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On a Categorical Approach to BV

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Abstract

The BV formalism (cf. [BV81; BV83]) is a cohomological symplectic framework for the perturbative quantization of gauge theories, generalizing the Faddeev–Popov method (cf. [FP67]) and the BRST formalism (cf. [BRS76; Tak79]). Its classical version is based on the graded symplectic structure of the BV space of fields and the associated Cartan calculus.

The category SmthSet of smooth sets, namely the topos of sheaves on Cartesian spaces, naturally encodes many classical bosonic Lagrangian field-theoretic constructions (cf. [GS25a]), and, in particular, the Cartan calculus on the field space when restricted to well-behaved de Rham forms. Its infinitesimal thickening ThSmthSet (cf. [GS25b]), given by enlarging the site with an infinitesimal structure, recovers all relevant tangent bundles via synthetic tangent bundle constructions. This framework naturally extends to fermionic Lagrangian field theories by further enriching the site with super-structure (cf. [Gio25; GS26]).

In this thesis, we show how the local classical BV formalism is naturally described within the category sThSmthSet of super thickened smooth sets. This yields a rigorous formulation of the Cartan calculus on the infinite-dimensional BV space of fields. As an example, we present the AKSZ construction of the BV extension of classical BF theory in $d = 4$. The AKSZ formalism provides a systematic construction of topological field theories compatible with the BV formalism (cf. [Ale+97]). BF theory is a topological field theory that exists in any dimension and can be used to define invariants of knots and manifolds (cf. [Cat97]). In different dimensions, it is related to different field theories (cf. [Mne19][ft. 100, p. 140]). However, both a complete treatment of the AKSZ formalism and a global formulation of BV theories require further extensions of sThSmthSet, leaving several interesting questions open.

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Contents

Introduction

Modern classical field theories are typically formulated in the Lagrangian framework, where the action principle determines the dynamics of fields. This principle, first applied to classical optical systems by Hamilton (cf. [Ham34]), is grounded in the successive formulations of the principle of least action by Maupertuis, Lagrange, and Euler, and ultimately traces its origins to Fermat's principle (1662). The latter states that light travels along the path that can be traversed in the shortest time. Hamilton's idea is that the Euler–Lagrange equations characterize the critical points of the action functional, thereby describing the dynamics of a physical system entirely in terms of its Lagrangian.
In modern classical field theories, this formulation remains fundamental for four main reasons. First, symmetries of physical systems are naturally encoded as symmetries of the action. Second, the manifest covariance of the action ensures that the Euler–Lagrange equations take equivalent forms in different coordinate systems. Third, within the Lagrangian framework, the potential replaces the force. While the force is a classical concept that does not generalize naturally to quantum mechanics, the potential does. Finally, it provides a natural route to quantization through Feynman's path integral formulation, which may be regarded as the quantum counterpart of the classical action principle (cf. [Fey48]).(2) The latter heuristically describes the propagator of the quantum particle as a sum over classical paths, each weighted by an exponential whose phase is the classical action of that path divided by $\hbar$.

The path integral is not rigorously defined from a measure-theoretic perspective (cf. [Maz09]), but it can be understood perturbatively around a classical configuration via the stationary phase formula. This allows one to define the path integral as an asymptotic series in $\hbar \to 0$, whose coefficients are Feynman diagrams. Crucially, the stationary phase formula requires critical points of the action, i.e. classical solutions, to be isolated. This guarantees that the Hessian at a critical point is non-degenerate. However, if the theory is a gauge theory, there exists a tangential distribution on the space of fields that preserves the action. Consequently, critical points lie on gauge orbits generated by this distribution and are therefore not isolated. The Faddeev–Popov method (cf. [FP67]), the BRST formalism (cf. [BRS76; Tak79]), and the BV formalism (cf. [BV81; BV83]) allow one to solve this issue and perturbatively quantize gauge theories whose gauge symmetry is described by a Lie group, a Lie algebroid, and a non-integrable distribution, respectively. The BV formalism subsumes the other two. Although the last is rigorously understood in the finite-dimensional case, the discussion becomes more delicate when the space of fields is infinite-dimensional (cf. [CMS25]). In particular, one lacks the notion of Cartan calculus and of a measure on the space of fields, essential for the classical and quantum versions of the BV formalism.

The category SmthSet of smooth sets, namely the topos of sheaves on Cartesian spaces, naturally encodes the action principle for local bosonic classical Lagrangian field theories together with many related constructions (cf. [GS25a]). In particular, it allows one to formalize the Cartan calculus on the field space when restricted to well-behaved de Rham forms. The underlying philosophy of this topos-theoretic approach is operational: geometric spaces are not understood as sets of points equipped with additional structure, but rather through the families of probes used to investigate them. In physics, these probes may be particle trajectories or, more generally, extended objects such as branes. By taking Cartesian spaces as probes, one can perform ordinary finite-dimensional differential-geometric constructions pointwise and thereby extend them to smooth sets.
The infinitesimal thickening ThSmthSet (cf. [GS25b]), given by enlarging the site with an infinitesimal structure, subsumes the SmthSet approach and further recovers all relevant tangent bundles via synthetic tangent bundle constructions. The latter naturally extends to describe fermionic Lagrangian field theories

(2) Feynman was inspired by the idea, presented by Dirac in [Dir33], that the Lagrangian formulation of quantum mechanics would have provided a way to interpret quantum dynamics as a superposition of classical paths.

by further enlarging the site with super-structure (cf. [Gio25; GS26]).
The aim of this thesis is to study the BV formalism (cf. [Mne19]), the categories SmthSet and ThSmthSet (cf. [GS25a; GS25b]) and to show how the super extension of the latter (cf. [Gio25; GS26]), needed to describe the odd nature of ghost fields, encodes the local classical BV formalism (cf. [Gri23]). The goal is to obtain a rigorous formulation of the Cartan calculus on the infinite-dimensional BV space of fields. As an example, we present the AKSZ construction of the BV extension of classical BF theory in $d = 4$. The AKSZ formalism is a systematic way to obtain topological field theories compatible with the BV formalism (cf. [Ale+97]). BF theory is a topological field theory defined in arbitrary dimension and can be used to construct invariants of knots and manifolds (cf. [Cat97]). In different dimensions, it is related to different field theories (cf. [Mne19][ft. 100, p. 140]). Unfortunately, the full treatment of the AKSZ formalism (see Rmk. 3.4.3) and the global approach to BV theories (see Rmk. 3.4.2) both need further generalization of sThSmthSet, leaving many interesting questions open.

We start in **Chapter** 1 with an overview of category theory, sheaf theory, graded geometry, and symplectic geometry. The basic notions of category, functor, and natural transformation are presented to state the fundamental Yoneda lemma (Prop. 1.1). This result is crucial to define smooth sets consistently in Section 3.1. Then, the general definition of limits and colimits is provided, along with some examples, and the relationships between limits and relevant functors are recalled. Consequently, petit and gros sheaves and topoi, together with the Grothendieck topology, are introduced in Section 1.2. Some results on categories of presheaves are stated. In Section 1.3, we build up to the notion of Cartan calculus on graded manifolds starting from the basic definitions of graded algebra. Graded vector bundles are introduced, and an extensive discussion regarding the Euler vector field and the total grading is given for completeness. The latter will be crucial to define the ghost grading within sThSmthSet. Finally, the basics of symplectic and graded symplectic geometry are introduced (see Sec. 1.4), together with some results and a motivating example from classical mechanics.

Chapter 2 starts with recalling local Lagrangian bosonic theories and the action principle. The absence of a fully rigorous formalism motivates the structural requirements that the category of smooth sets must satisfy to formalize the geometry of the action principle rigorously. Remark 2.1.1 discusses the simplified treatment of gauge theories used in the thesis. In Section 2.2, we briefly recall the path integral quantization and the degeneracy arising for gauge theories. Then, the finite-dimensional versions of the Faddeev–Popov method, BRST formalism, and BV formalism are discussed together with their relations. These are gauge-fixing formalisms that allow one to solve the degeneracy arising for gauge theories. While the finite-dimensional case is well understood, the infinite-dimensional case is more delicate and relies on the formal notion of Cartan calculus and of a measure on the infinite-dimensional space of fields. While the latter is not discussed in this thesis, the first is rigorously understood within sThSmthSet. Finally, we present the AKSZ formalism (see Sec. 2.3) and its application to BF theory in $d = 4$.

Chapter 3 introduces the category sThSmthSet of super thickened smooth sets and shows how it naturally encodes the local BV formalism, rigorously making sense of the Cartan calculus on the space of fields. In Section 3.1, we define the category SmthSet of smooth sets both heuristically and rigorously and demonstrate how different field-theoretical objects are naturally encoded in it. In particular, the field space, vector fields, and differential forms can be treated uniformly within SmthSet. However, while the wedge product and the contraction are well defined, we lack a notion of de Rham differential. To address this, we introduce the infinite jet bundle and the differential geometry on it. Starting from the definition of finite order jet bundles, we then define the infinite jet bundle as their projective limit within LocProMfd, a full subcategory of Fréchet manifolds (Def. 3.1.13), and consequently embed it into SmthSet. We then show that there is a canonical splitting of its tangent bundle into vertical and horizontal subbundles by means of the Cartan connection (Prop. 3.1.9). The splitting is essential to formulate the action principle via the local variational bicomplex and the related local Cartan calculus.[(3)] With some extra restrictions, the latter can be transgressed to the space of fields (see Prop. 3.1.13). In section 3.2, we present the thickened extension ThSmthSet and show how the ad hoc tangent bundle constructions in SmthSet are rigorously recovered via synthetic tangent bundle constructions. Consequently, we hint at the super extension of the latter (see Sec. 3.3) following [Gio25; GSS24]. sThSmthSet should be fully presented in [GS26]. Finally, we use this setting to make a rigorous sense of the local BV formalism (cf. [Gri23]) presenting the example of the local BV extension of the BF theory in $d = 4$ (see Sec. 3.4).

[(3)] The infinite jet bundle encodes an explicit global (i.e. chart-independent) description of local Lagrangians, fields, and forms. We talk about local objects if they factor through the infinite jet bundle.

Chapter 1

Preliminaries

This chapter summarizes the notions and results of category theory, sheaf theory, graded geometry, and symplectic geometry used in the thesis. We refer to [Lei14; Rie17] and to [MM12] for a detailed discussion of category theory and sheaf theory, respectively. A detailed discussion of graded geometry and symplectic geometry can be found, respectively, in [CS11; CM20; Keß19; Vys23; Sma24] and [Sil01].

1.1 Elements of Category Theory

In this section, the basic notions of category, functor, natural transformation, and the universal properties of representable and adjoint functors are introduced to state the Yoneda lemma. Afterward, three useful examples of limits are presented to build up to the general definition. Consequently, we state the relations of the Yoneda embedding, the hom-functor, and adjoint functors with limits. Finally, some significant results for categories of presheaves are recalled.

Categories, Functors, Adjoints, Natural Transformations and Yoneda Lemma

Definition 1.1.1. A **category** $\mathscr{A}$ consists of:

i. a class $\mathrm{ob}(\mathscr{A})$ of **objects** (we equivalently write $A \in \mathscr{A}$ or $A \in \mathrm{ob}(\mathscr{A})$);

ii. for each $A, B \in \mathrm{ob}(\mathscr{A})$, a class $\mathrm{Hom}_{\mathscr{A}}(A, B)$ of **morphisms** from A to B;

iii. for each $A \in \mathrm{ob}(\mathscr{A})$, an **identity** $1_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$;

iv. for each $A, B, C \in \mathrm{ob}(\mathscr{A})$ a **composition** map

$$\mathrm{Hom}_{\mathscr{A}}(B, C) \times \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{A}}(A, C), \ (f, g) \mapsto f \circ g$$

such that for each $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{A}}(B, C)$, $h \in \mathrm{Hom}_{\mathscr{A}}(C, D)$, the following holds:

$$(h \circ g) \circ f = h \circ (g \circ f) \ \text{ and } \ f \circ 1_A = f = 1_B \circ f.$$

A **subcategory** $\mathscr{B}$ of $\mathscr{A}$ is a sub-class of objects and morphisms closed under composition.

This definition enables a uniform treatment of mathematical entities that may be fundamentally different. For instance, consider the class **Set** of sets together with maps, the class **Grp** of groups and group homomorphisms, and the class **SmthMfd** of smooth manifolds and smooth maps. They all satisfy Definition 1.1.1 and share universal properties that can be studied categorically.
More generally, category theory studies the universal properties of objects[(1)] that are related to the structure of the category, regardless of its specific realization. For this purpose, it is useful to introduce commutative diagrams in which objects are connected by arrows representing morphisms. For example, the diagram

[(1)] The classical approaches to describe a universal property are representables, adjoint functors, and limits. For a formal definition and a detailed discussion, see [Rie17, Ch. 2.2] and [Lei14].

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ B \arrow[rr, "h"'] & & C \end{tikzcd}$$

is said to commute if $f = h \circ g$. It is straightforward to generalize the concepts of surjective and injective morphisms between objects.

Definition 1.1.2. Let $\mathscr{A}$ be a category and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$.
Then f is called **epic** if for all $C \in \mathscr{A}$ and all $g, g' \colon B \to C$, the following holds:

$$g \circ f = g' \circ f \;\Rightarrow\; g = g'.$$

In contrast, f is called **monic** if for all $C \in \mathscr{A}$ and all $h, h' \colon C \to A$, the following holds:

$$f \circ h = f \circ h' \;\Rightarrow\; h = h'.$$

Finally, f is an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathscr{A}}(B, A)$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Then, $g = f^{-1}$ is called the **inverse** of f and we write $A \cong B$.

There are two particularly relevant constructions: the dual or opposite category and the product category.

Definition 1.1.3. Given a category $\mathscr{A}$, the **opposite** category $\mathscr{A}^{\mathrm{op}}$ is such that $\mathrm{ob}(\mathscr{A}) = \mathrm{ob}(\mathscr{A}^{\mathrm{op}})$ and $\mathrm{Hom}_{\mathscr{A}}(A, B) = \mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(B, A)$ for all $A, B \in \mathrm{ob}(\mathscr{A}^{\mathrm{op}})$.

Definition 1.1.4. Given two categories $\mathscr{C}_1$ and $\mathscr{C}_2$, the **product category** $\mathscr{C}_1 \times \mathscr{C}_2$ is such that:

i. $\mathrm{ob}(\mathscr{C}_1 \times \mathscr{C}_2) = \mathrm{ob}(\mathscr{C}_1) \times \mathrm{ob}(\mathscr{C}_2)$;

ii. for all $A, B \in \mathrm{ob}(\mathscr{C}_1)$ and $A', B' \in \mathrm{ob}(\mathscr{C}_2)$,
$\mathrm{Hom}_{\mathscr{C}_1 \times \mathscr{C}_2}((A, A'), (B, B')) = \mathrm{Hom}_{\mathscr{C}_1}(A, B) \times \mathrm{Hom}_{\mathscr{C}_2}(A', B')$;

iii. for all $A \in \mathrm{ob}(\mathscr{C}_1)$ and $A' \in \mathrm{ob}(\mathscr{C}_2)$, $1_{(A,A')} = (1_A, 1_{A'})$;

iv. the composition is the componentwise composition from the contributing categories;

Remark 1.1.1. Informally, we can think of the opposite category as the original category with the arrows reversed. This is an application of the principle of duality for which every categorical definition or statement has a dual counterpart obtained by reversing the arrows.

We can further consider the category of categories[(2)] defining the morphisms between them as follows.

Definition 1.1.5. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **covariant functor** $F \colon \mathscr{A} \to \mathscr{B}$ consists of:

i. a function $F \colon \mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B})$, $A \mapsto F(A)$;

ii. for each $A, A' \in \mathscr{A}$, a function $F \colon \mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A'))$, $f \mapsto F(f)$;

such that $F(1_A) = 1_{F(A)}$ for all $A \in \mathscr{A}$ and $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$.
F is said to be **faithful** (**full**) if for all $A, A' \in \mathscr{A}$, the map

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A')),\; f \mapsto F(f)$$

is injective (surjective).
A **contravariant functor** $F \colon \mathscr{A} \to \mathscr{B}$ is a covariant functor $F \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

Functors can be composed, and the identity functor $1_{\mathscr{C}}$ can be trivially defined for each category $\mathscr{C}$. Hence, the class **CAT** of categories and functors forms indeed a category.[(3)] Some concrete examples of functors are the forgetful functor Grp $\to$ Set, that forgets the group structure on the underlying set, and the inclusion functor AbGrp $\to$ Grp, where **AbGrp** is the subcategory of abelian groups.
Many universal properties are described through adjoint functors, which also have relevant relations with limits.[(4)]

(2) This idea conceals subtleties and set-theoretical issues related to the size of the considered categories (cf. [Lei14, Ch. 3.2]). For the sake of simplicity, we ignore them throughout this thesis.
(3) CAT is a large 2-category whose 1-morphisms are functors and whose 2-morphisms are natural transformations.
(4) Adjoints do not always exist, but if they do, they are unique up to isomorphism, i.e. if G and G' are right adjoints to F, there is a natural isomorphism $G \Rightarrow G'$ that preserves the adjoint structure.

Definition 1.1.6. Let $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$ be categories and functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F \dashv G$, if

$$\mathrm{Hom}_{\mathscr{B}}(F(A), B) \cong \mathrm{Hom}_{\mathscr{A}}(A, G(B)) \tag{1.1}$$

naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Naturality here means that there is a specified bijection (see Eq. (1.1)) for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and that it satisfies the following:

$$\overline{(F(A) \xrightarrow{g} B \xrightarrow{q} B')} = (A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')) \quad \forall g \in \mathrm{Hom}_{\mathscr{B}}(F(A), B),\ q \in \mathrm{Hom}_{\mathscr{B}}(B, B'),$$
$$\overline{(A' \xrightarrow{p} A \xrightarrow{f} G(B))} = (F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B) \quad \forall p \in \mathrm{Hom}_{\mathscr{A}}(A', A),\ f \in \mathrm{Hom}_{\mathscr{A}}(A, G(B)),$$

where the bar denotes the correspondence of Equation (1.1):

$$(F(A) \xrightarrow{g} B) \mapsto (A \xrightarrow{\bar{g}} G(B))$$
$$(F(A) \xrightarrow{\bar{f}} B) \leftarrow\!\shortmid (A \xrightarrow{f} G(B)).$$

In Set, for example, the Cartesian product (see Def. 1.1.10) is left adjoint to the exponential (see Rmk. 1.1.4)

$$\begin{array}{ccc} - \times B\colon \mathrm{Set} \to \mathrm{Set} & \dashv & (-)^B\colon \mathrm{Set} \to \mathrm{Set} \\ A \mapsto A \times B & & A \mapsto A^B. \end{array}$$

The notion of a functor also enables us to establish some relationships between categories.

Definition 1.1.7.

1. A subcategory $\mathscr{B} \subseteq \mathscr{A}$ is **embedded** in $\mathscr{A}$ if the inclusion functor $i\colon \mathscr{B} \to \mathscr{A}$ is fully faithful and injective on objects, i.e. for every $B_1, B_2 \in \mathscr{B}$ with $B_1 \neq B_2$, then $i(B_1) \neq i(B_2)$.
2. Two categories $\mathscr{A}, \mathscr{B}$ are **isomorphic** if they are isomorphic as objects in CAT. That is, there exist two functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$ that are mutually inverse, i.e. such that $FG = 1_{\mathscr{A}}$ and $GF = 1_{\mathscr{B}}$. Here $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ are the identity functors respectively for $\mathscr{A}$ and $\mathscr{B}$.
3. Two categories $\mathscr{A}$, $\mathscr{B}$ are **equivalent** if there exists a fully faithful functor $F\colon \mathscr{A} \to \mathscr{B}$ such that for all $B \in \mathscr{B}$, there is $A \in \mathscr{A}$ with $F(A) \cong B$, i.e. F is **essentially surjective on objects**.

Since the conditions for a categorical isomorphism are often too restrictive, we are typically more interested in categorical equivalence instead. The latter can also be characterized via quasi-inverse functors and natural isomorphisms, i.e. isomorphisms of functors (cf. [Lei14, Def. 1.3.15 and Thm. 1.3.18]).

Definition 1.1.8. Let $\mathscr{A}$ and $\mathscr{B}$ be categories, and let $F, G\colon \mathscr{A} \to \mathscr{B}$ be functors. A **natural transformation** $\eta\colon F \Rightarrow G$ is a family of morphisms $\{\eta_A\colon F(A) \to G(A)\}_{A \in \mathscr{A}}$ in $\mathscr{B}$, called the **components** of η, such that for every $f\colon A \to A'$ in $\mathscr{A}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(A') \arrow[d, "\eta_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A'). \end{tikzcd}$$

That is, $\eta_{A'} \circ F(f) = G(f) \circ \eta_A$. We write the following diagram to indicate that η is a natural transformation from F to G:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{B}. \end{tikzcd}$$

η is a **natural isomorphism** between F and G if and only if all its components are isomorphisms in $\mathscr{B}$.

Consider the vertical composition of natural transformations, i.e. componentwise (cf. [Rie17, Lem. 1.7.1]), and take the trivial natural transformation as the identity. This allows us to define the category of functors $[\mathscr{A},\mathscr{B}]$ between two categories $\mathscr{A}$ and $\mathscr{B}$. We say that two functors between $\mathscr{A}$ and $\mathscr{B}$ are naturally isomorphic if they are isomorphic in this category.
To state the Yoneda lemma, we first need to introduce the following notions.

Definition 1.1.9. The **hom-functor** of $\mathscr{C}$ is the functor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$, defined as follows:

i. it sends $(C, C') \in \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(C, C')$;

ii. it sends $(f, g)\colon (C, C') \to (D, D')$ to $\mathrm{Hom}_{\mathscr{C}}(C, C') \to \mathrm{Hom}_{\mathscr{C}}(D, D')$, $q \mapsto g \circ q \circ f$, i.e.

$$\begin{tikzcd}
(C, C') \arrow[r, mapsto] & \mathrm{Hom}_{\mathscr{C}}(C, C') \arrow[d, "g\circ - \circ f"] \\
(D, D') \arrow[u, shift left, "f"] \arrow[r, mapsto] & \mathrm{Hom}_{\mathscr{C}}(D, D').
\end{tikzcd}$$

It is said to be **internal** if it is of the form $[-,-]\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$.
Given a hom-functor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$, for any object $C \in \mathscr{C}$, one defines:

1. the functor $H^C \coloneqq \mathrm{Hom}_{\mathscr{C}}(C, -)\colon \mathscr{C} \to \mathrm{Set}$;
2. the functor $H_C \coloneqq \mathrm{Hom}_{\mathscr{C}}(-, C)\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$.

Functors $\mathscr{C} \to \mathrm{Set}$ are called **copresheaves**, and those naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(C, -)$ for some $C \in \mathscr{C}$ are called **representable copresheaves**.
Functors of the form $\mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ are called **presheaves**, and those naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, C)$ for some $C \in \mathscr{C}$ are called **representable presheaves**.
Finally, we define the **Yoneda embedding**, $H_\bullet\colon \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, as $H_\bullet(C) \coloneqq H_C$ and $H_\bullet(f) \coloneqq H_f$ for all $C, C' \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$, where $H_f\colon H_C \Rightarrow H_{C'}$ is the natural transformation defined by post-composition via f.

It is injective on objects by construction and is fully faithful thanks to the following fundamental result (cf. [Lei14, Cor. 4.3.7]).

Theorem 1.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category. For any presheaf $F\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ and any object $C \in \mathscr{C}$, there is a bijection*

$$\begin{aligned} \Phi\colon [H_C, F] &\longrightarrow F(C) \\ \alpha &\longmapsto \alpha_C(1_C) \end{aligned}$$

where $[H_C, F]$ is the set of natural transformations between H_C and F.
Equivalently, $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}](H_C, F) \cong F(C)$ naturally in both $C \in \mathscr{C}$ and $F \in [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$.

A proof can be found in [Lei14, Thm. 4.2.1]. The dual version follows by inverting the arrows.
This result tells us how any presheaf in $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ relates to the representable presheaf H_C for any $C \in \mathscr{C}$. In particular, it shows how H_C is mapped to any other presheaf F via a natural transformation $H_C \Rightarrow F$. It turns out that such natural transformations can be seen as elements of $F(C)$. The naturality in the statement amounts to the following pair of assertions, where upper and lower $*$ denote precomposition and postcomposition.

Naturality in F: for all $\beta\colon F \Rightarrow G$, the following diagram commutes:

$$\begin{tikzcd}
{[\mathscr{C}(-, C), F]} \arrow[r, "\cong", "\Phi_F"'] \arrow[d, "\beta_*"'] & FC \arrow[d, "\beta_C"] \\
{[\mathscr{C}(-, C), G]} \arrow[r, "\cong", "\Phi_G"'] & GC.
\end{tikzcd}$$

Naturality in C: for all $f\colon C' \to C$, the following diagram commutes:

$$\begin{tikzcd}
{[\mathscr{C}(-, C), F]} \arrow[r, "\cong", "\Phi_F"'] \arrow[d, "H_f^*"'] & FC \arrow[d, "Ff"] \\
{[\mathscr{C}(-, C'), F]} \arrow[r, "\cong", "\Phi_F"'] & FC'.
\end{tikzcd}$$

Limits and Colimits

Let us now move to limits. We build up to the definition via three relevant examples.

Definition 1.1.10. Let $\mathscr{A}$ be a category and $A, B \in \mathscr{A}$. The **product** of A and B consists of a triple (P, p_1, p_2), where P is an object and p_1, p_2 are maps from P to A, B respectively, such that for any other such triple (C, f_1, f_2), there exists a unique map $\bar{f}\colon C \to P$ that makes the following diagram commute:

$$\begin{tikzcd}
 & C \arrow[d, dashed, "\bar{f}"'] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
 & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
A & & B.
\end{tikzcd}$$

Examples of products include the Cartesian product in Set or the intersection in the poset of Set ordered by inclusions. The dual concept can be thought of as a sum (cf. [Lei14, Ch. 5.2]).

Definition 1.1.11. Let $\mathscr{A}$ be a category, and $s, t\colon A \to B$ in $\mathscr{A}$. An **equalizer** of s and t is a couple (E, i) of an object E and a map $i\colon E \to A$ with $s \circ i = t \circ i$ such that for any other such couple (C, f) there exists a unique map $\bar{f}\colon C \to E$ that makes the following diagram commute:

$$\begin{tikzcd}
C \arrow[d, dashed, "\bar{f}"'] \arrow[dr, "f"] & & \\
E \arrow[r, "i"] & A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & B.
\end{tikzcd}$$

Given a homomorphism $\theta\colon G \to H$, the kernel $\operatorname{Ker}\theta = \{g \in G \,|\, \theta(g) = 0_H\}$ is an equalizer in Grp. Similarly for $E = \{x \in A | s(x) = t(x)\} \subseteq A$ in Set.

Definition 1.1.12. Let $\mathscr{A}$ be a category and $B \xrightarrow{g} C \xleftarrow{f} A$ in $\mathscr{A}$. A **pullback** of this diagram is a triple (P, p_1, p_2) of an object $P \in \mathscr{A}$ and two maps $p_1\colon P \to A$ and $p_2\colon P \to B$ with $f \circ p_1 = g \circ p_2$ such that for any other such triple (D, f_1, f_2), there is a unique map $\bar{f}\colon D \to P$ that makes the following commute:

$$\begin{tikzcd}
D \arrow[dr, dashed, "\bar{f}"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
 & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
 & A \arrow[r, "f"'] & C.
\end{tikzcd}$$

An example is the preimage in Set.

Remark 1.1.2. All of the above definitions dualize by inverting the arrows. Additionally, products, equalizers, pullbacks, and their duals may not exist in some categories; however, when they do, they are unique up to isomorphism. This observation holds in general for any limit (cf. [Rie17, Prop. 3.1.7]).

For each of the above definitions, the initial data can be represented as functors, respectively, from each of the following three categories to the category $\mathscr{A}$.

$$\bullet \qquad \bullet \qquad\qquad \bullet \rightrightarrows \bullet \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

Note that these categories are made of arrows and dots. Moreover, they are small, i.e. the class of objects is a set, and for every two objects, the class of morphisms between them is also a set.

Definition 1.1.13. Let $\mathscr{A}$ be a category and I a small category of dots and arrows. A functor $I \to \mathscr{A}$ is called a **diagram in $\mathscr{A}$ of shape I**.

The next definition encompasses all three examples above, providing the general concept of a limit.

Definition 1.1.14. Let $\mathscr{A}$ be a category, I a small category of dots and arrows, and $D\colon I \to \mathscr{A}$ a diagram in $\mathscr{A}$.

1. A **cone** on D is an object $A \in \mathscr{A}$ (the **vertex** of the cone) together with a family $(A \xrightarrow{f_i} D(i))_{i\in I}$ of maps in $\mathscr{A}$ such that for all maps $i \xrightarrow{\alpha} j$ in I, the triangle

$$\begin{tikzcd} & A \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D(i) \arrow[rr, "D(\alpha)"'] & & D(j) \end{tikzcd}$$

   commutes. The **cocone** is the dual concept. From now on $D(i) = D_i$.

2. A **limit**[5] of D is a cone $(L \xrightarrow{p_i} D(i))_{i\in I}$ such that for any other cone $(A \xrightarrow{f_i} D(i))_{i\in I}$ on D, there exists a unique map $\bar{f}\colon A \to L$ with $p_i \circ \bar{f} = f_i$ for all $i \in I$. The maps p_i are called the **projections** of the limit. The **colimit** is the dual concept.

Remark 1.1.3. We say that a category is (co)complete if it has all (co)limits. Interestingly, to assert that a category is (co)complete, it is sufficient to show that it admits (co)products and (co)equalizers. Indeed, any other limit can be constructed from these two (cf. [Lei14, Prop. 5.1.26]). Among the above examples,[6] Set and Grp are (co)complete, while SmthMfd is not since the intersection, or pullback, of two smooth manifolds does not need to be a smooth manifold.

It is relevant to understand the interactions between functors and limits.

Definition 1.1.15. Let I be a small category of dots and arrows, and $F\colon \mathscr{A} \to \mathscr{B}$ a functor between the categories $\mathscr{A}$ and $\mathscr{B}$.

1. F **preserves limits of shape** I if the following holds: for all diagrams $D\colon I \to \mathscr{A}$ and all cones $(A \xrightarrow{\lambda_i} D(i))_{i\in I}$ on D, if $(A \xrightarrow{\lambda_i} D(i))_{i\in I}$ is a limit cone on D in $\mathscr{A}$, then $(F(A) \xrightarrow{F\lambda_i} F(D(i)))_{i\in I}$ is a limit cone on $F \circ D$ in $\mathscr{B}$. F **preserves limits** if it preserves limits of shape I for all small categories I.

2. **Reflection of limits** is defined as in 1., but with the opposite implication in the condition.

3. F **creates limits** of shape I if, whenever $D\colon I \to \mathscr{A}$ is a diagram in $\mathscr{A}$, for any limit cone $(B \xrightarrow{q_i} F(D(i)))_{i\in I}$ on the diagram $F \circ D$ in $\mathscr{B}$, there exists a unique limit cone $(A \xrightarrow{p_i} D(i))_{i\in I}$ on D in $\mathscr{A}$ such that $F(A) = B$ and $F(p_i) = q_i$ for all $i \in I$. F **creates limits** if it creates limits of shape I for all small categories I.

The same terminology applies to colimits. The following result is particularly useful.

Proposition 1.1.1. *Let $\mathscr{C}$ be a category. Then:*

1. *The hom-functor* $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Set}$ (see Def. 1.1.9) *preserves limits in both its arguments recalling that a limit in $\mathscr{C}^{\mathrm{op}}$ is a colimit in $\mathscr{C}$.*

2. *If it exists, the internal hom-functor* $[-,-]\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ (see Def. 1.1.9) *preserves limits in the second variable, and contravariantly sends colimits in the first variable to limits.*

3. *The Yoneda embedding* $H_\bullet\colon \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ (see Def. 1.1.9) *preserves limits.*

The first and last statements are proven in [Lei14, Prop. 6.2.2 & Rmk. 6.2.3] and [Lei14, Cor. 6.2.12] respectively. For the second, one uses the fact that for all $C \in \mathscr{C}$, $[C,-]$ is a right adjoint (cf. [Mac98, Ch. IV, Sec. 6]), that right adjoints preserve limits (see Thm. 1.2) and conclude dually.

[5] The terminology can be understood showing that a limit is a terminal object in an opportune category, i.e. in the category of cones over D (cf. [Rie17, Def. 3.1.6]).

[6] For a proof, see [Rie17, Thm. 3.2.6] and [Lei14, Lem. 5.3.6] with the following comment.

Remark 1.1.4. The basic arithmetic operations on numbers and sets are $+$, $\times$, and exponentiation. Categorically, $+$ and $\times$ can be seen as coproduct and product, respectively. If it exists, the exponential Z^X in a category $\mathscr{C}$ can be described via the natural bijection

$$\mathrm{Hom}_{\mathscr{C}}(Y \times X, Z) \cong \mathrm{Hom}_{\mathscr{C}}(Y, Z^X). \tag{1.2}$$

This bijection completely determines Z^X up to isomorphism and defines the exponential functor $(-)^{(-)}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ of the category $\mathscr{C}$.

The following theorem states the relationship between adjoints and limits.

Theorem 1.2. *Let $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{A} \to \mathscr{B}$ be categories and functors.*

1. *If $F \dashv G$, then F preserves colimits and G preserves limits.*
2. *If $\mathscr{B}$ is complete and $\mathscr{A}, \mathscr{B}$, and G satisfy certain further conditions, then G has left adjoint if and only if G preserves limits.*

The first statement is proved in [Lei14][Thm. 6.3.1] while the second, known as the adjoint functor theorem, is proved in [Lei14][Thm. 6.3.10].
Finally, since we mainly work with categories of presheaves, let us understand limits in this context. In particular, any category of presheaves is complete and cocomplete, with both limits and colimits being computed objectwise.

Proposition 1.1.2. *Let $\mathscr{C}$ be a category and write $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ for its category of presheaves. Let I be a small category and consider any functor $F\colon I \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, that is, a I-shaped diagram in the category of presheaves. Then:*

1. *The **limit** of F exists, and it is the presheaf that, for each object $C \in \mathscr{C}$, is given by the limit in* Set *of the values of F at C:*
$$\left(\lim_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} F(i)\right)(C) \cong \lim_{i\in I}^{\mathrm{Set}} F(i)(C);$$
2. *The **colimit** of F exists, and it is the presheaf that, for each object $C \in \mathscr{C}$, is given by the colimit in* Set *of the values of F at C:*
$$\left(\mathrm{colim}_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} F(i)\right)(C) \cong \mathrm{colim}_{i\in I}^{\mathrm{Set}} F(i)(C).$$

A proof of the proposition can be found in [MM12, pp. 22-23].
Fortunately, representable presheaves characterize the entire category of presheaves.

Proposition 1.1.3. *Let $\mathscr{C}$ be a category and write $[\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ for its category of presheaves. Every presheaf $G \in [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$ is a colimit of representable presheaves, i.e. $G \cong \mathrm{colim}_{i\in I}^{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]} H_{C_i}$.*

A proof of the proposition can be found in [MM12, Ch. 1, Sec. 5, Prop. 1].

1.2 Elements of Sheaf Theory

In this section, the concepts of petit and gros sheaves are introduced along with the Grothendieck topology. Afterward, some relevant properties are recalled.

Consider a fixed topological space X and the category $\mathbf{X_{cl}}$[(7)] of open subsets of X together with inclusions between them. We want to encode the data of X_{cl} in the simpler category of sets Set via a presheaf (see Def. 1.1.9) $F\colon X_{\mathrm{cl}}^{\mathrm{op}} \to \mathrm{Set}$. However, this alone does not guarantee consistency at the intersections of open subsets. The additional locality and gluing conditions characterize sheaves and are necessary to uniquely obtain global information from local data.

(7) The suffix cl stands for classical topology.

Definition 1.2.1. Let X be a topological space. A presheaf of sets F on X consists of a set $F(U)$ for each open set $U \subseteq X$ and a restriction map $\mathrm{res}^U_V \colon F(U) \to F(V)$ for each inclusion of open sets $V \subseteq U$, satisfying the functoriality axioms:

i. for every open set $U \subseteq X$, the restriction map res^U_U is the identity on $F(U)$;

ii. for open sets $W \subseteq V \subseteq U$, we have $\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W$.

A **sheaf on** X is a presheaf $F \colon X_{\mathrm{cl}}^{\mathrm{op}} \to \mathrm{Set}$ that satisfies two additional axioms:[8]

1. let $U \subseteq X$ be open and $\{U_i\}_{i\in I}$ an open cover of U. If $s, t \in F(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$; **(Locality condition)**
2. let $\{U_i\}_{i\in I}$ be an open cover of $U \subseteq X$ and $\{s_i \in F(U_i)\}_{i\in I}$ be a family such that for all $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Then there exists $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. **(Gluing condition)**

For instance, the presheaf $C^0(-)$ assigning to each open $U \subseteq X$ the set $C^0(U)$ of real-valued continuous functions, with restriction maps being the usual restriction of functions, is also a sheaf on X. On the contrary, the constant presheaf $\mathbb{R}^{\mathrm{psh}}$, which assigns to every open set $U \subseteq X$ the set of constant functions from U to $\mathbb{R}$, does not satisfy the two additional sheaf axioms. For a smooth (C^∞) n-manifold M, we need to consider smooth functions, instead of continuous ones, to capture the smooth structure. The sheaf of C^∞-functions on M with the usual restriction is denoted by $\mathcal{O}_M$.
We now need the notion of sheaf morphisms to construct the category of sheaves on X.

