an object in the A_∞-category associated to ∂N.

Remark 3.8.3 (A way of proving Conjecture 3.8.1)**.** In [DF17], Daemi and Fukaya have proposed a proof for the Atiyah–Floer conjecture. In particular, they construct a different version of Lagrangian Floer homology and translate the Atiyah–Floer conjecture to the equivalent conjecture which states that this new version is actually isomorphic to $H_{\mathrm{inst}}F_\bullet$. They proposed a solution to this equivalent conjecture by considering a mixture of the moduli space of anti self-dual connections and the moduli space of pseudo-holomorphic curves. Moreover, they gave a formulation in terms of A_∞-categories in order to state

[16]An A_∞-category is, roughly speaking, a category whose associativity condition for the composition of morphisms is relaxed in a higher unbounded homotopical way (see e.g. [KS08a]). Usually, the morphisms are given by chain complexes as for the meaning of a *linear category*.

conjectures for stronger versions of some properties for instanton Floer homology of 2- and 3-manifolds.

Remark 3.8.4 (Fukaya category)**.** The A_∞-category whose objects are given by certain Lagrangian submanifolds of a given symplectic manifold (Σ,ω) and morphisms between objects are given by the Lagrangian Floer chain complexes is called the *Fukaya category* [Fuk93] and is denoted by $\mathrm{Fuk}(\Sigma,\omega)$ (see also [Fuk+09a; Fuk+09b; Kon94a; Aur14]). In particular, one could also construct a more general version of this category by using Lagrangian foliations as the objects. Such a construction would be interesting in order to understand the boundary structure when combining with the methods of geometric quantization as for the bounary state space construction in the BV-BFV setting.

Remark 3.8.5 (Homological mirror symmetry)**.** Based on the mirror symmetry conjecture (A- and B-model mirror construction) considered in string theory for Calabi–Yau 3-folds (see e.g. [Yau92; Hor+03]), Kontsevich formulated a homological version in terms of equivalences of triangulated categories. For some projective variety X, let $D^b\mathrm{Coh}(X)$ denote the derived category of coherent sheaves on X. Kontsevich's conjecture is then formulated as follows:

Conjecture 3.8.6 (Homological mirror symmetry[Kon94a])**.** Let X and Y be mirror dual Calabi–Yau varieties. Then there is an equivalence of triangulated categories

$$\mathrm{Fuk}(X)\cong D^b\mathrm{Coh}(Y), \tag{3.8.1}$$

Conjecture 3.8.6 has been proven by various people for different types of mirror varieties (e.g. toric 3-folds, quintic 3-folds). However, a general direct proof still remains open.

4. The BV-BFV formalism

In this section, we want to recall the most important concepts of [CMR17]. Good introductory references for the BV and BV-BFV formalisms are [Cos11; Mne19; CM20]. In order to avoid large formulae, we will not always write out the exterior product between differential forms, hence we remark that from now on this is automatically understood.

4.1. BV formalism.

Let us start with the BV construction on a closed d-dimensional source manifold Σ.

Definition 4.1.1 (BV manifold)**.** A *BV manifold* is a triple $(\mathcal{F},\omega,\mathcal{S})$, where $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold[17], ω an odd symplectic form on $\mathcal{F}$ of degree -1 and $\mathcal{S}$ an even function on $\mathcal{F}$ of degree 0 such that

$$(\mathcal{S},\mathcal{S})=0, \tag{4.1.1}$$

where $(\ ,\ )$ denotes the odd Poisson bracket induced by the odd symplectic form ω.

Equation (4.1.1) is called *Classical Master Equation (CME)*.

Remark 4.1.2. We call $\mathcal{F}$ the *BV space of fields*[18], ω the *BV symplectic form* and $\mathcal{S}$ the *BV action (functional)*. Moreover, $(\ ,\ )$ is often called *BV bracket or anti-bracket*. In the physics literature, the $\mathbb{Z}$-grading is called *ghost number*. We will denote the ghost number by gh. When considering differential forms, we will denote the form degree by deg.

[17] Recall that a *supermanifold* is a locally ringed space $(\mathcal{F},\mathcal{O}_\mathcal{F})$ such that the structure sheaf is locally, for some open set $U\subset\mathbb{R}^d$, given by $C^\infty(U)\otimes\bigwedge^\bullet V^*$, where V is some finite-dimensional real vector space.

[18] Usually, the BV space of fields is given by the (-1)-shifted cotangent bundle of the BRST space of fields, i.e. $\mathcal{F}_{\mathrm{BV}}=T^*[-1]\mathcal{F}_{\mathrm{BRST}}$. The fiber fields are usually called *anti-fields*. For each field Φ, there is a corresponding notion of anti-field through duality, denoted by $\Phi^\dagger$. Recall that $\mathrm{gh}(\Phi)+\mathrm{gh}(\Phi^\dagger)=-1$. Moreover, if $\mathcal{F}_\Sigma=\Omega^\bullet(\Sigma)$, then $\deg(\Phi)+\deg(\Phi^\dagger)=\dim\Sigma$.

Definition 4.1.3 (BV theory)**.** The assignment of a source manifold Σ to a BV manifold

$$(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{S}_\Sigma)$$

is called a *BV theory*.

4.1.1. *Quantization.* For the quantum picture one considers a canonical second-order differential operator Δ on half-densities on $\mathcal{F}$ with the properties

$$\Delta^2 = 0, \tag{4.1.2}$$

$$\Delta(fg) = \Delta fg \pm f\Delta g \pm (f,g), \quad \forall f,g \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}), \tag{4.1.3}$$

where $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ denotes the space of half-densities on $\mathcal{F}$. The operator Δ is called *BV Laplacian.* If Φ^i and $\Phi^\dagger_i$ denote field and anti-field respectively, we have

$$\Delta f = \sum_i (-1)^{\mathrm{gh}(\Phi^i)+1} f \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}, \frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i} \right\rangle, \quad f \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}).$$

We denote by $\frac{\overleftarrow{\delta}}{\delta\Phi}$ and $\frac{\overrightarrow{\delta}}{\delta\Phi}$ the *left-* and *right-derivatives.* Namely, we have

$$\frac{\overrightarrow{\delta}}{\delta\Phi^i} f = (-1)^{\mathrm{gh}(\Phi^i)(\mathrm{gh}(f)+1)} f \frac{\overleftarrow{\delta}}{\delta\Phi^i},$$
$$\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger_i} f = (-1)^{(\mathrm{gh}(\Phi^i)+1)(\mathrm{gh}(f)+1)} f \frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i}.$$

Remark 4.1.4. In particular, one can show that for any odd symplectic supermanifold $\mathcal{F}$, there is a supermanifold $\mathcal{N}$ such that $\mathcal{F} \cong T^*[1]\mathcal{N}$. Hence, functions on $\mathcal{F}$ are given by multivector fields on $\mathcal{N}$. Moreover, the Berezinian bundle on $\mathcal{F}$ is given by

$$\mathrm{Ber}(\mathcal{F}) \cong \bigwedge^{\mathrm{top}} T^*\mathcal{N} \otimes \bigwedge^{\mathrm{top}} T^*\mathcal{N}.$$

Thus, half-densities on $\mathcal{F}$ are defined by

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) := \Gamma\left(\mathrm{Ber}(\mathcal{F})^{\frac{1}{2}}\right).$$

One can show that there exists a canonical operator $\Delta^{\frac{1}{2}}_{\mathcal{F}}$ on $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ which squares to zero [Khu04]. Then, fixing a non-vanishing, $\Delta^{\frac{1}{2}}_{\mathcal{F}}$-closed reference half-density $\sigma \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$, one can define a Laplacian on functions on $\mathcal{F}$ by

$$\Delta_\sigma f := \frac{1}{\sigma}\Delta^{\frac{1}{2}}_{\mathcal{F}}(f\sigma). \tag{4.1.4}$$

It is easy to check that Δ_σ squares to zero. For convenience, we write Δ when we actually mean Δ_σ in order to not emphasize on the choice of reference half-density. It is important to mention that this construction needs a suitable regularization in the case where $\mathcal{F}$ is infinite-dimensional. See also [Šev06] for describing the BV Laplacian naturally through a spectral sequence approach of a double complex using the odd symplectic structure.

Theorem 4.1.5 (Batalin–Vilkovisky–Schwarz[BV81; Sch93])**.** *Let $f,g \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ be two half-densities on $\mathcal{F}$. Then*

*(1) if $f=\Delta g$ (*BV exact*), we get that*

$$\int_{\mathcal{L}} f = 0,$$

for any Lagrangian submanifold $\mathcal{L}\subset\mathcal{F}$.

*(2) if $\Delta f = 0$ (*BV closed*), we get that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t} f = 0,$$

for any smooth family $(\mathcal{L}_t)$ of Lagrangian submanifolds of $\mathcal{F}$.

For the application to quantum field theory, we want to consider[19]

$$f = \exp(\mathrm{i}\mathcal{S}/\hbar).$$

The choice of Lagrangian submanifold is in fact equivalent to the choice of gauge-fixing. Thus, the second point of Theorem 4.1.5 tells us that the the gauge-independence condition is encoded in the equation

$$\Delta\exp(\mathrm{i}\mathcal{S}/\hbar) = 0,$$

which is equivalent to

$$(\mathcal{S},\mathcal{S}) - 2\mathrm{i}\hbar\Delta\mathcal{S} = 0. \tag{4.1.5}$$

Equation (4.1.5) is called *Quantum Master Equation (QME)*. Note that the QME reduces to the CME for $\hbar\to 0$.

Example 4.1.6 (AKSZ theory[Ale+97])**.** A natural example of BV theories was formulated by Alexandrov, Kontsevich, Schwarz and Zaboronsky in [Ale+97]. A *differential graded symplectic manifold* of degree d consists of a graded manifold $\mathcal{M}$ endowed with an exact symplectic form $\omega=\mathrm{d}_{\mathcal{M}}\alpha$ of degree d and a smooth Hamiltonian function Θ on $\mathcal{M}$ of degree $d+1$ satisfying

$$\{\Theta,\Theta\}=0,$$

where $\{\ ,\ \}$ is the Poisson bracket induced by ω. Such a triple $(\mathcal{M},\omega=\mathrm{d}_{\mathcal{M}}\alpha,\Theta)$ is sometimes also called a *Hamiltonian manifold*. A d-dimensional *AKSZ sigma model* with target a Hamiltonian manifold $(\mathcal{M},\omega=\mathrm{d}_{\mathcal{M}}\alpha,\Theta)$ of degree $d-1$ is the BV theory, which associates to a d-manifold Σ the BV manifold $(\mathcal{F}_\Sigma,\omega_\Sigma,\mathcal{S}_\Sigma)$, where[20]

$$\mathcal{F}_\Sigma = \mathrm{Map}(T[1]\Sigma,\mathcal{M}), \tag{4.1.6}$$

the BV symplectic form ω_Σ is of the form

$$\omega_\Sigma = \int_{T[1]\Sigma}\boldsymbol{\mu}_d\omega_{\mu\nu}\delta\Phi^\mu\delta\Phi^\nu, \tag{4.1.7}$$

and the BV action is given by

$$\mathcal{S}_\Sigma = \int_{T[1]\Sigma}\boldsymbol{\mu}_d\left(\alpha_\mu(\Phi)\mathrm{d}_\Sigma\Phi^\mu + \Theta(\Phi)\right). \tag{4.1.8}$$

we have denoted by $\Phi\in\mathcal{F}_\Sigma$ the superfields, by $\omega_{\mu\nu}$ the components of the symplectic form ω, by α_μ the components of α and Φ^μ are the components of Φ in local coordinates (u,θ). Moreover, $\boldsymbol{\mu}_d := \mathrm{d}^d u\mathrm{d}^d\theta$ denotes the measure on $T[1]\Sigma$ and δ is the de Rham differential on

[19] More precisely, we want to consider $f=\exp(\mathrm{i}\mathcal{S}/\hbar)\rho$, where ρ is some Δ-closed reference half-density on $\mathcal{F}$.

[20] This is the infinite-dimensional graded manifold adjoint to the Cartesian product (internal morphisms).

$\mathcal{F}_\Sigma$. Indeed, one can check that ω_Σ is symplectic of degree -1 and that $\mathcal{S}_\Sigma$ is of degree 0 satisfying the CME

$$(\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) = 0.$$

This is obtained by considering the *transgression* map

$$\mathscr{T} : \Omega^\bullet(\mathcal{M}) \to \Omega^\bullet(\mathrm{Map}(T[1]\Sigma, \mathcal{M})),$$

given by the following diagram:

$$\begin{tikzcd} \mathrm{Map}(T[1]\Sigma, \mathcal{M}) \times T[1]\Sigma \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_1"'] & \mathcal{M} \arrow[dl, dashed] \\ \mathrm{Map}(T[1]\Sigma, \mathcal{M}) & \end{tikzcd}$$

where π_1 denotes the projection to the first factor and ev denotes the evaluation map. The transgression map is then defined as $\mathscr{T} := (\pi_1)_*\mathrm{ev}^*$. Note that the map $(\pi_1)_*$ denotes fiber integration. The (-1)-shifted symplectic form on $\mathrm{Map}(T[1]\Sigma, \mathcal{M})$ is then given by $\omega_\Sigma = \mathscr{T}(\omega) = (\pi_1)_*\mathrm{ev}^*\omega$. Many theories of interest are in fact of AKSZ-type such as e.g. Chern–Simons theory [Wit89; AS91; AS94], the Poisson sigma model [CF00; CF01], Witten's A- and B-model [Wit88b; Ale+97] and 2-dimensional Yang–Mills theory [IM19].

4.2. **BV algebras.** We want to recall some notions on BV algebras as in [Get94], and how it is related to the original BV formalism. A *braid algebra* $\mathcal{B}r$ is a commutative DG algebra endowed with a Lie bracket $[\ ,\ ]$ of degree $+1$ satisfying the Poisson relations

$$[a, bc] = [a, b]c + (-1)^{|a|(|b|-1)}b[a, c], \qquad \forall a, b, c \in \mathcal{B}r \tag{4.2.1}$$

An identity element in $\mathcal{B}r$ is an element $\mathbf{1}$ of degree 0 such that it is an identity for the product and $[\mathbf{1},\ ] = 0$. A *BV algebra* $\mathcal{BV}$ is a commutative DG algebra endowed with an operator $\Delta\colon \mathcal{BV}_\bullet \to \mathcal{BV}_{\bullet+1}$ such that $\Delta^2 = 0$ and

$$\begin{aligned} \Delta(abc) &= \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &\quad - \Delta abc - (-1)^{|a|}a\Delta bc - (-1)^{|a|+|b|}ab\Delta c, \qquad \forall a, b, c \in \mathcal{BV}. \end{aligned} \tag{4.2.2}$$

An identity in $\mathcal{BV}$ is an element $\mathbf{1}$ of degree 0 such that it is an identity for the product and $\Delta\mathbf{1} = 0$. One can show that a BV algebra is in fact a special type of a braid algebra. More precisely, a BV algebra is a braid algebra endowed with an operator $\Delta\colon \mathcal{BV}_\bullet \to \mathcal{BV}_{\bullet+1}$ such that $\Delta^2 = 0$ and such that the bracket and Δ are related by

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta ab - a\Delta b, \qquad \forall a, b \in \mathcal{BV}. \tag{4.2.3}$$

Moreover, in a BV algebra we have

$$\Delta([a, b]) = [\Delta a, b] + (-1)^{|a|-1}[a, \Delta b], \qquad \forall a, b \in \mathcal{BV}. \tag{4.2.4}$$

The motivation for such an algebra structure comes exactly from the approach of the BV formalism in quantum field theory. Let $(\mathcal{F}, \omega)$ be an odd symplectic (super)manifold. Let $f \in \mathcal{O}_\mathcal{F}$, where $\mathcal{O}_\mathcal{F}$ denotes the space of functions on $\mathcal{F}$, and consider its Hamiltonian vector field X_f. One can check that $\mathcal{O}_\mathcal{F}$ endowed with the BV anti-bracket

$$(f, g) := (-1)^{|f|-1}X_f(g) \tag{4.2.5}$$

is a braid algebra. Let $\mu \in \Gamma(\mathrm{Ber}(\mathcal{F}))$ be a nowhere-vanishing section of the Berezinian bundle of $\mathcal{F}$. As we have seen before, this represents a density which is characterized by the integration map $\int : \Gamma_0(\mathcal{F}, \mathrm{Ber}(\mathcal{F})) \to \mathbb{R}$, where Γ_0 denotes sections with compact support. Hence, μ induces an integration map on functions with compact support

$$\mathcal{O}_{\mathcal{F}} \ni f \mapsto \int_{\mathcal{L}\subset\mathcal{F}} f\mu^{1/2}, \tag{4.2.6}$$

for some Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, where the integral exists. Then one can define a divergence operator $\mathrm{div}_\mu X$ by

$$\int_{\mathcal{F}} (\mathrm{div}_\mu X) f\mu = -\int_{\mathcal{F}} X(f)\mu. \tag{4.2.7}$$

Lemma 4.2.1. *For a vector field X, define $X^* := -X - \mathrm{div}_\mu X$. Then*

$$\int_{\mathcal{F}} fX(g)\mu = (-1)^{|f||X|} \int_{\mathcal{F}} X^*(f)g\mu. \tag{4.2.8}$$

Moreover, $\mathrm{div}_\mu(fX) = f\,\mathrm{div}_\mu X - (-1)^{|f||X|}X(f)$ and if $\mathcal{S} \in \mathcal{O}_{\mathcal{F}}$ is an even function, then

$$\mathrm{div}_{\exp(\mathcal{S})\mu} X = \mathrm{div}_\mu X + X(\mathcal{S}).$$

One can then define the BV Laplacian Δ to be the odd operator on $\mathcal{O}_{\mathcal{F}}$ given by

$$\Delta f = \mathrm{div}_\mu X_f. \tag{4.2.9}$$

A BV (super)manifold $(\mathcal{F}, \omega, \mu)$ is then an odd symplectic (super)manifold with Berezinian μ such that $\Delta^2 = 0$.

Proposition 4.2.2 (Getzler[Get94])**.** *Let $(\mathcal{F}, \omega, \mu)$ be a BV (super)manifold.*

1. *The algebra $(\mathcal{O}_{\mathcal{F}}, (\ ,\ ), \Delta)$ is a BV algebra, where Δ is given as in (4.2.9) and $(\ ,\ )$ is the odd Poisson bracket coming from the odd symplectic form ω as in (4.2.5).*
2. *The Hamiltonian vector field associated to some $f \in \mathcal{O}_{\mathcal{F}}$ is given by the formula $X_f = -[\Delta, f] + \Delta f$, where $[\ ,\ ]$ denotes the commutator of operators.*
3. *If $\mathcal{S} \in \mathcal{O}_{\mathcal{F}}$ and $\Delta_{\mathcal{S}}$ is the operator associated to the Berezinian $\exp(\mathcal{S})\mu$, then $\Delta_{\mathcal{S}} = \Delta - X_{\mathcal{S}}$ and $\Delta^2_{\mathcal{S}} = X_{\Delta\mathcal{S}+\frac{1}{2}(\mathcal{S},\mathcal{S})}$.*

Note that point (3) of Proposition 4.2.2 is exactly the case that we have in quantum field theory. Moreover, if

$$\Delta\mathcal{S} + \frac{1}{2}(\mathcal{S}, \mathcal{S}) = 0, \tag{4.2.10}$$

we get that $\Delta^2_{\mathcal{S}} = 0$, which ensures a BV algebra structure. Note that here we have set $i\hbar = 1$.

4.3. **BFV formalism.** We want to continue with the Hamiltonian approach of the BFV formalism for closed source manifolds.

Definition 4.3.1 (BFV manifold)**.** A *BFV manifold* is a triple $(\mathcal{F}^\partial, \omega^\partial, Q^\partial)$ such that $\mathcal{F}^\partial$ is a $\mathbb{Z}$-graded supermanifold, ω^∂ is an even symplectic form of degree 0 and Q^∂ is a cohomological ($Q^2 = 0$) and symplectic ($L_Q\omega = 0$) vector field on $\mathcal{F}$ of degree $+1$. Moreover, the Hamiltonian function $\mathcal{S}^\partial$ of Q^∂ defined by the equation $\iota_{Q^\partial}\omega^\partial = \delta\mathcal{S}^\partial$, is required to satisfy the CME

$$\{\mathcal{S}^\partial, \mathcal{S}^\partial\} = 0, \tag{4.3.1}$$

where $\{\ ,\ \}$ denotes the Poisson bracket of degree 0 induced by the even symplectic form ω^∂.

Note that we have denoted by δ the de Rham differential on $\mathcal{F}^\partial$. Moreover, we can, equivalently, express Equation (4.3.1) as $Q^\partial\mathcal{S}^\partial=0$. Similarly, for a BV manifold $(\mathcal{F},\omega,\mathcal{S})$ we can consider the Hamiltonian vector field Q associated to $\mathcal{S}$ through the equation $\iota_Q\omega=\delta\mathcal{S}$ (by abuse of notation, we also denote the de Rham differential on $\mathcal{F}$ by δ). Hence, we can express Equation (4.1.1) as $Q\mathcal{S}=0$.

Definition 4.3.2 (Exact BFV manifold)**.** We call a BFV manifold $(\mathcal{F}^\partial,\omega^\partial,Q^\partial)$ *exact*, if ω^∂ is exact, i.e. there is a 1-form α on $\mathcal{F}^\partial$ such that $\omega^\partial=\delta\alpha^\partial$.

Definition 4.3.3 (BFV theory)**.** The assignment of a manifold Σ to a BFV manifold

$$(\mathcal{F}^\partial_{\partial\Sigma},\omega^\partial_{\partial\Sigma},Q^\partial_{\partial\Sigma})$$

is called a *BFV theory*.

Remark 4.3.4. Given an AKSZ theory $(\mathcal{F}_\Sigma=\mathrm{Map}(T[1]\Sigma,\mathcal{M}),\omega_\Sigma,\mathcal{S}_\Sigma)$ as in Example 4.1.6 associated to a manifold with boundary Σ, we can easily construct its BFV data by restriction to the boundary. The BFV action thus given by

$$\mathcal{S}^\partial_{\partial\Sigma}=\mathcal{S}_\Sigma\big|_{\partial\Sigma}.$$

Similarly we can obtain the BFV space of boundary fields

$$\mathcal{F}^\partial_{\partial\Sigma}=\mathrm{Map}(T[1]\partial\Sigma,\mathcal{M})=\mathcal{F}_\Sigma\big|_{\partial\Sigma}$$

and the BFV symplectic form

$$\omega^\partial_{\partial\Sigma}=\omega_\Sigma\big|_{\partial\Sigma}.$$

4.4. **BV-BFV formalism.** Consider now a source manifold Σ with boundary $\partial\Sigma$.

Definition 4.4.1 (BV-BFV manifold)**.** A *BV-BFV manifold* over an exact BFV manifold $(\mathcal{F}^\partial,\omega^\partial=\delta\alpha^\partial,Q^\partial)$ is a quintuple $(\mathcal{F},\omega,\mathcal{S},Q,\pi)$ such that $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold, ω is an odd symplectic form of degree -1 on $\mathcal{F}$, $\mathcal{S}$ is an even functional of degree 0, Q is a cohomological, symplectic vector field on $\mathcal{F}$ and $\pi\colon\mathcal{F}\to\mathcal{F}^\partial$ a surjective submersion such that

(1) $\delta\pi Q=Q^\partial$,
(2) $\iota_Q\omega=\delta\mathcal{S}+\pi^*\alpha^\partial$,

where $\delta\pi$ denotes the tangent map of π. In fact, condition (1) and (2) together imply the *modified Classical Master Equation (mCME)*

$$Q\mathcal{S}=\pi^*(2\mathcal{S}^\partial-\iota_{Q^\partial}\alpha^\partial). \tag{4.4.1}$$

Definition 4.4.2 (BV-BFV theory)**.** The assignment of a source manifold with boundary Σ to a BV-BFV manifold $(\mathcal{F}_\Sigma,\omega_\Sigma,\mathcal{S}_\Sigma,Q_\Sigma,\pi_\Sigma)$ over an exact BFV manifold $(\mathcal{F}^\partial_{\partial\Sigma},\omega^\partial_{\partial\Sigma},Q^\partial_{\partial\Sigma})$ is called a *BV-BFV theory*.

4.4.1. *Quantization.* For the quantization, one chooses a polarization[21] $\mathcal{P}$ on the boundary and assume a symplectic splitting[22] of the BV space of fields with respect to it

$$\mathcal{F}=\mathcal{B}^{\mathcal{P}}\times\mathcal{Y}. \tag{4.4.2}$$

[21] This is an involutive ω^∂-Lagrangian subbundle of $T\mathcal{F}^\partial$.

[22] Actually, this is only needed locally, so one can also allow $\mathcal{F}$ to be a BV bundle over $\mathcal{B}^{\mathcal{P}}$. A fiber bundle $\mathcal{F}$ over a base $\mathcal{B}$ with odd-symplectic fiber $(\mathcal{Y},\omega)$ is called a *BV bundle* if the transition maps of $\mathcal{F}$ are given by locally constant fiberwise symplectomorphisms. An even more general setting is given by a *hedgehog fibration* where the fibers of the fiber bundle are given by *hedgehogs* [CMR17].

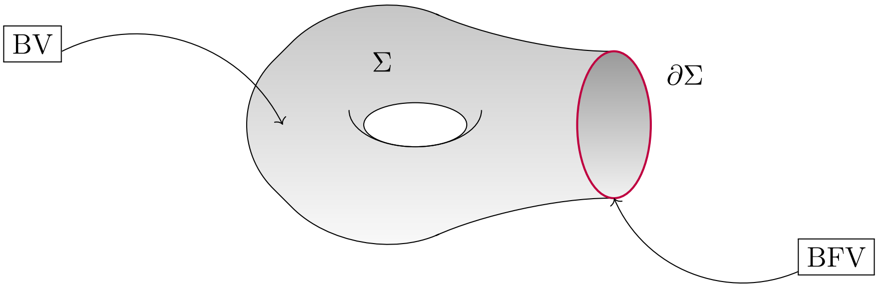

FIGURE 4.4.1. The BV theory is associated to the bulk and the BFV theory to the boundary such that we have a coherent coupling.

Here $\mathcal{B}^{\mathcal{P}}$ denotes the leaf space for the polarization $\mathcal{P}$ which we assume to be smooth and $\mathcal{Y}$ is a complement. Note that we can always split the BV space of fields in such a way but it is important to assume that it is symplectic, i.e. that the BV symplectic form ω is constant on $\mathcal{B}^{\mathcal{P}}$. In fact, we consider ω to be a weakly nondegenerate[23] 2-form on $\mathcal{Y}$ which extends to the product $\mathcal{B}^{\mathcal{P}} \times \mathcal{Y}$. For BF-like theories[24], which includes AKSZ-type theories, one can think of $\mathcal{B}^{\mathcal{P}}$ to be the fields restricted to the boundary and $\mathcal{Y}$ to be the bulk part. Using the fact that the BFV space of boundary fields $(\mathcal{F}^\partial, \omega^\partial)$ is a symplectic manifold, we consider a *geometric quantization* (see e.g. [Woo97; BW12; Mos20a]) on the boundary to obtain a vector space $\mathcal{H}^{\mathcal{P}}$. In order to take care of regularization, we consider another splitting of the bulk part

$$\mathcal{Y} = \mathcal{V} \times \mathcal{Y}', \tag{4.4.3}$$

where $\mathcal{V}$ denotes the space of *residual fields*[25]. Moreover, let $\mathcal{Z} := \mathcal{B}^{\mathcal{P}} \times \mathcal{V}$ denote the bundle of residual fields over $\mathcal{B}^{\mathcal{P}}$. Note that we also assume a splitting of the symplectic manifold $(\mathcal{Y}, \omega)$ into two symplectic manifolds

$$\begin{aligned} \mathcal{Y} &= \mathcal{V} \times \mathcal{Y}', \\ \omega &= \omega_{\mathcal{V}} + \omega_{\mathcal{Y}'}. \end{aligned}$$

Gauge-fixing is then equivalent to the choice of a Lagrangian submanifold $\mathcal{L}$ of $(\mathcal{Y}', \omega_{\mathcal{Y}'})$. Define $\widehat{\mathcal{H}}^{\mathcal{P}} := \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Z}) = \mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}}) \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V})$ and the BV Laplacian $\widehat{\Delta}^{\mathcal{P}} := \mathrm{id} \otimes \Delta_{\mathcal{V}}$ on $\widehat{\mathcal{H}}^{\mathcal{P}}$. We have denoted by $\Delta_{\mathcal{V}}$ the BV Laplacian on $\mathcal{V}$. Note that the space $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}})$ coincides with the vector space $\mathcal{H}^{\mathcal{P}}$ constructed by geometric quantization on the boundary. Thus we have

$$\widehat{\mathcal{H}}^{\mathcal{P}} = \mathcal{H}^{\mathcal{P}} \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}).$$

Remark 4.4.3. Another important assumption is that for any $\Phi \in \mathcal{Z}$, the restriction of the action to $\{\Phi\} \times \mathcal{L}$ has isolated critical points on $\{\Phi\} \times \mathcal{L}$.

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[23]In the finite-dimensional case the BV-BFV formalism is not consistent with the nondegeneracy of ω on the whole space and thus one has to exactly assume nondegeneracy along the fibers.

[24]These are perturbations of BF theories.

[25]$\mathcal{V}$ is often also called the space of *low energy fields* or *zero modes* which we assume to be finite-dimensional. The complement $\mathcal{Y}'$ is accordingly often called space of *high energy fields* or *fluctuation fields*

The state is then defined through the perturbative expansion into Feynman graphs of the BV integral

$$\mathsf{Z}^{\text{BV}-\text{BFV}}(\Phi)=\int_{\mathcal{L}}\exp\left(\mathrm{i}\mathcal{S}(\Phi)/\hbar\right)\mathscr{D}[\Phi]\in\widehat{\mathcal{H}}^{\mathcal{P}},\quad \Phi\in\mathcal{Z}. \tag{4.4.4}$$

Remark 4.4.4. If one uses a different choice of residual fields $\mathcal{V}'$ such that $\mathcal{V}$ fibers over $\mathcal{V}'$ as a *hedgehog* (see [CMR17] for the definition of a hedgehog fibration), then the corresponding quantum theories are *BV equivalent* in the sense of [CMR17]. however, there is a minimal choice for the residual fields for which the assumption in Remark 4.4.3 is satisfied by a good choice of Lagrangian submanifold $\mathcal{L}$. For the case of abelian BF theory, the minimal space of residual fields is given by the de Rham cohomology of the underlying source manifold.

If $\Delta\mathcal{S}=0$, which we usually want to assume, the QME is replaced by the *modified* QME (mQME)

$$\left(\hbar^2\Delta+\Omega^{\mathcal{P}}\right)\exp\left(\mathrm{i}\mathcal{S}/\hbar\right)=0, \tag{4.4.5}$$

where, for local coordinates $(b_i)\in\mathcal{B}^{\mathcal{P}}$, we have that

$$\Omega^{\mathcal{P}}:=\mathcal{S}^\partial\left(b_i,-\mathrm{i}\hbar\frac{\delta}{\delta b_i}\right)$$

is the standard ordering quantization of $\mathcal{S}^\partial$. If $\mathcal{S}$ depends on $\hbar$ and (or) $\Delta\mathcal{S}\neq 0$, we get the mQME from the assumption of the QME in the bulk by defining $\mathcal{S}^\partial_\hbar:=\mathcal{S}^\partial+O(\hbar)$ via the equation

$$\pi^*\mathcal{S}^\partial_\hbar=\frac{1}{2}(\mathcal{S},\mathcal{S})-\mathrm{i}\hbar\Delta\mathcal{S}$$

and setting $\Omega^{\mathcal{P}}$ to be the standard ordering quantization of $\mathcal{S}^\partial_\hbar$. Note that the operator $\hbar^2\Delta+\Omega^{\mathcal{P}}$ is of order $+1$. Moreover, we assume that the operator $\hbar^2\Delta+\Omega^{\mathcal{P}}$ squares to zero in order to have a well-defined BV cohomology. In the finite-dimensional setting we automatically get that $\widehat{\mathsf{Z}}$ solves the mQME:

$$\left(\hbar^2\Delta_{\mathcal{V}}+\Omega^{\mathcal{P}}\right)\widehat{\mathsf{Z}}=0. \tag{4.4.6}$$

In the infinite-dimensional setting, where we have to compute the Feynman graphs instead of an integral, the mQME is only expected to hold and requires a separate checking.

4.5. **Gluing of BV-BFV partition functions.** Let Σ be a d-manifold and consider two d-manifolds Σ_1,Σ_2 such that $\Sigma=\Sigma_1\cup_N\Sigma_2$ where N is the $(d-1)$-manifold which is identified with the common boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$ (see Figure 4.5.1). Then in general, following Atiyah's TQFT axioms [Ati88], we can define the glued partition function by the pairing

$$\mathsf{Z}_\Sigma=\int_N\mathsf{Z}_{\Sigma_1,\partial\Sigma_1}\mathsf{Z}_{\Sigma_2,\partial\Sigma_2}. \tag{4.5.1}$$

We want to see how this is adapted to the BV-BFV partition function. Let $(\mathcal{Y},\omega)$ be a direct product of two odd symplectic manifolds $(\mathcal{Y}',\omega')$ and $(\mathcal{Y}'',\omega'')$, i.e. $\mathcal{Y}=\mathcal{Y}'\times\mathcal{Y}''$, $\omega=\omega'+\omega''$. Then the space of half-densities on $\mathcal{Y}$ factorizes as

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y})=\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}')\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}'').$$

If we consider BV integration on the second factor for some Lagrangian submanifold $\mathcal{L}\subset\mathcal{Y}''$, we can define a pushforward map on half-densities

$$\int_{\mathcal{L}}\colon\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y})\xrightarrow{\mathrm{id}\otimes\int_{\mathcal{L}}}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{Y}').$$

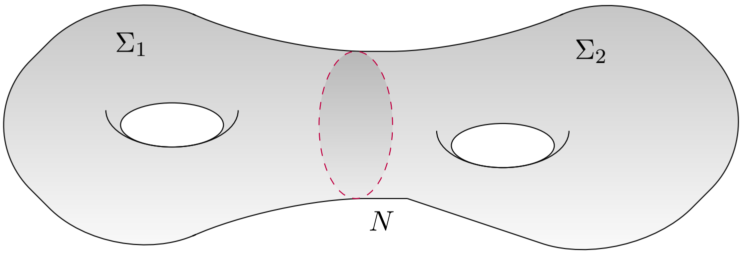

FIGURE 4.5.1. Gluing of Σ_1 and Σ_2 along N.

If we consider the space of fields $\mathcal{F} = \mathcal{B}\times\mathcal{Y}$ given by a product as in Section 4.4, and assuming that $\mathcal{Y}$ splits into a product of two odd symplectic manifolds $(\mathcal{V}, \omega_{\mathcal{V}})$ and $(\mathcal{Y}', \omega_{\mathcal{Y}'})$, we can consider a version of the BV pushforward in terms of families over $\mathcal{B}$ by

$$\int_{\mathcal{L}}\colon \mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) \xrightarrow{\mathrm{id}\otimes\int_{\mathcal{L}}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{Z}),$$

where $\mathcal{Z} = \mathcal{B}^{\mathcal{P}} \times \mathcal{V}$ for some chosen polarization $\mathcal{P}$ and $\mathcal{L} \subset \mathcal{Y}'$ is a Lagrangian submanifold. In particular, we have the coboundary operator $\hbar^2\Delta_{\mathcal{V}} + \Omega^{\mathcal{P}}$ on $\mathcal{Z}$, where $\Delta_{\mathcal{V}}$ is the canonical BV Laplacian on $\mathcal{V}$. In fact, the BV-BFV partition function is then defined through the family version of the BV pushforward.
The gluing of BV-BFV partition functions as in (4.4.4) is given by

$$\mathsf{Z}_{\Sigma}^{\mathrm{BV}} = \int_{\mathcal{L}} \left\langle \mathsf{Z}_{\Sigma_1,\partial\Sigma_1}^{\mathrm{BV-BFV}}, \mathsf{Z}_{\Sigma_2,\partial\Sigma_2}^{\mathrm{BV-BFV}} \right\rangle_N \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_{\Sigma}), \tag{4.5.2}$$

where $\int_{\mathcal{L}}$ denotes the *BV pushforward* with respect to the map

$$\mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2} \to \mathcal{V}_{\Sigma},$$

for some Lagrangian submanifold $\mathcal{L}$ in the bulk complement of $\mathcal{V}_{\Sigma}$ and $\langle\ ,\ \rangle_N$ is the pairing in $\mathcal{H}_N^{\mathcal{P}}$. Note that the product of the space of residual fields $\mathcal{V}_{\Sigma_1} \times \mathcal{V}_{\Sigma_2}$ is a *hedgehog fibration* as in [CMR17]. The BV-BFV partition function depends only on residual fields when Σ is closed. If we assume that the space of residual fields $\mathcal{V}_{\Sigma}$ is finite-dimensional, which is the case for many theories of interest including AKSZ theories, we can compute the number valued partition function by integrating out the residual fields $\int_{\mathcal{V}_{\Sigma}} \mathsf{Z}_{\Sigma}^{\mathrm{BV}} \in \mathbb{C}$. If the glued manifold Σ has boundary itself (see Figure 4.5.2), the BV-BFV partition function depends on the boundary fields $\mathcal{B}_{\partial\Sigma}^{\mathcal{P}}$. In particular, we get

$$\mathsf{Z}_{\Sigma,\partial\Sigma}^{\mathrm{BV-BFV}} = \int_{\mathcal{L}} \left\langle \mathsf{Z}_{\Sigma_1,\partial\Sigma_1}^{\mathrm{BV-BFV}}, \mathsf{Z}_{\Sigma_2,\partial\Sigma_2}^{\mathrm{BV-BFV}} \right\rangle_N \in \mathcal{H}_{\partial\Sigma}^{\mathcal{P}} \widehat{\otimes} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_{\Sigma}).$$

4.6. **Example: abelian** BF **theory.** Let Σ be a d-manifold with compact boundary $\partial\Sigma$. The BV space of fields is then given by $\mathcal{F}_{\Sigma} = \Omega^{\bullet}(\Sigma)[1] \oplus \Omega^{\bullet}(\Sigma)[d-2]$. Denote a field in $\mathcal{F}_{\Sigma}$ by $\mathbf{X} \oplus \mathbf{Y}$, where $\mathbf{X} \in \Omega^{\bullet}(\Sigma)[1]$ and $\mathbf{Y} \in \Omega^{\bullet}(\Sigma)[d-2]$. The BV symplectic form is then given by

$$\omega_{\Sigma} = \int_{\Sigma} \sum_i \delta\mathbf{Y}_i \delta\mathbf{X}^i,$$

the BV action by

$$\mathcal{S}_{\Sigma} = \int_{\Sigma} \sum_i \mathbf{Y}_i \mathrm{d}_{\Sigma}\mathbf{X}^i,$$

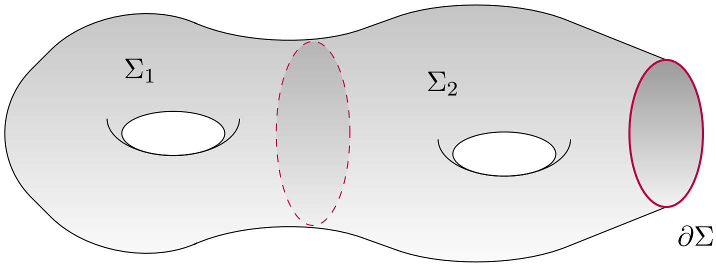

FIGURE 4.5.2. Example of the case when $\Sigma = \Sigma_1 \cup \Sigma_2$ has boundary.

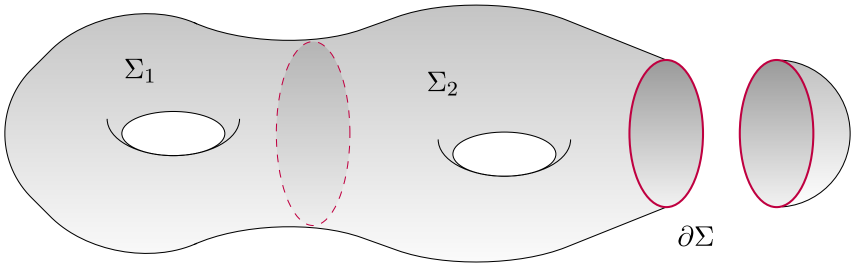

FIGURE 4.5.3. Closing the manifold Σ.

and the cohomological vector field by

$$Q_\Sigma = \int_\Sigma \sum_i \left(\mathrm{d}_\Sigma \mathbf{Y}_i \frac{\delta}{\delta \mathbf{Y}_i} + \mathrm{d}_\Sigma \mathbf{X}^i \frac{\delta}{\delta \mathbf{X}^i} \right),$$

where δ denotes the de Rham differential on $\mathcal{F}_\Sigma$ and d_Σ the one on Σ. The exact BFV manifold $(\mathcal{F}^\partial_{\partial\Sigma}, \omega^\partial_{\partial\Sigma} = \delta\alpha^\partial_{\partial\Sigma}, Q^\partial_{\partial\Sigma})$ assigned to the boundary $\partial\Sigma$ is given by the BFV space of fields $\mathcal{F}^\partial_{\partial\Sigma} = \Omega^\bullet(\partial\Sigma)[1] \oplus \Omega^\bullet(\partial\Sigma)[d-2]$, the primitive 1-form

$$\alpha^\partial_{\partial\Sigma} = (-1)^d \int_{\partial\Sigma} \sum_i \mathbb{Y}_i \delta \mathbb{X}^i,$$

the BFV boundary action

$$\mathcal{S}^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \sum_i \mathbb{Y}_i \mathrm{d}_{\partial\Sigma} \mathbb{X}^i,$$

and cohomological vector field

$$Q^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \sum_i \left(\mathrm{d}_{\partial\Sigma} \mathbb{Y}_i \frac{\delta}{\delta \mathbb{Y}_i} + \mathrm{d}_{\partial\Sigma} \mathbb{X}^i \frac{\delta}{\delta \mathbb{X}^i} \right).$$

We have denoted by $\mathbb{X} \oplus \mathbb{Y} = i^*_\Sigma(\mathbf{X} \oplus \mathbf{Y}) \in \mathcal{F}^\partial_{\partial\Sigma}$ for $\mathbf{X} \oplus \mathbf{Y} \in \mathcal{F}_\Sigma$, where $i_\Sigma \colon \partial\Sigma \hookrightarrow \Sigma$ denotes the inclusion. The surjective submersion $\pi_\Sigma \colon \mathcal{F}_\Sigma \to \mathcal{F}^\partial_{\partial\Sigma}$ is given by restricting to the boundary, i.e. $\pi_\Sigma = i^*_\Sigma$.
Consider the case where $\partial\Sigma$ is given by the disjoint union of two compact manifolds $\partial_1\Sigma$ and $\partial_2\Sigma$ such that we can consider a splitting $\mathcal{F}^\partial_{\partial\Sigma} = \mathcal{F}^\partial_{\partial_1\Sigma} \times \mathcal{F}^\partial_{\partial_2\Sigma}$. Then we can consider a polarization $\mathcal{P}$ on $\mathcal{F}^\partial_{\partial\Sigma}$ which is given by a direct product of polarizations on each factor. One can choose the convenient $\frac{\delta}{\delta\mathbb{Y}}$-polarization on $\partial_1\Sigma$ and identify the quotient leaf space

with $\mathcal{B}_{\partial_1\Sigma} \cong \Omega^\bullet(\partial_1\Sigma)[1]$ whose coordinates are given by the $\mathbb{X}$ fields. On $\partial_2\Sigma$ one can choose the convenient $\frac{\delta}{\delta\mathbb{X}}$-polarization and similarly as before identify the quotient leaf space with $\mathcal{B}_{\partial_2\Sigma} \cong \Omega^\bullet(\partial_2\Sigma)[d-2]$ whose coordinates are given by the $\mathbb{Y}$ fields. The leaf space associated to the boundary polarization is then given by $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma} = \mathcal{B}_{\partial_1\Sigma} \times \mathcal{B}_{\partial_2\Sigma}$.

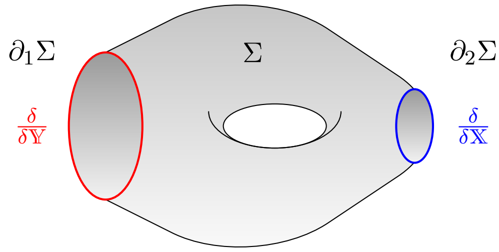

FIGURE 4.6.1. Example of Σ with two boundaries $\partial_1\Sigma$ and $\partial_2\Sigma$. On $\partial_1\Sigma$ we choose the $\frac{\delta}{\delta\mathbb{Y}}$-polarization (i.e. the $\mathbb{X}$-representation) and on $\partial_2\Sigma$ we choose the $\frac{\delta}{\delta\mathbb{X}}$-polarization (i.e. the $\mathbb{Y}$-representation).

The case when we have more than two boundary components is illustrated and explained in Figure 4.6.2. The case for boundary components with mixed polarization is explained in Figure 4.6.3. However, we will not consider this case in this paper[26].
In particular, we want to assume for the leaf space $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$ the property that for a suitably chosen local functional $f^{\mathcal{P}}_{\partial\Sigma}$, the restriction of the adapted BFV 1-form $\alpha^{\partial,\mathcal{P}}_{\partial\Sigma} := \alpha^{\partial}_{\partial\Sigma} - \delta f^{\mathcal{P}}_{\partial\Sigma}$ to the fibers of the polarization $\mathcal{P}$ vanishes. In this case we can identify the space of boundary states $\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}$ with $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{B}^{\mathcal{P}}_{\partial\Sigma})$ by multiplying with $\exp(\mathrm{i} f^{\mathcal{P}}_{\partial\Sigma}/\hbar)$. Moreover, we can then modify the BV action $\mathcal{S}_\Sigma$ to $\mathcal{S}^{\mathcal{P}}_\Sigma := \mathcal{S}_\Sigma + \pi^*_\Sigma f^{\mathcal{P}}_{\partial\Sigma}$. In our case, we consider the functional

$$f^{\mathcal{P}}_{\partial\Sigma} = (-1)^{d-1} \int_{\partial_2\Sigma} \sum_i \mathbb{Y}_i \mathbb{X}^i,$$

the adapted BFV 1-form

$$\alpha^{\partial,\mathcal{P}}_{\partial\Sigma} = (-1)^d \int_{\partial_1\Sigma} \sum_i \mathbb{Y}_i \delta\mathbb{X}^i - \int_{\partial_2\Sigma} \sum_i \delta\mathbb{Y}_i \mathbb{X}^i$$

and the modified BV action

$$\mathcal{S}^{\mathcal{P}}_\Sigma = \int_\Sigma \sum_i \mathbb{Y}_i \mathrm{d}_{\partial\Sigma}\mathbb{X} + (-1)^{d-1} \int_{\partial_2\Sigma} \sum_i \mathbb{Y}_i \mathbb{X}^i.$$

Denote by $\widetilde{\mathbb{X}} \oplus \widetilde{\mathbb{Y}} := \pi^*_\Sigma(\mathbb{X} \oplus \mathbb{Y})$ an extension of $\mathbb{X} \oplus \mathbb{Y}$ to $\mathcal{F}_\Sigma$ and denote by $\widehat{\mathbf{X}} \oplus \widehat{\mathbf{Y}}$ the complement such that

$$\mathbf{X} = \widetilde{\mathbb{X}} + \widehat{\mathbf{X}},$$
$$\mathbf{Y} = \widetilde{\mathbb{Y}} + \widehat{\mathbf{Y}}.$$

[26]This case was studied for the Poisson sigma model (2-dimensional AKSZ theory) in [CMW20] for the quantization of the relational symplectic groupoid. Another (a bit different) approach was considered for 2-dimensional Yang–Mills theory in [IM19].

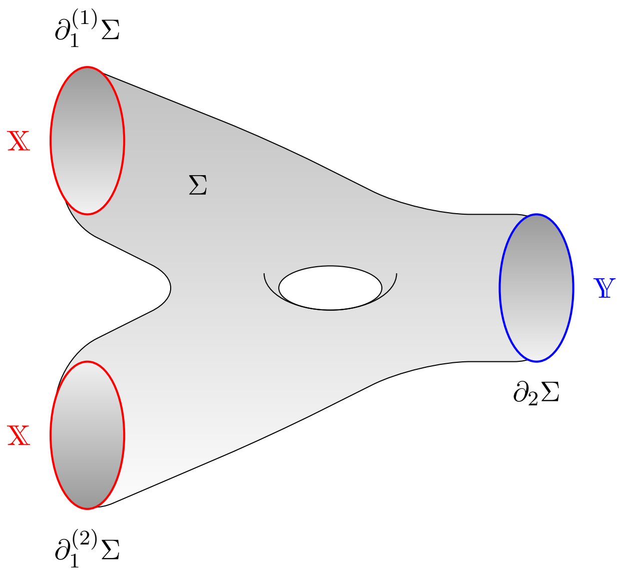

FIGURE 4.6.2. Example of a manifold with three boundary components where two components have the $\mathbb{X}$-representation and one component has the $\mathbb{Y}$-representation. Note that in this case we have $\partial_1\Sigma = \partial_1^{(1)}\Sigma \cup \partial_1^{(2)}\Sigma$. In general we can have r_1 boundary components in the $\mathbb{X}$-representation and r_2 boundary components in the $\mathbb{Y}$-representation. In this case we have $\partial_1\Sigma = \bigcup_{k=1}^{r_1} \partial_1^{(k)}\Sigma$ and $\partial_2\Sigma = \bigcup_{k=1}^{r_2} \partial_2^{(k)}\Sigma$.

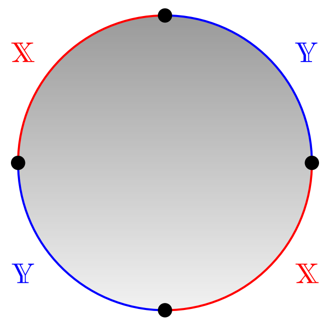

FIGURE 4.6.3. Example of a boundary component with mixed boundary polarization.

Note that we require that $i_1^*\widehat{\mathbf{X}} = 0$ and $i_2^*\widehat{\mathbf{Y}} = 0$, where $i_j\colon \partial_j\Sigma \hookrightarrow \Sigma$ denotes the inclusion of the corresponding boundary components for $j = 1, 2$. Let us choose this section of the bundle $\mathcal{F}_\Sigma \to \mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$. The modified BV action is then given by

$$\mathcal{S}^{\mathcal{P}}_\Sigma = \int_\Sigma \sum_i \left(\widetilde{\mathbb{Y}}_i \mathrm{d}_{\partial\Sigma}\widetilde{\mathbb{X}}^i + \widetilde{\mathbb{Y}}_i \mathrm{d}_\Sigma \widehat{\mathbf{X}}^i + \widehat{\mathbf{Y}}_i \mathrm{d}_{\partial\Sigma}\widetilde{\mathbb{X}}^i + \widehat{\mathbf{Y}}_i \mathrm{d}_\Sigma \widehat{\mathbf{X}}^i\right) + (-1)^{d-1}\int_{\partial_2\Sigma}\sum_i \left(\mathbb{Y}_i\widetilde{\mathbb{X}}^i + \mathbb{Y}_i\widehat{\mathbf{X}}^i\right).$$

Consider the last bulk term which we denote by $\widehat{\mathcal{S}}_\Sigma := \int_\Sigma \sum_i \widehat{\mathbf{Y}}_i \mathrm{d}_\Sigma \widehat{\mathbf{X}}^i$. Note that, by the vanishing boundary conditions, we get the critical points defined by the equations $\mathrm{d}_\Sigma \widehat{\mathbf{X}} = \mathrm{d}_\Sigma \widehat{\mathbf{Y}} = 0$. Moreover, consider the subcomplex of the de Rham complex $\Omega^\bullet(\Sigma)$ given by

$$\Omega^\bullet_{\mathrm{D}j}(\Sigma) := \{\gamma \in \Omega^\bullet(\Sigma) \mid i_j^*\gamma = 0\},$$

where D stands for *Dirichlet*. The residual fields are then given by

$$\mathcal{V}_\Sigma = H^\bullet_{\mathrm{D}1}(\Sigma)[1] \oplus H^\bullet_{\mathrm{D}2}(\Sigma)[d-2],$$

which is a finite-dimensional BV manifold. Note that we have a natural identification with relative cohomology $H^\bullet_{\mathrm{D}j}(\Sigma) \cong H^\bullet(\Sigma, \partial_j\Sigma)$ for $j = 1, 2$. Hence, we can define a canonical BV Laplacian by choosing coordinates $(z^j, z_j^\dagger)$ on $\mathcal{V}_\Sigma$. Choosing a basis $([\kappa_j])$ of $H^\bullet_{\mathrm{D}1}(\Sigma)$ with representative $\kappa_j \in \Omega^\bullet_{\mathrm{D}1}(\Sigma)$ and the corresponding dual basis $([\kappa^j])$ of $H^\bullet_{\mathrm{D}2}(\Sigma)$ with representative $\kappa^j \in \Omega^\bullet_{\mathrm{D}2}(\Sigma)$ satisfying the relation $\int_\Sigma \kappa^i\kappa_j = \delta^i_j$, we can write the residual fields as

$$\mathsf{x} = \sum_j z^j \kappa_j, \qquad \mathsf{y} = \sum_j z_j^\dagger \kappa^j. \tag{4.6.1}$$

The BV Laplacian on $\mathcal{V}_\Sigma$ is given by

$$\Delta_{\mathcal{V}_\Sigma} = \sum_j (-1)^{(d-1)\,\mathrm{gh}(z^j)+1} \frac{\partial^2}{\partial z^i \partial z_j^\dagger}$$

and the BV symplectic form on $\mathcal{V}_\Sigma$ is given by

$$\omega_{\mathcal{V}_\Sigma} = \sum_j (-1)^{(d-1)\,\mathrm{gh}(z^j)+1} \delta z_j^\dagger \delta z^j.$$

Note that $\mathrm{gh}(z^j) = 1 - \mathrm{gh}(\kappa_j)$ and $\mathrm{gh}(z_j^\dagger) = \mathrm{gh}(\kappa_j) - 2$. Consider another splitting of the fields

$$\widehat{\mathbf{X}} = \mathsf{x} + \mathscr{X},$$
$$\widehat{\mathbf{Y}} = \mathsf{y} + \mathscr{Y},$$

where $\mathscr{X} \oplus \mathscr{Y}$ denotes the corresponding complement of the field $\mathsf{x} \oplus \mathsf{y} \in \mathcal{V}_\Sigma$ as elements of a symplectic complement $\mathcal{Y}'_\Sigma$ of $\mathcal{V}_\Sigma$. Note that $i_1^*\mathscr{X} = i_2^*\mathscr{Y} = 0$. Using this, we can see that $\widehat{\mathcal{S}}_\Sigma = \int_\Sigma \mathscr{Y} \mathrm{d}_\Sigma \mathscr{X}$ which can be regarded as a quadratic function on $\Omega^\bullet_{\mathrm{D}1}(\Sigma)[1] \oplus \Omega^\bullet_{\mathrm{D}2}(\Sigma)[d-2]$ and has critical points given by closed forms. One can now choose a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{Y}'_\Sigma$ where $\widehat{\mathcal{S}}_\Sigma$ has isolated critical points at the origin, which is to say that the de Rham differential d has trivial kernel. This can be done by Hodge theory for manifolds with boundary. Consider the Hodge star operator $*$ for a chosen metric on Σ with product structure near the boundary, i.e. there is a diffeomorphism $\varphi\colon \partial\Sigma \supset U \to \partial\Sigma \times [0, \varepsilon)$ for some $\varepsilon > 0$ such that the restriction of φ to $\partial\Sigma$ is the identity on $\partial\Sigma$ and the metric on Σ restricted to the neighborhood U has the form $\varphi^*(g_{\partial\Sigma} + \mathrm{d}t^2)$, where $g_{\partial\Sigma}$ denotes the metric on the boundary and $t \in [0, \varepsilon)$ the coordinate on $\partial\Sigma \times [0, \varepsilon)$. One can define the Lagrangian submanifold

$$\mathcal{L} = (\mathrm{d}^*\Omega^{\bullet+1}_{\mathrm{N}2}(\Sigma) \cap \Omega^\bullet_{\mathrm{D}1}(\Sigma))[1] \oplus (\mathrm{d}^*\Omega^{\bullet+1}_{\mathrm{N}1}(\Sigma) \cap \Omega^\bullet_{\mathrm{D}2}(\Sigma))[d-2],$$

where $\mathrm{d}^* := *\mathrm{d}*$ and

$$\Omega^\bullet_{\mathrm{N}j}(\Sigma) := \{\gamma \in \Omega^\bullet(\Sigma) \mid i_j^* * \gamma = 0\}$$

is the space of *Neumann* forms relative to $\partial_j\Sigma$. One can check that the restriction of $\widehat{\mathcal{S}}_\Sigma$ to $\mathcal{L}$ is nondegenerate. Moreover, one can show (see [CMR17] for more details) that $\mathcal{L}$ is is indeed

Lagrangian in the symplectic complement $\mathcal{Y}'_\Sigma$ of $\mathcal{V}_\Sigma$. One can then construct a *propagator* $\mathscr{P}$ as the integral of the chain contraction K of $\Omega^\bullet_{\mathrm{D}1}(\Sigma)$ onto $H^\bullet_{\mathrm{D}1}(\Sigma)$. The gauge-fixing Lagrangian is then given as

$$\mathcal{L}=\mathrm{im}(K)[1]\oplus\mathrm{im}(K^*)[d-2].$$

This can be done by choosing a Hodge chain contraction $K\colon\Omega^\bullet_{\mathrm{D}1}(\Sigma)\to\Omega^{\bullet-1}_{\mathrm{D}1}(\Sigma)$. We call a differential form $\beta\in\Omega^\bullet(\Sigma)$ *ultra-Dirichlet relative to* $\partial_j\Sigma$ if the pullbacks to $\partial_j\Sigma$ of all even normal derivatives of β and pullbacks of all odd normal derivatives of $*\beta$ vanish. We call a differential form $\beta\in\Omega^\bullet(\Sigma)$ *ultra-Neumann relative to* $\partial_j\Sigma$ if pullbacks to $\partial_j\Sigma$ of all even normal derivatives of $*\beta$ and pullbacks of all odd normal derivatives of β vanish. Denote by $\Omega^\bullet_{\widehat{\mathrm{D}}j}(\Sigma)$ and $\Omega^\bullet_{\widehat{\mathrm{N}}j}(\Sigma)$ the space of ultra-Dirichlet and ultra-Neumann forms, respectively. We call a differential form $\beta\in\Omega^\bullet(\Sigma)$ *ultra-harmonic* if it is closed with respect to d and d^*. Denote by $\widehat{\mathrm{Harm}}^\bullet(\Sigma)$ the space of ultra-harmonic forms on Σ. Note that $\Omega^\bullet_{\widehat{\mathrm{D}}j}(\Sigma)$ and $\Omega^\bullet_{\widehat{\mathrm{N}}j}(\Sigma)$ are both subcomplexes of $\Omega^\bullet(\Sigma)$ with respect to d and d^*, respectively. One can easily see that ultra-harmonic Dirichlet forms are ultra-Dirichlet and ultra-harmonic Neumann forms are ultra-Neumann. Let $\Delta_{\mathrm{Hodge}}:=\mathrm{d}\mathrm{d}^*+\mathrm{d}^*\mathrm{d}$ denote the Hodge Laplacian. The Hodge chain contraction is then given by $K=\mathrm{d}^*/(\Delta_{\mathrm{Hodge}}+\pi_{\mathrm{Harm}})$, where π_{Harm} denotes the projection to (ultra-)harmonic forms. The propagator $\mathscr{P}$ is then a smooth form on the compactified configuration space $\overline{\mathrm{Conf}_2(\Sigma)}:=\overline{\{(u_1,u_2)\in\Sigma\times\Sigma\mid u_1\neq u_2\}}$. Moreover, consider the space $\mathfrak{D}:=\{(u_1,u_2)\in(\partial_1\Sigma\times\Sigma)\cup(\partial_2\Sigma\times\Sigma)\mid u_1\neq u_2\}$ (see Appendix B for more on configuration spaces on manifolds with boundary). Then we get that $\mathscr{P}\in\Omega^{d-1}(\overline{\mathrm{Conf}_2(\Sigma)},\mathfrak{D})=\{\gamma\in\Omega^{d-1}(\overline{\mathrm{Conf}_2(\Sigma)})\mid i^*_{\mathfrak{D}}\gamma=0\}$, where $i_{\mathfrak{D}}\colon\mathfrak{D}\hookrightarrow\overline{\mathrm{Conf}_2(\Sigma)}$ denotes the inclusion. We can write the propagator as a path integral

$$\mathscr{P}=\frac{1}{T_\Sigma}\frac{(-1)^d}{\mathrm{i}\hbar}\int_{\mathcal{L}}\exp\left(\mathrm{i}\widehat{\mathcal{S}}_\Sigma/\hbar\right)\pi_1^*\mathscr{X}\pi_2^*\mathscr{Y},$$

where $\pi_j\colon\Sigma\times\Sigma\to\Sigma$ for $j=1,2$ denotes the projection onto the first and second factor, respectively and $T_\Sigma=\int_{\mathcal{L}}\exp(\mathrm{i}\widehat{\mathcal{S}}_\Sigma/\hbar)\in\mathbb{C}\otimes\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)/\{\pm1\}$ is given in terms of the Ray–Singer torsion [RS71] which is defined through zeta regularization which does not depend on the choice of $\mathcal{L}$ since the Ray–Singer torsion does not depend on the choice of metric. Let N be a $(d-1)$-manifold and let $\mathcal{H}^n_{N,\ell}$ be the vector space of n-linear functionals on $\Omega^\bullet(N)[\ell]$ of the form

$$\Omega^\bullet(N)[\ell]\ni\mathbb{D}\mapsto\int_{N^n}\gamma\pi_1^*\mathbb{D}\cdots\pi_n^*\mathbb{D}$$

up to multiplication with a term given by the Reidemeister torsion [Rei35] of N. We have denoted by γ a distributional form on N^n. The boundary state space $\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}$ is then given by

$$\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}=\prod_{n_1,n_2=0}^{\infty}\mathcal{H}^{n_1}_{\partial_1\Sigma,1}\widehat{\otimes}\mathcal{H}^{n_2}_{\partial_2\Sigma,d-2}$$

and $\widehat{\mathcal{H}}^{\mathcal{P}}_\Sigma=\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$. Perturbatively, the BV-BFV partition function is asymptotically given by

$$T_\Sigma\exp(\mathrm{i}\mathcal{S}^{\mathrm{eff}}_\Sigma/\hbar)\times$$
$$\times\sum_{j\geq0}\hbar^j\sum_{n_1,n_2\geq0}\int_{(\partial_1\Sigma)^{n_1}\times(\partial_2\Sigma)^{n_2}}R^j_{n_1n_2}(\mathsf{x},\mathsf{y})\pi^*_{1,1}\mathbb{X}^{i_1}\cdots\pi^*_{1,n_1}\mathbb{X}^{i_n}\pi^*_{2,1}\mathbb{Y}_{i_1}\cdots\pi^*_{2,n_2}\mathbb{Y}_{i_n},$$

where $R^j_{n_1n_2}(\mathsf{x},\mathsf{y})$ denotes distributional forms on $(\partial_1\Sigma)^{n_1}\times(\partial_2\Sigma)^{n_2}$ with values in $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$. Note that we sum over $i_1,\dots,i_n$. Moreover, we have denoted by $\mathcal{S}^{\mathrm{eff}}_\Sigma$ the effective action[27] given by

$$\mathcal{S}^{\mathrm{eff}}_\Sigma=(-1)^{d-1}\left(\int_{\partial_2\Sigma}\sum_i\mathbb{Y}_i\mathsf{x}^i-\int_{\partial_1\Sigma}\sum_i\mathsf{y}_i\mathbb{X}^i\right)-(-1)^{2d}\int_{\partial_2\Sigma\times\partial_1\Sigma}\sum_i\pi_1^*\mathbb{Y}_i\mathscr{P}\pi_2^*\mathbb{X}^i.$$

The BFV boundary operator acting on $\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}$ is then given by the ordered standard quantization of the BFV boundary action $\mathcal{S}^\partial_{\partial\Sigma}$:

$$\Omega^{\mathcal{P}}_{\partial\Sigma}=(-1)^d\mathrm{i}\hbar\left(\int_{\partial_2\Sigma}\sum_i\mathrm{d}_{\partial\Sigma}\mathbb{Y}_i\frac{\delta}{\delta\mathbb{Y}_i}+\int_{\partial_1\Sigma}\sum_i\mathrm{d}_{\partial\Sigma}\mathbb{X}^i\frac{\delta}{\delta\mathbb{X}^i}\right).$$

Suppose now that we have two smooth d-manifolds Σ_1 and Σ_2 with common boundary component $(d-1)$-manifold N. We can then compute the glued partition function $\mathsf{Z}^{\mathrm{BV}}_\Sigma$ for the glued manifold $\Sigma=\Sigma_1\cup_N\Sigma_2$ (see Figure 4.5.1) out of the BV-BFV partition functions $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_1,\partial\Sigma_1}$ on Σ_1 and $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_2,\partial\Sigma_2}$ on Σ_2. We consider the transversal polarization on $\mathcal{F}^\partial_N$. In particular, we decompose the boundaries $\partial\Sigma_1=\partial_1\Sigma_1\sqcup\partial_2\Sigma_1$ and $\partial\Sigma_2=\partial_1\Sigma_2\sqcup\partial_2\Sigma_2$ such that $N\subset\partial_1\Sigma_1$ and $N^{\mathrm{opp}}\subset\partial_2\Sigma_2$. Let $\mathbb{X}^N_1$ and $\mathbb{Y}^N_2$ be the coordinates on $\Omega^\bullet(N)[1]$ and $\Omega^\bullet(N)[d-2]$, respectively. The glued state is then given by

$$\mathsf{Z}^{\mathrm{BV}}_\Sigma=\int_{\{\mathbb{X}^N_1,\mathbb{Y}^N_2\}}\exp\left(\frac{\mathrm{i}}{\hbar}(-1)^{d-1}\int_N\sum_i(\mathbb{Y}^N_2)_i(\mathbb{X}^N_1)^i\right)\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_1,\partial\Sigma_1}\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_2,\partial\Sigma_2}.$$

The glued partition function can be computed by considering the glued effective action and then considering $\mathsf{Z}^{\mathrm{BV}}_\Sigma=T_{\Sigma_1}T_{\Sigma_2}\exp(\mathrm{i}\mathcal{S}^{\mathrm{eff}}_\Sigma/\hbar)$.

Remark 4.6.1. This construction can be generalized to the case of BF-like theories (perturbations of abelian BF theory), see [CMR17] for more details. In particular, this is important since AKSZ theories often appear as BF-like theories. This will be relevant especially for treating DW theory in the BV-BFV setting by regarding it as an AKSZ theory, which will be the content of Section 5. An important result regarding the gluing for BF-like theories is given by the following proposition.

Proposition 4.6.2 (Cattaneo–Mnev–Reshetikhin[CMR17])**.** *Let Σ be cut along a codimension-one submanifold N into Σ_1 and Σ_2. Let $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_1,N}$ and $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_2,N^{\mathrm{opp}}}$ be the boundary states for Σ_1 and Σ_2 with a choice of residual fields and propagators and transverse ($\mathbb{X}$ vs. $\mathbb{Y}$) polarizations on N. Then the gluing of $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_1,N}$ and $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma_2,N^{\mathrm{opp}}}$ is the state $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}$ for Σ with the consequent choice of residual fields and propagators.*

4.7. **BV-BF^{k}V extension and shifted symplectic structures.** If we move to higher defects, the symplectic form will be shifted by degree $+1$. In particular, if we consider a codimension k submanifold $N_k\subset\Sigma$, then the symplectic gauge formalism associated to N_k will be $(k-1)$-shifted. We will call the theory associated to a codimension k submanifold a BF^{k}V theory and the coupling for each contiguous codimension (fully extended) a BV-BF^{k}V theory associated to the d-manifold Σ. The underlying mathematical theory for shifted symplectic structures was developed first in [Pan+13] by using methods and the language of derived algebraic geometry and studied further by various people. In [Pan+13], they define first a symplectic form on a smooth scheme over some base ring k of characteristic

[27] One can easily check that the partition function $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}=T_\Sigma\exp(\mathrm{i}\mathcal{S}^{\mathrm{eff}}_\Sigma/\hbar)$, defined through the effective action $\mathcal{S}^{\mathrm{eff}}_\Sigma$, satisfies the mQME (4.4.5)

zero to be a 2-form $\omega \in H^0(X, \Omega^{2,cl}_{X/k})$ which moreover is required to be nondegenerate, i.e. it induces an isomorphism Θ_ω between the tangent and cotangent bundles of X. Then one can define an *n-shifted symplectic form* on a derived Artin stack X (in particular, they consider X to be the solution of a *derived moduli problem* (see also [Toë14])) to be a closed 2-form $\omega \in H^n(X, \bigwedge^2 \mathbb{L}_{X/k})$ of degree n on X such that the corresponding morphism $\Theta_\omega \colon \mathbb{T}_{X/k} \to \mathbb{L}_{X/k}[n]$ is an isomorphism in the derived category of quasi-coherent sheaves $D^b\mathrm{QCoh}(X)$ on X. Here we have denoted by $\mathbb{L}_{X/k}$ the cotangent complex of X and by $\mathbb{T}_{X/k}$ its dual, the tangent complex of X. Of course, it is important to mention what *closed* in this setting actually means. They define closedness of general p-forms by interpreting sections of $\bigwedge^2 \mathbb{L}_{X/k}$ as functions on the derived loop stack $\mathcal{L}X$ and consider it as some type of S^1-equivariance property. An important result of [Pan+13] is an existence result concerning a derived algebraic version of the AKSZ formulation in this setting. In particular, they prove the following theorem:

Theorem 4.7.1 (Pantev–Toën–Vaquié–Vezzosi[Pan+13])**.** *Let X be a derived stack endowed with an $\mathcal{O}$-orientation of dimension d, and let (F, ω) be a derived Artin stack with an n-shifted symplectic structure ω. Then the derived mapping stack $\mathbf{Map}(X, F)$ carries a natural $(n-d)$-shifted symplectic structure.*

The BV-BFV formalism, as described in Section 4, can be easily extended to higher codimensions in the classical setting, but needs more sophisticated methods in the quantum setting [CMR14; Mos21]. As discussed in [Mos21], the quantum setting needs to couple the methods of deformation quantization to the methods of geometric quantization, both in the shifted setting. Fortunately, also the methods for shifted Poisson structures and shifted deformation quantization have been developed in [Cal+17; Saf17]. The setting of geometric quantization in the shifted picture has been recently considered in [Saf20]. The methods developed in [Pan+13] have been considered for the setting of codimension 1 structures (BV-BFV) in [Cal15] for the setting of AKSZ topological field theories by using Lagrangian correspondences and have been recently (fully) extended by Calaque, Haugseng and Scheimbauer [CHS].

Example 4.7.2 (Chern–Simons theory)**.** As an example we want to discuss the 3-dimensional case of Chern–Simons theory (see Section 3.2). Recall that the phase space is given by the moduli space of flat G connections for some compact Lie group G. In the codimension 0 case, we are working over a closed oriented 3-manifold Σ (see Figure 4.7.1) which carries the induced BV symplectic structure, i.e. a (-1)-shifted structure. In the codimension 1 case we are working over a closed oriented 2-manifold $N_1 \subset \Sigma$ (e.g. some Riemann surface Σ_g of genus g), to which we can associate a phase space endowed with the *Atiyah–Bott symplectic structure* [AB83], which is a 0-shifted structure. In the codimension 2 case we are working over a closed oriented 1-manifold $S^1 \cong N_2 \subset \Sigma$, for which the phase space is given by the stack of conjugacy classes $[G/G]$ and we can consider a 1-shifted symplectic structure by the canonical 3-form on G. Finally, in the codimension 3 case we consider a point pt (0-manifold), for which the phase space is given by the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ endowed with the 2-shifted symplectic form given by the invariant pairing (Killing form) on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

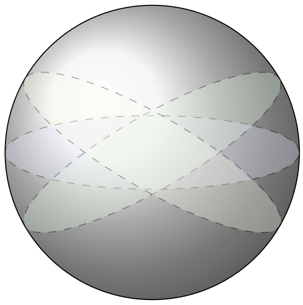

FIGURE 4.7.1. Examples of 2-dimensional defects sitting inside the 3-ball.

5. AKSZ formulation of DW theory

In this section we want to describe a way of obtaining the DW action functional as in Section 2.3 in terms of an AKSZ theory by choosing a suitable gauge-fixing Lagrangian submanifold $\mathcal{L}$ [BS89; Ike11; Bon+20].

5.1. AKSZ data.

Let $\mathfrak{h}$ be a Lie algebra and consider its *Weil model*[28] $(\mathcal{W}(\mathfrak{h}), \mathrm{d}_{\mathcal{W}})$. Note that $\mathcal{W}(\mathfrak{h})$ can be endowed with a natural symplectic form of degree $+3$ if $\mathfrak{h}$ is endowed with an invariant, nondegenerate symmetric pairing $\langle\ ,\ \rangle$. We want to consider this symplectic manifold to be the target for a 4-dimensional AKSZ theory. The graded vector space $\mathfrak{h}[1]\oplus\mathfrak{h}[2]$ is endowed with the symplectic structure

$$\omega = \langle \delta X, \delta Y\rangle,$$

where X denotes the coordinate of degree $+1$ and Y denotes the coordinate of degree $+2$. We can define a Hamiltonian of degree $+4$ by

$$\Theta(X,Y) = \frac{1}{2}\langle Y,Y\rangle + \frac{1}{2}\langle Y,[X,X]\rangle.$$

One can check that the Hamiltonian vector field of Θ with respect to ω is given by the Weil differential $\mathrm{d}_{\mathcal{W}}$. We have

$$\mathrm{d}_{\mathcal{W}}X = Y + \frac{1}{2}[X,X], \qquad \mathrm{d}_{\mathcal{W}}Y = [X,Y]. \tag{5.1.1}$$

Let now Σ be a 4-manifold and consider the BV space of fields

$$\mathcal{F}_\Sigma = \mathrm{Map}(T[1]\Sigma, \mathfrak{h}[1]\oplus\mathfrak{h}[2]) \cong \Omega^\bullet(\Sigma)\otimes\mathfrak{h}[1]\oplus\Omega^\bullet(\Sigma)\otimes\mathfrak{h}[2].$$

The superfields are given by

$$\mathbf{X}(u,\theta) = X + A\theta^{i_1} + \chi\theta^{i_1}\theta^{i_2} + \psi^\dagger\theta^{i_1}\theta^{i_2}\theta^{i_3} + Y^\dagger\theta^{i_1}\theta^{i_2}\theta^{i_3}\theta^{i_4}, \tag{5.1.2}$$

$$\mathbf{Y}(u,\theta) = Y + \psi\theta^{i_1} + \chi^\dagger\theta^{i_1}\theta^{i_2} + A^\dagger\theta^{i_1}\theta^{i_2}\theta^{i_3} + X^\dagger\theta^{i_1}\theta^{i_2}\theta^{i_3}\theta^{i_4}. \tag{5.1.3}$$

for local coordinates (u_i,θ^i) on $T[1]\Sigma$. The BV symplectic form is given by

$$\omega_\Sigma = \int_{T[1]\Sigma}\boldsymbol{\mu}_4\langle\delta\mathbf{X},\delta\mathbf{Y}\rangle,$$

[28] For a Lie algebra $\mathfrak{h}$, the Weil model is given by the algebra $\mathcal{W}(\mathfrak{h}) := \bigwedge^\bullet(\mathfrak{h}^*\oplus\mathfrak{h}^*[1])$ together with a differential $\mathrm{d}_{\mathcal{W}}\colon\mathcal{W}(\mathfrak{h})\to\mathcal{W}(\mathfrak{h})$ such that $(\mathcal{W}(\mathfrak{h}),\mathrm{d}_{\mathcal{W}})$ is a differential graded algebra. The differential is defined to act on $\mathfrak{h}^*$ as the differential for the Chevalley–Eilenberg algebra of $\mathfrak{h}$ plus the degree shift differential. In particular $\mathrm{d}_{\mathcal{W}} = \mathrm{d}_{\mathrm{CE}(\mathfrak{h})} + \mathrm{d}_s$, where d_s acts by degree shift $\mathfrak{h}^*\to\mathfrak{h}^*[1]$ for elements in $\mathfrak{h}$ and by 0 for elements in $\mathfrak{h}^*[1]$.

and the AKSZ-BV action is given by

$$\mathcal{S}_\Sigma=\int_{T[1]\Sigma}\boldsymbol{\mu}_4\left(\langle\mathbf{Y},\mathrm{d}_\Sigma\mathbf{X}\rangle+\frac{1}{2}\langle\mathbf{Y},\mathbf{Y}\rangle+\frac{1}{2}\langle\mathbf{Y},[\mathbf{X},\mathbf{X}]\rangle\right).$$

We have denoted by $\boldsymbol{\mu}_4:=\mathrm{d}^4u\mathrm{d}^4\theta$ the supermeasure on $T[1]\Sigma$. If we expand things into components and perform Berezinian integration, we get

$$\omega_\Sigma=\int_\Sigma\mathrm{d}^4u\left(\delta X\delta X^\dagger+\delta A\delta A^\dagger+\delta\chi\delta\chi^\dagger+\delta\psi^\dagger\delta\psi+\delta Y^\dagger\delta Y\right)\tag{5.1.4}$$

and

$$\begin{aligned}\mathcal{S}_\Sigma=\int_\Sigma\mathrm{d}^4u\bigg(&\langle\psi,\mathrm{d}_A\chi\rangle+\frac{1}{2}\langle Y,[\chi,\chi]\rangle+\langle\psi^\dagger,(\mathrm{d}_AY+[X,\psi])\rangle\\&+\langle\chi^\dagger,(F_A+[X,\chi])\rangle+\langle A^\dagger,(\psi+\mathrm{d}_AX)\rangle+\langle Y^\dagger,[X,Y]\rangle\\&+\left\langle X^\dagger,\left(\frac{1}{2}[X,X]\right)\right\rangle+\frac{1}{2}\langle\chi^\dagger,\chi^\dagger\rangle\bigg),\end{aligned}\tag{5.1.5}$$

where $\mathrm{d}_A=\mathrm{d}_\Sigma+[A,\ ]$ denotes the covariant derivative of A and $F_A=\mathrm{d}_\Sigma A+\frac{1}{2}[A,A]$. The BV transformations on the superfields are then given by

$$Q_\Sigma\mathbf{X}=\mathrm{d}_\Sigma\mathbf{X}+\mathbf{Y}+\frac{1}{2}[\mathbf{X},\mathbf{X}],\tag{5.1.6}$$

$$Q_\Sigma\mathbf{Y}=\mathrm{d}_\Sigma\mathbf{Y}+[\mathbf{X},\mathbf{Y}],\tag{5.1.7}$$

i.e., in superfield notation, the cohomological vector field is given by

$$Q_\Sigma=\int_{T[1]\Sigma}\boldsymbol{\mu}_4\left(\left(\mathrm{d}_\Sigma\mathbf{X}+\mathbf{Y}+\frac{1}{2}[\mathbf{X},\mathbf{X}]\right)\frac{\delta}{\delta\mathbf{X}}+\left(\mathrm{d}_\Sigma\mathbf{Y}+[\mathbf{X},\mathbf{Y}]\right)\frac{\delta}{\delta\mathbf{Y}}\right)\tag{5.1.8}$$

In component fields and after Berezinian integration, we get the cohomological vector field

$$\begin{aligned}Q_\Sigma=\int_\Sigma\mathrm{d}^4u\bigg(&\left(Y+\frac{1}{2}[X,X]\right)\frac{\delta}{\delta X}+(\psi+\mathrm{d}_AX)\frac{\delta}{\delta A}+(\chi^\dagger+F_A+[X,\chi])\frac{\delta}{\delta\chi}\\&+(\mathrm{d}_A\chi+A^\dagger+[X,\psi^\dagger])\frac{\delta}{\delta\psi^\dagger}+(\mathrm{d}_A\psi^\dagger+X^\dagger+[X,Y^\dagger])\frac{\delta}{\delta Y^\dagger}+[X,Y]\frac{\delta}{\delta Y}\\&+(\mathrm{d}_AY+[X,\psi])\frac{\delta}{\delta\psi}+(\mathrm{d}_A\psi+[X,\chi^\dagger]+[\chi,Y])\frac{\delta}{\delta\chi^\dagger}\\&+(\mathrm{d}_A\chi^\dagger+[X,A^\dagger]+[\psi^\dagger,Y]+[\chi.\psi])\frac{\delta}{\delta A^\dagger}\\&+(\mathrm{d}_AA^\dagger+[X,X^\dagger]+[Y^\dagger,Y]+[\psi^\dagger,\psi]+[\chi,\chi^\dagger])\frac{\delta}{\delta X^\dagger}\bigg),\end{aligned}\tag{5.1.9}$$

where we have used the relation in (5.1.1).

5.2. **A suitable gauge-fixing.** Consider a Riemannian metric on Σ such that we can take a splitting of the fields $\chi,\chi^\dagger$ into self-dual and anti self-dual parts

$$\chi=\chi^++\chi^-,\tag{5.2.1}$$

$$\chi^\dagger=\chi^{+\dagger}+\chi^{-\dagger}.\tag{5.2.2}$$

We can take the gauge-fixing where $\chi^-=\chi^{+\dagger}=0$ and all other forms are coexact when using Hodge decomposition. This will imply that $X^\dagger=Y^\dagger=0$. Introduce new fields $\bar{X}$ and

b with $\deg(\bar{X}) = \deg(b) = 0$ and $\mathrm{gh}(\bar{X}) = -1$ and $\mathrm{gh}(b) = 0$ together with corresponding anti-fields $\bar{X}^\dagger$ and $b^\dagger$ with $\deg(\bar{X}^\dagger) = \deg(b^\dagger) = 4$, such that $\mathrm{gh}(\bar{X}^\dagger) = 0$ and $\mathrm{gh}(b^\dagger) = -1$, respectively. Moreover, we consider the term

$$\mathcal{S}_\Sigma^{(1)} = \int_\Sigma \Big(\langle \bar{X}^\dagger, (b + [X, \bar{X}]) \rangle + \langle b^\dagger, ([X, b] - [Y, \bar{X}]) \rangle \Big).$$

Additionally, introduce more new fields $(\bar{Y}, \eta)$ with $\deg(Y) = \deg(\eta) = 0$ and $\mathrm{gh}(\bar{Y}) = -2$ and $\mathrm{gh}(\eta) = -1$ together with corresponding anti-fields $\bar{Y}^\dagger$ and $\eta^\dagger$ with $\deg(\bar{Y}^\dagger) = \deg(\eta^\dagger) = 4$, such that $\mathrm{gh}(\bar{Y}^\dagger) = +1$ and $\mathrm{gh}(\eta^\dagger) = 0$. We consider the term

$$\mathcal{S}_\Sigma^{(2)} = \int_\Sigma \Big(\langle \bar{Y}^\dagger, (\eta + [X, \bar{Y}] \rangle + \langle \eta^\dagger, ([X, \eta] + [\bar{Y}, Y]) \rangle \Big).$$

The gauge-fixed BV action is then obtained by

$$\mathcal{S}_\Sigma^{\mathrm{gf}} = \mathcal{S}_\Sigma + \mathcal{S}_\Sigma^{(1)} + \mathcal{S}_\Sigma^{(2)}$$

with gauge-fixing fermion

$$\Psi^{\mathrm{gf}} = \int_\Sigma \langle \bar{X}, \mathrm{d} * A \rangle + \int_\Sigma \langle \bar{Y}, \mathrm{d} * \psi \rangle. \tag{5.2.3}$$

such that $\mathcal{S}_\Sigma^{\mathrm{gf}} = \mathcal{S}_\Sigma + Q_\Sigma \Psi^{\mathrm{gf}}$. The auxiliary fields $\chi^{-\dagger}$ have ghost number zero and can be actually integrated out. One can check that the gauge-transformations for each field squares to zero except the one for χ^+. Namely, we have

$$\delta^2 \chi^+ = (\mathrm{d}_A \psi)^+ + [Y, \chi^+],$$

which is the equation of motion for this setting. Hence, the tuple

$$(A, Y, \psi, \chi^+, \bar{Y}, \eta)$$

is the same as in the construction of Section 2.3.

5.3. **BV-BFV formulation.** As an AKSZ theory, there a nice way of treating DW theory as a gauge-theory on 4-manifolds with boundary. Let Σ be a 4-manifold with 3-dimensional boundary $\partial\Sigma$. As obtained in Section 5.1, the BV theory of the AKSZ construction of DW theory is given by

$$(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{S}_\Sigma).$$

One can then easily formulate a BFV theory on the boundary $\partial\Sigma$ by setting

$$\mathcal{F}_{\partial\Sigma}^\partial := \mathrm{Map}(T[1]\partial\Sigma, \mathfrak{h}[1] \oplus \mathfrak{h}[2]) \cong \Omega^\bullet(\partial\Sigma) \otimes \mathfrak{h}[1] \oplus \Omega^\bullet(\partial\Sigma) \otimes \mathfrak{h}[2], \tag{5.3.1}$$

$$\omega_{\partial\Sigma}^\partial := \int_{T[1]\partial\Sigma} \mu_3 \langle \delta\mathbb{X}, \delta\mathbb{Y} \rangle, \tag{5.3.2}$$

$$\mathcal{S}_{\partial\Sigma}^\partial := \int_{T[1]\partial\Sigma} \mu_3 \left(\langle \mathbb{Y}, \mathrm{d}_{\partial\Sigma} \mathbb{X} \rangle + \frac{1}{2} \langle \mathbb{Y}, \mathbb{Y} \rangle + \frac{1}{2} \langle \mathbb{Y}, [\mathbb{X}, \mathbb{X}] \rangle \right), \tag{5.3.3}$$

where $\mu_3 := \mathrm{d}^3 u \mathrm{d}^3 \theta$ denotes the supermeasure on $T[1]\partial\Sigma$ for local coordinates (u_i, θ^i) on $T[1]\partial\Sigma$. The boundary superfields are given by

$$\mathbb{X}(u, \theta) = X + A\theta^{i_1} + \chi\theta^{i_1}\theta^{i_2} + \psi^\dagger \theta^{i_1}\theta^{i_2}\theta^{i_3}, \tag{5.3.4}$$

$$\mathbb{Y}(u, \theta) = Y + \psi\theta^{i_1} + \chi^\dagger \theta^{i_1}\theta^{i_2} + A^\dagger \theta^{i_1}\theta^{i_2}\theta^{i_3}. \tag{5.3.5}$$

The cohomological vector field $Q_{\partial\Sigma}^\partial$ is given by the Hamiltonian vector field of $\mathcal{S}_{\partial\Sigma}^\partial$:

$$\iota_{Q_{\partial\Sigma}^\partial} \omega_{\partial\Sigma}^\partial = \delta \mathcal{S}_{\partial\Sigma}^\partial.$$

It is easy to see that $\omega^\partial_{\partial\Sigma}$ is exact with primitive 1-form

$$\alpha^\partial_{\partial\Sigma}=\int_{T[1]\partial\Sigma}\mu_3\langle\mathbb{Y},\delta\mathbb{X}\rangle.$$

If we expand things into components and perform Berezinian integration, we get

$$\omega^\partial_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{d}^3\mathbb{u}\left(\delta\mathbb{X}\delta\mathbb{A}^\dagger+\delta\mathbb{A}\delta\chi^\dagger+\delta\chi\delta\psi+\delta\psi^\dagger\delta\mathbb{Y}\right)\tag{5.3.6}$$

and

$$\begin{aligned}\mathcal{S}^\partial_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{d}^3\mathbb{u}\Bigg(&\langle\psi,\mathbb{F}_{\mathbb{A}}\rangle+\langle\chi^\dagger,\mathrm{d}_{\mathbb{A}}\mathbb{X}\rangle+\langle\mathbb{Y},\mathrm{d}_{\mathbb{A}}\mathbb{X}\rangle\\&+\left\langle\mathbb{A}^\dagger,\mathbb{Y}+\frac{1}{2}[\mathbb{X},\mathbb{X}]\right\rangle+\langle\mathbb{Y},[\mathbb{X},\psi^\dagger]\rangle+\langle\psi,\chi^\dagger+[\mathbb{X},\chi]\rangle\Bigg),\end{aligned}\tag{5.3.7}$$

where $\mathrm{d}_{\mathbb{A}}=\mathrm{d}_{\partial\Sigma}+[\mathbb{A},\ ]$ denotes the covariant derivative of $\mathbb{A}$ and $\mathbb{F}_{\mathbb{A}}=\mathrm{d}_{\partial\Sigma}\mathbb{A}+\frac{1}{2}[\mathbb{A},\mathbb{A}]$. The BV transformations on the superfields are then given by

$$Q^\partial_{\partial\Sigma}\mathbb{X}=\mathrm{d}_{\partial\Sigma}\mathbb{X}+\mathbb{Y}+\frac{1}{2}[\mathbb{X},\mathbb{X}],\tag{5.3.8}$$

$$Q^\partial_{\partial\Sigma}\mathbb{Y}=\mathrm{d}_{\partial\Sigma}\mathbb{Y}+[\mathbb{X},\mathbb{Y}],\tag{5.3.9}$$

i.e., in superfield notation, the cohomological vector field is given by

$$Q^\partial_{\partial\Sigma}=\int_{T[1]\partial\Sigma}\mu_3\left(\left(\mathrm{d}_{\partial\Sigma}\mathbb{X}+\mathbb{Y}+\frac{1}{2}[\mathbb{X},\mathbb{X}]\right)\frac{\delta}{\delta\mathbb{X}}+\left(\mathrm{d}_{\partial\Sigma}\mathbb{Y}+[\mathbb{X},\mathbb{Y}]\right)\frac{\delta}{\delta\mathbb{Y}}\right)\tag{5.3.10}$$

In component fields and after Berezinian integration, we get the cohomological vector field

$$\begin{aligned}Q^\partial_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{d}^3\mathbb{u}\Bigg(&\left(\mathbb{Y}+\frac{1}{2}[\mathbb{X},\mathbb{X}]\right)\frac{\delta}{\delta\mathbb{X}}+(\psi+\mathrm{d}_{\mathbb{A}}\mathbb{X})\frac{\delta}{\delta\mathbb{A}}+(\chi^\dagger+\mathbb{F}_{\mathbb{A}}+[\mathbb{X},\chi])\frac{\delta}{\delta\chi}\\&+(\mathrm{d}_{\mathbb{A}}\chi+\mathbb{A}^\dagger+[\mathbb{X},\psi^\dagger])\frac{\delta}{\delta\psi^\dagger}+(\mathrm{d}_{\mathbb{A}}\psi^\dagger+[\mathbb{X},\mathbb{Y}])\frac{\delta}{\delta\mathbb{Y}}\\&+(\mathrm{d}_{\mathbb{A}}\mathbb{Y}+[\mathbb{X},\psi])\frac{\delta}{\delta\psi}+(\mathrm{d}_{\mathbb{A}}\psi+[\mathbb{X},\chi^\dagger]+[\chi,\mathbb{Y}])\frac{\delta}{\delta\chi^\dagger}\\&+(\mathrm{d}_{\mathbb{A}}\chi^\dagger+[\mathbb{X},\mathbb{A}^\dagger]+[\psi^\dagger,\mathbb{Y}]+[\chi,\psi])\frac{\delta}{\delta\mathbb{A}^\dagger}\Bigg).\end{aligned}\tag{5.3.11}$$

By setting $\pi_\Sigma\colon\mathcal{F}_\Sigma\to\mathcal{F}^\partial_{\partial\Sigma}$ to be the restriction of the fields to the boundary, we get that the modified CME

$$Q_\Sigma\mathcal{S}_\Sigma=\pi^*_\Sigma(2\mathcal{S}^\partial_{\partial\Sigma}-\iota_{Q^\partial_{\partial\Sigma}}\alpha^\partial_{\partial\Sigma})$$

is satisfied and that the cohomological vector field Q_Σ is indeed projectable, i.e. we have

$$\delta\pi_\Sigma Q_\Sigma=Q^\partial_{\partial\Sigma}.$$

6. Quantization of DW theory in the BV-BFV setting

6.1. **The BFV boundary operator.** Assume now that we can write the boundary as a disjoint union $\partial\Sigma=\partial_1\Sigma\sqcup\partial_2\Sigma$ where $\partial_1\Sigma$ and $\partial_2\Sigma$ have opposite orientation. We choose the convenient polarization $\mathcal{P}$ on $\partial\Sigma$ consisting of choosing the $\frac{\delta}{\delta\mathbb{Y}}$-polarization ($\mathbb{X}$-representation) on $\partial_1\Sigma$ and the $\frac{\delta}{\delta\mathbb{X}}$-polarization ($\mathbb{Y}$-representation) on $\partial_2\Sigma$ (see Figure 4.6.1). As we have seen in Section 4.4, we can compute the BFV boundary operator

$\Omega^{\mathcal{P}}_{\partial\Sigma}$ as the ordered standard quantization of the boundary action $\mathcal{S}^{\partial}_{\partial\Sigma}$ with respect to the chosen polarization. We get

$$\Omega^{\mathcal{P}}_{\partial\Sigma}=\Omega^{\mathcal{P}}_0+\Omega^{\mathcal{P}}_{\text{pert}},$$

where

$$\Omega^{\mathcal{P}}_0:=\underbrace{\mathrm{i}\hbar\int_{\partial_1\Sigma}\sum_i \mathrm{d}_{\partial_1\Sigma}\mathbb{X}^i\frac{\delta}{\delta\mathbb{X}^i}}_{=:\Omega^{\mathbb{X}}_0}+\underbrace{\mathrm{i}\hbar\int_{\partial_2\Sigma}\sum_i \mathrm{d}_{\partial_2\Sigma}\mathbb{Y}_i\frac{\delta}{\delta\mathbb{Y}_i}}_{=:\Omega^{\mathbb{Y}}_0},\tag{6.1.1}$$

and

$$\Omega^{\mathcal{P}}_{\text{pert}}=\Omega^{\mathbb{X}}_{\text{pert}}+\Omega^{\mathbb{Y}}_{\text{pert}},\tag{6.1.2}$$

with

$$\Omega^{\mathbb{X}}_{\text{pert}}:=\int_{\partial_1\Sigma}\left(-\frac{\hbar^2}{2}\sum_{i,j}\frac{\delta^2}{\delta\mathbb{X}^i\delta\mathbb{X}^j}+\frac{\mathrm{i}\hbar}{2}\sum_{i,j,k}c^k_{ij}\mathbb{X}^i\mathbb{X}^j\frac{\delta}{\delta\mathbb{X}^k}\right),\tag{6.1.3}$$

and

$$\Omega^{\mathbb{Y}}_{\text{pert}}:=\int_{\partial_2\Sigma}\left(\frac{1}{2}\sum_{i,j}\delta^i_j\mathbb{Y}_i\mathbb{Y}_j-\frac{\hbar^2}{2}\sum_{i,j,k}c^k_{ij}\mathbb{Y}_k\frac{\delta^2}{\delta\mathbb{Y}_i\delta\mathbb{Y}_j}\right),\tag{6.1.4}$$

where δ^i_j denotes the Kronecker delta and c^k_{ij} the structure constants of the Lie bracket $[\ ,\ ]$.

6.2. **Composite fields.** In order to regularize the higher functional derivatives, one needs to introduce the concept of *composite fields* as in [CMR17]. In particular, we want to regard the higher functional derivates as a single first order derivative with respect to the composite field. In order to get it coherent to the naive interpretation where a higher functional derivative concentrates the fields on some diagonal, we should also understand the product of integrals as containing the diagonal contributions for the corresponding composite field. We consider the following bullet product of integrals

$$\begin{aligned}&\left(\int_{\partial_k\Sigma}\sum_i\alpha_i\Phi^i\right)\bullet\left(\int_{\partial_k\Sigma}\sum_j\beta_j\Phi^j\right):=\\&(-1)^{\mathrm{gh}(\Phi^i)(\mathrm{gh}(\beta_j)-3)+3\mathrm{gh}(\alpha_i)}\left(\int_{\overline{\mathrm{Conf}_2(\partial_k\Sigma)}}\sum_{i,j}\pi_1^*\alpha_i\pi_2^*\beta_j\pi_1^*\Phi^i\pi_2^*\Phi^j+\int_{\partial_k\Sigma}\sum_{i,j}\alpha_i\beta_j[\Phi^i\Phi^j]\right),\end{aligned}\tag{6.2.1}$$

where α and β are smooth forms which depend on the bulk and residual fields and we denote by $[\Phi^i\Phi^j]$ the composite field. We can now interpret the operator $\int_{\partial_k\Sigma}\phi^{ij}\frac{\delta^2}{\delta\Phi^i\delta\Phi^j}$ as $\int_{\partial_k\Sigma}\phi^{ij}\frac{\delta}{\delta[\Phi^i\Phi^j]}$. Hence, we get

$$\int_{\partial_k\Sigma}\phi^{ij}\frac{\delta^2}{\delta\Phi^i\delta\Phi^j}\left(\left(\int_{\partial_k\Sigma}\alpha_i\Phi^i\right)\bullet\left(\int_{\partial_k\Sigma}\beta_j\Phi^j\right)\right)=\int_{\partial_k\Sigma}\alpha_i\beta_j\phi^{ij}.$$

More general, let $I:=(i_1,\ldots,i_n)$ be a multi-index and note that we can replace the higher derivative $\frac{\delta^n}{\delta\Phi^{i_1}\cdots\delta\Phi^{i_n}}$ by the first order derivative $\frac{\delta}{\delta[\Phi^I]}:=\frac{\delta}{\delta[\Phi^{i_1}\cdots\Phi^{i_n}]}$ with respect to the composite field $[\Phi^I]:=[\Phi^{i_1}\cdots\Phi^{i_n}]$. The higher functional derivatives appearing in the BFV boundary operator are then regularized by using composite fields and it acts on the state space given by *regular functionals*, i.e. on the algebra which is generated by linear combinations of expressions of the form

$$\int_{\overline{\mathrm{Conf}_{m_1}(\partial_1\Sigma)}\times\overline{\mathrm{Conf}_{m_2}(\partial_2\Sigma)}} \mathsf{Z}^{J_1^1\cdots J_1^{\ell_1}J_2^1\cdots J^{\ell_2}\cdots}_{I_1^1\cdots I_1^{r_1}I_2^1\cdots I_2^{r_2}\cdots}\pi_1^*\prod_{j=1}^{r_1}\left[\mathbb{X}^{I_1^j}\right]\cdots\pi_{m_1}^*\prod_{j=1}^{r_{m_1}}\left[\mathbb{X}^{I^j_{m_1}}\right]\times \\ \times\pi_1^*\prod_{j=1}^{\ell_1}\left[\mathbb{Y}_{J_1^j}\right]\cdots\pi_{m_2}^*\prod_{j=1}^{\ell_{m_2}}\left[\mathbb{Y}_{J^j_{m_2}}\right], \tag{6.2.2}$$

where I_i^j and J_i^j are multi-indices and $\mathsf{Z}^{J_1^1\cdots J_1^{\ell_1}J_2^1\cdots J^{\ell_2}\cdots}_{I_1^1\cdots I_1^{r_1}I_2^1\cdots I_2^{r_2}\cdots}$ is a smooth form on the product $\overline{\mathrm{Conf}_{m_1}(\partial_1\Sigma)}\times\overline{\mathrm{Conf}_{m_2}(\partial_2\Sigma)}$. If we denote by $\exp_\bullet$ the exponential defined through the bullet product $\bullet$ constructed as in (6.2.1), we can see that

$$\int_{\partial_k\Sigma}\phi^I\frac{\delta}{\delta\Phi^I}\Big\langle\exp_\bullet\left(\mathcal{S}^{\mathrm{res}}_\Sigma+\mathcal{S}^{\mathrm{source}}_\Sigma\right)\Big\rangle=\Big\langle\int_{\partial_k\Sigma}\phi^I\frac{\delta}{\delta\Phi^I}\exp_\bullet\left(\mathcal{S}^{\mathrm{res}}_\Sigma+\mathcal{S}^{\mathrm{source}}_\Sigma\right)\Big\rangle,$$

where

$$\mathcal{S}^{\mathrm{res}}_\Sigma:=\int_{\partial_1\Sigma}\langle \mathsf{y},\mathbb{X}\rangle-\int_{\partial_2\Sigma}\langle\mathbb{Y},\mathsf{x}\rangle,$$
$$\mathcal{S}^{\mathrm{source}}_\Sigma:=\int_{\partial_1\Sigma}\langle \mathscr{Y},\mathbb{X}\rangle-\int_{\partial_2\Sigma}\langle\mathbb{Y},\mathscr{X}\rangle,$$

The *full boundary state* is then defined through the regularization of composite fields by using (6.2.2):

$$\mathsf{Z}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}:=\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}\Big\langle\exp_\bullet\left(\mathcal{S}^{\mathrm{res}}_\Sigma+\mathcal{S}^{\mathrm{source}}_\Sigma\right)\Big\rangle, \tag{6.2.3}$$

where $\langle\ \rangle$ denotes the expectation value with respect to $\mathcal{S}_\Sigma$, i.e. for an observable $O\in\mathcal{O}_\mathcal{F}$, we have

$$\langle O\rangle:=\int\exp(\mathrm{i}\mathcal{S}_\Sigma(\mathbf{X},\mathbf{Y})/\hbar)O(\mathbf{X},\mathbf{Y})\mathscr{D}[\mathbf{X}]\mathscr{D}[\mathbf{Y}].$$

Let $I_1,\ldots,I_{m_1}$ and $J_1,\ldots,J_{m_2}$ be multi-indices for $m_1,m_2\geq 1$. Denote by Γ'_k for $k=1,2$ the graphs with m_1 vertices on $\partial_k\Sigma$, where vertex s has valency $|I_s|\geq 1$, with adjacent half-edges oriented inwards and decorated with boundary fields $\left[\Phi_k^{I_1}\right],\ldots,\left[\Phi_k^{I_{m_1}}\right]$ all evaluated at the point of collapse $\mathbb{u}\in\partial_k\Sigma$. We also want them to have $|J_1|+\cdots+|J_{m_2}|$ outward leaves if $k=1$ and $|J_1|+\cdots+|J_{m_2}|$ inward leaves if $k=2$, decorated with functional derivatives with respect to the boundary fields:

$$\mathrm{i}\hbar\frac{\delta}{\delta\left[\Phi_k^{J_1}\right]},\ldots,\mathrm{i}\hbar\frac{\delta}{\delta\left[\Phi_k^{J_{m_2}}\right]}$$

evaluated at the point of collapse $\mathbb{u}\in\partial_k\Sigma$. Moreover, there are no outward leaves if $k=2$ and no inward leaves if $k=1$. Denote by $\sigma_{\Gamma'_i}$ the differential form given by

$$\sigma_{\Gamma'_k}:=\int_{\overline{C_{\Gamma'_k}(\mathbb{H}^4)}}\omega_{\Gamma'_k},$$

where $\omega_{\Gamma'_k}$ denotes the product of limiting propagators at the point of collapse $\mathbb{u}\in\partial_k\Sigma$ and vertex tensors. Then we can construct the corresponding *full BFV boundary operator* through the regularization of composite fields as

$$\Omega_{\partial\Sigma}^{\bullet,\mathcal{P}} = \Omega_0^{\mathcal{P}} + \Omega_{\text{pert}}^{\bullet,\mathbb{X}} + \Omega_{\text{pert}}^{\bullet,\mathbb{Y}}, \tag{6.2.4}$$

where

(6.2.5)

$$\Omega_{\text{pert}}^{\bullet,\mathbb{X}} := \sum_{m_1,m_2\geq 0}\sum_{\Gamma_1'} \frac{(-\mathrm{i}\hbar)^{\ell(\Gamma_1')}}{|\mathrm{Aut}(\Gamma_1')|}\int_{\partial_1\Sigma}\left(\sigma_{\Gamma_1'}\right)_{I_1\cdots I_{m_1}}^{J_1\cdots J_{m_2}}\prod_{j=1}^{m_1}\left[\mathbb{X}^{I_j}\right]\left((-1)^{4m_2}(\mathrm{i}\hbar)^{m_2}\frac{\delta^{|J_1|+\cdots+|J_{m_2}|}}{\delta\left[\mathbb{X}^{J_1}\cdots\mathbb{X}^{J_{m_2}}\right]}\right)$$

(6.2.6)

$$\Omega_{\text{pert}}^{\bullet,\mathbb{Y}} := \sum_{m_1,m_2\geq 0}\sum_{\Gamma_2'} \frac{(-\mathrm{i}\hbar)^{\ell(\Gamma_2')}}{|\mathrm{Aut}(\Gamma_2')|}\int_{\partial_2\Sigma}\left(\sigma_{\Gamma_2'}\right)_{I_1\cdots I_{m_1}}^{J_1\cdots J_{m_2}}\prod_{j=1}^{m_1}\left[\mathbb{Y}_{I_j}\right]\left((-1)^{4m_2}(\mathrm{i}\hbar)^{m_2}\frac{\delta^{|J_1|+\cdots+|J_{m_2}|}}{\delta\left[\mathbb{Y}_{J_1}\cdots\mathbb{Y}_{J_{m_2}}\right]}\right)$$

6.3. **Feynman rules.** In order to formulate a BV-BFV quantization, i.e. for the boundary state and the BFV boundary operator, we need to consider the Feynman graphs on the source manifold Σ (see Figure 6.3.1 for an example) for DW theory. The graphs are determined by the Feynman rules of the theory and the corresponding degree count (see Section 6.5). The Feynman rules are determined by the AKSZ action functional $\mathcal{S}_\Sigma$ of DW theory. In particular, the interaction vertices are given by the ones as in Figure 6.3.2. The boundary vertices are given by Figure 6.3.3 and 6.3.4.

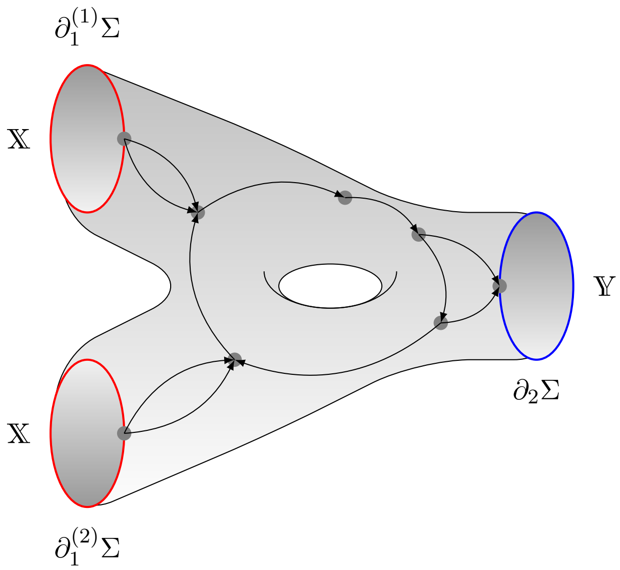

FIGURE 6.3.1. Example of a Feynman graph on the source manifold.

6.4. **The partition function.** The partition function (or boundary state) is then formally given by the functional integral where we integrate out the fluctuations, i.e. by

$$\mathsf{Z}_{\Sigma,\partial\Sigma}^{\mathrm{BV-BFV}}(\mathbb{X},\mathbb{Y},\mathsf{x},\mathsf{y};\hbar) = \int_{\mathcal{L}\subset\mathcal{Y}'}\exp\left(\mathrm{i}\mathcal{S}_\Sigma(\mathbf{X},\mathbf{Y})/\hbar\right)\mathscr{D}[\mathscr{X}]\mathscr{D}[\mathscr{Y}] \tag{6.4.1}$$

where the Lagrangian submanifold is given by the gauge-fixing as in Section 5.2, i.e.

$$\mathcal{L} = \mathrm{graph}(\mathrm{d}\Psi^{\mathrm{gf}}),$$

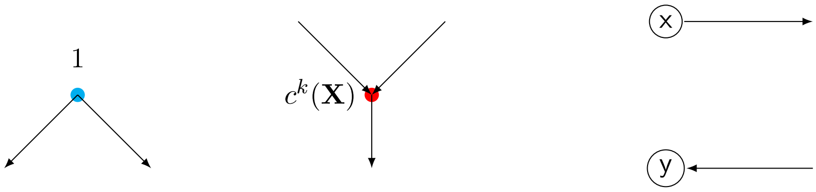

FIGURE 6.3.2. The Feynman rules for DW theory. The first type of vertex corresponds to the term $\frac{1}{2}\langle \mathbf{Y}, \mathbf{Y}\rangle = \frac{1}{2}\sum_i \mathbf{Y}_i\mathbf{Y}_i$. There are two outgoing arrows and no incoming arrow for the first vertex type. The second type of vertex corresponds to the term $\frac{1}{2}\langle \mathbf{Y}, [\mathbf{X}, \mathbf{X}]\rangle = \frac{1}{2}\sum_{i,j,k} c^k_{ij}\mathbf{X}^i\mathbf{X}^j\mathbf{Y}_k$, where c^k_{ij} are the structure constants of the Lie bracket and where we have denoted $c^k(\mathbf{X}) := c^k_{ij}\mathbf{X}^i\mathbf{X}^j$. There are at most two incoming arrows and exactly one outgoing arrow for the second vertex type. Finally, we also have leaves corresponding to the residual fields. Note that an arrow between vertices represents the propagator from the starting to the end point.

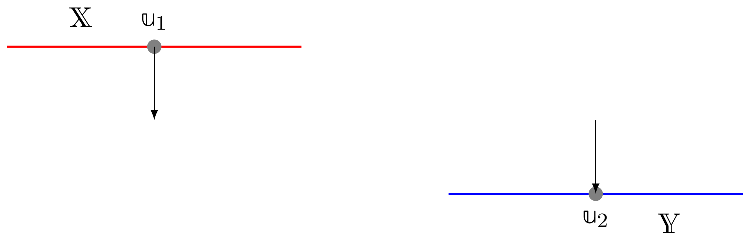

FIGURE 6.3.3. The Feynman rules for vertices on the boundary

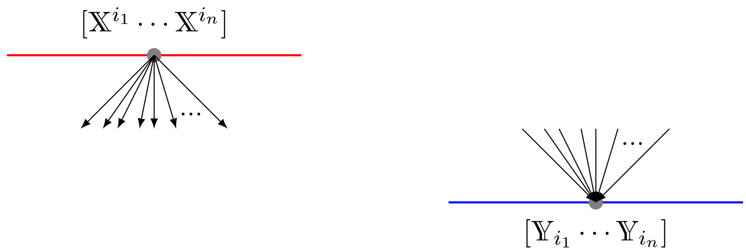

FIGURE 6.3.4. The Feynman rules for composite field vertices on the boundary

where Ψ^{gf} is the gauge-fixing fermion constructed as in (5.2.3). The integral (6.4.1) is defined perturbatively as a formal power series in $\hbar$ in terms of the Feynman graphs given by the according Feynman rules as in Section 6.3, i.e. we can rewrite it as a perturbative expansion around critical points of $\mathcal{S}_\Sigma$ as

$$\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}(\mathbb{X},\mathbb{Y},\mathsf{x},\mathsf{y};\hbar) = T_\Sigma \exp\left(\frac{\mathrm{i}}{\hbar}\sum_\Gamma \frac{(-\mathrm{i}\hbar)^{\ell(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\int_{\mathrm{Conf}_\Gamma(\Sigma)} \omega_\Gamma(\mathbb{X},\mathbb{Y},\mathsf{x},\mathsf{y})\right), \tag{6.4.2}$$

where we sum over connected Feynman graphs Γ and where $\ell(\Gamma)$ denotes the number of loops of Γ. Moreover, the integral in (6.4.2) is over the configuration space of the vertex set of Γ regarded as points in Σ where we integrate a differential form ω_Γ depending on the boundary fields and residual fields.

