Globalization Constructions for Perturbative Quantum Gauge Theories on Manifolds with Boundary

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Introduction to the methods of the thesis

This thesis deals with certain constructions in the intersection of mathematics (algebra, geometry, topology) and physics (quantum field theory). In quantum field theory (abbreviated QFT) we are interested in the mathematical formulation of elementary particle systems in space–time. Mathematically, a QFT comes with many different definitions. An important class of QFTs are given by topological theories, i.e. theories that are invariant under diffeomorphisms (and only have finitely many physical degrees of freedom). One point of view for a topological quantum field theory (TQFT) is given by a functorial approach proposed by Atiyah in [3] (and developed further by e.g. Witten [110] or Lurie [81]). There, a TQFT is defined as a functor

(1) $$Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\mathbb{C}}$$

from the category of n-cobordisms to the category of vector spaces over the complex numbers. The objects in $\mathbf{Cob}_n$ are $(n\ \ 1)$-manifolds (representing the boundary of a cobordism) with an additional in- and outgoing labeling (which can be also regarded as orientation) and the morphisms are given by the bounding n-manifolds. Composition of morphisms is given by gluing two $(n\ \ 1)$-manifolds with opposite labeling (see Figure 0.0.1). These n-manifolds represent space–time. Usually, one considers smooth manifolds, so the gluing procedure requires more data.

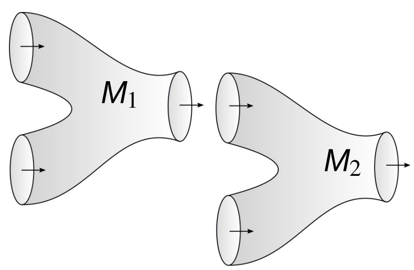

Figure 0.0.1. Illustration for the gluing of two manifolds M_1 and M_2 along the common boundary . The arrows at each boundary component represents the orientation *in* or *out*. Note that the gluing has to be done by gluing on M_1 with on M_2 endowed with the opposite orientation.

A topological field theory (TFT) can be more generally defined in a similar fashion where the target category is any symmetric monoidal category $(\mathbf{C}; \ )$. It is only a TQFT if the target category is given by $\mathbf{Vect}_{\mathbb{C}}$. Note that the symmetric monoidal structure on $\mathbf{Cob}_n$ is given by disjoint union of topological spaces and on $\mathbf{Vect}_{\mathbb{C}}$ is given by the usual tensor product . An important additional condition on Z is that it is a monoidal functor, i.e.

(2) $$Z\ {}_1 \sqcup {}_2 = Z\ {}_1 \quad Z\ {}_2; \quad \forall\ {}_1;\ {}_2 \in \mathbf{Cob}_n.$$

Consider orientation preserving diffeomorphisms $f\colon\ {}_1 \to\ {}_2$ and $g\colon\ {}_2 \to\ {}_3$ for ${}_1;\ {}_2;\ {}_3 \in \mathbf{Cob}_n$. Then the functorial property gives

(3) $$Z(f)\colon Z\ {}_1 \to Z\ {}_2; \qquad Z(g\ \ f) = Z(g)\ \ Z(f).$$

Moreover, if denotes with the opposite labeling, we get

(4) $$Z\ \ = Z\ ;$$

where Z_Σ^* denotes the dual space of $Z_\Sigma \in \mathbf{Vect}_\mathbb{C}$. Let M_1 and M_2 be morphisms in $\mathbf{Cob}_n$, such that $\Sigma_1 = \partial M_1$ and $\Sigma_2 = \partial M_2$. Then $Z(\tilde{f})\colon Z_{M_1} \mapsto Z_{M_2}$ for an extension $\tilde{f}$ of f to a diffeomorphism $M_1 \to M_2$. Moreover, if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3$, $\partial M_2 = \Sigma_2 \sqcup \Sigma_3^*$, and $M = M_1 \sqcup_{\Sigma_3} M_2$ (the manifold obtained by gluing M_1 and M_2 along Σ_3), then

$$Z_M = \langle Z_{M_1}, Z_{M_2} \rangle, \tag{5}$$

where $\langle\ ,\ \rangle$ denotes the pairing

$$Z_{\Sigma_1} \otimes Z_{\Sigma_3} \otimes Z_{\Sigma_3}^* \otimes Z_{\Sigma_2} \to Z_{\Sigma_1} \otimes Z_{\Sigma_2}. \tag{6}$$

Note that if $\Sigma_1 = \emptyset = \Sigma_2$, then the pairing gives a number since $Z_\emptyset = \mathbb{C}$.

This approach is using an important physical concept of nature which is called "locality". In particular, it says that the underlying space–time manifold can be cut into tiny little pieces where one can do computations and then glue again everything together to get the computation for the whole space–time. Another approach using the concept of locality is given by the notion of "perturbative quantization" within the Feynman path integral picture. Thus, another point of view for the definition of a QFT is given by a path integral. The theory is specified by a space of fields F_M on a space–time manifold M and a local action functional $S_M\colon F_M \to \mathbb{C}$ of the form

$$S_M[\phi] = \int_M \mathcal{L}(\phi, \partial\phi, \dots), \tag{7}$$

where $\phi \in F_M$ and $\mathcal{L}$ denotes the Lagrangian density depending on ϕ and higher derivatives. In the perturbative approach of QFT[1] we are interested in the "partition function" modeled in terms of a path integral using the classical theory described by S. Let $\Sigma := \partial M$ and denote by B_Σ the space of boundary values of the fields on M, i.e. we have a restriction map $F_M \to B_\Sigma$ and denote by $Z_\Sigma := \mathrm{Fun}_\mathbb{C}(B_\Sigma) \in \mathbf{Vect}_\mathbb{C}$ complex-valued functions on B_Σ. The partition function is given by

$$Z_M(\varphi; \hbar) = \int_{\{\phi \in F_M \mid \phi|_\Sigma := \varphi\}} e^{\frac{i}{\hbar} S_M[\phi]} \mathcal{D}[\phi] \in Z_\Sigma. \tag{8}$$

The gluing condition is simply translated to

$$Z_M = \int_{\varphi \in B_\Sigma} Z_{M_1}(\varphi) Z_{M_2}(\varphi) \mathcal{D}[\varphi] \tag{9}$$

if we glue M out of M_1 and M_2 along their common boundary Σ, i.e. $M = M_1 \sqcup_\Sigma M_2$. This is consistent with the Atiyah gluing construction, where the pairing of $Z_{M_1} \in Z_\Sigma$ and $Z_{M_2} \in Z_\Sigma^*$ gives a number in the case where M is closed. This number corresponds to the partition function on M. It is important to note that the symbol $\mathcal{D}$ denotes a "non-existing" measure on F_M, hence the expression above is mathematically not really defined. However, there are ways of dealing with these expressions by the methods of *stationary phase expansion* and *Feynman diagrams*. The *stationary phase formula* is given by

$$\int_X e^{\frac{i}{\hbar} f(x)} \xrightarrow{\hbar \to 0} \sum_{x_0 \in \{\text{critical points of } f\}} \frac{e^{\frac{i}{\hbar} f(x_0)}}{\sqrt{|\det(f''(x_0))|}} e^{\frac{i\pi}{4} \operatorname{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}}, \tag{10}$$

[1]In general, it does not have to be topological. However, we will mainly focus on topological theories.

where X is some compact manifold, μ a volume form on X and $f \in C^\infty(X)$. The formula (10) can be improved by corrections in $\hbar$ as follows:

$$\int_X e^{\frac{i}{\hbar} f(x)} \mu \underset{\hbar \to 0}{\sim} \sum_{x_0 \in \{\text{critical points of } f\}} \frac{e^{\frac{i}{\hbar} f(x_0)}}{\sqrt{|\det(f''(x_0))|}} e^{\frac{i\pi}{4} \operatorname{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}} \sum_\Gamma \hbar^{-\chi(\Gamma)} W_\Gamma; \tag{11}$$

where Γ denotes a *Feynman graph* with vertices of valence ≥ 3, $\chi(\Gamma) \leq 0$ denotes the Euler characteristic of a graph Γ and $W_\Gamma \in \mathbb{C}$ denotes the weight of a graph Γ. In particular, if we denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges of a graph Γ, we have

$$W_\Gamma = \frac{i^{|E(\Gamma)|+|V(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \sum_{i_1,\dots,i_{2|E(\Gamma)|} \in \{1,\dots,p\}} \prod_{e=(h_1,h_2)\in E(\Gamma)} f''(x_0)^{-1}_{i_{h_1} i_{h_2}} \prod_{v \in V(\Gamma)} \partial_{i_{h_1}} \cdots \partial_{i_{h_{\mathrm{val}(v)}}} f(x_0); \tag{12}$$

where $\mathrm{val}(v)$ denotes the valency of the vertex v and f'' denotes the Hessian of f. Formula (11) tells us that the expression of the partition function as defined in (8) can be understood perturbatively as a formal power series in $\hbar$.

However, there is a problem if one wants to apply the stationary phase construction to the case of general physical theories S, since it requires the critical points of the function f to be isolated, i.e. the Hessian of f has to be non-degenerate at each critical point. We are especially interested in theories where the Lagrangian L is invariant under some symmetries (diffeomorphisms of the space of fields), i.e. the action itself is preserved and there is no change of the physical theory. These type of theories are called *gauge theories* and the group of symmetries, denoted by G, is called the *gauge group*. It is well-known that the critical points of such theories are not isolated. Nevertheless, there is a way out of this problem by using the *Batalin–Vilkovisky (BV) formalism* (see Chapter 0.1), which is a formalism that allows us to deal with perturbative quantization of gauge theories. There one makes the replacement

$$\int_{F_M} e^{\frac{i}{\hbar} S_M[\phi]} \mathscr{D}[\phi] \to \int_{\mathcal{L} \subset \mathcal{F}_M} e^{\frac{i}{\hbar} \mathcal{S}_M[\phi]} \mathscr{D}[\phi]. \tag{13}$$

While the left hand side is not defined as mentioned before, the right hand side is defined for a suitable Lagrangian submanifold $\mathcal{L}$ of the $\mathbb{Z}$-graded supermanifold $\mathcal{F}_M$ which is now additionally endowed with a symplectic structure ω_M. Moreover, the classical action S_M is replaced by the *BV action* $\mathcal{S}_M$, which is an extension of the classical action that satisfies the *Classical Master Equation* (see Chapter 0.1). Similarly as for the stationary phase construction, one can now define the right hand side again as a power series using Feynman graphs by considering the asymptotic limit. The main point of this construction is that the right hand side is invariant under deformations of $\mathcal{L}$ (see Theorem 0.1.2.3) and thus the left hand side may be regarded as a bad choice for $\mathcal{L}$ for which the perturbative expansion is not defined. In fact, the bad choice on the right hand side is given by the left hand side times the vanishing integral on the corresponding ghost fields.

The focus of this thesis lies in a subset of a particular type of theories which are called *topological Sigma Models*. In particular, we consider Sigma Models of *AKSZ type* (see Chapter 1.2) and especially the *Poisson Sigma Model* (see Chapter 2.1). Since these theories are invariant under symmetries (gauge theories) we use the BV formalism to deal with the perturbative quantization construction. Moreover, we use its extension for manifolds with boundary coupled together with the construction in the bulk, which is called the BV-BFV formalism (see Chapter 0.1).

Abstract

This thesis is divided into three parts.

(1) In Part 1 we give a construction for globalization within the mathematical construction of the Batalin–Vilkovisky and Batalin–Fradkin–Vilkovisky formalisms (BV-BFV) for certain type of theories. This is done by using methods of formal geometry as in [17, 64]. In particular, the focus lies on AKSZ theories of a special type which we call "split". We extend the perturbative quantization on manifolds with boundary to a global type, where we replace the modified Quantum Master Equation, describing a gauge condition for quantization, to a differential version which we call the "modified differential Quantum Master Equation" and show that it is consistent with the cohomological methods described in [38]. Moreover, we explain how the state and the boundary operator in this setting behave for a change of depending choices.

(2) In Part 2 we consider the Poisson Sigma Model which is an example of a split AKSZ type Sigma Model. The Poisson Sigma Model is important in relation to deformation quantization and Kontsevich's star product construction and it is an example of a non-trivial theory where the Batalin–Vilkovisky formalism is in fact needed for quantization (the BRST formalism [13, 12, 11, 103], which is another way of dealing with gauge theories, is only valid for linear Poisson structures). We apply the constructions developed in the first part to the Poisson Sigma Model with the difference that we consider mixed boundary conditions (corners). This is in fact important for the quantization of the relational symplectic groupoid which is ultimately linked to the Poisson Sigma Model. However, considering these type of mixed conditions, the modified differential Quantum Master Equation does not hold. In fact, it is spoilt by "curvature terms" arising from Kontsevich's L_1-morphism. To fix this issue, one has to extend the constructions of the first part by adding some counter terms to the action and use a Fedosov-type globalization construction for Poisson manifolds. We call the extended construction the "twisted theory". We show that the modified differential Quantum Master Equation holds for this construction and that everything is again consistent with the cohomological methods as before.

(3) In Part 3 we construct a global version of a trace formula which was given by Cattaneo and Felder in [28] using the formal global extension of the Poisson Sigma Model on the disk. The globalization construction involves similar techniques as in Part 2, such as formal geometry, where we use a Fedosov-type equation to show the trace property. Moreover, we describe its connection to the Nest–Tsygan theorem (algebraic index theorem)[87] and the Tamarkin–Tsygan theorem [100]. We also show how this trace is reduced to a trace-construction for symplectic manifolds presented by Grady, Li and Li in [69] for the case of a cotangent bundle.

Original work

This thesis is based on my (collaboration) papers [41, 42, 84, 40]. Original work in this thesis is presented in Chapter 1.3, Chapter 1.4, Chapter 2.3, Chapter 2.4, and Chapter 2.5, which are based on original work from my collaboration papers [41, 42]. More original work is presented in Chapter 3.5 which is based on original work from my paper [84].

Self-plagiarism

Most of the thesis is taken from the above mentioned papers in a copy-paste manner. Chapter 0.1 is taken from my collaboration paper [42]. Minor parts of Chapter 0.1, and most of Appendices A and B are taken from my collaboration paper [40]. Most of Part 1 is taken from my collaboration paper [41]. Most of Part 2 is taken from my collaboration paper [42]. Most of Part 3 is taken from my paper [84]. Appendix C is taken from [84]. The Appendices D and E are taken from [42]. The papers [41, 42] are in collaboration with Alberto Cattaneo and Konstantin Wernli. We share equal credits (and responsibility) for both of them. The paper [40] is in collaboration with Alberto Cattaneo and we share equal credits (and responsibility) for it.

Main results

We would like to give a summary of the main results obtained in this thesis for each different parts.

Part 1. Let us summarize the main results of the first part. One of the main theorems of this thesis is the modified differential Quantum Master Equation for anomaly free, unimodular AKSZ theories:

Theorem (1.3.2.1). *Consider the full covariant perturbative state $\tilde{\psi}_{\Sigma,x}$ as a quantization of an anomaly free and unimodular split AKSZ theory with target $T^*[d-1]M$, where M is a graded manifold. Then*

$$\underbrace{\left(d_x - i\hbar\Delta_{\mathcal{V}_{\Sigma,x}} + \frac{i}{\hbar}\Omega_{\partial\Sigma}\right)}_{=:\nabla_G}\tilde{\psi}_{\Sigma,x} = 0, \tag{14}$$

where we denote by d_x the de Rham differential on $\overline{M}$, the body of the graded manifold M.

Another main result is that the quantum GBFV operator is a coboundary operator:

Theorem (1.3.4.1). *The operator ∇_G squares to zero, i.e.*

$$(\nabla_G)^2 \equiv 0. \tag{15}$$

We also show how the state and the BFV operator transform under change of data. This is captured in the following theorem:

Theorem (1.4.1.1). *Let Ω_t be defined as in Definition 0.1.4.25 and let $\tilde{\psi}_t$ be defined as in Definition 1.2.4.2 for all $t\in[0,1]$. Then we have*

$$\frac{d}{dt}\Big|_{t=0}\Omega_t = d_x\tau + [\Omega_{t=0},\tau], \tag{16}$$

$$\frac{d}{dt}\Big|_{t=0}\tilde{\psi}_t = \nabla_G(\tilde{\psi}_{t=0}\zeta) - \tau\tilde{\psi}_{t=0} \tag{17}$$

for some operator $\tau \in \Gamma(\mathrm{End}(\mathcal{H}_{tot}))$ and a section $\zeta \in \Gamma(\mathcal{H}_{tot})$.

Part 2. Let us summarize the main results of the second part. We show that the introduction of alternating boundary conditions introduces a quantum anomaly, i.e. a failure of the closedness of $\tilde{\psi}$. In fact, we have:

Proposition (2.4.3.1). *Consider the full state $\tilde{\psi}_{\Sigma,x}$ defined by $\hat{\psi}_{\Sigma,x}$ as in Definition 1.2.4.2. Then*

$$\nabla_G\tilde{\psi}_{\Sigma,x} = \exp\left(\frac{i}{\hbar}\int^{1}_{\partial_0\Sigma} F(R,R,T\varphi_x)(\mathbb{X})\right)\tilde{\psi}_{\Sigma,x}, \tag{18}$$

where we integrate out the X-fluctuation $\mathbb{X}$, which are the high energy part, in F along $\partial_0\Sigma$.

Here $F(R,R,T\varphi_x)$ is defined in Section 2.2.1 and is part of Kontsevich's L_∞-morphism, and $\partial_0\Sigma$ is a certain boundary component. Next, we show that by "twisting" the state and the operator ∇_G by an appropriate Maurer–Cartan element (see Chapter 2.3) the anomaly can be reduced to terms supported at the corners (i.e. points where boundary conditions change). We prove the following theorem:

THEOREM (2.4.4.3). *Consider the twisted full state $\tilde{\psi}_{\Sigma,x}$ defined in Definition 2.3.5.5 and the twisted quantum Grothendieck BFV operator ∇_G defined in Definition 2.4.4.1. Then*

$$\nabla_G \tilde{\psi}_{\Sigma,x} = \sum_{C\in\mathcal{C}_1} T(C)\tilde{\psi}_{\Sigma,x}, \tag{19}$$

where $T(C)$ are functionals on $\mathcal{B}^{\mathcal{P}}_{\partial}$ with values in $\Omega^1(P)$, depending only on the values of the fields at the corner point C.

We show that this twisted operator also squares to zero (Remark 2.4.4.2). However, we want again to interpret the state as a closed section with respect to a certain operator that squares to zero. In Chapter 2.5 we show that this can be done by enlarging the space of states (see Definition 2.5.1.2) and defining a new operator $\hat{\nabla}_G$ (see Equation (308)) on the new bundle of states. We show the following theorem:

THEOREM (2.5.3.1). *Let $\hat{\nabla}_G$ be given as in Equation (308), and consider the twisted full state $\tilde{\psi}_{\Sigma,x}$. Then*

$$\hat{\nabla}_G \tilde{\psi}_{\Sigma,x} = 0 \tag{20}$$

We also show that the new operator $\hat{\nabla}_G$ again squares to zero:

THEOREM (1.3.4.1). *The operator $\hat{\nabla}_G$ squares to zero, i.e. $(\hat{\nabla}_G)^2 = 0$.*

Part 3. Finally, we summarize the main results of the third part. Let (M,Π) be a Poisson manifold. One of the main results is the construction of the global trace using the formal global Poisson Sigma Model in terms of graphs and a morphism $\mathcal{V}$ of L_∞-modules. This is captured in the following Proposition:

PROPOSITION (3.3.3.2). *The map*

$$\mathrm{Tr}\colon f \mapsto \int_M \left(\int \exp\left(\frac{i}{\hbar}S^{\varphi}_{\Sigma,R} + \frac{i}{\hbar}\int_{\partial D}\langle x, \hat{X}\rangle\right)\mathcal{O}_{(T\varphi^* f)}\right)_0 \tag{21}$$

coincides with

$$\mathrm{Tr}\colon f \mapsto \sum_{n=0}^{\infty}\frac{1}{n!}\int_M \mathcal{V}^{\varphi}_n\left(\hbar(R \wedge \cdots \wedge R) \mid (T\varphi^* f)\right), \tag{22}$$

where we consider $\mathcal{V}^{\varphi}_n\left(\hbar(R \wedge \cdots \wedge R)\mid\right)$ to be defined on the negative cyclic complex for sections of the Weyl algebra $\mathcal{W}$.

Using a Fedosov-type equation (Equation (401)) for the appearing configuration integrals on the boundary when applying Stokes' theorem, we can show that the trace defined using the formal global action does indeed satisfy the trace property with respect to the global version of Kontsevich's star product. This is captured in the following theorem:

THEOREM (3.5.1.1). *The map*

$$\mathrm{Tr}\colon f \mapsto \sum_{n=0}^{\infty}\frac{1}{n!}\int_M \mathcal{V}^{\varphi}_n\left(\hbar(R \wedge \cdots \wedge R) \mid (T\varphi^* f)\right) \tag{23}$$

is a trace on the algebra $(C^\infty_c(M)[[\hbar]], \star_M)$.

Moreover, we show how the constructed globalized trace is related to the Tamarkin–Tsygan theorem which is the content of the following theorem:

Theorem (3.5.2.1). *The trace map*

$$\mathrm{Tr}\colon f \mapsto \prod_{n=0}^{\infty} \frac{1}{n!} \int_M \mathcal{V}_n\left(R, \ldots, R \mid (T\varphi)^* f\right) \tag{24}$$

evaluated at any periodic cyclic chain $c \in PC_m(C_c^{\infty}(M))[[\hbar]]$ is given by

$$\int_M \hat{A}_u(TM)\,\mathrm{Ch}(c)\exp\left(\pi_\hbar / u\right); \tag{25}$$

where $\pi_\hbar \in \hbar\,\Gamma\left(\bigwedge^2 TM\right)[[\hbar]]$ is a formal Poisson structures.

We also show how the global trace is related to the Nest–Tsygan theorem if the Poisson manifold is given by a cotangent bundle. This is captured by the following theorem:

Theorem (3.5.5.1). *If the Poisson manifold M is given by a cotangent bundle, the global trace*

$$\mathrm{Tr}\colon f \mapsto \prod_{n=0}^{\infty} \frac{1}{n!} \int_M \mathcal{V}_n\left(R, \ldots, R \mid (T\varphi)^* f\right) \tag{26}$$

satisfies

$$\mathrm{Tr}(1) = \int_M \hat{A}(TM)\exp(\omega_\hbar / \hbar). \tag{27}$$

Finally, we show how the global equivariant trace formula constructed by Grady, Li and Li for symplectic manifolds using topological quantum mechanics can be obtained from our global construction for Poisson manifolds if the Poisson manifold is a cotangent bundle. This is captured in the following Proposition:

Proposition (3.5.6.1). *The trace map*

$$\mathrm{Tr}\colon f \mapsto \prod_{n=0}^{\infty} \frac{1}{n!} \int_M \mathcal{V}_n\left(R, \ldots, R \mid (T\varphi)^* f\right) \tag{28}$$

reduces to the trace map

$$f \mapsto \mathrm{Tr}^{S^1}(f) := \int_M \int^{S^1} [f]^{S^1}_{\infty} \tag{29}$$

if the Poisson manifold M is given by a cotangent bundle.

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Nima Moshayedi
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"*Ein stiller Geist ist jahrelang geschäftig,*
die Zeit nur macht die feine Gärung kräftig."

Mephistopheles zu Faust, Hexenküche, Faust I
Johann Wolfgang von Goethe

Dedicated to the memory of Thi Oanh Pham

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CHAPTER 0.1

The BV-BFV formalism

The main constructions of this chapter are based on a special formalism treating gauge theories developed by Batalin and Vilkovisky during the 1970's and 1980's [10, 9, 7]; see also [95, 71, 2, 23, 22, 50, 89, 83] and references therein. A similar approach was developed by Batalin, Fradkin and Vilkovisky to path integrals for gauge theories using the Hamiltonian approach, whereas the formalism provided by Batalin and Vilkovisky was given in the Lagrangian setting using a Lagrangian modified by Faddeev–Popov ghost [54] and their BRST variations [13, 12, 11, 103], which add a factor to the path integral that divides out the gauge redundancy in a nice way. This formalism is known today as the BV formalism. The methods of gauge fixing were extended by Batalin, Fradkin, Fradkina and Vilkovisky in a series of papers [8, 6, 62, 61]. The Hamiltonian version is known today as the BFV formalism (see [8, 6, 62, 61] and also [99, 94]). The BV and BFV formalisms have been developed through time by the work of many different people. In [37] Cattaneo, Mnev and Reshetikhin studied for the first time the classical BV framework for gauge theories on space–time manifolds with boundary and extended everything to manifolds with corners. Recently, in [38, 36], they introduced a general perturbative quantization scheme for gauge theories on manifolds with boundary, compatible with cutting and gluing, in the cohomological symplectic formalism, which we call the BV-BFV formalism. Different types of theories are already explicitly described for this procedure, such as e.g. abelian BF theory [72, 95, 82], different AKSZ theories [1] (see Chapter 1.2) (e.g. Chern–Simons theory with a splitting of the Lie algebra [4, 5, 109, 39], the Poisson Sigma Model (see Chapter 2.1) [26, 31, 29, 32, 42] , or non-abelian BF theories [72, 82, 38]), and $2D$ Yang–Mills theory [74]. Another reference for learning about this formalism is [40].

0.1.1. Field theory

We start with the following definition of a classical field theory.

Definition 0.1.1.1 (Classical field theory). A d-dimensional *classical field theory* associates to every compact d-dimensional manifold M (possibly with boundary) a space of fields F_M and an action functional $S_M\colon F_M \to \mathbb{R}$.

Field theories are usually required to be *local*. For the purpose of the present paper, the following definition will suffice. When we refer to a "manifold" M, we implicitly allow M to come equipped with background fields (e.g. a metric) upon which the field theory is allowed to depend[1].

Definition 0.1.1.2 (Local field theory). We say that a field theory (F_M, S_M) is *local* if there is a fiber bundle $E \to M$ such that $F_M = \Gamma(E)$ and there is an integer k such that

$$S_M[\phi] = \int_M L[j^k \phi], \tag{30}$$

[1]This in particular will also allow us to consider 2D Yang–Mills theory in this formalism.

where j^k denotes the k-th jet prolongation, and $L\colon J^kE \to \mathrm{Dens}(M)$ is a function on the k-th jet bundle of E with values in densities of M. L is called the *Lagrangian* of the theory.

Let (F_M, S_M) be a local field theory. If $\partial M \neq \emptyset$ and we do not fix any boundary conditions, there is a 1-form $\alpha_{\partial M} \in \Omega^1(F_{\partial M})$ (the *Noether* 1-form) such that the variation of the action S_M is given by

$$\delta S_M := \mathrm{EL}_M + \pi_M^*\alpha_{\partial M}, \tag{31}$$

where $\pi_M\colon F_M \to F_{\partial M}$ is the natural surjective submersion from the space of fields F_M onto the space of fields $F_{\partial M}$ on the boundary ∂M. $F_{\partial M}$ is given by restrictions of bulk fields and their normal jets to the boundary. We denote by EL_M the 1-form[2] coming from the *Euler-Lagrange equations* (EL equations). The classical solutions are given by the critical points of S_M, i.e. by solutions of $\delta S_M = 0$. One can define a presymplectic form $\omega_{\partial M}$ on $F_{\partial M}$ by setting $\omega_{\partial M} := \delta\alpha_{\partial M}$ (we think of δ as the de Rham differential on the space of fields). By techniques of symplectic geometry, such as *symplectic reduction*, one can obtain a symplectic manifold $(F^{\partial}_{\partial M}, \omega^{\partial}_{\partial M})$. Moreover, this construction is compatible with cutting and gluing [37, 36]. It leads to a nice quantum formulation in the guise of path integrals after choosing a suitable polarization [38]. We will discuss these issues in this chapter.

Remark 0.1.1.3. Note that if $\partial M = \emptyset$ we get the usual Euler–Lagrange equations from $\delta S_M = 0$.

0.1.2. Finite-dimensional BV theory

Let M be a closed manifold and let F_M denote the space of fields associated to M. If we consider a regular[3] local field theory $S_M\colon F_M \to \mathbb{R}$ the partition function in the path integral approach is

$$Z_M = \int_{\phi\in F_M} e^{\frac{i}{\hbar}S_M[\phi]}D[\phi]. \tag{32}$$

Usually, F_M is infinite-dimensional, and one cannot define[4] $D[\phi]$. The way out is usually to translate the formal asymptotics as $\hbar \to 0$ of finite-dimensional integrals to the infinite-dimensional case. The terms in the asymptotic expansion are convenienetly labeled by Feynman diagrams [59, 58, 88]. If the critical points of the action functional S_M are degenerate, one needs to gauge-fix the theory before one can use the formal asymptotics. The most powerful gauge fixing formalism is the BV formalism. We briefly review its finite-dimensional version. Further references for gauge theories, different gauge fixing formalisms (including BV) and their perturbative quantization are [82, 83, 90].
The start is the following definition:

Definition 0.1.2.1 (BV manifold). A *BV manifold* is a triple $(\mathcal{F}, \omega, \mathcal{S})$, where $\mathcal{F}$ is a supermanifold with $\mathbb{Z}$-grading (see Appendix B for an exposition on graded geometry), ω an odd symplectic form of degree -1 on $\mathcal{F}$, and $\mathcal{S}$ is an even function of degree zero on $\mathcal{F}$, such that

$$(\mathcal{S}, \mathcal{S}) = 0. \tag{33}$$

Here, following Batalin and Vilkovisky [10, 9], we denote the Poisson bracket induced by the odd symplectic form with round brackets $(\ ,\ )$.

[2] EL_M is the term that depends only on the variations of the fields but not on higher jets.
[3] This means that the Hessian of the Lagrangian is weakly non degenerate.
[4] Only in special situations, i.e. $\dim M = 1$, and some examples discussed in [68].

Remark 0.1.2.2 (Grading on $\mathcal{F}$). Note that we have two different gradings on $\mathcal{F}$, the $\mathbb{Z}_2$-grading from the supermanifold structure and an additional $\mathbb{Z}$-grading. In physics, the $\mathbb{Z}$-grading is referred to as *ghost number* and the parity corresponds to bosonic and fermionic particles. Since we consider only bosonic theories, the $\mathbb{Z}_2$-grading coincides with the reduction of the $\mathbb{Z}$-grading.

In a Darboux chart (q^i, p_i), we can define the *BV Laplacian* by

$$\Delta_{\text{loc}} := \sum_i (-1)^{|q^i|} \frac{\partial^2}{\partial q^i \partial p_i}. \tag{34}$$

Then we get that $(\Delta_{\text{loc}})^2 = 0$ and for two functions f, g, $\Delta_{\text{loc}}(fg) = \Delta_{\text{loc}} f g \pm f \Delta_{\text{loc}} g \pm (f, g)$. This extends to a well-defined global operator Δ on half-densities [76, 96].
Moreover, given a half-density f and a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, we can define a *BV integral* $\int_{\mathcal{L}} f$ by restricting the half-density to the Lagrangian where it becomes a density and can be integrated. The main result in the Batalin–Vilkovisky formalism is the following theorem.

Theorem 0.1.2.3 (Batalin–Vilkovisky [10]). *If we assume that the integrals converge, then*

- *If $f = \Delta g$, then $\int_{\mathcal{L}} f = 0$,*
- *If $\Delta f = 0$ and $(\mathcal{L}_t)$ is a smoothly varying family of Lagrangians, then $\frac{d}{dt}\int_{\mathcal{L}_t} f = 0$.*

Remark 0.1.2.4. The second point of Theorem 0.1.2.3 tells us that if we have an ill-defined integral $\int_{\mathcal{L}_0} f$ for some Lagrangian submanifold $\mathcal{L}_0$, but we know that $\Delta f = 0$, then we can define the value of the integral by a well-defined one $\int_{\mathcal{L}_1} f$ for some Lagrangian submanifold $\mathcal{L}_1$, and this does not depend on the choice of $\mathcal{L}_1$ as long as we deform it smoothly.

The replacement of $\mathcal{L}_0$ by $\mathcal{L}_1$ is called *gauge-fixing*. This construction can be extended to any (super)manifold. Moreover, considering $f = e^{\frac{i}{\hbar}S}$, two other conditions arise, which are given by

$$(S, S) = 0, \tag{35}$$

$$(S, S) - 2i\hbar \Delta S = 0. \tag{36}$$

We call (35) the *Classical Master Equation* (CME) and (36) the *Quantum Master Equation* (QME). The latter one is equivalent to $\Delta e^{\frac{i}{\hbar}S} = 0$. The former one is the classical limit of the latter one for $\hbar \to 0$, and motivates the definition of a BV manifold as given above.

0.1.3. Classical BV-BFV formalism

We now turn to the infinite-dimensional case and review the main definitions of reference [37]. We first consider the classical BV formalism in field theory and its extension to manifolds with boundary.

Definition 0.1.3.1 (BV theory). A d-dimensional *BV theory* is the association of a BV manifold $M \mapsto (\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ to every closed d-manifold M.

Remark 0.1.3.2. These BV manifolds are typically infinite-dimensional. This means that neither the BV Laplacian nor the BV integral are defined (at least not without further work).

Definition 0.1.3.3 (BV extension). We say that a BV theory $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ is a *BV extension* of a local field theory $M \mapsto (F_M, S_M)$ if for all closed d-manifolds M, we have that the degree 0 part $(\mathcal{F}_M)_0$ of $\mathcal{F}_M$ satisfies $(\mathcal{F}_M)_0 = F_M$ and $\mathcal{S}_M|_{(\mathcal{F}_M)_0} = S_M$. Moreover, we want $\mathcal{F}_M$, $\mathcal{S}_M$ and ω_M to be local.

To extend the BV formalism to manifolds with boundary one needs its Hamiltonian counterpart, the BFV formalism [8, 6, 62, 61].

Definition 0.1.3.4 (BFV manifold). A *BFV manifold* is a triple

$$\mathcal{F}^\partial := (\mathcal{F}^\partial, \omega^\partial, Q^\partial) \tag{37}$$

where $\mathcal{F}^\partial$ is a graded manifold, ω^∂ an even symplectic form of degree 0, and Q^∂ a degree 1 cohomological, symplectic vector field on $\mathcal{F}^\partial$. If $\omega^\partial = \delta\alpha^\partial$ is exact, the BFV manifold is called *exact*.

Remark 0.1.3.5. For degree reasons, Q^∂ automatically has a Hamiltonian function that we denote by $\mathcal{S}^\partial$, and call it the *boundary action*. This is the reason that the boundary action $\mathcal{S}^\partial$ is not included in the data of a BFV manifold.

Again, we denote by δ the de Rham differential on the space of fields. The notion of BV theory can be extended to manifolds with boundary as was shown in [37, 38]. On the boundary we will use the BFV formalism. The compatibility between the BV formalism and the BFV formalism is captured in the following definition.

Definition 0.1.3.6 (BV-BFV manifold). A *BV-BFV manifold* over a given exact BFV manifold $\mathcal{F}^\partial = (\mathcal{F}^\partial, \omega^\partial = \delta\alpha^\partial, Q^\partial)$ is a quintuple

$$\mathcal{F} := (\mathcal{F}, \omega, \mathcal{S}, Q, \pi), \tag{38}$$

where

- $\mathcal{F}$ is a graded manifold,
- ω is an even symplectic form of degree 0,
- $\mathcal{S}$ is an even function of degree 0,
- Q is a degree 1 cohomological vector field,
- $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion

such that

$$\iota_Q\omega = \delta\mathcal{S} + \pi^*\alpha^\partial \tag{39}$$

and $Q^\partial = \delta\pi Q$ where $\delta\pi$ denotes the differential of π.

Remark 0.1.3.7. If $\mathcal{F}^\partial$ is a point, we get that $(\mathcal{F}_M, \omega_M, \mathcal{S}_M)$ is a BV manifold. The shorthand notation for a BV-BFV manifold is $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$.

Note that by Remark 0.1.3.7, the following notion generalizes the one of a BV theory.

Definition 0.1.3.8 (BV-BFV theory). A d-dimensional *BV-BFV theory* associates

- to every closed $(d-1)$-dimensional manifold Σ a BFV manifold $\mathcal{F}^\partial_\Sigma$,
- to a d-dimensional manifold M with boundary ∂M a BV-BFV manifold $\pi_M\colon \mathcal{F}^\partial_M \to \mathcal{F}^\partial_{\partial M}$.

Remark 0.1.3.9. Formally, for the Hamiltonian vector field Q of $\mathcal{S}$, one can write $(\mathcal{S},\mathcal{S}) = \iota_Q\iota_Q\omega = Q(\mathcal{S})$. If we consider a BV-BFV theory for a manifold M with boundary ∂M, one can prove [37, Proposition 3.1] that

$$Q(\mathcal{S}) = \pi^*(2\mathcal{S}^\partial - \iota_{Q^\partial}\alpha^\partial). \tag{40}$$

Together with (39) this implies that

$$\iota_Q\iota_Q\omega = 2\pi^*\mathcal{S}^\partial. \tag{41}$$

We call (41) the *modified Classical Master Equation*.

It was shown in [37] that *abelian BF theory* is an example of a BV-BFV theory.

EXAMPLE 0.1.3.10 (ABELIAN BF THEORY). *Abelian BF theory* is given by the following data: To a $(d-1)$-dimensional manifold Σ, we associate the BFV manifold

$$\mathcal{F}^\partial = \Omega^\bullet(\Sigma)[1] \oplus \Omega^\bullet(\Sigma)[d-2] \ni (\mathsf{X},\boldsymbol{\eta}) \tag{42}$$

$$\omega^\partial = \int_\Sigma \delta\mathsf{X} \wedge \delta\boldsymbol{\eta} \tag{43}$$

$$Q^\partial = (-1)^d \int_\Sigma \left(d\boldsymbol{\eta} \wedge \frac{\delta}{\delta\boldsymbol{\eta}} + d\mathsf{X} \wedge \frac{\delta}{\delta\mathsf{X}} \right) \tag{44}$$

The Hamiltonian function of Q^∂ is given by $\mathcal{S}^\partial = \int_\Sigma \boldsymbol{\eta} \wedge d\mathsf{X}$.
To a d-dimensional manifold M with boundary ∂M, we associate the BV-BFV manifold over $\mathcal{F}^\partial_{\partial M}$ given by

$$\mathcal{F}_M = \Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[d-2] \ni (\mathsf{X},\boldsymbol{\eta}) \tag{45}$$

$$\omega_M = \int_M \delta\mathsf{X} \wedge \delta\boldsymbol{\eta} \tag{46}$$

$$\mathcal{S}_M = \int_M \boldsymbol{\eta} \wedge d\mathsf{X} \tag{47}$$

$$Q_M = (-1)^d \int_M \left(d\boldsymbol{\eta} \wedge \frac{\delta}{\delta\boldsymbol{\eta}} + d\mathsf{X} \wedge \frac{\delta}{\delta\mathsf{X}} \right) \tag{48}$$

$$\tag{49}$$

and the map $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$ is given by restriction, i.e. $\pi = \iota^*_{\partial M}$, where $\iota_{\partial M}\colon \partial M \to M$ is the inclusion.

DEFINITION 0.1.3.11 (BF-LIKE THEORIES). We say that a BV-BFV theory is *BF-like* if

$$\mathcal{F}_M = \left(\Omega^\bullet(M) \otimes V[1]\right) \oplus \left(\Omega^\bullet(M) \otimes V^*[d-2]\right) \tag{50}$$

$$\mathcal{S}_M = \int_M \left(\langle \boldsymbol{\eta}, d\mathsf{X} \rangle + \mathcal{V}(\mathsf{X},\boldsymbol{\eta}) \right), \tag{51}$$

where V is a graded vector space, $\langle\ ,\ \rangle$ denotes the pairing between V^* and V, and $\mathcal{V}$ denotes some density-valued function of the fields X and $\boldsymbol{\eta}$ whose value $\mathcal{V}(x)$ at $x \in M$ depends only on $\mathsf{X}(x), \boldsymbol{\eta}(x)$[5], such that $\mathcal{S}_M$ satisfies the Classical Master Equation for M without boundary.

EXAMPLE 0.1.3.12 (QUANTUM MECHANICS). Consider M to be a 1-dimensional manifold, i.e. $d = 1$ and $V = W[-1]$ with W concentrated in degree zero. Denote by P and Q the degree-zero form components of X and $\boldsymbol{\eta}$, respectively. Choose a volume form dt on M and a function H on T^*W. Set $\mathcal{V}(\mathsf{X},\boldsymbol{\eta}) := H(\mathsf{X},\boldsymbol{\eta})dt = H(Q,P)dt$. Then

$$\mathcal{S}_M = \int_M \left(\sum_i P_i \dot{Q}^i + H(Q,P) \right) dt, \tag{52}$$

is the action of classical mechanics in the Hamiltonian formalism.

REMARK 0.1.3.13. The Poisson Sigma Model, which is the main theory regarded in this thesis, is an example of a BF-like AKSZ theory.

[5] In particular, $\mathcal{V}$ does not depend on derivatives of the fields.

Example 0.1.3.14 (BF-like AKSZ theories [1]). Assume we are given a function Θ on $T^*[d-1](V[1]) = V[1] \oplus V^*[d-2]$ that is of degree d such that $\{\Theta, \Theta\} = 0$, where $\{\ ,\ \}$ is the canonical Poisson structure on the shifted cotangent bundle. Set $\mathcal{V}(X, \Theta)$ to be the top degree part of $\Omega(X, \Theta)$.

0.1.4. Quantum BV-BFV formalism

In [38] the notion of a quantum BV-BFV theory was given and it was shown how to perturbatively quantize a classical BV-BFV theory[6]. Let us briefly review this[7].

Definition 0.1.4.1 (Quantum BV-BFV theory). A d-dimensional *quantum BV-BFV theory* associates

- To every closed $(d-1)$-dimensional manifold Σ a graded $\mathbb{C}[[\hbar]]$-module $\mathcal{H}_\Sigma$,
- To every d-dimensional manifold (possibly with boundary) M a finite-dimensional BV manifold $\mathcal{V}_M$, a degree 1 coboundary operator $\Omega_{\partial M}$ on $\mathcal{H}_{\partial M}$ and a homogeneous element[8]

$$\psi_M \in \hat{\mathcal{H}}_M := \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}_M) \otimes \mathcal{H}_{\partial M}, \tag{53}$$

where $\mathrm{Dens}^{\frac{1}{2}}(M)$ denotes the space of half-densities on some manifold M,

such that

$$(\hbar^2 \Delta_{\mathcal{V}_M} + \Omega_{\partial M})\psi_M = 0. \tag{54}$$

Remark 0.1.4.2. The shorthand notation for a quantum BV-BFV theory is

$$M \mapsto (\hat{\mathcal{H}}_M, \psi_M, \Delta_{\mathcal{V}_M}, \Omega_{\partial M}). \tag{55}$$

Let us introduce some terminology: We call $\mathcal{V}_M$ the *space of residual fields*, $\mathcal{H}_{\partial M}$ the *space of boundary states* and ψ_M the *quantum state*. $\Delta_{\mathcal{V}_M}$ denotes the canonical BV Laplacian on half-densities on the BV manifold $\mathcal{V}_M$. Recall that $\Delta^2_{\mathcal{V}_M} = 0$. Hence, $\hat{\mathcal{H}}_M$ carries the two commuting differentials $\Delta_{\mathcal{V}_M}$ and $\Omega_{\partial M}$ which gives it the structure of a bicomplex. We call $\Omega_{\partial M}$ the quantum BFV boundary operator. The condition (54) is called the *modified Quantum Master Equation*.

Remark 0.1.4.3 (Terminology). The space $\mathcal{H}_\Sigma$ is called the space of states because it arises as a quantization of the symplectic manifold of boundary fields (see also the discussion in 0.1.4.1 below). An element of $\mathcal{H}_\Sigma$ is then called a state. In the absence of residual fields, ψ_M is the state produced by the bulk. It is what is usually called a state in the literature, see e.g. [109] if ∂M has a single connected component[9]. In case M is a cylinder, ψ_M is actually an evolution operator that can be viewed as a generalized state (note that we never insist on $\mathcal{H}_\Sigma$ being a Hilbert space). If the boundary is empty (and there are no residual fields), then ψ_M is what is usually called the partition function. It is in general useful (and often necessary) to make a choice of "slow" or "low energy" fields, which we prefer to call residual fields, and to integrate on a complement. Then ψ_M will be properly a state only after integrating out the residual fields (which is not always possible, cf. the discussions in [37, Appendix F], [16], [74]), but by abuse of notation we prefer to call it the state anyway.

[6] We have to assume certain conditions which are in particular satisfied for BF-like theories.

[7] We slightly changed the definition of quantum BV-BFV theory so that in principle it does not depend on a classical BV-BFV theory.

[8] Typically, ψ will have degree 0. This is the case when the gauge-fixing Lagrangian (see below) has degree zero, in the sense that its Berezinian bundle has degree zero. This is the case in all examples we consider.

[9] Note that this is a particular state induced by the bulk and not some choice of vacuum state.

Definition 0.1.4.4 (Equivalence). We say that two quantum BV-BFV theories $(\hat{\mathcal{H}}_M, \Delta_{V_M}, \Omega_{\partial M}, \psi_M)$ and $(\hat{\mathcal{H}}'_M, \Delta_{V'_M}, \Omega'_{\partial M}, \psi'_M)$ are equivalent if for every manifold M with boundary ∂M there is a quasi-isomorphism of bicomplexes

(56) $$I_M : (\hat{\mathcal{H}}_M, \Delta_{V_M}, \Omega_{\partial M}) \to (\hat{\mathcal{H}}'_M, \Delta_{V'_M}, \Omega'_{\partial M})$$

such that $I_M(\psi_M) = \psi'_M$.

Definition 0.1.4.5 (Change of data). Another equivalence relation among theories is the following: We say that two quantum BV-BFV theories $(\hat{\mathcal{H}}_M, \Delta_{V_M}, \Omega_{\partial M}, \psi_M)$ and $(\hat{\mathcal{H}}_M, \Delta_{V'_M}, \Omega'_{\partial M}, \psi'_M)$ are related by *change of data* if there is an operator τ of degree 0 on $\mathcal{H}_{\partial M}$ and an element $\zeta_M \in \hat{\mathcal{H}}_M$ with $\deg(\zeta_M) = \deg(\psi_M) - 1$ such that

(57) $$\begin{aligned} \Omega'_{\partial M} &= [\Omega_{\partial M}, \tau] \\ \psi'_M &= (\hbar^2 \Delta_{V_M} + \Omega_{\partial M})\zeta_M - \psi_M \end{aligned}$$

Let us now explain how to produce a quantum BV-BFV theory by perturbative quantization of a classical BV-BFV theory. Fix a classical BV-BFV theory $\pi : \mathcal{F} \to \mathcal{F}^\partial$. For simplicity we shall assume that $\mathcal{F}$ and $\mathcal{F}^\partial$ are always *vector spaces*, which is sufficient for the present paper. For a general discussion see [38].

0.1.4.1. The space of states. Consider a $(d-1)$-dimensional manifold Σ. Then the BV-BFV theory associates to it a symplectic vector space $(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma, Q^\partial_\Sigma)$. Morally, we want to construct $\mathcal{H}_\Sigma$ and Ω_Σ as a geometric quantization of this symplectic vector space. More precisely, the construction proceeds as follows. We require the data of a polarization[10] $\mathcal{P}$ of this symplectic vector space. For our purposes, a splitting

(58) $$\mathcal{F}^\partial \cong \mathcal{B}^\mathcal{P} \times \mathcal{K}^\mathcal{P}$$

of $\mathcal{F}^\partial$ into Lagrangian subspaces is sufficient. Here $\mathcal{K}^\mathcal{P}$ is thought of as the Lagrangian distribution on $\mathcal{F}^\partial$ and $\mathcal{B}^\mathcal{P}$ is identified with the leaf space of the polarization. Given a polarization $\mathcal{P}$ the associated space of states $\mathcal{H}_{\partial M}$ is a certain space of functionals on $\mathcal{B}^\mathcal{P}$. We will discuss the space of states for BF-like theories in 0.1.4.3.

0.1.4.2. Splitting the space of fields. To define the quantum state we proceed with the following constructions. Consider a d-manifold M (possibly with boundary) and the associated BV-BFV manifold $(\mathcal{F}_M, \omega_M, \mathcal{S}_M, Q_M, \pi_M)$ over the exact BFV manifold $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M} = \delta\alpha^\partial_{\partial M}, Q^\partial_{\partial M})$. Then, choosing a polarization $\mathcal{P}$ on ∂M, we choose a splitting

(59) $$\mathcal{F}_M \cong \mathcal{B}^\mathcal{P}_{\partial M} \times \mathcal{Y},$$

where $\mathcal{Y}$ denotes some complement. This splitting is subject to the following assumption[11].

Assumption 0.1.4.6 ([38]). There is a weakly symplectic form $\omega_\mathcal{Y}$ on $\mathcal{Y}$ such that ω_M is the extension of $\omega_\mathcal{Y}$ to $\mathcal{F}_M$.

Formally, we can think of $\mathcal{B}^\mathcal{P}_{\partial M}$ as the space of *boundary fields* and $\mathcal{Y}$ the space of *bulk fields*. Depending on the boundary polarization, we split $\mathcal{Y}$ into residual fields and some complement, i.e. we choose a splitting

(60) $$\mathcal{Y} \cong \mathcal{V}^\mathcal{P}_M \times \mathcal{Y}'$$

[10] We have only considered the case of real polarizations so far.

[11] This assumption forces one to choose singular extensions of boundary fields.

subject to the following assumption[12]

Assumption 0.1.4.7. We assume the following hold:

(1) $\mathcal{V}^{\mathcal{P}}_M, \mathcal{Y}'$ are BV manifolds,
(2) $\mathcal{V}^{\mathcal{P}}_M$ is finite-dimensional
(3) $\omega_{\mathcal{Y}} = \omega_{\mathcal{V}^{\mathcal{P}}_M} + \omega_{\mathcal{Y}'}$.

We call the complement $\mathcal{Y}'$ the space of *fluctuation fields*. Residual fields are also called *low energy fields* or *slow fields* and fluctuation fields are also called *high energy fields* or *fast fields*. Typically we choose $\mathcal{V}^{\mathcal{P}}_M$ as the solutions of $\delta\mathcal{S}^0_M = 0$ modulo gauge transformations, where $\mathcal{S}^0_M$ denotes the quadratic part of the action $\mathcal{S}_M$. This is the minimal choice, and is typically called the space of *zero modes*. Other choices are related by the equivalence relations above.

Definition 0.1.4.8 (Good splitting). A splitting

$$\mathcal{F}_M \cong \mathcal{B}^{\mathcal{P}}_{\partial M} \times \mathcal{V}^{\mathcal{P}}_M \times \mathcal{Y}' \tag{61}$$

is called *good* if it satisfies Assumptions 0.1.4.6 and 0.1.4.7.

Remark 0.1.4.9 (Connection to Atiyah's TQFT formulation). From the point of view of topological quantum field theories (TQFTs) as functors $\mathbf{Cob}_n \to \mathbf{Vect}_{\mathbb{C}}$ from the n-cobordism category (objects are $(n-1)$-manifolds bounding an n-manifold and morphisms are exactly the bounding n-manifolds connecting the objects) to the category of vector spaces over the complex numbers, it is clear that the quantum state should depend on the bulk. This can be seen by using the fact that the state represents exactly the bounding manifold between the objects and thus a morphism of the cobordism category. This also makes sense for manifolds without boundary, in which case the state is given by a partition function $Z\colon \mathbb{C} \to \mathbb{C}$, where as a morphism in $\mathbf{Cob}_n$ it represents any closed n-manifold, seen as a bounding manifold connecting the empty $(n-1)$-manifold, i.e. as a morphism $\emptyset \to \emptyset$.

0.1.4.3. The quantum state in BF-like theories. The quantum state in BF-like theories is defined perturbatively in terms of Feynman graphs by considering integrals defined on the configuration space of these graphs. In BF-like theories there are two preferred polarizations, namely the $\frac{\delta}{\delta \mathsf{X}}$- and $\frac{\delta}{\delta \mathsf{E}}$-polarization. We specify a polarization by splitting the boundary ∂M of the manifold M into two parts $\partial_1 M$ and $\partial_2 M$, where we choose the $\frac{\delta}{\delta \mathsf{E}}$-polarization on $\partial_1 M$ and the $\frac{\delta}{\delta \mathsf{X}}$-polarization on $\partial_2 M$. We denote the X-leaf by $\mathbb{X} \in \mathcal{B}^{\frac{\delta}{\delta \eta}}_{\partial M}$ and the E-leaf by $\mathbb{E} \in \mathcal{B}^{\frac{\delta}{\delta \mathsf{X}}}_{\partial M}$.
For BF-like theories, the polarization determines the first splitting as

$$\mathcal{B}^{\mathcal{P}}_{\partial M} = \Omega^\bullet(\partial_1 M) \otimes V[1] \oplus \Omega^\bullet(\partial_2 M) \otimes V^*[d-2] \tag{62}$$

$$\mathcal{Y} = \Omega^\bullet(M, \partial_1 M) \otimes V[1] \oplus \Omega^\bullet(M, \partial_2 M) \otimes V^*[d-2] \tag{63}$$

The minimal space of residual fields is isomorphic to

$$\mathcal{V}^{\mathcal{P}}_M \cong H^\bullet(M, \partial_1 M) \otimes V[1] \oplus H^\bullet(M, \partial_2 M) \otimes V^*[d-2], \tag{64}$$

for some graded vector space V. A good splitting is then determined by a splitting of the complex of de Rham forms with relative boundary conditions into a subspace $\mathcal{V}^{\mathcal{P}}_M$ isomorphic to cohomology and a complement $\mathcal{Y}'$ in a way compatible with the symplectic structure. One possibility to do so is to use a Riemannian metric and embed the cohomology as harmonic forms.
Before we can introduce the quantum state we need to introduce the concept of *composite fields*,

[12] This assumption is rather strong but can be slightly relaxed to the notion of *hedgehog fibration*.

which we denote by square brackets $[\ ]$, e.g. for a boundary field $\mathbb{A}$ we will write $[\mathbb{A}^{i_1}\cdots\mathbb{A}^{i_k}]$. They can be understood as a *regularization* of higher functional derivatives: the higher functional derivative $\frac{\delta^k}{\delta\mathbb{A}^{i_1}\cdots\delta\mathbb{A}^{i_k}}$ gets replaced by a first order functional derivative $\frac{\delta}{\delta[\mathbb{A}^{i_1}\cdots\mathbb{A}^{i_k}]}$. Concretely, this corresponds to introducing additional boundary vertices as in Figure 0.1.2.

Remark 0.1.4.10. In fact, this concept will not be needed for the definition of the principal part of the quantum state. We will use this concept to define the full part of the quantum state where we need to make sure that it will be compatible with the quantum BFV boundary operator, where higher functional derivatives do indeed appear as we will see.

Definition 0.1.4.11 (Regular functional). A *regular functional* on the space of base boundary fields is a linear combination of expressions of the form

$$\int_{C_{m_1}(\partial_1 M)\times C_{m_2}(\partial_2 M)} L^{J_1^1\dots J_1^{\ell_1}J_2^1\dots J_2^{\ell_2}\dots}_{I_1^1\dots I_1^{r_1}I_2^1\dots I_2^{r_2}\dots}\wedge\prod_{j=1}^{r_1}\left[\mathbb{X}^{I_1^j}\right]\wedge\cdots\wedge\prod_{j=1}^{r_{m_1}}\left[\mathbb{X}^{I_{m_1}^j}\right]\wedge\prod_{j=1}^{\ell_1}\left[\mathbb{E}_{J_1^j}\right]\wedge\cdots\wedge\prod_{j=1}^{\ell_{m_2}}\left[\mathbb{E}_{J_{m_2}^j}\right],\tag{65}$$

where I_i^j and J_i^j are (target) multi-indices and $L^{J_1^1\dots J_1^{\ell_1}J_2^1\dots J_2^{\ell_2}\dots}_{I_1^1\dots I_1^{r_1}I_2^1\dots I_2^{r_2}\dots}$ is a smooth differential form on the direct product of compactified configuration spaces (see Appendix D) $C_{m_1}(\partial_1 M)\times C_{m_2}(\partial_2 M)$ depending on residual fields. A regular functional is called *principal* if all multi-indices have length one.

Definition 0.1.4.12 (Full space of boundary states). The *full space of boundary states* $\mathcal{H}^{\mathcal{P}}_{\partial M}$ is given by the linear combinations of regular functionals of the form (65).

Definition 0.1.4.13 (Principal space of boundary states). We define the *principal space of boundary states* $\mathcal{H}^{\mathcal{P},\text{princ}}_{\partial M}$ as the subspace of $\mathcal{H}^{\mathcal{P}}_{\partial M}$, where we only consider principal regular functionals.

The state is defined in terms of Feynman graphs and rules. We briefly explain what these terms mean in the BV-BFV context (for perturbations of abelian BF theory).

Definition 0.1.4.14 ((BF) Feynman graph). A (BF) *Feynman graph* is an oriented graph with three types of vertices $V(\Gamma)=V_{\text{bulk}}(\Gamma)\sqcup V_{\partial_1}\sqcup V_{\partial_2}$, called bulk vertices and type 1 and 2 boundary vertices, such that

- bulk vertices can have any valence,
- type 1 boundary vertices carry any number of incoming half-edges (and no outgoing half-edges),
- type 2 boundary vertices carry any number of outgoing half-edges (and no incoming half-edges),
- multiple edges and loose half-edges (leaves) are allowed but not short loops (tadpoles).

A *labeling* of a Feynman graph is a function from the set of half-edges to $\{1,\dots,\dim V\}$.

Definition 0.1.4.15 (Principal graph). A Feynman graph is called *principal* if all boundary vertices (type 1 and type 2) are univalent or zero valent.

For a set S and a manifold M, the open configuration space of S in M is

(66) $$\mathrm{Conf}_S(M) := \{\iota : S \hookrightarrow M \mid \text{injection}\}.$$

Let Γ be a Feynman graph and M a manifold with boundary $\partial M = \partial_1 M \sqcup \partial_2 M$ and denote

(67) $$\mathrm{Conf}_\Gamma(M) := \mathrm{Conf}_{V_{\text{bulk}}}(M) \times \mathrm{Conf}_{V_{\partial_1}}(\partial_1 M) \times \mathrm{Conf}_{V_{\partial_2}}(\partial_2 M)$$

The Feynman rules are a map that associate to a Feynman graph Γ a differential form $\omega_\Gamma \in \Omega(\mathrm{Conf}_\Gamma(M))$.

Definition 0.1.4.16 ((BF) Feynman rules). Let Γ be a labeled Feynman graph, and choose a configuration $\iota : V(\Gamma) \to \mathrm{Conf}(\Gamma)$ (that respects the decompositions). We decorate the graph according to the following rules (called *Feynman rules*):

- Bulk vertices in M decorated by "vertex tensors"

(68) $$\mathcal{V}^{j_1\ldots j_t}_{i_1\ldots i_s} := \frac{\partial^{s+t}}{\partial X^{i_1}\cdots\partial X^{i_s}\partial\eta_{j_1}\cdots\partial\eta_{j_t}}\bigg|_{X=\eta=0}\mathcal{V}(X,\eta),$$

  where s, t are the in- and out- valencies of the vertex and $i_1, \ldots, i_s$ and $j_1, \ldots, j_t$ are the labels of the in (resp. out-)oriented half-edges.
- Boundary vertices $v \in V_{\partial_1}(\Gamma)$ with outgoing half-edges labeled $i_1, \ldots, i_k$ and no incoming half-edges are decorated by a composite field $[X^{i_1}\ldots X^{i_k}]$ evaluated at the point (vertex location) $\iota(v)$ on $\partial_1 M$.
- Boundary vertices $v \in V_{\partial_2}(\Gamma)$ on $\partial_2 M$ with incoming half-edges labeled $j_1, \ldots, j_l$ are decorated by $[\mathsf{E}_{j_1}\ldots\mathsf{E}_{j_l}]$ evaluated at the point on $\partial_2 M$.
- Edges between vertices v_1, v_2 are decorated with the propagator $\eta(\iota(v_1), \iota(v_2))\cdot\delta^j_i$, where η is the propagator induced by $\mathcal{L}\subset\mathcal{Y}'$, the chosen gauge-fixing Lagrangian.
- Loose half-edges (leaves) attached to a vertex v and labeled i are decorated with the residual fields x^i (for out-orientation), e_i (for in-orientation) evaluated at the point $\iota(v)$.

We denote the differential forms given by the decorations collectively by ω_d. The differential form ω_Γ at ι is then defined by multiplying all decorations and summing over all labelings:

(69) $$\omega_\Gamma = \sum_{\substack{\text{labelings}\\ \text{of } \Gamma}} \prod_{\substack{\text{decorations}\\ d \text{ of } \Gamma}} \omega_d$$

The Feynman rules are summarized in Figures 0.1.1 and 0.1.2.