Definition 1.2.2. Let F and G be two sheaves on a topological space X. A **morphism** $\varphi \colon F \to G$, i.e. a natural transformation, consists of a morphism $\varphi_U \colon F(U) \to G(U)$ of sets for each open set $U \subseteq X$, such that it is compatible with restrictions. Here compatibility means that for every open subset $V \subseteq U$, the following diagram commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}^U_V"'] & G(U) \arrow[d, "\mathrm{res}'^U_V"] \\ F(V) \arrow[r, "\varphi_V"'] & G(V). \end{tikzcd}$$

For example, the derivative induces a morphism $\mathcal{O}^n_{\mathbb{R}} \to \mathcal{O}^{n-1}_{\mathbb{R}}$ of sheaves on $\mathbb{R}$.
Consider the whole category **Top** of topological spaces and continuous functions between them. We can think of a sheaf on it as a presheaf $F \colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ such that for every $X \in \mathrm{Top}$, the restriction F_X to the subcategory X_{cl} is a sheaf in the sense of Definition 1.2.1. To formalize this, we need to introduce the Grothendieck topology on the category $\mathrm{Top}^{\mathrm{op}}$. The idea is to equip a category with covering families.[9]

Definition 1.2.3. Let $\mathscr{C}$ be a category, $C \in \mathrm{Ob}(\mathscr{C})$. A **sieve** S on C is a subfunctor $S \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ of $\mathrm{Hom}_{\mathscr{C}}(-, C)$. That is, for all $C' \in \mathrm{Ob}(\mathscr{C})$, $S(C') \subseteq \mathrm{Hom}_{\mathscr{C}}(C', C)$ and for all arrows $f \colon C'' \to C'$, with $C', C'' \in \mathrm{Ob}(\mathscr{C})$, $S(f)$ is the restriction to $S(C')$ of $\mathrm{Hom}_{\mathscr{C}}(-, C)(f)$, where the latter is given by the precomposition with f.

Alternatively, a sieve S on C may be described as a family of morphisms in $\mathscr{C}$, all having codomain C, such that $f \circ g \in S$ whenever $f \in S$ and the composition $f \circ g$ is defined. Moreover, if S is a sieve on C and $h \colon D \to C$ is any arrow in $\mathscr{C}$, then the pullback sieve $h^*(S)$ on D is defined by

$$h^*S = \{\, g \mid \mathrm{cod}(g) = D,\ h \circ g \in S \,\}.$$

Definition 1.2.4. A **Grothendieck topology** on a category $\mathscr{C}$ is a function J that assigns to each object $C \in \mathrm{Ob}(\mathscr{C})$ a collection $J(C)$ of sieves on C, in such a way that:

(8) These axioms can be encoded in an equalizer: in the literature this universal definition of sheaf is usually used (cf. [Vis07, Ch. 2.3.3], [Rie17, App. E.4] and [MM12, Ch. 2]).

(9) We talk about open covers when working on a fixed topological space with traditional topology on it. We now introduce the terminology of covering families (or coverings) to discuss the generalization introduced in Definition 1.2.5. This allows us to define a "topology" on a general category. An open cover on X defines a covering family on X_{cl} that we call the usual Grothendieck pretopology.

i the maximal sieve $t_C = \{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$;

ii if $S \in J(C)$, then the pullback sieve $h^*(S) \in J(D)$ for any arrow $h\colon D \to C$; **(Stability under pullback)**

iii if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h\colon D \to C$ in S, then $R \in J(C)$. **(Transitivity)**

A category $\mathscr{C}$ equipped with a Grothendieck topology J is called a **site**. If a sieve $S \in J(C)$, we say that S covers C. S on C is said to cover $f\colon D \to C$ if $f^*(S)$ covers D.

We now introduce the Grothendieck pretopology. As this notion is more intuitive, and every pretopology induces[10] a topology, we shall work with pretopologies to avoid unnecessary additional complexity (cf. [MM12, Ch. 3, Sec. 2]).

Definition 1.2.5. Let $\mathscr{C}$ be a category. A **Grothendieck pretopology** on $\mathscr{C}$ is the assignment to each object $U \in \mathscr{C}$ of a collection of families of morphisms $\{U_i \to U\}$, called **coverings** of U, satisfying the following conditions:

i. if $V \to U$ is an isomorphism in $\mathscr{C}$, then $\{V \to U\}$ is a covering; **(Isomorphism condition)**

ii. if $\{U_i \to U\}$ is a covering and $V \to U$ is any morphism in $\mathscr{C}$, then the fiber products $U_i \times_U V$, i.e. the pullbacks of $U_i \to U \leftarrow V$, exist and the family of projections $\{U_i \times_U V \to V\}$ is a covering of V; **(Stability under pullback)**

iii. if $\{U_i \to U\}$ is a covering, and for each index i there is a covering $\{V_{ij} \to U_i\}$, then the collection of composites $\{V_{ij} \to U_i \to U\}$ is a covering of U. **(Transitivity)**

For any topological space X, the category X_{cl} with open cover, i.e. covering family of the usual Grothendieck pretopology, is a site. Moreover, Top becomes a site if we take jointly surjective collections of local homeomorphisms $U_i \to U$ as coverings. For SmthMfd, a good choice of coverings is obtained by considering jointly surjective collections of local diffeomorphisms; however, later we use a more convenient choice to define the category of SmthSet (see Def. 3.1.1).

Definition 1.2.6. Let $\mathscr{C}$ be a site, $F\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$[11] be a presheaf and $\{U_i \xrightarrow{u_i} U\}_{i\in I}$ be a covering in $\mathscr{C}$. Consider a family $\{a_i \in F(U_i)\}_{i\in I}$ and the canonical projections of the fibered product $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$. Finally, assume $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in F(U_i \times_U U_j)$, for all $i, j \in I$. Then F is a **sheaf over** $\mathscr{C}$ if there exists a unique[12] element $a \in F(U)$ such that $u_i^* a = a_i \in FU_i$ for all $i \in I$.
Let F, G be sheaves on a site $\mathscr{C}$. A **morphism of sheaves** $\varphi\colon F \to G$ is a natural transformation of functors. $\mathrm{Sh}(\mathscr{C})$ is the **category of sheaves** over the site $\mathscr{C}$ with such morphisms.

This definition is a generalization of Definition 1.2.1 since the two coincide when specializing the second to the site X_{cl}.[13] We introduce the following terminology to highlight the qualitative differences between these two cases of the same mathematical definition of a sheaf. A sheaf on a topological space X, i.e. over the site X_{cl}, is called petit, while a sheaf over a more general site is called gros. The category of petit sheaves over X_{cl} is a petit topos, while the category of sheaves over a more general site is a gros topos.
The following summarizes the most important properties of a category of sheaves over a site. As for presheaves, (co)limits in the category of sheaves are computed objectwise.

Proposition 1.2.1. *Let $\mathscr{C}$ be a site with fixed Grothendieck topology and $\mathrm{Sh}(\mathscr{C})$ the category of sheaves over it. Then:*

1. *$\mathrm{Sh}(\mathscr{C})$ is (co)complete and closed under limits. Moreover, if the site is subcanonical,[14] every sheaf is a colimit of representable sheaves;*
2. *$\mathrm{Sh}(\mathscr{C})$ has an internal hom-functor and it has an exponential.*

Proofs of the statements can be found in [MM12, Cor. 3, p. 42 & Ch. III, Sec. 4-6].

[10] Different pretopologies can give the same topology, but sheaf theory only depends on the topology. Therefore, we can work with pretopologies up to a certain equivalence (cf. [Vis07, Rmk. 2.25]).
[11] We could alternatively store data defining a (pre)sheaf using a generic category instead of Set (cf. [Rot09, Ch. 5]).
[12] Here, existence and uniqueness correspond respectively to the gluing and locality conditions (Def. 1.2.1).
[13] The connection between Definition 1.2.1 and Definition 1.2.6 is more evident when stating the sheaf condition in terms of an equalizer (cf. [Vis07, Ch. 2.3.3.]).
[14] A site is **subcanonical** if every representable presheaf on it is also a sheaf.

1.3 Elements of Graded Geometry

In this section, some notions and results of graded geometry[15] are recalled. Throughout the thesis, if both gradings are present, they are compatible mod 2. Notice that the classical DeWitt concrete approach to $\mathbb{Z}_2$-graded geometry, or super-geometry, is not the same as the algebraic approach described in this section. Nevertheless, the two approaches are equivalent (cf. [Rog07][Ch. 8.4]). We refer to [Cat18; Lee03; GSM09; Nak18] for a full discussion of the equivalent non-graded notions of what follows.

Graded Algebra

We can obtain the graded version of any category built on sets by splitting any would-be graded structure into minimally intersecting homogeneous-degree structures and prescribing a graded commutation rule.

Definition 1.3.1. A unital ring $(R, +, \cdot)$ is **$\mathbb{Z}$-graded commutative** whenever

$$R = \bigoplus_{i\in\mathbb{Z}} R_1, \quad R_i \cdot R_j \subseteq R_{i+j} \ \forall i, j, \quad R_i \cap R_j = 0 \ \forall i \neq j, \quad R_i \neq 0 \text{ only for finitely many } i \in \mathbb{Z},$$

and for all $r \in R_{\deg_{\mathbb{Z}}(r)}, s \in R_{\deg_{\mathbb{Z}}(s)}$ the following holds:

$$rs = (-1)^{\deg_{\mathbb{Z}}(r)\deg_{\mathbb{Z}}(s)} sr. \tag{1.3}$$

Conventionally, $\deg_{\mathbb{Z}}(0) = \deg_{\mathbb{Z}}(1) = 0$. Elements that (anti)commute are said to be of (**odd**) **even parity**.
The $\mathbb{Z}_2$ case is given by substituting $\mathbb{Z}$ with $\mathbb{Z}_2$ and understanding the sum of subscripts modulo 2.
A **$\mathbb{Z}_2$-graded R-algebra** A is a graded commutative ring homomorphism $\eta \colon R \to A$.[16]

When dealing with different gradings, it is convenient to consider the total degree.

Definition 1.3.2. Given n different gradings $\deg_1, \dots, \deg_n$, we define the total degree $|\cdot|$ as the sum of all the individual gradings

$$|\cdot| := \sum_{i=1}^{n} \deg_i(\cdot).$$

Importantly, the total degree always induces a $\mathbb{Z}_2$ grading if understood modulo 2. In what follows, unless otherwise specified, we define graded structures with respect to a total underlying grading that could be $\mathbb{Z}$, $\mathbb{Z}_2$[17] or a sum of different gradings. In the thesis, we specify the grading in which we are working if it is not evident from the context.
The following is the guiding idea to extend conventional structures to any graded setting. Whenever two adjacent graded elements of any kind exchange position, the sign changes according to their total degree, apart from any extra sign that the non-graded analogous structure includes. Furthermore, for each individual grading that two graded elements have in common, each respective degree of their product should be equal to the sum of their individual degrees. This also works for gradings that are not shared by trivially extending the lacking grading structure. Indeed, any structure always admits a trivial grading, so that any element in it has 0 degree in that specific grading.

Definition 1.3.3. A **graded module** V over a graded commutative ring R is a left and right module such that for all $r \in R$ and $v \in V$

$$rv = (-1)^{|r||v|} vr, \quad |rv| = \begin{cases} |r| + |v| & v \neq 0 \text{ and } r \neq 0 \\ 0 & v = 0 \text{ or } r = 0. \end{cases}$$

[15] More generally, one may define a G-graded geometry, where G is a set. Then, all the definitions of this section make sense if G is assumed to have some nice properties. In particular, G should be an abelian group (cf. [Cat18, Rmk. 9.28]).
[16] Since we are only considering unital $\mathbb{Z}_{(2)}$-graded commutative rings R, this definition gives unital associative commutative $\mathbb{Z}_{(2)}$-graded algebras. They can alternatively be understood as a $\mathbb{Z}_{(2)}$-graded module over R with additional associative product compatible (i.e. such that Equation (1.3) holds and the grade of the product is the sum of the grades of the two factors) with the graded structure and that distributes over the module operations. The associative product is induced by the homomorphism as follows $R \times A \to A$, $(r, a) \mapsto \eta(r)a$. The category of graded algebras over $\mathbb{R}$, with structure preserving morphisms, is denoted by $\mathbb{Z}_{(2)}\mathbf{CAlg}_{\mathbb{R}}$.
[17] When considering $\mathbb{Z}_2$-grading, every grading should be understood modulo 2.

All the following discussion also works for $\mathbb{Z}_{(2)}$**-graded vector spaces** defined as a collection of vector spaces $\{V_k\}_{k\in\mathbb{Z}_{(2)}}$ or, equivalently, as $\mathbb{Z}_{(2)}$-graded modules over a $\mathbb{Z}_{(2)}$-graded field, i.e. a $\mathbb{Z}_{(2)}$-graded commutative ring whose non-zero homogeneous elements are invertible.

Definition 1.3.4. Given a $\mathbb{Z}_{(2)}$-graded R-module V and any $k \in \mathbb{Z}$, the **k-shift** of V is a $\mathbb{Z}_{(2)}$-graded R-module $V[k]$ whose elements are the same as those of V, but with grading increased by k,[18] that is:

$$v \in V,\ \deg_{\mathbb{Z}_{(2)}}(x) = n \Longleftrightarrow v \in V[k],\ \deg_{\mathbb{Z}_{(2)}}(x) = n + k \pmod 2.$$

Definition 1.3.5. Given a graded R-module V, a linearly independent generating set $\{e_{(i)}\}$ forms a **basis**. We say that two subsets $\{l^i_v\}_i$ and $\{r^i_v\}_i$ of R are respectively the **left** and **right coordinates** of v whenever we have that

$$v = \sum_i l^i_v e_{(i)} = \sum_i e_{(i)} r^i_v.$$

A graded module with a basis is **free**, and the commutativity of R implies that all bases of V have the same cardinality, which we call its **dimension**.[19].

From now on, we assume that modules are free and work exclusively with left coordinates.
Given any two graded objects, a graded map between them has homogeneous degree $|f|$ if $|f(\cdot)| = |f|+|\cdot|$. With this in mind, we define the following.

Definition 1.3.6. Let $V_1, \ldots, V_n, W$ be graded R-modules. A map $f\colon V_1 \times \ldots \times V_n \to W$ is **graded n-linear** if it is additive in each argument and $\forall r \in R,\ v_k \in V_k$ the following holds:

$$f(\ldots, rv_k, \ldots) = (-1)^{|r|(|v_1|+\cdots+|v_{k-1}|+|f|)} r f(\ldots, v_k, \ldots).$$

If $n = 1$, f is called **graded linear**.

This allows us to define the notion of dual of a graded R-module.

Definition 1.3.7. Let V be a graded R-module. Its **dual** V^* is the graded module of graded linear maps $V \to R$. Given a basis $\{e_i\}$ of V, the associated **dual basis** $\{e^i\}$, with $|e^i| = -|e_i|$, is defined by

$$e^i(e_j) = \delta^i_j \quad \forall i, j \in \{1, \ldots, \dim V\}.$$

Once the grading and R are fixed, we consider the category of graded R-modules taking linear grading-preserving maps (i.e. of zero degree) as morphisms. In this category, we define the direct sum and the tensor product.[20]

Definition 1.3.8. Given a grading ($\mathbb{Z}$ or $\mathbb{Z}_2$) and a graded ring R, consider two R-modules V, W. Then,

1. $(V \oplus W)_i \coloneqq (V)_i \oplus (W)_i$ defines the **direct sum**.
2. $(V \otimes W)_i \coloneqq \bigoplus_{i=p+q \pmod 2} (V)_p \otimes_R (W)_q$ defines the **tensor product**.
3. $V_p \otimes_R W_q \to W_q \otimes_R V_p,\ v \otimes w \mapsto (-1)^{pq} w \otimes v$ defines the **symmetric braiding** $V \otimes W \to W \otimes V$.
4. When considering a $\mathbb{Z}_2$-grading, there exists an endofunctor Π, called **change of parity**, defined by $\Pi(V)_0 \coloneqq V_1$ and $\Pi(V)_1 \coloneqq V_0$ for any $V = V_0 \oplus V_1$. On linear maps $m\colon V \to W$, it acts by defining a new map $\Pi m\colon \Pi V \to \Pi W$ such that $\Pi m(\Pi V) \coloneqq \Pi(m(V))$.

Remark 1.3.1. For any vector space V, the parity-shifted superspace ΠV corresponds to the 1-shifted superspace $V[1]$.

The braiding supplies the notions of symmetric and exterior graded algebras.

(18) At the level of decomposition, this means $(V[k])_n \coloneqq (V)_{n+k \pmod 2}$, for any $n \in \mathbb{Z}_{(2)}$.
(19) Considering $\mathbb{Z}_2$ grading, we say that $V = V_0 \oplus V_1$ has dimension $(\dim V_0, \dim V_1)$ with $\dim V_0 + \dim V_1 = \dim V$.
(20) They are compatible, meaning the category is an additive symmetric monoidal category (cf. [Sac08]).

Definition 1.3.9. Let V be a graded R-module.

1. The **tensor algebra** is given by $T(V) \coloneqq \bigoplus_{k=0}^{\infty} V^{\otimes k}$, where $V^{\otimes k} \coloneqq \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}}$ and $V^{\otimes 0} \coloneqq R$. Besides any internal grading that V might have, $T(V)$ is endowed with an additional $\mathbb{N}_0$-degree, called **tensor degree**, such that
$$\deg_T t = k \Longleftrightarrow k \in V^{\otimes k}.$$
The multiplication is the tensor product: $t \otimes t' \in V^{\otimes(n+m)}$ for any $t \in V^{\otimes n}$, $t' \in V^{\otimes m}$.

2. Consider the two-sided ideal[21] generated by (anti-)symmetric elements of $T(V)$
$$I_{\pm} \coloneqq \langle \{u \otimes v \pm (-1)^{|u||v|} v \otimes u \mid u, v \in V \text{ are homogeneous}\} \rangle.$$
Then, the **symmetric algebra** $\mathrm{Sym}(V)$ and **exterior algebra** $\bigwedge(V)$ of V are given by
$$\mathrm{Sym}(V) \coloneqq {T(V)}\big/{I_-} \quad \text{and} \quad \bigwedge V \coloneqq {T(V)}\big/{I_+}.$$
Analogously, we define the two-sided $I_{\pm}^n$ within $V^{\otimes n}$, and in particular the **n-th symmetric** and **exterior powers** of V by
$$\mathrm{Sym}^n(V) \coloneqq {V^{\otimes n}}\big/{I_-^n} \quad \text{and} \quad \bigwedge^n V \coloneqq {V^{\otimes n}}\big/{I_+^n}.$$
The inherited products $\odot$ and $\wedge$ fulfill
$$t \odot t' = (-1)^{|t||t'|} t' \odot t \quad \text{and} \quad t \wedge t' = (-1)^{||t|| ||t'||} t' \wedge t. \tag{1.4}$$
where $||\cdot|| = |\cdot| + \deg_T(\cdot)$ is introduced to distinguish the total grading that considers the tensor degree from the intrinsic total grading $|\cdot|$ on V. Note that $\deg_T(t \odot t') = \deg_T(t \wedge t') = \deg_T(t) + \deg_T(t')$ and $|t \odot t'| = |t \wedge t'| = |t| + |t'|$.[22]

Remark 1.3.2. Let V be a $\mathbb{Z}_2$-graded module, then $\mathrm{Sym}^n(V_0 \oplus 0) = \mathrm{Sym}^n(V_0)$, $\mathrm{Sym}^n(0 \oplus V_1) = \bigwedge^n(V_1)$ and therefore
$$\mathrm{Sym}^n(V_0 \oplus V_1) = \bigoplus_{0 \leqslant k \leqslant n} \left(\mathrm{Sym}^k V_0 \otimes \bigwedge^{n-k} V_1 \right).$$

Finally, we introduce the notion of graded Lie algebra.

Definition 1.3.10. A **graded Lie algebra** A is a graded R-module A with a bilinear product $[-,-] \colon A \times A \to A$ that is **graded antisymmetric**, that is
$$[a, b] = -(-1)^{|a||b|} [b, a],$$
and satisfies the **graded Jacobi identity**, that is
$$[a, [b, c]] + (-1)^{|a|(|b|+|c|)} [b, [c, a]] + (-1)^{|c|(|b|+|a|)} [c, [a, b]] = 0$$
for all $a, b, c \in A$. Moreover, $|[a, b]| = |a| + |b|$.

(21) This defines a two-sided ideal since it is an additive subgroup of the ring $(T(V), \oplus, \otimes)$ that is closed under left and right multiplication by elements of $T(V)$.

(22) The definition of commutativity of the exterior product (see Eq. (1.4)) is conventional. If one wants to stick to the guiding idea above (cf. [Keß19][Principle 2.0.1 & p. 30]), the appropriate rule would follow the Deligne and Freed convention instead, that is $t \wedge t' = (-1)^{|t||t'| + \deg_T(t)\deg_T(t')} t' \wedge t$. The two conventions are actually equivalent (cf. [Del+99][App. to Ch. 1, pp. 62-64]) and choosing one of the two is a matter of taste and convenience. Indeed, choosing one convention or the other amounts to a different definition of braiding, resulting in a different additive symmetric monoidal category (this is a category of graded objects on graded modules, and therefore this does not change the underlying structure of graded modules defined in Definition 1.3.8). Crucially, there is an equivalence of categories given by a re-grading functor, and therefore the two points of view are equivalent. For our purposes, the total grading convention is more convenient. Indeed, the calculations we perform on the graded variational bicomplex (see Sec. 3.4) are drastically simplified by using the total grading, since we have to deal with three gradings simultaneously.

Graded Derivations and Differential Forms

Following the guiding idea above, we get the graded differential calculus as follows.

Definition 1.3.11. Let A be a graded R-algebra and V an R-module. A **graded derivation** from A with values in V is a graded linear map $D\colon A \to V$ satisfying the **graded Leibniz rule**:

$$D(ab) = D(a)b + (-1)^{|D||a|} aD(b), \quad \forall a, b \in A. \tag{1.5}$$

By the definition of a graded map, we have $|Da| = |D| + |a|$.

The collection of such graded derivations forms a graded A-module $\mathrm{Der}_R(A, E)$, which we denote by $\mathrm{Der}_R(A)$ if $A = E$. In this case, $\mathrm{Der}_R(A)$ forms a graded Lie algebra (cf. [Keß19][Ex. 2.11.4]) with the graded bracket

$$[X, Y](a) := X \circ Y(a) - (-1)^{|X||Y|} Y \circ X(a), \quad \forall X, Y \in \mathrm{Der}_R(A),\ a \in A. \tag{1.6}$$

Definition 1.3.12. A **graded differential n-form** on a graded R-algebra A is an element of

$$\Omega^n(A) := (\mathrm{Der}(A)^*)^{\wedge n}$$

for any $n \in \mathbb{N}$. The exterior A-algebra of these spaces is $\Omega^\bullet(A) := \bigoplus_{n=0}^\infty \Omega^n(A)$ and to it we associate the **cohomological degree**[(23)] $\deg_\Omega(\omega) = n \Leftrightarrow \omega \in \Omega^n(A)$.

We define three important derivations in $\mathrm{Der}_R(\Omega^\bullet(A))$.

Definition 1.3.13. Consider a graded R-algebra A with total intrinsic grading $|\cdot| := \sum_{i=1}^n \deg_i(\cdot)$. Define the total grading that considers the cohomological degree $||\cdot|| = |\cdot| + \deg_\Omega(\cdot)$.

1. The **de Rham differential** is the unique $\mathrm{d}_\Omega \in \mathrm{Der}_R(\Omega^\bullet(A))$ such that

$$\deg_\Omega(\mathrm{d}_\Omega) = 1, \quad \deg_i(\mathrm{d}_\Omega \alpha) = \deg_i(\alpha) \ \ \forall i,$$
$$\mathrm{d}_\Omega \Omega^n(A) \subseteq \Omega^{n+1}(A), \quad \mathrm{d}_\Omega^2 = 0, \quad \mathrm{d}_\Omega f(X) = X(f),$$

   for all homogeneous $\alpha \in \Omega^\bullet(A)$, $X \in \mathrm{Der}(A)$, $f \in \Omega^0(A) = A$. Note that, $||\mathrm{d}_\Omega|| = 1$. Furthermore, for all homogeneous $\alpha, \beta \in \Omega^\bullet(A)$ it holds that

$$\mathrm{d}_\Omega(\alpha \wedge \beta) = \mathrm{d}_\Omega(\alpha) \wedge \beta + (-1)^{||\alpha||} \alpha \wedge \mathrm{d}_\Omega(\beta).$$

2. The **contraction w.r.t.** $X \in \mathrm{Der(A)}$ is the unique $\iota_X \in \mathrm{Der}_R(\Omega^\bullet(A))$ such that

$$\deg_\Omega(\iota_X) = -1, \quad \deg_i(\iota_X) = \deg_i(X) \ \ \forall i,$$
$$\iota_X \Omega^n(A) \subseteq \Omega^{n-1}(A), \quad \iota_X f = 0, \quad \iota_X \mathrm{d}_\Omega f = X(f),$$

   for all homogeneous $f \in A$, $X \in \mathrm{Der}(A)$. Note that, $||\iota_X|| = ||X|| - 1$. Furthermore, for all homogeneous $\alpha, \beta \in \Omega^\bullet(A)$ we have

$$\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{||\alpha|| ||\iota_X||} \alpha \wedge \iota_X(\beta).$$

   The associated map

$$\iota\colon \mathrm{Der}(A) \times \Omega^\bullet(A) \to \Omega^\bullet(A),\ (X, \omega) \mapsto \iota_X \omega = \omega(X, -, \cdots, -)$$

   is called the **contraction**. It is graded 2-linear of degree $\deg_i(\iota) = 0 \ \ \forall i$ and $\deg_\Omega(\iota) = -1$.

3. The **Lie derivative w.r.t.** $X \in \mathrm{Der(A)}$ is the unique $\mathbb{L}_X \in \mathrm{Der}_R(\Omega^\bullet(A))$ defined by

$$\mathbb{L}_X := [\iota_X, \mathrm{d}_\Omega] = \iota_X \mathrm{d}_\Omega - (-1)^{(|X|-1)1} \mathrm{d}_\Omega \iota_X = \iota_X \mathrm{d}_\Omega + (-1)^{|X|} \mathrm{d}_\Omega \iota_X,$$

   where the bracket is defined as in Equation (1.6) with total grading $||\cdot||$. Furthermore, we have that $\deg_i(\mathbb{L}_X) = \deg_i(X) \ \ \forall i$, $\deg_\Omega(\mathbb{L}_X) = 0$, $||\mathbb{L}_X|| = ||X||$ and $\mathbb{L}_X \Omega^n(A) \subseteq \Omega^n(A)$. Importantly, the Lie derivative also acts on derivations via the Lie bracket $\mathbb{L}_X(Y) = [X, Y]$, for all $X, Y \in \mathrm{Der}(A)$.[(24)]

[(23)] This is exactly the tensorial degree of Definition 1.3.9. Furthermore, once one sets all the underlying gradings of A to zero, we recover the classical definition of differential forms.

[(24)] This can be checked by evaluating $\mathbb{L}_X(Y(f))$ for all $X, Y \in \mathrm{Der}(A)$ and $f \in A$ and using the graded Leibniz rule.

Crucially, these definitions and the following result match the classical case once we consider all the gradings to be trivial except for the cohomological one.

Theorem 1.1. *The following relations hold for all $X, Y \in \mathrm{Der}(A)$:*

$$[\mathrm{d}_\Omega, \mathrm{d}_\Omega] = 0, \quad [\iota_X, \mathrm{d}_\Omega] = \mathbb{L}_X, \quad [\mathbb{L}_X, \mathrm{d}_\Omega] = 0,$$
$$[\iota_X, \iota_Y] = 0, \quad [\mathbb{L}_X, \iota_Y] = \iota_{[X,Y]}, \quad [\mathbb{L}_X, \mathbb{L}_Y] = \mathbb{L}_{[X,Y]}.$$

Proof. $[\mathrm{d}_\Omega, \mathrm{d}_\Omega] = 0$ and $[\iota_X, \mathrm{d}_\Omega] = \mathbb{L}_X$ are true by Definition 1.3.13.[25]
Using the total degree $||\cdot|| = |\cdot| + \deg_\Omega(\cdot)$, $||\mathrm{d}_\Omega|| = 1$, $||\iota_X|| = ||X|| - 1$, and $||\mathbb{L}_X|| = ||X||$. Then, by graded Jacobi (see Def. 1.3.10) and assuming that the fifth relation holds, we get the following

$$\begin{aligned}
[\mathbb{L}_X, \mathrm{d}_\Omega] &= -(-1)^{||\mathbb{L}_X||||\mathrm{d}_\Omega||}[\mathrm{d}_\Omega, \mathbb{L}_X] = -(-1)^{||X||||\mathrm{d}_\Omega||}[\mathrm{d}_\Omega, [\iota_X, \mathrm{d}_\Omega]] = \\
&= -(-1)^{||X||||\mathrm{d}_\Omega||}\Big(-(-1)^{||\mathrm{d}_\Omega||(||\iota_X||+||\mathrm{d}_\Omega||)}[\iota_X, [\mathrm{d}_\Omega, \mathrm{d}_\Omega]] - (-1)^{||\mathrm{d}_\Omega||(||\iota_X||+||\mathrm{d}_\Omega||)}[\mathrm{d}_\Omega, [\mathrm{d}_\Omega, \iota_X]]\Big) = \\
&= -(-1)^{||\mathrm{d}_\Omega||||\iota_X||}[\mathrm{d}_\Omega, [\iota_X, \mathrm{d}_\Omega]] = -(-1)^{||\mathrm{d}_\Omega||||\iota_X||}\Big(-(-1)^{||\mathbb{L}_X||||\mathrm{d}_\Omega||}[\mathbb{L}_X, \mathrm{d}_\Omega]\Big) = -[\mathbb{L}_X, \mathrm{d}_\Omega] = 0, \\
\mathbb{L}_{[X,Y]} &= [\iota_{[X,Y]}, \mathrm{d}_\Omega] = -(-1)^{||\iota_{[X,Y]}||||\mathrm{d}_\Omega||}[\mathrm{d}_\Omega, [\mathbb{L}_X, \iota_Y]] = \\
&= -(-1)^{(||X||+||Y||-1)||\mathrm{d}_\Omega||}\Big(-(-1)^{||\mathrm{d}_\Omega||(||\mathbb{L}_X||+||\iota_Y||)}[\mathbb{L}_X, [\iota_Y, \mathrm{d}_\Omega]] + \\
&\quad -(-1)^{||\iota_Y||(||\mathbb{L}_X||+||\mathrm{d}_\Omega||)}[\iota_Y, [\mathrm{d}_\Omega, \mathbb{L}_X]]\Big) = [\mathbb{L}_X, \mathbb{L}_Y].
\end{aligned}$$

We show the third and the fifth on generators, i.e. $f \in A$ and exact 1-forms $\mathrm{d}_\Omega f \in \Omega^1(A)$. Using that $\iota_X(f) = 0$, $X(f) \in A$, and $\iota_X(\mathrm{d}_\Omega f) = X(f)$ for all $f \in A$, $X \in \mathrm{Der}(A)$, we get the following

$$\begin{aligned}
[\iota_X, \iota_Y](f) &= \iota_X\iota_Y(f) - (-1)^{||\iota_X||||\iota_Y||}\iota_Y\iota_X(f) = 0, \\
[\iota_X, \iota_Y](\mathrm{d}_\Omega f) &= \iota_X\iota_Y(\mathrm{d}_\Omega f) - (-1)^{||\iota_X||||\iota_Y||}\iota_Y\iota_X(\mathrm{d}_\Omega f) = \iota_X Y(f) - (-1)^{||\iota_X||||\iota_Y||}\iota_Y X(f) = 0, \\
[\mathbb{L}_X, \iota_Y](f) &= \mathbb{L}_X\iota_Y(f) - (-1)^{||\mathbb{L}_X||||\iota_Y||}\iota_Y\mathbb{L}_X(f) = -(-1)^{||\mathbb{L}_X||||\iota_Y||}\iota_Y(\iota_X\mathrm{d}_\Omega(f) + (-1)^{||X||}\mathrm{d}_\Omega\iota_X(f)) = \\
&= -(-1)^{||\mathbb{L}_X||||\iota_Y||}(-1)^{||X||}\iota_Y\,(X(f)) = 0 = \iota_{[X,Y]}(f). \\
[\mathbb{L}_X, \iota_Y](\mathrm{d}_\Omega f) &= \mathbb{L}_X\iota_Y(\mathrm{d}_\Omega f) - (-1)^{||\mathbb{L}_X||||\iota_Y||}\iota_Y\mathbb{L}_X(\mathrm{d}_\Omega f) = (\iota_X\mathrm{d}_\Omega + (-1)^{||X||}\mathrm{d}_\Omega\iota_x)Y(f) + \\
&\quad -(-1)^{||\mathbb{L}_X||||\iota_Y||}\iota_Y(\iota_X\mathrm{d}_\Omega + (-1)^{||X||}\mathrm{d}_\Omega\iota_X)(\mathrm{d}_\Omega f) = XY(f) - (-1)^{||X||||Y||}YX(f) = \iota_{[X,Y]}(\mathrm{d}_\Omega f).
\end{aligned}$$

∎

Many properties of these three differentials are analogous to those in the non-graded context (cf. [Keß19]). For more explicit expressions in the case of $\mathbb{Z}_2$-grading, we refer to [GSM09]. Mind that in this reference, the Deligne and Freed convention is used (see ft. (22), p. 14).

Definition 1.3.14. A $\mathbb{Z}$-graded R-module V with a linear graded map $\delta\colon V \to V$ of degree ± 1 is called a **complex**. A $\mathbb{Z}$-graded R-algebra A with a derivation $\delta\colon A \to A$ of degree ± 1 is called a **differential commutative graded algebra**.

An example of a complex is the **de Rham complex** over A $(\Omega^\bullet(A), \mathrm{d}_\Omega)$. Importantly, $(\Omega^\bullet(A), \wedge)$ is a $\mathbb{Z}$-graded R-algebra, and therefore $(\Omega^\bullet(A), \wedge, \mathrm{d}_\Omega)$ forms a DGCA.

Graded Manifolds

Before giving the global definition of a $\mathbb{Z}_{(2)}$-graded manifold, we present a short description of the local picture for the super case.
The prototype for a smooth n-manifold is the Cartesian space $\mathbb{R}^n$. Locally, we consider coordinates (x^i) on an open $U \subset \mathbb{R}^n$, which is dually (see Prop. 3.2.1) described by the algebra of smooth maps $C^\infty(U)$. The latter is algebraically characterized by the commutativity of the coordinates

$$x^i x^j = x^j x^i.$$

[25] One could define the graded Lie derivative by its action on functions, derivations, and forms. Then, the proof of the Cartan magic formula, which we use here as a definition instead, follows verbatim the one in the non-graded setting, once one keeps track of the gradings.

Working in this direction, we want to add odd (anti-commuting) coordinates such that algebraically we have

$$\theta^\mu x^i = x^i\theta^\mu \quad \text{and} \quad \theta^\mu\theta^\nu = -\theta^\nu\theta^\mu.$$

We define the algebra generated by such coordinates as $\mathbb{R}[x,\theta]\diagup\sim$. The quotient is with respect to the equivalence relation $\sim$ given by the (anti)-commutation of the generators. This algebra is called the **Grassmann algebra** and can be written as

$$\mathbb{R}[x,\theta]\diagup\sim = C^\infty(U)\otimes\bigwedge V^* =: \mathcal{O}(U\times\Pi V),$$

for some vector space V. More concretely, suppose $\dim V = m$. Then, any function $f \in \mathcal{O}(U\times\Pi V)$ is locally of the form

$$f(x,\theta) = \sum_{0\leqslant k\leqslant m}\sum_{\mu_1,\dots,\mu_k=1}^{m} \frac{f_{\mu_1\dots\mu_k}(x)}{k!}\otimes\theta^{\mu_1}\cdots\theta^{\mu_k} \tag{1.7}$$

for some collection of smooth[(26)] real functions $\{f_{\mu_1\dots\mu_k} \subset C^\infty(U)\}$ that are fully antisymmetric in $\mu_1,\dots,\mu_k$. We want to define smooth supermanifolds as the globalization of this local model. Consider a diffeomorphism between patches

$$\varphi\colon U\times\Pi V \to \tilde{U}\times\Pi\tilde{V}. \tag{1.8}$$

It is dually given by the graded $\mathbb{R}$-algebra morphism defined on the generators (cf. [Mne19][Ex. 4.2.7]) by

$$\begin{aligned}\varphi^*\colon \mathcal{O}(\tilde{U}\times\Pi\tilde{V}) &\to \mathcal{O}(U\times\Pi V)\\ \tilde{x}^{\tilde{i}} &\mapsto \varphi^*\left(\tilde{x}^{\tilde{i}}\right) \quad \text{even on } U\times\Pi V\\ \tilde{\theta}^{\tilde{\mu}} &\mapsto \varphi^*\left(\tilde{\theta}^{\tilde{\mu}}\right) \quad \text{odd on } U\times\Pi V.\end{aligned}$$

Patching $U\times\Pi V$ together, we get an (n,m)-supermanifold, globally described as follows. The bridge between the local and global pictures is described by Equation (1.9).