6.5. **Degree count and Feynman graphs.** Note that we want the dimension of the configuration space $\overline{\mathrm{Conf}_\Gamma(\Sigma)}$ to match the form degree of the differential form ω_Γ in order to perform the integrals in (6.4.2). Note that the dimension of the configuration space is given by

$$\dim \overline{\mathrm{Conf}_\Gamma(\Sigma)} = 4n + 3m,$$

where n denotes the amount of vertices in the bulk and m the amount of vertices on the boundary. Using the fact that the propagator $\mathscr{P} \in \Omega^3(\overline{\mathrm{Conf}_2(\Sigma)})$ is a 3-form on the configuration space of two points, the form degree of the differential form ω_Γ is given by $6 \cdot \#I + 3 \cdot \#II$, where $\#I$ denotes the amount of first type vertices and $\#II$ the amount of second type vertices as in Figure 6.3.2. Thus we get the system of equations

$$\begin{aligned} 4n + 3m &= 6 \cdot \#I + 3 \cdot \#II, \\ n &= \#I + \#II. \end{aligned} \tag{6.5.1}$$

Now if $m = 0$, we get $2 \cdot \#I = \#II$. Hence, one can check that the diagrams which give a contribution are either wheels with an even amount of type II vertices and $\frac{1}{2} \cdot \#I$ vertices attached to them as in Figure 6.5.1.

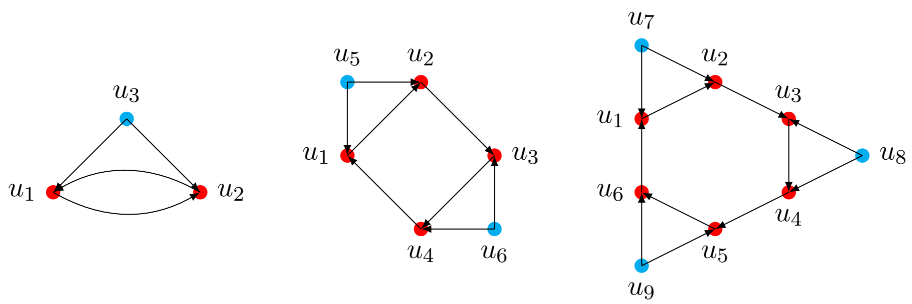

FIGURE 6.5.1. Example of wheel graphs appearing due to the degree count. In the first graph we have $\#II = 2$ and $\#I = 1$, in the second graph we have $\#II = 4$ and $\#I = 2$ and in the third graph we have $\#II = 6$ and $\#I = 3$. In fact, the first graph does not give any contribution, since by Kontsevich's lemma [Kon03] all the graphs with *double edges* (i.e. graphs where there exist two vertices which have exactly two arrows (propagators) connecting them) vanish. This can be seen by using the angle form on $\mathbb{H}^4$.

When $m \neq 0$, the system of equations (6.5.1) gives us $2 \cdot \#I - \#II - 3m = 0$. Example of Feynman graphs in this setting are given in Figure 6.5.3.

6.6. **The modified Quantum Master Equation.** We can now prove the following theorem:

Theorem 6.6.1 (mQME for DW theory)**.** *The BV-BFV partition function* $\mathsf{Z}^{\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}$ *for DW theory satisfies the modified Quantum Master Equation, i.e.*

$$\left(\hbar^2 \Delta_{\mathcal{V}_\Sigma} + \Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}\right)\mathsf{Z}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma} = 0,$$

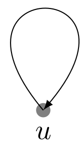

FIGURE 6.5.2. Short loop graphs (tadpoles) are not allowed since the propagator is singular on the diagonal. By a unimodularity condition which needs to be satisfied on $\mathfrak{h}[1] \oplus \mathfrak{h}[2]$, one can actually exclude these graphs. The condition is that the structure constants c^k_{ij} of the Lie bracket on $\mathfrak{h}[1] \oplus \mathfrak{h}[2]$ satisfy $\sum_i c^i_{ij} = 0$. In fact, the unimodularity condition can be dropped if the Euler characteristic of Σ vanishes.

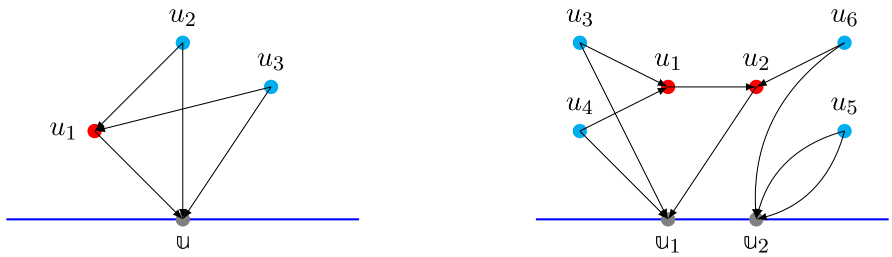

FIGURE 6.5.3. Example of Feynman graphs when $m \neq 0$. In the left graph we have $m = 1$, $\#I = 2$ and $\#II = 2$. In the right graph we have $m = 2$, $\#I = 4$ and $\#II = 2$. Similarly as befor, the second graph does not give any contribution since by Kontsevich's lemma [Kon03] all the graphs with double edges vanish.

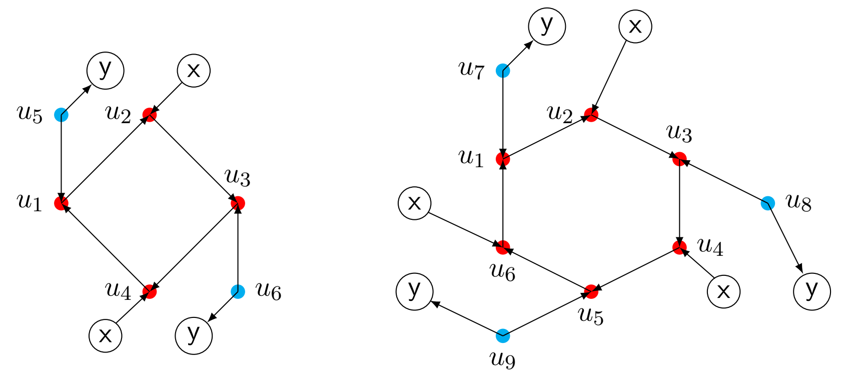

FIGURE 6.5.4. Example of Feynman graphs with leaves (residual fields). Note that the y-leaves give a contribution of form degree 2, whereas the x-leaves give a contribution of form degree 1. Gluing residual fields will induce a propagator of form degree 3.

where $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma} = \Omega^{\mathcal{P}}_0 + \Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}$ *with* $\Omega^{\mathcal{P}}_0$ *being the unperturbed quantization part as defined in* (6.1.1) *and* $\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}} := \Omega^{\bullet,\mathbb{X}}_{\mathrm{pert}} + \Omega^{\bullet,\mathbb{Y}}_{\mathrm{pert}}$ *is fully determined by the boundary configuration space integrals.*

Proof of Theorem 6.6.1. The proof is on the level of graphs. First, we want to describe the construction of $\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}$ in terms of boundary configuration space integrals. Consider the compactified configuration space $\overline{\mathrm{Conf}}_\Gamma$ and ω_Γ the corresponding differential form, for some Feynman graph Γ. Then, using Stokes' theorem, we get $\int_{\overline{\mathrm{Conf}}_\Gamma} \mathrm{d}\omega_\Gamma = \int_{\partial\overline{\mathrm{Conf}}_\Gamma} \omega_\Gamma$. When we apply the de Rham differential d to ω_Γ on the left-hand-side, it can either act on an $\mathbb{X}$-field, $\mathbb{Y}$-field or on the propagator $\mathscr{P}$. Clearly, the part acting on the $\mathbb{X}$- or $\mathbb{Y}$-fields corresponds to the action of $\frac{1}{\mathrm{i}\hbar}\Omega^{\mathcal{P}}_0$, whereas the part acting on the propagator will correspond to the action of $-\mathrm{i}\hbar\Delta_{\mathcal{V}_\Sigma}$ on the partition function $\mathsf{Z}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}$. The integral over the boundary configuration space on the right-hand-side contains terms of different possible strata for the corresponding manifold with corners. It either contains integrals over boundary components where two vertices collapse in the bulk, which we denote as situation (a), or integrals over boundary components where more than two vertices collapse in the bulk[29], which we denote by situation (b), or integrals over boundary components where two or more (bulk and/or boundary) vertices collapse at the boundary or one single bulk vertex collapses to the boundary, which we denoted by situation (c). For situation (a), we can note that since we assume the CME to hold which equivalently translates to the fact that

$$\frac{1}{2}\sum_{i=1}^{4}\pm\frac{\delta}{\delta\mathbf{X}^i}\left(\langle\mathbf{Y},\mathbf{Y}+[\mathbf{X},\mathbf{X}]\rangle\right)\cdot\frac{\delta}{\delta\mathbf{Y}_i}\left(\langle\mathbf{Y},\mathbf{Y}+[\mathbf{X},\mathbf{X}]\rangle\right)=0,$$

the combinatorics of the Feynman graphs in the perturbative expansion leads to the cancellation of such terms when summing over all graphs. For situation (b), one can use the usual vanishing theorems [Kon03; Kon93; Bot96; BT94] in order to get rid of faces where all vertices of a connected component of a Feynman graph collapse. For situation (c), note that we can split such integrals into an integral over a subgraph $\Gamma'\subset\Gamma$ and an integral over the graph which can be obtained by identifying all the vertices of Γ' and deleting all edges of Γ' which we denote by Γ/Γ'. Hence, one can define the action of $\frac{\mathrm{i}}{\hbar}\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}$ as the sum of the boundary contributions coming from the subgraphs $\Gamma'\subset\Gamma$ and thus we have

$$\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}\mathsf{Z}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}:=\sum_{\Gamma}\sum_{\Gamma'\subset\Gamma}\int_{\overline{C_{\Gamma'}(\mathbb{H}^4)\times\mathrm{Conf}_{\Gamma/\Gamma'}(\Sigma)}}\omega_{\Gamma'}.$$

□

Remark 6.6.2 (Principal part)**.** It is important to note that in the setting of DW theory, there are no higher corrections for the definition of the *principal part* of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$, i.e. the term $\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}$ which is in general of the form (6.2.5)+(6.2.6). This is due to the fact that we are working with 4-dimensional source manifolds and by the result of the following lemma.

Lemma 6.6.3 (Cattaneo–Mnev–Reshetikhin[CMR17])**.** *Let Σ be a d-dimensional source manifold. If d is even, then the principal part of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ is directly given by the ordered standard quantization of the boundary action $\mathcal{S}^\partial_{\partial\Sigma}$. If d is odd, the principal part of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ is given by the ordered standard quantization of the modified boundary action*

$$\tilde{\mathcal{S}}^\partial_{\partial\Sigma}:=\mathcal{S}^\partial_{\partial\Sigma}-\mathrm{i}\hbar\sum_{j=0}^{\left[\frac{d-3}{4}\right]}\int_{\partial\Sigma}\gamma_j\,\mathrm{Tr}\,\mathrm{ad}^{d-4j}_{\mathbf{X}},$$

where γ_j is a closed $4j$-form on $\partial\Sigma$ which is an invariant polynomial, with universal coefficients, of the curvature of the connection used in the construction of the propagator.

[29]Usually called *hidden faces*.

Moreover, we can prove the following theorem:

Theorem 6.6.4 (Flatness of BFV boundary operator)**.** *The full BFV boundary operator $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ for DW theory squares to zero and hence we have a well-defined BV-BFV cohomology for DW theory, i.e. the operator $\hbar^2\Delta_{\mathcal{V}_\Sigma}+\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ squares to zero.*

Proof of Theorem 6.6.4. It is easy to see that if $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ squares to zero, so does $\hbar^2\Delta_{\mathcal{V}_\Sigma}+\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ since $\Delta_{\mathcal{V}_\Sigma}$ squares to zero and $[\Delta_{\mathcal{V}_\Sigma},\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}]=0$. Consider again a subgraph $\Gamma'\subset\Gamma$ as in the definition of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ in the proof of Theorem 6.6.1 with corresponding differential form $\sigma_{\Gamma'}$ over the configuration space $\overline{\mathrm{C}_{\Gamma'}(\mathbb{H}^4)}$. Using Stokes' theorem, we get $\int_{\overline{\mathrm{C}_{\Gamma'}(\mathbb{H}^4)}}\mathrm{d}\sigma_{\Gamma'}=\int_{\partial\overline{\mathrm{C}_{\Gamma'}(\mathbb{H}^4)}}\sigma_{\Gamma'}$. Similarly as before, we can obtain that the part where the de Rham differential acts on the boundary fields in $\sigma_{\Gamma'}$ corresponds to the action of $\frac{1}{\mathrm{i}\hbar}\Omega^{\mathcal{P}}_0$. Moreover, we can obtain the similar three situations as in the proof of Theorem 6.6.1. In particular, the terms of situation (a) cancel out when summing over all graphs, the terms of situation (b) are excluded by the vanishing theorems and the terms of situation (c) lead to the action of $\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}$ when summing over all graphs. Hence, we have

$$\Omega^{\mathcal{P}}_0\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}+\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}}\Omega^{\mathcal{P}}_0+(\Omega^{\bullet,\mathcal{P}}_{\mathrm{pert}})^2=0,$$

since $(\Omega^{\mathcal{P}}_0)^2=0$. □

Remark 6.6.5. Instead of considering the boundary operator in terms of boundary configuration space integrals, one can also take the explicit form of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ as in (6.2.4) in order to prove that it squares to zero. In particular, for the full perturbation part, one can take the sum of the corresponding full versions of (6.1.3) and (6.1.4).

Remark 6.6.6. By Witten's approach [Wit88a] (see also Section 3.6), the expectation value of the observable O as in (3.6.3) with respect the BV-BFV partition function for the DW AKSZ-BV-action $\mathcal{S}_\Sigma$ should be given by a Floer cohomology class associated to the boundary after integrating out the residual fields (see [CMR17] for a possible integration theory on $\mathcal{V}_\Sigma$). The resulting boundary state would then be of the form

$$\Psi_{\partial\Sigma}(\mathbb{X},\mathbb{Y})=\int_{\mathcal{V}_\Sigma}\underbrace{\int_{\mathcal{L}}\exp(\mathrm{i}\mathcal{S}_\Sigma(\mathbf{X},\mathbf{Y})/\hbar)O(\mathbf{X},\mathbf{Y})\mathscr{D}[\mathscr{X}]\mathscr{D}[\mathscr{Y}]}_{=\left\langle\prod_{j=1}^d O^{(\gamma_j)}\right\rangle=\left\langle\prod_{j=1}^d\int_{\gamma_j}W_{k_j}\right\rangle}\in HF^\bullet(\partial\Sigma)\subset\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}.$$

For a particular polarization adapted to the boundary condition given by the Floer cohomology class $\Psi_{\partial\Sigma}$, the boundary state space $\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}$ in this case should include the Floer cohomology classes and thus the BFV boundary operator $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ should be given in terms of the Floer differential. Note that here we want $\mathfrak{h}=\mathfrak{su}(2)$. The gauge-fixing Lagrangian $\mathcal{L}$ is given by the one considered in Section 5.2.

6.7. **The modified differential Quantum Master Equation.** Following the construction in [CMW19], we can perform *globalization* on the target of the DW AKSZ theory by using methods of *formal geometry* as developed in [GF69; GF70; GK71; Bot10]. For a manifold M and an open neighborhood $U\subset TM$ of the zero section, we can consider a *generalized exponential map* $\varphi\colon U\to M$ such that $\varphi(x,y):=\varphi_x(y)$, i.e. φ satisfies the properties:

- $\varphi_x(0)=x,\quad\forall x\in M,$
- $\mathrm{d}\varphi_x(0)=\mathrm{id}_{T_xM},\quad\forall x\in M.$

Locally, we get

$$\varphi^i_x(y) = x^i + y^i + \frac{1}{2}\varphi^i_{x,jk}y^jy^k + \frac{1}{3!}\varphi^i_{x,jk\ell}y^jy^ky^\ell + \cdots \tag{6.7.1}$$

where (x^i) are coordinates on the base and (y^i) are coordinates on the fiber. A *formal exponential map* is given by the equivalence class of generalized exponential maps with respect to the equivalence relation which identifies two generalized exponential maps when their jets agree to all orders. By abuse of notation, we will also denote formal exponential maps by φ. We can define a *flat* connection D on $\widehat{\mathrm{Sym}}(T^*M)$, where $\widehat{\mathrm{Sym}}$ denotes the *completed* symmetric algebra. The connection is called *Grothendieck connection* [Gro68] and can be locally written as $D = \mathrm{d} + R$, where d denotes the de Rham differential on M and R denotes a 1-form with values in derivations of the completed symmetric algebra of the cotangent bundle[30]. For a section $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M))$, we get that R acts on σ through the Lie derivative, i.e. we have $R(\sigma) = L_R\sigma$. Locally, we can express $R = R_\ell \mathrm{d}x^\ell$, where $R_\ell := R^j_\ell(x,y)\frac{\partial}{\partial y^j}$ and

$$R^j_\ell(x,y) := -\frac{\partial\varphi^k}{\partial x^\ell}\left(\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right)^j_k = -\delta^j_\ell + \mathrm{O}(y).$$

Thus, for a section $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M))$, we have

$$R(\sigma) = L_R\sigma = R_\ell(\sigma)\mathrm{d}x^\ell = -\frac{\partial\sigma}{\partial y^j}\frac{\partial\varphi^k}{\partial y^\ell}\left(\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right)^j_k \mathrm{d}x^\ell.$$

Moreover, one can extend the Grothendieck connection D to the complex $\Gamma(\bigwedge^\bullet T^*M \otimes \widehat{\mathrm{Sym}}(T^*M))$ consisting of $\widehat{\mathrm{Sym}}(T^*M)$-valued forms.

Proposition 6.7.1 (e.g. [Kon03; Fed94; BCM12; Mos20b])**.** *A section $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M))$ is D-closed if and only if $\sigma = \mathsf{T}\varphi^*f$ for some $f \in C^\infty(M)$, where T denotes the Taylor expansion around the fiber coordinates at zero. Moreover, the D-cohomology is concentrated in degree zero and is given by*

$$H^0_D(\widehat{\mathrm{Sym}}(T^*M)) = \mathsf{T}\varphi^*C^\infty(M) \cong C^\infty(M).$$

The main part of the proof of Proposition 6.7.1 uses methods from *cohomological perturbation theory*. let us now consider the AKSZ construction of DW theory as in Section 5. In particular, the techniques of [CMW19] can be used here since the target is of the form $\mathfrak{h}[1]\oplus\mathfrak{h}[2]$. Consider a 4-dimensional source manifold (possibly with boundary) and recall the BV space of fields of DW theory

$$\mathcal{F}_\Sigma = \mathrm{Map}(T[1]\Sigma, \mathfrak{h}[1]\oplus\mathfrak{h}[2]) \cong \mathrm{Map}(T[1]\Sigma, (T[1]\mathfrak{h})[1]).$$

Consider then the formal exponential map given by

$$\begin{aligned}\varphi\colon T\mathfrak{h} \cong \mathfrak{h}\oplus\mathfrak{h} &\to \mathfrak{h},\\ (x,y) &\mapsto \varphi_x(y) := x + y.\end{aligned}$$

Note also that a particularly easy class of solutions of AKSZ theories is given by constant ones of the type $x = (x,0)\colon T[1]\Sigma \to \mathfrak{h}[1]\oplus\mathfrak{h}[2]$, where $x = \mathrm{const}$. Thus, we can consider the *linearized* space of fields at a constant solution x, given by

$$\varphi^*_x\mathcal{F}_\Sigma := \mathrm{Map}(T[1]\Sigma, (T[1]T_x\mathfrak{h})[1]).$$

[30] Note that flatness of D is equivalent to the Maurer–Cartan equation $\mathrm{d}R + \frac{1}{2}[R,R] = 0$.

Let $(\mathbf{X},\mathbf{Y})\in\mathcal{F}_\Sigma$. Then the corresponding lifts by φ_x are given by

$$\widehat{\mathbf{X}}:=\varphi_x^{-1}(\mathbf{X}),\qquad \widehat{\mathbf{Y}}:=(\mathrm{d}\varphi_x)^*\mathbf{Y}.$$

The Grothendieck connection can be computed by noticing that $R^j_\ell(x,y)=-\delta^j_\ell$. We can then define the *formal global action* by

$$\widehat{\mathcal{S}}_{\Sigma,x}:=\int_\Sigma\left(\widehat{\mathbf{Y}}_i\mathrm{d}_\Sigma\widehat{\mathbf{X}}^i+\widehat{\Theta}_x(\widehat{\mathbf{X}},\widehat{\mathbf{Y}})-\widehat{\mathbf{Y}}_\ell\mathrm{d}_{\mathcal{EL}}x^\ell\right).$$

Here we have denoted $\widehat{\Theta}_x(\widehat{\mathbf{X}},\widehat{\mathbf{Y}}):=\mathsf{T}\widehat{\varphi}_x^*\Theta(\mathbf{X},\mathbf{Y})$, where $\widehat{\varphi}_x\colon\varphi_x^*\mathcal{F}_\Sigma\to\mathcal{F}_\Sigma$. Note also that $\mathrm{d}_{\mathcal{EL}}$ denotes the de Rham differential on the moduli space $\mathcal{EL}$ of classical solutions, which can be identified with the target by considering constant solutions. One can then check that the *differential CME* (dCME) holds:

$$\mathrm{d}_x\widehat{\mathcal{S}}_{\Sigma,x}+\frac{1}{2}(\widehat{\mathcal{S}}_{\Sigma,x},\widehat{\mathcal{S}}_{\Sigma,x})=0. \tag{6.7.2}$$

If Σ has boundary, one can show that a *differential* version of the mCME (4.4.1) is also satisfied [CMW19]. For the quantum case, we have the following theorem:

Theorem 6.7.2 (mdQME for DW theory)**.** *The formal global BV-BFV partition function for DW theory, given formally by the functional integral*

$$\widehat{\mathsf{Z}}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}=\int_{\mathcal{L}}\exp(\mathrm{i}\widehat{\mathcal{S}}_{\Sigma,x}/\hbar),$$

satisfies the modified differential QME *(mdQME)*

$$\left(\mathrm{d}_x-\mathrm{i}\hbar\Delta_{\mathcal{V}_\Sigma}+\frac{\mathrm{i}}{\hbar}\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}\right)\widehat{\mathsf{Z}}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}=0. \tag{6.7.3}$$

Moreover, using similar methods as in [CMW19], we can show the following theorem:

Theorem 6.7.3 (Flatness of qGBFV operator for DW theory)**.** *The* quantum Grothendieck BFV (qGBFV) operator

$$\nabla_{\mathsf{G}}:=\mathrm{d}_x-\mathrm{i}\hbar\Delta_{\mathcal{V}}+\frac{\mathrm{i}}{\hbar}\Omega^{\bullet,\mathcal{P}}$$

is flat and behaves well under change of data. Moreover, it defines a cohomology theory on the globally extended state space.

Proof of Theorem 6.7.3. This proof is similar to the one in [CMW19]. In particular, the flatness of ∇_{G} is equivalent to the equation

$$\mathrm{i}\hbar\mathrm{d}_x\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}-\frac{1}{2}\left[\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma},\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}\right]=0. \tag{6.7.4}$$

Using the explicit expression of $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ through configuration space integrals by using the perturbed parts (6.2.5) and (6.2.6), we can apply Stokes' theorem

$$\mathrm{d}_x\int_{\overline{\mathrm{Conf}_\Gamma(\Sigma)}}\omega_\Gamma=\int_{\overline{\mathrm{Conf}_\Gamma(\Sigma)}}\mathrm{d}\omega_\Gamma\pm\int_{\partial\overline{\mathrm{Conf}_\Gamma(\Sigma)}}\omega_\Gamma, \tag{6.7.5}$$

where $\mathrm{d}=\mathrm{d}_x+\mathrm{d}_1+\mathrm{d}_2$ is the differential on $(\mathfrak{h}[1]\oplus\mathfrak{h}[2])\times\overline{\mathrm{Conf}}_\Gamma(\Sigma)$ with d_1 the differential acting on the propagator, i.e. on the residual fields, and d_2 the differential acting on the boundary fields. Then by [CMW19, Lemma 4.9], we can observe that the boundary face where more than two bulk vertices in a subgraph $\Gamma'\subset\Gamma$ collapse to the boundary was shown to be $\frac{1}{2}\left[\Omega^{\bullet,\mathbb{Y}}_{\mathrm{pert}},\Omega^{\bullet,\mathbb{Y}}_{\mathrm{pert}}\right]$ (and similarly for the $\mathbb{X}$-representation). For the case where exactly two bulk vertices collapse, we can observe that these faces cancel with $\mathrm{d}_x\omega_{\Gamma'}$ by using the dCME (6.7.2). □

Proof of Theorem 6.7.2. Again, using the configuration space integral formulation of the partition function as in (6.4.2), we can apply Stokes' theorem (6.7.5) in order to obtain similar relations of d_1 with $\Delta_{\mathcal{V}_\Sigma}$, d_2 with $\Omega_0^{\mathcal{P}}$ and d_x with the boundary contribution when applied to the partition function $\widehat{\mathsf{Z}}^{\bullet,\mathrm{BV-BFV}}_{\Sigma,\partial\Sigma}$ as in [CMW19]. □

7. Nekrasov's partition function, equivariant BV formalism and equivariant Floer (co)homology

7.1. **Seiberg–Witten theory.** In [SW94b; SW94a], Seiberg and Witten have formulated a way of describing low-energy behaviour for special supersymmetric gauge field theories. Moreover, they have formulated topological invariants of 4-manifolds which can be shown to be equivalent to the Donaldson polynomials but in general easier to compute. The gauge group is fixed to be SU(2) in this setting. Recall that an $\mathcal{N}=2$ *chiral multiplet* (or *vector multiplet*) includes a gauge field A_μ, two Weyl fermions[31] λ,ψ and a scalar field ϕ in the adjoint representation, i.e. we have something of the form

$$((A_\mu,(\lambda,\psi),\phi)),$$

where the brackets $((\ \ ))$ here indicate that we have singlets A_μ,ϕ and a doublet (λ,ψ). When $\mathcal{N}=1$, we can express this in one vector multiplet W_α (containing A_μ and λ) and a chiral multiplet Φ (containing ϕ and ψ). In $\mathcal{N}=1$ superspace, one can express the fermionic low-energy effective action in terms of a holomorphic function F as

$$S^{\mathrm{SW,eff}}=\frac{1}{4\pi}\mathrm{Im}\left(\int \mathrm{d}^4\theta\frac{\partial\mathsf{F}(A)}{\partial A}\bar{A}+\int\mathrm{d}^2\theta\frac{1}{2}\frac{\partial^2\mathsf{F}(A)}{\partial A^2}W_\alpha W^\alpha\right), \tag{7.1.1}$$

where θ denotes the angle for the global SU(2)-action, A denotes the $\mathcal{N}=1$ chiral multiplet in the $\mathcal{N}=2$ vector multiplet whose scalar component is given by some complex parameter which labels the vacua. The holomorphic function F is in fact the free energy which can be expressed in terms of the Seiberg–Witten formula for periods[32] of some differential $\mathrm{d}\mathsf{S}$ on some algebraic curve C (*Seiberg–Witten curve*).

7.2. **Nekrasov's partition function.** In [Nek03], Nekrasov constructed a regularized partition function $\mathsf{Z}^{\mathrm{Nek}}$ for the supersymmetric gauge field theories appearing in the construction of Seiberg–Witten, i.e. based on $\mathcal{N}=2$ supersymmetric Yang–Mills theory. The idea was to use *equivariant integration* with respect to a natural symmetry group for a long-distance cut-off regularization with parameter ε. Moreover, he proposed that for $\varepsilon\to 0$, asymptotically we get $\log\mathsf{Z}^{\mathrm{Nek}}\sim -\frac{1}{\varepsilon^2}\mathsf{F}$. The proof was given by Nekrasov and Okounkov for $\mathrm{U}(r)$-gauge theories with matter fields in fundamental and adjoint representations of the gauge group and for 5-dimensional theories compactified on the circle. Consider a connection A on a trivial rank $r>1$ bundle over $\Sigma=\mathbb{R}^4$ such that the Yang–Mills action functional $S^{\mathrm{YM}}_\Sigma(A)=\int_\Sigma\|F_A\|^2<\infty$. In this case, as we have seen in Section 2.1, it is bounded from below

$$8\pi^2k_A\leq S^{\mathrm{YM}}_\Sigma(A)=\int_\Sigma\|F_A\|^2.$$

[31] Recall that a *Weyl fermion* (or *Weyl spinor*) is a spinor which satisfies the *Weyl equations* $\sigma^\mu\partial_\mu\psi=0$, where σ^μ denotes the Pauli matrices for $\mu=0,\dots,3$.

[32] Recall that a *period* of a closed differential form ω over some n-cycle C is given by $\int_C\omega$.

Moreover, as we have also seen in Section 2.1, if A is an instanton (anti self-dual connection), we get equality. Donaldson showed in [Don84] that the moduli space of anti self-dual connections $\mathcal{M}_{\text{ASD}}$ is equal to the moduli space of holomorphic bundles on $\Sigma = \mathbb{C}^2 \cong \mathbb{R}^4$ which are trivial at infinity. One can describe Nekrasov's theory as a $(\mathrm{Aut}(\Sigma)\times \mathrm{GL}(r))$-equivariant integration over a certain partial compactification

$$\mathcal{M}_{\text{ASD}} \subset \overline{\mathcal{M}}_r = \{\text{framed torsion-free sheaves on } \Sigma \text{ of rank } r\}.$$

Note that each element of $\mathrm{Aut}(\Sigma)\times \mathrm{GL}(r)$ can always be considered in the form

$$\begin{pmatrix} \varepsilon_1 & \\ & \varepsilon_2 \end{pmatrix} \times \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}.$$

In particular, ε_1 and ε_2 can be considered as two rotations of $\mathbb{C}^2$ regarded as the generators of a torus $\mathbb{T}$. Let $\mathcal{M}$ be any smooth algebraic variety and consider a torus $\mathbb{T}$ acting on it and $\mathcal{E}$ a $\mathbb{T}$-equivariant coherent sheaf on $\mathcal{M}$. Furthermore, let $\overline{\Sigma}$ be a projective surface, e.g. $\mathbb{P}^2$, and choose an embedding $\Sigma \hookrightarrow \overline{\Sigma}$ with a framing[33] $\phi\colon \mathcal{E}|_D \to \mathcal{O}_D^{\oplus r}$ of $\mathcal{E}$ where $D = \overline{\Sigma}\setminus\Sigma$. Then one can consider the localization formula [CG97], given by

$$\chi(\mathcal{M},\mathcal{E}) = \chi\big(\mathcal{M}^{\mathbb{T}}, \mathcal{E}|_{\mathcal{M}^{\mathbb{T}}} \otimes \mathrm{Sym}\big(N^*\big(\mathcal{M}/\mathcal{M}^{\mathbb{T}}\big)\big)\big) \in \mathrm{K}_{\mathbb{T}}(\mathrm{pt})\left[\frac{1}{1-\varepsilon^{\nu}}\right] \subset \mathbb{Q}(\mathbb{T}),$$

where ε^{ν} denote the weights of the normal bundle $N\big(\mathcal{M}/\mathcal{M}^{\mathbb{T}}\big)$ and where Sym denotes the symmetric algebra. Here we have denoted by $\mathrm{K}_{\mathbb{T}}(\mathrm{pt})$ the $\mathbb{T}$-equivariant K-theory over a point and $\mathbb{Q}(\mathbb{T})$ denotes the representation ring of $\mathbb{T}$.

Remark 7.2.1. If a point $p \in \mathcal{M}^{\mathbb{T}}$ is isolated, and if the tangent space at that point is given by $T_p\mathcal{M} = \sum_i \varepsilon^{\nu_i}$ as a $\mathbb{T}$-module, then

$$\mathrm{Sym}\big(N^*\big(\mathcal{M}/p\big)\big) = \prod_i \frac{1}{1-\varepsilon^{\nu_i}}.$$

This product is given by the character of the $\mathbb{T}$-action on functions on the formal neighborhood of $p \in \mathcal{M}$.

We will only consider the case where all fixed points are isolated, hence $\chi\big(\mathcal{M}_r^{\mathbb{T}}\big)$ is a finite sum over r-tuples of partitions. Consider the *Hilbert scheme* (see also [Nak99] for the definition of a Hilbert scheme)

$$\mathrm{Hilb}(\Sigma,k) := \{\mathcal{I} \subset \mathbb{C}[u_1,u_2] \mid \mathcal{I} \text{ ideal of codimension } k,\ u_1,u_2 \in \Sigma\}$$

and let $\varepsilon := (\varepsilon_1,\varepsilon_2)$ acting on it by $f(u) \mapsto f(\varepsilon^{-1}\cdot u)$. In particular, if $\mathcal{I}$ is fixed, we get[34]

$$\begin{aligned}\mathrm{Hilb}(\Sigma,k)^{\varepsilon} &= \{\mathcal{I} \subset \mathbb{C}[u_1,u_2] \mid \mathcal{I} \text{ monomial ideal of codimension } k,\ u_1,u_2 \in \Sigma\}\\ &\cong \{\text{partitions of } k\}.\end{aligned}$$

Table 7.2.1 gives a good illustration for the relation between these ideals and the partitions. Consider the collection

$$\mathcal{E} := \mathcal{I}_{\lambda^{(1)}} \oplus \cdots \oplus \mathcal{I}_{\lambda^{(r)}} \in \mathcal{M},$$

[33] Note that $\mathrm{GL}(r) \cong \mathrm{Aut}(\mathcal{O}_D^{\oplus r})$ acts on ϕ which is the same as the action of constant gauge transformations on instantons.

[34] A partition of a number k is a monotone sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_{\ell(\lambda)} \geq 0)$ consisting of nonnegative integers whose sum is equal to k. Most of the times, we denote by $|\lambda| := \sum_{j=1}^{\ell(\lambda)} \lambda_j = k$ the *size* and $\ell(\lambda)$ the *length* of a partition λ.

1	u_1	u_1^2	u_1^3	u_1^4	u_1^5	u_1^6	u_1^7	$\boldsymbol{u_1^8}$	u_1^9
u_2	u_1u_2	$u_1^2u_2$	$u_1^3u_2$	$u_1^4u_2$	$u_1^5u_2$	$\boldsymbol{u_1^6u_2}$	$u_1^7u_2$	$u_1^8u_2$	$u_1^9u_2$
u_2^2	$u_1u_2^2$	$u_1^2u_2^2$	$u_1^3u_2^2$	$\boldsymbol{u_1^4u_2^2}$	$u_1^5u_2^2$	$u_1^6u_2^2$	$u_1^7u_2^2$	$u_1^8u_2^2$	$u_1^9u_2^2$
u_2^3	$u_1u_2^3$	$u_1^2u_2^3$	$\boldsymbol{u_1^3u_2^3}$	$u_1^4u_2^3$	$u_1^5u_2^3$	$u_1^6u_2^3$	$u_1^7u_2^3$	$u_1^8u_2^3$	$u_1^9u_2^3$
u_2^4	$\boldsymbol{u_1u_2^4}$	$u_1^2u_2^4$	$u_1^3u_2^4$	$u_1^4u_2^4$	$u_1^5u_2^4$	$u_1^6u_2^4$	$u_1^7u_2^4$	$u_1^8u_2^4$	$u_1^9u_2^4$
u_2^5	$u_1u_2^5$	$u_1^2u_2^5$	$u_1^3u_2^5$	$u_1^4u_2^5$	$u_1^5u_2^5$	$u_1^6u_2^5$	$u_1^7u_2^5$	$u_1^8u_2^5$	$u_1^9u_2^5$
$\boldsymbol{u_2^6}$	$u_1u_2^6$	$u_1^2u_2^6$	$u_1^3u_2^6$	$u_1^4u_2^6$	$u_1^5u_2^6$	$u_1^6u_2^6$	$u_1^7u_2^6$	$u_1^8u_2^6$	$u_1^9u_2^6$

TABLE 7.2.1. Illustration for the ideal $\mathcal{I}_\lambda \subset \mathcal{O}_\Sigma$ generated by monomials $u_1^8, u_1^6u_2, \dots, u_2^6$ which corresponds to the partition $\lambda = (8,6,4,3,1,1)$ of $k = 23$. The entries in the table for λ correspond to a basis of $\mathcal{O}_\Sigma/\mathcal{I}_\lambda$.

where $\mathcal{I}_{\lambda^{(j)}}$ denotes the ideal corresponding to the partition $\lambda^{(j)}$. To compute the tangent space $T_\mathcal{E}\mathcal{M}$, we can use the modular interpretation of $\mathcal{M}$. Generally, we can compute the tangent space to the moduli space of (coherent) sheaves by using the Ext^1-groups[35]. So we get

$$
\begin{aligned}
T_\mathcal{E}\mathcal{M} &= \mathrm{Ext}^1_\Sigma(\mathcal{E},\mathcal{E}(-D)) \\
&= \bigoplus_{1\leq i,j\leq r} a_j/a_i \otimes \mathrm{Ext}^1_\Sigma\big(\mathcal{I}_{\lambda^{(i)}}, \mathcal{I}_{\lambda^{(j)}}(-D)\big).
\end{aligned} \tag{7.2.1}
$$

In fact, we get

$$
\mathrm{Sym}(T^*_\mathcal{E}\mathcal{M}) = \prod_{i,j} \mathbb{E}\big(\lambda^{(i)}, \lambda^{(j)}, a_j/a_i\big)^{-1},
$$

with

$$
\mathbb{E}(\lambda,\mu,u) := \prod_{w \text{ weights of } u\otimes \mathrm{Ext}^1_\Sigma(\mathcal{I}_\lambda, \mathcal{I}_\mu(-D))} (1-w^{-1}). \tag{7.2.2}
$$

Lemma 7.2.2. *For a partition λ, consider the generating function*

$$
\mathbb{G}_\lambda = \chi^{\mathbb{C}^2}(\mathcal{O}_\Sigma/\mathcal{I}_\lambda) = \sum_{u_i^a u_j^b \notin \mathcal{I}_\lambda} \varepsilon_1^{-a}\varepsilon_2^{-b},
$$

where the sum corresponds to the boxes $\square = (a+1, b+1)$ in the table of λ. Moreover, define[36] *the* arm-length *for the partition λ of a box $\square = (j,i)$ by $a_\lambda(\square) := \lambda_i - j$ and the* leg-length *by $\ell_\lambda(\square) := \lambda'_j - i$, where λ' denotes the transposed table of λ. Then the character*

[35] Recall first that, for $D = \mathrm{Spec}(k[t]/t^2)$, a *deformation* over D of a coherent sheaf $\mathcal{E}$ over a scheme X is defined to be a coherent sheaf $\mathcal{E}'$ on $X' := X \times D$, flat over D, together with a homomorphism $\mathcal{E}' \to \mathcal{E}$ such that the induced map $\mathcal{E}' \otimes_D k \to \mathcal{E}$ is an isomorphism. Then there is a theorem (see e.g. [Har10]) which says that deformations of $\mathcal{E}$ over D are in natural one-to-one correspondence with elements of $\mathrm{Ext}^1_X(\mathcal{E},\mathcal{E})$, with the zero element corresponding to the trivial deformation. Finally, by the universal property of the moduli space $\mathcal{M}$, its tangent space at $\mathcal{E}$ consists of the deformations of $\mathcal{E}$ over D.

[36] Note that these numbers will be negative for $\square \notin \lambda$.

in (7.2.2) *is given by*

$$\operatorname{Ext}^1_\Sigma(\mathcal{I}_\lambda,\mathcal{I}_\mu(-D)) = \mathbb{G}_\mu + \varepsilon_1\varepsilon_2\overline{\mathbb{G}}_\lambda - (1-\varepsilon_1)(1-\varepsilon_2)\mathbb{G}_\mu\overline{\mathbb{G}}_\lambda \tag{7.2.3}$$

$$= \sum_{\square\in\mu}\varepsilon_1^{-a_\mu(\square)}\varepsilon_2^{\ell_\lambda(\square)+1} + \sum_{\square\in\lambda}\varepsilon_1^{a_\lambda(\square)+1}\varepsilon_2^{-\ell_\mu(\square)}. \tag{7.2.4}$$

Proof. The proof of (7.2.3) is given in [Oko19, Lemma 3.1] and the proof of (7.2.4) is given in [CO12, Lemma 3]. □

Remark 7.2.3. Note that $\overline{\varepsilon^d} = \varepsilon^{-d}$ denotes the usual duality for representations and characters. Moreover, in particular, there are $|\lambda|+|\mu|$ factors in (7.2.2).

Define now

$$\overline{\mathcal{M}} := \prod_i \overline{\mathcal{M}}_{r_i} \supset \mathcal{M}_{\mathrm{ASD}}(\Sigma,\mathrm{U}(r_i)). \tag{7.2.5}$$

Then we can define a *preliminary* partition function as

$$\begin{aligned}\mathsf{Z}^{\mathrm{pre}}_\Sigma &:= \chi\left(\overline{\mathcal{M}}, \prod_i z_i^{c_2(\mathcal{E}_i)}\chi_{\mathrm{top}}(\mathbb{M})\right)\\ &= \sum_{r\text{-tuple of partitions}} z^{\#\square} \prod_{\substack{\eta,\nu\ r\text{-tuples}\\ \text{interactions with mass } u_k}} \mathbb{E}(\eta,\nu,u_k)^{\pm1},\end{aligned} \tag{7.2.6}$$

where $r=\sum_i r_i$ denotes the total rank and

$$z^{\#\square} := \prod_i z_i^{c_2(\mathcal{E}_i)} = z_1^{\sum_i|\lambda^{(i)}|}z_2^{\sum_j|\mu^{(j)}|}\cdots$$

We have denoted by $\mathbb{M}$ in (7.2.6) the matter that is described by fermions which are defined through their representations of the gauge group of the form $(\mathbb{C}^{r_i})^*\otimes\mathbb{C}^{r_j}$.
Using the discussions before, we can define *Nekrasov's partition function* by

$$\mathsf{Z}^{\mathrm{Nek}}_\Sigma := \mathsf{Z}^{\mathrm{pert}}\mathsf{Z}^{\mathrm{pre}}_\Sigma\Big|_{\mathbb{E}\mapsto\widehat{\mathbb{E}}}, \tag{7.2.7}$$

where $\mathbb{E}$ is defined as in (7.2.2),

$$\widehat{\mathbb{E}}(\lambda,\mu,u) := \prod_{\substack{w \text{ weights of } u\otimes\operatorname{Ext}^1_\Sigma(\mathcal{I}_\lambda,\mathcal{I}_\mu(-D))\\ \text{same } w}} (w^{1/2}-w^{-1/2})$$

and $\mathsf{Z}^{\mathrm{pert}}$ denotes a perturbation factor given by

$$\mathsf{Z}^{\mathrm{pert}}(\varepsilon,a_1,\dots,a_r) \risingdotseq \prod_{1\leq i_1,i_2\leq r}\ \prod_{j_1,j_2\geq1} \mathrm{i}(a_{i_1}-a_{i_2}+\varepsilon(j_1-j_2)), \tag{7.2.8}$$

where $\risingdotseq$ denotes the equality up to regularization of the product on the right-hand-side. A suitable regularization is given by (see e.g. [NO06; Oko06]) using *Barne's double* Γ*-function* [Rui00]. Define

$$\zeta_2(s,w\mid c_1,c_2) := \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{d}t}{t}t^s\frac{\exp(-wt)}{\prod_i(1-\exp(-c_it))},\quad c_1,c_2\in\mathbb{R},\quad \mathrm{Re}\,(w)\gg0, \tag{7.2.9}$$

which has a meromorphic extension in s with poles at $s=1,2$. Define now

$$\Gamma_2(w\mid c_1,c_2) := \exp\left(\frac{\mathrm{d}}{\mathrm{d}s}\zeta_2(s,w\mid c_1,c_2)\right)\Bigg|_{s=0}.$$

Using the difference equation

$$w\Gamma_2(w)\Gamma_2(w+c_1+c_2)=\Gamma_2(w+c_1)\Gamma_2(w+c_2),$$

we can see that it extends to a meromorphic function of w. Moreover, define

$$\Lambda := \exp(-4\pi^2\beta/r) \tag{7.2.10}$$

for some parameter $\beta>0$ and the *scaled* perturbation factor

$$\mathsf{Z}^{\mathrm{pert}}(\varepsilon,a_1,\dots,a_r,\Lambda) := \prod_{1\leq i_1,i_2\leq r}\Gamma_2\left(\frac{\mathrm{i}(a_{i_1}-a_{i_2})}{\Lambda}\,\middle|\,\frac{\mathrm{i}\varepsilon}{\Lambda},\frac{-\mathrm{i}\varepsilon}{\Lambda}\right)^{-1}, \tag{7.2.11}$$

where Γ_2 is analytically continued to imaginary arguments by using

$$\Gamma_2(xw\mid xc,-xc)=x^{\frac{w^2}{2c^2}-\frac{1}{12}}\Gamma_2(w\mid c,-c),\quad \forall x\notin(-\infty,0].$$

Note also that

$$\Gamma_2(0\mid 1,1)=\exp(-\zeta'(-1)).$$

The instanton equations (ASD equations) are conformally invariant and can be translated to a punctured 4-sphere $S^4=\mathbb{R}^4\cup\{\infty\}$ by stereographic projection. Recall that Uhlenbeck's theorem for removing singularities (Theorem 2.2.1) implies that any instanton on $\mathbb{R}^4$ extends to one on S^4. Hence, there is an interpretation for an instanton at ∞. Consider the group

$$\mathcal{G}_0 := \{g\colon S^4\to \mathrm{U}(r)\mid g(\infty)=1\}$$

and define $\mathcal{M}(r,k) := \mathcal{M}^k_{\mathrm{ASD}}(S^4,\mathrm{U}(r))/\mathcal{G}_0$. Note that $\dim_{\mathbb{R}}\mathcal{M}(r,k)=4rk<\infty$ but nevertheless $\mathcal{M}(r,k)$ is not compact and thus has infinite volume. This is basically due to two problems: An element of $\mathcal{M}(r,k)$ is, roughly speaking, a nonlinear superposition of k instantons with charge $+1$. Some can be point-like[37], others can go to ∞. Uhlenbeck's construction will only solve the first problem by replacing point-like instantons by points in $\mathbb{R}^4$, but it does not solve the second problem, while Nekrasov's idea of using *equivariant integration* (see Appendix A) gives a suitable regularization procedure for those instantons. We can rewrite Nekrasov's partition function as

$$\mathsf{Z}^{\mathrm{Nek}}_{\Sigma\to S^4}(\varepsilon,a_1,\dots,a_r,\Lambda)=\mathsf{Z}^{\mathrm{pert}}(\varepsilon,a_1,\dots,a_r,\Lambda)\sum_{k\geq 0}\Lambda^{2rk}\int_{\overline{\mathcal{M}(r,k)}}1, \tag{7.2.12}$$

where the integral can be computed via equivariant localization (see Appendix A, Equation (A.1.2)) by using the formula

$$\int_\Sigma 1=\sum_{u\in\Sigma^{\mathbb{C}^\times}}\frac{1}{\det \xi|_{T_u\Sigma}}.$$

Here we consider the group $\mathbb{C}^\times$ acting on a complex manifold Σ and $\Sigma^{\mathbb{C}^\times}$ denotes the isolated fixed points with respect to this action. Moreover, $\xi=(\mathrm{diag}(-\mathrm{i}\varepsilon,\mathrm{i}\varepsilon),\mathrm{diag}(\mathrm{i}a_1,\dots,\mathrm{i}a_r))\in \mathrm{SU}(2)\times\mathrm{SU}(r)$ and $\mathsf{Z}^{\mathrm{pert}}$ is defined as in (7.2.8).

Remark 7.2.4 (Gluing)**.** A way of expressing the gluing of Nekrasov's partition function for 5-manifolds with boundary has been studied in [Qiu+15].

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[37] This means that their curvature might be concentrated in a peak with respect to the Dirac delta function.

7.3. **Equivariant BV formalism.** The equivariant formulation of DW theory and Nekrasov's construction are a main motivation to formulate a gauge formalism which deals with field theories in the equivariant setting. The equivariant extension of the BV formalism, which has been considered in [Bon+20], is a suitable method to deal with the AKSZ construction of DW theory as in Section 5 by extending it naturally to equivariant solutions of the CME, as it was also shown in [Bon+20]. We want to give the main ideas of this approach.

7.3.1. *Equivariant (co)homology.* consider a Lie algebra $\mathfrak{g}$ and define a $\mathfrak{g}$-differential algebra $\mathcal{O}$ to be a dg algebra $(\mathcal{O}, \mathrm{d})$ with Lie derivative and contraction $L_\xi \in \mathrm{Der}^0\mathcal{O}$ and $\iota_\xi \in \mathrm{Der}^{-1}\mathcal{O}$, respectively for all $\xi \in \mathfrak{g}$ which is linear in ξ and satisfies the usual equations in Cartan's calculus. Define the subalgebra $\mathcal{O}_b := \{O \in \mathcal{O} \mid L_\xi O = \iota_\xi O = 0,\ \forall \xi \in \mathfrak{g}\}$. Choosing a basis (e_a) of $\mathfrak{g}$, we can define its Weil model $\mathcal{W}(\mathfrak{g}) := (\bigwedge^\bullet \mathfrak{g}^* \otimes \mathrm{Sym}(\mathfrak{g}^*), \mathrm{d}_\mathcal{W})$ where the differential is given by

$$\mathrm{d}_\mathcal{W}\theta^i := u^i + \frac{1}{2}[\theta,\theta]^i, \tag{7.3.1}$$

$$\mathrm{d}_\mathcal{W}u^i := [\theta, u]^i. \tag{7.3.2}$$

Here, θ denote the odd coordinates of degree $+1$ on $\bigwedge^\bullet \mathfrak{g}^*$ and u the even coordinates of degree $+2$ on $\mathrm{Sym}(\mathfrak{g}^*)$. Moreover, we have $\iota_{e_i} = \frac{\partial}{\partial\theta^i}$ and $L_{e_i} = \{\iota_{e_i}, \mathrm{d}_\mathcal{W}\}$. Note that the subcomplex $\mathcal{O}_b$ is given by $\mathfrak{g}$-invariant polynomials in u endowed with the differential given by $\mathrm{d}_\mathcal{W}$ restricted to $\mathcal{O}_b$, i.e. we have $(\mathbb{C}[u]^\mathfrak{g} \cong \mathrm{Sym}(\mathfrak{g}^*)^\mathfrak{g}, \mathrm{d}_\mathcal{W}|_{\mathcal{O}_b})$. Let $H^\bullet_G(\mathcal{O})$ the cohomology of the subcomplex $\mathcal{O}_G := (\mathcal{O} \otimes \mathcal{W}(\mathfrak{g}))_b$. This is called the Weil model for $H^\bullet_G(\mathcal{O})$. Similarly, we can consider the Cartan model for $H^\bullet_G(\mathcal{O})$ by considering the graded algebra $\mathcal{O}[u] := \mathcal{O} \otimes \mathrm{Sym}(\mathfrak{g}^*)$ together with the differential $\mathrm{d}_G := \mathrm{d} - u^i\iota_{e_i}$ and the diagonal $\mathfrak{g}$-action. It is easy to check that $(\mathrm{d}_G)^2 = u^iL_{e_i}$. Thus, we have a dg algebra $(\mathcal{O}[u]^\mathfrak{g}, \mathrm{d}_G)$ with $\mathcal{O}[u]^\mathfrak{g} := \{O_u \in \mathcal{O}[u] \mid L_\xi O_u = 0,\ \forall \xi \in \mathfrak{g}\}$. In fact, we have an isomorphism between the cohomology of $(\mathcal{O}[u]^\mathfrak{g}, \mathrm{d}_G)$ and $H^\bullet_G(\mathcal{O})$. Indeed, one can check that the map

$$\mathrm{Exp}^\mathfrak{g} := \exp\left(-(\iota_{e_a} \otimes \theta^a)\right) : \mathcal{O} \otimes \mathcal{W}(\mathfrak{g}) \to \mathcal{O} \otimes \mathcal{W}(\mathfrak{g})$$

can be restricted to an isomorphism of dg-algebras $\mathrm{Exp}^\mathfrak{g} \colon \mathcal{O}_G \to \mathcal{O}[u]^\mathfrak{g}$.

7.3.2. *Equivariant extension for AKSZ-BV theories.* We can formulate an equivariant extension of the AKSZ construction as follows: Let Σ be a d-manifold togeher with a Lie algebra $\mathfrak{g}$ acting on it by the vector fields v_ξ for some $\xi \in \mathfrak{g}$. Let $\mathcal{M}$ be a graded manifold together with a symplectic form ω of degree $d-1$ and a smooth Hamiltonian function Θ on $\mathcal{M}$ of degree d. The Hamiltonian vector field of Θ is denoted by D_Θ. The space of fields is given by $\mathcal{F}_\Sigma = \mathrm{Map}(T[1]\Sigma, \mathcal{M})$ and the cohomological vector field by

$$Q_\Sigma = \hat{\mathrm{d}}_\Sigma + \hat{D}_\Theta = (\mathcal{S}_\Sigma,\ \ ),$$

where $\mathcal{S}_\Sigma = \mathcal{S}_0 + \mathcal{S}_\Theta$ with $\mathcal{S}_0$ and $\mathcal{S}_\Theta$ being the Hamiltonian functions of $\hat{\mathrm{d}}_\Sigma$ and $\hat{D}_\Theta$, respectively. The "hat" denotes the lift to a vector field on the mapping space $\mathrm{Map}(T[1]\Sigma, \mathcal{M})$ of the corresponding vector field either on the source or on the target. It is easy to check that since $(D_\Theta)^2 = 0$ and $(Q_\Sigma)^2 = 0$, the CME $(\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) = 0$ holds. Denote by $\mathcal{O}_{\mathcal{F}_\Sigma}$ the $\mathfrak{g}$-dg algebra of functionals on $\mathcal{F}_\Sigma$ endowed with the differential $\hat{\mathrm{d}}_\Sigma + \hat{Q}$. Consider then the contraction $\hat{\iota}_{v_\xi}$ and Lie derivative $\hat{L}_{v_\xi}$ on $\mathcal{O}_{\mathcal{F}_\Sigma}$ for $\xi \in \mathfrak{g}$ and their corresponding Hamiltonian functions $\mathcal{S}_{\hat{\iota}_{v_\xi}}$ and $\mathcal{S}_{\hat{L}_{v_\xi}}$, respectively. Define also $\mathcal{O}_{\mathcal{F}_\Sigma}[u] := \mathcal{O}_{\mathcal{F}_\Sigma} \otimes \mathrm{Sym}(\mathfrak{g}^*)$. If we choose a basis (e_i) of $\mathfrak{g}$, we can consider the equivariant Cartan model of the BV action as

$$\mathcal{S}^{\mathrm{Cartan}}_\Sigma = \mathcal{S}_\Sigma - u^i\mathcal{S}_{\hat{\iota}_{v_i}},$$

and therefore we get an equivariant model for the cohomological vector field as

$$Q_\Sigma^{\mathrm{Cartan}} = (\mathcal{S}_\Sigma^{\mathrm{Cartan}},\ ) = \hat{\mathrm{d}}_\Sigma + \hat{D}_\Sigma - u^i \hat{\iota}_{v_i}.$$

Define now $L_\xi := -\xi^i c_{ij}^k u^j \frac{\partial}{\partial u^k}$ and $\mathcal{L}_\xi := L_\xi + \hat{L}_{v_\xi}$ and note that then $\mathcal{L}_\xi \mathcal{S}_\Sigma^{\mathrm{Cartan}} = 0$. This implies that $\mathcal{S}_\Sigma^{\mathrm{Cartan}} \in \mathcal{O}[u]^{\mathfrak{g}}$ and it satisfies an equivariant version of the CME

$$\frac{1}{2}(\mathcal{S}_\Sigma^{\mathrm{Cartan}}, \mathcal{S}_\Sigma^{\mathrm{Cartan}}) + u^i \mathcal{S}_{\hat{L}_{v_i}} = 0. \tag{7.3.3}$$

We further define

$$\mathcal{T}_\Sigma := \frac{1}{2}(\mathcal{S}_\Sigma^{\mathrm{Cartan}}, \mathcal{S}_\Sigma^{\mathrm{Cartan}}) - \mathrm{i}\hbar\Delta\mathcal{S}_\Sigma^{\mathrm{Cartan}} = -u^i \mathcal{S}_{\hat{L}_{v_i}} - \mathrm{i}\hbar\Delta\mathcal{S}_\Sigma^{\mathrm{Cartan}},$$

such that $\mathcal{T}_\Sigma = -u^i(\mathcal{S}_{\hat{L}_{v_i}} + \mathrm{i}\hbar\Delta\mathcal{S}_{\hat{\iota}_{v_i}}) - \mathrm{i}\hbar\Delta\mathcal{S}_\Sigma^{\mathrm{Cartan}}$. If we apply Δ to $\mathcal{S}_0$ and $\mathcal{S}_{\hat{\iota}_{v_i}}$, we can use the fact that they are quadratic in the fields and thus we get

$$(\Delta\mathcal{S}_0,\ ) = (\Delta\mathcal{S}_{\hat{\iota}_{v_\xi}},\ ) = 0, \quad \forall X \in \mathfrak{g}. \tag{7.3.4}$$

This gives us that

$$\Delta\mathcal{S}_{\hat{L}_{v_\xi}} = \Delta(\mathcal{S}_0, \mathcal{S}_{\hat{\iota}_{v_\xi}}) = (\Delta\mathcal{S}_0, \mathcal{S}_{\hat{\iota}_{v_\xi}}) \pm (\mathcal{S}_0, \Delta\mathcal{S}_{\hat{\iota}_{v_\xi}}) = 0,$$

which needs to hold for consistency with the definition of a BV algebra as in Section 4.2. Hence, $\Delta\mathcal{T}_\Sigma = 0$ and thus $[\Delta, \hat{L}_{v_\xi}] = 0$. We can then check that $Q_\Sigma^{\mathrm{Cartan}}\mathcal{T}_\Sigma = 0$.
Let us briefly consider a more general case. For an observable O, note that we want the equation $\Delta O + \frac{\mathrm{i}}{\hbar} Q_\Sigma O = 0$, with $Q_\Sigma = (\mathcal{S}_\Sigma,\ )$, to be satisfied in order for point (2) of Theorem 4.1.5 to hold. This equation can be rephrased by using the twisted BV Laplacian $\Delta_{\mathcal{S}_\Sigma} := \exp(-\mathrm{i}\mathcal{S}_\Sigma/\hbar)\Delta\exp(\mathrm{i}\mathcal{S}_\Sigma/\hbar)$ according to the definition of the BV Laplacian on functions on the space of fields by Equation (4.1.4). In this case, O is called a *BV observable*. However, without assuming the Quantum Master Equation, we can define

$$\mathcal{T}_\Sigma := \left(\frac{\mathrm{i}}{\hbar}\right)^2 \exp(-\mathrm{i}\mathcal{S}_\Sigma/\hbar)\Delta\exp(\mathrm{i}\mathcal{S}_\Sigma/\hbar) = \frac{1}{2}(\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) - \mathrm{i}\hbar\Delta\mathcal{S}_\Sigma$$

and note that the twisted operator is given by

$$\Delta_{\mathcal{S}_\Sigma, \mathcal{T}_\Sigma} := \exp(-\mathrm{i}\mathcal{S}_\Sigma/\hbar)\Delta(\exp(\mathrm{i}\mathcal{S}_\Sigma/\hbar)O) = \Delta O + \frac{\mathrm{i}}{\hbar}Q_\Sigma O + \left(\frac{\mathrm{i}}{\hbar}\right)^2 \mathcal{T}_\Sigma O.$$

The observation is that then

$$\Delta\mathcal{T}_\Sigma + \frac{\mathrm{i}}{\hbar}Q_\Sigma\mathcal{T}_\Sigma = 0,$$

since $\Delta_{\mathcal{S}_\Sigma, \mathcal{T}_\Sigma}\mathcal{T}_\Sigma = 0$ and $(\mathcal{T}_\Sigma)^2 = 0$.
Moreover, if we apply Δ to $(\mathcal{S}_{\hat{L}_{v_i}}, \mathcal{S}_{\hat{\iota}_{v_j}})$ we get $c_{ij}^k \Delta\mathcal{S}_{\hat{\iota}_{v_k}} = 0$. Using (7.3.4), we also get $(\mathcal{T}_\Sigma, O) = (\mathcal{T}'_\Sigma, O)$ with $\mathcal{T}'_\Sigma := -u^i\mathcal{S}_{\hat{L}_{v_i}} - \mathrm{i}\hbar\Delta\mathcal{S}_\Theta$. Define now

$$\mathcal{N}_{\mathcal{T}_\Sigma} := \{O \in \mathcal{O}_{\mathcal{F}_\Sigma}[u] \mid (\mathcal{T}'_\Sigma, O) \in \mathcal{I}_{\mathcal{T}_\Sigma}\},$$

where $\mathcal{I}_{\mathcal{T}_\Sigma}$ denotes the ideal generated by $\mathcal{T}_\Sigma$ in $\mathcal{O}_{\mathcal{F}_\Sigma}[u]$ and consider the subalgebra

$$\mathcal{N}'_{\mathcal{T}_\Sigma} := \{O \in \mathcal{O}_{\mathcal{F}_\Sigma}[u] \mid \mathcal{L}_\xi O = 0, \forall \xi \in \mathfrak{g}\} \subset \mathcal{N}_{\mathcal{T}_\Sigma}.$$

Then, one can show that if (7.3.4) holds, $\mathcal{N}'_{\mathcal{T}_\Sigma}$ is a Poisson subalgebra which is invariant under both $Q_\Sigma^{\mathrm{Cartan}}$ and $\Delta_{\mathcal{S}_\Sigma, \mathcal{T}_\Sigma}$ and $\mathcal{T}_\Sigma \in \mathcal{N}'_{\mathcal{T}_\Sigma}$. Moreover, it is easy to see that the ideal $\mathcal{I}'_{\mathcal{T}_\Sigma} \subset \mathcal{N}'_{\mathcal{T}_\Sigma}$ generated by $\mathcal{T}_\Sigma$ is a $\Delta_{\mathcal{S}_\Sigma, \mathcal{T}_\Sigma}$-invariant Poisson ideal. Now one can define the algebra of *quantum equivariant preobservables* as $\mathcal{O}'_{\mathcal{F}_\Sigma} := \mathcal{N}'_{\mathcal{T}_\Sigma}/\mathcal{I}'_{\mathcal{T}_\Sigma}$ together with the induced differential

$$\Delta_{\mathcal{S}_\Sigma,\mathcal{T}_\Sigma}[O] := [\Delta_{\mathcal{S}_\Sigma,\mathcal{T}_\Sigma}O] = \left[\left(\Delta + \frac{\mathrm{i}}{\hbar}Q_\Sigma^{\mathrm{Cartan}}\right)O\right]. \tag{7.3.5}$$

A *quantum equivariant observable* can then be defined as a quantum equivariant preobservable which is additionally $\Delta_{\mathcal{S}_\Sigma,\mathcal{T}_\Sigma}$-closed.

Remark 7.3.1 (Gauge-fixing for (graded) linear targets)**.** If the target $\mathcal{M}$ is a graded vector space V, we can write the space of fields as $\mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)\otimes V$. Consider an invariant metric on Σ and define a the submanifold $\mathcal{L} := \Omega^\bullet_{\mathrm{coex}}(\Sigma)\otimes V$, where $\Omega^\bullet_{\mathrm{coex}}(\Sigma)$ denotes the subspace of *coexact* forms. Note that, in general, $\mathcal{L}$ is only isotropic due to harmonic forms. However, by invariance of the metric, we have $[L_{v_\xi}, \mathrm{d}^*] = 0$, and thus $i^*_{\mathcal{L}}\mathcal{S}_{\hat{L}_{v_\xi}} = 0$, where $i_{\mathcal{L}}\colon \mathcal{L}\hookrightarrow\mathcal{F}_\Sigma$. The foliation then defined by $(\mathcal{T}_\Sigma,\ )$ is the same as the infinitesimal $\mathfrak{g}$-action and hence we have to require that the Lie group G is compact.

7.4. **Equivariant DW theory.** Consider the set-up of Section 5.1 and let $\mathfrak{g}$ be the Lie algebra of a Lie group G acting on the 4-manifold Σ with vector fields v_ξ for $\xi\in\mathfrak{g}$. If we replace now d_Σ with the equivariant differential $\mathrm{d}_G := \mathrm{d}_\Sigma - u^i\iota_{v_i}$, we get the equivariant extension of the action as

$$\begin{aligned}\mathcal{S}_\Sigma^{\mathrm{Cartan}} &= \int_{T[1]\Sigma}\boldsymbol{\mu}_4\left(\langle\mathbf{Y},\mathrm{d}_G\mathbf{X}\rangle + \frac{1}{2}\langle\mathbf{Y},\mathbf{Y}\rangle + \frac{1}{2}\langle\mathbf{Y},[\mathbf{X},\mathbf{X}]\rangle\right)\\ &= \mathcal{S}_\Sigma - u^i\int_\Sigma\left(\psi\iota_{v_i}Y^\dagger + \chi^\dagger\iota_{v_i}\psi^\dagger + A^\dagger\iota_{v_i}\chi + X^\dagger\iota_{v_i}A\right).\end{aligned} \tag{7.4.1}$$

The BV transformations of the superfields are then given by

$$Q_\Sigma^{\mathrm{Cartan}}\mathbf{X} = \mathrm{d}_G\mathbf{X} + \mathbf{Y} + \frac{1}{2}[\mathbf{X},\mathbf{X}], \tag{7.4.2}$$

$$Q_\Sigma^{\mathrm{Cartan}}\mathbf{Y} = \mathrm{d}_G\mathbf{Y} + [\mathbf{X},\mathbf{Y}], \tag{7.4.3}$$

i.e. in superfield notation, the equivariant cohomological vector field is given by

$$Q_\Sigma^{\mathrm{Cartan}} = \int_{T[1]\Sigma}\boldsymbol{\mu}_4\left(\left(\mathrm{d}_G\mathbf{X} + \mathbf{Y} + \frac{1}{2}[\mathbf{X},\mathbf{X}]\right)\frac{\delta}{\delta\mathbf{X}} + (\mathrm{d}_G\mathbf{Y} + [\mathbf{X},\mathbf{Y}])\frac{\delta}{\delta\mathbf{Y}}\right). \tag{7.4.4}$$

In component fields and after Berezinian integration, we get the equivariant cohomological vector field

$$\begin{aligned}Q_\Sigma^{\mathrm{Cartan}} = \int_\Sigma\Bigg(&\left(Y + \frac{1}{2}[X,X] - u^i\iota_{v_i}A\right)\frac{\delta}{\delta X} + (\psi + \mathrm{d}_AX - u^i\iota_{v_i}\chi)\frac{\delta}{\delta A}\\ &+ (\chi^\dagger + F_A + [X,\chi] - u^i\iota_{v_a}\psi^\dagger)\frac{\delta}{\delta\chi} + ([X,Y] - u^i\iota_{v_i}\psi)\frac{\delta}{\delta Y}\\ &+ (\mathrm{d}_AY + [X,\psi] - u^i_{\iota_{v_i}}\chi^\dagger)\frac{\delta}{\delta\psi}\Bigg).\end{aligned} \tag{7.4.5}$$

7.4.1. *Relation to Nekrasov's partition function.* We can construct then a gauge-fixed solution for the equivariant CME (7.3.3) similarly as before whenever the chosen metric on Σ is invariant. In particular, we have

$$\mathcal{S}_\Sigma^{\text{Cartan,gf}} = \mathcal{S}_\Sigma^{\text{Cartan}} + \int_\Sigma \Big(\langle \bar{X}^\dagger, (b + [X, \bar{X}]) \rangle + \langle b^\dagger, (L_v \bar{X} + [X, b] - [Y, \bar{X}]) \rangle \Big) + \int_\Sigma \Big(\langle \bar{Y}^\dagger, (\eta + [X, \bar{Y}]) \rangle + \langle \eta^\dagger, (L_v \bar{Y} + [X, \eta] + [\bar{Y}, Y]) \rangle \Big). \tag{7.4.6}$$

If we redefine the fields, we can see that the transformations are the same as in [Nek03; NO06]. In particular, one can take v^i to be the two rotations of $\mathbb{C}^2$ and consider instead of u^i the parameters[38] ε^i. In fact, the (regularized) path integral quantization of this equivariant theory leads to Nekrasov's partition function (7.2.12), i.e. we have

$$\mathsf{Z}_\Sigma^{\text{BV}} = \int_{\mathcal{L}} \exp(\mathrm{i}\mathcal{S}_\Sigma^{\text{Cartan}}(\mathbf{X}, \mathbf{Y})/\hbar) \mathscr{D}[\mathscr{X}] \mathscr{D}[\mathscr{Y}] = \mathsf{Z}_\Sigma^{\text{Nek}},$$

where $\mathcal{L}$ denotes the gauge-fixing Lagrangian as in (7.4.6). It is important to note that in the mentioned construction, we are considering the perturbation around constant solutions rather than general instantons. Moreover, the methods presented in this section are only considered for the finite-dimensional case.

Remark 7.4.1. Similar equivariant extensions have been studied e.g. in [CF10; Mos19] for a (global) S^1-equivariant setting of the Poisson sigma model on the disk, and in [Get19] where a classical BV-equivariance under source diffeomorphisms for 1-dimensional systems is considered. Generalizations of this construction would lead to equivariant cohomological methods similarly as for the methods which appear in [Pes12].

7.5. **Equivariant Floer (co)homology and boundary structure.** Since the perturbative expectation value for DW theory of an observable as in [Wit88a] reproduces the Floer (co)homology groups as the boundary states on the corresponding boundary 3-manifold (see also Section 2.3), we expect to produce similarly in a perturbative way an equivariant version of Floer (co)homology as the boundary state when considering equivariant DW theory on a 4-manifold with boundary. An equivariant extension of Floer (co)homology groups in the setting of 4-manifold topology was considered in [AB96]. The convenient model chosen there is the Cartan model similarly as discussed in Section 7.3. In particular, for $G = \mathrm{SU}(2)$ and $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{su}(2)$ we can construct equivariant Floer homology and cohomology as follows: For any G-manifold N, define complexes

$$\Omega_{G,\bullet}(N) := (\mathrm{Sym}(\mathfrak{g}) \otimes \Omega^{\dim N - \bullet}(N))^G,$$
$$\Omega_G^\bullet(N) := (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^\bullet(N))^G,$$

endowed with the boundary and coboundary operators ∂_G and d_G, respectively. They define equivariant homology and cohomology $H_{G,\bullet}(N)$ and $H_G^\bullet(N)$, respectively. Consider the SU(2)-invariant Chern–Simons action functional $S_N^{\text{CS}}\colon \mathcal{A}/\tilde{\mathcal{G}}_0 \to S^1$, where $\tilde{\mathcal{G}}_0$ denotes the group of based gauge transformations of degree 0 and denote by A and B critical orbits of the perturbed action $S_N^{\text{CS}} + f$ for which the moduli space of gradient lines between A and B then becomes a smooth SU(2)-manifold $\widetilde{\mathcal{M}}_0(A, B)$ (here f denotes an element of a suitable class of perturbations). For $0 < p < \infty$ and $\delta \in \mathbb{R}$, denote by $L_\delta^p(E)$, for some

[38] This is true whenever the evaluation map which evaluates u^i at ε^i commutes with the path integral with the additional assumption that ε^i are not on the support of the equivariant cohomology class induced by the path integral.

bundle $E\to\Sigma$ over a 4-manifold Σ, the Banach space completion of smooth and compactly supported sections of E by using the norm

$$\|\sigma\|_{L^p_\delta}:=\left(\int_\Sigma \mathrm{dvol}_\Sigma(\exp(\tau\delta)|\sigma|^p)\right)^{1/p},$$

where $\tau\colon \mathrm{End}(\Sigma)\to(0,\infty)$ denotes a distance function on Σ defined as in [Tau87]. Furthermore, for $1\leq p<\infty$, $0\leq\ell<\infty$ and $\delta\in\mathbb{R}$, define the weighted Sobolev space $L^p_{\ell,\delta}(E)$ to be the completion of smooth and compactly supported sections of E with respect to the norm

$$\|\sigma\|_{L^p_{\ell,\delta}}:=\left(\int_\Sigma \mathrm{dvol}_\Sigma\left(\exp(\tau\delta)\sum_{0\leq k\leq\ell}|\nabla^k\sigma|^p\right)\right)^{1/p},$$

where $\nabla^k:=\nabla\nabla\cdots\nabla$ k-times and $\nabla\colon\Gamma^\infty_0(E\otimes_q T^*\Sigma)\to\Gamma^\infty_0(E\otimes_{q+1}T^*\Sigma)$ is the covariant derivative from the *end-periodic*[39] connections on E and $T^*\Sigma$. We have denoted by Γ^∞_0 the space of smooth sections with compact support.
We can then consider the Atiyah–Hitchin–Singer deformation theory [AHS77; AHS78] to the moduli space $\mathcal{M}_\varepsilon(A,B)$ for $\varepsilon<0$, which is locally a smooth $(\mathcal{G}(N)^A\times\mathcal{G}(N)^B)$-manifold whenever the cokernel of the following deformation operator vanishes. Define the *deformation* and *gauge-fixing operator* at some instanton A_t as

$$a\mapsto\left(\frac{\mathrm{d}}{\mathrm{d}t}a+*\mathrm{d}_{A_t}a+\partial^2 f_{A_t}a,\ \frac{\mathrm{d}}{\mathrm{d}t}a+\mathrm{d}^*_{A_t}a\right)$$

and note that this indeed defines a Fredholm operator

$$L^p_{\ell,\varepsilon}(\Omega^1(N\times\mathbb{R},\mathrm{ad}\,P))\to L^p_{\ell-1,\varepsilon}((\Omega^0\oplus\Omega^1)(N\times\mathbb{R},\mathrm{ad}\,P)).$$

Denote by $\mathrm{sf}(A,B)$ the *index*[40] of this operator and note that we then get

$$\dim\mathcal{M}_0(A,B)=\mathrm{sf}(A,B)+3-\dim\mathcal{G}(N)^A-\dim\mathcal{G}(N)^B.$$

Define then the *index* of a critical orbit $A\in\mathcal{A}/\tilde{\mathcal{G}}_0$ of the perturbed Chern–Simons action functional S^{CS}_N+f as

$$i_N(A):=-\mathrm{sf}(\theta,A)+\dim\mathcal{G}(N)^A,$$

where θ denotes a preferred trivial connection in $\mathcal{A}/\tilde{\mathcal{G}}_0$ and ε is chosen to be negative and sufficiently small. Fix a homotopy of the corresponding principal $\mathrm{SU}(2)$-bundle P over N and note that

$$\dim\widetilde{\mathcal{M}}_0(A,B)=i_N(A)-i_N(B)+\dim A-1.$$

For the 3-manifold N, we can then define graded vector spaces

$$CF_{\mathcal{G},k}:=\bigoplus_{i_N(A)+j=k}\Omega_{G,j}(A),$$
$$CF^k_{\mathcal{G}}:=\bigoplus_{i_N(A)+j=k}\Omega^j_G(A),$$

[39]The definition of *end-periodic* connections is rather technical and we refer to [Tau87] for a definition of *end-periodic* connections and manifolds. In fact, these objects are only defined for end-periodic 4-manifolds.
[40]This is again the spectral flow as defined in Section 3.4.

with boundary and coboundary operators given by

$$(\delta_{\mathcal{G}})_{A,B}\Psi := \begin{cases} \partial_G\Psi, & \text{if } A=B, \\ (-1)^{\deg\Psi}(e_B^-)_*(e_A^+)^*\Psi, & \text{if } i_N(A)>i_N(B), \\ 0, & \text{otherwise.} \end{cases} \tag{7.5.1}$$

and

$$(D_{\mathcal{G}})_{A,B}\Psi := \begin{cases} \mathrm{d}_G\Psi, & \text{if } A=B, \\ (-1)^{\deg\Psi}(e_A^+)_*(e_B^-)^*\Psi, & \text{if } i_N(A)>i_N(B), \\ 0, & \text{otherwise.} \end{cases} \tag{7.5.2}$$

We have used the *endpoint maps*

$$e_A^+\colon \widetilde{\mathcal{M}}_0(A,B)\to A,$$
$$e_B^-\colon \widetilde{\mathcal{M}}_0(A,B)\to B.$$

It is easy to see that the equivariant Floer complexes are actually filtered by index

$$CF_{\mathcal{G},k}^{(m)} := \bigoplus_{\substack{i_N(A)+j=k\\ i_N(A)<m}} \Omega_{G,j}(A), \tag{7.5.3}$$

$$CF_{\mathcal{G}}^{k,(m)} := \bigoplus_{\substack{i_N(A)+j=k\\ i_N(A)\geq m}} \Omega_G^j(A), \tag{7.5.4}$$

with the corresponding relative equivariant (co)homology groups

$$HF_{\mathcal{G},\bullet}^{(m)} := H_\bullet(CF_{\mathcal{G},\bullet}/CF_{\mathcal{G},\bullet}^{(m)}), \tag{7.5.5}$$

$$HF_{\mathcal{G}}^{\bullet,(m)} := H^\bullet(CF_{\mathcal{G}}^\bullet/CF_{\mathcal{G}}^{\bullet,(m)}). \tag{7.5.6}$$

Theorem 7.5.1 (Austin–Braam[AB96])**.** *The operators $\delta_{\mathcal{G}}$ and $D_{\mathcal{G}}$ square to zero and define equivariant Floer homology (resp. cohomology) $HF_{\mathcal{G},\bullet}(N)$ (resp. $HF_{\mathcal{G}}^\bullet(N)$) of the 3-manifold N. Moreover, there is a pairing $HF_{\mathcal{G},\bullet}(N)\times HF_{\mathcal{G}}^\bullet(N)\to\mathbb{R}$ which is defined on the level of chains by $\bigoplus_A\langle\ ,\ \rangle_A$. Furthermore, these groups are independent of the metric on N and the choice of perturbation withing a class of perturbations, up to natural isomorphisms. An orientation preserving diffeomorphism of N induces a natural map on equivariant Floer (co)homology.*

Remark 7.5.2. Equivariant Floer homology and cohomology are both modules over equivariant polynomials $\mathrm{Sym}(\mathfrak{g}^*)^G\cong\mathbb{R}[u]$ and the action of $\mathrm{Sym}(\mathfrak{g}^*)^G$ is symmetric with respect to the pairing. Moreover, the complexes defining equivariant homology and cohomology are filtered by index and there is an associated spectral sequence, whose E^1 and E_1 terms are the equivariant (co)homology of the critical locus.

Example 7.5.3 (3-sphere)**.** The equivariant Floer groups for S^3 are simple since there is only one flat connection, the trivial one. Hence, we get

$$HF_{\mathcal{G},\bullet}(S^3)=H_{G,\bullet}(\mathrm{pt})=\mathbb{R}[u],$$
$$HF_{\mathcal{G}}^\bullet(S^3)=H_G^\bullet(\mathrm{pt})=\mathbb{R}[\hat{u}].$$

Example 7.5.4 (Poincaré 3-sphere)**.** In the case of the Poincaré 3-sphere S_P^3, there are two irreducible flat connections A and B. These are indexed by $i_{S_P^3}(A)=0$ and $i_{S_P^3}(B)=4$ by using the standard orientation induced from S^3. The first term in the spectral sequence of the equivariant Floer cohomology complex has then generators endowed with index 0 and 4 coming from the connections A and B, together with a tower of $H^\bullet_G(\mathrm{pt})$ at index 0 which represents the trivial connection. Then the dimension of the nonzero terms will give lead to the fact that all higher differentials vanish and thus, after taking the quotient with respect to the $\mathbb{Z}$-translations, we get

$$HF^\bullet_{\mathcal{G}}(S^3_P) = H^\bullet_G(\mathrm{pt}) \oplus H^\bullet(\mathrm{pt}) \oplus H^{\bullet-4}(\mathrm{pt}),$$
$$HF_{\mathcal{G},\bullet}(S^3_P) = H_{G,\bullet}(\mathrm{pt}) \oplus H_\bullet(\mathrm{pt}) \oplus H_{\bullet-4}(\mathrm{pt}).$$

Consider now a $\mathrm{U}(2)$-bundle $P \to \Sigma = \Sigma_1 \cup_N \Sigma_2$ over an oriented 4-manifold Σ with $\partial\Sigma_1 = N$ and $\partial\Sigma_2 = N^{\mathrm{opp}}$. Moreover, consider the moduli space $\mathcal{M}_0(\Sigma, \mathrm{U}(2))$ of projectively anti self-dual connections on P with fixed central parts. Moreover, assume that $\mathcal{M}_0(\Sigma, \mathrm{U}(2))$ is compact. Define now $\mathcal{M}_0(P_1, A)$ and $\mathcal{M}_0(P_2, A)$ to be the moduli spaces of anti self-dual connections modulo gauge transformations asymptotic to the flat connection defined by A. Then we have

$$\dim \mathcal{M}_0(P_1, A) = C_{\Sigma_1} - i_N(A), \tag{7.5.7}$$
$$\dim \mathcal{M}_0(P_2, A) = C_{\Sigma_2} + 3 - i_N(A), \tag{7.5.8}$$

for some constants C_{Σ_1} and C_{Σ_2}. For the framed moduli space, we get the $\mathrm{SU}(2)$-invariant endpoint maps

$$e^1_A \colon \mathcal{M}_0(P_1, A) \to A,$$
$$e^2_A \colon \mathcal{M}_0(P_2, A) \to A.$$

If P_1 and P_2 are trivial bundles such that $c_2(P_1) + c_2(P_2) = c_2(P)$, we get a gluing map with open image

$$\mathcal{M}_0(P_1, A) \times \mathcal{M}_0(P_2, A) \to \mathcal{M}_0(\Sigma, \mathrm{U}(2)).$$

Theorem 7.5.5 (Austin–Braam[AB96]). *Let $\mathcal{M}_0(\Sigma, \mathrm{U}(2))$ be compact and let P_1, P_2 run over bundles such that $c_2(P_1) + c_2(P_2) = c_2(P)$. Assume that for a generic one parameter family of metrics on Σ_1 no reducible connections exist on P_1 for any A. Assume further that $b^2_+(\Sigma_2) = 0$ or $b^2_+(\Sigma_2) > 1$.*

(1) Let $a \in H^\bullet_G(\mathcal{A}(P_1)/\mathcal{G}_0(P_1))$ and denote by $\gamma := [\mathcal{M}_0(P_1, A)]_G \in H_{G,\bullet}(\mathcal{M}_0(P_1, A))$ the fundamental class. Then

$$\Psi_{\Sigma_1}(a) := \bigoplus_{i_N(A)\geq m} (e^1_A)_*(\gamma \cap a) \in CF_{\mathcal{G},\bullet}(N)$$

defines an element of $HF^{(m)}_{\mathcal{G},\bullet}(N)$ whenever $m > C_{\Sigma_1} - 8$; that is, it is closed and, up to boundaries, independent of the choice of metric on Σ_1.

(2) Let $b \in H^\bullet_G(\mathcal{A}(P_2)/\mathcal{G}_0(P_2))$. The integration over the fiber gives an element

$$\mathcal{D}(\Sigma_2)(b) := \bigoplus_A (e^2_A)_* b \in CF^\bullet_{\mathcal{G}}(N)$$

which is coclosed and up to coboundaries independent of the metric on Σ_2. Moreover, it also defines an element of $HF^{\bullet,(m)}_{\mathcal{G}}(N)$ for all $m < -3 - C_{\Sigma_2}$:

$$\int_{\mathcal{M}_0(\Sigma,\mathrm{U}(2))} a \cup b = \langle \Psi_{\Sigma_1}(a), \mathcal{D}(\Sigma_2)(b)\rangle_{HF^{(m)}_{\mathcal{G}}}.$$

7.5.1. *Irreducible connection on one side.* Assume now that the $2d$-dimensional moduli space of anti self-dual connections on P splits uniquely as

$$\mathcal{M}_0(\Sigma,\mathrm{U}(2))=\mathcal{M}_0(P_1,A)\times_A\mathcal{M}_0(P_2,A),$$

along a $\mathbb{Z}/2$ connection A. Then, we get that $H^\bullet_G(A,\mathbb{R})=\mathbb{R}[u]$ such that u is of degree 4. Moreover, suppose that $\mathcal{M}_0(P_2,A)\simeq S^2$ only consists of a single reducible connection which defines a line bundle reduction $\mathbb{L}\oplus\mathbb{L}^{\mathrm{opp}}\to\Sigma_2$. Recall from Section 2.2 that the Donaldson polynomials are given by

$$\mathcal{D}(\Sigma)([C],\dots,[C])=\int_{\mathcal{M}_0(\Sigma,\mathrm{U}(2))}\mu([C])^d,$$

for a 2-dimensional homology class $[C]\in H_2(\Sigma,\mathbb{Z})$. Let us first look at the Σ_2 side. There, we get that $\mathcal{M}_0(P_2,A)\simeq S^2$ corresponds to a single reducible connection with endpoint map $e^2_A\colon\mathcal{M}_0(P_2,A)\to A$ which sends S^2 to one single point. We can extend the map μ to equivariant cohomology $\mu\colon H_2(\Sigma_2,\mathbb{Z})\to H^2_G(\mathcal{A}(P_2)/\mathcal{G}_0(P_2))$ and hence

$$\mu([C])=-2\int_C c_1(\mathbb{L})v\in H^2_G(\mathcal{M}_0(P_2,A))=\mathbb{R}[u],$$

where v is of degree 2 and such that $v^2=u$. Note that we have a push-forward $(e^2_A)_*\colon\mathbb{R}[v]\to\mathbb{R}[u]$ such that

$$\begin{cases}v^k\mapsto u^{(k-1)/2}, & k\text{ odd},\\ 0, & \text{otherwise.}\end{cases}$$

Finally, the relative Donaldson polynomial in the equivariant Floer cohomology of N defined by Σ_2 is given by

$$\mathcal{D}(\Sigma_2)(\mu([C])^d)=\begin{cases}(-2)^d\left(\displaystyle\int_C c_1(\mathbb{L})\right)^d u^{(d-1)/2}, & d\text{ odd},\\ 0, & \text{otherwise.}\end{cases}\tag{7.5.9}$$

Now we look at the Σ_1 side. There we need $\Psi_{\Sigma_1}(\mathbf{1}):=(e^1_A)_*(\gamma)\in CF_{\mathcal{G},\bullet}(N)$ in the equivariant Floer homology of N. In particular, we get

$$\Psi_{\Sigma_1}(\mathbf{1})=\int_{\mathcal{M}_0(P_1,A)}\gamma=\frac{2^{(d-1)/2}}{(d-1)!}p_1^{(d-1)/2}\hat{u}^{(d-1)/2}\in H_{G,\bullet}(A)=\mathbb{R}[\hat{u}],$$

where $H_{G,\bullet}(\mathrm{pt})=\mathbb{R}[\hat{u}]$. In fact, the framed moduli space of anti self-dual connections on Σ_1 is an SO(3)-bundle and thus has a Pontryagin class p_1.

Remark 7.5.6. Note that the latter construction only holds if $d<4$, otherwise the compactness of the moduli spaces can be violated. Moreover, as it was mentioned in [AB96], this construction also coincides with the Friedman–Morgan construction [FM94a] if Σ is the blow-up of an algebraic surface, in particular, $\Sigma_2=\mathbb{CP}^2$ and $N=S^3$, thus the moduli space on Σ has to split along the trivial connection.

Remark 7.5.7. Similarly as mentioned in Remark 6.6.6, this construction is expected to be consistent with a possible equivariant extension of the BV-BFV formalism in parallel to the BV case for closed manifolds. In the same way as the BV partition function of equivariant DW theory on closed manifolds yields Nekrasov's partition function, one should be able to produce the equivariant Floer groups as a BV-BFV partition function for equivariant DW theory in the presence of boundary. This yields a boundary state of the form

$$\Psi^{\text{eq.}}_{\partial\Sigma}(\mathbb{X},\mathbb{Y})=\int_{\mathcal{V}_\Sigma}\underbrace{\int_{\mathcal{L}}\exp(\mathrm{i}\mathcal{S}^{\text{Cartan}}_\Sigma(\mathbf{X},\mathbf{Y})/\hbar)\widetilde{O}(\mathbf{X},\mathbf{Y})\mathscr{D}[\mathscr{X}]\mathscr{D}[\mathscr{Y}]}_{=\left\langle\prod_{j=1}^d\widetilde{O}^{(\gamma_j)}\right\rangle_{\text{eq.}}=\left\langle\prod_{j=1}^d\int_{\widetilde{\gamma}_j}\widetilde{W}_{k_j}\right\rangle_{\text{eq.}}}\in HF^\bullet_{\mathcal{G}}(\partial\Sigma)\subset\mathcal{H}^{\mathcal{P}}_{\partial\Sigma},$$

where $\widetilde{O},\widetilde{\gamma}_j$ and $\widetilde{W}_{k_j}$ are equivariant extensions of the respective objects as in Section 3.6 and $\langle\ \rangle_{\text{eq.}}$ denotes the expectation for the BV-BFV partition function with respect to $\mathcal{S}^{\text{Cartan}}_\Sigma$. In particular, the boundary state space $\mathcal{H}^{\mathcal{P}}_{\partial\Sigma}$ in this case should include the equivariant Floer cohomology and thus a possible equivariant extension of the BFV boundary operator $\Omega^{\bullet,\mathcal{P}}_{\partial\Sigma}$ should be given in terms of the equivariant Floer differential. Moreover, the polarization needs to be adapted to the boundary condition defined by the state $\Psi^{\text{eq.}}_{\partial\Sigma}$. Note that here we want $\mathfrak{h}=\mathfrak{su}(2)$.

8. Relation to Donaldson–Thomas theory

Donaldson–Thomas theory, starting first in [DT98], was motivated by formulating Donaldson's construction of invariants of 4-manifolds in higher dimensions. In [Tho00], Thomas constructed a holomorphic *Casson invariant* (see Definition8.1.2) counting bundles on a Calabi-Yau 3-fold, by using the holomorphic Chern–Simons action functional (see Section 3.2). He developed the deformation theory necessary to obtain certain virtual moduli cycles in moduli spaces of stable sheaves whose higher obstruction groups vanish, which gives Donaldson- and Gromov–Witten- (see Section 9) like invariants of Fano 3-folds. Moreover, he defined the holomorphic Casson invariant of a Calabi–Yau 3-fold X and proved that it is deformation invariant. In particular, when considering real Chern–Simons theory as in Section 3.2, one can show that the Hessian of the action functional is symmetric and hence the deformation complex is defined by its critical locus, given by flat connections, is self-dual. Thus, by Poincaré duality, we get

$$H^i(\operatorname{ad}E,A)\cong(H^{3-i}(\operatorname{ad}E,A))^*$$

and hence the virtual dimension of the moduli space of flat connections $\mathcal{M}_{\text{flat}}$ is given by

$$\operatorname{vdim}\mathcal{M}_{\text{flat}}=\sum_{i=0}^3(-1)^{i+1}\dim H^i(\operatorname{ad}E,A)=0.$$

A similar observation is also true when considering holomorphic Chern–Simons theory (see Section 3.2). In particular, instead of Poincaré duality one can use Serre duality to obtain that the virtual dimension of the moduli space of holomorphic bundles is zero. This observation in fact leads to the countig of bundles over Calabi–Yau 3-folds and more generally curves in algebraic 3-folds.

8.1. **DT invariants.**

Let X be a Calabi–Yau 3-fold. Define

$$\mathrm{Hilb}_\beta(X,k):=\{X_0\subset X\mid[X_0]=\beta,\ \chi_{\text{hol}}(\mathcal{O}_{X_0})=k\}$$

to be the Hilbert scheme depending on the 1-dimensional subschemes in the curve class $\beta\in H_2(X,\mathbb{Z})$ with *holomorphic*[41] Euler characteristic $k\in\mathbb{Z}$. By Behrend's construction [Beh09],

[41] Recall that this is defined through the alternating sum of the dimension of sheaf cohomology $\chi_{\text{hol}}(\mathcal{O}_X):=\sum_{i=0}^{\dim X}(-1)^i\dim H^i(X,\mathcal{O}_X)$.

one can define the *Donaldson–Thomas (DT) invariants* as a weighted Euler characteristic of the underlying moduli space. In particular, it is defined as

$$\mathrm{DT}^X_{\beta,k} := \chi_{\mathrm{top}}\left(\mathrm{Hilb}_\beta(X,k), \nu^X_{\beta,k}\right) := \sum_{l\in\mathbb{Z}} l\chi_{\mathrm{top}}\left(\left(\nu^X_{\beta,k}\right)^{-1}(l)\right), \tag{8.1.1}$$

where χ_{top} denotes the *topological* Euler characteristic and

$$\nu^X_{\beta,k}\colon \mathrm{Hilb}_\beta(X,k) \to \mathbb{Z} \tag{8.1.2}$$

denotes Behrend's constructible[42] function [Beh09].

Remark 8.1.1. A result of Brav, Bussi, Dupont and Joyce was to prove that the coarse moduli space of simple perfect complexes of coherent sheaves, with fixed determinant, on a Calabi–Yau 3-fold admits, locally, for the analytic topology, a potential, i.e. it is isomorphic to the critical locus of a function. As it was shown in [PT14], such a result is not true for general symmetric obstruction theories. Thus, the existence of such local potentials will crucially depend on the existence of a (-1)-shifted symplectic structure (in the setting of [Pan+13], see also Section 4.7) on the derived moduli stack of simple perfect complexes on a Calabi–Yau 3-fold X. As it was shown in [Ben+15; BBJ19], there is an étale local version of Darboux's theorem for k-shifted symplectic structures with $k<0$. However, extending such a local structure theorem to general n-shifted symplectic structures, especially for the case of interest where $n\geq 0$, might lead to the existence of DT invariants for higher dimensional Calabi–Yau manifolds. Moreover, the results of [Ben+15] might lead to a categoryfied version of DT theory and a motivic version of DT invariants similarly as in [KS08b].

8.1.1. *Relation to instanton Floer homology.* DT invariants can be interpreted as a complex version of a certain invariant of homology 3-spheres, called *Casson invariant*, and then related to the instanton Floer homology construction as it was shown in [Tau90]. Let us first recall the definition of a Casson invariant.

Definition 8.1.2 (Casson invariant[Sav99])**.** Let $\mathscr{S}$ be the class of oriented integral homology 3-spheres. A *Casson invariant* is a map $\lambda\colon \mathscr{S}\to\mathbb{Z}$ such that:

(1) $\lambda(S^3)=0$, and $\lambda(\mathscr{S})$ is not contained in any proper subgroup of $\mathbb{Z}$.
(2) For any homology sphere N and knot $k\subset N$, the difference

$$\lambda'(k) := \lambda\left(N+\frac{1}{m+1}\cdot k\right) - \lambda\left(N+\frac{1}{m}\cdot k\right),\quad m\in\mathbb{Z}, \tag{8.1.3}$$

is independent of m.
(3) Let $k\cup\ell$ be a link in a homology sphere N with linking number $\mathrm{lk}(k,\ell)=0$ and note that then for any integers m,n the manifold

$$N+\frac{1}{m}\cdot k+\frac{1}{n}\cdot\ell$$

is a homology 3-sphere. We have

$$\lambda''(k,\ell) := \lambda'\left(\ell\subset N+\frac{1}{m+1}\cdot k\right) - \lambda'\left(\ell\subset N+\frac{1}{m}\cdot k\right)=0, \tag{8.1.4}$$

for any boundary link $k\cup\ell$ in a homology sphere N.

[42] That is to say for a quasi-projective proper moduli scheme with a symmetric obstruction theory, in our case $\mathrm{Hilb}_\beta(X,k)$, the weighted Euler characteristic $\chi_{\mathrm{top}}(\mathrm{Hilb}_\beta(X,k),\nu^X_{\beta,k})$ is the degree of the *virtual fundamental class* $[\mathrm{Hilb}_\beta(X,k)]^{\mathrm{vir}}$ (see Footnote 44 for a definition of the virtual fundamental class).

Theorem 8.1.3 (Casson)**.** *There is a Casson invariant λ which is unique up to sign. Moreover, it has the following properties:*

(i) $\lambda'(k_{3_1})=\pm 1$, *where k_{3_1} denotes the trefoil knot.*
(ii) $\lambda'(k\subset N)=\frac{1}{2}\Delta''_{k\subset N}(1)\cdot\lambda'(k_{3_1})$ *for any knot $k\subset N$,*
(iii) $\lambda(-N)=-\lambda(N)$ *where $-N$ stands for N with reversed orientation,*
(iv) $\lambda(N_1\# N_2)=\lambda(N_1)+\lambda(N_2)$,
(v) $\lambda(N)=\mu(N)$ *mod 2, where μ denotes the* Rohlin invariant[43].

In [Tau90], Taubes has shown that the Casson invariant as in Theorem 8.1.3 can be defined by using an infinite-dimensional generalization of the topological Euler characteristic using instanton Floer homology for an oriented homology 3-sphere. Let G be a finitely presented group and N a homology 3-sphere. Denote by $R(G)=\mathrm{Hom}(G,\mathrm{SU}(2))$ where G is assumed to have the discrete topology. Note that one can turn $R(G)$ into a real algebraic variety by regarding $\mathrm{SU}(2)$ as S^4. The quotient $\mathcal{R}(G):=R(G)/\mathrm{SU}(2)$ under the $\mathrm{SU}(2)$-action given by conjugation, is the $\mathrm{SU}(2)$-character variety of G. Denote the open set of irreducible representations by $R^*(G)$. Then we can consider the quotient $\mathcal{R}^*(G):=R^*(G)/\mathrm{SO}(3)\subset\mathcal{R}(G)$. Now let $\mathcal{R}^*(N):=\mathcal{R}^*(\pi_1(N))$, where $\pi_1(N)$ denotes the *fundamental group* of N. Moreover, for some admissible perturbation function $f\colon\mathcal{A}^*/\mathcal{G}\to\mathbb{R}/8\pi^2\mathbb{Z}$, define the *perturbed moduli space*

$$\mathcal{R}_f(N):=\{A\in\mathcal{A}\mid *F_A-4\pi^2(\nabla f)(A)=0\}/\mathcal{G}.$$

Finally, define $\mathcal{R}^*_f(N)$ to be the subset of $\mathcal{R}_f(N)$ consisting of the orbits of irreducible perturbed flat connections. Then, we have the following theorem;

Theorem 8.1.4 (Taubes[Tau90])**.** *The Euler characteristic*

$$\chi_{\mathrm{top}}(N)=\frac{1}{2}\sum_{A\in\mathcal{R}^*_f(N)}(-1)^{\mu(A)}$$

with Floer index $\mu(A):=\mathrm{sf}(\theta,A)\bmod 8$, *where θ denotes the trivial connection, is independent of the holonomy perturbation f and the metric on N and equals up to an overall sign the Casson invariant of N.*

However, working with a homology 3-sphere does simplify things tremendously since it prevents the existence of non-trivial reducible flat connections before and after small perturbations. In fact, this allows to use the standard cobordism argument to show that the alternating sum is indeed independent of the choice of metric. We refer also to [Sav01] for an excellet overview.

8.2. **DT partition function.** Let X be a Calabi–Yau 3-fold. Using the DT invariants (8.1.1), we can define the *DT partition function* as a generating function by

$$\mathsf{Z}^{\mathrm{DT}}_X(q):=\sum_{(\beta,k)\in H_2(X,\mathbb{Z})\oplus\mathbb{Z}}\mathrm{DT}^X_{\beta,k}q^\beta(-z)^k, \tag{8.2.1}$$

where $q^\beta:=q_1^{\beta_1}\cdots q_n^{\beta_n}$ with $\beta=\beta_1C_1+\dots+\beta_nC_n$ where $\{C_1,\dots,C_n\}$ denotes a basis of $H_2(X,\mathbb{Z})$ which is chosen such that $\beta_i\geq 0$ for any effective curve class. Note that $\mathsf{Z}^{\mathrm{DT}}_X(q)$ is a formal power series in $q_1,\dots,q_n$ with coefficients in formal Laurent series in the complex

[43] For an oriented integral homology 3-sphere N, there exists a smooth simply-connected oriented 4-manifold Σ with even intersection form such that $\partial\Sigma=N$. Then the signature of Σ is divisible by 8 and $\mu(N):=\frac{1}{8}\mathrm{sign}\,\Sigma$ mod 2 is independent of the choice of Σ [Sav99]. The number $\mu(N)$ is called *Rohlin invariant* of N.

variable z. In particular, since X is a 3-fold, we can split the Hilbert scheme into components according to the degree and the arithmetic genus of C when $\beta=[C]$. We get

$$([C],\chi_{\mathrm{hol}}(\mathcal{O}_C))\in H_2(X,\mathbb{Z})\oplus H_0(X,\mathbb{Z}).$$

Consider a smooth divisor $D\subset X$ which algebraically takes the place of the boundary when X is a smooth manifold. We can define the relative DT partition function corresponding to the pair (X,D) as

$$\mathsf{Z}^{\mathrm{DT}}_{X/D}(q):=\sum_{(\beta,k)\in H_2(X,\mathbb{Z})\oplus H_0(X,\mathbb{Z})}q^\beta(-z)^k\int_{\mathrm{Hilb}_\beta(X,k)}\Xi(D),\tag{8.2.2}$$

where $\Xi(D)$ denotes a collection of forms constructed from the boundary conditions. We need to make sense of the integral in (8.2.2). The integral localizes to a *virtual fundamental class*[44] $[\mathcal{M}_{\mathrm{DT}}]^{\mathrm{vir}}\in H_{2\mathrm{vdim}}(X)$, where $\mathcal{M}_{\mathrm{DT}}$ denotes the moduli space of *supersymmetric configurations*. In particular, for the Hilbert scheme, we get the virtual dimension[45]

$$\mathrm{vdim}\,\mathrm{Hilb}_\beta(X,k)=\int_\beta c_1(X).$$

Using this construction, we can formulate a more precise version of (8.2.2) by

$$\mathsf{Z}^{\mathrm{DT}}_{X/D}(q)=\sum_{(\beta,k)\in H_2(X,\mathbb{Z})\oplus H_0(X,\mathbb{Z})}q^\beta(-z)^k\int_{[\mathrm{Hilb}_\beta(X,k)]^{\mathrm{vir}}}[\Xi](D),\tag{8.2.3}$$

where $[\Xi](D)$ denotes the collection of cohomology classes constructed from the boundary conditions.

Remark 8.2.1. Note that in fact, using the virtual fundamental class, we can equivalently express the DT invariants (8.1.1) as a *virtual count* by

$$\mathrm{DT}^X_{\beta,k}=\int_{\left[\mathrm{Hilb}_\beta(X,k)\right]^{\mathrm{vir}}}1.$$

8.3. **Gluing of DT partition functions.** We want to describe the gluing procedure for partition functions in DT theory. Let X be given by two parts X_1 and X_2. The Hilbert scheme of the singular variety

$$X_0:=X_1\cup_D X_2$$

is not nice (although well-defined) since if C intersects D in a nontransverse way, it will lead to different problems including the failure for the obstruction theory. However, one can resolve this by using expanded degenerations. There we allow X_0 to insert *bubbles* $\mathbb{B}$ such that

$$X_0[\ell]=X_1\cup_{D_1}\underbrace{\mathbb{B}\cup_{D_2}\cdots\cup_{D_\ell}\mathbb{B}}_{\ell}\cup_{D_{\ell+1}}X_2,$$

where $\mathbb{B}:=\mathbb{P}(N(X_1/D)\oplus\mathcal{O}_D)$ is a $\mathbb{P}^1$-bundle over D associated to the rank 2 vector bundle $N(X_1/D)\oplus\mathcal{O}_D$, where $N(X_1/D)$ denotes the normal bundle of D in X_1. The divisors

[44]Following [Beh09], for a scheme (or Deligne–Mumford stack) X with cotangent complex $\mathbb{L}_X$, the virtual fundamental class is defined through a perfect obstruction theory of $E\to\mathbb{L}_X$, where $E\in D(\mathcal{O}_X)$ (here $D(\mathcal{O}_X)$ denotes the derived category of sheaves of $\mathcal{O}_X$-modules). In particular, E defines a vector bundle stack $\mathfrak{C}$ over X. The *virtual fundamental class* $[X]^{\mathrm{vir}}\in A_{\mathrm{rk}\,E}(X)$ is then defined as the intersection of the fundamental class of the *intrinsic normal cone* $[\mathfrak{C}_X]$ (the perfect obstruction theory $E\to\mathbb{L}_X$ induces a closed immersion of cone stacks $\mathfrak{C}_X\hookrightarrow\mathfrak{C}$) with the zero section of $\mathfrak{C}$, i.e. $[X]^{\mathrm{vir}}:=0^!_{\mathfrak{C}}[\mathfrak{C}_X]$. Note that we have denoted by $A_r(X)$ the *Chow group* of r-cycles modulo rational equivalence on X with values in $\mathbb{Z}$.

[45]Typically computed by using some version of the Riemann–Roch theorem.

$D_1 \cong D_2 \cong \cdots \cong D_{\ell+1} \cong D$ are all copies of D which appear together with the bubbles $\mathbb{B}$. We consider now subschemes $C \subset X_0[\ell]$ of the form $C = C_0 \cup C_1 \cup \cdots \cup C_\ell \cup C_{\ell+1}$ with components $\{C_i\}$ all being transverse to the divisors $\{D_i\}$. We can glue them together by the setting

$$C_i \cap D_{i+1} = D_i \cap C_{i+1}.$$

This produces a new bubble each time the intersection does not appear to be transversal. The moduli spaces for different ℓ can be captured into one single orbifold

$$\mathrm{Hilb}(X_0[\bullet]) = \bigcup_{\ell\geq 0} \mathrm{Hilb}(X_0[\ell])_{\text{semistable}}\Big/(\mathbb{C}^\times)^\ell,$$

where $\mathbb{C}^\times$ acts on the $\mathbb{P}^1$-bundle. Note that when talking about the smooth setting for a manifold Σ, we consider tubular neighborhoods of the boundary $\partial\Sigma \times [0,1]$ in order to glue the two boundary pieces $\partial\Sigma \times \{0\}$ and $\partial\Sigma \times \{1\}$, which in the algebraic setting corresponds to the gluing of two distinguished divisors, namely $D_0 = \mathbb{P}(N(X/D)) \subset \mathbb{B}$ and $D_\infty = \mathbb{P}(\mathcal{O}_X) \subset \mathbb{B}$. In fact, the $\mathbb{C}^\times$-action on $\mathbb{P}^1$ preserves the divisors D_0 and D_∞. The DT gluing formula is then given by setting

$$\int_{[\mathrm{Hilb}(X)]^{\mathrm{vir}}} [\Xi](D) = \int_{[\mathrm{Hilb}(X_0[\bullet])]^{\mathrm{vir}}} [\Xi](D), \tag{8.3.1}$$

where $[\Xi](D)$ denotes a cohomology class defined by using boundary conditions away from D or in any other way such that it is well-defined as a cohomology class on the whole family of the Hilbert schemes. Additionally, we need to require that the integral on the right-hand-side of (8.3.1) can be computed in terms of DT counts in X_1 and X_2 relative to the divisor D. In particular, this means

$$\mathsf{Z}_X^{\mathrm{DT}} = \left\langle \mathsf{Z}^{\mathrm{DT}}_{X_1/D}, (-z)^{|\cdot|}\mathsf{Z}^{\mathrm{DT}}_{X_2/D}\right\rangle_{H_\bullet(\mathrm{Hilb}(D))}, \tag{8.3.2}$$

where $\mathsf{Z}^{\mathrm{DT}}_{X_i/D}$ for $i=1,2$ is defined as in (8.2.2) and $|\cdot|$ denotes the grading on $H_\bullet(\mathrm{Hilb}(D)) = \bigoplus_k H_\bullet(\mathrm{Hilb}(D,k))$. We put the extra weight $(-z)^{|\cdot|}$ since

$$\chi_{\mathrm{hol}}(\mathcal{O}_C) = \chi_{\mathrm{hol}}(\mathcal{O}_{C_1}) + \chi_{\mathrm{hol}}(\mathcal{O}_{C_2}) - \chi_{\mathrm{hol}}(\mathcal{O}_{C_1\cap D})$$

whenever $C = C_1 \cup_{C_1\cap D} C_2$ is a transverse union along the common intersection with D.

9. Relation to Gromov–Witten theory

9.1. **The moduli space of stable maps.** Gromov–Witten (GW) theory [Gro85; Wit91; Beh97] of a non-singular projective variety X deals with the *moduli spaces of stable maps*[46] constructed by Kontsevich (first appeared in [Kon95; KM94]), which is a generalization for the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves[47] of genus g (see also [Pan99; FP97; Hor+03]). As a set, $\overline{\mathcal{M}}_{g,n}$ is the set of isomorphism classes of n-pointed stable curves of genus g. It actually turns out that $\overline{\mathcal{M}}_{g,n}$ is a quasi-projective variety of dimension $3g-3+n$

[46]Recall that a stable map on a non-singular projective variety X is a morphism f from a pointed nodal curve C to X such that if f is constant on any component of C, then that component is required to have at least three distinguished points. The distinguished points are either marked points, or points lying over nodes in the normalization of C.

[47]This is a complete algebraic curve whose automorphism group, as an n-pointed curve, is finite.

and is given as a projective compacification[48] of the moduli space given by the isomorphism classes of smooth n-pointed stable curves of genus g, which is usually denoted by $\mathcal{M}_{g,n}$. In particular, the moduli space of stable maps from n-pointed nodal curves[49] of genus g to a non-singular projective variety X representing[50] the class $\beta \in H_2(X,\mathbb{Z})$ is denoted by $\overline{\mathcal{M}}_{g,n}(X,\beta)$. In fact, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a Deligne–Mumford stack. It is easy to see that if X is a point, we get that

$$\overline{\mathcal{M}}_{g,n}(X,0)=\overline{\mathcal{M}}_{g,n}.$$

A natural cohomology class on the moduli space of stable maps is given by the pull-back of X (see below), i.e. $\mathrm{ev}_i^*(\gamma)$ where $\gamma \in H^\bullet(X,\mathbb{Z})$ for $i=1,\dots,n$. At each point $[C,p_1,\dots,p_n,f]\in\overline{\mathcal{M}}_{g,n}(X,\beta)$, the cotangent space to C at each point p_i is a 1-dimensional vector space. Gluing all these spaces together, we get a line bundle $\mathbb{L}_i$ called *i-th tautological line bundle*. Denote by $\psi_i:=c_1(\mathbb{L}_i)$ its first Chern class. Then we have the *string equation* for $\overline{\mathcal{M}}_{g,n}$

$$\int_{\overline{\mathcal{M}}_{g,n+1}}\prod_{i=1}^n\psi_i^{\beta_i}=\sum_{i=1}^n\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\beta_1}\cdots\psi_i^{\beta_i-1}\cdots\psi_n^{\beta_n}. \tag{9.1.1}$$

One can then easily prove the *dilaton equation* for $\overline{\mathcal{M}}_{g,n}$

$$\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_{n+1}\prod_{i=1}^n\psi_i^{\beta_i}=(2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{\beta_i} \tag{9.1.2}$$

for $2g-2+n>0$. Moreover, define

$$\langle\tau_{\beta_1},\dots,\tau_{\beta_n}\rangle_g:=\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{\beta_i}.$$

The virtual dimension of the moduli space of stable maps [BF97] is given by

$$\mathrm{vdim}\,\overline{\mathcal{M}}_{g,n}(X,\beta)=\int_\beta c_1(X)+(\dim X-3)(1-g)+n. \tag{9.1.3}$$

Similarly as for the DT construction, one can also consider the virtual fundamental class

$$\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}\in H_{2\mathrm{vdim}}(X,\mathbb{Q}).$$

In particular, if there is no obstruction for all stable maps, we get that the virtual fundamental class is equal to the ordinary fundamental class. As already mentioned, for $i=1,...,n$ there are evaluation maps $\mathrm{ev}_i\colon\overline{\mathcal{M}}_{g,n}(X,\beta)\to X$ such that $\mathrm{ev}_i(f)=f(x_i)$ for $x_i\in C$. Thus, the classes $\gamma\in H^\bullet(X,\mathbb{Z})$ can be pulled back to classes in $H^\bullet(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$ by the map $\mathrm{ev}_i^*\colon H^\bullet(X,\mathbb{Z})\to H^\bullet(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$.