Remark 0.1.4.17 (Configuration spaces). We will work with the Fulton–MacPherson/Axelrod–Singer compactification of configuration spaces on manifolds with boundary and corners (FMAS-compactification, see Appendix D). It is a non-trivial analytic statement (proven first by Axelrod and Singer [5]) that the propagator, *a priori* defined only on the open configuration space $\mathrm{Conf}_2(M)$, extends to the compactification $\mathsf{C}_2(M)$. It follows that also ω_Γ, for all Feynman graphs Γ, extends to the compactification $\mathsf{C}_\Gamma(M)$ of $\mathrm{Conf}_\Gamma(M)$. Since integrals remain unchanged by adding strata of lower codimension, this immediately proves that all integrals in Equation (70) below are finite. Moreover, the combinatorics of the stratification can be used for various computations using Stokes' theorem.

Definition 0.1.4.18 (Principal quantum state). Let M be a manifold, possibly with boundary. Given a BF-like BV-BFV theory $\pi_M : \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$, a polarization $\mathcal{P}$ on $\mathcal{F}^\partial_{\partial M}$, a good splitting

$\mathcal{F}_M = \mathcal{B}^{\mathcal{P}}_{\partial M} \times \mathcal{V}^{\mathcal{P}}_M \times \mathcal{Y}'$, and a gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{Y}'$, we define the *principal part of the quantum state* by the formal power series

$$\psi_M(\mathbb{X}, \mathbb{E}; \mathsf{x}, \mathsf{e}) := T_M \exp\left(\frac{i}{\hbar}\sum_{\Gamma}\frac{(-i\hbar)^{\mathrm{loops}(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\int_{C_\Gamma(M)}\omega_\Gamma(\mathbb{X}, \mathbb{E}; \mathsf{x}, \mathsf{e})\right), \tag{70}$$

where ω_Γ is given as in (69) and where we denote for an element $(\mathsf{X}, \mathsf{E}) \in \mathcal{F}_M$ the split by

$$\mathsf{X} = \mathbb{X} + \mathsf{x} + \mathcal{X}, \tag{71}$$
$$\mathsf{E} = \mathbb{E} + \mathsf{e} + \mathcal{E}. \tag{72}$$

Here the sum is taken over all *connected*, oriented, *principal BF* Feynman graphs Γ, $\mathrm{Aut}(\Gamma)$ denotes the set of all automorphisms of Γ, and $\mathrm{loops}(\Gamma)$ denotes the number of all loops of Γ.

The coefficient T_M is related to the Reidemeister torsion of M, but its precise nature is irrelevant for the purpose of a present paper. For a definition see [37].

Remark 0.1.4.19. The formal power series (70) is our definition of the formal perturbative expansion of the BV integral

$$\psi_M = \int_{\mathcal{L}\subset\mathcal{Y}'} e^{\frac{i}{\hbar}\mathcal{S}_M[(\mathsf{X}, \mathsf{E})]} \in \hat{\mathcal{H}}^{\mathcal{P}}_M := \hat{\mathcal{H}}^{\mathcal{P}}_{\partial M} \otimes \mathrm{Dens}^{\frac{1}{2}}(\mathcal{V}^{\mathcal{P}}_M). \tag{73}$$

It was observed in [38] that, given a good splitting of the form (61), one can decompose the action as

$$\mathcal{S}^{\mathcal{P}}_M := \hat{\mathcal{S}}_{M,0} + \hat{\mathcal{S}}_{M,\mathrm{pert}} + \mathcal{S}^{\mathrm{res}} + \mathcal{S}^{\mathrm{source}} \tag{74}$$

with

$$\hat{\mathcal{S}}_{M,0} := \int_M \langle \mathcal{E}, d\mathcal{X}\rangle \tag{75}$$

$$\hat{\mathcal{S}}_{M,\mathrm{pert}} := \int_M \mathcal{V}(\mathcal{X}, \mathcal{E}) \tag{76}$$

$$\mathcal{S}^{\mathrm{res}} := (-1)^d \left(\int_{\partial_1 M}\langle \mathbb{E}, \mathsf{x}\rangle + \int_{\partial_2 M}\langle \mathbb{X}, \mathsf{e}\rangle\right) \tag{77}$$

$$\mathcal{S}^{\mathrm{source}} := (-1)^d \left(\int_{\partial_1 M}\langle \mathbb{E}, \mathcal{X}\rangle + \int_{\partial_2 M}\langle \mathbb{X}, \mathcal{E}\rangle\right) \tag{78}$$

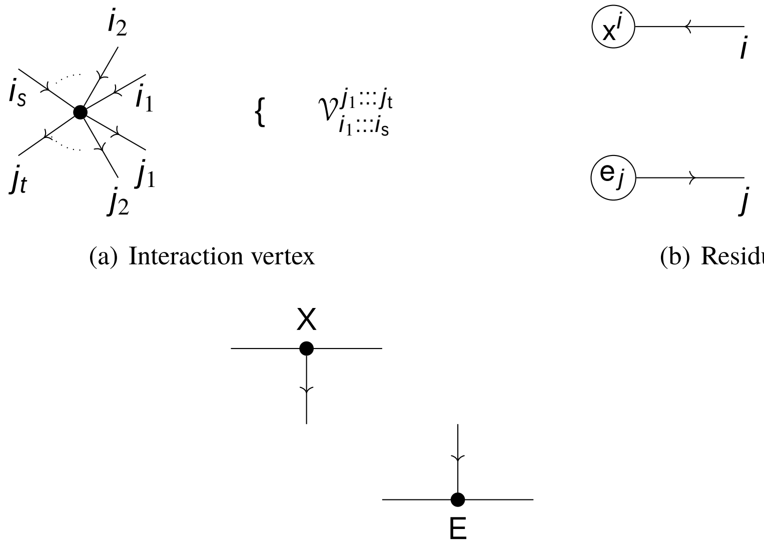
In that way we can rewrite

$$\psi_M = T_M \left\langle e^{\frac{i}{\hbar}(\mathcal{S}^{\mathrm{res}} + \mathcal{S}^{\mathrm{source}})}\right\rangle \tag{79}$$

where $\langle\ \rangle$ denotes the expectation value with respect to the bulk theory ($\hat{\mathcal{S}}_{M,0} + \hat{\mathcal{S}}_{M,\mathrm{pert}}$), i.e. formally

$$\left\langle e^{\frac{i}{\hbar}(\mathcal{S}^{\mathrm{res}} + \mathcal{S}^{\mathrm{source}})}\right\rangle = \int_{\mathcal{L}\subset\mathcal{Y}'} e^{\frac{i}{\hbar}(\hat{\mathcal{S}}_{M,0}[(\mathsf{X}, \mathsf{E})] + \hat{\mathcal{S}}_{M,\mathrm{pert}}[(\mathsf{X}, \mathsf{E})])} e^{\frac{i}{\hbar}(\mathcal{S}^{\mathrm{res}}[(\mathsf{X}, \mathsf{E})] + \mathcal{S}^{\mathrm{source}}[(\mathsf{X}, \mathsf{E})])}. \tag{80}$$

Remark 0.1.4.20. Note that we sum over connected graphs, such that the sum is given by the *effective action*.



(a) Interaction vertex (b) Residual fields

(c) Boundary vertices

FIGURE 0.1.1. Summary of Feynman graphs and rules

(a) Boundary vertex on ∂_1 (b) Boundary vertex on ∂_2

FIGURE 0.1.2. Composite field vertices.

Using composite fields, one can construct the *bullet product* on the full space of states as in [38]. For instance, the bullet product of $\int_{\partial_1 M} u_i \wedge \mathbb{X}^i$ and $\int_{\partial_1 M} v_i \wedge \mathbb{X}^i$ is

$$\left(\int_{\partial_1 M} u_i \wedge \mathbb{X}^i\right) \bullet \left(\int_{\partial_1 M} v_j \wedge \mathbb{X}^j\right) := \tag{81}$$

$$(-1)^{(|\mathbb{X}^i|-d-1+|v_j|)+|u_i|(-d-1)} \int_{C_2(\partial_1 M)} \pi_1^* u_i \wedge \pi_2^* v_j \wedge \pi_1^* \mathbb{X}^i \wedge \pi_2^* \mathbb{X}^j + \int_{\partial_1 M} u_i \wedge v_j \wedge [\mathbb{X}^i \mathbb{X}^j];$$

where u and v are smooth differential forms depending on the bulk and residual fields.

REMARK 0.1.4.21. Consider an operator $\int_{\partial_1 M} F^{ij} \frac{\delta^2}{\delta \mathbb{X}^i \delta \mathbb{X}^j}$. Such an operator is (by definition) interpreted as $\int_{\partial_1 M} F^{ij} \frac{\delta}{\delta [\mathbb{X}^i \mathbb{X}^j]}$, so one gets

$$\left(\int_{\partial_1 M} F^{ij} \frac{\delta^2}{\delta \mathbb{X}^i \delta \mathbb{X}^j}\right) \left(\int_{\partial_1 M} u_i \wedge \mathbb{X}^i\right) \bullet \left(\int_{\partial_1 M} v_j \wedge \mathbb{X}^j\right) = \int_{\partial_1 M} u_i v_j F^{ij}; \tag{82}$$

in accordance with our naive expectation.

DEFINITION 0.1.4.22 (FULL QUANTUM STATE). Let M be a manifold, possibly with boundary. Given a BF-like BV-BFV theory $\pi_M \colon \mathcal{F}_M \to \mathcal{F}^{\partial}_{\partial M}$, a polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M}$, a good splitting $\mathcal{F}_M$

$\mathcal{B}^{\mathcal{P}}_{\partial M} \times \mathcal{V}^{\mathcal{P}}_M \times \mathcal{Y}'$, and a gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{Y}'$, we define the *full quantum state* (similarly as in (70)) by the formal power series

$$\psi_M(\mathbb{X},\mathbb{E};\mathsf{x},\mathsf{e}) := T_M \exp\left(\frac{i}{\hbar}\sum_{\Gamma}\frac{(-i\hbar)^{\text{loops}(\Gamma)}}{|\text{Aut}(\Gamma)|}\int_{C_\Gamma(M)}\omega_\Gamma(\mathbb{X},\mathbb{E};\mathsf{x},\mathsf{e})\right), \tag{83}$$

where we also sum over graphs as in Figure 0.1.2 representing composite fields.

Remark 0.1.4.23. The full state can be interpreted as an expectation value with help of the bullet product:

$$\psi_M = T_M\left\langle e_\bullet^{\frac{i}{\hbar}(\mathcal{S}^{\text{res}}+\mathcal{S}^{\text{source}})}\right\rangle \tag{84}$$

where $e_\bullet$ denotes the exponential with respect to the bullet product.

0.1.4.4. The BFV boundary operator. We want to define the quantum BFV boundary operator for BF-like theories according to [38]. Similarly to the state, we will express at first its principal part and then extend it to a regularization using the notion of composite fields. The quantum BFV boundary operator is constructed as a quantization of the BFV action such that Theorem 0.1.4.26 below holds.

Definition 0.1.4.24 (Principal part of the BFV boundary operator). The *principal part* of the BFV boundary operator is given by

$$\Omega^{\text{princ}} = \underbrace{\Omega_0^{\mathbb{X}} + \Omega_0^{\mathbb{E}}}_{=:\Omega_0} + \underbrace{\Omega^{\mathbb{X}}_{\text{pert}} + \Omega^{\mathbb{E}}_{\text{pert}}}_{=:\Omega^{\text{princ}}_{\text{pert}}}, \tag{85}$$

where

$$\Omega_0^{\mathbb{X}} := (-1)^d i\hbar \int_{\partial_1 M} d\mathbb{X}\frac{\delta}{\delta\mathbb{X}}, \tag{86}$$

$$\Omega_0^{\mathbb{E}} := (-1)^d i\hbar \int_{\partial_2 M} d\mathbb{E}\frac{\delta}{\delta\mathbb{E}}, \tag{87}$$

$$\Omega^{\mathbb{X}}_{\text{pert}} := \sum_{n,k\geq 0}\sum_{\Gamma'_1}\frac{(-i\hbar)^{\text{loops}(\Gamma'_1)}}{|\text{Aut}(\Gamma'_1)|}\int_{\partial_1 M}\left(\sigma_{\Gamma'_1}\right)^{j_1\dots j_k}_{i_1\dots i_n}\wedge \mathbb{X}^{i_1}\wedge\dots\wedge\mathbb{X}^{i_n}\left((-1)^d i\hbar\frac{\delta}{\delta\mathbb{X}^{j_1}}\right)\cdots\left((-1)^d i\hbar\frac{\delta}{\delta\mathbb{X}^{j_k}}\right), \tag{88}$$

$$\Omega^{\mathbb{E}}_{\text{pert}} := \sum_{n,k\geq 0}\sum_{\Gamma'_2}\frac{(-i\hbar)^{\text{loops}(\Gamma'_2)}}{|\text{Aut}(\Gamma'_2)|}\int_{\partial_2 M}\left(\sigma_{\Gamma'_2}\right)^{j_1\dots j_k}_{i_1\dots i_n}\wedge \mathbb{E}^{i_1}\wedge\dots\wedge\mathbb{E}^{i_n}\left((-1)^d i\hbar\frac{\delta}{\delta\mathbb{E}^{j_1}}\right)\cdots\left((-1)^d i\hbar\frac{\delta}{\delta\mathbb{E}^{j_k}}\right), \tag{89}$$

where, for $\mathbb{F}_1 = \mathbb{X}$ and $\mathbb{F}_2 = \mathbb{E}$ and $\ell \in \{1,2\}$, Γ'_ℓ runs over graphs with

- n vertices on $\partial_\ell M$ of valence 1 with adjacent half-edges oriented inwards and decorated with boundary fields $\mathbb{F}_\ell^{i_1},\dots,\mathbb{F}_\ell^{i_n}$ all evaluated at the point of collapse $p \in \partial_\ell M$,

k outward leaves if $\ell = 1$ and k inward leaves if $\ell = 2$, decorated with variational derivatives in boundary fields

$$\left((-1)^d i\hbar\frac{\delta}{\delta \mathbb{F}_\ell^{j_1}}\right), \ldots, \left((-1)^d i\hbar\frac{\delta}{\delta \mathbb{F}_\ell^{j_k}}\right)$$

at the point of collapse,

no outward leaves if $\ell = 2$ and no inward leaves if $\ell = 1$ (graphs with them do not contribute).

The form Ω_Γ is obtained as the integral over the compactified configuration space $\tilde{C}_\Gamma(\mathbb{H}^d)$, where $\mathbb{H}^d$ denotes the d-dimensional upper half-plane, given by

$$\Omega_\Gamma = \int_{\tilde{C}_\Gamma(\mathbb{H}^d)} \omega_\Gamma, \tag{90}$$

with ω_Γ being the product of limiting propagators at the point p of collapse and vertex tensors.

We want to roughly describe the construction of the BFV boundary operator with composite fields (see [38] for a more detailed discussion). First, we need to define the following notion.
On a regular functional as in (65), we get a term L replaced by $\mathrm{d}L$ plus all the terms corresponding to the boundary of the configuration space. As L is smooth, its restriction to the boundary is also smooth and can be integrated on the fibers yielding a smooth form on the base configuration space; for example

$$\Omega_0\left(\int_{\partial_1 M} L_{IJ}\wedge[\mathbb{X}^I]\wedge[\mathbb{X}^J]\right) = -i\hbar\int_{\partial_1 M} \mathrm{d}L_{IJ}\wedge[\mathbb{X}^I]\wedge[\mathbb{X}^J], \tag{91}$$

$$\begin{aligned}\Omega_0\left(\int_{C_2(\partial_1 M)} L_{IJK}\wedge\pi_1^*([\mathbb{X}^I]\wedge[\mathbb{X}^J])\wedge\pi_2^*[\mathbb{X}^K]\right)\\ = -i\hbar\int_{C_2(\partial_1 M)} \mathrm{d}L_{IJK}\wedge\pi_1^*([\mathbb{X}^I]\wedge[\mathbb{X}^J])\wedge\pi_2^*[\mathbb{X}^K] - i\hbar\int_{\partial_1 M}\underline{L_{IJK}}\wedge[\mathbb{X}^I]\wedge[\mathbb{X}^J]\wedge[\mathbb{X}^K],\end{aligned} \tag{92}$$

with $\underline{L_{IJK}} = \pi^\partial_* L_{IJK}$, where $\pi^\partial\colon \partial C_2(\partial_1 M) \to \partial_1 M$ is the canonical projection.
Notice that for any two regular functionals S_1 and S_2 we have

$$\Omega_0(S_1 S_2) = \Omega_0(S_1)\, S_2 \pm S_1\, \Omega_0(S_2).$$

The other generators that we allow are products of expressions of the form

$$\int_{\partial_1 M} L^J_{I^1\ldots I^r}\,\mathbb{X}^{I_1}\wedge\cdots\wedge\mathbb{X}^{I_r}\frac{\delta^{|J|}}{\delta[\mathbb{X}^J]} \tag{93}$$

$$\int_{\partial_2 M} L_I^{J^1\ldots J^\ell}\,\mathbb{E}_{J_1}\wedge\cdots\wedge\mathbb{E}_{J_\ell}\frac{\delta^{|I|}}{\delta[\mathbb{E}_I]}. \tag{94}$$

Definition 0.1.4.25 (Full BFV boundary operator). The *full BFV boundary operator* is given by

$$\Omega_{\partial M} := \Omega_0 + \underbrace{\Omega^{\mathbb{X}}_{\text{pert}} + \Omega^{\mathbb{E}}_{\text{pert}}}_{\Omega_{\text{pert}}}, \tag{95}$$

where

(96)
$$\mathbb{X}^{\mathrm{pert}} := \sum_{n,k\geq 0}\sum_{\Gamma'_1}\frac{(-i\hbar)^{\mathrm{loops}(\Gamma'_1)}}{|\mathrm{Aut}(\Gamma'_1)|}\int_{\partial_1 M}\left(\sigma_{\Gamma'_1}\right)^{J_1\ldots J_k}_{I_1\ldots I_n}\wedge\mathbb{X}^{I_1}\wedge\cdots\wedge\mathbb{X}^{I_n}\left((-1)^{kd}(-i\hbar)^k\frac{\delta^{|J_1|+\cdots+|J_k|}}{\delta\mathbb{X}^{J_1}\cdots\delta\mathbb{X}^{J_k}}\right);$$

(97)
$$\mathbb{E}^{\mathrm{pert}} := \sum_{n,k\geq 0}\sum_{\Gamma'_2}\frac{(-i\hbar)^{\mathrm{loops}(\Gamma'_2)}}{|\mathrm{Aut}(\Gamma'_2)|}\int_{\partial_2 M}\left(\sigma_{\Gamma'_2}\right)^{I_1\ldots I_n}_{J_1\ldots J_k}\wedge\mathbb{E}_{I_1}\wedge\cdots\wedge\mathbb{E}_{I_n}\left((-1)^{kd}(-i\hbar)^k\frac{\delta^{|J_1|+\cdots+|J_k|}}{\delta\mathbb{E}_{J_1}\cdots\delta\mathbb{E}_{J_k}}\right)$$

where, for $\mathbb{F}_1=\mathbb{X}$ and $\mathbb{F}_2=\mathbb{E}$ and $\ell\in\{1,2\}$, Γ'_ℓ runs over graphs with

- n vertices on $\partial_\ell M$, where vertex s has valence $|I_s|\geq 1$, with adjacent half-edges oriented inwards and decorated with boundary fields $[\mathbb{F}_\ell^{I_1}],\ldots,[\mathbb{F}_\ell^{I_n}]$ all evaluated at the point of collapse $p\in\partial_\ell M$,
- $|J_1|+\cdots+|J_k|$ outward leaves if $\ell=1$ and $|J_1|+\cdots+|J_k|$ inward leaves if $\ell=2$, decorated with variational derivatives in boundary fields
$$(-1)^{d}(-i\hbar)\frac{\delta}{\delta[\mathbb{F}_\ell^{J_1}]},\ldots,(-1)^{d}(-i\hbar)\frac{\delta}{\delta[\mathbb{F}_\ell^{J_k}]}$$
at the point of collapse,
- no outward leaves if $\ell=2$ and no inward leaves if $\ell=1$ (graphs with them do not contribute).

The form $\sigma_{\Gamma'}$ is obtained as the integral over the compactified configuration space $\tilde{C}_{\Gamma'}(\mathbb{H}^d)$, where $\mathbb{H}^d$ denotes the d-dimensional upper half-plane, given by

(98)
$$\sigma_{\Gamma'} = \int_{\tilde{C}_{\Gamma'}(\mathbb{H}^d)}\omega_{\Gamma'};$$

with $\omega_{\Gamma'}$ being the product of limiting propagators at the point p of collapse and vertex tensors.

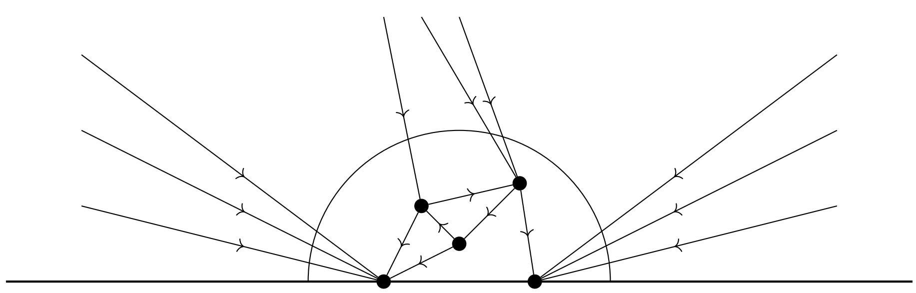


FIGURE 0.1.3. Example of a graph collapsing to the boundary with three bulk and two boundary vertices. The semicircle represents the collapsing of the graph.

THEOREM 0.1.4.26 ([38]). *Let M be a smooth manifold (possibly with boundary). Then the following hold:*

(1) The full covariant state ψ_M satisfies the modified Quantum Master Equation*:*

(99)
$$\left(\hbar^2\Delta_M + \Omega_{\partial M}\right)\psi_M = 0.$$

(2) *The full BFV boundary operator $\Omega_{\partial M}$ squares to zero:*

$$\left(\Omega_{\partial M}\right)^2 = 0. \tag{100}$$

(3) *A change of propagator or residual fields leads to a theory related by change of data as in 0.1.4.5.*

Part 1

Globalization of Nonlinear Split AKSZ Sigma Models on Manifolds with Boundary

Introduction

The goal of this part is to construct perturbative partition functions of certain AKSZ theories - on manifolds with and without boundary - that vary in a "covariant fashion" as one changes the point of expansion. This is achieved combining the BV-BFV formalism, as described in Chapter 0.1, with methods of formal geometry [64, 17, 34, 16]. The globalization method in the case of a field theory on manifolds with boundary has been considered so far only in [43], of which the current part is a far reaching generalization. In [43] this task was performed for a particular example of an AKSZ theory, the Poisson Sigma model [73, 93, 92], which we will return to in Part 2 of this thesis, with constant Poisson structure. We briefly introduce the main players.

Recall that the *BV-BFV formalism* is a method for the perturbative quantization of gauge theories on manifolds with boundary compatible with cutting and gluing; see also [95, 71, 50] and references therein. Moreover, recall that a classical BV-BFV theory associates to every manifold Σ of a fixed dimension - possibly with boundary - the data of a *BV-BFV manifold* (see Chapter 0.1), the space of fields $\mathcal{F}_\Sigma$ (plus extra data). Classical BV-BFV theories can be quantized by the construction described in Chapter 0.1. This procedure associates to Σ a bicomplex $\mathcal{H}_\Sigma$ with two commuting coboundary operators $\Delta_{\mathcal{V}_\Sigma}$ (the BV Laplacian) and $\Omega_{\partial\Sigma}$ (the BFV boundary operator). The *modified Quantum Master Equation* (mQME) is the statement that the partition function Z_Σ is closed with respect to the coboundary operator $\hbar^2\Delta_{\mathcal{V}_\Sigma} + \Omega_{\partial\Sigma}$, i.e.

$$(101)\qquad \left(\hbar^2\Delta_{\mathcal{V}_\Sigma} + \Omega_{\partial\Sigma}\right) Z_\Sigma = 0.$$

However, this construction works only if the space of fields is linear, i.e. a vector space. If the space of fields is nonlinear one has to linearize it, which amounts to working with a formal neighbourhood of a classical solution in the space of fields. In this part we show how this can be done consistently for a large set of solutions at once for AKSZ theories.

AKSZ theories were introduced by Alexandrov, Kontsevich, Schwarz and Zaboronsky in [1]. They form a large class of topological BV theories that naturally admit BV-BFV extensions, as was shown in [37] and is recalled in Chapter 1.2. In AKSZ theories the space of fields $\mathcal{F}_\Sigma$ is a space of (graded) maps with target a fixed graded manifold $\mathcal{M}$. If the target is a vector space, then also the space of fields has a vector space structure, but in many examples one is interested in the case where the target is nonlinear (a prominent one being the Poisson Sigma Model, which we will consider in Part 2; see [29]). In this case, the quantization is constructed by linearizing around constant maps.

In this part, we use methods of formal geometry, described in Chapter 1.1 (see also [30, 34, 16], and [80] for the case where the moduli space of solutions is graded) to define a "covariant partition function" $\tilde{Z}_\Sigma$. It is an inhomogenoeous differential form with values in the vector bundle $\hat{\mathcal{H}}^{\Sigma,tot}$ over (the body of) the target with fiber over x the space of states of the BV-BFV quantization around x. In Chapter 1.3 we show that it satisfies the following generalization of the mQME that we call "mdQME" (for *modified differential Quantum Master Equation*):

$$\left(d_x - i\hbar\Delta_{\mathcal{V}} + \frac{i}{\hbar}\Omega_{\partial}\right)\widetilde{\psi} = 0. \tag{102}$$

We also show that the *quantum Grothendieck BFV (GBFV) operator* $\nabla_{\mathsf{G}} := d_x - i\hbar\Delta_{\mathcal{V}} + \frac{i}{\hbar}\Omega_{\partial}$ squares to zero. The operator ∇_{G} can be thought of as a "connection" on the total space $\widehat{\mathcal{H}}_{\Sigma,tot}$ and hence we can think of it as a flat connection on $\widehat{\mathcal{H}}_{\Sigma,tot}$ (see Section 1.3.4). If one interprets ∇_{G} as a quantum version of the Grothendieck connection (105), Equation (102) says that $\widetilde{\psi}$ corresponds to the Taylor expansion of a globally defined object on M.

One of the goals of this construction is to go further towards the deformation quantization of the relational symplectic groupoid [48, 24, 25]. Part 2 of the thesis will be an extension of the results obtained in this chapter to the Poisson Sigma Model with alternating boundary conditions. However, we also hope to deepen the understanding of how perturbative partition functions depend on the point of expansion. In AKSZ Sigma Models, there is a nice smooth part of the moduli space of classical solutions given by constant maps. But e.g. in Chern–Simons theory the body of the target is a point, and one is interested in expanding around points representing equivalence classes of flat connections. This will be the subject of further investigation.

CHAPTER 1.1

Formal geometry

We are interested in how perturbative expansions change if one changes the point of expansion. The language of formal geometry [64, 17] provides adequate tools to study how the coefficients of Taylor expansions change if one changes coordinates. In this chapter we recollect some notions of formal geometry. We follow the expositions of [30] and [16], and refer to these papers for proofs of the statements. Another good reference is [53].

1.1.1. Formal power series on vector spaces

We begin with a very short review of formal power series on vector spaces. If V is a finite-dimensional vector space, the polynomial algebra on V is the symmetric algebra of the dual vector space

$$S V^* = \bigoplus_{k=0}^{\infty} S^k V^* .$$

If $e_1, \ldots, e_n$ is a basis of V, with dual basis $y^1, \ldots, y^n$, then elements $f \in S V^*$ can be represented by

$$f(y) = \sum_{i_1,\ldots,i_n=1}^{\infty} f_{i_1,\ldots,i_k} y_1^{i_1} \cdots y_n^{i_n} = \sum_I f_I y^I ,$$

with only finitely many non-vanishing f_I. Here $I = \{i_1, \ldots, i_k\}$ is a multi-index and we understand $y^I = y^{i_1} \cdots y^{i_k}, y^{\emptyset} := 1$.

We can complete this algebra to the algebra of formal power series $\widehat{S}V^*$, where infinitely many coefficents f_I can be nonzero. Both $S V^*$ and $\widehat{S}V^*$ are commutative algebras with the multiplication of polynomials or formal power series respectively, generated by V^*. Derivations of these algebras are specified by their value on these generators, hence the map

$$\begin{aligned} V \otimes S V^* &\to \mathrm{Der}(S V^*) \\ v \otimes f &\mapsto (V^* \ni \alpha \mapsto \alpha(v) f) \end{aligned} \tag{103}$$

is an isomorphism with inverse

$$\begin{aligned} \mathrm{Der}(S V^*) &\to V \otimes S V^* \\ D &\mapsto \sum_{i=1}^{n} e_i \otimes D(y^i) \end{aligned}$$

In coordinates, it simply amounts to sending $e^i \mapsto \frac{\partial}{\partial y^i}$.

1.1.2. Formal exponential maps

Let M be a smooth manifold. Let $\varphi \colon U \to M$ where $U \subset TM$ is a open neighbourhood of the zero section. For $x \in M, y \in T_xM \cap U$ we write $\varphi(x,y) = \varphi_x(y)$. We say that φ is a *generalized exponential map* if for all $x \in M$ we have that $\varphi_x(0) = x, \mathrm{d}\varphi_x(0) = \mathrm{id}_{T_xM}$. In local coordinates we can write

$$\varphi^i_x(y) = x^i + y^i + \frac{1}{2}\varphi^i_{x;jk}y^jy^k + \frac{1}{3!}\varphi^i_{x;jk\ell}y^jy^ky^\ell + \cdots$$

where the x^i are coordinates on the base and the y^i are coordinates on the fibers. We identify two generalized exponential maps if their jets agree to all orders. A *formal exponential map* is an equivalence class of generalized exponential maps. It is completely specified by the sequence of functions $\left(\varphi^i_{x;i_1\ldots i_k}\right)_{k=0}^{\infty}$. By abuse of notation, we will denote equivalence classes and their representatives by φ. From a formal exponential map φ and a function $f \in C^\infty(M)$, we can produce a section $\sigma \in \Gamma(\hat{S}T^*M)$ by defining $\sigma_x = \mathsf{T}\varphi_x^* f$, where T denotes the Taylor expansion in the fiber coordinates around $y = 0$ and we use any representative of φ to define the pullback. We denote this section by $\mathsf{T}\varphi^* f$, it is independent of the choice of representative, since it only depends on the jets of the representative.

1.1.3. Grothendieck connection

One can define a flat connection D_G on $\hat{S}T^*M$ with the property that $D_G\sigma = 0$ if and only if $\sigma = \mathsf{T}\varphi^* f$ for some $f \in C^\infty(M)$. Namely, $D_G = \mathrm{d} + L_R$ where $R \in \Gamma(T^*M \otimes TM \otimes \hat{S}T^*M)$ is a 1-form with values in derivations of $\hat{S}T^*M$, which we identify with $\Gamma(TM \otimes \hat{S}T^*M)$ using the isomorphism (103)[1]. Here we denote by L the Lie derivative (see Section A.3). The 1-form R can be defined in local coordinates by $R = R_i\mathrm{d}x^i$ and

(104) $$R_i(x;y) = -\left(\left(\frac{\partial\varphi_x}{\partial y}\right)^{-1}\right)^k_j \frac{\partial\varphi^j_x}{\partial x^i}\frac{\partial}{\partial y^k} =: Y^k_i(x;y)\frac{\partial}{\partial y^k}$$

so that

$$R(x;y) = R_i(x;y)\mathrm{d}x^i = Y^k_i(x;y)\frac{\partial}{\partial y^k}\mathrm{d}x^i;$$

where we use the Einstein summation convention. For $\sigma \in \Gamma(\hat{S}T^*M)$, $L_R\sigma = R(\sigma)$ is given by the Taylor expansion (in the y coordinates) of

$$-\mathrm{d}_y\sigma \circ (\mathrm{d}_y\varphi)^{-1} \circ \mathrm{d}_x\varphi \colon \Gamma(TM) \to \Gamma(\hat{S}T^*M);$$

this shows that R does not depend on the choice of coordinates. For a vector field $\xi = \xi^i\frac{\partial}{\partial x^i}$ we get

(105) $$(D_G)_\xi = \xi + \hat{\xi},$$

where

(106) $$\hat{\xi}(x;y) = \iota_\xi R(x;y) = \xi^i(x)Y^k_i(x;y)\frac{\partial}{\partial y^k}.$$

[1] This is slightly confusing, since the basis of T_xM is usually denoted $\frac{\partial}{\partial x^i}$, which under this isomorphism gets sent to $\frac{\partial}{\partial y^i}$.

The connection D_G is called the *classical Grothendieck connection*. Its flatness can be expressed by

$$d_x R + \frac{1}{2}[R, R] = 0.$$

By the Poincaré Lemma it can be shown that its cohomology is concentrated in degree 0 and is given by

$$H^0_{D_G}(\hat{S}T^*M) = T\varphi^* C^\infty(M) \cong C^\infty(M).$$

1.1.4. Formal vertical tensor fields

Now, let $E \to M$ be any tensorial[2] bundle, e.g. $E = \bigwedge^k TM$. Its sections are called *tensor fields of type E*. Then its associated *formal vertical bundle* is $\hat{E} := E \otimes \hat{S}T^*M$, and sections of this bundle are called *formal vertical tensors of type E*. One can think of these bundles as tensors of the same type on TM where the dependence on fiber directions is formal. The formal exponential map defines an injective map $T\varphi^* : E \to \hat{E}$ by taking the Taylor expansion of a tensor field pulled back[3] to U by φ. We can let R act on formal vertical tensors by Lie derivative. Thus we get a Grothendieck connection $D_G = d + L_R$ on any formal vertical tensor bundle. Again, it is flat, and the flat sections are precisely the ones in the image of $T\varphi^*$. Moreover the cohomology is concentrated in degree 0 and given by the flat sections, i.e. $\hat{E}$-valued 0-forms.

1.1.5. Changing the formal exponential map

Let φ be a family of formal exponential maps depending on a parameter t belonging to an open interval I. Then we can associate to this family a formal exponential map ψ for the manifold $M \times I$ by $\psi(x, t; y, \tau) := (\varphi_{x,t}(y), t + \tau)$, where τ denotes the tangent variable to t. We want to define the associated connection $\tilde{R}$: on a section σ of $\hat{S}T^*(M \times I)$ we have, by definition

$$\tilde{R}(\sigma) = -(d_y\sigma, d_\tau\sigma)\begin{pmatrix}(d_y\varphi)^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}d_x\varphi & \dot{\varphi}\\ 0 & 1\end{pmatrix}. \tag{107}$$

So we can write $\tilde{R} = R + C\,dt + T$ with R defined as in Section 1.1.3 (but now t-dependent),

$$C(\sigma) = -d_y\sigma\,(d_y\varphi)^{-1}\dot{\varphi},$$

and $T = -dt\frac{\partial}{\partial\tau}$. We can formulate the Maurer–Cartan equation for $\tilde{R}$ observing that $d_x T = d_t T = 0$ and that T commutes with both R and C. The $(2, 0)$-form component over $M \times I$ yields again the Maurer–Cartan equation for R, whereas the $(1, 1)$-component reads

$$\dot{R} = d_x C + [R, C].$$

Hence, under a change of formal exponential map, R changes by a gauge transformation with generator the section C of $\hat{\mathfrak{X}}(TM) := TM \otimes \hat{S}T^*M$. Finally, if σ is a section in the image of $T\varphi^*$, then by a simple computation one gets

$$\dot{\sigma} = -L_C\,\sigma,$$

which can be interpreted as the associated gauge transformation for sections.

[2] I.e. any bundle which is a tensor product or antisymmetric or symmetric product of the tangent or cotangent bundle, or a direct sum thereof.

[3] Since φ is a local diffeomorphism we can define the pullback of contravariant tensors as the pushforward of the inverse.

1.1.6. Extension to graded manifolds

The results of the previous sections can be generalized to the category of graded manifolds (see Appendix B) using the algebraic reformulation of formal exponential maps developed in [80]. Namely, given a formal exponential map φ on a smooth manifold $\overline{M}$, one can construct a map

$$\mathrm{pbw}\colon \Gamma(\widehat{S}T\overline{M}) \to \mathcal{D}(\overline{M}) \tag{108}$$

from sections of the completed symmetric algebra of the tangent bundle to the algebra of differential operators $\mathcal{D}(\overline{M})$ by defining

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n)(f) = \frac{\mathrm{d}}{\mathrm{d}t_1}\Big|_{t_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}t_n}\Big|_{t_n=0} f(\varphi(t_1X_1 + \dots + t_nX_n)). \tag{109}$$

This map can be defined also in the category of graded manifolds by choosing a torsion-free connection ∇ on the tangent bundle of a graded manifold M. In particular, there still exists an element $R^\nabla \in \Omega^1(M; TM \otimes \widehat{S}T^\vee M)$ with the property that $D_G = \mathrm{d}_M + L_{R^\nabla}$ is a flat connection on $\widehat{S}T^\vee M$, namely

$$R^\nabla = \delta + \Gamma + A^\nabla; \tag{110}$$

where in local coordinates x^i on M and corresponding coordinates y^i on TM we have (see [53, 80])

$$\delta = \mathrm{d}x^i \frac{\partial}{\partial y^i}; \tag{111}$$

$$\Gamma = -\mathrm{d}x^i\, \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k}; \tag{112}$$

$$A^\nabla = \mathrm{d}x^i \sum_{|J|\geq 2} A^k_{i;J}(x) y^J \frac{\partial}{\partial y^k}. \tag{113}$$

Here Γ^k_{ij} are the Christoffel symbols of ∇. We define $R_i \in \Gamma(M; \widehat{S}T^\vee M \otimes TM)$ and $Y_i^k \in \Gamma(M; \widehat{S}T^\vee M)$ by

$$R^\nabla = R_i(x;y)\mathrm{d}x^i = Y_i^k(x;y)\mathrm{d}x^i \frac{\partial}{\partial y^k}. \tag{114}$$

D_G extends to a differential on $\Omega(M; \widehat{S}T^\vee M)$. The "Taylor expansion" of a function $f \in C^\infty(M)$ can be defined as ([80])

$$T\varphi^* f := \sum_I \frac{1}{I!} y^I\, \mathrm{pbw}\left(\partial_x^I\right)(f) \tag{115}$$

where

$$\partial_x^I = \underbrace{\partial_{x_1} \cdots \partial_{x_1}}_{i_1} \cdots \underbrace{\partial_{x_n} \cdots \partial_{x_n}}_{i_n}.$$

One can prove that (115) still has the same properties, namely, the image of $T\varphi^*$ consists precisely of the D_G-closed sections of $\widehat{S}T^\vee M$.

One can describe how the formal exponential map varies under the choice of connection mimicking the construction for the smooth case described in 1.1.5. Namely, assume we have a smooth family

∇^t of connections on TM, then we can associate to that family a connection $\tilde\nabla$ on $M\times I$. The corresponding $R^{\tilde\nabla}$ still can be split as

$$R^{\tilde\nabla} = R^{\nabla^t} + C^{\nabla^t}dt + T$$

as in section 1.1.5, where $C\in\Omega^1(M,\widehat{S}T^*M)$. The fact that $(D_G)^2=0$ translates into the same equations as before, namely, we have

(116) $$\dot{R}^{\nabla^t} = d_M R^{\nabla^t} + \left[C^{\nabla^t}, R^{\nabla^t}\right]$$

and for any section σ in the image of $T\varphi$ we have

(117) $$\dot\sigma = -L_{C^{\nabla^t}}\sigma.$$

1.1.7. Lifting formal exponential maps to cotangent bundles

We want to consider the case were the base manifold is given by a cotangent bundle.

PROPOSITION 1.1.7.1. *If the base manifold is given by a cotangent bundle T^*M, the vector field $\bar R$, defined by the lift of the formal exponential map, is linear in the fiber coordinate of $T_{(q,p)}T^*M$ for any $(q,p)\in T^*M$.*

PROOF. Let M be a smooth manifold and consider a formal exponential map $\varphi\colon TM\to M$. Moreover, let $\tilde\varphi\colon TT^*M\to T^*M$ be the lift of the formal exponential map to the cotangent bundle of M. Explicitly, for $(q,p)\in T^*M$, we have

$$\tilde\varphi_{(q,p)}\colon T_{(q,p)}T^*M\cong T_qT_q^*M\oplus T_pT_q^*M\cong T_qM\oplus T_q^*M\to T^*M.$$

Let $(\bar q,\bar p)\in T_{(q,p)}T^*M$, and hence $\bar q\in T_qM$ and $\bar p\in T_q^*M$. Note that $\varphi_q\colon T_qM\to M$, and thus

$$\left(d_{\bar q}\varphi_q\right)^{*,-1}\colon T_q^*M\to T^*_{\varphi_q(\bar q)}M,$$

since $d_{\bar q}\varphi_q\colon T_{\bar q}T_qM\cong T_qM\to T_{\varphi_q(\bar q)}M$. Then we can write the lift of the exponential map as

$$\tilde\varphi_{(q,p)}(\bar q,\bar p) = \left(\varphi_q(\bar q),\ \left(d_{\bar q}\varphi_q\right)^{*,-1}\bar p\right)\in T^*M.$$

For $x=(q,p)\in T^*M$ and $y=(\bar q,\bar p)\in T_xT^*M = T_{(q,p)}T^*M$, we want to compute

$$d_y(\tilde\varphi_x)^{-1},\qquad d_x(\tilde\varphi_x).$$

We write $\tilde q := \varphi_q(\bar q)$ and $\tilde p := \left(d_{\bar q}\varphi_q\right)^{*,-1}\bar p$. Hence we get

(118) $$d_y\tilde\varphi_x = \begin{pmatrix}\frac{\partial\tilde q}{\partial\bar q} & \frac{\partial\tilde q}{\partial\bar p}\\ \frac{\partial\tilde p}{\partial\bar q} & \frac{\partial\tilde p}{\partial\bar p}\end{pmatrix} = \begin{pmatrix}\frac{\partial\tilde q}{\partial\bar q} & 0\\ \frac{\partial\tilde p}{\partial\bar q} & \frac{\partial\tilde p}{\partial\bar p}\end{pmatrix},\qquad d_x\tilde\varphi_x = \begin{pmatrix}\frac{\partial\tilde q}{\partial q} & \frac{\partial\tilde q}{\partial p}\\ \frac{\partial\tilde p}{\partial q} & \frac{\partial\tilde p}{\partial p}\end{pmatrix} = \begin{pmatrix}\frac{\partial\tilde q}{\partial q} & 0\\ \frac{\partial\tilde p}{\partial q} & 0\end{pmatrix}.$$

Moreover, we have

(119) $$\left(d_y\tilde\varphi_x\right)^{-1} = \begin{pmatrix}\left(\frac{\partial\tilde q}{\partial\bar q}\right)^{-1} & 0\\ -\left(\frac{\partial\tilde p}{\partial\bar p}\right)^{-1}\frac{\partial\tilde p}{\partial\bar q}\left(\frac{\partial\tilde q}{\partial\bar q}\right)^{-1} & \left(\frac{\partial\tilde p}{\partial\bar p}\right)^{-1}\end{pmatrix}.$$

Thus, we get

(120) $$\left(d_y\tilde\varphi_x\right)^{-1}d_x\tilde\varphi_x = \begin{pmatrix}\left(\frac{\partial\tilde q}{\partial\bar q}\right)^{-1}\frac{\partial\tilde q}{\partial q} & 0\\ -\left(\frac{\partial\tilde p}{\partial\bar p}\right)^{-1}\frac{\partial\tilde p}{\partial\bar q}\left(\frac{\partial\tilde q}{\partial\bar q}\right)^{-1}\frac{\partial\tilde q}{\partial q} + \left(\frac{\partial\tilde p}{\partial\bar p}\right)^{-1}\frac{\partial\tilde p}{\partial q} & 0\end{pmatrix}.$$

If we consider the lift $\bar{R} = \quad \mathrm{d}_{\mathsf{y}} \quad \mathrm{d}_{\mathsf{y}} \llcorner_{x} \quad {}^{1} \quad \mathrm{d}_{x} \llcorner_{x}$, we get that $\bar{R}$ is linear in $\bar{p}$ as claimed.

REMARK 1.1.7.2. Proposition 1.1.7.1 will simplify the graphs in the Feynman graph expansion of the formal global Poisson Sigma Model for cotangent targets as we will see later on.

CHAPTER 1.2

Perturbative quantization of AKSZ Sigma Models

In [1] Alexandrov, Kontsevich, Schwarz and Zaboronsky have formulated a construction for Sigma Models using certain mapping spaces as the space of fields to get solutions of the Classical Master Equation. These type of theories are known today as AKSZ theories. Moreover, they have shown that this particular construction gives rise to a BV theory. Many known theories are of the AKSZ type, e.g. Chern–Simons theory, the Poisson Sigma Model, Rozansky–Witten theory [91], BF theory, and the A- and B-twisted Sigma Models constructed by Witten [111]. In this chapter, we will recall some aspects of the AKSZ construction and how it naturally gives a BV theory. Furthermore, we will introduce the notion of "split" AKSZ Sigma Models and its coordinatization construction using formal geometry (see Chapter 1.1). Moreover, we will discuss some aspects within its quantum picture.

1.2.1. Review of AKSZ Sigma Models in the BV-BFV formalism

1.2.1.1. AKSZ Sigma Model. We will begin with a brief review of AKSZ Sigma Models [1] as described in [37].

Definition 1.2.1.1 (Differential graded symplectic manifold). A *differential graded symplectic manifold* of degree d is a graded manifold $\mathcal{M}_d$ endowed with an exact symplectic form $\omega_d = \delta\alpha_d \in \Omega^2(\mathcal{M}_d)$ of degree d, where $\alpha_d \in \Omega^1(\mathcal{M}_d)$ denotes a primitive 1-form and δ is the de Rham differential on $\mathcal{M}_d$ and a Hamiltonian function $\Theta_d \in C^\infty(\mathcal{M}_d)$ of degree $d+1$ satisfying $\{\Theta_d,\Theta_d\} = 0$, where $\{\ ,\ \}$ is the Poisson bracket induced by ω_d.

Remark 1.2.1.2. We can moreover consider the Hamiltonian vector field $Q_d \in \mathfrak{X}(\mathcal{M}_d)$ of Θ_d, defined by the formula $\iota_{Q_d}\omega_d = \delta\Theta_d$, with the properties $[Q_d,Q_d] = 0$ (cohomological) and $L_{Q_d}\omega_d = 0$ (symplectic). A quadruple $(\mathcal{M}_d,\omega_d = \delta\alpha_d,\Theta_d,Q_d)$ as in Definition 1.2.1.1 is also called a *Hamiltonian Q-manifold*. The function Θ_d is also called the *Hamiltonian* of the system.

Definition 1.2.1.3 (AKSZ Sigma Model). The *AKSZ Sigma Model* with target a Hamiltonian Q-manifold $(\mathcal{M}_{d-1},\omega_{d-1} = \delta\alpha_{d-1},\Theta_{d-1},Q_{d-1})$ is the BV theory, which associates to a d-manifold Σ_d the BV manifold $(\mathcal{F}_{\Sigma_d},\omega_{\Sigma_d},\mathcal{S}_{\Sigma_d})$, where the space of fields $\mathcal{F}_{\Sigma_d} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d,\mathcal{M}_{d-1})$ is given by graded manifold maps between $T[1]\Sigma_d$ and $\mathcal{M}_{d-1}$, the symplectic form ω_{Σ_d} is of the form $\omega_{\Sigma_d} = \int_{\Sigma_d}\omega_{\mu\nu}(\mathcal{A})\,\delta\mathcal{A}^\mu \wedge \delta\mathcal{A}^\nu$, and the action functional $\mathcal{S}_{\Sigma_d}$ is of the form $\mathcal{S}_{\Sigma_d} = \int_{\Sigma_d}\alpha_\mu(\mathcal{A})\mathrm{d}_{\Sigma_d}\mathcal{A}^\mu + \int_{\Sigma_d}\Theta_{d-1}(\mathcal{A})$, where $\mathcal{A} \in \mathcal{F}_{\Sigma_d}$, $\omega_{\mu\nu} \in C^\infty(\mathcal{F}_{\Sigma_d})$ are the components of the symplectic form ω_{d-1}, $\alpha_\mu \in C^\infty(\mathcal{F}_{\Sigma_d})$ are the components of the primitive 1-form α_{d-1} and $\mathcal{A}^\mu \in C^\infty(\Sigma_d) \otimes \bigwedge^\bullet T^*\Sigma_d$ are the components of $\mathcal{A}$ in local coordinates.

1.2.1.2. AKSZ-BV construction. We want to recall the construction of obtaining a BV theory out of the AKSZ construction of [1]. Fix a Hamiltonian Q-manifold $(\mathcal{M}_{d-1},\omega_{d-1} = \delta\alpha_{d-1},\Theta_{d-1},Q_{d-1})$ of degree $d-1$. Let $\mathcal{F}_{\Sigma_d} = \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d,\mathcal{M}_{d-1})$, where Σ_d is a d-dimensional source manifold (possibly with boundary). Let (x^μ) be local coordinates on $\mathcal{M}_{d-1}$ and

(u^i) be local coordinates on Σ_d with $\theta^i = du^i$ the corresponding degree $+1$ fiber coordinates on $T[1]\Sigma_d$. For $\mathcal{A} \in \mathcal{F}_{\Sigma_d}$ we get the components

$$\mathcal{A} = \mathcal{A}_{i_1\dots i_k}(u)\,\theta^{i_1}\wedge\dots\wedge\theta^{i_k} \in C^\infty(\Sigma_d)\,\widehat{\otimes}\, T_{\Sigma_d}; \tag{121}$$

where $\mathcal{A}_{i_1\dots i_k} \in C^\infty(\Sigma_d)$ are of degree $|x| - k$. We want to give more details on obtaining the BV symplectic form and the BV action, which are given by

$$\omega_{\Sigma_d} = \int_{\Sigma_d} \omega_{\alpha\beta}(\mathcal{A})\,\delta\mathcal{A}^\alpha \wedge \delta\mathcal{A}^\beta; \tag{122}$$

$$\mathcal{S}_{\Sigma_d} = \int_{\Sigma_d} \alpha(\mathcal{A})\,d_{\Sigma_d}\mathcal{A} + \Theta_{d-1}(\mathcal{A}). \tag{123}$$

Indeed, considering the push-pull diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d,\mathcal{M}_{d-1}) \times T[1]\Sigma_d & \xrightarrow{\ \mathrm{ev}\ } & \mathcal{M}_{d-1} \\ \downarrow p & & \\ \mathrm{Map}_{\mathrm{GrMnf}}(T[1]\Sigma_d,\mathcal{M}_{d-1}) & & \end{array}$$

such that ev denotes the evaluation and p the projection map. We can construct the *transgression map* by

$$\mathcal{T} := p_*\,\mathrm{ev}^*\colon \Omega^\bullet(\mathcal{M}_{d-1}) \to \Omega^\bullet(\mathcal{F}_{\Sigma_d}). \tag{124}$$

Note that p_* is given by integration over the $T[1]\Sigma_d$ fibers. Hence, we get a symplectic form ω_{Σ_d} on $\mathcal{F}_{\Sigma_d}$ by

$$\omega_{\Sigma_d} := \mathcal{T}(\omega_{d-1}) = p_*\,\mathrm{ev}^*\,\omega_{d-1} \in \Omega^2(\mathcal{F}_{\Sigma_d}). \tag{125}$$

One can check that this form is of degree $(d-1) - d = -1$, as we wanted. Let Q_{d-1} be the Hamiltonian vector field of Θ_{d-1}. Then one can define a cohomological vector field Q_{Σ_d} on $\mathcal{F}_{\Sigma_d}$ by lifting Q_{d-1} and the de Rham differential d_{Σ_d} on Σ_d (regarded as a cohomological vector field on $T[1]\Sigma_d$) to the mapping space $\mathcal{F}_{\Sigma_d}$. We denote the lifts by $\hat{Q}_{d-1}$ and $\hat{d}_{\Sigma_d}$ respectively. Hence, one can write down a BV action functional on $\mathcal{F}_{\Sigma_d}$ by

$$\mathcal{S}_{\Sigma_d} := \iota_{\hat{d}_{\Sigma_d}}\mathcal{T}(\alpha_{d-1}) + \mathcal{T}(\Theta_{d-1}) \in C^\infty(\mathcal{F}_{\Sigma_d}). \tag{126}$$

The cohomological vector field on $\mathcal{F}_{\Sigma_d}$ is locally given by

$$Q_{\Sigma_d} := \hat{d}_{\Sigma_d} + \hat{Q}_{d-1} = \int_{\Sigma_d}\left((d_{\Sigma_d}\mathcal{A})_{i_1\dots i_k}(u)\frac{\delta}{\delta\mathcal{A}_{i_1\dots i_k}(u)} + Q_{d-1}(\mathcal{A}(u))\frac{\delta}{\delta\mathcal{A}(u)}\right) \in \mathfrak{X}(\mathcal{F}_{\Sigma_d}). \tag{127}$$

Remark 1.2.1.4. If $\mathcal{M}_{d-1}$ is a graded vector space we can cover it by a single coordinate chart and identify the mapping space with the space of $\mathcal{M}_{d-1}$-valued nonhomogeneous forms on the source, i.e.

$$\mathcal{F}_{\Sigma_d} \cong \Omega^\bullet(\Sigma_d)\otimes\mathcal{M}_{d-1}. \tag{128}$$

This is a consequence of the fact that $C^\infty(T[1]\Sigma_d) \cong \Omega^\bullet(\Sigma_d)$ (see Appendix B).

Remark 1.2.1.5. Here "$\mathrm{Map}_{\mathrm{GrMnf}}$" denotes the right adjoint functor to the Cartesian product of graded manifolds (with a fixed factor). On objects we have $\mathrm{Hom}(X, \mathrm{Map}_{\mathrm{GrMnf}}(Y, Z)) = \mathrm{Hom}(X \times Y, Z)$, where Hom denotes the set of graded manifold morphisms. We will just write "Map" whenever it is clear.

1.2.2. Split AKSZ Sigma Models

Let us fix a source d-manifold Σ. In this part we will especially be interested in the case where $\mathcal{M} = \mathcal{M}_{d-1} = T^*[d-1]M$, with M a graded manifold, such that the symplectic form is given by the canonical symplectic structure.

Definition 1.2.2.1 (Split AKSZ Sigma Model). We call an AKSZ Sigma Model *split*, if the target is of the form

$$\mathcal{M}_{d-1} = T^*[d-1]M \tag{129}$$

with canonical symplectic structure, where M is a graded manifold.

Coordinates on the space of fields can be considered as a pair (X, η), where X and η are the base and fiber components of the map respectively. The action can be written as

$$\mathcal{S}_\Sigma[(\mathsf{X}, \eta)] = \mathcal{S}^{\mathrm{kin}}_\Sigma[(\mathsf{X}, \eta)] + \mathcal{S}^{\mathrm{int}}_\Sigma[(\mathsf{X}, \eta)] \tag{130}$$

where the kinetic and interaction terms are given by

$$\mathcal{S}^{\mathrm{kin}}_\Sigma[(\mathsf{X}, \eta)] := \int_\Sigma \langle \eta, d\mathsf{X} \rangle,$$

$$\mathcal{S}^{\mathrm{int}}_\Sigma[(\mathsf{X}, \eta)] := \int_\Sigma \Theta(\mathsf{X}, \eta),$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between tangent and cotangent bundle of M and we think of elements of $C^\infty(T[1]\Sigma)$ as differential forms in the usual way, i.e. of elements in $\Omega^\bullet(\Sigma)$. In [37] it was shown that these data define a BV-BFV theory as in Definition 0.1.3.8 in Chapter 0.1.

1.2.3. Coordinatization of split AKSZ theories

In this chapter we want to quantize split AKSZ theories as perturbations of abelian BF theory. This can be done by "coordinatizing the target", i.e. replacing the space of fields with the formal neighbourhood of a constant field. Using methods of *formal geometry* as in [64, 17, 30, 16, 43] one can do this consistently for all constant solutions at once. In Chapter 1.1 we recall this procedure and its extension to graded manifolds, which is discussed in [80]. For more details we refer to [30, 16].

1.2.3.1. Coordinatizing the AKSZ construction. The idea now is to expand the theory around critical points of the kinetic part of the action. Denote by $\overline{M}$ the body of the graded manifold M, and let $x \in \overline{M}$. We will work in formal neighbourhoods of constant maps

$$x = (\mathsf{X}, \eta) \equiv (x, 0) \in \mathrm{Map}(T[1]\Sigma, \mathcal{M})$$

Let φ be a formal exponential map (see Chapter 1.1.2) on M. This induces a map

$$\varphi_x \colon T_xM \to M$$

which lifts to a map

$$\tilde\phi_x\colon \mathcal{F}_{\Sigma,x} := \mathrm{Map}(T[1]\Sigma, T^*[d-1]T_xM) \to \mathrm{Map}(T[1]\Sigma,\mathcal{M})$$
$$(\hat{X}, b) \mapsto (X, \eta)$$

by taking post-composition with the cotangent lift. Notice that $\tilde\phi$ is a local symplectomorphism and that

$$S^{\mathrm{kin}}_{\Sigma,x} := T\tilde\phi_x^* S^{\mathrm{kin}} = \int_\Sigma \langle b, \mathrm{d}\hat{X}\rangle, \tag{131}$$

where T denotes the Taylor expansion as in 1.1.2. If we define

$$S^{\mathrm{int}}_{\Sigma,x} := T\tilde\phi_x^* S^{\mathrm{int}} = \int_\Sigma T\tilde\phi_x^* \Theta(X, \eta) \tag{132}$$

and

$$S_{\Sigma,x} := S^{\mathrm{kin}}_{\Sigma,x} + S^{\mathrm{int}}_{\Sigma,x} = \int_\Sigma \langle b, \mathrm{d}\hat{X}\rangle + T\tilde\phi_x^*\Theta(X,\eta) \tag{133}$$

then the pair $(\mathcal{F}_{\Sigma,x}, S_{\Sigma,x})$ is a BF-like theory in the sense of [38], i.e. the kinetic part of the action is a sum of copies of the kinetic part of abelian BF-theory and for every $x \in \overline{M}$ it satisfies the mCME (see Equation (41) in Chapter 0.1)

$$\iota_{Q_{\Sigma,x}}\omega_{\Sigma,x} - \delta S_{\Sigma,x} = \pi^*\alpha_{\partial\Sigma,x}.$$

where $Q_{\Sigma,x}$ is the Hamiltonian vector field of $S_{\Sigma,x}$. Moreover, $\omega_{\Sigma,x}$ and $\alpha_{\partial\Sigma,x}$ are the corresponding symplectic form and boundary 1-form of the BV-BFV manifold associated to the space of fields $\mathcal{F}_{\Sigma,x}$. Notice that it could be obtained from the AKSZ construction with target $T^*[d-1]T_xM$ and Hamiltonian function $\Theta_x := T\tilde\phi_x^*\Theta$. We regard Θ_x as a formal function on $T^*[d-1]T_xM$ and we will write

$$\Theta_x(y^1, \dots, y^r; \eta_1, \dots, \eta_r) = \sum_{k,l=0}^{\infty} \Theta^{j_1\dots j_l}_{i_1\dots i_k}(x)\, y^{i_1}\cdots y^{i_k}\,\eta_{j_1}\cdots\eta_{j_l} \tag{134}$$

where $r = \dim M$ and the η_i are the cotangent coordinates of the coordinates y^i.