Definition 1.3.15. A **supermanifold** $\mathcal{M}$ of dimension (m,n) is a pair $(M,\mathcal{O}_M)$, where the **structure sheaf** $\mathcal{O}_M$ is a sheaf[(27)] of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras on the smooth manifold M, the **body** of $\mathcal{M}$, such that

$$\mathcal{O}_M(U) \cong_{\mathbb{Z}_2\mathrm{CAlg}_\mathbb{R}} C^\infty(U)\otimes\bigwedge(\mathbb{R}^m)^* \tag{1.9}$$

where $U\subset\mathbb{R}^n$ is open.[(28)] The **supercartesian space** of dimension (n,m) is the supermanifold $(\mathbb{R}^{n|m},\mathcal{O}_{n|m})$ where $\mathcal{O}_{n|m} := C^\infty(\mathbb{R}^n)\otimes\bigwedge\mathbb{R}^m$.

A **morphism of supermanifolds** $(M,\mathcal{O}_M)\to(N,\mathcal{O}_N)$ is a pair $(f,f^\#)$, where $f\colon M\to N$ is a smooth map and $f^\#\colon\mathcal{O}_N\to\mathcal{O}_M$ a f-map of sheaves of local $\mathbb{Z}_2$-graded commutative rings.[(29)] More concretely, $f^\#$ is a collection of morphisms of $\mathbb{Z}_2$-graded commutative rings $\{f^\#_V\colon\mathcal{O}_N(V)\to\mathcal{O}_M(f^{-1}(V))\}_{V\subset N}$ open that preserve the maximal ideal such that the following commutes for all $V'\subset V\subset N$ open

$$\begin{tikzcd}
\mathcal{O}_N(V) \arrow[r, "f^\#_V"] \arrow[d, "\mathrm{res}^V_{V'}"'] & \mathcal{O}_M(f^{-1}(V)) \arrow[d, "\mathrm{res}'^{f^{-1}(V)}_{f^{-1}(V')}"] \\
\mathcal{O}_N(V) \arrow[r, "f^\#_{V'}"'] & \mathcal{O}_M(f^{-1}(V')).
\end{tikzcd}$$

We denote the category of supermanifolds by sSmthMfd and the category of supercartesian spaces by sCartSp.

(26) One could alternatively, but not equivalently, define the category considering only polynomials in the even coordinates.

(27) In the sense of Definition 1.2.1, when enriching the set $F(U)$ with extra structure.

(28) This means that $(M,\mathcal{O}_M)$ is locally isomorphic to $(U,C^\infty(U)\otimes\bigwedge(\mathbb{R}^m)^*)$ in the category of locally ringed spaces. In particular, the parity $\bigoplus_{k\geqslant0}C^\infty(U)\otimes\bigwedge^k(\mathbb{R}^m)^*\to\mathbb{Z}_2$, $f\otimes\theta\mapsto|f\otimes\theta| := |\theta| = k$ should be preserved. This means that the algebra $C^\infty(\mathcal{M})$ of global sections of $\mathcal{O}_M$ is a $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra such that for two homogeneous elements $f,g\in\mathcal{O}_M(M) =: C^\infty(\mathcal{M})$, one has $fg = (-1)^{\deg_{\mathbb{Z}_2}(f)\deg_{\mathbb{Z}_2}(g)}gf$.

(29) A $\mathbb{Z}_2$-graded commutative ring $R = R_0\oplus R_1$ is **local** if it has unique maximal homogeneous ideal $I = I_0\oplus I_1$. Since all the odd elements are nilpotent, they are in any homogeneous maximal ideal. Therefore, a $\mathbb{Z}_2$-graded commutative ring $R = R_0\oplus R_1$ is local if and only if its even part is local, i.e. if it has a unique maximal ideal (cf. [Var04][Lemma 4.1.1]). Similarly for the $\mathbb{Z}$-case. From now on, when dually describing morphisms of graded manifolds as morphisms in $\mathbb{Z}_2\mathrm{CAlg}_\mathbb{R}$, we always assume them to preserve this local structure.

Observe that smooth n-manifolds are supermanifolds of dimension $(n, 0)$. Furthermore, in our setting, that is the smooth setting, we can equivalently (dually) define morphisms of supermanifolds as $\mathbb{Z}_2\mathrm{CAlg}_{\mathbb{R}}$ morphisms of the superalgebras of global sections of the corresponding structure sheaves [Var04][pp. 129, 138-143]. More explicitly, assume that locally $\mathcal{M}$ and $\mathcal{N}$ are given by $U \oplus \Pi V$ and $\bar{U} \oplus \Pi \bar{V}$. Then,

$$\mathrm{Hom}_{\mathrm{sSmthMfd}}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathbb{Z}_2\mathrm{CAlg}_{\mathbb{R}}}(C^\infty(\mathcal{N}), C^\infty(\mathcal{M})) \cong \left((\bar{U} \oplus \Pi\bar{V}) \otimes C^\infty(\mathcal{M})\right)_0$$

where $C^\infty(M) := \mathcal{O}_M(M)$ is the superalgebra of global sections of the structure sheaf and the last space represents the even part of the corresponding superspace. In other words, it is sufficient to know the restriction of a morphism $C^\infty(\bar{U} \oplus \Pi\bar{V}) \to C^\infty(\mathcal{M})$ to the subspace of linear functions in order to recover the morphism itself. This means that we can work in coordinates.
Two good examples are the odd tangent and cotangent bundles of a smooth manifold M. They are the parity-reversed tangent and cotangent bundles ΠTM and ΠT^*M, dually described by their superalgebras

$$C^\infty(\Pi TM) = \Gamma_M\left(\bigwedge\nolimits^\bullet T^*M\right) = \Omega^\bullet(M) \quad \text{and} \quad C^\infty(\Pi T^*M) = \Gamma_M\left(\bigwedge\nolimits^\bullet TM\right) = \mathfrak{X}_{\mathrm{mult}}(M).$$

The symbol $\mathfrak{X}_{\mathrm{mult}}(M)$ denotes the algebra of multivector fields in M. More generally, this holds for any non-graded vector bundle $E \to M$ of rank m over a smooth n-manifold M. In this case, we can define $C^\infty(\Pi E) := \Gamma_M(\bigwedge^\bullet E^*)$ or equivalently we can take M as the body and $\mathcal{O}_{\Pi E}(U) \cong \Gamma_U(\bigwedge^\bullet E|_U^*)$ as the structure sheaf for any open $U \subset M$. In this case, $\dim(\Pi E) = (n, m)$. Any (real) supermanifold is non-canonically isomorphic to ΠE for some vector bundle $E \to M$ (cf. [Rog07][Batchelor's Thm. 8.2.1]). The definition of a $\mathbb{Z}$-graded manifold is an extension of the previous definition.

Definition 1.3.16. A **$\mathbb{Z}$-graded manifold** $\mathcal{M}$ is a pair $(M, \mathcal{O}_M)$, where the **structure sheaf** $\mathcal{O}_M$ is a sheaf of $\mathbb{Z}$-graded $\mathbb{R}$-algebras on the smooth manifold M, the **body** of $\mathcal{M}$, such that

$$\mathcal{O}_M(U) \cong_{\mathbb{Z}\mathrm{CAlg}_{\mathbb{R}}} C^\infty(U) \otimes \mathrm{Sym}(V^*),$$

where $U \subset \mathbb{R}^n$ is open and V is a $\mathbb{Z}$-graded vector space.
A **morphism of $\mathbb{Z}$-graded manifolds** $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a pair $(f, f^\#)$, where $f\colon M \to N$ is a smooth map and $f^\#\colon \mathcal{O}_N \to \mathcal{O}_M$ an f-map of sheaves of local $\mathbb{Z}$-graded commutative rings.

The algebra of smooth functions $C^\infty(\mathcal{M})$ of a graded manifold $(M, \mathcal{O}_M)$ (i.e. algebra of global sections) automatically inherits a $\mathbb{Z}$-grading. In the smooth setting, we can equivalently consider morphisms of the $\mathbb{Z}$-graded algebra of smooth functions.
Any smooth $\mathbb{Z}$-graded manifold is isomorphic to a $\mathbb{Z}$-graded manifold associated to a $\mathbb{Z}$-graded vector bundle E over a smooth manifold M, i.e. a collection of ordinary vector bundles $\bigoplus_{k\in\mathbb{Z}} E_k$ over M. This can be checked by noticing that the sheaf $U \mapsto \Gamma_U(\mathrm{Sym}(\mathrm{E}|_{\mathrm{U}}^*))$ defines a $\mathbb{Z}$-graded manifold. The example above is a sub-example of the latter. Indeed, we recover (cf. [Mne19][Ex. 4.2.14 & 4.2.15]) the odd (co)tangent bundles above by

$$C^\infty(T[1]M) \cong \Omega^\bullet(M) \quad \text{and} \quad C^\infty(T^*[-1]M) \cong \mathfrak{X}_{\mathrm{mult}}(M).$$

Remark 1.3.3. In all the following discussion the $\mathbb{Z}$ and $\mathbb{Z}_2$ gradings are compatible. This means that the $\mathbb{Z}$ grading modulo 2 gives the $\mathbb{Z}_2$ grading. Practically, we achieve this by demanding that the supermanifold is locally isomorphic to $(U, C^\infty(U) \otimes \bigwedge V^*)$ with

$$V = \bigoplus_{k\in 2\mathbb{Z}+1} V_k \quad \text{and} \quad \mathrm{Span}\{x^i(U)\} \cong \bigoplus_{k\in 2\mathbb{Z}} V_k.$$

In any local chart, the local coordinates (x^i, θ^μ) preserve this extra grading. This means that local functions taking the form of Equation (1.7) are inhomogeneous in $\mathbb{Z}$-degree, which we call **ghost degree**. One could potentially write a similar expression that further restricts the ghost degree of the function so that f becomes homogeneous in both gradings.

Graded Vector Bundles

We recall the notion of a conventional vector bundle and extend it to the graded setting.

Remark 1.3.4. Recall that a fiber bundle is a smooth manifold that locally looks like a Cartesian product of smooth manifolds. More in detail, a **fiber bundle** (F, π_M^F, M, E) consists of three smooth manifolds F, M and E, called the **total space**, **base space** , and **typical fiber**, and a smooth surjection $\pi_M^F\colon F \to M$ called the **projection** with $(\pi_M^F)^{-1}(p) = E_p \cong E$, so that the following holds:

i. there exists an open covering[30] $\{U_i\}$ of M with diffeomorphisms, called **local trivialisations**, $\phi_i\colon\ U_i \times E \longrightarrow (\pi_M^F)^{-1}(U_i)$ such that $\pi_M^F \circ \phi_i(p,t) = p$,

ii. At each point $p \in M$, $\phi_{i,p}(t) \equiv \phi_i(p,t)$ is a diffeomorphism $\phi_{i,p}\colon E \longrightarrow E_p$. On each overlap $U_i \cap U_j \neq \varnothing$, we require the **transition map** $g_{ij}(p) \coloneqq \phi_{i,p}^{-1} \circ \phi_{j,p}\colon E \longrightarrow E$ to be a diffeomorphism and $g_{ij}: p \mapsto g_{ij}(p)$ to be smooth such that $\phi_j(p,t) = \phi_i\big(p,\, g_{ij}(p)\, t\big)$.

If the typical fiber is a real k-dimensional vector space $E \cong \mathbb{R}^k$, the restrictions of the local trivializations $\phi_{i,p}$ are linear isomorphisms, and the transition maps g_{ij} are smooth functions assigning to each point $p \in U_i \cap U_j$ an invertible matrix in $\mathrm{GL}(k,\mathbb{R})$, then the fiber bundle is called a **vector bundle**. From the definition, it follows that the functions g_{ij} satisfy the cocycle condition $g_{ij}(p) = g_{ik}(p) \circ g_{kj}(p)$ for all $p \in U_i \cap U_k \cap U_j$. If the bundle is globally trivial, i.e. $F \cong M \times E$, the fiber bundle is called a **trivial bundle**. A **global section** $s\colon M \to F$ of a fiber bundle $\pi_M^F\colon F \to M$ is a smooth map that satisfies $\pi_M^F \circ s = \mathrm{id}_M$. Clearly, $s(p) = s|_p$ is an element of $E_p = (\pi_M^F)^{-1}(p)$. The set of smooth global sections on M is denoted by $\Gamma_M(F) = \Gamma(M,F)$. Not all fiber bundles admit global sections. If $U \subset M$, **local sections** are sections $s\colon U \to F$ defined only on U. The condition reads $\pi_M^F \circ s = \mathrm{id}_U$. $\Gamma(U,F)$ denotes the set of local sections on U. Note that $\pi_M^F \circ s = \mathrm{id}_M$ and compositions of sections preserve smoothness. Remarkably, $\Gamma_M(F)$ is a $C^\infty(M)$-module when F is a vector bundle. Moreover, it forms a sheaf of $C^\infty(M)$-modules over M in the sense of Definition 1.2.1[31] due to pointwise addition and multiplication of sections, and compatibility with the usual restriction to subsets $U \subset M$.
Finally, we define the category of fiber (vector) bundles with the following morphisms. Let $\pi_M^F\colon F \to M$ and $\pi_{M}^{F'}\colon F' \to M'$ be fiber (vector) bundles. A smooth map $\bar f\colon F' \to F$ is called a **bundle map** if it (linearly) maps each fiber E'_p of F' onto E_q of F. Then $\bar f$ naturally induces a smooth map $f\colon M' \to M$ such that $f(p) = q$. That is, the following diagram commutes:

$$\begin{tikzcd} F' \arrow[r, "\bar f"] \arrow[d, "\pi_M^{F'}"'] & F \arrow[d, "\pi_M^F"] \\ M' \arrow[r, "f"'] & M \end{tikzcd} \qquad \left(\begin{tikzcd} u \arrow[r, mapsto, "\bar f"] \arrow[d, mapsto, "\pi_M^{F'}"'] & \bar f(u) \arrow[d, mapsto, "\pi_M^F"] \\ p \arrow[r, mapsto, "f"'] & q \end{tikzcd}\right) \tag{1.10}$$

A smooth map $\bar f\colon F' \to F$ is not necessarily a bundle map. It may map $u, v \in E'_p$ of F' to $\bar f(u)$ and $\bar f(v)$ on different fibers of F, so that $\pi_M^F(\bar f(u)) \neq \pi_M^F(\bar f(v))$.

Working in this direction, we now define graded vector bundles from the local point of view.

Definition 1.3.17. Consider a $\mathbb{Z}_{(2)}$-graded vector space K, called the **fiber**, two $\mathbb{Z}_{(2)}$-graded manifolds $\mathcal{E}$ and $\mathcal{M}$, called the **total manifold** and the **base manifold**, and a morphism of $\mathbb{Z}_{(2)}$-graded manifolds $\pi_{\mathcal{M}}^{\mathcal{E}}\colon \mathcal{E} \to \mathcal{M}$, called **projection**, such that $\bar\pi_{\mathcal{M}}^{\mathcal{E}}\colon E \to M$ is surjective. A **$\mathbb{Z}_{(2)}$-graded vector bundle** (cf. [Vys22][Ch. 5.5]) is a tuple $(\mathcal{E}, \pi_{\mathcal{M}}^{\mathcal{E}}, \mathcal{M}, K)$ such that the following holds:

i. There exists a collection $\{(U_i, \varphi_i)\}_{i\in I}$, called **local trivialization**, such that $\{(U_i)\}_{i\in I}$ is an open cover of M, and each $\varphi_i\colon \mathcal{E}|_{V_i} \to \mathcal{M}|_{U_i} \times K$[32] is an isomorphism of $\mathbb{Z}_{(2)}$-graded manifolds satisfying $\pi_1 \circ \varphi_i = \pi_{\mathcal{M}}^{\mathcal{E}}|_{V_i}$, where $V_i \coloneqq (\bar\pi_{\mathcal{M}}^{\mathcal{E}})^{-1}(U_i)$ and $\pi_1\colon \mathcal{M}|_{U_i} \times K \to \mathcal{M}|_{U_i}$.

ii. The **transition maps**

$$\varphi_{ij} \coloneqq \varphi_i|_{V_i\cap V_j} \circ (\varphi_j|_{V_i \cap V_j})^{-1}\colon \mathcal{M}|_{U_i\cap U_j} \times K \to \mathcal{M}|_{U_i \cap U_j} \times K$$

are linear in the fiber coordinates.

[30]Strictly speaking, the definition of a fiber bundle should be independent of the special covering $\{U_i\}$ of M. In the mathematical literature, this definition is employed to define a coordinate bundle $(F, \pi_M^F, M, E, \{U_i\}, \{\phi_i\})$. Then, two coordinate bundles $(F, \pi_M^F, M, E, \{U_i\}, \{\phi_i\})$ and $(F, \pi_M^F, M, E, \{V_j\}, \{\psi_j\})$ are said to be equivalent if $(F, \pi_M^F, M, E, \{U_i\} \cup \{V_j\}, \{\phi_i\} \cup \{\psi_j\})$ is again a coordinate bundle. A fiber bundle is defined as an equivalence class of coordinate bundles. We shall assume a definite covering and make no distinction between a coordinate bundle and a fiber bundle.
[31]Even more, the category of real vector bundles M can be shown to be equivalent to the category of locally free and finitely generated sheaves of $C^\infty(M)$-modules. What follows is justified more properly if one works in this setting, but this would exceed the intended goal of the thesis. We refer to [Vak25][Ch.14.1] for a detailed discussion of this.
[32]The product is understood in the category of $\mathbb{Z}_{(2)}$-graded smooth manifolds. Indeed, K can be seen as a $\mathbb{Z}_{(2)}$-graded manifold locally isomorphic to $\Big(U \subset X_0, \Pi_{p=0}^{\infty} C^\infty_{X_0}(U) \otimes_{\mathbb{R}} \mathrm{Sym}^p(Y)\Big)$, where $C^\infty_{X_0}$ is the sheaf of smooth functions over X_0 and Y is the graded vector space defined by $Y_0 \coloneqq \{0\}$, $Y_k \coloneqq (X_{-k})^*$ for any $k \neq 0$ (cf. [Vys22][Ex. 3.23]). The product takes the form $\mathcal{M} \times \mathcal{N} \coloneqq \big(M \times N, C^\infty_{\mathcal{M}\times\mathcal{N}}\big)$: an explicit construction of the sheaf can be found in [Vys22][Ch. 3.6].

A **homogeneous section** of degree $n \in \mathbb{Z}_{(2)}$ is a map $s\colon \mathcal{M} \to \mathcal{E}[n]$[(33)] such that $\pi^{\mathcal{E}}_{\mathcal{M}}[n] \circ s = \mathrm{id}_{\mathcal{M}}$, where $\pi^{\mathcal{E}}_{\mathcal{M}}[n]\colon \mathcal{E}[n] \to \mathcal{M}$. The sheaf of sections is $\Gamma_{\mathcal{M}} := \bigoplus_n \Gamma_{\mathcal{M}}(\mathcal{E})_n$, where $\Gamma_{\mathcal{M}}(\mathcal{E})_n$ is the space of sections of degree n (cf. [Meh06][Def. 2.2.14]).
Consider the trivial $\mathbb{Z}_{(2)}$-graded vector bundles $(\mathcal{E}' = \mathcal{M}' \times K', \pi^{\mathcal{E}'}_{\mathcal{M}'}, \mathcal{M}', K')$ and $(\mathcal{E} = \mathcal{M} \times K, \pi^{\mathcal{E}}_{\mathcal{M}}, \mathcal{M}, K)$, and a morphism of $\mathbb{Z}_{(2)}$-graded manifolds $F\colon \mathcal{M}' \times K' \to \mathcal{M} \times K$. Let $\varphi\colon \mathcal{M}' \to \mathcal{M}$ and $h\colon \mathcal{M}' \times K' \to K$ be two morphisms of $\mathbb{Z}_{(2)}$-graded manifolds. In particular, h is such that for each $m' \in \mathcal{M}'$ the map $h\colon \{m'\} \times K' \to K$ is a morphism of $\mathbb{Z}_{(2)}$-graded vector spaces. Then, F is called a **morphism of trivial $\mathbb{Z}_{(2)}$-graded vector bundles** $\varphi\colon \mathcal{M}' \to \mathcal{M}$ if $F = (\varphi \circ \pi^{\mathcal{E}'}_{\mathcal{M}'}, h)$ and the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}' \times K' \arrow[r, "F"] \arrow[d, "\pi^{\mathcal{E}'}_{\mathcal{M}'}"'] & \mathcal{M} \times K \arrow[d, "\pi^{\mathcal{E}}_{\mathcal{M}}"] \\
\mathcal{M}' \arrow[r, "\varphi"] & \mathcal{M}.
\end{tikzcd}$$

Morphisms of $\mathbb{Z}_{(2)}$-graded vector bundles are locally morphisms of trivial $\mathbb{Z}_{(2)}$-graded vector bundles (cf. [Keß19][Def. 4.1.1 & 4.2.1] and [ŠV25][Def. 5.3]).

The definition of morphisms can be made more rigorous once one works dually (and globally) with the sheaves of sections. To understand this, let us first go back to the category of conventional vector bundles.

Remark 1.3.5. In the spirit of Proposition 3.2.1 and in view of the $\mathbb{Z}_{(2)}$-graded extension, we would like to dually characterize morphisms of vector bundles (see Eq. (1.10)) as morphisms of the appropriate sheaves of sections. It turns out that the natural way to do so is to construct a map $F\colon \Gamma'_M(F') \to \Gamma_M(F)$ given by a pair $F = (f, f^\dagger)$, where $f\colon M' \to M$ is the above smooth map and $f^\dagger$ is a morphism of sheaves of $C^\infty(M)$-modules[(34)]

$$f^\dagger\colon \Gamma_M(F^*) \to f_*\Gamma_{M'}(F'^*), \quad \sigma \mapsto f^\dagger(\sigma) \quad \text{with} \quad f^\dagger(\sigma)(x') := \sigma\big(f(x')\big) \circ \bar{f}_{x'} \quad \forall x' \in M'.$$

Here, F^* is the dual vector bundle obtained by taking the dual fiber-wise. $f_*\Gamma_{M'}(F'^*)$ is the pushforward sheaf over M defined by $f_*\Gamma_{M'}(F'^*)(U) := \Gamma_{M'}(F'^*)(f^{-1}(U)) \equiv \Gamma_{f^{-1}(U)}(F'^*)$, with $U \subset M$ open and $\bar{f}_{x'}\colon F'_{x'} \to F_{f(x')}$. Note that $f_*\Gamma_{M'}(F'^*)$ is not automatically a sheaf of $C^\infty(M)$-modules, but can be made into one by setting, for any $U \subset M$ open, $g \in C^\infty(M)|_U$, and $s \in f_*\Gamma_{M'}(F'^*)$, the product

$$g * s := f^*_U(g) \cdot s$$

where on the right-hand side there is the original $C^\infty(M')|_{f^{-1}(U)}$-module action.

Following the above remark (cf. [Vys22][Rmk. 5.9 & Def. 5.10]), we now globally define morphisms of graded vector bundles.

Definition 1.3.18. Given two $\mathbb{Z}_{(2)}$-graded vector bundles $(\mathcal{E}', \pi^{\mathcal{E}'}_{\mathcal{M}'}, \mathcal{M}', K')$ and $(\mathcal{E}, \pi^{\mathcal{E}}_{\mathcal{M}}, \mathcal{M}, K)$, a **$\mathbb{Z}_{(2)}$-graded vector bundle morphism** $F\colon \mathcal{E}' \to \mathcal{E}$ over a morphism of $\mathbb{Z}_{(2)}$-graded manifolds $\varphi\colon \mathcal{M}' \to \mathcal{M}$ is a pair $(\varphi, F^\dagger)$ such that

$$F^\dagger\colon \Gamma_{\mathcal{M}}(\mathcal{E}^*) \to \varphi_*\Gamma_{\mathcal{M}'}(\mathcal{E}'^*)$$

is a morphism of sheaves of $\mathbb{Z}_{(2)}$-graded $C^\infty(\mathcal{N})$-modules. Here, $\mathcal{E}^*$ is the dual $\mathbb{Z}_{(2)}$-graded vector bundle obtained by taking the dual fiber-wise, $\varphi_*\Gamma_{\mathcal{M}'}(\mathcal{E}'^*)$ is the pushforward sheaf over $\mathcal{M}$ defined by $\varphi_*\Gamma_{\mathcal{M}'}(\mathcal{E}'^*)(U) := \Gamma_{\mathcal{M}'}(\mathcal{E}'^*)(\varphi^{-1}(U)) \equiv \Gamma_{\varphi^{-1}(U)}(\mathcal{E}'^*)$, with $U \subset M$ open. Note that the latter is not automatically a sheaf of $\mathbb{Z}_{(2)}$-graded $C^\infty(\mathcal{M})$-modules, but can be made into one by setting, for any $U \subset M$ open, $g \in C^\infty(\mathcal{M})|_U$, and $s \in \varphi_*\Gamma_{\mathcal{M}'}(\mathcal{E}'^*)$, the product

$$g * s := \varphi^*_U(g) \cdot s$$

where on the right-hand side there is the original $\mathbb{Z}_{(2)}$-graded $C^\infty(\mathcal{M}')|_{\varphi^{-1}(U)}$-module action.

(33) The shift only acts on the fiber as in Definition 1.3.4. This produces a new $\mathbb{Z}_{(2)}$-vector bundle $\pi^{\mathcal{E}}_{\mathcal{M}}[n]\colon \mathcal{E}[n] \to \mathcal{M}$.
(34) That means each $f^\dagger_U$ is $C^\infty(M)$-linear, i.e. for each $U \in M$, $\sigma \in \Gamma_U(F^*)$ and $g \in C^\infty(U)$, then $f^\dagger_U(g \cdot \sigma) = g f^\dagger_U(\sigma)$.

The two notions of morphisms are compatible in the sense that they define equivalent categories [ŠV25][Thm. 5.10]. In the category of $\mathbb{Z}_{(2)}$-graded vector bundles, the direct sum, the tensor product, and the pull-back bundle can be defined as in the category of vector bundles, by the respective universal properties. Working within this category amounts to using the same algebraic definitions once the grading is properly accounted for. The following are local versions of the definitions; a more rigorous and global description can be found in [Vys22][Ch. 5].

Definition 1.3.19. Let $(\mathcal{M}\times K,\pi,\mathcal{M},K)$, $(\mathcal{M}\times K',\pi,\mathcal{M},K')$ be two trivial $\mathbb{Z}_{(2)}$-graded vector bundles and $(\mathcal{E}',\pi^{\mathcal{E}'}_{\mathcal{M}'},\mathcal{M},K')$, $(\mathcal{E},\pi^{\mathcal{E}}_{\mathcal{M}},\mathcal{M},K)$ two more general $\mathbb{Z}_{(2)}$-graded vector bundles. We omit the upper and lower indices on π,π' for ease of notation.

1. Given a morphism of $\mathbb{Z}_{(2)}$-graded manifolds $\varphi\colon\mathcal{N}\to\mathcal{M}$, the **trivial pullback $\mathbb{Z}_{(2)}$-graded vector bundle** is $(\varphi^*(\mathcal{M}\times K)\coloneqq\mathcal{N}\times K,\pi_1,\mathcal{N},K)$. We can further define the bundle map $\bar{\varphi}$ over φ by

$$\begin{tikzcd} \varphi^*(\mathcal{M}\times K) \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}\times K \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "\varphi"'] & \mathcal{M}, \end{tikzcd}$$

where π_1 is the projection into the first component of the direct product. The **pullback $\mathbb{Z}_{(2)}$-graded vector bundle** is $(\varphi^*(\mathcal{E}),\pi^{\varphi^*(\mathcal{E})}_{\mathcal{N}},\mathcal{N},K)$ is locally given as a trivial pullback $\mathbb{Z}_{(2)}$-graded vector bundle (cf. [Vys22][Prop. 5.15] and [ŠV25][Ex. 5.5]). For any $s\in\Gamma_{\mathcal{M}}(\mathcal{E})$ there exists a **pullback section** $\varphi^*s=(\mathrm{id}_{\mathcal{N}},s\circ\varphi)$ in $\Gamma_{\mathcal{N}}(\varphi^*\mathcal{E})$.

2. The **direct sum, tensor product, dual** and **k-shift** of trivial $\mathbb{Z}_{(2)}$-graded vector bundles are defined as one would expect, that is, acting fiberwise

$$\begin{aligned}(\mathcal{M}\times K)\oplus(\mathcal{M}\times K')&\coloneqq\mathcal{M}\times(K\oplus K')\quad\text{with}\quad\pi_{\oplus}\colon\mathcal{M}\times(K\oplus K')\to\mathcal{M},\\(\mathcal{M}\times K)\otimes(\mathcal{M}\times K')&\coloneqq\mathcal{M}\times(K\otimes K')\quad\text{with}\quad\pi_{\otimes}\colon\mathcal{M}\times(K\otimes K')\to\mathcal{M},\\(\mathcal{M}\times K)^*&\coloneqq\mathcal{M}\times K^*\quad\text{with}\quad\pi^*\colon\mathcal{M}\times K^*\to\mathcal{M},\\(\mathcal{M}\times K)[k]&\coloneqq\mathcal{M}\times K[k]\quad\text{with}\quad\pi[k]\colon\mathcal{M}\times K[k]\to\mathcal{M}.\end{aligned}$$

Then, $\mathcal{E}\oplus\mathcal{E}'$, $\mathcal{E}\otimes\mathcal{E}'$, $\mathcal{E}^*$ and $\mathcal{E}[k]$ look locally as the trivial versions above.
Finally, it is possible to show that

$$\Gamma_{\mathcal{M}}(\mathcal{E}\oplus\mathcal{E}')=\Gamma_{\mathcal{M}}(\mathcal{E})\oplus\Gamma_{\mathcal{M}}(\mathcal{E}'),\quad\Gamma_{\mathcal{M}}(\mathcal{E}\otimes\mathcal{E}')=\Gamma_{\mathcal{M}}(\mathcal{E})\otimes_{C^\infty(\mathcal{M})}\Gamma_{\mathcal{M}}(\mathcal{E}'),$$
$$\Gamma_{\mathcal{M}}(\mathcal{E}^*)=\Gamma_{\mathcal{M}}(\mathcal{E})^*,\quad\Gamma_{\mathcal{M}}(\mathcal{E}[k])=\Gamma_{\mathcal{M}}(\mathcal{E})[k].$$

Graded Vector Fields and Differential Forms

The differential calculus on smooth manifolds straightforwardly generalizes to the graded differential calculus on graded manifolds.

Definition 1.3.20. Given a $\mathbb{Z}_{(2)}$-graded manifold $\mathcal{M}=(M,\mathcal{O}_M)$, we define the following:

1. A **$\mathbb{Z}_{(2)}$-graded vector field of grade** k (mod 2) is a derivation of grade k (mod 2) in $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_M)$. We denote the Lie graded algebra of vector fields by $(\mathfrak{X}(\mathcal{M}),[\ ,\ ]_{\mathfrak{X}(\mathcal{M})})$, where the graded bracket is defined as in Equation (1.6). The vector bundle whose sheaf of sections is $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_M)$ is called the **graded tangent bundle** $T\mathcal{M}$.

2. A **graded n-differential form of degree** k is an element of $\Omega^n(\mathcal{M})\coloneqq(\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_M)^*)^{\wedge n}$. It can be viewed as a map
$$\omega\colon T\mathcal{M}^{\times n}\to\mathbb{R}[k\ (\mathrm{mod}\ 2)]$$
where the n-folded cartesian product is understood as in footnote (32) at page 17. The de Rham differential, the contraction and the Lie derivative are defined as in Definition 1.3.13. Furthermore, the full Cartan calculus holds (see Thm. 1.1). We denote by $(\Omega(\mathcal{M}),\mathrm{d})$ the de Rham complex.

Importantly, it can be equivalently described by $C^\infty(T[1]\mathcal{M})$[35] equipped with a **cohomological vector field** Q, i.e. a graded vector field of degree $+1$ such that $[Q,Q] = 2(Q \circ Q) = 0$.[36]

Let us work out a local description of graded vector fields and differential forms. Consider a $\mathbb{Z}_{(2)}$-graded manifold $\mathcal{M}$ with homogeneous local coordinates $\{z^i\}$ and total degree $|\cdot| = \deg_{\mathbb{Z}_{(2)}}(\cdot) + \deg_\Omega(\cdot)$.

Remark 1.3.6. Recall that above, to gain full generality, we used $|\cdot|$ to denote the intrinsic total grading and $||\cdot|| = |\cdot| + \deg_\Omega(\cdot)$ to denote the total grading that further considers the cohomological grading. For ease of notation, as the relevant intrinsic grading is $\mathbb{Z}$ or $\mathbb{Z}_2$, from now on we use the notation $|\cdot| = \deg_{\mathbb{Z}_{(2)}}(\cdot) + \deg_\Omega(\cdot)$.

We can locally define the Lie algebra of vector fields and the exterior algebra of differential forms. This amounts to adding the coordinates $\{\partial_{z^i}\}$ ($|\partial_{z^i}| = -|z^i|$) and $\{\mathrm{d}z^i\}$ ($|\mathrm{d}z^i| = |z^i| + 1$) defined by

$$\partial_{z^i}(z^j) \coloneqq \delta_i^j \quad \text{such that} \quad \partial_{z^i}(fg) = \partial_{z^i}(f)g + (-1)^{-|z^i||f|} f \partial_{z^i}(g) \tag{1.11}$$

$$\mathrm{d}z^i(\partial_{z^j}) \coloneqq \iota_{\partial_{z^j}} \mathrm{d}z^i \coloneqq \delta_j^i. \tag{1.12}$$

By degree considerations ($|\delta_j^i| = 0$), the contraction is required in Equation (1.12). Then, a graded vector field of degree $k \pmod 2$ is a linear combination of the form

$$X = \sum_i X^i \partial_{z^i}$$

where each $X^i \in C^\infty(\mathcal{M})_{k+|z^i| \pmod 2}$ is a function of homogeneous degree $k + |z^i| \pmod 2$ such that $|X(f)| = |k| + |f| \pmod 2$ for all homogeneous functions $f \in C^\infty(\mathcal{M})_{m \pmod 2}$ of degree $m \pmod 2$. Equation (1.11) ensures that X satisfies the graded Leibniz rule (see Eq. (1.5)). Furthermore, the graded commutator takes the local expression

$$[X,Y] = \sum_{i,j} X^i \partial_{z^i}(Y^j)\partial_{z^j} - (-1)^{|X||Y|} \sum_{i,j} Y^j \partial_{z^j}(X^i)\partial_{z^i}. \tag{1.13}$$

A graded n-differential form of degree $k \pmod 2$ locally takes the form

$$\omega = \sum_{i_1,\dots,i_n} \frac{\omega_{i_1\dots i_n}(z)}{n!} \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_n}.$$

The coefficients $\omega_{i_1\dots i_k} \in C^\infty(\mathcal{M})_{k \pmod 2}$ are fully antisymmetric in the indices $i_1,\dots,i_k$. The classical Cartan calculus extends to the graded setting (see Def. 1.3.13). In particular, in local coordinates

$$\mathrm{d} \coloneqq \sum_i \mathrm{d}z^i \partial_{z^i}, \quad \iota_{\partial_{z^i}} \mathrm{d}z^j \coloneqq \delta_i^j \;\text{ with } \iota_{\partial_{z^i}} \text{ a graded derivation of } \Omega(\mathcal{M}), \quad \mathbb{L}_{\partial_{z^i}} \coloneqq [\iota_{\partial_{z^i}}, \mathrm{d}].$$

By the definition of the local coordinate, d and $\mathbb{L}_{\partial_{z^i}}$ are automatically graded derivations of $\Omega^\bullet(\mathcal{M})$ of the right degree, and $|\iota| = -1$. The coordinate expressions of the action of d, ι_X, and $\mathbb{L}_X$ on forms are the same as the usual ones, once one takes care of possible extra minus signs due to the grading structure (cf. [GSM09][Ch. 3] for the super case and [CS11; Meh06] for the $\mathbb{Z}$-graded case).

Remark 1.3.7.

1. If z^i is odd, then $(\mathrm{d}z^i)^2 \neq 0$. This means that, if there exists at least one odd coordinate, the degree of the forms is unbounded since we may have a polynomial of every degree in that coordinate. Moreover, it turns out that graded differential forms do not come along with a nice integration theory. In the last part of this section, we introduce some basic notions of integration theory on a supermanifold without referring to differential forms.

[35] The shift acts only on the fibers, after omitting the possible cohomological degree.