[48]One can show that $\mathcal{M}_{g,n}$ is not compact by noticing that in general a family of smooth curves over a noncomplete base is not extendable to a family of smooth curves over a completion of it. The completion can be realized by allowing fibers that have nodes at worst and using the *stable reduction theorem*, hence compactness of $\overline{\mathcal{M}}_{g,n}$.

[49]Recall that a *nodal curve* is a complete algebraic curve such that each of its points is either smooth or locally complex-analytically isomorphic to a neighborhood of the origin within the locus of the equation $xy=0$ in $\mathbb{C}^2$ [ACG11].

[50]Any stable map $f\colon C\to X$ represents a homology class $\beta\in H_2(X,\mathbb{Z})$ whenever $f_*[C]=\beta$.

9.2. **GW invariants.** Given cohomology classes $\gamma_1,\dots,\gamma_n\in H^\bullet(X,\mathbb{Z})$, we can define the *GW invariants* by

$$\mathrm{GW}^X_{g,\beta}(\gamma_1,\dots,\gamma_n):=\int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i). \tag{9.2.1}$$

Remark 9.2.1. According to Witten's notation, we may sometimes also write $\langle\prod_{i=1}^n\gamma_i\rangle^X_{g,\beta}=\langle\gamma_1\cdots\gamma_n\rangle^X_{g,\beta}$ instead of $\mathrm{GW}^X_{g,\beta}(\gamma_1,\dots,\gamma_n)$.

One can also define a generalized version of the GW invariants, called the *gravitational descendant invariants* or just *descendant invariants*, defined by

$$\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\right\rangle^X_{g,\beta}:=\int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}}\prod_{i=1}^n\mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}, \tag{9.2.2}$$

where $\gamma_i\in H^\bullet(X,\mathbb{Z})$ and $a_i\in\mathbb{Z}_{>0}$. We can extend the string (9.1.1) and dilaton (9.1.2) equations to the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The string equation is given by

$$\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\mathbf{1}\right\rangle^X_{g,\beta}=\sum_{j=1}^n\left\langle\prod_{i=1}^{j-1}\tau_{a_i}(\gamma_i)\tau_{a_j-1}(\gamma_j)\prod_{i=j+1}^n\tau_{a_i}(\gamma_i)\right\rangle^X_{g,\beta} \tag{9.2.3}$$

where $\mathbf{1}\in H^\bullet(X,\mathbb{Z})$ denotes the unit. The dilaton equation is given by

$$\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\tau_1(\mathbf{1})\right\rangle^X_{g,\beta}=(2g-2+n)\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\right\rangle^X_{g,\beta}. \tag{9.2.4}$$

It is easy to see that (9.2.3) reduces to (9.1.1) and (9.2.4) to (9.1.2) if X is a point.

9.3. **GW partition function.** One can then define the *GW partition function* as

$$\begin{aligned}\mathsf{Z}^{\mathrm{GW}}_X(q,u)&:=\sum_{(\beta,g)\in H_2(X,\mathbb{Z})\oplus\mathbb{Z}}\mathrm{GW}^X_{\beta,g}(\gamma_1,\dots,\gamma_n)q^\beta u^{2g-2}\\&=\sum_{(\beta,g)\in H_2(X,\mathbb{Z})\oplus\mathbb{Z}}q^\beta u^{2g-2}\int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}}\prod_{i=1}^n\mathrm{ev}_i^*(\gamma_i)\end{aligned} \tag{9.3.1}$$

Remark 9.3.1. GW theory can be regarded as some A-mirror since the invariants can be thought of an A-model with path integral over the space of fields given through pseudo-holomorphic curves as for the construction of Lagrangian Floer homology (see Section 3.7). We refer to the excellent reference [Hor+03] for more details on GW theory and its connection to mirror symmetry.

Remark 9.3.2 (Korteweg-de-Vries (KdV) hirarchy)**.** Consider the generating series

$$\mathsf{F}(t_0,t_1,\dots):=\sum_{\substack{g\geq 0,n\geq 0\\2g-2+n\geq 0}}\frac{1}{n!}\sum_{\beta_1,\dots,\beta_n\geq 0}\left(\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{\beta_i}\right)t_0^{\beta_1}t_1^{\beta_2}\cdots t_n^{\beta_n}. \tag{9.3.2}$$

In [Wit91], Witten conjectured that the function $u:=\frac{\partial^2\mathsf{F}}{\partial t_0^2}$ satisfies the *KdV hirarchy*, i.e.

$$
\begin{aligned}
u_{t_1}&=uu_x+\frac{1}{12}u_{xxx},\\
u_{t_2}&=\frac{1}{2}u^2u_x+\frac{1}{12}(2u_xu_xx+uu_{xxx})+\frac{1}{240}u_{xxxxx}\\
&\ \ \vdots
\end{aligned}
\tag{9.3.3}
$$

where $x:=t_0$. Using the string equation (9.1.1), we get

$$
\frac{\partial\mathsf{F}}{\partial t_0}=\sum_{i\geq 0}t_{i+1}\frac{\partial\mathsf{F}}{\partial t_i}+\frac{t_0^2}{2}.
\tag{9.3.4}
$$

If F satisfies condition (9.3.3) together with (9.3.4), one can uniquely determine the generating series F. Witten's conjecture was proven by Kontsevich in [Kon92].

9.4. **Topological recursion.** Topological recursion started with the work of Eynard–Orantin in [EO09] which uses the recursive methods of *symplectic invariants* to solve matrix model loop equations. In particular, as an application they show how one can obtain e.g. a proof of Witten's conjecture (see Remark 9.3.2), Mirzakhani's recursive methods of deriving Weil–Petersson volumes [Mir07] and certain constructions of topological string theories. In particular, one is interested in the data given by a spectral curve $C=(\Sigma_g,x,y,B)$, where Σ_g denotes a Riemann surface of genus $g\geq 0$, x,y are two meromorphic functions on Σ_g and B is some 2-form on $\Sigma_g\times\Sigma_g$. Assume moreover that the zeros of $\mathrm{d}x$ are all simple and do not coincide with the zeros of $\mathrm{d}y$. Then, as a result of topological recusrsion, one obtains symmetric multidifferentials $\omega_{g,n}\in H^0(K_{\Sigma_g}(*D)^{\otimes n},(\Sigma_g)^n)^{S_n}$ for $g\geq 0$ and $n\geq 1$ such that $2g-2+n>0$. These differentials are usually called *correlators*. We have denoted by D the divisor of zeros of $\mathrm{d}x=0$ and by S_n the symmetric group of order n. Note that then the multidifferentials $\omega_{g,n}$ will be holomorphic outside of $\mathrm{d}x=0$ and can have poles of any order when the given variables approach the divisor D. The key is to define the correlators recursively. In particular, we define first the exceptional cases

$$
\omega_{0,1}(p_1)=y(p_1)\mathrm{d}x(p_1),
\tag{9.4.1}
$$

$$
\omega_{0,2}(p_1,p_2)=B(p_1,p_2).
\tag{9.4.2}
$$

Then, the recursive relation for the correlators in general is given by

$$
\begin{aligned}
\omega_{g,n}(p_1,\boldsymbol{p}_I)=&\sum_{\mathrm{d}x(\alpha)=0}\mathrm{Res}_{p=\alpha}K(p_1,p)\times\\
&\times\left\{\omega_{g-1,n+1}(p,\sigma_\alpha(p),\boldsymbol{p}_I)+\widehat{\sum_{\substack{h+h'=g\\ J\sqcup J'=I}}^{\omega_{0,1}}}\omega_{h,1+|J|}(p,\boldsymbol{p}_J)\omega_{h',1+|J'|}(\sigma_\alpha(p),\boldsymbol{p}_{J'})\right\}.
\end{aligned}
\tag{9.4.3}
$$

We have used the notation $I=\{2,3,\dots,n\}$ and $\boldsymbol{p}_J=\{p_{j_1},\dots,p_{j_k}\}$ for $J=\{j_1,j_2,\dots,j_k\}\subseteq I$. Note that the holomorphic function $p\mapsto\sigma_\alpha(p)$ is the non-trivial involution which is locally defined at the ramification point α and satisfies $x(\sigma_\alpha(p))=x(p)$. Moreover, we have denoted by $\widehat{\sum^{\omega_{0,1}}}$ in the bracket the sum which leaves out the terms involving $\omega_{0,1}$. The *recursion kernel* is given by

$$
K(p_1,p):=\frac{1}{2}\frac{1}{(y(p)-y(\sigma_\alpha(p)))\mathrm{d}x(p)}\int_{\sigma_\alpha(p)}^{p}\omega_{0,2}(p_1,\ \ ).
\tag{9.4.4}
$$

Note that the recursion will only depend on the local behaviour of y near the zeros of $\mathrm{d}x$ up to functions which are even with respect to the involution σ_α and thus it only depends on $\mathrm{d}y$. If $2g-2+n>0$, one can observe that $\omega_{g,n}(p_1,\ldots,p_n)$ are meromorphic forms on $(\Sigma_g)^n$ with poles at $p_i\in D$. Therefore, they can be expressed in terms of polynomials for a basis of 1-forms with poles only in D and divergent part being odd with respect to each local involution σ_α. If we denote such a basis by (ξ_k^α), we can write down a partition function for the spectral curve C as

$$\mathsf{Z}_C(\{t_k^\alpha\};\hbar):=\exp\left(\sum_{\substack{g\geq 0,\, n\geq 1\\ 2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\omega_{g,n}\big|_{\xi_k^\alpha=t_k^\alpha}\right). \tag{9.4.5}$$

Remark 9.4.1. The relation to GW theory was considered in [Dun+14] and studied further in [BN19]. In particular, in [Dun+14], they show how this construction fits into the cohomological field theory encoding GW invariants of $\mathbb{P}^1$ which corresponds to the spectral curve given by

$$C_{\mathbb{P}^1}=\left(\mathbb{P}^1, x=z+\frac{1}{z}, \mathrm{d}y=\frac{\mathrm{d}z}{z}, B=\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2}{(z_1-z_2)^2}\right).$$

Remark 9.4.2. An interesting approach to the formulation of topological recursion would be to combine it with the methods of [Pan+13] (see Section 4.7) in order to talk about a *shifted* version of the multidifferentials (9.4.3), i.e. to consider the behaviour of forms in $\omega_{g,n}\in H^k(K_{\Sigma_g}(*D)^{\otimes n},(\Sigma_g)^n)^{S_n}$, or more generally a shifted version of the symplectic invariants as defined in [EO09]. This would lead to a relation of the BV-BF^{k}V formalism by considering enumerative methods, such as GW invariants, for stratified spaces in the setting of perturbative gauge theories.

Remark 9.4.3. In [Bor+21], it was recently also shown how one can obtain Nekrasov's partition function by using methods of topological recursion.

9.5. **GW/DT correspondence.** Based on a duality construction involving topological strings in the limit of the large string coupling constant described in terms of a classical statistical mechanical model of crystal melting considered in [ORV06], Maulik, Nekrasov, Okounkov and Panharipande have formulated a correspondence between GW and DT partition functions in [Mau+06a; Mau+06b]. In particular, in [ORV06] one considers the crystal to be a discretization of the toric base of some Calabi–Yau manifold. Moreover, a more general duality involving the A-model string on a toric 3-fold was proposed. For this, let X be a nonsingular projective 3-fold and define the generating function

$$\widetilde{\mathsf{Z}}_X^{\mathrm{GW}}\left(u,\prod_{i=1}^n\tau_{a_i}(\gamma_i)\right)_\beta:=\sum_{g\in\mathbb{Z}}\left\langle\!\!\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\right\rangle\!\!\right\rangle_{g,\beta}^X u^{2g-2} \tag{9.5.1}$$

for a fixed nontrivial class $\beta\in H_2(X,\mathbb{Z})$. Here we have defined

$$\left\langle\!\!\left\langle\prod_{i=1}^n\tau_{a_i}(\gamma_i)\right\rangle\!\!\right\rangle_{g,\beta}^X:=\int_{\left[\overline{\mathcal{M}}'_{g,n}(X,\beta)\right]^{\mathrm{vir}}}\prod_{i=1}^n\mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i},$$

with $\overline{\mathcal{M}}'_{g,n}(X,\beta)$ the moduli space of maps with possibly disconnected domain curves C of genus g where there are no collapsed components, hence each component C represents a nontrivial[51] class $\beta \in H_2(X,\mathbb{Z})$. Let $I_k(X,\beta)$ be the moduli space of ideal sheaves[52] satisfying $\chi_{\mathrm{hol}}(\mathcal{O}_Y) = k$ and $[Y] = \beta \in H_2(X,\mathbb{Z})$ for some subscheme $Y \subset X$. The descendent fields $\tilde{\tau}_a(\gamma)$ in DT theory correspond to $(-1)^{a+1}\mathrm{ch}_{a+2}(\gamma)$, where

$$\mathrm{ch}_{a+2}(\gamma)\colon H_\bullet(I_k(X,\beta),\mathbb{Q}) \to H_{\bullet-2a+2-\ell}(I_k(X,\beta),\mathbb{Q}) \tag{9.5.2}$$

for $\gamma \in H^\ell(X,\mathbb{Z})$. One can express the map $\mathrm{ch}_{a+2}(\gamma)$ by considering the projections π_1 and π_2 to the first and second factor of $I_k(X,\beta)\times X$, respectively and the universal sheaf $\mathfrak{I} \to I_k(X,\beta)\times X$. One can show that there is[53] a finite resolution of $\mathfrak{I}$ by locally free sheaves on $I_k(X,\beta)\times X$. This will guarantee that the Chern classes of $\mathfrak{I}$ are well-defined and thus one can express the map (9.5.2) by

$$\mathrm{ch}_{a+2}(\gamma)(\xi) = (\pi_1)_*\big(\mathrm{ch}_{a+2}(\mathfrak{I})(\pi_2)^*(\gamma)\cap(\pi_1)^*(\xi)\big).$$

We can then define the descendent invariants by

$$\left\langle \prod_{i=1}^n \tilde{\tau}_{a_i}(\gamma_i)\right\rangle_{k,\beta}^X = \int_{\left[I_k(X,\beta)\right]^{\mathrm{vir}}} \prod_{i=1}^n (-1)^{a_i+1}\mathrm{ch}_{a_i+2}(\gamma_i). \tag{9.5.3}$$

Using this construction, and noticing that the moduli space $I_k(X,\beta)$ is canonically isomorphic to the Hilbert scheme $\mathrm{Hilb}_\beta(X,k)$, we can define the DT partition function by

$$\mathsf{Z}_X^{\mathrm{DT}}\left(q,\prod_{i=1}^n\tilde{\tau}_{a_i}(\gamma_i)\right)_\beta = \sum_{k\in\mathbb{Z}}\left\langle \prod_{i=1}^n \tilde{\tau}_{a_i}(\gamma_i)\right\rangle_{k,\beta}^X q^k.$$

We then define the *reduced* DT partition function by formally removing the degree zero contributions

$$\widetilde{\mathsf{Z}}_X^{\mathrm{DT}}\left(q,\prod_{i=1}^n\tilde{\tau}_{a_i}(\gamma_i)\right)_\beta := \frac{\mathsf{Z}_X^{\mathrm{DT}}\left(q,\prod_{i=1}^n\tilde{\tau}_{a_i}(\gamma_i)\right)_\beta}{\mathsf{Z}_X^{\mathrm{DT}}(q)_0}.$$

For simplicity, we state the GW/DT correspondence for *primary fields* $\tau_0(\gamma)$ and $\tilde{\tau}_0(\gamma)$.

Conjecture 9.5.1 (Maulik–Nekrasov–Okounkov–Pandharipande[Mau+06a])**.** Considering the change-of-variables $\exp(\mathrm{i}u) = -q$, we have

$$(-\mathrm{i}u)^d\widetilde{\mathsf{Z}}_X^{\mathrm{GW}}\left(u,\prod_{i=1}^n\tau_0(\gamma_i)\right)_\beta = (-q)^{-d/2}\widetilde{\mathsf{Z}}_X^{\mathrm{DT}}\left(q,\prod_{i=1}^n\tilde{\tau}_0(\gamma_i)\right)_\beta,$$

where $d := \int_\beta c_1(X)$.

Remark 9.5.2. Conjecture 9.5.1 has been proven for X being a toric 3-fold in [Mau+11].

9.5.1. *Relative GW/DT correspondence.* An important concept for us is the case for *relative theories* which corresponds to the algebraic case when considering defects of a manifold in the smooth setting. Let X be a nonsingular projective 3-fold and let $D\subset X$ be a nonsingular divisor. The *relative* GW invariants are then defined by

$$\left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i)\,\middle|\,\eta\right\rangle\!\!\right\rangle_{g,\beta}^{X/D} := \frac{1}{|\mathrm{Aut}\,(\eta)|}\int_{\left[\overline{\mathcal{M}}'_{g,n}(X/D,\beta,\eta)\right]^{\mathrm{virt}}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}\prod_{j=1}^m \mathrm{ev}_j^*(\delta_j), \tag{9.5.4}$$

[51] In particular, C has to represent a nonzero class.

[52] These are torsion-free sheaves of rank 1 with trivial determinant.

[53] This is due to the fact that $\mathfrak{I}$ is π_1-flat and X is nonsingular.

where $\beta \in H_2(X,\mathbb{Z})$ is such that $\int_\beta [D] \geq 0$, $\eta = (\eta_j)$ a partition whose components satisfy $\sum_j \eta_j = \int_\beta [D]$ together with a certain ordering condition[54] with respect to a basis $\delta_1, \ldots, \delta_m$ of $H^\bullet(D,\mathbb{Q})$, $\overline{\mathcal{M}}'_{g,n}(X/D,\beta,\eta)$ denotes the moduli of stable relative maps with possibly disconnected domains and relative multiplicities determined by η, and $\mathrm{ev}_j \colon \overline{\mathcal{M}}'_{g,n}(X/D,\beta,\eta) \to D$ are determined by the relative points. Hence, the *relative* GW partition function is given by

$$\widetilde{\mathsf{Z}}^{\mathrm{GW}}_{X/D}\left(u, \prod_{i=1}^n \tau_{a_i}(\gamma_i)\right)_{\beta,\eta} = \sum_{g\in\mathbb{Z}} \left\langle\!\!\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i) \,\middle|\, \eta \right\rangle\!\!\right\rangle^{X/D}_{g,\beta} u^{2g-2}. \tag{9.5.5}$$

Similarly as we have seen in Section 8.2, we can construct a relative version of the DT partition function by integration over the moduli space of relative ideal sheaves. Let η be a cohomology weighted partition with respect to the basis $\delta_1,\ldots,\delta_m \in H^\bullet(D,\mathbb{Q})$ and let $|\eta| := \sum_j \eta_j$. The relative moduli space[55] $I_k(X/D,\beta)$ in fact has still dimension $\int_\beta c_1(X)$. The relative DT partition function is then given by

$$\mathsf{Z}^{\mathrm{DT}}_{X/D}\left(q, \prod_{i=1}^n \tilde{\tau}_{a_i}(\gamma_i)\right)_{\beta,\eta} = \sum_{k\in\mathbb{Z}} \left\langle \prod_{i=1}^n \tilde{\tau}_{a_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{k,\beta} q^k, \tag{9.5.6}$$

where the descendent invariants in the relative DT theory are given by

$$\left\langle \prod_{i=1}^n \tilde{\tau}_{a_i}(\gamma_i) \,\middle|\, \eta \right\rangle_{k,\beta} = \int_{\left[I_k(X/D,\beta)\right]^{\mathrm{vir}}} \prod_{i=1}^n (-1)^{a_i+1}\mathrm{ch}_{a_i+2}(\gamma_i) \cap \epsilon^* C_\eta$$

with the canonical intersection map $\epsilon\colon I_k(X/D,\beta) \to \mathrm{Hilb}\left(D, \int_\beta [D]\right)$ to the Hilbert scheme of points and

$$C_\eta := \frac{1}{\prod_i \eta_i |\mathrm{Aut}(\eta)|} \prod_{j=1}^m P_{\delta_j}[\eta_j] \cdot \mathbf{1} \in H^\bullet\left(\mathrm{Hilb}\left(D, |\eta|\right), \mathbb{Q}\right),$$

where we follow the notation of [Nak99]. The collection $(C_\eta)_{|\eta|=k}$ is called *Nakajima basis* of the cohomology of the Hilbert scheme $\mathrm{Hilb}(D,k)$. The Nakajima basis is *orthogonal* with respect to the Poincaré pairing on the cohomology

$$\int_{\mathrm{Hilb}(D,k)} C_\eta \cup C_\nu = \frac{(-1)^{k-\ell(\eta)}}{\prod_i \eta_i |\mathrm{Aut}(\eta)|} \delta_{\nu,\eta^\vee},$$

---

[54] In fact, we can consider a *weighted* partition η which consists of tuples $(\eta_j, \delta_{\ell_j})$. One then chooses the *standard order*, i.e. $(\eta_j, \delta_{\ell_j}) > (\eta_{j'}, \delta_{\ell_{j'}})$ if $(\eta_j > \eta_{i'})$ or if $\eta_i = \eta_{i'}$ and $\ell_i > \ell_{i'}$. Then η is basically the partition $(\eta_1,\ldots,\eta_m)$ that is obtained from the standard order.

[55] We can construct a proper moduli space $I_k(X/D,\beta)$ consisting of stable ideal sheaves relative on the degenerations $X[\ell]$ of X. We define an ideal sheaf on $X[\ell]$ to be *predeformable* if for each singular divisor $D_l \subset X[\ell]$, the induces map

$$\mathscr{I} \otimes_{\mathcal{O}_{X[\ell]}} \mathcal{O}_{D_l} \to \mathcal{O}_{X[\ell]} \otimes_{\mathcal{O}_{X[\ell]}} \mathcal{O}_{D_l}$$

is injective. Moreover, we call an ideal sheaf $\mathscr{I}$ on $X[\ell]$ relative to D_l *stable* if $\mathrm{Aut}(\mathscr{I})$ is finite. We can then define the moduli space $I_k(X/D,\beta)$ by the parametrization of stable, predeformable, ideal sheaves $\mathscr{I}$ on degenerations $X[\ell]$ relative to D_l satisfying $\chi_{\mathrm{hol}}(\mathcal{O}_Y) = k$ and $\pi_*[Y] = \beta \in H_2(X,\mathbb{Z})$ with $\pi\colon X[\ell] \to X$ denoting the canonical stabilization map. In particular, $I_k(X/D)$ is a complete Deligne–Mumford stack together with a canonical perfect obstruction theory.

where $\eta^\vee$ denotes the dual[56] partition to the weighted partition η and $\ell(\eta)$ is the length of the partition η. We can define the *reduced* relative DT partition function by

$$\widetilde{\mathsf{Z}}^{\mathrm{DT}}_{X/D}\left(q,\prod_{i=1}^n\tilde{\tau}_{a_i}(\gamma_i)\right)_{\beta,\eta}:=\frac{\mathsf{Z}^{\mathrm{DT}}_{X/D}(q,\prod_{i=1}^n\tilde{\tau}_{a_i}(\gamma_i))_{\beta,\eta}}{\mathsf{Z}^{\mathrm{DT}}_{X/D}(q)_0}. \tag{9.5.7}$$

Again, for simplicity, we restrict the relative GW/DT correspondence to primary fields.

Conjecture 9.5.3 (Maulik–Nekrasov–Okounkov–Pandharipande[Mau+06b])**.** Considering the change-of-variables $\exp(\mathrm{i}u)=-q$, we have

$$(-\mathrm{i}u)^{d+\ell(\eta)-|\eta|}\widetilde{\mathsf{Z}}^{\mathrm{GW}}_{X/D}\left(u,\prod_{i=1}^n\tau_0(\gamma_i)\right)_{\beta,\eta}=(-q)^{-d/2}\widetilde{\mathsf{Z}}^{\mathrm{DT}}_{X/D}\left(q,\prod_{i=1}^n\tilde{\tau}_0(\gamma_i)\right)_{\beta,\eta},$$

where $d:=\int_\beta c_1(X)$.

9.6. **GW invariants on (graded) supermanifolds.** Recall that in the BV-BFV formalism we consider the space of boundary fields given by the BFV space $(\mathcal{F}^\partial_{\partial M},\omega^\partial_{\partial M})$ associated to a smooth variety M, which is a graded symplectic supermanifold. It would be interesting to understand the GW invariants on (graded) supermanifolds. Recently, an approach to this aim has been done in [KSY20] for *super Riemann surfaces*, i.e. a complex supermanifold M of dimension $1\mid 1$ endowed with a holomorphic distribution $\mathcal{D}\subset TM$ of rank $0\mid 1$ such that $\mathcal{D}\otimes\mathcal{D}\cong TM/\mathcal{D}$ (the isomorphism should be induced from the commutator of vector fields). Denote by I the almost complex structure on TM. They define the the notion of *super J-holomorphic curve* (recall Section 3.7 for the case of usual J-holomorphic curves) to be a map $\Phi\colon M\to N$ for some almost Kähler manifold (N,ω,J), if the differential $\bar{D}_J\Phi:=\frac{1}{2}(1+I\otimes J)\mathrm{d}\Phi\Big|_{\mathcal{D}}\in\Gamma(\mathcal{D}^*\otimes\Phi^*TN)$ vanishes. It was shown in [KSY20] that under certain assumptions the moduli space of super J-holomorphic curves $\Phi\colon M\to N$, denoted by $\mathcal{M}^{\mathrm{super}}([\Phi],J)$ is a smooth supermanifold of dimension

$$2n(1-g)+2\int_C c_1(N)\;\Big|\;2\int_C c_1(N),$$

where g denotes the topological genus of M and $C\in H_2(N)$ denotes the homology class of the image of the reduction $\Phi_{\mathrm{red}}\colon M_{\mathrm{red}}\to N$ (see [KSY20] for a detailed discussion). Using the construction of GW invariants through the moduli space of stable curves, it is expected that also there we would be able to obtain a smooth supermanifold structure for some non-singular projective variety X by using similar methods.

Appendix A. Equivariant localization

In this appendix we want to recall the most important notions on equivariant localization techniques [Sza00; BV82; BV83b], notions of equivariant (co)homolgy according to [AB84] and how they fit into the field theory setting, especially to the BV formalism [Ner93; Sza00].

A.1. **The ABBV method.** A way of treating (finite-dimensional) path integrals in QFT, similarly as through the saddle point approximation, is given by a construction that uses the symplectic structure of the underlying manifold, which is known today as the Atiyah–Bott–Berline–Vergne (ABBV) equivariant localization [AB84; BV82; BV83b] based on the

[56] We define the dual partition by taking the Poincaré dual of the cohomology weights.

Duistermaat–Heckman theorem [DH82] for symplectic manifolds (see also [Sza00]). Formally, as already seen, we describe the partition function as a functional integral of the form

$$\mathsf{Z}(\hbar)=\int_{\mathcal{M}}\frac{\omega^n}{n!}\exp(\mathrm{i}\Theta/\hbar),$$

where we consider a Hamiltonian G-space $(\mathcal{M},\omega,\Theta,G)$. Assume moreover that the Hamiltonian Θ is a Morse function as in Section 3.1 and let v_Θ be the Hamiltonian vector field of Θ. Using the *stationary phase expansion formula*, we can expand Z around critical points of Θ given by

$$\mathsf{Z}(\hbar)\sim(2\pi\mathrm{i}\hbar)^n\sum_{\substack{p\in\mathcal{M}\\ p\text{ critical point of }\Theta}}(-\mathrm{i})^{\mathrm{ind}_p\Theta}\exp(\mathrm{i}\Theta(p)/\hbar)\sqrt{\frac{\det\omega(p)}{\det\partial^2\Theta(p)}}+\mathrm{O}(\hbar^{n+1}),$$

where $\partial^2\Theta(x)=\left(\frac{\partial^2\Theta}{\partial x^i\partial x^j}(x)\right)$ denotes the Hessian of Θ. In [DH82], Duistermaat and Heckman found a class of Hamiltonian G-spaces for which the $\mathrm{O}(\hbar^{n+1})$ term vanishes in the above formula. Suppose that $\mathcal{M}$ is a compact symplectic manifold of dimension $2n$ endowed with a Riemannian metric and suppose that v_Θ, the Hamiltonian vector field of Θ, generates the global Hamiltonian action of a torus $\mathbb{T}$ on $\mathcal{M}$ (we can also work with the circle S^1 for simplicity) and is the Killing vector for the metric. The simplicity of considering the circle action implies that $\omega+\Theta$ is the equivariant extension of the symplectic form, i.e. it is closed with respect to the equivariant differential $\mathrm{d}_{v_\Theta}:=\mathrm{d}+\iota_{v_\Theta}$. Then the partition function can be written by

$$\mathsf{Z}(\hbar)=\int_{\mathcal{M}}\alpha(\hbar),$$

where

$$\alpha(\hbar):=(-\mathrm{i}\hbar)^n\exp(\mathrm{i}(\omega+\Theta)/\hbar)=(-\mathrm{i}\hbar)^n\exp(\mathrm{i}\Theta/\hbar)\sum_{k=0}^n\left(\frac{\mathrm{i}}{\hbar}\right)^k\frac{\omega^k}{k!}.$$

Note that since $\omega+\Theta$ is equivariantly closed, we have $\mathrm{d}_{v_\Theta}\alpha=0$. This allows us to apply the Berline–Vergne localization formula [BV82; BV83b] which gives

$$\int_{\mathcal{M}}\alpha(\hbar)=(2\pi\mathrm{i}\hbar)^n\sum_{p\text{ critical point of }\Theta}\frac{\exp(\mathrm{i}\Theta(p)/\hbar)}{\mathrm{Pfaff}\,\mathrm{d}V(p)},\tag{A.1.1}$$

where $\mathrm{d}V(p):=\omega^{-1}(p)\partial^2\Theta(p)$. Consider now the action of $\mathbb{C}^\times$ on some complex manifold Σ and denote by $\Sigma^{\mathbb{C}^\times}$ the set of isolated fixed points. Moreover, assume that the action of $\mathrm{U}(1)\subset\mathbb{C}^\times$ is generated by some Hamiltonian Θ with respect to a symplectic form ω. Then the Atiyah–Bott–Duistermaat–Heckman equivariant localization formula is given by

$$\int_\Sigma\exp(\omega-2\pi\xi\Theta)=\sum_{u\in\Sigma^{\mathbb{C}^\times}}\frac{\exp(-2\pi\xi\Theta(u))}{\det\xi|_{T_u\Sigma}}.\tag{A.1.2}$$

Here ξ is considered to be an element of the Lie algebra of $\mathbb{C}^\times$ given by $\mathbb{C}$, so it can act in the complex tangent space $T_u\Sigma$ to some fixed point $u\in\Sigma$. An important remark is that (A.1.2) can also work for non-compact manifolds Σ.

A.2. **Relation to the BV-BFV formalism.** We have seen in Section 4.3 that in the BFV formalism we are considering a $\mathbb{Z}$-graded supermanifold $\mathcal{F}^\partial$ together with a symplectic form ω^∂ of degree 0, thus we have a usual symplectic manifold $(\mathcal{F}^\partial, \omega^\partial)$. Hence, we can use the equivariant localization construction without corrections. In the case of the BV formalism the symplectic structure ω is odd of degree -1 inducing the anti-bracket $(\ ,\ )$ which requires the equivariant localization construction to be adapted to this case. We refer to [Ner93; Sza00] for such a BV formulation.

Appendix B. Configuration spaces on manifolds with boundary

In this appendix we want to recall some of the most important notions on configuration spaces on manifolds with boundary and their compactification. We refer to [Cam+18; BT94; Bot96; Kon93] for an excellent introduction on this subject.

B.1. **Open configuration spaces.** Let Σ be a closed d-manifold. Denote the open configuration space of n points in Σ by

$$\mathrm{Conf}_n(\Sigma) := \{(u_1, \dots, u_n) \in \Sigma^n \mid u_i \neq u_j, \text{ for } i \neq j\}. \tag{B.1.1}$$

This is a manifold with corners of dimension $d \cdot n$. If Σ has boundary, we consider can consider the configuration space of n points in the bulk and m points on the boundary given by

(B.1.2)

$$\begin{aligned}\mathrm{Conf}_{n,m}(\Sigma) := \{(u_1, \dots, u_n, \mathbb{u}_1, \dots, \mathbb{u}_m) \in \Sigma^n \times (\partial\Sigma)^m \mid u_i \neq u_j, \text{ for } i \neq j \text{ with } 1 \leq i, j \leq n \\ \text{and } \mathbb{u}_\ell \neq \mathbb{u}_k, \text{ for } \ell \neq k \text{ with } 1 \leq \ell, k \leq m\}.\end{aligned}$$

Moreover, we have

$$\dim \mathrm{Conf}_{n,m}(\Sigma) = d \cdot n + (d-1) \cdot m.$$

B.2. **Local formulation.** A special case is when we consider the configuration of points locally on $\mathbb{R}^d$. In particular, there is a symmetry group acting on $\mathrm{Conf}_n(\mathbb{R}^d)$, which is given by the $(d+1)$-dimensional Lie group $G^{(d+1)}$ consisting of scaling and translation, i.e. $u \mapsto au + b$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$. The quotient $\mathrm{C}_n(\mathbb{R}^d) := \mathrm{Conf}_n(\mathbb{R}^d)/G^{(d+1)}$ is then a manifold of dimension $d \cdot n + (d-1) \cdot m - (d+1)$. For the case with boundary, i.e. when considering the d-dimensional upper half-space $\mathbb{H}^d = \{(u_1, \dots, u_d) \in \mathbb{R}^d \mid u_d \geq 0\}$, one can see that there is a d-dimensional Lie group $G^{(d)}$ acting on $\mathrm{Conf}_{n,m}(\mathbb{H}^d)$ also by scaling and translation, i.e. $u \mapsto au + b$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^{d-1}$. The quotient $\mathrm{C}_{n,m}(\mathbb{H}^d) := \mathrm{Conf}_{n,m}(\mathbb{H}^d)/G^{(d)}$ is then a manifold of dimension $d \cdot n + (d-1) \cdot m - d$.

B.3. **Compactification.** There is a natural way of compactifying the open configuration space. The compactification construction was first formulated for algebraic varieties by Fulton–MacPherson [FM94b] and later adapted to the smooth setting of manifolds by Axelrod–Singer [AS91; AS94]. This compactification is often called *FMAS compactification* (see also [Sin04] for an introduction for the approach on manifolds). Let us give some ideas of the Fulton–MacPherson construction. Let S be a finite set and consider the space $\mathrm{Map}(S, \Sigma)$ of maps from S to Σ. Moreover, consider the smooth blow up $\mathrm{B}\ell(\mathrm{Map}(S, \Sigma), \Delta_S)$, where Δ_S denotes the diagonal $\Delta_S \subset \mathrm{Map}(S, \Sigma)$, consisting of constant maps $S \to \Sigma$. Denote by $\mathrm{Conf}_S(\Sigma)$ the space of embeddings of S into Σ. One can then observe that for every inclusion $K \subset S$ there are natural projections $\mathrm{Map}(S, \Sigma) \to \mathrm{Map}(K, \Sigma)$ and corresponding arrows $\mathrm{Conf}_S(\Sigma) \to \mathrm{Conf}_K(\Sigma)$ by restriction of maps from S to K as a functorial approach. Further, one can show that the inclusions $\mathrm{Conf}_S(\Sigma) \subset \mathrm{Map}(S, \Sigma)$ can be lifted to inclusions

$\mathrm{Conf}_S(\Sigma) \subset \mathrm{B}\ell(\mathrm{Map}(S,\Sigma),\Delta_S)$ since these sets avoid all diagonals. Thus, for a finite set X, we have a canonical inclusion

$$\mathrm{Conf}_X(\Sigma) \hookrightarrow \bigotimes_{\substack{S\subset X\\ |S|\geq 2}} \mathrm{B}\ell(\mathrm{Map}(S,\Sigma),\Delta_S)\times \mathrm{Map}(S,\Sigma).$$

The Fulton–MacPherson compactification, denoted by $\overline{\mathrm{Conf}_X(\Sigma)}$, is then defined as the closure of $\mathrm{Conf}_X(\Sigma)$ in this embedding. It turns out that the compactified configuration space is a *manifold with corners* and comes with equivariant functorial properties under embeddings and that the propagators do indeed extend smoothly to this compactification in certain important cases, e.g. when $\Sigma=\mathbb{R}^3$, and $\mathscr{P}_{ij}\colon \mathrm{C}_n(\mathbb{R}^3)\to S^2$, $\mathscr{P}_{ij}(x_1,\dots,x_n) := \frac{x_j-x_i}{\|x_j-x_i\|}$ for $1\leq i,j\leq n$, such an extensions holds, which is important for different aspects in the theory of Vassiliev's knot invariants arising from configuration space integrals [Kon93; Kon94b; Bot96; BC98].

Remark B.3.1 (Graphs)**.** If we consider a graph Γ in Σ, i.e. a graph whose vertex set $V(\Gamma)$ is contained in Σ, we will write $\mathrm{Conf}_\Gamma(\Sigma) := \mathrm{Conf}_{V(\Gamma)}(\Sigma)$ for the configuration space of points in Σ which are vertices of Γ. Moreover, if $V(\Gamma)=V_b(\Gamma)\cup V_\partial(\Gamma)$ with $V_b(\Gamma)$ the set of vertices of Γ lying in the bulk and $V_\partial(\Gamma)$ de set of vertices of Γ lying on the boundary, we have $\mathrm{Conf}_\Gamma(\Sigma) := \mathrm{Conf}_{|V_b(\Gamma)|,|V_\partial(\Gamma)|}(\Sigma)$.

B.4. **Boundary structure.** Since $\overline{\mathrm{Conf}_{n,m}(\Sigma)}$ is a manifold with corners, we can consider its boundary. The boundary is given by the collapsing of points in different situations, which we call *boundary strata*. In particular, since we have a manifold with corners, the boundary of the configuration space will consist of two different types of strata:

- (Strata of type S1) These are strata where $i\geq 2$ points in the bulk collapse to a point in the bulk. Elements of such a stratum can be described as points in $\overline{\mathrm{Conf}_i(\Sigma)}\times \overline{\mathrm{C}_{n-i+1,m}(\mathbb{H}^4)}$,
- (Strata of type S2) These are strata where $i>0$ points in the bulk and $j>0$ points on the boundary with $2i+j-2\geq 0$ collapse to a point on the boundary. Elements of such a stratum can be described as points in $\overline{\mathrm{Conf}_{i,j}(\Sigma)}\times \overline{\mathrm{C}_{n-i,m-j+1}(\mathbb{H}^4)}$.

References

[AB83] M. F. Atiyah and R. Bott. "The Yang-Mills equations over Riemann surfaces". In: *Phil. Trans. R. Soc. London* 308 (1983), pp. 523–615.

[AB84] M. F. Atiyah and R. Bott. "The moment map and equivariant cohomology". In: *Topology* 23 (1984), pp. 1–28.

[AB96] D. M. Austin and P. J. Braam. "Equivariant Floer Theory and Gluing Donaldson Polynomials". In: *Topology* 35.1 (1996), pp. 167–200.

[ACG11] E. Arbarello, M. Cornalba, and P. A. Griffiths. "Nodal Curves". In: *Geometry of Algebraic Curves*. Grundlehren der mathematischen Wissenschaft 268 (2011), pp. 79–166.

[AHS77] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. "Deformations of instantons". In: *PNAS* 47.7 (1977), pp. 2662–2663.

[AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. "Self-duality in four-dimensional Riemannian geometry". In: *Proceedings of the Royal Society A* 362.1711 (1978), pp. 425–461.

[Ale+97] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky. "The geometry of the master equation and topological quantum field theory". In: *Internat. J. Modern Phys. A* 12.7 (1997), pp. 1405–1429.

[APS76] M. F. Atiyah, V. Patodi, and I. M. Singer. "Sepctral asymmetry and Riemannian geometry I, II, III". In: *Math. Proc. Cambridge Philos. Soc.* 78,79 (1975,1976), pp. 405–432, 71–99.

[AS91] S. Axelrod and I. M. Singer. "Chern-Simons perturbation theory". In: *Differential geometric methods in theoretical physics, Proceedings, New York*. Vol. 1. 1991, pp. 3–45.

[AS94] S. Axelrod and I. M. Singer. "Chern-Simons perturbation theory. II". In: *J. Differ. Geom.* 39.1 (1994), pp. 173–213.

[Ati+78] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Y. I. Manin. "Construction of Instantons". In: *Phys. Lett A* 65 (1978), pp. 185–187.

[Ati87] M. F. Atiyah. "New invariants of three-dimensional and four-dimensional manifolds". In: *The Mathematical Heritage of Hermann Weyl, Proceedings*. 1987, pp. 285–299.

[Ati88] M. F. Atiyah. "Topological quantum field theories". In: *Publ.math. IHÉS* 68.1 (1988), pp. 175–186.

[Aur14] D. Auroux. *A Beginner's Introduction to Fukaya Categories*. Vol. 26. Contact and Symplectic Topology, Bolyai Society Mathematical Studies. Springer, Cham., 2014, pp. 85–136.

[BBJ19] C. Brav, V. Bussi, and D. Joyce. "A Darboux theorem for derived schemes with shifted symplectic structure". In: *J. Amer. Math. Soc.* 32 (2019), pp. 399–443.

[BC98] R. Bott and A. S. Cattaneo. "Integral invariants of 3-manifolds". In: *J. Differ. Geom.* 48.1 (1998), pp. 91–133.

[BCM12] F. Bonechi, A. S. Cattaneo, and P. Mnev. "The Poisson sigma model on closed surfaces". In: *J. High Energy Phys.* 1 (2012), pp. 099, 26.

[BD95a] J. C. Baez and J. Dolan. "Higher dimensional algebra and topological quantum field theory". In: *J. Math. Phys.* 36 (1995), pp. 6073–6105.

[BD95b] P. J. Braam and S. K. Donaldson. *Fukaya-Floer homology and gluing formulae for polynomial invariants*. Vol. 133. The Floer Memorial Volume. Progress in Mathematics. Birkhäuser Basel, 1995, pp. 257–281.

[Beh09] K. Behrend. "Donaldson-Thomas type invariants via microlocal geometry". In: *Ann. of Math.* 170.3 (2009), pp. 1307–1338.

[Beh97] K. Behrend. "Gromov-Witten invariants in algebraic geometry". In: *Invent. Math.* 127 (1997), pp. 601–617.

[Ben+15] O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce. "A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications". In: *Geom. Topol.* 19.3 (2015), pp. 1287–1359.

[BF83] I. A. Batalin and E. S. Fradkin. "A generalized canonical formalism and quantization of reducible gauge theories". In: *Phys. Lett. B* 122.2 (1983), pp. 157–164.

[BF86] I. A. Batalin and E. S. Fradkin. "Operator quantization and abelization of dynamical systems subject to first-class constraints". In: *La Rivista Del Nuovo Cimento Series 3* 9.10 (1986), pp. 1–48.

[BF97] K. Behrend and B. Fantechi. "The intrinsic normal cone". In: *Invent. Math.* 128 (1997), pp. 45–88.

[BN19] G. Borot and P. Norbury. “Loop Equations for Gromov-Witten Invariant of $\mathbb{P}^1$”. In: *SIGMA* 061 (2019), p. 29.

[Bon+20] F. Bonechi, A. S. Cattaneo, J. Qiu, and M. Zabzine. “Equivariant Batalin-Vilkovisky formalism”. In: *J. Geom. Phys.* 154 (2020).

[Bor+21] G. Borot, V. Bouchard, N. K. Chidambaram, and T. Creutzig. *Whittaker vectors for $\mathcal{W}$-algebras from topological recursion*. 2021. arXiv: 2104.04516.

[Bot10] R. Bott. “Some Aspects of Invariant Theory in Differential Geometry”. In: *Differential Operators on Manifolds*. Springer Berlin Heidelberg, 2010, pp. 49–145.

[Bot96] R. Bott. “Configuration spaces and imbedding invariants”. In: *Turkish Journal of Mathematics* 20 (1996), pp. 1–17.

[Bra91] P. J. Braam. “Floer homology groups for homology three-spheres”. In: *Adv. Math.* 88.2 (1991), pp. 131–144.

[BRS74] C. Becchi, A. Rouet, and R. Stora. “The abelian Higgs Kibble model, unitarity of the S-operator”. In: *Phys. Lett. B* 52.3 (1974), pp. 344–346.

[BRS75] C. Becchi, A. Rouet, and R. Stora. “Renormalization of the abelian Higgs-Kibble model”. In: *Commun. Math. Phys.* 42.2 (1975), pp. 127–162.

[BS89] L. Baulieu and I. M. Singer. “The topological sigma model”. In: *Commun. Math. Phys.* 125.2 (1989), pp. 227–237.

[BT94] R. Bott and C. Taubes. “On the self-linking of knots”. In: *J. Math. Phys.* 35 (1994), pp. 5247–5287.

[BV77] I. A. Batalin and G. A. Vilkovisky. “Relativistic S-matrix of dynamical systems with boson and fermion constraints”. In: *Phys. Lett. B* 69.3 (1977), pp. 309–312.

[BV81] I. A. Batalin and G. A. Vilkovisky. “Gauge algebra and quantization”. In: *Phys. Lett. B* 102.1 (1981), pp. 27–31.

[BV82] N. Berline and M. Vergne. “Classes Characteristiques Equivariantes”. In: *C. R. Acad. Sci. Paris* 295 (1982), p. 513.

[BV83a] I. A. Batalin and G. A. Vilkovisky. “Quantization of gauge theories with linearly dependent generators”. In: *Phys. Rev. D* 28.10 (1983), pp. 2567–2582.

[BV83b] N. Berline and M. Vergne. “Zeros d’un Champ de Vecteurs et Classes Characteristiques Equivariantes”. In: *Duke Math. J.* 50.2 (1983), pp. 513–549.

[BW12] S. Bates and A. Weinstein. *Lectures on the Geometry of Quantization*. University Reprints, 2012.

[Cal+17] D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. “Shifted Poisson structures and deformation quantization”. In: *J. Topol.* 10.2 (2017), pp. 483–584.

[Cal15] D. Calaque. “Lagrangian structures on mapping stacks and semi-classical TFTs”. In: *Stacks and Categories in Geometry, Topology, and Algebra, Contemp. Math., Amer. Math. Soc., Providence, RI* 643 (2015), pp. 1–23.

[Cam+18] R. Campos, N. Idrissi, P. Lambrechts, and T. Willwacher. *Configuration Spaces on Manifolds with Boundary*. 2018. arXiv: 1802.00716.

[CF00] A. S. Cattaneo and G. Felder. “A path integral approach to the Kontsevich quantization formula”. In: *Commun. Math. Phys.* 212 (2000), pp. 591–611.

[CF01] A. S. Cattaneo and G. Felder. “On the AKSZ formulation of the Poisson sigma model”. In: *Lett. Math. Phys.* 56.2 (2001), pp. 163–179.

[CF10] A. S. Cattaneo and G. Felder. “Effective Batalin–Vilkovisky Theories, Equivariant Configuration Spaces and Cyclic Chains”. In: *Higher Structures in Geometry and Physics* 287 (2010), pp. 111–137.

[CG97] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry.* Modern Birkhäuser Classics. Birkhäuser, 1997.

[Che52] S.-S. Chern. “Differential geometry of fiber bundles”. In: *Amer. Math. Soc., Providence, R. I.* 2 (1952), pp. 397–411.

[CHS] D. Calaque, R. Haugseng, and C. Scheimbauer. *In preparation.*

[CM08] A. S. Cattaneo and P. Mnev. “Remarks on Chern-Simons Invariants”. In: *Commun. Math. Phys.* 293 (2008), pp. 803–836.

[CM20] A. S. Cattaneo and N. Moshayedi. “Introduction to the BV-BFV formalism”. In: *Rev. Math. Phys.* 32 (2020), p. 67.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Classical BV Theories on Manifolds with Boundary”. In: *Commun. Math. Phys.* 332.2 (2014), pp. 535–603.

[CMR17] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. “Perturbative Quantum Gauge Theories on Manifolds with Boundary”. In: *Commun. Math. Phys.* 357.2 (2017), pp. 631–730.

[CMW19] A. S. Cattaneo, N. Moshayedi, and K. Wernli. “Globalization for Perturbative Quantization of Nonlinear Split AKSZ Sigma Models on Manifolds with Boundary”. In: *Commun. Math. Phys.* 372.1 (2019), pp. 213–260.

[CMW20] A. S. Cattaneo, N. Moshayedi, and K. Wernli. “On the Globalization of the Poisson Sigma Model in the BV-BFV Formalism”. In: *Commun. Math. Phys.* 375.1 (2020), pp. 41–103.

[CO12] E. Carlsson and A. Okounkov. “Exts and vertex operators”. In: *Duke Math. J.* 161.9 (2012), pp. 1797–1815.

[Cos11] K. Costello. *Renormalization and Effective Field Theory.* Vol. 170. Mathematical Surveys and Monographs. American Mathematical Society (AMS), 2011.

[CS74] S.-S. Chern and J. Simons. “Characteristic Forms and Geometric Invariants”. In: *Ann. of Math. (2)* 99.1 (1974), pp. 48–69.

[DF17] A. Daemi and K. Fukaya. *Atiyah-Floer Conjecture: a Formulation, a Strategy to Prove and Generalizations.* 2017. arXiv: 1707.03924.

[DH82] J. J. Duistermaat and G. Heckman. “On the variation in the cohomology of the symplectic form of the reduced phase space”. In: *Invent. Math.* 69 (1982), pp. 259–268.

[Don02] S. K. Donaldson. *Floer Homology Groups in Yang-Mills Theory.* Cambridge Tracts in Mathematics. Cambridge University Press, 2002.

[Don83] S. K. Donaldson. “An Application of Gauge Theory to Four Dimensional Topology”. In: *J. Differ. Geom.* 18.2 (1983), pp. 279–315.

[Don84] S. K. Donaldson. “Instantons and Geometric Invariant Theory”. In: *Commun. Math. Phys.* 93.4 (1984), pp. 453–460.

[Don90] S. K. Donaldson. “Polynomial invariants for smooth four-manifolds”. In: *Topology* 29.3 (1990), pp. 257–315.

[DT98] S. K. Donaldson and R. P. Thomas. “Gauge theory in higher dimensions”. In: *The geometric universe (Oxford, 1996), Oxford Univ. Press* (1998), pp. 31–47.

[Dun+14] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz. “Identification of the Givental formula with the spectral curve topological recursion procedure”. In: *Commun. Math. Phys.* 328 (2014), pp. 669–700.

[EO09] B. Eynard and N. Orantin. "Topological recursion in enumerative geometry and random matrices". In: *J. Phys. A: Math. Theor.* 42 (2009), p. 117.

[Fed94] B. V. Fedosov. "A simple geometrical construction of deformation quantization". In: *J. Differ. Geom.* 40.2 (1994), pp. 213–238.

[FF78] E. S. Fradkin and T. E. Fradkina. "Quantization of relativistic systems with boson and fermion first- and second-class constraints". In: *Phys. Lett. B* 72.3 (1978), pp. 343–348.

[Flo88] A. Floer. "Morse theory for Lagrangian intersections". In: *J. Differ. Geom.* 28 (1988), pp. 513–547.

[Flo89a] A. Floer. "An instanton-invariant for 3-manifolds". In: *Commun. Math. Phys.* 118.2 (1989), pp. 215–240.

[Flo89b] A. Floer. "Witten's complex and infinite-dimensional Morse theory". In: *J. Differ. Geom.* 30 (1989), pp. 207–221.

[FM94a] R. Friedman and J. W. Morgan. *Smooth Four-Manifolds and Complex Surfaces.* Vol. 27. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer-Verlag Berlin Heidelberg, 1994, p. 522.

[FM94b] W. Fulton and R. MacPherson. "A compactification of configuration spaces". In: *Ann. of Math. (2),* 139.1 (1994), pp. 183–225.

[FP67] L. D. Faddeev and V. N. Popov. "Feynman diagrams for the Yang-Mills field". In: *Phys. Lett. B* 25.1 (1967), pp. 29–30.

[FP97] W. Fulton and R. Pandharipande. "Notes on stable maps and quantum cohomology". In: *Algebraic Geometry–Santa Cruz 1995, Part 2 of Proc. Sympos. Pure Math., Amer. Math. Soc.* 62 (1997), pp. 45–96.

[Fre82] M. H. Freedman. "The topology of four-dimensional manifold". In: *J. Differ. Geom.* 17 (1982), pp. 357–453.

[FU84] D. S. Freed and K. K. Uhlenbeck. *Instantons and four-manifolds.* M. S. R. I. publications, Springer, New York, 1984.

[Fuk+09a] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I.* Vol. 46. AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc. (AMS), 2009.

[Fuk+09b] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II.* Vol. 46. AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc. (AMS), 2009.

[Fuk93] K. Fukaya. *Morse Homotopy, A_∞-category, and Floer Homologies.* Vol. 94. Math. Sciences Research Inst. (MSRI), 1993.

[FV75] E. S. Fradkin and G. A. Vilkovisky. "Quantization of relativistic systems with constraints". In: *Phys. Lett. B* 55.2 (1975), pp. 224–226.

[Get19] E. Getzler. "Covariance in the Batalin–Vilkovisky formalism and the Maurer–Cartan equation for curved Lie algebras". In: *Lett. Math. Phys.* 109 (2019), pp. 187–224.

[Get94] E. Getzler. "Batalin-Vilkovisky algebras and two-dimensional topological field theories". In: *Commun. Math. Phys.* 159.2 (1994), pp. 265–285.

[GF69] I. M. Gelfand and D. B. Fuks. "The cohomology of the Lie algebra of vector fields on a smooth manifold". In: *J. Funct. Analysis* 33 (1969), pp. 194–210.

[GF70] I. M. Gelfand and D. B. Fuks. "The cohomology of the Lie algebra of formal vector fields". In: *Izv. AN SSR* 34 (1970), pp. 110–116.

[GK71] I. M. Gelfand and D. A. Kazhdan. “Some problems of the differential geometry and the calculation of cohomologies of Lie algebras of vector fields”. In: *Dokl. Akad. Nauk Ser. Fiz.* 200 (1971), pp. 269–272.

[Gro68] A. Grothendieck. “Crystals and the de Rham cohomology of schemes”. In: *Dix exposes sur la cohomologie des schemas* (1968), pp. 306–358.

[Gro85] M. Gromov. “Pseudo holomorphic curves in symplectic geometry”. In: *Invent. Math.* 82 (1985), pp. 307–347.

[Har10] R. Hartshorne. *Deformation Theory*. Vol. 257. Graduate Texts in Mathematics. Springer-Verlag New York, 2010.

[Hor+03] K. Hori et al. *Mirror Symmetry*. Vol. 1. Clay Mathematics Monographs. American Mathematical Society, Clay Mathematics Institute, 2003.

[HZ94] H. Hofer and E. Zehnder. *The Arnold conjecture, Floer homology and symplectic homology*. Symplectic Invariants and Hamiltonian Dynamics. Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser Basel, 1994, pp. 193–271.

[Ike11] N. Ikeda. *Donaldson Invariants and Their Generalizations from AKSZ Topological Field Theories*. 2011. arXiv: 1104.2100.

[IM19] R. Iraso and P. Mnev. “Two-Dimensional Yang-Mills Theory on Surfaces with Corners in Batalin-Vilkovisky Formalism”. In: *Commun. Math. Phys.* 370 (2019), pp. 637–702.

[Khu04] H. M. Khudaverdian. “Semidensities on Odd Symplectic Supermanifolds”. In: *Commun. Math. Phys.* 247.2 (2004), pp. 353–390.

[KM94] M. Kontsevich and Y. Manin. “Gromov-Witten classes, quantum cohomology, and enumerative geometry”. In: *Commun. Math. Phys.* 164.3 (1994), pp. 525–562.

[Kon03] M. Kontsevich. “Deformation quantization of Poisson manifolds”. In: *Lett. Math. Phys.* 66.3 (2003), pp. 157–216.

[Kon92] M. Kontsevich. “Intersection theory on the moduli space of curves and the matrix Airy function”. In: *Commun. Math. Phys.* 147.1 (1992), pp. 1–23.

[Kon93] M. Kontsevich. “Vassiliev’s Knot Invariants”. In: *Advances in Soviet Mathematics* 16 (1993).

[Kon94a] M. Kontsevich. *Homological Algebra of Mirror Symmetry*. 1994. arXiv: alg-geom/9411018.

[Kon94b] M. Kontsevich. “Feynman Diagrams and Low-Dimensional Topology”. English. In: *First European Congress of Mathematics Paris, July 6–10, 1992*. Ed. by A. Joseph, F. Mignot, F. Murat, B. Prum, and R. Rentschler. Vol. 120. Progress in Mathematics. Birkhäuser Basel, 1994, pp. 97–121.

[Kon95] M. Kontsevich. “Enumeration of Rational Curves Via Torus Actions”. In: *The Moduli Space of Curves. Progress in Mathematics* 129 (1995), pp. 335–368.

[KS08a] M. Kontsevich and Y. Soibelman. “Notes on A_∞-Algebras, A_∞-Categories and Non-Commutative Geometry”. In: *Homological Mirror Symmetry*. Lecture Notes in Physics 757 (2008), pp. 1–67.

[KS08b] M. Kontsevich and Y. Soibelman. *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*. 2008. arXiv: 0811.2435.

[KSY20] E. Keßler, A. Sheshmani, and S.-T. Yau. *Super quantum cohomology I: Super stable maps of genus zero with Neveu-Schwarz punctures*. 2020. arXiv: 2010.15634.

[KT79] J. Kijowski and W. M. Tulczyjew. *A Symplectic Framework for Field Theories*. Springer Lecture Notes in Physics, 1979.

[Lur09] J. Lurie. “On the classification of topological field theories”. In: *Current Developments in Mathematics* 2008 (2009), pp. 129–280.

[LW15] J. Li and B. Wu. “Good degeneration of Quot-schemes and coherent systems”. In: *Comm. Anal. Geom.* 23.4 (2015), pp. 841–921.

[Mau+06a] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. “Gromov-Witten theory and Donaldson-Thomas theory. I”. In: *Compos. Math.* 142.5 (2006), pp. 1263–1285.

[Mau+06b] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. “Gromov-Witten theory and Donaldson-Thomas theory. II”. In: *Compos. Math.* 142.5 (2006), pp. 1286–1304.

[Mau+11] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande. “Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds”. In: *Invent. Math.* 186.2 (2011), pp. 435–479.

[Mir07] M. Mirzakhani. “Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces”. In: *Invent. Math.* 167 (2007), pp. 179–222.

[Mne19] P. Mnev. *Quantum Field Theory: Batalin–Vilkovisky Formalism and Its Applications*. Vol. 72. University Lecture Series. American Mathematical Society (AMS), 2019, p. 192.

[Mos19] N. Moshayedi. *On Globalized Traces for the Poisson Sigma Model*. 2019. arXiv: 1912.02435.

[Mos20a] N. Moshayedi. *Notes on Geometric Quantization*. 2020. arXiv: 2010.15419.

[Mos20b] N. Moshayedi. “Formal Global AKSZ Gauge Observables and Generalized Wilson Surfaces”. In: *Ann. Henri Poincaré* 21 (2020), pp. 2951–2995.

[Mos21] N. Moshayedi. “On Quantum Obstruction Spaces and Higher Codimension Gauge Theories”. In: *Phys. Lett. B* 815 (2021).

[Nak99] H. Nakajima. *Lectures on Hilbert Scheme of Points on Surfaces*. Vol. 18. University Lectures Series. AMS, 1999.

[Nek03] N. Nekrasov. “Seiberg-Witten prepotential from instanton counting”. In: *Adv. Theor. Math. Phys.* 7.5 (2003), pp. 831–864.

[Ner93] A. Neressian. “Antibrackets and Localization of (Path) Integrals”. In: *JETP Lett.* 58 (1993), pp. 66–70.

[Nic00] L. I. Nicolaescu. *Notes on Seiberg–Witten Theory*. Vol. 28. Graduate Studies in Mathematics. Amer. Math. Soc., 2000.

[NO06] N. Nekrasov and A. Okounkov. *Seiberg-Witten Theory and Random Partitions*. Vol. 244. The Unity of Mathematics, Progress in Mathematics. Birkhäuser Boston, 2006, pp. 525–596.

[Oko06] A. Okounkov. *Random partitions and instanton counting*. 2006. arXiv: math-ph/0601062.

[Oko19] A. Okounkov. “Takagi Lectures on Donaldson–Thomas theory”. In: *Jpn. J. Math.* 14 (2019), pp. 67–133.

[ORV06] A. Okounkov, N. Reshetikhin, and C. Vafa. “Quantum Calabi-Yau and Classical Crystals”. In: *The Unity of Mathematics*. Progress in Mathematics 244 (2006), pp. 597–618.

[Pan+13] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. “Shifted symplectic structures”. In: *Publications mathématiques de l’IHÉS* 117 (2013), pp. 271–328.

[Pan99] R. Pandharipande. “Intersection of $\mathbf{Q}$-divisors on Kontsevich’s moduli space $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry”. In: *Trans. Amer. Math. Soc.* 351 (1999), pp. 1481–1505.

[Pes12] V. Pestun. "Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops". In: *Commun. Math. Phys.* 313.71–129 (2012), pp. 785–816.

[PT14] R. Pandharipande and R. P. Thomas. "Almost closed 1-forms". In: *Glas. Math. J.* 56.1 (2014), pp. 169–182.

[Qiu+15] J. Qiu, L. Tizziano, J. Winding, and M. Zabzine. "Gluing Nekrasov Partition Functions". In: *Commun. Math. Phys.* 337 (2015), pp. 785–816.

[Rei35] K. Reidemeister. "Homotopieringe und Linsenräume". In: *Abh. Math. Sem. Univ. Hamburg* 11 (1935), pp. 102–109.

[RS71] D. B. Ray and I. M. Singer. "R-Torsion and the Laplacian on Riemannian manifolds". In: *Adv. Math.* 7.2 (1971), pp. 145–210.

[RT91] N. Y. Reshetikhin and V. G. Turaev. "Invariants of 3-manifolds via link polynomials and quantum groups". In: *Invent. Math.* 103.1 (1991), pp. 547–597.

[Rui00] S. Ruijsenaars. "On Barnes' multiple zeta and gamma functions". In: *Adv. Math.* 156.1 (2000), pp. 107–132.

[Saf17] P. Safronov. *Lectures on shifted Poisson geometry*. 2017. arXiv: 1709.07698.

[Saf20] P. Safronov. *Shifted geometric quantization*. 2020. arXiv: 2011.05730.

[Sal95] D. Salamon. "Lagrangian intersections, 3-manifolds with boundary and the Atiyah-Floer conjecture". In: *Proceedings of the International Congress of Mathematicians (Zürich 1994)*. Birkhäuser, Basel 1,2 (1995), pp. 526–536.

[Sav01] N. Saveliev. *Invariants for Homology 3-Spheres*. Vol. 1. Encyclopaedia of Mathematical Sciences. Springer-Verlag Berlin Heidelberg, 2001, pp. XII, 223.

[Sav99] N. Saveliev. *Lectures on the topology of 3-manifolds: An introduction to the Casson Invariant*. Graduate. de Gruyter, 1999.

[Sch93] A. Schwarz. "Geometry of Batalin-Vilkovisky quantization". In: *Commun. Math. Phys.* 155.2 (1993), pp. 249–260.

[Seg88] G. B. Segal. "The definition of conformal field theory". In: *Differential geometrical methods in theoretical physics*. Vol. 250. 1988, pp. 165–171.

[Šev06] P. Ševera. "On the Origin of the BV Operator on Odd Symplectic Supermanifolds". In: *Lett. Math. Phys.* 78.1 (2006), pp. 55–59.

[Sin04] D. P. Sinha. "Manifold-theoretic compactifications of configuration spaces". In: *Sel. Math.* 10.391 (2004).

[SW08] D. Salamon and K. Wehrheim. "Instanton Floer homology with Lagrangian boundary conditions". In: *Geom. Topol.* 12.2 (2008), pp. 747–918.

[SW94a] N. Seiberg and E. Witten. "Monopole Condensation, And Confinement In $N = 2$ Supersymmetric Yang-Mills Theory". In: *Nucl. Phys. B* 426 (1994), pp. 19–52.

[SW94b] N. Seiberg and E. Witten. "Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD". In: *Nucl. Phys. B* 431 (1994), pp. 484–550.

[Sza00] R. J. Szabo. *Equivariant Cohomology and Localization for Path Integrals*. Vol. 63. Lecture Notes in Physics Monographs. Springer-Verlag Berlin Heidelberg, 2000.

[Tau82] C. H. Taubes. "Self-dual connections on non-self-dual 4-manifolds". In: *J. Differ. Geom.* 17 (1982), pp. 139–170.

[Tau87] C. H. Taubes. "Gauge theory on asymptotically periodic 4-manifold". In: *J. Differ. Geom.* 25 (1987), pp. 363–430.

[Tau90] C. H. Taubes. "Casson's invariant and gauge theory". In: *J. Differ. Geom.* 31.2 (1990), pp. 547–599.

[Tho00] R. P. Thomas. "A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations". In: *J. Differ. Geom.* 54.2 (2000), pp. 367–438.

[Toë14] B. Toën. *Derived Algebraic Geometry*. 2014. arXiv: 1401.1044.

[Tyu76] I. V. Tyutin. "Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism". In: *Preprints of P.N. Lebedev Physical Institute, No. 39* (1976).

[Uhl82a] K. K. Uhlenbeck. "Connections with L^p bounds on curvature". In: *Commun. Math. Phys.* 83.1 (1982), pp. 31–42.

[Uhl82b] K. K. Uhlenbeck. "Removable Singularities in Yang-Mills Fields". In: *Commun. Math. Phys.* 83.1 (1982), pp. 11–29.

[Weh05a] K. Wehrheim. "Anti-self-dual instantons with Lagrangian boundary conditions. I. Elliptic theory". In: *Commun. Math. Phys.* 254.1 (2005), pp. 45–89.

[Weh05b] K. Wehrheim. "Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah-Floer conjecture". In: *J. Symplectic Geom.* 3.4 (2005), pp. 747–918.

[Wei49] A. Weil. "Géométrie différentielle des espaces fibres". In: *André Weil Oeuvres Scientifiques / Collected Papers, Springer* 1 (1949), pp. 422–436.

[Wit82] E. Witten. "Supersymmetry and Morse theory". In: *J. Differ. Geom.* 17.14 (1982), pp. 661–692.

[Wit88a] E. Witten. "Topological quantum field theory". In: *Commun. Math. Phys.* 117.3 (1988), pp. 353–386.

[Wit88b] E. Witten. "Topological sigma models". In: *Commun. Math. Phys.* 118.3 (1988), pp. 411–449.

[Wit89] E. Witten. "Quantum field theory and the Jones polynomial". In: *Commun. Math. Phys.* 121.3 (1989), pp. 351–399.

[Wit91] E. Witten. "Two-dimensional gravity and intersection theory on moduli space". In: *Surveys in differential geometry* (1991), pp. 243–310.

[Woo97] N. Woodhouse. *Geometric Quantization*. Oxford University Press (OUP), 1997.

[Yau92] S.-T. Yau. *Essays on mirror manifolds*. Hong Kong: International Press, 1992.

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Chapter 2

Formal Global Perturbative Quantization of the Rozansky–Witten Model in the BV-BFV Formalism

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Formal global perturbative quantization of the Rozansky–Witten model in the BV-BFV formalism

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We describe a globalization construction for the Rozansky–Witten model in the BV-BFV formalism for a source manifold with and without boundary in the classical and quantum case. We define an AKSZ sigma model, which, upon globalization through notions of formal geometry extended appropriately to our case, is shown to reduce to the Rozansky–Witten model. The relations with other relevant constructions in the literature are discussed. Moreover, we split the model as a BF-like theory and we construct a perturbative quantization of the model in the quantum BV-BFV framework. In this context, we are able to prove the modified differential Quantum Master Equation and the flatness of the quantum Grothendieck BFV operator. Additionally, we provide a construction of the BFV boundary operator in some cases.

1. Introduction

1.1. Overview and motivation

An important class of field theories in physics is represented by *gauge theories*. These are theories containing a redundant number of degrees of freedom which causes physical quantities to be invariant under certain local transformations, called *gauge symmetries*. Indeed the presence of gauge symmetries lead to challenging problems from the definition of path integral to the general problem of understanding the perturbative quantization of a gauge theory. Since the physical information about a classical field theory is encoded in the set of solutions of the Euler–Lagrange equations (the *critical locus*), a possible solution to deal with such problems is to consider the critical locus modulo the gauge symmetries. The fields are then constructed as functions on this quotient. However, this is not feasible since it turns out that these quotients are, in general, singular. Batalin and Vilkovisky introduced a method, which is known today as the *BV formalism* [10–12], that employs symplectic (co)homological tools [48] to treat these field theories, in particular it overcomes difficulties connected to the singularity of the quotient by taking homological resolution of the critical locus. A crucial observation in the BV formalism is also that gauge-fixing then corresponds to the choice of a Lagrangian submanifold. Another method developed around the same time is the *BFV formalism* by Batalin, Fradkin and Vilkovisky [8,9,36], which deals with gauge theories in the Hamiltonian setting, while the BV construction is formulated in the Lagrangian approach.

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Recently, the study of gauge theories on spacetime manifolds with boundary lead Cattaneo, Mnev and Reshetikhin [22,23] to relate these two formulations in order to develop the *BV-BFV formalism*. Their idea was that, under certain conditions, BV theories in the bulk can induce a BFV theory on the boundary. This approach was successfully applied to a large number of physical theories such as e.g. electrodynamics, Yang-Mills theory, scalar field theory and BF-theories [23]. In particular, the *AKSZ construction*, developed in [2], produces naturally a large variety of theories which satisfy automatically the BV-BFV axioms as it was shown in [23]. This is quite remarkable since many theories of interest are actually of AKSZ-type, such as e.g. Chern–Simons (CS) theory, BF-theory and the Poisson sigma model (PSM) [23].

In [24], a perturbative quantization scheme for gauge theories in the BV-BFV framework was introduced, which was called *quantum BV-BFV formalism*. The importance of this method relies on its compatibility with cutting and gluing in the sense of *topological quantum field theories* (TQFTs). The quantum BV-BFV formalism has been applied successfully in various physically relevant theories such as e.g. BF-theory and the PSM [24], split CS theory [25] and CS theory [18], the relational symplectic groupoid[1] [26] and 2D Yang–Mills theory on manifolds with corners [41,42].

An important effort has been spent to study TQFT within the quantum BV-BFV framework. Indeed, the method has been introduced to accomplish the goal of constructing perturbative topological invariants of manifolds with boundary compatible with cutting and gluing for topological field theories. During the years, two prominent TQFTs have been studied in detail: CS theory [4,5] in [25,73] and the PSM [40,67] in [28].

In [27] a globalized version of the (quantum) BV-BFV formalism in the context of nonlinear split AKSZ sigma models on manifolds with and without boundary by using methods of formal geometry à la Bott [16], Gelfand and Kazhdan [37] (see also [15] for an application of the globalization procedure for the PSM in the context of a closed source manifold) was developed. Their construction is able to detect changes of the quantum state when one modifies the constant map around which the perturbation is developed. This required them to formulate a "differential" version of the *(modified) Classical Master Equation* and the *(modified) Quantum Master Equation*, which are the two key equations in the BV(-BFV) formalism. As an example, this procedure was applied to the PSM on manifolds with boundary and extended to the case of corners in [28].

In this paper, we continue the effort in analyzing TQFTs within the quantum BV-BFV formalism by studying the *Rozansky–Witten (RW) theory*. The RW model is a topological sigma model with a source 3-dimensional manifold Σ_3, which was introduced by Rozansky and Witten in [63] through a *topological twist* of a 6-dimensional supersymmetric sigma model with target a hyperKähler manifold M. Of particular interest is the perturbative expansion of the RW partition function. Rozansky and Witten obtained this expansion as a combinatorial sum in terms of Feynman diagrams Γ, which are shown to be trivalent graphs:

$$Z_M(\Sigma_3) = \sum_{\Gamma} b_\Gamma(M) I_\Gamma(\Sigma_3), \tag{1.1.1}$$

the $b_\Gamma(M)$ are complex valued functions on trivalent graphs constructed from the target manifold, while $I_\Gamma(\Sigma_3)$ contains the integral over the propagators of the theory and depends on the source manifold. There are evidences which suggest that $I_\Gamma(\Sigma_3)$ are the *LMO invariants* of Le, Murakami and Ohtsuki [51]. On the other hand, Rozansky and Witten showed that $b_\Gamma(M)$ satisfy the famous AS (which is reflected in the absence of tadpoles diagrams) and IHX relations. As a result, $b_\Gamma(M)$ constitute the *Rozansky–Witten weight system* for the *graph homology*, the space of linear combinations of equivalence classes of trivalent graphs (modulo the AS and IHX relations). This means that the RW weights can be used to construct new finite type topological invariants for 3-dimensional manifolds [7].

The RW theory opened up a new branch of research which was undertaken by many mathematicians and physicists (e.g. [39,70]). Shortly after the original paper, Kontsevich understood that the RW invariants could be obtained by the characteristic classes of foliations and Gelfand-Fuks cohomology [49]. Inspired by the work of Kontsevich, Kapranov reformulated the weight system in cohomological terms (instead of using differential forms) in [44]. This idea relies on the fact that one can replace the Riemann curvature tensor by the Atiyah class [3], which is the obstruction to the existence of a global holomorphic connection. As a consequence of Kontsevich's and Kapranov's approaches, the RW weights were understood to be invariant under the hyperKähler metric on M: in fact, the model could be constructed more generally with target a holomorphic symplectic manifold. In this way, the RW weights were also called *RW invariants*[2] of M (see [66] for a detailed exposition). On the other hand, the possibility to consider as target manifold a holomorphic symplectic manifold was later interpreted in the context of topological sigma models by Rozansky and Witten in the appendix of [63].

In the last 20 years, the RW model has been the focus of intense research in order to formulate it as an extended TQFT (see [62,65]), in order to investigate its boundary conditions and defects [46,47], and in order to construct its globalization formulation [30,43,57].

1.2. Our contribution

The main contribution of this paper is to add the RW theory to the list of TQFTs which have been studied successfully within the globalized version of the quantum BV-BFV framework [27]. This will be a step towards the higher codimension

[1] The relational symplectic groupoid was first defined in [19].

[2] The terminology is unfortunate as in reality the proper invariants should be the products of the weights with $I_\Gamma(\Sigma_3)$.

quantization of RW theory, which will possibly lead to new insights towards the 3-dimensional correspondence between CS theory [74] and the Reshetikhin–Turaev construction [60] from the point of view of (perturbative) extended field theories described by Baez–Dolan [6] and Lurie [52]. Moreover this could also help in understanding (generalizations of a globalized version of the) Berezin–Toeplitz quantization (star product) [68] through field-theoretic methods using cutting and gluing similarly as it was done for Kontsevich's star product [50] in the case of the PSM in [28].

We construct the BV-BFV extension of an AKSZ model having a 3-dimensional manifold Σ_3 (possibly with boundary) as source and a holomorphic symplectic manifold M as target with holomorphic symplectic form Ω. Following [44], we define a formal holomorphic exponential map φ. This is used to linearize the space of fields of our model obtaining

$$\tilde{\mathcal{F}}_{\Sigma_3,x} = \Omega^\bullet(\Sigma_3)\otimes T^{1,0}_x M, \tag{1.2.1}$$

where $\Omega^\bullet(\Sigma_3)$ denotes the complex of de Rham forms on the source manifold and $T^{1,0}_x M$ is the holomorphic tangent space on the target. In order to vary the constant solution around which we perturb, we define a classical Grothendieck connection which can be seen as a complex extension of the Grothendieck connection used in [27,28]. In this way, we construct a *formal global action* for our model, i.e.

$$\tilde{\mathcal{S}}_{\Sigma_3,x} := \int\limits_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i d\hat{\mathbf{X}}^j + \left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^j + \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}\right) \tag{1.2.2}$$

with $\hat{\mathbf{X}}^i$ the coordinates of the spaces of fields $\tilde{\mathcal{F}}_{\Sigma_3,x}$ organized as superfields, x is the constant map over which we expand, $\left(\hat{R}^i_{\Sigma_3}\right)_j$ and $\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}$ the components of the Grothendieck connection given by

$$\begin{aligned} R^i_j(x;y)dx^j &:= -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial\varphi^p}{\partial x^j}dx^j, \\ R^i_{\bar{j}}(x;y)dx^{\bar{j}} &:= -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial\varphi^p}{\partial x^{\bar{j}}}dx^{\bar{j}}, \end{aligned} \tag{1.2.3}$$

where $\{y^i\}$ are the generators of the fiber of $\widehat{\mathrm{Sym}}^\bullet(T^{\vee 1,0}M)$. The formal action is such that the *differential Classical Master Equation* (dCME) is satisfied, namely

$$d_M\tilde{\mathcal{S}}_{\Sigma_3,x} + \frac{1}{2}(\tilde{\mathcal{S}}_{\Sigma_3,x},\tilde{\mathcal{S}}_{\Sigma_3,x}) = 0, \tag{1.2.4}$$

with $d_M = d_x + d_{\bar{x}}$ the sum of holomorphic and antiholomorphic Dolbeault differentials on M. The dCME presented here is different from the one presented in e.g. [15,27,28] since there d_M was the de Rham differential on the body of the target manifold.

The globalized model is then shown to be a globalization of the RW model [63], which reduces to the RW model itself in the appropriate limits. Our globalization of the RW model is compared with other globalization constructions as the one developed in [30] for a closed source manifold by using Costello's approach [31,32] to *derived geometry* [55,71,72], the procedure in [69] which extends the work of [30] to manifolds with boundary and the procedure in [43,57]. In general, our model is compatible with all these apparently different views. In particular, we give a detailed account of the similarities between our method and the one in [30], thus confirming the claim in Remark 3.6 in [27] about the equivalence between Costello's approach and ours.

In order to quantize the theory according to the quantum BV-BFV formalism, we formulate a split version of our globalized RW model. Since the globalization is controlled by an L_∞-algebra, following [69] and inspired by the work of Cattaneo, Mnev and Wernli for CS theory [25], we assume that we can split the L_∞-algebra in two isotropic subspaces. The action of the *globalized split RW model* is then

$$\tilde{\mathcal{S}}^s_{\Sigma_3,x} = \langle\hat{\mathbf{B}}, D\hat{\mathbf{A}}\rangle + \left\langle\left(\hat{R}_{\Sigma_3}\right)_j(x;\hat{\mathbf{A}}+\hat{\mathbf{B}})dx^j, \hat{\mathbf{A}}+\hat{\mathbf{B}}\right\rangle + \left\langle\left(\hat{R}_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{A}}+\hat{\mathbf{B}})dx^{\bar{j}}, \hat{\mathbf{A}}+\hat{\mathbf{B}}\right\rangle, \tag{1.2.5}$$

where $\langle -,-\rangle$ denotes the BV symplectic form on the space of fields $\tilde{\mathcal{F}}^s_{\Sigma_3,x}$ with values in the Dolbeault complex of M, $\hat{\mathbf{A}}^i$ and $\hat{\mathbf{B}}_i$ are the fields found from the splitting of the field $\hat{\mathbf{X}}^i$, and D denotes the superdifferential. Note that d is the de Rham differential on the target, not on the source.

Finally, we quantize the globalized split RW model within the quantum BV-BFV formalism framework. Here, we obtained the following two theorems.

Theorem *(Flatness of the qGBFV operator (Theorem 7.4.3)). The* quantum Grothendieck BFV (qGBFV) operator ∇_G *for the anomaly-free globalized split RW model squares to zero, i.e.*

$$(\nabla_G)^2 \equiv 0, \tag{1.2.6}$$

where

$$\nabla_G = d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3} = d_x + d_{\bar{x}} - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}, \tag{1.2.7}$$

with d_M the sum of the holomorphic and antiholomorphic Dolbeault differentials on the target M, $\Delta_{\mathcal{V}_{\Sigma_3,x}}$ the BV Laplacian and $\mathbf{\Omega}_{\partial\Sigma_3}$ the full BFV boundary operator.

Theorem *(mdQME for anomaly-free globalized split RW model (Theorem 7.5.1)). Consider the full covariant perturbative state $\hat{\psi}_{\Sigma_3,x}$ as a quantization of the anomaly-free globalized split RW model. Then*

$$\left(d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}\right)\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R} = 0. \tag{1.2.8}$$

The proof of both the theorems is very similar to the ones exhibited in [27] for non linear split AKSZ sigma models. Hence, we refer to [27] when the procedure is the same whereas we remark when there are differences (which are related to the presence of the sum of the holomorphic and antiholomorphic Dolbeault differentials in the quantum Grothendieck BFV operator instead of the de Rham differential as in [27]).

We provide an explicit expression for the BFV boundary operator up to one bulk vertices in the $\mathbb{B}$-representation by adapting to our case the degree counting techniques of [27]. Unfortunately, due to some complications related to the number of Feynman rules, we are not able to provide an explicit expression of the BFV boundary operator in the $\mathbb{B}$-representation in the case of a higher number of bulk vertices. See [64] for a limited example of graphs that appear when there are three bulk vertices.

This paper is structured as follows:

- In Section 2 we define an AKSZ model which upon globalization can be reduced to the RW model.
- In Section 3 we compare our construction to the original construction by Rozansky and Witten.
- In Section 4 we compare our globalization construction with other globalization constructions of the RW model.
- In Section 5 we give a BF-like formulation by a splitting of the fields of the RW model in order to be able to give a suitable description of its quantization.
- In Section 6 we quantize the globalized split RW model according to the quantum BV-BFV formalism.
- In Section 7 we introduce the *quantum Grothendieck BFV operator* for the globalized split RW model, we prove that it is flat and, in the end, we use it to prove the modified differential Quantum Master Equation.
- Finally, in Section 8 we present some possible future directions.

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2. Classical formal globalization

The idea is to construct a 3-dimensional topological sigma model, which, when globalized, reduces to the original RW model. In particular, we are interested in the formulation of the RW model with target a holomorphic symplectic manifold[3] (i.e. a complex symplectic manifold with a holomorphic symplectic form, see also the appendix in [63]). In this case, the space of maps is

$$\mathcal{F}_{\Sigma_3} := \mathrm{Maps}\left(T[1]\Sigma_3, M\right), \tag{2.0.1}$$

with M a holomorphic symplectic manifold endowed with coordinates X^i and $X^{\bar{i}}$ and holomorphic symplectic form $\Omega = \Omega_{ij}\delta X^i\delta X^j$; Σ_3 is a 3-dimensional manifold.

On the source manifold, we choose bosonic coordinates $\{u\}$ (ghost degree 0) on Σ_3 and fermionic odd coordinates $\{\theta\}$ (ghost degree 1) on the fibers of $T[1]\Sigma_3$. Moreover, by picking up local coordinates X^i on M, maps in $\mathcal{F}_{\Sigma_3}$ can be described by a superfield $\mathbf{X}$, whose components are chosen as:

[3] This construction reduces to the RW model considered in the bulk of [63], when we consider as target a hyperKähler manifold.

$$\mathbf{X}^i = X^i(u) + \theta^\mu X^i_\mu(u) + \frac{1}{2!}\theta^\mu\theta^\nu X^i_{\mu\nu}(u) + \frac{1}{3!}\theta^\mu\theta^\nu\theta^\chi X^i_{\mu\nu\chi}(u), \tag{2.0.2}$$

where X^i is a 0-form, X^i_μ is a 1-form etc. To these maps maps $X^i, X^i_\mu, \dots$ we assign ghost degrees such that the ghost degree of $\mathbf{X}$ is equal to the one of X^i (that is 0), for example $X^i_{\mu\nu}$ has form degree 2 and ghost degree -2.

Now, by assigning ghost degree[4] 2 to Ω_{ij}, we can form a BV manifold

$$\begin{aligned}
\omega_{\Sigma_3} &= \int\limits_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\delta\mathbf{X}^i\delta\mathbf{X}^j\right)\\
\mathcal{S}_{\Sigma_3} &= \int\limits_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\mathbf{X}^i D\mathbf{X}^j\right)\\
Q_{\Sigma_3} &= \int\limits_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(D\mathbf{X}^i\frac{\delta}{\delta\mathbf{X}^i}\right)
\end{aligned} \tag{2.0.3}$$

where δ denotes the de Rham differential on the space of fields, $D = \theta^\mu\frac{\partial}{\partial u^\mu}$ is the differential on $T[1]\Sigma_3$, μ_{Σ_3} is the canonical Berezinian on $T[1]\Sigma_3$ of degree -3. When Σ_3 is a closed manifold, the action $\mathcal{S}_{\Sigma_3}$ satisfies the CME

$$(\mathcal{S}_{\Sigma_3}, \mathcal{S}_{\Sigma_3}) = 0, \tag{2.0.4}$$

where we denote the odd Poisson bracket associated to Ω with round brackets $(-,-)$ and we call it the *BV bracket* (or *anti bracket*). Equivalently the CME can be written as

$$\iota_{Q_{\Sigma_3}}\omega_{\Sigma_3} = \delta S_{\Sigma_3}. \tag{2.0.5}$$

In the presence of boundaries, the model can be extended to a BV-BFV theory by associating the BV-BFV manifold

$$(\mathcal{F}^\partial_{\partial\Sigma_3}, \omega^\partial_{\partial\Sigma_3} = \delta\alpha^\partial_{\partial\Sigma_3}, \mathcal{S}^\partial_{\partial\Sigma_3}, Q^\partial_{\partial\Sigma_3}) \tag{2.0.6}$$

over the BV manifold $(\mathcal{F}_{\Sigma_3}, \omega_{\Sigma_3}, Q_{\Sigma_3})$, with the following set of data

$$\begin{aligned}
\mathcal{F}^\partial_{\partial\Sigma_3} &= \mathrm{Maps}(T[1]\partial\Sigma_3, X),\\
\mathcal{S}^\partial_{\partial\Sigma_3} &= \int\limits_{T[1]\partial\Sigma_3} \mu_{\partial\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\mathbf{X}^i D\mathbf{X}^j\right),\\
\alpha^\partial_{\partial\Sigma_3} &= \int\limits_{T[1]\partial\Sigma_3} \mu_{\partial\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\mathbf{X}^i \delta\mathbf{X}^j\right),\\
Q^\partial_{\partial\Sigma_3} &= \int\limits_{T[1]\partial\Sigma_3} \mu_{\partial\Sigma_3}\left(D\mathbf{X}^i\frac{\delta}{\delta\mathbf{X}^i}\right),
\end{aligned} \tag{2.0.7}$$

with $\mu_{\partial\Sigma_3}$ the Berezinian on the boundary $\partial\Sigma_3$ of degree -2. The data is such that

$$\iota_{Q_{\Sigma_3}}\omega_{\Sigma_3} = \delta\mathcal{S}_{\Sigma_3} + \pi^*\alpha^\partial_{\partial\Sigma_3}. \tag{2.0.8}$$

Remark 2.0.1. A possible modification of the model consists in coupling the target manifold with $\mathfrak{g}^\vee[1]\otimes\mathfrak{g}[1]$ or $\mathfrak{g}[1]$, forming thus the "BF-RW" model and the "CS-RW" model [43], respectively. In this way, after globalization, one should get an extension of the results obtained by Källén, Qiu and Zabzine [43].

2.1. Globalization

In the last section, we introduced a very simple AKSZ sigma model. Here we globalize that construction using methods of formal geometry [16,37] following [27]. First, we expand around critical points of the kinetic part of the action. The Euler–Lagrange equations for our model are simply $d\mathbf{X}^i = 0$, which means that the component of $\mathbf{X}^i$ of ghost degree 0 is a constant map: we denote it by x^i and we think of it as a *background field* [53]. Moreover, since we want to vary x itself, we lift the fields as the pullback of a formal exponential map at x. We also note that the fields $\mathbf{X}^{\bar{i}}$ are just *spectators*, which

[4] More precisely, we add a formal parameter q of ghost degree 2 in front of Ω_{ij}. The parameter is immediately suppressed from the notation.

means that they do not contribute to the action, hence we can think of taking constant maps also in the antiholomorphic direction.

The above allows to linearize the space of fields $\mathcal{F}_{\Sigma_3}$ by working in the formal neighborhoods of the constant map $x \in M$. We define the following *holomorphic formal exponential map*

$$\begin{aligned}\varphi:\ & T^{1,0}M \to M\\ & (x,y) \mapsto \varphi^i(x,y) = x^i + y^i + \frac{1}{2}\varphi^i_{jk}(x^i, x^{\bar{i}})y^j y^k + \dots\end{aligned} \tag{2.1.1}$$

Remark 2.1.1. We think about the holomorphic formal exponential map here defined as an extension to the complex case of the formal exponential map used in e.g. [21]. This notion should correspond to the "canonical coordinates" introduced in [14] and the holomorphic exponential map applied by Kapranov to the RW case in [44].

The formal exponential map lifts $\mathcal{F}_{\Sigma_3}$ to

$$\begin{aligned}\tilde{\varphi}_x:\ \tilde{\mathcal{F}}_{\Sigma_3,x} := \mathrm{Maps}(T[1]\Sigma_3, T^{1,0}_x M) &\to \mathrm{Maps}(T[1]\Sigma_3, M)\\ \hat{\mathbf{X}} &\mapsto \mathbf{X}\end{aligned} \tag{2.1.2}$$

which is given by precomposition with φ_x^{-1}, i.e. $\tilde{\mathcal{F}}_{\Sigma_3,x} = \varphi_x^{-1} \circ \mathcal{F}_{\Sigma_3}$ and $\mathbf{X} = \varphi_x(\hat{\mathbf{X}})$. Now, since the target is linear, we can write the space of fields as

$$\tilde{\mathcal{F}}_{\Sigma_3,x} = \Omega^\bullet(\Sigma_3) \otimes T^{1,0}_x M. \tag{2.1.3}$$

Consequently, we lift the BV action, the BV 2-form and the primitive 1-form obtaining:

$$\begin{aligned}\mathcal{S}_{\Sigma_3,x} &:= \mathrm{T}\tilde{\varphi}_x^* \mathcal{S}_{\Sigma_3} = \int_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i D\hat{\mathbf{X}}^j\right),\\ \omega_{\Sigma_3,x} &:= \tilde{\varphi}_x^* \omega_{\Sigma_3} = \int_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\delta\hat{\mathbf{X}}^i \delta\hat{\mathbf{X}}^j\right),\\ \alpha^\partial_{\partial\Sigma_3,x} &:= \tilde{\varphi}_x^* \alpha^\partial_{\partial\Sigma_3} = \int_{T[1]\partial\Sigma_3} \mu_{\partial\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i \delta\hat{\mathbf{X}}^j\right),\end{aligned} \tag{2.1.4}$$

where T denotes the Taylor expansion around the fiber coordinates $\{y\}$ at zero. This set of data satisfies the mCME for any $x \in M$:

$$\iota_{Q_{\Sigma_3,x}}\omega_{\Sigma_3,x} = \delta\mathcal{S}_{\Sigma_3,x} + \pi^*\alpha^\partial_{\partial\Sigma_3,x}, \tag{2.1.5}$$

with $Q_{\Sigma_3,x} = \int_{T[1]\Sigma_3}\mu_{\Sigma_3}\left(D\hat{\mathbf{X}}^i \frac{\delta}{\delta\hat{\mathbf{X}}^i}\right)$. Hence, we have a BV-BFV manifold associated to the space of fields $\tilde{\mathcal{F}}_{\Sigma_3,x}$.

The next remark introduces an important ingredient to write down the globalized action.

Remark 2.1.2. The constant map $x : T[1]\Sigma_3 \to M$ in $\mathcal{F}_{\Sigma_3}$ can be thought of as an element in M. Hence, we have a natural inclusion $M \hookrightarrow \mathcal{F}_{\Sigma_3}$. Hence, we can define a 1-form:

$$R_{\Sigma_3} = \left(R_{\Sigma_3}\right)_j(x;\mathbf{X})dx^j + \left(R_{\Sigma_3}\right)_{\bar{j}}(x;\mathbf{X})dx^{\bar{j}} \in \Omega^1\left(M, \mathrm{Der}\left(\widehat{\mathrm{Sym}^\bullet}(T^{\vee 1,0}M)\right)\right). \tag{2.1.6}$$

As before, we lift this 1-form to $\tilde{\mathcal{F}}_{\Sigma_3,x}$. This lift, denoted by $\hat{R}_{\Sigma_3}$, is locally written as:

$$\hat{R}_{\Sigma_3} = \left(\hat{R}_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})dx^j + \left(\hat{R}_{\Sigma_3}\right)_{\bar{j}}(x;\mathbf{X})dx^{\bar{j}}. \tag{2.1.7}$$

2.2. Variation of the classical background

So far, the classical background x has been fixed. However, our aim is to vary x and construct a global formulation of the action. Hence, we understand the collection $\{\mathcal{S}_{\Sigma_3,x}\}_{x\in M}$ as a map $\hat{\mathcal{S}}_{\Sigma_3}$ to be given by $\hat{\mathcal{S}}_{\Sigma_3} : x \mapsto S_{\Sigma_3,x}$ and we compute how it changes over M. In order to accomplish this task, inspired by [15,27,28,43], choosing a background field $x \in M$, we define

$$\mathcal{S}_{\Sigma_3,x,R} := \int_{\Sigma_3}\left(\left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^j + \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}\right) = \mathcal{S}_R + \mathcal{S}_{\bar{R}}. \tag{2.2.1}$$

The integrand is a well defined term of degree 3, since we assigned degree 2 to the symplectic form and $\hat{R}_{\Sigma_3}$ is a 1-form on M. After integration, $\mathcal{S}_{\Sigma_3,x,R}$ is then of total degree 0.

The term $\hat{R}_{\Sigma_3}$ has been introduced in Remark 2.1.2. However, its connection with the globalization procedure is not clear. To explain it, we introduce the *classical Grothendieck connection* adapted to our case (see Appendix [21]).

Definition 2.2.1 *(Classical Grothendieck connection).* Given a holomorphic formal exponential map φ, we can define the associated *classical Grothendieck connection* on $\widehat{\mathrm{Sym}}^{\bullet}(T^{\vee 1,0}M)$, given by $\mathcal{D}_{\mathrm{G}} := d_M + R$, where d_M is the sum of the holomorphic and antiholomorphic Dolbeault differentials on M and $R \in \Omega^1\left(M, \mathrm{Der}\left(\widehat{\mathrm{Sym}}^{\bullet}(T^{\vee 1,0}M)\right)\right)$. By using local coordinates $\{x\}$ on the basis and $\{y\}$ on the fibers, we have $R = R_j(x;y)dx^j + R_{\bar{j}}(x;y)dx^{\bar{j}}$, where $R_j = R^i_j(x;y)\frac{\partial}{\partial y}$ and $R_{\bar{j}} = R^i_{\bar{j}}(x;y)\frac{\partial}{\partial y}$ with

$$\begin{aligned} R^i_j(x;y)dx^j &:= -\left[\left(\frac{\partial \varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial \varphi^p}{\partial x^j}dx^j, \\ R^i_{\bar{j}}(x;y)dx^{\bar{j}} &:= -\left[\left(\frac{\partial \varphi}{\partial y}\right)^{-1}\right]^i_p \frac{\partial \varphi^p}{\partial x^{\bar{j}}}dx^{\bar{j}}. \end{aligned} \tag{2.2.2}$$

Note that $R^i_j(x;y)$ and $R^i_{\bar{j}}(x;y)$ are formal power series in the second argument, namely

$$\begin{aligned} R^i_j(x;y) &= \sum_{k=0}^{\infty} R^i_{j;j_1,\ldots,j_k}(x) y^{j_1}\ldots y^{j_k}, \\ R^i_{\bar{j}}(x;y) &= \sum_{k=0}^{\infty} R^i_{\bar{j};j_1,\ldots,j_k}(x) y^{j_1}\ldots y^{j_k}. \end{aligned} \tag{2.2.3}$$

Remark 2.2.2. The classical Grothendieck connection has a couple of important properties:

- It is flat, which can be rephrased by saying that the following equation is satisfied

$$d_M R + \frac{1}{2}[R,R] = 0. \tag{2.2.4}$$

- A section σ is closed under $\mathcal{D}_{\mathrm{G}}$ i.e. $\mathcal{D}_{\mathrm{G}}\sigma = 0$ if and only if $\sigma = \mathrm{T}\varphi^*_x f$, where $f \in \mathcal{C}^{\infty}(M)$.

In more down-to-Earth terms, the second property says that the classical Grothendieck connection selects those sections which are global.

Finally, we can clarify the relation between $\hat{R}_{\Sigma_3}$ and the Grothendieck connection. The components $\left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})$ and $\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})$ are given by the components of the classical Grothendieck connection $R^i_j(x;y)$ and $R^i_{\bar{j}}(x;y)$ evaluated in the second argument at $\hat{\mathbf{X}}$.

Having set up all the necessary tools, we can compute how $\hat{\mathcal{S}}_{\Sigma_3}$ varies when we change the background $x \in M$. On a closed manifold, we have

$$d_M \hat{\mathcal{S}}_{\Sigma_3} = -(\mathcal{S}_{\Sigma_3,x,R}, \hat{\mathcal{S}}_{\Sigma_3}), \tag{2.2.5}$$

which follows from the Grothendieck connection and that $\mathcal{S}_{\Sigma_3,x} = \mathrm{T}\varphi^*_x\mathcal{S}_{\Sigma_3}$.

The above identities can be collected in a nicer way via the following definition.

Definition 2.2.3. (Formal global action) The *formal global action* for the model is defined by

$$\begin{aligned} \tilde{\mathcal{S}}_{\Sigma_3,x} &:= \int_{\Sigma_3} \left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i d\hat{\mathbf{X}}^j + \left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^j + \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}\right) \\ &= \hat{\mathcal{S}}_{\Sigma_3} + \underbrace{\mathcal{S}_R + \mathcal{S}_{\bar{R}}}_{:=\mathcal{S}_{\Sigma_3,x,R}}. \end{aligned} \tag{2.2.6}$$

By using the formal global action, the *differential Classical Master Equation* (dCME) is satisfied

$$d_M\tilde{\mathcal{S}}_{\Sigma_3,x} + \frac{1}{2}(\tilde{\mathcal{S}}_{\Sigma_3,x}, \tilde{\mathcal{S}}_{\Sigma_3,x}) = 0. \tag{2.2.7}$$

Remark 2.2.4. Note that $\tilde{\mathcal{S}}_{\Sigma_3,x}$ is an inhomogeneous form over M, where $\hat{S}_{\Sigma_3}$ is a 0-form and $S_{\Sigma_3,x,R}$ is a 1-form. Therefore, Eq. (2.2.7) has a 0-form, a 1-form and a 2-form part. Specifically, the 0-form part

$$(\hat{\mathcal{S}}_{\Sigma_3}, \hat{\mathcal{S}}_{\Sigma_3}) = 0, \tag{2.2.8}$$

is the usual CME. The 1-form part:

$$d_M\hat{\mathcal{S}}_{\Sigma_3} + (\mathcal{S}_{\Sigma_3,x,R}, \hat{\mathcal{S}}_{\Sigma_3}) = 0, \tag{2.2.9}$$

means that $\hat{\mathcal{S}}_{\Sigma_3}$ is a global object (see Remark 2.2.2). The 2-form part

$$d_M\mathcal{S}_{\Sigma_3,x,R} + \frac{1}{2}(\mathcal{S}_{\Sigma_3,x,R}, \mathcal{S}_{\Sigma_3,x,R}) = 0, \tag{2.2.10}$$

means that the operator $\mathcal{D}_G$ is flat connection (see Eq. (2.2.4)). Explicitly, we have

$$d_x\mathcal{S}_R + \frac{1}{2}(\mathcal{S}_R, \mathcal{S}_R) = 0, \tag{2.2.11}$$

$$d_x\mathcal{S}_{\bar{R}} + \frac{1}{2}(\mathcal{S}_R, \mathcal{S}_{\bar{R}}) = 0, \tag{2.2.12}$$

$$d_{\bar{x}}\mathcal{S}_R + \frac{1}{2}(\mathcal{S}_{\bar{R}}, \mathcal{S}_R) = 0, \tag{2.2.13}$$

$$d_{\bar{x}}\mathcal{S}_{\bar{R}} + \frac{1}{2}(\mathcal{S}_{\bar{R}}, \mathcal{S}_{\bar{R}}) = 0. \tag{2.2.14}$$

Let Σ_3 be (again) a manifold with boundary. The BV-BFV theory on $\tilde{\mathcal{F}}_{\Sigma_3,x}$ furnishes the cohomological vector field $Q_{\Sigma_3,x}$. Moreover, by using the lift of $\hat{R}_{\Sigma_3}$, we can define

$$\tilde{Q}_{\Sigma_3,x} = Q_{\Sigma_3,x} + \hat{R}_{\Sigma_3}. \tag{2.2.15}$$

Then, the *modified differential Classical Master Equation* (mdCME) is satisfied:

$$\iota_{\tilde{Q}_{\Sigma_3,x}}\omega_{\Sigma_3,x} = \delta\tilde{\mathcal{S}}_{\Sigma_3,x} + \pi^*\alpha^{\partial}_{\partial\Sigma_3,x}, \tag{2.2.16}$$

where

$$\begin{aligned}\tilde{Q}_{\Sigma_3,x} = \int_{\Sigma_3}\Bigg(&- d\hat{\mathbf{X}}\frac{\delta}{\delta\hat{\mathbf{X}}} - \Omega^{pq}\frac{\delta\left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})}{\delta\hat{\mathbf{X}}^p}\Omega_{il}\hat{\mathbf{X}}^l dx^j\frac{\delta}{\delta\hat{\mathbf{X}}^q} - \left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})dx^j\Omega_{ip}\frac{\delta}{\delta\hat{\mathbf{X}}^p} \\ &- \Omega^{pq}\frac{\delta\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})}{\delta\hat{\mathbf{X}}^p}\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}\frac{\delta}{\delta\hat{\mathbf{X}}^q} - \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})dx^{\bar{j}}\Omega_{ip}\frac{\delta}{\delta\hat{\mathbf{X}}^p}\Bigg).\end{aligned} \tag{2.2.17}$$

In preparation for the comparisons we will draw in the following section, we redefine the components $\left(\hat{R}^i_{\Sigma_3}\right)_j$ and $\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}$ by a multiplicative factor $1/k!$ as

$$\begin{aligned}\left(\hat{R}^i_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}}) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\hat{R}^i_{j;j_1,\dots,j_k}(x)\hat{\mathbf{X}}^{j_1}\dots\hat{\mathbf{X}}^{j_k}, \\ \left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}}) &= \sum_{k=0}^{\infty}\frac{1}{(k+1)!}\hat{R}^i_{\bar{j};j_1,\dots,j_k}(x)\hat{\mathbf{X}}^{j_1}\dots\hat{\mathbf{X}}^{j_k}.\end{aligned} \tag{2.2.18}$$

Table 3.0.1
Comparison between kinetic term and interaction term for the RW theory and our model.

	Kinetic term	Interaction term
Original RW model	$\frac{1}{2}\frac{1}{\sqrt{h}}\epsilon^{\mu\nu\rho}\Omega_{IJ}\chi^I_\mu\nabla_\nu\chi^J_\rho$	$-\frac{1}{3!}\Omega_{IJ}R^J_{KL\bar{M}}\chi^I_\mu\chi^K_\nu\chi^L_\rho\eta^{\bar{M}}$
Our model	$\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i d\hat{\mathbf{X}}^j - \frac{1}{2}\Gamma^i_{jk}\hat{\mathbf{X}}^k\Omega_{il}\hat{\mathbf{X}}^l dx^j$	$\frac{1}{3!}R^i_{ks\bar{j}}\hat{\mathbf{X}}^k\hat{\mathbf{X}}^s\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}}$

3. Comparison with the original Rozansky–Witten model

In this section, we show that the globalized model we have just constructed reduces to the RW model and, moreover it provides a globalization of the former.

In order to compare effectively these models, we need to be more explicit about the terms involved in the classical Grothendieck connection. First, we discuss the choice of holomorphic formal exponential map in more detail. Since our target is a symplectic manifold, we choose the formal exponential map which preserves the symplectic form considered in [58] and we adapt it to our case, i.e.

$$\varphi^i = x^i + y^i - \frac{1}{2}\Gamma^i_{jk}y^j y^k + \left\{-\frac{1}{6}\partial_c\Gamma^i_{jk} + \frac{1}{3}\Gamma^i_{mc}\Gamma^m_{jk} - \frac{1}{24}R^i_{cjk}\right\}y^c y^j y^k + O(y^4), \tag{3.0.1}$$

where $R^i_{cjk} = (\Omega^{-1})^{bi}R_{bc}{}^a{}_k\Omega_{aj}$.

The Grothendieck connection is then

$$\mathcal{D}_{\mathrm{G}} = dx^i\frac{\partial}{\partial x^i} + dx^{\bar{i}}\frac{\partial}{\partial x^{\bar{i}}} + dx^j\left(R_{\Sigma_3}\right)_j + dx^{\bar{j}}\left(R_{\Sigma_3}\right)_{\bar{j}}, \tag{3.0.2}$$

where the third term on the right hand side was computed in [58],

$$\left(\hat{R}^i_{\Sigma_3}\right)_j dx^j = -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p\frac{\partial\varphi^p}{\partial x^j}dx^j = -\left[dx^j\left(\delta^i_j + \Gamma^i_{kj}y^k - \left(\frac{1}{8}R_j{}^i{}_{ks} + \frac{1}{4}R_{jk}{}^i{}_s\right)y^k y^s\right)\frac{\partial}{\partial y^i} + \dots\right], \tag{3.0.3}$$

whereas the fourth term is

$$\begin{aligned}\left(\hat{R}^i_{\Sigma_3}\right)_{\bar{j}} dx^{\bar{j}} &= -\left[\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]^i_p\frac{\partial\varphi^p}{\partial x^{\bar{j}}}dx^{\bar{j}} = \left[-\frac{1}{2}\Gamma^p_{ab,\bar{j}}y^a y^b dx^{\bar{j}}\right][\delta^i_p + \dots] = -\left[-\frac{1}{2}\Gamma^i_{ab,\bar{j}}y^a y^b dx^{\bar{j}} + \dots\right] \\ &= R^i_{ab\bar{j}}y^a y^b dx^{\bar{j}} - \dots.\end{aligned} \tag{3.0.4}$$

Considering the terms coming from the classical Grothendieck connection and the redefinition (2.2.18), we can re-write the formal global action (2.2.6) as

$$\tilde{\mathcal{S}}_{\Sigma_3,x} = \int_{\Sigma_3}\left(\frac{1}{2}\Omega_{ij}\hat{\mathbf{X}}^i d\hat{\mathbf{X}}^j - \frac{1}{2}\Gamma^i_{jk}\hat{\mathbf{X}}^k\Omega_{il}\hat{\mathbf{X}}^l dx^j - \delta^i_j\Omega_{il}\hat{\mathbf{X}}^l dx^j + \dots + \frac{1}{3!}R^i_{ks\bar{j}}\hat{\mathbf{X}}^k\hat{\mathbf{X}}^s\Omega_{il}\hat{\mathbf{X}}^l dx^{\bar{j}} + \dots\right). \tag{3.0.5}$$

For convenience, we recall the RW action [63]

$$S_{\mathrm{RW}} = \int_{\Sigma_3}\frac{1}{2}\frac{1}{\sqrt{h}}\epsilon^{\mu\nu\rho}\left(\Omega_{IJ}\chi^I_\mu\nabla_\nu\chi^J_\rho - \frac{1}{3}\Omega_{IJ}R^J_{KL\bar{M}}\chi^I_\mu\chi^K_\nu\chi^L_\rho\eta^{\bar{M}} + \frac{1}{3}(\nabla_L\Omega_{IK})(\partial_\mu\phi^I_\perp)\chi^K_\nu\chi^L_\rho\right), \tag{3.0.6}$$

where we refer to the original paper for an explanation of the various terms involved in the action. If we assume that the connection is compatible with the symplectic form, the third term in the RW action (3.0.6) drops. We are left with the first two terms. By associating $\hat{\mathbf{X}}^i \leftrightarrow \chi^I$ and $dx^{\bar{j}} \leftrightarrow \eta^{\bar{M}}$, we can sum up the comparison in Table 3.0.1.

The sign discrepancy comes from having defined the connection as $\nabla = d - \Gamma$ which gives a negative sign in front of the Γ^i_{jk} (see Eq. (3.0.1)).

Moreover, when the curvature misses the $(2,0)$-part (which could happen when we have a Hermitian metric), the remaining terms in our model are just the perturbative expansion of $R^i_{ks\bar{j}}$ around x. If we cut off the expansion at the first order, we are left with the original RW model.

4. Comparison with other globalization constructions

In the next sections, we are going to compare our globalization model with other three constructions: the first by [30] uses tools of derived geometry to linearize the space of fields in the neighborhood of a constant map as well as the Fedosov connection [35], the second [69] is an extension to a manifold with boundary of the first procedure, while the third [43,56,57] uses an approach similar to ours.

4.1. Comparison with the CLL construction

We compare our model with the formulation of the RW model constructed in [30] in the setting of derived geometry.

Let Σ_3 be a closed 3 dimensional manifold and M be a holomorphic symplectic manifold with a non-degenerate holomorphic 2-form ω.

To determine fields we use the language of L_∞-spaces (see [31,32] for an introduction) and we define the space of fields as

$$\mathrm{Maps}(\Sigma_{3,\mathrm{dR}}, M_{\bar\partial}), \tag{4.1.1}$$

where $\Sigma_{3,\mathrm{dR}}$ is the elliptic ringed space equipped with a sheaf of differential forms over Σ_3, i.e. $\Omega^\bullet(\Sigma_3)$ and $M_{\bar\partial} = (M, \mathfrak{g}_M)$ is a sheaf of L_∞-algebras, where $\mathfrak{g}_M = \Omega^{\bullet,\bullet}(M) \otimes T^{1,0}M[-1]$ with $T^{1,0}M$ the holomorphic tangent bundle. Since the critical points of the action functional are constant maps from Σ_3 to M, we are going to study $\mathrm{Maps}(\Sigma_{3,\mathrm{dR}}, M_{\bar\partial})$ in the neighborhood of a constant map $x \in \mathrm{Maps}(\Sigma_{3,\mathrm{dR}}, M_{\bar\partial})$, namely

$$\mathcal{F}_{\mathrm{CLL}} := \widehat{\mathrm{Maps}}(\Sigma_{3,\mathrm{dR}}, M_{\bar\partial}) = \Omega^\bullet(\Sigma_3) \otimes \mathfrak{g}_M[1], \tag{4.1.2}$$

with $\widehat{\mathrm{Maps}}(\Sigma_{3,\mathrm{dR}}, M_{\bar\partial})$ defined as in [31,32].

Having specified the space of fields, the shifted symplectic structure is given by

$$\begin{aligned} \langle -,-\rangle : \mathcal{F}_{\mathrm{CLL}} \otimes_{\Omega^{\bullet,\bullet}(M)} \mathcal{F}_{\mathrm{CLL}} &\to \Omega^{\bullet,\bullet}(M)[-1] \\ \langle \alpha\otimes g_1, \beta\otimes g_2\rangle &:= \underbrace{\omega(g_1,g_2)}_{\text{sympl. struct. on } M} \int_{\Sigma_3} \alpha\wedge\beta, \end{aligned} \tag{4.1.3}$$

where $\Omega^{\bullet,\bullet}(M) = \Gamma\left(\bigwedge^\bullet T^\vee M\right)$ is a section of the cotangent bundle.