1.2.3.2. Varying the classical background. We now define the map $\tilde{S}_\Sigma$ to be given by $\tilde{S}_\Sigma\colon x \mapsto S_{\Sigma,x}$. In local coordinates (x^i) on $\overline{M}$, we define

$$S_{\Sigma,x,R} := \int_\Sigma Y^j_i(x;\hat{X})\, b_j \wedge \mathrm{d}x^i, \tag{135}$$

where $Y \in \Omega^1(\overline{M}, \hat{S}T^*M)$ is defined in Chapter 1.1, which is also a formal power series in the second argument, hence we can express Y as

$$Y^j_i(x;y) = \sum_{k=0}^{\infty} Y^j_{i;i_1,\dots,i_k}(x)\, y^{i_1}\cdots y^{i_k}. \tag{136}$$

Notice that here we pull back to the body $\overline{M}$ of M via the zero section of $M \to \overline{M}$. Moreover, on a closed manifold we have

$$\mathrm{d}_x\tilde{S}_\Sigma = (S_{\Sigma,x,R}, \tilde{S}_\Sigma).$$

Definition 1.2.3.1 (Formal global action). The *formal global action* for a split AKSZ theory is defined by

$$\tilde{\mathcal{S}}_{\Sigma,x} := \int_\Sigma \left(\mathsf{b}_i \wedge \mathrm{d}\hat{\mathsf{X}}^i + \theta_x(\hat{\mathsf{X}}, \mathsf{b}) + Y_i^j(x;\hat{\mathsf{X}})\mathsf{b}_j \wedge \mathrm{d}x^i \right) = \mathcal{S}_{\Sigma,x} + \mathcal{S}_{\Sigma,x,R}. \tag{137}$$

Using the formal global action, we get

$$\mathrm{d}_x \tilde{\mathcal{S}}_{\Sigma,x} + \frac{1}{2}(\tilde{\mathcal{S}}_{\Sigma,x}, \tilde{\mathcal{S}}_{\Sigma,x}) = 0. \tag{138}$$

This condition is called the *differential Classical Master Equation* (dCME) (see [16, 37, 38, 43]). On a manifold with boundary, we get the cohomological vector field $\tilde{Q}_{\Sigma,x}$ from the BV-BFV theory on $\mathcal{F}_{\Sigma,x}$. Recall the construction of a vector field R in the setting of formal geometry as in Section 1.1.3. For a section $\sigma \in \Gamma(\hat{S}T^*M)$ we have

$$R(\sigma) = -\mathrm{d}_y \sigma \circ (\mathrm{d}_y \varphi)^{-1} \circ \mathrm{d}_x \varphi.$$

Indeed, we can lift the vector field R to a vector field $\tilde{R}$ on $\mathcal{F}_\Sigma$ and define $\tilde{Q}_{\Sigma,x} = \check{Q}_\Sigma + \tilde{R}$, where $\check{Q}_\Sigma$ is the Hamiltonian vector field for $\check{\mathcal{S}}_\Sigma$. Then we have

$$\iota_{\tilde{Q}_{\Sigma,x}} \omega_{\Sigma,x} - \delta \tilde{\mathcal{S}}_{\Sigma,x} = \pi^*_\Sigma \alpha_{\partial\Sigma}, \tag{139}$$

the *modified differential Classical Master Equation (mdCME)*.

Remark 1.2.3.2. A similar approach to globalization for closed manifolds was done by Grady–Gwilliam, Costello, Grady–Li–Li ([70, 49, 69]). Their construction is based on the idea that one can replace the target by an L_∞ equivalent one, whereas the one introduced in [16] before was based on the idea of using formal geometry to define a symplectomorphism on a neighborhood of each solution in the space of fields to start the perturbation theory. The two approaches are essentially equivalent. However, in [70, 49, 69] they only get BF_1 theories since they start with theories of a particular simple type. We consider more general theories that do not fit into this. Here BF_1 means that one of the two fields appears at most linearly, but this is not the case in our setting (e.g., in the Poisson Sigma Model for a nonlinear Poisson structure). Moreover, in principle one should work around more general solutions than just the constant ones. In principle, one should do formal geometry on the moduli space of solutions. Note also that this construction can in principle be generalized to non AKSZ models.

1.2.4. Quantization

We now have a bundle of BF-like theories over the body $\overline{M}$ of M. In every fiber we can apply a perturbative BV-BFV quantization as in [38]. That is, we define a splitting of the space of fields

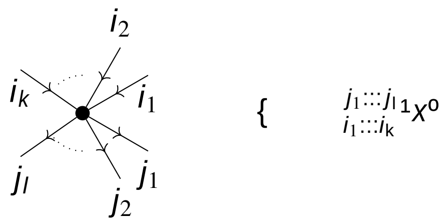
$$\mathcal{F}_{\Sigma,x} = \mathcal{B}^{\mathcal{P}}_{\partial\Sigma} \times \mathcal{V}_{\Sigma,x} \times \mathcal{Y}'$$

as in (61) and split the fields accordingly as

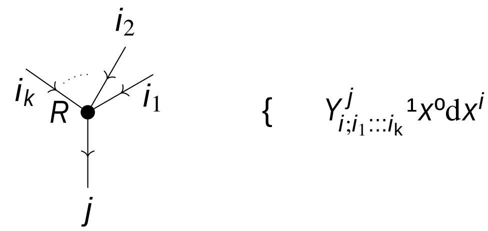
$$\hat{\mathsf{X}} = \mathbb{X} + \mathsf{x} + \mathcal{X},$$
$$\mathsf{b} = \mathbb{E} + \mathsf{e} + \mathcal{E},$$

where $\mathcal{B}^{\mathcal{P}}_{\partial\Sigma}$ is the base space of a polarization $\mathcal{P}$ of boundary fields,

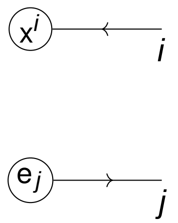
$$\mathcal{V}_{\Sigma,x} = H^\bullet(\Sigma, \partial_1\Sigma) \otimes T_xM \oplus H^\bullet(\Sigma, \partial_2\Sigma) \otimes T^*_xM$$



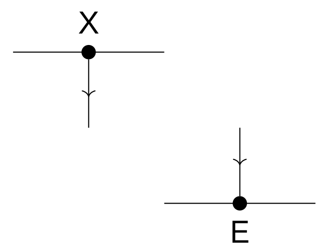
(a) Interaction vertex



(b) R vertex



(c) Residual fields



(d) Boundary vertices

FIGURE 1.2.1. Summary of Feynman graphs and rules

is the space of residual fields and $\mathcal{Y}$ is a symplectic complement of $\mathcal{B}^{\mathcal{P}}_{@}$ $\mathcal{V}_{;x}$. The polarizations that we consider are defined by splitting the boundary $@ = @_1 \sqcup @_2$ and choosing the X-representation on $@_1$ and the E-representation on[1] $@_2$. Let us denote by $\mathcal{H}^{\mathcal{P}}_{@ ;x}$ the boundary state space as in Definition 0.1.4.12. Using the definition of the formal global action $\mathfrak{S}_{;x}$ and Definition 0.1.4.18, we can define a covariant version of the principal part of the quantum state.

DEFINITION 1.2.4.1 (PRINCIPAL COVARIANT QUANTUM STATE). The *principal covariant quantum state* $e_{;x}$ is defined as in Definition 0.1.4.18, using the Feynman rules given in Figure 1.2.1 coming from the formal global action $\mathfrak{S}_{;x}$.

1.2.4.1. Feynman graphs and rules. The Feynman graphs and rules are the same as in [38, 39, 43], but there are additional interaction vertices given by $\mathcal{S}_{;x;R}$. Namely, to an interaction vertex with k incoming and l outgoing half-edges labeled by $i_1; : : : ; i_k$ and $j_1; : : : ; j_l$ respectively we associate $^{j_1:::j_l}_{i_1:::i_k}{}^1x^o$ as defined in (134). To a vertex labeled by R, with k incoming half-edges labeled $i_1; : : : ; i_k$ and one outgoing edge labeled j, we associate $Y^j_{i;i_1;:::;i_k}{}^1x^o$ as in (136). Half-edges can start at e zero modes and boundary vertices on $@_1$ and end at x zero modes or boundary vertices on $@_2$. See Figure 1.2.1.

1.2.4.2. The full covariant state. As it was discussed in Chapter 0.1, we need to deal with composite fields in order to regularize higher functional derivatives, hence we also need a covariant version of the full state.

DEFINITION 1.2.4.2 (FULL COVARIANT QUANTUM STATE). We define the *full covariant quantum state* $e_{;x}$ as in Definition 0.1.4.22, using the Feynman rules in Figure 1.2.1 coming from the formal global action $\mathfrak{S}_{;x}$ and additionally with the rules for the boundary vertices as in Figure 0.1.2.

[1]This simply means that we choose the $\frac{}{\mathsf{X}}$-polarization on $@_2$ and the $\frac{}{\mathsf{E}}$-polarization on $@_1$

CHAPTER 1.3

The modified differential Quantum Master Equation (mdQME)

The mQME, as a condition to hold in the BV-BFV formalism on manifolds with boundary, needs to be modified for a globalized AKSZ theory. The more general condition is called the *modified differential Quantum Master Equation (mdQME)*. The classical and quantum aspects of this modification are discussed in [16, 37], and first discussed for manifolds with boundary in [43]. We want to think of the operator

$$\nabla_G := d_x - i\hbar\Delta_{\mathcal{V}_{\Sigma,x}} + \frac{i}{\hbar}\Omega_{\partial M}$$

as a connection on the total bundle of spaces of states over (a part of) the moduli space of classical solutions of the theory. We call this operator the *quantum Grothendieck BFV (GBFV) operator*.

Remark 1.3.0.1. As already mentioned, the quantum GBFV operator ∇_G is an operator on forms valued in sections of the total bundle of states, which is a graded vector space and it is an operator of total degree $+1$, but not of form degree 1. We will call it an operator instead of a connection, since it can be misleading to think of $-i\hbar\Delta_{\mathcal{V}_{\Sigma,x}} + \frac{i}{\hbar}\Omega_{\partial M}$ as a connection 1-form. Rather, it defines a Maurer–Cartan element in the differential graded Lie algebra of differential forms with values in sections of the endomorphism bundle of the total state space.

The goal of this chapter can be rephrased as showing that the state gives a well-defined ∇_G-cohomology class. For this we have to show that:

(1) The state defines a closed section with respect to ∇_G (the mdQME),
(2) The operator ∇_G is a coboundary operator, i.e. $\nabla_G^2 = 0$,
(3) The cohomology class of $\tilde{\psi}_{\Sigma,x}$ is independent of the choices made, i.e. if we alter any of these choices, the state changes in a controlled way.

This will be the program of this chapter. Heuristically, this result can be interpreted as saying that the state comes from a well-defined function on (a part of) the moduli space of classical solutions of the theory.

1.3.1. Assumptions on the theory

The proof for the program of this chapter depends on two important assumptions on our theory, which we will discuss in this section.

1.3.1.1. No hidden faces anomalies. Let Γ be a Feynman graph and denote by $V(\Gamma)$ the set of its vertices; it decomposes into bulk vertices $V_B(\Gamma)$ and boundary vertices $V^{\partial}(\Gamma)$. The boundary of the configuration space is a union of several faces. We will denote by F_{ij} the faces where two bulk vertices $i, j \in V_B(\Gamma)$ collapse in the bulk. By $F_{\geq 3}$ we denote the union of the faces where at least three bulk vertices collapse in the bulk, usually called "hidden faces". By $F^{\partial}_{i_1,\ldots,i_k,j_1,\ldots,j_l}$ we denote faces where the bulk vertices $i_1, \ldots, i_k \in V_B(\Gamma)$ and the boundary vertices $j_1, \ldots, j_l \in V^{\partial}(\Gamma)$ collapse at a point in the boundary; the union of all these faces is denoted by F^{∂}.

Definition 1.3.1.1. We say that a theory is *(hidden faces) anomaly free* if for every graph Γ we have that

$$\int_{F_\Gamma} \omega_\Gamma = 0, \tag{140}$$

i.e. all possible contributions of hidden faces vanish.

Remark 1.3.1.2. A theory that is famously not anomaly free is Chern–Simons theory, see [4, 5] and [18], where the first ansatz for the quantum theory depends on the choice of gauge fixing. In this case one can get away of the anomaly with introducing a framing and a framing-dependent counterterm for the dependence on the gauge fixing. On the other hand, there are many examples of anomaly free theories. In particular, Kontsevich's result [77] implies that any 2-dimensional theory is anomaly-free, e.g. the Poisson Sigma Model ([73, 93, 92, 26]).

Remark 1.3.1.3 (Counterterms). A general ansatz to deal with theories with anomalies is the addition of counterterms to the action. If the differential form which results from integrating over a hidden face is exact, one can add the corresponding primitive to the action, thus producing new vertices cancelling the anomaly. In Chern–Simons theory, this produces the "framing" anomaly, since the only hidden faces contribution comes from faces where all vertices in a graph collapse. The resulting differential form is a representative of the relative Pontryagin class of $M \times I$, whose primitive is the Chern–Simons form of the flat connection used to construct the propagator.

1.3.1.2. Unimodularity. In the quantization of general AKSZ theories one can have tadpoles, also called short loops, i.e. arrows starting and ending at the same vertex. They need to be treated seperately and can in principle spoil the mdQME. The best way to get around them is to assume that the theory satisfies a "unimodularity" condition.

Definition 1.3.1.4 (Unimodularity). We say that a given theory is *unimodular* if any contraction of the vertex tensor with itself is zero.

1.3.2. The modified differential Quantum Master Equation

One of the main results, and the first point of the program is the following theorem:

Theorem 1.3.2.1 (mdQME for split AKSZ theories). *Consider the full covariant perturbative state $\tilde\psi_{\Sigma,x}$ as a quantization of an anomaly free and unimodular split AKSZ theory with target $T^*[d-1]M$, where M is a graded manifold. Then*

$$\left(d_x - i\hbar\Delta_{\mathcal{V}_{\Sigma,x}} + \frac{i}{\hbar}\Omega_{\partial\Sigma}\right)\tilde\psi_{\Sigma,x} = 0, \tag{141}$$

where we denote by d_x the de Rham differential on $\overline{M}$, the body of the graded manifold M.

We will prove this by considering the Feynman graphs of the theory analogously to the proof of the mQME in [38].

Proof. For the following computation we consider Feynman graphs which also have vertices, of any possible valency, on the boundary deriving the functions attached there. Let G denote the set of Feynman graphs of the theory. Then $\tilde\psi_{\Sigma,x}$ can be written as

$$\tilde\psi_{\Sigma,x} = T\prod_{\Gamma\in G}\int_{C_\Gamma}\omega_\Gamma, \tag{142}$$

where we include the combinatorial prefactor $\frac{(-i\hbar)^{\text{loops}(\Gamma)}}{|\text{Aut}(\Gamma)|}$ in ω_Γ (here loops(Γ) denotes the number of loops of a graph Γ). Moreover, we denote the configuration space $C(\Gamma)$ by C_Γ for simplicity. Note that ω_Γ is a ($\mathcal{V}_{\Sigma,x}$-dependent) differential form on $C_\Gamma \times \overline{M}$. Now recall Stokes' theorem for integration along a compact fiber with corners:

$$\mathrm{d}\pi_* = \pi_*\mathrm{d} - \pi_*^{\partial}. \tag{143}$$

The integrals in (142) are fiber integrals, hence we can apply (143) to yield

$$\mathrm{d}_x \int_{C_\Gamma} \omega_\Gamma = \int_{C_\Gamma} \mathrm{d}\,\omega_\Gamma - \int_{\partial C_\Gamma} \omega_\Gamma. \tag{144}$$

Here d inside the integral is the total differential on $\overline{M} \times C_\Gamma$, and thus we can split it as

$$\mathrm{d} = \mathrm{d}_x + \mathrm{d}_1 + \mathrm{d}_2 \tag{145}$$

Here d_1 denotes the part of the de Rham differential acting on the propagators in ω_Γ, and d_2 the part acting on X and E fields. Let us introduce some more notation: The set of edges of Γ will be denoted by $E(\Gamma)$. We denote by $E_k(\Gamma)$ the set of edges e whose removal increases the number of connected components by k. Clearly $E(\Gamma) = E_0(\Gamma) \sqcup E_1(\Gamma)$, and $e \in E_1(\Gamma)$ if and only if e is not part of a loop in Γ.

Proposition 1.3.2.2. *The following hold:*

(i) The action of the BV Laplacian on the state is given by

$$i\hbar\Delta_{\mathcal{V}_{\Sigma,x}}\psi_{\Sigma,x} = T\sum_{\Gamma\in G}\int_{C_\Gamma}\mathrm{d}_1\omega_\Gamma \tag{146}$$

(ii) The action of Ω_0 on the state is given by

$$\frac{i}{\hbar}\Omega_0\psi_{\Sigma,x} = T\sum_{\Gamma\in G}\int_{C_\Gamma}\mathrm{d}_2\omega_\Gamma \tag{147}$$

(iii) For the boundary contributions we have

$$\sum_{\Gamma\in G}\sum_{i,j\in V_B(\Gamma)}\int_{F_{ij}}\omega_\Gamma = \sum_{\Gamma\in G}\int_{C_\Gamma}\mathrm{d}_x\omega_\Gamma; \tag{148}$$

where we denote by F_{ij} the boundary faces where two bulk vertices $i,j \in V_B(\Gamma)$ collapse in the bulk, and furthermore

$$T\sum_{\Gamma\in G}\int_{F^\partial}\omega_\Gamma = \frac{i}{\hbar}\Omega_{\text{pert}}\psi_{\Sigma,x}, \tag{149}$$

where F^∂ is the union of the $F^\partial_{i_1,\ldots,i_k,j_1,\ldots,j_l}$, which denote the boundary faces where the bulk vertices $i_1,\ldots,i_k \in V_B(\Gamma)$ and the boundary vertices $j_1,\ldots,j_l \in V^\partial(\Gamma)$ collapse at a point in the boundary.

Proposition 1.3.2.2 immediately implies the mdQME for anomaly free theories by the following simple computation:

$$\begin{aligned} d_X e_{;x} &= T\left(\sum_{\Gamma\in G} d_X \int_C \omega\right) = T\left(\sum_{\Gamma\in G} \int_C d\,\omega - \int_{\partial C} \omega\right) \\ &= T\left(\sum_{\Gamma\in G} \int_C (d_X \omega + d_1 \omega + d_2 \omega) - \int_{\partial C}\omega\right) \\ &= i\hbar\, \Delta_{\mathcal{V};x}\, e_{;x} - \frac{i}{\hbar}\, \Omega_0\, e_{;x} - \frac{i}{\hbar}\, \Omega_{\text{pert}}\, e_{;x} \end{aligned}$$

1.3.3. Proof of Proposition 1.3.2.2

The proof of Theorem 1.3.2.1 is relied on the proof of Proposition 1.3.2.2. We split the proof into four parts. Namely, we show the Equations (146), (147), (148) and (149) seperately and conclude.

1.3.3.1. Proof of Equation (146). Consider a propagator[1] as in [38] and denote by $\eta_{ij} := (u_i, u_j)$ for $u_i, u_j \in$. Moreover, let $_i$ denote a representative for the basis of $\mathcal{V}_{;x}$ and denote by ^{i} a representative of its dual basis. Hence, in local coordinates, we can write the residual fields, x and e as $x = \sum_i z^i\, _i$ and $e = \sum_i z_i^+\, ^i$. Then we have the identity

$$d\,\eta_{12} = \sum_k \pi_1^*\, _k\; \pi_2^*\, ^k = \Delta_{\mathcal{V};x}(x_1 e_2), \tag{150}$$

where π_1 and π_2 denote the projections to the first and second factor of $C_2( )$ respectively and where $x_i := x(u_i)$ and $e_j := e(u_j)$. Recall also that the BV Laplacian is given by

$$\Delta_{\mathcal{V};x} = \sum_k \frac{\partial^2}{\partial z^k \partial z_k^+}$$

where $\deg \frac{\partial}{\partial z_k^+} = -\deg z_k^+ = \deg z^k - 1 = 1 - \deg\, _k - 1 = -\deg\, _k$. Since $\deg x = 1$ we get

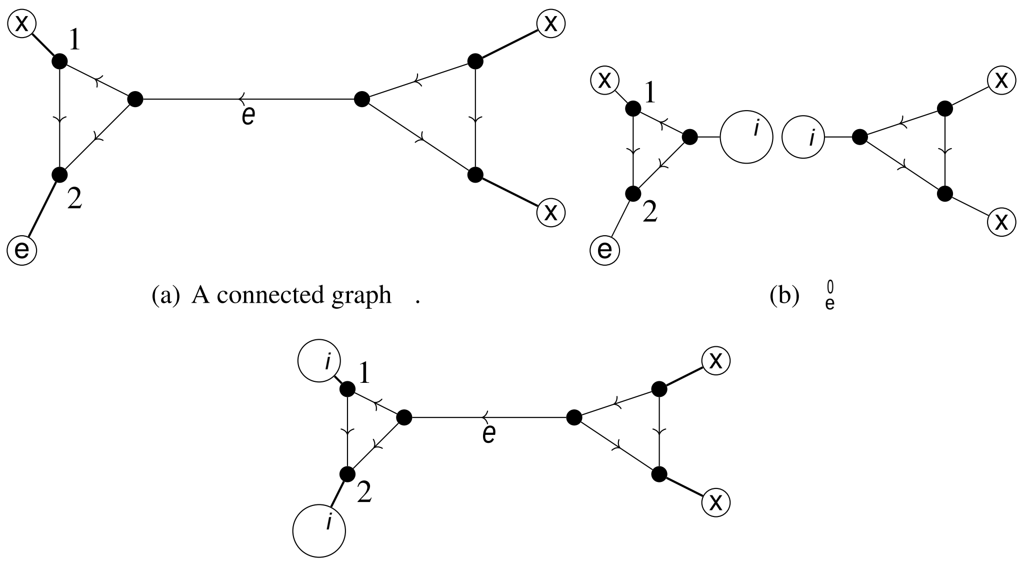
$$\Delta_{\mathcal{V};x}(\pi_1 x\; \pi_2 e) = \sum_k \pi_1\, _k\; \pi_2\, ^k.$$

Let us introduce some more notation for certain operations on graphs. For any graph Γ, let denote either an edge $e = (i, j)$ or a pair of residual fields[2] x_i, e_j. Denote by Γ' the graph resulting from removing the component labeled by and replacing it with a diagonal class between points i and j, i.e. the sum $\sum_k\, _i\, _k\, _j\, ^k$ (see also Figure 1.3.1).
Clearly, we have

$$\sum_{\Gamma\in G} \int_C d_1 \omega = \sum_{\Gamma\in G} \sum_{e\in E( )} \int_C \omega_{\Gamma'_e} \tag{151}$$

[1]This means that we want to be given by $\frac{1}{T}\frac{1}{i\hbar}\int_{\mathcal{L}} e^{\frac{i}{\hbar}S}\, X \wedge E$, where $T = \int_{\mathcal{L}} e^{\frac{i}{\hbar}S}$ and $S = \int E \wedge dX$. Here the Lagrangian $\mathcal{L}$ is given by the direct sum of the image of the Hodge-theoretic chain contraction and the image of its dual (with the correct shift).

[2]Note that an edge denotes a contracted X-E-pair, so denotes either an X-E or an x-e-pair.



(a) A connected graph Γ. (b) Γ'_e (c) $\Gamma'_{x_1 e_2}$

FIGURE 1.3.1. Explanation of the operation Γ'.

On the other hand, the properties of the BV Laplacian imply

$$\Delta_{\mathcal{V}_{\Gamma,x}} \omega_\Gamma = \sum_{\substack{\text{pairs of}\\ \text{residual fields }(x_i,e_j)\text{ in }\Gamma}} \omega_{\Gamma'_{x_i e_j}},$$

which we can interpret as a first order differential operator on a product $z_i^+ z^i$. By construction we get that if the edge e starts at i and ends at j, we have

$$\omega_{\Gamma'_e} = (-i\hbar)\,\omega_{\Gamma'_{x_i e_j}}, \tag{152}$$

since each edge comes with a factor of $(-i\hbar)$. Now consider the action of $\Delta_{\mathcal{V}_{\Gamma,x}}$ on $e_{\Gamma,x}$, and note that

$$\Delta_{\mathcal{V}_{\Gamma,x}} e_{\Gamma,x} = T\sum_{\Gamma\in\mathcal{G}_C} \Delta_{\mathcal{V}_{\Gamma,x}} \omega_\Gamma = T\sum_{\Gamma\in\mathcal{G}_C} \sum_{\substack{\text{pairs of}\\ \text{residual fields }(x_i,e_j)\text{ in }\Gamma}} \omega_{\Gamma'_{x_i e_j}} \tag{153}$$

The sum in Equation (151) above can be seen as summing over all graphs with one egde marked - we will denote this set by $\mathcal{G}^E$. In the sum in Equation (153) above we sum over all graphs with one pair of residual fields marked - we will denote this set by $\mathcal{G}^{\text{pair}}$. Now define a map

$$\mathcal{G}^E \to \mathcal{G}^{\text{pair}}$$
$$\Gamma \mapsto \Gamma'_e$$

which exchanges the marked edge for a marked pair of residual fields. Clearly this map is invertible and its inverse exchanges the marked pair of residual fields for a marked edge. The contributions to sum labeled by Γ and Γ'_e agree up to a factor by Equation (152). We conclude the proof of (146).

1.3.3.2. Proof of Equation (147)**.** Recall from Chapter 0.1 that Ω_0 is given by

$$\Omega_0 = (-1)^{\dim} i\hbar \left(\int_{\partial_1} dX^i \frac{\delta}{\delta X^i} + \int_{\partial_2} dE_i \frac{\delta}{\delta E_i} \right).$$

Hence Equation (147) follows immediatly from the definition of the de Rham differential on the X and E fields. Moreover, note that ω_Γ is given as a product of propagators, residual fields, boundary fields and vertex tensors as in Chapter 0.1 Equation (70).

1.3.3.3. Proof of Equation (148)**.** First notice that all the x-dependence of ω_Γ lies in the bulk vertex tensors. There are two types of bulk vertex tensors, arising from $\mathcal{V}_{\Pi,x}(\psi, b) := T\phi^*_x(X, \eta)$ and $\mathcal{S}_{\Pi,x,R}$ respectively, we will call them type I and type II vertices. Let us analyse them in more detail.

First, recall that

$$\mathcal{V}_{\Pi,x}(\psi, b) = \sum_{k,l=0}^{\infty} \Theta^{i_1\dots i_k}_{j_1\dots j_l}(x) \psi_{i_1} \cdots \psi_{i_k} b^{j_1} \cdots b^{j_l} \tag{154}$$

(the Θ's are exactly one set of vertex tensors in the Feynman graphs). The fact that $(\mathcal{V}_{\Pi,x}, \mathcal{V}_{\Pi,x}) = 0$ is equivalent to $\sum_r \frac{\partial}{\partial \psi_r} \mathcal{V}_{\Pi,x}(\psi, b) \frac{\partial}{\partial b^r} \mathcal{V}_{\Pi,x}(\psi, b) = 0$. In terms of the vertex tensors it reads as

$$\sum_{\substack{k'+k''=k+1 \\ l'+l''=l+1}} \sum_{\substack{1 \le s' \le k' \\ 1 \le s'' \le k''}} \sum_r \delta^r_{i'_{s'}} \delta^{j''_{s''}}_r \Theta'^{i'_1\dots i'_{k'}}_{j'_1\dots j'_{l'}}(x) \Theta''^{i''_1\dots i''_{k''}}_{j''_1\dots j''_{l''}}(x) = 0 \tag{155}$$

for every $k, l \ge 0$. This can be understood as follows: From a (k', l')-tensor Θ' and a (k'', l'')-tensor Θ'' we can form a $(k, l) = (k' + k'' - 1, l' + l'' - 1)$-tensor Θ by contracting exactly one index. We will say Θ has been merged from Θ' and Θ''. If we sum over all possibilities of constructing (k, l)-tensors this way, the result vanishes. Now, suppose we have a graph Γ with an edge e between two type I vertices v' and v'', with vertex tensors Θ' and Θ'', respectively. The boundary of C_Γ contains a face where these two vertices collapse; by normalization of the propagator, the integral over the corresponding fiber yields ± 1. We are left with a new graph Γ / e where the edge has been collapsed into a new marked vertex. The vertex tensor at this new vertex has been merged from Θ', Θ''. Now, sum over all graphs, and the corresponding boundary contributions of edges between type I vertices. Then we will sum over all ways of merging a vertex in Γ / e. Hence these contributions vanish by (155). Similarly, one can argue for edges between type II vertices, since also $(\mathcal{S}_{\Pi,x,R}, \mathcal{S}_{\Pi,x,R}) = 0$. For edges between type I and type II vertices, the relation $d_x \mathcal{V}_{\Pi,x} = (\mathcal{S}_{\Pi,x,R}, \mathcal{V}_{\Pi,x})$ implies (148).

1.3.3.4. Proof of Equation (149)**.** Recall that we have $\Omega_{\text{pert}} = \Omega^X_{\text{pert}} + \Omega^E_{\text{pert}}$, where Ω^X_{pert} is constructed as follows. Denote by Γ a Feynman graph of the theory, and let Γ' be a subgraph of Γ (we use the notation $\Gamma' \subset \Gamma$) which contains only bulk vertices and vertices on the boundary component, say, where we work in the X-representation. Then there is a corresponding contribution $\Omega_{\Gamma'}$ to Ω_∂ given as follows. If Γ' has inward leaves (i.e. there is an arrow from some vertex in $\Gamma \setminus \Gamma'$ to a vertex in Γ') then $\Omega_{\Gamma'}$ vanishes. Suppose the l outward leaves are labeled by $j_1, \dots, j_l$ and suppose Γ' has k boundary vertices with boundary fields $[X^{I_j}], j = 1, \dots, k$. Then

$$\Omega_{\Gamma'} = \frac{(-i\hbar)^{\text{loops}(\Gamma')}}{|\text{Aut}(\Gamma')|} \int_{\partial_1} \sigma_{\Gamma'}{}^{J}_{I^1\dots I^k} [X^{I_1}] \cdots [X^{I_k}] \frac{\delta^{|J|}}{\delta [X^J]} \tag{156}$$

where $\sigma_{\Gamma'}$ is the differential form on $\partial_1 M$ whose value at $x \in \partial_1 M$ is given by integrating the limiting propagators over the compactified configuration space $\tilde{C}_{\Gamma'}(\mathbb{H})$ in the upper half-space (see

Definition 0.1.4.25), which we denote simply by $\tilde{C}_{\Gamma'}$ for simplicity, as in Appendix D. Recall that in $\tilde{C}_{\Gamma'}$ we take the quotient by translation and scaling. Put differently, there is a boundary face $\partial_{1,\Gamma'}C_\Gamma$ of C_Γ corresponding to the collapse of Γ' at ∂_1, that face is given by

$$\partial_{1,\Gamma'}C_\Gamma \cong \tilde{C}_{\Gamma'} \times C_{\Gamma \bullet \Gamma'},$$

where we denote by $\Gamma \bullet \Gamma'$ the collapse of Γ' in Γ to a new boundary vertex in ∂_1. Let Γ'_{amp} be the "amputated" graph Γ' where we cut off all the outward leaves and the ones containing residual fields. Then $\Omega_{\Gamma'}$ is given as follows. Let $\tilde{e}_{\Gamma'}$ be the pushforward of $\omega_{\Gamma'_{amp}}$ along the map $\pi: \tilde{C}_{\Gamma'} \times C_{\Gamma \bullet \Gamma'} \to C_{\Gamma \bullet \Gamma'}$. Since we take the amputated Γ', this pushforward is a basic form in $p: C_{\Gamma \bullet \Gamma'} \to \partial_1$, and the corresponding form on ∂_1 is $\Omega_{\Gamma'}$, i.e. $\tilde{e}_{\Gamma'} = p^*\Omega_{\Gamma'}$. Then

$$\Omega^X_{pert} = \sum_{\Gamma'} \Omega_{\Gamma'}, \tag{157}$$

where the sum runs over all Feynman graphs Γ of the theory and all their subgraphs Γ'.

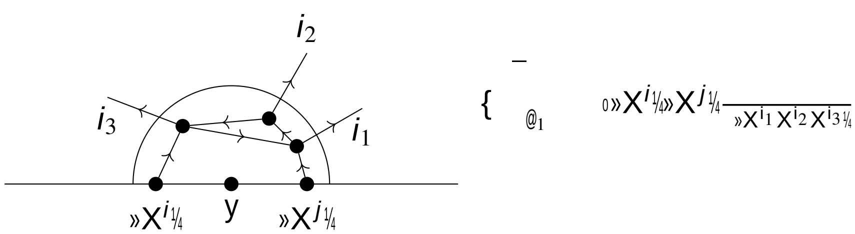


FIGURE 1.3.2. An example of a term in Ω_∂.

We can see that

$$\int_{\partial_{\Gamma'}C_\Gamma} \omega_\Gamma = \int \Omega_{\Gamma'} \int_{C_{\Gamma \bullet \Gamma'}} \omega_{\Gamma \bullet \Gamma'},$$

and hence we conclude Equation (149) by summing over all graphs. One can construct Ω^E_{pert} analogously.

1.3.4. Flatness of the quantum GBFV operator

We have the following theorem:

THEOREM 1.3.4.1. *The quantum GBFV operator* ∇_G *squares to zero, i.e.*

$$(\nabla_G)^2 \equiv 0. \tag{158}$$

PROOF. Note that condition (158) is the same as saying that

$$i\hbar\,\Delta_{\mathcal{V}_{\Sigma,x}}(d_x\Omega_\partial + \Omega_\partial d_x) = \Omega_\partial^2 \tag{159}$$

since $\Delta_{\mathcal{V}_{\Sigma,x}}\Omega_\partial + \Omega_\partial\Delta_{\mathcal{V}_{\Sigma,x}} = d_x\Delta_{\mathcal{V}_{\Sigma,x}} + \Delta_{\mathcal{V}_{\Sigma,x}}d_x = 0$. Here we interpret d_x and Ω_∂ as operators on $\mathcal{H}_{tot}$-valued differential forms. Equivalently, we can interpret Ω_∂ as an element of the differential graded Lie algebra of sections of $\bigwedge T^*M \otimes \mathrm{End}(\mathcal{H}_{tot})$ and rewrite Equation (159) in the following intriguing fashion:

$$i\hbar\, d_x\Omega_\partial - \frac{1}{2}[\Omega_\partial, \Omega_\partial] = 0. \tag{160}$$

Equation (160) shows that $\frac{i}{\hbar}\Omega_\partial$ is a Maurer–Cartan element in this differential graded Lie algebra. One can prove this equation by using Stokes' theorem for the definition with Feynman diagrams, similarly as in the proof of flatness in the mQME section of [38]. The crucial point is the following lemma.

Lemma 1.3.4.2. *We have that*

$$\left(\Omega^{\mathbb{X}}_{\text{pert}}\right)^2 = \sum_{\Gamma'' \, \Gamma'} \left(\int_{\partial_1} \omega_{\Gamma'' \cup \Gamma'} \right) [\mathbb{X}^{I_1}] \cdots [\mathbb{X}^{I^k}] \frac{\partial}{\partial [\mathbb{X}^{J}]}; \tag{161}$$

where $I_1; \ldots; I_k$ are the boundary vertices of Γ' and the outward leaves of Γ' are labeled by $J = j_1; \ldots; j_l$. An analogous statement holds in the $\mathbb{E}$-representation.

Proof of Lemma 1.3.4.2. Since $\Omega^{\mathbb{X}}_{\text{pert}}$ has degree 1 we can write

$$\left(\Omega^{\mathbb{X}}_{\text{pert}}\right)^2 = \frac{1}{2}\left[\Omega^{\mathbb{X}}_{\text{pert}}, \Omega^{\mathbb{X}}_{\text{pert}}\right] = \frac{1}{2}\sum_{\Gamma'_1, \Gamma'_2} \left[\Gamma'_1 \ \ _1; \Gamma'_2 \ \ _2\right]:$$

By definition, Ω_{∂} contains only first order derivatives (with respect to composite fields). Hence in the commutator the quadratic terms cancel and we are left with the terms where the derivatives act on the coefficients. The bracket

$$\left[\Gamma'_1 \ \ _1; \Gamma'_2 \ \ _2\right]$$

is nonzero if and only if the outward leaves of Γ'_1 exactly match the composite field at one of the vertices of Γ'_2 (or vice versa). In the first case, the corresponding contribution is (for simplicity we assume that the corresponding vertex is labeled by 1 in Γ'_2)

$$\int_{\partial_1} \Gamma'_1 \ \Gamma'_2 [\mathbb{X}^{I^1_1}] \cdots [\mathbb{X}^{I^1_{k_1}}][\mathbb{X}^{I^2_2}] \cdots [\mathbb{X}^{I^2_{k_2}}] \frac{\partial}{\partial [\mathbb{X}^{J}]};$$

where the composite fields at the vertices of Γ'_i are labeled by $I^i_j; 1 \leq j \leq k_i$, and the outward leaves of Γ'_2 are labeled by J. "Blowing up" the corresponding vertex i (we denote this operation by $\circ_i$) by replacing it by Γ'_1, from Γ_2 we obtain a new graph Γ, and from Γ'_2 a subgraph Γ' of Γ. Denoting the subgraph $\Gamma'_1 \subset \Gamma'$ by Γ'', we obtain that $\Gamma'_2 = \Gamma' \bullet \Gamma''$. In this way we obtain all possible graphs Γ with all possible combinations of subgraphs $\Gamma'' \subset \Gamma' \subset \Gamma$. See also Figure 1.3.3.

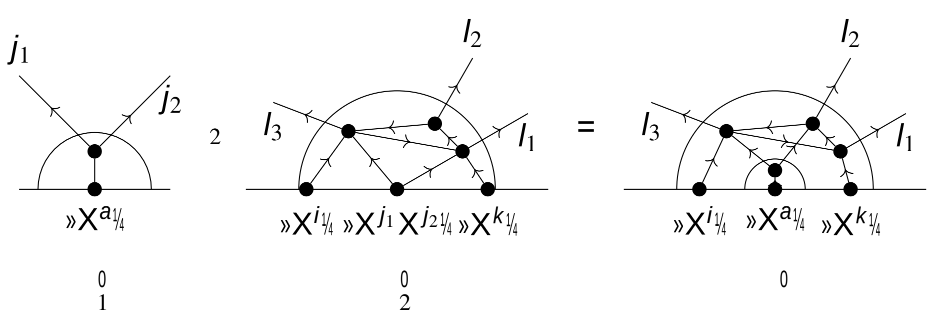


Figure 1.3.3. An example of a term in Ω^2.

Now the proof that Ω_{∂} satisfies the Maurer–Cartan equation can be done very similarly to the original one in [38].

Proof of Equation (160). We prove the equation for $\Omega^{\mathbb{X}}$, but the proof for $\Omega^{\mathbb{E}}$ is analogous and then the claim follows because $\Omega^{\mathbb{X}}$ and $\Omega^{\mathbb{E}}$ anticommute. We use again Stokes' theorem (twice). Suppose we apply d_X to a summand $\Phi_{\Gamma_0} = \int_{\partial_1} \omega_{\Gamma_0}[X^{I_1}] \cdots [X^{I_k}]\frac{\delta}{\delta[X^J]}$. Then, applying Stokes' theorem we find

$$d_X \Phi_{\Gamma_0} = \int_{\partial_1} d\,\omega'_{\Gamma_0}[X^{I_1}] \cdots [X^{I_k}]\frac{\delta}{\delta[X^J]} + [\Omega_0, \Phi_{\Gamma_0}]$$

(the second term is produced when d_X acts on the X fields). Now, we have

$$d_X \omega_{\Gamma_0} = d_X \int_{C_{\Gamma_0}} \omega_{\Gamma_0} = \int_{C_{\Gamma_0}} d\,\omega_{\Gamma_0} \pm \int_{\partial C_{\Gamma_0}} \omega_{\Gamma_0}.$$

Since the limiting propagator on C_{Γ_0} is closed we have $d\omega = d_X\omega$. In the boundary integral, we have again three classes of faces. The faces where two bulk points collapse cancel out with $d_X\omega$ by the mCME. The terms where more than two bulk points collapse vanish by our assumption that the theory is anomaly free. The terms where a subgraph of Γ' collapses at the boundary produce exactly $\frac{1}{2}[\Omega^X_{\text{pert}}, \Omega^X_{\text{pert}}]$ by Lemma 1.3.4.2.

Now since we have shown that (160) holds, we can conclude that ∇_G squares to zero.

CHAPTER 1.4

Dependence on choices

The definition of the state depends on the choices of

- the propagator,
- the residual fields,
- the formal exponential map.

In this chapter we will explicitly show how the state and the BFV boundary operator transform under a change in any of these choices.

1.4.1. Covariant gauge transformation

The definition of the state depends on the choices of Similarly to Definition 0.1.4.5 we have the following theorem:

THEOREM 1.4.1.1 (COVARIANT CHANGE OF DATA). *Let $_t$ be defined as in Definition 0.1.4.25 and let $\tilde{e}_t$ be defined as in Definition 1.2.4.2 for all $t \in [0,1]$. Then we have*

$$\frac{d}{dt}\Big|_{t=0} {}_t = d_x + [\ {}_{t=0}; \ ] \tag{162}$$

$$\frac{d}{dt}\Big|_{t=0} \tilde{e}_t = \nabla_G(\tilde{e}_{t=0} \ \%) \ \ \tilde{e}_{t=0} \tag{163}$$

for some operator $\in (\mathrm{End}(\mathcal{H}_{tot}))$ and a section $\% \in (\mathcal{H}_{tot})$. Recall that is the product constructed as in (81)

REMARK 1.4.1.2. In particular if is zero, the operator ∇_G does not change and the state changes by a ∇_G-exact term. Theorem 1.4.1.1 shall be seen as the behaviour of the full covariant state and the full BFV boundary operator under infinitesimal gauge transformation.

1.4.1.1. Possible choices. We have three different choices of how we can mark the graphs according to the change of the state. One possibility is to mark the leaves of a graph , which corresponds to the change of residual fields and the propagator, another one is to mark the edges which corresponds to the change of the propagator and the last choice is to mark the vertices, which corresponds to the change of the formal exponential map.

1.4.2. Changing the residual fields

We have the following proposition:

PROPOSITION 1.4.2.1 (CHANGE OF DATA: RESIDUAL FIELDS). *Fix some representatives $_i$ and ^{i} and consider their change by exact forms as*

$$\hat{}_i = d\ _i; \qquad _i \in {}^{\deg}_{D_1}{}^{\ i\ -1}(\ )$$

$$\hat{}^i = d\ ^i; \qquad ^i \in {}^{\deg}_{D_2}{}^{\ i\ -1}(\ );$$

where $_{D_i}(\ ) = \{\ \in\ (\ )\,|\ _i\ = 0\}$ *with* $_i$ *the inclusion map from* ∂_i *into* *for* $i \in \{1; 2\}$ *(D stands for Dirichlet). Then the family of states* (e_t) *changes by*

$$(164)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e_t = \nabla_G(e_{t=0}\ \ \%)\quad e_{t=0};$$

where $= 0$ *and*

$$(165)\qquad \% = \sum_{\substack{\Gamma^l_m \text{ marked}\\ \text{and connected graph}}} \%_{\Gamma^l_m} = \sum_{\substack{\Gamma^l_m \text{ marked}\\ \text{and connected graph}}} \int_{C_{\Gamma^l_m}} \omega'_{\Gamma^l_m} \in (\mathcal{H}_{tot});$$

where Γ^l_m *denotes a marked connnected graph with* $l \in \{x; e\}$, *i.e. a graph with a labeled* l *leaf and* $\omega'_{\Gamma^l_m}$ *is the form constructed with the usual Feynman rules where we place* $z_i^+\ ^i$ *at the marked leaf.*

Proof. The propagator changes by

$$(166)\qquad \dot{\eta} = \sum_i \ \ _1\ _i\ _2\ ^i\ \ \sum_i\ \ _1\ _i\ _2\ ^i:$$

Let $e = \int_C \omega$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} e = \frac{\mathrm{d}}{\mathrm{d}t}\sum \int_C \omega = \sum \int_C \dot{\omega} = \sum_{\Gamma^x_m} \omega_{\Gamma^x_m} + \sum_{\Gamma^e_m} \omega_{\Gamma^e_m} + \sum_{\Gamma^e_m} \omega^e_{\Gamma^e_m} + \omega^e_{\Gamma^e_m};$$

where by Γ^x_m we mean graphs with a labeled x leaf and $\omega_{\Gamma^x_m}$ is the usual ω but with $z^i\,\dot{\eta}_i$ placed at the marked leaf. Similarly, by Γ^e_m we mean graphs with a labeled e leaf and by $\omega_{\Gamma^e_m}$ the usual ω but with $z_i^+\,\dot{\eta}^i$ placed at the marked leaf. These terms arise when the time derivative hits a leaf. Finally, Γ^e_m denotes graphs with a marked edge $e \in E(\ )$, and $\omega^e_{\Gamma^e_m}$ denotes the usual ω, where at the marked edge we place $\ _1\ _i\ _2\ ^i$. These terms arise when the time derivative hits a propagator. We call them "edge splits" since the corresponding propagator in ω is split, see Figure 1.4.1. Define

$$\% := \sum_{\substack{\Gamma^l_m \text{ marked}\\ \text{and connected graph}}} \%_{\Gamma^l_m} = \sum_{\substack{\Gamma^l_m \text{ marked}\\ \text{and connected graph}}} \int_{C_{\Gamma^l_m}} \omega'_{\Gamma^l_m} \in (\mathcal{H}_{tot}).$$

Here $l \in \{x; e\}$. Again Γ^l_m denotes a graph with a marked l leaf. In $\omega'_{\Gamma^l_m}$ however we replace the marked leaf by $z^i\ _i$ or $z_i^+\ ^i$ respectively.

Now we show that

$$(167)\qquad \frac{\mathrm{d}}{\mathrm{d}t} e = \left(\mathrm{d}_x\ \ i\hbar\ \nu + \frac{i}{\hbar}\ \partial\right)(e\ \ \%).$$

The easiest way to see this is by considering graphs. We can expand

$$(168)\qquad e\ \ \% = \sum_{\Gamma^l_m} \int_{C_{\Gamma^l_m}} \omega'_{\Gamma^l_m}$$

where now the sum runs over all graphs, either connected or not. Now, we again consider Stokes' theorem for integrals along the fiber

$$(169)\qquad \mathrm{d}_x(e\ \ \%) = \sum_{\Gamma^l_m} \int_{C_{\Gamma^l_m}} \mathrm{d}\omega'_{\Gamma^l_m} \ \ \int_{\partial C_{\Gamma^l_m}} \omega'_{\Gamma^l_m}$$

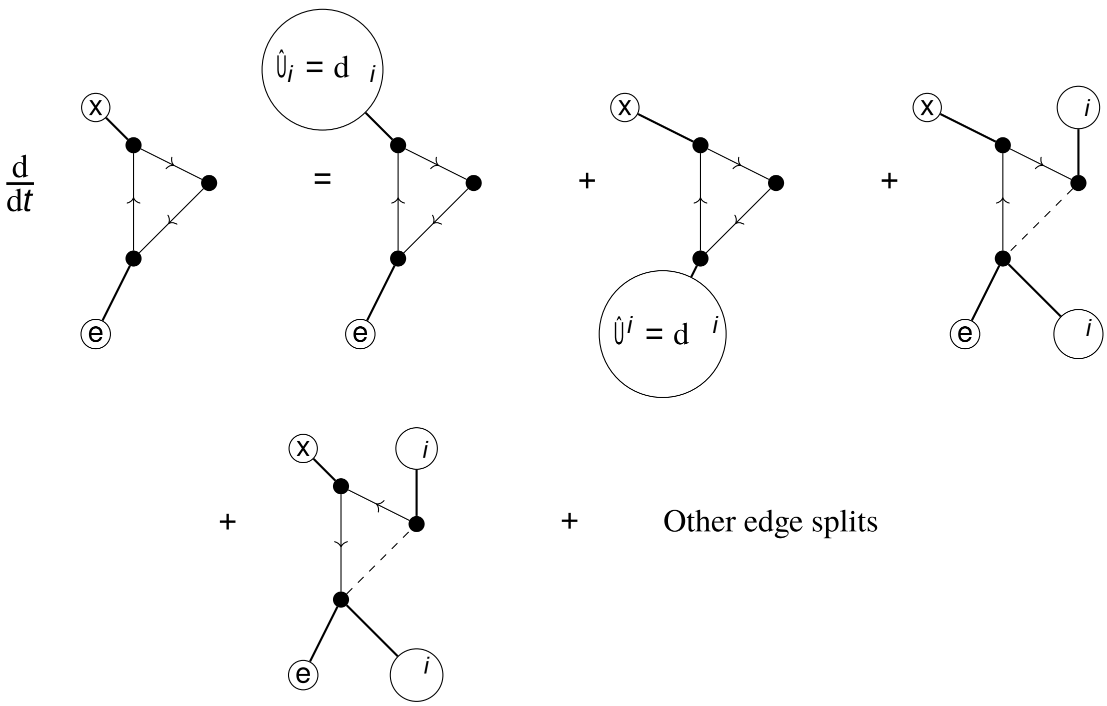


FIGURE 1.4.1. Graphical illustration of the time derivative, the first two terms come from time derivatives of leaves, all other terms - “edge splits” - from time derivatives of the propagator.

One can check that the edge split in $\frac{d}{dt}$ e corresponds to ν applied to a graph (up to some signs) having the same amount of leaves and carrying the same residual fields with the difference that for the two leafs splitting an edge, where one leaf carries a residual field x $= z^i$ $_i$ (e $= z_i^+$ ^{i}) and the other one a primitve field z_i^+ ^{i} (z^i $_i$). Moreover, one can check that the graphs with the d $_i$ (d ^{i}) leaves on the left hand side get produced by applying the de Rham differential on the configuration space to graphs with a $_i$ (^{i}) leaf. The rest of terms cancels out as in the proof of the mdQME.

1.4.3. Changing the propagator

We have the following proposition:

PROPOSITION 1.4.3.1 (CHANGE OF DATA: PROPAGATOR). *Suppose we change the propagator by an exact form 2 ^{n} ^{2}₁C$_2$¹ ºº with the appropriate boundary conditions (but keep the residual fields fixed),*

$$\hat{U} = d \,: \tag{170}$$

Then the family of states ¹e$_t$º *and the familiy of BFV boundary operators* ¹ $_t$º *change by*

$$\frac{d}{dt}_{t=0} \;_t = d_X \; + » \;_{t=0}; \; ¼ \tag{171}$$

$$\frac{d}{dt}_{t=0} e_t = r_G{}^1 e_{t=0} \; \%º \; e_{t=0} \tag{172}$$

where $= {}^{X} + {}^{E}$ *with*

(173)
$$ {}_{X} = \sum_{n,k\ 0} \sum_{\substack{e^0\\m}} \frac{1 i\sim \mathrm{oloops}^1\ {}^{e^0}_{m} \mathrm{o}}{|\mathrm{Aut}^1\ {}^{e^0}_{m} \mathrm{o}|} {}^{1}_{@_1} \quad {}^{e^0}_{m}\ {}^{J_1:::J_k}_{I_1:::I_n} \wedge X^{I_1} \wedge \quad \wedge X^{I_n} \quad 1 \quad 1^{okd} 1 i\sim^{ok} \left(\frac{|J_1|+ \quad +|J_k|}{X^{J_1} \quad X^{J_k}} \right)! ;$$

(174)
$$ {}_{E} = \sum_{n,k\ 0} \sum_{\substack{e^0\\m}} \frac{1 i\sim \mathrm{oloops}^1\ {}^{e^0}_{m} \mathrm{o}}{|\mathrm{Aut}^1\ {}^{e^0}_{m} \mathrm{o}|} {}^{1}_{@_2} \quad {}^{e^0}_{m}\ {}^{I_1:::I_n}_{J_1:::J_k} \wedge E_{I_1} \wedge \quad \wedge E_{I_n} \quad 1 \quad 1^{okd} 1 i\sim^{ok} \left(\frac{|J_1|+ \quad +|J_k|}{E_{J_1} \quad E_{J_k}} \right)! ;$$

where ${}_{\substack{e^0\\m}}$ *is given similarly as in Definition 0.1.4.25, with the di erence that we place at the marked edge, and*

(175)
$$\% = \sum_{\substack{e\\m}\ \text{marked and connected graph}} \% _{\substack{e\\m}} = \sum_{\substack{e\\m}\ \text{marked and connected graph}} \frac{1}{C_{\substack{e\\m}}} !^{e}_{\substack{e\\m}} 2 \ {}^{1}\mathcal{H}_{tot}{}^{o};$$

where ${}^{e}_{m}$ *denotes a marked connnected graph with edge* e *labeled by . and* $!^{e}_{\substack{e\\m}}$ *is the form constructed with the usual Feynman rules where we place at the marked edge* e*.*

Proof. Let us consider first. Let ${}_{t}$ be a family of propagators with $\hat{U} = \frac{d}{dt}|_{t=0} \ = d-$ and ${}_{t}$ corresponding family of BFV boundary operators, which we loosely write as ${}_{t} = {}^{0}{}_{@}\ {}^{0}{}_{;t}$. (Here the prime on ^{0} should remember us that we are taking "boundary graphs".) ${}_{0}$ is constructed from the propagators, boundary composite fields and derivatives with respect to composite fields. The only thing that depends on t is the propagator, hence the time derivative satisfies

$$\frac{d}{dt}_{t=0}\ {}_{t} = \sum_{\substack{e^0\\m}} {}^{1}{}_{@}\ {}_{\substack{e^0\\m};t} :$$

Here ${}^{e^0}_{m}$ is a graph with a marked edge e and in ${}_{e^0}$ we evaluate the marked edge to d . Using Stokes' theorem for fiber integration $d_X \ = \ d \ {}_{@}$, we can pull out the de Rham differential on of the integration. This gives the term d_X . The other terms from Stokes' theorem are of three kinds: The first kind are terms where the de Rham differential hits a propagator. These vanishe, because the limiting propagator is closed. The second kind are terms where the de Rham differential hits a boundary field, this corresponds to » ; ${}_{0}$¼. Finally, the boundary terms assemble to » ; ${}_{\mathrm{pert};t=0}$¼, similar to the proof of Lemma 1.3.4.2. This proves Equation (171).

The proof for the derivative of the state works in a similar way. In this case % is given by the sum of all connected Feynman graphs with one marked edge, evaluated using the usual Feynman rules, but placing at the marked edge. Now, observe that

$$\frac{d}{dt} ! \ = \sum_{e 2 E^1 \ o} !^{e};$$

where $!^{e}$ is the form obtained by placing d at the edge e. Again, we integrate by parts using Stokes' theorem. We get eight different types of terms.

(1) First, d can come out of the integral. This corresponds to $d_X {}^1e \quad {}^{o}$ ($^1e \quad {}^{o}$ is precisely given by summing over all graphs (not necessarily connected) with a single marked edge that evaluates to).
(2) Terms where d hits a propagator correspond to the action of $i\sim \ {}_{\gamma}$ on ${}^1e \quad {}^{o}$.
(3) Terms where d hits a boundary field correspond to the action of ${}_{0}$ on ${}^1e \quad {}^{o}$.

(4) Terms where d hits a vertex, they cancel with boundary terms just below:
(5) Boundary terms corresponding to the collapse of two vertices in the bulk with a single edge between them. If this edge is marked, the pushforward over the boundary sphere vanishes[1]. The terms without marked edges cancel out with the terms where d hits a vertex.
(6) Boundary terms where a subgraph with more that two vertices collapses in the bulk, these vanish by assumption.
(7) Boundary terms where a subgraph *without* a marked edge collapses on the boundary. These correspond to the action of $\Omega_{\text{pert};t=0}$ on (e).
(8) Finally, in this case we can have graphs with a marked edge collapsing at the boundary, which correspond to the action of to e.

This completes the proof of Equation (172).

1.4.4. Changing the formal exponential map

We can also change the connection on the graded manifold M used to construct the formal exponential map, which is described in Section 1.1.5 and 1.1.6. From a multivector field Y on M we can construct a functional

$$\mathcal{S}_{;Y} = \sum^{\infty} \frac{1}{k!} \int Y^{i_1,\dots,i_k}(X) \,{}_{i_1} \wedge \cdots \wedge {}_{i_k}. \tag{176}$$

We can do the same construction pointwise for formal vertical multivector fields $\hat{Y}$, yielding

$$\mathcal{S}_{;\hat{Y}} = \sum^{\infty} \frac{1}{k!} \int \hat{Y}^{i_1,\dots,i_k}(x;\hat{X}) \mathsf{b}_{i_1} \wedge \cdots \wedge \mathsf{b}_{i_k}. \tag{177}$$

Moreover, writing $e = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}_{;x}}$, for some Lagrangian submanifold $\mathcal{L}$ of the space of fields, one formally obtains ([16]) that

$$\frac{\mathrm{d}}{\mathrm{d}t} e = (\mathrm{d}_x - i\hbar \Delta_{\mathcal{V}_{;x}}) \int_{\mathcal{L}} e^{\frac{i}{\hbar}\hat{\mathcal{S}}_{;x}} \frac{i}{\hbar}\mathcal{S}_{;C} \tag{178}$$

if is a closed manifold, where $C \in (M; TM \hat{\otimes} \hat{S}T M)$ is a generator of the gauge transformation applied to the formal exponential map; see Section 1.1.5 and 1.1.6. Equation (178) motivates to introduce graphs with one marked vertex, labeled by C, with vertex tensor coming from the formal Taylor expansion of

$$\mathcal{S}_{;C}[X;\mathsf{b}] = \int C^{k}(x;X)\mathsf{b}_k = \sum^{\infty} \int C^{k}_{i_1\dots i_k}(x)\hat{X}^{i_1} \cdots \hat{X}^{i_k}\mathsf{b}_k. \tag{179}$$

Proposition 1.4.4.1 (Change of data: formal exponential map). *Let C be the generator of a gauge transformation of the Grothendieck connection as in Chapter 1.1. Then the family of states (e_t) and the familiy of BFV boundary operators (Ω_t) change by*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Omega_t = \mathrm{d}_x + [\Omega_{t=0}, \ ] \tag{180}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e_t = \nabla_{G}(e_{t=0}) \ \ e_{t=0} \tag{181}$$

[1]The propagators $_t$ are normalised to integrate to 1 over this sphere, hence the integral of must vanish.

where $\Omega = \Omega^{\mathbb{X}} + \Omega^{\mathbb{E}}$ *with*

(182)
$$\Omega^{\mathbb{X}} = \sum_{n,k\geq 0}\sum_{\Gamma^{v_0}_m} \frac{(-i\hbar)^{\mathrm{loops}(\Gamma^{v_0}_m)}}{|\mathrm{Aut}(\Gamma^{v_0}_m)|}\left(\int_{\partial_1} \omega_{\Gamma^{v_0}_m}{}^{J_1\cdots J_k}_{I_1\cdots I_n} \wedge \mathbb{X}^{I_1}\wedge\cdots\wedge \mathbb{X}^{I_n}\right)(-1)^{kd}(-i\hbar)^k \frac{\partial^{|J_1|+\cdots+|J_k|}}{\partial \mathbb{X}^{J_1}\cdots\partial\mathbb{X}^{J_k}};$$

(183)
$$\Omega^{\mathbb{E}} = \sum_{n,k\geq 0}\sum_{\Gamma^{v_0}_m} \frac{(-i\hbar)^{\mathrm{loops}(\Gamma^{v_0}_m)}}{|\mathrm{Aut}(\Gamma^{v_0}_m)|}\left(\int_{\partial_2} \omega_{\Gamma^{v_0}_m}{}^{I_1\cdots I_n}_{J_1\cdots J_k} \wedge \mathbb{E}_{I_1}\wedge\cdots\wedge \mathbb{E}_{I_n}\right)(-1)^{kd}(-i\hbar)^k \frac{\partial^{|J_1|+\cdots+|J_k|}}{\partial \mathbb{E}_{J_1}\cdots\partial\mathbb{E}_{J_k}};$$

where $\omega_{\Gamma^{v_0}_m}$ *is given similarly as in Definition 0.1.4.25, with the difference that we place C at the marked vertex, and*

(184)
$$\varrho = \sum_{\substack{\Gamma^{v}_m \text{ marked}\\ \text{and connected graph}}} \varrho_{\Gamma^{v}_m} = \sum_{\substack{\Gamma^{v}_m \text{ marked}\\ \text{and connected graph}}} \int_{C_{\Gamma^{v}_m}} \omega'_{\Gamma^{v}_m} \in \Omega(\mathcal{H}_{tot});$$

where Γ^{v}_m *denotes a marked connnected graph with vertex* v *labeled by C, i.e. a graph and* $\omega^{v}_{\Gamma^{v}_m}$ *is the form constructed with the usual Feynman rules where we place C at the marked vertex* v.

Proof. If we vary the formal exponential map, the vertex tensors at interaction and R vertices change according to the formulas

(185)
$$\hat{\delta}\mathcal{S}_{\Sigma,x} = L_C\mathcal{S}_{\Sigma,x}$$

(186)
$$\hat{\delta}R = \mathrm{d}_x C + [R, C]$$

Since we have $L_C S_{\Sigma,x} = (S_{\Sigma,C}, S_{\Sigma,x})$ and $S_{[R,C]} = (S_{\Sigma,R}, S_{\Sigma,C})$, in terms of Taylor expansions the time derivatives are obtained by contracting the terms in the Taylor expansion ($R^{i_1\cdots i_k}_{j_1\cdots j_l}$ or $Y^i_{j;i_1\cdots i_k}$) with $C^i_{j;i_1\cdots i_k}$ in all possible ways (plus taking the differential in the case of Y's). Keeping this in mind, the proof proceeds completely analogously to the proofs before: In the term $\nabla_G(e^{\Omega}\varrho)$ on left hand side of (181), the usual cancellations apply. Terms which survive are:

(1) Terms where d_x hits the C vertex.
(2) Boundary terms corresponding to the collapse of a single edge with the C vertex at one endpoint, and an R vertex at the other endpoint.
(3) Boundary terms corresponding to the collapse of a single edge with the C vertex at one endpoint, and an interaction ($\mathcal{S}$) vertex at the other endpoint.
(4) Boundary terms where a subgraph containing the C vertex collapses.