[36] One could define the category of **dg-manifolds** as the category of $\mathbb{Z}$-graded manifolds equipped with a cohomological vector field. A morphism $(f, f^\#)\colon (\mathcal{M}, Q_\mathcal{M}) \to (\mathcal{N}, Q_\mathcal{N})$ is a morphism of $\mathbb{Z}$-manifolds such that $Q_\mathcal{M} f^\# = f^\# Q_\mathcal{N}$.

2. Any cohomological vector field on a $\mathbb{Z}_{(2)}$-graded manifold $\mathcal{M}$ corresponds to a differential on $C^\infty(\mathcal{M})$. For example, consider the odd tangent bundle $T[1]M \cong \Pi TM$ for some smooth manifold M and the cohomological vector field $Q \in \mathfrak{X}(T[1]M)_1$[37] defined in local coordinates $\{z^i, t^i\}$ by
$$Q \coloneqq \sum_i t^i \partial_{z^i}.$$
Interpreting t^i as $\mathrm{d}z^i$, one can see that the action of Q on $C^\infty(T[1]M)$ and that of d on $\Omega^\bullet(M)$ are equivalent. The two points of view are equivalent, and choosing one is a matter of convenience. For the general AKSZ model (see Sec. 2.3), working with functions over the shifted tangent bundle is more convenient, while in the example of BV theory within sThSmthSet (see Sec. 3.4), we use forms over M.

Many other constructions are identical to the ones in differential geometry, once one employs the graded version of the corresponding algebraic structures. Some of the more relevant constructions for the thesis are recalled in the super case in the following chapters. Before recalling the integration theory on a supermanifold, let us define the Euler vector field.

Definition 1.3.21. Consider a $\mathbb{Z}_{(2)}$-graded manifold $\mathcal{M}$. The **Euler vector field** $\mathbb{E} \in \mathfrak{X}(\mathcal{M})_0$ is the even vector field defined by
$$\mathbb{E}(f) \coloneqq \deg_{\mathbb{Z}_{(2)}}(f) f$$
for any homogeneous function $f \in C^\infty(\mathcal{M})$.

In local coordinates $\{z^i\}$, it takes the form
$$\mathbb{E} = \sum_i \deg_{\mathbb{Z}_{(2)}}(z^i) z^i \partial_{z^i}. \tag{1.14}$$
Furthermore, for any $X \in \mathfrak{X}(\mathcal{M})_{k \ (\mathrm{mod}\ 2)}$ and $\omega \in \Omega^n(\mathcal{M})_{k \ (\mathrm{mod}\ 2)}$ it holds that
$$[\mathbb{E}, X] = (k \ (\mathrm{mod}\ 2))\, X \quad \text{and} \quad \mathbb{L}_{\mathbb{E}}\omega = (k \ (\mathrm{mod}\ 2))\, \omega.$$
Indeed, using Equations (1.13) and (1.14), and the Taylor expansion of the components, we have
$$\begin{aligned}[\mathbb{E}, X] &= \sum_{i,j} \deg_{\mathbb{Z}_{(2)}}(z^i) z^i \partial_{z^i}(X^j)\partial_{z^j} - \sum_{i,j} X^j \partial_{z^j}(\deg_{\mathbb{Z}_{(2)}}(z^i) z^i)\partial_{z^i} = \\ &= \sum_j (\deg_{\mathbb{Z}_{(2)}}(X^j) - \deg_{\mathbb{Z}_{(2)}}(z^j)) X^j \partial_{z^j} = (k \ (\mathrm{mod}\ 2))\, X,\end{aligned}$$
and on generators of forms, we get
$$\mathbb{L}_{\mathbb{E}}(z^i) = \deg_{\mathbb{Z}_{(2)}}(z^i) z^i \quad \text{and} \quad \mathbb{L}_{\mathbb{E}}\mathrm{d}z^i = \mathrm{d}\mathbb{L}_{\mathbb{E}} z^i = \deg_{\mathbb{Z}_{(2)}}(z^i)\mathrm{d}z^i.$$
This means that this Euler vector field sees only the underlying degree, and not the cohomological degree. It is not possible to define a total Euler vector field $\mathbb{E}_{\mathrm{tot}} = \mathbb{E} + \mathbb{D} \in \mathfrak{X}(\mathcal{M})_0$ that gives the total degree $|\cdot| = \deg_{\mathbb{Z}_{(2)}} + \deg_\Omega$. Indeed, asking for $\mathbb{D}$ to only see the cohomological degree implies $\mathbb{D}(f) = 0$ for all homogeneous functions $f \in C^\infty(\mathcal{M})$, which in turns implies $\mathbb{D} \equiv 0$. Instead, recalling that $\Omega^\bullet(\mathcal{M}) \cong C^\infty(T[1]\mathcal{M})$, we define the vector field $\mathbb{D} \in \mathfrak{X}(T[1]\mathcal{M})_0$ given in local coordinates $(z^i, t^i) \in T[1]\mathcal{M}$ by $\mathbb{D} \coloneqq \sum_i t^i \partial_{t^i}$. Crucially, it only sees the cohomological degree, indeed
$$\mathbb{D}(z^i) = 0 \quad \text{and} \quad \mathbb{D}(t^i) = t^i.$$
The action of $\mathbb{D}$ on $C^\infty(T[1]\mathcal{M})$ corresponds to a derivation of degree 0 on the algebra $\Omega^\bullet(\mathcal{M})$ that, by abuse of notation, we also denote by $\mathbb{D} \in \mathrm{Der}_{\mathbb{R}}(\Omega^\bullet(\mathcal{M}))$
$$\mathbb{D}\colon \Omega^n(\mathcal{M}) \to \Omega^n(\mathcal{M}), \quad \omega \mapsto \mathbb{D}(\omega) = n\omega.$$
Therefore, to work in total degree, one considers the derivation $\mathbb{L}_{\mathbb{E}} + \mathbb{D}$ such that
$$(\mathbb{L}_{\mathbb{E}} + \mathbb{D})\, \omega = (k \ (\mathrm{mod}\ 2) + n)\, \omega = |\omega|\omega, \quad \forall \omega \in \Omega^n(\mathcal{M})_k. \tag{1.15}$$
From now on, we consider the total grading $|\cdot|$ as this greatly simplifies the calculations.

[37] The subscript means we are considering the space of vector fields of degree 1.

Integration Theory

We now recall the basics of integration theory on superspaces and supermanifolds. For a more complete exposition, we refer to [Rog07; Sch08]. When integrating over $\mathbb{Z}$-graded manifolds, only the underlying superstructure plays a role. For simplicity, we assume the base manifold M of any $\mathbb{Z}_{(2)}$-graded manifold to be orientable and equipped with orientation, that is, we assume M to be oriented.

Definition 1.3.22. Consider a $\mathbb{Z}_2$-graded commutative ring R. A matrix M of size $(p+q)\times(r+s)$ with entries in R is called a **supermatrix** of size $(p|q)\times(r|s)$ if it has a decomposition in blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.16}$$

such that A, D consist of even entries and B, C consist of odd entries. The set $\mathrm{Mat}_{(p|q)\times(r|s)}(R)$ of supermatrices of size $(p|q)\times(r|s)$ forms a free $\mathbb{Z}_2$-graded R-module. When $p = r$ and $q = s$, considering the usual matrix multiplication, we denote the **$\mathbb{Z}_2$-graded R-algebra of supermatrices** of size $(p|q)\times(p|q)$ by $\mathrm{Mat}_{p|q}(R)$ and the group of **invertible supermatrices** of size $(p|q)\times(p|q)$ by $\mathrm{GL}_{p|q}(R) \subset \mathrm{Mat}_{p|q}(R)$.

It turns out that a supermatrix $M \in \mathrm{Mat}_{p|q}(R)$ is invertible if and only if the two even blocks are invertible (cf. [Keß19][Lemma 2.3.4 & Cor. 2.3.5]). This is crucial for the definition of Berezinian, as it implies that the even blocks of an invertible supermatrix are invertible and that their determinants are nonzero.

Definition 1.3.23. Let $M \in \mathrm{GL}_{p|q}(R)$ be as in Equation (1.16). The **Berezinian** of M is given by

$$\mathrm{Ber}(M) := \frac{\det\left(A - BD^{-1}C\right)}{\det(D)} \in R_0,$$

where the determinant is the usual one, since the entries in $A - BD^{-1}C$ and D are even.

Crucially, Berezinians behave well with respect to supermatrix multiplication (cf. [Keß19][Lemma 2.3.8]).

Proposition 1.3.1. *Let $M, N \in \mathrm{GL}_{p|q}(R)$. Then* $\mathrm{Ber}(M\cdot N) = \mathrm{Ber}(M)\mathrm{Ber}(N)$.

To define the integration on supermanifolds, we start locally by considering integration over superspaces. Let $\mathcal{M} \ni (x^i, \theta^\mu)_{1\leqslant i\leqslant n|1\leqslant\mu\leqslant m}$ be a $(n|m)$ supermanifold and define the **Berezinian integration** along θ^μ according to the following rules

$$\int \mathrm{d}\theta^\mu = 0, \quad \int \theta^\mu \mathrm{d}\theta^\nu = \delta_{\mu\nu}, \quad \text{Fubini's theorem holds,}$$
$$\text{and} \quad \int f \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^m = f_{\mu_1\ldots\mu_k} =: f_{\mathrm{top}}$$

for any $f \in C_c^\infty(\mathcal{M})$ given locally as Equation (1.7). Note that $\mathrm{d}\theta^\mu$ is not a 1-form as in Definition 1.3.20, but a measure that under change of coordinates transforms with the Berezinian (see Def. 1.3.23) of the Jacobian of the coordinate change. It is called Berezinian, even though, strictly speaking, it is not a Berezinian in the sense of Definition 1.3.23. Consider now a supermanifold of the form ΠV where V is a vector space. For any choice of coordinates on it, the integration defines a map

$$\int_{\Pi V} = \int_{\Pi V}\Big|_{\bigwedge^m V^*} : \bigwedge^m V^* \to \mathbb{R},$$

since for any $g \in \bigwedge^{k<m} V^*$ we get $\int_{\Pi V} g = 0$. Therefore, on a coordinate patch $U \times \Pi V$ of $\mathcal{M}$, we get

$$\int_{U\times\Pi V} f := \int_U \int_{\Pi V} f = \int_U f_{\mathrm{top}} \mathrm{d}x^1 \cdots \mathrm{d}x^n.$$

To define how the integral map behaves under a change of coordinates (see Eq. (1.8)), first notice that locally it can always be written as a linear isomorphism using its Jacobian $J\varphi$, which is a supermatrix. Then, it is possible to prove (cf. [Rog07][Thm. 11.2.3]) that for any $h \in C_c^\infty(\mathcal{M})$ with compact support in $\tilde{U} \times \Pi\tilde{V}$, the following holds

$$\int_{\tilde{U}\times\Pi\tilde{V}} h(\tilde{x}, \tilde{\theta})\mathrm{d}^n\tilde{x}\mathrm{d}^m\tilde{\theta} = \int_{U\times\Pi V} (\varphi^* f)(x,\theta)\mathrm{Ber}(J\varphi(x,\theta))\mathrm{d}^n x \mathrm{d}^m\theta.$$

This can be described globally by building the Berezin line bundle (cf. [Sch08]). Without loss of generality (cf. [Rog07][Batchelor's Thm. 8.2.1]), let us consider $\mathcal{M} = \Pi E$ for a vector bundle $E \to M$ of rank m over a smooth n-manifold M. Then the **Berezin line bundle** is defined as

$$\mathrm{Ber}(M) \coloneqq \bigwedge^n T^*M \otimes \bigwedge^m E \to M.$$

The symbolic measure $\mathrm{d}x^n \mathrm{d}\theta^m$ can be interpreted as a local section of such a bundle $\mu \in \Gamma_M(\mathrm{Ber}(\mathcal{M}))$. Then, the $\mathbb{R}$-linear integration map is given by

$$\int_{\mathcal{M}} \bullet \cdot \mu \colon C_c^\infty(\mathcal{M}) \to \mathbb{R} \quad \text{by} \quad \int_{\mathcal{M}} f\mu \coloneqq \int_{\mathcal{M}} \langle (f)_{\mathrm{top}}, \mu \rangle$$

where $\langle , \rangle$ is the fiberwise pairing between line bundles $\bigwedge^m E$ and $\bigwedge^m E^*$, and $(f)_{\mathrm{top}}$ is the component of $f \in C^\infty(\Pi E) \cong \Gamma_M(\bigwedge^\bullet E^*)$ in the top exterior power of E^*. After pairing, what remains is a top form over M, which can be integrated in the usual way. This gives us measures that are constant along the fiber directions of $\Pi E \to M$. We can define more general measures as sections of the super vector bundle $\mathrm{Ber}(\mathcal{M}) \otimes \bigwedge^\bullet E^*$, where the tensor product is taken in the category of super vector bundles over an even base, which means defining the **integration map** as

$$\int_{\mathcal{M}} \colon \mathrm{BER}(\mathcal{M}) \to \mathbb{R} \quad \text{where} \tag{1.17}$$
$$\mathrm{BER}(\mathcal{M}) \coloneqq \Gamma_M\left(\mathrm{Ber}(\mathcal{M}) \otimes \bigwedge^\bullet E^*\right) \cong \Gamma_M(\mathrm{Ber}(\mathcal{M})) \otimes_{C^\infty(\mathcal{M})} C^\infty(\mathcal{M}).$$

This can also be understood as the space of sections of the pullback bundle $p^*\mathrm{Ber}(\mathcal{M})$ along $p \colon \Pi E \to M$.

Remark 1.3.8. The shifted tangent bundle $T[1]M$ for a smooth oriented n-manifold M carries a canonical Berezinian $\mu_{T[1]M} = \mathrm{d}^n x \mathrm{d}^n \theta \in \mathrm{BER}(T[1]M)$. Observe that, for all $f \in C^\infty(T[1]M) \cong \Omega^\bullet(M)$, the latter satisfies

$$\int_{T[1]M} f\mu_{T[1]M} = \int_M f_{\mathrm{top}}.$$

On the right-hand side, the integration is the ordinary integration over M.

Finally, we present the extended definition of divergence.

Definition 1.3.24. Consider a graded vector field $X \in \mathfrak{X}(\mathcal{M})$ and a Berezinian $\mu \in \mathrm{BER}(\mathcal{M})$. The **divergence** $\mathrm{div}_\mu(X) \in C^\infty(\mathcal{M})$ of X with respect to μ is defined by

$$\int_{\mathcal{M}} X(f)\mu = -\int_{\mathcal{M}} \mathrm{div}_\mu(X) f\mu$$

for any $f \in C_c^\infty(\mathcal{M})$.

This definition is motivated by looking at the usual definition of divergence and the Stokes theorem (cf. [Mne19; Cat18]).

1.4 Elements of Symplectic Geometry

In this section, some basic notions and results of symplectic geometry are recalled, together with a motivating example from classical mechanics. Afterward, the symplectic structure on graded manifolds is defined.

Symplectic Geometry Rudiments

Definition 1.4.1. A **symplectic vector space** is a pair (V, ω), where V is a finite-dimensional $\mathbb{R}$-vector space and the **symplectic form** $\omega \in \bigwedge^2 V^*$ is such that $\omega^\flat \colon V \to V^*, v \mapsto \omega(v, -)$ is a vector space isomorphism. A linear isomorphism $\varphi \colon V \to V'$ between symplectic vector spaces (V, ω), (V', ω') such that $\varphi^*\omega' = \omega$ is called a **symplectomorphism**.

Equivalently, we can impose ω to be an antisymmetric, non-degenerate bilinear form on V.

Definition 1.4.2. Let (V, ω) be a symplectic vector space. Let Y be a subspace of V and define the **symplectic orthogonal complement** of Y by

$$Y^{\perp} := \{v \in V | \omega(v, y) = 0, \ \forall y \in Y\}.$$

Then

1. Y is **isotropic** if $Y \subseteq Y^{\perp}$, i.e. $\omega|_{Y \times Y} = 0$.
2. Y is **coisotropic** if $Y^{\perp} \subseteq Y$, i.e. all elements annihilated by Y are in Y.
3. Y is **Lagrangian** if it is isotropic and coisotropic, or equivalently if Y is isotropic and $\dim Y = \frac{1}{2}\dim V$.
4. Y is **symplectic** if $\omega|_{Y \times Y}$ is non-degenerate.

In a convenient basis, usually called the Darboux basis, ω has a canonical expression.

Proposition 1.4.1. *Let (V, ω) be a symplectic vector space.*
Then, there exists a basis, called ***Darboux basis****, $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \textit{and} \quad \omega(e_i, f_j) = \delta_{ij}$$

for all $i, j \in \{1, 2, \ldots, n\}$. Hence, we can write $\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*$ w.r.t. this basis.

A proof can be found in [Sil01][Thm. 1.1]. In particular, this result implies that a symplectic vector space necessarily has an even dimension.
This notion can be straightforwardly generalized to smooth manifolds by defining a pointwise symplectic structure on the tangent space.

Definition 1.4.3. A **symplectic manifold** is a pair (M, ω), where M is a smooth manifold ($\dim M = 2n$) and ω is a 2-form on M such that

1. ω is closed, i.e. $\mathrm{d}\omega = 0$,
2. ω is non-degenerate, i.e. for all $q \in M$, $\omega^{\flat} \colon T_qM \to T_q^*M$ is bijective.

A **symplectomorphism** is a diffeomorphism $\varphi \colon M \to M'$ such that $\varphi^*\omega' = \omega$.

Definition 1.4.4. Let $M = T^*N$. The **tautological 1-form** on M is defined by

$$\alpha_{(x,p)} := \left(\mathrm{d}\pi_{(x,p)}\right)^* p \in T^*_{(x,p)}M,$$

where $\pi \colon M = T^*N \to N$.

In a local Darboux chart $\{x^1, \ldots, x^n, p_1, \ldots, p_n\}$, we can write

$$\alpha = \sum_i p_i \mathrm{d}x^i \quad \text{and} \quad \omega = -\mathrm{d}\alpha = \sum_i \mathrm{d}x^i \wedge \mathrm{d}p_i.$$

This means that pointwise $\{\partial_{x^i}, \partial_{p_i}\}$ forms a Darboux basis on the fiber $T_{(x,p)}M$. Definition 1.4.2 extends as follows.

Definition 1.4.5. Let (M, ω) be a symplectic manifold and $N \subseteq M$ be a submanifold. Then

1. N is **isotropic** if T_xN is an isotropic subspace of T_xM, for all $x \in M$.
2. N is **coisotropic** if T_xN is a coisotropic subspace of T_xM, for all $x \in M$.
3. N is **Lagrangian** if T_xN is a Lagrangian subspace of T_xM, for all $x \in M$.
4. N is **symplectic** if $\omega|_N$ is symplectic.

Definition 1.4.6. A vector field X on a symplectic manifold (M, ω) is called **Hamiltonian** if there exists a function $H_X \in C^\infty(M)$, called **Hamiltonian for** X, such that $\iota_X\omega = \mathrm{d}H_X$.[(38)]

Remark 1.4.1. Since ω is not degenerate, every function is Hamiltonian and its Hamiltonian vector field is uniquely determined.

The mathematical structure of classical mechanics in the Hamiltonian framework is characterized by the symplectic structure in the phase space.

Example 1.4.1. In classical mechanics, the dynamics of a system is encoded in the action functional

$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\mathrm{d}t$$

where $L(q, \dot{q}) = \frac{1}{2}m\|\dot{q}\|^2 - V(q)$ is called the Lagrangian function of the path $q \colon [t_0, t_1] \to \mathbb{R}^n$, with some function $V \in C^\infty(\mathbb{R}^n)$ depending on q, which is called the potential energy. Physically, this is a particle of mass m moving in a configuration space $\mathbb{R}^n$ where there is a potential V. Using variational calculus, it is possible to show that the stationary point of S on the space of paths $\mathrm{Path}_{(x,y)}^{[t_0,t_1]}(\mathbb{R}^n)$, such that $q(t_0) = x$ and $q(t_1) = y$, gives the classical trajectory of the system (see Principle 2.1.1). The resulting equations are called the Euler–Lagrange equations, and in the case where the Lagrangian is as above, i.e. that of a particle of mass m in a potential V, they are equivalent to Newton's equations $F = m\ddot{q}(t) = -\nabla V$ (cf. [Mor23]). Moving to the Hamiltonian picture means considering the space of all possible states of the system, called phase space. This is given by $T^*\mathbb{R}^n \ni (q, p)$, where $p = \frac{\partial L}{\partial \dot{q}}$.[(39)], and it is endowed with a symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

This results in a Legendre transformation of the Lagrangian into the Hamiltonian function $H(q, p) = \frac{\|p\|^2}{2m} + V(q)$. The trajectory of the system is a solution of the Hamilton equations, which are equivalent to the Euler–Lagrange equations (cf. [Mor23][Thm. 11.6]). It can be expressed as an integral curve of the Hamiltonian vector field associated with the Hamiltonian of the system. Indeed, if we define $X_H \coloneqq \sum_i \left(\partial_{p_i} H \partial_{q^i} - \partial_{q_i} H \partial_{p_i}\right)$, then

$$\iota_{X_H}\omega = \sum_{i=1}^{n} \left[(\iota_{X_H}\mathrm{d}q^i) \wedge \mathrm{d}p_i - \mathrm{d}q^i \wedge (\iota_{X_H}\mathrm{d}p_i)\right] = \sum_{i=1}^{n} \left(\partial_{p_i} H \mathrm{d}p_i + \partial_{q_i} H \mathrm{d}q^i\right) = \mathrm{d}H.$$

Then, the conditions for $\rho_t = (q(t), p(t))$ to be an integral curve for X_H are the Hamilton equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) = \partial_{p_i} H \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) = -\partial_{q_i} H.$$

For the system above, $X_H = \sum_i \left(\frac{p_i}{m}\partial_{q^i} - \partial_{q_i} V \partial_{p_i}\right)$.

A classical system described by an action functional as in the above example can be quantized via the path-integral formalism (see Sec. 2.2). If the system exhibits redundancies, called gauge symmetries, this formalism fails (see the discussion after Eq. (2.6)). As described in Section 2.2, we overcome this issue by enlarging the space of configurations (or states) to obtain a graded symplectic structure. This is the idea behind the BV formalism (see Def. 2.2.4 and the following discussion).

Graded Symplectic Geometry

Following Section 1.3, we generalize the above to the graded setting.

Definition 1.4.7. A $\mathbb{Z}_{(2)}$**-graded symplectic** form $\omega \in \Omega^2(\mathcal{M})$ of degree k on a $\mathbb{Z}_{(2)}$-graded manifold $\mathcal{M}$ is a closed ($\mathrm{d}\omega = 0$) non-degenerate 2-form

$$\omega \colon T\mathcal{M} \to T^*[k]\mathcal{M}, \tag{1.18}$$

(38) This structure can further be studied by considering the flow of X, called the Hamiltonian action (cf. [Sil01; CM20]).
(39) This defines a bundle map $TM \to T^*\mathbb{R}^n, (q, \dot{q}) \mapsto (q, p)$ over M, called Legendre transformation.

where non-degeneracy means requiring such a map to be bijective. In local coordinates $\{z^i\}$ it reads

$$\omega = \sum_{i,j} \mathrm{d}z^i \wedge \omega_{ij}(z) \wedge \mathrm{d}z^j$$

with $(\omega_{ij}(z))$ an invertible matrix of local functions on $\mathcal{M}$. The pair $(\mathcal{M}, \omega)$ is called $\mathbb{Z}_{(2)}$**-graded symplectic manifold**. In the infinite-dimensional case, the non-degeneracy requirement is too strong: we talk about **weakly symplectic structure** when the map in Equation (1.18) is only required to be injective.

Remark 1.4.2. The results and definitions summarized above for non-graded symplectic geometry can be straightforwardly extended to the graded setting. In particular, the extension of Proposition 1.4.1 implies that graded odd (even) symplectic $\mathbb{Z}_2$-graded vector space need to be of dimension $(m|m)$ for some $m \in \mathbb{N}$ ($(2k, n)$ for some $k, n \in \mathbb{N}$). Similarly, a Lagrangian submanifold of an odd-symplectic $(n|n)$-supermanifold has dimension $(k|n-k)$ for some $k \in \mathbb{N}$. Note that we define Hamiltonian vector fields using the sign convention for which $\iota_X \omega = \mathrm{d}H_X$.

The following result is crucial for justifying the AKSZ formalism, as it shows that any odd symplectic supermanifold can always be locally identified with the shifted cotangent bundle of a smooth manifold.

Theorem 1.1. *Any odd-symplectic supermanifold $(\mathcal{M}, \omega)$ admits a system of local Darboux coordinates (x^i, θ^i), i.e. such that pointwise $(\partial_{x^i}, \partial_{\theta^i})$ form a Darboux basis on the fibers of $T\mathcal{M}$. Furthermore, there exists a global symplectomorphism to $(\Pi T^* M, \omega_{\mathrm{can}} = \sum_i \mathrm{d}x^i \wedge \mathrm{d}\theta^i)$, where M is the body of $\mathcal{M}$.*

A proof can be found in [Sch93]. The graded structure imposes some additional constraints (cf. [Roy02]).

Proposition 1.4.2. *Let $(\mathcal{M}, \omega)$ be a k-symplectic $\mathbb{Z}_{(2)}$-graded manifold with $k \neq 0$. Then*

1. *ω is exact.*
2. *If $X \in \mathfrak{X}(\mathcal{M})_l$ such that $k + l \neq 0$ and $\mathbb{L}_X \omega = 0$, then X is Hamiltonian.*

Proof. To show the first statement, one computes

$$k\omega = \mathbb{L}_{\mathbb{E}}\omega = \mathrm{d}\iota_{\mathbb{E}}\omega \quad \Rightarrow \quad \omega = \frac{\mathrm{d}\iota_{\mathbb{E}}\omega}{k}$$

where $\mathbb{E}$ is the Euler vector field. For the second, by definition

$$[\mathbb{E}, X] = lX \quad \text{and} \quad \mathbb{L}_X \omega = (-1)^{|X|}\mathrm{d}\iota_X \omega = 0.$$

Set $H := \iota_{\mathbb{E}}\iota_X\omega$ and observe that

$$\mathrm{d}H = \mathrm{d}\iota_{\mathbb{E}}\iota_X\omega = \mathbb{L}_{\mathbb{E}}\iota_X\omega - \iota_{\mathbb{E}}\mathrm{d}\iota_X\omega = \iota_{[\mathbb{E},X]}\omega + \iota_X\mathbb{L}_{\mathbb{E}}\omega = (l+k)\iota_X\omega \quad \Rightarrow \quad \iota_X\omega = \frac{\mathrm{d}H}{k+l}.$$

Note that any Hamiltonian vector field satisfies $\mathbb{L}_{X_f}\omega = (-1)^{|X_f|}\mathrm{d}\iota_{X_f}\omega = (-1)^{|X_f|}\mathrm{d}^2 f = 0.$ ■

Observe that the exceptional cases are the relevant ones. Indeed, in application to the BFV formalism, one considers an even symplectic form of degree 0, while in the BV formalism, one considers an odd symplectic form of degree -1 and a cohomological vector field Q of degree 1. Finally, we give an explicit definition of the graded Poisson bracket induced by a graded symplectic form and state a proposition that is a direct consequence of this structure.

Definition 1.4.8. Consider an k-symplectic $\mathbb{Z}_{(2)}$-graded manifold $(\mathcal{M}, \omega)$. The **graded Poisson bracket** $\{-,-\}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ induced by ω via

$$\{f, g\} := (-1)^{\deg_{\mathbb{Z}_{(2)}}(f)} X_f(g) = (-1)^{\deg_{\mathbb{Z}_{(2)}}(X_f)+k \pmod 2} \iota_{X_f}\iota_{X_g}\omega \quad \forall f, g \in C^\infty(\mathcal{M})$$

where X_f is the Hamiltonian vector field associated with f.
It follows that $|\{-,-\}| = \deg_{\mathbb{Z}_{(2)}}(\{-,-\}) = -k$. Finally, it satisfies

$$\begin{aligned} \{f, g\} &= -(-1)^{(\deg_{\mathbb{Z}_{(2)}}(f)+k)(\deg_{\mathbb{Z}_{(2)}}(g)+k)}\{g, f\} \\ \{f, \{g, h\}\} &= \{\{f, g\}, h\} + (-1)^{(\deg_{\mathbb{Z}_{(2)}}(f)+k)(\deg_{\mathbb{Z}_{(2)}}(g)+k)}\{g, \{f, h\}\} \end{aligned} \tag{1.19}$$

Proposition 1.4.3. *Let $(\mathcal{M},\omega)$ be a k-symplectic $\mathbb{Z}_{(2)}$-graded manifold and $f,g \in C^\infty(\mathcal{M})$. Then $\{f,g\} = \mathrm{const}$ if and only if $[X_f, X_g] = 0$. This implies that, considering a cohomological vector field $Q \in \mathfrak{X}(\mathcal{M})_1$ and its Hamiltonian $S \in C^\infty(\mathcal{M})_{k+1}$ such that $Q = (-1)^{|S|}\{S,-\}$, if $k \neq -2$ then*

$$Q^2 = 0 \Leftrightarrow \{S,S\} = 0.$$

Proof. The first statement is a consequence of Equation (1.19) and of the definition of the Hamiltonian vector field. Indeed,

$$\begin{aligned} X_{\{f,g\}} &= (-1)^{\deg_{\mathbb{Z}_{(2)}}(\{f,g\})}(\{\{f,g\},\cdot\}) = (-1)^{\deg_{\mathbb{Z}_{(2)}}(f)+\deg_{\mathbb{Z}_{(2)}}(g)+k}\Big(\{f,\{g,\cdot\}\}+ \\ &\quad - (-1)^{(\deg_{\mathbb{Z}_{(2)}}(f)+k)(\deg_{\mathbb{Z}_{(2)}}(g)+k)}\{g,\{f,\cdot\}\}\Big) = \\ &= (-1)^k\Big(X_fX_g - (-1)^{(\deg_{\mathbb{Z}_{(2)}}(f)+k)(\deg_{\mathbb{Z}_{(2)}}(g)+k)}X_gX_f\Big) = (-1)^k[X_f,X_g]. \end{aligned} \tag{1.20}$$

We conclude using Equation (1.19) and Equation (1.20).
Indeed, assuming $\{S,S\} = 0$ and using that $Q^2(f) = (-1)^{|S|}\{S,(-1)^{|S|}\{S,f\}\} = \{S,\{S,f\}\}$ for any $f \in C^\infty(\mathcal{M})$, we get

$$Q^2(f) = \{S,\{S,f\}\} \overset{(1.19)}{=} \Big(\{\{S,S\},f\} + (-1)^{(2k+1)(2k+1)}\{S,\{S,f\}\}\Big) = -\{S,\{S,f\}\} = 0.$$

Assuming instead $Q^2 = \frac{1}{2}[Q,Q] = 0$, we get

$$0 = Q^2(f) = \frac{1}{2}[Q,Q] \overset{(1.20)}{=} \frac{(-1)^k}{2}X_{\{S,S\}},$$

for any $f \in C^\infty(\mathcal{M})$. By the definition of a Hamiltonian vector field, this implies $\{S,S\} = \mathrm{const}$. Since constants are of degree 0, $k \neq -2$ implies $\{S,S\} = 0$ by noticing that $\deg_{\mathbb{Z}_{(2)}}\{S,S\} = 2(k+1) - k = k+2$. ∎

Chapter 2

Bosonic Gauge Field Theories

In this chapter, we introduce classical Lagrangian bosonic field theories. We hint at the idea behind path-integral quantization and explain the problems that arise when the theory exhibits redundancy in the Lagrangian, known as gauge symmetries. Consequently, we present three methods for path-integral quantization of gauge theories in the finite-dimensional setting: the Faddeev–Popov method, the BRST formalism, and the BV formalism. Finally, we present the AKSZ formalism, a systematic way to obtain topological field theories compatible with the BV formalism. As an example, we briefly discuss (non)-abelian BF theory. The main reference for this chapter is [Mne19][Ch. 4].

2.0.1 Conventions and Notation

For convenience, we recall here all the relevant conventions and notation for Chapters 2 and 3 (see Sec. 3.3, Sec. 3.4 and Rmk. 3.4.1). Some of these conventions are discussed in Chapter 1. Note that some of the references we refer to may use different conventions.

- The total degree is denoted by $|\cdot| = \deg_{\mathbb{Z}_{(2)}}(\cdot) + \deg_{\Omega}(\cdot)$, where the second is the cohomological degree (see Rmk. 1.3.6). This grading governs the Koszul sign, i.e. the parity is given by $|\cdot| \bmod 2$.
- The derivations of the Cartan calculus on a graded manifold have total degree $|\mathrm{d}| = 1$, $|\iota_X| = |X|-1$, and $|\mathbb{L}_X| = |X|$ for some graded vector field X (see ft. (22), p. 14). Moreover, $\mathbb{L}_X \coloneqq [\iota_X, \mathrm{d}]$ (see Prop. 1.1).
- Unless parentheses are present, derivations act on the first element to their right.
- The k-shift of a graded vector space V ($k \in \mathbb{Z}$) is the graded vector space $V[k]$ such that $(V[k])_n \coloneqq (V)_{n+k \pmod 2}$, for any $n \in \mathbb{Z}_{(2)}$ (see Def. 1.3.4). For any vector space V, the parity-shifted superspace ΠV corresponds to the 1-shifted superspace $V[1]$. On graded vector bundles, the shift only acts on the fibers.
- Given a graded symplectic manifold $(\mathcal{M}, \omega)$ of degree k, the Hamiltonian vector field of a function $H_X \in C^\infty(\mathcal{M})$ is defined by $\iota_X\omega = \mathrm{d}H_X$. The induced graded Poisson bracket is given by

$$\{f, g\} \coloneqq (-1)^{\deg_{\mathbb{Z}_{(2)}}(f)} X_f(g) = (-1)^{\deg_{\mathbb{Z}_{(2)}}(X_f)+k \pmod 2} \iota_{X_f}\iota_{X_g}\omega \quad \forall f, g \in C^\infty(\mathcal{M})$$

  where $X_{f(g)}$ is the Hamiltonian vector field associated with $f(g)$. It follows that $|\{-,-\}| = -k$.
- We denote the lifted vector fields from $\mathcal{M}$ and $\mathcal{N}$ to $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ both by lift. More explicitly, recalling that for any $f \in \mathrm{Map}(\mathcal{M}, \mathcal{N})$ the tangent space is $T_f\mathrm{Map}(\mathcal{M}, \mathcal{N}) \cong \Gamma_{\mathcal{M}}(f^*T\mathcal{N})$, we get for all $D \in \mathfrak{X}(\mathcal{M})$ and $P \in \mathfrak{X}(\mathcal{N})$

$$D^{\mathrm{lift}}_f(x) \coloneqq \mathrm{d}_{\mathcal{M}} f(x)(D_x) \quad \text{and} \quad P^{\mathrm{lift}}_f(x) \coloneqq P(f(x))$$

  with $|D| = |D^{\mathrm{lift}}|$, $|P| = |P^{\mathrm{lift}}|$ and $[D^{\mathrm{lift}}, Q^{\mathrm{lift}}] = 0$.
- When used, Einstein summation is understood for repeated upper and lower indices $x^i y_i = \sum_i x^i y_i$.

- For an n-tuple of variables $x = (x_1, \dots, x_n)$ and a multi-index $I \in \mathbb{N}^n$, we write $x^I = x_1^{I_1} \cdots x_n^{I_n}$ and $\partial_x^I = (\partial_{x_1})^{I_1} \cdots (\partial_{x_n})^{I_n}$.
- Unless otherwise specified, M is an oriented closed d-manifold. In local coordinates, we write $\mathrm{d}_H = \mathrm{d}x^\nu \partial_{x^\nu}$ and, for some k-form ω,

$$\mathrm{d}_H \omega = \sum_{1 \leqslant \nu, \mu_1, \dots, \mu_k \leqslant d} \mathrm{d}x^\nu \wedge \frac{\partial_{[\nu}\omega_{\mu_1,\dots,\mu_k]}}{k!} \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}.$$

Here, the anticommutation of the indices has a factor of $(k+1)!$ at the denominator.

2.1 Lagrangian Field Theories

In this section, we recall the Lagrangian approach to bosonic classical field theories and the classical action principle (cf. [Blo24]). The absence of a rigorous formalism motivates the structural requirements that the category of SmthSet must satisfy to rigorously formalize the geometry of the action principle.

Classically, the state of a system can be described by assigning the value of some physical observable to any point of a geometric space, typically spacetime. This assignment is supposed to be smooth.

Definition 2.1.1. Let M be a smooth oriented compact d-manifold without boundary[(1)] called **spacetime**. A **bosonic field** is a smooth section of a smooth fiber bundle (see Rmk. 1.3.4) $\pi_M^F \colon F \to M$, called **configuration bundle**. The **space of fields** is the space of sections $\mathcal{F} \coloneqq \Gamma_M(F)$. [(2)]

The simplest example is the space of scalar fields $\mathcal{F} = C^\infty(M)$ where the fields are sections of the trivial fiber bundle $F \coloneqq M \times \mathbb{R} \to M$.
Physical laws are usually expressed as field equations of the form $f(\varphi) = 0$, where $f \colon \mathcal{F} \to V$ is a map to a vector space V that is typically a differential operator. The solutions of the field equations are called on-shell fields, and their set is denoted $\mathcal{F}_{\text{shell}} \subset \mathcal{F}$. $\mathcal{F}_{\text{shell}}$ is usually complicated to study: the action principle is particularly helpful in characterizing it.