Since $C^\bullet(\mathfrak{g}_M) := \widehat{\mathrm{Sym}}^\bullet_{\Omega^{\bullet,\bullet}(M)}(\mathfrak{g}_M^\vee[1]) = \Omega^{\bullet,\bullet}(M) \otimes_{C^\infty(M)} \widehat{\mathrm{Sym}}^\bullet_{C^\infty(M)}(T^{\vee 1,0}M)$, to construct the action functional and to find our L_∞-algebra we can use a procedure similar to the Fedosov's construction of a connection on a symplectic manifold [34]. Let us denote the sections of the *holomorphic Weyl bundle* on M by

$$\mathcal{W} = \Omega^{\bullet,\bullet}(M) \otimes_{C^\infty(M)} \widehat{\mathrm{Sym}}^\bullet(T^{\vee 1,0}M)[[\hbar]] \tag{4.1.4}$$

where $\widehat{\mathrm{Sym}}^\bullet(T^{\vee 1,0}M)[[\hbar]]$ is the completed symmetric algebra over $T^{\vee 1,0}M$, the holomorphic cotangent bundle which has a local basis $\{y^i\}$ with respect to the local holomorphic coordinates $\{x^i\}$. We call the sub-bundle $\Omega^{p,q}(M) \otimes \mathrm{Sym}^r(T^{\vee 1,0}M)$ of $\mathcal{W}$ its (p,q,r) component, in particular we refer to r as *weight*. To $\hbar$ is assigned a weight of 2.

Proposition 4.1.1 *([30]). There is a connection on the holomorphic Weyl bundle of the following form*

$$\mathcal{D}_{\mathrm{F}} = \nabla - \delta + \frac{1}{\hbar}[I,-]_{\mathcal{W}}, \tag{4.1.5}$$

which is flat modulo $\hbar$ and $[-,-]_{\mathcal{W}}$ is defined as in [30]. Here I is a 1-form valued section of the Weyl bundle of weight ≥ 3, i.e. $I \in \bigoplus_{r\geq 3}\Gamma(\mathrm{Sym}^r(T^{\vee 1,0}M)\otimes T^\vee M)$, ∇ is the extension to $\mathcal{W}$ of a connection on $T^{1,0}M$ which is compatible with the complex structure as well as with the holomorphic symplectic form and torsion free, $\delta = dx^i \wedge \frac{\partial}{\partial y^i}$ is an operator on $\mathcal{W}$.

The connection $\mathcal{D}_{\mathrm{F}}$ is called *Fedosov connection* and it provides the L_∞-structure on $\mathfrak{g}_M$. In these terms the action can be written as

$$\mathcal{S}_{\mathrm{CLL}} = \frac{1}{2}\langle d_{\Sigma_3}\alpha, \alpha\rangle + \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\langle \ell_k(\alpha^{\otimes k}), \alpha\rangle \tag{4.1.6}$$

with $\alpha \in \mathcal{F}_{\mathrm{CLL}}$, $\langle -,-\rangle$ defined as in (4.1.3) and ℓ_k are the higher brackets in the L_∞-algebra, and d_{Σ_3} the de Rham differential on the source Σ_3. We can read ℓ_0 from the Fedosov connection in (4.1.5), i.e.

$$\ell_0 = -dx^i \frac{\partial}{\partial y^i} \tag{4.1.7}$$

The L_∞-products ℓ_1 and ℓ_2 are computed in the next section, when we compare the Fedosov connection with the classical Grothendieck connection.

Remark 4.1.2. The action in (4.1.6) satisfies the CME $(\mathcal{S}_{\mathrm{CLL}}, \mathcal{S}_{\mathrm{CLL}}) = 0$ (see [30, Proposition 2.16]). Moreover, in [30] it was observed that this construction is the formal version of the original RW model in the case when the $(2,0)$-part of the curvature is zero (see [30, Section 2.3]).

4.1.1. Comparison between the Fedosov connection and the classical Grothendieck connection

The sufficient condition for the flatness of $\mathcal{D}_{\mathrm{F}}$ (see the proof of Proposition 4.1.1 in [30]) implies that I satisfies

$$I=\delta^{-1}(R+\nabla I)+\frac{1}{\hbar}\delta^{-1}I^2, \tag{4.1.8}$$

where $\delta^{-1}=y^i\cdot\iota_{\partial_{x^i}}$ (up to a normalization factor) is another operator on $\mathcal{W}$ and R is the curvature tensor.

Remark 4.1.3. Since I is a 1-form valued section of $\mathcal{W}$, we can decompose it into its holomorphic and antiholomorphic component respectively. In particular the antiholomoprhic part component is the Taylor expansion of the Atiyah class as noted in [30]. In the case $R^{2,0}=0$, the L_∞-algebra is fully encoded by Taylor expansion of the Atiyah class as first noted by Kapranov in [44].

Since the operator δ^{-1} increases the weight by 1, while ∇ preserves the weight and I has at least weight 3, we can find a solution of the above equation with the following leading term (cubic term)[5]:

$$\delta^{-1}R=\frac{1}{8}\big[-\Gamma_{ijk,r}+\Gamma_{sir}\Gamma_{pjk}\Omega^{sp}\big]y^iy^jy^rdx^k+\frac{1}{6}R_{\bar{k}rij}y^iy^jy^rdx^{\bar{k}}=\delta^{-1}R_t+\delta^{-1}\bar{R}. \tag{4.1.9}$$

Since the Fedosov connection requires the computation of $\frac{1}{\hbar}[I,-]_{\mathcal{W}}$, we compute this commutator for the leading order term of I, which is the cubic term we have just found. For the first term on the right hand side of Eq. (4.1.9) we have

$$\begin{aligned}\frac{1}{\hbar}[\delta^{-1}R_t,-]_{\mathcal{W}}&=\left[\frac{1}{8}\Big(-\Gamma_{rjk,q}+\Gamma_{srq}\Gamma_{pjk}\Omega^{sp}\Big)+\frac{1}{4}\Big(-\Gamma_{qjk,r}+\Gamma_{sqk}\Gamma_{pjr}\Omega^{sp}\Big)\right]\Omega^{qi}y^jy^rdx^k\frac{\partial}{\partial y^i}\\&=\left[\frac{1}{8}\Big(-\Omega_{mr}\Gamma^m_{jk,q}+\Omega_{mr}\Gamma^m_{sk}\Gamma_{pjk}\Omega^{sp}\Big)+\frac{1}{4}\Big(-\Omega_{mq}\Gamma^m_{jk,r}+\Omega_{mq}\Gamma^m_{sk}\Gamma_{pjr}\Omega^{sp}\Big)\right]\times\\&\qquad\times\Omega^{qi}y^jy^rdx^k\frac{\partial}{\partial y^i}\\&=\left[\frac{1}{8}R_{k\ rj}^{\ i}+\frac{1}{4}R_{kr\ j}^{\ \ i}\right]y^jy^rdx^k\frac{\partial}{\partial y^i}.\end{aligned} \tag{4.1.10}$$

For the second term we have

$$\frac{1}{\hbar}[\delta^{-1}\bar{R},-]_{\mathcal{W}}=\frac{1}{2}\Gamma^i_{jk,\bar{r}}y^jy^kdz^{\bar{r}}\frac{\partial}{\partial y^i}. \tag{4.1.11}$$

After renaming some indices, the Fedosov connection is then

$$\mathcal{D}_{\mathrm{F}}=d_x+d_{\bar{x}}-dx^j\frac{\partial}{\partial y^j}-dx^j\Gamma^i_{kj}y^k\frac{\partial}{\partial y^i}+dx^j\left(\frac{1}{8}R_{j\ ks}^{\ i}+\frac{1}{4}R_{jk\ s}^{\ \ i}\right)y^ky^s\frac{\partial}{\partial y^i}+\frac{1}{2}dx^{\bar{j}}\Gamma^i_{ks,\bar{j}}y^ky^s\frac{\partial}{\partial y^i}+\dots. \tag{4.1.12}$$

More explicitly,

$$\begin{aligned}\ell_1&=-dx^j\Gamma^i_{kj}y^k\frac{\partial}{\partial y^i},\\ \ell_2&=dx^j\left(\frac{1}{8}R_{j\ ks}^{\ i}+\frac{1}{4}R_{jk\ s}^{\ \ i}\right)y^ky^s\frac{\partial}{\partial y^i}+\frac{1}{2}dx^{\bar{j}}\Gamma^i_{ks,\bar{j}}y^ky^s\frac{\partial}{\partial y^i}.\end{aligned} \tag{4.1.13}$$

Remark 4.1.4. The first terms in the Fedosov connection, explicitly written in (4.1.12) coincide with the first terms for the classical Grothendieck connection (3.0.2). Furthermore, by substituting the explicit expressions of ℓ_1 and ℓ_2 in the action $\mathcal{S}_{\mathrm{CLL}}$ (4.1.6), we can see that it coincides with the action $\tilde{\mathcal{S}}_{\Sigma_3,x}$ (3.0.5).

4.1.2. Comparison between the CLL space of fields and globalization space of fields

By rephrasing the argument of [54, Section 6.1] to our context, we can extend the classical Grothendieck connection $\mathcal{D}_{\mathrm{G}}$ to the complex

$$\Gamma\Big(\bigwedge\nolimits^{\bullet}T^\vee M\otimes\widehat{\mathrm{Sym}}^\bullet(T^{\vee 1,0}M)\Big), \tag{4.1.14}$$

which is the algebra of functions on the formal graded manifold

[5] See also [33,38].

$$T[1]M \bigoplus T^{1,0}M. \tag{4.1.15}$$

This graded manifold is turned into a differential graded manifold by the classical Grothendieck connection $\mathcal{D}_G$. Moreover, since $\mathcal{D}_G$ vanishes on the body of the graded manifold, we can linearize at $x \in M$ and we get

$$T_x[1]M \bigoplus T_x^{1,0}M. \tag{4.1.16}$$

On this graded manifold, we have a curved L_∞-structure (which is the same as $\mathfrak{g}_M[1]$) and Eq. (4.1.14) can be interpreted as the Chevalley–Eilenberg complex of the aforementioned L_∞-algebra. Then, the space of fields for the globalized theory can be rewritten as

$$\tilde{\mathcal{F}}_{\Sigma_3,x} = \Omega^\bullet(\Sigma_3) \otimes \Omega^{\bullet,\bullet}(M) \otimes T_x^{1,0}M \tag{4.1.17}$$

which is the same as $\mathcal{F}_{\mathrm{CLL}}$ by linearizing at $x \in M$ the holomorphic tangent bundle as $\mathcal{D}_F$ vanishes on M.

Remark 4.1.5. The idea that the classical Grothendieck connection and the Fedosov connection coincide is not new, in particular see [27, Remark 3.6] and [30, Section 2.3].

Remark 4.1.6. Finally note that in [30] the source manifold Σ_3 was considered to be a closed manifold. As explained above (see Section 2.2) our construction is valid also when $\partial\Sigma_3 \neq \emptyset$. In the next section, we tackle this last setting by comparing our approach with [69], where the derived geometric framework was implemented for manifolds with boundary.

4.2. Comparison with Steffens' construction

In [69], Steffens applied the same derived geometry approach we have seen in the last section to what he calls *AKSZ theories of Chern–Simons type*: CS theory and RW theories. In particular, his BV formulation of the RW model is completely analogue to the one in [30]: same space of fields, L_∞-algebra, action, etc. However, he takes a step further. He proves a *formal AKSZ theorem* [69, Theorem 2.4.1] in the context of derived geometry. His RW model is then shown to be an AKSZ theory by attaching degree 2 to the holomorphic symplectic form (as we did ourselves in Section 2). Consequently, he provides a BV-BFV formulation for the RW model. The BFV action found in [69] is analogous to the action in (4.1.6) in one dimension less (as it is customary with AKSZ theories). Even if the L_∞ products are not explicit in his construction, by using the ones in (4.1.13), his BV-BFV formulation of the RW model is visibly identical to ours.

4.3. Comparison with the (K)QZ construction

Let Σ_3 be a 3-dimensional manifold and M a hyperKähler manifold with holomorphic symplectic form Ω. Consider the symplectic graded manifold $\mathcal{M} := T^{\vee 0,1}[2]T^{\vee 0,1}[1]M$ constructed out of M. It has the following coordinates: $X^i, X^{\bar{i}}$ of degree 0 parametrizing M, $V^{\bar{i}}$ of degree 1 parametrizing the fiber $T^{0,1}M$ and dual coordinates $P_{\bar{i}}$, $Q_{\bar{i}}$ of degree 2 and 1, respectively. The symplectic form is

$$\omega_{\mathcal{M}} = dP_{\bar{i}} \wedge dX^{\bar{i}} + dQ_{\bar{i}} \wedge dV^{\bar{i}} + \frac{1}{2}\Omega_{ij} dX^i \wedge dX^j. \tag{4.3.1}$$

In order to have a ghost degree 2 symplectic form, the authors assign degree 2 to Ω. With this setup, in [43,56,57], Källén, Qiu and Zabzine construct an AKSZ model

$$\mathcal{F}_{\mathrm{QZ}} := \mathrm{Maps}(T[1]\Sigma_3, T^{\vee 0,1}[2]T^{\vee 0,1}[1]M) \tag{4.3.2}$$

$$\mathcal{S}_{\mathrm{QZ}} = \int_{T[1]\Sigma_3} d^3z d^3\theta \left(\mathbf{P}_{\bar{i}} D\mathbf{X}^{\bar{i}} + \mathbf{Q}_{\bar{i}} D\mathbf{V}^{\bar{i}} + \frac{1}{2}\Omega_{ij}\mathbf{X}^i D\mathbf{X}^j + \mathbf{P}_{\bar{i}}\mathbf{V}^{\bar{i}} \right) \tag{4.3.3}$$

endowed with a cohomological vector field

$$Q = \int_{T[1]\Sigma_3} d^3z d^3\theta \left(D\mathbf{P}_{\bar{i}} \frac{\partial}{\partial \mathbf{P}_{\bar{i}}} + D\mathbf{Q}_{\bar{i}} \frac{\partial}{\partial \mathbf{Q}_{\bar{i}}} + D\mathbf{V}^{\bar{i}} \frac{\partial}{\partial \mathbf{V}^{\bar{i}}} + D\mathbf{X}^i \frac{\partial}{\partial \mathbf{X}^i} + D\mathbf{X}^{\bar{i}} \frac{\partial}{\partial \mathbf{X}^{\bar{i}}} + \mathbf{P}_{\bar{i}} \frac{\partial}{\partial \mathbf{Q}_{\bar{i}}} + \mathbf{V}^{\bar{i}} \frac{\partial}{\partial \mathbf{X}^{\bar{i}}} \right), \tag{4.3.4}$$

where to the source manifold $T[1]\Sigma_3$, we assign coordinates $\{z^i\}$ of ghost degree 0 and coordinates $\{\theta^i\}$ of degree 1.

Remark 4.3.1. With a suitable gauge-fixing consisting on a particular choice of Lagrangian submanifolds, the action $\mathcal{S}_{\mathrm{QZ}}$ reduces to the RW model up to a factor of $\hbar$ (see [56, Section 4]):

$$\mathcal{S}_{\mathrm{QZ}}\Big|_{\mathrm{GF}} = \frac{1}{2}\int d^3z \left(\Omega_{ij} X^i_{(1)} \wedge d^\nabla X^j_{(1)} - \frac{1}{3} R^i_{k\bar{k}j} X^k_{(1)} \wedge \Omega_{li} X^l_{(1)} \wedge X^j_{(1)} V^{\bar{k}}_{(0)} \right), \tag{4.3.5}$$

with $d^{\nabla}X^i_{(1)} = dX^i_{(1)} + \Gamma^i_{jk} dX^j_{(0)} X^k_{(1)}$. Note that the only fields left are the even scalar $X^i_{(0)}$, the odd 1-form $X^i_{(1)}$ and the odd scalar $V^{\bar{k}}_{(0)}$. A quick glance to our expression for the RW model in (3.0.5) (assume again the $(2,0)$ part of the curvature is zero as well as we cut off the perturbative expansion of the $(1,1)$ part at the $R^i_{ks\bar{j}}$) suggests the association $V^{\bar{k}}_{(0)} \Leftrightarrow dx^{\bar{k}}$. We will comment more on this later.

By expanding $\mathbf{X}^i$ through the geodesic exponential map and by pulling back $\omega_{\mathcal{M}}$ as well as S_{QZ} through it, the authors find

$$\exp^* \omega_{\mathcal{M}} = dP_{\bar{i}} \wedge dX^{\bar{i}} + dQ_{\bar{i}} \wedge dV^{\bar{i}} + \frac{1}{2}\Omega_{ij}(x) dy^i \wedge dy^j - \delta X^{\bar{i}} \delta\Theta_{\bar{i}} \tag{4.3.6}$$

$$\exp^* \mathcal{S}_{\mathrm{QZ}}\Big|_{\tilde{\mathbf{P}}} = \int\limits_{T[1]\Sigma_3} d^3z d^3\theta \left(\tilde{\mathbf{P}}_{\bar{i}} D\mathbf{X}^{\bar{i}} + \mathbf{Q}_{\bar{i}} D\mathbf{V}^{\bar{i}} + \frac{1}{2}\Omega_{ij}\mathbf{y}^i D\mathbf{y}^j - \tilde{\mathbf{P}}_{\bar{i}}\mathbf{V}^{\bar{i}} + \Theta_{\bar{i}}(x;\mathbf{y})\mathbf{V}^{\bar{i}} \right) \tag{4.3.7}$$

where $\Theta_{\bar{i}}$ is of degree 2 and given by

$$\Theta_{\bar{i}}(x;y) = \sum_{n=3}^{\infty} \frac{1}{n!} \nabla_{l_4} \dots \nabla_{l_n} R_{\bar{i}l_1\ l_3}^{\ \ \ k} \Omega_{kl_2}(x) y^{l_1} \dots y^{l_n} \tag{4.3.8}$$

and $\tilde{P}_{\bar{i}} := P_{\bar{i}} + \Theta_{\bar{i}}$.

After removing the *spectator fields* (see [43,57]), the action becomes

$$\int\limits_{T[1]\Sigma_3} d^3z d^3\theta \left(\frac{1}{2}\Omega_{ij}\mathbf{y}^i D\mathbf{y}^j + \Theta_{\bar{i}}(x;\mathbf{y})\mathbf{V}^{\bar{i}} \right), \tag{4.3.9}$$

which further reduces to

$$\int\limits_{T[1]\Sigma_3} d^3z d^3\theta \left(\frac{1}{2}\Omega_{ij}\mathbf{y}^i D\mathbf{y}^j + \Theta_{\bar{i}}(x;\mathbf{y}) V^{\bar{i}}_{(0)} \right) \tag{4.3.10}$$

for degree reasons ($V^{\bar{i}}_{(0)}$ is an odd scalar). This action fails the CME by a $\bar{\partial}$-exact term due to Θ satisfying the Maurer–Cartan equation

$$\bar{\partial}_{[\bar{i}}\Theta_{\bar{j}]} = -(\Theta_{\bar{i}}, \Theta_{\bar{j}}), \tag{4.3.11}$$

where $\bar{\partial}$ is the Dolbeault differential and $[\bar{i}\bar{j}]$ denotes antisymmetrization over the indices $\bar{i}$ and $\bar{j}$.

The hyperKähler structure is then relaxed. A new connection which still preserves Ω (crucial for the perturbative approach through the exponential map above) is found. However, since the connection is not Hermitian, the curvature of Γ exhibits also a $(2,0)$-component. This complicates the exponential map which can not be worked out at all orders as in (4.3.8). In [43], the authors argue that a solution to this problem should originate from principles related to the globalization issues discussed in [13] and the application of Fedosov connection in order to deal with perturbation theory on curved manifold [21]. In the realm of this paper, we furnish an affirmative answer to both their ideas. In particular, as we have seen in Section 4.1, the Fedosov connection allowed to compute the terms in the L_∞-algebra and thus to work out the exponential map. In Section 2.2, we have seen the Grothendieck connection to accomplish the same in the context of formal geometry.

Remark 4.3.2. We can compare the procedure above with our globalization construction by associating $V^{\bar{i}}_{(0)}$ with $dx^{\bar{i}}$. First, note that $\left(R_{\Sigma_3}\right)_{\bar{i}}$ in Eq. (4.3.8) matches with the second term in Eq. (2.2.1). Second, the action in (4.3.10) coincides with our globalized action in (2.2.6) if we "forget" the $(2,0)$-part of the curvature. In particular, by associating $\bar{\partial}$ with $dx^{\bar{i}}\frac{\partial}{\partial x^{\bar{i}}}$, we can interpret the failure of (4.3.10) to satisfy the CME due to the term (4.3.11) as a consequence of the action satisfying the $(1,1)$-part of the dCME (Eq. (2.2.7)).

We reserve the last remark of the section to precise the association between $V^{\bar{i}}_{(0)}$ and $dx^{\bar{i}}$ as well as their "meaning" as we promised in Remark 4.3.1.

Remark 4.3.3. As we have seen above, $V^{\bar{i}}_{(0)}$ and $dx^{\bar{i}}$ arise in two different contexts: the first is an odd scalar coordinate parametrizing the fibers of $T^{0,1}M$, while the second is introduced through the classical Grothendieck connection as well as the perturbative expansion.

Nevertheless, the association makes sense considering that $V^{\bar{i}}_{(0)}$ is interpreted as an odd harmonic zero mode in [43]. In fact, recall from Section 2.1, x is the zero mode obtained from the Euler–Lagrange equation $D\mathbf{X} = 0$. If we enlarge the complex (see Eq. (4.1.14)), the space of fields becomes (4.1.17) meaning that $dx^{\bar{i}} \in T^{\vee 0,1}M$, i.e. an odd zero mode. This association was pointed out first by Qiu and Zabzine in [59].

The presence of these quantities has been known in the literature since the early days of the RW model and has deep consequences. Since they are odd, there can be as many as the dimension of M. As such, the perturbative expansion can not be infinite, but it can only stop at a certain order. This is a crucial difference between the CS and the RW theory, which was originally spotted in [63] and attributed to the need for the RW theory to saturate the zero modes. According to Kontsevich in [49], as a result the RW model can be understood as an AKSZ model with "parameters" (these parameters are $V^{\bar{i}}_{(0)}$ or $dx^{\bar{i}}$). In the same article, he presented a different perspective on this subject by pointing out that the RW invariants come from characteristic classes of holomorphic connections.

5. *BF*-like formulation of the Rozansky–Witten model

In order to quantize our globalized version of the RW model in the quantum BV-BFV framework [24], we need to formulate the model as a BF-like theory. This can be done by exploiting the similarities between the RW theory and the CS theory. These similarities have also been crucial in the construction of [69]. There it was argued that RW could be split following a similar approach to the one of Cattaneo, Mnev and Wernli for the CS theory in [25] (see also [73] for a more detailed exposition).

As shown in [30] (see Eq. (4.1.3)), we have a pairing on $\tilde{\mathcal{F}}_{\Sigma_3,x}$ given by the BV symplectic form which can be defined on homogeneous elements $\hat{\mathbf{Y}} \otimes g_1$ and $\hat{\mathbf{Z}} \otimes g_2$ as

$$\begin{aligned}&\langle -,-\rangle : \tilde{\mathcal{F}}_{\Sigma_3,x} \otimes \tilde{\mathcal{F}}_{\Sigma_3,x} \to \Omega^{\bullet,\bullet}(M),\\ &\qquad \langle \hat{\mathbf{Y}} \otimes g_1, \hat{\mathbf{Z}} \otimes g_2\rangle := \underbrace{\Omega(g_1,g_2)}_{\text{sympl. struct. on } M} \int_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\hat{\mathbf{Y}} \wedge \hat{\mathbf{Z}}\right).\end{aligned} \tag{5.0.1}$$

By expanding $\hat{\mathbf{X}} \in \tilde{\mathcal{F}}_{\Sigma_3,x}$ as $\hat{\mathbf{X}} = \hat{\mathbf{X}}^i e_i$, we have

$$\langle \hat{\mathbf{X}}, \hat{\mathbf{X}}\rangle = \Omega(e_i, e_j) \int_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\hat{\mathbf{X}}^i \wedge \hat{\mathbf{X}}^j\right) = \int_{T[1]\Sigma_3} \mu_{\Sigma_3}\left(\Omega_{ij}\hat{\mathbf{X}}^i \wedge \hat{\mathbf{X}}^j\right). \tag{5.0.2}$$

We can rewrite the globalized action (2.2.6) in the same way as in [30] (see the action in (4.1.6)), we have

$$\tilde{S}_{\Sigma_3,x} = \frac{1}{2}\left\langle \hat{\mathbf{X}}, D\hat{\mathbf{X}}\right\rangle + \left\langle \left(\hat{R}_{\Sigma_3}\right)_j(x;\hat{\mathbf{X}})dx^j, \hat{\mathbf{X}}\right\rangle + \left\langle \left(\hat{R}_{\Sigma_3}\right)_{\bar{j}}(x;\hat{\mathbf{X}})dx^{\bar{j}}, \hat{\mathbf{X}}\right\rangle, \tag{5.0.3}$$

with

$$\begin{aligned}\left(\hat{R}_{\Sigma_3}\right)_j(x,\hat{\mathbf{X}}) &= \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\left(\hat{R}_k\right)_j(\hat{\mathbf{X}}^{\otimes k}),\\ \left(\hat{R}_{\Sigma_3}\right)_{\bar{j}}(x,\hat{\mathbf{X}}) &= \sum_{k=2}^{\infty} \frac{1}{(k+1)!}\left(\hat{R}_k\right)_{\bar{j}}(\hat{\mathbf{X}}^{\otimes k}).\end{aligned} \tag{5.0.4}$$

Now, similarly to the approach in [25], we assume that we can split the L_∞-algebra as

$$\mathfrak{g}[1] = \Omega^{\bullet,\bullet}(M) \otimes T^{\vee 1,0}M = \Omega^{\bullet,\bullet}(M) \otimes V \oplus \Omega^{\bullet,\bullet}(M) \otimes W, \tag{5.0.5}$$

with V and W two isotropic subspaces. We identify $W \cong V^\vee$ via the pairing (in particular thanks to the holomorphic symplectic form). Consequently, the superfield splits as $\hat{\mathbf{X}} = \hat{\mathbf{A}} + \hat{\mathbf{B}} = \hat{\mathbf{A}}^i\xi_i + \xi^i\hat{\mathbf{B}}_i$ with $\xi_i \in V$ and $\xi^i \in W$. Concerning the assignment of degrees, we make the following choices. Since Ω has ghost degree 2 (and as such Ω^{-1} has ghost degree -2), we assign total degree 0 to $\mathbf{A}^i$ and ξ_i, total degree 2 to B_i and total degree -2 to ξ^i. We refer to Table 5.0.1 for an explanation of the ghost degrees for the components of the superfields $\hat{\mathbf{A}}^i$ and $\hat{\mathbf{B}}_i$. Then $\hat{\mathbf{A}}^i \oplus \hat{\mathbf{B}}_i \in \Omega^\bullet(\Sigma_3) \oplus \Omega^\bullet(\Sigma_3)[2]$, which is a BF-like theory.

Remark 5.0.1. As explained in [69, Remark 4.2.2], the splitting of the target $T^{\vee 1,0}_x M$ into two transversal holomorphic Lagrangian subbundles is not possible when M is a K3 surface. Instead, it is possible when $M = T^\vee Y$, with Y any complex manifold. In this case M with the standard holomorphic symplectic form will have a vertical as well as a horizontal polarization.

Table 5.0.1
Explanation for the form degree and ghost degree for the components of the superfields $\hat{\mathbf{A}}^i$ and $\hat{\mathbf{B}}_i$.

	Form degree	Ghost degree
$A^i_{(0)}$	0	0
$A^i_{(1)}$	1	−1
$A^i_{(2)}$	2	−2
$A^i_{(3)}$	3	−3
$B_{(0)i}$	0	2
$B_{(1)i}$	1	1
$B_{(2)i}$	2	0
$B_{(3)i}$	3	−2

To sum up, the space of fields is split as

$$\tilde{\mathcal{F}}^{s}_{\Sigma_3,x} = \Omega^{\bullet}(\Sigma_3) \otimes \Omega^{\bullet,\bullet}(M) \otimes V \oplus \Omega^{\bullet}(\Sigma_3)[2] \otimes \Omega^{\bullet,\bullet}(M) \otimes W. \tag{5.0.6}$$

Definition 5.0.2 *(Globalized split RW action).* The *globalized split RW action* is defined as

$$\begin{aligned}\tilde{\mathcal{S}}^{s}_{\Sigma_3,x} &:= \left\langle \hat{\mathbf{B}}, D\hat{\mathbf{A}} \right\rangle + \left\langle \left(\hat{R}_{\Sigma_3}\right)_j (x; \hat{\mathbf{A}} + \hat{\mathbf{B}}) dx^j, \hat{\mathbf{A}} + \hat{\mathbf{B}} \right\rangle + \left\langle \left(\hat{R}_{\Sigma_3}\right)_{\bar{j}} (x; \hat{\mathbf{A}} + \hat{\mathbf{B}}) dx^{\bar{j}}, \hat{\mathbf{A}} + \hat{\mathbf{B}} \right\rangle \\ &= \hat{\mathcal{S}}^{s}_{\Sigma_3,x} + \mathcal{S}^{s}_{\Sigma_3,x,R} + \mathcal{S}^{s}_{\Sigma_3,x,\bar{R}}.\end{aligned} \tag{5.0.7}$$

We refer to [64] for an explicit expression of the second and third terms in the action. We call the model associated with the action (5.0.7), *globalized split RW model.*

If Σ_3 is a closed manifold, the globalized split RW action satisfies the dCME:

$$d_M \tilde{\mathcal{S}}^{s}_{\Sigma_3,x} + \frac{1}{2}(\tilde{\mathcal{S}}^{s}_{\Sigma_3,x}, \tilde{\mathcal{S}}^{s}_{\Sigma_3,x}) = 0,$$

with $d_M = d_x + d_{\bar{x}}$ the sum of the holomorphic and antiholomorphic Dolbeault differentials on the target manifold M. In the presence of boundary, the globalized split action satisfies the mdCME:

$$\iota_{\tilde{Q}^{s}_{\Sigma_3,x}} \omega^{s}_{\Sigma_3,x} = \delta \tilde{\mathcal{S}}^{s}_{\Sigma_3,x} + \pi^* \alpha^{s,\partial}_{\partial\Sigma_3,x}, \tag{5.0.8}$$

with

$$\begin{aligned}\tilde{Q}^{s}_{\Sigma_3,x} = \int_{T[1]\Sigma_3} \mu_{\Sigma_3} \Bigg(&- d\hat{\mathbf{A}}^i \frac{\delta}{\delta \hat{\mathbf{A}}^i} - d\hat{\mathbf{B}}_i \frac{\delta}{\delta \hat{\mathbf{B}}_i} + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\hat{R}_k\right)^i_j ((\hat{\mathbf{A}} + \hat{\mathbf{B}})^{\otimes k}) dx^j \frac{\delta}{\delta \hat{\mathbf{A}}^i} \\ &- \sum_{k=0}^{\infty} \frac{1}{k!} \left(\hat{R}_k\right)^l_j ((\hat{\mathbf{A}} + \hat{\mathbf{B}})^{\otimes k}) dx^j \Omega_{li} \frac{\delta}{\delta \hat{\mathbf{B}}_i} + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\hat{R}_k\right)^i_{\bar{j}} ((\hat{\mathbf{A}} + \hat{\mathbf{B}})^{\otimes k}) dx^{\bar{j}} \frac{\delta}{\delta \hat{\mathbf{A}}^i}, \\ &- \sum_{k=0}^{\infty} \frac{1}{k!} \left(\hat{R}_k\right)^l_{\bar{j}} ((\hat{\mathbf{A}} + \hat{\mathbf{B}})^{\otimes k}) dx^{\bar{j}} \Omega_{li} \frac{\delta}{\delta \hat{\mathbf{B}}_i} \Bigg),\end{aligned} \tag{5.0.9}$$

$$\omega^{s}_{\Sigma_3,x} = \int_{T[1]\Sigma_3} \mu_{\Sigma_3} \left(\delta \hat{\mathbf{B}}_i \delta \hat{\mathbf{A}}^i \right), \tag{5.0.10}$$

$$\alpha^{s,\partial}_{\partial\Sigma_3,x} = \int_{T[1]\partial\Sigma_3} \mu_{\partial\Sigma_3} \left(\hat{\mathbf{B}}_i \delta \hat{\mathbf{A}}^i \right). \tag{5.0.11}$$

6. Perturbative quantization of the globalized split Rozansky–Witten model

In the last section, we have formulated our globalized RW model as a BF-like theory. This allows us to quantize perturbatively the newly constructed globalized split RW model according to the Quantum BV-BFV framework [24]. The quantization of the kinetic part of the action is analogous to the example of section 3 in [24], since the theory reduces to the abelian BF theory. Hence we will be rather quick in the exposition referring to [24] for further details. We will focus our attention to the interacting part of the action (in our case this is actually just the globalization term), which has a rich, as well as complicated, structure. In particular, we will draw some comparison with the PSM, which has been considered in [27].

6.1. Polarization

The recipe to perturbatively quantize a BF-like theory according to the quantum BV-BFV formalism starts by requiring the data of a polarization.

Following the result of Section 5, in the globalized split RW theory, the space of boundary fields splits as

$$\tilde{\mathcal{F}}^{s,\partial}_{\partial\Sigma_3,x} = \Omega^\bullet(\partial\Sigma_3)\otimes\Omega^{\bullet,\bullet}(M)\otimes V \oplus \Omega^\bullet(\partial\Sigma_3)[2]\otimes\Omega^{\bullet,\bullet}(M)\otimes W. \tag{6.1.1}$$

Since we split $T^{1,0}M$ into isotropic subspaces, by the isotropy condition the subspaces are, in particular, Lagrangian. Therefore, either of them can be used as a base or fiber of the polarization.

Notation 6.1.1. From now on we will drop the hat from the notation of the "globalized" superfields (e.g. $\hat{\mathbf{A}}^i$). Moreover, we will denote the coordinates on the base of the polarization by $\mathbb{A}^i$ or $\mathbb{B}^i$ and refer to this choice as $\mathbb{A}$- or $\mathbb{B}$-representation.

Let us choose a decomposition of the boundary $\partial\Sigma_3 = \partial_1\Sigma_3 \sqcup \partial_2\Sigma_3$, where $\partial_1\Sigma_3$ and $\partial_2\Sigma_3$ are two compact manifolds. Here, we can define a polarization $\mathcal{P}$ by choosing the $\mathbb{A}$-representation on $\partial_1\Sigma_3$ and the $\mathbb{B}$-representation on $\partial_2\Sigma_3$. The space of leaves of the associated foliations are $\mathcal{B}_1 := \Omega^\bullet(\partial_1\Sigma_3)$ and $\mathcal{B}_2 := \Omega^\bullet(\partial_2\Sigma_3)[2]$, respectively. The space of boundary fields is $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma_3} = \mathcal{B}_1\times\mathcal{B}_2 \ni (\mathbb{A}^i, \mathbb{B}_i)$.

The BFV 1-form is

$$\alpha^{s,\partial,\mathcal{P}}_{\partial\Sigma_3,x} = \int_{\partial_1\Sigma_3} \mathbf{B}_i\delta\mathbf{A}^i - \int_{\partial_2\Sigma_3} \delta\mathbf{B}_i\mathbf{A}^i \tag{6.1.2}$$

and the quadratic part of the action (5.0.7) is

$$\hat{\mathcal{S}}^{s,\mathcal{P}}_{\Sigma_3,x} = \int_{\Sigma_3} \mathbf{B}_i d\mathbf{A}^i - \int_{\partial_2\Sigma_3} \mathbf{B}_i\mathbf{A}^i. \tag{6.1.3}$$

6.2. Extraction of boundary fields

We split the space of fields as

$$\begin{aligned} \tilde{\mathcal{F}}^{s}_{\Sigma_3,x} &\to \tilde{\mathcal{B}}^{\mathcal{P}}_{\partial\Sigma_3}\oplus\mathcal{Y} \\ (\mathbf{A}^i,\mathbf{B}_i) &\mapsto (\tilde{\mathbb{A}}^i,\tilde{\mathbb{B}}_i)\oplus(\underline{\mathbf{A}}^i,\underline{\mathbf{B}}_i), \end{aligned} \tag{6.2.1}$$

where $\tilde{\mathcal{B}}^{\mathcal{P}}_{\partial\Sigma_3}$ denotes the bulk extension of $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma_3}$ to $\tilde{\mathcal{F}}^{s}_{\Sigma_3,x}$ with $\tilde{\mathbb{A}}^i$ and $\tilde{\mathbb{B}}_i$ the extensions of the boundary fields $\mathbb{A}^i$ and $\mathbb{B}$ to the bulk space of fields $\tilde{\mathcal{F}}^{s}_{\Sigma_3,x}$; $\underline{\mathbf{A}}^i$ and $\underline{\mathbf{B}}_i$ are the bulk fields, which are required to restrict to zero on $\partial_1\Sigma_3$ and $\partial_2\Sigma_3$, respectively. Here, the extensions are chosen to be singular: $\tilde{\mathbb{A}}^i$ and $\tilde{\mathbb{B}}_i$ are required to restrict to zero outside the boundary (a choice pointed out first in [24]). The action reduces to

$$\hat{\mathcal{S}}^{s,\mathcal{P}}_{\Sigma_3,x} = \int_{\Sigma_3} \underline{\mathbf{B}}_i d\underline{\mathbf{A}}^i - \left(\int_{\partial_2\Sigma_3} \mathbb{B}_i\underline{\mathbf{A}}^i - \int_{\partial_1\Sigma_3} \underline{\mathbf{B}}_i\mathbb{A}^i \right). \tag{6.2.2}$$

6.3. Construction of Ω_0

At this point, we can construct the coboundary operator Ω_0 by canonical quantization: we consider the boundary action and we replace any $\underline{\mathbf{B}}_i$ by $(-i\hbar\frac{\delta}{\delta\mathbb{A}^i})$ on $\partial_1\Sigma_3$, any $\underline{\mathbf{A}}^i$ by $(-i\hbar\frac{\delta}{\delta\mathbb{B}_i})$ on $\partial_2\Sigma_3$. We obtain

$$\Omega_0 = -i\hbar\left(\int_{\partial_2\Sigma_3} d\mathbb{B}_i\frac{\delta}{\delta\mathbb{B}_i} + \int_{\partial_1\Sigma_3} d\mathbb{A}^i\frac{\delta}{\delta\mathbb{A}^i} \right). \tag{6.3.1}$$

6.4. Choice of residual fields

The bulk contribution in the space of fields $\mathcal{Y}$ is further split into the space of residual fields $\mathcal{V}_{\Sigma_3}$ and a complement, the space of fluctuations fields $\mathcal{Y}'$, namely

$$\mathcal{Y} \to \mathcal{V}_{\Sigma_3}\oplus\mathcal{Y}' \tag{6.4.1}$$

$$(\underline{\mathbf{A}}^i,\underline{\mathbf{B}}_i) \mapsto (\mathsf{a}^i,\mathsf{b}_i)\oplus(\alpha^i,\beta_i), \tag{6.4.2}$$

where a^i and b_i are the residual fields, whereas α^i and β_i are the fluctuations. Note that the fluctuation α^i is required to restrict to zero on $\partial_1\Sigma_3$ while β_i is required to restrict to zero on $\partial_2\Sigma_3$. In our case, the minimal space of residual fields is

$$\mathcal{V}_{\Sigma_3} = H^\bullet(\Sigma_3, \partial_1\Sigma_3)[0] \oplus H^\bullet(\Sigma_3, \partial_2\Sigma_3)[2] \ni (\mathsf{a}^i, \mathsf{b}_i). \tag{6.4.3}$$

Here we can also define the BV Laplacian. To do it, pick a basis $\{[\chi_i]\}$ of $H^\bullet(\Sigma_3, \partial_1\Sigma_3)$ and its dual basis $\{[\chi^i]\}$ of $H^\bullet(\Sigma_3, \partial_2\Sigma_3)$ with representatives χ_i in $\Omega^\bullet(\Sigma_3, \partial_1\Sigma_3)$ and χ^i in $\Omega^\bullet(\Sigma_3, \partial_2\Sigma_3)$, with $\int_{\Sigma_3}\chi_i\chi^j = \delta_i^j$. We can write the residual fields in a basis as

$$\begin{aligned} \mathsf{a}^i &= \sum_k (z^k\chi_k)^i, \\ \mathsf{b}_i &= \sum_k (z_k^+\chi^k)_i, \end{aligned} \tag{6.4.4}$$

where $\{z^k, z_k^+\}$ are canonical coordinates on $\mathcal{V}_{\Sigma_3}$ with BV symplectic form

$$\omega_{\mathcal{V}_{\Sigma_3}} = \sum_i (-1)^{\deg z^k}\delta z_k^+\delta z^k. \tag{6.4.5}$$

Finally, the BV Laplacian on $\mathcal{V}_{\Sigma_3}$ is

$$\Delta_{\mathcal{V}_{\Sigma_3}} = \sum_i (-1)^{\deg z^k}\frac{\partial}{\partial z^k}\frac{\partial}{\partial z_k^+}. \tag{6.4.6}$$

6.5. Gauge-fixing and propagator

We now have to fix a Lagrangian subspace $\mathcal{L}$ of $\mathcal{Y}'$. In the case of abelian BF theory, in [24], the authors proved that such Lagrangian can be obtained from a *contracting triple* (ι, p, K) for the complex $\Omega^\bullet_{\underline{D}}(\Sigma_3)$.

In particular, the integral kernel of K is the propagator, which we call η. Since K is actually the inverse of an elliptic operator (as shown in [24]), the propagator is singular on the diagonal of $\Sigma_3\times\Sigma_3$. Hence, we will define it as follows. Let

$$\mathrm{Conf}_2(\Sigma_3) = \{(x_1, x_2)\in\Sigma_3 \mid x_1\neq x_2\}, \tag{6.5.1}$$

and let $\iota_{\mathfrak{D}}$ be the inclusion of

$$\mathfrak{D} := \{x_1\times x_2 \in (\partial_1\Sigma_3\times\Sigma_3)\cup(\Sigma_3\times\partial_2\Sigma_3) \mid x_1\neq x_2\} \tag{6.5.2}$$

into $\mathrm{Conf}_2(\Sigma_3)$. Then the propagator is the 2-form $\eta\in\Omega^2(\mathrm{Conf}_2(\Sigma_3), \mathfrak{D})$, where

$$\Omega^\bullet(\mathrm{Conf}_2(\Sigma_3), \mathfrak{D}) = \{\gamma\in\Omega^\bullet(\mathrm{Conf}_2(\Sigma_3)) \mid \iota_{\mathfrak{D}}^*\gamma = 0\}. \tag{6.5.3}$$

Explicitly,

$$\eta(x_1, x_2) = \frac{1}{T_{\Sigma_3}}\frac{1}{i\hbar}\int_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}^{\mathsf{s},\mathcal{P}}_{\Sigma_3,x}}\pi_1^*\alpha^i(x_1)\pi_2^*\beta_i(x_2), \tag{6.5.4}$$

with π_1, π_2 the projections from $M\times M$ to its first and second factor. The coefficient T_{Σ_3} is related to the Reidemeister torsion on Σ_3 as shown in [24]. However, its precise nature is irrelevant for the purposes of the present paper.

6.6. The quantum state

We can sum up the splittings we have made so far as

$$\begin{aligned} \tilde{\mathcal{F}}^{\mathsf{s}}_{\Sigma_3,x} &\to \mathcal{B}^{\mathcal{P}}_{\partial\Sigma_3}\times\mathcal{V}^{\mathcal{P}}_{\Sigma_3}\times\mathcal{Y}' \\ (\mathbf{A}^i, \mathbf{B}_i) &\mapsto (\mathbb{A}^i, \mathbb{B}_i) + (\mathsf{a}^i, \mathsf{b}_i) + (\alpha^i, \beta_i). \end{aligned} \tag{6.6.1}$$

Remark 6.6.1. As a result of the procedure detailed in [24], this is referred to as *good splitting*.

According to the splitting of the space of fields, the action decomposes as

$$\mathcal{S}^{\mathsf{s},\mathcal{P}}_{\Sigma_3,x} = \hat{\mathcal{S}}^{\mathsf{s},\mathcal{P}}_{\Sigma_3,x} + \hat{\mathcal{S}}^{\mathrm{pert}} + \mathcal{S}^{\mathrm{res}} + \mathcal{S}^{\mathrm{source}}, \tag{6.6.2}$$

where

$$\hat{\mathcal{S}}^{\mathrm{s},\mathcal{P}}_{\Sigma_3,x} = \int\limits_{\Sigma_3} \beta_i d\alpha^i, \tag{6.6.3}$$

$$\hat{\mathcal{S}}^{\mathrm{pert}} = \int\limits_{\Sigma_3} \mathcal{V}(\underline{\mathbf{A}}, \underline{\mathbf{B}}), \tag{6.6.4}$$

$$\mathcal{S}^{\mathrm{res}} = -\Bigg(\int\limits_{\partial_2\Sigma_3} \mathbb{B}_i \mathsf{a}^i - \int\limits_{\partial_1\Sigma_3} \mathsf{b}_i \mathbb{A}^i \Bigg), \tag{6.6.5}$$

$$\mathcal{S}^{\mathrm{source}} = -\Bigg(\int\limits_{\partial_2\Sigma_3} \mathbb{B}_i \alpha^i - \int\limits_{\partial_1\Sigma_3} \beta_i \mathbb{A}^i \Bigg), \tag{6.6.6}$$

where $\hat{\mathcal{S}}^{\mathrm{pert}}$ is an interacting term made up by a density-valued function $\mathcal{V}$ which depends on the fields but not on their derivatives (by assumption).

The state is given by:

$$\begin{aligned} \hat{\psi}_{\Sigma_3}(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) &= \int\limits_{(\alpha,\beta)\in\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}^{\mathrm{s},\mathcal{P}}_{\Sigma_3,x}(\mathbb{A}+\mathsf{a}+\alpha, \mathbb{B}+\mathsf{b}+\beta)} \mathscr{D}[\alpha]\mathscr{D}[\beta] \\ &= e^{\frac{i}{\hbar}\mathcal{S}^{\mathrm{res}}} \int\limits_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}^{\mathrm{s},\mathcal{P}}_{\Sigma_3,x}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}^{\mathrm{pert}}} e^{\frac{i}{\hbar}\mathcal{S}^{\mathrm{source}}}, \end{aligned} \tag{6.6.7}$$

where we denote by $\mathscr{D}$ a formal measure on $\mathcal{L}$. The idea here is to compute the integral through a perturbative expansion, hence let us expand the exponentials as

$$\begin{aligned} \hat{\psi}_{\Sigma_3}(\mathbb{A}, \mathbb{B}, \mathsf{a}, \mathsf{b}) = \sum_{k,l,m} \frac{1}{k!l!m!}(-1)^{k+m} &\Bigg(\int\limits_{\partial_2\Sigma_3} \mathbb{B}_i \mathsf{a}^i - \int\limits_{\partial_1\Sigma_3} \mathsf{b}_i \mathbb{A}^i \Bigg)^k \times \\ \times \int\limits_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}^{\mathrm{s},\mathcal{P}}_{\Sigma_3,x}} &\Bigg(\int\limits_{\Sigma_3} \mathcal{V}(\underline{\mathbf{A}}, \underline{\mathbf{B}}) \Bigg)^l \Bigg(\int\limits_{\partial_2\Sigma_3} \mathbb{B}_i \alpha^i - \int\limits_{\partial_1\Sigma_3} \beta_i \mathbb{A}^i \Bigg)^m. \end{aligned} \tag{6.6.8}$$

In the globalized split RW model, the interaction term is actually given by the globalization terms (the second and third terms in the action (5.0.7)). After having expanded the globalization terms in residual fields and in fluctuations, the integration over $\mathcal{L}$ can be solved by using the *Wick theorem*.

6.7. Feynman rules

In this section, we are going to introduce the Feynman rules needed to define precisely the quantum state of our theory. Before going into details, we take a short detour and we introduce the *composite fields*.

Since our aim is to prove the mdQME for the globalized split RW model, we will need to take care of the quantum Grothendieck BFV operator. This is a coboundary operator in which higher functional derivatives may appear (and as we will see they will indeed be present). As explained in [24], higher functional derivatives requires a sort of “regularization”. This is provided by the composite fields, which we denote by square brackets [ ] (e.g. for the boundary field $\mathbb{B}$, we will write $[\mathbb{B}_{i_1} \dots \mathbb{B}_{i_k}]$). The regularization works as follow: a higher functional $\frac{\delta^k}{\delta\mathbb{B}_{i_1}\dots\delta\mathbb{B}_{i_k}}$ is replaced by a first order functional derivative $\frac{\delta^k}{[\delta\mathbb{B}_{i_1}\dots\delta\mathbb{B}_{i_k}]}$. For further details see [24].

Definition 6.7.1 *(Globalized split RW Feynman graph).* A *globalized split RW Feynman graph* is an oriented graph with three types of vertices $V(\Gamma) = V_{\mathrm{bulk}}(\Gamma) \sqcup V_{\partial_1} \sqcup V_{\partial_2}$, called bulk vertices and type 1 and 2 boundary vertices, such that

- bulk vertices can have any valence,
- type 1 boundary vertices carry any number of incoming half-edges (and no outgoing half-edges),
- type 2 boundary vertices carry any number of outgoing half-edges (and no incoming half-edges),
- multiple edges and loose half-edges (leaves) are allowed.

A labeling of a Feynman graph is a function from the set of half-edges to $\{1, \dots, \dim V\}$.

In our case our source manifold Σ_3 has boundary $\partial\Sigma_3 = \partial_1\Sigma_3 \sqcup \partial_2\Sigma_3$, let Γ be a Feynman graph and define

$$\mathrm{Conf}_\Gamma(\Sigma_3) := \mathrm{Conf}_{V_{\mathrm{bulk}}}(\Sigma_3) \times \mathrm{Conf}_{V_{\partial_1}}(\partial_1\Sigma_3) \times \mathrm{Conf}_{V_{\partial_2}}(\partial_2\Sigma_3). \tag{6.7.1}$$

The Feynman rules are given by a map associating to a Feynman graph Γ a differential form $\omega_\Gamma \in \Omega^\bullet(\mathrm{Conf}_\Gamma(\Sigma_3))$.

Definition 6.7.2 *(Globalized split RW Feynman rules).* Let Γ be a labeled Feynman graph. We choose a configuration $\iota : V(\Gamma) \to \mathrm{Conf}(\Gamma)$, such that decompositions are respected. Then, we *decorate* the graph according to the following rules, namely, the *Feynman rules*:

- Bulk vertices in Σ_3 decorated by "globalized vertex tensors"

$$\begin{aligned}\left(\hat{R}_k\right)^{i_1\dots i_s}_{j;j_1\dots j_t} dx^j &:= \frac{\partial^{s+t}}{\partial\underline{\mathbf{A}}^{i_1}\dots\partial\underline{\mathbf{A}}^{i_s}\partial\underline{\mathbf{B}}_{j_1}\dots\partial\underline{\mathbf{B}}_{j_t}}\Bigg|_{\underline{\mathbf{A}}=\underline{\mathbf{B}}=0}\left(\hat{R}_k\right)^i_j((\underline{\mathbf{A}}+\underline{\mathbf{B}})^{\otimes k})(\Omega_{il}\underline{\mathbf{A}}^l+\underline{\mathbf{B}}_i)dx^j\\ \left(\hat{R}_k\right)^{i_1\dots i_s}_{\bar{j};j_1\dots j_t} dx^{\bar{j}} &:= \frac{\partial^{s+t}}{\partial\underline{\mathbf{A}}^{i_1}\dots\partial\underline{\mathbf{A}}^{i_s}\partial\underline{\mathbf{B}}_{j_1}\dots\partial\underline{\mathbf{B}}_{j_t}}\Bigg|_{\underline{\mathbf{A}}=\underline{\mathbf{B}}=0}\left(\hat{R}_k\right)^i_{\bar{j}}((\underline{\mathbf{A}}+\underline{\mathbf{B}})^{\otimes k})(\Omega_{il}\underline{\mathbf{A}}^l+\underline{\mathbf{B}}_i)dx^{\bar{j}}\end{aligned} \tag{6.7.2}$$

  where s, t are the out- and in-valencies of the vertex and $i_1, \dots, i_s$ and $j_1, \dots, j_t$ are the labels of the out (respectively in-)oriented half-edges.
- Boundary vertices $v \in V_{\partial_1}(\Gamma)$ with incoming half-edges labeled $i_1, \dots, i_k$ and no out-going half-edges are decorated by a composite field $[\mathbb{A}^{i_1}\dots\mathbb{A}^{i_k}]$ evaluated at the point (vertex location) $\iota(v)$ on $\partial_1\Sigma_3$.
- Boundary vertices $v \in V_{\partial_2}$ on $\partial_2 M$ with outgoing half-edges labeled $j_1, \dots, j_l$ are decorated by $[\mathbb{B}_{j_1}\dots\mathbb{B}_{j_l}]$ evaluated at the point on $\partial_2\Sigma_3$.
- Edges between vertices v_1, v_2 are decorated with the propagator $\eta(\iota(v_1), \iota(v_2)) \cdot \delta^i_j$, with η the propagator induced by $\mathcal{L} \subset \mathcal{Y}'$, the gauge-fixing Lagrangian.
- Loose half-edges (leaves) attached to a vertex v and labeled i are decorated with the residual fields a^i (for out-orientation), b_i (for in-orientation) evaluated at the point $\iota(v)$.

The Feynman Rules are represented in Figs. 6.7.1, 6.7.2 and 6.7.3.

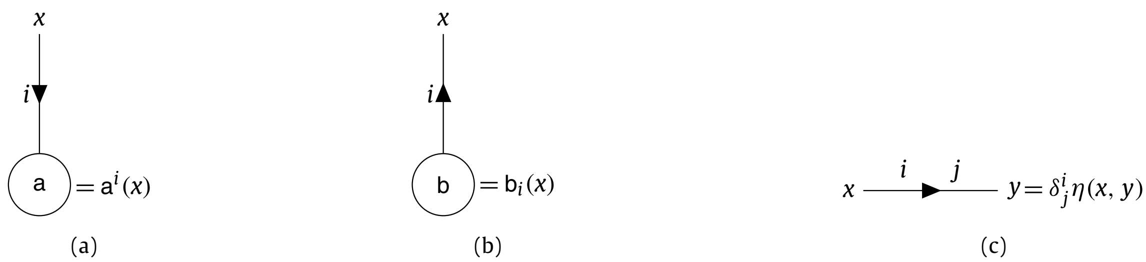


Fig. 6.7.1. Feynman rules for residual fields and propagator.

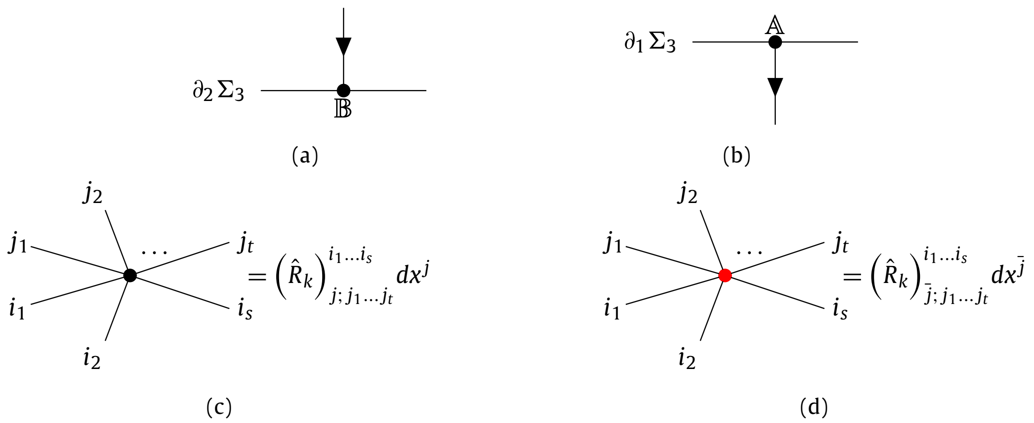


Fig. 6.7.2. Feynman rules for boundary fields and interaction vertices: we denote with a black dot the vertices arising from the (2, 0) part of the curvature (i.e. the terms corresponding to the term $\mathcal{S}_R$ in the action) and with a red dot the ones coming from the (1, 1) part (i.e. the terms corresponding to the term $\mathcal{S}_{\bar{R}}$ in the action). Informally, we will call the first type of vertices "black" vertex and the second one "red" vertex. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

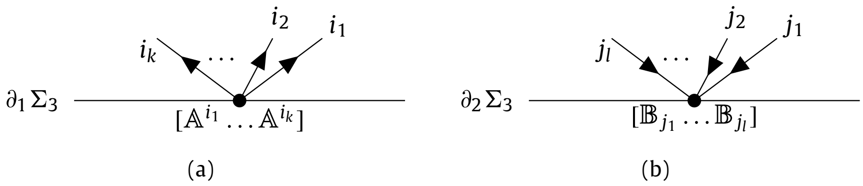


Fig. 6.7.3. Feynman rules for the composite fields.

The full covariant quantum state for globalized split RW theory is defined analogously as in [24].

Definition 6.7.3 *(Full quantum state for the globalized split RW theory).* Let Σ_3 be a 3-dimensional manifold with boundary. Consider the data of a globalized split RW theory which consists of the globalized split space of fields $\tilde{\mathcal{F}}^{s}_{\Sigma_3,x}$ as in (5.0.6), the globalized split space of boundary fields $\tilde{\mathcal{F}}^{s,\partial}_{\partial\Sigma_3,x}$ as in (6.1.1), a polarization $\mathcal{P}$ on $\tilde{\mathcal{F}}^{s,\partial}_{\partial\Sigma_3,x}$ a good splitting $\tilde{\mathcal{F}}^{s}_{\Sigma_3,x} = \mathcal{B}^{\mathcal{P}}_{\partial\Sigma_3} \times \mathcal{V}^{\mathcal{P}}_{\Sigma_3} \times \mathcal{Y}'$ and $\mathcal{L} \subset \mathcal{Y}'$, the gauge-fixing Lagrangian. We can define the *full quantum state for the globalized split RW theory* by the formal power series

$$\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R}(\mathbb{A},\mathbb{B};\mathsf{a},\mathsf{b}) = T_{\Sigma_3}\exp\left(\frac{i}{\hbar}\sum_{\Gamma}\frac{(-i\hbar)^{\mathrm{loops}(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\int\limits_{\mathsf{C}_\Gamma(\Sigma_3)}\omega_\Gamma(\mathbb{A},\mathbb{B};\mathsf{a},\mathsf{b})\right). \tag{6.7.3}$$

7. Proof of the modified differential Quantum Master Equation

In the BV-BFV formalism on manifolds with boundary we expect the mQME to hold. This is a condition which requires the quantum state to be closed under a certain coboundary operator (see [24]). However, in the context of a globalized AKSZ theory, this condition becomes more complicated. The new condition is called *modified differential Quantum Master Equation* (mdQME). We refer to [15,23] for a discussion of the classical and quantum aspects of this condition. An extension for this discussion for manifolds with boundary was provided in [26]. Finally, in [27] the mdQME for anomaly-free, unimodular split AKSZ theories was proven, and later on in [28] for the globalized PSM.

Our aim in this section is to prove the mdQME for the globalized split RW model, namely

$$\nabla_{\mathrm{G}}\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R} = 0, \tag{7.0.1}$$

where ∇_{G} is the *quantum Grothendieck BFV (qGBFV) operator* and $\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R}$ is the full covariant quantum state for the globalized split RW theory. As we will see, the proof follows almost verbatim from the proof of the mdQME in [27]. Before addressing the proof, we focus on the qGBFV operator and we discuss the construction of the full BFV boundary operator.

7.1. The quantum Grothendieck BFV operator

Definition 7.1.1 *(qGBFV operator for the globalized split RW model).* Inspired by [27], we define the *qGBFV operator for the globalized split RW model* as

$$\nabla_{\mathrm{G}} := \left(d_x + d_{\bar{x}} - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\boldsymbol{\Omega}_{\partial\Sigma_3}\right), \tag{7.1.1}$$

with $\boldsymbol{\Omega}_{\partial\Sigma_3}$ the full BFV boundary operator

$$\boldsymbol{\Omega}_{\partial\Sigma_3} = \boldsymbol{\Omega}^{\mathbb{A}}_{\partial\Sigma_3} + \boldsymbol{\Omega}^{\mathbb{B}}_{\partial\Sigma_3} = \Omega^{\mathbb{A}}_0 + \boldsymbol{\Omega}^{\mathbb{A}}_{\mathrm{pert}} + \Omega^{\mathbb{B}}_0 + \boldsymbol{\Omega}^{\mathbb{B}}_{\mathrm{pert}}, \tag{7.1.2}$$

where

$$\begin{aligned}\Omega^{\mathbb{A}}_0 &= -i\hbar\int\limits_{\partial_1\Sigma_3} d\mathbb{A}^i\frac{\delta}{\delta\mathbb{A}^i},\\ \Omega^{\mathbb{B}}_0 &= -i\hbar\int\limits_{\partial_2\Sigma_3} d\mathbb{B}_i\frac{\delta}{\delta\mathbb{B}_i}.\end{aligned} \tag{7.1.3}$$

and $\boldsymbol{\Omega}^{\mathbb{A}}_{\mathrm{pert}}$ and $\boldsymbol{\Omega}^{\mathbb{B}}_{\mathrm{pert}}$ are given by Feynman diagrams collapsing to the boundary in the $\mathbb{A}$-representation and $\mathbb{B}$-representation, respectively.

Remark 7.1.2. Note that ∇_{G} and $\Omega_{\partial\Sigma_3}$ are inhomogeneous forms on the holomorphic symplectic manifold M since the globalized term in the action is a 1-form on M. Explicitly, for example in the $\mathbb{B}$-representation, we can decompose the $\mathbf{\Omega}^{\mathbb{B}}_{\text{pert}}$ as

$$\mathbf{\Omega}^{\mathbb{B}}_{\text{pert}} = \underbrace{\mathbf{\Omega}^{\mathbb{B}}_{1,0} + \mathbf{\Omega}^{\mathbb{B}}_{0,1}}_{:=\mathbf{\Omega}^{\mathbb{B}}_{(1)}} + \underbrace{\mathbf{\Omega}^{\mathbb{B}}_{2,0} + \mathbf{\Omega}^{\mathbb{B}}_{1,1} + \mathbf{\Omega}^{\mathbb{B}}_{0,2}}_{:=\mathbf{\Omega}^{\mathbb{B}}_{(2)}} + \dots. \tag{7.1.4}$$

and similarly in the $\mathbb{A}$-representation.

In the next section, we proceed to give an explicit expression for the BFV boundary operator in the $\mathbb{B}$ and $\mathbb{A}$ representation. We start with the former.

7.2. BFV boundary operator in the $\mathbb{B}$-representation

Let us remind the reader about the general form of the BFV boundary operator in the $\mathbb{B}$-representation for a split AKSZ theory [27]:

$$\mathbf{\Omega}^{\mathbb{B}}_{\text{pert}} := \sum_{n,k\geq 0}\sum_{\Gamma} \frac{(i\hbar)^{\text{loops}(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} \int_{\partial_2 M} \left(\sigma_\Gamma\right)^{I_1\dots I_n}_{J_1\dots J_k} \wedge \mathbb{B}_{I_1} \wedge \dots \wedge \mathbb{B}_{I_n} \left((-1)^{kd}(i\hbar)^k \frac{\delta^{|J_1|+\dots+|J_k|}}{\delta[\mathbb{B}_{J_1}\dots\mathbb{B}_{J_k}]}\right) \tag{7.2.1}$$

In order to find an explicit expression for the BFV boundary operator, we adopt the strategy in [28] to find the BFV boundary operator in the $\mathbb{E}$-representation for the PSM. Their idea was to use the *degree counting*. Indeed, in general, the form σ_Γ is obtained as the integral over the compactification $\tilde{\mathsf{C}}_\Gamma(\mathbb{H}^d)$ of the open configuration space modulo scaling and translation, with $\mathbb{H}^d$ the d-dimensional upper half-space:

$$\sigma_\Gamma = \int_{\tilde{\mathsf{C}}_{\Gamma(\mathbb{H}^d)}} \omega_\Gamma, \tag{7.2.2}$$

where ω_Γ is the product of limiting propagators at the point p of collapse and vertex tensors. Note that in order for the integral (7.2.2) not to vanish the form degree of ω_Γ has to be the same as the dimension of $\tilde{\mathsf{C}}_\Gamma(\mathbb{H}^d)$. This gives constraints to the number of points in the bulk as well as points in the boundary admitted. We will apply this degree counting to our case, where, since we have $d = 3$, the dimension of the compactified configuration space $\tilde{\mathsf{C}}_\Gamma(\mathbb{H}^3)$ is $\dim \tilde{\mathsf{C}}_\Gamma(\mathbb{H}^3) = 3n + 2m - 3$, with n the number of bulk vertices and m the number of boundary vertices in Γ.

By using this procedure, in [28] it was possible to find an explicit expression for the BFV boundary operator in the $\mathbb{E}$-representation for the PSM. As we will see, for us this is not possible. One could say that the cause is the nature of the RW model reflected in a dramatic increment in the number of Feynman rules as we go on in the k-index for the globalized terms in the action (see Eq. (5.0.4)). To see this in practice, let us show explicitly the Feynman rules for the globalization terms in (5.0.7), which we sum up in Table 7.2.1. Notice how the structure of the Feynman rules repeats similarly at each order (e.g. for R_0 we have 2 Feynman rules with degrees 0 and 2, respectively, while for R_1 we have 3 graphs with degrees 0, 2, 4). Hence, it is easy to understand how this works for higher order terms. From there, one can notice that we have two types of vertices:

- vertices which are 1-forms in dx^i: we will denote them by a black dot ($\bullet$) and refer to them as black vertices;
- vertices which are 1-forms in $dx^{\bar{i}}$: we will denote them by a red dot ($\color{red}\bullet$) and refer to them as red vertices.

In our computations, we will limit ourselves to the Feynman rules in Table 7.2.1, these are already enough to get a feeling about what is going on and even understand the behavior of higher order terms, when possible. By using the names in the Table, since $n = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} + \mathrm{VIII} + \mathrm{IX} + \mathrm{X} + \mathrm{XI} + \mathrm{XII} + \mathrm{XIII}$ is the sum of the vertices, the degree counting produces the following equation

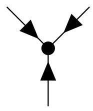
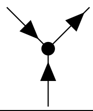
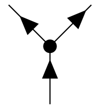
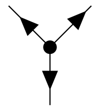
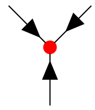
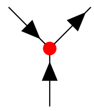
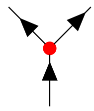
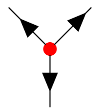
$$\begin{aligned} &\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} + \mathrm{VIII} + \mathrm{IX} + \mathrm{X} + \mathrm{XI} + \mathrm{XII} + \mathrm{XIII} + 2m - 3 = \\ &\qquad 2\mathrm{II} + 2\mathrm{IV} + 4\mathrm{V} + 2\mathrm{VII} + 4\mathrm{VIII} + 6\mathrm{IX} + 2\mathrm{XI} + 4\mathrm{XII} + 6\mathrm{XIII}, \end{aligned} \tag{7.2.3}$$

where on the right hand side we are taking into account that in the $\mathbb{B}$-representation, the arrows leaving the globalization vertex have to stay inside the collapsing subgraph. If this is not the case, by the boundary conditions on the propagator [24], the result would be zero.

First, let us focus on the black vertices (i.e. vertices I–IX). The equation reduces to

$$3\mathrm{I} + \mathrm{II} + 3\mathrm{III} + \mathrm{IV} - \mathrm{V} + 3\mathrm{VI} + \mathrm{VII} - \mathrm{VIII} - 3\mathrm{IX} + 2m - 3 = 0. \tag{7.2.4}$$

Table 7.2.1
Feynman rules for the globalization terms in the action (5.0.7).

Vertex	Feynman rule	Total degree	Name
$(\hat{R}_0)^i_j$		0	I
$(\hat{R}_0)^i_j$		2	II
$\left(\hat{R}_1\right)^i_{j;s}$		0	III
$\left(\hat{R}_1\right)^{si}_j$		2	IV
$\left(\hat{R}_1\right)^{is}_j$		4	V
$\left(\hat{R}_2\right)^i_{j;sm}$		0	VI
$\left(\hat{R}_2\right)^{mi}_{j;s}$		2	VII
$\left(\hat{R}_2\right)^{sim}_j$		4	VIII
$\left(\hat{R}_2\right)^{ism}_j$		6	IX
$\left(\hat{R}_2\right)^i_{\bar{j};sm}$		0	X
$\left(\hat{R}_2\right)^{mi}_{\bar{j};s}$		2	XI
$\left(\hat{R}_2\right)^{smi}_{\bar{j}}$		4	XII
$\left(\hat{R}_2\right)^{ism}_{\bar{j}}$		6	XIII

The Feynman diagrams contributing to the BFV boundary operator are those whose vertices solve the equation (7.2.4). Hence, let us solve the equation case-by-case. Up to one bulk vertex, with the Feynman rules I–IX we have one diagram (see Fig. 7.2.1).

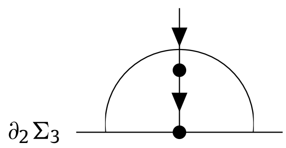


Fig. 7.2.1. First graph with a single black vertex contributing to the BFV boundary operator.

From Fig. 7.2.1, we notice that in order to have a degree 1 operator which satisfies the degree counting for higher order terms we need vertices with an even number of heads and tails. We show the first higher order contributions in Fig. 7.2.2, while a general diagram contributing to the BFV operator is exhibited in Fig. 7.2.3.

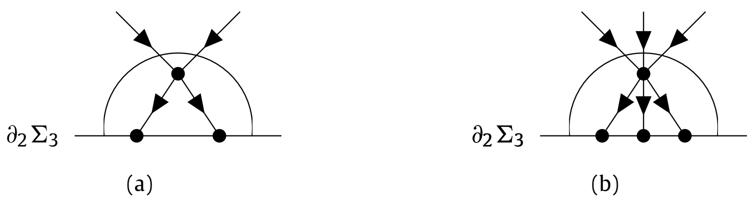


Fig. 7.2.2. Second and third graph with a single black vertex contributing to the BFV boundary operator.

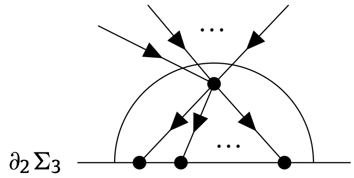


Fig. 7.2.3. A general Feynman diagram contributing to the BFV operator in the $\mathbb{B}$-representation up to one black bulk vertex.

Concerning the red vertices, the graphs contributing to the BFV operator up to one bulk vertex will start to appear from the vertices associated to the term $(R_3)_{\bar{j}}dx^{\bar{j}}$ (coming from the third term in the action (5.0.7)). Taking this into account, the general term of the diagrams with a red vertex is shown in Fig. 7.2.4.

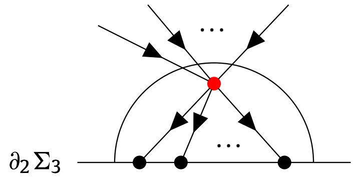


Fig. 7.2.4. A general Feynman diagram contributing to the BFV operator in the $\mathbb{B}$-representation up to one red bulk vertex. In particular, the graph with a total number of 4 arrows (2 entering and 2 leaving the red vertex) is the first non-zero contribution.

These considerations prove the following proposition.

Proposition 7.2.1. *Consider the globalized split RW model, in the $\mathbb{B}$-representation, the first contribution to $\mathbf{\Omega}^{\mathbb{B}}_{\text{pert}}$ is given by $\mathbf{\Omega}^{\mathbb{B}}_{(1)} = \mathbf{\Omega}^{\mathbb{B}}_{1,0} + \mathbf{\Omega}^{\mathbb{B}}_{0,1}$ with*

$$
\begin{aligned}
\mathbf{\Omega}^{\mathbb{B}}_{1,0} &= \sum_{\substack{k\geq 1, S_1,\dots S_k \\ i_1,\dots,i_k,j_1,\dots,j_k}} \frac{(-i\hbar)^k}{(k+S_1+\dots+S_k)!} \int_{\partial_2\Sigma_3} \left(\hat{R}_{2k-1}\right)^{i_1\dots i_k}_{j;j_1\dots j_k} dx^j [\mathbb{B}_{i_1}\mathbb{B}_{S_1}]\dots \\
&\qquad \times [\mathbb{B}_{i_k}\mathbb{B}_{S_k}] \frac{\delta^{|j_1+\dots+j_k|+|S_1|+\dots+|S_k|}}{\delta[\mathbb{B}_{j_1}\dots\mathbb{B}_{j_k}][\delta\mathbb{B}_{S_1}]\dots[\delta\mathbb{B}_{S_k}]} \\
\mathbf{\Omega}^{\mathbb{B}}_{0,1} &= \sum_{\substack{k\geq 2, S_1,\dots S_k \\ i_1,\dots,i_k,j_1,\dots,j_k}} \frac{(-i\hbar)^k}{(k+S_1+\dots+S_k)!} \int_{\partial_2\Sigma_3} \left(\hat{R}_{2k-1}\right)^{i_1\dots i_k}_{\bar{j};j_1\dots j_k} dx^{\bar{j}} [\mathbb{B}_{i_1}\mathbb{B}_{S_1}]\dots \\
&\qquad \times [\mathbb{B}_{i_k}\mathbb{B}_{S_k}] \frac{\delta^{|j_1+\dots+j_k|+|S_1|+\dots+|S_k|}}{\delta[\mathbb{B}_{j_1}\dots\mathbb{B}_{j_k}][\delta\mathbb{B}_{S_1}]\dots[\delta\mathbb{B}_{S_k}]}.
\end{aligned}
\tag{7.2.5}
$$

For $n > 1$, the situation gets more complicated. We solve equation (7.2.4) numerically. Empirically, for an even number of bulk vertices, we witness the absence of solutions. This implies immediately $\mathbf{\Omega}^{\mathbb{B}}_{(2)} = 0$.

In the case $n = 3$, the number of Feynman diagrams for the vertices I–IX increases dramatically with respect to the $n = 1$ case. This increment is tamed since the necessity of having a degree 1 operator will decrease their number. However, we are not able to provide an explicit as well as general form for the BFV operator along the same lines as in Proposition 7.2.1. We rely on examples which we show in [64].

Remark 7.2.2. Here we are assuming that the dimension of our target manifold M is at least 4, if this would not be the case, then we would not have the 3 bulk vertices contribution to the BFV boundary operator. Hence, the number of bulk vertices allowed is bounded by the dimension of M. This was already noticed in [28]. The difference here is that this reflects the "odd Grassmanian nature" of the RW model with respect to CS theory (see Remark 4.3.3).

7.3. BFV boundary operator in the $\mathbb{A}$-representation

In the $\mathbb{A}$-representation, the arrows coming from the globalized vertices are allowed to leave the collapsing subgraph. Therefore, our arguments about the degree counting are not valid here. However since the coboundary operator has a total degree 1, while $\mathbb{A}^i$ has total degree 0, we can have at most 1 bulk vertex, i.e. $\mathbf{\Omega}^{\mathbb{A}}_{\text{pert}} = \mathbf{\Omega}^{\mathbb{A}}_{1,0} + \mathbf{\Omega}^{\mathbb{A}}_{0,1}$ with

$$\begin{aligned}\mathbf{\Omega}^{\mathbb{A}}_{1,0} &= \sum_{k\geq 0} \int_{\partial_1 \Sigma_3} \sum_{J_1,\dots,J_r,I_1,\dots,I_s} \frac{(-i\hbar)^{|I_1|+\dots+|I_s|}}{(|I_1|+\dots+|I_s|)!} \left(\hat{R}_k\right)^{I_1\dots I_s}_{j;J_1\dots J_r} dx^j \prod_{r=1,s=1}^{r+s=k+1} [\mathbb{A}^{J_r}] \frac{\delta^{|I_s|}}{\delta[\mathbb{A}^{I_s}]},\\ \mathbf{\Omega}^{\mathbb{A}}_{0,1} &= \sum_{k\geq 3} \int_{\partial_1 \Sigma_3} \sum_{J_1,\dots,J_r,I_1,\dots,I_s} \frac{(-i\hbar)^{|I_1|+\dots+|I_s|}}{(|I_1|+\dots+|I_s|)!} \left(\hat{R}_k\right)^{I_1\dots I_s}_{\bar{j};J_1\dots J_r} dx^{\bar{j}} \prod_{r=1,s=1}^{r+s=k+1} [\mathbb{A}^{J_r}] \frac{\delta^{|I_s|}}{\delta[\mathbb{A}^{I_s}]},\end{aligned} \tag{7.3.1}$$

where we label by the multiindex J_r the arrows emanating from a boundary vertex towards the globalized vertex, by the multiindex I_s the leaves emanating from the bulk vertex. The sum of r and s has to be $k+1$ since these are the total number of arrows leaving and arriving at a globalized vertex $(R_k)_j dx^j$ (or $(R_k)_{\bar{j}} dx^{\bar{j}}$).

7.4. Flatness of the qGBFV operator for the globalized split RW model

In this section, we prove that the qGBFV operator for the globalized split RW model squares to zero. The proof follows along the same lines as in [27], we will remark where there are differences and refer to their work when the procedure is identical. Before entering into the details of the proof, we should mention that their proof (and the proof of the mdQME) depends on two assumptions: *unimodularity* and *absence of hidden faces* (*anomaly-free* condition). The first means that tadpoles are not allowed. In the case of the globalized split RW model, we notice that this assumption is not needed since tadpoles vanish [63].

Assumption 7.4.1. We assume that the globalized split RW model is *anomaly-free*, i.e. for every graph Γ, we have that

$$\int_{F_{\geq 3}} \omega_\Gamma = 0, \tag{7.4.1}$$

where by $F_{\geq 3}$, we denote the union of the faces where at least three bulk vertices collapse in the bulk (also called *hidden faces* [17]).

Remark 7.4.2. It is well known that Chern–Simons theory is *not* an anomaly-free theory [4,5]. The construction of the quantum theory there depends on the choice of gauge-fixing. The appearance of anomalies can be resolve by choosing a framing and framing-dependent counter terms for the gauge-fixing. A famous example of an anomaly-free theory is given by the Poisson sigma model [20] since by the result of Kontsevich [50] any 2-dimensional theory is actually anomaly-free. A general method for dealing with theories that do have anomalies is to add counter terms to the action. If the differential form ω_Γ, which is integrated over the hidden faces, is exact, one can use the primitive form to cancel the anomalies by the additional vertices that appear.

Since the integrals we will consider are fiber integrals, we will apply of Stokes' theorem for integration along a compact fiber with corners, i.e.

$$d\pi_* = \pi_* d - \pi_*^{\partial}, \tag{7.4.2}$$

where π_* denotes the fiber integration. In particular, the application of Stokes' theorem to a fiber integral yields

$$(d_x + d_{\bar{x}}) \int_{C_\Gamma} \omega_\Gamma = \int_{C_\Gamma} (d + d_{\bar{x}})\omega_\Gamma - \int_{\partial C_\Gamma} \omega_\Gamma, \tag{7.4.3}$$

where d is the differential on $M \times C_\Gamma$.

Theorem 7.4.3 *(Flatness of the qGBFV operator). The qGBFV operator ∇_G for the anomaly-free globalized split RW model squares to zero, i.e.*

$$(\nabla_G)^2 \equiv 0, \tag{7.4.4}$$

where

$$\nabla_G = d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3} = d_x + d_{\bar{x}} - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}. \tag{7.4.5}$$

Proof. According to [27], the flatness of ∇_G is equivalent to the equation

$$i\hbar d_M \mathbf{\Omega}_{\partial\Sigma_3} - \frac{1}{2}\left[\mathbf{\Omega}_{\partial\Sigma_3}, \mathbf{\Omega}_{\partial\Sigma_3}\right] = 0. \tag{7.4.6}$$

This equation was proven for a globalized split AKSZ theory in [27], in which the d_M is just the de Rham differential on the body of the target manifold. However, in our case, d_M is the sum of the holomorphic and antiholomorphic Dolbeault differentials on M.

We prove Eq. (7.4.6) for $\mathbf{\Omega}^{\mathbb{B}}$. For $\mathbf{\Omega}^{\mathbb{A}}$, the proof is analogous as discussed in [27]. Suppose we apply d_M to a term of the form

$$\mathbf{\Omega}^{\mathbb{B}}_{\Gamma} = \int\limits_{\partial_2\Sigma_3} \sigma_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right)^I_{J_1\dots J_s} [\mathbb{B}^{J_1}]\dots[\mathbb{B}^{J_s}]\frac{\delta}{\delta[\mathbb{B}^I]}, \tag{7.4.7}$$

where k could be any number greater than 0. Here, we chose the easiest term to express with more clarity what is going on. As in [27], we apply Stokes' theorem. However, this is different to the corresponding situation in [27] since in our theory we have also red vertices,[6] which is portrayed by the fact that σ_Γ depends also on $\left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}$. We obtain

$$(d_x + d_{\bar{x}})\mathbf{\Omega}^{\mathbb{B}}_{\Gamma} = \int\limits_{\partial_2\Sigma_3} \left\{(d_x + d_{\bar{x}})\sigma_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right)\right\}^I_{J_1\dots J_s} [\mathbb{B}^{J_1}]\dots[\mathbb{B}^{J_s}]\frac{\delta}{\delta[\mathbb{B}^I]} + [\Omega^{\mathbb{B}}_0, \mathbf{\Omega}^{\mathbb{B}}_{\Gamma}], \tag{7.4.8}$$

where the second term is produced when d_x acts on the $\mathbb{B}$ fields (we do not have a corresponding term for $d_{\bar{x}}$ since we do not have fields $\mathbb{B}^{\bar{i}}$ terms to act on). By applying again Stokes' theorem, we have:

$$\begin{aligned}(d_x + d_{\bar{x}})\sigma_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right) &= (d_x + d_{\bar{x}}) \int\limits_{\tilde{C}_\Gamma} \omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right) \\ &= \int\limits_{\tilde{C}_\Gamma} (d + d_{\bar{x}})\omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right) \\ &\quad \pm \int\limits_{\partial\tilde{C}_\Gamma} \omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j; \left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right).\end{aligned} \tag{7.4.9}$$

Remark 7.4.4. In principle, d is the differential on $M \times \mathrm{C}_\Gamma$, hence it can be decomposed as $d = d_x + d_1 + d_2$, where d_1 denotes the part of the differential acting on the propagator and d_2 the part acting on $\mathbb{B}$ fields (and, more generally, on $\mathbb{A}$ fields). We do not have a corresponding antiholomorphic differential on $M \times \mathrm{C}_\Gamma$ since the propagators and the fields are all holomorphic. This is different with respect to the case considered in [27].

As in [27], we have $d\omega_\Gamma = d_x\omega_\Gamma$ and in the boundary integral we have three classes of faces. The first two types of faces, where more than two bulk points collapse and where a subgraph Γ collapse at the boundary, can be proved as in [27]. In particular, the former vanishes by our assumptions that the theory is anomaly-free (see Assumption 7.4.1), while the second produces exactly the term $\frac{1}{2}\left[\mathbf{\Omega}^{\mathbb{B}}_{\text{pert}}, \mathbf{\Omega}^{\mathbb{B}}_{\text{pert}}\right]$ by [27, Lemma 4.9]. On the other hand, the third case, when two bulk vertices collapse, has some differences with respect to the analogous situation in [27] due to the already mentioned further presence of red vertices. Here we distinguish four cases:

- when a red vertex collapses with a black vertex, then these faces cancel out with $d_x\omega_\Gamma\left(\left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right)$ by the dCME (2.2.12);
- when a black vertex collapses with a red vertex, then these faces cancel out with $d_{\bar{x}}\omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j\right)$ by the dCME (2.2.14);
- when two black vertices collapse, then these faces cancel out with $d_x\omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j\right)$ by the dCME (2.2.11);
- when two red vertices collapse, then these faces cancel out with $d_{\bar{x}}\omega_\Gamma\left(\left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right)$ by the dCME (2.2.13).

By $\omega_\Gamma\left(\left(\hat{R}_k\right)_{\bar{j}} dx^{\bar{j}}\right)$ or $\omega_\Gamma\left(\left(\hat{R}_k\right)_j dx^j\right)$, we mean the part of the subgraph Γ', which contains a red or black vertex.

This proves (7.4.6), thus $(\nabla_G)^2 \equiv 0$. □

[6] For the sake of clarity, we stress again that in [27], the vertices are only "black" since d_M is the de Rham differential on the body of the target manifold.

7.5. Proof of the mdQME for the globalized split RW model

In this section, we are going to prove the mdQME for the globalized split RW model. The proof follows similarly as in [27]. As before, we will refer to their work when the situation is identical and point out eventual differences.

Theorem 7.5.1 *(mdQME for anomaly-free globalized split RW model). Consider the full covariant perturbative state $\hat{\psi}_{\Sigma_3,x}$ as a quantization of the anomaly-free globalized split RW model. Then*

$$\left(d_M - i\hbar\Delta_{\mathcal{V}_{\Sigma_3,x}} + \frac{i}{\hbar}\mathbf{\Omega}_{\partial\Sigma_3}\right)\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R} = 0. \tag{7.5.1}$$

Proof. Let $\mathcal{G}$ denote the set of Feynman graphs of the theory. Then, we can write the full covariant quantum state for the globalized split RW model as

$$\hat{\boldsymbol{\psi}}_{\Sigma_3,x,R} = T_{\Sigma_3}\sum_{\Gamma\in\mathcal{G}}\int_{\mathsf{C}_\Gamma}\omega_\Gamma\left(\hat{R}_j dx^j; \hat{R}_{\bar{j}} dx^{\bar{j}}\right), \tag{7.5.2}$$

where the combinatorial prefactor $\frac{(-i\hbar)^{\text{loops}(\Gamma)}}{|\text{Aut}(\Gamma)|}$ is included in ω_Γ (by loops we denote the number of loops of a graph Γ) and we denote the configuration space $\mathsf{C}_\Gamma(\Sigma_3)$ by C_Γ for simplicity. We note that ω_Γ is a ($\mathcal{V}_{\Sigma_3,x}$-dependent) differential form on $\mathsf{C}_\Gamma \times M$. Again, following [27], we can apply Stokes' theorem (7.4.3) and we get

$$d_M\int_{\mathsf{C}_\Gamma}\omega_\Gamma\left(\hat{R}_j dx^j; \hat{R}_{\bar{j}} dx^{\bar{j}}\right) = \int_{\mathsf{C}_\Gamma}(d+d_{\bar{x}})\omega_\Gamma\left(\hat{R}_j dx^j; \hat{R}_{\bar{j}} dx^{\bar{j}}\right) - \int_{\partial\mathsf{C}_\Gamma}\omega_\Gamma\left(\hat{R}_j dx^j; \hat{R}_{\bar{j}} dx^{\bar{j}}\right). \tag{7.5.3}$$

As mentioned in Remark 7.4.4, the d inside the integral is the total differential on $\mathsf{C}_\Gamma(\Sigma_3)\times M$, and thus we can split it as

$$d = d_x + d_1 + d_2, \tag{7.5.4}$$

where d_1 denotes the part of the differential acting on the propagators in ω_Γ and d_2 is the part acting on $\mathbb{B}$ and $\mathbb{A}$ fields.

With this setup, which is basically analogous to the one in [27], except for the presence of the red vertices and $d_{\bar{x}}$ already extensively discussed, Eq. (7.5.1) is verified by proving three relations

- a relation between the application of d_1 and of $\Delta_{\mathcal{V}_{\Sigma_3,x}}$ to the quantum state,
- a relation between the application of d_2 and of Ω_0 to the quantum state,
- a relation between the application of d_M and of the boundary contributions to the quantum state.

The proofs of these relations can be carried from [27] over to the globalized split RW model without any problem. The only difference is when they prove that the contributions in $\partial\mathsf{C}_\Gamma$ consisting of diagrams with two bulk vertices collapsing vanish (which is needed for the third relation). In our case one should consider again three contributions: when two bulk black vertices collapse, when two bulk red vertices collapse, when a red vertex and a black one collapse. The vanishing of these terms follows from Eqs. (2.2.11), (2.2.12), (2.2.13), (2.2.14). The rest of the procedure is identical to [27]. □

8. Outlook and future direction

Our globalization construction leads to an interesting extension of some aspects in the program presented in [29] for manifolds with boundary and cutting-gluing techniques. In particular, it would be of interest to understand some relations to the deformation quantization of Kähler manifolds in the guise of [61], especially using the constructions of [28], and Berezin–Toeplitz quantization as presented in [68] (possibly for the noncompact case). It also leads to a more general globalization construction of an algebraic index theory formulation by using the BV formalism together with Fedosov's globalization approach as presented in [38]. Moreover, it might also be related to a case of twisted topological field theories, known as Chern–Simons–Rozansky–Witten TFTs, constructed by Kapustin and Saulina in [45]. In particular, they use the BRST formalism to produce interesting observables as Wilson loops and thus one might be able to combine it with ideas of [1,54]. Another direction would be the study of the RW invariants through our construction for hyperKähler manifolds. We guess that this would require studying observables of RW theory in the BV-BFV formulation, but the globalization procedure should tell something about these 3-manifold invariants. We hope that this might also be compatible with some generalizations of RW invariants in the non-hyperKähler case as discussed in [62].

References

[1] A. Alekseev, Y. Barmaz, P. Mnev, Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism, J. Geom. Phys. 67 (2013) 1–15.
[2] M. Alexandrov, A. Schwarz, O. Zaboronsky, M. Kontsevich, The geometry of the master equation and topological quantum field theory, Int. J. Mod. Phys. A 12 (07) (Mar. 1997) 1405–1429.
[3] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Am. Math. Soc. 85 (1) (1957) 181–207.
[4] S. Axelrod, I.M. Singer, Chern-Simons perturbation theory, in: International Conference on Differential Geometric Methods in Theoretical Physics, vols. 1, 2, 1991, pp. 3–45.
[5] S. Axelrod, I.M. Singer, Chern-Simons perturbation theory. II, J. Differ. Geom. 39 (1) (1994) 173–213.
[6] J.C. Baez, J. Dolan, Higher-dimensional algebra and topological quantum field theory, J. Math. Phys. 36 (11) (Nov. 1995) 6073–6105.
[7] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (2) (1995) 423–472.
[8] I.A. Batalin, E.S. Fradkin, A generalized canonical formalism and quantization of reducible gauge theories, Phys. Lett. B 122 (1983) 157–164.
[9] I.A. Batalin, E.S. Fradkin, Operator quantization and abelization of dynamical systems subject to first class constraints, Riv. Nuovo Cimento 9 (10) (1986) 1–48.
[10] I.A. Batalin, G.A. Vilkovisky, Relativistic S matrix of dynamical systems with boson and fermion constraints, Phys. Lett. B 69 (1977) 309–312.
[11] I.A. Batalin, G.A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1981) 27–31.
[12] I.A. Batalin, G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (1983) 2567–2582.
[13] L. Baulieu, A.S. Losev, N.A. Nekrasov, Target space symmetries in topological theories I, J. High Energy Phys. 2002 (02) (Feb. 2002) 021.
[14] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (2) (Oct. 1994) 311–427.
[15] F. Bonechi, A.S. Cattaneo, P. Mnev, The Poisson sigma model on closed surfaces, J. High Energy Phys. 2012 (1) (Jan. 2012).
[16] R. Bott, Some aspects of invariant theory in differential geometry, in: Differential Operators on Manifolds, Springer, Berlin, Heidelberg, 2011, pp. 49–145.
[17] R. Bott, A.S. Cattaneo, Integral invariants of 3-manifolds, J. Differ. Geom. 48 (1) (1998) 91–133.
[18] A.S. Cattaneo, P. Mnev, K. Wernli, Quantum Chern-Simons theories on cylinders: BV-BFV partition functions, arXiv:2012.13983 [hep-th], 2021.
[19] A.S. Cattaneo, I. Contreras, Relational symplectic groupoids, Lett. Math. Phys. 105 (5) (Apr. 2015) 723–767.
[20] A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (3) (Aug. 2000) 591–611.
[21] A.S. Cattaneo, G. Felder, On the globalization of Kontsevich's star product and the perturbative Poisson sigma model, Prog. Theor. Phys. Suppl. 144 (Dec. 2001) 38–53.
[22] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Classical and quantum Lagrangian field theories with boundary, PoS C0RFU2011 (2011) 044.
[23] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Classical BV theories on manifolds with boundary, Commun. Math. Phys. 332 (2) (Aug. 2014) 535–603.
[24] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Perturbative quantum gauge theories on manifolds with boundary, Commun. Math. Phys. 357 (2) (Dec. 2017) 631–730.
[25] A.S. Cattaneo, P. Mnev, K. Wernli, Split Chern-Simons theory in the BV-BFV formalism, in: Quantization, Geometry and Noncommutative Structures in Mathematics and Physics, Springer International Publishing, Cham, 2017, pp. 293–324.
[26] A.S. Cattaneo, N. Moshayedi, K. Wernli, Relational symplectic groupoid quantization for constant Poisson structures, Lett. Math. Phys. 107 (9) (Apr. 2017) 1649–1688.
[27] A.S. Cattaneo, N. Moshayedi, K. Wernli, Globalization for perturbative quantization of nonlinear split AKSZ sigma models on manifolds with boundary, Commun. Math. Phys. 372 (1) (Nov. 2019) 213–260.
[28] A.S. Cattaneo, N. Moshayedi, K. Wernli, On the globalization of the Poisson sigma model in the BV-BFV formalism, Commun. Math. Phys. 375 (1) (Mar. 2020) 41–103.
[29] K. Chan, N.C. Leung, Q. Li, Quantization of Kähler manifolds, arXiv:2009.03690, 2020.
[30] K. Chan, N.C. Leung, Q. Li, BV quantization of the Rozansky-Witten model, Commun. Math. Phys. 355 (1) (June 2017) 97–144.
[31] K.J. Costello, A geometric construction of the Witten genus, II, arXiv:1112.0816 [math.QA], 2011.
[32] K.J. Costello, Renormalization and Effective Field Theory, Mathematical Surveys and Monographs, vol. 170, Amer. Math. Soc., 2011.
[33] B.V. Fedosov, The Atiyah-Bott-Patodi method in deformation quantization, Commun. Math. Phys. 209 (2000) 691–728.
[34] B.V. Fedosov, A simple geometrical construction of deformation quantization, J. Differ. Geom. 2 (1994) 213–238.
[35] R.P. Feynman, Space-time approach to quantum electrodynamics, Phys. Rev. 76 (Sept. 1949) 769–789.
[36] E.S. Fradkin, G.A. Vilkovisky, Quantization of relativistic systems with constraints equivalence of canonical and covariant formalisms in quantum theory of gravitational field, Phys. Lett. B 55 (1977) 224–226.
[37] I.M. Gelfand, D.A. Kazhdan, Some problems of the differential geometry and the calculation of cohomologies of Lie algebras of vector fields, Dokl. Akad. Nauk Ser. Fiz. 200 (1971) 269–272.
[38] R.E. Grady, Q. Li, S. Li, Batalin-Vilkovisky quantization and the algebraic index, Adv. Math. 317 (Sept. 2017) 575–639.
[39] N. Habegger, G. Thompson, The universal perturbative quantum 3 manifold invariant, Rozansky-Witten invariants, and the generalized Casson invariant, arXiv:math/9911049, Nov. 1999.
[40] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, Ann. Phys. 235 (2) (Nov. 1994) 435–464.
[41] R. Iraso, Two-dimensional gauge theories in BV formalism and gluing-cutting, Ph.D. Thesis, SISSA, 2018.
[42] R. Iraso, P. Mnev, Two-dimensional Yang-Mills theory on surfaces with corners in Batalin-Vilkovisky formalism, Commun. Math. Phys. 370 (2) (Mar. 2019) 637–702.
[43] J. Källén, J. Qiu, M. Zabzine, Equivariant Rozansky-Witten classes and TFTs, J. Geom. Phys. 64 (Feb. 2013) 222–242.
[44] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compos. Math. 115 (1) (1999) 71–113.
[45] A. Kapustin, N. Saulina, Generalisations of Rozansky-Witten invariants, Nucl. Phys. B 823 (2009) 403–427.
[46] A. Kapustin, L. Rozansky, Three-dimensional topological field theory and symplectic algebraic geometry II, arXiv:0909.3643 [math.AG], 2009.
[47] A. Kapustin, L. Rozansky, N. Saulina, Three-dimensional topological field theory and symplectic algebraic geometry I, Nucl. Phys. B 816 (3) (Aug. 2009) 295–355.
[48] J. Kijowski, W. Tulczyjew, A Symplectic Framework for Field Theories, Lecture Notes in Physics, Springer, Berlin, Heidelberg, ISBN 9783540095385, 1979.
[49] M. Kontsevich, Rozansky-Witten invariants via formal geometry, Compos. Math. 115 (1999) 115–127.
[50] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (3) (Dec. 2003) 157–216.
[51] T.T.Q. Le, J. Murakami, T. Ohtsuki, On a universal perturbative invariant of 3-manifolds, Topology 37 (3) (1998) 539–574.
[52] J. Lurie, On the classification of topological field theories, Curr. Dev. Math. 2008 (2009) 129–280.
[53] N. Moshayedi, On quantum obstruction spaces and higher codimension gauge theories, Phys. Lett. B 815 (2021).
[54] N. Moshayedi, Formal global AKSZ gauge observables and generalized Wilson surfaces, Ann. Henri Poincaré 21 (9) (July 2020) 2951–2995.
[55] T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, Shifted symplectic structures, Publ. Math. IHES (Apr. 2013) 54.
[56] J. Qiu, M. Zabzine, On the AKSZ formulation of the Rozansky-Witten theory and beyond, J. High Energy Phys. 2009 (09) (Sept. 2009) 24.
[57] J. Qiu, M. Zabzine, Odd Chern-Simons theory, Lie algebra cohomology and characteristic classes, Commun. Math. Phys. 300 (3) (Aug. 2010) 789–833.

[58] J. Qiu, M. Zabzine, Knot weight systems from graded symplectic geometry, arXiv:1110.5234 [math.QA], 2011.
[59] J. Qiu, M. Zabzine, Knot invariants and new weight systems from general 3D TFTs, J. Geom. Phys. 62 (2) (Feb. 2012) 242–271.
[60] N.Y. Reshetikhin, V.G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1) (1991) 547–597.
[61] N. Reshetikhin, L. Takhtajan, Deformation quantization of Kähler manifolds, arXiv:math/9907171, 1999.
[62] J. Roberts, J. Sawon, Generalisations of Rozansky-Witten invariants, Geom. Topol. Monogr. 4 (2002) 263–279.
[63] L. Rozansky, E. Witten, HyperKähler geometry and invariants of three manifolds, Sel. Math. 3 (1997) 401–458.
[64] D. Saccardo, Globalization of the Rozansky-Witten model in the BV-BFV formalism, Master Thesis, ETH, Zurich, 2021, https://sites.google.com/view/nimamoshayedi/teaching.
[65] J. Sawon, Topological quantum field theory and hyperKähler geometry, Turk. J. Math. 25 (2001) 169–194.
[66] J. Sawon, Rozansky-Witten invariants of hyperKähler manifolds, arXiv:math/0404360v1 [math.DG], 2004.
[67] P. Schaller, T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 09 (33) (Oct. 1994) 3129–3136.
[68] M. Schlichenmaier, Berezin-Toeplitz quantization for compact Kähler manifolds: a review of results, Adv. Math. Phys. 2010 (2010) 32.
[69] P. Steffens, Functorial field theory from perturbative BV-BFV quantization, Master Thesis, University of Amsterdam, 2017.
[70] G. Thompson, Holomorphic vector bundles, knots and the Rozansky-Witten invariants, Adv. Theor. Math. Phys. 5 (2000) 457–481.
[71] B. Toën, Higher and derived stack: a global overview, in: Algebraic Geometry-Seattle 2005, in: Proc. Sympos. Pure Math, vol. 80, May 2006.
[72] B. Toën, Derived algebraic geometry, arXiv:1401.1044 [math.AG], 2014.
[73] K. Wernli, Perturbative quantization of split Chern-Simons theory on handlebodies and Lens spaces by the BV-BFV formalism, Ph.D. Thesis, University of Zurich, Dec. 2018, http://user.math.uzh.ch/cattaneo/wernli.pdf.
[74] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (3) (1989) 351–399.

Chapter 3

Convolution Algebras for Relational Symplectic Groupoids and Reduction

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CONVOLUTION ALGEBRAS FOR RELATIONAL GROUPOIDS AND REDUCTION

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CONVOLUTION ALGEBRAS FOR RELATIONAL GROUPOIDS AND REDUCTION

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We introduce the notions of relational groupoids and relational convolution algebras. We provide various examples arising from the group algebra of a group G and a given normal subgroup H. We also give conditions for the existence of a Haar system of measures on a relational groupoid compatible with the convolution, and we prove a reduction theorem that recovers the usual convolution of a Lie groupoid.

1. Introduction

Motivation. Symplectic groupoids are fundamental objects in Poisson geometry. Every symplectic groupoid $G \rightrightarrows M$ induces a Poisson structure on M [Coste et al. 1987]. Such Poisson manifolds are called *integrable*, and G is also called a *symplectic realization* of M. It is a well-known fact that not all Poisson manifolds are integrable and that there are explicit obstructions to the integration [Crainic and Fernandes 2004]. However, one can associate to every Poisson manifold M a *relational symplectic groupoid* [Contreras 2013; Cattaneo and Contreras 2015], which is an infinite-dimensional symplectic manifold equipped with Lagrangian submanifolds that model the structure maps of a symplectic groupoid. Hawkins [2008] showed

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that a Poisson manifold can be quantized[1] via a twisted polarized convolution C^*-algebra of a symplectic groupoid integrating that Poisson manifold.

The first main idea behind this paper is to generalize Hawkins' approach to arbitrary Poisson manifolds (integrable or not) by generalizing this construction to relational symplectic groupoids. In order to achieve this objective, we introduce the notion of *relational groupoids* and their corresponding *relational convolution algebras*, which is an analogue of the convolution algebra in the partial category **Rel** of sets and relations.

We study various examples of relational convolution algebras that arise from extending Haar systems of measures to relational groupoids, and we prove our main result: a *reduction theorem for relational convolution algebras*, which recovers the usual groupoid convolution algebra. This is also the first step towards proving the "quantization commutes with reduction" conjecture by Guillemin and Sternberg [1982] in the setting of groupoid quantization.

In particular, this result serves as the first step towards reduction of its quantization (convolution algebras for relational groupoids). The next step is to construct the polarized algebra for relational symplectic groupoids. In addition to this, we hope to use relational convolution algebras to recover the C*-algebra quantization of Poisson pencils via reduction, recovering the results obtained in [Bonechi et al. 2014] regarding the Bohr–Sommerfeld groupoid.

The second main idea in this paper is that the relational symplectic groupoids could be used to study the relation between groupoid quantization and deformation quantization in a field-theoretic way, as follows. Relational symplectic groupoids were introduced in [Contreras 2013; Cattaneo and Contreras 2015] in order to describe the groupoid structure of the phase space of a 2-dimensional topological field theory, the *Poisson Sigma Model* (PSM) [Ikeda 1994; Schaller and Strobl 1994; Cattaneo and Felder 2001a], *before* gauge reduction [Cattaneo and Felder 2001b]. Cattaneo and Felder [2000] have shown that the perturbative quantization of the PSM using the Batalin–Vilkovisky (BV) formalism [Batalin and Vilkovisky 1977; Batalin and Vilkovisky 1981; Batalin and Vilkovisky 1983] yields Kontsevich's star product [Kontsevich 2003], a deformation quantization associated to any Poisson manifold.

It was shown by Cattaneo, Mnev and Reshetikhin [Cattaneo et al. 2018] that the BV formalism can be extended to deal with the perturbative quantization of gauge theories on manifolds with boundary by coupling the Lagrangian approach of the Batalin–Vilkovisky construction [Batalin and Vilkovisky 1977; Batalin and Vilkovisky 1981; Batalin and Vilkovisky 1983] in the bulk to the Hamiltonian approach of the Batalin–Fradkin–Vilkovisky construction [Fradkin and Vilkovisky

[1] Loosely speaking, a "quantization" of a Poisson manifold is a noncommutative deformation of the algebra of functions, or a Lie subalgebra of it, subject to a subset of certain axioms put forward by Dirac [1930].

1975; Batalin and Fradkin 1983; Stasheff 1997] on the boundary. This is known today as the *BV-BFV formalism* [Cattaneo and Moshayedi 2020]. Recently, this formalism has been applied to the relational symplectic groupoid for constant Poisson structures, linking the BV-BFV perturbative quantization of the relational symplectic groupoid and Kontsevich's star product in this case by methods of cutting and gluing for Lagrangian evolution relations [Cattaneo et al. 2017].

These constructions have been partially extended to a wider class of Poisson structures and source manifolds in [Cattaneo et al. 2020] and more general AKSZ theories [Alexandrov et al. 1997] in [Cattaneo et al. 2019]. We expect these results to be generalized to yield a BV-BFV description of a *global* deformation quantization for general Poisson manifolds, not necessarily Kontsevich's star product, which might produce some interesting algebraic structures.

A clear and explicit connection between geometric quantization (in terms of C^*-algebras) of the reduced phase space and deformation quantization of Poisson manifolds, via the PSM [Cattaneo and Felder 2000], remains an open question. By constructing relational convolution algebras for the PSM, in the future we hope to connect Kontsevich's and Hawkins' approaches via BV-BFV quantization of the relational symplectic groupoid. Eventually, these techniques might also help to generalize Kontsevich's star product to higher genera. In particular, the convolution algebra of a relational symplectic groupoid is the first step towards prescribing a field-theoretic interpretation of the C^*-algebra quantization of Poisson manifolds in terms of the nonperturbative PSM [Bonechi et al. 2006].

Further motivation for this paper stems from the connection between groupoids and Frobenius objects in a dagger monoidal category. For instance, a representative example of a relational convolution algebra is the relational group algebra, a version *up to equivalence*, of the group algebra of a group G. Group algebras are particular cases of Frobenius algebras, so relational convolution algebras provide a new class of examples of Frobenius objects in the category of sets and relations, which are also in correspondence with groupoids [Heunen et al. 2013; Mehta and Zhang 2020]. In a recent work [Contreras et al. $\geq$ 2021], Frobenius objects in the category of spans are considered. A prominent class of examples comes from groupoids, via simplicial sets.

Notation and conventions. We will denote groups or groupoids by usual letters G, H, K and relational groups or relational groupoids by calligraphic letters $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$. Moreover, we will use Greek letters to denote elements in the set of morphisms of a groupoid. Latin letters g, h, k (or $g_1, g_2, \dots$) will be used for elements of a relational groupoid $(\mathcal{G}, L, I)$. We will put an underline for a (relational) group(oid) to express the corresponding space observed after reduction. Underlined Latin letters $\underline{g}, \underline{h}, \underline{k}$ (or $\underline{g}_1, \underline{g}_2, \dots$) will denote that the given object is obtained by reduction.

A slashed arrow between two sets $A \nrightarrow B$ denotes a relation from A to B, i.e., a subset of $A \times B$. In this manuscript we treat relations as subsets of the Cartesian product, and the domain and codomain of the relation are prescribed in case it is ambiguous. Latin letters x, y, z (or $x_1, x_2, \dots$) will denote elements in the space of objects (base) of a (relational) groupoid. Functions will be denoted by $f_1, f_2, \dots$ to avoid confusion with elements of a (relational) group(oid).

2. Background material

Groupoids. Recall that a *groupoid* is a small category whose morphisms are invertible. We denote a groupoid by $G \rightrightarrows M$, endowed with source map $s\colon G \to M$ and target map $t\colon G \to M$, where G is the set of morphisms and M is the set of objects. We denote by $G^{(k)} \subset G^{\times k}$ the subset of k-composable morphisms, that is,

$$\begin{aligned} G^{(k)} &= \{(\alpha_1, \dots, \alpha_k) \in G^{\times k} \mid t(\alpha_{i+1}) = s(\alpha_i),\ i = 1, \dots, k-1\} \\ &= \underbrace{G \times_{(s,t)} \times \cdots \times_{(s,t)} G}_{k}. \end{aligned} \tag{2-1}$$

We denote by $m\colon G^{(k)} \to G$ the multiplication (composition of morphisms).

Definition 2.1. A *Lie groupoid* is a groupoid where M and G are smooth manifolds and all structure maps are smooth. The source and target maps are surjective submersions, which guarantees that the spaces of k-composable morphisms are smooth manifolds.

A particular case of interest is Lie groupoids with a symplectic structure [Weinstein 1987].

Definition 2.2. A *symplectic groupoid* is a Lie groupoid $G \rightrightarrows M$, where the space of morphisms is endowed with a symplectic form $\omega \in \Omega^2(G)$ such that the graph of the multiplication $m\colon G \times G \to G$ is a Lagrangian submanifold of $(G, \omega) \times (G, \omega) \times (G, -\omega)$.

The definition above is equivalent to saying that the symplectic form ω is multiplicative, i.e.,

$$m^*(\omega) = \pi_1^*(\omega) + \pi_2^*(\omega), \tag{2-2}$$

where π_1 and π_2 are projections of $G^{(2)} = G \times_{(s,t)} G$ onto its first and second component, respectively. Definition 2.2 is restrictive, e.g., one can show that there are no *symplectic groups*. Furthermore, the next theorem [Weinstein 1987] holds:

Theorem 2.3. *Let $(G, \omega) \rightrightarrows M$ be a symplectic groupoid.*

(i) *There is a unique Poisson structure Π on M such that the source map s is a Poisson map.*

(ii) *If ε denotes the unit map, then $\varepsilon(M)$ is a Lagrangian submanifold of G.*

(iii) *If ι denotes the inverse map, then the graph of ι is a Lagrangian submanifold of $G \times G$.*

Groupoid convolution algebras. There are several equivalent ways to define a convolution algebra on groupoids. They differ on the choice of the spaces in which the measures are defined. We first recall the construction of a Haar system on source fibers [Connes 1994; Higson 2004], and then we describe an equivalent system of measures on $(s \times t)$-fibers. The latter is more suitable for the generalization to relational groupoids. In the sequel, $C_c(G)$ denotes the space of continuous functions on G with compact support.

Definition 2.4. A *right Haar system* on a Lie groupoid $G \rightrightarrows M$ is a smooth family of smooth measures $(\mu_x)_{x\in M}$ on the source fibers $G_x := s^{-1}(x)$ such that:

(i) For all $f \in C_c(G)$, $s_* f(x) = \int_{G_x} f\, d\mu_x$ defines a smooth function $s_* f \in C_c(M)$.

(ii) For $\gamma\colon x \to y$, the right-multiplication diffeomorphism $R_\gamma\colon G_y \to G_x$ is measure-preserving, i.e., $(R_\gamma)_*\mu_y = \mu_x$.

Definition 2.5. Let $G \rightrightarrows M$ be a Lie groupoid with a right Haar system $(\mu_x)_{x\in M}$. Then its *groupoid convolution algebra* is $(C_c(G, \mathbb{C}), \star)$, continuous functions with compact support on G with values in $\mathbb{C}$, equipped with the groupoid convolution product

$$\star\colon C_c(G,\mathbb{C}) \times C_c(G,\mathbb{C}) \to C_c(G,\mathbb{C}) \tag{2-3}$$

defined by

$$(f_1 \star f_2)(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta)\, d\mu_{s(\gamma)}(\eta). \tag{2-4}$$

Proposition 2.6. *The convolution product $\star$, defined as in (2-4), is associative.*

Proof. Let $f_1, f_2, f_3 \in C_c(G,\mathbb{C})$ and consider, on the one hand,

$$\begin{aligned}((f_1\star f_2)\star f_3)(\gamma) &= \int_{G_{s(\gamma)}} (f_1 \star f_2)(\gamma\circ\eta^{-1}) f_3(\eta)\, d\mu_{s(\gamma)}(\eta)\\ &= \int_{G_{s(\gamma)}}\int_{G_{s(\gamma\circ\eta^{-1})}} f_1(\gamma\circ\eta^{-1}\circ\beta^{-1}) f_2(\beta)\, d\mu_{s(\gamma)}(\beta) f_3(\eta)\, d\mu_{s(\gamma)}(\eta).\end{aligned}$$

Now set $\tau = \beta\circ\eta$. We have $R_\eta\colon G_{s(\gamma\circ\eta^{-1})} \to G_{s(\gamma)}$, and using right-invariance of the measure, it follows that the above expression equals

$$\begin{aligned}&\int_{G_{s(\gamma)}}\int_{G_{s(\gamma)}} f_1(\gamma\circ\eta^{-1}\circ(\tau\circ\eta^{-1})^{-1}) f_2(\tau\circ\eta^{-1})\, d\mu_{s(\gamma\circ\eta^{-1})}(\beta)\, f_3(\eta) d\mu_{s(\gamma)}(\eta)\\ &\quad= \int_{G_{s(\gamma)}}\int_{G_{s(\gamma)}} f_1(\gamma\circ\tau^{-1}) f_2(\tau\circ\eta^{-1})\, d\mu_{s(\gamma)}(\tau) f_3(\eta)\, d\mu_{s(\gamma)}(\eta)\end{aligned} \tag{2-5}$$

On the other hand,

$$\begin{aligned}(f_1 \star (f_2 \star f_3))(\gamma) &= \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1})(f_2 \star f_3)(\eta)\, d\mu_{s(\gamma)}(\eta) \\ &= \int_{G_{s(\gamma)}} \int_{G_{s(\eta)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta \circ \tau^{-1}) f_3(\tau)\, d\mu_{s(\eta)}(\tau)\, d\mu_{s(\gamma)}(\eta).\end{aligned}$$

This expression equals (2-5) upon exchanging η and τ. □

The following equivalent definition of groupoid Haar system can be found in [Westman 1968]:

Definition 2.7. Let

$$G_{(x,y)} = \{g \in G \mid s(g) = x, t(g) = y\}.$$

A *Haar system* on these fibers is defined similarly as in Definition 2.4, with the requirement that, for $f \in C_c(G_{(x,y)}, \mathbb{C})$, if

$$\mu_{xy}(f) = \int_G f(g_{x,y})\, d\mu(g_{x,y}),$$

the function

$$\mu(f) : M \times M \to \mathbb{C}, \quad (x, y) \mapsto \mu_{x,y}(f|_{G_{x,y}})$$

is in $C_c(M \times M)$ whenever $f \in C_c(G)$.