The first and the second type of terms yield the time derivative of R vertices. The third type of terms yield time the derivative of an interaction vertex. Finally, the last type of terms yield the action of ϱ on e^{Ω}. This completes the proof of (181). The proof of (180) is entirely the same, using the method of the proof of (171).

Remark 1.4.4.2. In this paper we only considered free boundaries. We could also consider a boundary component $\partial_{\mathrm{fix}}\Sigma$ where we put boundary condition. As explained in [29] boundary conditions compatible with the BV formalism are Q-invariant Lagrangian submanifolds of the boundary space of fields. As we prove the dQME (and the mdQME) by Stokes' theorem, we have also to take boundary contributions on $\partial_{\mathrm{fix}}\Sigma$ into account. The classical boundary conditions mentioned above make the contributions corresponding to a single bulk point approaching $\partial_{\mathrm{fix}}\Sigma$ vanish. If the terms where two or more bulk points collapsing at $\partial_{\mathrm{fix}}\Sigma$ do not vanish, the theory needs quantum boundary corrections (similarly to what happens in the Landau–Ginzburg model [75, 19, 78]). We will consider this in the case of the Poisson Sigma Model in the second part of this thesis.

Part 2

Globalization of the Poisson Sigma Model on Manifolds with Boundary

Introduction

Symplectic groupoids are an important concept in Poisson and symplectic geometry [105]. A *groupoid* is a small category whose morphisms are invertible. We denote a groupoid by $G \rightrightarrows M$, where M is the set of objects and G the set of morphisms. A *Lie groupoid* is, roughly speaking, a groupoid where M and G are smooth manifolds and all structure maps are smooth. Finally, a *symplectic groupoid* is a Lie groupoid with a symplectic form $\omega \in \Omega^2(G)$ such that the graph of the multiplication is a Lagrangian submanifold of $(G, \omega) \times (G, \omega) \times (G, -\omega)$. The manifold of objects M has an induced Poisson structure uniquely determined by requiring that the source map $G \to M$ is Poisson. A Poisson manifold M that arises this way is called integrable. Not every Poisson manifold is integrable.

The *Poisson Sigma Model*, [93, 92, 73] is a 2-dimensional topological Sigma Model with target a Poisson manifold (P, π). The *reduced phase space* of the Poisson Sigma Model on an interval with target a Poisson manifold P is the source simply connected symplectic groupoid of P if P is integrable and otherwise a topological groupoid arising by singular symplectic reduction [32].

In [48, 24, 25] it was shown that the space of classical boundary fields always has an interesting structure called *relational symplectic groupoid*. A relational symplectic groupoid is, roughly speaking, a groupoid in the "extended category" of symplectic manifolds where morphisms are canonical relations. Recall that a *canonical relation* between two symplectic manifolds (M_1, ω_1) to (M_2, ω_2) is an immersed Lagrangian submanifold of $(M_1, \omega_1) \times (M_2, -\omega_2)$. The main structure of a relational symplectic groupoid $(\mathcal{G}, \omega)$ is then given by an immersed Lagrangian submanifold $\mathcal{L}_1$ of $(\mathcal{G}, \omega)$, which plays the role of unity, and by an immersed Lagrangian submanifold $\mathcal{L}_3$ of $(\mathcal{G}, \omega) \times (\mathcal{G}, \omega) \times (\mathcal{G}, -\omega)$, which plays the role of associative multiplication[1]. (In addition, there is also an antisymplectomorphism $\mathcal{J}$ of $\mathcal{G}$ that plays the role of the inversion map.)

The goal of this part is another step towards deformation quantization of the relational symplectic groupoid through the Poisson Sigma Model, using the BV-BFV formalism for the quantization of gauge theories on manifolds with boundary [37, 38]. This possible application of the BV-BFV formalism was first discussed in [38]. In [43] it was explained how the quantization of the relational symplectic groupoid can be achieved in the case of constant Poisson structures. In Part 1, we generalized the methods of formal geometry [17, 64] used in [16, 43] to describe the perturbative quantization of any nonlinear split AKSZ theory [1], possibly on manifolds with boundary. In that

[1] There is also an immersed Lagrangian submanifold $\mathcal{L}_2 \in (\mathcal{G}, \omega) \times (\mathcal{G}, -\omega)$ representing the identity. The composition of $\mathcal{L}_3$ with $\mathcal{L}_2$ also defines an immersed Lagrangian submanifold of $(\mathcal{G}, \omega) \times (\mathcal{G}, \omega) \times (\mathcal{G}, -\omega)$ that induces an equivalence relation and a quotient space which is precisely the symplectic reduction, so the symplectic groupoid G in case M is integrable.

picture, the quantum state of the Poisson Sigma Model[2] with target P is described by a section e of a certain bundle over P which is closed with respect to an operator r_G:

$$r_G e = 0: \tag{187}$$

In this part we apply the results of Part 1 to the Poisson Sigma Model, and extend them to the more general case when we consider, in addition, boundary pieces with fixed boundary conditions. Typically we allow the different types to occur on different pieces of a single connected component of the boundary of the source manifold . We speak of *alternating boundary conditions* and *corners*. These boundary conditions are required to define the relational symplectic groupoid on boundary fields of the Poisson Sigma Model.

[2]We consider the Poisson Sigma Model as a perturbation around the trivial Poisson structure, so the moduli space of classical solutions on which e is defined is identified with the target P.

The Poisson Sigma Model (PSM)

The Poisson Sigma Model is a 2-dimensional topological field theory, with important relation to deformation quantization, see also Section 2.2.3, and in particular a special case of an AKSZ Sigma Model. It was first considered independently by Ikeda [73] and Schaller–Strobl [93, 92] to understand the relation of certain 2-dimensional gravity models and Yang–Mills theories. The importance of this model was given later by the work of Cattaneo and Felder, where they have shown its relation to Kontsevich's deformation quantization construction [77]. In particular, they have shown that Kontsevich's star product can viewd as the semiclassical expansion of a Feynman path integral using the Poisson Sigma Model on the disk [26]. More properties and constructions for the Poisson Sigma model have been given by Cattaneo and Felder in a series of papers [31, 30, 29, 32].
In this chapter we will review some aspects of the classical Poisson Sigma Model, discuss its BV-BFV extension, and give some insights on its equivariant BV formulation.

2.1.1. Classical Poisson Sigma Model

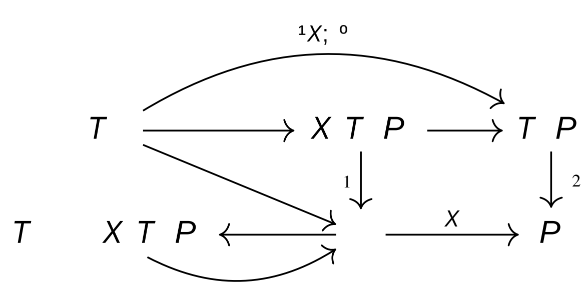
Let us fix a Poisson manifold (P, Π).

Definition 2.1.1.1 (Classical Poisson Sigma Model). The classical Poisson Sigma Model associates to a smooth, oriented, compact and connected 2-manifold Σ (usually called the worldsheet) the space of fields $F_\Sigma = \mathrm{Map}_{\mathrm{VecBun}}(T\Sigma, T^*P)$ of vector bundle maps from $T\Sigma$ to T^*P. An element of F_Σ will be identified with a pair (X, η) where $X\colon \Sigma \to P$ is the base map and $\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*P)$ is a 1-form on Σ with values in X^*T^*P. The Poisson Sigma Model action functional is given by

$$S_\Sigma[(X, \eta)] = \int_\Sigma \langle \eta, dX \rangle + \frac{1}{2}\langle \Pi(X), \eta \wedge \eta \rangle \in C^\infty(F_\Sigma), \tag{188}$$

where $\langle\ ,\ \rangle$ denotes the pairing between vectors and covectors.

We get the following diagram.



REMARK 2.1.1.2. In local coordinates x^i on P, we can write $\eta = \eta_i dx^i$ and $X = (X^1, \dots, X^n)$. Then the action reads

$$S[(X, \eta)] = \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j \in C^\infty(F_\Sigma), \tag{189}$$

where we use the Einstein summation convention.

2.1.2. BV-BFV extension

The Poisson Sigma Model is a *gauge theory*, in the sense that the Lagrangian is invariant under infinitesimal gauge transformations. More precisely there is a distribution on the space of fields which leaves the action invariant and closes on shell, i.e. once the equations of motions are imposed. In particular, the infinitesimal symmetries for the Poisson Sigma Model are given by the following gauge transformations

$$\delta X^i = \pi^{ij}(X)\beta_j, \tag{190}$$

$$\delta \eta_i = -d\beta_i - \partial_i \pi^{jk}(X)\eta_j \beta_k, \tag{191}$$

where β_i is an infinitesimal parameter that is a section of X^*T^*M. If $\partial\Sigma \neq \emptyset$, we also want that β_i vanishes on $\partial\Sigma$ since one wants β to vanish on the boundary.
Because the gauge symmetries only close on shell, the BRST formalism fails, and one needs to revert to the BV formalism [26, 31] on closed surfaces and to the BV-BFV formalism on surfaces with boundary [37, 38].

2.1.2.1. BV extension. The BV extended action and space of fields for the Poisson Sigma Model can be constructed from the AKSZ formalism as discussed in [29].

DEFINITION 2.1.2.1 (BV EXTENDED POISSON SIGMA MODEL). The BV theory associated to the Poisson Sigma Model is given by the triple

$$(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{S}_\Sigma),$$

where the *BV space of fields* is given by

$$\mathcal{F}_\Sigma := \mathrm{Map}_{\mathrm{SupMnf}}(T[1]\Sigma, T^*[1]P) \ni (\mathsf{X}, \boldsymbol{\eta}) \tag{192}$$

with $\mathsf{X}\colon T[1]\Sigma \to P$ a map and $\boldsymbol{\eta}$ a section of $\mathsf{X}^*T^*[1]P$, the *BV action* is given by

$$\mathcal{S}_\Sigma[(\mathsf{X}, \boldsymbol{\eta})] := \int_{T[1]\Sigma} \langle \boldsymbol{\eta}, \mathbf{D}\mathsf{X}\rangle + \frac{1}{2}\langle \pi(\mathsf{X}), \boldsymbol{\eta} \wedge \boldsymbol{\eta}\rangle \in C^\infty(\mathcal{F}_\Sigma), \tag{193}$$

where $\mathbf{D} = \theta^\mu \frac{\partial}{\partial u^\mu}$ is the superdifferential on $T[1]\Sigma$ for local even coordinates (u^μ) on Σ and odd coordinates (θ^μ), and the *BV symplectic form* is given by

$$\omega_\Sigma := \int_{T[1]\Sigma} \delta\mathsf{X} \wedge \delta\boldsymbol{\eta}. \tag{194}$$

One can write out the components of the superfields in terms of fields, antifields and ghosts as follows

$$\mathsf{X}^i = X^i + \theta^\mu \eta^{+,i}_\mu + \frac{1}{2}\theta^\mu\theta^\nu \beta^{+,i}_{\mu\nu}, \tag{195}$$

$$\boldsymbol{\eta}_i = \beta_i + \theta^\mu \eta_{i,\mu} + \frac{1}{2}\theta^\mu\theta^\nu X^+_{i,\mu\nu}, \tag{196}$$

where β denotes the ghost field. For a field Φ we denote by Φ^+ its antifield. Note that we have the relation $\mathrm{gh}(\Phi) + \mathrm{gh}(\Phi^+) = -1$ and $\deg(\Phi) + \deg(\Phi^+) = 2$, where "gh" denotes the *ghost number* which corresponds to the $\mathbb{Z}$-grading on $\mathcal{F}_\Sigma$, and "deg" denotes the form degree. Thus we get

$$\begin{aligned} &\deg(X) = 0; \quad \deg(X^+) = 2; \quad \mathrm{gh}(X) = 0; \quad \mathrm{gh}(X^+) = -1\\ &\deg(\eta) = 1; \quad \deg(\eta^+) = 1; \quad \mathrm{gh}(\eta) = 0; \quad \mathrm{gh}(\eta^+) = -1\\ &\deg(\beta) = 0; \quad \deg(\beta^+) = 2; \quad \mathrm{gh}(\beta) = 1; \quad \mathrm{gh}(\beta^+) = -2 \end{aligned}$$

REMARK 2.1.2.2. In local coordinates on $\mathcal{P}_\Sigma$ we can write

(197) $$\mathcal{S}_\Sigma[(X,\eta)] := \int_\Sigma \left(\eta_i \wedge dX^i + \frac{1}{2}\pi^{ij}(X)\eta_i \wedge \eta_j \right) \in C^\infty(\mathcal{F}_\Sigma).$$

On closed surfaces, this action satisfies the Classical Master Equation

(198) $$(\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) = 0.$$

Here $(\ ,\ )$ is the odd Poisson bracket (BV bracket) associated to the odd symplectic form ω_Σ. One can reformulate the Classical Master Equation in terms of the cohomological vector field as $Q_\Sigma(\mathcal{S}_\Sigma) = 0$ where $Q_\Sigma = (\mathcal{S}_\Sigma, \ )$ in local coordinates on $\mathcal{P}_\Sigma$ is given by

(199) $$Q_\Sigma = \int_\Sigma \left(\left(dX^i + \pi^{ij}(X) \wedge \eta_j\right) \wedge \frac{\delta}{\delta X^i} + \left(d\eta_i + \frac{1}{2}\frac{\partial}{\partial x^i}\pi^{jk}(X)\eta_j \wedge \eta_k\right) \wedge \frac{\delta}{\delta \eta_i} \right) \in \mathfrak{X}(\mathcal{F}_\Sigma).$$

2.1.2.2. Equivariant BV formulation. Consider a Lie algebra $\mathfrak{g}$ acting on Π via a vector field v_X for some $X \in \mathfrak{g}$. Note that the cohomological vector field is given by

(200) $$Q_\Sigma = d_{\mathcal{F}_\Sigma} + \hat{\pi}_\Sigma = (\mathcal{S}_\Sigma, \ ),$$

where $d_{\mathcal{F}_\Sigma}$ and $\hat{\pi}_\Sigma$ are the Hamiltonian vector fields for the Hamiltonians

(201) $$\mathcal{S}_{0,\Sigma}[(X,\eta)] := \int_\Sigma \langle \eta, dX \rangle \quad \text{and} \quad \mathcal{S}_{\pi,\Sigma}[(X,\eta)] := \frac{1}{2}\int_\Sigma \langle \pi(X), \eta \wedge \eta \rangle$$

respectively. Then one can check that the Classical Master Equation $Q_\Sigma(\mathcal{S}_\Sigma) = (\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) = 0$ holds. Consider some variable u of cohomological degree 2 and define a $\mathfrak{g}$-DG algebra $C^\infty(\mathcal{F}_\Sigma)[u] := C^\infty(\mathcal{F}_\Sigma) \otimes S\mathfrak{g}^*[u]$. We can define the equivariant extension of the BV action in the Cartan model as

(202) $$\mathcal{S}^C_\Sigma = \mathcal{S}_\Sigma + uS_{v_X},$$

for $X \in \mathfrak{g}$. Choosing a basis (e_j) of $\mathfrak{g}$, we get

(203) $$\mathcal{S}^C_\Sigma = \mathcal{S}_\Sigma + u^j \mathcal{S}_{\hat{v}_j},$$

where $\mathcal{S}_{\hat{v}_j}$ is the Hamiltonian of $\hat{v}_j$ which is the vector field on $\mathcal{F}_\Sigma$ obtained from the vector field v_j, such that

(204) $$Q^C_\Sigma = (\mathcal{S}^C_\Sigma, \ ) = d_{\mathcal{F}_\Sigma} + \hat{\pi}_\Sigma - u^j \hat{v}_j,$$

is the differential of the Cartan model of equivariant cohomology. Hence $\mathcal{S}^C_\Sigma \in C^\infty(\mathcal{F}_\Sigma)[u]^{\mathfrak{g}}$. Moreover, the Classical Master Equation extends to the *equivariant Classical Master Equation*

(205) $$\frac{1}{2}(\mathcal{S}^C_\Sigma, \mathcal{S}^C_\Sigma) + u^j \mathcal{S}_{\hat{v}_j} = 0,$$

where $\mathcal{S}_{L_{v_j}}$ is the Hamiltonian of the vector field $\hat{L}_{v_j}$ which is the vector field on $\mathcal{F}_\Sigma$ defined by L_{v_j}. The *equivariant Quantum Master Equation* is then given by

(206) $$u^j\left(\mathcal{S}_{\hat{L}_{v_j}} + i\hbar\Delta\mathcal{S}_{\hat{L}_{v_j}}\right) - i\hbar\Delta\mathcal{S} = 0.$$

For the case where Σ is the disk $D \subset \mathbb{R}^2$ we have an S^1-action and hence we can consider the S^1-equivariant theory. We will come back to this construction for Part 3. For more details on the equivariant BV construction see [15].

2.1.2.3. Multivector field extension. Let us consider for a multivector field $\gamma_k \in \Gamma\left(\bigwedge^k TM\right)$ the local functional[1]

(207) $$\mathcal{S}_{\gamma}[(X,\eta)] := \frac{1}{k!}\int_\Sigma \gamma^{i_1\dots i_k}(X)\,\eta_{i_1}\wedge\dots\wedge\eta_{i_k} \in C^\infty(\mathcal{F}_\Sigma).$$

Note that for any $k \geq 0$ we have $Q(\mathcal{S}_\gamma) = (\mathcal{S}, \mathcal{S}_\gamma) = 0$. If we denote by $\gamma^\varphi := T\varphi\,\gamma$, we can observe $T\varphi\,\mathcal{S}_\gamma = \mathcal{S}_{\gamma^\varphi}$. The Quantum Master equation is not satisfied in general. It can be shown that if π is divergence free (unimodular), the Quantum Master Equation $\Delta\exp\left(\frac{i}{\hbar}\mathcal{S}\right) = 0$ holds. Another case would be if the Euler-characteristic of Σ is zero (e.g. the torus). The choice of a unimodular Poisson structure can be seen as a renormalization procedure. One form of renormalization is to impose that there are no tadpoles (short loops), which results in the fact that

(208) $$\Delta(X(s)\eta(s)) = \sum_j (-1)^{|x^j|} x^j(s)\wedge e_j(s) =: \phi(s), \quad \forall s \in \Sigma,$$

where Δ is the BV Laplacian acting on the coefficients of the residual fields. If we choose a volume form Ω on M, we can define a divergence operator div_Ω and thus a *renormalized BV Laplacian* by setting (see also Appendix C.3)

(209) $$\Delta\mathcal{S}_\gamma = \int_\Sigma (\mathrm{div}_\Omega\gamma)^{i_1\dots i_{k-1}}(X)\,\eta_{i_1}\wedge\dots\wedge\eta_{i_{k-1}}.$$

Note that $\Delta\mathcal{S}_\gamma = 0$ if $\mathrm{div}_\Omega\gamma = 0$.

2.1.2.4. BFV extension. In the BV-BFV formalism the boundary conditions are left unspecified and hence the Classical Master Equation no longer makes sense. However, one can still define the symplectic form ω_Σ by (194), the action by (193) and the vector field Q_Σ by (199).

DEFINITION 2.1.2.3 (BV-BFV EXTENDED POISSON SIGMA MODEL). The BV-BFV theory associated to the Poisson Sigma Model is given by associating to a manifold Σ with boundary $\partial\Sigma$ the BV-BFV manifold

$$\left(\mathcal{F}^\partial_{\partial\Sigma}, \omega^\partial_{\partial\Sigma} = \delta\alpha^\partial_{\partial\Sigma}, \mathcal{S}^\partial_{\partial\Sigma}, Q^\partial_{\partial\Sigma}\right)$$

over the BV manifold $(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{S}_\Sigma)$, where

(210) $\mathcal{F}^\partial_{\partial\Sigma} := \mathrm{Map}(T[1]\partial\Sigma, T^*[1]M)$;

(211) $\omega^\partial_{\partial\Sigma} := \int_{\partial\Sigma}\delta\eta\wedge\delta X \in \Omega^2(\mathcal{F}^\partial_{\partial\Sigma})$;

(212) $Q^\partial_{\partial\Sigma} := \int_{\partial\Sigma}\left(\left(dX^i + \pi^{ij}(X)\eta_j\right)\wedge\frac{\delta}{\delta X^i} + \left(d\eta_i + \frac{1}{2}\frac{\partial}{\partial x^i}\pi^{jk}(X)\eta_j\wedge\eta_k\right)\wedge\frac{\delta}{\delta\eta_i}\right) \in \mathfrak{X}(\mathcal{F}^\partial_{\partial\Sigma})$;

(213) $\mathcal{S}^\partial_{\partial\Sigma} := \int_{\partial\Sigma}\left(\langle\eta, dX\rangle + \frac{1}{2}\langle\pi(X), \eta\wedge\eta\rangle\right) \in C^\infty(\mathcal{F}^\partial_{\partial\Sigma})$;

---

[1] Note that we are secretly using a formal exponential map φ. In particular, γ_k should be $\varphi^*\gamma_k$.

and the map $: \mathcal{F} \to \mathcal{F}^{\partial}_{\partial}$ given by restriction of maps.

As shown in [37], these then satisfy the axioms of a BV-BFV theory[2]:

(214) $$Q^2 = 0,$$

(215) $$(Q) = Q^{\partial}_{\partial},$$

(216) $$_{Q}\,\omega = \mathcal{S} + {}^{\partial}_{\partial}.$$

Moreover, $_{Q^{\partial}_{\partial}}\,\omega^{\partial}_{\partial} = \mathcal{S}^{\partial}_{\partial}$. As in Section 0.1.3, we have the modified Classical Master Equation

$$_{Q}\,_{Q}\,\omega = 2\,\mathcal{S}^{\partial}_{\partial}.$$

[2]This is automatic for theories which admit an AKSZ formulation.

CHAPTER 2.2

Deformation quantization and the Poisson Sigma Model

In this chapter we recollect some aspects of Kontsevich's star product [77, 20, 45], its globalization construction [34, 30, 16, 53], and recall the relation with the Poisson Sigma Model [26, 31].

2.2.1. Kontsevich's formality map on $\mathbb{R}^d$

Kontsevich's formality map is an L_∞ (quasi-iso)morphism (see also Section 3.1.1) from multivector fields $\mathcal{T}_{poly}(\mathbb{R}^d) := \bigoplus \bigwedge T\mathbb{R}^d$ to multidifferential operators $\mathcal{D}_{poly}(\mathbb{R}^d)$ on $\mathbb{R}^d$. As such it consists of a family of maps

$$
\begin{aligned}
\mathcal{U}_n\colon \left(\bigwedge^{k_1} T\mathbb{R}^d\right) \otimes \cdots \otimes \left(\bigwedge^{k_n} T\mathbb{R}^d\right) &\to \mathcal{D}^n_{poly}(\mathbb{R}^d)\\
(\xi_1,\ldots,\xi_n) \mapsto \mathcal{U}_n(\xi_1,\ldots,\xi_n) &:= \sum_{\Gamma\in\mathcal{G}_{n,\ell}} w_\Gamma B_{\Gamma,\xi_1,\ldots,\xi_n},
\end{aligned} \tag{217}
$$

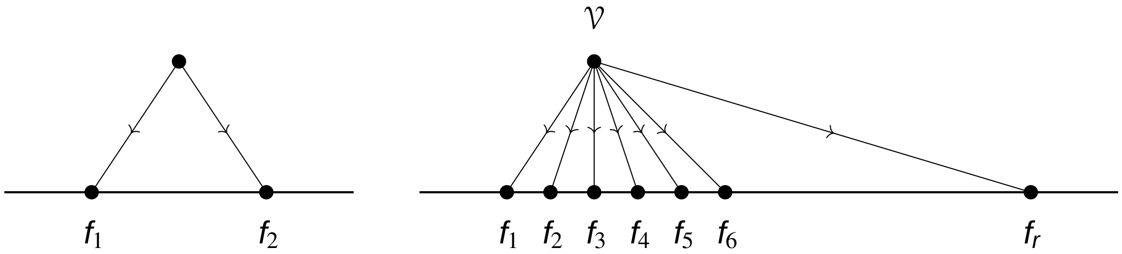
where $\mathcal{G}_{n,\ell}$ is the set of graphs with $n+\ell$ numbered vertices, with $\ell := 2-2n+\sum_{i=1}^n k_i$, such that the jth vertex for $1\le j\le n$ emanates exactly k_j arrows (without short loops). Here k_i represents the degree of the multivector field ξ_i. Note that $\mathcal{U}_n(\xi_1,\ldots,\xi_n)$ acts on ℓ functions. Here $B_{\Gamma,\xi_1,\ldots,\xi_n}$ are multidifferential operators, depending a graph Γ and also on the vector fields $\xi_1,\ldots,\xi_n$, and the w_Γ are weights corresponding to a graph Γ as in [77]. For a vector field ξ (i.e. ξ is of degree 1) and a bivector field π (i.e. π is of degree 2) we can define

$$P(\pi) := \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\mathcal{U}_j(\pi,\ldots,\pi), \tag{218}$$

$$A(\xi,\pi) := \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\mathcal{U}_{j+1}(\xi,\pi,\ldots,\pi), \tag{219}$$

$$F(\xi_1,\xi_2,\pi) := \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!}\mathcal{U}_{j+2}(\xi_1,\xi_2,\pi,\ldots,\pi), \tag{220}$$

which are formal power series in a formal parameter ε. We have chosen the letters in this way, because later we will think of P to be Kontsevich's star product for π a given Poisson tensor, A as a connection 1-form and F as its curvature. Let us take a look at some of the graphs appearing for some chosen multivector fields. For example, for a bivector field π, we get that the term $\mathcal{U}_1(\pi)$ corresponds to the first graph of Figure 2.2.1, whereas for a multivector field $\mathcal{V}$ of degree r we get for $\mathcal{U}_1(\mathcal{V})$ the second graph of Figure 2.2.1. Let now ξ be a vector field. Note that the number ℓ for $\mathcal{U}_n(\xi,\pi,\ldots,\pi)$ will always be 1 for every n, which implies that $A(\xi,\pi)$ takes a smooth map f as an argument.

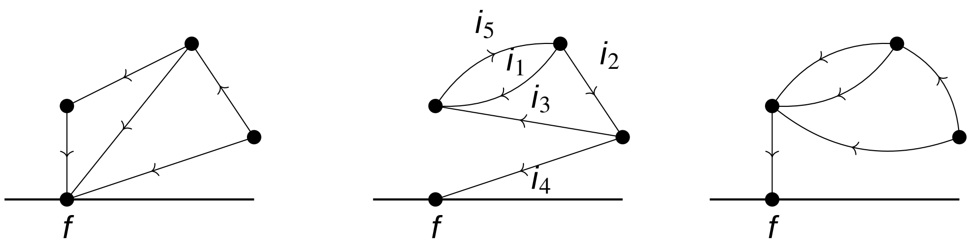


(a) Graph corresponding to a bivector field

(b) Graph corresponding to a multivector field $\mathcal{V}$ of degree r, where $f_1, \dots, f_r \in C^\infty(M)$

FIGURE 2.2.1. The graphs $\mathcal{U}_1(\pi)$ and $\mathcal{U}_1(\mathcal{V})$.

We want to look at graphs appearing for higher terms in A. We can, e.g., consider the $n = 3$ term, i.e. $\mathcal{U}_3(\pi, \pi, \pi)$. Some example of graphs in $\mathcal{G}_{3,1}$, which are taken in account for the sum, are given in Figure 2.2.2.



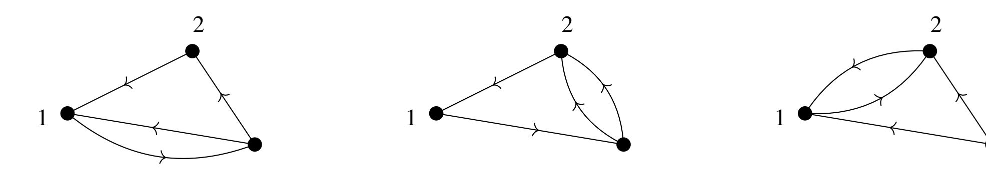
(a) (b) (c)

FIGURE 2.2.2. Example of graphs in $\mathcal{G}_{3,1}$.

We can also explicitly say what the differential operator given by a graph will be. E.g. for the graph as in Figure 2.2.2 (b) we get

$$\partial_{i_1}\partial_{i_3}\pi^{i_5}\partial_{i_2}\partial_{i_2}\pi^{i_3 i_4}\partial_{i_5}\pi^{i_1 i_2}\partial_{i_4}(f). \tag{221}$$

By definition of F, for every n we get that $\ell = 0$, i.e. the image of $\mathcal{U}_n$ will be a differential operator of degree zero, which is a smooth function. Some examples for graphs in $\mathcal{G}_{3,0}$ are given in Figure 2.2.3.



(a) (b) (c)

FIGURE 2.2.3. Example of graphs in $\mathcal{G}_{3,0}$.

2.2.2. Globalization

2.2.2.1. Notions of formal geometry. We recall the most important notions of formal geometry as in Chapter 1.1. For a smooth manifold P we can consider a formal exponential map φ on P, such that for $x \in P$ we have $\varphi_x \colon T_xP \to P$, and we define a vector field $R \in \Gamma(T^*P \otimes TP \otimes \hat{S}T^*P)$, which is a 1-form with values in derivations of $\hat{S}T^*P$. Recall that $\hat{S}$ denotes the completed symmetric algebra and that in local coordinates we have $R = R_i \mathrm{d}x^i$ with

$$R_i(x;y) = -\left[\left(\frac{\partial \varphi_x}{\partial y}\right)^{-1}\right]^k_j \frac{\partial \varphi_x^j}{\partial x^i}\frac{\partial}{\partial y^k} =: Y_i^k(x;y)\frac{\partial}{\partial y^k} \tag{222}$$

Moreover, recall the classical Grothendieck connection $D_G := \mathrm{d} + L_R$, which is a flat connection on $\hat{S}T^*M$.

2.2.2.2. Generalization to Poisson manifolds. Now let us describe how to generalize the above procedure to an arbitrary Poisson manifold (P,π). Namely, let $x \in P$, and φ a formal exponential map on P. Then $\mathcal{T}\varphi_x^*\pi$, the Taylor expansion of π around x defined using φ, is a Poisson tensor on $\hat{S}T_xP$. Any choice of coordinates on T_xM now allows us to identify $\hat{S}T_xP \cong \mathbb{R}[[y_1,\dots,y_d]]$ and define Kontsevich's star product $P(\mathcal{T}\varphi_x^*\pi)$. See [34] for a discussion of the equivariance of this construction in the choice of coordinates. In this way we get a new bundle $\mathcal{E} := \hat{S}T^*P[[\varepsilon]]$ of $\star$-algebras for some formal parameter[1] ε. One can use the Grothendieck connection defined in Chapter 1.1 to give a description of a subalgebra $\mathcal{A} \subset \Gamma(\mathcal{E})$ which is a deformation quantization of $C^\infty(P)$ seen as a subalgebra of $\Gamma(\mathcal{E})$. Formally we have

$$\Gamma(\mathcal{E}) \supset C^\infty(P) \xrightarrow{\text{Deformation Quantization}} \mathcal{A} \subset \Gamma(\mathcal{E}). \tag{223}$$

The algebra $\mathcal{A}$ is given by closed sections under a deformation of the Grothendieck connection, which is defined in two steps: For a tangent vector $\xi \in T_xP$, we let

$$\mathcal{D}_G := \xi + A(\hat{\xi}, \mathcal{T}\varphi_x^*\pi) = D_G + O(\varepsilon), \tag{224}$$

where again we denote by $\mathcal{T}\varphi_x^*\pi$ the Poisson tensor π lifted to a formal neighborhood and $\hat{\xi}$ is defined as in (106). One can write

$$\mathcal{D}_G = \mathrm{d} + A(R, \mathcal{T}\varphi^*\pi) \tag{225}$$

interpreting $A(R, \mathcal{T}\varphi^*\pi)$ as a one-form valued in differential operators on $\mathcal{E}$. At some point $x \in P$, in coordinates x^i around x, it is given by

$$A(R, \mathcal{T}\varphi_x^*\pi) = \mathrm{d}x^i A(R_i(x;y), \mathcal{T}\varphi_x^*\pi) = \mathrm{d}x^i A\left(Y_i^k(x;y)\frac{\partial}{\partial y^k}, \mathcal{T}\varphi_x^*\pi\right). \tag{226}$$

One can then show [34] that $\mathcal{D}_G$ is a globally defined connection on $\Gamma(\mathcal{E})$, a derivation, and that $(\mathcal{D}_G)^2$ is an inner derivation, i.e.

$$(\mathcal{D}_G)^2 = [F^P, \lambda]_\star := F^P \star \lambda - \lambda \star F^P, \tag{227}$$

for any $\lambda \in \Gamma(\mathcal{E})$, where F^P is the *Weyl curvature* tensor of $\mathcal{D}_G$ given by $F^P(\xi_1,\xi_2) := F(\hat{\xi}_1, \hat{\xi}_2, \mathcal{T}\varphi^*\pi)$, where $\xi_1, \xi_2 \in T_xP$ are two tangent vectors on P. More, precisely, F^P is a 2-form valued in sections of $\mathcal{E}$ which in local coordinates can be expressed as

$$F^P_x = \mathrm{d}x^i \wedge \mathrm{d}x^j F(R_i(x;y), R_j(x;y), \mathcal{T}\varphi_x^*\pi). \tag{228}$$

[1] In quantum field theory we use $\varepsilon = (i\hbar)/2$.

For the Weyl tensor we get $\mathcal{D}_G F^P = 0$. The task is to modify the globalized connection $\mathcal{D}_G$ slightly more, so that it becomes flat but still remaining a derivation. One can set[2]

$$\overline{\mathcal{D}}_G := \mathcal{D}_G + [\gamma, \cdot]_\star, \tag{229}$$

and observe that for any 1-form $\gamma \in \Omega^1(P, \mathcal{E})$ this connection is a derivation. Moreover, its Weyl curvature tensor is then given by

$$\overline{F}^P = F^P + \mathcal{D}_G \gamma + \gamma \star \gamma. \tag{230}$$

We call (224) the *deformed Grothendieck connection* and (229) the *modified deformed Grothendieck connection*. One then needs to find $\gamma \in \Omega^1(P, \mathcal{E})$ such that $\overline{F}^P = 0$, which implies that $(\overline{\mathcal{D}}_G)^2 = 0$, so that $\overline{\mathcal{D}}_G$-closed sections will form the algebra $\mathcal{A}$ as a deformation quantization of $C^\infty(P)$. If we compute $(\overline{\mathcal{D}}_G)^2$ explicitly, by using (229) we get

$$(\overline{\mathcal{D}}_G)^2 = \underbrace{(\mathcal{D}_G)^2}_{:=[F^P, \cdot]_\star} + \mathcal{D}_G[\gamma, \cdot]_\star + [\gamma, [\gamma, \cdot]_\star]_\star. \tag{231}$$

More precisely, γ has to satisfy

$$F^P + \mathcal{D}_G \gamma + \gamma \star \gamma = 0. \tag{232}$$

The existence of such a γ was shown in [30, 34] by homological perturbation theory. One can actually construct γ to be a solution of the more general equation given by

$$\overline{F}^P_\omega := F^P + \varepsilon \omega + \mathcal{D}_G \gamma + \gamma \star \gamma = 0, \tag{233}$$

where $\omega \in \Omega^2(P, \mathcal{E})$ such that $\mathcal{D}_G \omega = 0$ and $[\omega, \cdot]_\star = 0$ [34].

Remark 2.2.2.1. We will say that a connection is *compatible* if its extension to differential forms $\Omega(M, \mathcal{E})$ is a derivation of degree $+1$ with respect to the fiber wise star product on $\mathcal{E}$. A compatible connection on $\Omega(\mathcal{E})$ is called a *Fedosov connection* if it is an inner derivation with respect to its Weyl curvature tensor and it satisfies the Bianchi identity. By the constructions above, the deformed Grothendieck connection $\mathcal{D}_G$ is a Fedosov connection as well as any symplectic connection ∇ on the tangent bundle of a symplectic manifold as in Fedosov's construction. Note that, as we have seen, if $\mathcal{D}_G$ is a Fedosov connection, then $\overline{\mathcal{D}}_G = \mathcal{D}_G + [\gamma, \cdot]_\star$ is also a Fedosov connection.

2.2.2.3. Grothendieck connection on symplectic manifolds. Let (M, ω) be a symplectic manifold which can be considered as a special case of a Poisson manifold with Poisson structure coming from the symplectic form. Moreover, let $\mathcal{E} := \hat{S}T^*M[[\varepsilon]]$. By Darboux's theorem, we consider a constant symplectic form $\omega^\varphi := \varphi_x^* \omega$ lifted to the formal construction for any $x \in M$. Note that in this case R is a 1-form on M with values in formal Hamiltonian vector fields[3] for the corresponding Hamiltonian functions h_x such that $h_x|_{y=0} = 0$. For any $x \in M$, h_x is a 1-form with values in $\hat{S}T^*M$ and for any section $\sigma \in \Omega(\mathcal{E})$ we get $\mathcal{D}_G \sigma_x = d_x \sigma_x + \frac{1}{2\varepsilon}[h_x, \sigma_x]_\star$. For the Weyl curvature tensor we get $F_x^M(\xi, \zeta) = \frac{1}{4\varepsilon^2}([\langle h_x, \xi\rangle, \langle h_x, \zeta\rangle]_\star - 2\varepsilon\{\langle h_x, \xi\rangle, \langle h_x, \zeta\rangle\})$. Consider a symplectic connection ∇ on TM, which induces a connection $\mathcal{D}_G$ on $\Omega(\mathcal{E})$, which acts as a derivation on each fiber of $\mathcal{E}$. Its curvature is then given by $(\mathcal{D}_G)^2 = \frac{1}{2\varepsilon}[F^M, \cdot]$ with $F^M \in \Omega^2(M, \mathcal{E})$ given

[2] For any two $\mathcal{E}$-valued 1-forms $\alpha = \alpha_i dx^i$, $\beta = \beta_j dx^j \in \Omega^1(P, \mathcal{E})$ one defines their star product by $\alpha \star \beta := (\alpha_i \star \beta_j) dx^i \wedge dx^j$.

[3] Recall that for a function f one can construct a unique vector field X_f, such that $\iota_{X_f}\omega = -df$; the vector field X_f is called the Hamiltonian vector field of f and f is called the Hamiltonian function of X_f.

by $F^M = \frac{1}{2}\omega(\nabla^2 y, y)$ is the quadratic form on TM associated to the curvature ∇^2 of ∇. Fedosov showed [55] that for a closed 2-form $\omega_\varepsilon = \frac{1}{2\varepsilon}\omega + \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots \in \Omega^2(M;\mathbb{R})[[\varepsilon]]$ the equation

(234) $$\nabla^2 + \nabla\gamma_\varepsilon + \gamma_\varepsilon \star \gamma_\varepsilon = \omega_\varepsilon$$

has a solution $\gamma_\varepsilon = \frac{1}{2\varepsilon}\omega_{ij}y^i dx^j + \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \dots \in \Omega^1(M;\mathcal{E})$ such that $\gamma_\varepsilon|_{y=0} = 0$. Moreover, we consider the formal exponential map coming from the symplectic connection ∇

(235) $$\varphi^i_x(y) = x^i + y^i + \frac{1}{2}\sum_{k,\ell}\Gamma^i_{k\ell}y^k y^\ell + \dots$$

Then the connection $\nabla + [\gamma_\varepsilon, \ ]_\star$ is given by $\overline{\mathcal{D}}_G = \mathcal{D}_G + [\gamma, \ ]_\star$ with $\gamma_\varepsilon = \frac{1}{2\varepsilon}h_x + \gamma$, where γ is a solution of (233) with $\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$. The star product constructed in this way, using a closed two form $\omega_\varepsilon \in \Omega^2(M;\mathbb{R})[[\varepsilon]]$, is equivalent to the one constructed by Fedosov associated to the class $\frac{1}{2\varepsilon}\omega + \omega_\varepsilon$. Note that the deformations of the symplectic form are in one-to-one correspondence with their characteristic classes, which are formal power series $\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$, with $\omega_i \in H^2(M;\mathbb{R})$ such that ω_0 is the class of the symplectic form ω. For more details on these constructions see [33].

2.2.2.4. Globalization of Kontsevich's star product. Consider again a Poisson manifold (P,Π) and the deformed bundle $\mathcal{E} := \hat{S}T^*P[[\varepsilon]]$. As already mentioned, the algebra of smooth functions on P is isomorphic to the subalgebra of D_G-closed sections of $\mathcal{E}$. Denote by $H^0_{\overline{\mathcal{D}}_G}(\Gamma(\mathcal{E}))$ the subalgebra of $\Gamma(\mathcal{E})$ consisting of $\overline{\mathcal{D}}_G$-closed sections of $\mathcal{E}$. Since D_G and $\overline{\mathcal{D}}_G$ are flat connections we have natural cochain complexes $(\Gamma(\hat{S}T^*P), D_G)$ and $(\Gamma(\mathcal{E}), \overline{\mathcal{D}}_G)$.

PROPOSITION 2.2.2.2. *The subalgebra $H^0_{\overline{\mathcal{D}}_G}(\Gamma(\mathcal{E}))$ provides a deformation quantization of (P,Π).*

More precisely, we can construct a *cochain map*

(236) $$\tau\colon (\Gamma(\hat{S}T^*P), D_G) \to (\Gamma(\mathcal{E}), \overline{\mathcal{D}}_G),$$

which implies a *quantization map*

(237) $$\tau\colon C^\infty(P) \cong H^0_{D_G}(\Gamma(\hat{S}T^*P)) \to H^0_{\overline{\mathcal{D}}_G}(\Gamma(\mathcal{E})).$$

This map induces an isomorphism $C^\infty(P)[[\varepsilon]] \to H^0_{\overline{\mathcal{D}}_G}(\Gamma(\mathcal{E}))$, since there are no cohomological obstructions. Note that this is the analogue of the symbol map as in Fedosov's quantization. Moreover, there is a unique τ for each $\overline{\mathcal{D}}_G$ such that $\tau|_{y=0} = \mathrm{id}$. Using this map, one can define a global version of Kontsevich's star product, defined on the whole Poisson manifold P by

(238) $$f \star_P g := \left[\tau^{-1}(\tau(T\varphi^* f) \star \tau(T\varphi^* g))\right]_{y=0}.$$

Indeed, the map τ sends D_G-flat sections to $\overline{\mathcal{D}}_G$-flat sections since τ is a cochain map, i.e. we have $\tau \circ D_G = \overline{\mathcal{D}}_G \circ \tau$, and by compatibility with the star product, one can obtain that $J := \tau(T\varphi^* f) \star \tau(T\varphi^* g)$ is again $\overline{\mathcal{D}}_G$-closed because $T\varphi^* f$ is D_G-closed for all $f \in C^\infty(P)[[\varepsilon]]$. But since J is $\overline{\mathcal{D}}_G$-closed, we know that it has to lie in the image of τ. Hence there exists some $j \in \Gamma(\hat{S}T^*P)$ such that $\tau(j) = J$. This implies that j is D_G-closed and thus of the form $j = T\varphi^*\tilde{j}$ for some $\tilde{j} \in C^\infty(P)[[\varepsilon]]$. Setting the formal variables $y = 0$ one finds a global construction for the star product.

This approach generalizes Fedosov's construction for the Moyal product, to the globalization of Kontsevich's star product. It can be translated into field theoretic concepts using the Grothendieck connection together with the Poisson Sigma Model as we will see.
We want to look at two important examples of Poisson structures.

2.2.2.5. Constant Poisson structure. The situation of a constant Poisson structure is a first example to think about. Let (P, Π) be a Poisson manifold with constant Poisson structure Π and $\xi \in T_x P$ for $x \in P$ be a fixed tangent vector. By the definition of A, and the fact that each vertex has only one outgoing and no incoming arrow, we get $A(\hat{\xi}, T\varphi) = \hat{\xi}$, which leads to the fact that

$$\mathcal{D}_{\mathsf{G}} = (\delta + \hat{\xi}) = D_{\mathsf{G}}. \tag{239}$$

Therefore we get $(\mathcal{D}_{\mathsf{G}})^2 = 0$ and thus $F^P_M = 0$. We can then choose $\gamma = 0$.

2.2.2.6. Linear Poisson structure. Let now $(P = \mathfrak{g}^*, \Pi)$ be a Poisson manifold with linear Poisson structure $\Pi(x) = f^{ij}_k x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, where f^{ij}_k represent the structure constants of a Lie algebra $\mathfrak{g}$, and $\xi \in T_x P$ for $x \in P$ be a fixed tangent vector. As in the constant case, we observe that $A(\hat{\xi}, T\varphi) = \hat{\xi}$, which is the case since the integral of a bulk vertex with one incoming and one outgoing arrow is zero, and since there is at most one incoming arrow for each vertex. Again we may choose $\gamma = 0$.

2.2.3. Connection to the Poisson Sigma Model

In [26] and [31] it was shown that Kontsevich's formality map on $\mathbb{R}^d$ can be intepreted as the perturbative computation of expectation values of observables of the Poisson Sigma Model on the upper half-plane (or respectively the disk) with values in $\mathbb{R}^d$. The graphs which appear in the construction of Kontsevich's star product on Poisson manifolds [77] are given on the upper half-plane, where they can collapse, according to the boundary of the configuration space, on the boundary of the upper half-plane. This means that the graphs that appear in the Poisson Sigma Model are exactly the graphs that appear for Kontsevich's star product. More precisely, if one considers the disk $\mathbb{D}$ in $\mathbb{R}^2$ and the classical action of the Poisson Sigma Model on $\mathbb{D}$ given by $S_{\mathbb{D}}[(X, \eta)] = \int_{\mathbb{D}} \langle \eta, dX \rangle + \frac{1}{2}\langle \Pi(X), \eta \wedge \eta \rangle$, we can asymptotically write Kontsevich's star product for two smooth maps f and g as a perturbative expansion of the following path integral:

$$f \star g(x) = \int_{X(\infty)=x} f(X(0))\, g(X(1))\, e^{\frac{i}{\hbar} S_{\mathbb{D}}[(X,\eta)]}, \tag{240}$$

where $0, 1, \infty$ represent some marked points on the boundary of $\mathbb{D}$ (see Figure 2.2.4). Note that $x \in \mathrm{Map}(\mathbb{D}, \mathbb{R}^d)$ is a constant map, i.e. the we get a local representation of the star product. If one considers a general Poisson manifold (P, Π), one can consider the constant map $x \in \mathrm{Map}(\mathbb{D}, P)$ as a point sitting in P giving a local product on each fiber. As already described in Section 2.2.2, one can then algebraically construct the star product on all of P.

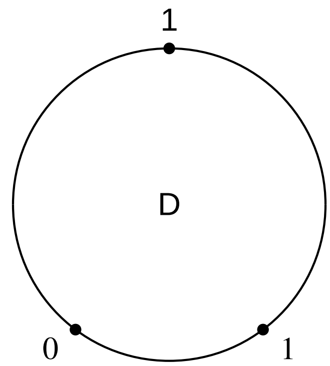


FIGURE 2.2.4. Cyclically ordered points on $S^1 = \partial D$

CHAPTER 2.3

Globalized BV-BFV quantization of the Poisson Sigma Model

In this chapter, we analyze in detail the construction explained in the first part of the thesis applied to the Poisson Sigma Model. In particular, we want to describe the BFV boundary operator Ω_∂ for the Poisson Sigma Model in the case of a worldsheet where we have a single boundary component endowed with a certain polarization (see Figure 2.3.1). As a preparation for the remainder of the part, we also discuss how the boundary operator behaves under certain modifications of the formal globalized action.

2.3.1. Globalization at the classical level

We consider the Poisson Sigma Model action as a perturbation of the quadratic part of the action,

$$\mathcal{S}_{0,\Sigma} = \int_\Sigma \langle \eta, dX \rangle. \tag{241}$$

Recall that we expand around critical points of $\mathcal{S}_{0,\Sigma}$, which in particular satisfy $dX = 0$. Hence the ghost number 0 component of X is a constant map, which we denote by its image $x \in P$. As discussed in [30, 16, 43] and Appendix F of [38], it makes sense to perform perturbative quantization around points in the moduli space of classical solutions. Since the Euler–Lagrange equations for the Poisson Sigma Model are given by $dX + \pi^\sharp(X)\eta = 0$, we will perturb around the classical solution $X = x = const.$ and $\eta = 0$ and gauge equivalent solutions. Hence for the Poisson Sigma Model the appropriate moduli space is given by

$$\mathcal{M}_0 = \{(x, 0) \mid x \text{ const map to } P\} \cong P. \tag{242}$$

In this special case we have $\mathcal{M}_0 \subset \mathcal{F}_\Sigma$. Instead of fixing a single classical solution $x \in \mathcal{M}_0$ and expanding around it, we want to vary x itself. As in the first part of the thesis we consider the fields $\hat{X}$ and $\hat\eta$ given by

$$X = \varphi_x(\hat{X}), \qquad \eta = (d\varphi_x)^{*,-1}\hat\eta. \tag{243}$$

We get a *formally globalized action* for the Poisson Sigma Model as in Chapter 1.2 by

$$\tilde{\mathcal{S}}_{\Sigma,x}[(\hat{X},\hat\eta)] = \underbrace{\int_\Sigma \hat\eta_i \wedge d\hat{X}^i}_{=:\hat{\mathcal{S}}_{0,\Sigma}} + \underbrace{\frac{1}{2}\int_\Sigma T\varphi_x^*\pi^{ij}(\hat{X})\hat\eta_i \wedge \hat\eta_j}_{=:\hat{\mathcal{S}}_{\Sigma,\pi,x}} + \underbrace{\int_\Sigma Y_i^j(x;\hat{X})\hat\eta_j \wedge d_{\mathcal{M}_0}x^i}_{=:\hat{\mathcal{S}}_{\Sigma,x,R}}, \tag{244}$$

where we denote by $d_{\mathcal{M}_0}$ and d_Σ the de Rham differentials on $\mathcal{M}_0$ and Σ respectively (we only write it once and leave out the indication every time it is clear).

2.3.2. The boundary BFV operator

In this section we want to see how Ω_∂ is constructed for a formal linearized action but without any globalization term, i.e. for $\hat{\mathcal{S}}_{\Sigma,x} = \hat{\mathcal{S}}_{0,\Sigma} + \hat{\mathcal{S}}_{\Sigma,\pi,x}$ in the notation of Equation (244). We can formulate the boundary operator Ω_∂ for the Poisson Sigma Model by the usual construction of

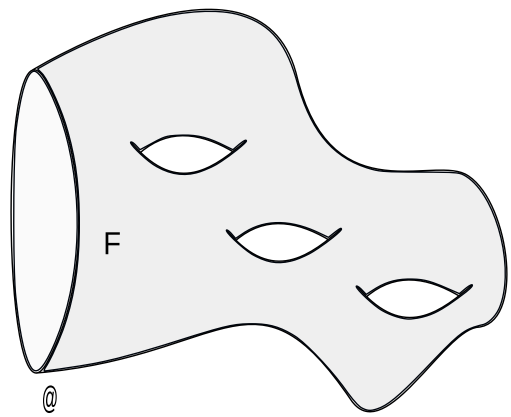


FIGURE 2.3.1. Example of a higher genus worldsheet with one connected boundary component and different polarization. Here F denotes either X or E.

the collapsing of subgraphs Γ' using Definition 0.1.4.25 for the non-globalized theory. We briefly review the results of [38, Section 4.8], where the boundary operator of the non-globalized theory was computed. Recall the splitting of the space of fields as in (61)

$$\begin{aligned} \mathcal{F} &\to \mathcal{B}^{\mathcal{P}}_{\partial} \times \mathcal{V}^{\mathcal{P}} \times \mathcal{Y}' \\ (X, \eta) &\mapsto (\mathsf{X}, \mathsf{E}) + (x, e) + (\mathcal{X}, \mathcal{E}). \end{aligned} \tag{245}$$

We now describe the BFV boundary operator for the different representations[1].

2.3.2.1. E-representation. We look first at the E-representation.

PROPOSITION 2.3.2.1. *In the E-representation, the boundary operator is given by*

$$\Omega^{\mathsf{E}}_{\partial} = \Omega^{\mathsf{E}}_{0} + \Omega^{\mathsf{E}}_{\text{pert}}, \tag{246}$$

where

$$\Omega^{\mathsf{E}}_{\text{pert}} = \int_{\partial} \sum_{I,J,K,R,S} \frac{(-i\hbar)^{|K|-|I|-|J|+1}}{(|K|+|R|+|S|)!} \partial_K B^{IJ}(T\varphi_x)[\mathsf{E}_I\mathsf{E}_R][\mathsf{E}_J\mathsf{E}_S]\frac{\delta^{|K|+|R|+|S|}}{\delta[\mathsf{E}_K]\,\delta[\mathsf{E}_R]\,\delta[\mathsf{E}_S]}, \tag{247}$$

where the B^{IJ}s are defined as the coefficients in the star product on $C^\infty(\mathbb{R}^n)[[\hbar]]$ by

$$f \star g = fg + \sum_{I,J} B^{IJ}\frac{\partial^{|I|}f}{\partial x^I}\frac{\partial^{|J|}g}{\partial x^J} = fg - \frac{i\hbar}{2}\sum_{ij}(T\varphi_x)\pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} + O(\hbar^2), \tag{248}$$

where I, J are multi-indices and i and j are indices and $B^{IJ} = 0$ if $|I| = 0$ or $|J| = 0$, and $\star$ denotes Kontsevich's star product [77].

PROOF. Consider a graph Γ' with n bulk vertices and k boundary vertices collapsing on the E-boundary. Note that we have $\dim \mathsf{C}_{\Gamma'}(\mathbb{H}^d) = 2n + k - 2$, which has to be the same as the form degree of $\omega_{\Gamma'}$ so that the integral

$$\sigma_{\Gamma'} = \int_{\mathsf{C}_{\Gamma'}(\mathbb{H}^d)} \omega_{\Gamma'} \tag{249}$$

does not vanish.

[1] We call the $\frac{\delta}{\delta\mathsf{E}}$-polarization the X-representation and vice versa.

Thus we need to have $2n + k - 2 = 2n$, since n is the number of points in the bulk which represent the Poisson tensor, i.e. emitting two arrows that have to remain inside the collapsing subgraph (otherwise the contribution vanishes by the boundary condition on the propagator). Hence we get $k = 2$, i.e. the graph has exactly two boundary vertices. We label one boundary vertex by u_0 and the other one by u_1. Let L be a multiindex labeling the inward leaves of Γ'. We decompose L as $L = (R, K, S)$, where R, K, S are again multiindices, representing different types of inward leaves. R labels the leaves arriving directly at u_0, S labels the leaves arriving directly to u_1 and K labels the leaves arriving at some bulk vertices of Γ'. Moreover, we label by the multiindex I the arrows arriving at u_0 from some bulk vertices of Γ' and by the multiindex J the arrows arriving at u_1 from some bulk vertices of Γ' (see Figure 2.3.2). Since we have exactly two boundary vertices ($k = 2$), the graphs when considered without leaves are given by the same graphs as in Kontsevich's star product. If we sum over all graphs having the same multiindices K, I, J, we obtain the K'th derivative of the B^{IJ} coefficient in the star product, since the limiting propagator coincides with Kontsevich's propagator, and hence we get (246).

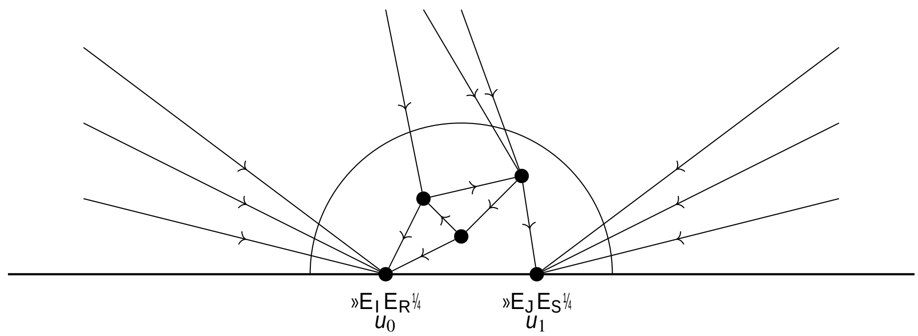


FIGURE 2.3.2. An example of a subgraph collapsing as in the description. Here we have three incoming arrows to the boundary for the collapsing graph Γ' on the right side corresponding to the index S, three incoming arrows to the boundary on the left side corresponding to the index R, three incoming arrows to Γ' corresponding to the index K, two incoming arrows to u_0 from Γ' corresponding to the index I and one incoming arrow to u_1 from Γ' corresponding to the index J.

To analyze the BFV boundary operator, we introduce the notion of certain multiplication operators appearing from collapsing graphs on the boundary endowed with the E-representation. Therefore we give the following definition:

DEFINITION 2.3.2.2 (EXPONENTIAL MULTIPLICATION OPERATOR). The *exponential multiplication operator* for the boundary field E is given by the map

$$e^{\frac{i}{\hbar}\langle E, y\rangle}\colon \mathfrak{H} \to \mathfrak{H}[[y]] \tag{250}$$

$$\mapsto e^{\frac{i}{\hbar}\langle E, y\rangle} := \sum_{k\geq 0}\left(\frac{i}{\hbar}\right)^k \sum_{|I|=k} y^I \left(E_I\right) \tag{251}$$

On the total space $\mathfrak{H}_{\Sigma,\mathrm{tot}}$, the multiplication operator is given by a map

$$\mathfrak{H}_{\Sigma,\mathrm{tot}} \to \mathfrak{H}_{\Sigma,\mathrm{tot}} \,\widehat{\otimes}\, \widehat{S}T P. \tag{252}$$

Remark 2.3.2.3. Note that the exponential multiplication operator takes regular functionals to regular functionals. The construction in [30], recalled in Chapter 2.2, yields a bundle $\mathcal{E} = \widehat{S}T^*\mathcal{P}[[\hbar]]$ of $\star$-algebras on $\mathcal{P}$ by applying Kontsevich's deformation quantization in every tangent space. Thus, we can define a map

$$\star\colon \Gamma(\mathcal{H}_{\partial,\mathrm{tot}}\otimes \widehat{S}T^*\mathcal{P})\otimes \Gamma(\mathcal{H}_{\partial,\mathrm{tot}}\otimes \widehat{S}T^*\mathcal{P})\to \Gamma(\mathcal{H}_{\partial,\mathrm{tot}}\otimes \widehat{S}T^*\mathcal{P}), \tag{253}$$

given by multiplication in $\Gamma(\mathcal{H}_{\partial,\mathrm{tot}})$ and the fiber wise star product in $\widehat{S}T^*\mathcal{P}[[\hbar]]$, i.e. we consider the tensor product of the two algebra bundles $\mathcal{H}_{\partial,\mathrm{tot}}$ and $\widehat{S}T^*\mathcal{P}$ over $\mathbb{C}[[\hbar]]$.

Remark 2.3.2.4. Note that we can define a map from $\Gamma(\mathcal{H}_{\partial,\mathrm{tot}}\otimes \widehat{S}T^*\mathcal{P})$ to the space of operators, by replacing the fiber coordinates y^I by functional derivatives $-i\hbar\frac{\delta}{\delta[\mathbb{E}_I]}$. Thus, if we have a section of $\mathcal{H}_{\partial,\mathrm{tot}}$, we can define the boundary operator $\Omega^{\mathbb{E}}_{\mathrm{pert}}$ by

$$\Omega^{\mathbb{E}}_{\mathrm{pert}} = \left(e^{-\frac{i}{\hbar}[\mathbb{E}]y}\star e^{\frac{i}{\hbar}[\mathbb{E}]y}\right)\Big|_{y=-i\hbar\frac{\delta}{\delta[\mathbb{E}]}}. \tag{254}$$

Then one can check that (246) is given by the standard quantization[2] of the boundary action

$$\mathcal{S}^{\partial}_{\partial} = \int_{\partial}\left(\langle b, d\mathbb{X}\rangle + \frac{1}{2}\left[e^{\frac{i}{\hbar}b}, e^{-\frac{i}{\hbar}b}\right]_{\star}(\mathbb{X})\right), \tag{255}$$

where $\langle\ ,\ \rangle$ denotes the canoncial pairing of $T_x^*\mathcal{P}$ with $T_x\mathcal{P}$, where x is the constant background field $\phi\colon \Sigma\to\mathcal{P}$, $[\ ,\ ]_\star$ is the star commutator, and $\star$ is the star product in $T_x\mathcal{P}$. Note that the interesting part here is that we can view the BFV boundary operator as the standard quantization of a deformed boundary action.