Principle 2.1.1. *There is a smooth function $S \colon \mathcal{F} \to \mathbb{R}$, called the **action**, such that $\varphi \in \mathcal{F}$ is a solution of the field equation if and only if it is a critical point of S.*

The action allows us to study the field theory more easily, also exploiting the symmetries of the system that are manifest as symmetries of the action. Physical action functionals are usually given in an integral form:[(3)]

$$S(\varphi) \coloneqq \int_M \mathcal{L}(\varphi) \tag{2.1}$$

where the Lagrangian $\mathcal{L}$ is defined as follows.

Definition 2.1.2. A **local Lagrangian** $\mathcal{L} \colon \mathcal{F} \to \Omega^d(M)$ is a smooth map of sections from the field space to the space of d-differential forms on M (of dimension d), such that $\mathcal{L}(\varphi)$ at $m \in M$ depends smoothly on m and on the partial derivatives of φ only up to order k for some $k \geqslant 0$.[(4)]

Note that the degree of the Lagrangian form has to match the dimension of the manifold $\dim(M) = d$ for $S(\varphi)$ not to be trivially zero. The dynamics of a physical theory is characterized by the Lagrangian and the type of fields we are considering in it.

[(1)] Unless otherwise specified, in these notes M is always assumed to have these properties. If the manifold is non-compact the definitions involving an integral over M should be adjusted to only allow for integration of compactly supported differential top forms, or more generally compactly supported densities (cf. [Cat18]).

[(2)] This definition should be extended to account for gauge fields defined as connection 1-forms on a principal G-bundle, fermionic fields, and fields as elements in more general mapping spaces (cf. [Mne19; Ham17]). In this thesis, we only consider bosonic gauge theories whose gauge structure is described by trivial principal bundles (see Rmk. 2.1.1). Furthermore, we do not consider fermionic theories with fields that are sections of a spinor bundle. Nevertheless, odd fields might still arise from the gauge-fixing formalism we introduce in Section 1.2.

[(3)] When the base manifold M is compact, this integral is well defined. If M is not compact, this integral is usually divergent or not well defined (cf. [Blo24]), but in practice one does not need to compute it to characterize the on-shell field space.

[(4)] The locality condition, here stated in a local chart, can be globally characterized using the infinite jet bundle (see Def. 3.1.12 and Def. 3.1.21). It is necessary to ensure that the critical points are solutions of a PDE. Physically, this is equivalent to asking the Lagrangian not to include non-local interactions.

Definition 2.1.3. A **Lagrangian field theory** (LFT) consists of a smooth bundle $F \to M$ and a local Lagrangian $\mathcal{L}\colon \mathcal{F} \to \Omega^d(M)$.

Examples of the Lagrangian formulation of bosonic theories, such as a particle in a potential and of classical electromagnetism, can be found in [Blo24, Ch. 1].
Using the locality of the Lagrangian, the action principle translates to:

$$\varphi \in \mathcal{F}_{\text{shell}} \ \text{ iff } \ \delta S|_{\varphi} = \int_M \langle \mathcal{EL}(\varphi), \delta\varphi \rangle = 0.$$

Here $\mathcal{EL}$ is the Euler–Lagrange differential operator symbolically given in local coordinates by

$$\mathcal{EL}(\varphi) = \sum_{|I|=0}^{\infty} (-1)^{|I|} \partial_I \left(\frac{\delta \mathcal{L}(\varphi)}{\delta(\partial_I \varphi)} \right),$$

where I is the usual multi-index.[(5)] Globally, this is a map of sections

$$\mathcal{EL} \coloneqq \mathcal{F} \longrightarrow \Gamma_M(V^*F \otimes \wedge^d T^*M),$$

where $V^*F \to F$ is the dual vertical bundle[(6)] The infinitesimal variation $\delta\varphi$ is identified as a section $\delta\varphi \in \Gamma_M(VF)$ to account only for the field variation along a fiber while the base manifold is kept fixed. The natural pairing $\langle -, - \rangle$ is the one induced by the fiber-wise non-degenerate duality pairing of VF with V^*F. Thus, φ is critical if and only if it solves the following equations, called Euler–Lagrange equations:

$$\mathcal{EL}(\varphi) = 0_{\varphi} \in \Gamma_M(V^*F \otimes \wedge^d T^*M).$$

The latter is the field equation mentioned above, and the shell space can be thought of as an intersection, i.e. as the following pullback in some abstract category:

$$\begin{tikzcd}
\mathcal{F}_{\text{shell}} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, "\mathcal{EL}"] \\
\mathcal{F} \arrow[r, "0_{\mathcal{F}}"] & T^*_{\text{var}}\mathcal{F}.
\end{tikzcd}$$

Here $T^*_{\text{var}}\mathcal{F} \coloneqq \Gamma_M(V^*F \otimes \wedge^d T^*M) \to \mathcal{F}$ is the bundle of variational densities, not to be confused with the cotangent bundle of $\mathcal{F}$.
The above formulas and statements are not fully rigorous. Indeed, these results hold in the case where all the spaces involved are nice smooth manifolds; however, $\mathcal{F}$ and $\mathcal{F}_{\text{shell}}$ typically do not belong to SmthMfd. Considering the simple case of scalar fields over a compact smooth manifold M, i.e. taking $F = M \times \mathbb{R}$, $\mathcal{F}$ is a Fréchet manifold (see Def. 3.1.2) and $\mathcal{F}_{\text{shell}} = f^{-1}(0)$ is generally even more complicated. Moreover, physically we are usually interested in non-compact spacetime, and in this case $\mathcal{F}$ is not even a Fréchet manifold. Finally, the action principle, as stated in Principle 2.1.1, is not rigorously true (cf. [Blo24, Ch. 1.3]).
We aim to make sense of the above in terms of "smooth" spaces and maps within an appropriate category. The following are some requirements that the category of such generalized smooth spaces must satisfy.

1. Smooth manifolds and field-theoretical spaces such as the bundle F, the field space $\mathcal{F}$ or the d-dimensional differential forms $\Omega^d(M) = \Gamma_M(\wedge^d T^*M)$, should be objects of the category having therefore a comparable smooth structure.[(7)] To encode locality in global terms, the infinite jet bundle $J^\infty_M F$ and its space of sections should also be objects in it (see Ch. 3).

2. The Lagrangian, the integral operation, and therefore the action, should be smooth maps in the category. For the same reason as above, infinite jet prolongation $j^\infty\colon \Gamma_M(F) \to \Gamma_M(J^\infty_M F)$ should be smooth.

(5) For simplicity of notation, we are assuming a simple scalar theory, i.e. $\varphi \in C^\infty(M)$. Note that the infinite sum and the functional derivative are used here informally and would require a rigorous definition.
(6) Let $\pi^{TF}_F\colon TF \to F$ be the tangent bundle of a fiber bundle $\pi^F_M\colon F \to M$. The **vertical** subbundle $TF \supset VF = \ker(\mathrm{d}\pi^F_M)$, where $\mathrm{d}\pi^F_M\colon TF \to TM$, consists of the vectors tangent to the fibers of F.
(7) As recalled later, this is one of the advantages in considering the category of all sheaves SmthSet and not only the concrete ones, i.e. the category **DflSp** (cf. [Blo24]).

3. The category must include a notion of smooth path of fields $\varphi_t\colon \mathbb{R} \to \mathcal{F}$, so that compositions like $S \circ \varphi_t$ are smooth in the classical sense.[(8)] The infinitesimal variation of the action functional at $\varphi = \varphi_0$ should then be rigorously defined via the usual derivative. Consequently, the action principle should be derived via the former so that

$$\partial_t(S \circ \varphi_t)|_{t=0} = \int \langle \mathcal{EL}(\varphi), \dot{\varphi}_0 \rangle, \tag{2.2}$$

where $\dot{\varphi}_0 = \partial_t \varphi_t|_{t=0} \in T_\varphi \mathcal{F} \to \Gamma_M(VF)$ defines the infinitesimal variation at $\varphi \in \mathcal{F}$ as the corresponding vertical tangent vector over it. Note that the variation formula (2.2) is only formal and becomes rigorous once $\Gamma_M(F)$ is interpreted as a smooth set and the integral as a natural transformation.

4. $\mathcal{F}_{\text{shell}}$ should inherit a natural smooth subspace structure, i.e. the category should have pullbacks.[(9)]

In Section 3.1, we introduce the category of smooth sets and demonstrate that it meets these demands.

Remark 2.1.1. Due to Principle 2.1.1, the dynamics of a theory depends only on the action. Therefore, any symmetry of the latter leaves the physics unchanged. These symmetries are local (in the sense of Definition 2.1.2) maps $\mathcal{D}\colon \mathcal{F} \to \mathcal{F}$ such that $\mathcal{D}^*S = S$, or such that $\mathcal{D} \circ \mathcal{L} = \mathcal{L} + \mathrm{d}_M \mathcal{K}$ for some local map $\mathcal{K}\colon \mathcal{F} \to \Omega^{d-1}(M)$. Some symmetries arise from physical principles, such as Lorentz symmetry, but others are redundancies needed to formulate certain theories, such as Yang–Mills theory. We call the latter **gauge symmetries**. In an important class of examples, they are described by the action of a Lie group G on the space of fields. To be fully rigorous, we should extend Definition 2.1.1 to consider gauge fields, i.e. connection 1-forms on principal G-bundles (cf. [Ham17]). In this setting, the gauge symmetries are intrinsically described in the geometry of the principal G-bundle, as they correspond to principal bundle automorphisms. They form a group and look locally like $\theta_\alpha\colon M \ni U_\alpha \to G$. For simplicity, we only consider trivial principal G-bundles, meaning that the approach we are using is only local (perturbative) for more general gauge theories. This greatly simplifies the definitions and computations. Indeed, consider a compact Lie group G and the trivial principal G-bundle $G \times M \to M$. Then, 1-form connections on it correspond to Lie algebra valued 1-forms over M

$$\Omega^1(M) \otimes \mathfrak{g} \cong \Gamma_M(T^*M \otimes (M \times \mathfrak{g})),$$

falling within Definition 2.1.1. Then the gauge transformations are (cf. [Ham17][Thm. 5.4.2 & Thm. 5.4.4])

$$A \mapsto g^{-1}Ag + g^{-1}\mathrm{d}_M g,$$

where $\mathrm{Gauge}_{M,G} \coloneqq C^\infty(M, G) \ni g\colon M \to G$. Locally, they take the form

$$A \mapsto \mathrm{d}_A \alpha \coloneqq \mathrm{d}_M \alpha + [A, \alpha],$$

where $\mathrm{gauge}_{M,G} \coloneqq \Gamma_M(M \times \mathfrak{g}) \ni \alpha\colon M \to \mathfrak{g}$.
This assumption can be relaxed in the classical setting by considering more general principal G-bundles. However, to formulate a categorical description of gauge theories and the BV formalism, one should extend SmthSet to the category of sheaves of groupoids (cf. [GS25a][Ch. 8]), which falls outside the scope of this thesis (see Rmk. 3.4.2). Therefore, we already restrict ourselves to this simplified local setting.

2.2 Faddeev–Popov, BRST and BV Formalisms

In this section, we recall the path-integral quantization, the problems that arise when the theory exhibits additional symmetries of the Lagrangian, and three methods to address them. We focus on the finite-dimensional case as in [Mne19][Ch. 4]. The rise of anti-commuting fields highlights the need to work in a super extension of SmthSet. Note that the infinite-dimensional BV classical formalism relies on a rigorous understanding of the infinite-dimensional Cartan calculus on the space of fields, which is addressed in Chapter 3.

(8) This assumes that the Cartesian space $\mathbb{R}^n$, for all $n \in \mathbb{N}$, should also be viewed as a generalized smooth space.
(9) This is guaranteed for SmthSet since limits exist and are computed objectwise (see Prop. 1.2.1).

Remark 2.2.1. The three formalisms we are presenting are rigorously defined in the finite-dimensional case. The discussion is more delicate in the infinite-dimensional case, which is of interest for field-theoretical applications. In particular, the classical BV formalism relies on the differential calculus on the space of fields. On top of this, the quantum BV formalism requires a well-defined integration on the space of fields. The category we introduce in the next chapter allows us to rigorously define the relevant notions of differential geometry on the space of fields, in particular, the local Cartan calculus and the relative variational bicomplex. This is enough to attempt a categorical description of the local BV formalism. As there is no notion of measure on the infinite-dimensional space of fields, we should make a formal sense of it by the perturbative expansion in the limit $\hbar \to 0$ (cf. [CMS25] and see Eq. (2.6)). Similarly, the infinite-dimensional classical BRST formalism requires the notion of Cartan calculus on the space of fields, while the infinite-dimensional Faddeev–Popov method and quantum BRST formalism rely on a notion of measure on it. For clarity, we specify that the following parts of this section are based on infinite-dimensional heuristic arguments of differential-geometry or measure theory:

- Equation (2.4), Equation (2.5) and the discussion following Equation (2.6). Indeed, while the latter rigorously recall the stationary phase formula for the finite-dimensional case, the discussion later heuristically extends it to the infinite-dimensional case;
- Remark 2.2.2;
- Definition 2.2.1 and Definition 2.2.3, unless $\mathcal{F}$ is finite dimensional. Consequently, Remark 2.2.3 is fully rigorous only when $\mathcal{F}$ is finite-dimensional;
- Definition 2.2.4, Remark 2.2.5, Definition 2.2.5 and following discussion, Definition 2.2.6, Theorem 2.1, Remark 2.2.6 and Definition 2.2.7, unless $\mathcal{F}$ is finite dimensional.

Path Integral Quantization

The first axiomatic formulation of quantum mechanics was developed by Dirac (cf. [Dir30]) and von Neumann (cf. [Von32]) in analogy with classical Hamiltonian mechanics. In [Dir33], Dirac suggested an alternative Lagrangian approach of quantum theory, later developed by Feynman (cf. [Fey48]) via the path integral formulation. The idea was to generalize the concept of probability amplitude to a completely specified trajectory. Consequently, exploiting the superposition of the probability amplitudes, Feynman described the dynamics of the quantum particle as a sum over the classical paths, each weighted by an exponential whose imaginary phase was the classical action of that path in units of $\hbar$, the parameter of quantization. More specifically, referring to Example 1.4.1, the canonical quantization of the system is obtained by taking as the space of states the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable functions, and as observables the operators on it. In particular, the canonical quantization of the classical Hamiltonian is the operator $\hat{H} \coloneqq -\frac{\hbar^2}{2m}\Delta + V$ and the exponential $\exp(\frac{i}{\hbar}\hat{H})$ governs the dynamical evolution of the system. We define the quantum mechanical propagator $K(x,y)$ by

$$\left(e^{\frac{i}{\hbar}\hat{H}}\psi\right)(x) = \int K(x,y)\psi(y)dy,$$

where $\psi \in L^2(\mathbb{R}^n)$. This can be represented as a path integral

$$K(x,y) = \int_{\mathrm{Path}^{[t_0,t_1]}_{(x,y)}(\mathbb{R}^n)} e^{\frac{i}{\hbar}S(q)}\mathcal{D}q, \tag{2.3}$$

where $\mathcal{D}q$ is a symbol for the reference measure on $\mathrm{Path}^{[t_0,t_1]}_{(x,y)}(\mathbb{R}^n)$. This approach can also be used to quantize classical field theories. The idea is to build the partition function of the theory over a fixed spacetime M as

$$Z(\hbar) \coloneqq \int_{\mathcal{F}} e^{\frac{i}{\hbar}S(\varphi)}\mathcal{D}\varphi. \tag{2.4}$$

Here $\mathcal{D}\varphi$ is a symbol for the reference measure on the space of fields $\mathcal{F}$ and S is the action of the theory (see Eq. (2.1)). Then, given a collection of observables $\mathcal{O}_1, \dots, \mathcal{O}_N \in C^\infty(\mathcal{F})$, the expectation value is given by

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle \coloneqq \frac{1}{Z}\int_{\mathcal{F}} \mathcal{O}_1 \cdots \mathcal{O}_N e^{\frac{i}{\hbar}S(\varphi)}\mathcal{D}\varphi. \tag{2.5}$$

Unfortunately, the above equations (see Eq. (2.3), Eq. (2.4), and Eq. (2.5)) are ill-defined from a measure-theoretic point of view (cf. [Maz09]). We can still make sense of the path integral through its perturbative expansion in formal power series. The latter are asymptotic series in $\hbar \to 0$ whose coefficients are Feynman diagrams. To understand how this is done, let us consider the finite-dimensional case. Consider a manifold X without boundary, a fixed volume form μ_X on it, and a function $f \in C_c^\infty(X)$. The stationary phase formula (cf. [Mne19])

$$\int_X \mu_X e^{\frac{i}{\hbar}f(x)} \underset{\hbar\to 0}{\sim} \sum_{x_0\in\{x\in X\,|\,\mathrm{d}f(x)=0\}} e^{\frac{i}{\hbar}f(x_0)}\left|\det\big(f''(x_0)\big)\right|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\mathrm{sign}(f''(x_0))}(2\pi\hbar)^{\frac{\dim X}{2}} \tag{2.6}$$

is an asymptotical ($\hbar \to 0$) way of computing the value of the finite-dimensional oscillating integral on the left-hand side. The idea is that the rapid change in the phase of the integrand when $\hbar \to 0$ cancels out all contributions but those around a critical point of f, where the oscillations slow down. Following Principle 2.1.1, these are the classical solutions of the theory described by the action f. This is the reason why we can understand the path integral as a way to interpret the semiclassical limit. The formula can be improved considering corrections in powers of $\hbar$, given diagrammatically in terms of Feynman graphs (cf. [Mne19][Ch. 3]). Then, we define the path integral as such a series in $\hbar$, which means that we expand perturbatively in $\hbar \to 0$ around a critical point of f. Crucially, the stationary phase formula requires critical points to be isolated. This guarantees that the Hessian at a critical point is non-degenerate. However, if the theory is a gauge theory, there exists a tangential distribution[(10)] $\mathcal{G} \subset TX$ which preserves the action. In the simple case of Remark 2.1.1, this is described by the local action of gauge$_{M,G}$, but in more general cases it could be given by Lie algebroid (see Def. 2.2.2) or by a non-integrable distribution. Then, critical points of f come in $\mathcal{G}$-orbits and therefore they are not isolated. That is, the Hessian is degenerate along $\mathcal{G}$-orbits. To apply the stationary phase formula in this case, we must first address this issue. The idea is that, since the gauge symmetry is a redundancy of the theory, this can be solved by fixing the gauge parameter consistently with the path integral quantization. Naively, we could think of doing so by working on the quotient $\mathcal{F}/G$, but $\mathcal{F}/G$ might not be smooth, that is, a space of sections of a fiber bundle. Faddeev–Popov method, BRST formalism, and BV formalism are gauge-fixing methods compatible with path-integral quantization for theories with gauge symmetry respectively given by a Lie group, a Lie algebroid, or a non-integrable distribution. The advantage of the last is that it subsumes the other two, it can be used in more general cases, and it can be extended to the case of manifolds with boundaries (cf. [CM20; CMR14; CMR18]).

Faddeev–Popov method

This method was first developed in [FP67] to solve the problem of degeneracy of critical points of the Yang–Mills action. We recall the finite-dimensional case. Let G be a finite-dimensional compact connected semisimple Lie group of dimension m acting smoothly and freely[(11)] on a closed oriented smooth n-manifold X. The action is given by the map $\gamma\colon G \times X \to X$, $(g, x) \mapsto \gamma_g(x)$. Denote by $\{T_a\}$ the basis of the Lie algebra $\mathfrak{g}$ of G, and by $v_a \in \mathfrak{X}(X)$ the fundamental vector fields defined by

$$v_{a,x} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\exp(-tT_a)\cdot x\big), \quad \forall x \in X. \tag{2.7}$$

Note that by definition and by the Lie correspondence theorem, the Lie algebra is also finite-dimensional and semisimple. Let $S \in C^\infty(X)^G$ and $\mu_X \in \Omega_c^n(X)^G$ be G-invariant, that is, $\gamma_g^* S = S$ and $\gamma_g^*\mu_X = \mu_X$ for any $g \in G$. Then, considering $p\colon X \to X/G$[(12)], it follows that

$$I = \int_X e^{\frac{i}{\hbar}S}\mu_X = \mathrm{Vol}(G)\int_{X/G} e^{\frac{i}{\hbar}\tilde{S}}\tilde{\mu}_X.$$

Here, $\tilde{S} \in C^\infty(X/G)$ and $\tilde{\mu}_X \in \Omega^{n-m}(X/G)$ are defined by

$$S =: p^*\tilde{S} \quad \text{and} \quad \mu_X =: p^*\tilde{\mu}_X \wedge \chi,$$

(10) A **distribution** is a smooth assignment of a vector subspace of the tangent space to each point of the manifold $x \mapsto \Delta_x \subset T_xX$. It is said to be **integrable** if, through any point $x \in X$, there is an integral manifold, that is, a submanifold $N \subset X$ such that $T_xN = \Delta_x$ for any $x \in N$.

(11) We refer to [Kir08; Ham17] for a complete exposition of Lie groups and Lie algebras, in which we assume the reader to have basic knowledge.

(12) Note that the quotient X/G and the map p are respectively a smooth $(n-m)$-manifold and a submersion. This is due to the action being free and proper (cf. [Ham17][Cor. 3.7.30]), the latter being a consequence of the compactness of the Lie group (cf. [Ham17][Prop. 3.7.22]).

where $\chi \in \Omega^m(X)$ satisfies $\iota_{v_m} \cdots \iota_{v_1}\chi = 1$ and, when restricted to G-orbits in X, yields the volume form on the orbits induced by the Haar measure on G.
Let the **gauge-fixing function** $\phi\colon X \to \mathfrak{g}$ be smooth and such that:

1. $0 \in \mathfrak{g}$ is a regular value of ϕ, that is, it is not the image of a critical point of ϕ;
2. $\sigma := \phi^{-1}(0) \subset X$ intersects every G-orbit transversally exactly N times for some $N \geqslant 1$.(13)

Then, using the distributional m-form $\delta^{(m)}(\phi) := \phi^*\delta_D \cdot \bigwedge_a \mathrm{d}\phi^a$ supported at σ, we get(14)

$$I = \frac{\mathrm{Vol}(G)}{N}\int_\sigma e^{\frac{i}{\hbar}S|_\sigma} p^*\tilde{\mu}_X = \frac{\mathrm{Vol}(G)}{N}\int_X \left(e^{\frac{i}{\hbar}S}\cdot \delta^{(m)}(\phi)\right) p^*\tilde{\mu}_X =: \frac{\mathrm{Vol}(G)}{N}\int_X \left(e^{\frac{i}{\hbar}S}\cdot \phi^*\delta_D \cdot J\right)\mu_X,$$

where the latter is taken as a definition of $J \in C^\infty(X)$.

Proposition 2.2.1. *We can express the coefficient $J \in C^\infty(X)$ by the **Faddeev–Popov determinant***

$$J(x) = \det_{\mathfrak{g}}(FP(x)),$$

where $FP(x) = \mathrm{d}_x\phi \circ \mathrm{d}_{1,x}\gamma\colon \mathfrak{g} \to \mathfrak{g}$, with $\mathrm{d}_{1,x}\gamma$ the infinitesimal action of G on X viewed as a derivative of the group action γ, and the determinant is calculated in the adjoint representation of $\mathfrak{g}$. In components, we have

$$FP(x)^a_b = \langle \mathrm{d}\phi(x)^a, v_b(x)\rangle = v_b(\phi^a)\big|_x.$$

where the pairing is the dual pairing of T^*_xX and T_xX. This implies that

$$I = \int_X e^{\frac{i}{\hbar}}\mu_X = \frac{\mathrm{Vol}(G)}{N}\int_X \left(e^{\frac{i}{\hbar}S}\cdot \phi^*\delta_D \cdot \det_{\mathfrak{g}}(FP(x))\right)\mu_X.$$

A proof can be found in [Mne19][Lemma 4.1.2]. Note that the transversality condition is essential to ensure non-degeneracy of the FP matrix, which physically means that there is no residual gauge freedom left.
Then, we introduce auxiliary integration variables $\lambda \in \mathfrak{g}^*$, $c \in \Pi\mathfrak{g}$ and $\bar{c} \in \Pi\mathfrak{g}^*$, called respectively **Lagrange multiplier variable**, **ghost**, and **antighost**. By standard Fourier analysis and using the property of the graded integration recalled in Section 1.3, the following hold

$$\delta_D(\phi(x)) = \frac{1}{(2\pi\hbar)^m}\int_{\mathfrak{g}^*} e^{\frac{i}{\hbar}\langle\lambda,\phi(x)\rangle}\mathrm{d}^m\lambda,$$
$$\det_{\mathfrak{g}}(FP(x)) = \left(\frac{\hbar}{i}\right)^m \int_{\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} e^{\frac{i}{\hbar}\langle\bar{c},FP(x)c\rangle}\mathrm{d}^m c\cdot \mathrm{d}^m\bar{c},$$

where the pairing is the dual pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$. Using the above, we obtain

$$I = \int_X e^{\frac{i}{\hbar}S}\mu_X = \frac{\mathrm{Vol}(G)}{N(2\pi i)^m}\int_{X\times\mathfrak{g}\times\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} e^{\frac{i}{\hbar}S_{FP}(x,\lambda,c,\bar{c})}\mu_X\cdot \mathrm{d}^m\lambda\cdot\mathrm{d}^m c\cdot \mathrm{d}^m\bar{c} \tag{2.8}$$
$$\text{with}\quad S_{FP}(x,\lambda,c,\bar{c}) := S(x) + \langle\lambda,\phi(x)\rangle + \langle\bar{c},FP(x)c\rangle.$$

The right-hand side of the latter has isolated critical points. As a consequence, the Hessian of the extended Faddeev–Popov action S_{FP} is non-degenerate and the stationary phase formula (see Eq. (2.6)) is applicable (cf. [Mne19][Thm. 4.1.5]). This gives a perturbative solution around a classical field configuration solving the extended Faddeev–Popov Euler–Lagrange equations

$$c = \bar{c} = 0, \quad \partial_{x_i}S(x) + \langle\lambda,\partial_{x_i}\phi\rangle = 0, \quad \phi(x) = 0.$$

Remark 2.2.2. In the infinite-dimensional case, the odd variables $c, \bar{c}$ are sections of a $\mathbb{Z}_{(2)}$-graded vector bundle $\mathcal{Y} \to M$ with purely odd fiber V_{odd} (see Def. 1.3.18). To simultaneously consider bosonic and ghost fields in a classical field theory, we should build the composite bundle $\mathcal{Y} \to F \to M$ as in Definition 3.3.3 (cf. [GSM09]). In this case, the integral should be understood heuristically as an asymptotic series in $\hbar \to 0$.

(13) σ can be understood as the image of a local section of the principal bundle $p\colon X \to X/G$. Thanks to transversality, every orbit sufficiently close to the base point intersects σ in exactly one point. In this sense, ϕ gives a way of gauge fixing, that is, to locally pick a unique representative of the orbit.
(14) We are assuming a consistent orientation for all the intersections, but usually this is not the case. In general, the factor in the denominator should be the algebraic sum of the number of intersections. Typically, this is less N and often 1.

BRST formalism

The BRST formalism, which was independently developed in [Tyu75; BRS76], is a cohomological formalism to treat gauge symmetry. Informally, one observes that, after gauge-fixing Yang–Mills theory via the Faddeev–Popov method, there is an additional symmetry governed by the BRST operator. We can think of the BRST formalism as a way to deal with both the gauge symmetry and this additional symmetry.

Definition 2.2.1. A **classical BRST theory** is a tuple $(\mathcal{F}, Q, S)$ where $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold (see Def. 1.3.16), the **BRST operator** $Q \in \mathfrak{X}(\mathcal{F})_1$ is a cohomological vector field (see Def. 1.3.20), and the action $S \in C^\infty(\mathcal{F})_0$ is a gauge invariant even function, that is $Q(S) = 0$.

Starting from the finite-dimensional Faddeev–Popov data, the BRST classical package is given by

$$\left(\mathcal{F} = X \times \mathfrak{g}[1],\ Q = \mathrm{d}_{CE},\ S \in C^\infty(X)^G\right),$$

where d_{CE} is the Chevalley-Eilenberg differential of $\mathfrak{g}$ twisted by $C^\infty(X)$ (see Eq. (2.13)). In local coordinates $(x^i, c^a) \in X \times \mathfrak{g}[1]$, we can write

$$Q = \frac{1}{2}\sum_{a,b,c} f^c_{ab} c^a c^b \partial_{c^c} - \sum_{a,i} c^a v^i_a(x)\partial_{x^i}, \tag{2.9}$$

where $\sum_c f^c_{ab} T_c = [T_a, T_b]$ is the structure constant of $\mathfrak{g}$. Since $\deg_{\mathbb{Z}}(x^i) = 0$ and $\deg_{\mathbb{Z}}(c^a) = 1$, it follows that $\deg_{\mathbb{Z}}(Q) = 1$. Note that S has degree 0 as it is zero along c^a and $Q(S) = 0$ is equivalent to gauge invariance of the action $\gamma_g^* S = S$. Indeed, using the definition of fundamental vector fields v_a (see Eq. (2.7)), the latter can be expressed in terms of local action as $v_a(S) = 0$. The last thing to check is that $Q^2 = 0$. To see this, let us recast the latter discussion in terms of a Lie algebroid.

Definition 2.2.2. A **Lie algebroid** is a vector bundle $E \to M$ with an $\mathbb{R}$-bilinear skew-symmetric map

$$[-,-]\colon \Gamma_M(E) \times \Gamma_M(E) \to \Gamma_M(E),$$

satisfying the Jacobi identity. Additionally, the vector bundle is endowed with a bundle map $\rho\colon E \to TM$ over M, called **anchor map**, such that for all $\alpha, \beta \in \Gamma_M(E)$, $f \in C^\infty(M)$, the following holds

$$[\alpha, f\cdot\beta] = f\cdot[\alpha,\beta] + \rho(\alpha)(f)\cdot\beta. \tag{2.10}$$

This equation implies that the anchor map descends to a Lie algebra morphism $\rho\colon \Gamma_M(E) \to \mathfrak{X}(M)$, i.e.

$$\rho([\alpha,\beta]) = [\rho(\alpha), \rho(\beta)]. \tag{2.11}$$

Indeed, for all $\alpha, \beta, \delta \in \Gamma_M(E)$, $f \in C^\infty(M)$, Jacobi identity and Equation (2.10) imply

$$\begin{aligned}
f[[\alpha,\beta],\delta] + \rho([\alpha,\beta])(f)\delta &= [[\alpha,\beta], f\delta] = [\alpha,[\beta,f\delta]] - [\beta,[\alpha,f\delta]] = [\alpha, f[\beta,\delta] + \rho(\beta)(f)\delta] + \\
&- [\beta, f[\alpha,\delta] + \rho(\alpha)(f)\delta] = f[\alpha,[\beta,\delta]] + \cancel{\rho(\alpha)(f)[\beta,\delta]} + \cancel{\rho(\beta)(f)[\alpha,\delta]} + \rho(\alpha)(\rho(\beta)(f))\delta + \\
&- f[\beta,[\alpha,\delta]] - \cancel{\rho(\beta)(f)[\alpha,\delta]} - \cancel{\rho(\alpha)(f)[\beta,\delta]} - \rho(\beta)(\rho(\alpha)(f))\delta = f[[\alpha,\beta],\delta] + [\rho(\alpha),\rho(\beta)](f)\delta.
\end{aligned}$$

Let us consider $M = X$, $E = \mathfrak{g} \times X$, the Lie bracket of section defined as

$$[\alpha,\beta](x) \coloneqq [\alpha(x),\beta(x)]_{\mathfrak{g}} + \rho(\alpha(x))(\beta)(x) - \rho(\beta(x))(\alpha)(x), \tag{2.12}$$

where the anchor map is given by $\rho \coloneqq \mathrm{d}_{1,-}\gamma\colon E = \mathfrak{g}\times X \to TX$. This defines the **action Lie algebroid**. The bracket is clearly bilinear and skew-symmetric. Moreover, using the definition of ρ and the fact that the Lie brackets on $\mathfrak{g}$ and $\mathfrak{X}(X)$ satisfy the Jacobi identity, it is possible to show that the whole bracket satisfies it. Finally, using the linearity of ρ in $\mathfrak{g}$ and the Leibniz rule for the regular derivation induced by ρ, we get for all $x \in X$

$$\begin{aligned}
[\alpha, f\cdot\beta](x) &= [\alpha(x), f(x)\cdot\beta(x)]_{\mathfrak{g}} + \rho(\alpha(x))(f\beta)(x) - \rho(f(x)\beta(x))(\alpha)(x) = \\
&= f(x)\cdot\Big([\alpha(x),\beta(x)]_{\mathfrak{g}} + \rho(\alpha(x))(\beta)(x) - \rho(\beta(x))(\alpha)(x)\Big) + \rho(\alpha(x))(f)(x)\cdot\beta(x) = \\
&= [f\cdot[\alpha,\beta] + \rho(\alpha)(f)\cdot\beta](x).
\end{aligned}$$

Consider the corresponding $\mathbb{Z}_2$-graded manifold $\Pi E \cong E[1] = X \times \mathfrak{g}[1]$ defined by the $\mathbb{Z}_2$-graded algebra of functions
$$C^\infty(E[1]) = \bigwedge\nolimits^\bullet \mathfrak{g}^* \otimes C^\infty(X). \tag{2.13}$$
On it, we can define a cohomological vector field that corresponds in local coordinates to Equation (2.9). By doing so, we construct the Chevalley-Eilenberg complex $(C^\infty(E[1]), Q)$ of $\mathfrak{g}$ with coefficients in the module $C^\infty(X)$, where the module structure is given by $T_a \otimes f \mapsto v_a(f)$. It is possible to see that $Q^2 = 0$ is equivalent to the Jacobi identity for the Lie algebroid bracket (see Eq. (2.12)) and the fact that the anchor map descends to a Lie algebra morphism (see Eq. (2.11)). Indeed, we get

$$\begin{aligned}
Q^2(c^c) &= Q\Bigg(\sum_{a,b} \frac{1}{2} f^c_{ab} c^a c^b\Bigg) = \sum_{a,b,d,e} \frac{1}{4} f^c_{ab}(f^a_{de} c^d c^e c^b - f^b_{de} c^d c^e c^a) = -\sum_{a,b,d,e} \frac{1}{2} f^c_{ba} f^a_{de} c^d c^e c^b = 0 \\
Q^2(x^i) &= Q\Bigg(-\sum_a c^a v^i_a(x)\Bigg) = \sum_{j,a,d} c^d v^j_d(x) c^a \partial_{x^j} v^i_a(x) - \sum_{a,d,e} \frac{f^a_{de}}{2} c^d c^e v^i_a(x) = \\
&= \sum_{d,e} \frac{c^d c^e}{2}([v_d, v_e]^i(x) - v^i_{[d,e]}(x)) = 0.
\end{aligned}$$

In the last step of the first equation, we anti-symmetrized in d, e, b obtaining the Jacobi identity at the level of structure constants $\sum_a (f^c_{ba} f^a_{de} + f^c_{da} f^a_{eb} + f^c_{ea} f^a_{bd}) = 0$. In the second-to-last step of the second equation, we anti-symmetrized in d, a and we used linearity of ρ in $\mathfrak{g}$ to get
$$v^i_{[d,e]}(x) = \rho^i([T_d, T_e], x) = \sum_a f^a_{de} \rho^i(T_a, x) = \sum_a f^a_{de} v^i_a(x).$$

We concluded noticing that what we obtained is nothing more than Equation (2.11) in components.