Example 2.8. Let G be a locally compact group acting continuously on a locally compact Hausdorff space X, then $G \times X$ (as an *action groupoid*) admits a (right) Haar system $\{\delta_x, \mu\}$, where μ is a Haar measure on G and δ_x is the Dirac measure at $x \in X$.

Remark 2.9. The groupoid convolution algebra has an involutive $*$-operation given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}. \tag{2-6}$$

In order to obtain groupoid C^*-algebras, we need to use completion with respect to a certain norm and a given convolution algebra representation.

Definition 2.10. The *left regular representation* of the groupoid convolution algebra is a map, for all $x \in M$,

$$\lambda_x : C_c(G) \to \mathcal{B}(L^2(G_x)),$$

which for $f \in C_c(G)$, $h \in L^2(G_x)$ and $\gamma \in G_x$ is given by

$$\begin{aligned}(\lambda_x(f)h)(\gamma) &= (f \star h)(\gamma) \\ &= \int_{G_{s(\gamma)}} f(\gamma \circ \eta^{-1}) h(\eta)\, d\mu_{s(\gamma)}(\eta).\end{aligned}$$

Definition 2.11. The *reduced* groupoid C^*-algebra of G is the completion of a groupoid convolution algebra $C_c(G)$ with respect to the norm

$$\|f\| = \sup_x \|\lambda_x(f)\|_{\mathcal{B}(L^2(G_x))}.$$

3. Relational groupoids

The category of relational groupoids.

Definition 3.1. A *relational groupoid* is a triple $(\mathcal{G}, L, I)$ such that:

(1) $\mathcal{G}$ is a set,

(2) L is a subset of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$,

(3) $I : \mathcal{G} \to \mathcal{G}$ is a function,

satisfying the following axioms:

A.1: L is cyclically symmetric, i.e., if $(g, h, k) \in L$, then $(h, k, g) \in L$.

A.2: I is an involution (i.e., $I^2 = \mathrm{id}$).

A.3: Let T denote the transposition map

$$T : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}, \quad (g, h) \mapsto (h, g).$$

Then

$$I \circ L = L \circ T \circ (I \times I). \tag{3-1}$$

A.4: Let $L_3 := I \circ L : \mathcal{G} \times \mathcal{G} \nrightarrow \mathcal{G}$. Then the following equality holds:

$$L_3 \circ (L_3 \times \mathrm{id}) = L_3 \circ (\mathrm{id} \times L_3). \tag{3-2}$$

A.5: Denoting by L_1 the morphism $L_1 := L_3 \circ I : * \nrightarrow \mathcal{G}$, then

$$L_3 \circ (L_1 \times L_1) = L_1. \tag{3-3}$$

A.6: If we define the morphism

$$L_2 := L_3 \circ (L_1 \times \mathrm{id}) : \mathcal{G} \nrightarrow \mathcal{G},$$

then the following equations hold:

$$L_2 = L_3 \circ (\mathrm{id} \times L_1). \tag{3-4}$$

$$L_2 \circ L_1 = L_1, \tag{3-5}$$

$$L_2 \circ L_2 = L_2, \tag{3-6}$$

$$L_2 \circ L_3 = L_3 \circ (L_2 \times L_2) = L_3, \tag{3-7}$$

$$I \circ L_2 = L_2 \circ I, \tag{3-8}$$

where (3-5)–(3-7) shows that L_2 leaves L_1, L_2 and L_3 invariant and (3-8) shows that L_2 is a symmetric relation.

Definition 3.2. A *relational Lie groupoid* is a relational groupoid $(\mathcal{G}, L, I)$ such that $\mathcal{G}$, L and I are smooth manifolds and smooth relations, respectively.

The next proposition says we can equally well define a relational groupoid through the relations L_1, L_2, L_3 defined in Definition 3.1 above.

Proposition 3.3. *The data* $(\mathcal{G}, L, I)$ *and* $(\mathcal{G}, I, L_1, L_2, L_3)$ *are equivalent.*

Proof. Clearly we are able to obtain the relations L_i from $(\mathcal{G}, L, I)$. Now, assume that $(\mathcal{G}, I, L_1, L_2, L_3)$ is given. Then L is recovered using that $L = I \circ L_3$, and therefore Axioms A.1–A.6 can be written just in terms of L_i and I. □

Definition 3.4. Let $(\mathcal{G}, L, I)$ be a relational groupoid. A relational groupoid $(\mathcal{H}, L_{\mathcal{H}}, I_{\mathcal{H}})$ is a *relational subgroupoid* of $\mathcal{G}, L, I$ if $\mathcal{H} \subseteq \mathcal{G}$, $L_{\mathcal{H}} \subseteq L$ and $I_{\mathcal{H}} \subseteq I$.

Definition 3.5. Let $(\mathcal{G}_1, L_1, I_1)$ and $(\mathcal{G}_2, L_2, I_2)$ be two relational groupoids. A morphism $F : \mathcal{G}_1 \to \mathcal{G}_2$ is a relational subgroupoid of $\mathcal{G}_1 \times \mathcal{G}_2$.

We can extend the category of groupoids **Grpd** to the *category* of *relational groupoids* **RelGrpd**.

Remark 3.6. The category **RelGrpd** is endowed with an involution

$$\dagger\colon (\mathbf{RelGrpd})^{op} \to \mathbf{RelGrpd}$$

that is the identity on objects and is the relational converse of morphisms, i.e., for $f\colon A \nrightarrow B$, we get $f^\dagger := \{(b, a) \in B \times A \mid (a, b) \in f\}$.

Graphical interpretation of the axioms.

- The cyclicity axiom, Axiom A.1, encodes the cyclic behavior of the multiplication and inversion maps for groups, namely, if g, h, k are elements of a group G with unit e such that $ghk = e$, then $gh = k^{-1}$, $hk = g^{-1}$, $kg = h^{-1}$.
- Axiom A.2 encodes the involutivity property of the inversion map of a group, i.e., $(g^{-1})^{-1} = g, \forall g \in G$.
- Axiom A.3 encodes the compatibility between multiplication and inversion:
$$(gh)^{-1} = h^{-1}g^{-1}, \quad \forall g, h \in G.$$
- Axiom A.4 encodes the associativity of the product: $g(hk) = (gh)k, \forall g, h, k \in G$.
- Axiom A.5 encodes the property of the unit of a group being idempotent: $ee = e$.
- Axiom A.6 states an important difference between the construction of relational groupoids and usual groupoids. The compatibility between the multiplication and the unit is defined up to an equivalence relation, denoted by L_2, whereas for groupoids such compatibility is strict; more precisely, for groupoids such an equivalence relation is the identity. In addition, the multiplication and the unit are equivalent with respect to L_2.

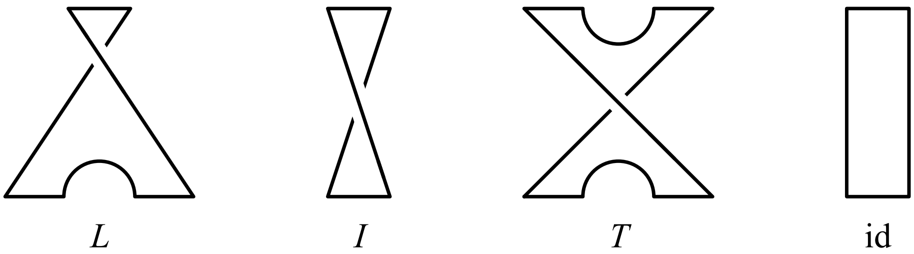

Figure 1. The special relations L, I, T and id.

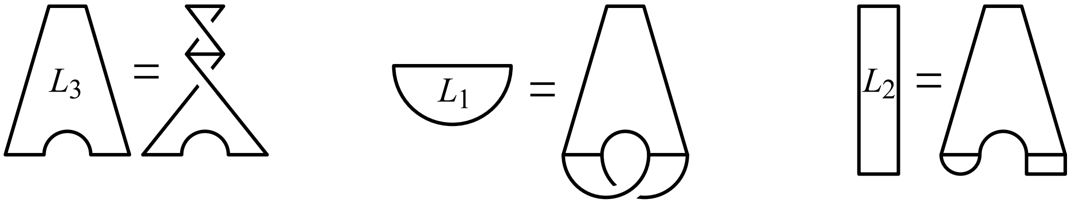

Figure 2. The structure relations L_1, L_2 and L_3.

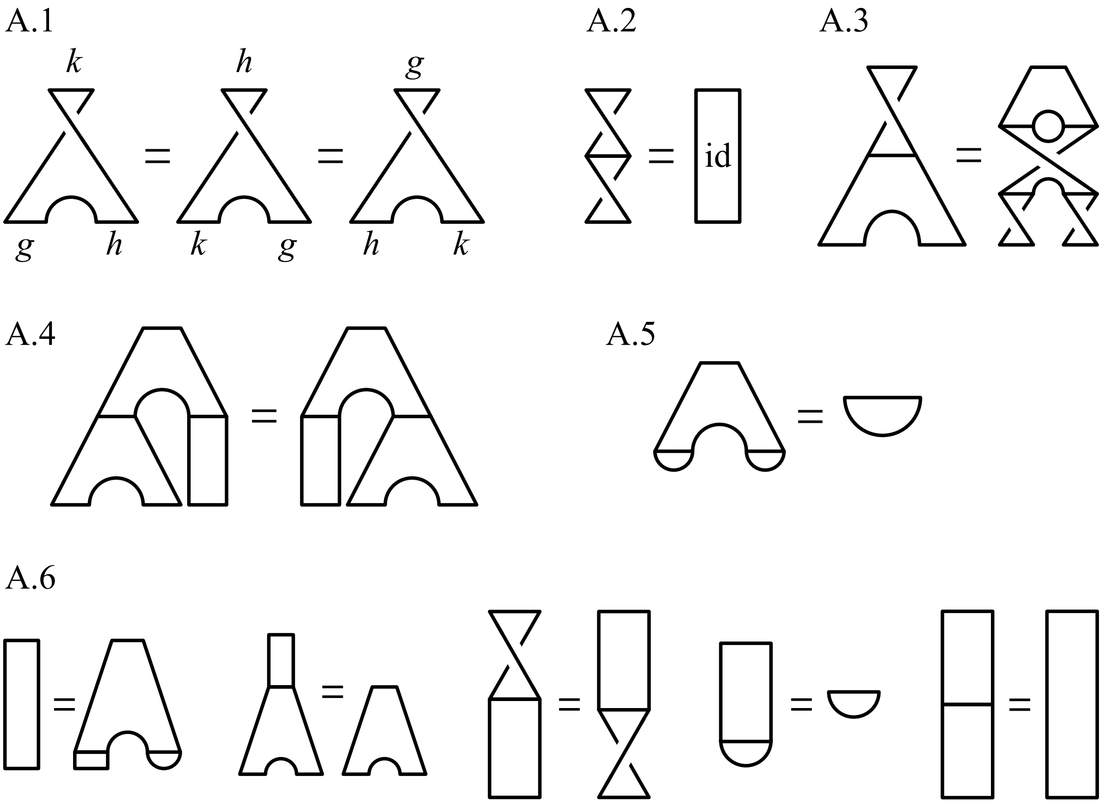

Figure 3. The axioms A.1–A.6.

Figures 1–3 illustrates the diagrammatics of the relational groupoid axioms.

Relational groups as relational groupoids. The next example is representative of relational groupoids: it is given by a group $\mathcal{G}$ and a normal subgroup $\mathcal{H}$ of $\mathcal{G}$.

Example 3.7. Let $\mathcal{G}$ be a group with multiplication defined by $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $\mathcal{H} \triangleleft \mathcal{G}$ a normal subgroup. Denote by $\sim_{\mathcal{H}} \subset \mathcal{G} \times \mathcal{G}$ the equivalence relation

$g_1 \sim_{\mathcal{H}} g_2 \Leftrightarrow \exists h \in \mathcal{H}$ such that $m(h, g_1) = g_2$. Moreover, define

$$L_3 := \{(g_1, g_2, m(m(g_1, g_2), h)) \mid g_1, g_2 \in \mathcal{G}, h \in \mathcal{H}\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}, \tag{3-9}$$

$$L_2 := \sim_{\mathcal{H}} \subset \mathcal{G} \times \mathcal{G}, \tag{3-10}$$

$$L_1 := \mathcal{H} \triangleleft \mathcal{G}, \tag{3-11}$$

$$I := g \mapsto g^{-1}. \tag{3-12}$$

Then the quintuple $(\mathcal{G}, I, L_1, L_2, L_3)$ defines a relational groupoid.

Proof. We need to check the axioms of a relational groupoid as in Definition 3.1 explicitly. We will not always write the multiplication map by m but instead $g_1 g_2 := m(g_1, g_2)$. To see Axiom A.1, let $(g_1, g_2, g_3) \in L$, we want to show $(g_3, g_1, g_2) \in L$ or $(g_3, g_1, g_2^{-1}) \in L_3$ Then $g_3 = (g_1 g_2 h)^{-1}$ for some $h \in \mathcal{H}$ and $g_3 g_1 = h^{-1} g_2^{-1}$. By normality, $h^{-1} g_2^{-1} = g_2^{-1} h'$, with $h' \in \mathcal{H}$, and so $(g_3, g_1, g_2^{-1}) \in L_3$. Clearly I is an involution, hence Axiom A.2 holds. Axiom A.3 follows from the fact that $L = I \circ L_3$ and that for group elements g_1, g_2, we have $I(g_1 g_2) = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1} = I(g_2) I(g_1)$ and normality. Next we want to show Axiom A.4. Consider an element $(g_1, g_2, g_1 g_2 h) \in L_3$ for some $h \in \mathcal{H}$. Consider the diagram of relations

$$\begin{tikzcd}
 & \mathcal{G} \times \mathcal{G} \arrow[dr, bend left, "L_3"] & \\
\mathcal{G} \times \mathcal{G} \times \mathcal{G} \arrow[ur, bend left, "L_3 \times \mathrm{id}"] \arrow[rr, bend left=15, "L_3 \circ (L_3 \times \mathrm{id})"'] \arrow[rr, bend right=15, "L_3 \circ (\mathrm{id} \times L_3)"] \arrow[dr, bend right, "\mathrm{id} \times L_3"'] & & \mathcal{G} \\
 & \mathcal{G} \times \mathcal{G} \arrow[ur, bend right, "L_3"'] &
\end{tikzcd} \tag{3-13}$$

Let us first look at the relation $L_3 \circ (L_3 \times \mathrm{id})$. It follows that $(g_1, g_2, g_3) \sim (g_1 g_2 h_1, g_3) \sim g_1 g_2 h_1 g_3 h_2$, with $h_1, h_2 \in \mathcal{H}$ and $g_1, g_2, g_3 \in \mathcal{G}$. Now using normality of $\mathcal{H}$ we get $h_1 g_3 = g_3 h_1'$, for some $h_1' \in \mathcal{H}$, and setting $\bar{h} := h_1' h_2$, we get $g_1 g_2 h_1 g_3 h_2 = g_1 g_2 g_3 \bar{h} \in g_1 g_2 g_3 \mathcal{H}$.

If we look at the relation $L_3 \circ (\mathrm{id} \times L_3)$, we get $(g_1, g_2, g_3) \sim (g_1, g_2 g_3 h_3) \sim g_1 g_2 g_3 h_3 h_4 \in g_1 g_2 g_3 \mathcal{H}$. Next we show Axiom A.5. Consider the relations

$$\begin{tikzcd}
* \arrow[r, "L_1 \times L_1"] \arrow[rr, bend left=60, "L_1"] \arrow[rr, bend right=60, "L_3 \circ (L_1 \times L_1)"'] & \mathcal{G} \times \mathcal{G} \arrow[r, "L_3"] & \mathcal{G}
\end{tikzcd} \tag{3-14}$$

where we have $* \sim (h_1, h_2)$ with $h_1, h_2 \in \mathcal{H}$ and then $(h_1, h_2) \sim h_1 h_2 h_3 =: \tilde{h} \in \mathcal{H}$. On the other hand, we have a relation L_1 for $* \sim h$ for any element h of $\mathcal{H}$. Next we show (3-4) of Axiom A.6. Let us look at

$$\begin{tikzcd}
\mathcal{G}\times * \arrow[r, "\mathrm{id}\times L_1", "/" marking] \arrow[rr, bend left=60, "L_2", "/" marking] \arrow[rr, bend right=60, "L_3\circ(\mathrm{id}\times L_1)", "/" marking] & \mathcal{G}\times\mathcal{G} \arrow[r, "L_3", "/" marking] & \mathcal{G}
\end{tikzcd} \tag{3-15}$$

and consider first the relation $L_3\circ(\mathrm{id}\times L_1)$. Take $g\in\mathcal{G}$, then we get $g\sim(g,h_1)$ for $h_1\in\mathcal{H}$ and $(g,h_1)\sim g\bar{h}\in g\mathcal{H}$ with $\bar{h}=h_1h_2\in\mathcal{H}$. On the other hand, if we consider the relation L_2 we get for $g\in\mathcal{G}$ and $h\in\mathcal{H}$ that $g\sim gh\in g\mathcal{H}$. Next we show (3-5) of Axiom A.6. Consider the relations

$$\begin{tikzcd}
* \arrow[r, "L_1", "/" marking] \arrow[rr, bend left=60, "L_1", "/" marking] \arrow[rr, bend right=60, "L_2\circ L_1", "/" marking] & \mathcal{G} \arrow[r, "L_2", "/" marking] & \mathcal{G}
\end{tikzcd} \tag{3-16}$$

and let us look at the relation $L_2\circ L_1$. The relation L_1 is $*\sim h$ for any element $h\in\mathcal{H}$. Then by L_2 we get $*\sim h\sim\tilde{h}:=h\bar{h}\in\mathcal{H}$. Since $\mathcal{H}$ is a subgroup, $*\sim h$ for any $h\in\mathcal{H}$, which is L_1. Finally, we show (3-6) of Axiom A.6. Then we have the relations

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "L_2", "/" marking] \arrow[rr, bend left=60, "L_2", "/" marking] \arrow[rr, bend right=60, "L_2\circ L_2", "/" marking] & \mathcal{G} \arrow[r, "L_2", "/" marking] & \mathcal{G}
\end{tikzcd} \tag{3-17}$$

and we have that L_2 gives for an element $g\in\mathcal{G}$ that $g\sim gh_1$ for $h_1\in\mathcal{H}$; and then $gh_1\sim gh_1h_2=g\bar{h}\in g\mathcal{H}$ with $\bar{h}=h_1h_2\in\mathcal{H}$, which gives the same relation as just L_2. Similarly, we can show for (3-7) of Axiom A.6 that the last diagram also commutes:

$$\begin{tikzcd}
& \mathcal{G}\times\mathcal{G} \arrow[dr, "L_3", "/" marking] & \\
\mathcal{G}\times\mathcal{G} \arrow[ur, "L_2\times L_2", "/" marking] \arrow[r, "L_3", "/" marking] \arrow[rr, bend left=50, "L_3", "/" marking] \arrow[rr, bend right=50, "L_2\circ L_3", "/" marking] & \mathcal{G} \arrow[r, "L_2", "/" marking] & \mathcal{G}
\end{tikzcd} \tag{3-18}$$

which completes the proof. □

For later use we record some simple examples of relational groups below.

Example 3.8 (a finite example). One can check that $\mathcal{G} = \mathbb{Z}_4$ with normal subgroup $\mathcal{H} = \mathbb{Z}_2 \triangleleft \mathbb{Z}_4$, together with the canonical relations as in Example 3.7, is a relational group and, hence, a relational groupoid.

Example 3.9 (discrete example: integers modulo n). We want to consider the example of $\mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$. Hence, let $G = \mathbb{Z}$, $L_1 = n\mathbb{Z}$, $L_2 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b \in n\mathbb{Z}\}$ and $L_3 = \{(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid \exists k \in \mathbb{Z}\colon a + b + nk = c\}$.

Example 3.10 (a continuous example). Let $\mathcal{G} = S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, and let $L_1 = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\} \triangleleft S^1$ be the normal subgroup of n-th roots of unity. We define the relation

$$L_2 = \left\{(z, w) \in S^1 \times S^1 \mid \exists k \in \{0, 1, \dots, n-1\} \text{ such that } \arg(w) - \arg(z) = \frac{2\pi k}{n}\right\}.$$

In particular, $z \sim w$ if and only if $\exists \zeta \in L_1$ such that $z \cdot \zeta = w$ for $z, w \in S^1$. We define $L_3 = \{(z, w, z \cdot w \cdot \zeta) \in S^1 \times S^1 \times S^1 \mid z, w \in S^1, \zeta \in L_1\} \subset S^1 \times S^1 \times S^1$. Moreover, we define the map I by complex conjugation $z \mapsto \bar{z}$. Then one can check that this is indeed a relational group as in Example 3.7 and, hence, a relational groupoid.

Additional examples. Here are some other simple examples of relational groupoids that are not relational groups.

Example 3.11. We can further extend Example 3.7 to a parametrized family of relational groups. The local model in this example is $\mathcal{G} \times \mathbb{R}^k$, where at each point p in $\mathbb{R}^k$ there is a fiber that we identify with $\mathcal{G}$. For instance, Example 3.10 can be extended to the relational bundle $S^1 \times \mathbb{R} \to \mathbb{R}$, where each fiber is isomorphic to the relational group $\mathcal{G} = S^1$ and at each fiber, the normal subgroup of n-th roots of unity is chosen.

Example 3.12. Of course, groupoids are also examples of relational groupoids. If (G, m) is a groupoid, then $L := \mathrm{Graph}(I \circ m)$, $I(g) = g^{-1}$ makes G into a relational groupoid. In this case, the special relations are given as follows: $L_1 \subset G$ is the unit section, $L_2 = \mathrm{diag}_G \subset G \times G$ is the diagonal and $L_3 = \mathrm{Graph}(m)$.

Relational symplectic groupoids.

Definition 3.13. A *relational symplectic groupoid* is a relational groupoid $(\mathcal{G}, L, I)$ such that:

(1) $\mathcal{G}$ is a weak symplectic manifold,[2]

[2] In the infinite-dimensional setting we restrict to the case of Banach manifolds (when the regularity type of fields is fixed) $\mathcal{G}$ endowed with a closed 2-form ω, such that the induced map $\omega^\sharp : T\mathcal{G} \to T^*\mathcal{G}$ is injective. The result also holds for smooth fields and Fréchet manifolds.

(2) L is an immersed Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ denotes $\mathcal{G}$ equipped with the negative of the given symplectic form,

(3) I is an antisymplectomorphism of $\mathcal{G}$.

Example 3.14. In [Contreras 2013; Cattaneo and Contreras 2015] it is proven that the phase space of the Poisson Sigma Model (PSM)[3] is an example of a infinite-dimensional relational symplectic groupoid.

Reduction of relational groupoids. One of the important properties of a relational groupoid is the fact that L_2 encodes the information of an equivalence relation. In general L_2 is not necessarily an equivalence relation on the whole of $\mathcal{G}$, but on a subset $\mathcal{C}$, called the *constraint set*.

Proposition 3.15. *Define*

$$\mathcal{C} := L_2 \circ \mathcal{G},$$

where $\mathcal{G}$ is considered as the relation $* \nrightarrow \mathcal{G}$. Then L_2 is an equivalence relation on $\mathcal{C}$.

Proof. The fact that L_2 is transitive and symmetric follows from Axiom A.6 ($L_2 \circ L_2 = L_2$ and $L_2^\dagger = L_2$). Reflexivity follows from the definition of $\mathcal{C}$. □

Theorem 3.16. *Let $(\mathcal{G}, I, L_1, L_2, L_3)$ be a relational groupoid. Then $\underline{\mathcal{G}} = \mathcal{C}/L_2$ is a groupoid, and the quotient map q is a morphism of relational groupoids.*

Proof. First, let us consider the following relations, that are the relational analogues of the source and target maps:

$$\mathcal{S} := \{(c, \ell) \in \mathcal{C} \times L_1 \mid \exists g \in \mathcal{G} \text{ s.t. } (\ell, c, g) \in L_3\}$$

and

$$\mathcal{T} := \{(c, \ell) \in \mathcal{C} \times L_1 \mid \exists g \in \mathcal{G} \text{ s.t. } (c, \ell, g) \in L_3\}.$$

It follows from the definition that $\mathcal{T} = I \circ \mathcal{S}$, and furthermore, if $M := L_1/L_2$:

$$s := \underline{\mathcal{S}} : \underline{\mathcal{G}} \to M$$

is a surjective map, where $\underline{\mathcal{S}} = q \circ \mathcal{S}$ is the reduction of $\mathcal{S}$. □

Example 3.17. Following Example 3.7, the reduction of a relational group is a (set theoretical) group. If, in addition, we impose the condition that G is a Lie group and H is closed subgroup, then the quotient is a Lie group.

Example 3.18. Let G be a set. Then one can define its *pair groupoid* by $G \times G \rightrightarrows G$, where s and t are given by projection to the first and second factor, respectively. For two elements (g, h) and (h, k), composition is given by $(g, h)(h, k) = (g, k)$. The inverse is defined by $(g, h)^{-1} = (h, g)$. We can define a *relational pair groupoid* in a similar way. For a relational version of this example, let $\mathcal{G}$ be a set and R

[3] Here, the source space is a disk, and the target space is a Poisson manifold.

be an equivalence relation. We then define the relational groupoid $\mathcal{G}\times\mathcal{G}$, where $L_1=\mathcal{G}$, $L_2=R\times R$ and L_3 is given by composition of relations. In the case that the equivalence relation R is the identity, we recover the pair groupoid after L_2-reduction.

Example 3.19. If M is an integrable Poisson manifold, then the reduction of the infinite-dimensional relational symplectic groupoid of Example 3.14 is a finite-dimensional symplectic groupoid integrating the Poisson manifold M. See [Contreras 2013; Cattaneo and Contreras 2015] for the details of the reduction procedure and how it coincides with the gauge reduction of the Poisson Sigma Model.

The previous results and examples allow us to connect different constructions on relational groupoids with standard notions on groupoids, via reduction. For instance, in the next subsection we introduce actions of relational groupoids that are necessary to describe the compatibility of measures (relational Haar systems) and the structure relations in Section 4.

Relational groupoid actions. First, following the characterization of the Haar measure on groups via right-invariance, we introduce the notion of relational right action. Right, left and adjoint group actions are natural examples of relational group actions, and Haar measures (on groups and relational groups) are invariant with respect to the right relational group action. Also, the conditions (i) and (ii) of a relational Haar system in Definition 4.2 encode the invariance with respect to relational right actions.

Definition 3.20. Let $(\mathcal{G},L,I)$ be a relational groupoid, and let Z be a set with an equivalence relation L_Z. A *relational right action* of $\mathcal{G}$ on Z is a relation $\rho\colon Z\times\mathcal{G}\nrightarrow Z$ such that:

(1) We have
$$\rho\circ(\rho\times\mathrm{id}_{\mathcal{G}})=\rho\circ(\mathrm{id}_Z\times L_3),$$
as relations $Z\times\mathcal{G}\times\mathcal{G}\nrightarrow Z$.

(2) The relation $\rho_{L_1}\colon Z\nrightarrow Z$ given by
$$\rho_{L_1}=\{(x,y)\in Z\times Z\mid \exists g\in L_1,\ (x,g,y)\in\rho\}$$
coincides with the equivalence relation L_Z on Z, i.e., $\rho_{L_1}=L_Z$.

We also say that $(\mathcal{G},L,I)$ acts on (Z,L_Z) by relations from the right. Sometimes we drop L_Z notation.

An obvious example is the action of a relational groupoid on itself from the right.

Example 3.21. Let $\mathcal{G}$ be a relational groupoid. Then setting $\rho=L_3$ defines a relational right action of $\mathcal{G}$ on $(\mathcal{G},L_1)$.

Proof. This follows directly from Axiom A.4 (associativity) and Axiom A.6 (unitality) of Definition 3.1. □

Of course, there is an analogous definition of left relational action, and relational groupoids also act on themselves from the left. We can generalize the relation ρ_{L_1} defined above to an arbitrary subset of $\mathcal{G}$.

Definition 3.22 (relational action). Let $(\mathcal{G}, L, I)$ be a relational groupoid and suppose that $\mathcal{G}$ acts on Z by relations, and let $S \subset \mathcal{G}$ be any subset of $\mathcal{G}$. Then we define the relation $R_S \colon Z \nrightarrow Z$ by

$$R_S := \{(z_1, z_2) \in Z \times Z \mid \exists g \in S, (z_1, g, z_2) \in \rho\}. \tag{3-19}$$

For $S = \{g\} \subset \mathcal{G}$, we write $R_S = R_g$.

For $g, h \in \mathcal{G}$, we denote $gh := \{x \in \mathcal{G} \mid (g, h, x) \in L_3\}$. With this notation, we have the following proposition:

Proposition 3.23. *Let $(\mathcal{G}, L, I)$ be a relational groupoid, and let $g, h \in \mathcal{G}$, which acts on a set Z from the right. Then*

$$R_h \circ R_g = R_{gh}. \tag{3-20}$$

4. Relational convolution algebras

Relational Haar systems. Let $(\mathcal{G}, L, I)$ be a relational groupoid. As for a usual groupoid, we denote by

$$\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \exists k \in \mathcal{G} \text{ s.t. } (g, h, k) \in L_3\} \tag{4-1}$$

the set of composable pairs in $\mathcal{G}$. For $k \in \mathcal{G}$, we denote by $\mathcal{G}_k^{(2)}$ the set of pairs that compose to k, i.e., we have

$$\mathcal{G}_k^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid (g, h, k) \in L_3\}. \tag{4-2}$$

Recall that we have the quotient groupoid $\underline{\mathcal{G}} := \mathcal{C}/L_2$, and there is a relation $q \colon \mathcal{G} \nrightarrow \underline{\mathcal{G}}$ which, restricted to $\mathcal{C} \times \underline{\mathcal{G}}$, is the graph of a surjective map that we will also denote q. It is clear from the definitions that for every $g \in \mathcal{C}$ we have $\mathcal{G}_g^{(2)}/(L_2 \times L_2) = \underline{\mathcal{G}}_{q(g)}^{(2)}$, and we denote[4] the quotient map $q_g \colon \mathcal{G}_g^{(2)} \to \underline{\mathcal{G}}_{\underline{g}}^{(2)}$, where $q(g) =: \underline{g}$.

We will need the following terminology from measure theory (see, e.g., [Chang and Pollard 1997; Ambrosio et al. 2005]).

Definition 4.1 (disintegrating measure). Let μ be a measure on a set Y, and let $q \colon Y \to X$ be a map. Moreover, let $\nu = q_* \mu$ be the pushforward measure on X. We say that μ *disintegrates with respect to* q if there exists a family of probability measures $(\mu_x)_{x \in X}$ such that:

[4] We are slightly abusing notation here, q_g is actually the restriction of $q \times q$ to $\mathcal{G}_g^{(2)}$.

- For all μ-measurable sets $E \subset Y$, the function $x \to \mu_x(E)$ is ν-measurable.
- $\mu_x(Y \setminus q^{-1}(x)) = 0$ for ν-almost every $x \in X$.
- For all μ-measurable functions $f : Y \to \mathbb{R}$, we have

$$\int_Y f\,d\mu = \int_X \int_{q^{-1}(x)} f(y)\,d\mu_x(y)\,d\nu(x).$$

A measure μ disintegrates under fairly general assumptions, e.g., when X, Y are Radon spaces and q is Borel-measurable. The family μ_x is uniquely determined ν-almost everywhere (and, in turn, determines the measure μ). See [Pachl 1978] for a detailed account.

We now define a relational Haar system as follows:

Definition 4.2. Let $\mathcal{G}$ be a relational groupoid, $\underline{\mathcal{G}}$ its quotient groupoid and $q : \mathcal{C} \to \underline{\mathcal{G}}$ the quotient map. A *relational Haar system* on $\mathcal{G}$ is a system of measures μ_g on $\mathcal{G}_g^{(2)}$, with $g \in \mathcal{G}$, such that $\mu_g = 0$ for $g \in \mathcal{G} \setminus \mathcal{C}$ and for $g \in \mathcal{C}$ we have:

(i) If $q(g) = q(g')$, then $(q_g)_* \mu_g = (q_{g'})_* \mu_{g'}$.

(ii) The system of measures $\nu_{\underline{g}} := (q_g)_* \mu_g$ on $\underline{\mathcal{G}}_{\underline{g}}^{(2)}$ defines a right Haar system on the quotient groupoid $\underline{\mathcal{G}}$ (recall that $\underline{g} = q(g)$).

(iii) μ_g disintegrates with respect to q_g.

Equivalently, a relational Haar system on $\mathcal{G}$ is determined by a right Haar system $(\nu_{\underline{g}})_{\underline{g}} \in \underline{\mathcal{G}}$ on the quotient groupoid $\underline{\mathcal{G}}$ and a family of probability measures $(\mu_g)_{\underline{g}_1 \underline{g}_2}$ on the fibers $q_g^{-1}(\underline{g}_1, \underline{g}_2)$ for $\underline{g}_1 \underline{g}_2 = \underline{g}$.

Remark 4.3. A relational Haar system is invariant under the relational right action of the relational groupoid on itself, in the following sense. Let $A \subset \mathcal{G}_g^{(2)}$ be an L_2-saturated set, i.e., $(L_2 \times L_2) \circ A = A$. Then, if we have $(g, h, k) \in L_3$, we have $\mu_g(A) = \mu_k((\mathrm{id} \times R_h) \circ A)$.

Notice also that the condition that μ_g vanishes for $g \notin \mathcal{C}$ is automatic, because in this case $\mathcal{G}_g^{(2)} = \varnothing$ (see also Proposition App.1). In some sense, the axioms presented above are the weakest possible set of axioms that ensure existence of a well-defined Haar system on the quotient. However, these measures can have extra properties with regard to the structure relations that define a relational groupoid.

Definition 4.4 (L_2-invariant measure). We say that a relational Haar system μ_g is L_2-*invariant* if $\mu_g = \mu_h$ whenever $(g, h) \in L_2$.

Notice that this condition is stronger than condition (i) of Definition 4.2, which merely demands that the pushforwards be the same.

Definition 4.5 (split relational Haar system). We say that a relational Haar system μ_g *splits* if, for all $g \in \mathcal{C}$, there is a family of probability measures $(\tau_{\underline{g}_1}^g)_{\underline{g}_1 \in \underline{\mathcal{G}}}$ on $\mathcal{G}$

with $\tau^g_{\underline{g}_1}$ supported on $q^{-1}(\underline{g}_1)$ such that for all $\underline{g}_1, \underline{g}_2 \in \underline{\mathcal{G}}$, with $\underline{g}_1\underline{g}_2 = q(g)$,

$$(\mu_g)_{\underline{g}_1\underline{g}_2} = \tau^g_{\underline{g}_1} \times \tau^g_{\underline{g}_2}. \tag{4-3}$$

Definition 4.6 (strongly split relational Haar system). We say that a relational Haar system μ_g *splits strongly* if there is a family of probability measures $(\tau_{\underline{g}_1})_{\underline{g}_1 \in \underline{\mathcal{G}}}$ on $\mathcal{G}$ with $\tau_{\underline{g}_1}$ supported on $q^{-1}(\underline{g}_1)$ such that for all $g \in \mathcal{G}$ and $\underline{g}_1, \underline{g}_2 \in \underline{\mathcal{G}}$, with $\underline{g}_1\underline{g}_2 = q(g)$,

$$(\mu_g)_{\underline{g}_1\underline{g}_2} = \tau_{\underline{g}_1} \times \tau_{\underline{g}_2}. \tag{4-4}$$

In particular, a strongly split relational Haar system is split and L_2-invariant.

Example 4.7. Consider the relational group $\mathcal{G} = \mathbb{Z}_4$ with normal subgroup $\mathcal{H} = \mathbb{Z}_2 \lhd \mathbb{Z}_4$, as in Example 3.8. The sets $\mathcal{G}^{(2)}_g$ are given by

$$\mathcal{G}^{(2)}_0 = \{(0,0), (1,1), (2,2), (3,3), (0,2), (2,0), (1,3), (3,1)\} = \mathcal{G}^{(2)}_2,$$
$$\mathcal{G}^{(2)}_1 = \{(1,0), (0,1), (1,2), (2,1), (3,0), (0,3), (2,3), (3,2)\} = \mathcal{G}^{(2)}_3,$$

and in the quotient we have

$$\underline{\mathcal{G}}^{(2)}_{\underline{0}} = \{(\underline{0},\underline{0}), (\underline{1},\underline{1})\} \quad \text{and} \quad \underline{\mathcal{G}}^{(2)}_{\underline{1}} = \{(\underline{1},\underline{0}), (\underline{0},\underline{1})\}.$$

Let μ_{count} denote the counting measure, $\mu_{\mathrm{count}}(A) = \#A$. The unique Haar system (up to normalization) on the quotient is given by letting $\nu_{\underline{0}} = \nu_{\underline{1}} = \frac{1}{2}\mu_{\mathrm{count}}$, and of course, this corresponds to the natural Haar measure μ_{count} on $\mathbb{Z}_2$. A relational Haar system can now be given by assigning, for $\underline{g}_1\underline{g}_2 = q(g) \in \underline{\mathcal{G}}$, probability measures $(\mu_g)_{\underline{g}_1\underline{g}_2}$ on the fibers

$$q_g^{-1}(\underline{g}_1, \underline{g}_2) = \{\underline{g}_1, \underline{g}_1 + 2\} \times \{\underline{g}_2, \underline{g}_2 + 2\}$$

for $\underline{g}_1\underline{g}_2 = q(g)$. An obvious choice is to assign $(\mu_g)_{\underline{g}_1\underline{g}_2} = \frac{1}{4}\mu_{\mathrm{count}}$. This yields the family of measures $(\mu_g = \frac{1}{8}\mu_{\mathrm{count}})_{g \in \mathbb{Z}_4}$, which is strongly split, with $\tau_{\underline{0}} = \tau_{\underline{1}} = \frac{1}{2}\mu_{\mathrm{count}}$. But other choices are possible: For instance, we can define a split, but not L_2-invariant measure by setting $\tau^g_{\underline{g}_1} = \delta_g$, the Dirac measure at $g \in q^{-1}(\underline{g}_1)$, whenever $q(g) = \underline{g}_1$ and $\tau^g_{\underline{g}_1} = \frac{1}{2}\mu_{\mathrm{count}}$ otherwise. Concretely, we have

$$\tau^0_{\underline{0}} = \delta_0, \qquad \tau^0_{\underline{1}} = \tfrac{1}{2}\mu_{\mathrm{count}}, \qquad \tau^2_{\underline{0}} = \delta_2, \qquad \tau^2_{\underline{1}} = \tfrac{1}{2}\mu_{\mathrm{count}},$$

and similarly for $\tau^1_{\underline{g}_1}$ and $\tau^3_{\underline{g}_1}$. Moreover, we can define an L_2-invariant system which is not split by letting $(\mu_0)_{\underline{g}_1\underline{g}_2} = (\mu_2)_{\underline{g}_1\underline{g}_2}$ any probability measure on $q^{-1}(\underline{g}_1, \underline{g}_2)$ which is not a product measure, for instance

$$(\mu_0)_{\underline{0}\underline{0}}(0,0) = (\mu_0)_{\underline{0}\underline{0}}(2,0) = (\mu_0)_{\underline{0}\underline{0}}(0,2) = \tfrac{1}{3} \quad \text{and} \quad (\mu_0)_{\underline{0}\underline{0}}(2,2) = 0;$$

and similarly for the other probability measures.

Example 4.8. Consider, again, the example of the relational group $\mathcal{G}$ corresponding to $n\mathbb{Z} \triangleleft \mathbb{Z}$ as in Example 3.9. In this case, assigning for $l \in \mathbb{Z}$ the counting measure μ_{count} on $\mathcal{G}_l^{(2)}$ does not provide an example of a relational Haar system, since it does not disintegrate in the sense of Definition 4.1. The point is that the fibers are infinite, and thus the "measures along the fibers" $(\mu_l)_{\underline{l}_1\underline{l}_2}$ are not probability measures. One possibility to define a (strongly split) relational Haar system is to define probability measures $\tau_{\underline{l}}$ on $l+n\mathbb{Z}$, for $\underline{l} \in \mathbb{Z}_n$ (for instance the Dirac measure supported at some representative). However, these measures will necessarily fail to be translation-invariant.

The relational convolution algebra. Given the definition of a relational Haar system, we now define a generalization of the convolution algebra to the case of relational groupoids. To define a set of functions on which the convolution product is well-defined, we will from now on assume that our relational groupoids are equipped with a topology. A relational Haar systems is assumed to be continuous[5] and given by Radon measures with respect to this topology. The following subspace of functions is a convenient one:

Definition 4.9. Let $\mathcal{G}$ be a topological relational groupoid and $\mu = (\mu_g)_{g\in\mathcal{G}}$ a Haar system given by Radon measures. Denote by $\underline{\mathcal{G}}$ the quotient groupoid of $\mathcal{G}$ equipped with the quotient topology. Then a continuous function f is *admissible* if it is bounded and there is a compact set $\underline{\mathcal{K}} \subset \underline{\mathcal{G}}$ such that $\operatorname{supp}(f) \cap \mathcal{C} \subset q^{-1}(\underline{\mathcal{K}})$. The set of admissible functions is denoted by $\mathcal{A}(\mathcal{G})$.

The point about admissible functions is that they have a well-defined convolution product which is again an admissible function.

Proposition 4.10. *Let $f_1, f_2 \in \mathcal{A}(\mathcal{G})$. Then the convolution product defined by*

$$(f_1 \star f_2)(g) = \int_{\mathcal{G}_g^{(2)}} f_1(h) f_2(k)\, d\mu_g(h,k) \tag{4-5}$$

converges for all $g \in \mathcal{G}$ and $g \mapsto (f_1 \star f_2)(g) \in \mathcal{A})(\mathcal{G})$.

Proof. First, notice that $(f_1 \star f_2)(g) = 0$ if $g \notin \mathcal{C}$. This follows from the fact that $\mathcal{G}_g^{(2)} = \varnothing$ in this case (see Proposition App.1). Otherwise, we use the axiom that the Haar measure μ_g disintegrates to write

$$(f_1 \star f_2)(g) = \int_{(\underline{g}_1,\underline{g}_2)\in\underline{\mathcal{G}}^{(2)}_{q(g)}} \underbrace{\int_{q^{-1}((\underline{g}_1,\underline{g}_2))} f_1(h) f_2(k)\, d\mu_{g\underline{g}_1\underline{g}_2}(h,k)}_{=:\tilde{f}(\underline{g}_1,\underline{g}_2)}\, d\nu_{\underline{g}}(\underline{g}_1,\underline{g}_2). \tag{4-6}$$

Since f_1 and f_2 are bounded, integrating $f_1 \cdot f_2$ along the fibers results in a bounded continuous function $\tilde{f}$ on the base $\underline{\mathcal{G}}^{(2)}_{q(g)}$ and $|\tilde{f}| \le |f_1||f_2|$. Let $\underline{\mathcal{K}}_i$ denote compact

[5] One way to phrase this is that for every continuous function f on $\mathcal{G}^{(3)}$ the function $k \mapsto \int_{(g,h)\in\mathcal{G}^{(2)}} f(g,h,k)\, d\mu_k$ is continuous

sets containing $q(\mathrm{supp}(f_i))$. The support of $\tilde{f}$ is contained in $\underline{\mathcal{K}}_1 \times \underline{\mathcal{K}}_2$, which is compact. Since $\mu_q(g)$ is also a Radon measure, the integral is finite and in fact $|(f_1 \star f_2)(g)| \leq \nu_{\underline{g}}(\underline{\mathcal{K}}_1 \times \underline{\mathcal{K}}_2)|f_1 \times f_2|$. Hence, $f_1 \star f_2$ is defined pointwise. By continuity of the Haar system, it follows that $f_1 \star f_2$ is also continuous. It remains to check that $f_1 \star f_2$ is admissible. Let m denote the multiplication in the quotient groupoid, and denote by $\underline{\mathcal{K}}$ the set $m((\underline{\mathcal{K}}_1 \times \underline{\mathcal{K}}_2) \cap \underline{\mathcal{G}}^{(2)})$. Then $\underline{\mathcal{K}}$ is compact since the multiplication is continuous. The arguments above imply that $\mathrm{supp}(f_1 \star f_2) \subset q^{-1}(\underline{\mathcal{K}})$, and hence in the preimage of a compact set. To see that $f_1 \star f_2$ is bounded, note that $|(f_1 \star f_2)| \leq |f_1||f_2| \sup_{\underline{g} \in \underline{\mathcal{K}}} \nu_{\underline{g}}(\underline{\mathcal{K}})$. By compactness, the supremum is obtained and $f_1 \star f_2$ is bounded, and hence admissible. □

The definition of the convolution algebra is now straightforward.

Definition 4.11. Let $(\mathcal{G}, L, I)$ be a relational groupoid with a relational Haar system $(\mu_g)_{g \in \mathcal{G}}$. Denote by $\mathcal{A}(\mathcal{G})$ the space of admissible functions. The relational convolution algebra is then $(\mathcal{A}(\mathcal{G}), \star)$, with the convolution product defined in (4-5).

Remark 4.12. Another possible domain for the convolution product is given by continuous functions which are just compactly supported. This is a subspace of the space of admissible functions. However, below we want to consider L_2-invariant functions, to recover the algebra of functions on the quotient. If the L_2-fibers are not compact (for instance, this happens for the relational group associated to $\mathcal{G} = \mathbb{Z}$, $\mathcal{H} = k\mathbb{Z}$), then such functions will never be compactly supported. This motivates our choice of space of admissible functions. Of course, for compact relational groupoids the spaces of admissible, compactly supported and continuous functions all coincide.

Associativity. A natural question when defining an algebra is whether or not it is associative. This is the question we investigate in this subsection. It turns out that in general, the convolution is associative only when restricted to L_2-invariant functions, or when the Haar system satisfies some restrictive condition.

Proposition 4.13. *The space of L_2-invariant functions is an associative subalgebra of $(\mathcal{A}(\mathcal{G}), \star)$.*

Notice that for an L_2-invariant function f there is a well-defined continuous function $\tilde{f}$ on the quotient groupoid $\underline{\mathcal{G}}$, defined by $\tilde{f}(\underline{g}) = f(g)$. By Axiom (ii) of Definition 4.2, to a Haar system μ on $\mathcal{G}$ there is an associated Haar system ν on $\underline{\mathcal{G}}$. Let us denote its convolution product on compactly supported functions by $\star_{\underline{\mathcal{G}}}$. The proof of Proposition 4.13 is a direct corollary of the following Lemma:

Lemma 4.14. *For L_2-invariant functions $f_1, f_2 \in \mathcal{A}(\mathcal{G})$, the convolution is given by*

$$f_1 \star f_2 = \begin{cases} q^*(\tilde{f}_1 \star_{\underline{\mathcal{G}}} \tilde{f}_2) & \text{on } \mathcal{C}, \\ 0 & \text{on } \mathcal{G} \setminus \mathcal{C}. \end{cases} \tag{4-7}$$

Proof. First, we recall that by definition the support of the convolution of any two admissible functions is contained in $\mathcal{C}$, since $\mathcal{G}_g^{(2)} = \varnothing$ for $g \in \mathcal{G}$. Notice further that if $f_1, f_2 \in \mathcal{A}(\mathcal{G})$ are L_2-invariant, then by definition their associated functions on $\underline{\mathcal{G}}$ are compactly supported; hence, the right-hand side of (4-7) is defined. Now, we use again the fact that the Haar system disintegrates to write

$$\begin{aligned}(f_1 \star f_2)(g) &= \int_{(\underline{g}_1,\underline{g}_2)\in\underline{\mathcal{G}}^{(2)}_{q(g)}} \int_{q^{-1}((\underline{g}_1,\underline{g}_2))} f_1(h) f_2(k)\, d\mu_{g\underline{g}_1\underline{g}_2}(h,k)\, d\nu_{\underline{g}}(\underline{g}_1,\underline{g}_2)\\ &= \int_{(\underline{g}_1,\underline{g}_2)\in\underline{\mathcal{G}}^{(2)}_{q(g)}} \tilde{f}_1(\underline{g}_1)\tilde{f}_2(\underline{g}_2)\, d\nu_{\underline{g}}(\underline{g}_1,\underline{g}_2)\\ &= (\tilde{f}_1 \star_{\underline{\mathcal{G}}} \tilde{f}_2)(\underline{g}).\end{aligned}$$

Here, the second equation follows from normalization of the disintegrating measures and the fact that L_2-invariance is equivalent to being constant on fibers of q. Integrating a constant function against a probability measure, we simply obtain that constant. □

By associativity of the convolution in the quotient groupoid, Proposition 4.13 holds. Next, we give a sufficient criterion on the Haar system for the convolution algebra to be associative.

Proposition 4.15. *Suppose that the relational Haar system μ is strongly split as in Definition 4.6. Then the convolution algebra $\mathcal{A}(\mathcal{G})$ is associative.*

Proof. Another way to write formula (4-7) is

$$f_1 \star f_2 = q^*(q_* f_1 \star_{\underline{\mathcal{G}}} q_* f_2), \tag{4-8}$$

where we denoted by $\tilde{f} =: q_* f$ the pushforward of L_2-invariant functions. Associativity of the convolution restricted to L_2-invariant functions follows from the property

$$q_* \circ q^* = \mathrm{id}, \tag{4-9}$$

i.e., q_* is defined as the left inverse of the injective map $q^* \colon C_c(\underline{\mathcal{G}}) \to \mathcal{A}(\mathcal{G})$ on its image. Thus, the convolution is associative on all functions if we can find an extension of the map q_* to all of $\mathcal{A}(\mathcal{G})$ in such a way that property (4-8) is still true. In this case, there is a family of probability measures $\tau_{\underline{g}}$ on $\mathcal{G}$, and we can extend q_* by setting

$$q_* f(\underline{g}) = \int_{q^{-1}(\underline{g})} f(g)\, d\tau_{\underline{g}}(g);$$

and again, we have properties (4-8) and (4-9). Thus, associativity follows from associativity of the convolution algebra of the quotient. □

Remark 4.16. In the two examples above, we were able to infer associativity from the fact that the convolution algebra on the quotient is associative. However, both of these examples are "well behaved" with respect to the quotient, since they satisfy the strongly split condition. Already on the simplest relational groupoid with nontrivial L_2 relation, one can find examples of (split, invariant) relational Haar systems such that the convolution algebra is not associative, see Example 4.17 below. However, since the L_3 relation in the relational groupoid is strictly associative, and L_2-invariant, one might ask whether there is a remnant of this fact that is visible at the level of the convolution algebra, i.e., whether the algebra is "associative up to homotopy" in general. This is an interesting question that the authors plan to address in a future paper.

The relational convolution algebra associated to a split L_2-invariant relational Haar system which is not strongly split is not necessarily associative, which follows from the counterexample below:

Example 4.17. We consider again the relational group given by $\mathcal{G} = \mathbb{Z}_4$, $\mathcal{H} = \mathbb{Z}_2$. Denote by δ_g the function with $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. Unraveling the definitions, we can compute

$$\begin{aligned}
\delta_0 \star \delta_0 &= \mu_0(0, 0)\delta_0 + \mu_2(0, 0)\delta_2,\\
\delta_0 \star \delta_1 &= \mu_1(0, 1)\delta_1 + \mu_3(0, 1)\delta_3,\\
\delta_2 \star \delta_1 &= \mu_1(2, 1)\delta_1 + \mu_3(2, 1)\delta_3,\\
\delta_0 \star \delta_3 &= \mu_1(0, 3)\delta_1 + \mu_3(0, 3)\delta_3,
\end{aligned}$$

$$\begin{aligned}
&\delta_0 \star (\delta_0 \star \delta_1)\\
&= (\underbrace{\mu_1(0, 1)^2 + \mu_3(0, 1)\mu_1(0, 3)}_{A}) \cdot \delta_1 + (\underbrace{\mu_1(0, 1)\mu_3(0, 1) + \mu_3(0, 1)\mu_3(0, 3)}_{B}) \cdot \delta_3,\\
&(\delta_0 \star \delta_0) \star \delta_1\\
&= (\underbrace{\mu_0(0,0)\mu_1(0,1)+\mu_2(0,0)\mu_1(2,1)}_{C})\cdot\delta_1+(\underbrace{\mu_0(0,0)\mu_3(0,1)+\mu_2(0,0)\mu_3(2,1)}_{D})\cdot\delta_3.
\end{aligned}$$

The condition for μ being a relational Haar system implies that

$$\mu_g(g_1, g_2) = \nu_{\underline{g}}(\mu_g)_{\underline{g}_1\underline{g}_2}(g_1, g_2) = \tfrac{1}{2}(\mu_g)_{\underline{g}_1\underline{g}_2}(g_1, g_2),$$

where $(\mu_g)_{\underline{g}_1\underline{g}_2}(g_1, g_2)$ is some probability measure on $q^{-1}(\underline{g}_1, \underline{g}_2)$. It is clear that, in general, A, B, C, D are pairwise different. If we suppose the measure is L_2-invariant, we obtain

$$\begin{aligned}
A = B &= \mu_1(0, 1)^2 + \mu_1(0, 1)\mu_1(0, 3),\\
C = D &= \mu_0(0, 0)\mu_1(0, 1) + \mu_1(2, 1).
\end{aligned}$$

Furthermore, if we suppose that μ is split, we obtain $\mu_0(0, 0) = \frac{1}{4}\tau^0_{\underline{0}}(0)^2$. Letting $\tau^0_{\underline{0}}$ be the measure on $\{0, 2\}$ supported at 2, we see that $C = D = 0$ (in fact, in this case $\delta_0 \star \delta_0 = 0$). On the other hand, $A = B$ depends only on $\tau^1_{\underline{0}}$ and $\tau^1_{\underline{1}}$. Thus, this is an example of a split L_2-invariant Haar system with nonassociative convolution algebra.

Reduction of the algebra of admissible functions. In this subsection, we show that the reduction of the relational convolution algebra is isomorphic to the convolution algebra of the groupoid which is the reduction of the relational groupoid. We will obtain this result via a two-step reduction: one reduction with respect to the constraint set, followed by a reduction with respect to the L_2 relation. Let $(\mathcal{G}, L, I)$ be a relational groupoid, and let $\mathcal{C}$ be its constraint set. We denote by $I_{\mathcal{C}}$ the subset of functions in $C_c(\mathcal{G})$ that vanish on $\mathcal{C}$. The set $I_{\mathcal{C}}$ is usually called the vanishing ideal, since it is an ideal inside $C_c(\mathcal{G})$ with the standard (commutative) product. Furthermore we obtain the following isomorphism of convolution algebras:

Proposition 4.18. *$I_{\mathcal{C}}$ is also an ideal in $\mathcal{A}(\mathcal{G})$, and*

$$\mathcal{A}(\mathcal{G})/I_{\mathcal{C}} \cong \mathcal{A}(\mathcal{C}).$$

Proof. The fact that $I_{\mathcal{C}}$ is an ideal of $\mathcal{A}(\mathcal{G})$ follows from the fact that if $f_1 \in I_{\mathcal{C}}$, then f_1 vanishes on $\mathcal{C}$; and hence, $f_1 \star f_2|_{\mathcal{C}} = 0$, for all $f_2 \in \mathcal{A}(\mathcal{G})$. □

Definition 4.19. Denote by $(\mathcal{A}(\mathcal{C}))^{L_2} \subset \mathcal{A}(\mathcal{C})$ the subspace of admissible functions on $\mathcal{C}$ that are constant along L_2, i.e., the subspace of functions $f \in \mathcal{A}(\mathcal{C})$ satisfying $(h, k) \in L_2 \Rightarrow f(h) = f(k)$.

Proposition 4.20. *$(\mathcal{A}(\mathcal{C})^{L_2}, \star)$ is a subalgebra of $(\mathcal{A}(\mathcal{C}), \star)$.*

Proof. It follows directly from Lemma 4.14. □

Definition 4.21. Let $\mathcal{G}$ be a relational groupoid. Its *reduced convolution algebra* (with respect to L_2) is

$$\underline{\mathcal{A}}(\mathcal{G}) = (\mathcal{A}(\mathcal{G})/I_{\mathcal{C}})^{L_2} \cong \mathcal{A}(\mathcal{C})^{L_2}. \tag{4-10}$$

We are now ready to state our main result.

Theorem 4.22. *The reduced convolution algebra of a relational groupoid is isomorphic to the (groupoid) convolution algebra of its reduction:*

$$\underline{\mathcal{A}}(\mathcal{G}) \cong \mathcal{A}(\underline{\mathcal{G}}). \tag{4-11}$$

Proof. Since $\underline{\mathcal{G}} = \mathcal{C}/L_2$, it follows that the sets $\underline{\mathcal{A}}(\mathcal{G})$ and $\mathcal{A}(\underline{\mathcal{G}})$ are in bijection. The fact that the map

$$\Phi : \underline{\mathcal{A}}(\mathcal{G}) \to \mathcal{A}(\underline{\mathcal{G}}), \quad f \mapsto \tilde{f}$$

is an isomorphism of convolution algebras follows from L_2-invariance of functions in $\underline{\mathcal{A}}(\mathcal{G})$ and Proposition 4.20. □

5. Future directions

Relational C*-algebras and higher structures. A natural next step is to introduce the C^*-completion of the relational convolution algebra. In particular, we will study the relational analogue of the algebra $M(n, \mathbb{C})$ of complex valued $n \times n$-matrices and the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H. We will also study the representations of relational convolution algebras and how this construction relates to C^*-algebras for 2-groups and higher-categorical versions of relational groupoids.

Field theory. In a follow-up work, we intend to describe the groupoid convolution algebra for the relational symplectic groupoid obtained via the Poisson Sigma Model. This would provide a positive answer for the Guillemin–Sternberg conjecture for this particular 2-dimensional TFT. In particular, we will use Hawkins' approach to C^*-algebra quantization of symplectic groupoids [Hawkins 2008], in the context of relational groupoids.

More precisely, Hawkins developed a C^*-algebra quantization procedure that is compatible with the groupoid structure maps, as well as with the multiplicative symplectic structure on the space of morphisms. The construction is based on prescribing the following data (each built based on the previous ones):

(1) a symplectic groupoid $G \rightrightarrows M$ integrating a Poisson manifold M,

(2) a prequantum line bundle (L, ∇) over G,

(3) a symplectic groupoid polarization $\mathcal{P}$ of G,

(4) a "half-form" bundle $\Omega^{1/2}_{\mathcal{P}}$,

(5) A twisted, polarized convolution algebra $\mathcal{C}^*_{\mathcal{P}}(G, \sigma)$ (where σ encodes the twisting data).

Our construction serves as a first step towards a field theoretic interpretation of the geometric quantization procedure introduced by Hawkins. In particular, we conjecture that the final object in step (5) can be obtained via transgression in the AKSZ formulation of the Poisson Sigma Model. A natural candidate for the relational Haar system comes from the path space construction, analogous to the Wiener measure on loop groups. This interpretation will help connect the perturbative quantization of the Poisson Sigma Model via relational symplectic groupoids [Cattaneo et al. 2017] and nonperturbative approaches, i.e., geometric quantization [Bonechi et al. 2006]. Such interpretation will be independent of the integrability of the Poisson manifold.

Relational convolution and split relations. In [Cattaneo and Contreras 2021], split relations were studied in the context of relational symplectic groupoids. A relation is called *split* if it is isotropic and it has a closed isotropic complement. It turns out that the structure relations of every finite-dimensional symplectic groupoid is split, as well as the infinite-dimensional relational symplectic groupoid obtained via the PSM. We will study the relationship between split relations and the convolution algebra for split relational groupoids.

Relational convolution and Frobenius algebras. In [Heunen et al. 2013], it has been proven that special dagger Frobenius objects in **Rel** (the category of sets and relations) are in one-to-one correspondence with groupoids. Recently in [Mehta and Zhang 2020], this result has been generalized using a characterization of Frobenius objects in **Rel** using simplicial sets. Relational convolution algebras are natural examples of Frobenius objects in **Rel**, in the same way that group algebras are a special class of Frobenius algebras. We expect to have a simplicial interpretation of these extension in the future [Contreras et al. $\geq$ 2021].

Appendix: Some results on relational groupoids

We will prove some results about the structure of the sets

$$\mathcal{G}_g^{(2)} = \{(h,k) \in \mathcal{G}^{(2)} \mid (h,k,g) \in L_3\}.$$

Moreover, let $\mathcal{C}$ be defined as in Proposition 3.15. Then we have:

Proposition App.1. *If $\mathcal{G}_g^{(2)}$ is nonempty, then $g \in \mathcal{C}$.*

Proof. Let $(h,k) \in \mathcal{G}_g^{(2)}$. Then $(h,k,g) \in L_3 = L_2 \circ L_3$ by L_2-invariance (3-7). It follows that there is g' such that $(g', g) \in L_2$. In particular, $g \in L_2 \circ \mathcal{G} = \mathcal{C}$. ☐

For a usual groupoid $G \rightrightarrows M$, these sets are the fibers of the fibration $G^{(2)} \to G$. In a relational groupoid, this is no longer true. Instead we have the following statement:

Proposition App.2. *For two distinct elements $g, g' \in \mathcal{G}$, the sets $\mathcal{G}_g^{(2)}$ and $\mathcal{G}_{g'}^{(2)}$ are equal if and only if $(g, g') \in L_2$ and disjoint otherwise.*

Proof. First, suppose that $(g, g') \in L_2$, and let $(h,k,g) \in L_3$. Then $(h,k,g') \in L_2 \circ L_3 = L_3$. Hence, $\mathcal{G}_g^{(2)} \subseteq \mathcal{G}_{g'}^{(2)}$, and symmetry of L_2 implies the other direction. Now, assume that $(h,k) \in \mathcal{G}_g^{(2)} \cap \mathcal{G}_{g'}^{(2)}$. It suffices to show that $(g, g') \in L_2$. Axiom A.1 and Axiom A.3 imply that $(h,k,g) \in L_3$ if and only if $(g, I(k), h) \in L_3$. Hence, $(g, I(k), k, g') \in L_3 \circ (L_3 \times \mathrm{id}) = L_3 \circ (\mathrm{id} \times L_3)$. Thus, there is $e \in \mathcal{G}$ such that $(I(k), k, e) \in L_3$ and $(g, e, g') \in L_3$. By definition of L_1 and L_2, we conclude that $e \in L_1$, and hence $(g, g') \in L_2$. ☐

An immediate corollary is the following:

Corollary App.3. *Let $(h,k) \in \mathcal{G}_g^{(2)}$, then the set $hk := \{g' \in \mathcal{G} \mid (h,k,g') \in L_3\}$ satisfies $hk = L_2(g)$. In particular, we have*

$$\mathcal{G}_{hk}^{(2)} = \bigcup_{g' \in hk} \mathcal{G}_{g'}^{(2)} = \mathcal{G}_g^{(2)}. \tag{App-1}$$

Analogous to the right relational action, we can define left relational actions. In particular, a relational groupoid acts on itself by left and right multiplication, and the sets $\mathcal{G}_g^{(2)}$ behave nicely under this action.

Proposition App.4. *Let* $(\mathcal{G}, L, I)$ *be a relational groupoid, and let* $(g, h) \in \mathcal{G}^{(2)}$. *Then, we have*

$$(\mathrm{id} \times R_h) \circ \mathcal{G}^{(2)}_g = \mathcal{G}^{(2)}_{gh}, \tag{App-2}$$

and similarly

$$(L_g \times \mathrm{id}) \circ \mathcal{G}^{(2)}_h = \mathcal{G}^{(2)}_{gh}, \tag{App-3}$$

Proof. For the first equation, we have

$$\begin{aligned}
(\mathrm{id} \times R_h) \circ \mathcal{G}^{(2)}_g &= \{(g_1, g_2) \mid \exists k, (g_1, k) \in G^{(2)}_g, (k, g_2) \in R_h\} \\
&= \{(g_1, g_2) \mid \exists k, (g_1, k) \in \mathcal{G}^{(2)}_g, (g_2, k) \in (R_h)^T = R_{I(h)}\} \\
&= \{(g_1, g_2) \mid \exists k, (g_1, k, g) \in L_3, (g_2, I(h), k) \in L_3\} \\
&= \{(g_1, g_2) \mid (g_1, g_2, I(h), g) \in L_3 \circ (\mathrm{id} \times L_3)\} \\
&= \{(g_1, g_2) \mid (g_1, g_2, I(h), g) \in L_3 \circ (L_3 \times \mathrm{id})\} \\
&= \{(g_1, g_2) \mid \exists k \in \mathcal{G} \colon (g_1, g_2, k) \in L_3, (k, I(h), g) \in L_3\} \\
&= \{(g_1, g_2) \mid \exists k \in \mathcal{G} \colon (g_1, g_2, k) \in L_3, (k, I(h), I(g)) \in L\} \\
&= \{(g_1, g_2) \mid \exists k \in \mathcal{G} \colon (g_1, g_2, k) \in L_3, (I(h), I(g), k) \in L\} \\
&= \{(g_1, g_2) \mid \exists k \in \mathcal{G} \colon (g_1, g_2, k) \in L_3, (I(h), I(g), I(k)) \in L_3\} \\
&= \{(g_1, g_2) \mid \exists k \in \mathcal{G} \colon (g_1, g_2, k) \in L_3, (g, h, k) \in L_3\} \\
&= \mathcal{G}^{(2)}_{gh}
\end{aligned}$$

The other equation is proven similarly. □

In particular, we have the following.

Corollary App.5. *Let* $(\mathcal{G}, L, I)$ *be a relational groupoid, and let* $(g, h) \in \mathcal{G}^{(2)}$. *Then, we have*

$$(L_{I(g)} \times R_h) \circ \mathcal{G}^{(2)}_g = \mathcal{G}^{(2)}_h. \tag{App-4}$$

Proof. By Proposition App.4, we have $(L_{I(g)} \times R_h) \circ \mathcal{G}^{(2)}_g = \mathcal{G}^{(2)}_{I(g)gh}$. By definition of L_2, this can be rewritten as $\mathcal{G}^{(2)}_{I(g)gh} = \mathcal{G}^{(2)}_{L_2(h)}$. We conclude the proof using Proposition App.4. □

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References

[Alexandrov et al. 1997] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, "The geometry of the master equation and topological quantum field theory", *Int. J. Modern Phys. A* **12**:7 (1997), 1405–1429. MR Zbl

[Ambrosio et al. 2005] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows: in metric spaces and in the space of probability measures*, Birkhäuser, Basel, 2005. MR Zbl

[Batalin and Fradkin 1983] I. A. Batalin and E. S. Fradkin, "A generalized canonical formalism and quantization of reducible gauge theories", *Phys. Lett. B* **122**:2 (1983), 157–164. MR Zbl

[Batalin and Vilkovisky 1977] I. A. Batalin and G. A. Vilkovisky, "Relativistic S-matrix of dynamical systems with boson and fermion constraints", *Phys. Lett. B* **69**:3 (1977), 309–312.

[Batalin and Vilkovisky 1981] I. A. Batalin and G. A. Vilkovisky, "Gauge algebra and quantization", *Phys. Lett. B* **102**:1 (1981), 27–31. MR

[Batalin and Vilkovisky 1983] I. A. Batalin and G. A. Vilkovisky, "Quantization of gauge theories with linearly dependent generators", *Phys. Rev. D* (3) **28**:10 (1983), 2567–2582. MR

[Bonechi et al. 2006] F. Bonechi, A. S. Cattaneo, and M. Zabzine, "Geometric quantization and non-perturbative Poisson sigma model", *Adv. Theor. Math. Phys.* **10**:5 (2006), 683–712. MR Zbl

[Bonechi et al. 2014] F. Bonechi, N. Ciccoli, J. Qiu, and M. Tarlini, "Quantization of Poisson manifolds from the integrability of the modular function", *Comm. Math. Phys.* **331**:2 (2014), 851–885. MR Zbl

[Cattaneo and Contreras 2015] A. S. Cattaneo and I. Contreras, "Relational symplectic groupoids", *Lett. Math. Phys.* **105**:5 (2015), 723–767. MR Zbl

[Cattaneo and Contreras 2021] A. Cattaneo and I. Contreras, "Split canonical relations", *Ann. H. Lebesgue* **4** (2021), 155–185. MR

[Cattaneo and Felder 2000] A. S. Cattaneo and G. Felder, "A path integral approach to the Kontsevich quantization formula", *Comm. Math. Phys.* **212**:3 (2000), 591–611. MR Zbl

[Cattaneo and Felder 2001a] A. S. Cattaneo and G. Felder, "On the AKSZ formulation of the Poisson sigma model", *Lett. Math. Phys.* **56**:2 (2001), 163–179. MR Zbl

[Cattaneo and Felder 2001b] A. S. Cattaneo and G. Felder, "Poisson sigma models and symplectic groupoids", pp. 61–93 in *Quantization of singular symplectic quotients*, edited by N. P. Landsman et al., Progr. Math. **198**, Birkhäuser, Basel, 2001. MR Zbl

[Cattaneo and Moshayedi 2020] A. S. Cattaneo and N. Moshayedi, "Introduction to the BV-BFV formalism", *Rev. Math. Phys.* **32**:9 (2020), art. id. 2030006. MR Zbl

[Cattaneo et al. 2017] A. S. Cattaneo, N. Moshayedi, and K. Wernli, "Relational symplectic groupoid quantization for constant Poisson structures", *Lett. Math. Phys.* **107**:9 (2017), 1649–1688. MR Zbl

[Cattaneo et al. 2018] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, "Perturbative quantum gauge theories on manifolds with boundary", *Comm. Math. Phys.* **357**:2 (2018), 631–730. MR Zbl

[Cattaneo et al. 2019] A. S. Cattaneo, N. Moshayedi, and K. Wernli, "Globalization for perturbative quantization of nonlinear split AKSZ sigma models on manifolds with boundary", *Comm. Math. Phys.* **372**:1 (2019), 213–260. MR Zbl

[Cattaneo et al. 2020] A. S. Cattaneo, N. Moshayedi, and K. Wernli, "On the globalization of the Poisson sigma model in the BV-BFV formalism", *Comm. Math. Phys.* **375**:1 (2020), 41–103. MR Zbl

[Chang and Pollard 1997] J. T. Chang and D. Pollard, "Conditioning as disintegration", *Stat. Neerl.* **51**:3 (1997), 287–317. MR Zbl

[Connes 1994] A. Connes, *Noncommutative geometry*, Academic, San Diego, CA, 1994. MR Zbl

[Contreras 2013] I. Contreras, *Relational symplectic groupoids and Poisson sigma models with boundary*, Ph.D. thesis, University of Zurich, 2013, available at https://arxiv.org/abs/1306.3943.

[Contreras et al. $\geq$ 2021] I. Contreras, M. Keller, and R. Mehta, "Frobenius objects in the category of spans", preprint. arXiv

[Coste et al. 1987] A. Coste, P. Dazord, and A. Weinstein, "Groupoïdes symplectiques", pp. 1–62 in *Publications du Département de Mathématiques*, *II*, Publ. Dép. Math. Nouvelle Sér. A **87**, Univ. Claude-Bernard, Lyon, 1987. MR Zbl

[Crainic and Fernandes 2004] M. Crainic and R. L. Fernandes, "Integrability of Poisson brackets", *J. Differential Geom.* **66**:1 (2004), 71–137. MR Zbl

[Dirac 1930] P. A. M. Dirac, *The principles of quantum mechanics*, Oxford Univ. Press, 1930. Zbl

[Fradkin and Vilkovisky 1975] E. S. Fradkin and G. A. Vilkovisky, "Quantization of relativistic systems with constraints", *Phys. Lett. B* **55**:2 (1975), 224–226. MR Zbl

[Guillemin and Sternberg 1982] V. Guillemin and S. Sternberg, "Geometric quantization and multiplicities of group representations", *Invent. Math.* **67**:3 (1982), 515–538. MR Zbl

[Hawkins 2008] E. Hawkins, "A groupoid approach to quantization", *J. Symplectic Geom.* **6**:1 (2008), 61–125. MR Zbl

[Heunen et al. 2013] C. Heunen, I. Contreras, and A. S. Cattaneo, "Relative Frobenius algebras are groupoids", *J. Pure Appl. Algebra* **217**:1 (2013), 114–124. MR Zbl

[Higson 2004] N. Higson, "Groupoids, C^*-algebras and index theory", Forschungsinst. Math., 2004, available at https://tinyurl.com/higsonfim.

[Ikeda 1994] N. Ikeda, "Two-dimensional gravity and nonlinear gauge theory", *Ann. Phys.* **235**:2 (1994), 435–464. MR Zbl

[Kontsevich 2003] M. Kontsevich, "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66**:3 (2003), 157–216. MR Zbl

[Mehta and Zhang 2020] R. A. Mehta and R. Zhang, "Frobenius objects in the category of relations", *Lett. Math. Phys.* **110**:7 (2020), 1941–1959. MR Zbl

[Pachl 1978] J. K. Pachl, "Disintegration and compact measures", *Math. Scand.* **43**:1 (1978), 157–168. MR Zbl

[Schaller and Strobl 1994] P. Schaller and T. Strobl, "Poisson structure induced (topological) field theories", *Modern Phys. Lett. A* **9**:33 (1994), 3129–3136. MR Zbl

[Stasheff 1997] J. Stasheff, "Homological reduction of constrained Poisson algebras", *J. Differential Geom.* **45**:1 (1997), 221–240. MR Zbl

[Weinstein 1987] A. Weinstein, "Symplectic groupoids and Poisson manifolds", *Bull. Amer. Math. Soc.* (*N.S.*) **16**:1 (1987), 101–104. MR Zbl

[Westman 1968] J. J. Westman, "Harmonic analysis on groupoids", *Pacific J. Math.* **27** (1968), 621–632. MR Zbl

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Chapter 4

On Quantum Obstruction Spaces and Higher Codimension Gauge Theories

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ABSTRACT

Using the quantum construction of the BV-BFV method for perturbative gauge theories, we show that the obstruction for quantizing a codimension 1 theory is given by the second cohomology group with respect to the boundary BRST charge. Moreover, we give an idea for the algebraic construction of codimension k quantizations in terms of $\mathbb{E}_k$-algebras and higher shifted Poisson structures by formulating a higher version of the quantum master equation.

1. Introduction

The Batalin–Vilkovisky formalism [11–13] is a powerful method to deal with perturbative quantizations of local gauge theories. The extension of this formalism to manifolds with boundary combines the Lagrangian approach of the Batalin–Vilkovisky (BV) formalism in the bulk with the Hamiltonian approach of the Batalin–Fradkin–Vilkovisky (BFV) formalism [10,33] on the boundary of the underlying source (spacetime) manifold. This construction is known as the BV-BFV formalism [23,24,26]. In particular, it describes a codimension 1 quantum gauge formalism. Within a classical gauge theory one is interested in describing the obstructions for it to be quantizable. The cohomological symplectic formulation suggests an operator quantization for the boundary action. To get a well-defined and consistent cohomology theory, one has to require that this induced operator squares to zero. This will lead to obstruction spaces for boundary theories by considering a deformation quantization of the boundary action in order to formulate a boundary version of the quantum master equation as the gauge-independence condition. We will show that the obstruction for the quantization of manifolds with boundary is controlled by the second cohomology group with respect to the cohomological vector field on the boundary fields. Moreover, we formulate a classical extension of higher codimension k theories as in [23] which we call BF^{k}V theories. The coupling for each stratum, in fact, is easily extended in the classical setting (BV-BF^{k}V theories), whereas for the quantum setting it might be rather involved. In order to formulate a fully extended topological quantum field theory in the sense of Baez–Dolan [8] or Lurie [48], the coupling is indeed necessary. Since one layer of the quantum picture, namely the quantum master equation, is described in terms of deformation quantization, we can formulate an algebraic approach for the higher codimension extension in terms of $\mathbb{E}_k$- and $\mathbb{P}_k$-algebras [49,61]. Here $\mathbb{E}_k$ denotes the ∞-operad of little k-dimensional disks [35,44,49]. Moving to one codimension higher corresponds to the shift of the Poisson structure by -1 since the symplectic form is shifted by $+1$ (see [55] for the shifted symplectic setting). This is controlled by the operad $\mathbb{P}_k$ on codimension k which corresponds to $(1-k)$-shifted Poisson structures [17,60]. Using this notion, we give some ideas for the quantization in higher codimension. Moreover, if one uses the notion of Beilinson–Drinfeld ($\mathbb{BD}$) algebras [15,29], in particular $\mathbb{BD}_0$- and $\mathbb{BD}_1$-algebras, one can try to consider the action of $\mathbb{P}_0\cong\mathbb{BD}_0/\hbar$ (for $\hbar\to 0$) on $\mathbb{P}_1\cong\mathbb{BD}_1/\hbar$ (for $\hbar\to 0$) in order to capture the algebraic structure of the classical bulk-boundary coupling (see also [60, Section 5]). Here $\cong$ denotes an isomorphism of operads. In general, one can define the

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$\mathbb{BD}_k$ operads to provide a certain interpolation between the $\mathbb{P}_k$ and $\mathbb{E}_k$ operads in the sense that they are graded Hopf [47] differential graded (dg) operads over $\mathbf{K}[\![\hbar]\!]$, where $\hbar$ is of weight $+1$ and $\mathbf{K}$ a field of characteristic zero, together with the equivalences

$$\mathbb{BD}_k/\hbar \cong \mathbb{P}_k, \qquad \mathbb{BD}_k[\![\hbar^{-1}]\!] \cong \mathbb{E}_k((\hbar)).$$

The formality of the $\mathbb{E}_k$ operad [35,44,69] implies the equivalence $\mathbb{BD}_k \cong \mathbb{P}_k[\![\hbar]\!]$. There is a formulation of a $\mathbb{BD}_2$-algebra in terms of brace algebras [18,61] and one can show that there is in fact a quasi-isomorphism $\mathbb{P}_2 \cong \mathbb{BD}_2/\hbar$ (for $\hbar \to 0$). However, the notion of a $\mathbb{BD}_k$-algebra for $k \geq 3$ in terms of braces is currently not defined, but there should not be any obstruction to do this. Using these operads, one can define a deformation quantization of a $\mathbb{P}_{k+1}$-algebra A to be a $\mathbb{BD}_{k+1}$-algebra $A_\hbar$ together with an equivalence of $\mathbb{P}_{k+1}$-algebras $A_\hbar/\hbar \cong A$ (see [17,53] for a detailed discussion).

Notation and conventions. We will denote functions on a manifold M by $\mathcal{O}(M)$. Vector fields on M will be denoted by $\mathfrak{X}(M)$ and the space of differential k-forms on M by $\Omega^k(M)$. We denote by $A[\![t]\!]$ the space of formal power series in a formal parameter t with coefficients in some algebra A. The imaginary unit is denoted by $\mathrm{i} := \sqrt{-1}$. If the manifolds are infinite-dimensional, they are usually *Banach* or *Fréchet* manifolds. The ring of integers will be denoted by $\mathbf{Z}$. Real and complex numbers will be denoted by $\mathbf{R}$ and $\mathbf{C}$ respectively. A general field of characteristic zero will be denoted by $\mathbf{K}$.

2. Obstruction spaces for quantization on manifolds with boundary

2.1. Classical BV theories

We start with the BV approach for the bulk theory. A BV manifold is a triple

$$(\mathcal{F}, \mathcal{S}, \omega)$$

such that $\mathcal{F}$ is a $\mathbf{Z}$-graded supermanifold, $\mathcal{S} \in \mathcal{O}(\mathcal{F})$ is an even function of degree 0, and $\omega \in \Omega^2(\mathcal{F})$ an odd symplectic form of degree -1. The $\mathbf{Z}$-grading corresponds to the *ghost number* which we will denote by "gh". The BV space of fields $\mathcal{F}$ is usually given as the (-1)-shifted cotangent bundle of the BRST space of fields, i.e. $\mathcal{F}_{\mathrm{BV}} := T^*[-1]\mathcal{F}_{\mathrm{BRST}}$. In many cases, $\mathcal{F}$ is an infinite-dimensional Fréchet manifold. Denote by Q the Hamiltonian vector field of $\mathcal{S}$ of degree $+1$, i.e. $\iota_Q\omega = \delta\mathcal{S}$, where δ denotes the de Rham differential on $\mathcal{F}$. If we denote by $(\ ,\ )$ the odd Poisson bracket induced by the odd symplectic form ω (also called the *anti bracket*, or *BV bracket*), we get

$$Q = (\mathcal{S},\ ).$$

Note that, by definition, Q is *cohomological*, i.e. $Q^2 = 0$. Moreover, Q is a *symplectic* vector field, i.e. $L_Q\omega = 0$, where L denotes the Lie derivative. For a BV theory we require the *classical master equation (CME)*

$$Q(\mathcal{S}) = (\mathcal{S}, \mathcal{S}) = 0 \tag{2.1}$$

to hold. The assignment $\Sigma \mapsto (\mathcal{F}_\Sigma, \mathcal{S}_\Sigma, \omega_\Sigma)$ of a (usually, closed compact oriented) manifold Σ to a BV manifold is called a *BV theory*. By the physical property of *locality*, given a BV theory, we usually want to work over *local functions* on $\mathcal{F}_\Sigma$, which we denote by $\mathcal{O}_{loc}(\mathcal{F}_\Sigma) \subset \mathcal{O}(\mathcal{F}_\Sigma)$. These are defined by functions on $\mathcal{F}_\Sigma$ of the form

$$\Phi \mapsto \int_{x\in\Sigma} \mathscr{L}\big(x, \Phi(x), \partial\Phi(x), \partial^2\Phi(x), \dots, \partial^N\Phi(x)\big),$$

where $\Phi \in \mathcal{F}_\Sigma$ denotes some field configuration and $\mathscr{L}$ denotes the *Lagrangian density* of the given theory which depends on Φ and higher derivatives for $N \in \mathbf{Z}_{>0}$.

2.2. Examples of classical BV theories

We want to give some examples of non-reduced classical BV theories. For the reduced case see [23].

2.2.1. Electrodynamics

We want to consider the (minimal) BV extension of classical Euclidean electrodynamics for a trivial $U(1)$-bundle. Let Σ be a smooth oriented n-dimensional Riemannian manifold. Denote by $*\colon \Omega^j(\Sigma) \to \Omega^{n-j}(\Sigma)$ the *Hodge star* induced by the metric on Σ. The BV space of fields is then given by the shifted cotangent bundle $T^*[-1]E_\Sigma$, where

$$E_\Sigma := \Omega^1(\Sigma) \oplus \Omega^{n-2}(\Sigma) \oplus \Omega^0(\Sigma)[1].$$

The first term of E_Σ denotes the space of connections A of a trivial $U(1)$-bundle over Σ, the second term denotes the space of the Hamiltonian counterpart of those connections (i.e. the momentum), which we denote by B, and the third term denotes the space of ghost fields c. Hence, we have

$$\mathcal{F}_\Sigma := T^*[-1]E_\Sigma = \Omega^1(\Sigma) \oplus \Omega^{n-2}(\Sigma) \oplus \Omega^0(\Sigma)[1] \oplus \Omega^{n-1}(\Sigma)[-1] \oplus \Omega^2(\Sigma)[-1] \oplus \Omega^n(\Sigma)[-2]$$

We denote a field in $\mathcal{F}_\Sigma$ by (A, B, c, A^+, B^+, c^+). Then the BV symplectic form is given by

$$\omega_\Sigma = \int_\Sigma \left(\delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+\right).$$

The BV action is given by

$$\mathcal{S}_\Sigma = \int_\Sigma \left(B \wedge F_A + \frac{1}{2} B \wedge *B + A^+ \wedge \mathrm{d}c \right),$$

where $F_A := \mathrm{d}A$, denotes the curvature of the connection A. The cohomological vector field is given by

$$Q_\Sigma = \int_\Sigma \left(\mathrm{d}c \wedge \frac{\delta}{\delta A} + \mathrm{d}B \wedge \frac{\delta}{\delta A^+} + (*B + \mathrm{d}A) \wedge \frac{\delta}{\delta B^+} + \mathrm{d}A^+ \wedge \frac{\delta}{\delta c^+} \right).$$

In particular, we have the following symmetries:

$$\begin{aligned} Q_\Sigma(A) &= \mathrm{d}c, \\ Q_\Sigma(A^+) &= \mathrm{d}B, \\ Q_\Sigma(B^+) &= *B + \mathrm{d}A, \\ Q_\Sigma(c^+) &= \mathrm{d}A^+, \end{aligned}$$

and $Q_\Sigma(B) = Q_\Sigma(c) = 0$. It is easy to show that $Q^2 = 0$ and that the CME is indeed satisfied, i.e. $Q_\Sigma(\mathcal{S}_\Sigma) = 0$.

2.2.2. Yang–Mills theory

Consider an n-dimensional closed oriented compact smooth Riemannian manifold Σ. Let $\mathfrak{g}$ be the Lie algebra of a finite-dimensional simply connected Lie group G endowed with a $\mathfrak{g}$-invariant inner product given by $\langle g, h \rangle := \mathrm{Tr}(gh)$. Moreover, let P be principal G-bundle over Σ and assume for simplicity that P is trivial. The BV space of fields is given by

$$\mathcal{F}_\Sigma := \Omega^1(\Sigma) \otimes \mathfrak{g} \oplus \Omega^{n-2}(\Sigma) \otimes \mathfrak{g} \oplus \Omega^0(\Sigma)\mathfrak{g}[1] \oplus \Omega^{n-1}(\Sigma) \otimes \mathfrak{g}[-1] \oplus \Omega^2(\Sigma) \otimes \mathfrak{g}[-1] \oplus \Omega^n(\Sigma) \otimes \mathfrak{g}[-2]$$

We denote a field in $\mathcal{F}_\Sigma$ by components (A, B, c, A^+, B^+, c^+). The BV symplectic form is given by

$$\omega_\Sigma = \int_\Sigma \mathrm{Tr}\big(\delta A \wedge \delta A^+ + \delta B \wedge \delta B^+ + \delta c \wedge \delta c^+\big)$$

and the BV action by

$$\mathcal{S}_\Sigma = \int_\Sigma \mathrm{Tr} \left(B \wedge F_A + \frac{1}{2} B \wedge *B + A^+ \wedge \mathrm{d}_A c + B^+ \wedge [B, c] + \frac{1}{2} c^+ \wedge [c, c] \right),$$

where $F_A := \mathrm{d}A + \frac{1}{2}[A, A]$ denotes the curvature of the connection A and d_A is the covariant derivative for A. The cohomological vector field is given by

$$\begin{aligned} Q_\Sigma = \int_\Sigma \Bigg(& \mathrm{d}_A c \wedge \frac{\delta}{\delta A} + [B, c] \wedge \frac{\delta}{\delta B} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} + (\mathrm{d}_A B + [A^+, c]) \wedge \frac{\delta}{\delta A^+} + (F_A + *B + [B^+, c]) \wedge \frac{\delta}{\delta B^+} + \\ & + (\mathrm{d}_A A^+ + [B, B^+] + [c, c^+]) \wedge \frac{\delta}{\delta c^+} \Bigg). \end{aligned}$$

In particular, we have the following symmetries:

$$\begin{aligned} Q_\Sigma(A) &= \mathrm{d}_A c, \\ Q_\Sigma(B) &= [B, c], \\ Q_\Sigma(c) &= \frac{1}{2}[c, c], \\ Q_\Sigma(A^+) &= \mathrm{d}_A B + [A^+, c], \\ Q_\Sigma(B^+) &= F_A + *B + [B^+, c], \\ Q_\Sigma(c^+) &= \mathrm{d}_A A^+ + [B, B^+] + [c, c^+]. \end{aligned}$$

2.2.3. Chern–Simons theory

Let Σ be a 3-dimensional closed compact oriented smooth manifold and let $\mathfrak{g}$ be the Lie algebra of a Lie group G endowed with an invariant inner product (e.g. a simple Lie algebra). Denote by $\mathrm{Tr}(gh)$ the Killing form for two elements $g, h \in \mathfrak{g}$. The space of fields is given by graded connections on a principal G-bundle. For simplicity, we assume that the bundle is trivial. Then the BV space of fields is given by

$$\mathcal{F}_\Sigma := \Omega^\bullet(\Sigma) \otimes \mathfrak{g}[1] = \bigoplus_{j=0}^{3} \Omega^j(\Sigma) \otimes \mathfrak{g}[1].$$

A field in $\mathcal{F}_\Sigma$ will be denoted by the tuple (c, A, A^+, c^+). Note that the ghost numbers are $1, 0, -1, -2$ respectively. Consider the *superfield* $\mathbf{A} := c + A + A^+ + c^+$. Then the BV symplectic form is given by

$$\omega_\Sigma = \frac{1}{2}\int\limits_\Sigma \mathrm{Tr}(\delta\mathbf{A} \wedge \delta\mathbf{A}) = \int\limits_\Sigma \mathrm{Tr}(\delta c \wedge \delta c^+ + \delta A \wedge \delta A^+).$$

The cohomological vector field is given by

$$Q_\Sigma = \int\limits_\Sigma \mathrm{Tr}\left(\left(\mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}]\right) \wedge \frac{\delta}{\delta\mathbf{A}}\right) = \int\limits_\Sigma \mathrm{Tr}\left(\mathrm{d}_A c \wedge \frac{\delta}{\delta A} + (F_A + [c, A^+]) \wedge \frac{\delta}{\delta A^+} + (\mathrm{d}_A A^+ + [c, c^+]) \wedge \frac{\delta}{\delta c^+} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c}\right).$$

In particular, we have the following symmetries:

$$\begin{aligned} Q_\Sigma(A) &= \mathrm{d}_A c, \\ Q_\Sigma(c) &= \frac{1}{2}[c, c], \\ Q_\Sigma(A^+) &= F_A + [c, A^+], \\ Q_\Sigma(c^+) &= \mathrm{d}_A A^+ + [c, c^+]. \end{aligned}$$

The BV action is given by

$$\mathcal{S}_\Sigma = \int\limits_\Sigma \mathrm{Tr}\left(\frac{1}{2}\mathbf{A} \wedge \mathrm{d}\mathbf{A} + \frac{1}{6}\mathbf{A} \wedge [\mathbf{A}, \mathbf{A}]\right) = \int\limits_\Sigma \mathrm{Tr}\left(\frac{1}{2}A \wedge \mathrm{d}A + \frac{1}{6}A \wedge [A, A] + \frac{1}{2}A^+ \wedge \mathrm{d}_A c + \frac{1}{2}c \wedge \mathrm{d}_A A^+ + \frac{1}{2}c^+ \wedge [c, c]\right)$$

2.2.4. (Abelian) BF theory

Let us first consider abelian BF theory. Let Σ be an n-dimensional closed compact oriented smooth manifold. The space of fields is given by

$$\mathcal{F}_\Sigma := \Omega^\bullet(\Sigma)[1] \oplus \Omega^\bullet(\Sigma)[n-2].$$

We denote the superfields by $\mathbf{A} \in \Omega^\bullet(\Sigma)[1]$ and $\mathbf{B} \in \Omega^\bullet(\Sigma)[n-2]$. The BV symplectic form is then given by

$$\omega_\Sigma = \int\limits_\Sigma \delta\mathbf{A} \wedge \delta\mathbf{B}.$$

The BV action is given by

$$\mathcal{S}_\Sigma = \int\limits_\Sigma \mathbf{B} \wedge \mathrm{d}\mathbf{A}.$$

The cohomological vector field is given by

$$Q_\Sigma = \int\limits_\Sigma \left(\mathrm{d}\mathbf{A} \wedge \frac{\delta}{\delta\mathbf{A}} + \mathrm{d}\mathbf{B} \wedge \frac{\delta}{\delta\mathbf{B}}\right).$$

Note that $Q_\Sigma(\mathbf{A}) = \mathrm{d}\mathbf{A}$ and $Q_\Sigma(\mathbf{B}) = \mathrm{d}\mathbf{B}$. Now let us consider the case of non-abelian BF theory, i.e. we consider a finite-dimensional Lie algebra $\mathfrak{g}$ with invariant inner product. The BV space of fields is then given by

$$\mathcal{F}_\Sigma := \Omega^\bullet(\Sigma) \otimes \mathfrak{g}[1] \oplus \Omega^\bullet(\Sigma) \otimes \mathfrak{g}[n-2] \ni (\mathbf{A}, \mathbf{B}).$$

The BV symplectic form is given by

$$\omega_\Sigma = \int\limits_\Sigma \mathrm{Tr}(\delta\mathbf{B} \wedge \delta\mathbf{A}).$$

The cohomological vector field is given by

$$Q_\Sigma = \int\limits_\Sigma \mathrm{Tr}\left(\left(\mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}]\right) \wedge \frac{\delta}{\delta\mathbf{A}} + \mathrm{d}_\mathbf{A}\mathbf{B} \wedge \frac{\delta}{\delta\mathbf{B}}\right).$$

In particular, we have the following symmetries:

$$\begin{aligned} Q_\Sigma(\mathbf{A}) &= \mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}], \\ Q_\Sigma(\mathbf{B}) &= \mathrm{d}_\mathbf{A}\mathbf{B}. \end{aligned}$$

The BV action is given by

$$\mathcal{S}_\Sigma = \int_\Sigma \mathrm{Tr}\left(\mathbf{B} \wedge \left(\mathrm{d}\mathbf{A} + \frac{1}{2}[\mathbf{A},\mathbf{A}]\right)\right).$$

Remark 2.1. Note that non-abelian BF theory reduces to the abelian one when $\mathfrak{g} = \mathbf{R}$. In fact, abelian BF theory is given by two copies of abelian Chern–Simons theory (i.e. the theory described in 2.2.3 when $\mathfrak{g} = \mathbf{R}$). Moreover, (abelian) BF theory and Chern–Simons theory are examples of a more general type of theory, called *AKSZ theory* [1], which forms a subclass for BV theories. Other examples of AKSZ theories include the *Poisson sigma model* [21,22,39,63], *Witten's A- and B-model* [1,77], *Rozansky–Witten theory* [56,59], *Donaldson–Witten theory* [40,78], the *Courant sigma model* [28,58], and *2D Yang–Mills theory* [24,41],

2.3. Obstruction space in the bulk

It is well known that the obstruction space for quantization in the BV formalism is given by the first cohomology group with respect to Q. See e.g. [9] and references therein.

Theorem 2.2. *The obstruction space for a BV theory to be quantizable is given by*

$$\mathrm{H}^1_Q(\mathcal{O}_{loc}(\mathcal{F})). \tag{2.2}$$

Proof. Consider a deformation of the BV action $\mathcal{S}$, denoted by $\mathcal{S}_\hbar$, depending on $\hbar$ and consider its expansion as a formal power series

$$\mathcal{S}_\hbar := \mathcal{S}_0 + \hbar\mathcal{S}_1 + \hbar^2\mathcal{S}_2 + O(\hbar^3) = \sum_{k\geq 0}\hbar^k\mathcal{S}_k \in \mathcal{O}_{loc}(\mathcal{F})[\![\hbar]\!], \tag{2.3}$$

where each $\mathcal{S}_k \in \mathcal{O}_{loc}(\mathcal{F})$ for all $k \geq 0$ and $\lim_{\hbar\to 0}\mathcal{S}_\hbar = \mathcal{S}$, i.e. $\mathcal{S}_0 := \mathcal{S}$. Note that $\mathrm{gh}\,\mathcal{S}_k = 0$ for all $k \geq 0$ since $\mathrm{gh}\,\mathcal{S} = 0$. For the quantum BV picture (see e.g. [57,64]) one should note that there is a canonical second order differential operator Δ on $\mathcal{O}_{loc}(\mathcal{F})$ such that $\Delta^2 = 0$. It is called BV Laplacian (see [42,76] for a mathematical exposure). In particular, if Φ^i and Φ^+_i denote field and anti-field respectively, one can define Δ as

$$\Delta f = \sum_i (-1)^{\mathrm{gh}\,\Phi^i+1} f\left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}, \frac{\overleftarrow{\delta}}{\delta\Phi^+_i}\right\rangle, \quad f \in \mathcal{O}_{loc}(\mathcal{F}).$$

We have denoted by $\frac{\overleftarrow{\delta}}{\delta\Phi^i}$ and $\frac{\overrightarrow{\delta}}{\delta\Phi^i}$ the left and right derivatives with respect to Φ^i. An analogous version also holds for the anti-fields Φ^+_i. In fact, we have

$$\frac{\overrightarrow{\delta}}{\delta\Phi^i} f = (-1)^{\mathrm{gh}\,\Phi^i(\mathrm{gh}\,f+1)} f\frac{\overleftarrow{\delta}}{\delta\Phi^i}, \tag{2.4}$$

$$\frac{\overrightarrow{\delta}}{\delta\Phi^+_i} f = (-1)^{(\mathrm{gh}\,\Phi^i+1)(\mathrm{gh}\,f+1)} f\frac{\overleftarrow{\delta}}{\delta\Phi^+_i}. \tag{2.5}$$

Remark 2.3. To be precise, for our constructions we want to consider a (global) BV Laplacian on *half-densities* on $\mathcal{F}$. It can be shown that for any odd symplectic supermanifold $\mathcal{F}$ there exists a supermanifold $\mathcal{M}$ such that $\mathcal{F} \cong T^*[1]\mathcal{M}$. Then $\mathcal{O}(\mathcal{F}) \cong \mathcal{O}(T^*[1]\mathcal{M}) = \Gamma(\bigwedge^\bullet T\mathcal{M})$. The *Berezinian bundle* on $\mathcal{F}$ is given by

$$\mathrm{Ber}(\mathcal{F}) \cong \bigwedge^{\mathrm{top}} T^*\mathcal{M} \otimes \bigwedge^{\mathrm{top}} T^*\mathcal{M}.$$

The half-densities on $\mathcal{F}$ are defined by

$$\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}) := \Gamma\left(\mathrm{Ber}(\mathcal{F})^{\frac{1}{2}}\right).$$

One can then show that there is a canonical operator $\Delta^{\frac{1}{2}}_{\mathcal{F}}$ on $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F})$ such that it squares to zero [42]. At this point one should mention that this is only canonical in the finite-dimensional setting. For the infinite-dimensional case this only holds after a suitable *renormalization*. We can define a Laplacian by

$$\Delta_\sigma f := \frac{1}{\sigma}\Delta^{\frac{1}{2}}_{\mathcal{F}}(f\sigma), \quad f \in \mathcal{O}(\mathcal{F}),$$

where σ is a non-vanishing reference half-density on $\mathcal{F}$ which is $\Delta^{\frac{1}{2}}_{\mathcal{F}}$-closed. Note that $(\Delta_\sigma)^2 = 0$. We usually just write $\Delta \equiv \Delta_\sigma$ without mentioning σ.

To observe gauge-independence in the BV formalism, one requires the *quantum master equation (QME)*

$$\Delta \exp(\mathcal{S}_\hbar/\hbar) = 0 \Longleftrightarrow (\mathcal{S}_\hbar, \mathcal{S}_\hbar) + 2\hbar\Delta\mathcal{S}_\hbar = 0 \tag{2.6}$$

to hold. Here we denote by Δ the BV Laplacian. Solving (2.6) for each order in $\hbar$, we get the system of equations

$$(\mathcal{S}_0, \mathcal{S}_0) = 0, \tag{2.7}$$

$$\Delta\mathcal{S}_0 = (\mathcal{S}_0, \mathcal{S}_1), \tag{2.8}$$

$$\Delta\mathcal{S}_1 = (\mathcal{S}_0, \mathcal{S}_2) + \frac{1}{2}(\mathcal{S}_1, \mathcal{S}_1) \tag{2.9}$$

$$\vdots \tag{2.10}$$

Note that Equation (2.7) is the CME which we assume to hold. Then, using the CME and the formula

$$\Delta(f, g) = (f, \Delta g) - (-1)^{\mathrm{gh}\, g}(\Delta f, g), \quad \forall f, g \in \mathcal{O}_{loc}(\mathcal{F}),$$

we get

$$0 = \Delta(\mathcal{S}_0, \mathcal{S}_0) = (\mathcal{S}_0, \Delta\mathcal{S}_0) = Q(\Delta\mathcal{S}_0).$$

Hence $\Delta\mathcal{S}_0$ is closed with respect to the coboundary operator $Q = (\mathcal{S}_0, \ )$. Moreover, if we assume that it is also Q-exact, we get that there is some $\mathcal{S}_1 \in \mathcal{O}_{loc}(\mathcal{F})$ such that $\Delta\mathcal{S}_0 = Q(\mathcal{S}_1) = (\mathcal{S}_0, \mathcal{S}_1)$, which is exactly the statement of Equation (2.8). This will automatically imply that all the higher order equations hold. Indeed, if $\Delta\mathcal{S}_1 = (\mathcal{S}_0, \mathcal{S}_1)$ for some $\mathcal{S}_1 \in \mathcal{O}_{loc}(\mathcal{F})$, we get

$$0 = \Delta\underbrace{(\mathcal{S}_0, \mathcal{S}_1)}_{\Delta\mathcal{S}_0} = (\Delta\mathcal{S}_0, \mathcal{S}_1) - (-1)^{\mathrm{gh}\,\mathcal{S}_0}(\mathcal{S}_0, \Delta\mathcal{S}_1) = ((\mathcal{S}_0, \mathcal{S}_1), \mathcal{S}_1) - (\mathcal{S}_0, \Delta\mathcal{S}_1), \tag{2.11}$$

where we used $\Delta^2 = 0$. Using the graded Jacobi formula for the BV bracket, we get

$$((\mathcal{S}_0, \mathcal{S}_1), \mathcal{S}_1) = (\mathcal{S}_0, (\mathcal{S}_1, \mathcal{S}_1)) - (-1)^{(\mathrm{gh}\,\mathcal{S}_0 - 1)(\mathrm{gh}\,\mathcal{S}_1 - 1)}(\mathcal{S}_1, (\mathcal{S}_0, \mathcal{S}_1)). \tag{2.12}$$

Furthermore, by graded commutativity of the BV bracket we have

$$((\mathcal{S}_0, \mathcal{S}_1), \mathcal{S}_1) = -(-1)^{(\mathrm{gh}(\mathcal{S}_0, \mathcal{S}_1) - 1)(\mathrm{gh}\,\mathcal{S}_1 - 1)}(\mathcal{S}_1, (\mathcal{S}_0, \mathcal{S}_1)). \tag{2.13}$$

Now since

$$\mathrm{gh}(\mathcal{S}_0, \mathcal{S}_1) = \mathrm{gh}\,\mathcal{S}_0 + \mathrm{gh}\,\mathcal{S}_1 + \mathrm{gh}(\ ,\ )$$

we get

$$2((\mathcal{S}_0, \mathcal{S}_1), \mathcal{S}_1) = (\mathcal{S}_0, (\mathcal{S}_1, \mathcal{S}_1)).$$

Hence, using Equation (2.11), we get

$$(\mathcal{S}_0, \Delta\mathcal{S}_1) = \left(\mathcal{S}_0, \frac{1}{2}(\mathcal{S}_1, \mathcal{S}_1)\right).$$

This will give us

$$\Delta\mathcal{S}_1 = \frac{1}{2}(\mathcal{S}_1, \mathcal{S}_1) + Q\text{-exact term},$$

so we can find some $\mathcal{S}_2 \in \mathcal{O}_{loc}(\mathcal{F})$ such that the Q-exact term is given by $Q(\mathcal{S}_2) = (\mathcal{S}_0, \mathcal{S}_2)$. This implies that Equation (2.9) holds. The higher order equations hold in a similar iterative computation. □

2.4. Classical BV-BFV theories

Let us describe the BFV approach for the space of boundary fields. A BFV manifold is a triple

$$\left(\mathcal{F}^\partial, \omega^\partial, Q^\partial\right),$$

where $\mathcal{F}^\partial$ is a **Z**-graded supermanifold, $\omega^\partial \in \Omega^2(\mathcal{F}^\partial)$ an even symplectic form of ghost number 0 and Q^∂ cohomological and symplectic vector field of degree $+1$ with odd Hamiltonian function $\mathcal{S}^\partial \in \mathcal{O}_{loc}(\mathcal{F}^\partial)$ of ghost number $+1$, i.e. $\iota_{Q^\partial}\omega^\partial = \delta\mathcal{S}^\partial$, where δ denotes the de Rham differential on $\mathcal{F}^\partial$. Moreover, we want

$$Q^\partial(\mathcal{S}^\partial) = \{\mathcal{S}^\partial, \mathcal{S}^\partial\} = 0.$$

We say that a BFV manifold is *exact*, if there exists a primitive 1-form α^∂, such that $\omega^\partial = \delta\alpha^\partial$. A BV-BFV manifold over an exact BFV manifold $(\mathcal{F}^\partial, \omega^\partial = \delta\alpha^\partial, Q^\partial)$ is a quintuple

$$(\mathcal{F}, \omega, \mathcal{S}, Q, \pi),$$

where $\pi : \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion such that

- $\delta\pi Q = Q^\partial$,
- $\iota_Q \omega = \delta\mathcal{S} + \pi^*\alpha^\partial$.

A consequence of this definition is

$$Q(\mathcal{S}) = \pi^*\left(2\mathcal{S}^\partial - \iota_{Q^\partial}\alpha^\partial\right) \tag{2.14}$$

which is called the *modified classical master equation (mCME)*. The assignment $\Sigma \mapsto \left(\mathcal{F}_\Sigma, \mathcal{S}_\Sigma, Q_\Sigma, \pi_\Sigma \colon \mathcal{F}_\Sigma \to \mathcal{F}^\partial_{\partial\Sigma}\right)$ of a manifold Σ with boundary $\partial\Sigma$ to a BV-BFV manifold is called a *BV-BFV theory*.

2.5. Examples of classical BV-BFV theories

2.5.1. Electrodynamics

Let everything be as in 2.2.1 with the difference that Σ now has non-vanishing boundary $\partial\Sigma$. The boundary BFV space of fields is then given by

$$\mathcal{F}^\partial_{\partial\Sigma} := \Omega^1(\partial\Sigma) \oplus \Omega^{n-2}(\partial\Sigma) \oplus \Omega^0(\partial\Sigma)[1] \oplus \Omega^{n-1}(\partial\Sigma)[-1].$$

A field in $\mathcal{F}_{\partial\Sigma}$ will be denoted by (A, B, c, A^+). If we denote by $i\colon \partial\Sigma \hookrightarrow \Sigma$ the inclusion of the boundary, the surjective submersion $\pi : \mathcal{F}_\Sigma \to \mathcal{F}_{\partial\Sigma}$ acts on the component fields as

$$\begin{aligned}
\pi_\Sigma(A) &= i^*(A) := \mathbb{A},\\
\pi_\Sigma(B) &= i^*(B) := \mathbb{B},\\
\pi_\Sigma(c) &= i^*(c) := \mathbb{c},\\
\pi_\Sigma(A^+) &= i^*(A^+) := \mathbb{A}^+,
\end{aligned}$$

and $\mathbb{B}^+ := \pi_\Sigma(B^+) = 0 = \pi_\Sigma(c^+) =: \mathbb{c}^+$. The BFV symplectic form is given by

$$\omega^\partial_{\partial\Sigma} = \delta\alpha^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \left(\delta\mathbb{B}\wedge\delta\mathbb{A} + \delta\mathbb{A}^+ \wedge \delta\mathbb{c}\right),$$

where

$$\alpha^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \left(\mathbb{B}\wedge\delta\mathbb{A} + \mathbb{A}^+ \wedge \delta\mathbb{c}\right).$$

The BFV charge $Q^\partial_{\partial\Sigma} = \delta\pi_\Sigma Q_\Sigma$ is given by

$$Q^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \left(\mathrm{d}\mathbb{B}\wedge\frac{\delta}{\delta\mathbb{A}^+} + \mathrm{d}\mathbb{c}\wedge\frac{\delta}{\delta\mathbb{A}}\right).$$

It is easy to check that the boundary action is given by

$$\mathcal{S}^\partial_{\partial\Sigma} = \int_{\partial\Sigma} \mathbb{c}\wedge\mathrm{d}\mathbb{B}.$$

Then the mCME is indeed satisfied, since we have

$$\iota_{Q_\Sigma}\omega_\Sigma = \int_\Sigma \left(\mathrm{d}c\wedge\delta A^+ + \delta A\wedge \mathrm{d}B + \delta B\wedge(*B + \mathrm{d}A) + \mathrm{d}A^+\wedge\delta c\right)$$

and

$$\delta\mathcal{S}_\Sigma = \int_\Sigma \left(\delta B\wedge\mathrm{d}A + B\wedge\mathrm{d}\delta A + \delta B\wedge *B + \delta A^+\wedge\mathrm{d}c + A^+\wedge\mathrm{d}\delta c\right).$$

Putting everything together and using Stokes' theorem, we get the claim.