Remark 2.3.2.5. The fact that $\left(\Omega^{\mathbb{E}}_{\partial}\right)^2 = 0$ is equivalent to the associativity of the Kontsevich star product.

2.3.2.2. $\mathbb{X}$-representation. Next, we consider the $\mathbb{X}$-representation.

Proposition 2.3.2.6. *In the $\mathbb{X}$-representation, the boundary operator is given by*

$$\Omega^{\mathbb{X}}_{\partial} = \Omega^{\mathbb{X}}_{0} + \Omega^{\mathbb{X}}_{\mathrm{pert}}, \tag{256}$$

where

$$\Omega^{\mathbb{X}}_{\mathrm{pert}} = \sum_{k=0}^{\infty}\frac{1}{k!}\int_{\partial}\sum_{L,I_1,\ldots,I_k,R_1,\ldots,R_k}\frac{(-i\hbar)^{|L|-(|I_1|+\dots+|I_k|)+1}}{(|L|+|R_1|+\dots+|R_k|)!}\partial^L a_{I_1,\ldots,I_k}(T\varphi_x)\left(\prod_{j=1}^{k}[\mathbb{X}^{I_j}\mathbb{X}^{R_j}]\right)\frac{\delta^{|L|+|R_1|+\dots+|R_k|}}{\delta[\mathbb{X}^L]\,\delta[\mathbb{X}^{R_1}]\cdots\delta[\mathbb{X}^{R_k}]}, \tag{257}$$

where $a_{I_1,\ldots,I_k}$ are given by the sum of the weights over all Feynman graphs with k boundary vertices and $|I_j|$ outgoing arrows for $1\leq j\leq k$.

Proof. In the $\mathbb{X}$-representation there can be arbitrarily many vertices on the boundary, since the arrows emanating from the bulk vertices can now leave the graph. Denote the number of vertices on the boundary by k. Then we have a similar construction as for the $\mathbb{E}$-representation, only with the difference that for each boundary vertex we can have arbitrarily many outgoing arrows either out of the collapsing graph Γ' (left or right) and arbitrarily many outgoing arrows going into Γ'.

[2] Choosing a leaf $b\in\mathcal{B}^{\mathcal{P}}_{\partial}$ one considers it's conjugated momentum $-i\hbar\frac{\delta}{\delta b}$.

Label the vertices u on the boundary by $1,\dots,k$. We denote by L the multiindex labeling the leaves, which emanate from bulk vertices of Γ', by I_j the multiindices labeling the arrows which start at u_j and end at some bulk vertices of Γ' for $1 \leq j \leq k$ and by R_j the multiindices labeling the outward leaves which start at u_j for $1 \leq j \leq k$ (see Figure 2.3.3). Summing over all such graphs Γ', we get (256).

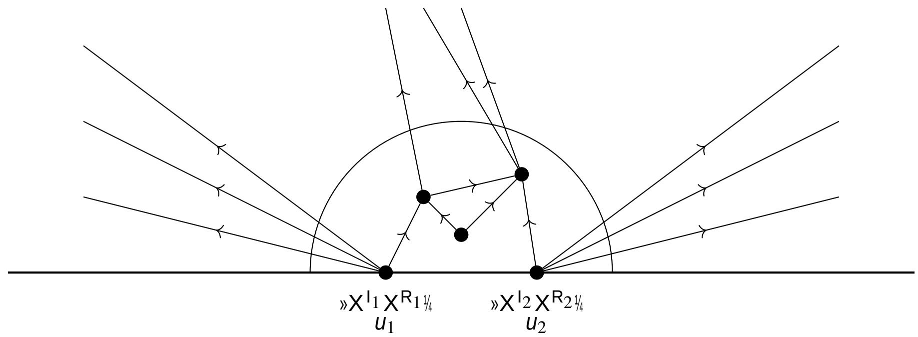


FIGURE 2.3.3. An example of a subgraph collapsing as in the description. We consider a term for $k = 2$ as before and we label them by u_1 and u_2. Here we have three outgoing arrows for the collapsing graph Γ' on the right side corresponding to the index R_1, three outgoing arrows on the left side corresponding to the index R_1, three outgoing arrows to Γ' corresponding to the index L and one incoming to Γ' out of each of the two boundary points corresponding to the indices I_1 and I_2.

REMARK 2.3.2.7. The coefficients $a_{I_1,\dots,I_k}$ can be regarded as the coefficients of an A_∞-algebra [27]. The fact that Ω^X_∂ squares to zero corresponds to the A_∞-relations.

2.3.3. The globalized BFV operator

We now give a formulation for Ω_∂ where we also consider the globalization term $\hat{S}_{;x;R}$. Recall that graphically this amounts to introducing new vertices emanating only a single arrow, representing the vector field R as explained in the Feynman rules of Chapter 1.2. This means that Ω_∂ now becomes an inhomogeneous form on the Poisson manifold P, since R is a 1-form on P. As before, we distinguish between the E- and the X-representation.

2.3.3.1. E-representation. We start with the E-representation.

PROPOSITION 2.3.3.1. *In the E-representation, the globalized boundary operator is given by*

$$\tilde{\Omega}^{\mathsf{E}}_\partial = \Omega^{\mathsf{E}}_{(0)} + \Omega^{\mathsf{E}}_{(1)} + \Omega^{\mathsf{E}}_{(2)}, \tag{258}$$

where

$$\Omega^{\mathsf{E}}_{(0)} := \Omega^{\mathsf{E}}_\partial, \tag{259}$$

$$\Omega^{\mathsf{E}}_{(1)} = \sum_{I,J,L}^{\infty} \frac{(-i\hbar)^{|L|-|I|}}{(|L|+|J|)!} \partial_L A^I(R;T\varphi_x)[\mathsf{E}_I\mathsf{E}_J] \frac{\partial^{|L|+|J|}}{\partial[\mathsf{E}_L]\,\partial[\mathsf{E}_J]}, \tag{260}$$

$$\Omega^{\mathsf{E}}_{(2)} = \sum_{L}^{\infty} \frac{(-i\hbar)^{|L|+1}}{|L|!} \partial_L F(R;R;T\varphi_x) \frac{\partial^{|L|}}{\partial[\mathsf{E}_L]}, \tag{261}$$

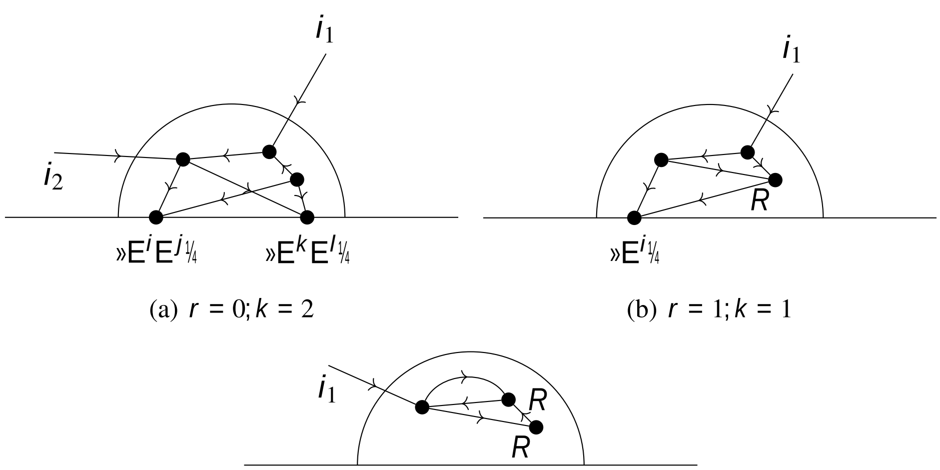
where A^I denotes the sum of weights of graphs with a single boundary vertex, where the incoming arrows at the boundary vertex are labeled by I, and F denotes the sum of weights of graphs with no boundary vertices.

Remark 2.3.3.2. Recall that in the globalization of the Poisson Sigma Model after [30], reviewed in Section 2.2.2, the choice of a formal exponential map on P induces a Fedosov connection $\mathcal{D}_G = d + A$ on the bundle of $\star$-algebras given by applying Kontsevich formality for $(T_x P, T'_x P)$ for every $x \in P$. Here $\mathcal{D}_G$ arises by "quantizing" the Grothendieck connection D_G. In particular, the graphs appearing in $\Omega^{\mathsf{E}}_{(1)}$ are exactly the ones appearing in the definition of the connection 1-form A as in (219). The connection $\mathcal{D}_G$ is not flat, $(\mathcal{D}_G)^2 = [F, \cdot]_\star$. The graphs appearing in $\Omega^{\mathsf{E}}_{(2)}$ are exactly the ones appearing in the Definition of the curvature 2-form F as in (220). Note that, by the notation as before, we can also write

$$\Omega^{\mathsf{E}}_{(1)} = A(R; T'_x)\left(e^{\frac{i}{\hbar}[\mathsf{E}]y}\right)\Big|_{y=\frac{\partial}{\partial[\mathsf{E}]}}, \tag{262}$$

$$\Omega^{\mathsf{E}}_{(2)} = F(R; R; T'_x)\left(\frac{\partial}{\partial[\mathsf{E}]}\right). \tag{263}$$

Proof of Proposition 2.3.3.1. We have seen that degree counting implies that there are exactly two boundary vertices in a collapsing graph. Now we have to take the R vertices into account. Consider a collapsing graph with n bulk and k boundary vertices. Then the dimension of the corresponding configuration space is $2n + k - 2$. On the other hand, there are now two types of bulk vertices: Suppose there are m vertices labeled by the Poisson bivector field (emitting two arrows) and r vertices labeled by the vector field R (emitting one arrow). Since arrows cannot leave the collapsing graph, the total form degree is $2m + r$, which has to be equal to $2n + k - 2$. Since $n = m + r$, this implies that $r + k = 2$. This means there can be either zero, one or two vertices labeled by R with two, one or zero boundary vertices respectively, as shown in Figure 2.3.4.



(a) $r = 0; k = 2$ (b) $r = 1; k = 1$ (c) $r = 2; k = 0$

Figure 2.3.4. Possible graphs in the E-representation.

The first contribution $r = 0$ and $k = 2$ is exactly the operator $\Omega^{\mathsf{E}}_{(0)}$ given in (246) from the non-globalized case. We get graphs with exactly one boundary vertex labeled by R and graphs with exactly two boundary vertices labeled by R.

In the case $r = 1$ and $k = 1$ we obtain precisely the graphs with a single boundary vertex and a single R bulk vertex (there can be an arbitrary number of vertices labeled by Π). This proves Equation (260). In the case $r = 2$ and $k = 0$ we obtain Equation (261).

2.3.3.2. $\mathbb{X}$-representation. Next, we consider the $\mathbb{X}$-representation.

Proposition 2.3.3.3. *In the $\mathbb{X}$-representation, the globalized boundary operator is given by*

$$\boldsymbol{\Omega}^{\mathbb{X}}_{\partial} = \sum_{j=0}^{\dim(P)} \Omega^{\mathbb{X}}_{(j)}, \tag{264}$$

where $\Omega^{\mathbb{X}}_{(0)} := \Omega^{\mathbb{X}}_{\partial}$ and $\Omega^{\mathbb{X}}_{(j)}$ is the sum of all graphs with j vertices labeled by R for $1 \leq j \leq \dim(P)$.

Proof. In the $\mathbb{X}$-representation, arrows can leave the collapsing graph, so we cannot do a degree count like in the $\mathbb{E}$-representation; in particular, the number of R vertices in a collapsing graph is only bounded by the dimension of P.

2.3.4. Algebraic structure in the flatness conditon for the BFV operator

We know from Part 1 that $(\nabla_G)^2 = 0$, and that this is equivalent to $d_x \boldsymbol{\Omega}_{\partial} + \frac{1}{2}[\boldsymbol{\Omega}_{\partial}, \boldsymbol{\Omega}_{\partial}] = 0$. For the Poisson Sigma Model it is interesting to see how this condition can be derived by looking at the explicit structure of $\boldsymbol{\Omega}_{\partial}$ as discussed in Section 2.3.3. We again consider the two different representations separately.

2.3.4.1. $\mathbb{E}$-represention. Recall that

$$\boldsymbol{\Omega}^{\mathbb{E}}_{\partial} = \Omega^{\mathbb{E}}_{(0)} + \Omega^{\mathbb{E}}_{(1)} + \Omega^{\mathbb{E}}_{(2)}, \tag{265}$$

where $\Omega^{\mathbb{E}}_{(j)}$ denotes the part of form degree j for $j \in \{0, 1, 2\}$.

Proposition 2.3.4.1. *We have the following equations:*

$$[\Omega^{\mathbb{E}}_{(0)}, \Omega^{\mathbb{E}}_{(0)}] = 0, \tag{266}$$

$$d_x \Omega^{\mathbb{E}}_{(0)} + [\Omega^{\mathbb{E}}_{(0)}, \Omega^{\mathbb{E}}_{(1)}] = 0, \tag{267}$$

$$d_x \Omega^{\mathbb{E}}_{(1)} + [\Omega^{\mathbb{E}}_{(0)}, \Omega^{\mathbb{E}}_{(2)}] + \frac{1}{2}[\Omega^{\mathbb{E}}_{(1)}, \Omega^{\mathbb{E}}_{(1)}] = 0, \tag{268}$$

$$d_x \Omega^{\mathbb{E}}_{(2)} + [\Omega^{\mathbb{E}}_{(1)}, \Omega^{\mathbb{E}}_{(2)}] = 0, \tag{269}$$

$$[\Omega^{\mathbb{E}}_{(2)}, \Omega^{\mathbb{E}}_{(2)}] = 0. \tag{270}$$

Proof. Proposition 2.3.4.1 follows from general arguments of Part 1, but here we give an independent proof. First we look at Equation (267).
The construction in [30], recalled in Chapter 2.2, yields a bundle $\mathcal{E} = \widehat{S}T^*P[[\hbar]]$ of $\star$-algebras on P by applying Kontsevich's deformation quantization in every tangent space. Picking a Grothendieck connection $D_G = d_x + L_R$ on P, and applying the Kontsevich formality map to R, one obtains a connection $\mathcal{D}_G = d_x + A$ on $\mathcal{E}$. In [30] it is shown that this connection is a derivation of $\Gamma(\mathcal{E})$, i.e. for $\sigma, \tau \in \Gamma(\mathcal{E})$, we have

$$\mathcal{D}_G(\sigma \star \tau) = (\mathcal{D}_G \sigma) \star \tau + \sigma \star (\mathcal{D}_G \tau). \tag{271}$$

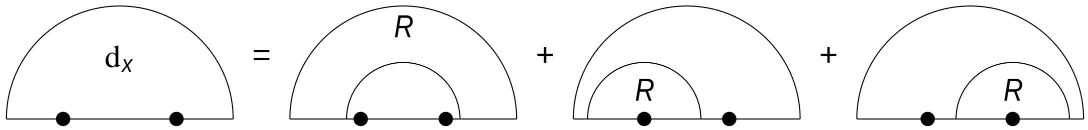


FIGURE 2.3.5. Schematics of the diagrammatic content of (271). and are arbitrary sections of $^1\mathcal{E}^o$. We sum over all possible graphs. By d_x we mean that we apply d_x to the result of the integration. An R means that there is precisely one vertex labeled by R in every graph.

We claim that this equation is equivalent to (267). This can be done directly by writing out (267) and (271) in coefficients, but it is best seen through Feynman diagrams (after all, A and the star product are defined through Feynman diagrams). First, rewrite (271) into

$$d_x{}^1 \; ? \; {}^o \quad d_x \; ? \qquad ? \, d_x \;=\; A^1 \; ? \; {}^o + {}^1A \; {}^o ? \;+\; ? \, {}^1A \; {}^o: \tag{272}$$

The left hand side of this equation is given by applying d_x to the coefficients of the star product. Schematically, we represent the diagrammatic content as in Figure 2.3.5.
Recall from Part 1 that the commutator $» \; {}^{E}_{10^o}; \; {}^{E}_{11^o}¼$ can be expressed by replacing the boundary vertices in the graphs defining ${}^{E}_{11^o}$ by the graphs appearing in ${}^{E}_{10^o}$ and vice versa. If we ignore possible arrows arriving at the boundary vertices from outside the graph, this yields precisely the graphs on the right hand side of Figure 2.3.5: The first term are the graphs of ${}^{E}_{10^o}$ placed at the boundary vertex of graphs appearing in ${}^{E}_{11^o}$, and the second and the third term represent the graphs of ${}^{E}_{11^o}$ placed at one of the boundary vertices of ${}^{E}_{10^o}$. Arriving arrows from outside the graph corresponds to taking derivatives of and . On the other hand, the left hand side yields precisely $d_x \; {}^{E}_{10^o}$.
The other equations are proven in a similar fashion, using the following relations:

Equation (266) holds, since the non-globalized boundary operator squares to zero (which is in turn a consequence of the Classical Master Equation, see [38] and Part 1).
Equation (270) holds, since there are no E-field contributions in ${}^{E}_{12^o}$.
Equation (269) corresponds to the *Bianchi identity*.
Equation (268) corresponds to the equation $d_x A + \frac{1}{2}»A; A¼ = »F; \; ¼?$.

In the last two points, the proof is similar to the proof of the degree 1 case (267).

REMARK 2.3.4.2. Note that the fact that the non-globalized BFV operator ${}_{@}$ (which depends on the constant background field x: ! P) gives rise to a family of star products, constructed from the Poisson structure $T\,'_x$ on T_xP. Moreover, the fact that it squares to zero corresponds to the associativity of these star products. Similarly the globalized BFV operator contains the data of a connection and its curvature and the fact that it is flat corresponds to the structural equations relating these objects. Hence we naturally recover the construction of the globalized version of Kontsevich's star product as it was discussed in [30]. In [30] the connection was twisted by a 1-form with values in the deformed jet bundle of ?-algebras to obtain a flat connection $\overline{\mathcal{D}}_G$, i.e. we have the following chain (see also Section 2.2.2)

$$D_G \;!\; \mathcal{D}_G \;!\; \overline{\mathcal{D}}_G: \tag{273}$$

This motivates the introduction of an additional term $\mathcal{S}_{\Sigma,\gamma}$ in the action to obtain Ω_∂ corresponding to the connection $\overline{\mathcal{D}}_G$ (see Section 2.3.5).

2.3.4.2. $\mathbb{X}$-representation. In the $\mathbb{X}$-representation, one can similarly decompose the boundary operator into form degrees $\Omega^{\mathbb{X}}_\partial = \sum_{j=0}^{\dim P} \Omega^{\mathbb{X}}_{(j)}$, and for every $k = 0,\ldots,r$ one obtains the equations

$$d_x \Omega^{\mathbb{X}}_{(k-1)} + \frac{1}{2}\sum_{i+j=k}\left[\Omega^{\mathbb{X}}_{(i)}, \Omega^{\mathbb{X}}_{(j)}\right] = 0. \tag{274}$$

The form degree zero part is again the fact that the non-globalized boundary operator squares to zero. It would be interesting to investigate whether there is an algebraic structure underlying the equations in the other form degrees, similar to the $\mathbb{E}$-representation.

2.3.5. Modification of the action

We modify the classical BV action by using results of [16, 30, 34] as we also discuss in Chapter 2.2. Let $\mathcal{E} := \hat{T}P[[\epsilon]]$ for some deformation parameter ϵ. Recall from Section 2.2.2, that given $\omega \in \Omega^2(P;\mathcal{E})$ such that $\mathcal{D}_G\omega = 0$ and $[\omega,\ \cdot\ ]_\star = 0$, we can always find $\gamma \in \Omega^1(P;\mathcal{E})$ such that

$$\overline{F}^P_\omega := F^P + \epsilon\omega + \mathcal{D}_G\gamma + \gamma\star\gamma = 0. \tag{275}$$

This is equivalent to Equation (233).
According to Remark 2.3.4.2, we now formulate a new "modified" action.

Definition 2.3.5.1 (Modified formal globalized action). Let $\gamma \in \Omega^1(P;\mathcal{E})$ be a solution of Equation (233) for $\omega \in \Omega^2(P;\mathcal{E})$ as above (here the formal parameter ϵ is given by $(-i\hbar)/2$). Then the *modified formal globalized action* $S_{\Sigma,x}$ is given by

$$S_{\Sigma,x} = \widehat{\mathcal{S}}_{\Sigma,x} + \mathcal{S}_{\Sigma,\gamma} + \mathcal{S}_{\Sigma,\omega}, \tag{276}$$

where[3]

$$\mathcal{S}_{\Sigma,\gamma} = \int_{\partial\Sigma}\gamma(x;\hat{\mathsf{A}}) = \sum_{k\geq 1}(-1)^k\left(\frac{i\hbar}{2}\right)^k\sum_l\int_{\partial\Sigma} dx^i\,\gamma^{(k)}_{i;l}(x)\hat{\mathsf{A}}^l, \tag{277}$$

$$\mathcal{S}_{\Sigma,\omega} = \int_\Sigma \omega(x;\hat{\mathsf{A}}) = \sum_{k\geq 1}(-1)^k\left(\frac{i\hbar}{2}\right)^k\sum_J\int_\Sigma dx^i\wedge dx^j\,\omega^{(k)}_{ij;J}(x)\hat{\mathsf{A}}^J. \tag{278}$$

Remark 2.3.5.2. Here we integrate the source 1-form part of $\hat{\mathsf{A}}$ along the boundary, which, since the $\hat{\mathsf{A}}$-fluctuation vanishes on components of the boundary in $\mathbb{X}$-representation, implies that for a single boundary with $\mathbb{X}$-representation $\mathcal{S}_{\Sigma,\gamma}$ does not give any contribution to $\Omega^{\mathbb{X}}_\partial$. Therefore we only need to look at the $\mathbb{E}$-representation. Moreover, note that $\gamma = O(\hbar)$, i.e. it is already a type of quantum counterterm which is not present classically, so it does not violate the modified Classical Master Equation.

Proposition 2.3.5.3. *The BFV boundary operator $\Omega^{\mathbb{E},\gamma}_\partial$ for the modified formal globalized action $S_{\Sigma,x}$ is given by*

$$\Omega^{\mathbb{E},\gamma}_\partial = \Omega^{\mathbb{E}}_\partial + \Omega^{\mathbb{E}}_{(1)} + \left[e^{\frac{i}{\hbar}[\mathbb{E}]y},\gamma\right]_\star\Big|_{y=\frac{\hbar}{i}\frac{\delta}{\delta[\mathbb{E}]}}, \tag{279}$$

[3] The reason why such counter terms always exists is due to the fact that the cohomology which would provide obstructions is trivial [30].

where ? denotes again the fiberwise star product on $\mathcal{E}$ as in Section 2.3.2.

PROOF. Considering again a degree counting, we get different cases of boundary vertex configurations. For the case ¹$r = 0; k = 2$º, we can either have two E-field boundary vertices, one E-field and one boundary vertex or two boundary vertices. For the case ¹$r = 1; k = 1$º, we can have either one E-field boundary vertex or one boundary vertex. For the case ¹$r = 2; k = 0$º we have the same contribution as before. In the case ! , 0, there is a configuration where ¹$r = k = 0$º, but there is a single ! vertex. These different diagrams contribute to different terms for the new boundary operator, which are:

$r = 0; k = 2$ (E; E on the boundary): Summing over all these graphs, this corresponds to the term

$$\frac{E}{@} ; \tag{280}$$

$r = 0; k = 2$ (; on the boundary): Summing over all these graphs, this corresponds to

$$1 \; ? \; 0 \; \frac{}{»E¼} ; \tag{281}$$

$r = 0; k = 2$ (E; on the boundary): Summing over all these graphs, this corresponds to

$$»e^{\frac{i}{~}»E¼y}; \; ¼? \; \Big|_{y=\frac{}{»E¼}} ; \tag{282}$$

$r = 1; k = 1$ (E on the boundary): Summing over all these graphs, this corresponds to the term

$$A¹R; T'_x \; º1 e^{\frac{i}{~}»E¼yº} \Big|_{y=\frac{}{»E¼}} ; \tag{283}$$

$r = 1; k = 1$ (on the boundary): Summing over all these graphs, this corresponds to the connection term

$$A¹R; T'_x \; º1 \; º = A¹R; T'_x \; º \; i \; \frac{}{»E¼} \; dx^i ; \tag{284}$$

$r = 2; k = 0$ (nothing on the boundary): Summing over all these graphs, this corresponds to the curvature term

$$F¹R; R; T'_x \; º; \tag{285}$$

$r = k = 0$ (just one ! vertex in the bulk): Summing over all graphs this just yields ! .

By Equation (233) and (225), we obtain that the terms (281), (282), (285) and possibly ! , can be put together as

$$A¹R; T'_x \; º1 \; º \quad F¹R; R; T'_x \; º \quad ? \quad ! = d_x \; : \tag{286}$$

Hence they do not contribute to the boundary operator, since they cancel the terms in d_x e $_{;x}$ in the modified differential Quantum Master Equation, where the full state is defined by the action S $_{;x}$.

REMARK 2.3.5.4. By Equation (229) and the fact that $d_x e^{\frac{i}{~}E} = 0$, the surviving terms will correspond to

$$\overline{\mathcal{D}}_G e^{\frac{i}{~}»E¼y} \Big|_{y=\frac{}{»E¼}} = \mathcal{D}_G e^{\frac{i}{~}»E¼y} \Big|_{y=\frac{}{»E¼}} + »e^{\frac{i}{~}»E¼y}; \; ¼? \; \Big|_{y=\frac{}{»E¼}} ; \tag{287}$$

where $\mathcal{D}_G e^{\frac{i}{\hbar}\langle E\rangle y}$ means that we apply $\mathcal{D}_G$ to the fiber coordinates y of $e^{\frac{i}{\hbar}\langle E\rangle y}$. Hence the boundary operator is given by

$$\Omega^{E,}_{\partial} = \Omega_{\partial} + \overline{\mathcal{D}}_G e^{\frac{i}{\hbar}\langle E\rangle y}\Big|_{y=\frac{\hbar}{i}\frac{\delta}{\delta\langle E\rangle}}. \tag{288}$$

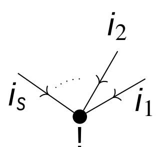
2.3.5.1. The twisted state. Using the modified action (276) one can define a state twisted by as follows.

DEFINITION 2.3.5.5 (TWISTED FULL COVARIANT QUANTUM STATE). Let be a manifold, possibly with boundary. Given a *BF*-like BV-BFV theory $: \mathcal{F} \to \mathcal{F}^{\partial}_{\partial}$, a polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial}$, a splitting $\mathcal{F} \cong \mathcal{B}^{\mathcal{P}}_{\partial} \times \mathcal{V}^{\mathcal{P}} \times \mathcal{Y}'$, and a gauge-fixing Lagrangian $\mathcal{L}_{,x} \subset \mathcal{Y}'$, we define the *twisted full covariant quantum state* by the formal perturbative expansion of the BV integral

$$\Omega_{,x}(\mathbb{X}, E; x, e') := \int_{\mathcal{L}_{,x}\subset\mathcal{Y}'} e^{\frac{i}{\hbar}S_{,x}\langle(\mathbb{X}, b'\rangle)} \in (\ ; \hat{\mathcal{H}}^{\mathcal{P}}_{,\mathrm{tot}}), \tag{289}$$

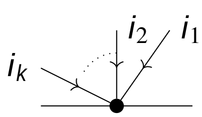
using the Feynman rules in Figure 0.1.1 and additionally with the rules for the boundary vertices as in Figure 0.1.2 and Figure 2.3.6.

The reason to introduce this state will become clear in the next two sections, when we analyze the anomaly arising from alternating boundary conditions. Essentially, the twist localizes the anomaly at the corners, where it can be canceled by changing the boundary operator (see Theorem 2.4.4.3 and 2.5.3.1 of Chapter 2.4 and 2.5 respectively).



$$\sum_{k\geq 1} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \mathrm{d}x^i \wedge \mathrm{d}x^j\, \omega^{(k)}_{ij;i_1\ldots i_k}(x)$$

(a) New interaction vertices in the bulk representing ω



$$\sum_{k\geq 1} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \mathrm{d}x^i\, \omega^{(k)}_{i;i_1\ldots i_k}(x)$$

(b) New boundary vertices representing

FIGURE 2.3.6. New vertices appearing in the Feynman rules

The twisted state is closed with respect to the operator

$$\nabla_G := \mathrm{d}_x - i\hbar\,\Delta_{\mathcal{V}} + \frac{i}{\hbar}\Omega^{E,}_{\partial}. \tag{290}$$

This is a consequence of Theorem 2.4.4.3 of Chapter 2.4.

2.3.5.2. Flatness. The following Proposition tells us that the twisted quantum Grothendieck BFV operator still remains a differential, i.e. squares to zero for $\Omega^{E,}_{\partial}$.

PROPOSITION 2.3.5.6. *The operator*

$$\nabla_G = \left(d_x - i\hbar\Delta + \frac{i}{\hbar}\Omega^{\mathbb{P}}_{\partial} \right) \tag{291}$$

on $\hat{\mathcal{H}}^{\mathcal{P}}_{\text{tot}}$ *squares to zero.*

PROOF. Note that the flatness condition of ∇_G, is equivalent to

$$d_x\Omega^{\mathbb{P}}_{\partial} + \frac{1}{2}\left[\Omega^{\mathbb{P}}_{\partial}, \Omega^{\mathbb{P}}_{\partial}\right] = 0. \tag{292}$$

Separating the equation by form degree in P this is equivalent to

$$\frac{1}{2}\left[\Omega^{0}_{\partial}, \Omega^{0}_{\partial}\right] = 0 \tag{293}$$

$$d_x\Omega^{0}_{\partial} + \left[\Omega^{0}_{\partial}, \bar{\mathcal{D}}_G e^{\frac{i}{\hbar}\langle E, y\rangle}\Big|_{y=\frac{\delta}{\delta E}}\right] = 0 \tag{294}$$

$$d_x\left(\bar{\mathcal{D}}_G e^{\frac{i}{\hbar}\langle E, y\rangle}\Big|_{y=\frac{\delta}{\delta E}}\right) + \left[\bar{\mathcal{D}}_G e^{\frac{i}{\hbar}\langle E, y\rangle}\Big|_{y=\frac{\delta}{\delta E}}, \bar{\mathcal{D}}_G e^{\frac{i}{\hbar}\langle E, y\rangle}\Big|_{y=\frac{\delta}{\delta E}}\right] = 0. \tag{295}$$

Equation (293) is just saying that the standard BFV boundary operator squares to zero. Equation (295) is true because $\bar{\mathcal{D}}_G$ is a flat connection. Equation (294) means that Ω^{0}_{∂} is a $\bar{\mathcal{D}}_G$-closed section. This comes from the fact that the coefficients of Ω^{0}_{∂} are the same as in the star product.

CHAPTER 2.4

Alternating boundary conditions and the modified differential QME

In this chapter we give a precise construction for the boundary conditions and explain the appearing anomaly appearing in this setting for the boundary BFV operator, such that the modified differential Quantum Master Equation does not hold anymore. Moreover, we explain the appearance of corners, construct a "twisted" version of the quantum GBFV operator, and show how the twisted version of the modified differential Quantum Master Equation will be expressed in terms of corners.

2.4.1. Consistent boundary conditions

In [26] it was shown that the perturbative expansion of the QFT given from of the Poisson Sigma Model on the disk coincides with Kontsevich's star product, where we expand around the gauge equivalent classical solutions of the given Euler–Lagrange equations, which are $\mathsf{X} = x = const.$, $\eta = 0$ (recall Section 2.3.1 and see also Chapter 2.2). The boundary conditions on the disk D are exactly set such that $\eta|_{\partial D} = 0$ in order to be consistent with these types of solutions.

2.4.2. Construction of boundary conditions

In Part 1, the globalization construction was only considered for boundaries with a single polarization. We want to extend the methods developed in Part 1 to describe deformation quantization of the relational symplectic groupoid [32, 24, 25] extending what was done in [43] in the case of a constant Poisson structure. This requires that we perform the BV-BFV quantization in the presence of "alternating" boundary conditions, which we can formulate for any worldsheet Σ: Let $\partial\Sigma = \bigsqcup_\ell \partial\Sigma^{(\ell)}$ and consider a partition into two distinguished components for every connected component $\partial\Sigma^{(\ell)}$ of the boundary given by $\partial\Sigma^{(\ell)} = \partial_0\Sigma^{(\ell)} \sqcup \partial_{\mathcal{P}}\Sigma^{(\ell)}$. Each $\partial\Sigma^{(\ell)}$ is given as a disjoint union of an even number of intervals $I_1^{(\ell)},\dots,I_n^{(\ell)}$, such that $\partial_0\Sigma^{(\ell)} = \bigsqcup_{\substack{1\le j\le n\\ j \text{ odd}}} I_j^{(\ell)}$ and $\partial_{\mathcal{P}}\Sigma^{(\ell)} = \bigsqcup_{\substack{1\le j\le n\\ j \text{ even}}} I_j^{(\ell)}$. Now the alternating condition is that on components of $\partial_0\Sigma^{(\ell)}$ we set $\mathsf{b} = 0$, and on components of $\partial_{\mathcal{P}}\Sigma^{(\ell)}$ we choose some polarization $\mathcal{P}_j$ for each I_j, and consider the corresponding boundary fields. We think of the endpoints of the intervals as "corners". Moreover, we denote by $\partial_1\Sigma$ the components of $\partial_{\mathcal{P}}\Sigma$ with the $\frac{\delta}{\delta\mathsf{E}}$-polarization and by $\partial_2\Sigma$ the components of $\partial_{\mathcal{P}}\Sigma$ with the $\frac{\delta}{\delta\mathsf{X}}$-polarization.

Remark 2.4.2.1. The choice of polarization imposes boundary conditions on the fluctuations. The boundary conditions corresponding to our polarizations for split AKSZ theories are some generalization of Dirichlet and Neumann conditions. Note that, even if fixing a field (to zero, in the case of a fluctuation) on the boundary looks like a Dirichlet boundary condition, it may also be thought as a Neumann one, for our theory is of first order.

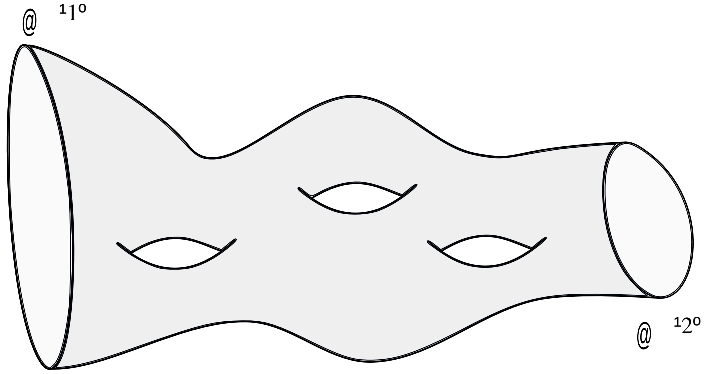


FIGURE 2.4.1. Example of a worldsheet manifold Σ with genus $g = 3$ and two disjoint boundary components $\partial\Sigma^{(1)}$ and $\partial\Sigma^{(2)}$.

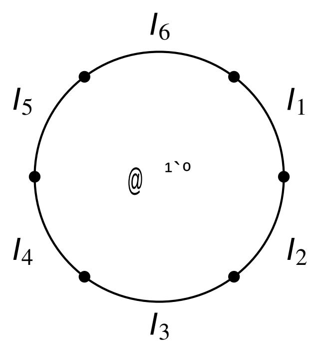


FIGURE 2.4.2. Example of a boundary component of Σ as in Figure 2.4.1, where the boundary $\partial\Sigma^{(\ell)}$ is split into $n = 6$ disjoint components, i.e. $\partial\Sigma^{(\ell)} = \bigsqcup_{1\leq j\leq 6} I_j^{(\ell)}$ with $\partial_0\Sigma^{(\ell)} = \bigsqcup_{j\in\{1,3,5\}} I_j^{(\ell)}$ and $\partial_{\mathcal{P}}\Sigma^{(\ell)} = \bigsqcup_{j\in\{2,4,6\}} I_j^{(\ell)}$. On $\partial_0\Sigma^{(\ell)}$ we set $b = 0$. On $I_2^{(\ell)}, I_4^{(\ell)}, I_6^{(\ell)}$ we choose polarizations and take the corresponding boundary conditions.

2.4.3. Curvature Anomaly

Unlike in the constant case discussed in [43], upon quantization[1] the modified differential Quantum Master Equation fails to be satisfied. This effect arises from the curvature of the deformed Grothendieck connection.

PROPOSITION 2.4.3.1. *Consider the full state $\tilde{\psi}_{\Sigma,x}$ defined by $\tilde{\hat{\psi}}_{\Sigma,x}$ as in Definition 1.2.4.2. Then*

$$\nabla_{G}\tilde{\psi}_{\Sigma,x} = \exp\left(\frac{i}{\hbar}\int_{\partial_0\Sigma} F(R;R;T\varphi_x)(\mathsf{X})\right)\tilde{\psi}_{\Sigma,x}, \tag{296}$$

where we integrate the X-fluctuation X in F along $\partial_0\Sigma$. Here F denotes the curvature of the deformed Grothendieck connection $\mathcal{D}_G$ defined in Section 2.2.2 and ∇_G is the quantum Grothendieck BFV operator defined as in (141).

[1]Note that we are not performing "extended" quantization of a manifold with corners in the sense of extended TQFTs, but simply apply BV-BFV quantization where we allow boundary conditions to change along connected components of the boundary.

Proof. If we try to prove the modified differential Quantum Master Equation as in Part 1, when integrating over the boundary of the compactified configuration space there are strata where a bulk graph collapses at a point $u \in \partial_0$, i.e. one of the boundary components where $b = 0$. The degree count as we have seen before, shows that we will only end up with graphs without any boundary vertices and precisely two R vertices in the bulk. Summing over all these graphs one obtains the curvature of the Grothendieck connection as in Chapter 2.2. However, since there are no boundary fields on ∂_0, these terms cannot be cancelled by a term in the BFV boundary operator.

Remark 2.4.3.2. This can be interpreted as a quantum anomaly, since this problem is not present at the classical level. To restore the modified differential Quantum Master Equation, we can add additional terms to the action, reminiscent to the addition of counterterms. This will yield new boundary terms, but they can be cancelled by adding appropriate terms to the BFV boundary operator as we have already seen in Section 2.3.5, if we allow for a slight extension of the space of states (see Section 2.5.1).

2.4.4. New boundary contributions in the proof of the mdQME

To cancel this anomaly we add quantum counterterms to the action, specifically, the terms $\mathcal{S}_{;}$ and $\mathcal{S}_{;!}$ defined in (277) and (278) respectively. The new terms in the action give rise to additional vertices. Namely, we now have vertices of arbitrary valence on components of the boundary where $b \neq 0$, i.e. on the $b = 0$ boundary components and the components of $\partial_{\mathcal{P}}$ in E-representation. At such a vertex we place the corresponding derivative of in the formal directions. Also, there are new bulk vertices labeled by $!$, which are similarly labeled by derivatives of $!$ in the formal directions.

Let C denote the set of all corner points of . There are two types of corners: Let $C_2 \subset C$ denote the subset containing those corner points which connect a $\overline{X}$-polarized connected component (i.e. a component in E-representation) of ∂_0 with a connected component of $\partial_{\mathcal{P}}$ and let $C_1 \subset C$ denote the subset containing those corner points which connect a $\overline{E}$-polarized connected component of ∂_0 with a connected component of $\partial_{\mathcal{P}}$.

(a) A corner in C_2 (b) A corner in C_1

Figure 2.4.3. The two types of corners.

Definition 2.4.4.1 (Twisted quantum Grothendieck BFV operator). We define the *twisted quantum Grothendieck BFV operator* by

$$\nabla_G := d_x - i\hbar \nu + \frac{i}{\hbar} \underbrace{\left(\Theta^{X}_{\partial_1} + \Theta^{E;}_{\partial_2}\right)}_{=:\Theta^{\mathcal{P};}_{\partial}} : \tag{297}$$

Remark 2.4.4.2. The twisted quantum Grothendieck BFV operator is also a coboundary operator. This follows from Proposition 2.3.5.6 and the fact that $\mathrm{e}^{X}_{\partial_1}$ also squares to zero.

We can now state the main theorem of this section.

Theorem 2.4.4.3. *Consider the twisted full state* $\mathrm{e}_{;x}$ *defined in Definition 2.3.5.5 and the twisted quantum Grothendieck BFV operator* r_{G} *defined in Definition 2.4.4.1. Then*

$$\mathrm{r}_{G}\,\mathrm{e}_{;x} = \sum_{C \in C_1} T^1 C^o\,\mathrm{e}_{;x}; \tag{298}$$

where $T^1 C^o$ *are functionals on* $\mathcal{B}^{\mathcal{P}}_{\partial}$ *with values in* ${}^{1\,1}P^{o}$*, depending only on the values of the fields at the corner point* C*.*

Remark 2.4.4.4. In particular, $T^1 C^o$ are non-regular functionals in the sense of Definition 0.1.4.11. In Chapter 2.5, we discuss an extension of the space of operators and states, which will allow us to rewrite Equation (298) as a closedness condition with respect to a differential on this extended space.

2.4.5. Proof of Theorem 2.4.4.3

If we try to proceed with the proof of the modified differential Quantum Master Equation as in Part 1, we get terms where a part of a graph collapses on ∂_0, i.e. the part of the boundary where $\mathrm{b} = 0$. We will now analyze these terms more closely. Let ^{0} be a subgraph that collapses on a point of the boundary, and denote by $\bullet^{0}$ the resulting graph. Suppose ^{0} has n bulk and k boundary vertices on ∂_0. Then the dimension of the corresponding boundary stratum is $2n + k - 2$ as we have seen before. The contribution of the graph is non-vanishing only if the form degree of $!_{0}$ is also $2n + k - 2$. The bulk vertices correspond to either or R, the former has two outgoing arrows, the latter only one. If one of these arrows points out of ^{0}, then $!_{\bullet 0} = 0$, since it contains a propagator with the tail evaluated on the $\mathrm{b} = 0$ boundary component. Hence all these arrows must point to another vertex in ^{0}. Suppose there are m vertices with two outgoing arrows and r vertices with one outgoing arrow. Then we must have the following system of equations:

$$2n + k - 2 = 2m + r \tag{299}$$

$$n = m + r; \tag{300}$$

which is equivalent to $r = 2 - k$ (m is arbitrary, and $n = m + r$). Since $r \geq 0$, we conclude that k is either $0; 1$, or 2. Let us analyze these possibilities in more detail.

2.4.5.1. Terms with $k = 0$. In these terms there are no boundary vertices. They are also present if we do not add $\mathcal{S}_{;}$ to the action. We have $r = 2 - k = 2$, so these terms are given by graphs with R at two vertices. Summing over all these terms yields the curvature of the Grothendieck connection, F (again, see Chapter 2.2 for details).

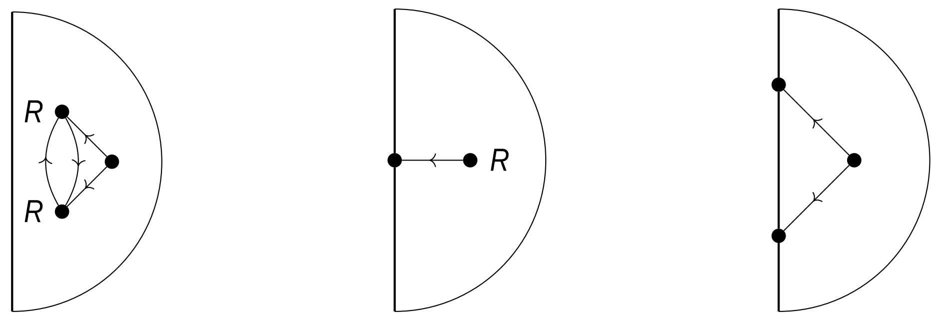
This is what spoils the modified differential Quantum Master Equation, since we cannot cancel it with terms in the BFV boundary operator, which can only cancel the boundary contributions on boundary components with free boundary fields. We are thus forced to add other terms to the action to cancel the appearance.

2.4.5.2. Terms with $k = 1$. In these terms there is one boundary vertex labeled by , and one bulk vertex labeled by the vector field R. If we sum over all such graphs, we get

$$A^1R;\mathrm{T}'_{x}\ {}^{01}\ {}^{0} = A^1R;\mathrm{T}'_{x}\ {}^{01}\ {}_i{}^{0}\mathrm{d}x^i \qquad (301)$$

by the Definition of A as in Chapter 2.2.

2.4.5.3. Terms with $k = 2$. In these terms there are two boundary vertices labeled by , and no vertices labeled by the vector field R. If we sum over all such terms, we get precisely the star product $?$.



(a) Graph with $r = 2$, $k = 0$ (b) Graph with $r = k = 1$ (c) Graph with $r = 0$, $k = 2$

FIGURE 2.4.4. Different contributions at the boundary

REMARK 2.4.5.1. For the case of symplectic Poisson structures, the Kontsevich weights of the connection and curvature graphs were computed explicitly in [85]. Moreover, it was shown that if the symplectic manifold is given by a cotangent bundle, the curvature expression will be significantly simplyfied. Using Equation (232), the computations in [85] should also allow to give an explicit for the special cases.

2.4.5.4. New contributions at the corners. Introducing alternating boundary conditions means that the compactification of the configuration space changes. Namely, there are new boundary strata corresponding to the collapse of vertices at one of the corners. Such a collapse can be modeled on a configuration of points on the upper right quadrant, with a choice of boundary conditions on both sides. Here there is no translation symmetry, so the dimension of the boundary stratum is different. Adding $\mathcal{S}_{;}$ to the action cancels the anomaly that comes from allowing for alternating boundary conditions. However, it results in new boundary contributions that come from graphs collapsing at the corners, as we will show presently. The propagator still vanishes when its tail is evaluated at one of the corners (this can be checked from the explicit formula for the propagator in Appendix E). For this reason, as above if some subgraph ^{0} of a graph collapses at a corner, the contribution is only non-vanishing if no arrows leave ^{0}. Let us start at a corner C in C_2. Then we cannot have propagators ending at the $\overline{\mathrm{X}}$-polarized boundary, since otherwise we need to evaluate the E-field at the corner point, which is equal to zero because of its boundary condition. So, any subgraph collapsing at C can only have bulk vertices, say $n = m + r$ of them, where m denotes the number of interaction and r the number of R vertices, and vertices and $@_{\mathcal{P}}$, say k of them. Counting the dimensions we arrive at the following system of equations:

$$2n + k \quad 1 = 2m + r \qquad (302)$$

$$n = m + r; \qquad (303)$$

which has the solutions $k = 0, r = 1$ and $k = 1, r = 0$, with m arbitrary. However, at these corners, graphs with bulk vertices do not contribute, this is the statement of the following lemma.

Lemma 2.4.5.2. *If Γ' is a subgraph of Γ containing bulk points, then the integral of ω_Γ, defined as in (69), over the boundary face of C_Γ where Γ' collapses at a corner $C \in C_2$ vanishes.*

Proof. The point is that at these corners the boundary conditions are the same on both sides, so we can map the configuration to a configuration of points on the upper half-plane, where we use the usual Kontsevich propagator, but without taking the quotient with respect to translations along the real axis. Instead we fix the image of the corner point to be a given point, e.g. 0. See also Figure 2.4.5. Now, observe that configurations with one bulk point evaluate to 0: These are either $k = 0, m = 0, r = 1$, but this case is ruled out because there are no tadpoles, or $k = 1, m = 1, r = 0$, but this is 0 because graphs cannot double edges. For more than two bulk points, note that the Kontsevich propagator depends on the the real parts of the points in the configuration only through their differences. Hence the product of propgators that is to be integrated has no component in the real part of the center of mass of the configuration, so integrating along this direction yields 0.

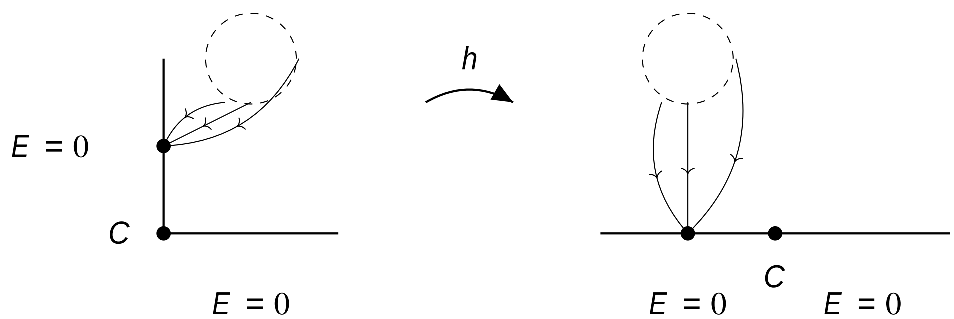
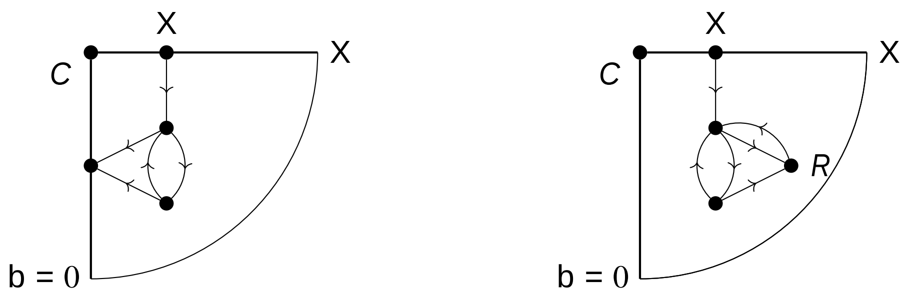


Figure 2.4.5. Here h represents the mapping of the corner with the interior to the upper half-plane, where the corner point is mapped to zero (with the same boundary conditions). The dashed circle represents some graph in the bulk with vertices corresponding to the Poisson structure π and the globalization term R, with some outgoing arrows deriving on the boundary. In particular, the map h is given by $z \mapsto z^2$ on the upper half-plane.

This means the only possibly nonzero contributions are those with $k = 1, n = 0$, i.e. subgraphs Γ' consisting of a single π vertex - possibly with any number of inward leaves - approaching the corner. This vertex can either lie on the ∂_0 or $\partial_{\mathcal{P}}$ component and the corresponding boundary faces have opposite orientation. Hence all terms cancel out: there are no extra contributions from corners in C_2.

Next let us turn to corners $C \in C_1$. Here the boundary conditions change, so the propagator does not have translation symmetry along the axis. However, by continuity, now it vanishes when either the head or the tail are evaluated at the point of collapse. This implies that a subgraph collapsing at C can have neither inward nor outward leaves, i.e. only entire connected components of graphs can collapse at corner $C \in C_1$. Counting dimensions as above, we see that there are again the two possibilities $r = 0, k = 1$ and $r = 1, k = 0$, with m arbitrary; in addition now we can have an arbitrary number b of vertices at the boundary with X-representation.

Since only connected components of a graph can collapse, the corresponding action on the state is a multiplication operator T^1C^0 that multiplies states with a functional of the values of X at corners



(a) Example of an $r = 0; k = 1$ configuration

(b) Example of an $r = 1; k = 0$ configuration

FIGURE 2.4.6. Possibilities for graphs collapsing at C 2 C_1.

in C_1, given by summing over all possible boundary contributions. Since and R are both 1-forms on P , T^1C^0 takes values in 1-forms on P . This is not a regular functional as in Definition 0.1.4.11, as it contains evaluation of fields on the corners. This completes the proof of Theorem 2.4.4.3.

CHAPTER 2.5

The mdQME for the globalized PSM with alternating boundary conditions

We have seen that the modified differential Quantum Master Equation fails if we impose alternating boundary conditions as in Proposition 2.4.3.1 and Theorem 2.4.4.3. Hence we need to extend the quantum Grothendieck BFV operator on an extended space of operators and states such that the modified differential Quantum Master Equation holds for the extended connection. The plan is to promote the corner terms $T(C)$ to multiplication operators on the state space. This requires the extensions of the state space to include functionals which evaluate fields at the corners.

2.5.1. Extension of states

There are two different terms in $T(C)$, namely the one where we have a single on the boundary approaching the corner and no vector field R, or no boundary vertex on the $\mathsf{b} = 0$ component and one single vector field R included in the graph from the bulk (see also Figure 2.4.6). To interpret them as multiplication operators we have to enlarge the space of states to allow for functionals evaluating boundary fields at corners.

Definition 2.5.1.1 (Space of corner states). For $C \in C_1$, we define the space of corner states by

$$\mathcal{H}_C := \left\{ F\colon \mathcal{B}^{\mathcal{P}}_{\partial} \to \mathbb{C}[[\hbar]] \;\middle|\; F(\mathbb{X}) = \sum_J B_J[\mathbb{X}^J(C)];\ \text{where } B_J \in \mathbb{C}[[\hbar]] \right\} \tag{304}$$

Definition 2.5.1.2 (Extended state space). We define the extended state space by

$$\hat{\mathcal{H}}^C_{\partial,x} := \hat{\mathcal{H}}_{\partial,x} \otimes \bigotimes_{C\in C_1} \mathcal{H}_C. \tag{305}$$

Moreover, the total space is given by $\hat{\mathcal{H}}^C_{\partial,\mathrm{tot}} := \prod_{x\in P} \hat{\mathcal{H}}^C_{\partial,x}$.

Now we can define a state to be given as a nonhomogeneous differential form on P with values in $\hat{\mathcal{H}}^C_{\partial,\mathrm{tot}}$, i.e. an element of $\Omega^\bullet(P, \hat{\mathcal{H}}^C_{\partial,\mathrm{tot}})$.

2.5.2. Extension of operators

Recall from [38] that the algebra of the operators is generated by $\Omega^{\mathrm{princ}}_0$, which is the standard quantization of $\mathcal{S}_{0,}$, and simple operators, which are of the form

$$\int_{\partial} L^J_{I_1\dots I_r}[\mathbb{X}^{I_1}]\cdots[\mathbb{X}^{I_r}]\frac{\delta^{|J|}}{\delta[\mathbb{X}]^J}, \tag{306}$$

where $L^J_{I_1\dots I_r} \in \Omega^\bullet(\partial)$ are some coefficients. Note that we can also have a similar expression for $\mathbb{E}$. We want to extend this space by the multiplication operators coming from the corners as described above. The space of operators is extended by the multiplication operators that appear in the case of corners. The algebra of boundary operators acts on the algebra of corner operators by commutators. E.g. $\partial_k \;\; ^{ij}X^k \frac{\ }{[X^iX^j]}$ is a boundary operator and $[X^iX^j](C)\partial_i\;\partial_j$ is a corner operator, with $C \in C_1$. Then the commutator is given by

$$\left[\partial_k \;\; ^{ij}X^k \frac{\ }{[X^iX^j]}, [X^iX^j](C)\partial_i\;\partial_j\right] = \partial_k \;\; ^{ij}X^k(C)\partial_i\;\partial_j \; . \tag{307}$$

The extended space now consists of operators taking a state in $(P;\hat{\mathcal{H}}^C_{\partial,\mathrm{tot}})$ and multiplying it with an element in $(P;\mathcal{H}_C)$.

2.5.3. The mdQME and flatness

Now we are able to define the extended operator as follows. Let $\;_C := \sum_{C\in C_1} T(C)$, where $T(C)$ is as in Theorem 2.4.4.3. The new operator ∇_{G} is then defined by

$$\nabla_{\mathsf{G}} := \mathrm{d}_x - i\hbar\;_{\mathcal{V}} + \frac{i}{\hbar}\,e^{\mathcal{P};}_{\partial} + \;_C \; . \tag{308}$$

2.5.3.1. The mdQME for alternating boundary conditions. We have the following theorem.

THEOREM 2.5.3.1 (MODIFIED DIFFERENTIAL QUANTUM MASTER EQUATION FOR ALTERNATING BOUNDARY CONDITIONS). *Let ∇_{G} be given as before, and consider the twisted full state $e\;_{;x}$. Then*

$$\nabla_{\mathsf{G}}\, e\;_{;x} = 0 \tag{309}$$

PROOF. This follows immediatley from Theorem 2.4.4.3.

2.5.3.2. Flatness. We have the following theorem.

THEOREM 2.5.3.2. *The operator ∇_{G} is a coboundary operator, i.e.* $(\nabla_{\mathsf{G}})^2 = 0$.

PROOF. The flatness condition is equivalent to the fact that $\;^{\mathrm{ext}} = e^{\mathcal{P};}_{\partial} + \;_C$ is a Maurer–Cartan element of the differential graded Lie algebra of differential forms with values in $\mathrm{End}(\hat{\mathcal{H}}^C_{\partial,\mathrm{tot}})$. Hence the proof of Theorem 2.5.3.2 is given by the Proposition 2.5.3.3.

PROPOSITION 2.5.3.3. $\mathrm{d}_x\;^{\mathrm{ext}} + \frac{1}{2}\left[\;^{\mathrm{ext}},\;^{\mathrm{ext}}\right] = 0$.

PROOF. First of all note that $\mathrm{d}_x e^{\mathcal{P};}_{\partial} + \frac{1}{2}\left[e^{\mathcal{P};}_{\partial}, e^{\mathcal{P};}_{\partial}\right] = 0$. This means we only need to prove

$$\mathrm{d}_x\;_C + \frac{1}{2}[\;_C,\;_C] + \left[\;_C, e^{\mathcal{P};}_{\partial}\right] = 0. \tag{310}$$

We can show this similarly to [38, 41]. Namely, since $e^{\mathcal{P};}_{\partial}$ and $\;_C$ are given as sum of integrals over the boundary of the configuration space of collapsing graphs, we can use Stokes' Theorem:

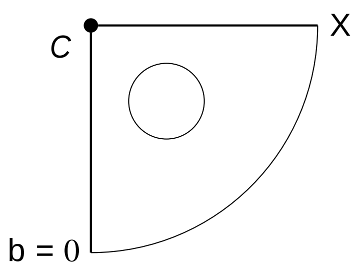
$$\mathrm{d}_x\;_C = \mathrm{d}_x \int_{C^{C}_{0}(\ )} \;_0 = \int_{C^{C}_{0}(\ )} \mathrm{d}\;_0 = \int_{\partial C^{C}_{0}(\ )} \;_0 \tag{311}$$

Here $C^{C}_{0}(\ )$ is the configuration space describing the relative position of the vertices of the subgraph collapsing to the corner. In the first, the differential can act on the propagators, the boundary fields, or the vertex tensors $T'_x(\ ;\ ;R)$. The restriction of the propagators to this boundary face is closed, see Appendix E. If the differential acts on the boundary fields, this yields $[\;^{\mathrm{princ}}_{0},\;_C]$. The differential acting on vertex tensors will be cancelled by boundary terms. Notice that on the boundary faces the dimension counting is different and we can have either two vertices labeled by R, one R vertex and one vertex on the boundary or two vertices on the boundary. A boundary face of $C^{C}_{0,1}(\;)$ corresponds to a collapse of a subgraph $\;'' \subset \;'$ to a single point. There are four distinct possibilities for that point (see Figure 2.5.1):

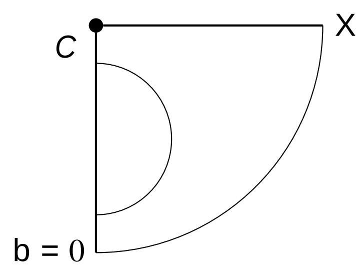
- The point can be in the bulk. If $\;''$ contains more than two vertices then the contribution is zero by a Kontsevich vanishing lemma. If it contains exactly two vertices, there is a cancellation similar to the proof of the modified differential Quantum Master Equation using the Classical Master Equation, the fact that vertex tensors are D_{G}-closed, and that $[R, R] = 0$.
- The point can be the corner. These terms yield $[\Omega_C, \Omega_C]$.
- The point can be at the boundary with the $\mathsf{b} = 0$ boundary condition. In that case there is a cancellation similar to one in the proof of the modified differential Quantum Master Equation in Section 2.4.4 using the equation

$$d_x \; + A(R; T\varphi_x)(\;^{01}) + \;?\; + F(R; R; T\varphi_x)(\;) = 0.$$

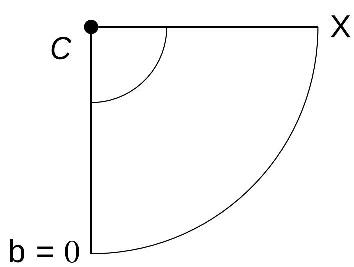
- The point can be on the upper boundary, this corresponds to $\left[e^{\mathcal{P};}_{@} ; \;_C \right]$, the action of the algebra of boundary operators on the algebra of corner operators.