Definition 2.2.3. A **quantum BRST theory** is a classical BRST theory $(\mathcal{F}, Q, S)$ with a Berezinian μ on $\mathcal{F}$ such that
$$\operatorname{div}_\mu Q = 0.$$

Remark 2.2.3. The latter definition is rigorous if $\mathcal{F}$ is finite-dimensional, for which we can make proper sense of the Berezinian. In this case, a direct consequence of Definition 1.3.24 is that
$$\int_{\mathcal{F}} Q(f)\mu = 0, \tag{2.14}$$
for any $f \in C^\infty(\mathcal{F})$ compactly supported or such that it rapidly decays at infinity, which means that the boundary contribution to the integral vanishes at infinity. We are interested in integrals of the form $\int_{\mathcal{F}} f\mu$ where $f \in C^\infty(\mathcal{F})$ is a gauge invariant function, i.e. $Q(f) = 0$. Such functions are also called **Q-cocycles**. By Equation (2.14), the integral $\int_{\mathcal{F}} f\mu$ is invariant under shifts of the integrand by a **Q-coboundary** $Q(g)$, with $g \in C^\infty(\mathcal{F})$. Then, the **BRST integral** is a map from the cohomology classes of Q to the real or complex field
$$\int_{\mathcal{F}} \mu \colon H^k_Q(C^\infty(\mathcal{F})) \coloneqq \frac{\{f \in C^\infty(\mathcal{F})_k | Q(f) = 0\}}{\{f \in C^\infty(\mathcal{F})_k | Q(g) = f,\ g \in C^\infty(\mathcal{F})_{k-1}\}} \to \mathbb{R} \quad (\text{or } \mathbb{C}), \tag{2.15}$$
for any $k \in \mathbb{Z}$. As a consequence, the partition function can be written as
$$Z = \int_{\mathcal{F}} e^{\frac{i}{\hbar}S}\mu = \int_{\mathcal{F}} e^{\frac{i}{\hbar}(S + Q(\Psi))}\mu$$
for some $\Psi \in C^\infty(\mathcal{F})_{-1}$. Indeed, the difference of the two integrands is Q-exact
$$e^{\frac{i}{\hbar}(S+Q(\Psi))} - e^{\frac{i}{\hbar}S} = e^{\frac{i}{\hbar}S} Q\big(\tfrac{i\Psi}{\hbar}\big) \sum_{n=1}^{\infty} \frac{\big(Q\big(\frac{i\Psi}{\hbar}\big)\big)^n}{(n+1)!} = e^{\frac{i}{\hbar}S} Q\big(\tfrac{i\Psi}{\hbar}\big) \frac{e^{Q(\frac{i\Psi}{\hbar})} - 1}{Q\big(\frac{i\Psi}{\hbar}\big)} = Q\left(e^{\frac{i}{\hbar}S} \frac{i\Psi}{\hbar} \frac{e^{Q(\frac{i\Psi}{\hbar})} - 1}{Q\big(\frac{i\Psi}{\hbar}\big)}\right).$$
In the last equality, we used $Q(S) = 0$ and $Q^2 = 0$. Observables[15] in this formalism are Q-cocycles $\mathcal{O} \in C^\infty(\mathcal{F})$, as this ensures the integrand of Equation (2.5) to be a Q-cocycle.[16] In the infinite-dimensional case, one has to make sense of it via functional integral measure or via the stationary phase

[15] Observables are usually Q-cocycles of ghost degree 0. However, we could define observables as combinations of Q-cocycles of non-null ghost degree such that the total ghost degree is 0. Examples of this are the surface observables in BF theory (cf. [Cat97]).

[16] Multiplication of Q-cocycles is a Q-cocycle by Leibniz rule and induction.

formula. In the latter case, one has to first take care of the degeneracy of the Hessian of S due to the gauge symmetry. A good gauge-fixing is a choice of Ψ, called **gauge fixing fermion**, so that the critical points of $S + Q(\Psi)$ are isolated. By Equation (2.14), the result is independent of this choice.

Let us consider the Faddeev–Popov data as above. The space of fields $\mathcal{F} = X \times \mathfrak{g}[1]$ does not have a non-zero element of degree -1 that could be taken as a gauge-fixing fermion. Furthermore, since S is null along $\mathfrak{g}[1]$, the would-be partition function

$$Z = \int_{\mathcal{F}} e^{\frac{i}{\hbar}S} \mu_X \mathrm{d}^m c$$

is identically null. The solution is to consider the following system

$$\left(\mathcal{F}_{\text{tot}} \coloneqq \mathcal{F} \times \mathcal{F}_{\text{aux}}, \quad Q_{\text{tot}} \coloneqq Q + Q_{\text{aux}}, \quad S \in C^\infty(X)^G, \quad \mu \coloneqq \mu_X \cdot \mathrm{d}^m c \cdot \mu_{\text{aux}}\right),$$

where $\mathcal{F}_{\text{aux}} \coloneqq \mathfrak{g}^*[-1] \oplus \mathfrak{g}^* \ni (\bar{c}_a, \lambda_a)$, $Q_{\text{aux}} \coloneqq \sum_a \lambda_a \partial_{\bar{c}_a}$, and $\mu_{\text{aux}} \coloneqq \mathrm{d}^m \bar{c} \cdot \mathrm{d}^m \lambda$. Note that

1. $Q^2_{\text{tot}} = 0$ since

$$\begin{aligned} Q^2_{\text{tot}}(x^i) = Q^2(x^i) = 0, \quad Q^2_{\text{tot}}(c^a) = Q^2(c^a) = 0 \\ Q^2_{\text{tot}}(\bar{c}_a) = Q(\lambda_a) + Q_{\text{aux}}(\lambda_a) = 0 \quad Q^2_{\text{tot}}(\lambda_a) = 0 \end{aligned}$$

2. $S \in C^\infty(\mathcal{F})_0$ and $Q_{\text{tot}}(S) = Q(S) = 0$ as above.
3. It is possible to check in coordinates (cf. [Mne19][Sec. 4.2.6]) that

$$\operatorname{div}_\mu Q_{\text{tot}} = \sum_a c^a \Big(\sum_b f^b_{ab} - \operatorname{div}_{\mu_X}(v_a) \Big) = 0$$

since the two terms vanish independently. Indeed, μ_X is G-invariant and thus fundamental vector fields v_a are divergence free w.r.t. it,(17) and $\sum_b f^b_{ab} = \operatorname{tr}_{\mathfrak{g}}[T_a, -]$ vanishes since the Lie group G is compact and connected.(18)

4. Due to freeness(19) of the G-action on X, the BRST cohomology $H^\bullet_{Q_{\text{tot}}} \cong H^\bullet_Q$ (see Eq. (2.15)) is concentrated in degree zero, with $H^0_Q \cong C^\infty(X)^G \cong C^\infty(X/G)$. In this sense, we may say that the BRST theory we just defined is a cohomological resolution of the quotient X/G. This can be understood in the setting of equivariant cohomology (cf. [GS99; Tu20; Kal93]). Indeed, when the G-action is free, the G-equivariant cohomology of X corresponds to the de Rham cohomology of G-invariant forms (cf. [Tu20][Cor. 9.6]) on X. We now show this more explicitly, referring to [BT82; Hat02] for a proper discussion of what follows.
To show that $H^\bullet_{Q_{\text{tot}}} \cong H^\bullet_Q$, or equivalently that $C^\infty(\mathcal{F}_{\text{tot}}), Q_{\text{tot}}$ and $C^\infty(\mathcal{F}), Q$ are quasi-isomorphic, we would like to show that the auxiliary part has no contribution.

   (a) First, we show that $C^\infty(\mathcal{F}_{aux}), Q_{aux}$ has the cohomology of a point,(20) that is

$$H^0_{Q_{\text{aux}}}(C^\infty(\mathcal{F}_{\text{aux}})) = \mathbb{R} \quad \text{and} \quad H^k_{Q_{\text{aux}}}(C^\infty(\mathcal{F}_{\text{aux}})) = 0, \quad \forall k \neq 0.$$

   In other words, we want to show that functions in $C^\infty(\mathcal{F}_{\text{aux}})$ contract to constants, which are functions on a point. Using the Einstein summation convention, we write any function as

$$C^\infty(\mathcal{F}_{\text{aux}}) \ni f(\lambda, \bar{c}) = \frac{g^{i_1 \dots i_n}(\lambda)}{n!} \bar{c}_{i_1} \cdots \bar{c}_{i_n} \quad \text{with} \quad n \leqslant m,$$

(17) The divergence $\operatorname{div}_\mu(v)$ measure how the flow by v locally changes volume, as measured using μ.
(18) The fact that the Lie group G is compact implies that G is unimodular (cf. [Kna05][Prop. 6.15]). G is **unimodular**, i.e. its left and right Haar measures agree, if and only if $|\det(\mathrm{Ad}(g))| = 1$ for all $g \in G$ (cf. [Dow23]). If G is connected, this condition needs only to be checked locally around the identity element of G and without absolute value. The fact that $\det(\mathrm{Ad}(g)) = 1$ is constant near the identity is equivalent to ask $\operatorname{tr}(\mathrm{ad}) = 0$ at the level of Lie algebra $\mathfrak{g}$.
(19) Note that freeness ensures both the smoothness of X/G (see ft. (12), p. 35) and that the cohomology is concentrated in degree zero. For example, consider the case of a Lie algebra acting trivially on a point. The latter is an extreme case of non-freeness. Then, the cohomology of $Q = \frac{1}{2}\sum_{a,b,c} f^c_{ab} c^a c^b \partial_{c^c}$ is the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the trivial module, which is generally nontrivial (cf. [HS71][Ch. VII]).
(20) This point can be understood as a particular extension of the algebraic proof of the Poincarè Lemma [Cat18][Lemma 9.33], where instead of the exponential in the integral we use an appropriate power of the integral variable.

where $g^{i_1\cdots i_n} \in C^\infty(\mathfrak{g}^*)$ are fully antisymmetric in the indices. Then, define

$$h(f) := \int_0^1 t^n \partial_{t\lambda_a} \frac{g^{i_1\dots i_n}(t\lambda)}{n!} \mathrm{d}t\bar{c}_a\bar{c}_{i_1}\cdots\bar{c}_{i_n} = \int_0^1 t^n \partial_{t\lambda_a} \frac{g^I(t\lambda)}{n!}\mathrm{d}t\bar{c}_a\bar{c}_I,$$

where $I = (i_1, \dots, i_n)$ is the usual multi-index. This defines a map of degree -1. Then,

$$\begin{aligned}\big(Q_{\mathrm{aux}}h + hQ_{\mathrm{aux}}\big)(f) &= Q\left(\int_0^1 t^n \frac{\partial_{t\lambda_a} g^I(t\lambda)}{n!}\mathrm{d}t\bar{c}_a\bar{c}_I\right) + h\left(n\frac{g^{bi_2\dots i_n}(\lambda)}{n!}\lambda_b\bar{c}_{i_2}\cdots\bar{c}_{i_n}\right) = \\ &= \int_0^1 t^n\frac{\partial_{t\lambda_a}g^I(t\lambda)}{n!}\mathrm{d}t\lambda_a\bar{c}_I - \int_0^1 t^n\frac{\partial_{t\lambda_a}g^I(t\lambda)}{n!}\mathrm{d}t\bar{c}_aQ(\bar{c}_I) + \\ &+ \int_0^1 nt^{n-1}\frac{\partial_{t\lambda_a}g^{bi_2\dots i_n}(t\lambda)}{n!}t\lambda_b\mathrm{d}t\bar{c}_a\bar{c}_{i_2}\cdots\bar{c}_{i_n} + \int_0^1 nt^{n-1}\frac{g^{ai_2\dots i_n}(t\lambda)}{n!}\mathrm{d}t\bar{c}_a\bar{c}_{i_2}\cdots\bar{c}_{i_n} = \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(t^n\frac{g^I(t\lambda)}{n!}\right)\mathrm{d}t\bar{c}_I = \left[\frac{g^I(\lambda)}{n!} - \lim_{t\to 0} t^n\frac{g^I(t\lambda)}{n!}\right]\bar{c}_I = \begin{cases} f(\lambda,\bar{c}) & ,\ n>0 \\ f(\lambda,\bar{c}) - f(0,\bar{c}), & n=0\end{cases}\end{aligned}$$

where we used $n\lambda_b\bar{c}_{i_2}\cdots\bar{c}_{i_n} = Q(\bar{c}_I)$ and the fundamental theorem of calculus. Therefore,

$$Q_{\mathrm{aux}}h + hQ_{\mathrm{aux}} = \mathrm{id} - \pi,$$

where $\pi(f(\lambda,\bar{c})) = f(0,0)$ is the projection into constant part in λ and $\bar{c}$. Consequently, for $n > 1$, any closed ($Q_{\mathrm{aux}}f = 0$) function $f \in C^\infty(\mathcal{F}_{\mathrm{aux}})_{-n}$ is also exact

$$f = \big(Q_{\mathrm{aux}}h + hQ_{\mathrm{aux}}\big)(f) = Q_{\mathrm{aux}}h(f), \quad h(f) \in C^\infty(\mathcal{F}_{\mathrm{aux}})_{-(n+1)}.$$

Since this algebra is only negatively graded, this means that $H^k_{Q_{\mathrm{aux}}}(C^\infty(\mathcal{F}_{\mathrm{aux}})) = 0, \quad \forall k \neq 0$. All the cohomology is concentrated in $n = 0$. In particular, any closed function $f \in C^\infty(\mathcal{F}_{\mathrm{aux}})_0$ is cohomologically characterized by its constant part

$$f = \big(Q_{\mathrm{aux}}h + hQ_{\mathrm{aux}} + \pi\big)(f) = Q_{\mathrm{aux}}h(f) + f(0,0).$$

This means that the cohomological class of f is characterized by $f(0,0) \in \mathbb{R}$, as their difference is exact. That is, $H^0_{Q_{\mathrm{aux}}}(C^\infty(\mathcal{F}_{\mathrm{aux}})) = \mathbb{R}$.

(b) We want to extend this to show that functions $C^\infty(\mathcal{F}_{\mathrm{tot}})$ contracts to functions on the original space of fields $C^\infty(\mathcal{F})$. To do this, define

$$H(f\otimes g) := (-1)^{-\deg_{\mathbb{Z}}(f)} f \otimes h(g)$$

for any $f \otimes g \in C^\infty(\mathcal{F}_{\mathrm{tot}}) = C^\infty(\mathcal{F}) \otimes C^\infty(\mathcal{F}_{\mathrm{aux}})$ of homogeneous degree $\deg_{\mathbb{Z}}(f\otimes g) = \deg_{\mathbb{Z}}(f) + \deg_{\mathbb{Z}}(g)$. On it, the cohomological vector field acts as $Q_{\mathrm{tot}}(f\otimes g) = Q(f)\otimes g + (-1)^{\deg_{\mathbb{Z}}(f)} f\otimes Q_{\mathrm{aux}}(g)$. Then,

$$\begin{aligned}\big(Q_{\mathrm{tot}}H + HQ_{\mathrm{tot}}\big)(f\otimes g) &= Q_{\mathrm{tot}}\big((-1)^{-\deg_{\mathbb{Z}}(f)} f\otimes h(g)\big) + H\big(Q(f)\otimes g + \\ &+ (-1)^{\deg_{\mathbb{Z}}(f)} f \otimes Q_{\mathrm{aux}}(g)\big) = (-1)^{-\deg_{\mathbb{Z}}(f)}Q(f)\otimes h(g) + f\otimes Q_{\mathrm{aux}}h(g) + \\ &+ (-1)^{-\deg_{\mathbb{Z}}(f)+1}Q(f)\otimes h(g) + f\otimes hQ_{\mathrm{aux}}(g) = f\otimes(g - \pi g) = f\otimes g(0,0).\end{aligned}$$

That is,

$$Q_{\mathrm{tot}}H + HQ_{\mathrm{tot}} = \mathrm{id} - \mathrm{id}_{C^\infty(\mathcal{F})}\otimes\pi.$$

In other words, the cohomological classes are defined by functions on the original space of fields $C^\infty(\mathcal{F})$. Indeed, for any closed $f\otimes g \in C^\infty(\mathcal{F}_{\mathrm{tot}})$, we have

$$f\otimes g - (\mathrm{id}_{C^\infty(\mathcal{F})}\otimes\pi)(f\otimes g) = Q_{\mathrm{tot}}H(f\otimes g).$$

Therefore, $f\otimes g \sim (\mathrm{id}_{C^\infty(\mathcal{F})}\otimes\pi)(f\otimes g) \sim f\otimes g(0,0) \sim g(0,0)f\otimes 1$. So $H^\bullet_{Q_{\mathrm{tot}}} \cong H^\bullet_Q$.

To show $H^0_Q \cong C^\infty(X)^G \cong C^\infty(X/G)$, first notice that the second relation is automatically true using $p^*\colon C^\infty(X/G) \to C^\infty(X)^G$ as isomorphism. To characterize H^0_Q remember that, as argued for the action S, for any zero degree function $F \in C^\infty(\mathcal{F})_0$, that is constant along c^a, closeness $Q(F) = 0$ is equivalent to G-invariance $\gamma_g^*F = F$. Indeed, using the definition of fundamental

vector fields v_a (see Eq. (2.7)), the latter can be expressed in terms of local action as $v_a(F) = 0$. As $C^\infty(\mathcal{F})$ is positively graded there is no function such that $Q(g) = f$.
Finally, $H^k_Q(C^\infty(\mathcal{F})) = 0, \quad \forall k \neq 0$ is true for any $k < 0$ as this algebra is positively graded. For $k > 0$, we should repeat the steps of point (a) above with an appropriate h. Alternatively, we could use the local triviality of X over X/G, given by freeness, and locally repeat the steps of point (a) only along the orbits. Then, to conclude, this should be trivially extended in the considered local patch of X/G. More generally, since the Lie algebra is finite-dimensional and semisimple, one could use Whitehead's theorem [AI95][Thm. 6.6.1]. Indeed, we are working over $\mathbb{R}$ or $\mathbb{C}$, which are fields of characteristic zero. Furthermore, we can split the $\mathfrak{g}$-module $C^\infty(X)$ as a sum of finite-dimensional irreducible representations of $\mathfrak{g}$. This carries with it a direct decomposition of the cohomology groups, to each of which we can apply the theorem if the representation is non-trivial. Thinking of X as a G-bundle over X/G, thanks to the freeness of the action, we can locally write $C^\infty(X) \cong C^\infty(X/G) \times C^\infty(G)$. Then, we use the Peter-Weyl theorem[(21)] to decompose $C^\infty(G)$ into a sum of finite-dimensional irreducible representations of $\mathfrak{g}$. Notice that the trivial representation appears once, corresponding to constant functions, while all other irreducibles are nontrivial and hence acyclic by Whitehead. Gluing this together, one gets that globally

$$H^0_Q \cong C^\infty(X/G) \quad \text{and} \quad H^{k\neq 0}_Q \cong 0.$$

Then the integral

$$I = \int_{\mathcal{F}_{\text{tot}}} e^{\frac{i}{\hbar}S}\mu = \int_{\mathcal{F}_{\text{tot}}} e^{\frac{i}{\hbar}(S+Q_{\text{tot}}(\Psi_\phi))}\mu \tag{2.16}$$

with $\Psi_\phi := \langle \bar{c}, \phi(x)\rangle \in C^\infty(\mathcal{F}_{\text{tot}})_{-1}$ so that $Q_{\text{tot}}(\Psi_\phi) = \langle \lambda, \phi(x)\rangle + \langle \bar{c}, FP(x)c\rangle$, giving exactly Equation (2.8) up to a constant factor. Thanks to Equation (2.14), the result is invariant under change of the gauge fixing fermion or of gauge fixing function $\phi\colon X \to \mathfrak{g}$.

Remark 2.2.4. The above can be cast in terms of the infinitesimal gauge symmetry described by the integrable distribution $\operatorname{im}(\mathrm{d}_{1,-}\gamma) \subset TX$ of the anchor map of the action Lie algebroid $\mathfrak{g} \times X \to X$ (see around Eq. (2.12)). The BRST formalism allows us to treat gauge theories whose symmetries are described by more general integrable distributions associated to certain Lie algebroids (cf. [Mne19][Rmk. 4.3.7]). In this sense, it subsumes the Faddeev–Popov method for gauge theories with symmetry groups given by Lie groups, while also extending the formalism to a broader class of symmetries encoded by integrable Lie algebroids. To consider even more complicated symmetries given by non-integrable distributions, we have to resort to the BV formalism.

BV formalism

In [BV81; BV83], Batalin and Vilkovisky developed a more general construction compatible with path-integral quantization of gauge theories with local symmetry given by a non-integrable distribution. Classically, this amounts to finding a suitable odd-symplectic extension of the classical gauge theory by ghosts, antifields, and antighosts. The gauge symmetry is given via a cohomological Hamiltonian vector field of the proper extended action as in the BRST formalism, while the gauge fixing is described as a choice of Lagrangian submanifold of the extended space of fields.[(22)]

Definition 2.2.4. A **classical BV theory** is a tuple $(\mathcal{F}, \omega, S)$ where $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold (see Def. 1.3.16), $\omega \in \Omega^2(\mathcal{F})_{-1}$ is an odd-symplectic form (see Def. 1.4.7) and the **BV action** $S \in C^\infty(\mathcal{F})_0$ is an even function satisfying the **classical master equation** (**CME**)

$$\{S, S\}_\omega = 0, \tag{2.17}$$

where the bracket is defined as in Equation (1.4.8) with respect to ω.

(21) This theorem gives a decomposition of $L^2(G)$ into a sum of finite-dimensional irreducible representations of $\mathfrak{g}$ (cf. [Kna86][Thm. 1.12]). Assuming rapid-decay condition, one can also show it for $C^\infty(G)$.
(22) This setting can be extended to a cohomological symplectic formalism (BV–BFV) that allows treating gauge theories on spacetime manifolds with boundaries [CMR14]. Its quantum version [CMR18] is a powerful tool, compatible with cutting and gluing, to quantize such gauge theories perturbatively.

Remark 2.2.5.

1. The Hamiltonian vector field generated by S
$$Q \coloneqq X_S = \{S, -\}_\omega \in \mathfrak{X}(\mathcal{F})_1$$
squares to zero as a consequence of CME (see Prop. 1.4.3). We call it the **BRST operator**. Moreover, $\mathbb{L}_Q\omega = 0$ follows from $\iota_Q\omega = \mathrm{d}S$.

2. Equivalently (see Prop. 1.4.3), we can define a BV theory as a tuple $(\mathcal{F}, \omega, S, Q)$ where $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold, $\omega \in \Omega^2(\mathcal{F})_{-1}$ is an odd-symplectic form, $S \in C^\infty(\mathcal{F})_0$ and $Q \in \mathfrak{X}(\mathcal{F})_1$ such that
$$\iota_Q\omega = \mathrm{d}S \quad \text{and} \quad [Q, Q] = 0,$$
where the latter is the Lie bracket in $\mathfrak{X}(\mathcal{F})$ (see Eq. (1.13)).

3. Using the odd-symplectic structure of $\mathcal{F}$ and the definition of Q, it is possible to show in local coordinates that the zero locus of Q is the critical locus of the BV action S. We speak of BV extension $(\mathcal{F}, \omega, S)$ of a classical theory $(\mathcal{F}_{\mathrm{cl}}, S_{\mathrm{cl}})$ when $\mathcal{F}_{\mathrm{cl}}$ is the non Q-exact ghost number zero part of $\mathcal{F}$ and $S_{\mathrm{cl}} = S|_{\mathcal{F}_{\mathrm{cl}}}$. Q-exact fields must be quotiented out, as they can only arise as auxiliary fields and therefore do not correspond to physical fields in the BV/BRST formalism. To understand this, see Theorem 2.1 (Equation (2.14)), and its role in defining physical BV (BRST) classical observables ($\hbar \to 0$). Then $H^0_Q = C^\infty(\{\delta S_{\mathrm{cl}} = 0\})$ corresponds to functions on the zero locus of the classical action S_{cl}[(23)]. This means that $\{Q = 0\} \cap \mathcal{F}_{\mathrm{cl}}$ is the critical locus of S_{cl}, that is, the space of fields solving the Euler–Lagrange equations.

4. In the infinite-dimensional case, asking ω to be non-degenerate is too restrictive. In general, it is enough to ask for it to be a weakly-symplectic form (see Def. 1.4.7). This only ensures the uniqueness of the Hamiltonian vector field, but not its existence. Therefore, the existence of the cohomological vector field Q should be regarded as additional global data of the BV theory, and the first condition in point 2. should be imposed to ensure that Q is the Hamiltonian vector field for S. For the application we are presenting in Section 3.4, weak non-degeneracy alone does not guarantee the existence of a Hamiltonian vector field for S, and therefore its existence is imposed as additional global data (see Def. 3.4.1).

5. The Hamiltonian counterpart of this formalism, called the BFV formalism (cf. [BF83]), is given considering $\omega \in \Omega^2(\mathcal{F})_0$, $S \in C^\infty(\mathcal{F})_1$. Note that the space of fields is not the same as for BV.

Geometrically, the classical critical locus of a Lagrangian field theory is the zero locus of the variational differential of the action functional (see Principle 2.1.1). For Lagrangian gauge field theories, this locus is typically singular, and the BV formalism replaces it with a derived critical locus. To encode gauge data, one considers the extended space of fields $\mathcal{F}$ that also include ghosts. Antifields then arise naturally as coordinates on the shifted cotangent bundle of the latter $T^*[-1]\mathcal{F}$, and encode the equations of motion together with their higher relations, including the Noether identities. More explicitly, considering the initial Faddeev–Popov data, we can construct one of the two BRST systems described above and further obtain their BV extensions as follows. Given a BRST system $(\mathcal{F}_{\mathrm{BRST}}, Q_{\mathrm{BRST}}, S_{\mathrm{BRST}})$, this entails considering
$$\left(\mathcal{F}_{\mathrm{BV}} \coloneqq T^*[-1]\mathcal{F}_{\mathrm{BRST}},\ \omega_{\mathrm{BV}},\ S_{\mathrm{BV}} \coloneqq p^*S_{\mathrm{BRST}} + \widetilde{Q_{\mathrm{BRST}}}\right).$$
Here, ω_{BV} is the standard symplectic structure of the cotangent bundle and $p\colon \mathcal{F}_{\mathrm{BV}} \to \mathcal{F}_{\mathrm{BRST}}$ is the projection to the base of the cotangent bundle. Finally, $\widetilde{Q_{\mathrm{BRST}}}$ is the lifting of Q_{BRST} from the base of the cotangent bundle to a function on the total space which is linear in the fibers. In local Darboux coordinates of fields and antifields $(\varphi^\alpha, \varphi^+_\alpha) \in T^*[-1]\mathcal{F}_{\mathrm{BRST}}$, we get
$$\omega_{\mathrm{BV}} = \sum_\alpha \mathrm{d}\varphi^\alpha \wedge \mathrm{d}\varphi^+_\alpha$$
$$S_{\mathrm{BV}}(\varphi, \varphi^+) = S_{\mathrm{BRST}}(\varphi) + \sum_\alpha \varphi^+_\alpha Q^\alpha_{\mathrm{BRST}}, \tag{2.18}$$

(23) Note that this is only an heuristic identification in the regular case. Strictly speaking, one usually obtains functions on the derived critical locus modulo gauge equivalence. Identifying this with ordinary functions on the Euler—Lagrange locus requires additional regularity assumptions and absence of residual gauge phenomena.

where $Q^\alpha_{\mathrm{BRST}} = \mathbb{L}_{Q_{\mathrm{BRST}}}\varphi^\alpha$. By definition, the above have the right grades. Moreover, $Q^2_{\mathrm{BRST}} = 0$ and $Q_{\mathrm{BRST}}(S_{\mathrm{BRST}}) = 0$ imply the CME. Indeed, working in local Darboux coordinates, we get

$$\{S_{\mathrm{BV}}, S_{BV}\}_{\omega_{\mathrm{BV}}} = 2\sum_\alpha \partial_{\varphi^\alpha}(S_{\mathrm{BV}})\partial_{\varphi^+_\alpha}(S_{\mathrm{BV}}) = 2\Bigg(\sum_\alpha \partial_{\varphi^\alpha}(S_{\mathrm{BRST}})Q^\alpha_{\mathrm{BRST}}(\varphi) + \\ + \sum_{\beta,\alpha}\varphi^+_\beta\partial_{\varphi^\alpha}(Q^\beta_{\mathrm{BRST}}(\varphi))Q^\alpha_{\mathrm{BRST}}(\varphi)\Bigg) = 2Q_{\mathrm{BRST}}(S_{\mathrm{BRST}}) + \sum_\beta \varphi^+_\beta[Q_{\mathrm{BRST}}, Q_{\mathrm{BRST}}]^\beta(\varphi) = 0. \tag{2.19}$$

The Hamiltonian vector field of S_{BV}, i.e. the BRST operator, has the form

$$Q_{\mathrm{BV}} = X_{p^*S_{\mathrm{BRST}}} + Q^{\mathrm{cot.lift}}_{\mathrm{BRST}},$$

where the second is the cotangent lift of Q_{BRST} on the base of the cotangent bundle to a vector field on the total space.(24) In coordinates, we get

$$Q_{\mathrm{BV}} = \sum_\alpha \partial_{\varphi^\alpha}(S_{\mathrm{BRST}}(\varphi))\partial_{\varphi^+_\alpha} + \\ + \sum_\alpha Q^\alpha(\varphi)\partial_{\varphi^\alpha} + \sum_{\alpha,\beta}(-1)^{(\deg_{\mathbb{Z}}(\varphi^\alpha)+1)\deg_{\mathbb{Z}}(\varphi^\beta)}\varphi^+_\alpha\partial_{\varphi^\beta}\big(Q^\alpha(\varphi)\big)\partial_{\varphi^+_\beta}.$$

To understand the quantum version, let us introduce the following definition and result.

Definition 2.2.5. Let $(\mathcal{M}, \omega)$ be an odd-symplectic $\mathbb{Z}_2$-graded $(m|m)$-manifold.(25) A Berezinian $\mu \in \mathrm{BER}(\mathcal{M})$ (see Def. 1.3.23) is **compatible** with ω if there exists an atlas of Darboux charts (x^i, ξ_i) on $\mathcal{M}$ such that locally $\mu = \mathrm{d}^m x \mathrm{d}^m \xi$ in all charts of the atlas, called **special Darboux charts**. Given $(\mathcal{M}, \omega, \mu)$ an odd-symplectic manifold with a compatible Berezinian, we define the **Batalin–Vilkovisky Laplacian** as the odd second-order operator $\Delta_\mu\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ given by

$$\Delta_\mu(f) := \frac{1}{2}\mathrm{div}_\mu(X_f).$$

The BV Laplacian squares to zero. Indeed, in local special Darboux coordinates, we get the following

$$\Delta_\mu = \sum_i \partial_{x^i}\partial_{\xi_i} \quad\Longrightarrow\quad \Delta^2_\mu = \sum_{i,j}\partial_{x^i}\partial_{x^j}\partial_{\xi_i}\partial_{\xi_j} = 0.$$

The latter expression is zero, since the summand changes sign under $i \leftrightarrow j$, making the sum over i, j vanish. In both the $\mathbb{Z}_2$-graded and $\mathbb{Z}$-graded case (in which $\deg_{\mathbb{Z}}(\omega) = -1$) the BV Laplacian has degree $+1$. The graded Poisson bracket with respect to ω can be understood as the failure of the BV Laplacian to be a derivative. Indeed, for all homogeneous $f, g \in C^\infty(\mathcal{M})$, the following holds

$$\Delta_\mu(fg) = (\Delta_\mu f)g + (-1)^{\deg_{\mathbb{Z}_{(2)}}(f)}f(\Delta_\mu g) + (-1)^{\deg_{\mathbb{Z}_{(2)}}(f)}\{f, g\},$$

as can be checked in local coordinates.

Definition 2.2.6. Consider an odd-symplectic manifold with a compatible Berezinian $(\mathcal{M}, \omega, \mu)$, a homogeneous function $f \in C^\infty_c(\mathcal{M})$ and an odd function $\Psi \in C^\infty(\mathcal{M})$ depending only on x-variables of a local special Darboux chart (x^i, ξ_i), called **gauge-fixing fermion**. Let us choose the Lagrangian submanifold $\mathcal{L}_\Psi \subset \Pi T^*\mathbb{R}^m$ defined by $\xi_i = \partial_{x^i}\Psi$, which is projectable onto the space of the x-variables.(26) Then the **BV integration map**(27) is defined by

$$\int_{\mathcal{L}_\Psi} f := \int f\Big|_{\xi_i = \partial_{x^i}\Psi}\mathrm{d}^m x.$$

The following theorem is the main result of Batalin and Vilkovisky, which guarantees the independence of the physical observables from the gauge fixing.

(24) Let (M, ω) be a symplectic manifold. The **cotangent lift** of a vector field $v \in \mathfrak{X}(M)$ is the unique vector field $v_\# \in \mathfrak{X}(T^*M)$ whose flow is the lift (cf. [Sil01][Sec. 2.4]) of the flow of v.

(25) If it is $\mathbb{Z}$-graded, we consider only the underlying Z_2-graded structure.

(26) $\mathcal{L}_\Psi$ is a projectable Lagrangian submanifold since it is the graph of $\mathrm{d}\Psi$ over the x-space. Then, $\dim \mathcal{L}_\Psi = \frac{1}{2}\dim \Pi T^*\mathbb{R}^m$, $\omega_{\mathrm{can}}|_{\mathcal{L}_\Psi \times \mathcal{L}_\Psi} = \sum_{i,j}\partial_{x^j}\partial_{x^i}(\Psi)\mathrm{d}x^i \wedge \mathrm{d}x^j = 0$, and $\mathcal{L}_\Psi$ is projectable, that is, the projection $(x, \xi) \mapsto x$ restricts to a bijection when projecting $\mathcal{L}_\Psi$. Note that implicitly we are using Schwarz Theorem (see Thm. 1.1).

(27) To describe this globally, we should work with half-densities (cf. [Mne19][Sec. 4.6.2]).

Theorem 2.1. *The following holds:*

1. *If $f = \Delta_\mu g$, then $\int_{\mathcal{L}_\Psi} f = 0$, for every Ψ such that the integral is defined.*
2. *If $\Delta_\mu f = 0$, then $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_{\Psi_t}} f = 0$, where $\{\Psi_t\}$ is a continuous family of gauge fixing fermions, such that the integral is defined for every t.*

A proof can be found in [CM20][Thm. 5.0.1] or [Mne19][Thm. 4.4.12].

Remark 2.2.6. The above theorem indeed allows us to speak of gauge independence. To see this, consider a continuous family $\{\Psi_t\}$ of gauge fixing fermions, let $\mathcal{L}_{\Psi_0} = \mathbb{R}^{n|m} \subset \Pi T^*\mathbb{R}^{n|m}$ such that $\int_{\mathcal{L}_{\Psi_0}} f$ does not converge for f a given Δ-cocycle ($\Delta f = 0$). Assume $\int_{\Psi_t} f$ converges for all $t \neq 0$ in a neighborhood of 0. Then we define

$$\int_{\mathcal{L}_{\Psi_0}} f \coloneqq \int_{\mathcal{L}_{\Psi_t}} f$$

for some t in this pointed neighborhood. The choice of Lagrangian submanifolds is equivalent to gauge-fixing, and the integral above is independent of such a choice by Theorem 2.1. When $f = e^{\frac{i}{\hbar}S}$ for some even function S, the integral above becomes some partition function (see Eq. (2.4)). In order to apply the stationary phase formula, we use the above as a gauge-fixing procedure to solve a potential Hessian degeneracy arising from gauge freedom. Note that, using the Taylor expansion of the exponential, we get

$$0 = \Delta_\mu f = \frac{i}{\hbar}\Delta_\mu S e^{\frac{i}{\hbar}S} + \left(\frac{i}{\hbar}\right)^2 \frac{1}{2}\{S,S\}_\omega e^{\frac{i}{\hbar}S}. \tag{2.20}$$

BV-observables $\mathcal{O} \in C^\infty(\mathcal{M})$ are δ_S-cocycles, with $\delta_S \coloneqq \{S,-\}_\omega - i\hbar\Delta_\mu$. The latter condition is equivalent to asking for the observable to be gauge-invariant, as it implies $\Delta_\mu(e^{\frac{i}{\hbar}S}\mathcal{O}) = 0$.