2.5.2. Yang–Mills theory

Let everything be as in 2.2.2 with the difference that Σ has non-vanishing boundary $\partial\Sigma$. Let us denote the pullback of the forms A, B, A^+, c with respect to the inclusion $i\colon \partial\Sigma \hookrightarrow \Sigma$ by $\mathbb{A}, \mathbb{B}, \mathbb{A}^+, \mathbb{c}$ respectively. Note that here we have $\pi_\Sigma := i^*\colon \mathcal{F}_\Sigma \to \mathcal{F}^\partial_{\partial\Sigma}$. The BFV space of fields is then given by

$$\mathcal{F}^\partial_{\partial\Sigma} = \Omega^1(\partial\Sigma)\otimes\mathfrak{g}[1] \oplus \Omega^{n-2}(\partial\Sigma)\otimes\mathfrak{g}[n-2] \oplus \Omega^0(\partial\Sigma)\otimes\mathfrak{g} \oplus \Omega^{n-1}(\partial\Sigma)\otimes\mathfrak{g}[n-2].$$

The BFV symplectic form is given by

$$\omega^{\partial}_{\partial\Sigma}=\delta\alpha^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\delta\mathbb{B}\wedge\delta\mathbb{A}+\delta\mathbb{A}^+\wedge\delta\mathbb{c}\right),$$

where

$$\alpha^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\mathbb{B}\wedge\delta\mathbb{A}+\mathbb{A}^+\wedge\delta\mathbb{c}\right).$$

The cohomological vector field is given by

$$Q^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\mathrm{d}_{\mathbb{A}}\mathbb{c}\wedge\frac{\delta}{\delta\mathbb{A}}+[\mathbb{B},\mathbb{c}]\wedge\frac{\delta}{\delta\mathbb{B}}+(\mathrm{d}_{\mathbb{A}}\mathbb{B}+[\mathbb{A}^+,\mathbb{c}])\wedge\frac{\delta}{\delta\mathbb{A}^+}+\frac{1}{2}[\mathbb{c},\mathbb{c}]\wedge\frac{\delta}{\delta\mathbb{c}}\right)$$

and the BFV action by

$$\mathcal{S}^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\mathbb{B}\wedge\mathrm{d}_{\mathbb{A}}\mathbb{c}+\frac{1}{2}\mathbb{A}^+\wedge[\mathbb{c},\mathbb{c}]\right).$$

2.5.3. Chern–Simons theory

Let everything be as in 2.2.3 with the difference that Σ has non-vanishing boundary $\partial\Sigma$. Let us denote the pullback of the forms A, A^+, c with respect to the inclusion $i\colon\partial\Sigma\hookrightarrow\Sigma$ by $\mathbb{A},\mathbb{A}^+,\mathbb{c}$ respectively. Note that here we have $\pi_\Sigma:=i^*\colon\mathcal{F}_\Sigma\to\mathcal{F}^{\partial}_{\partial\Sigma}$. We will denote the superfield on the boundary by $\mathfrak{A}:=\mathbb{c}+\mathbb{A}+\mathbb{A}^+$. The BFV space of boundary fields is then given by

$$\mathcal{F}^{\partial}_{\partial\Sigma}:=\Omega^\bullet(\partial\Sigma)\otimes\mathfrak{g}[1]=\bigoplus_{j=0}^{2}\Omega^j(\partial\Sigma)\otimes\mathfrak{g}[1]\ni\mathfrak{A}.$$

The BFV symplectic form is then given by

$$\omega^{\partial}_{\partial\Sigma}=\delta\alpha^{\partial}_{\partial\Sigma}=\frac{1}{2}\int_{\partial\Sigma}\mathrm{Tr}(\delta\mathfrak{A}\wedge\delta\mathfrak{A})=\int_{\partial\Sigma}\mathrm{Tr}\left(\frac{1}{2}\delta\mathbb{A}\wedge\delta\mathbb{A}+\delta\mathbb{c}\wedge\delta\mathbb{A}^+\right),$$

where

$$\alpha^{\partial}_{\partial\Sigma}=\frac{1}{2}\int_{\partial\Sigma}\mathrm{Tr}(\mathfrak{A}\wedge\delta\mathfrak{A})=\frac{1}{2}\int_{\partial\Sigma}\mathrm{Tr}\left(\mathbb{A}\wedge\delta\mathbb{A}+\mathbb{c}\wedge\delta\mathbb{A}^++\mathbb{A}^+\wedge\delta\mathbb{c}\right).$$

The cohomological vector field is given by

$$Q^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\left(\frac{1}{2}\mathrm{d}\mathfrak{A}+\frac{1}{2}[\mathfrak{A},\mathfrak{A}]\right)\wedge\frac{\delta}{\delta\mathfrak{A}}\right)=\int_{\partial\Sigma}\mathrm{Tr}\left(\mathrm{d}_{\mathbb{A}}\mathbb{c}\wedge\frac{\delta}{\delta\mathbb{A}}+(F_{\mathbb{A}}+[\mathbb{c},\mathbb{A}^+])\wedge\frac{\delta}{\delta\mathbb{A}^+}+\frac{1}{2}[\mathbb{c},\mathbb{c}]\wedge\frac{\delta}{\delta\mathbb{c}}\right)$$

The BFV action is given by

$$\mathcal{S}^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathrm{Tr}\left(\frac{1}{2}\mathfrak{A}\wedge\mathrm{d}\mathfrak{A}+\frac{1}{6}\mathfrak{A}\wedge[\mathfrak{A},\mathfrak{A}]\right)=\int_{\partial\Sigma}\mathrm{Tr}\left(\mathbb{c}\wedge F_{\mathbb{A}}+\frac{1}{2}[\mathbb{c},\mathbb{c}]\wedge\mathbb{A}^+\right).$$

2.5.4. (Abelian) BF theory

Let everything be as in 2.2.4 with the difference that Σ has non-vanishing boundary $\partial\Sigma$. Let $i\colon\partial\Sigma\hookrightarrow\Sigma$ be the inclusion and denote the pullback of the superfields to the boundary by $\mathfrak{A}:=i^*(\mathbf{A})$ and $\mathfrak{B}:=i^*(\mathbf{B})$. Note that here we have $\pi_\Sigma:=i^*\colon\mathcal{F}_\Sigma\to\mathcal{F}^{\partial}_{\partial\Sigma}$. Let us first look at the boundary theory for abelian BF theory. The BFV space of fields is given by

$$\mathcal{F}^{\partial}_{\partial\Sigma}:=\Omega^\bullet(\partial\Sigma)[1]\oplus\Omega^\bullet(\partial\Sigma)[n-2]\ni(\mathfrak{A},\mathfrak{B}).$$

The BFV symplectic form is given by

$$\omega^{\partial}_{\partial\Sigma}=\delta\alpha^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\delta\mathfrak{A}\wedge\delta\mathfrak{B},$$

where

$$\alpha^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\mathfrak{A}\wedge\delta\mathfrak{B}.$$

The cohomological vector field is given by

$$Q^{\partial}_{\partial\Sigma}=\int_{\partial\Sigma}\left(\mathrm{d}\mathfrak{A}\wedge\frac{\delta}{\delta\mathfrak{A}}+\mathrm{d}\mathfrak{B}\wedge\frac{\delta}{\delta\mathfrak{B}}\right).$$

The BFV action is given by

$$\mathcal{S}^{\partial}_{\partial\Sigma} = \int\limits_{\partial\Sigma} \mathfrak{B} \wedge \mathrm{d}\mathfrak{A}.$$

For the non-abelian case we have the BFV space of fields

$$\mathcal{F}^{\partial}_{\partial\Sigma} := \Omega^{\bullet}(\partial\Sigma) \otimes \mathfrak{g}[1] \oplus \Omega^{\bullet}(\partial\Sigma) \otimes \mathfrak{g}[n-2] \ni (\mathfrak{A}, \mathfrak{B}).$$

The BFV symplectic form is given by

$$\omega^{\partial}_{\partial\Sigma} = \delta\alpha^{\partial}_{\partial\Sigma} = \int\limits_{\partial\Sigma} \mathrm{Tr}\big(\delta\mathfrak{A} \wedge \delta\mathfrak{B}\big),$$

where

$$\alpha^{\partial}_{\partial\Sigma} = \int\limits_{\partial\Sigma} \mathrm{Tr}\big(\mathfrak{A} \wedge \delta\mathfrak{B}\big).$$

The cohomological vector field is given by

$$Q^{\partial}_{\partial\Sigma} = \int\limits_{\partial\Sigma} \mathrm{Tr}\left(\left(\mathrm{d}\mathfrak{A} + \frac{1}{2}[\mathfrak{A},\mathfrak{A}]\right) \wedge \frac{\delta}{\delta\mathfrak{A}} + \mathrm{d}_{\mathfrak{A}}\mathfrak{B} \wedge \frac{\delta}{\delta\mathfrak{B}}\right).$$

The BFV action is given by

$$\mathcal{S}^{\partial}_{\partial\Sigma} = \int\limits_{\partial\Sigma} \mathrm{Tr}\left(\mathfrak{B} \wedge \left(\mathrm{d}\mathfrak{A} + \frac{1}{2}[\mathfrak{A},\mathfrak{A}]\right)\right).$$

2.6. Obstruction space on the boundary

Similarly as for BV theories one can ask about the quantization obstruction for a BV-BFV theory, i.e. for a codimension 1 theory. In fact, we get the following theorem.

Theorem 2.4. *Let $\big(\mathcal{F}, \omega, \mathcal{S}, Q, \pi\colon \mathcal{F} \to \mathcal{F}^{\partial}\big)$ be a BV-BFV manifold over an exact BFV manifold $\big(\mathcal{F}^{\partial}, \omega^{\partial} = \delta\alpha^{\partial}, Q^{\partial}\big)$. The obstruction space for quantization on the underlying boundary BFV manifold $\mathcal{F}^{\partial}$ is given by*

$$\mathrm{H}^2_{Q^{\partial}}(\mathcal{O}_{loc}(\mathcal{F}^{\partial})), \tag{2.15}$$

where

$$Q^{\partial} = \{\mathcal{S}^{\partial},\ \}$$

with $\{\ ,\ \}$ the Poisson bracket induced by the symplectic form ω^{∂}.

Proof. Consider a deformation of the BFV action $\mathcal{S}^{\partial}$, denoted by $\mathcal{S}^{\partial}_{\hbar}$, depending on $\hbar$ and consider its expansion as a formal power series

$$\mathcal{S}^{\partial}_{\hbar} := \mathcal{S}^{\partial}_0 + \hbar\mathcal{S}^{\partial}_1 + \hbar^2\mathcal{S}^{\partial}_2 + O(\hbar^3) = \sum_{k\geq 0} \hbar^k \mathcal{S}^{\partial}_k \in \mathcal{O}_{loc}(\mathcal{F}^{\partial})[\![\hbar]\!], \tag{2.16}$$

where $\mathcal{S}^{\partial}_k \in \mathcal{O}_{loc}(\mathcal{F}^{\partial})$ for all $k \geq 0$ such that $\mathcal{S}^{\partial}_0 := \mathcal{S}^{\partial}$. Note that $\mathrm{gh}\,\mathcal{S}^{\partial}_k = +1$ since $\mathrm{gh}\,\mathcal{S}^{\partial} = +1$ and the corresponding symplectic form ω^{∂} is even of ghost number 0. In the BV-BFV construction one assumes a symplectic splitting of the BV space of fields

$$\mathcal{F} = \mathcal{B} \times \mathcal{Y} \tag{2.17}$$

where the BV symplectic form ω is constant on $\mathcal{B}$. One should think of $\mathcal{B}$ as the boundary part and $\mathcal{Y}$ as the bulk part of the fields. In fact, the space $\mathcal{B}$ is constructed as the leaf space for a chosen polarization on the space of boundary fields $\mathcal{F}^{\partial}$ (i.e. a Lagrangian subbundle of $T\mathcal{F}^{\partial}$ closed under the Lie bracket) and $\mathcal{Y}$ is just a symplectic complement. Using this splitting, we can write the mCME

$$\delta_{\mathcal{Y}}\mathcal{S} = \iota_{Q_{\mathcal{Y}}}\omega, \tag{2.18}$$

$$\delta_{\mathcal{B}}\mathcal{S} = -\alpha^{\partial}, \tag{2.19}$$

where $Q_{\mathcal{Y}}$ denotes the part of the cohomological vector field Q on $\mathcal{Y}$, $\delta_{\mathcal{Y}}$ and $\delta_{\mathcal{B}}$ denote the corresponding parts of the de Rham differential δ on the BV space of fields $\mathcal{F}$ according to the splitting (2.17). Note that we have dropped the pullback π^*. These two equations together with (2.14) imply

$$\frac{1}{2}(\mathcal{S},\mathcal{S})_{\mathcal{Y}} = \frac{1}{2}\iota_{Q_{\mathcal{Y}}}\iota_{Q_{\mathcal{Y}}}\omega = \mathcal{S}^{\partial}. \tag{2.20}$$

Choose Darboux coordinates (b^i, p_i) on $\mathcal{F}^\partial$ such that b^i denotes the coordinates on the base $\mathcal{B}$ and p_i on the leaves. In the case of an infinite-dimensional Banach manifold, locally one has Darboux's theorem by using Moser's trick, whenever the tangent spaces are split, i.e. there are two Lagrangian subspaces $\mathcal{L}_1, \mathcal{L}_2 \subset T_\beta\mathcal{F}^\partial$ such that $T_\beta\mathcal{F}^\partial = \mathcal{L}_1 \oplus \mathcal{L}_2$ for some $\beta \in \mathcal{F}^\partial$. This is in general not true for Fréchet manifolds, even if the tangent spaces are split. Note that such a splitting is guaranteed if each fiber is a Hilbert space (see [20] for similar discussions). However, for the quantization we want to perturb around each critical point, thus we only have to use the linear structure. Additionally, we have to assume that the tangent spaces are split and that there is Darboux's theorem if we work in the Fréchet setting (see [31] for discussions about Darboux's theorem on infinite-dimensional Fréchet manifolds). This allows us to write

$$\alpha^\partial = -\sum_i p_i \delta b^i.$$

Using Equation (2.19), we get

$$\frac{\overrightarrow{\delta}}{\delta b^i}\mathcal{S} = p_i, \quad \forall i.$$

Denote by $\Delta_\mathcal{Y}$ the BV Laplacian restricted to $\mathcal{Y}$. We will assume that $\Delta_\mathcal{Y}\mathcal{S} = 0$. For the closed case this means that we assume that $\mathcal{S}$ solves both, the CME and the QME. For the case with boundary, the BV Laplacian anyway only makes sense on $\mathcal{Y}$, so $\Delta = \Delta_\mathcal{Y}$. Next, we can obtain

$$\Delta_\mathcal{Y} \exp(\mathrm{i}\mathcal{S}/\hbar) = \left(\frac{\mathrm{i}}{\hbar}\right)^2 \frac{1}{2}(\mathcal{S}, \mathcal{S})_\mathcal{Y} \exp(\mathrm{i}\mathcal{S}/\hbar)$$

and by Equation (2.20), we get

$$-\hbar^2\Delta_\mathcal{Y} \exp(\mathrm{i}\mathcal{S}/\hbar) = \mathcal{S}^\partial \exp(\mathrm{i}\mathcal{S}/\hbar)\,. \tag{2.21}$$

Now consider the standard quantization $\widehat{p}_i := -\mathrm{i}\hbar\frac{\overrightarrow{\delta}}{\delta b^i}$. If $\widehat{p}_i$ acts on a function $\mathcal{S}$ on $\mathcal{B}$ parametrized by $\mathcal{Y}$, we get

$$\widehat{p}_i\mathcal{S} = -\mathrm{i}\hbar p_i, \quad p_i \in \mathcal{Y}.$$

Finally, considering the ordered standard quantization of $\mathcal{S}^\partial$ given by

$$\widehat{\mathcal{S}^\partial} := \mathcal{S}^\partial\left(b^i, -\mathrm{i}\hbar\frac{\overrightarrow{\delta}}{\delta b^i}\right),$$

where all the derivatives are placed to the right, and using Equation (2.21), we get the *modified quantum master equation (mQME)* [24]

$$\left(\hbar^2\Delta_\mathcal{Y} + \widehat{\mathcal{S}^\partial}\right)\exp(\mathrm{i}\mathcal{S}/\hbar) = 0. \tag{2.22}$$

In order to get a well-defined cohomology theory, we require that

$$\left(\hbar^2\Delta + \widehat{\mathcal{S}^\partial}\right)^2 = 0.$$

Since $\Delta^2 = 0$ and obviously the commutator $\left[\Delta, \widehat{\mathcal{S}^\partial}\right]$ vanishes, we have to assume that $\left(\widehat{\mathcal{S}^\partial}\right)^2 = 0$. This clearly follows if

$$\mathcal{S}^\partial_\hbar \star \mathcal{S}^\partial_\hbar = 0, \tag{2.23}$$

where

$$\star\colon \mathcal{O}(\mathcal{F}^\partial)[\![\hbar]\!] \times \mathcal{O}(\mathcal{F}^\partial)[\![\hbar]\!] \to \mathcal{O}(\mathcal{F}^\partial)[\![\hbar]\!]$$

denotes the star product (deformation quantization) induced by the BFV form ω^∂ and the standard ordering as mentioned above. Actually, the construction with the star product does not require the notion of a BV-BFV manifold and thus can be also considered independently for the BFV case. Moreover, the deformed boundary action $\mathcal{S}^\partial_\hbar$ satisfying (2.23) might spoil the mQME (2.22). Note that we can endow the deformed algebra $\mathcal{O}_{loc}(\mathcal{F}^\partial)[\![\hbar]\!]$ with a dg structure by considering the differential given by

$$Q^\partial_\hbar := \mathcal{S}^\partial_\hbar \star -. \tag{2.24}$$

Then we have

$$Q^\partial_\hbar(\mathcal{S}^\partial_\hbar) = \mathcal{S}^\partial_\hbar \star \mathcal{S}^\partial_\hbar = \mathcal{S}^\partial_\hbar\mathcal{S}^\partial_\hbar + \sum_{k\geq 1}\hbar^k B_k(\mathcal{S}^\partial_\hbar, \mathcal{S}^\partial_\hbar) = \mathcal{S}^\partial_\hbar\mathcal{S}^\partial_\hbar + \hbar\{\mathcal{S}^\partial_\hbar, \mathcal{S}^\partial_\hbar\} + \hbar^2 B_2(\mathcal{S}^\partial_\hbar, \mathcal{S}^\partial_\hbar) + O(\hbar^3), \tag{2.25}$$

where B_k denotes some bidifferential operator for all $k \geq 1$ with $B_1 := \{\ ,\ \}$. Moreover, note that we have

$$\{\mathcal{S}^\partial_\hbar, \mathcal{S}^\partial_\hbar\} = \{\mathcal{S}^\partial_0, \mathcal{S}^\partial_0\} + \hbar\{\mathcal{S}^\partial_0, \mathcal{S}^\partial_1\} + \hbar\{\mathcal{S}^\partial_1, \mathcal{S}^\partial_0\} + \hbar^2\{\mathcal{S}^\partial_1, \mathcal{S}^\partial_1\} + \hbar^2\{\mathcal{S}^\partial_0, \mathcal{S}^\partial_2\} + O(\hbar^3) \tag{2.26}$$

Using (2.16) and (2.26), we get

$$\begin{aligned}\mathcal{S}^\partial_\hbar\star\mathcal{S}^\partial_\hbar&=(\mathcal{S}^\partial_0+\hbar\mathcal{S}^\partial_1+\hbar^2\mathcal{S}^\partial_2+O(\hbar^3))\times(\mathcal{S}^\partial_0+\hbar\mathcal{S}^\partial_1+\hbar^2\mathcal{S}^\partial_2+O(\hbar^3))+\hbar(\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_0\}+\hbar\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_1\}+\hbar\{\mathcal{S}^\partial_1,\mathcal{S}^\partial_0\}+\\&\quad+\hbar^2\{\mathcal{S}^\partial_1,\mathcal{S}^\partial_1\}+\hbar^2\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_2\}+O(\hbar^3))+\hbar^2B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0)+O(\hbar^3)\\&=\mathcal{S}^\partial_0\mathcal{S}^\partial_0+\hbar(\mathcal{S}^\partial_1\mathcal{S}^\partial_0+\mathcal{S}^\partial_0\mathcal{S}^\partial_1+\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_0\})+\hbar^2(\mathcal{S}^\partial_0\mathcal{S}^\partial_2+\mathcal{S}^\partial_2\mathcal{S}^\partial_0+\mathcal{S}^\partial_1\mathcal{S}^\partial_1+\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_1\}+B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0))+O(\hbar^3)\\&=\hbar\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_0\}+\hbar^2(\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_1\}+B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0))+O(\hbar^3),\end{aligned}\tag{2.27}$$

where we have used the graded commutativity relation

$$\{f,g\}=-(-1)^{(\mathrm{gh}\,f+1)(\mathrm{gh}\,g+1)}\{g,f\},$$

the fact that $\{\ ,\ \}$ is even of ghost number 0 and that each $\mathcal{S}^\partial_k$ is odd of ghost number $+1$ for all $k\geq 0$. Note that by the CME for $\mathcal{S}^\partial_0$ the first term in (2.27) vanishes. Moreover, by the associativity of the star product we get

$$\{\mathcal{S}^\partial_0,B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0)\}=Q^\partial(B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0))=0,$$

and thus $B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0)$ is closed under the coboundary operator $Q^\partial=\{\mathcal{S}^\partial,\ \}$. If we assume that $B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0)$ is also Q^∂-exact, there exists some $\mathcal{S}^\partial_1\in\mathcal{O}_{loc}(\mathcal{F}^\partial)$, such that

$$B_2(\mathcal{S}^\partial_0,\mathcal{S}^\partial_0)=-\{\mathcal{S}^\partial_0,\mathcal{S}^\partial_1\}=-Q^\partial(\mathcal{S}^\partial_1).$$

Thus the coefficients in degree $+2$ vanish and one can check that by the construction of the star product all the higher coefficients will also vanish using a similar iterative procedure as we have seen before. □

More generally, in the quantum BV-BFV construction [24] one can construct a geometric quantization [14,43,79] on the space of boundary fields $\mathcal{F}^\partial$ using the symplectic form ω^∂ and the chosen polarization. This will give a vector space $\mathcal{H}$ (actually a chain complex $\big(\mathcal{H},\widehat{\mathcal{S}^\partial}\big)$ associated to the source boundary. In fact, we can construct $\mathcal{H}$ as the space of half-densities $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{B})$ on $\mathcal{B}$. We call $\widehat{\mathcal{H}}:=\mathcal{H}\hat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V})$ the *space of states*. We have denoted by $\hat{\otimes}$ a certain completion of the tensor product in order to deal with the infinite-dimensional case. Moreover, we have denoted by $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V})$ the space of half-densities on $\mathcal{V}$. In order to deal with high energy terms for a functional integral quantization, we assume another splitting

$$\mathcal{Y}=\mathcal{V}\times\mathcal{Y}',\tag{2.28}$$

where $\mathcal{V}$ denotes the space of classical solutions (critical points) of the quadratic part of the action modulo gauge symmetry and $\mathcal{Y}'$ is a complement. In fact, we assume that the BV Laplacian and the BV symplectic form split accordingly as

$$\Delta=\Delta_{\mathcal{V}}+\Delta_{\mathcal{Y}'},\tag{2.29}$$

$$\omega=\omega_{\mathcal{V}}+\omega_{\mathcal{Y}'}.\tag{2.30}$$

Such a splitting is guaranteed for many important theories, such as (perturbations of) abelian BF theories, by methods of *Hodge decomposition* [24]. In that special case, $\mathcal{B}$ is in fact given by the fields restricted to the boundary. The elements of $\mathcal{V}$ are given by the *zero modes* of the bulk fields and the elements of $\mathcal{Y}$ are given by the *high energy* parts of the bulk fields. Choosing a gauge-fixing Lagrangian submanifold $\mathcal{L}\subset\mathcal{Y}'$, a boundary state is given by

$$\widehat{\Psi}:=\int_{\mathcal{L}\subset\mathcal{Y}'}\exp(\mathrm{i}\mathcal{S}/\hbar)\in\widehat{\mathcal{H}},\tag{2.31}$$

where the functional integral is defined by its perturbative expansion. One can then extend (2.22) to elements of $\widehat{\mathcal{H}}$

$$\left(\hbar^2\Delta_{\mathcal{V}}+\widehat{\mathcal{S}^\partial}\right)\widehat{\Psi}=0.\tag{2.32}$$

Note that a state $\widehat{\Psi}$ depends on leaves in $\mathcal{B}$ and zero modes in $\mathcal{V}$. One can show that the space of zero modes is given by a finite-dimensional BV manifold $(\mathcal{V},\Delta_{\mathcal{V}},\omega_{\mathcal{V}})$ if we consider BF-like theories [24]. Note that in this case it makes sense to define $\Delta_{\mathcal{V}}$. As it was argued in [24], there is a way of integrating out the zero modes. Using cutting and gluing techniques on the source, motivated by the constructions of [2,65], we will obtain a number which corresponds to the value of the partition function for a closed manifold. Moreover, in [24] it was shown that there is always a quantization $\widehat{\mathcal{S}^\partial}$ of $\mathcal{S}^\partial$ that squares to zero and satisfies (2.32). It is fully described by integrals over the boundary of suitable configuration spaces determined by the underlying Feynman graphs.

2.7. Examples of quantum BV-BFV theories

One can extract the axiomatics for a quantum BV-BFV theory out of the computations we have seen before. A *quantum BV-BFV theory* consists of the following data:

(*i*) A graded vector space $\mathcal{H}_{\widetilde{\Sigma}}$ associated to each $(n-1)$-dimensional manifold $\widetilde{\Sigma}$ with a choice of polarization on $\mathcal{F}^\partial_{\widetilde{\Sigma}}$. It is constructed by geometric quantization of the symplectic manifold $(\mathcal{F}^\partial_{\widetilde{\Sigma}},\omega^\partial_{\widetilde{\Sigma}})$. The space $\mathcal{H}_{\widetilde{\Sigma}}$ is called *space of states*.

(*ii*) A coboundary operator $\Omega_{\widetilde{\Sigma}}$ on $\mathcal{H}_{\widetilde{\Sigma}}$ which is a quantization of the BFV action $\mathcal{S}^\partial_{\widetilde{\Sigma}}$. The operator $\Omega_{\widetilde{\Sigma}}$ is called *quantum BFV operator*.

(iii) A finite-dimensional manifold $\mathcal{V}_\Sigma$ associated to each n-dimensional manifold Σ, which is endowed with a degree -1 symplectic form $\omega_{\mathcal{V}_\Sigma}$ and a polarization on $\mathcal{F}^\partial_{\partial\Sigma}$. It is called the *space of residual fields*. Moreover, the space

$$\widehat{\mathcal{H}}_\Sigma := \mathcal{H}_{\partial\Sigma}\hat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$$

is endowed with two commuting coboundary operators $\widehat{\Omega}_{\partial\Sigma} := \Omega_{\partial\Sigma}\otimes \mathrm{id}$ and $\widehat{\Delta}_\Sigma := \mathrm{id}\otimes\Delta_{\mathcal{V}_\Sigma}$, where $\Delta_{\mathcal{V}_\Sigma}$ denotes the canonical BV Laplacian on half-densities on residual fields $\mathcal{V}_\Sigma$.

(iv) A *state* $\widehat{\Psi}_\Sigma \in \widehat{\mathcal{H}}_\Sigma$ which satisfies the *modified QME*

$$\big(\hbar^2\widehat{\Delta}_\Sigma + \widehat{\Omega}_{\partial\Sigma}\big)\widehat{\Psi}_\Sigma = 0.$$

2.7.1. Example: (perturbations of) abelian BF theory

Consider the abelian version of the classical BV theory as in 2.2.4 and its classical BV-BFV extension. Moreover, we want to consider a source manifold Σ whose boundary $\partial\Sigma$ splits into the disjoint union of two boundary components $\partial_1\Sigma$ and $\partial_2\Sigma$ representing the *incoming* and *outgoing* boundary components respectively. On $\partial_1\Sigma$ we choose the $\frac{\delta}{\delta\mathfrak{B}}$-polarization such that the quotient (leaf space) may be identified with $\mathcal{B}_1 := \Omega^\bullet(\partial_1\Sigma)[1] \ni \mathfrak{A}$ and on $\partial_2\Sigma$ we choose the $\frac{\delta}{\delta\mathfrak{A}}$-polarization such that the quotient may be identified with $\mathcal{B}_2 := \Omega^\bullet(\partial_2\Sigma)[n-2] \ni \mathfrak{B}$. The whole leaf space is given by the product $\mathcal{B}_{\partial\Sigma} = \mathcal{B}_1\times\mathcal{B}_2$. The space of residual fields is given by the finite-dimensional BV manifold

$$\mathcal{V}_\Sigma := \mathrm{H}^\bullet_{\mathrm{D}1}(\Sigma)[1]\oplus \mathrm{H}^\bullet_{\mathrm{D}2}(\Sigma)[n-2],$$

where $\mathrm{H}^\bullet_{\mathrm{D}j}(\Sigma)$ denotes the de Rham cohomology of the space $\Omega^\bullet_{\mathrm{D}j}(\Sigma) := \{\gamma\in\Omega^\bullet(\Sigma)\mid i_j^*\gamma = 0\}$, where i_j denotes the inclusion map $\partial_j\Sigma\hookrightarrow\Sigma$. Here D stands for *Dirichlet*. Note that by Poincaré duality we get $\mathcal{V}_\Sigma = T^*[-1](\mathrm{H}^\bullet_{\mathrm{D}1}(\Sigma)[1]) = T^*[-1](\mathrm{H}^\bullet_{\mathrm{D}2}(\Sigma)[n-2])$. The BV Laplacian $\Delta_{\mathcal{V}_\Sigma}$ can be defined by using a basis $([\chi_i])_i$ of $\mathrm{H}^\bullet_{\mathrm{D}1}(\Sigma)$ and its dual basis $([\chi^i])_i$ of $\mathrm{H}^\bullet_{\mathrm{D}2}(\Sigma)$ with chosen representatives $\chi_i\in\Omega^\bullet_{\mathrm{D}1}(\Sigma)$ and $\chi^i\in\Omega^\bullet_{\mathrm{D}2}(\Sigma)$. Note that we have

$$\int_\Sigma \chi^i\wedge\chi_j = \delta^i_j,$$

and we can write the residual fields in $\mathcal{V}_\Sigma$ by

$$\mathsf{a} = \sum_i z^i\chi_i,\quad \mathsf{b} = \sum_j z^+_j\chi^j,$$

with (z^i, z^+_j) being canonical coordinates on $\mathcal{V}_\Sigma$. The BV symplectic form on $\mathcal{V}_\Sigma$ is then given by

$$\omega_{\mathcal{V}_\Sigma} = \sum_i (-1)^{1+(n-1)\,\mathrm{gh}\,z^i}\delta z^+_i\wedge\delta z^i.$$

The BV Laplacian on $\mathcal{V}_\Sigma$ is then given by

$$\Delta_{\mathcal{V}_\Sigma} = \sum_i(-1)^{1+(n-1)\,\mathrm{gh}\,z^i}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^+_i}.$$

The quantum BFV operator $\widehat{\Omega}_{\partial\Sigma}$, acting on $\mathcal{B}_{\partial\Sigma}\times\mathcal{V}_\Sigma$, is given by the ordered standard quantization of $\mathcal{S}^\partial_{\partial\Sigma}$ relative to the chosen polarization:

$$\widehat{\Omega}_{\partial\Sigma} = \mathrm{i}\hbar(-1)^n\left(\int_{\partial_2\Sigma}\mathrm{d}\mathfrak{B}\wedge\frac{\delta}{\delta\mathfrak{B}} + \int_{\partial_1\Sigma}\mathrm{d}\mathfrak{A}\wedge\frac{\delta}{\delta\mathfrak{A}}\right).$$

Using the *effective action* given by

$$\mathcal{S}^{\mathrm{eff}}_\Sigma := (-1)^{n-1}\left(\int_{\partial_2\Sigma}\mathfrak{B}\wedge\mathsf{a} - \int_{\partial_1\Sigma}\mathsf{b}\wedge\mathfrak{A}\right) - (-1)^{2n}\int_{\partial_2\Sigma\times\partial_1\Sigma}\pi_1^*\mathfrak{B}\eta\pi_2^*\mathfrak{A},$$

where $\eta\in\Omega^{n-1}(C_2(\Sigma))$ is a chosen *propagator* on the compactified configuration space

$$C_2(\Sigma) := \overline{\{(x_1,x_2)\in\Sigma^2\mid x_1\neq x_2\}}^{\mathrm{FMAS}}$$

(here FMAS stands for the *Fulton–MacPherson/Axelrod–Singer* compactification of configuration spaces [4,36]), we get the state

$$\widehat{\Psi}_\Sigma = T_\Sigma\exp\left(\mathrm{i}\mathcal{S}^{\mathrm{eff}}_\Sigma/\hbar\right).$$

Here $T_\Sigma\in\mathbf{C}$ denotes a coefficient expressed in terms of the *Reidemeister torsion*. Indeed, one can then immediately check that the mQME is satisfied where $\widehat{\Delta}_\Sigma := \Delta_{\mathcal{V}_\Sigma}$ acts on the fibers of $\mathcal{B}_{\partial\Sigma}\times\mathcal{V}_\Sigma$. We can construct the state space to be

$$\widehat{\mathcal{H}}_\Sigma=\left(\prod_{j_1,j_2\geq 0}\mathcal{H}^{j_2,n-2}_{\partial_2\Sigma}\hat{\otimes}\mathcal{H}^{j_1,1}_{\partial_1\Sigma}\right)\hat{\otimes}\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma),$$

where $\mathcal{H}^{j,\ell}_{\partial\Sigma}$ is the vector space of j-linear functionals on $\Omega^\bullet(\partial\Sigma)[\ell]$ of the form

$$\Omega^\bullet(\partial\Sigma)[\ell]\ni\mathfrak{D}\mapsto\int_{(\partial\Sigma)^j}\gamma\pi_1^*\mathfrak{D}\wedge\cdots\wedge\pi_j^*\mathfrak{D}$$

times some prefactor (given in terms of the Reidemeister torsion), where γ is some distributional form on $(\partial\Sigma)^j$ and π_i denotes the projection to the i-th component. Considering perturbations, we asymptotically get that states are of the form

$$\widehat{\Psi}_\Sigma\sim T_\Sigma\exp\left(\mathrm{i}\mathcal{S}^{\mathrm{eff}}_\Sigma/\hbar\right)\times\sum_{k\geq 0}\hbar^k\sum_{j_1,j_2\geq 0}\int_{(\partial_1\Sigma)^{j_1}\times(\partial_2\Sigma)^{j_2}}R^k_{j_1j_2}(\mathsf{a},\mathsf{b})\pi^*_{1,1}\mathfrak{A}\wedge\cdots\wedge\pi^*_{1,j_1}\mathfrak{A}\wedge\pi^*_{2,1}\mathfrak{B}\wedge\cdots\wedge\pi^*_{2,j_2}\mathfrak{B},$$

where $\pi_{i,j}$ denotes the j-th projection of $(\partial_i\Sigma)^{j_i}$ and $R^k_{j_1j_2}$ denotes distributional forms on $(\partial_1\Sigma)^{j_1}\times(\partial_2\Sigma)^{j_2}$ with values in $\mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_\Sigma)$. Note that here $\mathcal{S}^{\mathrm{eff}}_\Sigma$ is replaced by the corresponding zero-loop effective action. We refer the reader to [24] for more examples and a detailed discussion of perturbative quantizations on manifolds with boundary.

An important version of a perturbation of abelian BF theory is given by *split Chern–Simons theory* [25]. Consider Chern–Simons theory as in 2.2.3 for a Lie algebra $\mathfrak{g}$ endowed with an invariant pairing $\langle\ ,\ \rangle$. Moreover, consider a suitable 3-manifold Σ. As we have seen, for $\mathbf{A}\in\Omega^\bullet(\Sigma)\otimes\mathfrak{g}[1]$, the BV action is given by

$$\mathcal{S}_\Sigma=\int_\Sigma\left(\frac{1}{2}\langle\mathbf{A},\mathrm{d}\mathbf{A}\rangle+\frac{1}{6}\langle\mathbf{A},[\mathbf{A},\mathbf{A}]\rangle\right).$$

Assume that the Lie algebra splits as $\mathfrak{g}=V\oplus W$ into maximally isotropic subspaces with respect to $\langle\ ,\ \rangle$, i.e. the pairing restricts to zero on V and W and $\dim V=\dim W=\frac{1}{2}\dim\mathfrak{g}$. Then one can identify $W\cong V^*$ by using the pairing and consider a decomposition $\mathbf{A}=\mathbf{V}+\mathbf{W}$ with $\mathbf{V}\in\Omega^\bullet(\Sigma)\otimes V[1]$ and $\mathbf{W}\in\Omega^\bullet(\Sigma)\otimes W[1]$. Then we can decompose the action $\mathcal{S}_\Sigma=\mathcal{S}^{\mathrm{kin}}_\Sigma+\mathcal{S}^{\mathrm{int}}_\Sigma$ into a *kinetic* and *interaction* part:

$$\begin{aligned}\mathcal{S}^{\mathrm{kin}}_\Sigma&=\frac{1}{2}\int_\Sigma\langle\mathbf{A},\mathrm{d}\mathbf{A}\rangle=\int_\Sigma\langle\mathbf{W},\mathrm{d}\mathbf{V}\rangle,\\ \mathcal{S}^{\mathrm{int}}_\Sigma&=\frac{1}{6}\int_\Sigma\langle\mathbf{A},[\mathbf{A},\mathbf{A}]\rangle\\ &=\frac{1}{6}\int_\Sigma\langle\mathbf{V}+\mathbf{W},[\mathbf{V}+\mathbf{W},\mathbf{V}+\mathbf{W}]\rangle.\end{aligned}$$

An important assumption on the theory is that $(\mathfrak{g},V,W)$ is in fact a *Manin triple*, i.e. V and W are actually Lie subalgebras of $\mathfrak{g}$. The quantum picture is then similar to the one of abelian BF theory (see [25] for a detailed construction). More general perturbations of abelian BF theory are given by *AKSZ theories* [1] as mentioned in Remark 2.1 (see also [23,24,27] for a detailed treatment of such theories in the BV-BFV formalism).

3. Higher codimension

3.1. Higher codimension gauge theories: BV-BF^{k}V theories

Since the BV-BFV construction is a codimension 1 formulation, we have an action of a dg algebra of observables, coming from the deformation quantization construction, to a chain complex (or vector space) associated to the boundary via geometric quantization with respect to the symplectic manifold $(\mathcal{F}^\partial,\omega^\partial)$. This corresponds to the action of the operator $\widehat{\mathcal{S}^\partial}\in\mathrm{End}(\mathcal{H})$ on $\widehat{\Psi}\in\widehat{\mathcal{H}}$. Classical BV-BFV theories can be extended to higher codimension manifolds [23]. One can define an exact BF^{k}V manifold to be a triple $\left(\mathcal{F}^{\partial^k},\omega^{\partial^k}=\delta\alpha^{\partial^k},Q^{\partial^k}\right)$ where $\mathcal{F}^{\partial^k}$ is a $\mathbf{Z}$-graded supermanifold, $\omega^{\partial^k}\in\Omega^2(\mathcal{F}^{\partial^k})$ is an exact symplectic form of ghost number $k-1$ with primitive 1-form α^{∂^k}, and $Q^{\partial^k}\in\mathfrak{X}(\mathcal{F}^{\partial^k})$ is a cohomological, symplectic vector field with Hamiltonian function $\mathcal{S}^{\partial^k}$ of ghost number k. A BV-BF^{k}V manifold over an exact BF^{k}V manifold $\left(\mathcal{F}^{\partial^k},\omega^{\partial^k}=\delta\alpha^{\partial^k},Q^{\partial^k}\right)$ is a quintuple

$$\left(\mathcal{F}^{\partial^{k-1}},\omega^{\partial^{k-1}},\mathcal{S}^{\partial^{k-1}},Q^{\partial^{k-1}},\pi\colon\mathcal{F}^{\partial^{k-1}}\to\mathcal{F}^{\partial^k}\right)$$

such that π is a surjective submersion and

- $\delta\pi Q^{\partial^{k-1}}=Q^{\partial^k}$,
- $\iota_{Q^{\partial^{k-1}}}\omega^{\partial^{k-1}}=\delta\mathcal{S}^{\partial^{k-1}}+\pi^*\alpha^{\partial^k}$.

Again, this will lead to a higher codimension version of the mCME

$$Q^{\partial^{k-1}}\left(\mathcal{S}^{\partial^{k-1}}\right)=\pi^*\left(2\mathcal{S}^{\partial^k}-\iota_{Q^{\partial^k}}\alpha^{\partial^k}\right).\tag{3.1}$$

3.1.1. Example: classical codimension 2 theory

Consider Yang–Mills theory as in 2.2.2. Let $\Sigma_2 \subset \Sigma$ be a codimension 2 stratum. The BV-BFV theory on Σ and $\partial\Sigma$ induces the following data associated on Σ_2: The space of fields

$$\mathcal{F}^{\partial^2}_{\Sigma_2} = \Omega^{n-2}(\Sigma_2) \otimes \mathfrak{g}[n-2] \oplus \Omega^0(\Sigma_2) \otimes \mathfrak{g}[1] \ni (\mathbb{B}_2, \mathbb{c}_2),$$

with $\mathrm{gh}\,\mathbb{B}_2 = 0$ and $\mathrm{gh}\,\mathbb{c}_2 = 1$. The BF^{2}V symplectic form is given by

$$\omega^{\partial^2}_{\Sigma_2} = \delta\alpha^{\partial^2}_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr}\left(\delta\mathbb{B}_2 \wedge \delta\mathbb{c}_2\right),$$

with

$$\alpha^{\partial^2}_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr}\left(\mathbb{B}_2 \wedge \delta\mathbb{c}_2\right).$$

The cohomological vector field is given by

$$Q^{\partial^2}_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr}\left([\mathbb{B}_2, \mathbb{c}_2] \wedge \frac{\delta}{\delta\mathbb{B}_2} + \frac{1}{2}[\mathbb{c}_2, \mathbb{c}_2] \wedge \frac{\delta}{\delta\mathbb{c}_2}\right).$$

The BF^{2}V action (which one can obtain from $Q^{\partial^2}_{\Sigma_2}$ by using the construction of [58]) is given by

$$\mathcal{S}^{\partial^2}_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr}\left(\frac{1}{2}\mathbb{B}_2 \wedge [\mathbb{c}_2, \mathbb{c}_2]\right).$$

Remark 3.1. The quantum extension is more difficult and requires certain algebraic constructions. Following the codimension 1 construction, one can try to formulate a similar procedure by considering a deformation quantization of Poisson structures with higher shifts. However, we will not consider a general coupling of higher codimension theories here, but rather describe the idea for quantization of the according BF^{k}V theory for a codimension k stratum. It is expected that a coupling on each codimension for general geometric situations is possible.

3.2. Algebraic and geometric structure for the quantization in higher codimension

3.2.1. Deformation quantization picture

Let us denote by $\mathbb{E}_k$ the topological operad of little k-dimensional disks and let $\mathbb{P}_k$ denote the operad controlling $(1-k)$-shifted (unbounded) Poisson dg algebras [49]. It is known that deformation quantization of $\mathbb{P}_1$-algebras corresponds to $\mathbb{E}_1$-algebras [44], which is the same as an A_∞-algebra (associative algebra). The higher codimension picture for deformation quantization is related to the higher version of the *Deligne conjecture* [44]. The Deligne conjecture, which is related to usual deformation quantization, states that there is a natural action of an $\mathbb{E}_2$-algebra over the category of chains complexes to the Hochschild cohomology generated by an arbitrary associative algebra (see e.g. [45,51] for a proof). This can be generalized to the $\mathbb{E}_k$ operad. Using the fact that the $\mathbb{E}_k$ operad is formal in the category of chain complexes, i.e. equivalent to its homology and that its homology is given by the $\mathbb{P}_k$ operad, there is an equivalence between the $\mathbb{E}_k$ and the $\mathbb{P}_k$ operad [35,44,68,69]. Thus for all $k \geq 2$, there exists a deformation quantization for a $\mathbb{P}_k$-algebra and there is a canonical Lie bracket $[\ ,\ ]_{\mathbb{E}_k}$ on an $\mathbb{E}_k$-algebra which corresponds to a $(1-k)$-shifted Poisson structure through the equivalence. One can then view $\mathcal{O}_{loc}(\mathcal{F}^{\partial^k})[\![\hbar]\!]$ as an $\mathbb{E}_k$-algebra endowed with a dg structure induced by the differential

$$Q^{\partial^k}_\hbar := \left[\mathcal{S}^{\partial^k}_\hbar,\ \right]_{\mathbb{E}_k}. \tag{3.2}$$

The higher shifted analog of the quantum master equation is then given by

$$\left[\mathcal{S}^{\partial^k}_\hbar, \mathcal{S}^{\partial^k}_\hbar\right]_{\mathbb{E}_k} = 0. \tag{3.3}$$

Note that for $k = 0$, we obtain a $+1$-shifted Poisson structure $(\ ,\ ) := [\ ,\ ]_{\mathbb{E}_0}$ as in the construction of a BV algebra, so there is an equivalence of the operad $\mathbb{BV}$, which controls the BV algebra structure, to the homology of the *framed* $\mathbb{E}_0$-operad. Note the difference of having a framing, i.e. we also allow *rotations* of little disks. The rotations indeed correspond to the BV Laplacian Δ in the homology of the operad over chain complexes.

In the category of chain complexes we can observe a chain complex of observables $\mathcal{O}_{loc}(\mathcal{F})$ with differential $Q = (\mathcal{S},\ )$. In fact, we will have the operations given by the differential Q (degree $+1$), usual multiplication of functions (degree 0), the Poisson structure $(\ ,\ )$ (degree $+1$) and the BV Laplacian Δ (degree $+1$) such that for all $f, g \in \mathcal{O}_{loc}(\mathcal{F})$ we have

$$\Delta(fg) = \Delta f g + (-1)^{\mathrm{gh}\, f} f \Delta g + (-1)^{\mathrm{gh}\, f}(f, g).$$

Deforming this operad will lead to the $\mathbb{BD}_0$ operad which is basically the same as the $\mathbb{BV}$ operad without emphasizing the BV Laplacian Δ but, for the purpose of QFT, include it into the differential and where everything is now over chain complexes of $\mathbf{K}[\![\hbar]\!]$-modules, i.e. we consider the algebra of deformed observables $\mathcal{O}_{loc}(\mathcal{F})[[\hbar]]$ together with the bracket $\hbar(\ ,\ )$ and a differential D of degree $+1$ such that for all $f, g \in \mathcal{O}_{loc}(\mathcal{F})[[\hbar]]$ we have

$$D(fg) = Dfg + (-1)^{\mathrm{gh}\, f} f Dg + (-1)^{\mathrm{gh}\, f} \hbar(f, g).$$

In application to QFT we have $D = Q + \hbar\Delta$ and one can check that we indeed have $D^2 = 0$ (see also [29] for the construction of Beilinson–Drinfeld algebras in connection to QFT and factorization algebras).

Remark 3.2. It is important to mention that the usual convention for the degrees in the definition of a BV algebra is the one where the Poisson bracket $(\ ,\ )$ is of degree -1 and hence Δ is also of degree -1. In this case the $\mathbb{BV}$ operad is equivalent to the homology of the framed $\mathbb{E}_2$ operad (little 2-disks) [37]. However, for $\mathbb{BD}_0$-algebras one still requires that the bracket is of degree $+1$, so one assigns a weight of $+1$ to $\hbar$.

For $k = 1$, this corresponds to usual deformation quantization [29]. In particular, for a $\mathbb{P}_1$-algebra $(A, \{\ ,\ \})$, the lift to a $\mathbb{BD}_1$-algebra, which is flat over $\mathbf{K}[[\hbar]]$, is the same as a deformation quantization of A in the usual sense. Indeed, to describe $\mathbb{BD}_1$-structures on $A[[\hbar]]$ compatible with the given $\mathbb{P}_1$-structure we can give an associative product on $A[[\hbar]]$, linear over $\mathbf{K}[[\hbar]]$, and which modulo $\hbar$ is given by the commutative product on A. The relations in the $\mathbb{BD}_1$ operad imply that the $\mathbb{P}_1$-structure on A is related to the associative product on $A[[\hbar]]$ by

$$\frac{1}{\hbar}(f \star g - g \star f) = \{f, g\} \bmod \hbar.$$

The higher codimension k version of the quantum master equation follows the picture of $\mathbb{BD}_k$-algebras.

This construction is also consistent with the k-dimensional version of the *Swiss-Cheese operad* [44,75] $\mathbb{SC}_{k,k-1}$ which couples the $\mathbb{E}_k$ operad to the $\mathbb{E}_{k-1}$ operad [44,50,71] by an action of $\mathbb{E}_k$-algebras on $\mathbb{E}_{k-1}$-algebras. Describe it as an operad of sets. This colored operad has two colors: points may be in the bulk or on the boundary. The set of colors is a poset, that is a category, rather than a set, and there are only operations compatible with this structure. The Swiss-Cheese operad is important when dealing with the coupling in contiguous codimension. A *Swiss-Cheese algebra* (i.e. an algebra over the Swiss-Cheese operad $\mathbb{SC}_{k,k-1}$) consists of a triple (A, B, ρ) such that A is a(n) (framed) $\mathbb{E}_k$-algebra, B a(n) (framed) $\mathbb{E}_{k-1}$-algebra and $\rho\colon A \to \mathsf{HC}^\bullet_{\mathsf{Disk}^{\mathrm{fr}}_{k-1}}(B)$ the coupling action. Here $\mathsf{HC}^\bullet_{\mathsf{Disk}^{\mathrm{fr}}_{k-1}}(B)$ denotes the *Hochschild cochain object* of the $\mathbb{E}_{k-1}$-algebra B. In fact, $\mathsf{HC}^\bullet_{\mathsf{Disk}^{\mathrm{fr}}_{k-1}}(B)$ carries the structure of an $\mathbb{E}_k$-algebra. Thus, ρ is a map of (framed) $\mathbb{E}_k$-algebras.

Unfortunately, it can be shown that the Swiss-Cheese operad is not formal [46] (still, one can show that there is a higher codimension version of the Swiss-Cheese operad which in fact is indeed formal [38]). However, there is an equivalence of $\mathbb{SC}_{k,k-1}$ to a classical notion for coupling contiguous codimensions by the $\mathbb{P}_{k,k-1}$ operad (see [52] for the definition of $\mathbb{P}_{k,k-1}$ and detailed discussions). Let us briefly discuss the coupling in each codimension from the point of view of *factorization homology for stratified spaces* as in [6]. A very good introduction to factorization homology in the topological field theory setting is [70]. Similarly, as the contiguous codimensions can be coupled together by the Swiss-Cheese operad through the coupling action ρ, we can extend this to a generalization for the coupling in each codimension. Denote by $\mathsf{Disk}^{\mathrm{fr}}_d$ the ∞-category with objects given by finite disjoint unions of framed d-dimensional disks and morphisms being smooth embeddings equipped with a compatibility of framings. A $\mathsf{Disk}^{\mathrm{fr}}_d$-algebra (or equivalently framed $\mathbb{E}_d$-algebra) A in a monoidal symmetric ∞-category $\mathcal{C}^\otimes$ is a symmetric monoidal functor $A\colon \left(\mathsf{Disk}^{\mathrm{fr}}_d\right)^{\coprod} \to \mathcal{C}^\otimes$, where $\coprod$ denotes the symmetric monoidal structure given by disjoint union of topological spaces. Denote by $\mathsf{Mnfld}^{\mathrm{fr}}_d$ the ∞-category whose objects are smooth framed d-dimensional manifolds and whose morphisms consist of all smooth embeddings equipped with a compatibility of framings.

Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category admitting all sifted colimits. Fix an $\mathbb{E}_d$-algebra in $\mathcal{C}^\otimes$, which we will denote by A. Then *factorization homology with coefficients in A* is the left Kan extension,

$$\begin{tikzcd} \mathsf{Disk}^{\mathrm{fr}}_d \arrow[r, "A"] \arrow[d, hook] & \mathcal{C}^\otimes \\ \mathsf{Mnfld}^{\mathrm{fr}}_d \arrow[ur, dashed, "\int A"'] & \end{tikzcd}$$

and for a framed d-manifold $X \in \mathsf{Mnfld}^{\mathrm{fr}}_d$, we denote by

$$\int_X A$$

the *factorization homology of X with coefficients in A*. It is called *homology* since it satisfies the *Eilenberg–Steenrod axioms* [32] for a homology theory. Note that we can also fix a manifold $Y \in \mathsf{Mnfld}^{\mathrm{fr}}_{n-d-1}$ and consider a map

$$\int_Y\colon \mathsf{Alg}_{\mathsf{Disk}^{\mathrm{fr}}_n}(\mathcal{C}^\otimes) \to \mathsf{Alg}_{\mathsf{Disk}^{\mathrm{fr}}_{d+1}}(\mathcal{C}^\otimes).$$

More general, for a symmetric monoidal ∞-category $\mathcal{C}^\otimes$ and an ∞-category B of *basics*, one can define the *absolute factorization homology* to be the left adjoint to

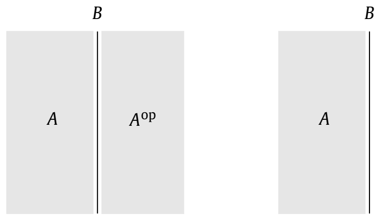


Fig. 1. Difference between the defect situation and the boundary situation.

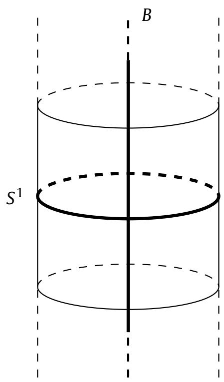


Fig. 2. Example of a 1-dimensional defect B sitting in $\mathbb{R}^3$. The orthogonal sphere is given by S^1 through homotopy equivalence of the tubular neighborhood around it.

$$\begin{tikzcd} \int : \mathsf{Alg}_{\mathsf{Disk(B)}}(\mathcal{C}^{\otimes}) \arrow[rr, bend left, dashed] & & \mathsf{Fun}^{\otimes}(\mathsf{Mnfld(B)}, \mathcal{C}^{\otimes}). \arrow[ll, bend left] \end{tikzcd}$$

The basics for our construction are given by framed disks of a given dimension. Moreover, for our purpose we want to consider the category $\mathcal{C}^{\otimes}$ to be given by chain complexes. For the situation where $\mathsf{B} = \mathsf{Disk}^{\mathrm{fr}}_{d\subset n}$ is defined such that B-manifolds are framed n-dimensional manifolds with a framed d-dimensional properly embedded submanifold such that the framing splits along this submanifold we have the following:

For an $\mathbb{E}_n$-algebra A and an $\mathbb{E}_d$-algebra B, we can observe an action [6, Proposition 4.8]

$$\rho\colon \int_{S^{n-d-1}} A \to \mathrm{HC}^{\bullet}_{\mathsf{Disk}^{\mathrm{fr}}_d}(B),$$

where S^{n-d-1} denotes the $(n-d-1)$-sphere. Here we are considering a d-dimensional defect sitting inside an n-manifold as locally described by the basics category $\mathsf{Disk}^{\mathrm{fr}}_{d\subset n}$. Note that ρ has to be a map of $\mathbb{E}_{d+1}$-algebras. This can be regarded as a *higher* version of Deligne's conjecture. In particular, for $d = n-1$, we can obtain the action given by

$$\rho\colon \int_{S^0} A = A \otimes A^{\mathrm{op}} \to \mathrm{HC}^{\bullet}_{\mathsf{Disk}^{\mathrm{fr}}_{n-1}}(B).$$

Note that this is not exactly the Swiss-Cheese case since here the domain is given by $A \otimes A^{\mathrm{op}}$ where as in the Swiss-Cheese case it is given by A. Intuitively, this is because the opposite half of the bulk is missing (see Fig. 1).

A nice interpretation for the integration was also given in [19,34]. The integral $\int_{S^{n-d-1}} A$ is by definition given by $\int_{S^{n-d-1}\times\mathbb{R}^{d+1}} A$, hence an $\mathbb{E}_{d+1}$-algebra since $S^{n-d-1}\times\mathbb{R}^{d+1}$ is an $\mathbb{E}_{d+1}$-algebra by considering the space of embeddings

$$\coprod_I S^{n-d-1}\times\mathbb{R}^{d+1} \to S^{n-d-1}\times\mathbb{R}^{d+1}.$$

This is in fact true for any $(n-d-1)$-manifold not just for S^{n-d-1}. Thus, one may think of the action of A on the defect B to be given by a $\int_{S^{n-d-1}} A$-module structure. Consider the local situation when there is a defect $B \subset \mathbb{R}^d$ sitting in $\mathbb{R}^n$. We can think of S^{n-d-1} as the *orthogonal sphere* and $S^{n-d-1}\times\mathbb{R}^{d+1}$ as some tubular neighborhood around $\mathbb{R}^d$ without $\mathbb{R}^d$. Then $\int_{S^{n-d-1}} A$ is the global algebra of observables on this neighborhood. In this interpretation, the action by $\int_{S^{n-d-1}} A$ is equivalent to restricting the bulk algebra to an infinitesimal neighborhood of B where it acts (see Fig. 2 for an illustration).

For $d = n-2$ we get an action

$$\rho\colon \int_{S^1} A = A \otimes_{A\otimes A^{\mathrm{op}}} A \to \mathrm{HC}^{\bullet}_{\mathsf{Disk}^{\mathrm{fr}}_{n-2}}(B).$$

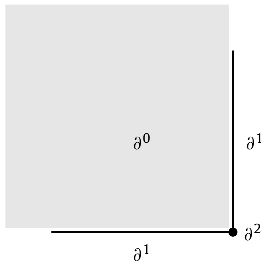


Fig. 3. A corner situation.

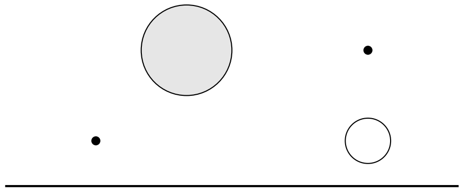


Fig. 4. Situation of defects sitting inside some n-manifold for which the mentioned construction applies.

In fact, in [5,34] it has been proven that for any symmetric monoidal category $\mathcal{C}^\otimes$ there is an equivalence

$$\mathrm{Mod}_A^{\mathsf{Disk}^{\mathrm{fr}}_{n-d}}(\mathcal{C}^\otimes) \cong \mathrm{Mod}_{\int_{S^{n-d-1}} A}(\mathcal{C}^\otimes),$$

which is used in the proof of [6, Proposition 4.8]. Note that in this picture, the d in $\mathbb{E}_d$ denotes the dimension of the corresponding submanifold (defect). For a classical theory on an n-manifold, the codimension k submanifolds of dimension $d = n - k$, give rise to $\mathbb{E}_d$-algebras, and are endowed with an additional $\mathbb{P}_k$-structure. In this case (i.e. with additional structure) all of the previous constructions hold but there will be extra constrains due to the $\mathbb{P}_k$-structure. In particular, it is not clear how the action is expressed in presence of additional structure.

Remark 3.3. An important remark at this point is to emphasize that above mentioned construction for the action of a different codimension is in fact just pairwise. In order to describe the coupling action in each codimension (i.e. not just pairwise) some modification has to be made. This means that a geometric situation as e.g. in Fig. 3 is not directly covered by the constructions of [6]. Note that the coupling depends on the given geometry and how the defects are relative to each other. There has to be some compatibility of the action of the ∂^0 part to the ∂^1 parts where these have to have again a coupling to the ∂^2 part. This has to be compatible with the action of the ∂^0 to the ∂^2 part. However, the extension to a situation as in Fig. 3 is known and a coupling on a general stratified space is expected to be possible [19]. (See Fig. 4.)

3.2.2. Geometric quantization picture

The phase space of an n-dimensional classical field theory associated to a closed d-dimensional submanifold is endowed with a $(n-d-1)$-shifted symplectic structure [55]. This also makes sense for the case when $d = n$, i.e. for the case of a (-1)-shifted symplectic structure which corresponds to the case of the BV formalism. Let us briefly consider the example of 3-dimensional classical Chern–Simons theory as in 2.2.3 [62]. For a compact Lie group G we associate the phase space which is given by the moduli space of flat G-connections. If Σ is a closed oriented 3-manifold, the phase space is given by the critical locus of the Chern–Simons functional, so it carries the induced BV symplectic structure. If Σ is a closed oriented 2-manifold, the phase space is endowed with the Atiyah–Bott symplectic structure [3]. If $\Sigma = S^1$ (1-dimensional closed compact oriented manifold), the phase space is the stack of conjugacy classes $[G/G]$ and the corresponding 1-shifted symplectic structure is given in terms of the canonical 3-form on G. If $\Sigma = \mathrm{pt}$ (0-dimensional case), the phase space is given by the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ and the corresponding 2-shifted symplectic structure is given in terms of the invariant symmetric bilinear form on the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ used in the definition of Chern–Simons theory (Killing form).

The analog of geometric quantization for k-shifted symplectic structures uses the notion of *higher categories* [49] and *derived algebraic geometry* [55,73]. It was recently shown that it corresponds to the notion of an (∞, k)-category [62]. Let us give a bit more insights on this construction. Recall that the data for geometric quantization of a symplectic manifold (M, ω), is given by a *prequantization*, i.e. a line bundle (usually called *prequantum line bundle*) $(\mathscr{L}, \nabla)$ together with a connection on M with curvature ω, and a *polarization*, i.e. a Lagrangian foliation $\mathscr{F} \subset TM$ which is a subbundle closed with respect to the Lie bracket of vector fields which is Lagrangian with respect to ω [43,54, 79]. The constructed vector space is then given by the space of ∇-flat sections $\Gamma_{\mathrm{flat}}(M, \mathscr{L}) \subset \Gamma(M, \mathscr{L})$ of $\mathscr{L}$ along the foliation $\mathscr{F}$. In the k-shifted case, one defines the analog of a prequantum line bundle, called a *prequantum k-shifted Lagrangian fibration*, to be given by a k-gerbe $\mathscr{G}$ on the base manifold together with an extension of the natural relative flat connection on the pullback of $\mathscr{G}$ to the fiber to a connective structure ∇. The polarization is encoded in a k-shifted Lagrangian foliation (see [16,74]). For a 1-shifted symplectic structure we get the ∞-category $\mathrm{QCoh}^{\mathscr{G}}$ of twisted quasi-coherent sheaves on the base and for the 2-shifted case the ∞-category of quasi-coherent sheaves on a certain category and so on, such that the output for k-shifted structures gives an (∞, k)-category (see [62] for a detailed construction).

Note that this indeed is compatible with the quantum BV-BFV construction which assigns a chain complex $(\mathcal{H}, \Omega)$ to the 0-shifted case on the boundary (rather than a vector space). It is also compatible with the BV construction in the bulk, i.e. for a (-1)-shifted symplectic structure. Namely, the constructions as in [64] give rise to a "geometric quantization". The expectation value, assigning to an observable a value in the ground field through a path integral of a half-density (usually $\exp(\mathrm{i}\mathcal{S}/\hbar)\sigma$ where σ is some Δ-closed reference half-density) over a chosen Lagrangian submanifold (gauge-fixing), is considered as some analog for the polarization. This can be seen by an extension of the result in [76]. The original form states that for an odd symplectic supermanifold $(\mathcal{F},\omega)$ there is a quasi-isomorphism between the complex $(\Omega^\bullet(\mathcal{F}),\omega\wedge)$, consisting of differential forms on $\mathcal{F}$ endowed with the differential given by wedging with ω, and the complex $(\mathrm{Dens}^{\frac{1}{2}}(\mathcal{F}),\Delta)$, consisting of half-densities on $\mathcal{F}$ endowed with the BV Laplacian (this is obviously related to the BV theorem as in [64]). Moreover, it states that the de Rham differential vanishes on the cohomology of $(\Omega^\bullet(\mathcal{F}),\omega\wedge)$ and that the BV Laplacian is given by $\Delta=(\mathrm{d}\circ(\omega\wedge)^{-1}\circ\mathrm{d})$. In fact, it is the third differential in the spectral sequence of the bicomplex $(\Omega^\bullet(\mathcal{F}),\omega\wedge,\mathrm{d})$ and all higher differentials are zero. The shifted analog states a similar quasi-isomorphism [62, Proposition 2.34]. Hence, we want to extend the outcome to "dg (∞,k)-categories". Let us first talk about the 1-categorical picture and give an informal construction for it. One can regard a dg k-category to be a k-category $\mathcal{C}$ [7] for which each set of morphisms $\mathrm{Hom}(X,Y)$ between two objects $X,Y\in\mathcal{C}$ forms a dg module, i.e. it is given by a direct sum

$$\mathrm{Hom}(X,Y)=\bigoplus_{n\in\mathbf{Z}}\mathrm{Hom}_n(X,Y),$$

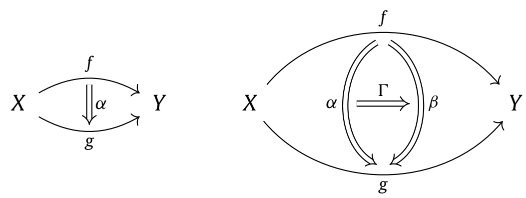
endowed with a differential

$$\mathrm{d}_{\mathcal{C}}^{(X,Y)}\colon\ \mathrm{Hom}_n(X,Y)\to\mathrm{Hom}_{n+1}(X,Y).$$

Composition of morphisms is given by maps of dg modules

$$\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z)\to\mathrm{Hom}(X,Z),\quad \forall X,Y,Z\in\mathcal{C}$$

satisfying some additional relations [30,72]. One should think of Hom as the space of 1-morphisms. Denote by $\mathrm{Hom}^{(k)}$ the space of k-morphisms, which again forms a dg module. Below an illustration of a 2-morphism α between morphisms $f,g\in\mathrm{Hom}^{(1)}(X,Y)$ and a 3-morphism Γ between two 2-morphisms α,β.



We require that they satisfy the same conditions as $\mathrm{Hom}=\mathrm{Hom}^{(1)}$, i.e. for two $(k-1)$-morphisms f,g, we want that the space of k-morphisms between them is given by a direct sum

$$\mathrm{Hom}^{(k)}(f,g)=\bigoplus_{n\in\mathbf{Z}}\mathrm{Hom}_n^{(k)}(f,g)$$

endowed with a differential

$$\mathrm{d}_{\mathcal{C}}^{(f,g)}\colon\ \mathrm{Hom}_n^{(k)}(f,g)\to\mathrm{Hom}_{n+1}^{(k)}(f,g).$$

The composition of k-morphisms should then, similarly as for higher categories, satisfy some Stasheff pentagon identity [66,67]. Formally, one can construct a *strict* dg k-category as an iteration (similarly as for defining higher categories) of enrichments over the category of chain complexes. That is, one defines a dg k-category as a category enriched over dg $(k-1)$-categories. This is more or less straightforward since the category of chain complexes can be endowed with a symmetric monoidal structure. The more interesting notion in this setting is the *non-strict* version. There one has to start with an ∞-category for the enrichment instead of just chain complexes. Although there should not be any obstacles in the construction, this will be rather involved. The diagram below illustrates the quantization for higher codimensions where we denote by Ch the category of chain complexes, by $\mathsf{Alg}_{\mathbb{P}_k}^{\mathbb{E}_d}(\mathsf{Ch}):=\mathsf{Alg}_{\mathbb{P}_k}(\mathsf{Alg}_{\mathbb{E}_d}(\mathsf{Ch}))$ the category of $\mathbb{P}_k$-algebras over $\mathbb{E}_d$-algebras in Ch and by $\mathsf{dgCat}_{(\infty,k)}$ the category of dg (∞,k)-categories. One should think of the horizontal arrows as passing to higher codimension and not as a functor in particular. The quantum picture on the level of deformation quantization focuses on the algebraic structure (shifted Poisson structure) on the space of (higher codimension) observables, whereas the picture on the level of geometric quantization focuses on the geometric structure induced by the space of (higher codimension) boundary fields, namely its (shifted) symplectic manifold structure.

$$\cdots\longleftarrow\partial^k\longleftarrow\cdots\longleftarrow\partial^2\longleftarrow\partial^1\longleftarrow\partial^0$$

$$\begin{tikzcd}
\boxed{\textbf{DefQuant}} & \cdots & \mathsf{Alg}_{\mathbb{BD}_k}^{\mathbb{E}_{n-k}}(\mathsf{Ch}) \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathsf{Alg}_{\mathbb{BD}_2}^{\mathbb{E}_{n-2}}(\mathsf{Ch}) \arrow[l] \arrow[d] & \mathsf{Alg}_{\mathbb{BD}_1}^{\mathbb{E}_{n-1}}(\mathsf{Ch}) \arrow[l] \arrow[d] & \mathsf{Alg}_{\mathbb{BD}_0}^{\mathbb{E}_{n}}(\mathsf{Ch}) \arrow[l] \arrow[d] \\
\boxed{\textbf{GeomQuant}} & \cdots & \mathsf{dgCat}_{(\infty,k-1)} \arrow[l] & \cdots \arrow[l] & \mathsf{dgCat}_{(\infty,1)} \arrow[l] & \mathsf{Ch} \arrow[l] & \mathbf{K} \arrow[l]
\end{tikzcd}$$

3.3. Obstruction spaces

Recall that for codimension 0 theories the quantum obstruction space was given by the first cohomology group with respect to the cohomological vector field in the bulk (Theorem 2.2) and for coboundary 1 theories it was given by the second cohomology group with respect to the cohomological vector field on the boundary (Theorem 2.4). A natural question is whether the obstruction space for the quantization of codimension k theories is given by

$$\mathrm{H}^{k+1}_{Q^{\partial^k}}(\mathcal{O}_{loc}(\mathcal{F}^{\partial^k})).$$

This is not clear at the moment. We plan to consider this more carefully in the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, O. Zaboronsky, The geometry of the master equation and topological quantum field theory, Int. J. Mod. Phys. A 12 (7) (1997) 1405–1429, https://doi.org/10.1142/S0217751X97001031.
[2] M.F. Atiyah, Topological quantum field theories, Publ. Math. IHÉS 68 (1) (January 1988) 175–186, https://doi.org/10.1007/bf02698547.
[3] M.F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. R. Soc. Lond. 308 (1983) 523–615, https://doi.org/10.1098/rsta.1983.0017.
[4] S. Axelrod, I.M. Singer, Chern-Simons perturbation theory. II, J. Differ. Geom. 39 (1) (1994) 173–213, https://doi.org/10.4310/jdg/1214454681.
[5] D. Ayala, J. Francis, Factorization homology of topological manifolds, J. Topol. 8 (4) (2015) 1045–1084, https://doi.org/10.1112/jtopol/jtv028.
[6] D. Ayala, J. Francis, H.L. Tanaka, Factorization homology of stratified spaces, Sel. Math. 23 (1) (2017) 293–362, https://doi.org/10.1007/s00029-016-0242-1.
[7] J.C. Baez, An introduction to n-categories, in: Category Theory and Computer Science, vol. 1290, 2005, pp. 1–33.
[8] J.C. Baez, J. Dolan, Higher dimensional algebra and topological quantum field theory, J. Math. Phys. 36 (1995) 6073–6105, https://doi.org/10.1063/1.531236.
[9] G. Barnich, F. Del Monte, Introduction to classical gauge field theory and to Batalin-Vilkovisky quantization, arXiv:1810.00442, 2018.
[10] I.A. Batalin, E.S. Fradkin, A generalized canonical formalism and quantization of reducible gauge theories, Phys. Lett. B 122 (2) (March 1983) 157–164, https://doi.org/10.1016/0370-2693(83)90784-0.
[11] I.A. Batalin, G.A. Vilkovisky, Relativistic S-matrix of dynamical systems with boson and fermion constraints, Phys. Lett. B 69 (3) (Aug 1977) 309–312, https://doi.org/10.1016/0370-2693(77)90553-6.
[12] I.A. Batalin, G.A. Vilkovisky, Gauge algebra and quantization, Phys. Lett. B 102 (1) (June 1981) 27–31, https://doi.org/10.1016/0370-2693(81)90205-7.
[13] I.A. Batalin, G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, Phys. Rev. D 28 (10) (November 1983) 2567–2582, https://doi.org/10.1103/physrevd.28.2567.
[14] S. Bates, A. Weinstein, Lectures on the Geometry of Quantization, University Reprints, 2012.
[15] A. Beilinson, V. Drinfeld, Chiral Algebras, Amer. Math. Society, Colloquium Publications, vol. 51, 2004.
[16] G. Borisov, A. Sheshmani, S.-T. Yau, Global shifted potentials for moduli stacks of sheaves on Calabi-Yau four-folds I (derived schemes), arXiv:1908.00651, 2019.
[17] D. Calaque, T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, Shifted Poisson structures and deformation quantization, J. Topol. 10 (2) (2017) 483–584, https://doi.org/10.1112/topo.12012.
[18] D. Calaque, T. Willwacher, Triviality of the higher formality theorem, Proc. Am. Math. Soc. 143 (2015) 5181–5193, https://doi.org/10.1090/proc/12670.
[19] N. Capacci, Ö. Tetik, in preparation.
[20] A.S. Cattaneo, I. Contreras, Split canonical relations, arXiv:1811.10107, 2018.
[21] A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591–611, https://doi.org/10.1007/s002200000229, arXiv:math/9902090.
[22] A.S. Cattaneo, G. Felder, On the AKSZ formulation of the Poisson sigma model, Lett. Math. Phys. 56 (2) (2001) 163–179, https://doi.org/10.1023/a:1010963926853.
[23] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Classical BV theories on manifolds with boundary, Commun. Math. Phys. 332 (2) (August 2014) 535–603, https://doi.org/10.1007/s00220-014-2145-3, arXiv:1201.0290.
[24] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Perturbative quantum gauge theories on manifolds with boundary, Commun. Math. Phys. 357 (2) (Dec 2017) 631–730, https://doi.org/10.1007/s00220-017-3031-6.
[25] A.S. Cattaneo, P. Mnev, K. Wernli, Split Chern–Simons theory in the BV-BFV formalism, in: Quantization, Geometry and Noncommutative Structures in Mathematics and Physics, Springer International Publishing, 2017, pp. 293–324.
[26] A.S. Cattaneo, N. Moshayedi, Introduction to the BV-BFV formalism, Rev. Math. Phys. 32 (2020) 67, https://doi.org/10.1142/S0129055X2030006X.
[27] A.S. Cattaneo, N. Moshayedi, K. Wernli, Globalization for perturbative quantization of nonlinear split AKSZ sigma models on manifolds with boundary, Commun. Math. Phys. 372 (1) (2019) 213–260, https://doi.org/10.1007/s00220-019-03591-5.
[28] A.S. Cattaneo, J. Qiu, M. Zabzine, 2D and 3D topological field theories for generalized complex geometry, Adv. Theor. Math. Phys. 14 (2) (2010) 695–725, https://doi.org/10.4310/ATMP.2010.v14.n2.a9.
[29] K. Costello, O. Gwilliam, Factorization algebras in quantum field theory, vol. 2, https://people.math.umass.edu/~gwilliam/, 2016.
[30] V. Drinfeld, DG quotients of DG categories, J. Algebra 272 (2) (2004) 643–691, https://doi.org/10.1016/j.jalgebra.2003.05.001.
[31] K. Eftekharinasab, On the generalization of the Darboux theorem, Proc. Int. Geom. Center 12 (2) (2019) 1–10, https://doi.org/10.15673/tmgc.v12i2.1436.
[32] S. Eilenberg, N.E. Steenrod, Axiomatic approach to homology theory, Proc. Natl. Acad. Sci. 31 (4) (1945) 117–120, https://doi.org/10.1073/pnas.31.4.117.
[33] E.S. Fradkin, G.A. Vilkovisky, Quantization of relativistic systems with constraints, Phys. Lett. B 55 (2) (Feb 1975) 224–226, https://doi.org/10.1016/0370-2693(75)90448-7.
[34] J. Francis, The tangent complex and Hochschild cohomology of $\mathcal{E}_n$-rings, Compos. Math. 149 (3) (2012) 430–480, https://doi.org/10.1112/S0010437X12000140.
[35] B. Fresse, T. Willwacher, The intrinsic formality of E_n-operads, J. Eur. Math. Soc. 22 (7) (2020) 2047–2133, https://doi.org/10.4171/JEMS/961.
[36] W. Fulton, R. MacPherson, A compactification of configuration spaces, Ann. Math. (2) 139 (1) (January 1994) 183–225, https://doi.org/10.2307/2946631.
[37] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories, Commun. Math. Phys. 159 (2) (1994) 265–285, https://doi.org/10.1007/BF02102639.

[38] N. Idrissi, Formality of a higher-codimensional Swiss-cheese operad, arXiv:1809.07667, 2018.
[39] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, Ann. Phys. 235 (2) (Nov 1994) 435–464, https://doi.org/10.1006/aphy.1994.1104.
[40] N. Ikeda, Donaldson invariants and their generalizations from AKSZ topological field theories, arXiv:1104.2100, 2011.
[41] R. Iraso, P. Mnev, Two-dimensional Yang-Mills theory on surfaces with corners in Batalin-Vilkovisky formalism, Commun. Math. Phys. 370 (2019) 637–702, https://doi.org/10.1007/s00220-019-03392-w.
[42] H.M. Khudaverdian, Semidensities on odd symplectic supermanifolds, Commun. Math. Phys. 247 (2) (May 2004) 353–390, https://doi.org/10.1007/s00220-004-1083-x.
[43] A. Kirillov, Geometric quantization, in: Dynamical Systems - 4, vol. 4, 1985, pp. 141–176.
[44] M. Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys. 48 (1999) 35–72, https://doi.org/10.1023/A:1007555725247.
[45] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture, in: Proceedings, Conference Moshe Flato: Quantization, Deformations, and Symmetries, I/II, 2000, pp. 255–308, arXiv:math/0001151 [math.QA].
[46] M. Livernet, Non-formality of the Swiss-cheese operad, J. Topol. 8 (4) (2015) 1156–1166, https://doi.org/10.1112/jtopol/jtv018.
[47] M. Livernet, F. Patras, Lie theory for Hopf operads, J. Algebra 319 (12) (2008) 4899–4920, https://doi.org/10.1016/j.jalgebra.2008.03.016.
[48] J. Lurie, On the classification of topological field theories, Curr. Dev. Math. 2008 (2009) 129–280.
[49] J. Lurie, Higher algebra, available at https://www.math.ias.edu/~lurie/, 2017.
[50] N. Markarian, Weyl n-algebras and the Swiss cheese operad, arXiv:2006.09709, 2020.
[51] J.E. McClure, J.H. Smith, A solution of Deligne's conjecture, arXiv:math/9910126, 1999.
[52] V. Melani, P. Safronov, Derived coisotropic structures I: affine case, Sel. Math. 24 (2018) 3061–3118, https://doi.org/10.1007/s00029-018-0406-2.
[53] V. Melani, P. Safronov, Derived coisotropic structures II: stacks and quantization, Sel. Math. 24 (2018) 3119–3173, https://doi.org/10.1007/s00029-018-0407-1.
[54] N. Moshayedi, Notes on geometric quantization, arXiv:2010.15419, 2020.
[55] T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, Shifted symplectic structures, Publ. Math. IHÉS 117 (2013) 271–328, https://doi.org/10.1007/s10240-013-0054-1.
[56] J. Qiu, M. Zabzine, On the AKSZ formulation of the Rozansky-Witten theory and beyond, J. High Energy Phys. 2009 (13) (2009), https://doi.org/10.1088/1126-6708/2009/09/024.
[57] N. Reshetikhin, Lectures on quantization of gauge systems, in: New Paths Towards Quantum Gravity, Springer Berlin Heidelberg, 2010, pp. 125–190.
[58] D. Roytenberg, AKSZ–BV formalism and Courant algebroid-induced topological field theories, Lett. Math. Phys. 79 (2007) 143–159, https://doi.org/10.1007/s11005-006-0134-y.
[59] L. Rozansky, E. Witten, Hyper-Kähler geometry and invariants of three-manifolds, Sel. Math. 3 (1997) 401–458.
[60] P. Safronov, Poisson reduction as a coisotropic intersection, Higher Struct. 1 (1) (2017) 87–121.
[61] P. Safronov, Braces and Poisson additivity, Compos. Math. 154 (8) (2018) 1698–1745, https://doi.org/10.1112/S0010437X18007212.
[62] P. Safronov, Shifted geometric quantization, arXiv:2011.05730, 2020.
[63] P. Schaller, T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 09 (33) (Oct 1994) 3129–3136, https://doi.org/10.1142/s0217732394002951.
[64] A. Schwarz, Geometry of Batalin-Vilkovisky quantization, Commun. Math. Phys. 155 (2) (Jul 1993) 249–260, https://doi.org/10.1007/bf02097392.
[65] G.B. Segal, The definition of conformal field theory, in: Differential Geometrical Methods in Theoretical Physics, vol. 250, 1988, pp. 165–171.
[66] J. Stasheff, Homotopy associativity of H-spaces I, Trans. Am. Math. Soc. 108 (2) (1963) 275–292, https://doi.org/10.2307/1993608.
[67] J. Stasheff, Homotopy associativity of H-spaces II, Trans. Am. Math. Soc. 108 (2) (1963) 293–312, https://doi.org/10.2307/1993609.
[68] D.E. Tamarkin, Another proof of M. Kontsevich formality theorem, arXiv:math/9803025, 1998.
[69] D.E. Tamarkin, Formality of chain operad of little discs, Lett. Math. Phys. 66 (2003) 65–72, https://doi.org/10.1023/B:MATH.0000017651.12703.a1.
[70] H.L. Tanaka, Lectures on Factorization Homology, ∞-Categories, and Topological Field Theories (with contributions from A. Amabel, A. Kalmykov, and L. Müller), SpringerBriefs in Mathematical Physics, vol. 39, Springer International Publishing, 2020.
[71] J. Thomas, Kontsevich's Swiss cheese conjecture, Geom. Topol. 20 (2016) 1–48, https://doi.org/10.2140/gt.2016.20.1.
[72] B. Toën, Lectures on DG-categories, in: Topics in Algebraic and Topological K-Theory, vol. 2008, 2011, pp. 243–302.
[73] B. Toën, Derived algebraic geometry, arXiv:1401.1044, 2014.
[74] B. Toën, G. Vezzosi, Algebraic foliations and derived geometry: the Riemann-Hilbert correspondence, arXiv:2001.05450, 2020.
[75] A.A. Voronov, The Swiss-cheese operad, in: Homotopy Invariant Algebraic Structures, Baltimore, MD, 1998, Amer. Math. Soc., 1999, pp. 365–373.
[76] P. Ševera, On the origin of the BV operator on odd symplectic supermanifolds, Lett. Math. Phys. 78 (1) (Jul 2006) 55–59, https://doi.org/10.1007/s11005-006-0097-z.
[77] E. Witten, Topological sigma models, Commun. Math. Phys. 118 (3) (1988) 411–449, http://projecteuclid.org/euclid.cmp/1104162092.
[78] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (3) (1989) 351–399, http://projecteuclid.org/euclid.cmp/1104178138.
[79] N. Woodhouse, Geometric Quantization, Oxford University Press (OUP), 1997.

Chapter 5

Formal Global AKSZ Gauge Observables and Generalized Wilson Surfaces

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Formal Global AKSZ Gauge Observables and Generalized Wilson Surfaces

Nima Moshayedi

Abstract. We consider a construction of observables by using methods of supersymmetric field theories. In particular, we give an extension of AKSZ-type observables constructed in Mnev (Lett Math Phys 105:1735–1783, 2015) using the Batalin–Vilkovisky structure of AKSZ theories to a formal global version with methods of formal geometry. We will consider the case where the AKSZ theory is "split" which will give an explicit construction for formal vector fields on base and fiber within the formal global action. Moreover, we consider the example of formal global generalized Wilson surface observables whose expectation values are invariants of higher-dimensional knots by using BF field theory. These constructions give rise to interesting global gauge conditions such as the differential quantum master equation and further extensions.

Contents

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Birkhäuser

1. Introduction

Observables play a fundamental role in theoretical and mathematical physics. They are used in several constructions, e.g., deformation quantization and factorization algebras. In [46], a method for constructing observables in the setting of AKSZ theories was introduced, where several examples, including Wilson-loop-type observables for different theories, have been addressed.

These constructions were given using the approach of supersymmetric field theory and methods of functional integrals. In particular, the focus lies within a special formalism dealing with gauge theories which is called the Batalin–Vilkovisky (BV) formalism. This formalism was developed by Batalin and Vilkovisky in a series of papers [5,7,8] during the 1970s and 1980s in order to deal with the functional integral quantization approach where the Lagrangian is invariant under certain symmetries and the integral is ill-defined. They have shown (later also formulated in a more mathematical language by Schwarz) that these issues can be resolved by replacing the ill-defined integral by a well-defined (after some regularization is also introduced) one without changing the final value. The mathematical structures of this powerful formalism have been studied since then by many different people.

AKSZ theories [1] (named after Alexandrov, Kontsevich, Schwarz and Zaboronsky) are a particular type of field theories where the space of fields is given by a mapping space between manifolds. It can be shown that these theories, regarded in a special setting, will give rise to field theories as formulated in the BV setting. Many interesting theories are in fact of AKSZ-type, e.g., Chern–Simons theory [3,4,18,22,58], the Poisson sigma model [16,35,51], Rozansky–Witten theory [50], the Courant sigma model [49], BF theory [20, 47], Witten's A- and B-twisted sigma models [59] and $2D$ Yang–Mills theory [36].

The globalization idea originates from a field-theoretic approach to globalization of Kontsevich's star product [39] in deformation quantization. The associated field theory is given by the Poisson sigma model. The Poisson sigma model is a two-dimensional bosonic string theory with target a Poisson manifold which was first considered by Ikeda [35] and Schaller–Strobl [51] by the attempt of studying 2D gravity theories and combine them to a common form with Yang–Mills theories. Using the Poisson sigma model on the disk, Cattaneo and Felder have proven that Kontsevich's star product is exactly given by the perturbative expansion of its functional integral quantization [15]. Regarding the fact that the Poisson sigma model is a gauge theory, it is interesting to note that it is a fundamental non-trivial theory where the BRST gauge formalism [9–11,56] does not work if the Poisson structure is not linear. In fact, to treat the Poisson sigma model and its quantization, one has to use the BV formalism. However, the field-theoretic construction of Kontsevich's star product was only considered locally since Kontsevich's formula was only given for the local picture on the upper half-plane. Later on, using techniques of formal geometry, developed by, e.g., Gelfand–Fuks [32,33], Gelfand–Kazhdan [34] or Bott [13], it was possible to construct a globalization, similar to the approach of Fedosov for symplectic manifolds which only covers the case of constant (symplectic) Poisson structures [30].

In [12,17], this approach was first extended to the field theoretic BV construction of the Poisson sigma model for closed source manifolds. In recent work [25] this construction was extended to the case of source manifolds with boundary. There one has to extend the BV formalism to the *BV-BFV formalism* which couples the boundary BFV theory to the bulk BV theory such that everything is consistent in the cohomological formalism. Here BFV stands for Batalin–Fradkin–Vilkovisky which formulated a Hamiltonian version of the BV construction in [6,31]. The bulk-boundary coupling (the BV-BFV formalism) was first introduced classically in [19,20] and extended to the quantum version in [21]. The globalization construction for the Poisson sigma model on manifolds with boundary was more generally extended in [24] to a special class of AKSZ theories which are called "split" where the case of the Poisson sigma model is an example.

The aim of this paper is to extend the constructions of [46] to a formal global construction. In fact, we will construct formal global observables by using the notion of a Hamiltonian Q-bundle [41] together with notions of formal geometry, and we will study the formal global extension of Wilson loop type observables for the Poisson sigma model.

Additionally, we discuss the formal global extension of Wilson surface observables which have been studied in [27] by using the AKSZ formulation of BF theories. We will show that these constructions lead to interesting gauge conditions such as the *differential quantum master equation* (and further extensions).

These constructions are expected to extend to manifolds with boundary by using the BV-BFV formalism as the globalization constructions have been studied for nonlinear split AKSZ theories on manifolds with boundary [24].

2. The Batalin–Vilkovisky (BV) Formalism

In this section, we will recall some aspects of the Batalin–Vilkovisky formalism as in [21,48]. An introductory reference for learning about the formalism is [23], which also covers the most important concepts of supergeometry and the case of manifolds with boundary (BV-BFV).

2.1. Classical BV Picture

Let us start with the classical setting of the BV formalism.

Definition 2.1 (*BV manifold*). A *BV manifold* is a triple

$$(\mathcal{F}, \mathcal{S}, \omega)$$

such that $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold[1], $\mathcal{S}$ is an even function on $\mathcal{F}$ of degree 0 and ω is an odd symplectic form on $\mathcal{F}$ of degree -1. Moreover, we want that $\mathcal{S}$ satisfies the Classical Master Equation (CME)

$$\boxed{\{\mathcal{S}, \mathcal{S}\}_\omega = 0,} \tag{1}$$

where $\{\ ,\ \}_\omega$ denotes the odd Poisson bracket induced by the odd symplectic form ω. This odd Poisson bracket is also called *BV bracket* and, according to Batalin and Vilkovisky, is often denoted by round brackets $(\ ,\ )$. We will call $\mathcal{F}$ the *BV space of fields*[2], $\mathcal{S}$ the *BV action* (sometimes also called the *master action*) and ω the *BV symplectic form.*

Remark 2.2. In physics, the $\mathbb{Z}$-grading is called the *ghost number*. We will denote the ghost number by gh and the form degree by deg.

Remark 2.3. The data of a BV manifold induces a *symplectic cohomological* vector field Q of degree $+1$ which is given by the Hamiltonian vector field of $\mathcal{S}$, i.e.,

$$\iota_Q \omega = \delta \mathcal{S}, \tag{2}$$

[1] Typically, this is an infinite-dimensional manifold. However, there are certain cases where this is a finite-dimensional manifold, e.g., if we consider the moduli of flat connections on a compact, oriented 2-manifold with holonomies on the boundary according to Atiyah and Bott [2] which is of importance regarding BF theory.

[2] Usually, the BV space of fields is given by the (-1)-shifted cotangent bundle of the BRST space of fields, i.e., $\mathcal{F}_{\mathrm{BV}} = T^*[-1]\mathcal{F}_{\mathrm{BRST}}$.

wher δ denotes the de Rham differential on $\mathcal{F}$. The cohomological property means that $[Q, Q] = 0$ and the symplectic property means $L_Q\omega = 0$, where L denotes the Lie derivative. Moreover, note that by definition

$$Q = \{\mathcal{S}, \ \}_\omega.$$

Definition 2.4 (*Exact BV manifold*). A BV manifold is called *exact* if $\omega = \delta\alpha$ for some primitive 1-form α.

In what will follow, we will mostly consider exact BV manifolds. According to the use of sigma models we want to consider space-time manifolds as the source manifolds for our theory. Moreover, in this paper, we will restrict ourself to topological theories.

Definition 2.5 (*BV theory*). A *BV theory* is an assignment of a manifold Σ to a BV manifold

$$\Sigma \mapsto (\mathcal{F}_\Sigma, \mathcal{S}_\Sigma, \omega_\Sigma, Q_\Sigma). \tag{3}$$

2.2. Quantum BV Picture

We continue with the quantum setting of the BV formalism.

Definition 2.6 (*Quantum BV manifold*). A *quantum BV manifold* is a quadruple $(\mathcal{F}, \omega, \mu, \mathcal{S})$ such that $\mathcal{F}$ is a $\mathbb{Z}$-graded supermanifold, ω a symplectic form on $\mathcal{F}$ of degree -1, μ a volume element[3] of $\mathcal{F}$ which is compatible with ω in the sense that the associated BV Laplacian

$$\Delta\colon f \mapsto \frac{1}{2}\operatorname{div}_\mu\{f, \ \}_\omega \tag{4}$$

satisfies

$$\Delta^2 = 0, \tag{5}$$

and $\mathcal{S}$ is a degree 0 function on $\mathcal{F}$ such that it satisfies the QME (8).

Remark 2.7. The BV Laplacian satisfies a generalized BV Leibniz rule. For two functions f, g on $\mathcal{F}$, we have

$$\Delta(fg) = \Delta(f)g \pm f\Delta(g) \pm \{f, g\}_\omega.$$

see also [38, 53] for a mathematical exposure to the origin of the BV Laplacian.

Moreover, define δ_{BV} to be the degree $+1$ operator given by

$$\delta_{\mathrm{BV}} := Q - \mathrm{i}\hbar\Delta \tag{6}$$

which satisfies

$$\delta_{\mathrm{BV}}^2 = 0. \tag{7}$$

The following theorem is one of the main statements in the formalism developed by Batalin and Vilkovisky. In its present form, it was stated by Schwarz on general manifolds [52].

Theorem 2.8 (Batalin–Vilkovisky). *For any half-density f on $\mathcal{F}$, we have:*

---

[3] We want the space of fields $\mathcal{F}$ to be endowed with a natural measure.

(1) *If* $f = \Delta g$, *then*

$$\int_{\mathcal{L}} f = 0,$$

for a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$,

(2) *If* $\Delta f = 0$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_t} f = 0,$$

for any continuous family $(\mathcal{L}_t)$ *of Lagrangian submanifolds of* $\mathcal{F}$.

Remark 2.9. The choice of Lagrangian submanifold is in fact equivalent to fixing a gauge. The second part of Theorem 2.8 tells us that if we have an integral over a Lagrangian submanifold which is ill-defined, but on the other hand $\Delta f = 0$, then we can deform the Lagrangian submanifold $\mathcal{L}$ continuously to a Lagrangian submanifold $\mathcal{L}'$ (choosing a different gauge) where the integral is well-defined. In application to quantum field theory, we have $f = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}$. Hence, for gauge independence, we need to impose

$$\boxed{\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}} = 0 \Longleftrightarrow \{\mathcal{S}, \mathcal{S}\}_\omega - 2\mathrm{i}\hbar \Delta \mathcal{S} = 0.} \tag{8}$$

The condition (8) is called the *Quantum Master Equation (QME)*. If we let $\mathcal{S}$ depend on $\hbar$, we can see that in order zero we get the CME $\{\mathcal{S}, \mathcal{S}\}_\omega = 0$. One can then solve (8) order by order.

2.3. L_∞-Structure

Recall that a Q-manifold with trivial body induces an L_∞-algebra structure (see, e.g., [45]). More generally, a Q-manifold with non-trivial body induces an L_∞-algebroid structure. Similarly, a BV manifold endowed with its Q-structure induces an L_∞-algebra structure on $\mathcal{F}$ [55]. This L_∞-algebra encodes all the relevant classical information of the field theory. Hence, at the classical level, Lagrangian field theories can be equivalently described in terms of the underlying (cyclic[4]) L_∞-algebra structure. Moreover, equivalent theories induce quasi-isomorphic L_∞-algebras. The unary operation ℓ_1 is in fact encoded in the linear part of the action $Q = \{\mathcal{S}, \ \}_\omega$ on the field corresponding to the image of ℓ_1. The higher brackets then make the linearized expressions covariant and to allow for higher interaction terms. The operator δ_{BV} in fact induces a quantum L_∞-algebra (or loop homotopy algebra) on the same graded space. In particular, by a direct application of the homological perturbation lemma, one can prove a similar decomposition theorem and compute its *minimal model* as for the classical case, which leads directly to a homotopy between a quantum L_∞-algebra and its minimal model in which the non-triviality of the action

[4] A *cyclic* L_∞-algebra [40] is an L_∞-algebra $\mathfrak{g}$ endowed with a non-degenerate, symmetric, bilinear pairing $\langle\ ,\ \rangle_{\mathfrak{g}} \colon \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{R}$ such that

$$\begin{aligned}\langle X_1, \ell_{n+1}(X_2, \dots, X_{n+1})\rangle_{\mathfrak{g}} &= (-1)^{n+n(\deg(X_1)+\deg(X_{n+1}))+\deg(X_{n+1})\sum_{j=1}^{n}\deg(X_j)} \\ &\qquad \langle X_{n+1}, \ell_n(X_1, \dots, X_n)\rangle_{\mathfrak{g}},\end{aligned}$$

for $X_1, \dots, X_{n+1} \in \mathfrak{g}$ and where (ℓ_n) denote the n-ary brackets on $\mathfrak{g}$. In the case of a Q-manifold the cyclic inner product corresponds to a symplectic structure.

is fully absorbed in the higher brackets. Moreover, the homotopy Maurer–Cartan theory[5] implies that for an arbitrary L_∞-algebra the BV complex of fields, ghosts and anti fields is just the L_∞-algebra itself. See, e.g., [37,54,55] for a more detailed discussion of L_∞-structures for BV field theories.

3. AKSZ Theories

3.1. Preliminaries

In [1], Alexandrov, Kontsevich, Schwarz, and Zaboronsky have proposed a class of local field theories which are compatibel with the Batalin–Vilkovisky gauge formalism construction, in the sense that the constructed local actions are solutions to the Classical Master Equation. Hence, these theories give a subclass of BV theories. In this section we want to recall the most important notions of AKSZ sigma models. We start with defining the ingredients.

Definition 3.1 (*Differential graded symplectic manifold*). A *differential graded symplectic manifold* of degree k is a triple

$$(\mathcal{M}, \Theta_\mathcal{M}, \omega_\mathcal{M} = \mathrm{d}_\mathcal{M}\alpha_\mathcal{M})$$

such that $\mathcal{M}$ is a $\mathbb{Z}$-graded manifold, $\Theta_\mathcal{M} \in C^\infty(\mathcal{M})$ is a function on $\mathcal{M}$ of degree $k+1$, and $\omega_\mathcal{M} \in \Omega^2(\mathcal{M})$ is an exact symplectic form of degree k with primitive 1-form $\alpha_\mathcal{M} \in \Omega^1(\mathcal{M})$, such that

$$\{\Theta_\mathcal{M}, \Theta_\mathcal{M}\}_{\omega_\mathcal{M}} = 0, \tag{9}$$

where $\{\ ,\ \}_{\omega_\mathcal{M}}$ is the odd Poisson bracket induced by $\omega_\mathcal{M}$. We have denoted by $\mathrm{d}_\mathcal{M}$ the de Rham differential on $\mathcal{M}$.

Remark 3.2. We denote by $Q_\mathcal{M} \in \mathfrak{X}(\mathcal{M})$ the Hamiltonian vector field of $\Theta_\mathcal{M}$, defined by the equation

$$\iota_{Q_\mathcal{M}}\omega_\mathcal{M} = \mathrm{d}_\mathcal{M}\Theta_\mathcal{M}$$

with the properties $[Q_\mathcal{M}, Q_\mathcal{M}] = 0$ (cohomological) and $L_{Q_\mathcal{M}}\omega_\mathcal{M} = 0$ (symplectic). Note that $Q_\mathcal{M}$ is of degree $+1$. A quadruple $(\mathcal{M}, Q_\mathcal{M}, \Theta_\mathcal{M}, \omega_\mathcal{M} = \mathrm{d}_\mathcal{M}\alpha_\mathcal{M})$ as in Definition 3.1 is also called a *Hamiltonian Q-manifold*.

[5] This is the theory induced by the action term $\mathcal{S}_{\mathrm{MC}}(\Psi) = \sum_{j\geq 1} \frac{1}{(j+1)!}\langle \Psi, \ell_j(\Psi,\dots,\Psi)\rangle_\mathfrak{g}$ for a cyclic L_∞-algebra $\mathfrak{g}$ endowed with an inner product $\langle\ ,\ \rangle_\mathfrak{g}$. Here (ℓ_j) denotes the family of j-ary brackets on $\mathfrak{g}$. The stationary locus of this action is given by solutions of the homotopy Maurer–Cartan equation $\sum_{j\geq 1}\frac{1}{j!}\ell_j(\Psi,\dots,\Psi) = 0$. In fact, the deformed Lagrangian (still classical)

$$\mathcal{S}(\Psi) = \frac{1}{2}\langle \Psi, Q(\Psi)\rangle_\mathfrak{g} + \sum_{j\geq 1}\frac{1}{(j+1)!}\langle \Psi, \ell_j(\Psi,\dots,\Psi)\rangle_\mathfrak{g}$$

satisfies the CME.

3.2. AKSZ Sigma Models

Let Σ_d be a d-dimensional compact, oriented manifold (possibly with boundary) and consider its shifted tangent bundle $T[1]\Sigma_d$. Moreover, fix a Hamiltonian Q-manifold

$$(\mathcal{M}, Q_\mathcal{M}, \Theta_\mathcal{M}, \omega_\mathcal{M} = \mathrm{d}_\mathcal{M}\alpha_\mathcal{M})$$

of degree $d-1$ for $d \geq 0$. We can consider the mapping space of graded manifolds from $T[1]\Sigma_d$ to $\mathcal{M}$ to be our space of fields:

$$\mathcal{F}^{\mathcal{M}}_{\Sigma_d} := \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, \mathcal{M}), \tag{10}$$

where $\mathrm{Map}_{\mathrm{GrMnf}}$ denotes the mapping space between graded manifolds.[6] We would like to endow $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ with a Q-manifold structure. This can be done by considering the lifts of the de Rham differential d_{Σ_d} on Σ_d and the cohomological vector field $Q_\mathcal{M}$ on the target $\mathcal{M}$ to the mapping space. Hence, we get a cohomological vector field

$$Q_{\Sigma_d} := \widehat{\mathrm{d}}_{\Sigma_d} + \widehat{Q}_\mathcal{M} \in \mathfrak{X}\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right), \tag{11}$$

where $\widehat{\mathrm{d}}_{\Sigma_d}$ and $\widehat{Q}_\mathcal{M}$ denote the corresponding lifts to the mapping space. Note that we can regard d_{Σ_d} as a cohomological vector field on $T[1]\Sigma_d$. Consider the following push-pull diagram

$$\mathcal{F}^{\mathcal{M}}_{\Sigma_d} \xleftarrow{\mathrm{p}} \mathcal{F}^{\mathcal{M}}_{\Sigma_d} \times T[1]\Sigma_d \xrightarrow{\mathrm{ev}} \mathcal{M}, \tag{12}$$

where p denotes the projection onto $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ and ev is the evaluation map. We can construct a *transgression* map

$$\mathscr{T}_{\Sigma_d} := \mathrm{p}_*\mathrm{ev}^* \colon \Omega^\bullet(\mathcal{M}) \to \Omega^\bullet\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right). \tag{13}$$

Note that the map p_* is given by fiber integration on $T[1]\Sigma_d$. Now we can endow the space of fields $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ with a symplectic structure ω_{Σ_d} by setting

$$\omega_{\Sigma_d} := (-1)^d \mathscr{T}_{\Sigma_d}(\omega_\mathcal{M}) \in \Omega^2\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right). \tag{14}$$

Moreover, we will get a solution $\mathcal{S}_{\Sigma_d}$ to the CME, the BV action functional, by

$$\mathcal{S}_{\Sigma_d} := \underbrace{\iota_{\widehat{\mathrm{d}}_{\Sigma_d}} \mathscr{T}_{\Sigma_d}(\alpha_\mathcal{M})}_{=:\mathcal{S}^{\mathrm{kin}}_{\Sigma_d}} + \underbrace{\mathscr{T}_{\Sigma_d}(\Theta_\mathcal{M})}_{=:\mathcal{S}^{\mathrm{target}}_{\Sigma_d}} \in C^\infty\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right). \tag{15}$$

Indeed, one can check that

$$\{\mathcal{S}_{\Sigma_d}, \mathcal{S}_{\Sigma_d}\}_{\omega_{\Sigma_d}} = 0. \tag{16}$$

Note that the symplectic form ω_{Σ_d} is of degree $(d-1)-d = -1$ as expected. Moreover, the action $\mathcal{S}_{\Sigma_d}$ is of degree 0. Thus this setting does indeed induce a BV manifold $\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}, \mathcal{S}_{\Sigma_d}, \omega_{\Sigma_d}\right)$. Consider local coordinates (x^μ) on $\mathcal{M}$ and let (u^i) be local coordinates on Σ_d for $1 \leq i \leq d$. Denote the odd

[6] More precisely, $\mathrm{Map}_{\mathrm{GrMnf}}$ denotes the right adjoint functor to the Cartesian product in the category of graded manifolds with a fixed factor. On objects X, Y, Z we have $\mathrm{Hom}(X, \mathrm{Map}_{\mathrm{GrMnf}}(Y, Z)) = \mathrm{Hom}(X \times Y, Z)$, where Hom denotes the set of graded manifold morphisms.

fiber coordinates of degree +1 on $T[1]\Sigma_d$ by $\theta^i = \mathrm{d}_{\Sigma_d} u^i$. Then, for a field $\mathcal{A} \in \mathcal{F}^{\mathcal{M}}_{\Sigma_d}$, we have the local expression

$$\mathcal{A}^\mu(u,\theta) = \sum_{\ell=0}^{d} \underbrace{\sum_{1\leq i_1<\cdots<i_\ell\leq d} \mathcal{A}^\mu_{i_1\ldots i_\ell}(u)\theta^{i_1}\wedge\cdots\wedge\theta^{i_\ell}}_{\mathcal{A}^\mu_{(\ell)}(u,\theta)}$$

$$\in \bigoplus_{\ell=0}^{d} C^\infty(\Sigma_d)\otimes\bigwedge^{\ell} T^*\Sigma_d. \tag{17}$$

The functions $\mathcal{A}^\mu_{i_1\ldots i_\ell} \in C^\infty(\Sigma_d)$ are of degree $\deg(x^\mu)-\ell$ on $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$. The local expression of the symplectic form $\omega_{\mathcal{M}}$ and its primitive 1-form $\alpha_{\mathcal{M}}$ on $\mathcal{M}$ are given by

$$\alpha_{\mathcal{M}} = \alpha_\mu(x)\mathrm{d}_{\mathcal{M}}x^\mu \in \Omega^1(\mathcal{M}), \tag{18}$$

$$\omega_{\mathcal{M}} = \frac{1}{2}\omega_{\mu_1\mu_2}(x)\mathrm{d}_{\mathcal{M}}x^{\mu_1}\wedge\mathrm{d}_{\mathcal{M}}x^{\mu_2} \in \Omega^2(\mathcal{M}). \tag{19}$$

Locally, using the expressions above, we get the following expression for the BV symplectic form, its primitive 1-form and the BV action functional:

$$\alpha_{\Sigma_d} = \int_{\Sigma_d}\alpha_\mu(\mathcal{A})\delta\mathcal{A}^\mu \in \Omega^1\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right), \tag{20}$$

$$\omega_{\Sigma_d} = (-1)^d\frac{1}{2}\int_{\Sigma_d}\omega_{\mu_1\mu_2}(\mathcal{A})\delta\mathcal{A}^{\mu_1}\wedge\delta\mathcal{A}^{\mu_2} \in \Omega^2\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right), \tag{21}$$

$$\mathcal{S}_{\Sigma_d} = \int_{\Sigma_d}\alpha_\mu(\mathcal{A})\mathrm{d}_{\Sigma_d}\mathcal{A}^\mu + \int_{\Sigma_d}\Theta_{\mathcal{M}}(\mathcal{A}) \in C^\infty\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right). \tag{22}$$

Note that we have denoted by δ the de Rham differential on $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$. If we consider Darboux coordinates on $\mathcal{M}$, we get that

$$\omega_{\mathcal{M}} = \frac{1}{2}\omega_{\mu_1\mu_2}\mathrm{d}_{\mathcal{M}}x^{\mu_1}\wedge\mathrm{d}_{\mathcal{M}}x^{\mu_2},$$

where the $\omega_{\mu_1\mu_2}$ are constant implying that $\alpha_{\mathcal{M}} = \frac{1}{2}x^{\mu_1}\omega_{\mu_1\mu_2}\mathrm{d}_{\mathcal{M}}x^{\mu_2}$. Hence we get the BV symplectic form

$$\begin{aligned}\omega_{\Sigma_d} &= \frac{1}{2}\int_{T[1]\Sigma_d}\mu_{\Sigma_d}\left(\omega_{\mu_1\mu_2}\delta\mathcal{A}^{\mu_1}\wedge\delta\mathcal{A}^{\mu_2}\right)\\ &= \frac{1}{2}\int_{\Sigma_d}\left(\omega_{\mu_1\mu_2}\delta\mathcal{A}^{\mu_1}\wedge\delta\mathcal{A}^{\mu_2}\right)^{\mathrm{top}}\end{aligned} \tag{23}$$

and the master action

$$\mathcal{S}_{\Sigma_d} = \int_{T[1]\Sigma_d}\mu_{\Sigma_d}\left(\frac{1}{2}\mathcal{A}^\mu\omega_{\mu_1\mu_2}\boldsymbol{D}_{\Sigma_d}\mathcal{A}^{\mu_2}\right) + (-1)^d\int_{T[1]\Sigma_d}\mu_{\Sigma_d}\mathcal{A}^*\Theta_{\mathcal{M}}, \tag{24}$$

where μ_{Σ_d} is a canonical measure on $T[1]\Sigma_d$ and $\boldsymbol{D}_{\Sigma_d} = \theta^j\frac{\partial}{\partial u_j}$ the superdifferential on $T[1]\Sigma_d$.

4. Hamiltonian Q-Bundles

We want to construct a combination of the notion of Q-manifolds and the concept of Hamiltonian vector fields together with the notion of vector bundles, where we want to extend most of our constructions on the fiber (see also [41]). We will see that the fiber will represent the target of an AKSZ theory for an embedded source manifold when lifted to an AKSZ-BV theory. We will call the fiber theory *auxiliary*. In this section we will give the main definitions as in [46]. Let us start with the definition of the trivial case.

Definition 4.1 (*Trivial Q-bundle*). Let $\mathcal{N}$ be a graded manifold and $(\mathcal{M}, Q_\mathcal{M})$ a graded Q-manifold. A *trivial Q-bundle* is a trivial bundle

$$\pi\colon \mathcal{E} := \mathcal{M}\times\mathcal{N}\to\mathcal{M} \tag{25}$$

such that $\mathrm{d}\pi(Q_\mathcal{E}) = Q_\mathcal{M}$, where $Q_\mathcal{E}$ denotes the Q-structure on the total space $\mathcal{E}$.

Remark 4.2. Note that this implies that

$$Q_\mathcal{E} = Q_\mathcal{M} + \mathcal{V},$$

where $\mathcal{V}\in\ker\mathrm{d}\pi\cong C^\infty(\mathcal{M})\widehat{\otimes}\mathfrak{X}(\mathcal{N})$ denotes the vertical part of $Q_\mathcal{E}$. The fact that $[Q_\mathcal{E}, Q_\mathcal{E}] = 0$ can be translated to

$$\underbrace{[Q_\mathcal{M}, Q_\mathcal{M}]}_{=0} + [Q_\mathcal{M}, \mathcal{V}] + \frac{1}{2}[\mathcal{V},\mathcal{V}] = 0. \tag{26}$$

Definition 4.3 (*Trivial Hamiltonian Q-bundle*). A *trivial Hamiltonian Q-bundle* of degree $n\in\mathbb{Z}$ is a trivial Q-bundle

$$\pi\colon \mathcal{E} := \mathcal{M}\times\mathcal{N}\to\mathcal{M}$$

as in Definition 4.1 with $Q_\mathcal{E} = Q_\mathcal{M} + \mathcal{V}$ such that the fiber $\mathcal{N}$ is endowed with an exact symplectic structure $\omega_\mathcal{N} = \mathrm{d}_\mathcal{N}\alpha_\mathcal{N}\in\Omega^2(\mathcal{N})$ of degree n with $\alpha_\mathcal{N}\in\Omega^1(\mathcal{N})$ and a Hamiltonian function $\Theta_\mathcal{E}\in C^\infty(\mathcal{E})$ of degree $n+1$ satisfying

$$\mathcal{V} = \{\Theta_\mathcal{E},\ \}_{\omega_\mathcal{N}} \tag{27}$$

$$Q_\mathcal{M}(\Theta_\mathcal{E}) + \frac{1}{2}\{\Theta_\mathcal{E},\Theta_\mathcal{E}\}_{\omega_\mathcal{N}} = 0. \tag{28}$$

We can now give the definition of a general Hamiltonian Q-bundle.

Definition 4.4. (*Hamiltonian Q-bundle*) A *Hamiltonian Q-bundle* is a Q-bundle $\pi\colon\mathcal{E}\to\mathcal{M}$ where the total space $\mathcal{E}$ is endowed with a degree n exact pre-symplectic form $\omega_\mathcal{E} = \mathrm{d}_\mathcal{E}\alpha_\mathcal{E}$ such that $\ker\omega_\mathcal{E}\subset T\mathcal{E}$ is transversal to the vertical distribution $T^{\mathrm{vert}}\mathcal{E}$ and hence $\ker\omega_\mathcal{E}$ defines a flat Ehresmann connection $\nabla_{\omega_\mathcal{E}}$. Moreover, there is a Hamiltonian function $\Theta_\mathcal{E}\in C^\infty(\mathcal{E})$ with

$$\iota_{Q_\mathcal{E}}\omega_\mathcal{E} = \mathrm{d}^{\mathrm{vert}}_\mathcal{E}\Theta_\mathcal{E},$$

where $\mathrm{d}^{\mathrm{vert}}_\mathcal{E}$ denotes the vertical part of the de Rham differential on $\mathcal{E}$ as a pullback by the natural inclusion $T^{\mathrm{vert}}\mathcal{E}\hookrightarrow T\mathcal{E}$. Finally, we also want that

$$\left(Q^{\mathrm{hor}}_\mathcal{E} + \frac{1}{2}Q^{\mathrm{vert}}_\mathcal{E}\right)(\Theta_\mathcal{E}) = 0, \tag{29}$$

where we split $Q_\mathcal{E} = Q_\mathcal{E}^{\mathrm{hor}} + Q_\mathcal{E}^{\mathrm{vert}}$ into its horizontal and vertical parts by using the Ehresmann connection $\nabla_{\omega_\mathcal{E}}$ defined by $\omega_\mathcal{E}$.

5. Observables in the BV Formalism

We want to define certain classes of observables arising within the BV construction which are compatible with the structure of an underlying Q-bundle. We will start with the classical setting.