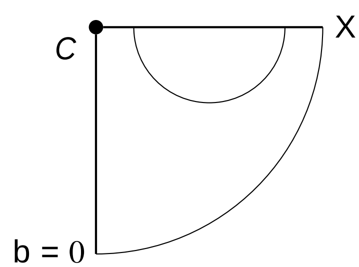
(a) Collapsing for a bulk point



(b) Collapsing for a point on the $\mathsf{b} = 0$ boundary



(c) Collapsing for a corner point



(d) Collapsing for a point on the polarized boundary

FIGURE 2.5.1. Illustration of the different cases for the collapsing

REMARK 2.5.3.4. The failure of the (modifed) differential Quantum Master Equation and its resolution is somehow similar to what happens in the Landau–Ginzburg model [75, 19, 78]. Namely, the classical boundary conditions turn out not to be compatible with quantization. The resolution consists in coupling the bulk theory with a boundary theory with action $\mathcal{S}_{;}$.

CHAPTER 2.6

Outlook

In this chapter we want to give an outlook to further aspects which are planed be considered in the future using the constructions presented in this part and Part 1.

2.6.1. Relational Symplectic Groupoid

2.6.1.1. Kontsevich's star product. One can construct the Moyal product [86] (deformation quantization) as the gluing of canonical relations as it was shown in [43]. It still remains to show that one can also use the gluing of the relational symplectic groupoid to construct a globalized version of Kontsevich's star product using the gluing formulas of the BV-BFV formalism. One can use the results of this paper to deal with the L_3 worldsheet structure, which is given as in Figure 2.6.1 with mixed boundary conditions.

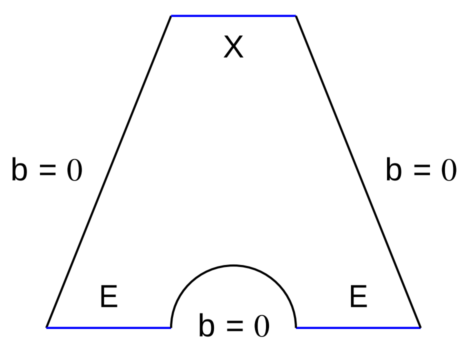


FIGURE 2.6.1. The canonical relation L_3 with its boundary structure. Here we have two $\frac{}{\mathsf{X}}$-polarized boundaries (the lower) and one $\frac{}{\mathsf{E}}$-polarized boundary (the upper), which would correspond to $@_2 L_3$ and the $\mathsf{b} = 0$ boundaries which are components of $@_1 L_3$.

2.6.1.2. Relational symplectic groupoid with handles. Another interesting aspect would be to consider the relational symplectic groupoid with handles. That is, one considers canonical relations L_3 with non vanishing genus. Since our theory is topological, we are able to move the handle in arbitrary directions, which means that one has to understand what happens when a hole will approach an observable for the gluing of the disk in Part 1. Moreover, one has to check what kind of structures appear for associativity.

2.6.1.3. Generalization of Kontsevich's star product. Kontsevich's star product arises from the computation of expectation values of observables in the Poisson Sigma Model for a genus zero wordsheet surface. As in string theory, one expects that we should sum over all genera. Since a particular gluing of the relational symplectic groupoid gives rise to Kontsevich's star product, one can relate this structure to the relational symplectic groupoid construction with handles.

2.6.2. Manifolds with corners

The methods developed in this part can be useful to give a description for the the quantization of manifolds with corners. Here the corners arose from the structure of mixed boundary conditions, but in principle the methods that we develop might be adapted to the general case. Another paper in this direction is [74].

2.6.3. Globalization of other theories

AKSZ theories have a particularly nice subset of classical solutions, the space of constant maps. This subset admits for a natural globalization, as was shown in Part 1. It would be interesting to see whether the methods we used carry over to more complicated moduli spaces of classical solutions. E.g. in Chern-Simons theory, this subset is just the trivial connection, since the body of the target in that case is just a point, but one would like to take non-trivial connections into account as well.

Part 3

Globalized Traces for the Poisson Sigma Model

Introduction

In [77] Kontsevich showed that the differential graded Lie algebra (DGLA) of multidifferential operators on a manifold M is L_∞-quasi-isomorphic to the DGLA of multivector fields on M. This is known as the formality theorem. The construction of Kontsevich's star product in deformation quantization is given by the special case of the formality theorem for bivector fields and bidifferential operators. In [26] it was shown that this star product can be written as a perturbative expansion of a path integral given by the Poisson Sigma Model [93, 73]. In [101] Tsygan formulated a formality conjecture for cyclic chains (which was motivated as a chain version of the Connes–Flato–Sternheimer cyclic cohomology construction [47]), which was partially proven by Shoikhet [97], Dolgushev [52] and Willwacher [107]. In [108] Willwacher and Calaque have proven the cyclic formaility conjecture of Kontsevich, which was the formulation for cyclic cochains.
A global geometrical picture of the star product coming from the Weyl quantization approach for symplectic manifolds, i.e. for a constant Poisson structure, was given by Fedosov in [55]. There one chooses a(n) (always existing) symplectic connection and its corresponding exponential map. This construction can be generalized to the local picture of Kontsevich's star product to produce a global version on any Poisson manifold [34, 33], where one uses notions of formal geometry [17, 64]. The symplectic connection (lifted to the Weyl bundle) can be replaced by the (deformed) Grothendieck connection which is constructed by using any (formal) exponential map (see also [30]). A globalized picture in the field theoretic approach using the Poisson Sigma Model in the Batalin–Vilkovisky (BV) formalism [10, 9, 7] was given in [16] for closed worldsheet manifolds and extended in Part 2 for manifolds with boundary using the BV-BFV formalism [37, 38, 41, 40].

An important object to study for closed star products [47] are trace maps.

Definition 3.0.0.1 (Closed star product). Let $A = C^\infty(M)$ for some Poisson manifold M. For a deformed algebra $(A[[\hbar]], \star)$, closedness means that the integration of elements of the deformed algebra is a trace with respect to $\star$. In particular, a star product is *closed* if and only if

$$\int_M (f \star g)\,\Omega = \int_M (g \star f)\,\Omega \qquad \forall f, g \in A, \tag{312}$$

where Ω is a volume form on M.

In [87] Nest and Tsygan showed an algebraic version of the Atiyah–Singer index theorem, where they made the link to a trace map with respect to the underlying star product and computed the index as the trace of the constant function 1 (see also [56] for Fedosov's construction). This construction is given for symplectic manifolds together with the globalization construction of the Moyal product. For a general Poisson manifold with Kontsevich's star product, Cattaneo and Felder constructed a trace map in terms of local field theoretic constructions using the Poisson Sigma Model on the disk for negative cyclic chains [28] in the presence of residual fields (a.k.a "slow" fields,"low energy" fields).

We will extend this construction to a global one by using a formal global version of the Poisson Sigma Model. This construction in fact combines Fedosov's globalization construction with field theoretic concepts on Poisson manifolds and the BV formulation. We also give the connection of the obtained globalized trace to the Tamarkin–Tsygan theorem, which can be seen as a cyclic equivariant extension of the Nest–Tsygan theorem for Poisson manifolds using formally extended Poisson structures. The connection can be understood by field theoretic concepts by looking at the Feynman graph expansion for the obtained trace formula, which geometrically gives rise to a deformed version of the Grothendieck connection and its curvature.

In [69] a global equivariant trace formula for symplectic manifolds was constructed by using a Fedosov connection and solutions to the Fedosov equation. The field theoretic construction was given by the effective theory of topological quantum mechanics on the circle S^1. We show that our trace formula reduces to this trace formula if we consider the Poisson Sigma Model with cotangent target. To show this, we use the fact that the vertices of our graphs in the expansion which arise from the Grothendieck connection are linear in the fiber coordinates if the underlying manifold is a cotangent bundle.

CHAPTER 3.1

Cyclic Formality

In this chapter we recall some details of Kontsevich's formality theorem and explain its cyclic extension given by Tsygan.

3.1.1. The Kontsevich Formality

Let $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_{SN}, \mathrm{d}=0)$ be the DGLA of multivector fields on $\mathbb{R}^d$ endowed with the Schouten–Nijenhuis bracket and the zero differential and let $(\mathcal{D}_{poly}(\mathbb{R}^d), [\ ,\ ]_G, \mathrm{d}_H)$ be the DGLA of multidifferential operators on $\mathbb{R}^d$ endowed with the Gerstenhaber bracket and the Hochschild differential. In [77] Kontsevich proved the celebrated formality theorem, which states that these two complexes are quasi-isomorphic as L_∞-algebras.

Theorem 3.1.1.1 (Kontsevich [77]). *There exists an L_∞-quasi-isomorphism*

$$\mathcal{U}\colon (\mathcal{T}_{poly}(\mathbb{R}^d), [\ ,\ ]_{SN}, \mathrm{d}=0) \to (\mathcal{D}_{poly}(\mathbb{R}^d), [\ ,\ ]_G, \mathrm{d}_H). \tag{313}$$

For the case of degree two, Theorem 3.1.1.1 implies a star product on $\mathbb{R}^d$ endowed with any Poisson structure. Moreover, Theorem 3.1.1.1 can be extended to a global version, where $\mathbb{R}^d$ can be replaced by any finite-dimensional manifold M. Let us briefly recall the main objects to understand the *formality theorem*.

3.1.1.1. The Hochschild complex and the Gerstenhaber bracket. Let A be a unital algebra with unit 1. One can consider the graded algebra $C_\bullet(A) := A\otimes \bar{A}^{\otimes \bullet}$, where $\bar{A} := A/\mathbb{R}1$. This space is endowed with a map

$$b([a_0\otimes\cdots\otimes a_m]) = \sum_{i=1}^{m-1} (-1)^i [a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_m] + (-1)^m [a_m a_0\otimes\cdots\otimes a_{m-1}]. \tag{314}$$

One can check that $b\colon C_\bullet(A) \to C_{\bullet-1}(A)$ is a differential, called the *Hochschild differential* and the tuple $(C_\bullet(A), b)$ is called the *Hochschild chain complex* of A. Here we denote by $[a_0\otimes\cdots\otimes a_m]$ the class of $a_0\otimes\cdots\otimes a_m$ in $C_\bullet(A)$. Moreover, we define $C_m(A) = 0$ for all $m < 0$. The DGLA of multidifferential operators $\mathcal{D}_{poly}(M)$, for a manifold M, can thus be seen as the subcomplex of the shifted complex $C^\bullet(A) := \mathrm{Hom}(A^{\otimes \bullet+1}, A)$, where $A = C^\infty(M)$, consisting of multilinear maps which are differential operators in each argument. The *Gerstenhaber bracket* of two multidifferential operators D, D' is given by

$$[D, D']_G := D\circ_G D' - (-1)^{|D||D'|} D'\circ_G D, \tag{315}$$

where $|D|$ denotes the degree of the multidifferential operator D and the *Gerstenhaber product* $\circ_G$ is given by

$$D\circ_G D' := \sum_{k=0}^{|D|} (-1)^{|D'|(|D|-k)} D(\mathrm{id}^{\otimes k}\otimes D'\otimes \mathrm{id}^{\otimes |D|-k}). \tag{316}$$

The differential on $C^\bullet(A)$ is given in terms of the Gerstenhaber bracket by $\mathrm{d}_H := [\mu, \,\cdot\,]_G$ for $\mu \in \mathrm{Hom}(A \otimes A, A)$ being the multiplication map of A. In fact, in [65] it was shown that the Hochschild cohomology $HH^\bullet(A)$ together with $\cup_G$ and $[\,,\,]_G$ is a Gerstenhaber algebra.

3.1.1.2. Multivector fields and the Schouten–Nijenhuis bracket. The space of multivector fields on a manifold M is given by $\Gamma(\bigwedge^\bullet TM)$. We define $\mathcal{T}_{poly}(M) := \bigoplus_{j \geq -1} \Gamma(\bigwedge^{j+1} TM)$, with the convention that $\mathcal{T}^{-1}_{poly}(M) = C^\infty(M)$, $\mathcal{T}^0_{poly}(M) = \Gamma(TM)$, $\mathcal{T}^1_{poly}(M) = \Gamma(\bigwedge^2 TM)$, etc. The *Schouten–Nijenhuis bracket* $[\,,\,]_{SN}$ is given by the usual Lie bracket extended to multivector fields by the Leibniz rule, i.e. for multivector fields ξ, η, γ we have

$$[\xi \wedge \eta, \gamma]_{SN} = \xi \wedge [\eta, \gamma]_{SN} + (-1)^{|\eta|(|\gamma|+1)} [\xi, \gamma]_{SN} \wedge \eta. \tag{317}$$

3.1.1.3. The Hochschild–Kostant–Rosenberg map. Consider vector fields $\xi_1, \ldots, \xi_n \in \mathcal{T}^0_{poly}(M)$ and $f_1, \ldots, f_n \in A$. One can construct a map, which for $n \geq 1$ is given by

$$\begin{aligned} \mathcal{T}^{n-1}_{poly}(M) &\to \mathcal{D}^{n-1}_{poly}(M) \\ \xi_1 \wedge \cdots \wedge \xi_n &\mapsto \left(f_1 \otimes \cdots \otimes f_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\, \xi_{\sigma(1)}(f_1) \cdots \xi_{\sigma(n)}(f_n) \right), \end{aligned} \tag{318}$$

and for $n = 0$ it is given by the identity on $C^\infty(M)$. Here S_n denotes the symmetric group of order n. This map is called *Hochschild–Kostant–Rosenberg (HKR) map*. One can check that it is indeed a chain map and a quasi-isomorphism of complexes, but does not respect the Lie bracket on the level of complexes. In fact Kontsevich's L_∞-quasi-isomorphism $\mathcal{U}$ gives a solution to this problem as a certain extension of the HKR map. In particular, the first Taylor component $\mathcal{U}_1$ of $\mathcal{U}$ is precisely the HKR map.

3.1.2. The Kontsevich–Tsygan Formality

One can generalize the formality construction to a cyclic version by considering *cyclic chains*. There is another differential, called the *Connes differential* [46, 47], of degree $+1$ on the Hochschild complex given by

$$B([a_0 \otimes \cdots \otimes a_m]) := \sum_{i=0}^{m} (-1)^{im} [1 \otimes a_i \otimes \cdots \otimes a_m \otimes a_0 \otimes \cdots \otimes a_{i-1}]. \tag{319}$$

Note that there is an HKR chain map

$$\begin{aligned} (C_\bullet(A), b) &\to (\Omega^\bullet(M; \mathbb{R}), \mathrm{d} = 0) \\ [a_0 \otimes \cdots \otimes a_m] &\mapsto \frac{1}{m!} a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_m. \end{aligned} \tag{320}$$

This map is also called the *Connes map* [47], which identifies cyclic and de Rham cohomology. Following Getzler [67], the *negative cyclic chain complex* is then given by

$$CC^-_\bullet(A) := C_\bullet(A)[u] \tag{321}$$

endowed with the differential $b + uB$. Here u denotes some formal variable of degree 2. Similarly to the negative cyclic chain complex, one can define the *periodic cyclic chain complex* by allowing negative powers of the formal parameter u, hence we have the formal Laurent polynomials $PC_\bullet(A) := C_\bullet(A)[u, u^{-1}]$. We can extend the HKR map by $\mathbb{R}[u]$-linearity and obtain a quasi-isomorphism

$$(CC_\bullet(A), b + uB) \to (\Omega^\bullet(M, \mathbb{R})[u], ud). \tag{322}$$

Consider a module W over the graded algebra $\mathbb{R}[u]$ of finite projective dimension and define $CC^W_\bullet(A) := C_\bullet(A)[u] \otimes_{\mathbb{R}[u]} W$. The formality for cyclic chains is given by the following theorem.

THEOREM 3.1.2.1 (KONTSEVICH–TSYGAN [101]). *There exists an L_∞-quasi-isomorphism*

$$\mathcal{U}^{cyc}\colon (CC^W_\bullet(A), b + uB) \to (\Omega^\bullet(M, \mathbb{R})[u] \otimes_{\mathbb{R}[u]} W, ud). \tag{323}$$

This was proven by Shoikhet, Willwacher and globally extended by Dolgushev using Fedosov resolution. Using Shoikhet's L_∞-quasi-isomorphism $\mathcal{U}^{Sh}$, one can obtain Theorem 3.1.2.1 as a corollary by obtaining $\mathcal{U}^{Sh} \circ b = \mathrm{d} \circ \mathcal{U}^{Sh}$ [107].

REMARK 3.1.2.2. This construction leads to a field theoretic construction using the Poisson Sigma Model on the disk. One can construct a trace map which uses an $\mathbb{R}[u]$-linear morphism of L_∞-modules over some suitable algebra.

CHAPTER 3.2

Fedosov's approach to deformation quantization

In this chapter we want to recall the most important notions and constructions for the global deformation quantization of symplectic manifolds given by Fedosov in [55].

3.2.1. Weyl algebra and Moyal product

Let (M,ω) be a symplectic manifold and let $\{x_i\}$ be local coordinates on M and $\{y^i\}$ coordinates on the corresponding fiber of the tangent bundle, i.e. $(x^i,y^i)\in M\times T_{x^i}M$. Consider the *Weyl bundle* $\mathcal{W}(M):=\widehat{S}T^*M[[\hbar]]$ associated to M, where $\widehat{S}$ denotes the completed symmetric algebra. The Weyl bundle can be regarded as a deformation of the bundle of formal functions on TM. We will write $\mathcal{W}$ instead of $\mathcal{W}(M)$ whenever it is clear. A section $a\in\Gamma(\mathcal{W})$ is locally given by[1]

$$a(x,y,\hbar)=\sum_{k,\ell}\hbar^k a_{k,i_1,\dots,i_\ell}(x)y^{i_1}\cdots y^{i_\ell}, \tag{324}$$

where $a_{k,i_1,\dots,i_\ell}\in C^\infty(M)$. In each fiber $\mathcal{W}_x$ for $x\in M$, one can construct an algebra structure by considering the associative product

$$\begin{aligned}\star\colon \mathcal{W}_x&\otimes\mathcal{W}_x\to\mathcal{W}_x,\\ (a(x,\hbar),b(x,\hbar))\mapsto (a\star b)(x,\hbar)&:=\exp\left(\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(y,\hbar)b(z,\hbar)\Big|_{z=y}\\ &=\sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k\frac{1}{k!}\omega^{i_1j_1}\cdots\omega^{i_kj_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial z^{j_1}\cdots\partial z^{j_k}}.\end{aligned} \tag{325}$$

Here we denote by (ω^{ij}) the components of the inverse ω^{-1} of the symplectic form. For any $x\in M$, the tuple $(\mathcal{W}_x,\star)$ is called the *Weyl algebra* and $\star$ is called the *Moyal product*. One can check that

$$\lim_{\hbar\to 0}\frac{1}{\hbar}(a\star b-b\star a)=\{a,b\}, \tag{326}$$

where $\{\ ,\ \}$ is the Poisson bracket coming from the symplectic structure ω, which makes sure that $\star$ is actually a deformation quantization of T_xM with constant Poisson structure ω_x^{-1}. Let $\Omega(M,\mathcal{W})$ denote the space of global differential forms on M with values in $\mathcal{W}$. A section $a\in\Gamma(\Omega(M,\mathcal{W}))$ is of the form

$$a(x,y,dx,\hbar)=\sum_{k,p,q}\hbar^k a_{k,i_1,\dots,i_p,j_1,\dots,j_q}(x)y^{i_1}\cdots y^{i_p}dx^{j_1}\wedge\cdots\wedge dx^{j_q}. \tag{327}$$

Moreover, we define the operators δ and δ^* according to [55] by

$$\delta a:=dx^k\wedge\frac{\partial a}{\partial y^k},\qquad \delta^* a:=y^k\iota_{\frac{\partial}{\partial x^k}}a, \tag{328}$$

where ι denotes the contraction. Define $\delta^{-1}:=\frac{1}{p+q}\delta^*$ for $p+q>0$ and zero if $p+q=0$.

[1] We will use the Einstein summation convention.

3.2.2. Symplectic connection and curvature

Consider now a symplectic connection ∇^{TM} on the tangent bundle TM, i.e. a torsion-free connection such that $\nabla^{TM}\omega = 0$. This induces directly a connection $\nabla^{\mathcal{W}}$ on $\mathcal{W}$ which we will just denote by ∇. The curvature of this connection is given by

$$F^{\nabla} = \frac{1}{2}F^i_{jk\ell}\,\mathrm{d}x^k \wedge \mathrm{d}x^\ell. \tag{329}$$

Moreover, consider the tensor

$$F := \frac{1}{4}F_{ijk\ell}\,\mathrm{y}^i\mathrm{y}^j\,\mathrm{d}x^k \wedge \mathrm{d}x^\ell, \quad F_{ijk\ell} := \omega_{im}F^m_{jk\ell}. \tag{330}$$

In [55] it was shown that the curvature of ∇ can be formulated as

$$\nabla^2 = \frac{1}{\hbar}[F,\ \cdot\ ]_\star, \tag{331}$$

where $[\ ,\ ]_\star$ denotes the commutator with respect to the Moyal product $\star$.

3.2.3. Fedosov's main theorems

Consider a connection

$$\bar{\nabla} := \nabla + \frac{1}{2\hbar}[\gamma,\ \cdot\ ]_\star \tag{332}$$

on $\mathcal{W}$, where $\gamma \in \Omega^1(M;\mathcal{W})$. One can check that $\bar{\nabla}$ is compatible with the Moyal product, i.e.

$$\bar{\nabla}(a \star b) = \bar{\nabla}(a) \star b + a \star \bar{\nabla}(b). \tag{333}$$

Theorem 3.2.3.1 (Fedosov [55]). *Consider a sequence $\{\omega_k\}_{k\geq 1}$ of closed 2-forms on M. Then there is a flat connection $\bar{\nabla}$ (that is $\bar{\nabla}^2 = 0$) defined as in (332) such that $\gamma = \sum_{i,j}\omega_{ij}\mathrm{y}^i\mathrm{d}x^j + r$, where $r \in \Omega^1(M;\mathcal{W})$ satisfying $\delta^{-1}r = 0$. Moreover, γ satisfies*

$$\bar{\nabla}\gamma = \nabla\gamma + \frac{1}{2\hbar}[\gamma,\gamma]_\star + F = \omega_\hbar, \quad \omega_\hbar := \omega + \sum_{k\geq 1}\hbar^k\omega_k. \tag{334}$$

Consider the *symbol map*

$$\sigma\colon \Gamma(\mathcal{W}) \to C^\infty(M)[[\hbar]],$$
$$a(x,\mathrm{y},\hbar) = \sum_{k,\ell}\hbar^k a_{k,i_1,\ldots,i_\ell}(x)\mathrm{y}^{i_1}\cdots\mathrm{y}^{i_\ell} \mapsto a(x,0,\hbar) = \sum_k \hbar^k a_{k,i_1,\ldots,i_\ell}(x), \tag{335}$$

which sends all the y^is to zero.

Theorem 3.2.3.2 (Fedosov [55]). *The symbol map induces an isomorphism*

$$\sigma\colon H^0_{\bar{\nabla}}(\Gamma(\mathcal{W})) \to C^\infty(M)[[\hbar]], \tag{336}$$

where $H^0_{\bar{\nabla}}(\Gamma(\mathcal{W}))$ denotes the space of flat sections of the Weyl bundle with respect to $\bar{\nabla}$. Moreover, since for any flat connection Equation (333) holds, we can construct a global star product on $C^\infty(M)[[\hbar]]$ by the formula

$$f \star_M g := \sigma\left(\sigma^{-1}(f) \star \sigma^{-1}(g)\right), \tag{337}$$

which defines a deformation quantization on (M,ω).

REMARK 3.2.3.3. Theorem 3.2.3.2 tells us the existence of a global version of the Moyal product for symplectic manifolds. There is a similar approach to globalization for any Poisson manifold, where we start with Kontsevich's star product on the local picture using elements of formal geometry, such as the construction of the Grothendieck connection as we have already discussed in Chapter 2.2. The deformed Grothendieck connection will replace the symplectic connection in Fedosov's picture. In fact, Fedosov's construction uses the exponential map of a symplectic connection, whereas the more general approach uses the notion of a general formal exponential map as we have discussed in Chapter 1.1.

CHAPTER 3.3

Traces and algebraic index theorem

In this chapter we recall the definition of a trace and state the Nest–Tsygan and Tamarkin–Tsygan theorems. We describe the construction of a trace given by Cattaneo and Felder using a morphism of L_∞-modules and explain its relation to the construction via the Poisson Sigma Model.

3.3.1. Algebraic index theorem

Recall that a *trace map* on a Poisson manifold (M, π) is a linear functional Tr on compactly supported functions $f, g \in C_c^\infty(M)$ with values in $\mathbb{R}((\hbar))$ such that

$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f) \tag{338}$$

(hence the name “trace"). There is a canonical trace associated to any star product coming from a symplectic manifold (M, ω) which is described within the local picture. Locally, all deformations are equivalent to the Weyl algebra and on the Weyl algebra there is a canonical trace which is constructed as an integral with respect to the Liouville measure [55]. If we consider functions with support in neighborhoods of any point of M, we set the trace equal to this canonical trace restricted to these functions. Let $\hat{A}(TM)$ denote the $\hat{A}$*-genus* of M, which is a characteristic class of the tangent bundle TM. One can express it by a de Rham representative as

$$\hat{A}(TM) = \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right), \tag{339}$$

where R denotes the curvature of any connection on TM.

Theorem 3.3.1.1 (Nest–Tsygan [87]). *Let (M, ω) be a compact symplectic manifold and let $\star$ be a star product with characteristic class $\omega_\hbar = \omega + \hbar\omega_1 + \hbar^2\omega_2 + \cdots$. Then the canonical trace associated to $\star$ obeys*

$$\mathrm{Tr}(1) = \int_M \hat{A}(TM)\exp(\omega_\hbar/\hbar). \tag{340}$$

Consider again a Poisson manifold (M, π). Let $A_\hbar := (C^\infty(M)[[\hbar]], \star)$ with star product coming from the Poisson structure π (e.g. Kontsevich's star product), and denote by $CH_\bullet(A_\hbar)$ the *cyclic homology* and by $PH_\bullet(A_\hbar)$ the *periodic cyclic homology*. One can show that $CH_0(A_\hbar) \cong HH_0(A_\hbar)$, where $HH_\bullet(A_\hbar)$ denotes the *Hochschild homology*. Moreover, as shown by Shoikhet and Dolgushev, the zeroth Hochschild homology is isomorphic to the zeroth *Poisson homology*[1] $HP_0(M)$. If we assume that there is a volume form Ω on M and that the Poisson structure π is unimodular and $\mathrm{div}_\Omega \pi = 0$, we can construct a map

$$\begin{aligned} HP_0(M) &\to \mathbb{R}, \\ f &\mapsto \int_M f\Omega. \end{aligned} \tag{341}$$

[1] The Poisson homology $HP_\bullet(M)$ of a Poisson manifold (M, π) is given by the homology of the complex $(\mathcal{T}_{\mathrm{poly}}(M), [\pi, \ ])$, where $[\pi, \ ]$ is the Poisson differential of degree -1 (note that we have taken the opposite grading).

Now we can define an integration map on the zeroth periodic cyclic homology by composition

$$\mathcal{I}\colon PH_0(A_\hbar) \to CH_0(A_\hbar) \to HP_0(M) \xrightarrow{\int_M} \mathbb{R}. \tag{342}$$

Let $\mathcal{R}$ be a DG ring with differential $\mathrm{d}_{\mathcal{R}}$. For a projective $\mathcal{R}$-module $\mathcal{M}$, one defines a connection to be a map

$$\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}, \tag{343}$$

where $\Omega^1_{\mathcal{R}} := \mathcal{R} \otimes \mathcal{R}[1]$, with the usual property

$$\nabla(r \cdot m) = \mathrm{d}_{\mathcal{R}} r \cdot m + (-1)^{|r|} r \nabla m, \quad \forall r \in \mathcal{R},\ m \in \mathcal{M}. \tag{344}$$

The *Atiyah class* of a connection ∇ is then defined by

$$\mathrm{At}(\nabla) := [\nabla, \mathrm{d}_{\mathcal{R}}] \in \Omega^1_{\mathcal{R}} \otimes \mathrm{End}_{\mathcal{R}}(\mathcal{M}). \tag{345}$$

In fact, $[\mathrm{At}(\nabla)]$ measures the the obstruction to find a $\mathrm{d}_{\mathcal{R}}$-compatible connection. We define the *Chern character* of a connection ∇ by

$$\mathrm{Ch}(\nabla) := \mathrm{Tr}\exp\left(\frac{1}{2\pi i}\mathrm{At}(\nabla)\right). \tag{346}$$

Moreover, one can then define more generally the $\hat{A}$-genus of a connection ∇ on $\mathcal{M}$ in terms of these classes by

$$\hat{A}(\nabla) = \exp\left(-\frac{1}{2}\mathrm{Ch}(\nabla)\right)\mathrm{Td}(\nabla), \tag{347}$$

where Td denotes the *Todd class*, defined by

$$\mathrm{Td}(\nabla) := \frac{\mathrm{Ch}(\nabla)}{1-\exp(-\mathrm{Ch}(\nabla))}. \tag{348}$$

Theorem 3.3.1.2 (Tamarkin–Tsygan [100]). *Let M be a compact manifold with formal Poisson structure $\pi \in \hbar\Gamma(\bigwedge^2 TM)[[\hbar]]$ and Ω a volume form on M with $\mathrm{div}_\Omega \pi = 0$ and $c \in PC_0(A_\hbar)$. Then*

$$\mathcal{I}(c) = \int_M \hat{A}_u(TM)\mathrm{Ch}(c)\exp(-\pi/u). \tag{349}$$

Here

$$\hat{A}_u(TM) := \hat{A}_0(TM) + u\hat{A}_1(TM) + u^2\hat{A}_2(TM) + \dots = \det{}^{1/2}\left(\frac{uR/2}{\sinh(uR/2)}\right), \tag{350}$$

where $\hat{A}_j(TM) \in H^{2j}(M)$ are the components of the $\hat{A}$-genus.

3.3.2. A trace map for negative cyclic chains

In [28] it was shown how one can obtain a trace map by constructing an L_∞-morphism from negative cyclic chains to multivector fields with an adjunction of the formal parameter u of degree 2. Moreover, the relation to the BV formulation of the Poisson Sigma Model and how the former formula can be interpreted as an expectation value with respect to the corresponding quantum field theory was shown. However, this construction was only given for open subsets M of $\mathbb{R}^d$. We will extend this construction to a global one using notions of formal geometry as we have seen before.

THEOREM 3.3.2.1 (CATTANEO–FELDER [28]). *Let M be an open subset of $\mathbb{R}^d$ and consider a volume form Ω on M. Denote by $\mathrm{div}^u_\Omega := u\,\mathrm{div}_\Omega$. Let $A = C^\infty(M)$ and let $(\mathcal{T}_{poly}(M), \mathrm{div}_\Omega)$ be the DG module over the DGLA $(\mathcal{T}_{poly}(M)[[u]], [\,,\,])$ with trivial $(\mathcal{T}_{poly}(M)[[u]], [\,,\,])$-action. Then there exists an $\mathbb{R}[[u]]$-linear morphism of L_∞-modules over $(\mathcal{T}_{poly}(M)[[u]], [\,,\,])$*

$$\mathcal{V}\colon (CC_-(A), b + uB) \to (\mathcal{T}_{poly}(M)[[u]], u\,\mathrm{div}_\Omega), \tag{351}$$

such that

(1) The zeroth Taylor component $\mathcal{V}_0$ of $\mathcal{V}$ vanishes on $CC_m(A)$, $m > 0$ and for $f \in A \subset CC_0(A)$, $\mathcal{V}_0(f) = f$.

(2) For $\gamma \in \Gamma(\bigwedge^k TM)$, $\ell \geq 0$, $a = [a_0 \otimes \cdots \otimes a_m] \in CC_m(A)$,

$$\mathcal{V}_1(\gamma u^\ell \mid a) = \begin{cases} (-1)^m u^s\, \gamma \lrcorner H(a), & \text{if } k \geq m \text{ and } s = k + \ell - m - 1 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{352}$$

where $\lrcorner\colon \mathcal{T}^k_{poly}(M) \otimes \Omega^m(M,\mathbb{R}) \to \mathcal{T}^{k-m}_{poly}(M)$ and H is the HKR map.

(3) The maps $\mathcal{V}_n$ are equivariant under linear coordinate transformations and[2]

$$\mathcal{V}_n(\gamma_1 \cdots \gamma_n \mid a) = \gamma_1 \wedge \mathcal{V}_{n-1}(\gamma_2 \cdots \gamma_n \mid a) \tag{353}$$

whenever $\gamma_1 = \sum (c^i_k x_k + d^i)\partial_i \in \mathcal{T}_{poly}(M) \subset (\mathcal{T}_{poly}(M)[[u]], [\,,\,])$ is an affine vector field and $\gamma_1, \ldots, \gamma_n \in (\mathcal{T}_{poly}(M)[[u]], [\,,\,])$.

The Taylor components of $\mathcal{V}$ are given by maps

$$\mathcal{V}_n\colon (S^n\mathcal{T}^{\bullet+1}_{poly}(M)[[u]], [\,,\,]) \otimes CC_-(A) \to \mathcal{T}^{n-1}_{poly}(M). \tag{354}$$

Note that an element of degree $+1$ in $(\mathcal{T}_{poly}(M)[[u]], [\,,\,])$ has the form $\tilde\pi = \pi + uh$, where π is a bivector field and h a function. The Maurer–Cartan equation $u\,\mathrm{div}_\Omega\tilde\pi + \frac{1}{2}[\tilde\pi, \tilde\pi] = 0$ translates to $[\pi, \pi] = 0$ and

$$\mathrm{div}_\Omega \pi - [h, \pi] = 0, \tag{355}$$

and hence π is Poisson and h corresponds to the Hamiltonian function of the Hamiltonian vector field $\mathrm{div}_\Omega \pi$. As we have seen, this is equivalent to the unimodularity condition.

REMARK 3.3.2.2. The morphism $\mathcal{V}$ is in fact related to Shoikhets morphism [97] in the proof of Tsygan's formality theorem on chains [101] for $M = \mathbb{R}^d$. It is a morphism of L_∞-modules over $\mathcal{T}^{\bullet+1}_{poly}(M)$ from $C_\bullet(A)$ to the DG module of differential forms $(\Omega^\bullet(M,\mathbb{R}), \mathrm{d} = 0)$ and extends to (323). The action of $\gamma \in \mathcal{T}^{\bullet+1}_{poly}(M)$ on $\Omega(M,\mathbb{R})$ is given by Lie derivative $L_\gamma = \mathrm{d}\,\iota_\gamma \pm \iota_\gamma\,\mathrm{d}$, where the internal multiplication of vector fields is extended to multivector fields by $\iota_{\gamma\wedge\delta} = \iota_\gamma \iota_\delta$. This construction was globalized by Dolgushev to any manifold M. Moreover, recall that a volume form $\Omega \in \Omega^d(M,\mathbb{R})$ defines an isomorphism

$$\begin{aligned} \mathcal{T}^k_{poly}(M) &\to \Omega^{d-k}(M) \\ \gamma &\mapsto \iota_\gamma\Omega, \end{aligned} \tag{356}$$

and thus we identify the differential d on $\Omega(M,\mathbb{R})$ by the divergence operator div_Ω on $\mathcal{T}_{poly}(M)$. By the fact that $\mathcal{V}$ is an L_∞-morphism we get $\iota_{\mathrm{div}_\Omega\gamma}\Omega = \mathrm{d}\,\iota_\gamma\Omega$.

[2] Note that we write $\gamma_1 \cdots \gamma_n$ for the symmetric tensor product $\gamma_1 \odot \cdots \odot \gamma_n$ of multivector fields $\gamma_1, \ldots, \gamma_n \in \mathcal{T}_{poly}(M)$.

Let

$$\tilde{\tilde{\gamma}} := \tilde{\gamma} + uh, \tag{357}$$

which is a Maurer–Cartan element if $\gamma + uh$ is a Maurer–Cartan element in $(\mathcal{T}_{poly}(M)[u]((\hbar)), \delta)$. If we consider the twist of $\mathcal{V}$ by $\tilde{\tilde{\gamma}}$, denoted by $\mathcal{V}^{\tilde{\tilde{\gamma}}}$, we can define a *trace map* [28]

$$\begin{aligned} \mathrm{Tr}\colon C_c^{\infty}(M)[\![\hbar]\!] &\to \mathbb{R}((\hbar)) \\ f &\mapsto \mathrm{Tr}(f) = \int_M \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{V}_n(\tilde{\tilde{\gamma}}, \dots, \tilde{\tilde{\gamma}} \mid f), \end{aligned} \tag{358}$$

since $\mathcal{V}^{\tilde{\tilde{\gamma}}}\colon (CC_{\bullet}(A_{\gamma}), b + uB) \to (\mathcal{T}_{poly}(M)[u]((\hbar)), u\,\mathrm{div}_{\Omega})$ is a chain map. We will elaborate on this fact a bit more in Section 3.5.

REMARK 3.3.2.3. For a d-manifold M denote by $V\mathcal{T}_{poly}(M) := \Omega^{d}(M, \bigwedge TM)$ differential forms of degree d with values in multivector fields. By the isomorphism as mentioned in Remark 3.3.2.2, we can construct a natural non-degenerate pairing by

$$\begin{aligned} \langle\ ,\ \rangle\colon V\mathcal{T}_{poly}(M) \otimes \Omega_c(M;\mathbb{R}) &\to \mathbb{R} \\ (\alpha \otimes \beta) &\mapsto \langle \alpha, \beta \rangle := \int_M \iota_{\alpha}(\beta), \end{aligned} \tag{359}$$

where α is a k-vector field and β is a k-form. Here Ω denotes again a chosen volume form on M. We have denoted by $\Omega_c(M;\mathbb{R})$ differential forms with compact support. It is obvious that this map can be extended u-bilinearly. Moreover, there is an isomorphism

$$\begin{aligned} \mathcal{T}_{poly}(M)[u] &\to V\mathcal{T}_{poly}(M)[u] \\ \alpha &\mapsto \alpha\Omega. \end{aligned} \tag{360}$$

3.3.3. Construction via the Poisson Sigma Model

3.3.3.1. Splitting of the space of fields. We consider a *symplectic splitting* of the space of fields into residual fields and fluctuations, which, as we have discussed in Chapter 0.1, exists by techniques of Hodge theory. We write

$$\mathcal{F} = \mathcal{M}_1 \times \mathcal{M}_2, \tag{361}$$

where $\mathcal{M}_1$ is the space of residual fields and $\mathcal{M}_2$ the space of fluctuation fields. We want to assume that $\mathcal{M}_1$ is finite-dimensional, which is the case for BF-like theories (such as the Poisson Sigma Model). In this case it is always possible to find a splitting of the BV Laplacian $\Delta = \Delta_1 + \Delta_2$, where Δ_j is a BV Laplacian on $\mathcal{M}_j$, $j = 1, 2$. Consider a half-density f on $\mathcal{F}$. Then for any Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}_2$ we get

$$\Delta_1 \int_{\mathcal{L}} f = \int_{\mathcal{L}} \Delta f. \tag{362}$$

Here $\int_{\mathcal{L}}$ denotes the BV pushforward as in Chapter 0.1, which is defined on half-densities by restricting the half-density to $\mathcal{L}$ which makes it a density and apply the Berezinian integral. As we have seen in Chapter 0.1, for BF-like theories, $\mathcal{F}$ is given as the direct sum of two complexes $\mathcal{C} \oplus \bar{\mathcal{C}}$ endowed with differentials δ and $\bar{\delta}$ respectively. We want them to be endowed with a non-degenerate pairing $\langle\ ,\ \rangle$ of degree -1 such that the differentials are related by $\langle \delta B, A \rangle = \langle \bar{\delta} B, A \rangle$ for all $A \in \mathcal{C}$ and $B \in \bar{\mathcal{C}}$. In that case $\mathcal{M}_1$ is given by the cohomology $\mathcal{H} \oplus \bar{\mathcal{H}}$ and $\mathcal{M}_2$ is just a complement in $\mathcal{F}_\partial$. For the case of the Poisson Sigma Model with boundary $(\partial\Sigma, \omega)$ such that the boundary is given by the disjoint union of two boundary components $\partial_1\Sigma$ and $\partial_2\Sigma$ we have

(363) $$\mathcal{F}_\partial = \Omega^\bullet(\partial_1\Sigma, T_xM) \oplus \Omega^\bullet(\partial_2\Sigma, T^*_xM)[1],$$

for a constant background field $x\colon \Sigma \to M$, and thus

(364) $$\mathcal{M}_1 = H^\bullet(\partial_1\Sigma, T_xM) \oplus H^\bullet(\partial_2\Sigma, T^*_xM)[1].$$

According to the splitting of the space of fields, we write $\mathsf{X} = \mathsf{x} + \mathcal{X}$ and $\eta = \mathsf{e} + \mathcal{E}$, where $\mathsf{x}, \mathsf{e} \in \mathcal{M}_1$ and $\mathcal{X}, \mathcal{E} \in \mathcal{M}_2$.

3.3.3.2. Obtaining a trace. Consider now the Poisson Sigma Model on the disk D. Let

(365) $$Z_0 := \int_{\mathcal{L}} \exp\left(\frac{i}{\hbar}\mathcal{S}_0\right);$$

and define the *vacuum expectation value* of an observable by the map

(366) $$\begin{aligned} \langle\ \rangle_0 \colon \mathcal{A}_\hbar &\to \mathbb{R}[[\hbar]] \\ f &\mapsto \langle f\rangle_0 := \frac{1}{Z_0}\int_{\mathcal{L}} \exp\left(\frac{i}{\hbar}\mathcal{S}_0\right) f. \end{aligned}$$

The map $\mathcal{V}_n$ can be expressed as the vacuum expectation of an observable $\mathcal{S}_{\gamma_1}\cdots\mathcal{S}_{\gamma_j} O_{a_0,\ldots,a_m}$, where

(367) $$O_{a_0,\ldots,a_m} := a_0(\mathsf{X}(t_0)) \int_{t_1<t_2<\cdots<t_m \in \partial D\setminus\{t_0\}} a_1(\mathsf{X}(t_1)) \cdots a_m(\mathsf{X}(t_m)).$$

For m points $t_1, \ldots, t_m \in \partial D$ we consider the ordering $t_0 < \cdots < t_m$, which means that if we start at t_1 and move counterclockwise on ∂D, we wil first meet t_2, then t_3, and so on. If we embed the disk into the complex plane, i.e. we have $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and set $t_0 = 1$, we can express the counterclockwise condition on ∂D by $0 < \arg(t_1) < \arg(t_2) < \cdots < \arg(t_m) < 2\pi$. The cohomology $H^\bullet(D)$ is 1-dimensional and concentrated in degree zero, while the relative cohomology $H^\bullet(D, \partial D)$ is 1-dimensional and concentrated in degree two. So, for $\mathcal{M}_1, \mathcal{H}, \bar{\mathcal{H}}$ as defined in Section 3.3.3.1, we get $\mathcal{H} = (\mathbb{R}^d)^*[1]$ and $\bar{\mathcal{H}} = \mathbb{R}^d$, thus $\mathcal{M}_1 = T^*[1]M$. Note that functions on $\mathcal{M}_1$ are then multivector fields on M with reversed degree and Ω_1 is given by the divergence operator div_μ for the constant volume form. Note that Ω_1 is an operator of degree $+1$. For a function $f \in C^\infty(M)$ and some Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}_2$, we have a map

(368) $$\mathrm{tr}(f) := \frac{1}{Z_0}\int_{\mathcal{L}} \exp\left(\frac{i}{\hbar}(\mathcal{S}_0 + \mathcal{S}_{\tilde\pi})\right) O_f = \left\langle \exp\left(\frac{i}{\hbar}\mathcal{S}_{\tilde\pi}\right) O_f \right\rangle_0;$$

given by the expectation value of the corresponding observable. Recall that $\tilde\pi$ is a Maurer–Cartan element for a unimodular Poisson structure and that we work with the boundary condition $\eta|_{\partial D} = 0$. For two functions $f, g \in C^\infty(M)$, we define $O_f(\mathsf{X}, \eta) := f(\mathsf{X}(1))$ for $1 \in \partial D$ and $O_g(\mathsf{X}, \eta) := g(\mathsf{X}(0))$ for $0 \in \partial D$. Moreover, define[3]

(369) $$\mathrm{tr}_2(f, g) := \left\langle \exp\left(\frac{i}{\hbar}\mathcal{S}_{\tilde\pi}\right) O_{f,g} \right\rangle_0 = \left\langle \exp\left(\frac{i}{\hbar}\mathcal{S}_{\tilde\pi}\right) f(\mathsf{X}(t)) \int_{s \in \partial D\setminus\{t\}} g(\mathsf{X}(s)) \right\rangle_0;$$

Then we observe

(370) $$\Omega_1 \mathrm{tr}_2(f, g) = \left\langle \exp\left(\frac{i}{\hbar}\mathcal{S}_{\tilde\pi}\right) O_{f,g} \right\rangle_0;$$

[3]Note that this does not depend on t, since we fix it on the boundary.

where δ was the differential on the complex $\mathcal{C}$ in the definition of the space of fields in Section 3.3.3.1. This follows from the *Ward identity*

$$\langle \delta \mathcal{O} \rangle_0 = \left\langle \mathcal{O}\, \frac{i}{\hbar}\, \mathcal{O} \right\rangle_0, \tag{371}$$

which is true by (362), the fact that Z_0 is constant on $\mathcal{M}_1$ and the Leibniz rule for the BV Laplacian

$$\Delta(fg) = \Delta f\, g + (-1)^{|f|} f\, \Delta g - (-1)^{|f|}\{f, g\}, \quad \forall f, g \in C^{\infty}(M) \tag{372}$$

(see also Appendix C.2, Equation (478)). Hence by (240) the two functions f, g can move under the trace map from both sides[4] to each other on ∂D. Thus we get

$$\hbar\, \mathrm{tr}_2(f, g) = \mathrm{tr}(f \star g) - \mathrm{tr}(g \star f). \tag{373}$$

Hence we get a trace on $C^{\infty}_c(M)$ by

$$\mathrm{Tr}(f) := \int_M \mathrm{tr}(f). \tag{374}$$

To globalize the construction, we want to consider the formal global action (244) and additional vertices in the *Feynman graph expansion*. In fact we will have two types of vertices in the bulk, the ones representing the formally lifted Poisson structure $\tilde{\pi}^{\varphi} := T\varphi^* \tilde{\pi} = \tilde{\pi}$ ... $uT\varphi^* h$ and the ones representing the R vector field coming from the definition of the Grothendieck connection. We will also consider additional vertices on the boundary where we place solutions of (232). Then we can consider the vacuum expectation value

$$\left\langle \exp\left(\frac{i}{\hbar} \mathcal{S}^{\varphi}_{\pi, R}\right) \mathcal{O}_{T\varphi^* f} \right\rangle_0, \tag{375}$$

where $\mathcal{S}^{\varphi}_{\pi,R} := \mathcal{S}^{\varphi}_{\tilde{\pi}} + \varphi^* \mathcal{S}_R$.

REMARK 3.3.3.1. Recall from Section 2.3.5 that $\mathcal{S}_{D,\gamma} = \int_{\partial D} \gamma(\mathsf{X}; \bar{\eta})$. The additional vertices labeled by a solution γ of (232) give rise to the twisted formal global action, i.e. an action functional of the form $\tilde{\mathcal{S}}_D + \mathcal{S}_{D,\gamma}$.

PROPOSITION 3.3.3.2. *The map*

$$\mathrm{Tr}\colon f \mapsto \int_M \left\langle \exp\left(\frac{i}{\hbar} S^{\varphi}_{\pi,R} + \frac{i}{\hbar} \mathcal{S}_{D,\gamma}\right) \mathcal{O}_{T\varphi^* f} \right\rangle_0 \tag{376}$$

coincides with

$$\mathrm{Tr}\colon f \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \int_M \mathcal{V}_n(R, \ldots, R \mid T\varphi^* f), \tag{377}$$

where we consider $\mathcal{V}_n(R, \ldots, R \mid \cdot)$ to be defined on the negative cyclic complex for sections of the Weyl algebra $\mathcal{W}$.

This can be seen by constructing the maps $\mathcal{V}_n$ in terms of graphs. We will do this in Section 3.4.1.

[4]This argument follows from Stokes' theorem and the "bubbling" concept of the Deligne–Mumford compactification on the disk.

REMARK 3.3.3.3. In fact, one can construct Kontsevich's star product directly by using a path integral quantization with respect to the formal global action $\tilde{\mathcal{S}}_D$ as in (244) on the disk, using a similar apporach as in [26], with the difference that the observables on the boundary are given by $\overline{\mathcal{D}}_G$-closed sections of the form $\mathcal{O}_{(T\varphi)^*f}$ (see Figure 3.3.1). Hence we can write it down as a path integral

$$f \star_M g(x) = \int_{\hat{X}(1)=x} (T\varphi_x^* f)(\hat{X}(0))\,(T\varphi_x^* g)(\hat{X}(1)) \exp\left(\frac{i}{\hbar}\tilde{\mathcal{S}}_{D,x}[\hat{X},\hat{\eta}]\right)\Bigg|_{y=0} \tag{378}$$

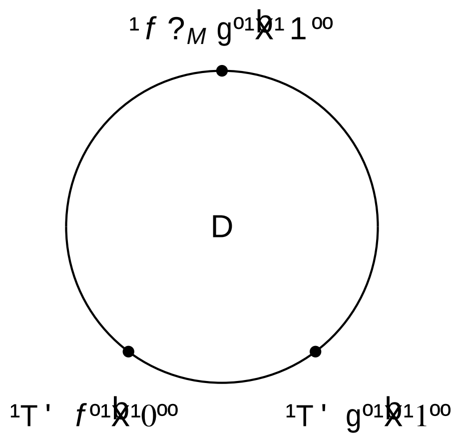


FIGURE 3.3.1. Cyclically ordered points on $S^1 = \partial D$

CHAPTER 3.4

Feynman graphs for the globalized action

In this chapter we will explain the construction of the morphism $\mathcal{V}$ in terms of graphs. Moreover, we will consider an equivariant extension for the underlying worldsheet symmetry given by the S^1-action. Moreover, we state an equivariant version of Stokes' theorem and describe the corresponding propagator.

3.4.1. Construction via graphs

We want to describe how the Taylor components of $\mathcal{V}$ are given in terms of graphs. In fact we have

$$\mathcal{V}_n(\gamma \mid a) = \sum_{\Gamma \in \mathcal{G}_{\mathbf{k},m}} w_\Gamma \, \mathcal{V}_\Gamma(\gamma \mid a); \tag{379}$$

where $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_n$, with $\gamma_i \in \Gamma(\bigwedge^{k_i} TM)[[\hbar]]$, $\mathbf{k} = (k_1, \ldots, k_n)$ and $a = [a_0 \otimes \cdots \otimes a_m] \in C_m(A)$. Here $w_\Gamma \in \mathbb{R}$ denotes the weight of a graph Γ according to the given Feynman rules, which can be computed as integrals over configuration spaces of points on the the interior of the disk and on the boundary. We want to recall the definition of the finite set $\mathcal{G}_{\mathbf{k},m}$ of oriented graphs as in [28].

For each graph[1] $\Gamma \in \mathcal{G}_{\mathbf{k},m}$ with $n+m$ vertices (n vertices in the bulk and m vertices on the boundary), we assign a vertex set $V(\Gamma) = V_1(\Gamma) \sqcup V_2(\Gamma) \sqcup V_w(\Gamma)$. We will distinguish between two different types of vertices which we call the *black vertices* $V_b(\Gamma) = V_1(\Gamma) \sqcup V_2(\Gamma)$ and the *white vertices* $V_w(\Gamma)$. Within the black vertices we will also distinguish between vertices of *type 1* and of *type 2* according to the following rules.

- There are n vertices in $V_1(\Gamma)$. There are exactly k_i edges originating at the ith vertex of $V_1(\Gamma)$.
- There are m vertices in $V_2(\Gamma)$. There are no edges originating at these vertices.
- There is exactly one edge pointing at each vertex in $V_w(\Gamma)$ and no edge originating from it.
- There are no edges starting and ending at the same vertex.
- For each pair of vertices (i, j) there is at most one edge from i to j.

Each multivector field γ_i can be endowed with a power of the formal parameter $\hbar^\ell$, which represent the residual field assigned to a black vertex.

Example 3.4.1.1. Let Γ be the graph constructed as in Figure 3.4.1 using the multivector fields $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(\bigwedge TM)$ with $|\gamma_1| = 5, |\gamma_2| = 4$, and $|\gamma_3| = 2$. Then we get

$$\begin{aligned}\mathcal{V}_\Gamma(\gamma_1\hbar^{\ell_1} \wedge \gamma_2\hbar^{\ell_2} \wedge \gamma_3\hbar^{\ell_3} \mid [a_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4]) \\ = \sum \gamma_1^{i_1i_2i_3i_4i_5}\partial_{i_1}\gamma_2^{j_1j_2j_3j_4}\partial_{j_1}\gamma_3^{m_1m_2}\partial_{i_2}a_0\partial_{i_3}a_1\partial_{i_4}a_2\partial_{j_2}a_3\partial_{m_1}a_4 \, \partial_{i_5} \wedge \partial_{j_3} \wedge \partial_{j_4} \wedge \partial_{m_2};\end{aligned} \tag{380}$$

where we sum over all indices and where we set $\partial_i := \frac{\partial}{\partial x^i}$ for local coordinates (x^i) on M.

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[1] In addition to the rules below we will also consider it modulo graph isomorphisms which respect the partition and the orderings. The set $\mathcal{G}_{\mathbf{k},m}$ is in principle given by the set of equivalence classes.

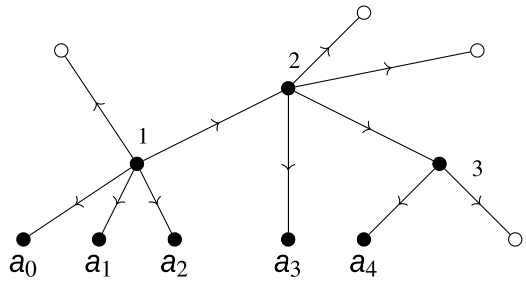

FIGURE 3.4.1. Example of a graph Γ.

To compute the configuration integrals, we want to make a degree count, i.e. we want the form degree to be equal to the dimension of the configuration space. Let Σ be a manifold with boundary and define the *configuration space* of n points in the bulk and m points on the boundary by

(381) $$\mathrm{Conf}_{n,m}(\Sigma) := \{(x_1,\dots,x_n,y_1,\dots,y_m) \in \mathrm{int}(\Sigma)^n \times (\partial\Sigma)^n \mid x_i \neq x_j,\ y_i \neq y_j\ \forall i \neq j\}.$$

Moreover, denote by $C_{n,m}(\Sigma)$ the *FMAS-compactification* [63, 5] of $\mathrm{Conf}_{n,m}(\Sigma)$ (or of its quotient with respect to the corresponding group action). Let now $\Sigma = D$ and fix the point 1 on ∂D. Then we have to work on the section space

(382) $$C^0_{n,m}(D) := \{(z,t) \in (\mathrm{int}(D))^n \times (\partial D)^m \mid z_i \neq z_j\ (i \neq j),\ 0 < \arg(t_1) < \dots < \arg(t_m) < 2\pi\}.$$

The space (382) has dimension $2n + m$. Moreover, the number m represents the amount of points on the boundary distinct from the fixed point 1, i.e. the total amount of points on the boundary is $m + 1$. In fact, (382) is equal to the set $\{(z,t) \in C_{n,m+1}(D) \mid t_0 = 1\}$ for $m \geq 1$.
As already mentioned, we have an S^1-action on the disk. Instead of working with the quotient of the configuration space by $PSL_2(\mathbb{R})$, we will work with *equivariant differential forms*, which arise from the equivariant BV construction of the Poisson Sigma Model within the Feynman graph expansion.

3.4.2. Equivariant differential forms and equivariant Stokes' theorem

We want to work with equivariant differential forms with respect to the S^1-action on the disk. We define them as

(383) $$\Omega_{S^1}(D) := \Omega(D)^{S^1}[u],$$

where the differential is given by $\mathrm{d}_{S^1} := \mathrm{d} - u\,\iota_{\mathbf{v}}$. Here $\mathbf{v} \in \Gamma(TD)$ denotes the image of the infinitesimal vector field $\frac{\mathrm{d}}{\mathrm{d}t}$, which is the generator of the infinitesimal action $\mathbb{R}\frac{\mathrm{d}}{\mathrm{d}t} \to \Gamma(TD)$. Now consider a differential form ω on the configuration space $C^0_{n,m}(D)$. We want to describe the boundary of the configuration space. Let S be a subset of $\bar{n} \geq 2$ points in the bulk which collapse at a point in the bulk of the disk. Then the *stratum of type I* is given by

(384) $$\partial_S C_{n,m}(D) \cong C_{\bar{n}}(\mathbb{C}) \times C^0_{n-\bar{n}+1,m}(D).$$

The *stratum of type II* is constructed as follows. Let S be the subset of $\bar{n}$ points in the bulk and T the subset of $\bar{m}$ points on the boundary which collapse at a point on the boundary of the disk. Hence we get the stratum

(385) $$\partial_{S,T} C_{n,m}(D) \cong C_{\bar{n},\bar{m}}(\mathbb{H}) \times C^0_{n-\bar{n},m-\bar{m}+1}(D),$$

where $\mathbb{H}$ denotes the upper half-plane.

THEOREM 3.4.2.1 (EQUIVARIANT STOKES [28]). *Let $\omega \in \Omega_{S^1}(C_{n,m+1}(D))$. Denote also by ω its restriction on $C^0_{n,m}(D) \subset C_{n,m+1}(D)$. Denote by ω^{∂} its restriction to the coboundary 1 strata $\partial_i C^0_{n,m}(D)$. Then*

$$\int_{C^0_{n,m}(D)} d_{S^1}\omega = \sum_i \int_{\partial_i C^0_{n,m}(D)} \omega^{\partial} - u \int_{C_{n,m+1}(D)} \omega \tag{386}$$

3.4.3. Weights of graphs

We will consider a *propagator* η on $D \times D \setminus \mathrm{diag}$, where $\mathrm{diag} := \{(z,z) \mid z \in D\} \subset D \times D$ denotes the diagonal on the disk. The propagator will be a 1-form on the configuration space of the disk. In particular we have

$$\eta(z,w) := \frac{1}{4\pi i}\left(d\log \frac{(z-w)(1-z\bar{w})}{(\bar{z}-\bar{w})(1-\bar{z}w)} + z\,d\bar{z} - \bar{z}\,dz \right). \tag{387}$$

Note that this propagator is equivariant under the S^1-action, hence $\eta \in \Omega^1(D)^{S^1}$.

REMARK 3.4.3.1. An important fact [18, 38] of the propagator is

$$d\eta(z_1,z_2) = \sum_j \pi_1^*\alpha_j \wedge \pi_2^*\beta^j = \pi_1^*(x_1 \wedge e_2), \tag{388}$$

where π_1, π_2 are the projections to the first and second factor respectively. Here α_j, β^j are representatives of the cohomology classes and their duals respectively, such that $\int_D \alpha_i \wedge \beta^j = \delta_i^j$.

Computing this directly, we get

$$d_{S^1}\eta = d\eta - u\,\iota_v\eta \tag{389}$$

$$= \frac{1}{4\pi i} d(z\,d\bar{z} - \bar{z}\,dz) - u\,\iota_v\eta \tag{390}$$

$$= \pi_1^*\left(\frac{i}{2\pi} dz \wedge d\bar{z} + u(1-|z|^2)\right). \tag{391}$$

The first term of (391) is a volume form on the disk and hence a representative of the cohomology class, hence the whole is a representative of the equivariant cohomology class.

Graphically, this corresponds to the fact that if the de Rham differential acts on an edge of a graph between two (black) vertices (which represents a propagator), it will split into residual fields (see Figure 3.4.2). This can be extended to the equivariant differential d_{S^1}. The white vertices mentioned in the graph construction before are actually represented by zero modes on D. More precisely, we have the following Lemma.

LEMMA 3.4.3.2 (E.G. [28, 38]). *Let $\Gamma_{\partial e}$ be the graph which is obtained from the graph Γ by adding a white vertex $\circ$ and replacing the edge $e \in E_b(\Gamma)$ connecting two black vertices by an edge originating at the same vertex as e but ending at the white vertex $\circ$. Then*

$$d_{S^1}\omega_\Gamma = \sum_{e \in E_b(\Gamma)} (-1)^{|E_b(\Gamma)|}\,\omega_{\Gamma_{\partial e}}. \tag{392}$$

The represented zero modes are parametrized by the formal variable u^{ℓ} attached to each vertex. The weight of a graph $\Gamma \in \mathcal{G}_{(k_1,\dots,k_n),m}$ is then computed by

$$w_\Gamma = \frac{1}{k_1!\cdots k_n!}\int_{C^0_{n,m}(D)} \omega_\Gamma. \tag{393}$$

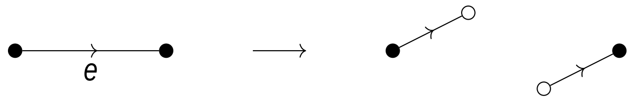

FIGURE 3.4.2. Edge split

The *equivariant cohomology* $H_{S^1}(D)$ is generated by the constant function 1. Moreover, the *relative equivariant* cohomology $H_{S^1}(D;\partial D)$ is generated by the class of

$$(z;u) := \frac{i}{2} dz \wedge d\bar{z} + u(1 - |z|^2). \tag{394}$$

REMARK 3.4.3.3. Note that with this notation we have $d_{S^1} = {}_1$.