Definition 2.2.7. A **quantum BV theory** is a tuple $(\mathcal{F},\omega,\mu,S)$ where $\mathcal{F}$ is a $\mathbb{Z}$-graded manifold, $\omega \in \Omega^2(\mathcal{F})_{-1}$ is an odd-symplectic form, $\mu \in \mathrm{BER}(\mathcal{F})$ is a Berezinian compatible with ω (see Def. 2.2.5) and the **BV action** $S \in C^\infty(\mathcal{F})_0$, that can be expanded as power series in $-i\hbar$, satisfies the **quantum master equation** (**QME**)

$$\frac{1}{2}\{S,S\}_\omega - i\hbar\Delta_\mu S = 0 \quad\iff\quad \Delta_\mu e^{\frac{i}{\hbar}}S = 0. \tag{2.21}$$

The bracket is defined as in Equation (1.4.8) with respect to ω.[28]

The latter definition is rigorous if $\mathcal{F}$ is finite-dimensional, for which we can make proper sense of the Berezinian.
Given the initial finite-dimensional Faddeev–Popov data, let us consider the quantum BRST system as above

$$\left(\mathcal{F}_{\mathrm{tot}} \coloneqq \mathcal{F}\times\mathcal{F}_{\mathrm{aux}},\quad Q_{\mathrm{tot}} \coloneqq Q + Q_{\mathrm{aux}},\quad S \in C^\infty(X)^G,\quad \mu \coloneqq \mu_X\cdot \mathrm{d}^m c\cdot \mathrm{d}^m\bar{c}\cdot \mathrm{d}^m\lambda\right).$$

We can build the classical BV extension following the general procedure described above. In particular,

$$\begin{aligned}
\mathcal{F}_{\mathrm{BV}} &\coloneqq T^*[-1]\mathcal{F}_{\mathrm{tot}} = T^*[-1]X \times \mathfrak{g}[1]\times\mathfrak{g}^*[-2]\times\mathfrak{g}^*[-1]\times\mathfrak{g}\times\mathfrak{g}^*\times\mathfrak{g}[-1] \\
&\quad \text{with local special Darboux coordinates} \quad \mathcal{F}_{\mathrm{BV}} \ni (x^i, x_i^+, c^a, c_a^+, \bar{c}_a, \bar{c}^{+a}, \lambda_a, \lambda^{+a}),\\
\omega_{\mathrm{BV}} &\coloneqq \sum_i \mathrm{d}x^i \wedge \mathrm{d}x_i^+ + \sum_a \mathrm{d}c^a\wedge \mathrm{d}c_a^+ + \sum_a \mathrm{d}\bar{c}_a \wedge \mathrm{d}\bar{c}^{+a} + \sum_a \mathrm{d}\lambda_a\wedge\mathrm{d}\lambda^{+a},\\
S_{\mathrm{BV}} &\coloneqq S(x) + \sum_{i,a} c^a v_a^i(x) x_i^+ + \frac{1}{2}\sum_{a,b,c} f_{ab}^c c^b c^c c_a^+ + \sum_a \lambda_a \bar{c}^{+a}.
\end{aligned}$$

The BV action contains the classical action as a term independent of antifields, and the gauge data, i.e. the local gauge transformations on classical fields and the algebra of gauge transformations, as terms

[28] The QME comes from Equation (2.20). Moreover, we can expand the BV action as $S = S_0 + (-i\hbar)S_1 + (-i\hbar)^2 S_2 + \ldots$. S_0 is the classical action of the original classical BV theory, and it satisfies the CME $\{S_0,S_0\} = 0$. The QME can be understood as an order-by-order (in $\hbar$) condition on the quantum action: quantization then equates to finding the quantum action expansion terms satisfying such conditions. For a more detailed discussion, we refer to [Mne19; CMS25].

linear in antifields. Higher terms in antifields would arise in the case we consider a non-integrable gauge symmetry. Furthermore, $\{Q_{\mathrm{BV}} = 0\} \cap X$ is precisely the critical locus of S. To see this, we need to build the quantum BV theory. To this end, consider the compatible Berezinian

$$\mu_{\mathrm{BV}} = \mathrm{d}^n x \cdot \mathrm{d}^n x^+ \cdot \mathrm{d}^m c \cdot \mathrm{d}^m c^+ \cdot \mathrm{d}^m \bar{c} \cdot \mathrm{d}^m \bar{c}^+ \cdot \mathrm{d}^m \lambda \cdot \mathrm{d}^m \lambda^+.$$

Note that S_{BV} is independent of $\hbar$, so the QME splits into CME, which holds as in Equation (2.19), and $\Delta_{\mu_{\mathrm{BV}}} S_{\mathrm{BV}} = 0$, which follows from the definitions of Δ_μ, Q_{BV} and $\mathrm{div}_\mu Q_{\mathrm{BRST}} = 0$, as can be checked in local coordinates. The gauge-fixing is given by the choice of the Lagrangian submanifold $\mathcal{L}_\Psi = \mathrm{graph}(\mathrm{d}\Psi) \subset \mathcal{F}_{\mathrm{BV}}$ for some $\Psi \in C^\infty(\mathcal{F}_{\mathrm{tot}})_{-1}$. Choosing $\Psi_\phi = \langle \bar{c}, \phi(x) \rangle$ as in the BRST case, locally corresponds to imposing the following

$$\mathcal{L}_\Psi : \begin{cases} x, c, \lambda, \bar{c} \text{ are free} \\ x^+ = -(\mathrm{d}_x \phi)^T(\bar{c}) \\ c^+ = 0 \\ \lambda^+ = 0 \\ \bar{c}^+ = \phi(x). \end{cases}$$

This yields exactly the right hand side of Equation (2.16), or of Equation (2.8) up to a constant factor,

$$\int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar} S_{\mathrm{BV}}} = \int_{\mathcal{F}_{\mathrm{tot}}} e^{\frac{i}{\hbar}\left(S(x) + \sum_{a,b} \bar{c}_b FP_a^b(x) c^a + \sum_a \lambda_a \phi(x)^a\right)} \mu = \int_{\mathcal{F}_{\mathrm{tot}}} e^{\frac{i}{\hbar}(S + Q_{\mathrm{tot}}(\Psi_\phi))} \mu.$$

The above integral is independent of the BV gauge-fixing (see Thm. 2.1), which, in this case, is equivalent to the BRST or Faddeev–Popov gauge-fixing. Finally, we can write

$$\begin{aligned} Q_{\mathrm{BV}} = &\sum_i \partial_{x^i} S(x) \partial_{x_i^+} - \sum_{i,b} c^b v_b^i(x) \partial_{x^i} + \frac{1}{2} \sum_{a,b,c} f_{bc}^a c^b c^c \partial_{c^a} + \sum_a \lambda_a \partial_{\bar{c}_a} + \\ &+ \sum_{j,i,b} x_j^+ c^b \partial_{x^i} v_b^j(x) \partial_{x_i^+} - \sum_{j,a} x_j^+ v_a^j(x) \partial_{c_a^+} + \sum_{a,b,c} c_a^+ f_{bc}^a c^c \partial_{c_b^+} + \sum_b \bar{c}^{+b} \partial_{\lambda^{+b}}. \end{aligned}$$

Therefore, $\{Q_{\mathrm{BV}} = 0\} \cap X = \{\mathrm{d}S = 0\}$ as expected. In particular, restricting to the ghost number zero in this case is not enough, since also λ and c^+ have ghost number zero. However, they are Q_{BV}-exact as $Q_{\mathrm{BV}}(\bar{c}_a) = \lambda_a$ and $Q_{\mathrm{BV}}(\lambda^{+a}) = \bar{c}^{+a}$.
For more general theories whose local gauge symmetry is given as a non-integrable distribution on the space of classical fields, it is possible to extend the above to construct a BV classical and quantum extension of the theory. We refer to [Mne19][Sec. 4.8.4] for a discussion of how to approach this in a general but simplified case.

Remark 2.2.7. In what follows, we only focus on classical infinite-dimensional BV theories, in particular AKSZ theories. Indeed, the category of super thickened smooth sets allows us to locally mimic the finite-dimensional case via the differential-geometry of the space of fields. However, this category does not provide a notion of measure on the latter, preventing us from mimicking the finite-dimensional approach to quantization, which we introduced in this chapter only for completeness. We do not discuss the infinite-dimensional case of the Faddeev–Popov method or the quantum BRST formalism, since they would also require a theory of integration on the infinite-dimensional space of fields. Also, we do not describe infinite-dimensional classical BRST theories, since they can always be formulated as BV theories.

2.3 AKSZ Formalism and BF Theory

In this section, we show how the AKSZ formalism (cf. [Ale+97]) provides a systematic construction of a solution of the classical master equation on the mapping space between two $\mathbb{Z}_{(2)}$-graded manifolds endowed with appropriate supergeometric data. As an example, we present (non)-abelian BF theory.

Remark 2.3.1. As discussed in Remark 2.2.1 and Remark 2.2.7, the infinite-dimensional classical BV formalism relies on a rigorous understanding of the Cartan calculus on the infinite-dimensional space of fields, which is addressed in Chapter 3. The AKSZ formalism, which constructs a BV theory on a mapping space between supermanifolds equipped with additional geometric structures (see Thm. 2.1), likewise relies on a notion of Cartan calculus on this infinite-dimensional mapping space. In this section (see Def. 2.3.2, Thm. 2.1, and following discussion), we proceed as if such a notion were given. More concretely, we do the usual graded Cartan calculus manipulations on the infinite-dimensional AKSZ space of fields, i.e. a suitable mapping space, as if it were finite-dimensional.[(29)] Assuming the target is a graded vector space, we rigorously define this in Chapter 3 (see Rmk. 3.4.3). Note also that Definition 2.3.1 and the following discussion rely on infinite-dimensional heuristic arguments that will be understood within sThSmthSet (see Ex. 3.3.1 and proof of Prop. 3.3.2) Therefore, the infinite-dimensional constructions are formal until Chapter 3 provides the right categorical framework.

Definition 2.3.1. Consider two $\mathbb{Z}_{(2)}$-graded manifolds $\mathcal{M},\mathcal{N}$ and the set of $\mathbb{Z}_{(2)}$-graded algebra morphisms $C^\infty(\mathcal{N})\to C^\infty(\mathcal{M})$ denoted by $\mathrm{Mor}(\mathcal{M},\mathcal{N})=\mathrm{Hom}_{\mathbb{Z}_{(2)}\mathrm{SmthMfd}}(\mathcal{M},\mathcal{N})$ (see Def. 1.3.1). Informally, the **mapping space** $\mathrm{Map}(\mathcal{M},\mathcal{N})$ is the extension of $\mathrm{Mor}(\mathcal{M},\mathcal{N})$ that includes non-zero graded maps.[(30)] More formally, the category of $\mathbb{Z}_{(2)}$-graded manifolds admits a monoidal structure given by the product as in [Vys22][Ch. 3.6].[(31)] Then, we can define[(32)] $\mathrm{Map}(\mathcal{M},\mathcal{N})$ as the (usually infinite-dimensional) $\mathbb{Z}_{(2)}$-graded manifold such that

$$\mathrm{Mor}(\mathcal{W}\times\mathcal{M},\mathcal{N}) =: \mathrm{Mor}(\mathcal{W},\mathrm{Map}(\mathcal{M},\mathcal{N})).$$

Note that $\mathrm{Mor}(\mathcal{M},\mathcal{N})$ is naturally identified with the degree zero submanifold of $\mathrm{Map}(\mathcal{M},\mathcal{N})$. If $\mathcal{N}$ is a $\mathbb{Z}_{(2)}$-graded vector space, then

$$\mathrm{Map}(\mathcal{M},\mathcal{N}) = C^\infty(\mathcal{M})\otimes\mathcal{N}. \tag{2.22}$$

As discussed in [CF01; CS11], we can lift geometric structures from $\mathcal{M}$ and $\mathcal{N}$ to $\mathrm{Map}(\mathcal{M},\mathcal{N})$. In particular, the groups of invertible maps $\mathrm{Diff}(\mathcal{M})$ and $\mathrm{Diff}(\mathcal{N})$ act on the mapping space by left and right composition, respectively. Furthermore, these two actions commute. Differentiation yields commuting infinitesimal left and right actions of $\mathfrak{X}(\mathcal{M})$ and $\mathfrak{X}(\mathcal{N})$, respectively, on $\mathfrak{X}(\mathrm{Map}(\mathcal{M},\mathcal{N}))$. This allows us to lift vector fields as the image of such actions. More explicitly, recalling that for any $f\in\mathrm{Map}(\mathcal{M},\mathcal{N})$ the tangent space is $T_f\mathrm{Map}(\mathcal{M},\mathcal{N})\cong\Gamma_{\mathcal{M}}(f^*T\mathcal{N})$,[(33)] we get for all $D\in\mathfrak{X}(\mathcal{M})$ and $P\in\mathfrak{X}(\mathcal{N})$

$$D^{\mathrm{lift}}_f(x) \coloneqq \mathrm{d}_{\mathcal{M}}f(x)(D_x) \quad\text{and}\quad P^{\mathrm{lift}}_f(x)\coloneqq P(f(x))$$

with $[D^{\mathrm{lift}},Q^{\mathrm{lift}}]=0$, $\deg_{\mathbb{Z}_{(2)}}(D)=\deg_{\mathbb{Z}_{(2)}}(D^{\mathrm{lift}})$ and $\deg_{\mathbb{Z}_{(2)}}(P)=\deg_{\mathbb{Z}_{(2)}}(P^{\mathrm{lift}})$.
Before introducing the notion of AKSZ theory, we remark that the entire section is written within the classical BV/AKSZ framework. The derived-geometric approach is not required for this discussion, even though it provides a modern interpretation of the constructions above and below.

Definition 2.3.2. An **AKSZ theory** in d-dimensions is a tuple $(M,\mathcal{N},Q_{\mathcal{N}},\omega_{\mathcal{N}},\alpha_{\mathcal{N}},\Theta_{\mathcal{N}})$ where the **source** M is a closed (compact without boundaries) smooth oriented d-manifold and the **target** $\mathcal{N}$ is a finite-dimensional $\mathbb{Z}_{(2)}$-graded manifold. The latter is endowed with a cohomological vector field $Q_{\mathcal{N}}\in\mathfrak{X}(\mathcal{N})_1$ (see Def. 1.3.20), a symplectic form $\omega_{\mathcal{N}}\in\Omega^2(\mathcal{N})_{d-1}$ of total degree[(34)] $d+1$ (see Def. 1.4.7) such that $\omega_{\mathcal{N}}=\mathrm{d}_{\mathcal{N}}\alpha_{\mathcal{N}}$, with $\alpha_{\mathcal{N}}\in\Omega^1(\mathcal{N})_{d-1}$ of total degree d, and a Hamiltonian $\Theta_{\mathcal{N}}\in C^\infty(\mathcal{N})_d$ of total degree d that generates $Q_{\mathcal{N}}$ as its associated Hamiltonian vector field and such that $\{\Theta_{\mathcal{N}},\Theta_{\mathcal{N}}\}_{\omega_{\mathcal{N}}}=0$.[(35)]

Then, consider the following mapping space as the **AKSZ space of fields**

$$\mathcal{F}\coloneqq\mathrm{Map}(T[1]M,\mathcal{N}).$$

If $\mathcal{N}$ is a $\mathbb{Z}_{(2)}$-graded vector space, using Equation (2.22) we get $\mathcal{N}$-valued differential forms

$$\mathcal{F}\coloneqq\mathrm{Map}(T[1]M,\mathcal{N})\cong C^\infty(T[1]M)\otimes\mathcal{N}\cong\Omega^\bullet(M,\mathcal{N})\cong\Gamma_M\Big(\bigwedge\nolimits^\bullet T^*M\otimes(M\times\mathcal{N})\Big). \tag{2.23}$$

(29) Note that the Cartan calculus on the base and on the target is well defined, as they are both assumed to be finite dimensional.
(30) It is necessary to work with the full mapping space, not just the set of morphisms, because the mapping space naturally inherits a graded structure. In contrast, the set of morphisms captures only degree-zero maps and thus does not contain the higher-degree fields needed in the BV formalism.
(31) This can be equivalently given as a coproduct of locally ringed spaces.
(32) The mapping space is the right adjoint functor to the Cartesian product of graded manifolds, here given on objects.
(33) We can interpret this as follows: for each $m\in\mathcal{M}$ the infinitesimal variation of $f(x)$ is given by $v(m)\in T_{f(m)}\mathcal{N}$. This assignment should be smooth in m so that the infinitesimal variation of f is given by $v\in\Gamma_{\mathcal{M}}(f^*T\mathcal{N})$.
(34) We consider the total degree as in Remark 1.3.6.
(35) Due to arbitrariness of d, we cannot use Proposition 1.4.3. The latter holds if $d\neq-1$.

Fields are elements of the AKSZ space of fields. Locally, they can be viewed as differential forms on the source manifold taking values in the graded vector space on which the target graded manifold is locally modeled. Consider the evaluation map $\mathrm{ev}\colon T[1]M \times \mathcal{F} \to \mathcal{N}$ which, given a point $(u, \theta) \in T[1]M$ and a field $\varphi \in \mathcal{F}$, assigns the value of φ at x, i.e. $\mathrm{ev}(x, \varphi) = \varphi(x) \in \mathcal{N}$. Let $\pi_2\colon T[1]M \times \mathcal{F} \to \mathcal{F}$ be the projection onto the second component. Then, following the push-pull diagram

$$\begin{tikzcd} T[1]M \times \mathcal{F} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"'] & \mathcal{N} \\ \mathcal{F} & \end{tikzcd}$$

we can pullback a form $\beta \in \Omega^p(\mathcal{N})_j$ to $\mathrm{ev}^*\beta \in \Omega^p(T[1]M \times \mathcal{F})$ choosing the bi-degree $(0, p)$ component according to the bi-grading of the de Rham complex of a direct product. That is, choosing the component in $\big(C^\infty(T[1]M) \otimes \Omega^p(\mathcal{N})\big)_j$. Define $(\pi_2)_*$ as the integration against the canonical Berezinian on $T[1]M$ of the component of $\mathrm{ev}^*\beta$ in $C^\infty(T[1]M)_d \otimes \Omega^p(\mathcal{F})_{j-d}$. Then, the **transgression map** is given by

$$\mathbb{T} \coloneqq (\pi_2)_* \mathrm{ev}^* \colon \Omega^p(\mathcal{N})_j \to \Omega^p(\mathcal{F})_{j-d}.$$

While both ev^* and $(\pi_2)_*$ preserve the cohomological degree, the latter changes the internal degree by $-d$ as it assigns real numbers to elements of $C^\infty(T[1]M)_d$, or equivalently integrates the top forms $\Omega^d(M)$ over M. Working with total grading, $|(\pi_2)_*| = |\mathbb{T}| = -d$.

Theorem 2.1. *For any AKSZ theory $(M, \mathcal{N}, Q_\mathcal{N}, \omega_\mathcal{N}, \alpha_\mathcal{N}, \Theta_\mathcal{N})$ there is a classical BV theory $(\mathcal{F}, \omega, Q, S)$.*

1. *The graded manifold is the AKSZ space of fields $\mathcal{F} \coloneqq \mathrm{Map}(T[1]M, \mathcal{N})$.*
2. *The odd-symplectic structure is $\omega \coloneqq (-1)^d \mathbb{T}\omega_\mathcal{N} \in \Omega^2(\mathcal{F})_{-1}$. Furthermore, $\mathrm{d}_\mathcal{F}\alpha = \omega$ with $\alpha \coloneqq \mathbb{T}\alpha_\mathcal{N} \in \Omega^1(\mathcal{F})_{-1}$.*
3. *The cohomological vector field is $Q \coloneqq \mathrm{d}_M^{\mathrm{lift}} + Q_\mathcal{N}^{\mathrm{lift}} \in \mathfrak{X}(\mathcal{F})_1$, where $\mathrm{d}_M \in \mathfrak{X}(T[1]M)_1$ and $Q_\mathcal{N} \in \mathfrak{X}(\mathcal{N})_1$ are cohomological vector fields.*[36]
4. *The action is $S \coloneqq \iota_{\mathrm{d}_M^{\mathrm{lift}}} \mathbb{T}\alpha_\mathcal{N} + \mathbb{T}\Theta_\mathcal{N} \in C^\infty(\mathcal{F})_0$. Here, the source term $\iota_{\mathrm{d}_M^{\mathrm{lift}}} \mathbb{T}\alpha_\mathcal{N}$ can be understood as the kinetic term, while the target term $\mathbb{T}\Theta_\mathcal{N}$ is the interaction term.*

Remark 2.3.2. It is useful to express the above definitions in local coordinates. Let $\{x^a\}$ be local coordinates on $\mathcal{N}$ and $\{u^i, \theta^i \equiv du^i\}$ be local coordinates on $T[1]M$. We define the coordinate functions $X^a_{i_1,\ldots,i_k}/k! \in C^\infty(M)$ on the mapping space so that they are fully antisymmetric in $i_1, \ldots, i_k$ and $\deg \mathbb{Z}_{(2)}(X^a_{i_1,\ldots,i_k}/k!) = \deg \mathbb{Z}_{(2)}(x^a) - k$. We can write the superfields

$$\mathrm{ev}^* x^a = X^a = \sum_{k=0}^{d} \sum_{1 \leqslant i_1,\ldots,i_k \leqslant d} \frac{X^a_{i_1\cdots i_k}(u)}{k!} \theta^{i_1} \cdots \theta^{i_k} \tag{2.24}$$

and understand them as coordinate functions on $\mathcal{F}$ taking values in forms on M. Consider

$$\omega_\mathcal{N} \coloneqq \sum_{a,b} \frac{\omega_{ab}(x)}{2} \mathrm{d}_\mathcal{N} x^a \wedge \mathrm{d}_\mathcal{N} x^b \quad \text{and} \quad \alpha_\mathcal{N} \coloneqq \sum_a \alpha_a(x) \mathrm{d}_\mathcal{N} x^a.$$

Notice that each superfield X^a can be decomposed into components $X^a_{i_1\cdots i_k}(u)/k!$ of fixed cohomological degree k. In local coordinates, following the push-pull diagram above corresponds to symbolically replacing the coordinates x^a with their associated superfields (see Def. 2.24), expanding in components of fixed degree and keeping only cohomological top-degree terms which are then integrated over M. More explicitly, recalling the definition of lifting (see discussion after Eq. (2.22)), we get

$$\omega = (-1)^d \mathbb{T}(\omega_\mathcal{N}) = (-1)^d \underbrace{\int_M}_{(\pi_2)_*} \underbrace{\sum_{a,b} \frac{\omega_{ab}(X)}{2} \mathrm{d}_\mathcal{F} X^a \wedge \mathrm{d}_\mathcal{F} X^b}_{\mathrm{ev}^*\omega_\mathcal{N}};$$

[36] Note that it is natural to define the AKSZ formalism within the category of dg-manifolds.

$$\begin{aligned}
\alpha &= \mathbb{T}(\alpha_{\mathcal{N}}) = \int_M \sum_a \alpha_a(X) \mathrm{d}_{\mathcal{F}} X^a \\
Q &= \int_M (\mathrm{d}_M X^a)_{i_1\cdots i_k}(u) \partial_{X^a_{i_1\cdots i_k}(u)} + Q_{\mathcal{N}}(X(u))^a \partial_{X^a(u)} \quad \text{with} \quad \mathrm{d}_M = \sum_i \theta^i \partial_{u^i}; \\
S &= \int_M \sum_a \alpha_a(X) \mathrm{d}_M X^a + \Theta_{\mathcal{N}}(X).
\end{aligned}$$

For an explicit low-dimensional example, see the case of Chern–Simon theory (cf. [Mne19][Sec. 4.9.2]).

Proof. The grade of the objects is an immediate consequence of the definitions. The graded nature of $\mathcal{F}$ is discussed in Definition 2.3.1. Since d_M and $Q_{\mathcal{N}}$ are cohomological, $Q^2 = 0$ follows by the definition of the lift. Let us now show that $\mathbb{T}\mathrm{d}_{\mathcal{N}}\varepsilon = (-1)^d \mathrm{d}_{\mathcal{F}}\mathbb{T}\varepsilon$ for any $\varepsilon \in \Omega^p(\mathcal{N})_j$. By coordinate inspection, it is clear that $\mathrm{ev}^*(\mathrm{d}_{\mathcal{N}}\varepsilon) = \mathrm{d}_{\mathcal{F}}\mathrm{ev}^*\varepsilon$. The de Rham differential on $T[1]M \times \mathcal{F}$ reads $\mathrm{d}_M \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d}_{\mathcal{F}}$ on the relevant components of $\mathrm{ev}^*\varepsilon$ in $(C^\infty(T[1]M) \otimes \Omega^p(\mathcal{F}))_j \cong (\Omega^\bullet(M) \otimes \Omega^p(\mathcal{F}))_j$. Moreover, recall that $\mathrm{id} \otimes \mathrm{d}_{\mathcal{F}}(\beta \otimes \gamma) = (-1)^{\deg_{\mathbb{Z}_{(2)}}(\beta)} \beta \otimes \mathrm{d}_{\mathcal{F}}\gamma$. Then,

$$\int_M \mathrm{ev}^*(\mathrm{d}_{\mathcal{N}}\varepsilon) = \int_M (\mathrm{d}_M \otimes \mathrm{id})\mathrm{ev}^*\varepsilon + \int_M (\mathrm{id} \otimes \mathrm{d}_{\mathcal{F}})\mathrm{ev}^*\varepsilon \overset{\text{Stokes}}{=} (-1)^d \mathrm{d}_{\mathcal{F}} \int_M \mathrm{ev}^*\varepsilon.$$

It follows that $\mathrm{d}_{\mathcal{F}}\alpha \coloneqq \mathrm{d}_{\mathcal{F}}\mathbb{T}\alpha_{\mathcal{N}} = (-1)^d \mathbb{T}\mathrm{d}_{\mathcal{N}}\alpha_{\mathcal{N}} = (-1)^d \mathbb{T}\omega_{\mathcal{N}} = \omega$, therefore ω is exact and closed. By inspection, it is evident that non-degeneracy of $\omega_{\mathcal{N}}$ implies that ω is weakly non-degenerate. If we show that S is the Hamiltonian of Q we conclude since this, together with $Q^2 = 0$, implies CME (see Prop. 1.4.3). Using $\iota_{Q^{\text{lift}}_{\mathcal{N}}} \mathbb{T}\omega_{\mathcal{N}} = \mathbb{T}(\iota_{Q_{\mathcal{N}}}\omega_{\mathcal{N}})$ we get

$$\begin{aligned}
\iota_Q \omega &= (-1)^d \big(\iota_{\mathrm{d}^{\text{lift}}_M} \mathbb{T}\omega_{\mathcal{N}} + \iota_{Q^{\text{lift}}_{\mathcal{N}}} \mathbb{T}\omega_{\mathcal{N}}\big) = \iota_{\mathrm{d}^{\text{lift}}_M} \mathrm{d}_{\mathcal{F}}\mathbb{T}\alpha_{\mathcal{N}} + (-1)^d \mathbb{T}(\iota_{Q_{\mathcal{N}}}\omega_{\mathcal{N}}) = \\
&\overset{\text{Cartan}}{=} \underbrace{\mathbb{L}_{\mathrm{d}^{\text{lift}}_M} \mathbb{T}\alpha_{\mathcal{N}}}_{0} + \mathrm{d}_{\mathcal{F}} \iota_{\mathrm{d}^{\text{lift}}_M} \mathbb{T}\alpha_{\mathcal{N}} + \mathrm{d}_{\mathcal{F}}\mathbb{T}\Theta_{\mathcal{N}} = \mathrm{d}_{\mathcal{F}} S,
\end{aligned}$$

where we used that $\mathbb{L}_{\mathrm{d}^{\text{lift}}_M} \mathbb{T}(\cdots) \sim \int_M \mathrm{d}_M(\cdots) \overset{\text{Stokes}}{=} 0$. ■

The BV theory constructed from an AKSZ theory as above can be regarded as a BV extension of the classical gauge theory[(37)]

$$\big(\mathcal{F}_{\text{cl}} \coloneqq \mathrm{Map}(T[1]M, \mathcal{N})_0 \cong \mathrm{Mor}(\mathrm{T}[1]\mathrm{M}, \mathcal{N}),\ S_{\text{cl}} \coloneqq S|_{X^a \to X^a_{\text{cl}}}\big).$$

Here, X^a_{cl} is the zero ghost degree part of X^a (see Eq. (2.24)), that is,

$$X^a_{\text{cl}} = \begin{cases} \sum_{1 \leqslant i_1, \ldots, i_k \leqslant d} \frac{X^a_{i_1\cdots i_k}(u)}{k!} \theta^{i_1} \cdots \theta^{i_k}, & \text{if} \quad 0 \leqslant \deg_{\mathbb{Z}_{(2)}}(x^a) = k \leqslant d \\ 0, & \text{otherwise.} \end{cases}$$

The gauge symmetry generators are in $\mathrm{Map}(T[1]M, \mathcal{N})_1$, and all the gauge data can be read from the action S. Finally, as in point 3. of Remark 2.2.5, the Euler–Lagrange locus is given by $\{Q = 0\} \cap \mathcal{F}_{\text{cl}}$. Elements in it can be thought of as anti-*dg* morphisms $\phi\colon (T[1]M, \mathrm{d}_M) \to (\mathcal{N}, Q_{\mathcal{N}})$ such that $\phi^* Q_{\mathcal{N}} = -\mathrm{d}_M \phi^*$, by definition of lift vector fields and $Q\phi = 0$. This explicitly shows how the AKSZ formalism is naturally understood within the category of *dg*-manifolds and how the AKSZ action geometrically encodes both source and target cohomological structures. In particular, the kinetic term encodes the source differential, while the interaction term encodes the target hamiltonian, or equivalently the source cohomological vector field.

As an example, we discuss the BF theory in d-dimensions. It is an example of a Schwarz-type topological field theory which exists in any dimension (see [Mne19][Ch. 4.9.4] for $d = 0, 1$). Consider a finite-dimensional compact Lie group G, its Lie algebra $\mathfrak{g}$, and a closed oriented smooth d-manifold M with $d \geqslant 2$. The theory is defined by

$$S_{\text{cl}} \coloneqq \int_M \langle B \,\overset{\wedge}{,}\, F_A \rangle,$$

(37) In case $\mathcal{N}$ is non negatively graded, that implies $0 \leqslant \deg_{\mathbb{Z}_{(2)}}(x^a) \leqslant \deg_{\mathbb{Z}_{(2)}}(\omega_{\mathcal{N}}) = (d-1)$, it is possible to obtain S form S_{cl} by substituting $X^a_{\text{cl}} \to X^a$, as we do for BF theory in $d \geqslant 2$ later on.

where $A \in \Omega^1(M,\mathfrak{g})$ is a 1-form connection on the trivial G-bundle on M,[(38)] $F_A = \mathrm{d}_A A \coloneqq \mathrm{d}A + \frac{1}{2}[A,A] \in \Omega^2(M,\mathfrak{g})$ is its curvature, and $B \in \Omega^{d-2}(M,\mathfrak{g}^*)$. Note that $[-,-]$ is the Lie bracket and $\langle -\,\overset{\wedge}{,}\,-\rangle$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ that further computes the wedge product. The Euler–Lagrange equations are[(39)]

$$F_A = 0 \quad \text{and} \quad \mathrm{d}_A B \coloneqq (\mathrm{d}_M + \mathrm{ad}^*_A)B = 0.$$

The gauge transformations leaving the action invariant are

$$(A,B) \mapsto (g^{-1}Ag + g^{-1}\mathrm{d}_M g, \mathrm{Ad}^*_{g^{-1}}(B)) \quad \text{and} \quad (A,B) \mapsto (a, B + \mathrm{d}_A t), \tag{2.25}$$

where $C^\infty(M,G) \ni g\colon M \to G$ and $t \in \Omega^{d-3}(M,\mathfrak{g}^*)$. Recalling that the gauge action on A induces $F_A \mapsto \mathrm{Ad}_{g^{-1}} F_A$ (cf. [Ham17][Thm. 5.6.3 & Thm. 7.3.2]), the first symmetry comes from $\langle \mathrm{Ad}^*_{g^{-1}}(B) \overset{\wedge}{,} \mathrm{Ad}_{g^{-1}}F_A\rangle = \langle B \overset{\wedge}{,} \mathrm{Ad}_g \mathrm{Ad}_{g^{-1}} F_A\rangle = \langle B \overset{\wedge}{,} F_A\rangle$. To show the second symmetry, we first split $\mathrm{d}_A t = \mathrm{d}_M t + \mathrm{ad}^*_A t$ so that the extra terms are $\langle \mathrm{d}_M t, F_A\rangle$ and $\langle \mathrm{ad}^*_A t, F_A\rangle$. Then, we conclude by using the (graded) Leibniz rule for d_M, the Stokes theorem, the definition of coadjoint action (see ft. (39), p. 49), and the Bianchi identity $\mathrm{d}_A F_A = 0$ (cf. [Ham17][Thm. 5.14.2]). Thus, the gauge group acting on the classical space of fields $\mathcal{F}_{\mathrm{cl}} = \Omega^1(M,\mathfrak{g}) \oplus \Omega^{d-2}(M,\mathfrak{g}^*)$ is the non-abelian group $\mathcal{G} = C^\infty(M,G) \ltimes \Omega^{d-3}(M,\mathfrak{g}^*)$. We can go further and, using $\mathrm{d}^2_A t' = -f^c_{ab} F^a_A t'_c T^b$ (cf. [Ham17][Ex. 5.15.12]) and (graded) Jacobi, notice that the stabilizer of a solution (A,B) contains the orbit of the unit of $\mathcal{G}$ given by the shift $(g,t) \mapsto (\tilde{g},\tilde{t}) = (g, t + \mathrm{d}_A t')$, with $t' \in \Omega^{d-4}(M,\mathfrak{g}^*) =: \mathcal{G}_2$. Indeed,

$$\mathcal{G}_{(A,B)} \coloneqq \{(g,t) \in \mathcal{G} | (g,t)\cdot(A,B) = (A,B)\} \supset \{t' \cdot (e,0) | t' \in \mathcal{G}_2\}.$$

Furthermore, these shifts are the same under the shift $t' \mapsto \tilde{t}' = t' + \mathrm{d}_A t''$, with $t'' \in \Omega^{d-5} =: \mathcal{G}_3$. We can continue like this up to $\mathcal{G}_{d-2}$ obtaining a tower of reducibility of gauge symmetry

$$\mathcal{G}_{d-2} \circlearrowright \cdots \circlearrowright \mathcal{G}_2 \circlearrowright \mathcal{G} \circlearrowright \mathcal{EL} \tag{2.26}$$

where $\mathcal{G}_{k\geqslant 2} = \Omega^{d-2-k}(M,\mathfrak{g}^*)$ are abelian and act on $\mathcal{G}_{k-1}$ by translation $t_{k-1} \mapsto t_{k-1} + \mathrm{d}_A t_k$. This means that in the BV extension, we need to introduce higher ghosts for ghosts to deal with this tower of gauge symmetries. The infinitesimal picture is given by an action Lie algebroid over $\mathcal{F}_{\mathrm{cl}}$ whose anchor ρ corresponds to the infinitesimal versions of the gauge transformations

$$(A,B) \mapsto (A + \epsilon \mathrm{d}_A\gamma, B - \epsilon\, \mathrm{ad}^*_\gamma(B)) \quad \text{and} \quad (A,B) \mapsto (a, B + \epsilon \mathrm{d}_A t),$$

where $(\gamma,t) \in \Omega^0(M,\mathfrak{g}) \oplus \Omega^{d-3}(M,\mathfrak{g}^*)$. The infinitesimal resolution of the tower above is a complex of vector bundles on $\mathcal{EL}$ (cf. [Mne19][Sec. 4.9.4] for more details)

$$\underbrace{\Omega^0(M,\mathfrak{g}^*)}_{\mathrm{Lie}(\mathcal{G}_{d-2})} \xrightarrow{\mathrm{d}_A} \cdots \xrightarrow{\mathrm{d}_A} \underbrace{\Omega^{d-4}(M,\mathfrak{g}^*)}_{\mathrm{Lie}(\mathcal{G}_2)} \xrightarrow{\mathrm{d}_A} \underbrace{\Omega^0(M,\mathfrak{g}) \otimes \Omega^{d-3}(M,\mathfrak{g}^*)}_{\mathrm{Lie}(\mathcal{G})} \xrightarrow{\rho} T\mathcal{EL}.$$

Consider the AKSZ theory $(M,\mathcal{N},Q_\mathcal{N},\omega_\mathcal{N},\alpha_\mathcal{N},\Theta_\mathcal{N})$ where the source M is as above and the target is $\mathcal{N} \coloneqq \mathfrak{g}[1] \oplus \mathfrak{g}^*[d-2]$ with coordinates (ψ^a,ξ_a) and generating functions $\psi = \sum_a \psi^a T_a \in C^\infty(\mathcal{N},\mathfrak{g})_1$ and $\xi = \sum_a \xi_a T^a \in C^\infty(\mathcal{N},\mathfrak{g}^*)_{d-2}$. The cohomological vector field is the Chevalley-Eilenberg differential in $C^\infty(\mathcal{N}) \cong C^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}^\bullet(\mathfrak{g}^*[d-2]))$ explicitly given by

$$Q_\mathcal{N} \coloneqq \frac{1}{2}\langle [\psi,\psi], \partial_\psi\rangle + \langle \mathrm{ad}^*_\psi(\xi), \partial_\xi\rangle = \sum_{a,b,c} \frac{1}{2} f^a_{bc}\psi^b\psi^c\partial_{\psi^a} - \sum_{a,b,c} f^a_{b,c}\psi^b\xi_a\partial_{\xi_c} \in \mathfrak{X}(\mathcal{N})_1.$$

It squares to zero by the Jacobi identity for the structure constants of $\mathfrak{g}$ once we use the coadjoint action as the anchor map of the Lie algebroid (see Eq. (2.13) and the following discussion). Finally, the symplectic form and the Hamiltonian are given by

$$\omega_\mathcal{N} \coloneqq \langle \mathrm{d}_\mathcal{N}\xi \overset{\wedge}{,} \mathrm{d}_\mathcal{N}\psi\rangle = \sum_a \mathrm{d}_\mathcal{N}\xi_a \wedge \mathrm{d}_\mathcal{N}\psi^a = \mathrm{d}_\mathcal{N} \sum_a \xi_a \mathrm{d}_\mathcal{N}\psi^a = \mathrm{d}_\mathcal{N} \underbrace{\langle \xi, \mathrm{d}_\mathcal{N}\psi\rangle}_{\alpha_\mathcal{N} \in \Omega^1(\mathcal{N})_{d-1}} \in \Omega^2(\mathcal{N})_{d-1};$$

$$\Theta_\mathcal{N} \coloneqq (-1)^{\deg_\mathbb{Z}(\theta_\mathcal{N})}\{Q_\mathcal{N}, -\}_{\omega_\mathcal{N}} = \frac{1}{2}\langle \xi, [\psi,\psi]\rangle = \sum_{a,b,c}\frac{1}{2} f^a_{bc}\xi_a\psi^b\psi^c \in C^\infty(\mathcal{N})_d.$$

(38) More generally, we could consider a non-trivial principal G-bundle over M (cf. [Mne19][Rmk. 4.9.6]).