5.1. Observables for Classical BV Manifolds

Definition 5.1 (*BV classical observable*) A *classical observable* for a BV manifold $(\mathcal{F}, \mathcal{S}, Q, \omega)$ is defined as a function $\mathcal{O} \in C^\infty(\mathcal{F})$ of degree 0 such that

$$Q(\mathcal{O}) = 0. \tag{30}$$

Definition 5.2 (*Equivalence of BV classical observables*). Two BV classical observables $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are said to be equivalent if

$$\widetilde{\mathcal{O}} - \mathcal{O} = Q(\Psi), \quad \Psi \in C^\infty(\mathcal{F}), \tag{31}$$

or equivalently, $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ have the same Q-cohomology class.

Definition 5.3 (*BV classical pre-observable*). For a classical BV theory

$$(\mathcal{F}, \mathcal{S}, Q, \omega)$$

we define a *pre-observable* to be a Hamiltonian Q-bundle over $\mathcal{F}$ of degree -1. We denote the fiber by $\mathcal{F}^{\mathrm{aux}}$ and call them the *space of auxiliary fields*, which itself is endowed with a symplectic structure ω^{aux} of degree -1 and an action functional $\mathcal{S}^{\mathrm{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}^{\mathrm{aux}})$ of degree 0 such that

$$Q(\mathcal{S}^{\mathrm{aux}}) + \frac{1}{2}\{\mathcal{S}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}\}_{\omega^{\mathrm{aux}}} = 0. \tag{32}$$

Using the notions of quantum BV manifolds as in Definition 2.6, we can define a fiber auxiliary version which is compatible with the Hamiltonian Q-bundle construction as in Definition 4.4.

Definition 5.4 (*BV semi-quantum pre-observable*). For a classical BV theory $(\mathcal{F}, \mathcal{S}, Q, \omega)$ we define a *BV semi-quantum pre-observable* to be a quadruple

$$(\mathcal{F}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}, \omega^{\mathrm{aux}}, \mu^{\mathrm{aux}})$$

such that μ^{aux} is a volume form on $\mathcal{F}^{\mathrm{aux}}$ compatible with ω^{aux}, i.e., the associated BV Laplacian on $C^\infty(\mathcal{F}^{\mathrm{aux}})$ given by

$$\Delta^{\mathrm{aux}} \colon f \mapsto \frac{1}{2}\mathrm{div}_{\mu^{\mathrm{aux}}}\{f, \ \}_{\omega^{\mathrm{aux}}} \tag{33}$$

satisfies $(\Delta^{\mathrm{aux}})^2 = 0$. Moreover, the action functional $\mathcal{S}^{\mathrm{aux}}$ satisfies

$$Q(\mathcal{S}^{\mathrm{aux}}) + \frac{1}{2}\{\mathcal{S}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}\}_{\omega^{\mathrm{aux}}} - \mathrm{i}\hbar\Delta^{\mathrm{aux}}\mathcal{S}^{\mathrm{aux}} = 0, \tag{34}$$

which is equivalent to

$$\delta_{\mathrm{BV}}^{\mathrm{aux}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{aux}}} := (Q - \mathrm{i}\hbar\Delta^{\mathrm{aux}})\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{aux}}} = 0. \tag{35}$$

Remark 5.5. The name "semi-quantum" is chosen since it is not a quantum observable yet, but rather the theory whose functional integral quantization will lead to a quantum observable in the sense that it is closed with respect to the infinitesimal symmetries.

We also want to extend the notion of equivalent pre-observables to the case of semi-quantum pre-observables.

Definition 5.6 (*Equivalent BV semi-quantum pre-observables*). Two BV semi-quantum pre-observables

$$(\mathcal{F}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}, \omega^{\mathrm{aux}}, \mu^{\mathrm{aux}}) \quad \text{and} \quad (\mathcal{F}^{\mathrm{aux}}, \widetilde{\mathcal{S}}^{\mathrm{aux}}, \omega^{\mathrm{aux}}, \mu^{\mathrm{aux}})$$

are said to be *equivalent* if there exists a function $f^{\mathrm{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}^{\mathrm{aux}})$ such that

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widetilde{\mathcal{S}}^{\mathrm{aux}}} - \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{aux}}} = (Q - \mathrm{i}\hbar\Delta^{\mathrm{aux}})\left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{aux}}} f^{\mathrm{aux}}\right). \tag{36}$$

Proposition 5.7 ([27,46]). *Let* $(\mathcal{F}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}, \omega^{\mathrm{aux}}, \mu^{\mathrm{aux}})$ *be a BV semi-quantum pre-observable. Define*

$$\mathcal{O}_{\mathcal{L}} := \int_{\mathcal{L}\subset\mathcal{F}^{\mathrm{aux}}} e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{aux}}}\sqrt{\mu^{\mathrm{aux}}}|_{\mathcal{L}} \in C^\infty(\mathcal{F}), \tag{37}$$

where $\mathcal{L} \subset \mathcal{F}^{\mathrm{aux}}$ *is a Lagrangian submanifold. Then* $\mathcal{O}_{\mathcal{L}}$ *is an observable, i.e.,* $Q(\mathcal{O}_{\mathcal{L}}) = 0$. *Moreover, if for two Lagrangian submanifolds* $\mathcal{L}$ *and* $\widetilde{\mathcal{L}}$ *there exists a homotopy between them, then the observables* $\mathcal{O}_{\mathcal{L}}$ *and* $\mathcal{O}_{\widetilde{\mathcal{L}}}$ *are equivalent. Also for two equivalent BV semi-quantum pre-observables* $\mathcal{S}^{\mathrm{aux}}$ *and* $\widetilde{\mathcal{S}}^{\mathrm{aux}}$*, the corresponding observables* $\mathcal{O}_{\mathcal{L}}$ *and* $\widetilde{\mathcal{O}}_{\mathcal{L}}$ *are equivalent.*

Definition 5.8 (*Good auxiliary splitting*). We say that a semi-quantum pre-observable $(\mathcal{F}^{\mathrm{aux}}, \mathcal{S}^{\mathrm{aux}}, \omega^{\mathrm{aux}}, \mu^{\mathrm{aux}})$ has a *good splitting* if there is a decomposition

$$\mathcal{F}^{\mathrm{aux}} = \mathsf{F}^{\mathrm{aux}} \times \mathscr{F}^{\mathrm{aux}}$$

such that

$$\omega^{\mathrm{aux}} = \omega_1^{\mathrm{aux}} + \omega_2^{\mathrm{aux}}, \tag{38}$$

$$\mu^{\mathrm{aux}} = \mu_1^{\mathrm{aux}} \otimes \mu_2^{\mathrm{aux}}, \tag{39}$$

where ω_1^{aux} is a symplectic form on $\mathsf{F}^{\mathrm{aux}}$, ω_2^{aux} is a symplectic form on $\mathscr{F}^{\mathrm{aux}}$, μ_1^{aux} is a volume form on $\mathsf{F}^{\mathrm{aux}}$ and μ_2^{aux} is a volume form on $\mathscr{F}^{\mathrm{aux}}$.

Remark 5.9. This is in fact the trivial case. The general version, called *hedgehog*, is discussed in [21].

Remark 5.10. We split the auxiliary fields into *high energy modes* $\mathscr{F}^{\mathrm{aux}}$ and *low energy modes* $\mathsf{F}^{\mathrm{aux}}$. This splitting can be done by using *Hodge decomposition* of differential forms into exact, coexact and harmonic forms (see Appendix A of [21]). Note that, in addition, we might also have *background fields*[7]. If Σ_d would have boundary, one can in general split the space of fields into three

[7] These are background choices for classical fields that are not fixed by the boundary conditions and the Euler–Lagrange equations.

parts, the low energy fields, the high energy fields and the boundary fields. The boundary fields are generally given by techniques of symplectic reduction as the leaves of a chosen polarization on the boundary. This is the content of the BV-BFV formalism [20, 21, 23].

Proposition 5.11 ([46]). *Let* $(\mathcal{F}^{\text{aux}}, \mathcal{S}^{\text{aux}}, \omega^{\text{aux}}, \mu^{\text{aux}})$ *be a semi-quantum pre-observable with a good splitting. Define* $\mathsf{S}^{\text{aux}} \in C^\infty(\mathcal{F} \times \mathsf{F}^{\text{aux}})$ *by*

$$\mathsf{S}^{\text{aux}} = -\mathrm{i}\hbar \log \int_{\mathscr{L} \subset \mathscr{F}^{\text{aux}}} e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{aux}}} \sqrt{\mu_2^{\text{aux}}}|_{\mathscr{L}}, \tag{40}$$

where $\mathscr{L}$ *is a Lagrangian submanifold of* $\mathscr{F}^{\text{aux}}$*. Then* $(\mathsf{F}^{\text{aux}}, \mathsf{S}^{\text{aux}}, \omega_1^{\text{aux}}, \mu_1^{\text{aux}})$ *defines a semi-quantum pre-observable for the same BV theory. Moreover, the observable for the BV theory induced by* S^{aux} *using Equation (37) with a Lagrangian submanifold* $\mathsf{L} \subset \mathsf{F}^{\text{aux}}$ *is equivalent to the one induced by* $\mathcal{S}^{\text{aux}}$ *using the Lagrangian submanifold* $\mathcal{L} \subset \mathcal{F}^{\text{aux}}$*, if there exists a homotopy between* $\mathcal{L}$ *and* $\mathsf{L} \times \mathscr{L}$ *in* $\mathcal{F}^{\text{aux}}$*.*

Remark 5.12. Note that Equation (40) means that S^{aux} is the *low energy effective action* (zero modes).

5.2. Observables for Quantum BV Manifolds

Definition 5.13 (*BV quantum observable*). A *BV quantum observable* for a quantum BV manifold is a function $\mathcal{O}$ on $\mathcal{F}$ of degree 0 such that

$$\delta_{\text{BV}}\mathcal{O} = 0 \Longleftrightarrow \Delta\left(\mathcal{O}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}}\right) = 0. \tag{41}$$

Definition 5.14 (*Equivalent BV quantum observables*). Two BV quantum observables $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are said to be equivalent if

$$\widetilde{\mathcal{O}} - \mathcal{O} = \delta_{\text{BV}}\Psi, \qquad \Psi \in C^\infty(\mathcal{F}), \tag{42}$$

or equivalently, $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ have the same δ_{BV}-cohomology class.

Definition 5.15 (*BV quantum pre-observable*). A *BV quantum pre-observable* for a BV manifold is a BV semi-quantum pre-observable

$$(\mathcal{F}^{\text{aux}}, \omega^{\text{aux}}, \mu^{\text{aux}}, \mathcal{S}^{\text{aux}})$$

where $\mathcal{S} + \mathcal{S}^{\text{aux}}$ satisfies the QME

$$(\Delta + \Delta^{\text{aux}})\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\mathcal{S}+\mathcal{S}^{\text{aux}})} = 0. \tag{43}$$

Proposition 5.16 ([46]). *Let* $(\mathcal{F}^{\text{aux}}, \omega^{\text{aux}}, \mu^{\text{aux}}, \mathcal{S}^{\text{aux}})$ *be a BV quantum pre-observable. Define*

$$\mathcal{O}_{\mathcal{L}} := \int_{\mathcal{L} \subset \mathcal{F}^{\text{aux}}} e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\text{aux}}} \sqrt{\mu^{\text{aux}}}|_{\mathcal{L}} \in C^\infty(\mathcal{F}), \tag{44}$$

where $\mathcal{L} \subset \mathcal{F}^{\text{aux}}$ *is a Lagrangian submanifold. Then* $\mathcal{O}_{\mathcal{L}}$ *is an observable, i.e.,* $\delta_{\text{BV}}\mathcal{O}_{\mathcal{L}} = 0$*. Moreover, if for two Lagrangian submanifolds* $\mathcal{L}$ *and* $\widetilde{\mathcal{L}}$ *there exists a homotopy between them, then the observables* $\mathcal{O}_{\mathcal{L}}$ *and* $\mathcal{O}_{\widetilde{\mathcal{L}}}$ *are equivalent.*

6. Formal Global Split AKSZ Sigma Models

The formal global construction for ASKZ sigma models is given by using methods of formal geometry (see [13,34] for the formal geometry part, and [24] for a detailed discussion of the formal global split AKSZ construction and its quantization) where one constructs a BV action that depends on a choice of classical background by adding an additional term to the AKSZ-BV action. This construction leads to modifications in the usual BV gauge-fixing condition if we apply the BV construction to this new formal global action. The globalization arises in an equivalent way as for the constructions involving the underlying curved[8] L_∞-structure for the space of fields (see, e.g., [42] for an exposition on curved ∞-structures and [29] for the field-theoretic concept).

In this section we want to recall some notions of formal geometry and describe the extension of AKSZ sigma models to a formal global version.

6.1. Notions of Formal Geometry

Let us introduce the main players.

Definition 6.1 (*Generalized exponential map*). Let M be a manifold and let $U \subset TM$ be an open neighborhood of the zero section of the tangent bundle. A *generalized exponential map* is a map $\phi\colon U \to M$ such that $\phi\colon (x,p) \mapsto \phi_x(p)$ with $\phi_x(0) = x$ and $\mathrm{d}\phi_x(0) = \mathrm{id}_{T_xM}$. Locally, we have

$$\phi_x^i(p) = x^i + p^i + \frac{1}{2}\phi^i_{x,jk}p^jp^k + \frac{1}{3!}\phi^i_{x,jk\ell}p^jp^kp^\ell + \cdots \tag{45}$$

where (x^i) are coordinates on the base and (p^i) are coordinates on the fiber.

Definition 6.2 (*Formal exponential map*). A *formal exponential map* is an equivalence class of generalized exponential maps, where we identify two generalized exponential maps if their jets agree to all orders.

One can define a flat connection D on $\widehat{\mathrm{Sym}}(T^*M)$, where $\widehat{\mathrm{Sym}}$ denotes the completed symmetric algebra. Such a flat connection D is called *classical Grothendieck* connection [17] and it is locally given by $D = \mathrm{d}_M + R$, where

$$R \in \Omega^1\left(M, \mathrm{Der}\left(\widehat{\mathrm{Sym}}(T^*M)\right)\right)$$

is a 1-form with values in derivations of the completed symmetric algebra of the cotangent bundle. Here R acts on sections $\sigma \in \Gamma\left(\widehat{\mathrm{Sym}}(T^*M)\right)$ by Lie derivative, that is $R(\sigma) = L_R\sigma$. Note that we have denoted by d_M the de Rham differential on M. In local coordinates we have $R = R_\ell \mathrm{d}_M x^\ell$, where $R_\ell = R_\ell^j(x,p)\frac{\partial}{\partial p^j}$ and

$$R_\ell^j(x,p) = -\frac{\partial \phi^k}{\partial x^\ell}\left(\left(\frac{\partial\phi}{\partial p}\right)^{-1}\right)_k^j = -\delta_\ell^j + O(p). \tag{46}$$

[8]An L_∞-algebra $\mathfrak{g}$ is called *curved* if there exists an operation $\ell_0\colon \mathbb{R} \to \mathfrak{g}$ of degree 0. In particular, the strong homotopy Jacobi identity implies that $\ell_1 \circ \ell_1 = \pm\ell_2(\ell_0, \ )$, meaning that the unary bracket ℓ_1 does not square to zero anymore, as it is the case for usual L_∞-algebras. In this case we say that ℓ_1 has non-vanishing curvature, thus the name "curved".

Hence, for $\sigma \in \Gamma\left(\widehat{\mathrm{Sym}}(T^*M)\right)$ we have

$$R(\sigma) := L_R(\sigma) = R_\ell(\sigma)\mathrm{d}_M x^\ell = -\frac{\partial\sigma}{\partial p^j}\frac{\partial\phi^k}{\partial x^\ell}\left(\left(\frac{\partial\phi}{\partial p}\right)^{-1}\right)_k^j \mathrm{d}_M x^\ell. \tag{47}$$

Note that we can extend the connection D to the complex

$$\Gamma\left(\bigwedge^\bullet T^*M \otimes \widehat{\mathrm{Sym}}(T^*M)\right)$$

of $\widehat{\mathrm{Sym}}(T^*M)$-valued differential forms[9]. The following proposition tells us that the D-closed sections are exactly given by smooth functions.

Proposition 6.3. *A section $\sigma \in \Gamma\left(\widehat{Sym}(T^*M)\right)$ is D-closed if and only if $\sigma = \mathsf{T}\phi^* f$ for some $f \in C^\infty(M)$, where T denotes the Taylor expansion around the fiber coordinates at zero. Moreover, the D-cohomology*

$$H_D^\bullet\left(\widehat{Sym}(T^*M)\right)$$

is concentrated in degree 0 and

$$H_D^0\left(\widehat{Sym}(T^*M)\right) = \mathsf{T}\phi^* C^\infty(M) \cong C^\infty(M). \tag{48}$$

Remark 6.4. Note that we use any representative of ϕ to define the pullback.

Proof of Proposition 6.3. If we use (45) and (47), We can see that $R = \delta + R'$ where $\delta = \mathrm{d}x^i \frac{\partial}{\partial p^i}$ and R' is a 1-form with values in vector fields vanishing at $p = 0$. Then we have $D = \delta + D'$ with

$$D' = \mathrm{d}x^i \frac{\partial}{\partial x^i} + R'. \tag{49}$$

One should note that δ is itself a differential and that it decreases the polynomial degree in p, whereas D' does not decrease the degree. We can show that the cohomology of δ consists of 0-forms which are constant in p. To show this, let

$$\delta^* = p^i \iota_{\frac{\partial}{\partial x^i}}$$

and note that

$$(\delta\delta^* + \delta^*\delta)\sigma = k\sigma, \tag{50}$$

where σ is an r-form of degree s in p such that $r + s = k$. By cohomological perturbation theory the cohomology of D is isomorphic to the cohomology of δ. □

[9]Since $\Gamma\left(\bigwedge^\bullet T^*M \otimes \widehat{\mathrm{Sym}}(T^*M)\right)$ is the algebra of functions on the formal graded manifold $T[1]M \oplus T[0]M$, the differential D turns this graded manifold into a differential graded manifold. In particular, since D vanishes on the body of the graded manifold, we can linearize at each $x \in M$ and obtain an L_∞-structure on $T_xM[1] \oplus T_xM$.

Note that in local coordinates we get for $f \in C^\infty(M)$

$$\mathsf{T}\phi_x^* f = f(x) + p^i \partial_i f(x) + \frac{1}{2} p^j p^k (\partial_j \partial_k f(x) + \phi^i_{x,jk} \partial_i f(x)) + \cdots \tag{51}$$

An interesting question is how the Grothendieck connection depends on the choice of formal exponential map. Let $I \subset \mathbb{R}$ be an open interval and let ϕ be a family of formal exponential maps depending on a parameter $t \in I$. This family may be associated to a family of formal exponential maps ψ on $M \times I$ by

$$\psi(x, t, p, \tau) = (\phi_{x,t}(p), t + \tau), \tag{52}$$

where τ denotes the tangent coordinate to t. The associated connection $\widetilde{R}$ is defined by

$$\begin{aligned} &\widetilde{R}\,(\widetilde{\sigma}) = -(\mathrm{d}_p \widetilde{\sigma}, \mathrm{d}_\tau \widetilde{\sigma}) \circ \begin{pmatrix} (\mathrm{d}_p \phi)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{d}_x \phi & \dot{\phi} \\ 0 & 1 \end{pmatrix}, \\ &\widetilde{\sigma} \in \Gamma\left(\widehat{\mathrm{Sym}}(T^*(M \times I))\right). \end{aligned} \tag{53}$$

Thus we can write $\widetilde{R} = R + C\mathrm{d}t + T$ with R defined as before with the difference that it now depends on t, C is given by

$$C(\widetilde{\sigma}) = -\mathrm{d}_p \widetilde{\sigma} \circ (\mathrm{d}_p \phi)^{-1} \circ \dot{\phi}, \tag{54}$$

and $T = -\mathrm{d}t \frac{\partial}{\partial \tau}$. Note that $\mathrm{d}_x T = 0$, $\mathrm{d}_t T = 0$ and $[T, R] = 0$, $[T, C] = 0$. Thus, using the Maurer–Cartan equation for $\widetilde{R}$ and for R, we get

$$\dot{R} = \mathrm{d}_x C + [R, C], \tag{55}$$

which shows that under a change of formal exponential map, R changes by a gauge transformation with generator C. Moreover, if $\sigma = \mathsf{T}\phi_x^* f$ for some $f \in C^\infty(M \times I)$, we get

$$\dot{\sigma} = -L_C \sigma. \tag{56}$$

This can be thought of as an associated gauge transformation for sections.

6.2. Formal Global AKSZ Sigma Models

Let Σ_d be a closed, oriented, compact d-manifold and consider a Hamiltonian Q-manifold

$$(\mathcal{M}, \omega_{\mathcal{M}} = \mathrm{d}_{\mathcal{M}} \alpha_{\mathcal{M}}, \Theta_{\mathcal{M}}, Q_{\mathcal{M}})$$

of degree $d-1$. As described in Sect. 3.2, we can consider its induced AKSZ theory with the space of fields

$$\mathcal{F}^{\mathcal{M}}_{\Sigma_d} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, \mathcal{M}). \tag{57}$$

Consider now a formal exponential map $\phi\colon T\mathcal{M} \to \mathcal{M}$. Then we can lift the space of fields by ϕ. For $x \in \mathcal{M}$ we denote the lifted space of fields by

$$\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d} := \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, T_x\mathcal{M}) \cong \Omega^\bullet(\Sigma_d) \otimes T_x\mathcal{M}. \tag{58}$$

Note that we have used the fact that

$$C^\infty(T[1]\Sigma_d) \cong \Omega^\bullet(\Sigma_d). \tag{59}$$

This construction gives us a linear space for the target and thus we can identify the fields with differential forms on Σ_d with values in the vector space $T_x\mathcal{M}$ for $x \in \mathcal{M}$. Consider the map

$$\widetilde{\phi}_x : \widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x} \to \mathcal{F}^{\mathcal{M}}_{\Sigma_d}, \tag{60}$$

which is given by composition with ϕ_x^{-1}, i.e., $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x} = \phi_x^{-1} \circ \mathcal{F}^{\mathcal{M}}_{\Sigma_d}$. We can lift the BV symplectic 2-form ω_{Σ_d}, the primitive 1-form α_{Σ_d} and the BV action $\mathcal{S}_{\Sigma_d}$ to the lifted space of fields. We will denote the lifts by

$$\widehat{\alpha}_{\Sigma_d,x} = \widetilde{\phi}^*_x \alpha_{\Sigma_d} \in \Omega^1\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\right), \tag{61}$$

$$\widehat{\omega}_{\Sigma_d,x} = \widetilde{\phi}^*_x \omega_{\Sigma_d} \in \Omega^2\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\right), \tag{62}$$

$$\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d,x} = \iota_{\widehat{\mathrm{d}}_{\Sigma_d}} \widetilde{\phi}^*_x \mathscr{T}_{\Sigma_d}(\alpha_{\mathcal{M}}) + \mathsf{T}\widetilde{\phi}^*_x \mathscr{T}_{\Sigma_d}(\Theta_{\mathcal{M}}) \in C^\infty\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\right). \tag{63}$$

Note that we can regard a constant map $x\colon T[1]\Sigma_d \to \mathcal{M}$ in $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ as an element of $\mathcal{M}$, hence there is a natural inclusion $\mathcal{M} \hookrightarrow \mathcal{F}^{\mathcal{M}}_{\Sigma_d}$. For a constant field x and $\mathcal{A} \in \mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ We can construct a 1-form

$$R_{\Sigma_d} = (R_{\Sigma_d})_\mu(x, \mathcal{A})\mathrm{d}_{\mathcal{M}} x^\mu \tag{64}$$

on $\mathcal{M}$ with values in differential operators on $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$. Moreover, we can lift this 1-form to $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d}$ and we denote the lift by $\widehat{R}_{\Sigma_d}$. Locally, we write

$$\widehat{R}_{\Sigma_d} = \left(\widehat{R}_{\Sigma_d}\right)_\mu \left(x, \widehat{\mathcal{A}}\right) \mathrm{d}_{\mathcal{M}} x^\mu. \tag{65}$$

It is important to recall that classical solutions for AKSZ sigma models, i.e., solutions of $\delta\mathcal{S}_{\Sigma_d} = 0$, are given by differential graded maps

$$(T[1]\Sigma_d, \mathrm{d}_{\Sigma_d}) \to (\mathcal{M}, Q_{\mathcal{M}}).$$

Hence we can consider the moduli space of classical solutions M_{cl} for AKSZ theories which is given by constant maps $x\colon T[1]\Sigma_d \to \mathcal{M}$ and thus we get an isomorphism $\mathrm{M}_{\mathrm{cl}} \cong \mathcal{M}$. We will refer to this constant solutions as being *background fields*. Choosing a background field $x \in \mathcal{M}$, we can define a formal global AKSZ action.

Definition 6.5 (*Formal global AKSZ action*). The *formal global AKSZ action* is given by

$$\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x} = \iota_{\widehat{\mathrm{d}}_{\Sigma_d}} \widetilde{\phi}^*_x \mathscr{T}_{\Sigma_d}(\alpha_{\mathcal{M}}) + \mathsf{T}\widetilde{\phi}^*_x \mathscr{T}_{\Sigma_d}(\Theta_{\mathcal{M}}) + \widehat{\mathcal{S}}_{\Sigma_d,R,x}, \tag{66}$$

where $\widehat{\mathcal{S}}_{\Sigma_d,R,x}$ is constructed locally such that

$$\widehat{\mathcal{S}}_{\Sigma_d,R,x}\left(\widehat{\mathcal{A}}\right) = \int_{\Sigma_d} \left(\widehat{R}_{\Sigma_d}\right)_\mu \left(x, \widehat{\mathcal{A}}\right) \mathrm{d}_{\mathcal{M}} x^\mu. \tag{67}$$

Hence locally we get we get

$$\begin{aligned}\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x} = &\int_{\Sigma_d} \widehat{\alpha}_\mu \left(\widehat{\mathcal{A}}\right) \mathrm{d}_{\Sigma_d} \widehat{\mathcal{A}}^\mu + \int_{\Sigma_d} \widehat{\Theta}_{\mathcal{M},x}\left(\widehat{\mathcal{A}}\right) \\ &+ \int_{\Sigma_d} \left(\widehat{R}_{\Sigma_d}\right)_\mu \left(x, \widehat{\mathcal{A}}\right) \mathrm{d}_{\mathcal{M}} x^\mu,\end{aligned} \tag{68}$$

where $\widehat{\alpha}_\mu$ are the coefficients of $\widehat{\alpha}_{\Sigma_d,x} := \widetilde{\phi}_x^*\alpha_{\Sigma_d}$ and $\widehat{\Theta}_{\mathcal{M},x} := \mathsf{T}\widetilde{\phi}_x^*\Theta_{\mathcal{M}}$.

Remark 6.6. This construction has to be understood in a formal way. The geometric meaning and the relation to a global construction is clear when using the relation of R_{Σ_d} to the Grothendieck connection D. This can be done if we start with a theory called *split* which we will introduce now.

6.3. Formal Global Split AKSZ Sigma Models

AKSZ theories can generally be more difficult to work with depending on the target differential graded symplectic manifold $\mathcal{M}$. Recall that, using the isomorphism (59), if the target is linear, we have an isomorphism

$$\mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, \mathcal{M}) \cong \Omega^\bullet(\Sigma_d) \otimes \mathcal{M}. \tag{69}$$

Moreover, we can split the space of fields by considering $\mathcal{M}$ to be the shifted cotangent bundle of a linear space. At first, however, we only want $\mathcal{M}$ to be the shifted cotangent bundle of any graded manifold M. This leads to the following definition of AKSZ theories.

Definition 6.7 (*Linear split AKSZ sigma model*). We call a d-dimensional AKSZ sigma model *linear split* if the target is of the form

$$\mathcal{M} = V \oplus V^*$$

for some vector space V.

Definition 6.8 (*Split AKSZ sigma model*). We call a d-dimensional AKSZ sigma model *split* if the target is of the form

$$\mathcal{M} = T^*[d-1]M$$

for some graded manifold M.

This space can be lifted to a formal construction using methods of formal geometry as in Sect. 6.1 to the shifted cotangent bundle of the tangent space of M at some constant background in M. Consider a d-dimensional split AKSZ sigma model with space of fields given by

$$\mathcal{F}_{\Sigma_d}^{\mathcal{M}} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, T^*[d-1]M), \tag{70}$$

for some graded manifold M, with its corresponding AKSZ-BV theory

$$\left(\mathcal{F}_{\Sigma_d}^{\mathcal{M}}, \mathcal{S}_{\Sigma_d}, \omega_{\Sigma_d}\right).$$

Note that, similarly as for general AKSZ theories, one type of classical solutions to the Euler–Lagrange equations for split AKSZ theories are given by fields of the form $(x, 0)$ where $x\colon \Sigma_d \to M$ is a constant background field. Note that the classical space of fields F_{Σ_d} is given by vector bundle maps $T\Sigma_d \to T^*M$, i.e.,

$$F_{\Sigma_d} = \mathrm{Map}_{\mathrm{VecBun}}(T\Sigma_d, T^*M).$$

Then the BV space of fields is given by (70). Thus, for the classical space of fields F_{Σ_d}, we have a moduli space of classical solutions

$$\mathrm{M}_{\mathrm{cl}} = \{(A, B) \in \mathrm{Map}(T\Sigma_d, T^*M) \mid A = x = const, B = 0\} \cong M. \tag{71}$$

Moreover, for a chosen formal exponential map $\phi\colon TM \to M$ and a constant background field $x\colon \Sigma_d \to M$ regarded as an element of the moduli space of classical solutions M_{cl}, one can consider the lifted space of fields

$$\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, T^*[d-1]T_xM), \tag{72}$$

which gives a *linearization* (or also *coordinatization*) of the space of fields in the target as we have seen before. Let $(\boldsymbol{A},\boldsymbol{B}) \in \mathcal{F}^{\mathcal{M}}_{\Sigma_d}$, where $\boldsymbol{A}\colon T[1]\Sigma_d \to M$ denotes the base superfield and $\boldsymbol{B} \in \Gamma(\Sigma_d, T^*\Sigma_d \otimes \boldsymbol{A}^*T^*[d-1]M)$ the fiber superfield. Consider the corresponding lifts by ϕ where the superfields are given by

$$\widehat{\boldsymbol{A}} := \phi_x^{-1}(\boldsymbol{A}), \qquad \widehat{\boldsymbol{B}} := (\mathrm{d}\phi_x)^*\boldsymbol{B} \tag{73}$$

The BV action functional $\mathcal{S}_{\Sigma_d}$ then lifts to a *formal global action.*

Definition 6.9 (*Formal global split AKSZ action*). The *formal global action* for the *split AKSZ sigma model* is given by

$$\boxed{\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x} := \int_{\Sigma_d} \widehat{\boldsymbol{B}}_\ell \wedge \mathrm{d}_{\Sigma_d}\widehat{\boldsymbol{A}}^\ell + \int_{\Sigma_d} \widehat{\Theta}_{\mathcal{M},x}\left(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}}\right) + \int_{\Sigma_d} R^j_\ell\left(x,\widehat{\boldsymbol{A}}\right)\widehat{\boldsymbol{B}}_j \wedge \mathrm{d}_M x^\ell.} \tag{74}$$

Remark 6.10. Note that in this case we get a lift of R as defined in Sect. 6.1 to the space of fields which splits into base and fiber fields by

$$\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x} \cong \Omega^\bullet(\Sigma_d) \otimes T_xM \oplus \Omega^\bullet(\Sigma_d) \otimes T^*_xM[d-1]. \tag{75}$$

Hence the induced 1-form $\widehat{R}_{\Sigma_d}$ is indeed given by

$$\widehat{R}_{\Sigma_d} = R^j_\ell\left(x,\widehat{\boldsymbol{A}}\right)\widehat{\boldsymbol{B}}_j \wedge \mathrm{d}_M x^\ell, \tag{76}$$

where R^j_ℓ are the components of $R \in \Omega^1\left(M, \mathrm{Der}\left(\widehat{\mathrm{Sym}}(T^*M)\right)\right)$.

The Q-structure is given by the Hamiltonian vector field of $\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x}$. Indeed, let $\widehat{R}_{\Sigma_d}$ denote the lift of the vector field R_{Σ_d} to $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ and let

$$\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d,x} := \int_{\Sigma_d} \widehat{\boldsymbol{B}}_\ell \wedge \mathrm{d}_{\Sigma_d}\widehat{\boldsymbol{A}}^\ell + \int_{\Sigma_d} \widehat{\Theta}_{\mathcal{M},x}\left(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}}\right) \tag{77}$$

$$\widehat{\mathcal{S}}_{\Sigma_d,R,x} := \int_{\Sigma_d} R^j_\ell\left(x,\widehat{\boldsymbol{A}}\right)\widehat{\boldsymbol{B}}_j \wedge \mathrm{d}_M x^\ell, \tag{78}$$

such that

$$\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x} = \widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d,x} + \widehat{\mathcal{S}}_{\Sigma_d,R,x}. \tag{79}$$

Denote by $\widehat{\omega}_{\Sigma_d,x} = \widetilde{\phi}^*_x\omega_{\Sigma_d}$ the lift of the symplectic form on $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ to a symplectic form on $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$. Then we can define a cohomological vector field $\widehat{Q}_{\Sigma_d,x}$ on $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ by

$$\widehat{Q}_{\Sigma_d,x} = \widehat{Q}^{\mathrm{AKSZ}}_{\Sigma_d,x} + \widehat{R}_{\Sigma_d}, \tag{80}$$

where $\widehat{Q}^{\mathrm{AKSZ}}_{\Sigma_d,x}$ is the Hamiltonian vector field of

$$\mathrm{Back}_{\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d}}\colon x\mapsto \widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d,x}, \tag{81}$$

and hence we have

$$\iota_{\widehat{Q}_{\Sigma_d,x}}\widehat{\omega}_{\Sigma_d,x}=\delta\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x}. \tag{82}$$

This is in fact true if the source manifold is closed, i.e., $\partial\Sigma_d=\varnothing$. We have denoted the map by "Back" to indicate the variation of the "background".

Proposition 6.11. *If $\partial\Sigma_d=\varnothing$, then*

$$\mathrm{d}_x\mathrm{Back}_{\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d}}=\left\{\widehat{\mathcal{S}}_{\Sigma_d,R,x},\mathrm{Back}_{\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_d}}\right\}_{\widehat{\omega}_{\Sigma_d,x}}, \tag{83}$$

where d_M denotes the de Rham differential on the moduli space space of classical solutions $\mathrm{M}_{\mathrm{cl}}\cong M$.

Using the formal global action, we get the following Proposition (see also Proposition 8.4 for the quantum version)

Proposition 6.12 (dCME). *The* differential Classical Master Equation *for the formal global split AKSZ action holds:*

$$\boxed{\mathrm{d}_x\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x}+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x},\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x}\right\}_{\widehat{\omega}_{\Sigma_d,x}}=0.} \tag{84}$$

Definition 6.13 (*Formal global split AKSZ sigma model*). The *formal global split AKSZ sigma model* is given by the AKSZ-BV theory for the quadruple

$$\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x},\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x},\widehat{\omega}_{\Sigma_d,x},\widehat{Q}_{\Sigma_d,x}\right). \tag{85}$$

Remark 6.14. Note that the CME has to be replaced by the dCME as in (84) in the formal global setting.

7. Pre-observables for AKSZ Theories

7.1. AKSZ Pre-observables

Let Σ_d be a closed and oriented source d-manifold and for some differential graded symplectic manifold $(\mathcal{N},\omega_{\mathcal{N}}=\mathrm{d}_{\mathcal{N}}\alpha_{\mathcal{N}})$ let

$$\pi\colon\mathcal{E}=\mathcal{M}\times\mathcal{N}\to\mathcal{M}$$

be a trivial Hamiltonian Q-bundle of degree n over some Hamiltonian Q-manifold $(\mathcal{M},\omega_{\mathcal{M}}=\mathrm{d}_{\mathcal{M}}\alpha,Q_{\mathcal{M}},\Theta_{\mathcal{M}})$ of degree $d-1$. Denote by $\Theta_{\mathcal{E}}\in C^\infty(\mathcal{E})$ the Hamiltonian on the total space $\mathcal{E}$ and by $\mathcal{V}_{\mathcal{E}}\in\ker\mathrm{d}\pi$ the vertical part of $Q_{\mathcal{E}}$, such that

$$Q_{\mathcal{E}}=Q_{\mathcal{M}}+\mathcal{V}_{\mathcal{E}}.$$

Consider the corresponding AKSZ-BV theory with BV manifold given by

$$\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d},\mathcal{S}_{\Sigma_d},\omega_{\Sigma_d},Q_{\Sigma_d}\right)$$

as it was constructed in Sect. 3. Let $i\colon \Sigma_k \hookrightarrow \Sigma_d$ be the embedding of a closed oriented submanifold of dimension $k \leq d$ and let the auxiliary space of fields be given by

$$\mathcal{F}^{\mathcal{N}}_{\Sigma_k} := \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_k, \mathcal{N}). \tag{86}$$

Moreover, consider the transgression maps

$$\mathscr{T}_{\Sigma_k} : \Omega^\bullet(\mathcal{N}) \to \Omega^\bullet\left(\mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right), \tag{87}$$
$$\mathscr{T}^{\mathcal{E}}_{\Sigma_k} : \Omega^\bullet(\mathcal{E}) \to \Omega^\bullet\left(\mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_k, \mathcal{E})\right) \tag{88}$$

corresponding to the fiber $\mathcal{N}$ and the total space $\mathcal{E}$. Define[10]

$$p := i^*\colon \underbrace{\mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, \mathcal{M})}_{=:\mathcal{F}^{\mathcal{M}}_{\Sigma_d}} \to \underbrace{\mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_k, \mathcal{M})}_{=:\mathcal{F}^{\mathcal{M}}_{\Sigma_k}}.$$

Furthermore, let $\widehat{\mathrm{d}}_{\Sigma_k} \in \mathfrak{X}\left(\mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right) \subset \mathfrak{X}^{\mathrm{vert}}\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d} \times \mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right)$, where $\mathfrak{X}^{\mathrm{vert}}$ denotes the space of vertical vector fields, and let

$$\widehat{\mathcal{V}}_{\mathcal{E}} \in \mathfrak{X}^{\mathrm{vert}}\left(\mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_k, \mathcal{E})\right)$$

be the lift of $\mathcal{V}_{\mathcal{E}} \in \mathfrak{X}^{\mathrm{vert}}(\mathcal{E})$ such that $p^*\widehat{\mathcal{V}}_{\mathcal{E}} \in \mathfrak{X}^{\mathrm{vert}}\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d} \times \mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right)$, where

$$p^*\colon C^\infty\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_k}\right) \to C^\infty\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right).$$

Proposition 7.1 ([46]). *Consider the data given by*

$$\mathcal{S}^{\mathcal{N}}_{\Sigma_k} = \iota_{\widehat{\mathrm{d}}_{\Sigma_k}} \mathscr{T}_{\Sigma_k}(\alpha_{\mathcal{N}}) + p^* \mathscr{T}^{\mathcal{E}}_{\Sigma_k}(\Theta_{\mathcal{E}}), \tag{89}$$
$$\omega^{\mathcal{N}}_{\Sigma_k} = (-1)^k \mathscr{T}_{\Sigma_k}(\omega_{\mathcal{N}}), \tag{90}$$
$$\mathcal{V}^{\mathcal{E}}_{\Sigma_k} = \widehat{\mathrm{d}}_{\Sigma_k} + p^*\widehat{\mathcal{V}}_{\mathcal{E}}. \tag{91}$$

Then the quadruple

$$\left(\mathcal{F}^{\mathcal{N}}_{\Sigma_k}, \mathcal{S}^{\mathcal{N}}_{\Sigma_k}, \omega^{\mathcal{N}}_{\Sigma_k}, \mathcal{V}^{\mathcal{E}}_{\Sigma_k}\right) \tag{92}$$

defines a pre-observable for the AKSZ-BV theory as in (85), that is we have

$$Q_{\Sigma_d}\left(\mathcal{S}^{\mathcal{N}}_{\Sigma_k}\right) + \frac{1}{2}\left\{\mathcal{S}^{\mathcal{N}}_{\Sigma_k}, \mathcal{S}^{\mathcal{N}}_{\Sigma_k}\right\}_{\omega^{\mathcal{N}}_{\Sigma_k}} = 0. \tag{93}$$

Remark 7.2. This pre-observable is invariant under reparamterizations of Σ_k and under diffeomorphism of the ambient manifold Σ_d. In fact, for $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}^{\mathcal{M}}_{\Sigma_d} \times \mathcal{F}^{\mathcal{N}}_{\Sigma_k}$, $\varphi_d \in \mathrm{Diff}(\Sigma_d)$ and $\varphi_k \in \mathrm{Diff}(\Sigma_k)$, one can immediately show that

$$\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A}, \mathcal{B}; \varphi_d \circ i \circ \varphi_k) = \mathcal{S}^{\mathcal{N}}_{\Sigma_k}\left(\varphi_d^*\mathcal{A}, (\varphi_k)^{-1}\mathcal{B}; i\right) \tag{94}$$

[10]Note that extending to the case with auxiliary fields $\mathcal{F} \times \mathcal{F}^{\mathrm{aux}}$, we can extend π to a map $\pi^{\mathcal{N}} = \pi \times \mathrm{id}_{\mathcal{F}^{\mathcal{N}}_{\Sigma_k}} : \mathcal{F}^{\mathcal{M}}_{\Sigma_d} \times \mathcal{F}^{\mathcal{N}}_{\Sigma_k} \to \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_k, \mathcal{E})$.

7.2. Formal Global AKSZ Pre-observables

We want to extend the constructions above to a formal global lift by using methods of formal geometry as in Sect. 6. It turns out that the formal global lift of the pre-observable constructed in the previous section is not automatically a pre-observable. In particular, it is spoilt by an obstruction which can be phrased as an equation that has to be satisfied. Hence we get the following theorem.

Theorem 7.3. *Let $\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}, \mathcal{S}_{\Sigma_d}, \omega_{\Sigma_d}, Q_{\Sigma_d}\right)$ be the AKSZ-BV theory constructed as before and let $i\colon \Sigma_k \hookrightarrow \Sigma_d$ be a submanifold of Σ_d. Moreover, consider constant background fields $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Then its formal global AKSZ construction*

$$\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}, \widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_d,x}, \widehat{\omega}_{\Sigma_d,x}, \widehat{Q}_{\Sigma_d,x}\right), \tag{95}$$

constructed by using a formal exponential map $T\mathcal{M} \to \mathcal{M}$, together with the formal global fiber

$$\left(\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}, \widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}, \widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y} = \mathrm{d}_{\mathcal{N}}\widehat{\alpha}^{\mathcal{N}}_{\Sigma_k,y}, \widehat{Q}^{\mathcal{N}}_{\Sigma_k,y}\right), \tag{96}$$

constructed using a formal exponential map $T\mathcal{N} \to \mathcal{N}$, defines a pre-observable if and only if

$$\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y} + \frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y}, \widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}_{\Sigma_k,y}} = 0. \tag{97}$$

Remark 7.4. Moreover, for an exponential map $\phi\colon T\mathcal{N} \to \mathcal{N}$, we set

$$\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y} = \underbrace{\widetilde{\phi}^*_y \iota_{\widehat{\mathrm{d}}_{\Sigma_k}} \mathscr{T}_{\Sigma_k}(\alpha_{\mathcal{N}})}_{=:\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}} + \underbrace{\mathsf{T}\widetilde{\phi}^*_y p^* \mathscr{T}^{\mathcal{E}}_{\Sigma_k}(\Theta_{\mathcal{E}})}_{=:\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}}, \tag{98}$$

and thus we have a decomposition, similarly as in (79), of the formal global action as

$$\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y} = \widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y} + \widehat{\mathcal{S}}_{\Sigma_k,R,y}. \tag{99}$$

The following Lemma is going to be useful for the proof of Theorem 7.3.

Lemma 7.5. *Let Σ be a compact, connected manifold and let $\mathcal{M}$ be a differential graded symplectic manifold. Moreover, let $X \in \mathfrak{X}(T[1]\Sigma)$, $Y \in \mathfrak{X}(\mathcal{M})$, $\Xi \in \Omega^\bullet(\mathcal{M})$ and denote the lifts of X and Y to the mapping space by $\widehat{X}$ and $\widehat{Y}$ respectively. Then*

$$L_{\widehat{X}}\mathscr{T}_\Sigma(\Xi) = 0, \tag{100}$$

$$L_{\widehat{Y}}\mathscr{T}_\Sigma(\Xi) = (-1)^{\mathrm{gh}(\widehat{Y})\dim\Sigma}\mathscr{T}_\Sigma(L_Y\Xi). \tag{101}$$

Proof of Theorem 7.3. First we note that the lift

$$\widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y} = \widehat{\mathrm{d}}_{\Sigma_k} + \widetilde{\phi}^*_y p^*\widehat{\mathcal{V}}_{\mathcal{E}} \tag{102}$$

of $\mathcal{V}^{\mathcal{E}}_{\Sigma_k}$ to $\mathfrak{X}^{\mathrm{vert}}\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\times\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}\right)$, the space of vertical vector fields on the lifted mapping spaces, is the Hamiltonian vector field for $\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}$, i.e., we have

$$\widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y}=\left\{\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y},\ \right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}. \tag{103}$$

Indeed, we have

$$\begin{aligned}\iota_{\widehat{\mathrm{d}}_{\Sigma_k}}\widehat{\omega}_{\Sigma_k,y}&=\iota_{\widehat{\mathrm{d}}_{\Sigma_k}}(-1)^k\widetilde{\phi}^*_y\mathscr{T}_{\Sigma_k}(\omega_{\mathcal{N}})=\widetilde{\phi}^*_y\iota_{\widehat{\mathrm{d}}_{\Sigma_k}}\delta\mathscr{T}_{\Sigma_k}(\alpha_{\mathcal{N}})\\&=\widetilde{\phi}^*_y\underbrace{L_{\widehat{\mathrm{d}}_{\Sigma_k}}\mathscr{T}_{\Sigma_k}(\alpha_{\mathcal{N}})}_{=0}+\widetilde{\phi}^*_y\delta\iota_{\widehat{\mathrm{d}}_{\Sigma_k}}\mathscr{T}_{\Sigma_k}(\alpha_{\mathcal{N}})=\delta\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}\\&=\delta^{\mathrm{vert}}\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y},\end{aligned}\tag{104}$$

where we have used Cartan's magic formula $L=\mathrm{d}\iota+\iota\mathrm{d}$, Lemma 7.5 and the fact that $\widetilde{\phi}^*_y\widehat{\mathrm{d}}_{\Sigma_k}=\widehat{\mathrm{d}}_{\Sigma_k}$. We have denoted by δ^{vert} the vertical part of the de Rham differential δ on the lifted total mapping space $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\times\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}$, i.e., in the fiber direction $\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}$. The last equality holds since $\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}$ is constant in the $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ direction. Similarly, we have

$$\begin{aligned}\iota_{\widetilde{\phi}^*_yp^*\widehat{\mathcal{V}}_{\mathcal{E}}}\widehat{\omega}_{\Sigma_k,y}&=\mathsf{T}\widetilde{\phi}^*_yp^*\iota_{\widehat{\mathcal{V}}_{\mathcal{E}}}(-1)^k\mathscr{T}^{\mathcal{E}}_{\Sigma_k}(\omega_{\mathcal{N}})\\&=(-1)^k\mathsf{T}\widetilde{\phi}^*_yp^*\mathscr{T}^{\mathcal{E}}_{\Sigma_k}(\underbrace{\iota_{\mathcal{V}_{\mathcal{E}}}\omega_{\mathcal{N}}}_{=\delta^{\mathrm{vert}}\Theta_{\mathcal{E}}})\\&=\delta^{\mathrm{vert}}\mathsf{T}\widetilde{\phi}^*_yp^*\mathscr{T}^{\mathcal{E}}_{\Sigma_k}(\Theta_{\mathcal{E}})=\delta^{\mathrm{vert}}\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}.\end{aligned}\tag{105}$$

Moreover, we have

$$\begin{aligned}&\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}\right)+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\&\quad=\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)+\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\&\qquad+\underbrace{\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}}_{=\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}.\end{aligned}\tag{106}$$

The first two terms of the left hand side of (106) are given by

$$\begin{aligned}\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)&=\widehat{Q}^{\mathrm{AKSZ}}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)\\&=\widehat{\mathrm{d}}_{\Sigma_d}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)+\widetilde{\phi}^*_y\widehat{Q}_{\mathcal{M}}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right),\end{aligned}\tag{107}$$

$$\begin{aligned}\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)&=\widehat{Q}^{\mathrm{AKSZ}}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)\\&=\widehat{\mathrm{d}}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)+\widetilde{\phi}^*_y\widehat{Q}_{\mathcal{M}}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right).\end{aligned}\tag{108}$$

Since $\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}$ and $\widehat{\mathcal{S}}_{\Sigma_k,R,y}$ are constant in direction of $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$, we get

$$\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right)=\underbrace{\widehat{\mathrm{d}}_{\Sigma_d}\left(\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right)}_{=0}+\widetilde{\phi}^*_y\widehat{Q}_{\mathcal{M}}\left(\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right), \quad (109)$$

$$\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)=0. \quad (110)$$

Using (107), (108), (109) and (110) we get

$$\begin{aligned}
&\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}\right)+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\
&=\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\widetilde{\phi}^*_y\widehat{Q}_{\mathcal{M}}\left(\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right)+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)\\
&\quad+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\left\{\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\
&=\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)\\
&\quad+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\mathrm{kin}}_{\Sigma_k,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\underbrace{\widehat{\mathrm{d}}_{\Sigma_k}\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}}_{=0}+\widetilde{\phi}^*_y\left(\widehat{Q}_{\mathcal{M}}+\frac{1}{2}p^*\widehat{\mathcal{V}}_{\mathcal{E}}\right)\left(\widehat{\mathcal{S}}^{\mathrm{target}}_{\Sigma_k,y}\right)\\
&=\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)\\
&\quad+\underbrace{(-1)^k\widetilde{\phi}^*_y p^*\mathscr{T}^{\mathcal{E}}_{\Sigma_k}\left(Q_{\mathcal{M}}(\Theta_{\mathcal{E}})+\frac{1}{2}\mathcal{V}_{\mathcal{E}}(\Theta_{\mathcal{E}})\right)}_{=(-1)^k\widetilde{\phi}^*_y p^*\mathscr{T}^{\mathcal{E}}_{\Sigma_k}\left(Q_{\mathcal{M}}(\Theta_{\mathcal{E}})+\frac{1}{2}\{\Theta_{\mathcal{E}},\Theta_{\mathcal{E}}\}_{\omega_{\mathcal{N}}}\right)=0\quad\text{(by definition of Hamiltonian Q-bundle)}}\\
&=\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)
\end{aligned} \quad (111)$$

Note that $\widehat{R}_{\Sigma_d}$ is a vector field on the lifted space $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ which implies that $\widehat{R}_{\Sigma_d}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)=0$ because $\widehat{\mathcal{S}}_{\Sigma_k,R,y}\in C^\infty\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\times\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}\right)$ is constant in the direction of $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ and the claim follows. □

Corollary 7.6. *An equivalent condition for the formal global AKSZ-BV theory as in Theorem 7.3 to be a pre-observable is given by*

$$\widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)=\widehat{\mathcal{S}}_{\Sigma_k,\mathrm{d}_yR,y}. \quad (112)$$

Proof. Note that we have

$$\begin{aligned}
&\mathrm{d}_y\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\
&\quad=\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k x}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}+\frac{1}{2}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}\\
&\quad=\widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y}\left(\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right)+\underbrace{\frac{1}{2}\widehat{\mathcal{S}}_{\Sigma_k,[R,R],x}}_{=\widehat{\mathcal{S}}_{\Sigma_k,\mathrm{d}_yR,y}},
\end{aligned} \quad (113)$$

where we have used that $\widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y}$ is the Hamiltonian vector field of $\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}$ and the fact that [12]

$$\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}_{\Sigma_k,R,y}\right\}_{\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}}=\widehat{\mathcal{S}}_{\Sigma_k,[R,R],y}. \tag{114}$$

the last equality (under the braces) follows from the fact that D is a flat connection on $\widehat{\mathrm{Sym}}(T^*\mathcal{N})$ which can be translated into

$$\mathrm{d}_y R+\frac{1}{2}[R,R]=0. \tag{115}$$

Moreover, it is easy to see that $\widehat{\mathcal{S}}_{\Sigma_k,\ell R,y}=\ell\widehat{\mathcal{S}}_{\Sigma_k,R,y}$ for any $\ell\in\mathbb{R}$. □

7.3. Formal Global Auxiliary Construction in Coordinates

We want to describe the auxiliary theory as well as its formal global extension in terms of coordinates. The description follows similarly from the description of the ambient theory as in Sect. 3.2 and its formal global extension as in Sect. 6.2. Let (v^j) be even local coordinates on Σ_k and consider the corresponding odd local coordinates $\xi^j=\mathrm{d}_{\Sigma_k}v^j$ for $1\leq j\leq k$. Then we can construct superfield coordinates

$$\begin{aligned}\mathcal{B}^\nu(v,\xi)&=\sum_{\ell=1}^k\underbrace{\sum_{1\leq j_1<\cdots<j_\ell\leq k}\mathcal{B}^\nu_{j_1\dots j_\ell}(v)\xi^{j_1}\wedge\cdots\wedge\xi^{j_\ell}}_{=\mathcal{B}^\nu_{(\ell)}(v,\xi)}\\&\in\bigoplus_{\ell=0}^k C^\infty(\Sigma_k)\otimes\bigwedge^\ell T^*\Sigma_k.\end{aligned} \tag{116}$$

associated to local homogeneous coordinates (y^ν) of $\mathcal{N}$. Note that locally we have

$$\alpha_{\mathcal{N}}=\alpha^{\mathcal{N}}_\nu(y)\mathrm{d}_{\mathcal{N}}y^\nu\in\Omega^1(\mathcal{N}), \tag{117}$$

$$\omega_{\mathcal{N}}=\frac{1}{2}\omega^{\mathcal{N}}_{\nu_1\nu_2}(y)\mathrm{d}_{\mathcal{N}}y^{\nu_1}\wedge\mathrm{d}_{\mathcal{N}}y^{\nu_2}\in\Omega^2(\mathcal{N}). \tag{118}$$

Hence we get

$$\alpha^{\mathcal{N}}_{\Sigma_k}=\int_{\Sigma_k}\alpha^{\mathcal{N}}_\nu(\mathcal{B})\delta\mathcal{B}^\nu\in\Omega^1\left(\mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right), \tag{119}$$

$$\omega^{\mathcal{N}}_{\Sigma_k}=(-1)^k\frac{1}{2}\int_{\Sigma_k}\omega^{\mathcal{N}}_{\nu_1\nu_2}(\mathcal{B})\delta\mathcal{B}^{\nu_1}\wedge\delta\mathcal{B}^{\nu_2}\in\Omega^2\left(\mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right), \tag{120}$$

and thus we get an action for the auxiliary fields as

$$\begin{aligned}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)=&\int_{\Sigma_k}\alpha^{\mathcal{N}}_\nu(\mathcal{B})\mathrm{d}_{\Sigma_k}\mathcal{B}^\nu\\&+\int_{\Sigma_k}\Theta_{\mathcal{E}}(i^*\mathcal{A},\mathcal{B})\in C^\infty\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\times\mathcal{F}^{\mathcal{N}}_{\Sigma_k}\right),\end{aligned} \tag{121}$$

These expressions can be lifted to the formal global construction. Indeed, consider a formal exponential map $\phi\colon T\mathcal{N}\to\mathcal{N}$. Let $\widehat{\mathcal{A}}=\phi_x^{-1}(\mathcal{A})$ be the lift of $\mathcal{A}$ to $\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ and $\widehat{\mathcal{B}} = \phi_y^{-1}(\mathcal{B})$ be the lift of $\mathcal{B}$ to $\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}$ for $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Then we get

$$\widehat{\alpha}^{\mathcal{N}}_{\Sigma_k,y} = \int_{\Sigma_k} \widehat{\alpha}_\nu \left(\widehat{\mathcal{B}}\right) \delta\widehat{\mathcal{B}}^\nu \in \Omega^1\left(\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}\right), \tag{122}$$

$$\widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y} = (-1)^k \frac{1}{2} \int_{\Sigma_k} \widehat{\omega}_{\nu_1\nu_2} \left(\widehat{\mathcal{B}}\right) \delta\widehat{\mathcal{B}}^{\nu_1} \wedge \delta\widehat{\mathcal{B}}^{\nu_2} \in \Omega^2\left(\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}\right), \tag{123}$$

where $\widehat{\alpha}^{\mathcal{N}}_\nu$ and $\widehat{\omega}^{\mathcal{N}}_{\nu_1\nu_2}$ are the coefficients of $\widehat{\alpha}_{\mathcal{N}} \in \Omega^1(T\mathcal{N})$ and $\widehat{\omega}_{\mathcal{N}} \in \Omega^2(T\mathcal{N})$ respectively. If we set $\widehat{\Theta}_{\mathcal{E},y} := \mathsf{T}\widetilde{\phi}_y^* \Theta_{\mathcal{E}}$, the auxiliary formal global AKSZ action is then given by

$$\begin{aligned}\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}\left(\widehat{\mathcal{A}},\widehat{\mathcal{B}};i\right) &= \underbrace{\int_{\Sigma_k} \widehat{\alpha}^{\mathcal{N}}_\nu \left(\widehat{\mathcal{B}}\right) \mathrm{d}_{\Sigma_k}\widehat{\mathcal{B}}^\nu + \int_{\Sigma_k} \widehat{\Theta}_{\mathcal{E},y}\left(i^*\widehat{\mathcal{A}},\widehat{\mathcal{B}}\right)}_{=\widehat{\mathcal{S}}^{\text{AKSZ}}_{\Sigma_k,y}} \\ &\quad + \underbrace{\int_{\Sigma_k} (\widehat{R}_{\Sigma_k})_\nu \left(y,\widehat{\mathcal{B}}\right) \mathrm{d}_{\mathcal{N}} y^\nu}_{=\widehat{\mathcal{S}}_{\Sigma_k,R,y}}. \end{aligned} \tag{124}$$

7.4. Formal Global Split Auxiliary Construction in Coordinates

If we consider a split AKSZ model with target $\mathcal{M} = T^*[d-1]M$, for some graded manifold M, for the ambient theory associated to Σ_d, we can consider a split construction for the auxiliary theory associated to the embedding $i\colon \Sigma_k \hookrightarrow \Sigma_d$. We set $\mathcal{N} = T^*[k-1]N$ for some graded manifold N. The description is analogously given by the one of the ambient theory as in Sect. 6.3. Hence we have

$$\mathcal{F}^{\mathcal{N}}_{\Sigma_k} = \mathrm{Map}(T[1]\Sigma_k, T^*[k-1]N), \tag{125}$$

and choosing a formal exponential map $\phi\colon TN \to N$ together with $y \in N$ we get

$$\begin{aligned}\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y} &= \mathrm{Map}(T[1]\Sigma_k, T^*[k-1]T_yN) \\ &\cong \Omega^\bullet(\Sigma_k) \otimes T_yN \oplus \Omega^\bullet(\Sigma_k) \otimes T^*_yN[k-1]. \end{aligned} \tag{126}$$

Then we can write $\widehat{\mathcal{A}} = \left(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}}\right) \in \widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}$ and $\widehat{\mathcal{B}} = \left(\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}}\right) \in \widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}$, thus we have an auxiliary formal global split AKSZ action given by

$$\begin{aligned}\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}\left(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}};i\right) &= \int_{\Sigma_k} \widehat{\boldsymbol{\beta}}_\ell \wedge \mathrm{d}_{\Sigma_k}\widehat{\boldsymbol{\alpha}}^\ell + \int_{\Sigma_k} \widehat{\Theta}_{\mathcal{E},y}\left(i^*\widehat{\boldsymbol{A}}, i^*\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}}\right) \\ &\quad + \int_{\Sigma_k} R^j_\ell\left(y,\widehat{\boldsymbol{\alpha}}\right)\widehat{\boldsymbol{\beta}}_j \wedge \mathrm{d}_N y^\ell, \end{aligned} \tag{127}$$

where $R \in \Omega^1\left(N, \mathrm{Der}\left(\widehat{\mathrm{Sym}}(T^*N)\right)\right)$.

8. From Pre-observables to Observables

8.1. AKSZ-Observables

We want to construct the observables for the AKSZ theories out of pre-obsrvables by integrating out means of auxiliary fields similarly as in Proposition 5.7. For a submanifold $i\colon \Sigma_k \hookrightarrow \Sigma_d$ we set

$$\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)=\int_{\mathcal{L}\subset\mathcal{F}_{\Sigma_k}}\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)}\in C^{\infty}\left(\mathcal{F}^{\mathcal{M}}_{\Sigma_d}\right). \tag{128}$$

There are several things to note. First, $\mathcal{O}_{\Sigma_k}$ depends only on the fields in $\mathcal{F}^{\mathcal{M}}_{\Sigma_d}$ via the pullback of $i\colon \Sigma_k \hookrightarrow \Sigma_d$, hence $Q_{\Sigma_d}(\mathcal{O}_{\Sigma_k})=0$ which is consistent with the definition of an observable. Moreover, the Q_{Σ_k}-cohomology class of $\mathcal{O}_{\Sigma_k}$ does not depend on deformations of the Lagrangian submanifold $\mathcal{L}\subset\mathcal{F}_{\Sigma_k}$ and is invariant under isotopies of Σ_k. We get the following Proposition.

Proposition 8.1. *Let* $\mathrm{Diff}_0(\Sigma_k)\subset\mathrm{Diff}(\Sigma_k)$ *be diffeomorphisms on* Σ_k *which are connected to the identity. Then for* $\varphi_k\in\mathrm{Diff}(\Sigma_k)$ *we have*

$$\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i\circ\varphi_k)=\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)+Q_{\Sigma_k}\textit{-exact}. \tag{129}$$

Proof. Indeed, we have

$$\begin{aligned}\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i\circ\varphi_k)&=\int_{\mathcal{L}\subset\mathcal{F}_{\Sigma_k}}\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i\circ\varphi_k)}\\&=\int_{\mathcal{L}}\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}\left(\mathcal{A},(\varphi_k^{-1})^*\mathcal{B};i\right)}\\&=\int_{(\varphi_k^{-1})^*\mathcal{L}}(\varphi_k^{-1})^*\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)}\\&=\int_{(\varphi_k^{-1})^*\mathcal{L}}\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)}\\&=\int_{\mathcal{L}}\mathscr{D}[\mathcal{B}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)}+Q_{\Sigma_k}\text{-exact}\\&=\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)+Q_{\Sigma_k}\text{-exact},\end{aligned} \tag{130}$$

where we think of $\int_{\mathcal{L}}\mathscr{D}[\mathcal{B}]$ to be in fact given by $\int_{\mathcal{L}}\sqrt{\mu}|_{\mathcal{L}}$, with μ being the functional integral measure on $\mathcal{F}^{\mathcal{N}}_{\Sigma_k}$. Moreover, we have used the isotopy property of φ_k to make sure that $\mathcal{L}$ and $(\varphi_k^{-1})^*\mathcal{L}$ are indeed homotopic. □

There is a similar invariance result for diffeomorphisms of the ambient manifold Σ_d which is the content of the following Proposition.

Proposition 8.2. *For a diffeomorphism* $\varphi_d\in\mathrm{Diff}(\Sigma_d)$ *we get*

$$\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};\varphi_d\circ i)=\mathcal{O}_{\Sigma_k}(\varphi_d^*\mathcal{A},\mathcal{B};i). \tag{131}$$

This is indeed true since $\mathcal{O}_{\Sigma_k}$ only depends of the ambient field $\mathcal{A}$ via the pullback by i.

Another important property is that the correlator of an observable should be invariant under ambient isotopies.

Proposition 8.3. *For $\varphi_d \in \mathrm{Diff}_0(\Sigma_d)$ we have*

$$\langle \mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};\varphi_d \circ i)\rangle = \langle \mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)\rangle. \tag{132}$$

Proof. Indeed, we have

$$\begin{aligned}\langle \mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};\varphi_d \circ i)\rangle &= \int_{\mathcal{L}\subset\mathcal{F}_{\Sigma_k}} \mathscr{D}[\mathcal{A}]\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};\varphi_d\circ i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\Sigma_d}(\mathcal{A})}\\ &= \int_{\mathcal{L}} \mathscr{D}[\mathcal{A}]\mathcal{O}_{\Sigma_k}(\varphi_d^*\mathcal{A},\mathcal{B};i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\Sigma_d}(\mathcal{A})}\\ &= \int_{\varphi_d^*\mathcal{L}} (\varphi_d^*)_*\mathscr{D}[\mathcal{A}]\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\Sigma_d}((\varphi^{-1})^*\mathcal{A})}\\ &= \int_{\varphi_d^*\mathcal{L}} \mathscr{D}[\mathcal{A}]\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\Sigma_d}(\mathcal{A})}\\ &= \int_{\mathcal{L}} \mathscr{D}[\mathcal{A}]\mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\Sigma_d}(\mathcal{A})}\\ &= \langle \mathcal{O}_{\Sigma_k}(\mathcal{A},\mathcal{B};i)\rangle,\end{aligned} \tag{133}$$

where we have used Proposition 8.2 and that the AKSZ action $\mathcal{S}_{\Sigma_d}$ (see Remark 7.2) and the functional integral measure $\mathscr{D}[\mathcal{A}]$ are invariant under diffeomorphisms for our theory is topological. □

8.2. Formal Global AKSZ-Observables

The construction above can be extended to a formal global one if we start with a formal global pre-observable. Then we have

$$\widehat{\mathcal{O}}_{\Sigma_k,y}\left(\widehat{\mathcal{A}},\widehat{\mathcal{B}};i\right) = \int_{\widehat{\mathcal{L}}\subset\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}} \mathscr{D}\left[\widehat{\mathcal{B}}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}(\widehat{\mathcal{A}},\widehat{\mathcal{B}};i)} \in C^\infty\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d}\right). \tag{134}$$

If we start with a split AKSZ theory we get

$$\begin{aligned}&\widehat{\mathcal{O}}_{\Sigma_k,y}\left(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}};i\right)\\ &\quad= \int_{\widehat{\mathcal{L}}\subset\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}} \mathscr{D}\left[\widehat{\boldsymbol{\alpha}}\right]\mathscr{D}\left[\widehat{\boldsymbol{\beta}}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}};i)} \in C^\infty\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d}\right).\end{aligned} \tag{135}$$

We have the following proposition (quantum version of (84)).

Proposition 8.4 (dQME). *The* differential quantum master equation *(dQME) for the formal global split AKSZ-observable holds:*

$$\boxed{\mathrm{d}_y\widehat{\mathcal{O}}_{\Sigma_k,y} - (-1)^d\mathrm{i}\hbar\Delta\widehat{\mathcal{O}}_{\Sigma_k,y} = 0.} \tag{136}$$

Proof. Note that we have

$$\begin{aligned}\mathrm{d}_y\widehat{\mathcal{O}}_{\Sigma_k,y} = -\frac{\mathrm{i}}{\hbar}\int_{\widehat{\mathcal{L}}\subset\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}} &\mathscr{D}\left[\widehat{\boldsymbol{\alpha}}\right]\mathscr{D}\left[\widehat{\boldsymbol{\beta}}\right]\\ &\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_k,y}(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}};i)}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}^{\mathrm{AKSZ}}_{\Sigma_k,y}\right\}_{\omega^{\mathcal{N}}_{\Sigma_k,y}},\end{aligned} \tag{137}$$

which we can write as

$$
\begin{aligned}
&-\frac{\mathrm{i}}{\hbar}\int_{\widehat{\mathcal{L}}\subset\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}}\mathscr{D}\left[\widehat{\boldsymbol{\alpha}}\right]\mathscr{D}\left[\widehat{\boldsymbol{\beta}}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}};i)}\left\{\widehat{\mathcal{S}}_{\Sigma_k,R,y},\widehat{\mathcal{S}}^{\text{AKSZ}}_{\Sigma_k,y}\right\}_{\omega^{\mathcal{N}}_{\Sigma_k,y}}\\
&\quad=-\Delta\int_{\widehat{\mathcal{L}}\subset\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}}\mathscr{D}\left[\widehat{\boldsymbol{\alpha}}\right]\mathscr{D}\left[\widehat{\boldsymbol{\beta}}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}(\widehat{\boldsymbol{A}},\widehat{\boldsymbol{B}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}};i)}\widehat{\mathcal{S}}_{\Sigma_k,R,y}
\end{aligned}\tag{138}
$$

if we assume that $\Delta\widehat{\mathcal{S}}_{\Sigma_k,R,y}=0$, which is true, e.g., if the Euler characteristic of Σ_k is zero or if $\mathrm{div}_{\mathsf{T}\phi^*\mu}R=0$, where μ is some volume form on $\mathcal{N}$. Note that $\mathrm{d}_y\mathsf{T}\phi^*\mu=-L_R\mathsf{T}\phi^*\mu$ which means that $\mathrm{div}_{\mathsf{T}\phi^*\mu}R=0$ if and only if $\mathrm{d}_y\mathsf{T}\phi^*\mu=0$. For any volume element μ it is always possible to find a formal exponential map ϕ such that the latter condition is satisfied. Note that this is then also translated into the *differential quantum master equation*

$$
\mathrm{d}_y\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}+\frac{1}{2}\left\{\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y},\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}\right\}_{\widehat{\omega}_{\Sigma_k,y}}-\mathrm{i}\hbar\Delta\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}=0,\tag{139}
$$

and by the assumption $\Delta\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}=0$, we obtain the differential CME as in (84). Hence the claim follows. $\square$

Remark 8.5. One can check that $\widehat{Q}_{\Sigma_d,x}\left(\widehat{\mathcal{O}}_{\Sigma_k,y}\right)=0$ and that Proposition 8.1 and 8.2 also hold for the formal global extension if we indeed start with a formal global pre-observable, i.e., that the assumption of Theorem 7.3 is satisfied.

Remark 8.6. The dQME as in (136) can be thought of as a descent equation for different form degrees. In fact we have

$$
\widehat{\delta}_{\mathrm{BV}}\widehat{\mathcal{O}}_{\Sigma_k,y}=(-1)^d\mathrm{d}_y\widehat{\mathcal{O}}_{\Sigma_k,y}\tag{140}
$$

since $\widehat{\mathcal{O}}_{\Sigma_k,y}$ is a formal global observable. We have set $\widehat{\delta}_{\mathrm{BV}}=\widehat{Q}_{\Sigma_d,x}-\mathrm{i}\hbar\Delta$.

Remark 8.7. Note that if $\mathcal{N}$ is a point, we have $\mathcal{V}_{\mathcal{E}}=0$, $\omega^{\mathcal{N}}_{\Sigma_k}=0$ and $\Theta_{\mathcal{E}}\in C^\infty(\mathcal{M})$. The associated pre-observable is then given by

$$
\mathcal{F}^{\mathcal{N}}_{\Sigma_k}=\mathrm{pt},\quad \mathcal{V}^{\mathcal{E}}_{\Sigma_k}=0,\quad \omega^{\mathcal{N}}_{\Sigma_k}=0,\quad \mathcal{S}^{\mathcal{N}}_{\Sigma_k}(\mathcal{A})=\int_{\Sigma_k}\Theta_{\mathcal{E}}(i^*\mathcal{A}).\tag{141}
$$

Hence, since there are no auxiliary fields $\mathcal{B}$, the constructed observable is given by

$$
\mathcal{O}_{\Sigma_k}(\mathcal{A};i)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\Sigma_k}\Theta_{\mathcal{E}}(i^*\mathcal{A})}.\tag{142}
$$

This can be easily lifted to a formal global pre-observable by

$$
\begin{gathered}
\widehat{\mathcal{F}}^{\mathcal{N}}_{\Sigma_k,y}=\mathrm{pt},\quad \widehat{\mathcal{V}}^{\mathcal{E}}_{\Sigma_k,y}=0,\quad \widehat{\omega}^{\mathcal{N}}_{\Sigma_k,y}=0,\\
\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_k,y}\left(\widehat{\mathcal{A}}\right)=\int_{\Sigma_k}\widehat{\Theta}_{\mathcal{E},y}\left(i^*\widehat{\mathcal{A}}\right),
\end{gathered}\tag{143}
$$

Thus, we get a formal global observable by

$$
\widehat{\mathcal{O}}_{\Sigma_k,y}\left(\widehat{\mathcal{A}};i\right)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\Sigma_k}\widehat{\Theta}_{\mathcal{E},y}(i^*\widehat{\mathcal{A}})}.\tag{144}
$$

8.3. Loop Observables

Let us consider the case where S^1 is embedded into Σ_d, i.e., $i\colon \Sigma_1 := S^1 \hookrightarrow \Sigma_d$ and assume that $\mathcal{N}$ is given by an ordinary symplectic manifold with symplectic structure $\omega_\mathcal{N} = \mathrm{d}_\mathcal{N}\alpha_\mathcal{N}$, which means that $\mathcal{N}$ is concentrated in degree zero. Let σ denote the coordinate on Σ_1. Then we can write the auxiliary field as

$$\mathcal{B}^\nu(\sigma, \mathrm{d}_{\Sigma_1}\sigma) = \mathcal{B}^\nu_{(0)}(\sigma) + \mathcal{B}^\nu_{(1)}\mathrm{d}_{\Sigma_1}\sigma, \tag{145}$$

and hence we get a pre-observable by

$$\mathcal{F}^\mathcal{N}_{\Sigma_1} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_1, \mathcal{N}), \tag{146}$$

$$\omega^\mathcal{N}_{\Sigma_1} = -\oint_{\Sigma_1} \omega^\mathcal{N}_{\nu_1\nu_2}\left(\mathcal{B}_{(0)}\right)\delta\mathcal{B}^{\nu_1}_{(0)} \wedge \delta\mathcal{B}^{\nu_2}_{(1)} + \oint_{\Sigma_1} \frac{1}{2}\mathcal{B}^{\nu_3}_{(1)}\partial_{\nu_3}\omega^\mathcal{N}_{\nu_1\nu_2}\left(\mathcal{B}_{(0)}\right)\delta\mathcal{B}^{\nu_1}_{(0)} \wedge \delta\mathcal{B}^{\nu_2}_{(0)}, \tag{147}$$

$$\mathcal{S}^\mathcal{N}_{\Sigma_1} = \oint_{\Sigma_1} \alpha^\mathcal{N}_\nu\left(\mathcal{B}_{(0)}\right)\mathrm{d}_{\Sigma_1}\mathcal{B}^\nu_{(0)} + \oint_{\Sigma_1}\Theta_\mathcal{E}(i^*\mathcal{A}, \mathcal{B}), \tag{148}$$

where $\omega^\mathcal{N}_{\nu_1\nu_2}$ are the coefficients of $\omega_\mathcal{N}$ and $\alpha^\mathcal{N}_\nu$ are the coefficients of $\alpha_\mathcal{N}$. Note that in this setting we have

$$\mathcal{F}^\mathcal{N}_{\Sigma_1} = \left\{\left(\mathcal{B}_{(0)}, \mathcal{B}_{(1)}\right)\,\middle|\, \mathcal{B}_{(0)}\colon \Sigma_1 \to \mathcal{N},\ \mathcal{B}_{(1)} \in \Gamma\left(\Sigma_1, T^*\Sigma_1 \otimes \mathcal{B}^*_{(0)}T^*\mathcal{N}\right)[-1]\right\}. \tag{149}$$

Hence we can construct the observable as

$$\mathcal{O}_{\Sigma_1}(\mathcal{A}; i) = \int_\mathcal{L} \mathscr{D}\left[\mathcal{B}_{(0)}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\oint_{\Sigma_1}\alpha^\mathcal{N}_\nu(\mathcal{B}_{(0)})\mathrm{d}_{\Sigma_1}\mathcal{B}_{(0)} + \frac{\mathrm{i}}{\hbar}\oint_{\Sigma_1}\Theta_\mathcal{E}\left(i^*\mathcal{A}, \mathcal{B}_{(0)}\right)}, \tag{150}$$

where we have chosen the natural Lagrangian submanifold

$$\mathcal{L} = \mathrm{Map}_{\mathrm{Mnf}}(\Sigma_1, \mathcal{N}) \subset \mathcal{F}^\mathcal{N}_{\Sigma_1}, \tag{151}$$

which is obtained by setting all odd variables $\mathcal{B}_{(1)}$ to zero.

Remark 8.8 (*Bohr–Sommerfeld*). If $\mathcal{N}$ is a differential graded symplectic manifold of degree different from zero, we know that the symplectic form $\omega_\mathcal{N}$ is always exact since we can write it as

$$\omega_\mathcal{N} = \mathrm{d}_\mathcal{N}(\iota_E\omega_\mathcal{N}),$$

(see [49]) where E is the *Euler vector field*. For the degree zero case, the symplectic form does not automatically have a primitive 1-form and hence one can not immediately define $\mathcal{S}^{\mathrm{kin}}_{\Sigma_1}$. However, one can also assume that $\omega_\mathcal{N}$ satisfies the *Bohr–Sommerfeld condition*, which says that

$$\frac{\omega_\mathcal{N}}{2\pi} \in H^2(\mathcal{N}, \mathbb{Z}).$$

Then the primitive 1-form can be understood as a Hermitian line bundle over $\mathcal{N}$ endowed with a $U(1)$-connection $\nabla_\mathcal{N}$ such that its curvature is given by $(\nabla_\mathcal{N})^2 = \omega_\mathcal{N}$. Thus we can define

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}^{\mathrm{kin}}_{\Sigma_1}(\mathcal{B})}$$

to be given by the holonomy of $\left(\mathcal{B}_{(0)}\right)^* \nabla_{\mathcal{N}}$ around Σ_1. Using Stokes' theorem we get

$$\mathcal{S}^{\mathrm{kin}}_{\Sigma_1}(\mathcal{B}) = \int_{\mathbb{D}} \left(\mathcal{B}^{\mathrm{ext}}_{(0)}\right)^* \omega_{\mathcal{N}}, \tag{152}$$

where $\mathbb{D}$ is a disk with $\partial\mathbb{D} = \Sigma_1$ and $\mathcal{B}^{\mathrm{ext}}_{(0)}$ is any extension of $\mathcal{B}_{(0)}$ to $\mathbb{D}$.

Remark 8.9. This construction can be obviously extended to the formal global case. The case of a dimension 1 submanifold gives the same auxiliary theory as for the case when our theory is split.

8.4. Formal Global Loop Observables

The following proposition is an extension of Proposition 5 in [46] to the formal global case.

Proposition 8.10. *Let $(\mathcal{N}, \omega_{\mathcal{N}})$ be a symplectic manifold and assume that it can be geometrically quantized to a complex vector space $\mathcal{H}$, the state space, and that the Hamiltonian $\Theta_{\mathcal{E}} \in C^\infty(\mathcal{E})$ can be quantized to an operator valued function $\mathbf{\Theta}_{\mathcal{E}} \in C^\infty(\mathcal{M}) \otimes \mathrm{End}(\mathcal{H})$. Moreover, for a formal exponential map $\phi\colon T\mathcal{M} \to \mathcal{M}$, let $\widehat{\mathbf{\Theta}}_{\mathcal{E},x} := \mathsf{T}\widetilde{\phi}^*_x \mathbf{\Theta}_{\mathcal{E}}$ and assume that*

$$\widehat{Q}_{\mathcal{M}}\left(\widehat{\mathbf{\Theta}}_{\mathcal{E},x}\right) + \widehat{R}_{\Sigma_d}\left(\widehat{\mathbf{\Theta}}_{\mathcal{E},x}\right) + \mathrm{i}\hbar\left(\widehat{\mathbf{\Theta}}_{\mathcal{E},x}\right)^2 = 0 \tag{153}$$

for $x \in \mathcal{M}$. Then for $\Sigma_1 := S^1$ we get that

$$\widehat{\mathcal{O}}_{\Sigma_1,x} = \mathrm{Tr}_{\mathcal{H}} \mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\oint_{\Sigma_1} \widehat{\mathbf{\Theta}}_{\mathcal{E},x}\left(i^*\widehat{\mathcal{A}}\right)\right) \tag{154}$$

is a formal global observable, where we have denoted by $\mathrm{Tr}_{\mathcal{H}}$ the trace map on $\mathcal{H}$ and $\mathcal{P}\exp$ denotes the path-ordered exponential.

Remark 8.11. Note that (153) is the formal global quantum version of (28).

Proof of Proposition 8.10. Let $\gamma\colon \Sigma_1 := [0,1] \to \Sigma_d$ be a path in Σ_d which is parametrized by $t \in [0,1]$. Denote by

$$\widehat{\boldsymbol{\psi}} := \widehat{\mathbf{\Theta}}_{\mathcal{E},x}\left(\gamma^*\widehat{\mathcal{A}}\right) \in \Omega^\bullet([0,1]) \otimes C^\infty\left(\widehat{\mathcal{F}}^{\mathcal{M}}_{\Sigma_d,x}\right) \otimes \mathrm{End}(\mathcal{H}).$$

Moreover denote by $\widehat{\boldsymbol{\psi}}_{(0)}(t)$ and $\widehat{\boldsymbol{\psi}}_{(1)}(t, \mathrm{d}t)$ the 0- and 1-form part of $\widehat{\boldsymbol{\psi}}$. Then, for the 1-form part, we get

$$\begin{aligned}\widehat{W}^{\gamma}_{\Sigma_1,x} &= \mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\int_0^1 \widehat{\boldsymbol{\psi}}_{(1)}\right)\\ &= \lim_{N\to\infty} \overleftarrow{\prod_{0\leq r\leq N}}\left(\mathrm{id}_{\mathcal{H}} + \frac{\mathrm{i}}{\hbar}\iota_{\frac{1}{N}\frac{\partial}{\partial t}}\widehat{\boldsymbol{\psi}}_{(1)}\left(\frac{r}{N}, \mathrm{d}t\right)\right) \in C^\infty\left(\widehat{\mathcal{F}}_{\Sigma_d,x}\right) \otimes \mathrm{End}(\mathcal{H})\end{aligned} \tag{155}$$

Then we get

$$\begin{aligned}\widehat{Q}_{\Sigma_d,x}\left(\widehat{W}^{\gamma}_{\Sigma_1,x}\right) &= -\mathrm{i}\hbar\int_0^1 \mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\int_t^1 \widehat{\psi}_{(1)}\right)\widehat{Q}_{\Sigma_d,x}\left(\widehat{\psi}(t,\mathrm{d}t)\right)\mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\int_0^t \psi_{(1)}\right)\\
&= -\mathrm{i}\hbar\int_0^1 \mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\int_t^1 \widehat{\psi}_{(1)}\right)\\
&\quad\left(\mathrm{d}t\frac{\partial}{\partial t}\widehat{\psi}_{(0)}(t)-\mathrm{i}\hbar\left[\widehat{\psi}_{(0)}(t),\widehat{\psi}_{(1)}(t,\mathrm{d}t)\right]\right)\mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\int_0^t \widehat{\psi}_{(1)}\right)\\
&= -\mathrm{i}\hbar\lim_{N\to\infty}\sum_{\ell=0}^{N-1}\overleftarrow{\prod_{\ell<r<N}}\left(\mathrm{id}_{\mathcal{H}}+\frac{\mathrm{i}}{\hbar}\iota_{\frac{1}{N}\frac{\partial}{\partial t}}\widehat{\psi}_{(1)}\left(\frac{r}{N},\mathrm{d}t\right)\right)\\
&\quad\times\left(\widehat{\psi}_{(0)}\left(\frac{\ell+1}{N}\right)\left(\mathrm{id}_{\mathcal{H}}+\frac{\mathrm{i}}{\hbar}\iota_{\frac{1}{N}\frac{\partial}{\partial t}}\widehat{\psi}_{(1)}\left(\frac{\ell}{N},\mathrm{d}t\right)\right)\right.\\
&\quad\left.-\left(\mathrm{id}_{\mathcal{H}}+\frac{\mathrm{i}}{\hbar}\iota_{\frac{1}{N}\frac{\partial}{\partial t}}\widehat{\psi}_{(1)}\left(\frac{\ell}{N},\mathrm{d}t\right)\right)\widehat{\psi}_{(0)}\left(\frac{\ell}{N}\right)\right)\\
&\quad\times\overleftarrow{\prod_{0\leq r<\ell}}\left(\mathrm{id}_{\mathcal{H}}+\frac{\mathrm{i}}{\hbar}\iota_{\frac{1}{N}\frac{\partial}{\partial t}}\widehat{\psi}_{(1)}\left(\frac{r}{N},\mathrm{d}t\right)\right)\\
&= -\mathrm{i}\hbar\left(\widehat{\psi}_{(0)}(1)\widehat{W}^{\gamma}_{\Sigma_1,x}-\widehat{W}^{\gamma}_{\Sigma_1,x}\widehat{\psi}_{(0)}(0)\right).\end{aligned}\tag{156}$$

We have used (154), which gives us

$$\widehat{Q}_{\Sigma_d,x}\left(\widehat{\psi}\right)=\mathrm{d}_{\Sigma_1}\widehat{\psi}-\mathrm{i}\hbar\left[\widehat{\psi},\widehat{\psi}\right],\tag{157}$$

where $[\ ,\ ]$ denotes the commutator of operators. Now if $\Sigma_1 := S^1$ we have $\gamma(0)=\gamma(1)$, and thus we get

$$\begin{aligned}\widehat{Q}_{\Sigma_d,x}\left(\widehat{O}_{\Sigma_1,x}\right) &= \mathrm{Tr}_{\mathcal{H}}\,\widehat{Q}_{\Sigma_d,x}\left(\widehat{W}^{\gamma}_{\Sigma_1,x}\right)\\
&= -\mathrm{i}\hbar\,\mathrm{Tr}_{\mathcal{H}}\left[\widehat{\boldsymbol{\Theta}}_{\mathcal{E},x}\left(\widehat{\mathcal{A}}_{(0)}(\gamma(0))\right),\widehat{W}^{\gamma}_{\Sigma_1,x}\right]=0,\end{aligned}\tag{158}$$

where $\widehat{\mathcal{A}}_{(0)}$ denotes the degree zero component of $\widehat{\mathcal{A}}$. □

Remark 8.12. The construction in Proposition 8.10 does not require $\omega_{\mathcal{N}}$ to be exact. It is in fact enough to require that $\omega_{\mathcal{N}}$ satisfies the Bohr–Sommerfeld condition as discussed in Remark 8.8. This is necessary for the assumption that $\mathcal{N}$ can be geometrically quantized.

8.5. Formal Global Loop Observables for the Poisson Sigma Model

The Poisson sigma model is an example of a 2-dimensional AKSZ theory which is split as in Definition 6.8. Let M be a Poisson manifold with Poisson bivector $\pi\in\Gamma\left(\bigwedge^2 TM\right)$. Moreover, consider a 2-dimensional source Σ_2. Let (x,p) be base and fiber coordinates on $T^*[1]M$. Then we can define a differential graded symplectic manifold as the target of the AKSZ theory by the data

$$\mathcal{M} = T^*[1]M, \tag{159}$$

$$Q_{\mathcal{M}} = \left\langle \pi(x), p\frac{\partial}{\partial x}\right\rangle + \frac{1}{2}\left\langle \frac{\partial}{\partial x}\pi(x), (p\wedge p)\otimes \frac{\partial}{\partial p}\right\rangle, \tag{160}$$

$$\omega_{\mathcal{M}} = \langle \delta p, \delta x\rangle, \tag{161}$$

$$\alpha_{\mathcal{M}} = \langle p, \delta x\rangle, \tag{162}$$

$$\Theta_{\mathcal{M}} = \frac{1}{2}\langle \pi(x), p\wedge p\rangle. \tag{163}$$

The corresponding 2-dimensional AKSZ-BV theory is given by the data

$$\begin{aligned}\mathcal{F}_{\Sigma_2} &= \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_2, T^*[1]M)\\ &\cong \Omega^\bullet(\Sigma_2)\otimes T_xM \oplus \Omega^\bullet(\Sigma_2)\otimes T^*_xM[1] \ni (\boldsymbol{X},\boldsymbol{\eta}),\end{aligned} \tag{164}$$

$$\omega_{\Sigma_2} = \int_{\Sigma_2}\langle \delta\boldsymbol{\eta}, \delta\boldsymbol{X}\rangle, \tag{165}$$

$$\mathcal{S}_{\Sigma_2} = \int_{\Sigma_2}\langle \boldsymbol{\eta}, \mathrm{d}_{\Sigma_2}\boldsymbol{X}\rangle + \frac{1}{2}\int_{\Sigma_2}\langle \pi(\boldsymbol{X}), \boldsymbol{\eta}\wedge\boldsymbol{\eta}\rangle. \tag{166}$$

Choosing a formal exponential map $\phi\colon TM \to M$ together with a background field $x\colon T[1]\Sigma_2 \to M$, the formal global action for the Poisson sigma model is given by

$$\begin{aligned}\widehat{\mathcal{S}}^{\mathrm{global}}_{\Sigma_2,x}\left(\widehat{\boldsymbol{X}},\widehat{\boldsymbol{\eta}}\right) &= \int_{\Sigma_2}\widehat{\boldsymbol{\eta}}_\ell\wedge \mathrm{d}_{\Sigma_2}\widehat{\boldsymbol{X}}^\ell + \frac{1}{2}\int_{\Sigma_2}\left(\mathsf{T}\widetilde{\phi}^*_x\pi\right)^{ij}\left(\widehat{\boldsymbol{X}}\right)\widehat{\boldsymbol{\eta}}_i\wedge\widehat{\boldsymbol{\eta}}_j\\ &\quad + \int_{\Sigma_2}R^j_\ell\left(x,\widehat{\boldsymbol{X}}\right)\widehat{\boldsymbol{\eta}}_j\wedge \mathrm{d}_M x^\ell.\end{aligned} \tag{167}$$

We want to construct a formal global Wilson loop like observables using the Poisson sigma model toegtehr with an auxiliary theory for an embedding $i\colon \Sigma_1 := S^1 \hookrightarrow \Sigma_2$. Consider an exact symplectic manifold $(\mathcal{N}, \omega_{\mathcal{N}} = \mathrm{d}_{\mathcal{N}}\alpha_{\mathcal{N}})$. We can construct a vertical vector field $\mathcal{V}$ on the trivial bundle $\mathcal{N}\times M\to\mathcal{N}$ which can be viewed as a map $\mathcal{N}\to\mathfrak{X}(M)$ with the property

$$\frac{1}{2}\{\mathcal{V},\mathcal{V}\}_{\omega_{\mathcal{N}}} + [\pi,\mathcal{V}]_{\mathrm{SN}} + R\wedge\mathcal{V} = 0,$$

where $[\ ,\ ]_{\mathrm{SN}}$ denotes the *Schouten–Nijenhuis* bracket defined on polyvector fields on M. We have a degree 0 Hamiltonian Q-bundle structure on

$$T^*[1]M\times\mathcal{N}\to T^*[1]M$$

with fiber $\mathcal{N}$ endowed with the structure

$$\mathcal{V}_{\mathcal{E}} = \langle p, \{\mathcal{V},\ \}_{\omega_{\mathcal{N}}}\rangle, \tag{168}$$

$$\Theta_{\mathcal{E}} = \langle p, \mathcal{V}\rangle, \tag{169}$$

where $\mathcal{E} = T^*[1]M\times\mathcal{N}$. If we use the notation of Sect. 8.3, we can associate a pre-observable to the Poisson sigma model given by the data (146) and (147) together with the auxiliary action

$$\mathcal{S}^{\mathcal{N}}_{\Sigma_1}(\boldsymbol{X},\boldsymbol{\eta},\mathcal{B};i) = \oint_{\Sigma_1}\alpha^{\mathcal{N}}_\nu(\mathcal{B})\mathrm{d}_{\Sigma_1}\mathcal{B}^\nu + \oint_{\Sigma_1}\langle i^*\boldsymbol{\eta}, \mathcal{V}(i^*\boldsymbol{X},\mathcal{B})\rangle. \tag{170}$$

Choosing a formal exponential map $\phi\colon T\mathcal{N} \to \mathcal{N}$ together with local coordinates we can lift this to a formal global auxiliary action

$$\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_1,y}\left(\widehat{\boldsymbol{X}},\widehat{\boldsymbol{\eta}},\widehat{\mathcal{B}};i\right) = \oint_{\Sigma_1} \alpha^{\mathcal{N}}_\nu\left(\widehat{\mathcal{B}}\right)\mathrm{d}_{\Sigma_1}\widehat{\mathcal{B}}^\nu + \oint_{\Sigma_1} \mathsf{T}\widetilde{\phi}^*_y\langle i^*\boldsymbol{\eta},\mathcal{V}(i^*\boldsymbol{X},\mathcal{B})\rangle \\ + \oint_{\Sigma_1}(\widehat{R}_{\Sigma_1})_\nu\left(y,\widehat{\mathcal{B}}\right)\mathrm{d}_{\mathcal{N}}y^\nu. \tag{171}$$

The corresponding auxiliary formal global observable is given by

$$\widehat{\mathcal{O}}_{\Sigma_1,y}\left(\widehat{\boldsymbol{X}},\widehat{\boldsymbol{\eta}};i\right) = \int_{\widehat{\mathcal{L}}}\mathscr{D}\left[\widehat{\mathcal{B}}_{(0)}\right]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\widehat{\mathcal{S}}^{\text{global}}_{\Sigma_1,y}\left(\widehat{\boldsymbol{X}},\widehat{\boldsymbol{\eta}},\widehat{\mathcal{B}}_{(0)};i\right)}, \tag{172}$$

where we use the gauge-fixing Lagrangian

$$\widehat{\mathcal{L}} = \mathrm{Map}_{\mathrm{Mnf}}(\Sigma_1,T_y\mathcal{N}) \subset \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_1,T_y\mathcal{N}) \cong \Omega^\bullet(\Sigma_1)\otimes T_y\mathcal{N}. \tag{173}$$

If we assume that $(\mathcal{N},\omega_{\mathcal{N}})$ can be geometrically quantized to a space of states $\mathcal{H}$ and $\mathcal{V}$ is quantized to an operator-valued vector field $\boldsymbol{\mathcal{V}} \in \mathrm{End}(\mathcal{H})\otimes\mathfrak{X}(M)$ such that $[\pi,\boldsymbol{\mathcal{V}}]_{\mathrm{SN}} + R\wedge\boldsymbol{\mathcal{V}} + \mathrm{i}\hbar\boldsymbol{\mathcal{V}}\wedge\boldsymbol{\mathcal{V}} = 0$, then we get that

$$\widehat{\mathcal{O}}_{\Sigma_1,x}\left(\widehat{\boldsymbol{X}},\widehat{\boldsymbol{\eta}};i\right) = \mathrm{Tr}_{\mathcal{H}}\mathcal{P}\exp\left(\frac{\mathrm{i}}{\hbar}\oint_{\Sigma_1}\langle i^*\widehat{\boldsymbol{\eta},\boldsymbol{\mathcal{V}}(i^*}\boldsymbol{X})\rangle\right), \tag{174}$$

which is, by Proposition 8.10, indeed a formal global observable. Here we have chosen an exponential map for the base of the target of the Poisson sigma model $TM \to M$ with background field $x \in M$.

9. Wilson Surfaces and Their Formal Global Extension

9.1. BF Theory and Wilson Surfaces

Let G be a Lie group and denote by $\mathfrak{g}$ its Lie algebra. Moreover, consider a principal G-bundle P over some d-manifold Σ_d and construct the adjoint bundle of P, denoted by $\mathrm{ad}P$, given as the frame bundle $P\times^{\mathrm{Ad}}\mathfrak{g}$ with respect to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ and let ad^*P denote its coadjoint bundle. Let $\mathscr{A}$ be the affine space of connection 1-forms on P and $\mathscr{G}$ the group of gauge transformations. For a connection $A \in \mathscr{A}$, let d_A be the covariant derivative on $\Omega^\bullet(\Sigma_d,\mathrm{ad}P)$ and $\Omega^\bullet(\Sigma_d,\mathrm{ad}^*P)$. Let $A \in \mathscr{A}$ and $B \in \Omega^{d-2}(M,\mathrm{ad}^*P)$ and define the BF action by

$$S(A,B) := \int_{\Sigma_d}\langle B,F_A\rangle, \tag{175}$$

where $\langle\ ,\ \rangle$ denotes the extension of the adjoint and coadjoint type for the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ to differential forms.

Remark 9.1 (*Abelian BF theory*). The *abelian BF* action, i.e., the action for the case where $\mathfrak{g} = \mathbb{R}$, in fact arises as the unperturbed part of many different AKSZ theories such as the Poisson sigma model or Chern–Simons theory. In fact, for the abelian case we have $(A,B) \in \Omega^\bullet(\Sigma_d)[1]\oplus\Omega^\bullet(\Sigma_d)[d-1]$ such that $F_A = \mathrm{d}A$ and thus we get an action $S = \int_{\Sigma_d} B\wedge\mathrm{d}A$.

The solutions to the Euler–Lagrange equations $\delta S = 0$ for S defined as in (226) are given by

$$\mathrm{M}_{\mathrm{cl}} = \left\{(A,B) \in \mathscr{A} \times \Omega^{d-2}(\Sigma_d, \mathrm{ad}^* P) \,\middle|\, F_A = 0, \mathrm{d}_A B = 0\right\} \tag{176}$$

Remark 9.2. One can check that the BF action is invariant under the action of

$$\widetilde{\mathscr{G}} := \mathscr{G} \rtimes \Omega^{d-3}(\Sigma_d, \mathrm{ad}^* P), \tag{177}$$

where $\mathscr{G}$ acts on $\Omega^{d-3}(\Sigma_d, \mathrm{ad}^* P)$ by the coadjoint action. For $(g,\sigma) \in \widetilde{\mathscr{G}}$ and $(A,B) \in \mathscr{A} \times \Omega^{d-2}(\Sigma_d, \mathrm{ad}^* P)$ we have an action

$$A \mapsto A^g, \qquad B \mapsto B^{(g,\sigma)} = \mathrm{Ad}^*_{g^{-1}} B + \mathrm{d}_{A^g}\sigma. \tag{178}$$

It is then easy to check that $S(A^g, B^{(g,\sigma)}) = S(A,B)$.

Consider an embedded submanifold $i\colon \Sigma_{d-2} \hookrightarrow \Sigma_d$ and consider the pull-back bundle of P by i according to the diagram

$$\begin{tikzcd} i^*P \arrow[r] \arrow[d] & P \arrow[d] \\ \Sigma_{d-2} \arrow[r, hook, "i"] & \Sigma_d \end{tikzcd}$$

We can now formulate an important type of classical action which is important for the study of higher-dimensional knots [27].

Definition 9.3 (*Wilson surface action*). The *Wilson surface action* is given by

$$W(\alpha,\beta,A,B;i) := \int_{\Sigma_{d-2}} \langle \alpha, \mathrm{d}_{i^*A}\beta + i^*B\rangle, \tag{179}$$

where $\alpha \in \Omega^0(\Sigma_{d-2}, \mathrm{ad}\, i^*P)$ and $\beta \in \Omega^{d-3}(\Sigma_{d-2}, \mathrm{ad}^*\, i^*P)$.

Definition 9.4 (*Wilson surface observable*). The *Wilson surface observable* is given by

$$\mathcal{W}_{\Sigma_{d-2}}(A,B;i) := \int \mathscr{D}[\alpha]\mathscr{D}[\beta]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}W(\alpha,\beta,A,B;i)} \tag{180}$$

Remark 9.5. The expectation values of Wilson surface observables in fact give certain higher-dimensional knot invariants [26]. These invariants are based on the construction of invariants by Bott [14] giving the generalization to a family of isotopy invariants for long knots $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$ for odd $n \geq 3$, which are based on constructions involving combinations of configuration space integrals. In [57] it was proven that these invariants are of finite type for the case of long ribbon knots and that they are related to the Alexander polynomial for these type of knots. Further generalizations based on this construction, in particular for rectifiable knots, have been given in [43,44].

9.2. BV Formulation of BF Theory

We can consider BF theory in terms of its BV extension. The BV space of fields is given by

$$\mathcal{F}_{\Sigma_d} = \Omega^\bullet(\Sigma_d, \mathrm{ad}\, P)[1] \oplus \Omega^\bullet(\Sigma_d, \mathrm{ad}^*\, P)[d-2], \tag{181}$$

where $\mathscr{A} = \Omega^1(\Sigma_d, \mathrm{ad}\, P)$. We will denote the superfields in $\mathcal{F}_{\Sigma_d}$ by $(\boldsymbol{A}, \boldsymbol{B})$. Note that there is an induced Lie bracket $[\![\ ,\ ]\!]$ on $\Omega^\bullet(\Sigma_d, \mathrm{ad}\, P)[1]$ which is induced by the Lie bracket on $\mathfrak{g}$.

Remark 9.6. If we consider local coordinates on $\mathfrak{g}$ with corresponding basis (e_i), we have

$$[\![a, b]\!] = (-1)^{\mathrm{gh}(a)\deg(b)} a^i b^j f_{ij}^k e_k, \tag{182}$$

where f_{ij}^k denotes the structure constants of $\mathfrak{g}$.

Moreover, for $\boldsymbol{A} \in \Omega^\bullet(\Sigma_d, \mathrm{ad}\, P)[1]$ we get the curvature

$$\boldsymbol{F}_{\boldsymbol{A}} = F_{A_0} + \mathrm{d}_{A_0}\boldsymbol{a} + \frac{1}{2}[\![\boldsymbol{a}, \boldsymbol{a}]\!], \tag{183}$$

where A_0 is any reference connection and $\boldsymbol{a} := \boldsymbol{A} - A_0 \in \Omega^\bullet(\Sigma_d, \mathrm{ad}\, P)[1]$.

Definition 9.7 (*BV action for BF theory*). The BV action for BF theory is defined by

$$\mathcal{S}_{\Sigma_d}(\boldsymbol{A}, \boldsymbol{B}) = \int_{\Sigma_d} \langle\!\langle \boldsymbol{B}, \boldsymbol{F}_{\boldsymbol{A}} \rangle\!\rangle, \tag{184}$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ is the extension to forms of the adjoint and coadjoint type of the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. For two forms a, b we have

$$\langle\!\langle a, b \rangle\!\rangle = (-1)^{\mathrm{gh}(a)\deg(b)} \langle a, b \rangle, \tag{185}$$

We can see that

$$\mathcal{F}_{\Sigma_d} = T^*[-1]\Omega^\bullet(\Sigma_d, \mathrm{ad}\, P)[1], \tag{186}$$

hence we have a canonical symplectic structure ω_{Σ_d} on $\mathcal{F}_{\Sigma_d}$. Similarly as before, let us denote the odd Poisson bracket induced by ω_{Σ_d} by $\{\ ,\ \}_{\omega_{\Sigma_d}}$ and note that $\mathcal{S}_{\Sigma_d}$ satisfies the CME

$$\{\mathcal{S}_{\Sigma_d}, \mathcal{S}_{\Sigma_d}\}_{\omega_{\Sigma_d}} = 0. \tag{187}$$

The cohomological vector field Q_{Σ_d} is given as the Hamiltonian vector field of $\mathcal{S}_{\Sigma_d}$, thus $Q_{\Sigma_d} = \{\mathcal{S}_{\Sigma_d},\ \}_{\omega_{\Sigma_d}}$. Note that

$$Q_{\Sigma_d}(\boldsymbol{A}) = (-1)^d \boldsymbol{F}_{\boldsymbol{A}}, \qquad Q_{\Sigma_d}(\boldsymbol{B}) = (-1)^d \mathrm{d}_{\boldsymbol{A}}\boldsymbol{B}. \tag{188}$$

If we choose a volume element μ which is compatible with ω_{Σ_d}, we can define the BV Laplacian by

$$\Delta\colon f \mapsto \frac{1}{2}\mathrm{div}_\mu\{f,\ \}_{\omega_{\Sigma_d}}. \tag{189}$$

Then we can show that the QME holds:

$$\delta_{\mathrm{BV}}\mathcal{S}_{\Sigma_d} = \{\mathcal{S}_{\Sigma_d}, \mathcal{S}_{\Sigma_d}\}_{\omega_{\Sigma_d}} - 2\mathrm{i}\hbar\Delta\mathcal{S}_{\Sigma_d} = 0. \tag{190}$$

This is in fact true since $\Delta\mathcal{S}_{\Sigma_d} = 0$. Moreover, as expected, we have $\delta_{\mathrm{BV}}^2 = 0$.

9.3. Formal Global BF Theory from the AKSZ Construction

Let us consider the case of abelian BF theory. Note that in this case the Wilson surface action is given by

$$W(\alpha,\beta,A,B;i) := \int_{\Sigma_{d-2}} \alpha(\mathrm{d}\beta + i^*B), \tag{191}$$

where d is the de Rham differential on $\mathbb{R}$. Solving the Euler–Lagrange equations for $\delta W = 0$, we get that the ciritical points are solutions to

$$\mathrm{d}\alpha = 0, \tag{192}$$

$$\mathrm{d}\beta + i^*B = 0. \tag{193}$$

We want to deal with B perturbatively, that means we can consider solutions to $\mathrm{d}\alpha = \mathrm{d}\beta = 0$ instead and hence we look at solutions of the form $\alpha = const$ and $\beta = 0$. This means that the constant field α is going to take the place of the background field. The Wilson surface observable is then given by

$$\mathcal{W}_{\Sigma_{d-2}}(A,B;i) = \int \mathscr{D}[\alpha]\mathscr{D}[\beta]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{\Sigma_{d-2}}\alpha\mathrm{d}\beta} \int_{x\in\mathbb{R}} \mu(x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}x\int_{\Sigma_{d-2}} i^*B}, \tag{194}$$

where μ is a volume element on the moduli space of classical solutions for the auxiliary theory which is given by

$$\mathrm{M}_{\mathrm{cl}} = \left\{(\alpha,\beta)\in\Omega^0(\Sigma_{d-2})\oplus\Omega^{d-3}(\Sigma_{d-2})\,\middle|\,\alpha = const,\ \beta = 0\right\} \cong \mathbb{R}. \tag{195}$$

By abbuse of notation we will also denote the perturbation of α around $x\in\mathbb{R}$ by α. Moreover, if we assume that P is a trivial bundle, not for the abelian case, we get

$$\mathcal{F}_{\Sigma_d} \cong \Omega^\bullet(\Sigma_d)\otimes\mathfrak{g}[1]\oplus\Omega^\bullet(\Sigma_d)\otimes\mathfrak{g}^*[d-2] \tag{196}$$

$$\cong \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d, \mathfrak{g}[1]\oplus\mathfrak{g}^*[d-2]). \tag{197}$$

Remark 9.8. The assumption that P is trivial is similar to a formal lift, whereas the background field is given by a constant critical point of the form $(x,0)$ with constant background field $x\colon T[1]\Sigma_d \to \mathfrak{g}[1]\oplus\mathfrak{g}^*[d-2]$. In fact, it induces a *linear split* theory as in Definition 6.7.

Remark 9.9 (*CE complex and L_∞-structure*). Let $\mathfrak{g}$ be a Lie algebra and consider the differential graded algebra

$$\mathrm{CE}(\mathfrak{g}) := \left(\bigwedge^{\bullet}\mathfrak{g}^*, \mathrm{d}_{\mathrm{CE}}\right) \cong (C^\infty(\mathfrak{g}[1]), Q)\,. \tag{198}$$

This is called the *Chevalley–Eilenberg algebra* of $\mathfrak{g}$ [28]. The real valued *Chevalley–Eilenberg complex* is given by

$$0\to \mathrm{Hom}\left(\bigwedge^0\mathfrak{g},\mathbb{R}\right) \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \mathrm{Hom}\left(\bigwedge^1\mathfrak{g},\mathbb{R}\right) \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \mathrm{Hom}\left(\bigwedge^2\mathfrak{g},\mathbb{R}\right) \xrightarrow{\mathrm{d}_{\mathrm{CE}}} \cdots \tag{199}$$

endowed with the *Chevalley–Eilenberg differential*

$$\mathrm{d}_{\mathrm{CE}}\colon \mathrm{Hom}\left(\bigwedge^n \mathfrak{g},\mathbb{R}\right)\to \mathrm{Hom}\left(\bigwedge^{n+1}\mathfrak{g},\mathbb{R}\right)$$

given by

$$\begin{aligned}(\mathrm{d}_{\mathrm{CE}}F)&(X_1,\dots,X_{n+1})\\ &:=\sum_{j=1}^{n+1}(-1)^{j+1}X_iF(X_1,\dots,\widehat{X}_j,\dots,X_{n+1})\\ &\quad+\sum_{1\leq j<k\leq n+1}(-1)^{j+k}F([X_j,X_k],X_1,\dots,\widehat{X}_j,\dots,\widehat{X}_k,\dots,X_{n+1}),\end{aligned}\tag{200}$$

where the hat means that these elements are omitted. Denote by (ξ^i) the coordinates on $\mathfrak{g}[1]$ of degree $+1$. Then Q has to be of the form

$$Q=-\frac{1}{2}f_{ij}^k\xi^i\xi^j\frac{\partial}{\partial\xi^k},$$

where f_{ij}^k are the structure constants of $\mathfrak{g}$. Note that a function $F\in\mathrm{Hom}\left(\bigwedge^n\mathfrak{g},\mathbb{R}\right)$ corresponds to an element in $C_n^\infty(\mathfrak{g}[1])$ such that the Chevalley–Eilenberg differential is indeed mapped to Q under the isomorphism

$$F(X_{j_1}\wedge\ldots\wedge X_{j_n})=:F_{j_1\ldots j_n}\longleftrightarrow\frac{1}{n!}\xi^{j_1}\cdots\xi^{j_n}F_{j_1\ldots j_n}.$$

In fact, for a graded vector space $\mathfrak{g}=\bigoplus_{k\in\mathbb{Z}}\mathfrak{g}_k$, the differential graded algebra $(C^\infty(\mathfrak{g}),Q)$ corresponds to an L_∞-algebra which is actually given by the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g}[-1])$ of the L_∞-algebra $\mathfrak{g}[-1]$. The dual of the cohomological vector field Q is given by a codifferential D of homogenous degree $+1$ on $\widehat{\mathrm{Sym}}(\mathfrak{g})\cong\widehat{\mathrm{Sym}}(\mathfrak{g}[-1])$. The isomorphism is induced by the *shift isomorphism* $s\colon\mathfrak{g}\xrightarrow{\sim}\mathfrak{g}[1]$. The codifferential D decomposes into a sum $D=\sum_{j\geq1}\bar{D}_j$ such that the restrictions

$$D_j:=\bar{D}_j\big|_{\widehat{\mathrm{Sym}}^j(\mathfrak{g})}\colon\widehat{\mathrm{Sym}}^j(\mathfrak{g})\to\mathfrak{g}$$

satisfy

$$\ell_j=(-1)^{\frac{1}{2}j(j-1)+1}s^{-1}\circ D_j\circ s^{\otimes j},\qquad\forall j\geq1.$$

Note that since $Q^2=0$, we get $D^2=0$. Such a codifferential induces a classical Grothendieck connection as in Sect. 6.1.