The differential form $\omega \in \Omega^{2n+m}_{S^1}(C^0_{n;m}(D))$ is given by

$$\omega = \prod_{i\in V_1}\prod_{(i;j)\in E_b} (z_i;z_j) \prod_{i\in V_1} (z_i;u)^{r_i}, \tag{395}$$

where the number r_i is given by the degree of the vertex i plus the amount of white vertices attached to it. Moreover, we have the following lemma.

LEMMA 3.4.3.4 ([77, 28]). *For all* $z, z' \in D$ *we have*

$$\int_{w\in D} (z;w) \wedge (w;z') = 0. \tag{396}$$

Moreover, for all $z \in D$ *we have*

$$\int_{w\in D} (z;w) \wedge (w;u) = 0. \tag{397}$$

FIGURE 3.4.3. The first picture corresponds to the integrand of (396) and the second picture corresponds to the one of (397). Graphs with such a vertex w vanish.

CHAPTER 3.5

Relation to the Nest–Tsygan and Tamarkin–Tsygan theorems

This chapter contains the main results of Part 3. First we show that the map constructed in (377) is indeed a trace with respect to the global version of Kontsevich's star product. Then we will give its relation to the Tamarkin–Tsygan and Nest–Tsygan theorems. Finally, we will show its relation to the global construction given by Grady, Li and Li for the symplectic case of topological quantum mechanics.

3.5.1. Proof of the trace property

THEOREM 3.5.1.1. *The map (377) is a trace on the algebra $(C_c^\infty(M)[[\hbar]], \star_M)$.*

PROOF. This follows by the fact that

$$\mathcal{V}^{\gamma}_{\hbar}\colon (CC_{-\bullet}(\mathcal{A}_\hbar), b+uB) \to (\mathcal{T}_{poly}^{-\bullet}(M)[u][[\hbar]], u\,\mathrm{div}_\Omega) \tag{398}$$

is a chain map, which follows from Theorem 3.4.2.1 and Lemma 3.4.3.2. Using the construction with the Poisson Sigma Model, the trace property follows from (240) and the constructions in Section 3.3.3.

Indeed, consider the observable $O_{(T\varphi)(f),(T\varphi)(g)}$ for $f,g\in C_c^\infty(M)[[\hbar]]$. Note that the configuration integrals are considered on the section space where the point 1 is fixed on the boundary, labeled by the observable $O_{(T\varphi)(f)}$. We consider another point 0 on the boundary, which is not fixed, labeled by the observable $O_{(T\varphi)(g)}$. Moreover, we have some additional $m-1$ boundary points labeled by γ. Note that there are boundary strata of the configuration space where g collides to f from the left and one where it collides from the right. Recall that the dimension of the configuration space $C^0_{n,m}(D)\subset C_{n,m+1}(D)$ is given by $2n+m$. Without the point 0 we would have that the dimension is equal to $2n+m-1$, which has to be the same as the form degree of the differential form ω_Γ within the configuration integral for any graph $\Gamma\in\mathcal{G}_{(k_1,\ldots,k_n),m}$. Hence, we look at its equivariant differential $d_{S^1}\omega_\Gamma$ and apply the equivariant Stokes' theorem (Theorem 3.4.2.1). Using (379) and (393), we can write

$$\mathcal{V}_n^{\gamma}(R\otimes\cdots\otimes R\mid (T\varphi)(f)) = \sum_{\Gamma\in\mathcal{G}_{\mathbf{k},m}}\int_{C^0(D)} d_{S^1}\omega_\Gamma\left(\mathcal{V}^{\gamma}(R\otimes\cdots\otimes R\mid (T\varphi)(f))\right) \tag{399}$$

$$=\sum_{\Gamma\in\mathcal{G}_{\mathbf{k},m}}\sum_{\Gamma'<\Gamma}\int_{C_{\Gamma'<\Gamma}(H)\times C^0_{n-n'}(D)}\omega_{\Gamma'}\left(\omega_{\Gamma/\Gamma'}-u\int_{C_{\tilde\Gamma}(D)}\omega_{\tilde\Gamma}\right)\mathcal{V}^{\gamma}(R\otimes\cdots\otimes R\mid (T\varphi)(f));$$

where $\mathbf{k}=(k_1,\ldots,k_n)$ and $\Gamma'<\Gamma$ is a subgraph of Γ, where $n'<n$ points collapse in the bulk and $m'<m$ collapse on the boundary. Moreover, $\tilde\Gamma$ is a graph whose vertex set satisfies $|V(\tilde\Gamma)|=|V(\Gamma)|+1$ with the same amount of vertices in the bulk and on the boundary plus an additional vertex on the boundary. Note that by setting $u=0$, Theorem 3.4.2.1 reduces to the usual Stokes' theorem for corners. The dimension of the configuration space $C_{n,m}(H)$ modulo scaling and translation is given by $2n+m-2$. This has to be equal to the form degree of the differential form

we want to integrate. Let p be the amount of vertices labeled by $\tilde{\pi}$ and r the amount of vertices labeled by R. Then we have

$$2n + m - 2 = 2p + r; \qquad n = p + r. \tag{400}$$

This implies three different cases (see Figure 3.5.1);

$r = 2; m = 0$,
$r = m = 1$,
$r = 0; m = 2$.

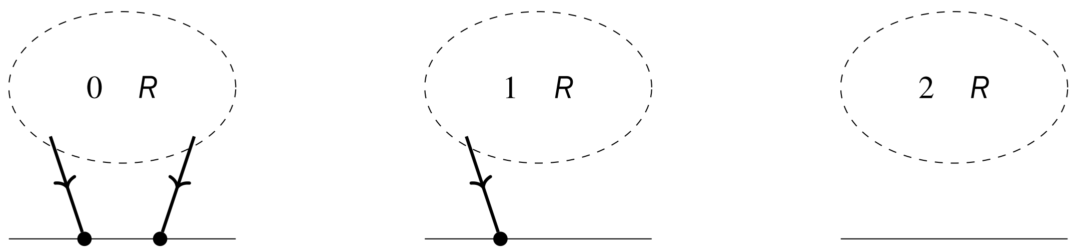

FIGURE 3.5.1. Illustration of the three different cases. The dashed ellipses represent a graph Γ in the bulk of the disk with either $r = 0$ (first picture), $r = 1$ (second picture), or $r = 2$ (third picture). Note that in each picture p can be arbitrary. The thick arrows denote the fact that there can be arbitrarily many incoming arrows, depending on the combinatorics.

Summing over all these graphs, the third picture will exactly correspond to $F^M = F^M(R, R)$, the curvature of the deformed Grothendieck connection $\mathcal{D}_G$, the second picture to $\mathcal{D}_G\Pi$ and the first picture is exactly the star product $\Pi \star \Pi$. Thus, summing them together we get a contribution

$$F^M + \mathcal{D}_G\Pi + \Pi \star \Pi = 0, \tag{401}$$

and hence these terms vanish. Hence the only strata that survive within the boundary of the configuration space are the ones where g approaches f from the left and from the right, so by [26] we get the boundary contribution $g \star f - f \star g$.

In fact, for any $\gamma = \gamma_1 \cdots \gamma_n \in (S^n\mathcal{T}^{+1}_{poly}(M))[\![u]\!]$ and $a \in CC_m(A)$, we have

$$\begin{aligned}\mathcal{V}_n(\gamma \mid a) &+ (-1)^{|\gamma|+m}\mathcal{V}_n(\gamma \mid (b+uB)a) \\ &+ \sum_{k=0}^{n}\sum_{\sigma\in S_{k,n-k}} (-1)^{|\gamma|-1}\epsilon(\sigma;\gamma)\mathcal{V}_k(\gamma_{\sigma(1)}\cdots\gamma_{\sigma(k)} \mid \mathcal{U}_{n-k}(\bar{\gamma}_{\sigma(k+1)}\cdots\bar{\gamma}_{\sigma(n)})\,a) \\ &+ \sum_{i<j}\epsilon_{ij}\mathcal{V}_{n-1}((-1)^{|\gamma|-1}[\gamma_i,\gamma_j]_{SN}\,\gamma_1\cdots\hat{\gamma}_i\cdots\hat{\gamma}_j\cdots\gamma_n \mid a) = \operatorname{div}\mathcal{V}_n(\gamma \mid a),\end{aligned} \tag{402}$$

where $\bar{\gamma}_i$ denotes the projection of γ_i to $\mathcal{T}^{+1}_{poly}(M)$, $S_{p,q} \subset S_{p+q}$ is the set of (p,q)-shuffles and the signs $\epsilon(\sigma;\gamma), \epsilon_{ij}$ are the Koszul signs coming from the permutation of the γ_i, and $|\gamma| = \sum_i |\gamma_i|$. Note that div is extended to a degree $+1$ derivation on $S\mathcal{T}^{+1}_{poly}(M)[\![u]\!]$. The maps $\mathcal{U}_k\colon S^k\mathcal{T}^{+1}_{poly}(M) \to \mathcal{D}^{+1}_{poly}(M)$ are the Taylor components of Kontsevich's L_∞-morphism.

Indeed, one can show that for any $a = [a_0 \otimes \cdots \otimes a_m] \in C_{-m}(A)$, $\Gamma \in \mathcal{G}_{\mathbf{k},m}$ and $\xi = \xi_1 \wedge \cdots \wedge \xi_n$ with $\xi_i \in \bigwedge^{k_i} T(M)$ we have[1]

$$\operatorname{div} \mathcal{V}_n(\xi \mid a) - \mathcal{V}_n(\operatorname{div} \xi \mid a) = \sum_{(\Gamma,e)} (-1)^{|E(\Gamma)|-|e|} w_{\partial_e \Gamma} \mathcal{V}_\Gamma(\xi \mid a), \tag{403}$$

by identifying $\bigwedge^n T(M)$ with $C^\infty(M)[\theta_1, \ldots, \theta_n]$, where θ_i are odd variables such that $\operatorname{div} = \sum_{1 \le i \le n} \frac{\partial^2}{\partial t_i \partial \theta_i}$. In fact, we have

$$\sum_{e \in E(\Gamma)} (-1)^{|E(\Gamma)|-|e|} w_{\partial_e \Gamma} = \sum_i \int_{\partial_i C^0_{n,m}(D)} \omega_\Gamma \pm \sum_{k=0}^{m} (-1)^{km} \int_{C^0_{n,m+1}(D)} j_k^* \omega_\Gamma, \tag{404}$$

where j_k is defined as follows: Define a map

$$\begin{aligned} j_0 \colon C^0_{n,m}(D) &\to C_{n,m}(D) \\ (z, 1, t_1, \ldots, t_m) &\mapsto (z, t_1, \ldots, t_m) \end{aligned} \tag{405}$$

Moreover, define a map

$$\begin{aligned} \tau \colon C^0_{n,m}(D) &\to C^0_{n,m}(D) \\ (z_1, \ldots, z_n, 1, t_1, \ldots, t_m) &\mapsto (z_1, \ldots, z_n, 1, t_m, t_1, \ldots, t_{m-1}) \end{aligned} \tag{406}$$

Then the collection $j_k := j_0 \circ \underbrace{\tau \circ \cdots \circ \tau}_{k \text{ times}}$, for $k = 0, 1, \ldots, m-1$ defines an embedding

$$j \colon C^0_{n,m}(D) \sqcup \cdots \sqcup C^0_{n,m}(D) \hookrightarrow C_{n,m}(D). \tag{407}$$

Moreover, note that

$$\int_{C_{n,m+1}(D)} \omega = \sum_{k=0}^{m} (-1)^{km} \int_{C^0_{n,m+1}(D)} j_k^* \omega; \tag{408}$$

and the second term on the right-hand side of (404) is given by $\mathcal{V}_{n+1}(\xi \mid Ba)$. Let us look at the boundary integral in the first term of the right hand side of (404). As argued in [28], one can show that treating the boundary strata of type I, the only remaining term will be the sum in (402) containing the Schouten–Nijenhuis bracket. The strata of type II will give a contribution as the sum in (402) containing Kontsevich's L_∞-morphism and a term $\mathcal{V}_{n-1}(\xi \mid ba)$.
Note that (404) together with (373), (388) and Lemma 3.4.3.2 ensure that $\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f)$ since $\operatorname{div} \varphi_\hbar = 0$.
Using Equation (402), we get that the twist of $\mathcal{V}$ by $\varphi_\hbar$ is indeed a chain map. Recall from Chapter 3.3 that the zeroth cyclic homology $CH_0(A_\hbar)$ is isomorphic to the zeroth Hochschild homology $HH_0(A_\hbar)$, which is again isomorphic to the zeroth Poisson homology $HP_0(M)$. Hence the chain map $\mathcal{V}^{\varphi_\hbar}$ induces a map

$$C^\infty(M)[[\hbar]][u, u^{-1}] \cong CH_0(A_\hbar) \cong HH_0(A_\hbar) \xrightarrow{\mathcal{V}^{\varphi_\hbar}} HP_0(M) \to \mathbb{R}((\hbar))[u, u^{-1}], \tag{409}$$

given by integration as in (341).

[1]We use the same notation as in Lemma 3.4.3.2.

3.5.2. Relation to the Tamarkin–Tsygan theorem

THEOREM 3.5.2.1. *The trace formula in (377) evaluated at any periodic cyclic chain $c \in PC_{-m}(A_\hbar)$ is given by (349).*

PROOF. Note that since the Lie derivative with respect to the Poisson tensor π is defined by $L_\pi := d\iota_\pi - \iota_\pi d$, we get an isomorphism of complexes

$$(410)\qquad (\Omega^{-\bullet}(M;\mathbb{R}[[\hbar]])((u)), u^{-1}L_\pi + ud) \to (\Omega^{-\bullet}(M;\mathbb{R}[[\hbar]])((u)), ud), \quad \omega \mapsto \exp(\iota_\pi/u)\omega.$$

Let A_t be a 1-parameter family of algebras given by $A[[t]]$ as an $\mathbb{R}[[t]]$-module. Denote by $\mathcal{V} := \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{V}_n$. Consider the *Gauss–Manin connection* on the periodic cyclic cohomology viewed as a vector bundle over the parameter space (see e.g [67, 102, 35]). In [35] it was shown that for a 1-parameter family π_t of solutions to the Maurer–Cartan equation, e.g. with polynomial dependence $\pi_t = t\pi$ for a Poisson tensor π, and a cyclic cycle $c_t \in PC_{-m}(A_t)$, which is horizontal with respect to the Gauss–Manin connection, the class of $((\exp(\iota_{\pi_t}/u)\mathcal{V}^{\pi_t})(c_t))$ in $\bigoplus_{j\geq 0} H^{m+2j}(M;\mathbb{R}[[\hbar]])u^j$ is independent of t. Denote by $\widetilde{\mathcal{V}}^{\pi_t} := \exp(\iota_{\pi_t}/u)\mathcal{V}^{\pi_t}$ the image of $\mathcal{V}^{\pi_t}$ under the isomorphism (410) with formal Poisson structure π_t. Since $[\widetilde{\mathcal{V}}^{\pi_t}(c_t)]$ is independent of t, we can set the Poisson tensor to be zero. In our case we have $t = \hbar$ and this will allow us to put $\hbar = 0$ in $\widetilde{\mathcal{V}}_\hbar$ and to merge all the π vertices at zero (see Figure 3.5.2). This will produce wheel graphs as considered e.g. in [104, 106]. Let us denote the weight for a wheel graph with j vertices by w_j.
Note that the curvature of the Grothendieck connection is contained in the R-vertices (see Section 2.2.2.3). Hence, considering the Propagator ϕ on the disk we can compute the weight of a wheel diagram with j black vertices. We will get

$$(411)\qquad w_j = \int_{C^0_{j+1;0}(D)} d\phi(z_1,z_2)\wedge\cdots\wedge d\phi(z_j,z_1)\wedge\bigwedge_{k=1}^{j} d\phi(0,z_k)$$

where $z_1,\ldots,z_j$ are the vertices labeled by R. Moreover, recall that in coordinates we have

$$(412)\qquad R^{j}_{\ell} = \frac{\partial}{\partial y^j}\left(\frac{\partial\varphi}{\partial y}\right)^{-1\,j}_{k}\frac{\partial\varphi^k}{\partial x^\ell}, \quad \forall \varphi\in\Gamma(ST^*M).$$

Define $R(x,y) := R_\ell(x,y)dx^\ell$, $R_\ell(x,y) := Y^j_\ell(x,y)\frac{\partial}{\partial y^j}$, $R^j(x,y) := Y^j_\ell(x,y)dx^\ell$, and

$$(413)\qquad Y^j_\ell := \left(\frac{\partial\varphi}{\partial y}\right)^{-1\,j}_{k}\frac{\partial\varphi^k}{\partial x^\ell} = \delta^j_\ell + O(y).$$

Hence, we get a differential form

$$(414)\qquad \mathcal{V}_j(R\cdots R\mid 1) = \sum \partial_{\ell_1}\partial_{k_j}Y^{k_1}_{\ell_1}(x,y)dx^{\ell_1}\wedge\partial_{\ell_2}\partial_{k_1}Y^{k_2}_{\ell_2}(x,y)dx^{\ell_2}\wedge\cdots\wedge\partial_{\ell_j}\partial_{k_{j-1}}Y^{k_j}_{\ell_j}(x,y)dx^{\ell_j},$$

where we sum over all indices. Thus permuting everything into the right place we get $\mathrm{Tr}(R^j)$, where by abuse of notation we also denote by R the appearing curvature. The permutation will give a sign

$$(415)\qquad \prod_{k=1}^{j-1}(-1)^s = (-1)^{\sum_{s=1}^{j-1}s} = (-1)^{j(j-1)/2}.$$

To get the correct form degree and be consistent with the isomorphism (410), we will need a factor of u^j. Indeed, note that $(0; u) = u$, which in fact appears for any -vertex and hence merging this j times we get a factor u^j.

One can easily see that the propagator will reduce to Kontsevich's angle propagator $(0; z_i) = \frac{1}{2\pi} d\arg(z_i)$ and hence ω_j vanishes if j is odd. Note that if j is even, we have

$$\int_D \bigwedge_{k=1}^{j} (0; z_k) = \frac{1}{(2\pi)^j} \int_D \bigwedge_{k=1}^{j} d\arg(z_k) = \frac{1}{(2\pi)^j} \prod_{k=1}^{j} \int_D d\arg(z_k) = \prod_{k=1}^{j} \frac{1}{k} = \frac{1}{j!}. \tag{416}$$

Therefore, as it was computed in [104, 106], we get

$$\omega_j = \int_{C^0_{j+1;0}(D)} (z_1; z_2) \wedge \cdots \wedge (z_j; z_1) \wedge \bigwedge_{k=1}^{j} (0; z_k) = (-1)^{j(j-1)/2} \frac{B_j}{(2j)j!}, \tag{417}$$

where B_j are the *Bernoulli numbers*[2]. Hence we have

$$[\Phi^{=0}(1)] = \left[\exp\left(-\sum_{j\geq 2} \omega_j u^j \mathcal{V}_j(R \cdots R \mid 1)\right)\right]. \tag{418}$$

Thus we get

$$[\Phi^{=0}(1)] = \left[\exp\left(-\sum_{j\geq 1} \frac{B_{2j}}{(4j)(2j)!} u^{2j}\, \mathrm{Tr}(R^{2j})\right)\right] = \det{}^{1/2}\left(\frac{uR/2}{\sinh(uR/2)}\right) = \hat{A}_u(TM). \tag{419}$$

Recall from Section 3.1.2 that the *Connes isomorphism* Co is given by

$$\begin{aligned} \mathrm{Co}\colon PH(A) &\to H(M; \mathbb{R})[u, u^{-1}] \\ [a_0 \otimes \cdots \otimes a_m] &\mapsto \frac{1}{m!} a_0\, da_1 \wedge \cdots \wedge da_m. \end{aligned} \tag{420}$$

Then for any $c \in PC(A)$ we can see that one gets

$$[\Phi^{=0}(c)] = \hat{A}_u(TM)\,\mathrm{Co}(c), \tag{421}$$

In fact, $\mathrm{Co}(c_0) = \mathrm{Ch}(c)$ is the Chern character of the cyclic cycle c. Note that if $c = 1$ we get $\mathrm{Co}(1) = 1$. The map I is given by integration $\int_M [\Phi(c)] \exp(\ /u)$ for a chosen volume form on M and thus

$$I(c) = \int_M \hat{A}_u(TM)\,\mathrm{Ch}(c) \exp(\ /u). \tag{422}$$

[2] Note that this makes indeed sense since the B_j is zero for odd $j \neq 1$. The case of $j = 1$ is not relevant since j has to be at least two.

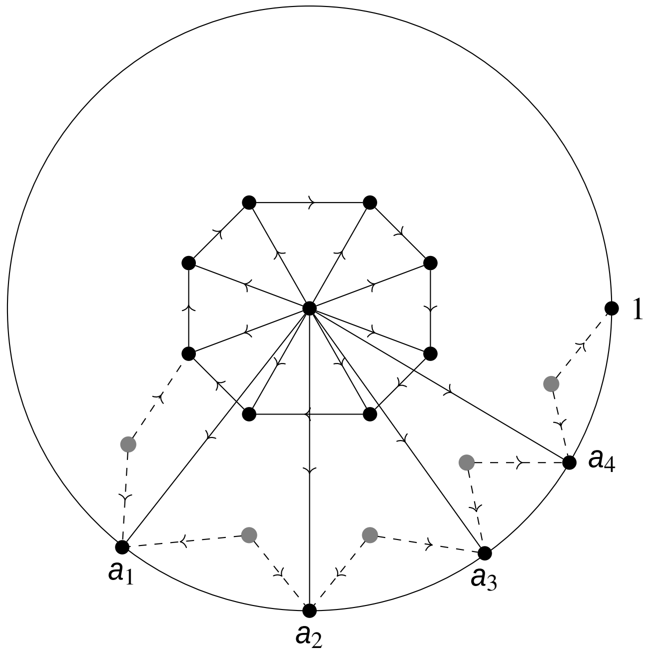

FIGURE 3.5.2. Illustration of the merging of all $\tilde{\varphi}$-vertices (illustrated in gray) to the center of the disk. The black vertices on the wheel contain the curvature coming from the Grothendieck connection by the 1-forms R.

3.5.3. The symplectic case

In [69] a similar construction was considered for symplectic manifolds. They formulate a global trace map using the 1-dimensional Chern–Simons theory[3] within the setting of the BV formalism, by considering solutions of the Quantum Master Equation, and solutions of Fedosov's equation (334). Moreover, they extend this map to an equivariant one with respect to the S^1-action. Let us give some more details for this construction. Let (M,ω) be a symplectic manifold of dimension $2d$ and consider a symplectic connection ∇ on TM. Let

$$\widehat{\mathcal{W}}(TM) := \widehat{S}T^*M \otimes \bigwedge T^*M, \qquad \bigwedge T^*M := \bigoplus_k \bigwedge^k (T^*M)[k]. \tag{423}$$

Moreover, define a map

$$\begin{aligned} \int_{Ber} &: \Omega^\bullet(\widehat{\mathcal{W}}(TM)) \to C^\infty(M) \\ a(x,y,\hbar) &\mapsto \int_{Ber} a(x,y,\hbar) := \frac{1}{d!}\left(\frac{1}{2}\omega^{ij}\partial_{y^i}\wedge\partial_{y^j}\right)^d a(x,y,\hbar)\Big|_{x=y=0}, \end{aligned} \tag{424}$$

where ω^{ij} are the components of ω^{-1}. Let $\Omega^\bullet(M,\widehat{\mathcal{W}}(TM))$ denote the complex of differential forms with values in $\widehat{\mathcal{W}}(TM)$. The symplectic connection can be extended to a map

$$\nabla\colon \Omega^k(M,\widehat{\mathcal{W}}(TM)) \to \Omega^{k+1}(M,\widehat{\mathcal{W}}(TM)). \tag{425}$$

[3] This is simply the case of topological quantum mechanics with action $S = \int_{S^1} p\,dq$.

A degree zero element $S \in \Gamma(M, \widehat{\bigwedge}(TM))[[\hbar]]$ is said to satisfy the Quantum Master Equation if

$$\left(\nabla + i\hbar\Delta + \frac{i}{\hbar}\mathrm{d}_{TM}F\right)\exp\left(\frac{i}{\hbar}S\right) = 0, \tag{426}$$

where d_{TM} is the de Rham differential on TM, $\Delta := L_\Pi = [\mathrm{d}_{TM}, \iota_\Pi]$ with Π the Poisson structure induced by ω (here L denotes the Lie derivative), and F is the Weyl curvature tensor given as in (330). In fact $(\widehat{\bigwedge}(TM), \Delta)$ is a BV algebra like as in Appendix C.2, which is why in [69] they call $\widehat{\bigwedge}(TM)$ the *BV bundle*. One can show that if (426) is satisfied, the operator $\nabla + i\hbar\Delta + \{S, \ \}$ is a differential on $\Gamma(M, \widehat{\bigwedge}(TM))$. Here $\{\ ,\ \}$ denotes the odd Poisson bracket defined by Δ. For a solution S of (426), we define the *twisted integration map*

$$\begin{aligned} \int_S : \Gamma(M, \widehat{\bigwedge}(TM)) &\mapsto \Gamma(M, \mathbb{R}((\hbar))) \\ a &\mapsto \int_S a := \int_{Ber} \exp\left(\frac{i}{\hbar}S\right) a. \end{aligned} \tag{427}$$

In fact, one can show that

$$\int_S : \left(\Gamma(M, \widehat{\bigwedge}(TM))[[\hbar]], \nabla + i\hbar\Delta + \{S, \ \}\right) \to \left(\Gamma(M, \mathbb{R}((\hbar))), \mathrm{d}\right) \tag{428}$$

is a cochain map and hence, by composition, we have a map

$$\int_M \int_S : H^0\left(\Gamma(M, \widehat{\bigwedge}(TM))[[\hbar]], \nabla + i\hbar\Delta + \{S, \ \}\right) \to \mathbb{R}((\hbar)). \tag{429}$$

Fix a solution γ of (334). Then one can construct a nilpotent[4] solution γ_∞ of (426) as an effective action

$$\gamma_\infty := -i\hbar \log \sum_{\Gamma \in \mathcal{G}^0} \frac{\hbar^{\ell(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \int_{C_\Gamma(S^1)} \omega(\Gamma; P_{S^1}), \tag{430}$$

where $\mathcal{G}^0$ denotes the set of all connected graphs, $\ell(\Gamma)$ denotes the number of loops of Γ, $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of Γ, and $\omega(\Gamma; P_{S^1})$ a differential form depending on a chosen propagator P_{S^1} on S^1 and γ.
Define a map

$$[\ ]_\infty : \Gamma(M, \mathcal{W}) \to \Gamma(M, \widehat{\bigwedge}(TM))[[\hbar]] \tag{431}$$

which represents a *factorization map* from local observables on the interval to global observables on S^1. The trace map in this setting is defined by

$$\begin{aligned} \mathrm{Tr}\colon C^\infty(M)[[\hbar]] &\to \mathbb{R}((\hbar)) \\ f &\mapsto \mathrm{Tr}(f) := \int_M \int_{\gamma_\infty} [\sigma(f)]_\infty, \end{aligned} \tag{432}$$

where σ is the symbol map (335).
For the equivariant formulation, extend the map σ to the BV bundle

$$\sigma : \Gamma(M, \widehat{\bigwedge}(TM)) \to \Gamma(M, \mathbb{R}), \tag{433}$$

[4]i.e. there is some $N \geq 0$ such that $\gamma_\infty^N = 0$, which is in fact true since the exponential map will terminate for some power.

by sending $y^i, dy^i \mapsto 0$, and define the S^1-equivariantly extended complexes

$$(434)\qquad \Gamma(M,\mathcal{W})^{S^1} := \left(\Gamma(M,\mathcal{W})[u,u^{-1}; dt],\ \nabla + \frac{1}{\hbar}[\gamma, \cdot] - u\,\iota_{\frac{d}{dt}}\right),$$

where t is the coordinate on S^1, and

$$(435)\qquad \Gamma(M,\widehat{\bigwedge} T^*M)^{S^1}[[\hbar]] := \left(\Gamma(M,\widehat{\bigwedge} T^*M)[u,u^{-1}],\ \nabla + i_\hbar + \{\Omega_1, \cdot\} + u\,d_{TM}\right).$$

Moreover, one can extend the map $\rho\tau_1$ to an equivariant version

$$(436)\qquad \rho\tau_1^{S^1}\colon \Gamma(M,\mathcal{W})^{S^1} \to \Gamma(M,\widehat{\bigwedge} T^*M)^{S^1}[[\hbar]],$$

and show that it still remains a cochain map for the equivariant differentials. Furthermore, one also defines an *equivariant twisted integration map*

$$(437)\qquad \begin{aligned}\int^{S^1}\colon \Gamma(M,\widehat{\bigwedge} T^*M)^{S^1}[[\hbar]] &\to \Gamma(M,\mathbb{R}((\hbar)))[u,u^{-1}]\\ a &\mapsto \left(u^{d}\exp(\hbar\,\bullet u^{0})\, a \exp\left(\frac{i}{\hbar}\Omega_1\right)\right).\end{aligned}$$

REMARK 3.5.3.1. In fact one can show that (437) extends (427) as

$$(438)\qquad \lim_{u\to 0}\int^{S^1} a = \int a, \qquad a \in \Gamma(M,\widehat{\bigwedge} T^*M).$$

Again, one can show that (437) remains a cochain map with respect to the extended complexes, and in particular the composition

$$(439)\qquad \int^{S^1}\rho\tau_1^{S^1}\colon \Gamma(M,\mathcal{W})^{S^1} \to \Gamma(M,\mathbb{R}((\hbar)))[u,u^{-1}]$$

is a cochain map. The S^1-equivariant trace map is then defined by

$$(440)\qquad \begin{aligned}\mathrm{Tr}^{S^1}\colon \Gamma(M,\mathcal{W})^{S^1} &\to \mathbb{R}((\hbar))[u,u^{-1}]\\ f &\mapsto \mathrm{Tr}^{S^1}(f) = \int_M\int^{S^1}\rho f\tau_1^{S^1}.\end{aligned}$$

Moreover, the relation to (432) is

$$(441)\qquad \mathrm{Tr}(f) = \mathrm{Tr}^{S^1}(dt\,(f)).$$

3.5.4. Feynman graphs for cotangent targets

Consider the case of the Poisson Sigma Model with target a cotangent bundle $M = T^*N$ for some manifold N. Then by Proposition 1.1.7.1 and Lemma 3.4.3.4 the graphs will reduce to a certain class of graphs. We have two different bulk vertices. There are vertices labeled by $\frac{\pi}{\hbar}$ and vertices labeled by R. The $\frac{\pi}{\hbar}$-vertices emanate two arrows, representing $\bar{q}$- and $\bar{p}$-derivatives as in Section 1.1.7, and there are no arrows arriving at them, since the Poisson structure is constant. The R-vertices emanate one arrow and there can be an arbitrary amount of arrows representing $\bar{q}$-derivatives arriving at them, but by Proposition 1.1.7.1 we can only have at most one arrow representing a $\bar{p}$-derivative arriving. We also consider vertices on the boundary representing solutions of (232). For each of them there are no arrows emanating and arbitrarily many arriving.

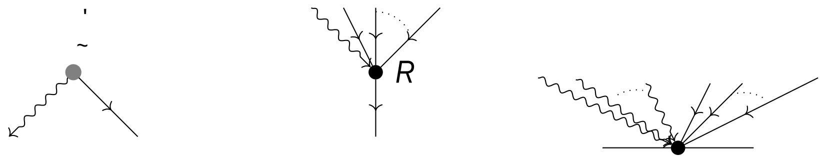

FIGURE 3.5.3. The interaction vertices appearing in the cotangent case. The straight arrows represent a $\bar{q}$-derivative and the wavy arrows represent a $\bar{p}$-derivative. There are no incoming arrows at the -vertices and exactly two emanating arrows. There are arbitrarily many incoming arrows representing the $\bar{q}$-derviatives for an R-vertex, but at most one arrow representing a $\bar{p}$-derivative and exactly one arrow emanating. For the -vertices we have arbitrarily many incoming $\bar{q}$- and $\bar{p}$-derivatives and no emanating arrows.

EXAMPLE 3.5.4.1. Examples of graphs appearing for cotangent targets are given in Figure 3.5.4 and 3.5.5.

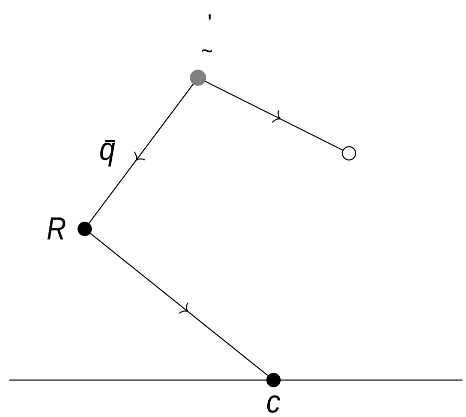

FIGURE 3.5.4. Example of a graph contributing to the trace formula for a cotangent target. Note there is no $\bar{p}$-derivative for the R-vertex. Moreover, one can check that it provides a correct degree count. Indeed, the amount of black vertices in the bulk is given by 2, hence $\dim \mathrm{C}^{0}{}^{1}\mathrm{D}^{\circ} = 4$ and the form degree of ω is given by $|R| + 1 + 2 = 4$.

3.5.5. Relation to the Nest–Tsygan theorem

Let $M = T^*N$ be the cotangent bundle for a manifold N endowed with its canonical symplectic form ω and consider the constant function 1 on the boundary of the disk. In this setting we get the following theorem.

THEOREM 3.5.5.1. *The trace formula* (377) *satisfies* (340).

PROOF. One can easily check that by Proposition 1.1.7.1 and degree reasons the only diagrams contributing within the trace formula are given by wheel-like loops as in Figure 3.5.6, and residual graphs as in Figure 3.5.7. Using the same construction as in Section 3.5.2, we can merge the gray vertices to the center, and obtain wheel graphs which again will give rise to $\hat{A}_u{}^{1}TM^{\circ}$. Recall that

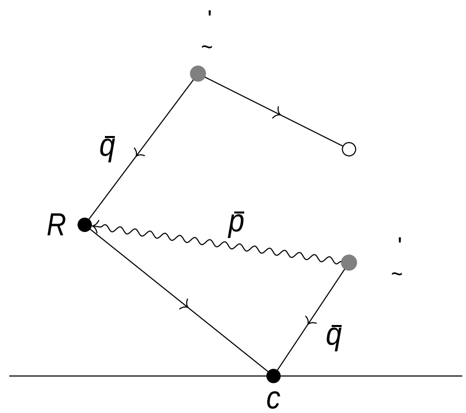

FIGURE 3.5.5. Example of a graph contributing to the trace formula for a cotangent target. Note there is only one $\bar{p}$-derivative for the R-vertex. Moreover, one can check that it provides a correct degree count. Indeed, the amount of black vertices in the bulk is given by 3, hence $\dim C^0(D) = 6$ and the form degree of ω is given by $|R| + |\tilde{\varphi}| + 1 + 2 = 6$.

$\hat{A}_u(TM) = \hat{A}_0(TM) + u\hat{A}_1(TM) + u^2\hat{A}_2(TM) + \dots$, where $\hat{A}_j(TM) \in H^{2j}(M)$. Note that we choose ω to be the symplectic volume form $\frac{\omega^d}{d!}$ and, using (349), we can see that if $c = 1$, the u's will all cancel eachother and thus it will not depend on u. Indeed, we have

$$(442)\qquad \exp\left(\epsilon u\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{u^n}\left(\epsilon\right)^n = \sum_{n=0}^{\infty}\frac{\hbar^n}{n!\,u^n}\prod_{k=1}^{n}\left(\omega^{i_kj_k}\right)\left(\sum_{1\leq i_1<j_1<\dots<i_n<j_n\leq 2d}\prod_{k=1}^{n}\partial_{i_k}\partial_{j_k}\right);$$

and therefore

$$(443)\qquad \exp\left(\epsilon u\right)\frac{\omega^d}{d!} = \sum_{n=0}^{\infty}\sum_{1\leq i<j\leq 2d}\frac{1}{u^n}\frac{\hbar^n}{n!\,d!}\prod_{k=1}^{n}\left(\omega^{i_kj_k}\right)\bigwedge_{k=1}^{n}dx^{i_k}\wedge dx^{j_k}.$$

By degree reasons, the only surviving terms in $\hat{A}_u(TM)\exp\left(\epsilon u\right)\frac{\omega^d}{d!}$ are

$$(444)\qquad \hat{A}(TM)\exp\left(\omega\epsilon\hbar\right)\sum_{k=1}^{\infty}\exp\left(\hbar^{k-1}\omega_k\right),\quad \omega_k\in\Omega^2(M)$$

From the field theoretical construction, it is easy to check that the sum over all residual graphs will exactly give a contribution $\exp(\omega\hbar\epsilon\hbar)$. Indeed, the integral

$$(445)\qquad \int_{D_\epsilon}\Theta(z;u)^s = \frac{i}{2}su^{s-1}\int_{D}(1-|z|^2)^{s-1}dz\wedge d\bar{z} = u^{s-1};\quad s\geq 1;$$

and for $s = 1$, we get $\int_D \Theta = 1$. Hence summing over all such graphs we get $\exp(\tilde{\varphi}) = \exp(\omega\hbar\epsilon\hbar)$. Putting everything together, we have

$$(446)\qquad \mathrm{Tr}(1) = \int_M \hat{A}(TM)\exp(\omega\hbar\epsilon\hbar).$$

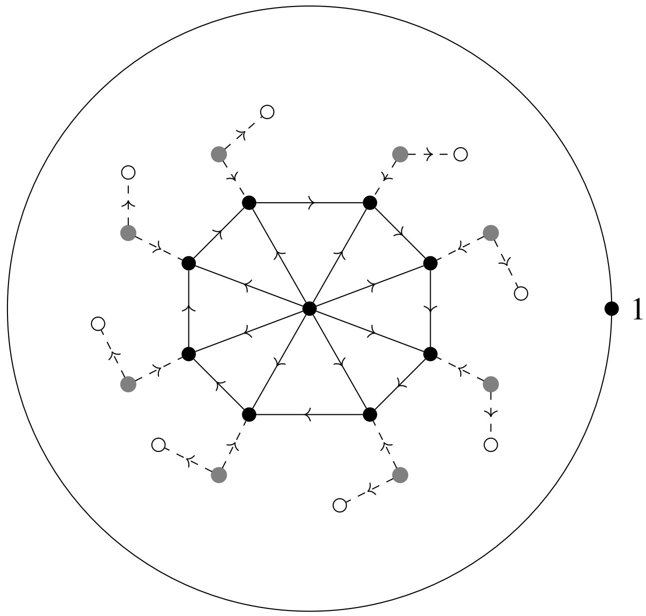

FIGURE 3.5.6. Example of a wheel graph that gives a contribution to the trace formula if we place the constant function 1 on the boundary. The $\tilde{\varphi}$-vertices are represented by the gray vertices and the R-vertices are represented by the black vertices. The picture without the center vertex and the corresponding arrows starting at the center is meant to be before merging. After merging we get the wheel with spokes pointing outwards.

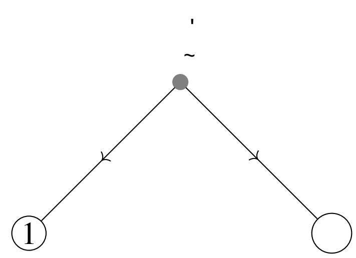

FIGURE 3.5.7. The appearing residual graphs. Here 1 and   both are regarded as the generators of the relative equivariant cohomology on the disk $H_{S^1}(D, \partial D)$.

3.5.6. Reduction of the trace formula for cotangent targets

PROPOSITION 3.5.6.1. *The trace map for the globalized Poisson Sigma Model with cotangent target reduces to the trace map* (440).

PROOF. Consider the Poisson Sigma Model with target a cotangent bundle $M = T^*N$ for some manifold N such that $\dim M = 2d$. The Poisson structure is then induced by the canonical symplectic form ω on M. Note first that (377) can be written as

$$\mathrm{Tr}(f) = \int_M (T\varphi^* f)|_{y=0} \exp(T\varphi^* h|_{y=0}) + O(\hbar) = \int_M f \exp(h) + O(\hbar); \tag{447}$$

where h was the Hamiltonian function for φ such that $\operatorname{div}_{\Omega}\varphi = [h, \pi] = 0$. Indeed, by considering the Feynman graph expansion of $\mathcal{V}_{n}^{\hbar}$, we get that

$$\mathrm{Tr}(f) = \int_M \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} P_n(T'\pi; T'h; (T')f) \exp(T'h); \tag{448}$$

where P_n are differential polynomials in $T'\pi$; $T'h$, and $(T')f$. Now, considering cotangent targets and choosing Ω to be the symplectic volume $\frac{\omega^d}{d!}$, we can see that the leading order $\hbar$ term of $\mathrm{Tr}(f)$ is given by

$$\int_M f\frac{\omega^d}{d!}. \tag{449}$$

Since Ω is constant, we get that $\operatorname{div}_{\Omega}\varphi = 0$ and hence $[h, \pi] = 0$, which implies that h is constant, e.g. $h = 0$. This is also compatible with the Nest–Tsygan theorem. One can compute

$$\begin{aligned} \mathrm{Tr}(1) &= \int_M \hat{A}(TM)\exp(\omega_\hbar \gamma_\hbar) = \int_M (\hat{A}_0(TM) + \hat{A}_1(TM) + \dots)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\omega_\hbar^n}{\hbar^n}\right) \\ &= \int_M (\hat{A}_0(TM) + \hat{A}_1(TM) + \dots)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{\hbar^n}\left(\omega + \sum_{k\geq 1}\hbar^k\omega_k\right)^n\right) \\ &= \frac{(-1)^d}{\hbar^d}\int_M (\hat{A}_0(TM) + \hat{A}_1(TM) + \dots)\frac{\omega^d}{d!} + O(\hbar) \\ &= \frac{(-1)^d}{\hbar^d}\int_M \hat{A}_0(TM)\frac{\omega^d}{d!} + O(\hbar) = \frac{(-1)^d}{\hbar^d}\int_M \frac{\omega^d}{d!} + O(\hbar); \end{aligned} \tag{450}$$

Note that we have used $\hat{A}_0(TM) = 1 \in H^0(M)$. Using the Feynman graphs for the corresponding effective theory together with the fact that, for a solution γ of (334), the leading $\hbar$ term of $\mathrm{d}_{TM}\gamma$ is given by $\omega_{ij}\mathrm{d}y^i \wedge \mathrm{d}x^j$, it can be seen that the leading $\hbar$ term of (440) is given by

$$\int_M \int_{\mathcal{L}_{Ber}} f\exp(\omega_{ij}\mathrm{d}y^i \wedge \mathrm{d}x^j \gamma_\hbar) = \frac{(-1)^d}{\hbar^d}\int_M f\frac{\omega^d}{d!}. \tag{451}$$

Here we use that the map μ_{Ber} will give rise to the symplectic volume form $\frac{\omega^d}{d!}$ on M. Therefore the $\hbar$ leading terms coincide. Moreover, in [69] they also show how the trace (432) is compatible with the Nest–Tsygan theorem. Note that the morphism $\mathcal{V}^{\hbar}$, which is given as the expectation value of the Fedosov-type formal global action, gives rise to the twisted integration map, where the effective action is indeed given in terms of a solution γ of (334) since (232) is reduced to (334) for the symplectic case as explained in Section 2.2.2.3. This action functional corresponds to ψ_1, which can be seen by using the corresponding Feynman rules on S^1. Moreover, the Grothedieck connection gives rise to the globalization map $\pi_1^{S^1}$ for observables and the quantization map reduces to the inverse of the symbol map σ.

Appendix

APPENDIX A

Elements of Symplectic Geometry

In this chapter we want to recall some standard definitions and properties of symplectic geometry. More on symplectic geometry can be found e.g. in [98]. We will start with the local picture and continue with the global structure.

A.1. Symplectic vector spaces

Let V be a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Denote by V^* the dual of V. An element of V^* is a $\mathbb{K}$-linear map $f\colon V \to \mathbb{K}$. Let $0 \leq m \leq \dim V$. Define

$$\bigwedge^m V^* := \Big\{ \alpha\colon \underbrace{V \times \cdots \times V}_{m} \to \mathbb{K} \mid \alpha \text{ is linear in each argument and } \alpha \text{ is alternating,}$$

$$\text{i.e. } \alpha(v_1, \ldots, v_j, v_{j+1}, \ldots, v_m) = -\alpha(v_1, \ldots, v_{j+1}, v_j, \ldots, v_m) \text{ for all } j = 1, 2, \ldots, m-1 \Big\}$$

Example A.1.1. Let $f, g \in V^*$. Then we can define $(f \wedge g) \in \bigwedge^2 V^*$ by

$$(f \wedge g)(v_1, v_2) = f(v_1)g(v_2) - f(v_2)g(v_1).$$

In fact, it can be shown that all the elements of $\bigwedge^2 V^*$ are finite linear combinations of such elements. Given $\omega \in \bigwedge^2 V^*$, we can define a map

$$\begin{aligned} \omega^\flat\colon V &\to V^* \\ v &\mapsto \omega^\flat(v), \end{aligned}$$

where $\omega^\flat(v)(w) := \omega(v, w)$.

Definition A.1.2 (Symplectic vector space). A *symplectic vector space* is a pair (V, ω), where V is a (finite-dimensional) vector space and $\omega \in \bigwedge^2 V^*$ such that $\omega^\flat$ is a vector space isomorphism.

Remark A.1.3. Since we are in the finite-dimensional setting, $\omega^\flat$ is a vector space isomorphism if and only $\omega^\flat$ is injective.

Remark A.1.4. $\omega^\flat$ is injective if and only if $\omega(v, w) = 0$ for all $w \in V$ implies $v = 0$.

Example A.1.5. Let $(W, \langle\ ,\ \rangle)$ be an inner product space. Consider $V = W \oplus W$, with

$$\omega((w_1, w_2), (w_1', w_2')) = \langle w_2', w_1 \rangle - \langle w_2, w_1' \rangle.$$

Then (V, ω) is a real symplectic vector space. More generally, $V = W \oplus W^*$ and $\omega_{can}((w, \alpha), (w', \alpha')) = \alpha'(w) - \alpha(w')$, then (V, ω_{can}) is a symplectic vector space.

Remark A.1.6. (V, ω_{can}) is a symplectic vector space.

Definition A.1.7 (Isotropic/Coisotropic/Lagrangian). Let (V, ω) be a symplectic vector space. Let Y be a subspace of V. Define the symplectic orthogonal complement of Y by $Y^{\perp} := \{v \in V \mid \omega(v, y) = 0, \forall y \in Y\}$. Then

- Y is *isotropic* if $Y \subset Y^{\perp}$,
- Y is *coisotropic* if $Y^{\perp} \subset Y$,
- Y is *Lagrangian* if Y is isotropic and Y is symplectic if $\omega|_{Y\times Y}$ is nondegenerate, i.e. $Y \cap Y^{\perp} = \{0\}$.

Example A.1.8. If $\dim Y = 1$, then Y is isotropic. If Y is isotropic, then $Y^{\perp}$ is coisotropic. If Y is symplectic, then so is $Y^{\perp}$. Moreover, $Y^{\perp\perp} = (Y^{\perp})^{\perp} = Y$.

Proposition A.1.9. *Let (V, ω) be a symplectic vector space. Then there is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that*

$$\omega(e_i, e_j) = 0,$$
$$\omega(f_i, f_j) = 0,$$
$$\omega(e_i, f_j) = \delta_{ij},$$

for all $i, j \in \{1, 2, \ldots, n\}$. Hence, we can write $\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*$.

Remark A.1.10. In fact, $\dim V = \dim Y + \dim Y^{\perp}$. Moreover, Y is Lagrangian if and only if $V \cong Y \oplus Y^{\perp}$.

Remark A.1.11. Y is a Lagrangian subspace if and only if Y is isotropic and $\dim Y = \frac{1}{2}\dim V$. Moreover, Y is Lagrangian if and only if Y is a maximal isotropic subspace.

A.2. Symplectic manifolds

To understand the mathematical structure of classical mechanics, it is necessary to understand the notion of a symplectic manifold.

Definition A.2.1 (Closed/exact). We call a k-form ω *closed*, if $d\omega = 0$. It is called *exact* if there is a $(k-1)$-form α such that $d\alpha = \omega$.

Example A.2.2. If ω is exact, then $d\omega = 0$, i.e. exact forms are closed as well. Let $M = \mathbb{R}^n$, then ω is closed if and only if ω is exact (this is given by the "Poincaré lemma") for $k > 0$.

Definition A.2.3 (Symplectic manifold). A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a 2-form on M such that

1. ω is closed, i.e. $d\omega = 0$,
2. ω is nondegenerate, i.e. for all $q \in M$, $\omega^{\flat}\colon T_qM \to T_q^*M$ is injective.

Definition A.2.4 (Tautological 1-form). Let $M = T^*N$. Define a 1-form α on M as

$$\alpha_{x,p}(X_{x,p}) := p(d\pi_{x,p}X_{x,p}),$$

where $p\colon T^*N \to N$, $\pi\colon TM \to M$ and $X_{x,p} \in T_{x,p}M$. The form α is called the *tautological 1-form* on T^*N.

Let X be an n-manifold, with $M = T^*X$ its cotangent bundle. If $x_1, \ldots, x_n$ are coordinates on $U \subseteq X$, with associated cotangent coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ on T^*U, then the tautological 1-form on T^*X is $\alpha = \sum_i \xi_i dx^i$ and the canonical 2-form is

$$\omega = d\alpha = \sum_i dx^i \wedge d\xi_i.$$

EXAMPLE A.2.5. Let $M = T^*\mathbb{R} \cong \mathbb{R}\times\mathbb{R} \ni (x,p)$. Let $\lambda = f\mathrm{d}x + g\mathrm{d}p$. Then $\lambda(\partial_x) = f$ and $\lambda(\partial_p) = 0$, thus $\lambda = f\mathrm{d}x$. on the other hand, $\lambda_{x,p}(\partial_x) = p$ and hence $\lambda = p\mathrm{d}x$. More generally, if $M = T^*\mathbb{R}^n \ni (x^1,\dots,x^n,p_1,\dots,p_n)$, then $\lambda = \sum_{1\le j\le n} p_j\mathrm{d}x^j$

REMARK A.2.6. Let (U,φ) be a local coordinate system on $M = T^*N$ given by

$$\varphi(q) = (x^1(q),\dots,x^n(q),p_1(q),\dots,p_n(q)).$$

One can check that $\lambda = \sum_{1\le j\le n} p_j\mathrm{d}x^j$. Moreover, $(T^*N, \omega = \mathrm{d}\lambda)$ is a symplectic manifold.

A.3. Lie derivative

The concept of a derivative can be generalized in a tensorial sense by the following definiton:

DEFINITION A.3.1 (LIE DERIVATIVE). Let $f \in C^\infty(M)$ and X be a vector field. The *Lie derivative* of f along X is defined as $L_X f = X(f)$. Let X and Y be two vector fields. Then we define $L_X Y = [X,Y]$. Moreover, let X be a vector field and α a 1-form. Then $L_X\alpha$ is a 1-form defined by the equation

$$L_X(\alpha(Y)) = (L_X\alpha)(Y) + \alpha(L_X Y).$$

More generally, if α is a k-form then $L_X\alpha$ is again a k-form defined by

$$(L_X\alpha)(Y_1,\dots,Y_k) = L_X(\alpha(Y_1,\dots,Y_k)) - \sum_{j=1}^{k}\alpha(Y_1,\dots,Y_{j-1},[X,Y_j],\dots,Y_k).$$

REMARK A.3.2. Given a k-form α, and a vector field X, $(L_X\alpha)(p)$ is the rate of change of α in the direction of the "flow" of x at p.

A.4. Lagrangian submanifolds and conormal bundles

A.4.1. Lagrangian submanifolds. Let (M,ω) be a $2n$-dimensional symplectic manifold.

DEFINITION A.4.1 (LAGRANGIAN SUBMANIFOLD). A submanifold Y of M is called *Lagrangian* if, at each $p \in Y$, T_pY is a Lagrangian subspace of T_pM (see Definiton A.1.7). Equivalently, if $i\colon Y \hookrightarrow M$ is the inclusion, then Y is Lagrangian if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$ (see also RemarkA.1.11).

DEFINITION A.4.2 (ZERO SECTION). The *zero section* of T^*X, defined by

$$X_0 = \{(x,\xi) \in T^*X \mid \xi = 0\}$$

is an n-dimensional submanifold of T^*X.

PROPOSITION A.4.3. *X_0 is a Lagrangian submanifold of T^*X.*

PROOF. Clearly λ vanishes on $X_0 \cap T^*U$. In particular, if $i_0\colon X_0 \hookrightarrow T^*X$ is the inclusion, we have $i_0^*\lambda = 0$. Hence, $i_0^*\omega = i_0^*\mathrm{d}\lambda = 0$, and X_0 is Lagrangian.

REMARK A.4.4. Similar to Lagrangian submanifolds, we can define *isotropic* and *coisotropic* submanifolds, by requiring that at each point of the submanifold the tangent space is isotropic or coisotropic (see Definiton A.1.7) respectively.

A.4.2. Conormal bundles. Let S be any k-dimensional submanifold of an n-manifold X.

Definition A.4.5 (Conormal space). The *conormal space* at $x \in S$ is given by

$$N_x^* S = \{\xi \in T_x^* X \mid \xi(v) = 0, \text{ for all } v \in T_x S\}. \tag{452}$$

Definition A.4.6 (Conormal bundle). The *conormal bundle* of S is given by

$$N^* S = \{(x, \xi) \in T^* X \mid x \in S, \xi \in N_x^* S\}. \tag{453}$$

Remark A.4.7. The conormal bundle N^*S is an n-dimensional submanifold of T^*X (one can check that by using coordinates on X adapted to S).

Proposition A.4.8. *Let $i \colon N^*S \hookrightarrow T^*X$ be the inclusion, and let α be the tautological 1-form on T^*X. Then $i^*\alpha = 0$.*

Proof. Let $(U, x_1, \ldots, x_n)$ be a coordinate system on X centered at $x \in S$ and adapted to S, so that $U \cap S$ is described by $x_{k+1} = \cdots = x_n = 0$. Let $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*U$ is then described by

$$x_{k+1} = \cdots = x_n = 0,$$
$$\xi_1 = \cdots = \xi_k = 0.$$

Since $\alpha = \sum_i \xi_i \mathrm{d}x^i$ on T^*U, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i \mathrm{d}x^i \Big|_{\mathrm{span}(\partial_{x_i}, i \leq k)} = 0$$

Corollary A.4.9. *A conormal bundle is a Lagrangian submanifold.*

A.5. Hamiltonian actions

The mathematical structure of classical mechanics, in the sense of Hamiltonian mechanics, is characterized by the symplectic structure of the phase space. The notion of a Hamiltonian action gives more insights for the symplectic structure.

A.5.1. Momentum and Comomentum Maps. Let (M, ω) be a symplectic manifold, and denote by $\mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M)$ the set of symplectomorphisms on M. Moreover, ler G be a Lie group, and $\phi \colon G \to \mathrm{Sympl}(M, \omega)$ a smooth symplectic action, i.e. a group homomorphism such that the evaluation map $\mathrm{ev}_\phi(g, p) := \phi_g(p)$ is smooth. Let us consider the case where $G = \mathbb{R}$. We have the following bijective correspondence:

$$\{\text{symplectic actions of } \mathbb{R} \text{ on } M\} \longleftrightarrow \{\text{complete symplectic vector fields on } M\}$$

$$\phi \mapsto X_p = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(p)$$

$$\phi = \exp(tX) \longleftarrow X$$

"flow of X" "vector field generated by ϕ"

Definition A.5.1 (Hamiltonian action (for $\mathbb{R}$-action)). The action ϕ is said to be *Hamiltonian* if there is a function $H \colon M \to \mathbb{R}$ such that $\mathrm{d}H = \iota_X \omega$ where X is the vector field on M generated by ϕ, and ι_X denotes the contraction with the vector field X.

Consider now the case where $G = S^1$. An action of S^1 is an action of $\mathbb{R}$ which is 2π-periodic, i.e. $\Phi_{2\pi} = \Phi_0$. The S^1-action is called Hamiltonian if the underlying $\mathbb{R}$-action is Hamiltonian. Let us now consider the general case. Let $\mathfrak{g}$ be the Lie algebra of G and denote its dual by $\mathfrak{g}^*$.

Definition A.5.2 (Hamiltonian action (general)). The action Φ is called *Hamiltonian* if there is a map

$$\mu\colon M \to \mathfrak{g}^*, \tag{454}$$

satisfying the following conditions:

(1) For each $X \in \mathfrak{g}$, let
- $\mu^X\colon M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X\rangle$, be the component of μ along X,
- $X^\sharp$ be the vector field on M generated by the 1-parameter subgroup $\{\exp(tX) \mid t \in \mathbb{R}\} \subseteq G$.

Then $d\mu^X = \iota_{X^\sharp}\omega$, i.e. μ^X is a Hamiltonian function for the vector field $X^\sharp$.

(2) μ is equivariant with respect to the given action Φ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, given by

$$\mu \circ \Phi_g = \mathrm{Ad}^*_g \circ \mu \tag{455}$$

for all $g \in G$.

Definition A.5.3 (Momentum map). The map μ given as in Definition A.5.2 is called *momentum map*.

Definition A.5.4 (Hamiltonian G-space). The Quadruple (M, ω, G, μ) given as in Definition A.5.2 is called *Hamiltonian G-space*.

Definition A.5.5 (Comomentum map). We call a map

$$\mu^*\colon \mathfrak{g} \to C^\infty(M) \tag{456}$$

a *comomentum map* if

(1) $\mu^*(X) := \mu^X$ is a Hamiltonian function for the vector field $X^\sharp$,

(2) μ^* is a Lie algebra homomorphism, i.e.

$$\mu^*([X, Y]) = \{\mu^*(X), \mu^*(Y)\},$$

for all $X, Y \in \mathfrak{g}$, where $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$ and $\{\ ,\ \}$ is the Poisson bracket on $C^\infty(M)$.

A.6. Symplectic reduction

The concept of a symplectic reduction is important for the treatment of gauge theories on manifolds with boundary. To be able to perform *geometric quantization* on the boundary, we have to make sure that the space of fields can be reduced to a symplectic space of boundary fields with certain conditions. The following theorem guarantees such a structure for the needed purposes in many examples.

Theorem A.6.1 (Marsden-Weinstein). *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i\colon \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*
- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi\colon \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

Proof. See e.g. [98].

Remark A.6.2. In field theory we work in the infinite-dimensional setting, so Theorem A.6.1 does not apply as it is. Also, for field theory, we normally consider weak symplectic vector spaces, i.e. the corresponding linear map from V to V^* is injective. Morover, there is a notable example, 4D gravity, where one has to deal with general coisotropic reduction and not just Marsden–Weinstein reduction.

Definition A.6.3 (Reduction). The pair (M_{red}, ω_{red}) is called the *reduction* of (M, ω) with respect to (G, μ), or the *reduced space*, or the *symplectic quotient*, or the *Marsden-Weinstein quotient*.

APPENDIX B

Elements of Supergeometry

Supergeometry is an essential tool to understand reduction in cohomological terms[1]. In particular the Faddeev–Popov [54], BRST [13, 12, 11, 103], or Batalin–Vilkovisky gauge formalism require the structure of a supermanifold and the notion of *odd* and *even* coordinates (in physics language they correspond to *fermionic*[2] (anticommuting) and *bosonic* (commuting) particles respectively). The mathematical theory of supergeometry goes back to the work of Berezin and Leites between 1975 and 1980 in [14, 79]. For our purposes we need a refinement, called graded geometry, where we give the variables an additional integer degree. More on graded geometry can be found e.g. in [21, 60, 51].

Remark B.0.1. We will denote the *exterior algebra* of a vector space V by $\bigwedge V = \bigoplus_k \bigwedge^k V$ and the algebra of *differential forms* on a manifold M by $\Omega(M) = \bigoplus_k \Omega^k(M)$. We use $\hat{\otimes}$ to indicate the topological tensor product, i.e. the unique tensor product such that $C^\infty(\mathbb{R}^n)\hat{\otimes}C^\infty(\mathbb{R}^m) \cong C^\infty(\mathbb{R}^{n+m})$.

B.1. Graded spaces

To understand the concept of a "supermanifold" we need to look first at the linear case, which is the concept of a graded space (or also graded vector space).

Definition B.1.1 (Graded vector space). A *$\mathbb{Z}$-graded vector space* (or often just *graded vector space*) is a collection[3] of vector spaces $V = (V_k)_{k\in\mathbb{Z}}$.

Definition B.1.2 (Shift). Let $V = (V_n)_{n\in\mathbb{Z}}$ be a graded vector space. For any integer k we can define the *k-shift* of V to be given by

$$(V[k])_n := (V_{k+n})_{n\in\mathbb{Z}}.$$

Definition B.1.3 (Graded linear map). A *graded linear map* $f\colon V \to W$ between two graded vector spaces V and W is given by a collection of linear maps $(f_k\colon V_k \to W_k)_{k\in\mathbb{Z}}$.

Remark B.1.4. A graded linear map of *degree k* between two graded vector spaces V and W is a graded linear map between V and $W[k]$.

Definition B.1.5 (Dual of a graded vector space). Let $V = (V_k)_{k\in\mathbb{Z}}$ be a graded vector space. The dual space V^* of V is defined as $V^* = (V^*_{-k})_{k\in\mathbb{Z}}$.

Definition B.1.6 (Superspace). A *superspace* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We denote by V_0 the vector space of *even* vectors and by V_1 the space of *odd* vectors.

[1]The concept of Supergeometry is also used for SUSY field theories, but this is not what we discuss here.

[2]Here *fermionic* just means anticommuting (which corresponds to Pauli's exlclusion principle) but not necessarily spinorial.

[3]Working with the "total space", i.e. the direct sum of the V_ks, create all sort of troubles when one start talking of morphisms, so it should be avoided, thus we just talk of the components V_k. However, one should mention that the "total space" picture is fine in the the finite-dimensional case (or just when only finitely many V_k are nontrivial.)

Definition B.1.7 (Parity/degree). The *parity* (or *degree*) is the $\mathbb{Z}_2$-grading of the superspace. The degree of a homogeneous element $x \in V$ is denoted by $|x|$ and is defined by

$$|x| := \begin{cases} even, & x \in V_{\bar{0}} \\ odd, & x \in V_{\bar{1}} \end{cases}$$

Remark B.1.8. In addition to the $\mathbb{Z}$-grading we introduce independently a parity in $\mathbb{Z}_2$ denoted by $|\ |$. Moreover, in this note we will only consider the case when the parity is the degree modulo 2.

Remark B.1.9. If we consider the superspace V as an ordinary vector space V together with an automorphism $P\colon V \to V$, such that $P^2 = \mathrm{id}$, we get that $V_{\bar{0}}$ is the 1-eigenspace and $V_{\bar{1}}$ is the (-1)-eigenspace. Moreover, $P(x) = (-1)^{|x|}x$.

Definition B.1.10 (Dimension). We define the *dimension of a superspace* to be given by

$$\dim V = (\dim V_{\bar{0}}, \dim V_{\bar{1}}),$$

and we also say that V is a $\dim V_{\bar{0}} \mid \dim V_{\bar{1}}$-dimensional superspace.

Remark B.1.11. We can always think of a vector space V to be a superspace with $P = \mathrm{id}$. In this case, one usually considers V to be the even part of the superspace $(V = V \oplus 0, \mathrm{id})$. We can also consider V to be given as the odd part of the superspace by the pair $(V = 0 \oplus V[1], -\mathrm{id})$, where we denote by $V[1]$ the odd degree which is simply given by $(V, -\mathrm{id})$.

Remark B.1.12. Note that, considering a vector space V as a superspace (V, P_V), we can consider its dual by $(V^*, P_{V^*} = (P_V)^*)$.

Consider a $\mathbb{Z}$-graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$, where only finitely many V_ks are nontrivial. We can consider V as a superspace by setting

$$\begin{aligned} V_{\bar{0}} &= \bigoplus_{k \in 2\mathbb{Z}} V_k, \\ V_{\bar{1}} &= \bigoplus_{k \in 2\mathbb{Z}+1} V_k. \end{aligned} \tag{457}$$

Definition B.1.13 (Morphism of superspaces). A *morphism* between two superspaces V and W is a linear map $\varphi\colon V \to W$ which preserves the grading. Considering V and W with their correpsonding automorphisms P_V and P_W we can equivalently say that φ is a morphisms between V and W if the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{P_V} & V \\ \downarrow & & \downarrow \\ W & \xrightarrow{P_W} & W \end{array}$$

B.2. Category of superspaces

One can observe that superspaces form a category (over some fixed field $\mathbb{K}$), which we denote by **SuperVect**$_{\mathbb{K}}$. One can define an *endofunctor* on **SuperVect**$_{\mathbb{K}}$

$$\Pi\colon (V, P) \mapsto (V, -P).$$

This functor is called *change of parity*. Thus we get

$$\begin{aligned} \Pi(V)_{\bar{0}} &= V_{\bar{1}}, \\ \Pi(V)_{\bar{1}} &= V_{\bar{0}}. \end{aligned} \tag{458}$$

Remark B.2.1. The superspace $V[1]$, coming from a vector space V, can also be written as ΠV.

Remark B.2.2. One can observe that $\mathbf{SuperVect}_{\mathbb{K}}$ is actually a symmetric monoidal category by

$$\begin{aligned}(V \otimes W)_{\bar{0}} &= (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}),\\ (V \otimes W)_{\bar{1}} &= (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).\end{aligned} \tag{459}$$

Moreover, $P_{V\otimes W} = P_V \otimes P_W$. One can also define an operation

$$\begin{aligned} V \otimes W &\to V \otimes W\\ x \otimes y &\mapsto (-1)^{|x||y|} y \otimes x,\end{aligned} \tag{460}$$

which is called the *braiding*.

Remark B.2.3. We have a natural embedding of symmetric monoidal categories:

$$\begin{aligned}\mathbf{Vect}_{\mathbb{K}} &\to \mathbf{SuperVect}_{\mathbb{K}}\\ V &\mapsto (V, \mathrm{id}),\end{aligned} \tag{461}$$

where $\mathbf{Vect}_{\mathbb{K}}$ denotes the category of $\mathbb{K}$-vector spaces.

Definition B.2.4 (Symmetric superspace). For any superspace V we can define for any positive integer n its nth symmetric power as

$$S^nV := V^{\otimes n}/(x_1 \otimes \cdots \otimes x_n - \sigma(x_1 \otimes \cdots \otimes x_n)),\quad \sigma \in S_n), \tag{462}$$

where S_n denotes the symmetric group of order n and $(\ )$ the ideal generated by some relation.

Remark B.2.5. We get $S^n(V_{\bar{0}} \oplus 0) = S^nV_{\bar{0}}$ and $S^n(0 \oplus V_{\bar{1}}) = \bigwedge^n V_{\bar{1}}$. Moreover

$$S^n(V_{\bar{0}} \oplus V_{\bar{1}}) = \bigoplus_{0\leq k\leq n} S^kV_{\bar{0}} \otimes \left(\bigwedge^{n-k} V_{\bar{1}}\right). \tag{463}$$

B.3. Short description for the local picture

Locally, we consider coordinates (x^i) on an open $U \subset \mathbb{R}^n$ and the algebra of smooth maps $C^\infty(U)$, which are algebraically described by the commutativity of the coordinates, i.e. we have the equivalence relation $x^ix^j = x^jx^i$. Considering this relation, we can add coordinates θ such that algebraically we have

$$\begin{aligned}x^i\theta^\mu &= \theta^\mu x^i\\ \theta^\mu\theta^\nu &= -\theta^\nu\theta^\mu,\end{aligned}$$

i.e. an anticommuting relation for the (θ). So, we can describe the algebra, generated by these coordinates, as $\mathcal{A} = \mathbb{R}[x;\theta]/\sim$, where $\sim$ is given by the commutative relation of the (x^i)-coordinates and the anticommutative relation of the (θ)-coordinates. Equivalently we can write

$$\mathcal{A} = C^\infty(U) \otimes \bigwedge V =: C^\infty(U, \bigwedge V),$$

for some vector space V.

B.4. Supermanifolds

We want to explain the globalization of the local picture given above, i.e. we want to describe a manifold structure such that locally we get the structure as we have seen before. Consider a diffeomorphism between patches

$$\varphi \colon U \times V \to \tilde{U} \times \tilde{V} \tag{464}$$

such that

$$\varphi^* \colon C^\infty(\tilde{U} \times \tilde{V}) \to C^\infty(U \times V)$$

or equivalently

$$\varphi^* \colon C^\infty(\tilde{U}) \otimes \bigwedge \tilde{V}^* \to C^\infty(U) \otimes \bigwedge V^*$$

is a superalgebra morphism. Let x and θ be coordinates on U and V respectively and $\tilde{x}$ and $\tilde{\theta}$ coordinates of $\tilde{U}$ and $\tilde{V}$ respectively. Then we can write

$$\tilde{x}^{\tilde{i}} = \varphi^*_{\tilde{i}}(x, \theta) \quad \text{even on} \quad C^\infty(U) \otimes \bigwedge^{even} V^*;$$

$$\tilde{\theta}^{\tilde{j}} = \varphi^*_{\tilde{j}}(x, \theta) \quad \text{odd on} \quad C^\infty(U) \otimes \bigwedge^{odd} V^*.$$

A supermanifold is then given by patching together $U \times V$. Mathematically more clearly, we have the following definition:

Definition B.4.1 (Supermanifold). A *supermanifold* $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_M)$, which is locally isomorphic to

$$\left(U, C^\infty(U) \otimes \bigwedge V^*\right),$$

where $U \subseteq \mathbb{R}^n$ is open and V is some finite-dimensional real vector space. We call M the *body* of $\mathcal{M}$ and $\mathcal{O}_M$ the *structure sheaf* of M.

Remark B.4.2. The isomorphism mentioned in Definition B.4.1 is in the category of $\mathbb{Z}_2$-graded algebras, which is the parity:

$$|\cdot| \colon \bigoplus_{k \geq 0} C^\infty(U) \otimes \bigwedge^k V^* \to \mathbb{Z}_2$$

$$f \mapsto |f| := |\cdot| = k \mod 2.$$

This induces that globally $C^\infty(\mathcal{M})$ is a graded commutative algebra. In particular, for two homogeneous elements $f, g \in C^\infty(\mathcal{M})$, we have $fg = (-1)^{|f||g|} gf$.

Remark B.4.3. The supermanifold denoted by $\mathbb{R}^{n|m}$ is the supermanifold with body $\mathbb{R}^n$ and sheaf of functions $C^\infty(\mathbb{R}^n) \otimes \bigwedge \mathbb{R}^m$.

B.5. Morphisms of supermanifolds

Let $\mathcal{M}$ and $\mathcal{N}$ be two supermanifolds. Then (in the smooth setting) we can define a morphism between $\mathcal{M}$ and $\mathcal{N}$ to be given by a morphism of *superalgebras* from $C^\infty(\mathcal{N})$ to $C^\infty(\mathcal{M})$. Note that a *superalgebra* is a $\mathbb{Z}_2$-graded algebra, i.e. an algebra over a commutative ring, where the multiplication preserves the grading. Hence the notion of a morphism is simillar as for superspaces. This idea is clear by the following definition:

Definition B.5.1 (Morphism of locally ringed spaces). Let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be two ringed spaces. A *morphism* $(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ is a pair $(f,f^\sharp)$, where $f\colon X\to Y$ is a continuous map and $f^\sharp\colon\mathcal{O}_Y\to f_*\mathcal{O}_X$ is a morphism of local rings (i.e. it respects the maximal ideal). Here f_* denotes the direct image functor for sheaves.

Remark B.5.2. Assume $\mathcal{N}$ to be given a local patch $\bar V\oplus\Pi\bar W$. Then one would get

$$\mathrm{Hom}(\mathcal{M},\mathcal{N})\cong\mathrm{Hom}(C^\infty(\mathcal{N}),C^\infty(\mathcal{M}))\cong\left((\bar V\oplus\Pi\bar W)\otimes C^\infty(\mathcal{M})\right)_0.$$

The last space represents the even part of the corresponding superspace.
We can also consider the algebra of *polynomial functions* on $V\oplus\Pi W$, where V and W are finite-dimensional vector spaces, which is given by $S(V^*)\otimes\bigwedge W^*$. We denote by S the symmetric algebra. Globally, the algebra of smooth functions on a supermanifold $C^\infty(\mathcal{M})$ is defined to be the algebra of global sections of the sheaf associated to $\mathcal{M}$. The parity can be extended to this algebra and thus it is a super commutative algebra with respect to this parity, i.e. for two homogeneous elements f and g of degree $|f|$ and $|g|$ respectively, one has $f\cdot g=(-1)^{|f||g|}g\cdot f$.

Example B.5.3 (Differential forms). Let M be a manifold. Then the algebra of differential forms $\Omega(M)$ is locally isomorphic to $C^\infty(U)\otimes\bigwedge T^*_xM$, where x is some point on U. This means that the sheaf of differential forms on a manifold corresponds to a supermanifold.

Example B.5.4. Let V be a vector space. If we regard V as a superspace (V,id), then one can consider its associated shifted (or odd) superspace $\Pi V=V[1]=(V,-\mathrm{id})$ such that $C^\infty(\Pi V)=C^\infty(V[1])=\bigwedge V^*$.

Example B.5.5 (Odd vector bundle). Let $E\to M$ be a vector bundle. Then we can associate to it the odd vector bundle ΠE, which is a supermanifold. Moreover, the smooth functions are given by

$$C^\infty(\Pi E)=\Gamma\left(\bigwedge E^*\right),\tag{465}$$

where Γ denotes the space of smooth sections.

Example B.5.6 (Chevalley–Eilenberg complex). Consider a real, finite-dimensional Lie algebra $\mathfrak{g}$. The cochains of the *Chevalley–Eilenberg* complex of $\mathfrak{g}$ are the elements of $\bigwedge\mathfrak{g}^*$, i.e. the vector spaces $\bigwedge^k\mathfrak{g}^*$, which is the same as the algebra of smooth functions on the supermanifold $\Pi\mathfrak{g}$. In particular, this is a subexample of the more general Example B.5.5.

Example B.5.7 (Odd (co)tangent bundle). This is again a subexample of Example B.5.5. Consider a smooth manifold M. Then we can consider the supermanifolds ΠTM and ΠT^*M, which we call the *odd tangent bundle* and *odd cotangent bundle* of M respectively. According to Example B.5.5, the smooth functions are then given by

$$C^\infty(\Pi TM)=\Gamma\left(\bigwedge T^*M\right)=\Omega(M),\tag{466}$$

$$C^\infty(\Pi T^*M)=\Gamma\left(\bigwedge TM\right)=\mathfrak{X}_{mult}(M),\tag{467}$$

where $\mathfrak{X}_{mult}(M)$ denotes the algebra of *multivector fields* on M endowed with the *Schouten–Nijenhuis bracket*.

Remark B.5.8. Since we mostly consider the the supermanifolds to be odd (co)tangent bundles, equations (466) and (467) are indeed helpful, since the fields are then given by differential forms or multivector fields.

B.6. Graded manifolds

We want to introduce another important manifold structure, which can be combined with the structure of supermanifolds.

Definition B.6.1 (Graded manifold). A *graded manifold* $\mathcal{M}$ is a manifold M, which locally looks like $(U, C^\infty(U)\otimes S(V^*))$, where $U\subset\mathbb{R}^n$ is open and V is a graded vector space.

Remark B.6.2. One can construct an isomorphism between the the structure sheaf of a supermanifold and the local model of a graded manifold, which will be in the category of $\mathbb{Z}$-graded algebras. Thus we consider a graded manifold as a tuple $(M,\mathcal{O}_M)$. We denote the $\mathbb{Z}$-grading by gh.

Remark B.6.3. In application to field theory, the space of fields arise with a graded manifold structure. Moreover, the $\mathbb{Z}$-grading in bosonic field theory, as defined before, is called the *ghost number*. Hence, we can combine the structure of a graded manifold and a supermanifold by considering a superspace of the form (457) and consider $S(V^*)$ in Definition B.6.1 to be given as in (463).

Definition B.6.4 (Graded vector bundle). A *graded vector bundle* E over a manifold M is a collection of ordinary vector bundles $\bigoplus_{k\in\mathbb{Z}} E_k$ over M.

Remark B.6.5. One can consider a sheaf $U\mapsto\Gamma(U,S(E^*|_U))$, which maps an open subset to a section with values in the graded symmetric algebra of the dual bundle restricted to U. Examples B.5.6 and B.5.7 have natural structures of graded manifolds, with ghost number corresponding to the usual degrees of the algebras associated to these examples.

Proposition B.6.6. *Any smooth graded manifold is isomorphic to a graded manifold associated to a graded vector bundle.*

B.7. Vector fields

We can now define the notion of a vector field on a supermanifold.

Definition B.7.1 (Supervector field). A *supervector field* is a vector field X on a supermanifold $\mathcal{M}$, with local coordinates (x^i,θ^μ), which is given by

$$X=\sum_{i,\mu} X^i\partial_{x^i}+Y^\mu\partial_{\theta^\mu},$$

such that $\partial_{\theta^\mu} x^i=0$ and $\partial_{\theta^\mu}\theta^\nu=\delta_\mu^\nu$.

Equivalently, vector fields on $U\times V$ are global derivations on $C^\infty(U)\otimes\bigwedge V^*$.

Remark B.7.2. The ∂_{x^i} are the usual position derivatives.

Definition B.7.3 (Graded supervector field). A *graded vector field* is a graded linear map $X\colon C^\infty(\mathcal{N})\to C^\infty(\mathcal{N})[k]$, where $\mathcal{N}$ is a graded manifold, which satisfies the *graded Leibniz rule*: for any two homogeneous functions $f,g\in C^\infty(\mathcal{N})$ we have

$$X(fg)=X(f)g+(-1)^{k|f|}fX(g). \tag{468}$$

Denote the space of graded vector fields on a given graded manifold (or supermanifold) $\mathcal{M}$ by $\mathfrak{X}(\mathcal{M})$. We can define a *graded* Lie bracket $[\ ,\ ]_{\mathfrak{X}(\mathcal{M})}\colon\mathfrak{X}(\mathcal{M})\times\mathfrak{X}(\mathcal{M})\to\mathfrak{X}(\mathcal{M})$ by

$$[X,Y]_{\mathfrak{X}(\mathcal{M})}:=X\circ Y-(-1)^{|X||Y|}Y\circ X.$$

The space $\mathfrak{X}(\mathcal{M})$ endowed with $[\ ,\ ]_{\mathfrak{X}(\mathcal{M})}$ becomes then a graded Lie algebra (since the bracket is graded).

Remark B.7.4. We will not always write the "super" in front of "vector field" and assume it is understood from the context.

B.8. Differential forms

Consider a graded manifold $\mathcal{M}$ with homogeneous local coordinates (x^i). Then we can locally form an algebra by adding coordinates $(\mathrm{d}x^i)$ to the coordinates we had before. Note that if x^i is a coordinate of odd degree, then $\mathrm{d}x^i$ is even and $(\mathrm{d}x^i)^2 \neq 0$. On an ordinary smooth manifold, differential forms have two important properties: they can be differentiated – hence the name differential forms – and they also provide the right objects for an integration theory on submanifolds. It turns out that on graded manifolds, the latter is no longer true, since the differential forms we introduced do not come along with a nice integration theory. Differential forms on a supermanifold $\mathcal{M}$ with local coordinates $(x^i;\theta^\mu)$ are locally generated by the 1-forms $\mathrm{d}x^i$ and $\mathrm{d}\theta^\mu$. We can take the algebra, which is generated by $x^i;\theta^\mu;\mathrm{d}x^i;\mathrm{d}\theta^\mu$, whereas now the $\mathrm{d}x^i$ are odd and the $\mathrm{d}\theta^\mu$ are even. In particular, for any local coordinate y of $\mathcal{M}$, we have $|\mathrm{d}y^j| = |y^j| + 1$. Thus, we have a de Rham differential on the algebra of differential forms

$$\mathrm{d}\colon\; x \mapsto \mathrm{d}x;\quad \theta \mapsto \mathrm{d}\theta;$$

with $\mathrm{d}^2 = 0$, which is a *graded differential*. Considering a supermanifold $\mathcal{M}$ with local coordinates $(x^i;\theta^\mu)$ we can look at the *shifted (odd) tangent bundle* $T[1]\mathcal{M}$ with local coordinates $(x^i;\theta^\mu;\mathrm{d}x^i;\mathrm{d}\theta^\mu)$. Moreover, the differential is then given by

$$\mathrm{d} = \sum_{i;\mu} \mathrm{d}x^i \partial_{x^i} + \mathrm{d}\theta^\mu \partial_{\theta^\mu}; \tag{469}$$

with the property

$$[\mathrm{d};\mathrm{d}] = 0. \tag{470}$$

Note that the differential is given as a vector field on $T[1]\mathcal{M}$. A vector field, such as in (469), satisfying (470) is called a *cohomological vector field*. Consider the *de Rham complex* $(\Omega^\bullet(\mathcal{M});\mathrm{d})$ for some given supermanifold $\mathcal{M}$. It is given by $C^\infty(T[1]\mathcal{M})$ equipped with a graded vector field Q of degree $+1$, i.e. $Q\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})[1]$, which satisfies the graded Leibniz rule (468). This becomes more clear by the following lemma:

Lemma B.8.1. *Every* cohomological vector field *on a supermanifold* $\mathcal{M}$ *corresponds to a* differential *on the graded algebra of smooth functions* $C^\infty(\mathcal{M})$

Proof. Since a cohomological vector field Q is a vector field of degree $+1$, which commutes with itself we have $[Q;Q] = 2Q\circ Q = 0$, and since it raises the degree of a function by 1, it corrsponds to a differential.

Example B.8.2 (Cohomological vector field on odd tangent bundles). Consider the odd tangent bundle $T[1]M$ for some smooth manifold M. Since $C^\infty(T[1]M)$ is given by $\Omega^\bullet(M)$, we get that the de Rham differential d_M on M is a cohomological vector field. In local coordinates $(x^i;\mathrm{d}x^i)$, we get

$$Q = \sum_i \mathrm{d}x^i \partial_{x^i}.$$

Example B.8.3 (Chevalley–Eilenberg differential). Let $(\mathfrak{g};[\ ;\ ])$ be a finite-dimensional Lie algebra. The graded manifold $\mathfrak{g}[1]$ carries a cohomological vector field Q, which corresponds to the Chevalley–Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^* = C^\infty(\mathfrak{g}[1])$. Let (e_i) be a basis of $\mathfrak{g}$, and let (f_{ij}^k) be the corresponding structure constants given by $[e_i, e_j] = \sum_k f_{ij}^k e_k$. Then, we get the cohomological vector field

$$Q = \frac{1}{2}\sum_{i,j,k} x^i x^j f_{ij}^k \partial_{x^k},$$

where (x^i) are the coordinates on $\Pi\mathfrak{g}$, which correspond to the basis dual to (e_i). In particular, one can check that $[Q, Q] = 0$ is equivalent to the fact that the bracket $[\ ,\ ]\colon \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ defined on the generators as above, satisfies the Jacobi identity.

B.9. Graded symplectic forms

A graded *symplectic form* of degree k on a graded manifold $\mathcal{M}$ is a closed (w.r.t. the de Rham differential), nondegenerate 2-form

$$\omega\colon T\mathcal{M} \to T^*[k]\mathcal{M},$$

which is, in local coordinates, given by

$$\omega = \sum_{i,j} dz^i \omega_{ij} dz^j, \tag{471}$$

where $z \in \{x^i, \ldots\}$.

Remark B.9.1. In application to the *BV formalism*, one considers ω to be *odd* and of degree -1, and for the *BFV formalism* to be *even* and of degree 0.

Example B.9.2. Let V be a real vector space. The contraction between V and V^* defines a nondegenerate pairing on $V \oplus V^*$. This includes a constant symplectic form of degree $k + \ell$ on $V[k] \oplus V^*[\ell]$.

Example B.9.3. Consider $\mathbb{R}[1]$ with the 2-form $\omega = dx dx$. This is a symplectic form of degree 2.

Next, we want to introduce some notation. Let V be a superspace and consider for a homogeneous element $v \in V$, and a function f on V, the so-called *left* and *right derivatives*, which are defined as

$$\overrightarrow{\partial}_v f := \partial_v f \tag{472}$$

$$f \overleftarrow{\partial}_v := (-1)^{|v|(|f|+1)} \partial_v f \tag{473}$$

Remark B.9.4. One can check that the right derivative satisfies the Leibniz rule from the right.

One can observe that a symplectic form as in (471), induces a *graded Poisson bracket*

$$\{\ ,\ \}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),$$

which is given by

$$\{f, g\} = f \overleftarrow{\partial}_i (\omega^{-1})^{ij} \overrightarrow{\partial}_j g,$$

which is a graded Poisson bracket. Similarly to ordinary manifolds, the Hamiltonian vector field X_H for a Hamiltonian function $H \in C^\infty(\mathcal{M})$ is given by the equation $\iota_{X_H}\omega = -dH$, which leads to the equation

$$\{H, G\} = (-1)^{|H|+1} X_H(G)$$

for some function $G \in C^\infty(\mathcal{M})$. Note that ι_X, for some vector field X, is a vector field on $\Pi T\mathcal{M}$. Moreover, if $\{\ ,\ \}$ is of degree k, we have

$$\{f, g\} = -(-1)^{(|f|+k)(|g|+k)}\{g, f\}$$

B.10. Lie derivative

We can also extend the definition of a *Lie derivative* (see Section A.3) L to supermanifolds, by noticing that with respect to X, we have

$$L_X = [\iota_X, d] \in \mathfrak{X}(\Pi T\mathcal{M}),$$

which is obtained by the *Cartan calculus*. Moreover, we have $[L_X, L_Y] = L_{[X,Y]}$ and $[\iota_X, L_Y] = \iota_{[X,Y]}$. In particular, L_X can be obtained by differentiating the *flow* of the vector field X.

B.11. Integration on superspaces

After defining the most important concepts around supermanifolds, we want to be able to perform integrals. Clearly, we need to extend the usual integration theory on manifolds to a more general picture. We want to start locally by considering integration over superspaces. Naturally, we define the integration for the even coordinates as usual. To get the correct integration theory, we need some notions on supermatrices and so-called *Berezinians*.

Definition B.11.1 (Supermatrix). A *supermatrix structure* is a matrix structure with parity attached to each row and column.

We usually arange a supermatrix structure in such a way that all the even rows and columns come first, and the odd ones second, so that it can be conveniently written in block form, e.g.

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are the matrices corresponding to the partition into even and odd rows and columns.

Definition B.11.2 (Order). If a supermatrix structure has p even rows and q odd rows, and r even columns and s odd columns, we call it a matrix structure of size $(p,q)\times(r,s)$. A $(p,q)\times(p,q)$ structure is said to have *order* (p,q).

Denote by $\mathrm{Mat}_{p,q}(\mathcal{A})$ the space of matrices of order (p,q) on a commutative superalgebra $\mathcal{A} = \mathcal{A}_{even}\oplus\mathcal{A}_{odd}$ and by $\mathrm{GL}_{p,q}(\mathcal{A})\subset\mathrm{Mat}_{p,q}(\mathcal{A})$ we denote the subset of invertible elements. We want to consider a homomorphism $\mathrm{GL}_{p,q}(\mathcal{A})\to\mathrm{GL}_{1,0}(\mathcal{A}) = \mathcal{A}_{even}^\times$, where $\mathcal{A}_{even}^\times$ denotes the group of invertible elements of $\mathcal{A}_{even}$. Such a map would be an analogue to the usual determinant.

Lemma B.11.3. *Consider*

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (\mathrm{Mat}_{p,q}(\mathcal{A}))_{even}.$$

Then X is invertible if and only if A and D are invertible.

Lemma B.11.3 is an immediate consequence of the following proposition.

Proposition B.11.4. *Let $\mathcal{A}$ be a commutative superalgebra, and*

$$\varepsilon\colon \mathcal{A}\to\bar{\mathcal{A}} = \mathcal{A}/(\mathcal{A}_{odd})$$

the natural homomorphism, and

$$\varepsilon\colon \mathrm{Mat}_n(\mathcal{A})\to\mathrm{Mat}_n(\bar{\mathcal{A}})$$

the corresponding homomorphism of matrix algebras (where the superstructure is ignored). Then $X\in\mathrm{Mat}_n(\mathcal{A})$ is invertible if and only if $\varepsilon(X)$ is invertible.

Definition B.11.5 (Berezinian). For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{p|q}(\mathcal{A})$ we define the *Berezinian* of X by

$$\mathrm{Ber}(X) := \frac{\det(A - BD^{-1}C)}{\det(D)}.$$

Remark B.11.6. Note that D is invertible by Lemma B.11.3. The entries in D and $A - BD^{-1}C$ lie in the commutative algebra $\mathcal{A}_{even}$ so that the determinants are well-defined and $\mathrm{Ber}(X) \in \mathcal{A}_{even}$.

Theorem B.11.7. *If $X, Y \in \mathrm{GL}_{p|q}(\mathcal{A})$, then*

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y).$$

Consider now a supermanifold $\mathcal{M}$ with local coordinates (x^i, θ_α) such that $1 \leq i \leq n$ and $1 \leq \alpha \leq m$. For the odd coordinates θ, we integrate the Berezinian $\mathrm{d}\theta$ according to the following rules:

- $\int \mathrm{d}\theta = 0$,
- $\int \theta\, \mathrm{d}\theta = 1$,
- Fubini's theorem holds.

Moreover, consider a function $f \in C^\infty(\mathcal{M})$ given locally as

$$f = \sum_{1 \leq \alpha_1 < \dots < \alpha_m \leq m} f_0 + f_{\alpha_1}\theta^{\alpha_1} + f_{\alpha_1\alpha_2}\theta^{\alpha_1}\theta^{\alpha_2} + \dots + f_{\alpha_1,\dots,\alpha_m}\theta^{\alpha_1}\cdots\theta^{\alpha_m}; \tag{474}$$

where f_0 and each $f_{\alpha_1,\dots,\alpha_k}$ are elements in $C^\infty(U)$ for U an open subset of $\mathbb{R}^n$ (for all $1 \leq k \leq m$). Then, according to the rules above, we get

$$\int f\, \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^n = f_{\alpha_1,\dots,\alpha_m} =: f_{top}.$$

Thus, for a supermanifold of the form ΠV, integration in general will give a map

$$\int_{\Pi V} : \widehat{\bigwedge} V \to \mathbb{R}; \tag{475}$$

such that for any $g \in \bigwedge^{<top} V$ we get $\int_{\Pi V} g = 0$, and hence the map (475) is given by

$$\int_{\Pi V} = \int_{\Pi V}\Big|_{\bigwedge^{top} V} : \widehat{\bigwedge}^{top} V \to \mathbb{R}; \tag{476}$$

for each choice of frame of V. If we take a coordinate patch $U \times \Pi V$ of $\mathcal{M}$, we get

$$\int_{U \times \Pi V} f = \int_U \int_{\Pi V} f = \int_U f_{top}\,\mathrm{d}x.$$

Consider now a linear map $A\colon V \to W$ between two vector spaces V and W. We can look at the map $\wedge A^* : \bigwedge W \to \bigwedge V$, which can be regarded as a map $\Pi V \to \Pi W$. Then, for any $f \in C^\infty(\Pi W) = \bigwedge W$, we have

$$\int_{\Pi V} \wedge A^* f = \det(A) \int_{\Pi W} f.$$

Moreover, consider a linear isomorphism

$$\varphi := D \times \wedge A : U \times \Pi V \to U' \times \Pi W;$$

corresponding to linear maps $D\colon U \to U'$ and $A\colon V \to W$. Then

$$\frac{\det(D)}{\det(A)} \int_{U \oplus V} \varphi f = \operatorname{Ber}\begin{pmatrix} D & 0 \\ 0 & A \end{pmatrix} \int_{U \oplus V} \varphi f = \int_{U' \oplus W} f.$$

The corresponding measure is given by

$$\frac{\det(D)}{\det(A)} = \mathrm{d}x^1 \cdots \mathrm{d}x^n \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^m;$$

We can apply this construction also to *Gaussian integrals* and observe

$$\int_{\mathbb{C}} \mathrm{e}^{\frac{\mathrm{i}}{2} z^{\mathrm{t}} M z} \mathrm{d}z = \frac{\text{const.}}{\operatorname{Ber}(M)},$$

where M is the matrix for some nondegenerate pairing. Using the diffeomorphism (464), we can obtain.

$$\int_{U \oplus V} \varphi f \operatorname{Ber}(\varphi\, \mathrm{d}) = \int_{U' \oplus V'} f.$$

B.12. Integration on supermanifolds

To perform integration on a manifold, we need the notion of a *density*, which is fairly standard for ordinary manifolds. A *density* for our purposes is a section of the *Berezinian bundle* tensor the orientation bundle of the underlying manifold $\mathcal{M}$, which locally means that they are functions transforming like $\operatorname{Ber}(\varphi\, \mathrm{d})$. Everything else is constructed in the same way as for ordinary manifolds.

APPENDIX C

BV algebras and relation to field theory

We want to recall some notions on BV algebras as in [66], and how it is related to the original gauge formalism developed by Batalin and Vilkovisky within quantum field theory.

C.1. Braid algebras

Let us first recall what a braid algebra is. A *braid algebra* B is a commutative DG algebra endowed with a Lie bracket $[\ ,\ ]$ of degree $+1$ satisfying the Poisson relations

$$[a, bc] = [a, b]c + (-1)^{|a|(|b|-1)} b[a, c], \qquad \forall a, b, c \in B \tag{477}$$

An identity element in B is an element $\mathbf{1}$ of degree 0 such that it is an identity for the product and $[\mathbf{1}, -] = 0$.

C.2. BV algebras

A *BV algebra* A is a commutative DG algebra endowed with an operator $\Delta \colon A_\bullet \to A_{\bullet+1}$ such that $\Delta^2 = 0$ and

$$\begin{aligned}\Delta(abc) = {} & \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{(|a|-1)|b|} b\Delta(ac) \\ & - \Delta a\, bc - (-1)^{|a|} a\Delta b\, c - (-1)^{|a|+|b|} ab\Delta c, \qquad \forall a, b, c \in A.\end{aligned} \tag{478}$$

An *identity* in A is an element $\mathbf{1}$ of degree 0 such that it is an identity for the product and $\Delta\mathbf{1} = 0$. One can show that a BV algebra is in fact a special type of a braid algebra. More precisely, a BV algebra is a braid algebra endowed with an operator $\Delta \colon A_\bullet \to A_{\bullet+1}$ such that $\Delta^2 = 0$ and such that the bracket and Δ are related by

$$[a, b] = (-1)^{|a|} \Delta(ab) - (-1)^{|a|} \Delta a\, b - a\Delta b, \qquad \forall a, b \in A. \tag{479}$$

Moreover, in a BV algebra we have

$$\Delta([a, b]) = [\Delta a, b] + (-1)^{|a|-1}[a, \Delta b], \qquad \forall a, b \in A. \tag{480}$$

C.3. Connection to field theory

We would like to explain the name "BV" algebra. This comes from the approach to deal with gauge theories in quantum field theory developed by Batalin–Vilkovisky in the setting of odd symplectic (super)manifolds. Let $(\mathcal{F}, \omega)$ be an odd symplectic (super)manifold. In physics, $\mathcal{F}$ is called the space of fields. Let $f \in C^\infty(\mathcal{F})$ and consider its Hamiltonian vector field X_f. One can check that $C^\infty(\mathcal{F})$ endowed with the Poisson bracket

$$(f, g) := (-1)^{|f|-1} X_f(g) \tag{481}$$

is a braid algebra. Let $\mu \in \Gamma(Ber(\mathcal{F}))$ be a nowhere-vanishing section of the Berezinian bundle of $\mathcal{F}$. This represents a density which is characterized by the integration map $\int\colon \Gamma_c(\mathcal{F}, Ber(\mathcal{F})) \to \mathbb{R}$. Hence μ induces an integration map on functions with compact support

$$C^\infty(\mathcal{F}) \ni f \mapsto \int_{\mathcal{L}\subset\mathcal{F}} f\mu^{1/2}, \tag{482}$$

for some Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, where the integral exists. Then one can define a divergence operator $\mathrm{div}_\mu X$ by

$$\int_{\mathcal{F}} (\mathrm{div}_\mu X) f \mu = -\int_{\mathcal{F}} X(f)\mu. \tag{483}$$

LEMMA C.3.1. *For a vector field X let $\tilde{X} = -X - \mathrm{div}_\mu X$. Then*

$$\int_{\mathcal{F}} fX(g)\mu = (-1)^{|f||X|}\int_{\mathcal{F}} \tilde{X}(f)g\mu. \tag{484}$$

Moreover, $\mathrm{div}_\mu(fX) = f\,\mathrm{div}_\mu X - (-1)^{|f||X|}X(f)$ *and if $S \in C^\infty(\mathcal{F})$ is an even function, then* $\mathrm{div}_{\exp(S)\mu} X = \mathrm{div}_\mu X + X(S)$.

One can then define Δ_μ to be the odd operator on $C^\infty(\mathcal{F})$ given by

$$\Delta_\mu f = \mathrm{div}_\mu X_f. \tag{485}$$

A BV (super)manifold $(\mathcal{F}, \omega, \mu)$ is then an odd symplectic (super)manifold with Berezinian μ such that $\Delta_\mu^2 = 0$.

PROPOSITION C.3.2. *Let $(\mathcal{F}, \omega, \mu)$ be a BV (super)manifold.*

(1) *The algebra $(C^\infty(\mathcal{F}), (\ ,\ ), \Delta_\mu)$ is a BV algebra, where Δ_μ is given as in (485) and $(\ ,\ )$ is the odd Poisson bracket coming from the odd symplectic form ω as in (481).*
(2) *The Hamiltonian vector field associated to some $f \in C^\infty(\mathcal{F})$ is given by the formula $X_f = [\Delta_\mu, f] + \Delta_\mu f$, where $[\ ,\ ]$ denotes the commutator of operators.*
(3) *If $S \in C^\infty(\mathcal{F})$ and Δ_S is the operator associated to the Berezinian $\exp(S)\mu$, then $\Delta_S = \Delta_\mu + X_S$ and $\Delta_S^2 = X_{\Delta_\mu S + \frac{1}{2}(S,S)}$.*

Note that point (3) is exactly the case that we have in quantum field theory. Moreover, if

$$\Delta_\mu S + \frac{1}{2}(S, S) = 0, \tag{486}$$

we get that $\Delta_S^2 = 0$, which ensures a BV algebra structure. In physics, the function S is called the action and Equation (486) is usually called the Quantum Master Equation[1].

[1]Here we have set $i\hbar = 1$, whereas in quantum field theory we want dependence on $\hbar$ as a formal variable and consider formal power series as Taylor expansions (cf. perturbative expansion of path integrals)

APPENDIX D

Configuration spaces and their compactifications

To define the quantum state, we need to recall the notion of configuration spaces and their compactification as in [5, 63] due to Fulton–MacPherson and Axelrod–Singer.

D.1. FMAS-compactification

We start with the definition of the configuration space.

Definition D.1.1 (Open configuration space). Let M be a manifold and S a finite set. The *open configuration space* of S in M is defined as

$$\mathrm{Conf}_S(M) := \{\iota\colon S \hookrightarrow M \mid \iota \text{ injection}\} \tag{487}$$

Elements of $\mathrm{Conf}_S(M)$ are called S-configurations. To give an explicit definition of the compactification that can be extended to manifolds with boundaries and corners, we introduce the concept of *collapsed configurations*. Intuitively, a collapsed S-configuration is the result of a collapse of a subset of the points in the S-configuration. However, we remember the relative configuration of the points before the collapse by directions in the tangent space. This is a configuration in the tangent space that is well-defined only up to translations and scaling. The difficulty is that one can imagine a limiting configuration where two points collapse first together and then with a third (see Figure D.1). This explains the recursive nature of the following definition. Recall that if X is a vector space, then $X \rtimes \mathbb{R}_{>0}$ acts on X by translations and scaling.

Definition D.1.2 (Collapsed configuration in M). Let M be a manifold, S a finite set and $P = \{S_1,\dots,S_k\}$ be a partition of S. A *P-collapsed configuration in M* is a k-tuple (p_α, c_α) such that $((p_\alpha, c_\alpha))_{\alpha=1}^k$ satisfies

(1) $p_\alpha \in M$ and $p_\alpha \neq p_{\alpha'}$, for $\alpha \neq \alpha'$,
(2) $c_\alpha \in \tilde{C}_{S_\alpha}(T_{p_\alpha}M)$, where for $|S| = 1$, $\tilde{C}_S(X) := \{pt\}$ and for $|S| \geq 2$

$$\tilde{C}_S(X) := \left(\bigsqcup_{\substack{P=\{S_1,\dots,S_k\}\\ S=\sqcup S_\alpha,\, k\geq 2}} \{(x_\alpha, c_\alpha)_1 \cdots {}_k \mid (x_\alpha, c_\alpha)\ P\text{-collapsed } S\text{-configuration in } X\}\right) \Big/ (X \rtimes \mathbb{R}_{>0}) \tag{488}$$

Here, $\phi \in X \rtimes \mathbb{R}_{>0}$ acts on (x_α, c_α) by $(x_\alpha, c_\alpha) \mapsto (\phi(x_\alpha), d\phi_{x_\alpha} c_\alpha)$.

Intuitively, given a partition $P = \{S_1,\dots,S_k\}$, a k-tuple (p_α, c_α) describes the collapse of the points in S_α to p_α. c_α remembers the relative configuration of the collapsing points. This relative configuration can itself be the result of a collapse of some points.

Definition D.1.3 (FMAS compactification). The *compactified configuration space* $\mathrm{C}_S(M)$ of S in M is given by

$$\mathrm{C}_S(M) := \bigsqcup_{\substack{S_1,\dots,S_k\\ S=\sqcup S_\alpha}} \{(p_\alpha, c_\alpha)_1 \cdots {}_k \mid (p_\alpha, c_\alpha)\ P\text{-collapsed } S\text{-configuration in } M\}. \tag{489}$$

D.2. Boundary strata

A precise description of the combinatorics of the stratification can be found in [63], where it is also shown that $C_S(M)$ is a manifold with corners and is compact if M is compact. For us, only strata in low codimensions are interesting. Let $S = \{s_1, \dots, s_k\}$. The stratum of codimension 0 corresponds to the partition $P = \{\{s_1\}, \dots, \{s_k\}\}$. For $\ell > 1$, strata of codimension 1 correspond to the collapse of exactly one subset $S' = \{s_1, \dots, s_\ell\} \subseteq S$ with no further collapses, i.e a partition $P = \{\{s_1, \dots, s_\ell\}, \{s_{\ell+1}\}, \dots, \{s_k\}\}$ and configuration (p, c) with c in the component of $\tilde{C}_{S'}(X)$ given by the partition $P = \{\{s_1\}, \dots, \{s_\ell\}\}$. This boundary stratum will be denoted by $\partial_{S'} C_S(M)$, in particular, we have

(490)
$$\partial C_S(M) = \bigcup_{S' \subseteq S} \partial_{S'} C_S(M).$$

There is a natural fibration $\partial_{S'} C_S(M) \to C_{S \setminus S' \cup \{pt\}}(M)$ whose fiber is $\tilde{C}_S(\mathbb{R}^{\dim M})$. Finally, we note that if $|S| = 2$, then $C_S(M) \cong Bl_\Delta(M \times M)$, the *differential-geometric blow-up* of the diagonal $\Delta \subset M \times M$, and $\tilde{C}_S(X) \cong S^{\dim X - 1}$. See Figure D.1 for an example of a configuration of points and coresponding boundary strata.

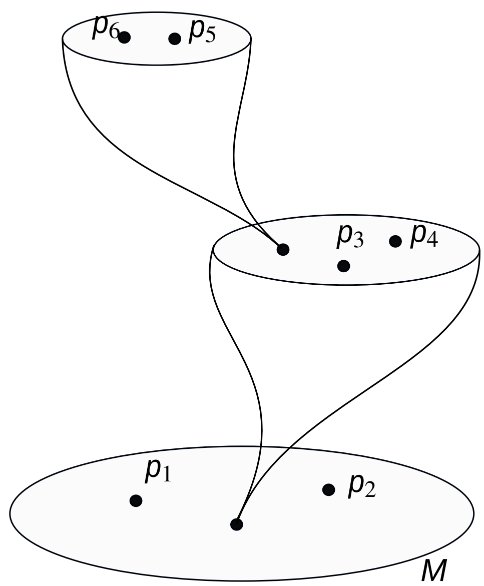

FIGURE D.1. An element of $C_S(M)$.

D.3. Configuration spaces for manifolds with boundary

We proceed to recall the definition of a compactified configuration space for manifolds with boundary. Let M be a compact manifold with boundary ∂M. Recall that for a manifold M with boundary ∂M, at points $p \in \partial M$ there is a well-defined notion of inward and outward half-space in $T_p M$. If $H \subset X$ is a half-space, then $\partial H \subset X$ is a hyperplane. $\partial H \rtimes \mathbb{R}_{>0}$ acts on H by translations and scaling.

DEFINITION D.3.1 (CONFIGURATION SPACES FOR MANIFOLDS WITH BOUNDARY). Let M be a manifold with boundary ∂M. For S, T finite sets, we define the *open configuration space* by

(491)
$$\mathrm{Conf}_{S,T}(M, \partial M) := \{(\iota, \iota') : S \sqcup T \hookrightarrow M \sqcup \partial M\}$$

Definition D.3.2 (Collapsed configuration on manifolds with boundary). Let $(M,\partial M)$ be a manifold with boundary. Let S,T be finite sets and $P=\{S_1,\dots,S_k\}$ a partition of $S\sqcup T$. Then, a P-collapsed (S,T)-configuration in M is a k-tuple of pairs (p_α,c_α) such that

(1) $p_\alpha\in M$ and $p_\alpha\neq p_{\alpha'}$, for all $\alpha\neq\alpha'$,
(2) $S_\alpha\cap T\neq\emptyset\Rightarrow p_\alpha\in\partial M$,
(3)
$$c_\alpha\in\begin{cases}\breve{C}_{S_\alpha}(T_{p_\alpha}M) & p_\alpha\in M\setminus\partial M\\ \breve{C}_{S\cap S_\alpha,T\cap S_\alpha}(\mathbb{H}(T_{p_\alpha}M)) & p_\alpha\in\partial M\end{cases}$$

where $\mathbb{H}(T_pM)\subset T_pM$ denotes the inward half-space in T_pM. Here, for a vector space X and a half-space $H\subset X$, $\breve{C}_{\emptyset,\{pt\}}(H):=\breve{C}_{\{pt\},\emptyset}(H):=\{pt\}$, and for $|S\sqcup T|\geq 2$,

(492)
$$\breve{C}_{S,T}(H):=\left(\coprod_{\substack{P=\{S_1,\dots,S_k\}\\ S\sqcup T=\sqcup S_\alpha,\,k\geq 2}}\{(v_1,c_1)\dots(v_k,c_k)\ P\text{-collapsed }(S,T)\text{-configuration in }H\}\right)\Big/(\partial H\rtimes\mathbb{R}_{>0})$$

Definition D.3.3 (FMAS compactification for manifolds with boundary). We define the *compactification* $C_{S,T}(M,\partial M)$ of $\mathrm{Conf}_{S,T}(M,\partial M)$ by

(493)
$$C_{S,T}(M,\partial M)=\coprod_{\substack{P=\{S_1,\dots,S_k\}\\ S\sqcup T=\sqcup S_\alpha}}\{(p_\alpha,c_\alpha)_1\dots(p_\alpha,c_\alpha)_k\ P\text{-collapsed }(S,T)\text{-configuration}\}$$

Again, this is a manifold with corners and is compact if M is compact. We proceed to describe the strata of low codimension. Let $U=\{u_1,\dots,u_k\}$, $V=\{v_1,\dots,v_k\}$. The codimension 0 stratum again is given by the partition $P=\{\{u_1\},\dots,\{u_k\},\{v_1\},\dots,\{v_\ell\}\}$. Let us describe the strata of codimension 1. We denote by $\partial^{I}_{S}C_{U,V}(M,\partial M)$ a boundary stratum where a subset $S\subset U$ collapses in the bulk, described in the same way as above. On manifolds with boundary, there are new boundary strata in the compactified configuration space given by the collapse of a subset of points to a point in the boundary. Concretely, given a subset $S=\{u_1,\dots,u_{k'},v_1,\dots,v_{\ell'}\}\subset U\sqcup V$, there is a boundary stratum $\partial^{II}_{S}C_{U,V}(M,\partial M)$ corresponding to the partition $P=\{S,\{u_{k'+1}\},\dots,\{u_k\},\{v_{\ell'+1}\},\dots,\{v_\ell\}\}$ and collapsed configurations (p_α,c_α) with $p_\alpha\in\partial M$ and c_α corresponding to the partition $P'=\{\{u_1\},\dots,\{u_k\},\{v_1\},\dots,\{v_\ell\}\}$. The boundary decomposes as

(494)
$$\partial C_{U,V}(M,\partial M)=\coprod_{S\subset U}\partial^{I}_{S}C_{U,V}(M,\partial M)\sqcup\coprod_{S\subset U\sqcup V}\partial^{II}_{S}C_{U,V}(M,\partial M)$$

D.4. Configuration spaces for manifolds with corners

Finally, we consider a manifold M with boundary ∂M and corners $\partial\partial M$. Note that around points in corners $p\in\partial\partial M$ there is a notion of inward quadrant $Q(T_pM)\subset T_pM$. It can be defined e.g. in coordinates, since the transition functions have to preserve both boundaries and corners. If $Q\subset X$ is any quadrant, its boundary is the union of two half-hyperplanes whose intersection is a $(\dim X-2)$-dimensional subspace W. This subspace acts on Q by translations. Again, $\mathbb{R}_{>0}$ acts on Q by scaling. Note that in this case, $\breve{C}_{\{pt\}}(Q)\cong I$, where I is an interval. Hence the definition of collapsed configurations should be adapted to this case. We want to compactify the open configuration spaces

(495)
$$\mathrm{Conf}^{C}_{S,T,U}(M,\partial M,\partial\partial M)$$

where M is a manifold with corners. We proceed to define collapsed configurations as above:

Definition D.4.1 (Collapsed configurations for manifolds with corners). Let $(M, \partial M, \partial\partial M)$ be a manifold with corners. Let S, T, U be finite sets and $\mathsf{P} = (S_1, \ldots, S_k)$ be a partition of $S \sqcup T \sqcup U$. Then a P-collapsed (S, T, U)-configuration in M is a k-tuple of pairs (p_α, c_α) such that

(1) $p_\alpha \in M$ and $p_\alpha \neq p_{\alpha'}$, for all $\alpha \neq \alpha'$,
(2) $S_\alpha \cap T \neq \emptyset \Rightarrow p_\alpha \in \partial M$,
(3) $S_\alpha \cap U \neq \emptyset \Rightarrow p_\alpha \in \partial\partial M$,
(4)
$$c_\alpha \in \begin{cases} \mathring{C}^C_{S_\alpha}(T_{p_\alpha} M) & p_\alpha \in M \setminus \partial M \\ \mathring{C}^C_{S \cap S_\alpha, T \cap S_\alpha}(\mathbb{H}(T_{p_\alpha} M)) & p_\alpha \in \partial M \setminus \partial\partial M \\ \mathring{C}^C_{S \cap S_\alpha, T \cap S_\alpha, U \cap S_\alpha}(\mathbb{Q}(T_{p_\alpha} M)) & p_\alpha \in \partial\partial M \end{cases}$$

where, for Y a quadrant of X, we have $\mathring{C}^C_{S,\emptyset,\emptyset}(Y) = \mathring{C}^C_{\emptyset,T,\emptyset}(Y) = \{pt\}$, $\mathring{C}^C_{\emptyset,\emptyset,\{pt\}}(Y) \cong I$, and for $|S \sqcup T \sqcup U| \geq 2$ we define

(496)
$$\mathring{C}^C_{S,T,U}(Y) := \bigsqcup_{\substack{\mathsf{P} = \{S_1, \ldots, S_k\} \\ S \sqcup T \sqcup U = \sqcup S_\alpha, k \geq 2}} \{(y_\alpha, c_\alpha) \mid (y_\alpha, c_\alpha)\ \mathsf{P}\text{-collapsed } (S,T,U)\text{-configuration in } Y\} / (\partial Y \rtimes \mathbb{R}_{>0})$$

This compactified configuration space has three types of boundary strata: Strata where a set of bulk points collapses in the bulk (called Type I strata), strata where a subset of bulk and boundary points collapses at the boundary (called Type II strata), and strata where a subset of all points collapses to a corner point (called Type III strata):

$$\partial C_{S,T,U}(M, \partial M, \partial\partial M) = \bigsqcup_{S' \subseteq S} \partial^{\mathrm{I}}_{S'} C_{S,T,U}(M, \partial M, \partial\partial M) \sqcup \bigsqcup_{\substack{S' \subseteq S \sqcup T \\ S' \not\subseteq S}} \partial^{\mathrm{II}}_{S'} C_{S,T,U}(M, \partial M, \partial\partial M) \sqcup \bigsqcup_{S' \subseteq S \sqcup T \sqcup U} \partial^{\mathrm{III}}_{S'} C_{S,T,U}(M, \partial M, \partial\partial M) \tag{497}$$

Remark D.4.2. At this point, one can generalize the definitions above to that of compactifications of configuration spaces on stratified manifolds, with strata of any codimension. This is required for the extension of perturbative quantization to fully extended theories.

Notation D.4.3. For a manifold M without boundary, we also denote the compactified configuration space of n points $C_{[n]}(M)$ on M by $C_n(M)$ (here $[n] = \{1, \ldots, n\}$). Moreover, for a manifold M with boundary, we denote the compactified configuration space $C_{[n],[m]}(M)$ of n points on the bulk of M and m points on the boundary ∂M of M by $C_{[n],[m]}(M, \partial M)$. We will also write $C_\Gamma(M)$ for $C_{[n],[m]}(M, \partial M)$, if Γ is a graph with $n + m$ vertices, n vertices in the bulk of M and m vertices on ∂M. Moreover, we will write $C^C_{n,m}(M)$ (or $C^C_\Gamma(M)$) for $C^C_{[n],[m],\emptyset}(M, \partial M, \partial\partial M)$, if M is a manifold with corners.

APPENDIX E

On the Propagator

We have an explicit propagator for the Poisson Sigma Model, i.e. using the superfields of it, on a disk with alternating boundary conditions, which was computed in [27], in [44] and, in full generality, in [57].

E.1. Construction of the branes

Consider an n-sided polygon $P_n = u(\mathbb{H}^+)$ where $u : \mathbb{H}^+ \to P_n$ is a suitable homeomorphism between the compactified complex upper half-plane $\mathbb{H}^+$ and P_n, depending on the number of the branes considered. Let G_{S_i}, be the relevant superpropagators for the Poisson Sigma Model with n branes defined by constraints $C_j = \{x^{\mu_j} = 0 \mid \mu_j \in I_j\}$ (also called *branes*) and index sets $S_1 = I_1^C \cap I_2 \cap I_3^C \cap \cdots \cap I_n$, $S_2 = I_1 \cap I_2^C \cap I_3 \cap \cdots \cap I_n^C$ for n even, and $S_1 = I_1^C \cap I_2 \cap I_3^C \cap \cdots \cap I_n^C$, $S_2 = I_1 \cap I_2^C \cap \cdots \cap I_n$ for n odd, which are called *relevant*. It turns out that the $C_i \subset P$ are coisotropic submanifolds of P [27].

E.2. Constructing integral kernels

The integral kernels $\theta(Q,P)_{S_i} := \frac{i}{\hbar}\langle \widehat{\mathsf{X}}(Q)\widehat{\mathsf{b}}(P)\rangle$ for the two brane case are given by:

$$\theta(Q,P)_{S_1} = \frac{1}{2\pi} d \arg \frac{(u-v)(\bar{u}-v)}{(\bar{u}+v)(u+v)}, \tag{498}$$

$$\theta(Q,P)_{S_2} = \frac{1}{2\pi} d \arg \frac{(u-v)(\bar{u}+v)}{(\bar{u}-v)(u+v)}, \tag{499}$$

where $P_2 := u(\mathbb{H}^+)$ with $u(z) = \sqrt{z}$, $v := u(w)$, $d = d_u + d_v$. We identify (P,Q) with the couple (u,v). Consider e.g. P_2 to be the worldsheet disk Σ with boundary $\partial\Sigma = \bigcup_{1 \leq j \leq 6} J_j$ (we denote the intervals here by J instead of I such that there is no confusion with the index sets) and the branes $C_1 = \{x^{\mu_1} = 0 \mid \mu_1 \in I_1 = \{1,\ldots,n\}\}$ and $C_2 = \{x^{\mu_2} = 0 \mid \mu_2 \in I_2 = \emptyset\}$, which correspond to the boundary conditions of $\partial_1\Sigma$ and $\partial_2^{\text{tot}}\Sigma$ respectively. The components $\partial_1\Sigma$ and $\partial_2^{\text{tot}}\Sigma$ are such that $\partial\Sigma = \overline{\partial_1\Sigma \sqcup \partial_2^{\text{tot}}\Sigma}$, where $\partial_1\Sigma$ is chosen to be some J_1 endowed with the $\frac{\delta}{\delta E}$-polarization and $\partial_2^{\text{tot}}\Sigma = \bigcup_{2 \leq j \leq 6} J_j$ such that J_j is endowed with the $\frac{\delta}{\delta X}$-polarization and with the boundary condition $\mathsf{b} \neq 0$ for j odd and even respectively. Now we get $S_1 = I_1^C \cap I_2 = \emptyset$ and $S_2 = I_1 \cap I_2^C = \{1,\ldots,n\}$. Now P_2 is defined by $P_2 = u(\mathbb{H}^+)$, where u is the map $z \mapsto \sqrt{z}$. Points $(P,Q) \in P_2 \times P_2$ are represented respectively by a pair of complex numbers (u,v) in the first quadrant, with $u = u(z)$, $v = u(w)$ for all $(z,w) \in \mathbb{H}^+ \times \mathbb{H}^+$. The boundary $\partial_1 P_2$ (corresponding to $\partial_1\Sigma$) is given by the positive imaginary axis, while $\partial_2 P_2$ (corresponding to $\partial_2^{\text{tot}}\Sigma$) is given by the positive real axis.

E.3. Construction of superpropagators

The boundary conditions imposed by the index sets S_i are $\phi(v; u \in \partial_1 P_2)_{S_1} = \phi(u \in \partial_2 P_2; u)_{S_1} = 0$, $\phi(v; u \in \partial_2 P_2)_{S_2} = \phi(v \in \partial_1 P_2; u)_{S_2} = 0$. Let

$$\psi(u; v)_{S_1} = \arg \frac{(u - v)(\bar{u} - v)}{(\bar{u} + v)(u + v)}, \tag{500}$$

$$\psi(u; v)_{S_2} = \arg \frac{(u - v)(\bar{u} + v)}{(\bar{u} - v)(u + v)}, \tag{501}$$

which satisfy the same boundary conditions as $\phi(v; u)_{S_i}$. Now for vanishing cohomology, we get the following Theorem.

Theorem E.3.1. *The integral kernels for the superpropagators G_{S_i} in presence of two branes are given by*

$$\phi(v; u)_{S_i} = \frac{1}{2} d\, \psi(u; v)_{S_i}, \tag{502}$$

with angle maps (500) *and* (501). *The integral kernels satisfy the additional boundary conditions $\phi(v; u)_{S_1} = \phi(v; \bar{u}) = \phi(-\bar{v}; u)_{S_1}$, $\phi(v; u)_{S_2} = \phi(v; -\bar{u})_{S_2} = \phi(\bar{v}; u)_{S_2}$, i.e. every boundary component of P_2 is labeled by a boundary condition for both the variables $(u; v)$. By construction $\phi(v; u)_{S_1} = \psi(u; v)_{S_2}$, $\phi(v; u)_{S_2} = \psi(u; v)_{S_1}$.*

E.4. Relation to Kontsevich's propagator

Let ϕ be Kontsevich's angle 1-form. Then, one can show that

$$\phi(v; u)_{\mathcal{A}_1} = \frac{1}{2} d \arg \frac{(u - v)(u + v)}{(u + \bar{v})(u - \bar{v})} = \frac{1}{2} d \arg \frac{(z - w)}{(z - \bar{w})} = \frac{1}{2} d\, \phi(z; w), \tag{503}$$

$$\phi(v; u)_{\mathcal{A}_2} = \frac{1}{2} d \arg \frac{(u - v)(u + v)}{(\bar{u} - v)(\bar{u} + v)} = \frac{1}{2} d \arg \frac{(z - w)}{(\bar{z} - w)} = \frac{1}{2} d\, \phi(w; z), \tag{504}$$

where $\mathcal{A}_1 = I_2 \setminus I_2$ and $\mathcal{A}_2 = I_1^C \setminus I_2^C$.

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