Remark 9.10 (L_∞*-structure on* $\Omega^\bullet$). If $\mathfrak{g}$ is endowed with a (curved) L_∞-structure, we can view

$$\Omega^\bullet(\Sigma_d,\mathfrak{g})=\bigoplus_{\substack{r+j=k\\0\leq r\leq d\\j\in\mathbb{Z}}}\Omega^r(\Sigma_d)\otimes\mathfrak{g}_j$$

as a (curved) L_∞-algebra. The L_∞-structure arises as the linear extension of the higher brackets

$$\hat{\ell}_1(\alpha_1 \otimes X_1) := \mathrm{d}_{\Sigma_d}\alpha_1 \otimes X_1 + (-1)^{\deg(\alpha_1)}\alpha_1 \otimes \ell_1(X_1) \tag{201}$$

$$\begin{aligned}\hat{\ell}_n(\alpha_1 \otimes X_1, \dots, \alpha_n \otimes X_n) &:= (-1)^{n\sum_{j=1}^n \deg(\alpha_j) + \sum_{j=0}^{n-2}\deg(\alpha_{n-j})\sum_{k=1}^{n-j-1}\deg(X_k)} \\ &\quad\times (\alpha_1 \wedge \dots \wedge \alpha_n) \otimes \ell_n(X_1, \dots, X_n)\end{aligned} \tag{202}$$

for $n \geq 2$, $\alpha_1, \dots, \alpha_n \in \Omega^\bullet(\Sigma_d)$ and $X_1, \dots, X_n \in \mathfrak{g}$. If $\mathfrak{g}$ is cyclic, and Σ_d is compact, oriented without boundary, there is a natural cyclic inner product on $\Omega^\bullet(\Sigma_d, \mathfrak{g})$ given by

$$\begin{aligned}&\langle \alpha_1 \otimes X_1, \alpha_2 \otimes X_2\rangle_{\Omega^\bullet(\Sigma_d,\mathfrak{g})} \\ &\quad = (-1)^{\deg(\alpha_2)\deg(X_1)}\int_{\Sigma_d}\alpha_1 \wedge \alpha_2 \langle X_1, X_2\rangle_{\mathfrak{g}}\end{aligned} \tag{203}$$

for $\alpha_1, \alpha_2 \in \Omega^\bullet(\Sigma_d)$ and $X_1, X_2 \in \mathfrak{g}$.

9.4. BV Extension of Wilson Surfaces

We will now construct the BV extended observable for the auxiliary codimension 2 theory in the case where P is a trivial bundle. Let

$$\mathcal{F}_{\Sigma_{d-2}} \cong \Omega^\bullet(\Sigma_{d-2}) \otimes \mathfrak{g}[1] \oplus \Omega^\bullet(\Sigma_{d-2}) \otimes \mathfrak{g}^*[d-2] \tag{204}$$

endowed with the symplectic form $\omega_{\Sigma_{d-2}}$ which induces the corresponding BV bracket $\{\ ,\ \}_{\omega_{\Sigma_{d-2}}}$. For auxiliary superfields $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{F}_{\Sigma_{d-2}}$ and ambient superfields $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{F}_{\Sigma_d}$ we have the following definition:

Definition 9.11 (*BV extended Wilson surface action*). The *BV extended Wilson surface action* is given by

$$\boldsymbol{W}^0_{\Sigma_{d-2}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{A}, \boldsymbol{B}; i) = \int_{\Sigma_{d-2}} \langle\!\langle \boldsymbol{\alpha}, \mathrm{d}_{i^*\boldsymbol{A}}\boldsymbol{\beta} + i^*\boldsymbol{B}\rangle\!\rangle. \tag{205}$$

Remark 9.12. As it was shown in [27], we can extend $\boldsymbol{W}^0_{\Sigma_{d-2}}$, regarded as a function on embeddings $i\colon \Sigma_{d-2} \hookrightarrow \Sigma_d$, to a form-valued function $\boldsymbol{W}_{\Sigma_{d-2}}$ on these embeddings by setting

$$\boldsymbol{W}_{\Sigma_{d-2}}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{A}, \boldsymbol{B}; i) := \pi_*\langle\!\langle \boldsymbol{\alpha}, \mathrm{d}_{\mathrm{ev}^*\boldsymbol{A}}\boldsymbol{\beta} + \mathrm{ev}^*\boldsymbol{B}\rangle\!\rangle, \tag{206}$$

where ev denotes the evaluation map of embeddings $\Sigma_{d-2} \hookrightarrow \Sigma_d$ and π_* denotes the integration along the fiber Σ_{d-2}.

Proposition 9.13 ([27]). *The Wilson surface action satisfies a modified version of the dCME, i.e., we have*

$$\mathrm{d}\boldsymbol{W}_{\Sigma_{d-2}} - (-1)^d\left\{\boldsymbol{W}_{\Sigma_{d-2}}, \boldsymbol{W}_{\Sigma_{d-2}}\right\}_{\omega_{\Sigma_d}} - \frac{1}{2}\left\{\boldsymbol{W}_{\Sigma_{d-2}}, \boldsymbol{W}_{\Sigma_{d-2}}\right\}_{\omega_{\Sigma_{d-2}}} = 0 \tag{207}$$

Remark 9.14. Proposition 9.13 is a consequense of the fact that

$$\mathrm{d}\int_{\Sigma_{d-2}} = (-1)^d\int_{\Sigma_{d-2}}\mathrm{d}$$

and (188).

Denote by $\Delta_{\Sigma_{d-2}}$ the BV Laplacian for the auxiliary theory. Then we get the following proposition.

Proposition 9.15 ([27]). *Define the vector field*

$$\boldsymbol{Q}_{\Sigma_{d-2}} = \left\{\boldsymbol{W}_{\Sigma_{d-2}},\ \right\}_{\omega_{\Sigma_{d-2}}}, \tag{208}$$

which acts on generators by

$$\boldsymbol{Q}_{\Sigma_{d-2}}(\boldsymbol{\alpha}) = (-1)^d \mathrm{d}_{\mathrm{ev}^*\boldsymbol{A}}\boldsymbol{\alpha}, \qquad \boldsymbol{Q}_{\Sigma_{d-2}}(\boldsymbol{\beta}) = (-1)^d(\mathrm{d}_{\mathrm{ev}^*\boldsymbol{A}}\boldsymbol{\beta} + \mathrm{ev}^*\boldsymbol{B}). \tag{209}$$

Assume that the formal measure $\mathscr{D}[\boldsymbol{\alpha}]\mathscr{D}[\boldsymbol{\beta}]$ is invariant with respect to the vector fields (209). Then we have

$$\begin{aligned} &\mathrm{d}\boldsymbol{W}_{\Sigma_{d-2}} - (-1)^d\left(\delta_{\mathrm{BV}}\boldsymbol{W}_{\Sigma_{d-2}} + \frac{1}{2}\left\{\boldsymbol{W}_{\Sigma_{d-2}}, \boldsymbol{W}_{\Sigma_{d-2}}\right\}_{\omega_{\Sigma_d}}\right)\\ &\quad + \frac{1}{2}\left(\left\{\boldsymbol{W}_{\Sigma_{d-2}}, \boldsymbol{W}_{\Sigma_{d-2}}\right\}_{\omega_{\Sigma_{d-2}}} - 2\mathrm{i}\hbar\Delta_{\Sigma_{d-2}}\boldsymbol{W}_{\Sigma_{d-2}}\right) = 0 \end{aligned} \tag{210}$$

Remark 9.16. Note that the assumption of invariance of the formal measure implies that $\Delta_{\Sigma_{d-2}}\boldsymbol{W}_{\Sigma_{d-2}} = 0$.

Definition 9.17 (*BV extended Wilson surface observable*). We define the *BV extended Wilson surface observable* as

$$\mathcal{W}_{\Sigma_{d-2}}(\boldsymbol{A}, \boldsymbol{B}; i) = \int \mathscr{D}[\boldsymbol{\alpha}]\mathscr{D}[\boldsymbol{\beta}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\boldsymbol{W}_{\Sigma_{d-2}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{A},\boldsymbol{B};i)} \tag{211}$$

9.5. Formulation by Hamiltonian $\boldsymbol{Q}$-Bundles

Let $\mathcal{M} = \mathfrak{g}[1]\oplus\mathfrak{g}^*[d-2]$. Denote by $x\colon \mathfrak{g}[1] \to \mathfrak{g}$ the degree 1 $\mathfrak{g}$-valued coordinate on $\mathfrak{g}[1]$ and let $x^*\colon \mathfrak{g}^*[d-2] \to \mathfrak{g}^*$ be the $\mathfrak{g}^*$-valued coordinate on $\mathfrak{g}^*[d-2]$ of degree $d-2$.

Then we can consider a trivial Hamiltonian Q-bundle over $\mathcal{M}$ given by the fiber data

$$\mathcal{N} = \mathfrak{g}\oplus\mathfrak{g}^*[d-3], \tag{212}$$

$$\mathcal{V}_{\mathcal{E}} = \left\langle [x,y], \frac{\partial}{\partial y}\right\rangle + \left\langle \mathrm{ad}^*_x\, y^*, \frac{\partial}{\partial y^*}\right\rangle + (-1)^d\left\langle x^*, \frac{\partial}{\partial y^*}\right\rangle, \tag{213}$$

$$\omega_{\mathcal{N}} = \langle\delta y^*, \delta y\rangle, \tag{214}$$

$$\alpha_{\mathcal{N}} = \langle y^*, \delta y\rangle, \tag{215}$$

$$\Theta_{\mathcal{E}} = \langle y^*, [x,y]\rangle + \langle x^*, y\rangle, \tag{216}$$

where y is the $\mathfrak{g}$-valued coordinate of degree 0 on $\mathfrak{g}$ and y^* is the $\mathfrak{g}^*$-valued coordinate of degree $d-3$ on $\mathfrak{g}^*[d-3]$. For an embedding $i\colon \Sigma_{d-2} \hookrightarrow \Sigma_d$ we get the auxiliary theory

$$\mathcal{F}^{\mathcal{N}}_{\Sigma_{d-2}} = \Omega^\bullet(\Sigma_{d-2})\otimes\mathfrak{g}\oplus\Omega^\bullet(\Sigma_{d-2})\otimes\mathfrak{g}^*[d-3], \tag{217}$$

$$\omega^{\mathcal{N}}_{\Sigma_{d-2}} = (-1)^d\int_{\Sigma_{d-2}}\langle\delta\boldsymbol{y}^*,\delta\boldsymbol{y}\rangle, \tag{218}$$

$$\mathcal{S}^{\mathcal{N}}_{\Sigma_{d-2}} = \int_{\Sigma_{d-2}}\langle\boldsymbol{y}^*,\mathrm{d}_{\Sigma_{d-2}}\boldsymbol{y}\rangle+\int_{\Sigma_{d-2}}\langle\boldsymbol{y}^*,[i^*\boldsymbol{A},\boldsymbol{y}]\rangle+\int_{\Sigma_{d-2}}\langle i^*\boldsymbol{B},\boldsymbol{y}\rangle. \tag{219}$$

Note that $\mathcal{M}$ is a differential graded symplectic manifold with the following data:

$$Q_{\mathcal{M}} = \left\langle\frac{1}{2}[x,x],\frac{\partial}{\partial x}\right\rangle+\left\langle\mathrm{ad}^*_x\,x^*,\frac{\partial}{\partial x^*}\right\rangle, \tag{220}$$

$$\omega_{\mathcal{M}} = \langle\delta x^*,\delta x\rangle, \tag{221}$$

$$\alpha_{\mathcal{M}} = \langle x^*,\delta x\rangle, \tag{222}$$

$$\Theta_{\mathcal{M}} = \frac{1}{2}\langle x^*,[x,x]\rangle\,. \tag{223}$$

Hence the ambient theory is given by

$$\mathcal{F}^{\mathcal{M}}_{\Sigma_d} = \Omega^\bullet(\Sigma_d)\otimes\mathfrak{g}[1]\oplus\Omega^\bullet(\Sigma_d)\otimes\mathfrak{g}^*[d-2]\ni(\boldsymbol{A},\boldsymbol{B}), \tag{224}$$

$$\omega_{\Sigma_d} = \int_{\Sigma_d}\langle\delta\boldsymbol{B},\delta\boldsymbol{A}\rangle, \tag{225}$$

$$\mathcal{S}_{\Sigma_d} = \int_{\Sigma_d}\langle\boldsymbol{B},\boldsymbol{F}_{\boldsymbol{A}}\rangle = \int_{\Sigma_d}\left\langle\boldsymbol{B},\mathrm{d}_{\Sigma_d}\boldsymbol{A}+\frac{1}{2}[\boldsymbol{A},\boldsymbol{A}]\right\rangle. \tag{226}$$

Note that (226) is exactly the BF action as in Definition 9.7. In the case of abelian BF theory, i.e., when $\mathfrak{g}=\mathbb{R}$, we get that $Q_{\mathcal{M}}=0$, $\Theta_{\mathcal{M}}=0$ and the ambient theory

$$\mathcal{F}^{\mathcal{M}}_{\Sigma_d} = \Omega^\bullet(\Sigma_d)[1]\oplus\Omega^\bullet(\Sigma_d)[d-2], \tag{227}$$

$$\omega_{\Sigma_d} = \int_{\Sigma_d}\delta\boldsymbol{B}\wedge\delta\boldsymbol{A}, \tag{228}$$

$$\mathcal{S}_{\Sigma_d} = \int_{\Sigma_d}\boldsymbol{B}\wedge\mathrm{d}_{\Sigma_d}\boldsymbol{A}. \tag{229}$$

Remark 9.18. The constructions presented in this paper are expected to extend to manifolds with boundary. Using the constructions as in [25] together with the quantum BV-BFV formalism [21,23], one can show how the formal global observables for split AKSZ sigma models on the boundary induce a more general gauge condition as the dQME which is called *modified differential Quantum Master Equation (mdQME)*. This condition also handles the boundary part which arises as the ordered standard quantization Ω of the boundary action $\mathcal{S}^\partial$ of dgeree $+1$, induced by the underlying BFV manifold, plus some higher degree terms. The mdQME is then given as some annihilation condition for the formal global boundary observable $\boldsymbol{O}^\partial$. In fact, it is annihilated by the quantum Grothendieck BFV operator $\nabla_{\mathsf{G}}:=\mathrm{d}-\mathrm{i}\hbar\Delta+\frac{\mathrm{i}}{\hbar}\Omega$ (see [24,25]), which means that $\nabla_{\mathsf{G}}\boldsymbol{O}^\partial=0$.

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References

[1] Alexandrov, M., Kontsevich, M., Schwarz, A., Zaboronsky, O.: The geometry of the master equation and topological quantum field theory. Int. J. Modern Phys. A **12**(7), 1405–1429 (1997)

[2] Atiyah, M.F., Bott, R.: The Yang–Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond. **308**, 523–615 (1983)

[3] Axelrod, S., Singer, I.M.: Chern–Simons perturbation theory. In: Differential Geometric Methods in Theoretical Physics, Proceedings, New York, vol. 1, pp. 3–45 (1991)

[4] Axelrod, S., Singer, I.M.: Chern-Simons perturbation theory. II. J. Differ. Geom. **39**(1), 173–213 (1994)

[5] Batalin, I.A., Vilkovisky, G.A.: Quantization of gauge theories with linearly dependent generators. Phys. Rev. D **28**(10), 2567–2582 (1983)

[6] Batalin, I., Fradkin, E.: A generalized canonical formalism and quantization of reducible gauge theories. Phys. Lett. B **122**(2), 157–164 (1983)

[7] Batalin, I., Vilkovisky, G.: Gauge algebra and quantization. Phys. Lett. B **102**(1), 27–31 (1981)

[8] Batalin, I., Vilkovisky, G.: Relativistic S-matrix of dynamical systems with boson and fermion constraints. Phys. Lett. B **69**(3), 309–312 (1977)

[9] Becchi, C., Rouet, A., Stora, R.: Renormalization of gauge theories. Ann. Phys. **98**(2), 287–321 (1976)

[10] Becchi, C., Rouet, A., Stora, R.: Renormalization of the abelian Higgs–Kibble model. Commun. Math. Phys. **42**(2), 127–162 (1975)

[11] Becchi, C., Rouet, A., Stora, R.: The abelian Higgs Kibble model, unitarity of the S-operator. Phys. Lett. B **52**(3), 344–346 (1974)

[12] Bonechi, F., Cattaneo, A.S., Mnev, P.: The Poisson sigma model on closed surfaces. J. High Energy Phys. **1**, 099 (2012). 26

[13] Bott, R.: Some aspects of invariant theory in differential geometry. In: Differential Operators on Manifolds, pp. 49–145. Springer, Berlin (2010)

[14] Bott, R.: Configuration spaces and imbedding invariants. Turkish J. Math. **20**, 1–17 (1996)

[15] Cattaneo, A.S., Felder, G.: A path integral approach to the Kontsevich quantization formula. Commun. Math. Phys. **212**, 591–611 (2000)

[16] Cattaneo, A.S., Felder, G.: On the AKSZ formulation of the Poisson sigma model. Lett. Math. Phys. **56**(2), 163–179 (2001)

[17] Cattaneo, A.S., Felder, G.: On the Globalization of Kontsevich's star product and the perturbative poisson sigma model. Progr. Theor. Phys. Suppl. **144**, 38–53 (2001)

[18] Cattaneo, A.S., Mnev, P.: Remarks on Chern–Simons invariants. Commun. Math. Phys. **293**, 803–836 (2008)

[19] Cattaneo, A.S., Mnev, P., Reshetikhin, N.: Classical and quantum Lagrangian field theories with boundary. In: PoS CORFU2011, p. 44 (2011)

[20] Cattaneo, A.S., Mnev, P., Reshetikhin, N.: Classical BV theories on manifolds with boundary. Commun. Math. Phys. **332**(2), 535–603 (2014)

[21] Cattaneo, A.S., Mnev, P., Reshetikhin, N.: Perturbative quantum Gauge theories on manifolds with boundary. Commun. Math. Phys. **357**(2), 631–730 (2017)

[22] Cattaneo, A.S., Mnev, P., Wernli, K.: Split Chern–Simons theory in the BV–BFV formalism. In: Quantization, Geometry and Noncommutative Structures in Mathematics and Physics, pp. 293–324. Springer, New York (2017)

[23] Cattaneo, A.S., Moshayedi, N.: Introduction to the BV-BFV formalism. Rev. Math. Phys. **32**, 67 (2020)

[24] Cattaneo, A.S., Moshayedi, N., Wernli, K.: Globalization for perturbative quantization of nonlinear split AKSZ sigma models on manifolds with boundary. Commun. Math. Phys. **372**(1), 213–260 (2019)

[25] Cattaneo, A.S., Moshayedi, N., Wernli, K.: On the globalization of the poisson sigma model in the BV-BFV formalism. Commun. Math. Phys. **375**(1), 41–103 (2020)

[26] Cattaneo, A.S., Rossi, C.: Higher-dimensional BF theories in the Batalin–Vilkovisky formalism: The BV action and generalized Wilson Loops. Commun. Math. Phys. **221**, 591–657 (2001)

[27] Cattaneo, A.S., Rossi, C.: Wilson surfaces and higher dimensional knot invariants. Commun. Math. Phys. **256**, 513–537 (2005)

[28] Chevalley, C., Eilenberg, S.: Cohomology theory of Lie groups and Lie algebras. Trans. Am. Math. Soc. **63**, 85–124 (1948)

[29] Costello, K.: A geometric construction of the Witten genus II (2011). arXiv:1112.0816v2

[30] Fedosov, B.V.: A simple geometrical construction of deformation quantization. J. Differ. Geom. **40**(2), 213–238 (1994)

[31] Fradkin, E., Vilkovisky, G.: Quantization of relativistic systems with constraints. Phys. Lett. B **55**(2), 224–226 (1975)

[32] Gelfand, I.M., Fuks, D.B.: The cohomology of the Lie algebra of formal vector fields. Izv. AN SSR **34**, 110–116 (1970)

[33] Gelfand, I.M., Fuks, D.B.: The cohomology of the Lie algebra of vector fields on a smooth manifold. J. Funct. Anal. **33**, 194–210 (1969)

[34] Gelfand, I.M., Kazhdan, D.A.: Some problems of the differential geometry and the calculation of cohomologies of Lie algebras of vector fields. Dokl. Akad. Nauk Ser. Fiz. **200**, 269–272 (1971)

[35] Ikeda, N.: Two-dimensional gravity and nonlinear Gauge theory. Ann. Phys. **235**(2), 435–464 (1994)

[36] Iraso, R., Mnev, P.: Two-dimensional Yang–Mills theory on surfaces with corners in Batalin–Vilkovisky formalism. Commun. Math. Phys. **370**, 637–702 (2019)

[37] Jurčo, B., Raspollini, L., Sämann, C., Wolf, M.: L1-algebras of classical field theories and the Batalin–Vilkovisky formalism. arXiv:1809.09899

[38] Khudaverdian, H.M.: Semidensities on odd symplectic supermanifolds. Commun. Math. Phys. **247**(2), 353–390 (2004)

[39] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)

[40] Kontsevich, M.: Feynman diagrams and low-dimensional topology. English. In: Joseph, A., Mignot, F., Murat, F., Prum, B., Rentschler, R. (eds.) First European Congress of Mathematics Paris, July 6–10, 1992, vol. 120, pp. 97–121. Progress in Mathematics. Birkhäuser, Basel (1994)

[41] Kotov, A., Strobl, T.: Characteristic classes associated to Q-bundles. Int. J, Geom. Methods Modern Phys. **12**(01), 23 (2015)

[42] Lazarev, A., Schedler, T.: Curved infinity-algebras and their characteristic classes. J. Topol. **5**, 503–528 (2012)

[43] Leturcq, D.: Generalized Bott–Cattaneo–Rossi invariants of high-dimensional long knots. arXiv:1907.01712

[44] Leturcq, D.: The Reidemeister torsion of high-dimensional long knots from configuration space integrals. arXiv:2003.01007

[45] Mehta, R., Zambon, M.: L1-algebra actions. Differ. Geom. Appl. **30**, 576–587 (2012)

[46] Mnev, P.: A construction of observables for AKSZ sigma models. Lett. Math. Phys. **105**, 1735–1783 (2015)

[47] Mnev, P.: Discrete BF theory. 2008. arXiv:0809.1160

[48] Mnev, P.: Quantum Field Theory: Batalin–Vilkovisky Formalism and Its Applications. vol. 72. University Lecture Series. American Mathematical Society (AMS), (2019)

[49] Roytenberg, D.: AKSZ-BV formalism and courant algebroid-induced topological field theories. Lett. Math. Phys. **79**, 143–159 (2007)

[50] Rozansky, L., Witten, E.: Hyper-Kähler geometry and invariants of three-manifolds. Selecta Math. **3**, 401–458 (1997)

[51] Schaller, P., Strobl, T.: Poisson structure induced (topological) field theories. Mod. Phys. Lett. A **09**(33), 3129–3136 (1994)

[52] Schwarz, A.: Geometry of Batalin-Vilkovisky quantization. Commun. Math. Phys. **155**(2), 249–260 (1993)

[53] Ševera, P.: On the origin of the BV operator on odd symplectic supermanifolds. Lett. Math. Phys. **78**(1), 55–59 (2006)

[54] Stasheff, J.: Homological reduction of constrained poisson algebras. J. Differ. Geom. **45**(1), 221–240 (1997)

[55] Stasheff, J.: The (secret?) homological algebra of the Batalin–Vilkovisky approach. arXiv:hep-th/9712157

[56] Tyutin, I.V.: Gauge invariance in field theory and statistical physics in operator formalism. In: Preprints of P.N. Lebedev Physical Institute, No. 39 (1976)

[57] Watanabe, T.: Configuration space integral for long n-knots and the Alexander Polynomial. Algebr. Geom. Topol. **7**(1), 47–92 (2007)

[58] Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**(3), 351–399 (1989)

[59] Witten, E.: Topological sigma models. Commun. Math. Phys. **118**(3), 411–449 (1988)

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Chapter 6

Computation of Kontsevich Weights of Connection and Curvature Graphs for Symplectic Poisson Structures

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Computation of Kontsevich weights of connection and curvature graphs for symplectic Poisson structures

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We give a detailed explicit computation of weights of Kontsevich graphs which arise from connection and curvature terms within the globalization picture as in [12] for the special case of symplectic manifolds. We will show how the weights for the curvature graphs can be explicitly expressed in terms of the hypergeometric function as well as by a much simpler formula combining it with the explicit expression for the weights of its underlined connection graphs. Moreover, we consider the case of a cotangent bundle, which will simplify the curvature expression significantly.

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1. Introduction

1.1. Motivation

In [17] Kontsevich proved that the differential graded Lie algebra (DGLA)[1] of multivector fields on an open subset $M \subset \mathbb{R}^d$ is L_∞-quasi-isomorphic to the DGLA[2] of multidifferential operators on functions on M, i.e. there exists an L_∞-quasi-isomorphism

$$\mathcal{U}\colon T_{poly}(M) \to D_{poly}(M), \tag{1}$$

such that its zeroth Taylor component $\mathcal{U}^{(0)}$ is given by the Hochschild–Kostant–Rosenberg map. This result is known as the *formality* theorem. If one restricts to the case of bivector fields and bidifferential operators, one can recover deformation quantization for Poisson manifolds. The resulting star product was also constructed in [17] by an explicit formula. In [5, 7, 12] a globalization picture was presented for this star product on any Poisson manifold M, including the construction of the local star product by using techniques of field theory, in particular the *Poisson Sigma Model* [4, 16, 20]. In [12] this construction was extended to manifolds with boundary as in the *BV-BFV formalism* [8–10] which is a perturbative quantum gauge formalism compatible with cutting and gluing. A similar approach, as the one presented by Fedosov in [13] for symplectic manifolds, was used, by considering notions of formal geometry. In particular, one starts with a *formal exponential map* ϕ on the manifold M and constructs a *flat* connection D_G, called the *classical Grothendieck connection*, on the completed symmetric algebra of the cotangent bundle $\widehat{\mathrm{Sym}}(T^*M)$. This construction can be deformed to the *Weyl bundle* $\widehat{\mathrm{Sym}}(T^*M)[[\hbar]]$ and, as it was shown in [5, 6, 12], it induces a similar equation as the one in Fedosov's construction. In [12] it was shown that the different terms of this equation are given by a certain class of graphs. We want to give an explicit computation of the weights for these graphs. Let $G_{n,m}$ denote the set of all *admissible graphs* as in [17] with n vertices in the bulk of the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and m vertices on $\mathbb{R}$. Define a map

$$\mathcal{U}_\Gamma\colon \bigwedge^n T_{poly}(M) \to D_{poly}(M)[1-n] \tag{2}$$

[1]Endowed with the zero differential and the Schouten–Nijenhuis bracket.
[2]Endowed with the Hochschild differential and the Gerstenhaber bracket.

using the L_∞-morphism $\mathcal{U}$. Let π be a Poisson structure on $\mathbb{R}^d$ and let ξ, ζ be any two vector fields on $\mathbb{R}^d$. Let us define

$$P(\pi) := \sum_{n\geq 0} \sum_{\Gamma\in G_{n,2}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\pi\wedge\cdots\wedge\pi), \tag{3}$$

$$A(\xi,\pi) := \sum_{n\geq 0} \sum_{\Gamma\in G_{n+1,1}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi\wedge\pi\wedge\cdots\wedge\pi), \tag{4}$$

$$F(\xi,\zeta,\pi) := \sum_{n\geq 0} \sum_{\Gamma\in G_{n+2,0}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi\wedge\zeta\wedge\pi\wedge\cdots\wedge\pi), \tag{5}$$

where $w_\Gamma \in \mathbb{R}$ denotes the Kontsevich weight of the graph Γ. The term (3) represents Kontsevich's star product, (4) represents the *deformed Grothendieck connection* $\mathcal{D}_G := D_G + O(\hbar)$ (see construction below) and (5) its curvature. Let us emphasize a bit more on the formal geometry construction.

1.2. Notions of formal geometry

Let M be a smooth manifold and let $\phi\colon U \to M$ be a map where $U \subset TM$ is an open neighbourhood of the zero section. For $x \in M, y \in T_xM \cap U$ we write $\phi_x(y) := \phi(x,y)$. We say that ϕ is a *generalized exponential map* if for all $x \in M$ we have that $\phi_x(0) = x$, and $\mathrm{d}\phi_x(0) = \mathrm{id}_{T_xM}$. In local coordinates we can write

$$\phi_x^i(y) = x^i + y^i + \frac{1}{2}\phi^i_{x,jk}y^jy^k + \frac{1}{3!}\phi^i_{x,jk\ell}y^jy^ky^\ell + \cdots \tag{6}$$

where the x^i are coordinates on the base and the y^i are coordinates on the fibers. We identify two generalized exponential maps if their jets agree to all orders. A *formal exponential map* is an equivalence class of generalized exponential maps. It is completely specified by the sequence of functions $\left(\phi^i_{x,i_1\dots i_k}\right)_{k=0}^{\infty}$. By abuse of notation, we will denote equivalence classes and their representatives by ϕ. From a formal exponential map ϕ and a function $f \in C^\infty(M)$, we can produce a section $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M))$ by defining $\sigma_x = \mathsf{T}\phi_x^* f$, where T denotes the Taylor expansion in the fiber coordinates around $y = 0$ and we use any representative of ϕ to define the pullback. We denote this section by $\mathsf{T}\phi^* f$, it is independent of the choice of representative, since it only depends on the jets of the representative.

As it was shown [3, 5, 11], one can define a *flat* connection D_G on $\widehat{\mathrm{Sym}}(T^*M)$ with the property that $D_G\sigma = 0$ if and only if $\sigma = \mathsf{T}\phi^* f$ for some $f \in C^\infty(M)$. As already mentioned before, this connection is called the *classical Grothendieck connection*. In fact, $D_G = \mathrm{d}_x + L_R$ where $R \in \Omega^1(M, \mathrm{Der}(\widehat{\mathrm{Sym}}(T^*M)))$ is a 1-form with values in derivations of $\widehat{\mathrm{Sym}}(T^*M)$, which we identify with $\Gamma(TM \otimes \widehat{\mathrm{Sym}}(T^*M))$. We have denoted by d_x the de Rham differential on M and by L the Lie derivative. In coordinates we have

$$R(\sigma)_\ell = -\frac{\partial \sigma}{\partial y^j}\left(\left(\frac{\partial \phi}{\partial y}\right)^{-1}\right)^j_k \frac{\partial \phi^k}{\partial x^\ell}. \tag{7}$$

Define $R(x,y) := R_\ell(x,y)\mathrm{d}x^\ell$, $R_\ell(x,y) := R^j_\ell(x,y)\frac{\partial}{\partial y^j}$, $R^j(x,y) := R^j_\ell(x,y)\mathrm{d}x^\ell$, and

$$R^j_\ell = -\left(\left(\frac{\partial \phi}{\partial y}\right)^{-1}\right)^j_k \frac{\partial \phi^k}{\partial x^\ell} = -\delta^j_\ell + O(y). \tag{8}$$

For $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M))$, $L_R\sigma$ is given by the Taylor expansion (in the y coordinates) of

$$-\mathrm{d}_y\sigma \circ (\mathrm{d}_y\phi)^{-1} \circ \mathrm{d}_x\phi\colon \Gamma(TM) \to \Gamma(\widehat{\mathrm{Sym}}(T^*M)),$$

where we denote by d_y the de Rham differential on the fiber. This shows that R does not depend on the choice of coordinates. One can generalize this also for any fixed vector $\xi = \xi^i(x)\frac{\partial}{\partial x^i} \in T_xM$, instead of just considering the de Rham differential d_x, by

$$D^\xi_G = \xi + \widehat{\xi}, \tag{9}$$

where

$$\widehat{\xi}(x,y) = \iota_\xi R(x,y) = \xi^i(x)R^j_\ell(x,y)\frac{\partial}{\partial y^j}. \tag{10}$$

Here $\xi^i(x)$ would replace the 1-form part $\mathrm{d}x^i$.

This paper is based on the master thesis [19].

Acknowledgements

We would like to thank Alberto Cattaneo and Konstantin Wernli for useful discussions.

2. Computation of Kontsevich weights

Let (M, ω) be a symplectic manifold regarded as a Poisson manifold with Poisson structure π induced by the symplectic form ω. Moreover let $\phi\colon TM \supset U \to M$ be a formal exponential map and denote by T the Taylor expansion in fiber coordinates around zero. Anticipating the computation of the star product $P(\mathsf{T}\phi^*\pi)$, the connection 1-form $A(R, \mathsf{T}\phi^*\pi)$ and its curvature 2-form $F(R, R, \mathsf{T}\phi^*\pi)$ as in [12], we will explicitly compute the Kontsevich weights of three families of graphs in this section. Throughout the paper we use the harmonic angle function

$$\varphi(u, v) = \arg\left(\frac{v-u}{v-\bar{u}}\right) = \frac{1}{2i}\log\left(\frac{(v-u)(\bar{v}-u)}{(v-\bar{u})(\bar{v}-\bar{u})}\right) \tag{11}$$

which measures the angle on $\mathbb{H} \cup \mathbb{R}$ as depicted in Figure 1.

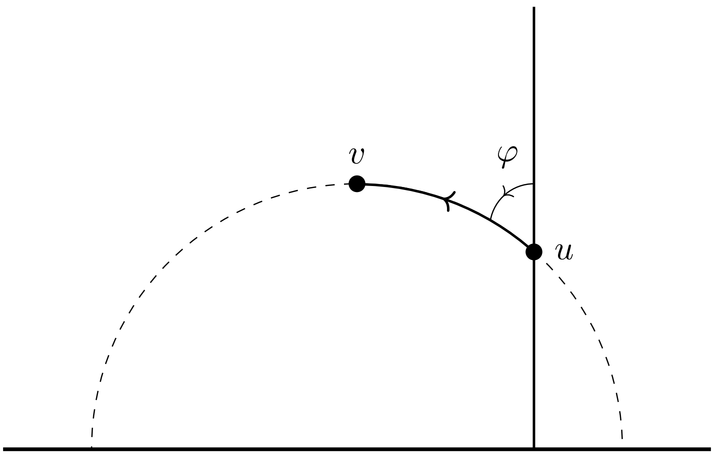


Figure 1: Illustartion of the angle function φ.

The propagator used in the computation of the Kontsevich weights is then simply given by $\mathrm{d}\varphi(u, v)$ and is usually called the *Kontsevich propagator*. Now let $\Gamma \in G_{n,m}$ be an admissible graph with n vertices of first type, m vertices of second type and $2n + m - 2$ edges. We use this propagator to compute the Kontsevich weight [17] w_Γ of Γ as

$$w_\Gamma = \int\limits_{\bar{C}_{n,m}} \omega_\Gamma. \tag{12}$$

Here $\bar{C}_{n,m}$ denotes the *Fulton–MacPherson/Axelrod–Singer (FMAS) compactification* [1, 15] of the configuration space $C_{n,m}$ of n points in $\mathbb{H}$ and m

points on $\mathbb{R}$ modulo scaling and translation. Let us briefly recall the construction of the needed configuration spaces. We define the open configuration space

$$\begin{aligned}\text{Conf}_{n,m} = \{(x_1, \ldots, x_n, q_1, \ldots, q_m) \in \mathbb{H}^n \times \mathbb{R}^m \mid \\ x_i \neq x_j,\ \forall i \neq j,\ q_1 < \ldots < q_m\}.\end{aligned} \tag{13}$$

The 2-dimensional real Lie group of orientation preserving affine transformations of the real line

$$G^{(1)} = \{z \mapsto az + b \mid a, b \in \mathbb{R},\ a > 0\} \tag{14}$$

acts freely on $\text{Conf}_{n,m}$. One can check that the quotient space $C_{n,m} := \text{Conf}_{n,m}/G^{(1)}$ is in fact a smooth manifold of dimension $2n + m - 2$. The differential form ω_Γ on $\bar{C}_{n,m}$ is given by

$$\omega_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \bigwedge_{\text{edges } e} \mathrm{d}\varphi_e, \tag{15}$$

where the wedge product is over all $2n + m - 2$ edges e of the graph Γ. Let $n \geq 2$ and define

$$\text{Conf}_n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j,\ \forall i \neq j\}. \tag{16}$$

We have an action on Conf_n by the 3-dimensional Lie group

$$G^{(2)} = \{z \mapsto az + b \mid a \in \mathbb{R},\ b \in \mathbb{C},\ a > 0\}. \tag{17}$$

Again, one can check that the quotient space $C_n := \text{Conf}_n/G^{(2)}$ is a smooth manifold of dimension $2n - 3$. Also here, we will denote its FMAS compactification by $\bar{C}_n$. We refer to [17] for a more detailed construction. To simplify the notation we will use graphical language, where the figure below corresponds to a factor of $\mathrm{d}(\varphi(u, v)^n)/(2\pi)^n$ in w_Γ. If there is no n above the arrow it simply means that $n = 1$.

$$u \bullet \xrightarrow{\quad n \quad} \bullet\, v$$

We know that the dimension of the configuration space $C_{n,m}$ is $2n + m - 2$, and since we work on a symplectic manifold M (with Darboux coordinates around each point $x \in M$), a vertex of first type is either a vertex representing the tensor $\mathsf{T}\phi_x^*\pi$, which we will call a $\mathsf{T}\phi_x^*\pi$-vertex, with precisely two outgoing and no incoming edges, or a vertex representing the 1-form R, which we will call an R-vertex, with precisely one outgoing edge and arbitrarily many incoming edges [12, 18]. So we may write $n = p + r$, where p is the number of $\mathsf{T}\phi_x^*\pi$-vertices and r is the number of R-vertices. We then have that $\deg(\omega_\Gamma) = 2p + r$, and in order for the integral (12) not to vanish, we must have that ω_Γ is a top form, i.e. that $2n + m - 2 = 2p + r$. This then implies that

$$r + m = 2 \tag{18}$$

So we have to distinguish three different cases, namely $(r, m) = (2, 0)$, $(r, m) = (1, 1)$ and $(r, m) = (0, 2)$, which we will treat separately in what follows.

Remark 2.1. Actually, we will see below that all the integrals over the non-compactified configuration spaces $C_{n,m}$ of the graphs we are considering converge and are thus finite. So it is not necessary to work with the compactifications.

2.1. Case 1: No boundary vertices

We will first treat the case $(r, m) = (2, 0)$, i.e. the case where we have no boundary vertices and exactely two R-vertices. In that case we get a family of graphs $(\Gamma_n)_{n\geq 0}$, where Γ_n is the graph with n wedges as in Figure 2(a) (stemming from n $\mathsf{T}\phi_x^*\pi$-vertices) attached to the wheel as in Figure 2(b) (stemming from the two R-vertices).

(a) wedge (b) wheel

Figure 2: Graphs in the case $(r, m) = (2, 0)$ consist of: (a) wedges stemming from $\mathsf{T}\phi_x^*\pi$-vertices attached to (b) a wheel stemming from the two R-vertices.

Examples of the graphs Γ_n are given in Figure 3 below for $n = 0, 1, 2$.

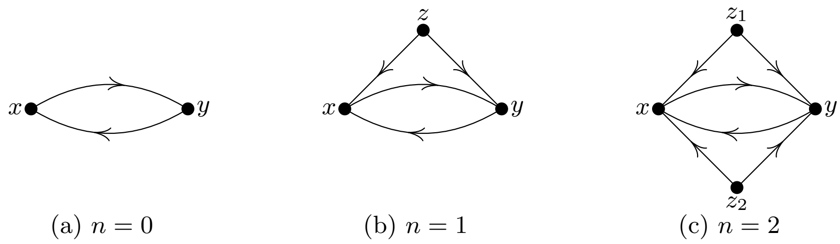


Figure 3: Graphs Γ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the wheel.

The Kontsevich weight of the graph Γ_n for $n \geq 0$ is given by

$$w_{\Gamma_n} = \frac{1}{(2\pi)^{2n+2}} \int\limits_{C_{n+2,0}} \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x)\mathrm{d}\varphi(z_1,x) \times \mathrm{d}\varphi(z_1,y)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,y). \tag{19}$$

Remark 2.2. We will omit the wedge product if it is clear. Moreover, for $n = 0$ we simply set $\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,y)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,y) = 1$ in the integral above.

Remark 2.3. The sign of the weight ω_Γ depends on the ordering of the edges of the graph Γ (i.e. the ordering of the propagator 1-forms in the integrand), and thus the ordering must always be specified. Throughout this whole section, we will stick to the ordering given in (19).

The goal now is to compute (19) explicitly. We will do this in several steps, mainly using Stokes' theorem as in [21].

2.1.1. Step 1. In a first step, we want to integrate out the wedges. More precisely, for a wedge as in Figure 2(a) we want to compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y), \tag{20}$$

i.e. we want to integrate out z (with $x, y \in \mathbb{H}$ fixed). To do this we make a branch cut such that $\varphi(z,x) \in (0, 2\pi)$ and use Stokes' theorem

$$\int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y) = \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,y), \tag{21}$$

where ∂ is the boundary of the integration domain depicted in Figure 4 below.

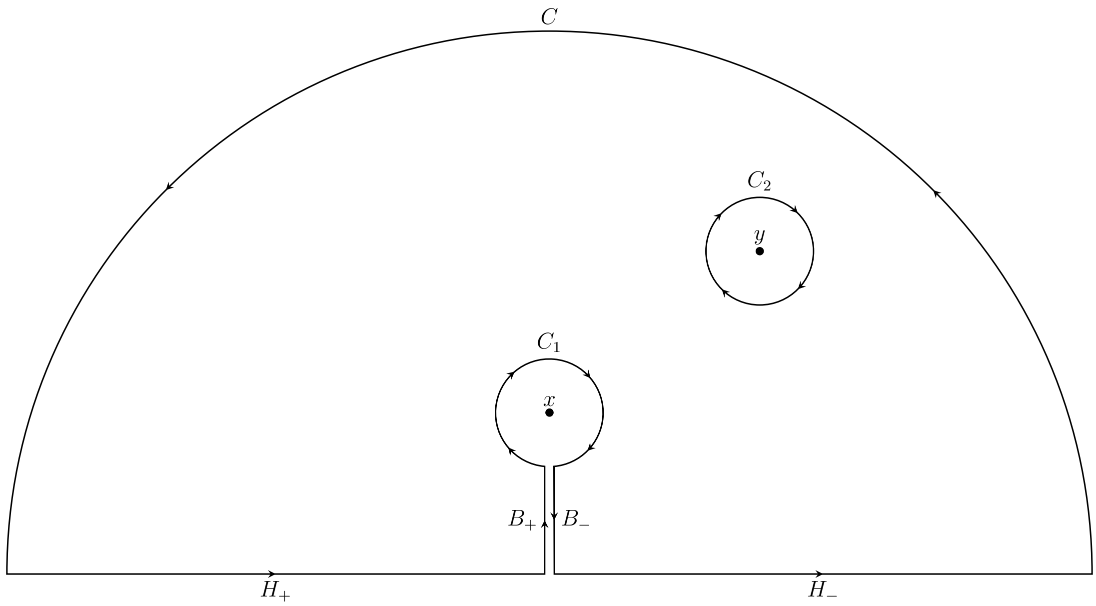


Figure 4: Boundary ∂ of the integration domain: C is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the circles C_1 and C_2 have infinitesimal radius and $H_+ \cup H_-$ is the real line.

Now using (11) we can discuss the different boundary components:

- On $H_+ \cup H_-$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z, y) = \mathrm{darg}(1) = 0$
- On B_+: $\varphi(z, x) = 2\pi$
- On B_-: $\varphi(z, x) = 0$
- On C_1: $z = x + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0 \implies \mathrm{d}\varphi(z, y) = \mathrm{darg}\left(\frac{y-x}{y-\bar{x}}\right) = 0$
- On C_2: $z = y + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in [0, 2\pi) \implies \varphi(z, x) \to \varphi(y, x)$, $\mathrm{d}\varphi(z, y) = -\mathrm{d}\theta$
- On C: $z = R\mathrm{e}^{i\theta}$ for $R \to \infty$ and $\theta \in [0, \pi] \implies \varphi(z, x) = \varphi(z, y) = 2\theta$, $\mathrm{d}\varphi(z, y) = 2\mathrm{d}\theta$

We then finally get

$$
\begin{aligned}
\int\limits_{z\in\mathbb{H}\setminus\{x,y\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,y) &= \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,y) \\
&= 2\pi \int\limits_{B_+} \mathrm{d}\varphi(z,y) + \int\limits_0^{\pi} 4\theta \mathrm{d}\theta - \varphi(y,x) \int\limits_0^{2\pi} \mathrm{d}\theta \\
&= 2\pi\big(\varphi(x,y) - \varphi(y,x) + [x;y]\pi\big),
\end{aligned}
\tag{22}
$$

where

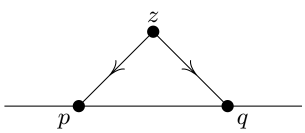
$$
[x;y] = \begin{cases} +1, & \text{if } \mathrm{Re}(x) > \mathrm{Re}(y) \\ -1, & \text{if } \mathrm{Re}(x) < \mathrm{Re}(y) \end{cases}
\tag{23}
$$

Dividing the result (22) by $(2\pi)^2$ (as in (20)), we get

$$
\frac{1}{2\pi}\big(\varphi(x,y) - \varphi(y,x)\big) \pm \frac{1}{2},
\tag{24}
$$

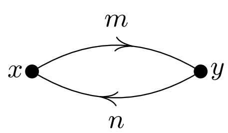
which agrees with the result given in [2, Lemma 3.3].

Finally, consider the limit $(x,y) \to (p,q)$ for $p,q \in \mathbb{R}$ with $p < q$. Using (24) and the fact that $\varphi(p,q) = 2\pi$ and $\varphi(q,p) = 0$, we can compute the Kontsevich weight of the graph below.



We get that the integral of the form representing this graph over $C_{1,2}$ is equal to $\frac{1}{2}$, which agrees with the result in [17, Section 6.4.3].

2.1.2. Step 2. In a second step we want to compute the weight of the graph



where $n, m \geq 1$ and we use the notation introduced above, i.e. we want to explicitly compute the integral

$$\frac{1}{(2\pi)^{n+m}} \int_{C_{2,0}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n. \tag{25}$$

As before we make the branch cut such that $\varphi(x,y) \in (0, 2\pi)$ and use Stokes' theorem

$$\int_{y \in \mathbb{H} \setminus \{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n = \int_{\partial} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \tag{26}$$

with boundary ∂ of the integration domain depicted in Figure 5 below.

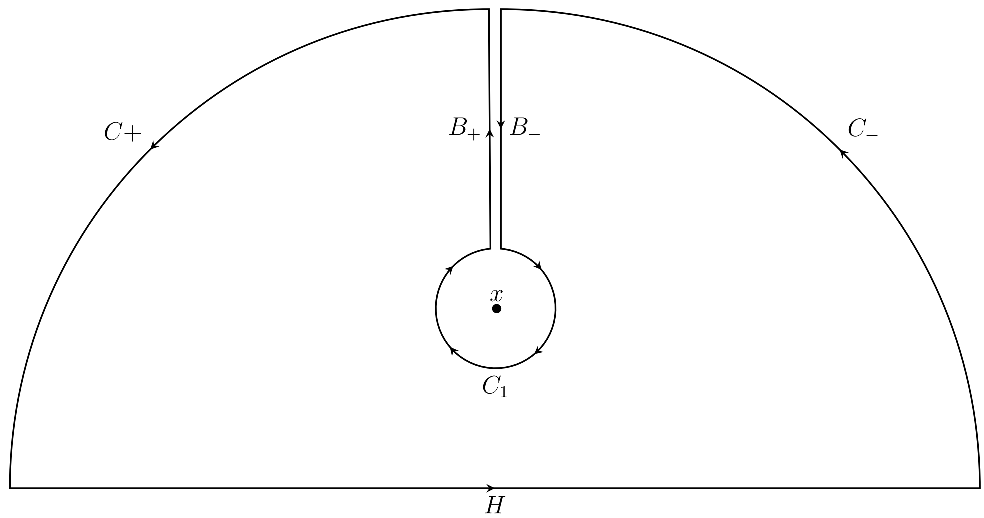


Figure 5: Boundary ∂ of the integration domain: $C_- \cup C_+$ is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the circle C_1 has infinitesimal radius and H is the real line.

Again, we discuss the different boundary components:

- On H: $y \in \mathbb{R} \implies \mathrm{d}\varphi(y,x) = \mathrm{darg}(1) = 0$
- On $B_- \cup B_+$: $\mathrm{d}\varphi(y,x) = 0$
- On C_-: $y = R\mathrm{e}^{i\theta}$ for $R \to \infty \implies \varphi(x,y) = 2\pi$
- On C_+: $y = R\mathrm{e}^{i\theta}$ for $R \to \infty \implies \varphi(x,y) = 0$

- On C_1: $y = x + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$ and
  $\varphi(y,x) = \begin{cases} \frac{\pi}{2} - \theta, & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{5\pi}{2} - \theta, & \text{for } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$

With this, we compute the integral

$$\begin{aligned}\int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n &= \int\limits_{\partial} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ &= (2\pi)^m \int\limits_{C_-} \mathrm{d}\varphi(y,x)^n + \int\limits_{C_1} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ &= (2\pi)^m \pi^n - n \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta \\ &\quad - n \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta.\end{aligned} \tag{27}$$

Now we use the substitution $a = \frac{\pi}{2} - \theta$ to compute

$$\begin{aligned}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta &= \int\limits_0^{\pi} (\pi + a)^m a^{n-1} \mathrm{d}a \\ &= \sum_{k=0}^{m} \binom{m}{k} \pi^k \int\limits_0^{\pi} a^{m+n-k-1} \mathrm{d}a = \sum_{k=0}^{m} \binom{m}{k} \frac{\pi^{m+n}}{m+n-k}.\end{aligned} \tag{28}$$

Similarly, we use the substitution $a = \frac{3\pi}{2} - \theta$ to compute

$$\begin{aligned}\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} \mathrm{d}\theta &= \int\limits_0^{\pi} a^m (\pi + a)^{n-1} \mathrm{d}a \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \pi^k \int\limits_0^{\pi} a^{m+n-k-1} \mathrm{d}a = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\pi^{m+n}}{m+n-k}.\end{aligned} \tag{29}$$

Putting everything together we get

$$\int_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ = \left(2^m - \sum_{k=0}^{m} \binom{m}{k} \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{n}{m+n-l}\right) \pi^{m+n}. \tag{30}$$

It is not hard to see that for $n = 1$ the above formula simplifies to

$$\int_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x) = 2^m \left(1 - \frac{2}{m+1}\right) \pi^{m+1}, \tag{31}$$

which agrees with the result in [21, Section 4].

2.1.3. Step 3. In a third step we want to compute an integral similar to (25), but with an additional factor $[x;y]$ as defined in (23). So we want to compute the integral

$$\frac{1}{(2\pi)^{n+m}} \int_{y\in\mathbb{H}\setminus\{x\}} [x;y]\mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \tag{32}$$

As usual, we use Stokes' theorem

$$\int_{y\in\mathbb{H}\setminus\{x\}} [x;y]\mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ = \int_{\substack{y\in\mathbb{H}\setminus\{x\} \\ \mathrm{Re}(y)<\mathrm{Re}(x)}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int_{\substack{y\in\mathbb{H}\setminus\{x\} \\ \mathrm{Re}(y)>\mathrm{Re}(x)}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ = \int_{\partial_+} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int_{\partial_-} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \tag{33}$$

with boundaries ∂_+ and ∂_- of the integration domain depicted in Figure 6 below.

As before, we discuss the different boundary components:

- On $H_+ \cup H_-$: $\mathrm{d}\varphi(y,x) = 0$
- On $B_\pm \cup L_\pm$: $\mathrm{d}\varphi(y,x) = 0$

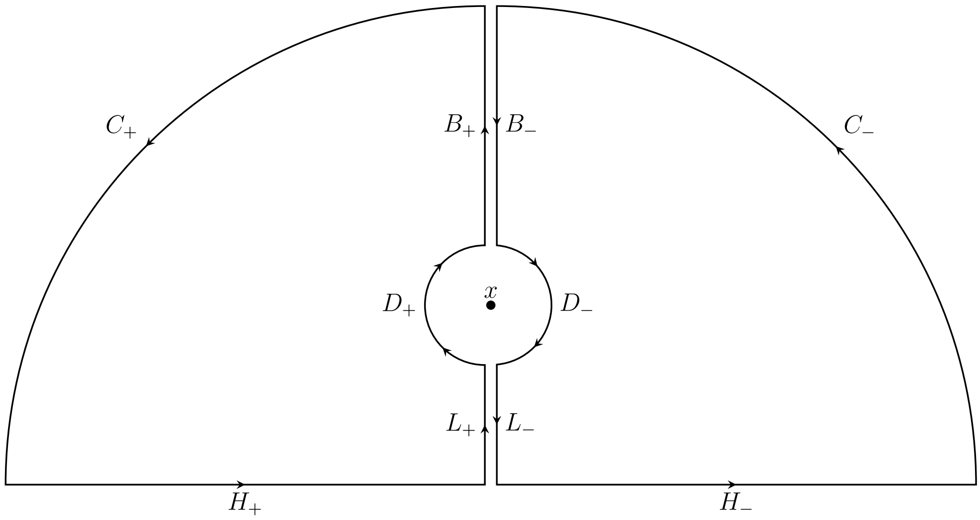


Figure 6: Boundaries ∂_+ (on the left) and ∂_- (on the right) of the integration domain: $C_- \cup C_+$ is the half-circle at infinity, B_+ and B_- as well as L_+ and L_- are infinitesimally close together, the circle $D_+ \cup D_-$ has infinitesimal radius and $H_+ \cup H_-$ is the real line.

- On C_-: $\varphi(x,y) = 2\pi$
- On C_+: $\varphi(x,y) = 0$
- On D_-: $y = x + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$, $\varphi(y,x) = \frac{\pi}{2} - \theta$
- On D_+: $y = x + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x,y) = \frac{3\pi}{2} - \theta$, $\varphi(y,x) = \frac{5\pi}{2} - \theta$

With this we compute the integral

$$
\begin{aligned}
&\int\limits_{y \in \mathbb{H} \setminus \{x\}} [x;y] \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\
&\quad = \int\limits_{\partial_+} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n - \int\limits_{\partial_-} \varphi(x,y)^m \mathrm{d}\varphi(y,x)^n
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&= -n \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{5\pi}{2} - \theta\right)^{n-1} d\theta - (2\pi)^m \pi^n \\
&\quad + n \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3\pi}{2} - \theta\right)^m \left(\frac{\pi}{2} - \theta\right)^{n-1} d\theta \\
&= \left(-2^m + \sum_{k=0}^{m} \binom{m}{k} \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{n}{m+n-l}\right) \pi^{m+n},
\end{aligned}
$$

where we used (28) and (29) in the last step.

2.1.4. Putting everything together. Finally, we are able to compute the Kontsevich weight (19) of the graphs Γ_n described at the beginning of Section 2.1. Integrating over z_i for $i = 1, \ldots, n$, and applying the result for (22) obtained in the first step we get

$$
\begin{aligned}
w_{\Gamma_n} &= \frac{1}{(2\pi)^{2n+2}} \int\limits_{C_{n+2,0}} \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x)\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,y) \cdots \\
&\qquad \times \mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,y) \\
&= \frac{1}{(2\pi)^{n+2}} \int\limits_{y\in\mathbb{H}\setminus\{x\}} \big(\varphi(x,y) - \varphi(y,x) + [x;y]\pi\big)^n \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x) \\
&= \frac{1}{(2\pi)^{n+2}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \\
&\quad \times (-1)^l \int\limits_{y\in\mathbb{H}\setminus\{x\}} \varphi(x,y)^{n-k-l} \varphi(y,x)^l ([x;y]\pi)^k \mathrm{d}\varphi(x,y)\mathrm{d}\varphi(y,x) \\
&= \frac{1}{2^{n+2}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \frac{(-1)^l}{\pi^{n-k+2}(n-k-l+1)(l+1)} \\
&\quad \times \int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1} \mathrm{d}\varphi(y,x)^{l+1}.
\end{aligned} \tag{35}
$$

Note that for even k we have

$$\begin{aligned}
&\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&\quad= \int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&\quad= \Bigg(2^{n-k-l+1} - \sum_{r=0}^{n-k-l+1}\binom{n-k-l+1}{r}\frac{l+1}{n-k-r+2} \\
&\qquad - \sum_{s=0}^{l}\binom{l}{s}\frac{l+1}{n-k-s+2}\Bigg)\pi^{n-k+2},
\end{aligned} \tag{36}$$

where we have used (30). Similarly, for odd k we get

$$\begin{aligned}
&\int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]^k \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&\quad= \int\limits_{y\in\mathbb{H}\setminus\{x\}} [x;y]\mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\
&\quad= \Bigg(-2^{n-k-l+1} + \sum_{r=0}^{n-k-l+1}\binom{n-k-l+1}{r}\frac{l+1}{n-k-r+2} \\
&\qquad - \sum_{s=0}^{l}\binom{l}{s}\frac{l+1}{n-k-s+2}\Bigg)\pi^{n-k+2},
\end{aligned} \tag{37}$$

where we have used (34).

We will now simplify the expressions we got. We start by observing a few things:

First of all, we clearly have that

$$\mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} = -\mathrm{d}\varphi(y,x)^{l+1}\mathrm{d}\varphi(x,y)^{n-k-l+1}. \tag{38}$$

Similarly, we also have that

$$[x;y] = -[y;x]. \tag{39}$$

Furthermore, we can obviously swap x and y in the integral and get the same result, i.e.

$$\begin{aligned}\int\limits_{y\in\mathbb{H}\setminus\{x\}} & [x;y]^m \mathrm{d}\varphi(x,y)^{n-k-l+1}\mathrm{d}\varphi(y,x)^{l+1} \\ &= \int\limits_{x\in\mathbb{H}\setminus\{y\}} [y;x]^m \mathrm{d}\varphi(y,x)^{n-k-l+1}\mathrm{d}\varphi(x,y)^{l+1}.\end{aligned} \tag{40}$$

Now assume that n is even. Applying (38), (39) and (40) to the last line of (35), it follows that

$$\begin{aligned} w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{\substack{k=0 \\ k \text{ even}}}^{n} & \binom{n}{k}\binom{n-k}{\frac{n-k}{2}} \frac{(-1)^{\frac{n-k}{2}}}{\pi^{n-k+2}\left(\frac{n-k}{2}+1\right)^2} \\ & \times \int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^{\frac{n-k}{2}+1}\mathrm{d}\varphi(y,x)^{\frac{n-k}{2}+1}.\end{aligned} \tag{41}$$

So most of the terms cancel for n even.
Now using (30) we observe that

$$\begin{aligned} & \frac{1}{\pi^{2m}} \int\limits_{y\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^m \\ & = 2^m - \sum_{k=0}^{m}\binom{m}{k}\frac{m}{2m-k} - \sum_{l=0}^{m-1}\binom{m-1}{l}\frac{m}{2m-l} \\ & = 2^m - \sum_{k=0}^{m-1}\left(\binom{m}{k}+\binom{m-1}{k}\right)\frac{m}{2m-k} - 1 \\ & = 2^m - \sum_{k=0}^{m-1}\binom{m}{k} - 1 = 2^m - \sum_{k=0}^{m}\binom{m}{k} = 0.\end{aligned} \tag{42}$$

Plugging this result into (41) with $m = \frac{n-k}{2}$ we finally get that

$$w_{\Gamma_n} = 0 \qquad \text{for even } n \geq 0. \tag{43}$$

For n odd the different terms in the last line of (35) do not cancel anymore. Instead, we will try to write (30) and (34) more compactly. To do this, let us introduce the so-called *hypergeometric function* ${}_2F_1(a,b;c;z)$. It is defined by the series

$$ {}_2F_1(a,b;z;c) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{44} $$

for $z \in \mathbb{C}$ with $|z| < 1$, where $(a)_k$ is the *Pochhammer symbol* given by

$$ (a)_k = \begin{cases} 1, & \text{if } k = 0 \\ a(a+1)\cdots(a+k-1), & \text{if } k > 0 \end{cases} \tag{45} $$

It is not hard to see that the series terminates if either a or b is a non-positive integer. In that case the hypergeometric function reduces to a polynomial and can therefore also be defined for $|z| \geq 1$.

We can now use the hypergeometric function to write

$$ \sum_{k=0}^{m} \binom{m}{k} \frac{1}{m+n-k} = \frac{{}_2F_1(-m,-m-n;1-m-n;-1)}{m+n}, \tag{46} $$

and

$$ \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{m+n-k} = \frac{{}_2F_1(1-n,-m-n;1-m-n;-1)}{m+n}. \tag{47} $$

This allows us to write (30) as

$$ \begin{aligned} &\int\limits_{y\in\mathbb{H}\backslash\{x\}} \mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ &\quad= \Bigg(2^m - \frac{n}{m+n}\big({}_2F_1(-m,-m-n;1-m-n;-1) \\ &\qquad + {}_2F_1(1-n,-m-n;1-m-n;-1)\big)\Bigg)\pi^{m+n}, \end{aligned} \tag{48} $$

and (34) as

$$ \begin{aligned} &\int\limits_{y\in\mathbb{H}\backslash\{x\}} [x;y]\mathrm{d}\varphi(x,y)^m \mathrm{d}\varphi(y,x)^n \\ &\quad= \Bigg(-2^m + \frac{n}{m+n}\big({}_2F_1(-m,-m-n;1-m-n;-1) \\ &\qquad - {}_2F_1(1-n,-m-n;1-m-n;-1)\big)\Bigg)\pi^{m+n}. \end{aligned} \tag{49} $$

Plugging those results into (35) we finally get for all $n \geq 0$

(50)
$$\begin{aligned} w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \frac{(-1)^l}{(n-k-l+1)(l+1)} \Big((-1)^k 2^{n-k-l+1} \\ - \frac{l+1}{n-k+2} \big({}_2F_1(-l, -n+k-2; -n+k-1; -1) \\ + (-1)^k {}_2F_1(-n+k+l-1, -n+k-2; -n+k-1; -1) \big) \Big). \end{aligned}$$

The Kontsevich weights of the first few graphs are given in Table 1 below.

n	0	1	2	3	4	5	6	7	8	9
w_{Γ_n}	0	$\frac{1}{24}$	0	$\frac{1}{320}$	0	$\frac{1}{2688}$	0	$\frac{1}{18432}$	0	$\frac{1}{112640}$

Table 1: Kontsevich weights of the graphs Γ_n for $n = 0, 1, \ldots, 9$

As a sanity check we have the following: For $n = 0$ the graph Γ_0 is just a wheel with two vertices (see Figure 3(a)) and its weight is zero according to [17, Lemma 7.3]. For $n = 1$ the graph Γ_1 is just a wheel with two spokes pointing outward (see Figure 3(b)) and its weight is $\frac{1}{24}$ according to [21, Proposition 1.1]. So at least for $n = 0, 1$ our formula (50) for the Kontsevich weights w_{Γ_n} produces the correct values.

2.2. Case 2: One Boundary vertex

We will now treat the case $(r, m) = (1, 1)$, i.e. the case where we have one boundary vertex and one R-vertex. In that case we get a family of graphs $(\Upsilon_n)_{n \geq 0}$, where Υ_n is the graph with n wedges as in Figure 7(a) (stemming from n $\mathsf{T}\phi_x^* \pi$-vertices) attached to the graph containing a single edge from the R-vertex to the boundary vertex as in Figure 7(b) below.

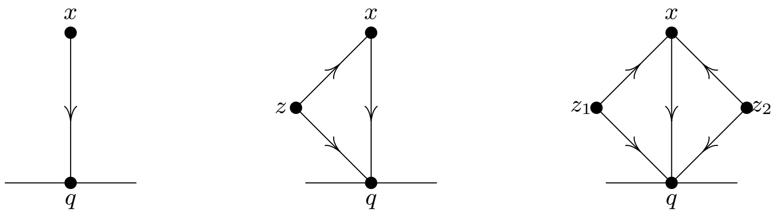
Examples of the graphs Υ_n are given in Figure 8 below for $n = 0, 1, 2$.

The Kontsevich weight of the graph Υ_n for $n \geq 0$ is given by

(51)
$$w_{\Upsilon_n} = \frac{1}{(2\pi)^{2n+1}} \int\limits_{C_{n+1,1}} \mathrm{d}\varphi(x, q) \mathrm{d}\varphi(z_1, x) \mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, x) \mathrm{d}\varphi(z_n, q).$$

(a) wedge (b) single edge

Figure 7: Graphs in the case $(r, m) = (1, 1)$ consist of: (a) wedges stemming from $\mathsf{T}\phi_x^*\pi$-vertices attached to (b) a single edge from the R-vertex to the boundary vertex



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 8: Graphs Υ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the single edge from the R-vertex to the boundary vertex

Remark 2.4. As before, the ordering of the edges of the graph Υ_n specified in (51) above determines the sign of w_{Υ_n}. Throughout this whole section we will stick to this ordering.

Again, the goal is to compute (51) explicitly. As before, we will do this in several steps.

2.2.1. Step 1. For a wedge as in Figure 7(a) we want to compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int\limits_{z\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(z, x)\mathrm{d}\varphi(z, q), \tag{52}$$

i.e. we want to integrate out z (with $x, q \in \overline{\mathbb{H}}$ fixed). The computation is almost the same as the one we have already done in Section 2.1.1 above: Again we make a branch cut such that $\varphi(z, x) \in (0, 2\pi)$ and use Stokes' theorem

$$\int_{z\in\mathbb{H}\setminus\{x\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,q) = \int_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,q), \tag{53}$$

where ∂ is the boundary of the integration domain depicted in Figure 9 below.

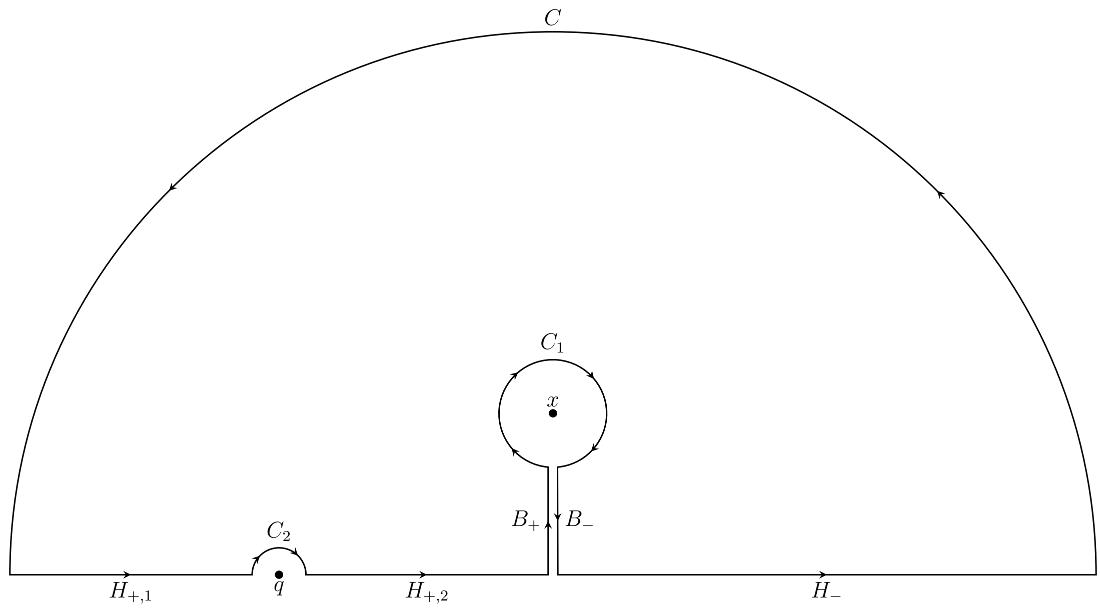


Figure 9: Boundary ∂ of the integration domain: C is the half-circle at infinity, B_+ and B_- are infinitesimally close together, the (half-)circles C_1 and C_2 have infinitesimal radius and $H_{+,1} \cup H_{+,2} \cup H_-$ is the real line.

Let us have a look at the different boundary components:

- On $H_{+,1} \cup H_{+,2} \cup H_-$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z,q) = 0$
- On B_+: $\varphi(z,x) = 2\pi$
- On B_-: $\varphi(z,x) = 0$
- On C_1: $z = x + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0 \implies \mathrm{d}\varphi(z,q) = \mathrm{darg}\left(\frac{q-x}{q-\bar{x}}\right) = 0$
- On C_2: $z = q + \varepsilon \mathrm{e}^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in [-\pi, 0] \implies \varphi(z,x) \to \varphi(q,x)$, $\varphi(z,q) = -2\theta$
- On C: $z = R\mathrm{e}^{i\theta}$ for $R \to \infty$ and $\theta \in [0,\pi] \implies \varphi(z,x) = \varphi(z,q) = 2\theta$

We can then compute the integral

$$
\begin{aligned}
\int\limits_{z\in\mathbb{H}\setminus\{x,q\}} \mathrm{d}\varphi(z,x)\mathrm{d}\varphi(z,q) &= \int\limits_{\partial} \varphi(z,x)\mathrm{d}\varphi(z,q) \\
&= 2\pi \int\limits_{B_+} \mathrm{d}\varphi(z,q) + \int\limits_0^\pi 4\theta \mathrm{d}\theta - 2\varphi(q,x)\int\limits_{-\pi}^0 \mathrm{d}\theta \\
&= 2\pi\big(\varphi(x,q) - \varphi(q,x) + [x;q]\pi\big)
\end{aligned} \tag{54}
$$

where

$$
[x;q] = \begin{cases} +1, & \text{if } \mathrm{Re}(x) > q \\ -1, & \text{if } \mathrm{Re}(x) < q \end{cases} \tag{55}
$$

Dividing the result (54) by $(2\pi)^2$ we get

$$
\frac{1}{2\pi}\big(\varphi(x,q) - \varphi(q,x)\big) \pm \frac{1}{2}, \tag{56}
$$

which agrees with the result in [21, Lemma 5.3].

Finally, observe that one obtains (54) by simply taking the limit $y \to q \in \mathbb{R}$ in (22).

2.2.2. Step 2. In a second step we want to compute the integral

$$
\frac{1}{(2\pi)^{m+n}} \int\limits_{C_{1,1}} \varphi(q,x)^m \mathrm{d}\varphi(x,q)^n \tag{57}
$$

for $n \geq 1$ and $m \geq 0$.

First note that $C_{1,1}$, shown in Figure 10 below, is a smooth manifold of dimension 1 which is homeomorphic to an open interval and $\bar{C}_{1,1}$ is homeomorphic to a closed interval.

Remark 2.5. We work with the standard orientation on $C_{1,1}$, which is induced by the standard orientation on the plane $\mathbb{R}^2$.

It is not hard to see that the boundary $\partial C_{1,1}$ is just a two-element set. More precisely $\partial C_{1,1} = \{(q,s),(q,t)\}$ with $s < q$ and $t > q$ (for a more detailed treatment, see [10, 12, 17]).

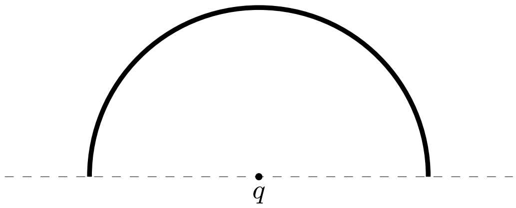


Figure 10: The manifold $C_{1,1}$ is the product of a (fixed) single point q on the real line and an open half circle

But now we have to make a branch cut such that $\varphi(q,x) \in (0,2\pi)$. Then the boundary ∂ of the integration domain, depicted in Figure 11 below, contains four points, namely

$$\partial = \{(q,s),(q,t),(q,y_+),(q,y_-)\}, \tag{58}$$

where y is the point on the half circle directly above q, i.e. with $\mathrm{Re}(y) = q$, and y_+ and y_- are the limits $x \to y$ on the half circle from the left (i.e. from the region $\mathrm{Re}(x) < q$ of the half-circle) and from the right (i.e. from the region $\mathrm{Re}(x) > q$) respectively.

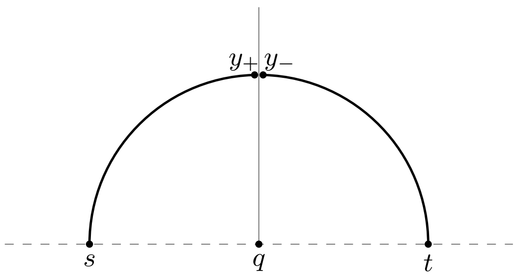


Figure 11: $C_{1,1}$ with branch cut and its boundary ∂ consisting of four points

Finally, using Stokes' theorem and the fact that $\mathrm{d}\varphi(q,x)=0$ for $q\in\mathbb{R}$, we get

$$\begin{aligned}\int\limits_{C_{1,1}}\varphi(q,x)^m\mathrm{d}\varphi(x,q)^n &= \int\limits_{\partial}\varphi(q,x)^m\varphi(x,q)^n\\ &= \varphi(q,s)^m\varphi(s,q)^n-\varphi(q,y_+)^m\varphi(y_+,q)^n\\ &\quad+\varphi(q,y_-)^m\varphi(y_-,q)^n-\varphi(q,t)^m\varphi(t,q)^n\\ &= \begin{cases}(2\pi)^n, & \text{if } m=0\\ 2^m\pi^{m+n}, & \text{if } m>0\end{cases}\end{aligned}\tag{59}$$

2.2.3. Step 3. Let us start with writing (54) differently as

$$2\pi\big(\varphi(x,q)-\varphi(q,x)+\pi[x;q]\big)=2\pi\big(\varphi(x,q)-\varphi(q,x)-\pi+2\pi(x;q)\big)\tag{60}$$

where

$$(x;q)=\begin{cases}+1, & \text{if } \mathrm{Re}(x)>q\\ 0, & \text{if } \mathrm{Re}(x)<q\end{cases}\tag{61}$$

In this step we then want to compute an integral similar to (57), but with an additional factor $(x;y)$ as defined above. So we want to compute

$$\frac{1}{(2\pi)^{m+n}}\int\limits_{C_{1,1}}(x;q)\varphi(q,x)^m\mathrm{d}\varphi(x,q)^n\tag{62}$$

for $n\geq 1$ and $m\geq 0$.

As before we use Stokes' theorem and find that

$$\begin{aligned}&\int\limits_{C_{1,1}}(x;q)\varphi(q,x)^m\mathrm{d}\varphi(x,q)^n\\ &\quad=\int\limits_{\substack{C_{1,1}\\ \mathrm{Re}(x)>q}}\varphi(q,x)^m\mathrm{d}\varphi(x,q)^n\\ &\quad=\varphi(q,y_-)^m\varphi(y_-,q)^n-\varphi(q,t)^m\varphi(t,q)^n=2^m\pi^{m+n}\end{aligned}\tag{63}$$

for all $m\geq 0$ and all $n\geq 1$.

2.2.4. Putting everything together. Now we can use the results from steps 1-3 to compute the Kontsevich weight (51) of the graphs Υ_n, $n\geq 0$, described at the beginning of Section 2.2. Integrating over z_i for $i = 1, \dots, n$, and applying (60), we get

(64)
$$
\begin{aligned}
w_{\Upsilon_n} &= \frac{1}{(2\pi)^{2n+1}} \int\limits_{C_{n+1,1}} \mathrm{d}\varphi(x,q)\mathrm{d}\varphi(z_1,x)\mathrm{d}\varphi(z_1,q)\cdots\mathrm{d}\varphi(z_n,x)\mathrm{d}\varphi(z_n,q) \\
&= \frac{1}{(2\pi)^{n+1}} \int\limits_{C_{1,1}} \big(\varphi(x,q) - \varphi(q,x) - \pi + 2\pi(x;q)\big)^n \mathrm{d}\varphi(x,q) \\
&= \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l} \binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}(-1)^{l+s} \\
&\quad \times \int\limits_{C_{1,1}} \varphi(x,q)^{n-k-l-s}\varphi(q,x)^s \pi^l (2\pi(x;q))^k \mathrm{d}\varphi(x,q) \\
&= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l} \binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s} \\
&\quad \times \frac{(-1)^{l+s}}{2^{n-k+1}\pi^{n-k-l+1}(n-k-l-s+1)} \\
&\quad \times \int\limits_{C_{1,1}} (x;q)^k \varphi(q,x)^s \mathrm{d}\varphi(x,q)^{n-k-l-s+1}.
\end{aligned}
$$

We note that for $k = 0$ we have

(65)
$$
\int\limits_{C_{1,1}} \varphi(q,x)^s \mathrm{d}\varphi(x,q)^{n-l-s+1} = \begin{cases} (2\pi)^{n-l+1}, & \text{if } s = 0 \\ 2^s \pi^{n-l+1}, & \text{if } s > 0 \end{cases}
$$

where we have used (59). Similarly, for $k \geq 1$ we have

(66)
$$
\begin{aligned}
&\int\limits_{C_{1,1}} (x;q)^k \varphi(q,x)^s \mathrm{d}\varphi(x,q)^{n-k-l-s+1} \\
&\qquad = \int\limits_{C_{1,1}} (x;q)\varphi(q,x)^s \mathrm{d}\varphi(x,q)^{n-k-l-s+1} = 2^s \pi^{n-k-l+1},
\end{aligned}
$$

where we have used (63).

Plugging the above two results into the last line of (64) we get

$$w_{\Upsilon_n} = \underbrace{\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l}\binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}\frac{(-1)^{l+s}}{2^{n-k-s+1}(n-k-l-s+1)}}_{=:A(n)} \tag{67}$$
$$-\underbrace{\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^{n+1}(n-l+1)}}_{=:B(n)}+\underbrace{\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{2^l(n-l+1)}}_{=:C(n)}.$$

As shown in Appendix A, we have that

$$\begin{aligned} A(n) &= \frac{(-1)^n}{2^{n+1}(n+1)}, \\ B(n) &= \frac{(-1)^n}{2^{n+1}(n+1)}, \\ C(n) &= \frac{1+(-1)^n}{2^{n+1}(n+1)}. \end{aligned} \tag{68}$$

Hence, we finally have

$$w_{\Upsilon_n} = \frac{1+(-1)^n}{2^{n+1}(n+1)}, \qquad n \geq 0. \tag{69}$$

In particular, we see that

$$w_{\Upsilon_n} = 0 \qquad \text{for odd } n \geq 1. \tag{70}$$

The Kontsevich weights of the first few graphs are given in Table 2 below.

n	0	1	2	3	4	5	6	7	8	9
w_{Υ_n}	1	0	$\frac{1}{12}$	0	$\frac{1}{80}$	0	$\frac{1}{448}$	0	$\frac{1}{2304}$	0

Table 2: Kontsevich weights of the graphs Υ_n for $n = 0, 1, \ldots, 9$

As a sanity check we have the following: For $n = 0$ the graph Υ_0 is just a single edge as in Figure 7(b) and its weight is zero according to [17, Section 6.4.3]. For $n = 1$ the graph Υ_1 is just a single edge with one wedge attached as in Figure 8(b) and its weight is 0 according to [14, Appendix B]. For $n = 2$

the graph Υ_2 is a single edge with two wedges attached as in Figure 8(c) and its weight is $\frac{1}{12}$ according to [21, Appendix A]. So at least for $n = 0, 1, 2$ our formula (69) for the Kontsevich weights w_{Γ_n} produces the correct values.

2.3. Case 3: Two Boundary vertices

Finally, we will treat the case $(r, m) = (0, 2)$, i.e. the case where we have two boundary vertices and no R-vertex. In that case we get a family of graphs $(\Lambda_n)_{n\geq 0}$, where Λ_n is the graph with n wedges as in Figure 12 (stemming from n $\mathsf{T}\phi_x^*\pi$-vertices) attached to the two boundary vertices.

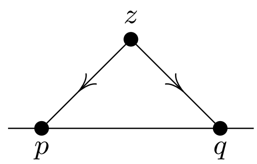
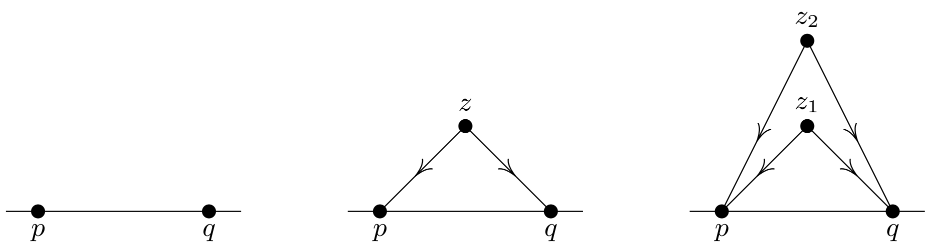


Figure 12: Graphs in the case $(r, m) = (0, 2)$ consist of wedges attached to the two boundary vertices

Examples of the graphs Λ_n are given in Figure 13 below for $n = 0, 1, 2$.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 13: Graphs Λ_n for (a) $n = 0$, (b) $n = 1$ and (c) $n = 2$ wedges attached to the two boundary vertices

The Kontsevich weight of the graph Λ_n for $n \geq 0$ is given by

$$w_{\Lambda_n} = \frac{1}{(2\pi)^{2n}} \int\limits_{C_{n,2}} \mathrm{d}\varphi(z_1, p)\mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, p)\mathrm{d}\varphi(z_n, q). \tag{71}$$

Remark 2.6. For $n = 0$ we simply set

$$\mathrm{d}\varphi(z_1, p)\mathrm{d}\varphi(z_1, q) \cdots \mathrm{d}\varphi(z_n, p)\mathrm{d}\varphi(z_n, q) = 1$$

in the integral above.

Remark 2.7. As before, the ordering of the edges of the graph Λ_n specified in (71) above determines the sign of w_{Λ_n}. Throughout this whole section we will stick to this ordering.

Remark 2.8. Since we work with the configuration space $\mathrm{Conf}_{n,m}$ as in (13), and in particular with the quotient $C_{n,m} = \mathrm{Conf}_{n,m}/G^{(1)}$, it follows that $C_{0,2}$ is a single point (and not a two-element set).

Finally, our goal is to compute (51) explicitly for the given family of graphs. However, this time the computation is much easier and shorter than before.

For the boundary vertices $p, q, \in \mathbb{R}$ with $p < q$ we have already computed the Kontsevich weight of a wedge as in Figure 12 at the end of Section 2.1.1. Our result was

$$\frac{1}{(2\pi)^2} \int\limits_{C_{1,2}} \mathrm{d}\varphi(z,p)\mathrm{d}\varphi(z,q) = \frac{1}{2}. \tag{72}$$

For the sake of completeness and to make sure that we get the same result, let us nonetheless do a direct computation. For a wedge as in Figure 12, we compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int\limits_{z\in\mathbb{H}} \mathrm{d}\varphi(z,p)\mathrm{d}\varphi(z,q), \tag{73}$$

with $p, q \in \mathbb{R}$, $p < q$ fixed. As before, we use Stokes' theorem

$$\int\limits_{z\in\mathbb{H}} \mathrm{d}\varphi(z,p)\mathrm{d}\varphi(z,q) = \int\limits_{\partial} \varphi(z,p)\mathrm{d}\varphi(z,q), \tag{74}$$

where ∂ is the boundary of the integration domain depicted in Figure 14 below.

As usual, let us have a look at the different boundary components:

- On $H_1 \cup H_2 \cup H_3$: $z \in \mathbb{R}$ and hence $\mathrm{d}\varphi(z,q) = 0$
- On C_1: $z = p + \varepsilon\mathrm{e}^{-i\theta}$ for $\varepsilon \to 0 \implies \mathrm{d}\varphi(z,q) = \mathrm{darg}(1) = 0$
- On C_2: $z = q + \varepsilon\mathrm{e}^{-i\theta}$ for $\varepsilon \to 0 \implies \varphi(z,p) \to \varphi(q,p) = 0$
- On C: $z = R\mathrm{e}^{i\theta}$ for $R \to \infty$ and $\theta \in [0,\pi] \implies \varphi(z,p) = \varphi(z,q) = 2\theta$

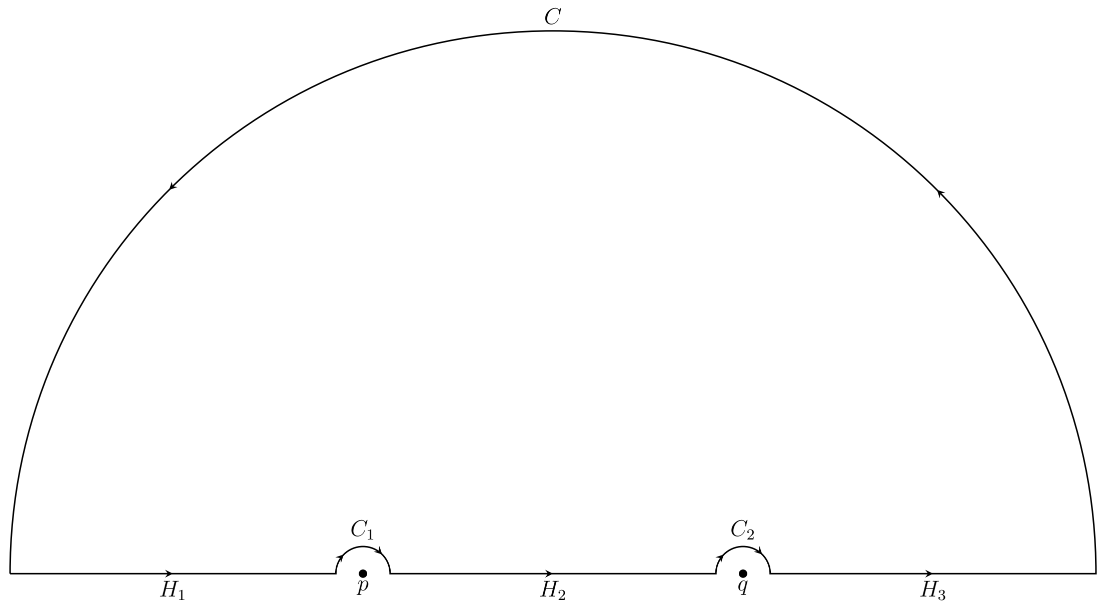


Figure 14: Boundary ∂ of the integration domain: C is the half-circle at infinity, the half circles C_1 and C_2 have infinitesimal radius and $H_1 \cup H_2 \cup H_3$ is the real line.

We can then compute the integral

$$\int_{z\in\mathbb{H}} \mathrm{d}\varphi(z,p)\mathrm{d}\varphi(z,q) = \int_{\partial} \varphi(z,p)\mathrm{d}\varphi(z,q) = \int_0^\pi 4\theta d\theta = 2\pi^2, \tag{75}$$

which indeed agrees with (72) after dividing by $(2\pi)^2$.

With this result at hand, it is now easy to compute the Kontsevich weight of the graph Λ_n for $n \geq 1$. We get

$$\begin{aligned} w_{\Lambda_n} &= \frac{1}{(2\pi)^{2n}} \int_{C_{n,2}} \mathrm{d}\varphi(z_1,p)\mathrm{d}\varphi(z_1,q)\cdots\mathrm{d}\varphi(z_n,p)\mathrm{d}\varphi(z_n,q) \\ &= \frac{1}{(2\pi)^{2n}}(2\pi^2)^n = \frac{1}{2^n}. \end{aligned} \tag{76}$$

For $n = 0$ it is not hard to see that

$$w_{\Lambda_0} = \int_{C_{0,2}} 1 = 1, \tag{77}$$

since $C_{0,2}$ is a single point (cf Remark 2.8). So all in all we finally have

$$w_{\Lambda_n} = \frac{1}{2^n}, \qquad n \geq 0. \tag{78}$$

2.4. Another approach for the explicit computation of w_{Γ_n} and w_{Υ_n}

We want to give a more fast and explicit approach for the computation of the weights w_{Γ_n} and w_{Υ_n}. The following approach has the advantage that it doesn't require the use of the hypergeometric function for w_{Γ_n} but rather gives an explicit expression in terms of w_{Υ_n}. First note that

$$\mathrm{d}\varphi(z,x) = \frac{1}{4\pi i}\mathrm{d}\log\left(\frac{(z-x)(z-\bar{x})}{(\bar{z}-x)(\bar{z}-\bar{x})}\right), \qquad \forall z,x \in \mathbb{H}\cup\mathbb{R}. \tag{79}$$

Moreover, recall from [2, Lemma 5.3] the formula

$$\int_{z\in\mathbb{H}} \mathrm{d}\varphi(z,x)\wedge \mathrm{d}\varphi(z,y) = \frac{1}{2\pi i}\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right), \qquad \forall x,y \in \mathbb{H}\cup\mathbb{R}. \tag{80}$$

Integrating out the z_i variables in w_{Υ_n} using (80), we get

$$\begin{aligned} w_{\Upsilon_n} &= \int_{C_{1,1}} \left(\frac{1}{2\pi i}\log\left(\frac{x-p}{p-\bar{x}}\right)\right)^n \frac{1}{2\pi i}\mathrm{d}\log\left(\frac{x-p}{\bar{x}-p}\right) \\ &= \frac{1}{n+1}\int_{C_{1,1}} \mathrm{d}\left(\frac{1}{2\pi i}\mathrm{d}\log\left(\frac{x-p}{p-\bar{x}}\right)\right)^{n+1} \end{aligned} \tag{81}$$

because $\mathrm{d}\log(\bar{x}-p) = \mathrm{d}\log(p-\bar{x})$. Now recall that $C_{1,1} = (\mathbb{H}\times\mathbb{R})/(\mathbb{R}^{>0}\ltimes\mathbb{R})$ is isomorphic to $\mathbb{R}$; for instance every point in the quotient can be represented uniquely by a pair (i,p) with i the imaginary unit and $p\in\mathbb{R}$. Hence we get

$$\begin{aligned} w_{\Upsilon_n} &= \frac{1}{n+1}\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{2\pi i}\log\left(\frac{i-p}{p+i}\right)\right)^{n+1}\mathrm{d}p \\ &= \left(\frac{1}{2\pi i}\log\left(\frac{i-p}{p+i}\right)\right)^{n+1}\Bigg|_{p=-\infty}^{p=\infty} \\ &= \frac{1+(-1)^n}{2^{n+1}(n+1)}, \end{aligned} \tag{82}$$

where we have used the boundary values

$$\lim_{p\to\pm\infty} \log\left(\frac{i-p}{p+i}\right) = \lim_{p\to\pm\infty} \log\left(-1+\frac{2i}{p}+O(1/p^2)\right) = \pm i\pi. \tag{83}$$

Now, using (80) and integrating out the z_i variables of w_{Γ_n}, we get

$$w_{\Gamma_n} = \int_{C_{2,0}} \left(\frac{1}{2\pi i}\log\left(\frac{x-\bar{y}}{z-\bar{x}}\right)\right)^n \mathrm{d}\varphi(x,y)\wedge \mathrm{d}\varphi(y,x). \tag{84}$$

Note that $\mathrm{d}\varphi(x,y) = \mathrm{d}\varphi(y,x) + \frac{1}{2\pi i}\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right)$, and thus

$$\begin{aligned} w_{\Gamma_n} &= \int_{C_{2,0}} \left(\frac{1}{2\pi i}\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right)\right)^n \frac{1}{2\pi i}\mathrm{d}\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right)\wedge \mathrm{d}\varphi(y,x) \\ &= \frac{1}{n+1}\int_{C_{2,0}} \mathrm{d}\left[\left(\frac{1}{2\pi i}\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right)\right)^{n+1}\mathrm{d}\varphi(y,x)\right]. \end{aligned} \tag{85}$$

By Stokes' theorem, the integral is reduced to the boundary of Kontsevich's eye $\bar{C}_{2,0}$ [17, Section 5.2.]. This boundary has three components:

- The iris of the eye which is isomorphic to S^1, corresponding to the collision $x \to y$. This gives zero contribution, since $\log\left(\frac{x-\bar{y}}{y-\bar{x}}\right) = 0$ for $x = y$.
- The upper eyelid, corresponding to the limit $|x| \to \infty$. This gives no contribution, since $\mathrm{d}\varphi(y,x) = 0$ in the limit.
- The lower eyelid, corresponding to the collision $x \to \bar{x}$ onto the real line.

The lower eyelid is isomorphic to $C_{1,1}$ and by comparison with the integrand of $w_{\Upsilon_{n+1}}$ in (81) with $p := x = \bar{x}$, we get

$$w_{\Gamma_n} = \frac{1}{n+1} w_{\Upsilon_{n+1}} = \frac{1-(-1)^n}{2^{n+2}(n+1)(n+2)}. \tag{86}$$

One can easily check that this formula produces the same values as in Table 1.

3. Including the weights

3.1. The product $P(\mathsf{T}\phi^*\pi)$

Let $\sigma, \tau \in \Gamma(\widehat{\mathrm{Sym}}(T^*M)[[\hbar]])$ be sections and let $x \in M$. Using the Kontsevich weights computed above, we get

$$
\begin{aligned}
(87)\qquad & P(\mathsf{T}\phi_x^*\pi)(\sigma_x \otimes \tau_x) \\
&= \sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n} n!} (\mathsf{T}\phi_x^*\pi)^{i_1 j_1} \cdots (\mathsf{T}\phi_x^*\pi)^{i_n j_n} (\sigma_x)_{,i_1 \cdots i_n} (\tau_x)_{,j_1 \cdots j_n}
\end{aligned}
$$

where we sum over all the indices $i_1, \ldots, i_n, j_1, \ldots, j_n$. Moreover, we use the notation, where the indices on the right of the comma denote derivatives with respect to the corresponding variable, e.g.

$$
R_{i,i_1 \cdots i_k} := \partial_{i_1} \cdots \partial_{i_k} R_i. \tag{88}
$$

3.2. The connection 1-form $A(R, \mathsf{T}\phi^*\pi)$

In Section 2.2 we have obtained the Kontsevich weights

$$
w_{\Upsilon_n} = \frac{1 + (-1)^n}{2^{n+1}(n+1)}, \qquad n \geq 0 \tag{89}
$$

of the family of graphs $(\Upsilon_n)_{n \geq 0}$. Let $\sigma \in \Gamma(\widehat{\mathrm{Sym}}(T^*M)[[\hbar]])$ be a section and fix $x \in M$. For R as in Section 1.2 we set $R_x(y) := R(x, y)$ and $(R_x)_i^k(y) := R_i^k(x, y)$. Using the Kontsevich weights above, we get

(90)

$$
\begin{aligned}
A(R_x, \mathsf{T}\phi_x^*\pi)(\sigma_x) &= \mathrm{d}x^i A\left((R_x)_i^k \frac{\partial}{\partial y^k}, \mathsf{T}\phi_x^*\pi \right)(\sigma_x) \\
&= \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \frac{1 + (-1)^n}{2^{n+1}(n+1)} \\
&\qquad \times (\mathsf{T}\phi_x^*\pi)^{i_1 j_1} \cdots (\mathsf{T}\phi_x^*\pi)^{i_n j_n} (R_x)^k_{i,i_1 \cdots i_n} (\sigma_x)_{,k j_1 \cdots j_n}
\end{aligned}
$$

where we again sum over all indices $i, k, i_1, \ldots, i_n, j_1, \ldots, j_n$.

This allows us to write down an explicit expression for the deformed Grothendieck connection, namely

(91)
$$
\begin{aligned}
(\mathcal{D}_G)_x &= \mathrm{d}_x + A(R_x, \mathsf{T}\phi_x^*\pi) = \left(\frac{\partial}{\partial x^i} + A\big((R_x)_i, \mathsf{T}\phi_x^*\pi\big)\right)\mathrm{d}x^i \\
&= \Bigg(\frac{\partial}{\partial x^i} + \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!}\frac{1+(-1)^n}{2^{n+1}(n+1)} \\
&\qquad \times (\mathsf{T}\phi_x^*\pi)^{i_1 j_1}\cdots(\mathsf{T}\phi_x^*\pi)^{i_n j_n}(R_x)^k_{i,i_1\cdots i_n}\frac{\partial^{n+1}}{\partial y^{j_n}\cdots\partial y^{j_1}\partial y^k}\Bigg)\mathrm{d}x^i
\end{aligned}
$$

3.3. The curvature 2-form $F(R, R, \mathsf{T}\phi^*\pi)$

In Section 2.1 we have obtained the Kontsevich weights in terms of the hypergeometric function and gave a more explicit formula in Section 2.4

$$
w_{\Gamma_n} = \frac{1-(-1)^n}{2^{n+2}(n+1)(n+2)}, \qquad n \geq 0 \tag{92}
$$

for the family of graphs $(\Gamma_n)_{n\geq 0}$. Using these weights above we then get for $x \in M$

$$
\begin{aligned}
F(R_x, R_x, \mathsf{T}\phi_x^*\pi) &= \mathrm{d}x^i \wedge \mathrm{d}x^j F\big((R_x)_i, (R_x)_j, \mathsf{T}\phi_x^*\pi\big) \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!}\frac{1-(-1)^n}{2^{n+2}(n+1)(n+2)} \\
&\quad \times (\mathsf{T}\phi_x^*\pi)^{i_1 j_1}\cdots(\mathsf{T}\phi_x^*\pi)^{i_n j_n}(R_x)^k_{i,l i_1\cdots i_n}(R_x)^l_{j,k j_1\cdots j_n}
\end{aligned} \tag{93}
$$

where, as usual, we sum over the indices $i, j, k, l, i_1, \ldots, i_n, j_1, \ldots, j_n$.

3.4. A Fedosov-type equation for Poisson manifolds

We can now write down the *modified deformed Grothendieck connection* as defined in [5, 12], namely

$$
\overline{\mathcal{D}}_G = \mathcal{D}_G + [\gamma, \ ]_\star, \tag{94}
$$

where the deformed Grothendieck connection $\mathcal{D}_G$ is explicitly given by (91), the star product is explicitly given by (87), $[\ ,\ ]_\star$ denotes the star commutator and $\gamma \in \Omega^1(M, \widehat{\mathrm{Sym}}(T^*M)[[\hbar]])$ is such that the following Fedosov-type equation holds:

$$F^M + \mathcal{D}_G\gamma + \gamma \star \gamma = 0, \tag{95}$$

with Weyl curvature $F^M = F(R, R, \mathsf{T}\phi^*\pi)$ explicitly given by (93). This equation appears in the globalization construction for deformation quantization of Poisson manifolds. The existence of such a γ was given in [5, 6]. Let us emphasize a bit more on this existence result. Since γ takes values in $\widehat{\mathrm{Sym}}(T^*M)[[\hbar]]$ we may write

$$\gamma = \gamma_0 + \hbar\gamma_1 + \hbar^2\gamma_2 + \dots \tag{96}$$

Similarly, for the deformed Grothendieck connection we may write

$$\mathcal{D}_G = D_G + \hbar^2\mathcal{D}_2 + \hbar^4\mathcal{D}_4 + \dots \tag{97}$$

where $D_G = \mathrm{d} + L_R$ is the classical Grothendieck connection and where we have used that the Kontsevich weights (89) satisfy $w_{\Upsilon_0} = 1$ and $w_{\Upsilon_n} = 0$ for all odd $n \geq 1$.

Finally, for the curvature we can write

$$F^M = \hbar F_1 + \hbar^3 F_3 + \hbar^5 F_5 + \dots \tag{98}$$

where we have used that that the Kontsevich weights (92) satisfy $w_{\Gamma_n} = 0$ for all even $n \geq 0$.

This now allows us to decompose Equation (95) into a system of equations depending on the order of $\hbar$.

In order $\hbar^0$ we get the equation

$$D_G\gamma_0 = 0, \tag{99}$$

which, according to Section 1.2, can be solved by $\gamma_0 = \mathsf{T}\phi^* f$ for some smooth function $f \in C^\infty(M)$, since the cohomology of the classical Grothendieck connection $H^\bullet_{D_G}(\Gamma(\widehat{\mathrm{Sym}}(T^*M)))$ is concentrated in degree zero (by the Poincaré Lemma) and

$$H^0_{D_G}(\Gamma(\widehat{\mathrm{Sym}}(T^*M))) \cong \mathsf{T}\phi^* C^\infty(M) \cong C^\infty(M). \tag{100}$$

In order $\hbar^1$ we get the equation

$$F_1 + D_G\gamma_1 + (\gamma_0 \star \gamma_0)_1 = 0. \tag{101}$$

Using the Bianchi identity we see that $D_G F_1 = 0$ and by Equation (99) it also immidiately follows that $D_G(\gamma_0 \star \gamma_0)_1 = 0$. So we get that $D_G\gamma_1$ is equal to a D_G-closed form, but the corresponding cohomology group is trivial. Hence it follows that $D_G\gamma_1$ is equal to a D_G-exact form, and thus it is possible to find a γ_1 that solves Equation (101).

By induction, one can show that in each order $\hbar^k$ for $k \geq 1$, $D_G\gamma_k$ is equal to a D_G-closed and hence D_G-exact form depending on the lower order coefficients of F^M and γ. In particular it follows that there exists a γ_k solving the equation for the corresponding order.

Remark 3.1. Note that in order to globalize Kontsevich's star product one may be tempted to define a bullet product

$$(f \bullet g)(x) := \big(P(\mathsf{T}\phi^*\pi)(\mathsf{T}\phi^* f \otimes \mathsf{T}\phi^* g)\big)(x;0). \tag{102}$$

This is indeed a well-defined global product on $C^\infty(M)[[\hbar]]$, but it is in general not associative. To make this product associative one has to introduce a quantization map (see e.g. [5])

$$\rho\colon H^0_{D_G}(\Gamma(\widehat{\mathrm{Sym}}(T^*M))) \to H^0_{\mathcal{D}_G}(\Gamma(\widehat{\mathrm{Sym}}(T^*M)[[\hbar]])) \tag{103}$$

which then again leads to the global star product

$$f \star_{_M} g := \big(\rho^{-1}\big(\rho(\mathsf{T}\phi^* f) \star \rho(\mathsf{T}\phi^* g)\big)\big)\Big|_{y=0}. \tag{104}$$

Here $\star$ denotes Kontsevich's star product and $\star_M$ its global version on M. Using the weights, we can also get an explicit expression for the bullet product (102) by

$$\begin{aligned}&\big(P(\mathsf{T}\phi^*\pi)(\mathsf{T}\phi^* f \otimes \mathsf{T}\phi^* g)\big)(x;0)\\ &\quad = \left(\sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n} n!} (\mathsf{T}\phi_x^*\pi)^{i_1 j_1} \cdots (\mathsf{T}\phi_x^*\pi)^{i_n j_n} (\mathsf{T}\phi_x^* f)_{,i_1\cdots i_n} (\mathsf{T}\phi_x^* g)_{,j_1\cdots j_n}\right)(0).\end{aligned} \tag{105}$$

3.5. The lifted curvature 2-form $F(\overline{R}, \overline{R}, \mathsf{T}\overline{\phi}^*\pi)$

Let M be a smooth manifold and let $\phi\colon TM \to M$ be a formal exponential map and consider the lift $\overline{\phi}\colon TN \to N$ to the cotangent bundle $N = T^*M$. We set $x = (q,p) \in N$ and $y = (\bar{q}, \bar{p}) \in T_xN$. Note that this is a particular case of a canonical symplectic manifold.

We will consider the lifted vector fields $\overline{R}$ to the cotangent case, which induce lifted interaction vertices within the Feynman graphs which appear in the computation of the connection 1-form and its curvature 2-form and see how these terms simplify. First we note that $A(\overline{R}, \mathsf{T}\overline{\phi}^*\pi)$ is still given by

$$(106)\quad A(\overline{R}_x, \mathsf{T}\overline{\phi}_x^*\pi)(\sigma_x) = \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \frac{1+(-1)^n}{2^{n+1}(n+1)} \times (\mathsf{T}\overline{\phi}_x^*\pi)^{i_1 j_1} \cdots (\mathsf{T}\overline{\phi}_x^*\pi)^{i_n j_n} (\overline{R}_x)^k_{i,i_1\cdots i_n}(\sigma_x)_{,kj_1\cdots j_n}.$$

The simplification in this case is a small one: All summands containing a term $(\overline{R}_x)^k_{i,i_1\cdots i_n}$ with more than one derivative with respect to $\bar{p}$ will vanish [18].

For the case of the curvature 2-form F^N the simplification is more interesting. Since for each non-vanishing coefficient $(\mathsf{T}\overline{\phi}_x^*\pi)^{ij}$ one of the two outgoing edges is always representing a $\bar{q}$-derivative and the other corresponding edge representing a $\bar{p}$-derivative (since we work with Darboux coordinates around $x \in N$), we see that the sum in (93) terminates at $n = 2$. Or put differently, we only have to consider the graphs Γ_n up to $n = 2$, i.e. with at most two wedges attached to the wheel consisting of two $\overline{R}$-vertices (cf. Figure 2 in Section 2.1). Moreover, since the Kontsevich weights (92) are, up to $n = 2$, given by $w_{\Gamma_0} = 0$, $w_{\Gamma_1} = \frac{1}{24}$ and $w_{\Gamma_2} = 0$, we get

$$(107)\qquad F_x^N = F(\overline{R}_x, \overline{R}_x, \mathsf{T}\overline{\phi}_x^*\pi) = \frac{\hbar}{48}(\mathsf{T}\overline{\phi}_x^*\pi)^{rs}(\overline{R}_x)^k_{i,lr}(\overline{R}_x)^l_{j,ks}\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

where we sum over the indices i, j, r, s, k, l and where again summands containing a term $(\overline{R}_x)^k_{i,lr}$ with more than one derivative with respect to $\bar{p}$ vanish. So in the case of a cotangent bundle we get a much simpler expression for the Weyl curvature F^N.

Appendix A. Binomial sums

Here we will treat the binomial sums appearing in the expression (67) and show that we indeed get the results stated in (68).

A.1. $B(n)$

Let us start with

$$B(n) = \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{2^{n+1}(n-l+1)}. \tag{A.1}$$

Using the well known identity

$$\sum_{l=0}^{n} (-1)^l \binom{n+1}{l} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{n+1}{n-l+1} = (-1)^n \tag{A.2}$$

it immidiately follows that

$$B(n) = \frac{(-1)^n}{2^{n+1}(n+1)}. \tag{A.3}$$

A.2. $C(n)$

Let us continue with

$$C(n) = \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{2^l(n-l+1)}. \tag{A.4}$$

Using the identity (A.2), we can write

$$(n+1)C(n) = \sum_{l=0}^{n} \binom{n+1}{l} \left(-\frac{1}{2}\right)^l. \tag{A.5}$$

Using the Binomial theorem we then find

$$\sum_{l=0}^{n} \binom{n+1}{l} \left(-\frac{1}{2}\right)^l = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}, \tag{A.6}$$

and hence

$$C(n) = \frac{1+(-1)^n}{2^{n+1}(n+1)}. \tag{A.7}$$

A.3. $A(n)$

Finally, let us treat the case

(A.8)
$$A(n)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{s=0}^{n-k-l}\binom{n}{k}\binom{n-k}{l}\binom{n-k-l}{s}\frac{(-1)^{l+s}}{2^{n-k-s+1}(n-k-l-s+1)}.$$

Write

(A.9)
$$A(n)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^{l}}{2^{n-k+1}}\sum_{s=0}^{n-k-l}\binom{n-k-l}{s}\frac{(-2)^{s}}{(n-k-l-s+1)}.$$

We first treat the innermost sum: Set $m=n-k-l$. Then

$$\sum_{s=0}^{n-k-l}\binom{n-k-l}{s}\frac{(-2)^{s}}{(n-k-l-s+1)}=\sum_{s=0}^{m}\binom{m}{s}\frac{(-2)^{s}}{(m-s+1)}. \tag{A.10}$$

Using $\binom{m+1}{s}=\frac{m+1}{m-s+1}\binom{m}{s}$, we find that

$$\sum_{s=0}^{m}\binom{m}{s}\frac{(-2)^{s}}{(m-s+1)}=\frac{1}{m+1}\sum_{s=0}^{m}\binom{m+1}{s}(-2)^{s}. \tag{A.11}$$

Applying the Binomial theorem we get that

$$\sum_{s=0}^{m}\binom{m+1}{s}(-2)^{s}=(-1)^{m+1}\big(1-2^{m+1}\big). \tag{A.12}$$

Plugging all of this into (A.9), we find

(A.13)
$$\begin{aligned}A(n)&=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}\frac{(-1)^{l}}{2^{n-k+1}}\frac{(-1)^{n-k-l+1}}{n-k-l+1}\big(1-2^{n-k-l+1}\big)\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k+1}}{2^{n-k+1}}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{1}{n-k-l+1}\big(1-2^{n-k-l+1}\big).\end{aligned}$$

Moreover, we have

$$\text{(A.14)}\quad \begin{aligned}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{1}{n-k-l+1}&=\frac{1}{n-k+1}\sum_{l=0}^{n-k}\binom{n-k+1}{l}\\&=\frac{1}{n-k+1}\left(2^{n-k+1}-1\right),\end{aligned}$$

and

$$\text{(A.15)}\quad \begin{aligned}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{2^{n-k-l+1}}{n-k-l+1}&=\frac{1}{n-k+1}\sum_{l=0}^{n-k}\binom{n-k+1}{l}2^{n-k-l+1}\\&=\frac{1}{n-k+1}\left(3^{n-k+1}-1\right).\end{aligned}$$

Hence

$$\text{(A.16)}\quad \begin{aligned}A(n)&=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k+1}}{2^{n-k+1}(n-k+1)}\left(2^{n-k+1}-3^{n-k+1}\right)\\&=(-1)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n-k+1}\left(1-\left(\frac{3}{2}\right)^{n-k+1}\right).\end{aligned}$$

Now

$$\text{(A.17)}\quad \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n-k+1}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}(-1)^k=\frac{(-1)^n}{n+1},$$

and

$$\text{(A.18)}\quad \begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n-k+1}\left(\frac{3}{2}\right)^{n-k+1}&=\frac{(-1)^{n+1}}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\left(-\frac{3}{2}\right)^{n-k+1}\\&=\frac{(-1)^n}{n+1}=\frac{1}{2^{n+1}(n+1)}+\frac{(-1)^n}{n+1}.\end{aligned}$$

Finally, we get

$$\text{(A.19)}\quad A(n)=(-1)^{n+1}\left(\frac{(-1)^n}{n+1}-\frac{1}{2^{n+1}(n+1)}-\frac{(-1)^n}{n+1}\right)=\frac{(-1)^n}{2^{n+1}(n+1)}.$$

References

[1] S. Axelrod and I. M. Singer, *Chern-Simons perturbation theory. II*, J. Differ. Geom. **39**.1 (1994), 173–213.

[2] P. Banks, E. Panzer, and B. Pym, *Multiple zeta values in deformation quantization*, Invent. Math. **222** (2020), 79–159. `doi:10.1007/s00222-020-00970-x`.

[3] F. Bonechi, A. S. Cattaneo, and P. Mnev, *The Poisson sigma model on closed surfaces*, J. High Energy Phys. **1** (2012), 099, 26.

[4] A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, Commun. Math. Phys. **212** (2000), 591–611.

[5] A. S. Cattaneo and G. Felder, *On the globalization of Kontsevich's star product and the perturbative Poisson sigma model*, Progress of Theoretical Physics Supplement **144** (2001), 38–53.

[6] A. S. Cattaneo, G. Felder, and L. Tomassini, *Fedosov connections on jet bundles and deformation quantization*, Halbout G. Deformation quantization (2002), 191–202.

[7] A. S. Cattaneo, G. Felder, and L. Tomassini, *From local to global deformation quantization of Poisson manifolds*, Duke Math J. **115**.2 (2002), 329–352.

[8] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, *Classical BV theories on manifolds with boundary*, Commun. Math. Phys. **332**.2 (2014), 535–603.

[9] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, *Perturbative quantum gauge theories on manifolds with boundary*, Commun. Math. Phys. **357**.2 (2017), 631–730.

[10] A. S. Cattaneo and N. Moshayedi, *Introduction to the BV-BFV formalism*, Rev. Math. Phys. **32** (2020), p. 67.

[11] A. S. Cattaneo, N. Moshayedi, and K. Wernli, *Globalization for perturbative quantization of nonlinear split AKSZ sigma models on manifolds with boundary*, Commun. Math. Phys. **372**.1 (2019), 213–260.

[12] A. S. Cattaneo, N. Moshayedi, and K. Wernli, *On the globalization of the Poisson sigma model in the BV-BFV formalism*, Commun. Math. Phys. **375**.1 (2020), 41–103.

[13] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differ. Geom. **40**.2 (1994), 213–238.

[14] G. Felder and T. Willwacher, *On the (Ir)rationality of Kontsevich Weights*, International Mathematics Research Notices **2010** (2009), 701–716.

[15] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2), **139**.1 (1994), 183–225.

[16] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, Ann. Phys. **235**.2 (1994), 435–464.

[17] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66**.3 (2003), 157–216.

[18] N. Moshayedi, *On globalized traces for the Poisson sigma model*, `arXiv:1912.02435`, (2019).

[19] F. Musio, *Computation of Kontsevich weights of connection and curvature graphs for symplectic Poisson structures*, Master Thesis, Universität Zürich, available at: `http://user.math.uzh.ch/cattaneo/theses.html`, (2019).

[20] P. Schaller and T. Strobl, *Poisson structure induced (topological) field theories*, Mod. Phys. Lett. A **09**.33 (1994), 3129–3136.

[21] M. Van den Bergh, *The Kontsevich weight of a wheel with spokes pointing outward*, Algebras and Representation Theory **12**.2-5 (2009), 443–479.

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