(39) Recall that $\mathrm{Ad}_g h \coloneqq ghg^{-1}$, $\langle \mathrm{Ad}^*_g k, h\rangle \coloneqq \langle k, \mathrm{Ad}_{g^{-1}}h\rangle$, $\mathrm{ad}_t h = [t,h]$, and $\langle \mathrm{ad}^*_t k, h\rangle \coloneqq \langle k, -\mathrm{ad}_t h\rangle$, for all $g \in G$, $t,h \in \mathfrak{g}$ and $k \in \mathfrak{g}^*$. Let $\{T_a\}$ ($\{T^a\}$) be a basis for $\mathfrak{g}^{(*)}$, then $\mathrm{ad}_t h = \sum_{a,b,c} f^c_{ab} t^a h^b T_c$ and $\mathrm{ad}^*_t k = -\sum_{a,b,c} f^c_{ab} t^a k_c T^b$. If $\mathfrak{g}$ is $\mathbb{Z}_{(2)}$-graded, the coordinate expressions remain valid as the Koszul sign appears twice.

Since $d \neq 1$, $Q_{\mathcal{N}}^2 = 0$ implies $\{\Theta_{\mathcal{N}}, \Theta_{\mathcal{N}}\} = 0$ (see Prop. 1.4.3). We could also show this by direct inspection using the Jacobi identity for the structure constant of $\mathfrak{g}$. Then, following the AKSZ procedure, we can build the BV extension of BF theory ($d \geqslant 2$). Consider the space of fields (see Eq. (2.22))

$$\mathcal{F} = \Omega^\bullet(M,\mathfrak{g})[1] \oplus \Omega^\bullet(M,\mathfrak{g})[d-2] \ni (\mathcal{A},\mathcal{B}) \quad \text{where} \tag{2.27}$$
$$\mathcal{A} \coloneqq \mathrm{ev}^*\psi = c + A + B^+ + \tau_1^+ + \cdots + \tau_{d-2}^+, \quad \mathcal{B} \coloneqq \mathrm{ev}^*\xi = \tau_{d-2} + \cdots + \tau_1 + B + A^+ + c^+.$$

The ghost and the cohomological degree of the fields are such that $|\mathcal{A}| = 1$ and $|\mathcal{B}| = d-2$.

	c	A	B^+	τ_1^+	$\cdots$	τ_{d-2}^+	τ_{d-2}	$\cdots$	τ_1	B	A^+	c^+
dR	0	1	2	3	$\cdots$	d	0	$\cdots$	d-3	d-2	d-1	d
gh	1	0	-1	-2	$\cdots$	1-d	d-2	$\cdots$	1	0	-1	-2

Ghost number 1 field components are the ghost fields for the gauge transformation (see Eq. (2.25)) of the classical fields of the theory. The latter are those with ghost number 0. The higher ghosts are ghosts for the translations given by $\mathcal{G}_{k\geqslant 2}$. Each field of degree (k,j) has an antifield of degree $(d-k,-1-j)$. Then, the BV extension is

$$\begin{aligned}
\omega =\ & (-1)^d \int_M \langle \mathrm{d}_{\mathcal{F}}\mathcal{B} \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}\mathcal{A}\rangle \overset{d=4}{=} (-1)^d \int_M \langle \mathrm{d}_{\mathcal{F}}A \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}A^+\rangle + \langle \mathrm{d}_{\mathcal{F}}B \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}B^+\rangle + \langle \mathrm{d}_{\mathcal{F}}c \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}c^+\rangle + \\
& \qquad + \langle \mathrm{d}_{\mathcal{F}}\tau_1 \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}\tau_1^+\rangle + \langle \mathrm{d}_{\mathcal{F}}\tau_2 \,\overset{\wedge}{,}\, \mathrm{d}_{\mathcal{F}}\tau_2^+\rangle; \qquad (2.28)\\
S =\ & \int_M \langle \mathcal{B} \,\overset{\wedge}{,}\, \mathrm{d}_M\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}]\rangle \overset{d=4}{=} \int_M \langle B \,\overset{\wedge}{,}\, F_A\rangle + \langle A^+ \,\overset{\wedge}{,}\, \mathrm{d}_A c\rangle + \langle B^+ \,\overset{\wedge}{,}\, \mathrm{ad}_c^*(B) + \mathrm{d}_A\tau_1\rangle + \\
& + \langle \tau_1^+ \,\overset{\wedge}{,}\, \mathrm{ad}_c^*(\tau_1) + \mathrm{d}_A\tau_2\rangle + \frac{1}{2}\langle c^+ \,\overset{\wedge}{,}\, [c,c]\rangle + \langle \tau_2^+ \,\overset{\wedge}{,}\, \mathrm{ad}_c^*(\tau_2)\rangle + \frac{1}{2}\langle \tau_2 \,\overset{\wedge}{,}\, [B^+,B^+]\rangle. \qquad (2.29)\\
Q =\ & \int_M \langle \mathrm{d}_M\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}], \partial_{\mathcal{A}}\rangle + \langle \mathrm{d}_M\mathcal{B} + \mathrm{ad}^*_{\mathcal{A}}\mathcal{B}, \partial_{\mathcal{B}}\rangle \overset{d=4}{=} \int_M \frac{1}{2}\langle [c,c], \partial_c\rangle + \langle \mathrm{d}_A c, \partial_A\rangle + \\
& + \langle F_A + [c,B^+], \partial_B^+\rangle + \langle \mathrm{d}_A B^+ + [c,\tau_1^+], \partial_{\tau_1^+}\rangle + \langle \mathrm{d}_A\tau_1^+ + [c,\tau_2^+] + \frac{1}{2}[B^+,B^+], \partial_{\tau_2^+}\rangle + \\
& \quad + \langle \mathrm{ad}_c^*\tau_2, \partial_{\tau_2}\rangle + \langle \mathrm{ad}_c^*\tau_1 + \mathrm{d}_A\tau_2, \partial_{\tau_1}\rangle + \langle \mathrm{ad}_c^* B + \mathrm{d}_A\tau_1 + \mathrm{ad}^*_{B^+}\tau_2, \partial_B\rangle + \\
& \qquad + \langle \mathrm{ad}_c^* A^+ + \mathrm{d}_A B + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2, \partial_{A^+}\rangle + \\
& \qquad\quad + \langle \mathrm{ad}_c^* c^+ + \mathrm{d}_A A^+ + \mathrm{ad}^*_{B^+} B + \mathrm{ad}^*_{\tau_1^+}\tau_1 + \mathrm{ad}^*_{\tau_2^+}\tau_2, \partial_{c^+}\rangle, \qquad (2.30)
\end{aligned}$$

which means Q acts on fields as $Qc = \frac{1}{2}[c,c]$, $QA = \mathrm{d}_A c$ and so on. To arrive at these explicit expressions, we used degree considerations, the definition of (co)adjoint action (see ft. (39), p. 49), the (graded) Leibniz rule for d_M, and the Stokes theorem. The case $d = 4$ is interesting: S contains a quadratic term in the antifields, contrary to any BV action given by the extension of a BRST theory (see Eq. (2.18)). Furthermore, the theory also contains one higher ghost τ_2. These characteristics are not present for $d < 4$ and only appear for $d \geqslant 4$. Notice that we could use the Faddeev–Popov method or the BRST formalism to treat BF in $d < 4$. However, the emergence of these characteristics indicates that the BV formalism is essential for treating BF in dimensions $d \geqslant 4$. This is ultimately because the gauge symmetry becomes complicated (see Eq. (2.26)).
We now introduce the category of sThSmthSet. In this category, at least locally and with some reasonable restriction, we can make sense of the notions of differential geometry on the infinite-dimensional space of fields $\mathcal{F}$, in particular the Cartan calculus, which so far we formally used as if $\mathcal{F}$ were finite-dimensional.

Chapter 3

Super Thickened Smooth Set Framework for Gauge Field Theories

In this chapter, we introduce the category sThSmthSet and show how it provides a natural setting for treating BV theories locally. In particular, working within this category, the differential geometry of the field space emerges formally as we would expect if it were a finite-dimensional $\mathbb{Z}_2$-graded smooth manifold.
First, following [GS25a], we lay out the natural formulation of bosonic variational field theory via the functorial geometry of smooth sets (see Def. 3.1.1). The category of smooth sets, namely the topos over the site of Cartesian spaces with smooth maps between them, is shown to satisfy the requirements in Section 2.1. Crucially, it provides a notion of the local Cartan calculus on the space of fields and the associated variational bicomplex.
Afterward, we recall how this category can be extended to the category ThSmthSet of infinitesimally thickened smooth sets (cf. [GS25b]). This extension allows us to treat, at the same time, the smoothness and the infinitesimal structure of the space of fields in a coordinate-free manner. Importantly, it subsumes all the constructions previously formulated within SmthSet. In particular, it recovers the ad hoc tangent bundles arising in traditional field-theoretic considerations via the synthetic tangent bundle construction. It also suggests the rigorous formalization of perturbative field theory[(1)] and further allows for generalizations for field theories over manifolds with boundaries and corners (cf. [GS25b, App. 4.2]).
Consequently, we sketch how the natural extension to the category sThSmthSet of super infinitesimally thickened smooth sets should unravel, following the hinted approach given in [Gio25; GSS24].
Finally, we use this setting to make a rigorous sense of the local picture of a BV system, taking the (non)-abelian BF theory in $d = 4$ as an example.

3.1 Smooth Sets

In this section, we define the category of smooth sets and show how it incorporates many field theoretical objects. In particular, the space of fields, the infinite jet bundle, the local variational bicomplex, and the associated Cartan calculus that, once transgressed, gives the notion of Cartan calculus on the space of fields. A complete description can be found in [GS25a].

Definition of Smooth Sets

An intuitive approach to smooth sets can be given from an operational physics perspective, in analogy with the string-theoretical concept of probe branes. Crucially, the Yoneda lemma for smooth sets ensures the consistency of the definition. The idea is that we can explore a generalized space $\mathcal{G}$ through the trajectories, also called plots, of known probe branes.[(2)] In general, a brane of dimension p sweeps out

[(1)] This is sketched only for 0-dimensional field theory by restricting to a synthetic infinitesimal neighborhood around a field configuration (cf. [GS25b, pp. 27-28]).
[(2)] For instance, we can sample the structure and geometry of spacetime via a one-dimensional trajectory given by the worldline of a point particle, which is a zero-dimensional brane.

a $(p+1)$-dimensional world volume. More formally, given any finite-dimensional manifold Σ, that is the probe brane's world volume, we call plot the smooth trajectory $\Sigma \to \mathcal{G}$. Instead of defining $\mathcal{G}$ as a set of points with extra structure, we follow the idea of characterizing $\mathcal{G}$ via the system of these plots. The following heuristic requirements describe the minimum structure needed for this to be feasible. They are formalized below by defining smooth sets as sheaves on the site CartSp.[(3)]

1. For each probe manifold Σ, there should be a set of Σ-shaped smooth plots $\mathrm{Plots}(\Sigma, \mathcal{G})$.

2. Precomposition of a plot with a smooth map between probe manifolds should be a plot. Precomposition with the identity map should be the identity on the set of plots, and precomposition with two successive maps should satisfy the following:

$$\mathrm{SmthMfd}^{\mathrm{op}} \xrightarrow{\mathrm{Plots}(-,\mathcal{G})} \mathbf{Set}$$

$$\begin{tikzcd}
\Sigma \arrow[r, mapsto] & \mathrm{Plots}(\Sigma, \mathcal{G}) \arrow[d, "f^*"] \arrow[dd, bend left=60, "(f\circ g)^*"] \\
\Sigma' \arrow[u, "f"] \arrow[r, mapsto] & \mathrm{Plots}(\Sigma', \mathcal{G}) \arrow[d, "g^*"] \\
\Sigma'' \arrow[u, "g"] \arrow[uu, bend left=60, "f\circ g"] \arrow[r, mapsto] & \mathrm{Plots}(\Sigma'', \mathcal{G}).
\end{tikzcd}$$

We can recognize in the above the conditions characterizing a contravariant functor. Therefore, $\mathrm{Plots}(-, \mathcal{G})$ should be a presheaf on SmthMfd.

3. We should have a notion of locality and a way of consistently gluing information to go from local to global. Consider a differentiably-good open cover of the probe manifold $\{\Sigma_i \xhookrightarrow{\iota_i} \Sigma\}_{i\in I}$, i.e. a collection $\{\Sigma_i \subset \Sigma\}_{i\in I}$ of open subsets of Σ such that $\Sigma = \bigcup_{i\in I} \Sigma_i$ and such that all Σ_i, together with all their inhabited finite intersections, are diffeomorphic to an open ball. Then, Σ-plots should be the same as I-tuples of the overlapping Σ_i-plots. That is, the following should be a bijective decomposition:

$$\begin{aligned}\mathrm{Plots}(\Sigma, \mathcal{G}) &\longrightarrow \left\{(p_i \in \mathrm{Plots}(\Sigma_i, \mathcal{G}))_{i\in I} \,\middle|\, \forall i,j \in I \text{ with } p_i|_{\Sigma_i\cap\Sigma_j} = p_j|_{\Sigma_i\cap\Sigma_j}\right\} \\ (\Sigma \xrightarrow{p} \mathcal{G}) &\longmapsto (\Sigma_i \xhookrightarrow{\iota_i} \Sigma \xrightarrow{p} \mathcal{G})_{i\in I}.\end{aligned} \tag{3.1}$$

The latter is the sheaf condition (see Def. 1.2.6) on the site SmthMfd with respect to the Grothendieck topology (see Def. 1.2.4) induced by the pretopology of differentiably good open covers (see Def. 1.2.5). The plots are maps from the defining site.

4. Morphisms should preserve the internal structure. Generalized spaces are fully determined by their consistent system of plots. Therefore, a morphism $h\colon \mathcal{G} \to \mathcal{H}$ is completely characterized by a set of maps between $\mathbb{R}^k$-shaped plots $h_{\mathbb{R}^k}\colon \mathcal{G}(\mathbb{R}^k) \to \mathcal{H}(\mathbb{R}^k)$ that must be consistent with respect to pullback of probes. That is, the diagram

$$\begin{tikzcd}
\mathcal{G}(\mathbb{R}^k) \arrow[r, "h_{\mathbb{R}^k}"] \arrow[d, "\mathcal{G}(f)"] & \mathcal{H}(\mathbb{R}^k) \arrow[d, "\mathcal{H}(f)"] \\
\mathcal{G}(\mathbb{R}^{\tilde{k}}) \arrow[r, "h_{\mathbb{R}^{\tilde{k}}}"] & \mathcal{H}(\mathbb{R}^{\tilde{k}})
\end{tikzcd}$$

should commute for all $\mathbb{R}^k$, $\mathbb{R}^{\tilde{k}}$, and $f\colon \mathbb{R}^{\tilde{k}} \to \mathbb{R}^k$. This is precisely the definition of a natural transformation.

Finally, note that by definition smooth n-manifolds are covered by Cartesian spaces $\mathbb{R}^n$. Therefore, using the sheaf condition of Eq. (3.1), we can show the equivalence of categories $\mathrm{Sh}(\mathrm{SmthMfd}) \cong \mathrm{Sh}(\mathrm{CartSp})$, i.e. it is sufficient to probe $\mathcal{G}$ using $\mathbb{R}^n$, $n \in \mathbb{N}$.

[(3)] **CartSp** is the category of Cartesian spaces with morphisms being smooth maps in the classical sense.

Definition 3.1.1. The category of **smooth sets** is the category of sheaves over the site of Cartesian spaces with respect to the differentiably-good open covers:

$$\mathbf{SmthSet} \coloneqq \mathrm{Sh}(\mathrm{CartSp}),\ \mathcal{G}(-) \coloneqq \mathrm{Plots}(-, \mathcal{G}).$$

Remark 3.1.1. We would like smooth manifolds to be smooth in the sense of SmthSet. This is true for any $M \in \mathrm{SmthMfd}$ by taking $\mathrm{Plots}(\mathbb{R}^k, M) \coloneqq C^\infty(\mathbb{R}^k, M)$, $k \in \mathbb{N}$, since for all $f\colon \mathbb{R}^{k'} \to \mathbb{R}^k$, $k' \in \mathbb{N}$, the corresponding pullback of plots is the usual precomposition $(-)\circ f$ of smooth maps in SmthMfd. There is, however, a potential inconsistency in the definition: we bootstrap Definition 3.1.1 assuming the existence of a map $\Sigma \to \mathcal{G}$, but now we know that these two objects live in the same category, so such map should now consistently send $\mathbb{R}^k$-plots of $y(M)$ to $\mathbb{R}^k$-plots of $\mathcal{G}$. In other words, the defining plots $\mathcal{G}(M)$ should coincide with the smooth maps $\mathrm{Hom}_{\mathrm{SmthSet}}(y(M), \mathcal{G})$, i.e. with the natural transformations between the two sheaves. This is the case due to Proposition 3.1.1, that is, the Yoneda Lemma for smooth sets.

Proposition 3.1.1. *For $\mathcal{G} \in \mathrm{SmthSet}$ and $M \in \mathrm{SmthMfd}$, there is a natural bijection between the M-plots of $\mathcal{G}$ and the smooth maps of smooth sets from $y(M)$ to $\mathcal{G}$:*

$$\mathcal{G}(M) \equiv \mathrm{Plots}(M, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{SmthSet}}(y(M), G).$$

Here, $y(M) \coloneqq \mathrm{Hom}_{\mathrm{SmthMfd}}(-, M)$ is the image via Yoneda embedding of the smooth manifold M into the category of smooth sets.
Choosing $\mathcal{G} = y(N) \in \mathrm{SmthSet}$ for an arbitrary manifold N, the above immediately implies that the embedding functor

$$y\colon \mathrm{SmthMfd} \hookrightarrow \mathrm{SmthSet},\ M \mapsto y(M) \coloneqq \mathrm{Hom}_{\mathrm{SmthMfd}}(-, M)$$

is fully faithful. Hence, any results and constructions on finite-dimensional smooth manifolds may equivalently be phrased in terms of their smooth set incarnation and vice versa.

A proof of the statement can be found in [GS25a, Prop. 2.5].
Consequently, Definition 3.1.1 is well posed and all the properties of categories of sheaves discussed in Chapter 1 apply. Note that this means that all limits and colimits exist in SmthSet and they are computed pointwise (see Prop. 1.2.1). Consequently, requirement 4 in Section 2.1 is already fulfilled as far as we can express the critical locus as such a pullback.[(4)]

Vector Fields and Differential Forms on the Field Space

The traditional approach to formalizing field spaces is to model them as infinite-dimensional Fréchet manifolds, i.e. modeled on Fréchet spaces instead of Cartesian spaces (cf. [KM97]).

Definition 3.1.2. A **Fréchet manifold** is a Hausdorff topological space X with an atlas of coordinate charts over Fréchet spaces whose transitions are smooth mappings in the sense of Michal–Bastiani calculus (cf. [Ham82, Def. 3.6.1. and Ch. I.4]). A Fréchet space is a Hausdorff topological vector space whose topology may be induced by a countable family of semi-norms that are complete with respect to this family.

Consider the trivial bundle $F = M \times N$ where $N, M \in \mathrm{SmthMfd}$. If M is compact, the space of fields is inside the category of Fréchet manifolds $C^\infty(M, N) \in \mathbf{FrMfd}$,[(5)] whereas if M is not compact, this is not the case and the manifold description via infinite-dimensional charts becomes tricky (cf. [KM97]). In this setting, one can give sense to the smoothness of field-theoretical tools as presented above; however, this does not generalize straightforwardly to the non-compact case and brings heavy functional-analytical baggage.
A way to bypass the problem of defining the smooth structure of the field space is to utilize the technology of infinite jet bundles to encode local theories, which are more physically relevant. More recently, attention has shifted to diffeological spaces (cf. [Blo24]), in which field theories and the technology of the infinite jet bundle are better understood. The category of diffeological spaces is the category of concrete sheaves,

[(4)] This goes beyond the scope of this report, and it is properly done in [GS25a, Ch. 5]. An intuitive approach is given in the example after Definition 3.1.4.
[(5)] One can see this considering the countable family of seminorms each given by the sum of the supremum norms of all the derivatives up to some order $k \in \mathbb{N}$. M being compact ensures that these seminorms remain finite.

i.e. with a notion of underlying set of points,[(6)] over CartSp with respect to the same Grothendieck topology we used to define SmthSet. However, this brings two drawbacks: the categorical background needed to work in DflSp is heavy and the latter does not allow for a uniform treatment of all field-theoretical objects (cf. [GS25a, Rmk. 3.8]).
SmthSet is a further abstraction in which SmthMfd, FrMfd and DflSp are fully faithfully included. It should allow us to solve these drawbacks by subsuming and combining these approaches, and should naturally generalize to include fermionic fields and infinitesimal structure (cf. [Gio25; GS25b]), as we outline in the following sections.
To build up to the general definitions in SmthSet, we start from FrMfd and then use the following result.

Proposition 3.1.2. *Consider the category* FrMfd *of infinite-dimensional Fréchet manifolds. The embedding along* CartSp $\hookrightarrow$ SmthMfd $\hookrightarrow$ FrMfd *defines a fully faithful embedding*

$$y\colon \mathrm{FrMfd} \longrightarrow \mathrm{SmthSet},\ G \longmapsto \mathrm{Hom}_{\mathrm{FrMfd}}(-, G)|_{\mathrm{CartSp}} \tag{3.2}$$

where on the right-hand side we consider smooth Fréchet maps.
Moreover, let M, N be finite-dimensional manifolds, with M compact. Then there exists a canonical bijection

$$\mathrm{Hom}_{\mathrm{FrMfd}}(S, C^\infty(M,N)_{\mathrm{FrMfd}}) \cong_{\mathrm{Set}} \mathrm{Hom}_{\mathrm{SmthMfd}}(S \times M, N),$$

for any smooth manifold $S \in \mathrm{SmthMfd}$, which is, moreover, natural in S.

For a proof of this proposition, see the references of [GS25a, Prop. 2.8 and Prop. 2.9].
The latter exponential law has to be consistent, via Yoneda embedding, with the one given by the honest internal hom-function (see Def. 1.1.9) of SmthSet. This is true if the smooth structure in the sense of Fréchet manifolds is equivalent to the one given by the hom-functor in SmthSet.

Definition 3.1.3. Let $\mathcal{G}, \mathcal{H} \in \mathrm{SmthSet}$, define the **smooth mapping set** $[\mathcal{G}, \mathcal{H}] \in \mathrm{SmthSet}$ by

$$[\mathcal{G}, \mathcal{H}](y(\mathbb{R}^k)) := \mathrm{Hom}_{\mathrm{SmthSet}}(y(\mathbb{R}^k) \times \mathcal{G}, \mathcal{H}).$$

The exponential property on SmthSet

$$\mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{X}, [\mathcal{G}, \mathcal{H}]) \cong_{\mathrm{SmthSet}} \mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{X} \times \mathcal{G}, \mathcal{H}),$$

descends from Yoneda lemma (see Prop. 3.1.1) and generalizes to all smooth sets using the relations between hom-functor and limits and the fact that any sheaf is a colimit of representable presheaves (see Prop. 1.1.1 and Prop. 1.2.1). Indeed, Yoneda preserves Fréchet mapping space.

Proposition 3.1.3. *Let $y(M), y(N) \in \mathrm{SmthSet}$ for $M, N \in \mathrm{SmthMfd}$ with M compact. The embedding of the Fréchet manifold $C^\infty(M,N)_{\mathrm{FrMfd}}$ in smooth sets is isomorphic to the mapping smooth set $[y(M), y(N)]$:*

$$y(C^\infty(M,N)_{\mathrm{FrMfd}}) \cong_{\mathrm{SmthSet}} [y(M), y(N)].$$

For a proof, see [GS25a, Prop. 2.11].
Thus, the internal hom-functor provides a means of defining the correct smooth structure within SmthSet, thereby bypassing functional analytical technology and extending the Fréchet mapping space even to the case when M is not compact.
We can now define smooth sets of smooth sections.

Definition 3.1.4. Let $\pi^F_M\colon F \to M$ be a fiber bundle of smooth manifolds, with set of smooth sections $\Gamma_M(F)$. The **smooth set of sections** $\mathcal{F} = \mathbf{\Gamma}_M(F) \in \mathrm{SmthSet}$ is defined by

$$\mathcal{F}(\mathbb{R}^k) \equiv \mathbf{\Gamma}_M(F)(\mathbb{R}^k) := \{\varphi^k\colon \mathbb{R}^k \times M \to F \mid \pi^F_M \circ \varphi^k = \mathrm{pr}_2\}$$

where $\mathbb{R}^k \in \mathrm{CartSp}$ and $\mathrm{pr}_2\colon \mathbb{R}^k \times M \to M$ is projection onto M. That is, $\varphi^k\colon \mathbb{R}^k \times M \to F$ is such that

[(6)] One can intuitively think of this as follows: the smooth set $\mathcal{G}$ is concrete if there exists a set of points $G_s \in \mathrm{Set}$ such that for each probe $\mathbb{R}^k \in \mathrm{CartSp}$, $\mathcal{G}(\mathbb{R}^k) \subset \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^k, G_s)$.

$$
\begin{tikzcd}
 & F \arrow[d, "\pi_M^F"] \\
\mathbb{R}^k \times M \arrow[ur, "\varphi^k"] \arrow[r, "\mathrm{pr}_2"] & M
\end{tikzcd}
$$

commutes, and so equivalently $\mathcal{F}(\mathbb{R}^k) \cong_{\mathrm{Set}} \Gamma_{M\times\mathbb{R}^k}(\mathrm{pr}_2^* F)$ where $\mathrm{pr}_2^* F$ denotes the pullback bundle (see Def. 1.3.19).

Remark 3.1.2. Note that, up to this point, $\mathcal{F} \equiv \Gamma_M(F)$ denotes the set-theoretic field space, i.e. the set of smooth sections of the fiber bundle $\pi_M^F \colon F \to M$, where F is the total space. From now on, unless stated otherwise, $\mathcal{F} \equiv \mathbf{\Gamma}_M(F)$ denotes the smooth set–theoretic field space. This is distinguished from the mere set of sections (encoded as $*$-plot of this generalized space) by the use of bold notation.

In other words,[7] $\mathcal{F}(\mathbb{R}^k)$ is the set of smoothly $\mathbb{R}^k$-parametrized sections of $F \to M$.
As a motivating example, let us consider the trivial bundle $F = M \times N$, where M and N are smooth manifolds, with M being compact. The space of off-shell fields is the Fréchet manifold $C^\infty(M, N)$, i.e. the space of smooth functions from M to N. To see that this is indeed a smooth set, we need to build $\mathbb{R}^k$-shaped plots into it. We define a plot as a smoothly $\mathbb{R}^k$-parametrized family of fields φ^k, that is, via the map $\varphi^k \colon \mathbb{R}^k \times M \to N, (x, m) \mapsto \varphi^k_x(m) \equiv \varphi^k(x, m)$, smooth in both variables. Therefore $\varphi^k \in \mathrm{Plots}\big(\mathbb{R}^k, y(C^\infty(M, N))\big) \coloneqq C^\infty(\mathbb{R}^k \times M, N)$.
This simple case hints at the next steps toward formalizing the variational approach to field theories. Considering a smooth 1-parameter family of fields $\varphi_t \equiv \varphi^1 \colon \mathbb{R} \times M \to N,\ (t, m) \mapsto \varphi^1_t(m)$, it is possible to see that for an action functional to be a smooth map, it must take 1-parameter families of fields to 1-parameter families of real numbers:

$$
\begin{aligned}
y(C^\infty(M, N)) \quad &\xrightarrow{S} \quad y(\mathbb{R}) \\
S^{\mathbb{R}}_* \colon C^\infty(\mathbb{R} \times M, N) &\longrightarrow C^\infty(\mathbb{R}, \mathbb{R}) \\
\varphi_t \quad &\longmapsto (t_0 \mapsto S(\varphi_{t=t_0}))\,.
\end{aligned}
$$

Through the usual derivative on $\mathbb{R}$, we can define the variation δS at φ_0, along the family φ_t, as $\delta_{\varphi_t} S \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} S(\varphi_t)\big|_{t=0}$. Consequently, the critical locus is the following:

$$
\mathrm{Crit}(S) = \left\{ \varphi \in C^\infty(M, N) \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} S(\varphi_t)\bigg|_{t=0} = 0,\ \forall \varphi_t \in C^\infty(\mathbb{R} \times M, N) \text{ such that } \varphi_{t=0} = \varphi \right\}.
$$

More generally, the notion of smooth sets allows us to perform standard operations of finite-dimensional differential geometry pointwise on ordinary manifolds, and thereby extend these notions to smooth sets like $\mathbf{\Gamma}_M(F)$. This idea is used in [GS25a] to construct the variation of the action and demonstrate how the action can indeed be viewed as a smooth map of smooth sets. In [GS25a, Ch. 5] the critical locus is also shown to have a natural smooth subset structure, in more detail, satisfying requirements 3 in Section 2.1.

Remark 3.1.3. An arbitrary fiber bundle $F \to M$ might have no global sections, so Definition 3.1.4 might be null. However, noticing that the assignment of local smooth sections $U \mapsto \Gamma_U(F)$ defines a petit sheaf on M with values in SmthSet, we might instead consider the functor

$$
\begin{aligned}
\mathbf{\Gamma}_{(-)}(F) \colon M_{\mathrm{cl}}^{\mathrm{op}} \times \mathrm{CartSp}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\
(U \times \mathbb{R}^k) &\longmapsto \Gamma_U(F)(\mathbb{R}^k) \cong \Gamma_{U\times\mathbb{R}^k}(\mathrm{pr}_2^* F).
\end{aligned}
$$

Since all statements and definitions regarding the smooth set of fields $\mathcal{F} = \mathbf{\Gamma}_M(F)$ functorially apply for each such open set U, we can continue to work with Definition 3.1.4.

We would now like to define the tangent vectors intuitively as first-order infinitesimal smooth curves[8] in the field space. First, notice that smooth real-valued functions on $\mathcal{F}$ are defined as maps of smooth

[7] This interpretation holds only for concrete smooth sets.
[8] This intuition is incorporated in a rigorous definition enriching the category with infinitesimal structure (see Def. 3.2.4).

sets $C^\infty(\mathcal{F}) = \mathrm{Hom}_{\mathsf{SmthSet}}(\mathcal{F}, y(\mathbb{R}))$ and therefore the algebra structure of $C^\infty(\mathcal{F})$ follows pointwise from that of $y(\mathbb{R})$. Moreover, we can define the induced derivation

$$C^\infty(\mathcal{F}) \longrightarrow \mathbb{R}, \ f \mapsto \partial_t(f \circ \varphi_t)|_{t=0},$$

since for any $\varphi_t \in \mathbf{\Gamma}_M(F)(\mathbb{R}) = \mathrm{Hom}_{\mathsf{SmthSet}}(y(\mathbb{R}), \mathbf{\Gamma}_M(F))$, the composition $f \circ \varphi_t$ defines a smooth map $\mathbb{R} \to \mathbb{R}$. Note that, by the section condition, for each $m \in M$ we have a smooth curve $\varphi_t(m) \colon \mathbb{R}_t \to F$ whose image is contained in the fiber over m. Thus,

$$\partial_t \varphi_t(m)|_{t=0} \in V_{\varphi_0(m)} F$$

defines a vertical tangent vector at $\varphi_0(m) \in F$. Varying over $m \in M$, we get a smooth section of the vertical bundle $\pi_F^{TF} \colon TF \to F$ (see ft. (6) p. 32)

$$\begin{tikzcd} & VF \arrow[d, "\pi_F^{TF}"] \\ M \arrow[ur, "\partial_t \varphi_t|_{t=0}"] \arrow[r, "\varphi_0"] & F. \end{tikzcd}$$

Equivalently, the above diagram defines a section of the pullback bundle $\varphi_0^* VF$. We therefore interpret $\Gamma_M(\varphi_0^* VF)$ as the tangent space at φ_0 getting the following:

$$T(\Gamma_M(F)) \coloneqq \bigcup_{\varphi_0 \in \Gamma_M(F)} \Gamma_M(\varphi_0^* VF) \cong_{\mathsf{Set}} \Gamma_M(VF).$$

The next result ensures that any tangent vector field is represented by a line-plot.

Proposition 3.1.4. *For any $Z_\varphi \in \Gamma_M(VF)$ covering $\varphi = \pi_F^{TF} \circ Z_\varphi$, $\pi_F^{TF} \colon TF \to F$, i.e. such that*

$$\begin{tikzcd} & VF \arrow[d, "\pi_F^{TF}"] \\ & F \arrow[d, "\pi_M^F"] \\ M \arrow[uur, "Z_\varphi"] \arrow[ur, "\varphi"] \arrow[r, "\mathrm{id}_M"] & M \end{tikzcd}$$

commutes, there exists a $\varphi_t \colon \mathbb{R} \times M \to F$ such that $\varphi_0 = \varphi$ and $\partial_t \varphi_t|_{t=0} = Z_\varphi$. That is, the following map is surjective:

$$\mathbf{\Gamma}_M(F)(\mathbb{R}) \to T(\Gamma_M(F)), \ \varphi_t \mapsto \partial_t \varphi_t|_{t=0}.$$

A proof can be found in [GS25a, Lem. 2.18]. The existence of φ_t, and hence the surjectivity of the above map, is guaranteed under suitable regularity assumptions on $F \to M$. In particular, since $F \to M$ is a fiber bundle, the result holds due to the geometrical and topological properties of its sections and its vector fields.

Remark 3.1.4. Unlike the finite-dimensional case, there is no obvious reason why the derivation should depend only on the corresponding tangent vector. More explicitly, the derivation might depend on the representative $\mathbb{R}$-plot (cf. [GS25a][Rmk. 2.19]). This highlights that our naive treatment of tangent bundles, vector fields, and, consequently, differential forms on the full space of fields is insufficient. In particular, it motivates the introduction of local structure and infinitesimally thickened smooth sets. First, this issue is resolved by working with local vector fields (see Def. 3.1.24) and local differential forms (see Def. 3.1.26). Indeed, the introduction of the infinite jet bundle allows us to restore a usable Cartan calculus (see Prop. 3.1.12). Second, the synthetic geometry within ThSmthSet provides a more natural understanding of the ad hoc tangent bundle (see Def. 3.1.5) via the synthetic tangent bundle (see Prop. 3.2.3). In this setting, path derivations depend only on tangent vectors (cf. [GS25b][Lemma 2.32]). Consequently, ThSmthSet yields a more intrinsic understanding of the local Cartan calculus.

This directly leads to the definition of the smooth tangent bundle to the field space.

Definition 3.1.5. The **smooth tangent bundle** to a field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined by

$$T\mathcal{F} \coloneqq \mathbf{\Gamma}_M(VF), \tag{3.3}$$

as the smooth set of sections of VF via Definition 3.1.4.

More concretely, $\mathbb{R}^k$-plots of the tangent bundle $T\mathcal{F}$ correspond to pairs $(Z_{\varphi_0^k}, \varphi_0^k)$ of $\mathbb{R}^k$-parametrized sections over M such that the following commutes:

$$\begin{tikzcd} & VF \arrow[d, "\pi_F^{TF}"] \\ \mathbb{R}^k \times M \arrow[ur, "Z_{\varphi_0^k}"] \arrow[r, "\varphi_0^k"'] & F. \end{tikzcd}$$

From this, we can see that the fiber-wise $\mathbb{R}$-linear structure of the plain set bundle $\Gamma_M(VF) \to \Gamma_M(F)$ extends plot-wise to a smooth $\mathbb{R}$-linear map

$$+_{T\mathcal{F}} \colon T\mathcal{F} \times_{\mathcal{F}} T\mathcal{F} \longrightarrow T\mathcal{F},\ (Z^1_{\varphi_0^k}, Z^2_{\varphi_0^k}) \mapsto Z^1_{\varphi_0^k} + Z^2_{\varphi_0^k}. \tag{3.4}$$

Similarly for the scalar multiplication. Furthermore, the projection $\Gamma_M(VF) \to \Gamma_M(F)$ extends to the smooth projection map

$$\pi_{\mathcal{F}}^{T\mathcal{F}} \colon T\mathcal{F} \to \mathcal{F},\ (Z_{\varphi_0^k}, \varphi_0^k) \mapsto \varphi_0^k.$$

This allows us to define vector fields on the field space as geometrical smooth sections of the tangent bundle.

Definition 3.1.6. The set of **smooth vector fields** on the field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as smooth sections of its tangent bundle

$$\mathfrak{X}(\mathcal{F}) \coloneqq \{Z \colon \mathcal{F} \to T\mathcal{F} \mid \pi_{\mathcal{F}}^{T\mathcal{F}} \circ Z = \mathrm{id}_{\mathcal{F}}\}.$$

That is, smooth maps $Z \colon \mathcal{F} \to T\mathcal{F}$ such that the following diagram of smooth sets commutes:

$$\begin{tikzcd} & T\mathcal{F} \arrow[d, "\pi_{\mathcal{F}}^{T\mathcal{F}}"] \\ \mathcal{F} \arrow[ur, "Z"] \arrow[r, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}. \end{tikzcd} \tag{3.5}$$

In concrete smooth sets, one can understand this as follows. On $*$-plots, such a section defines a vector field in the usual sense, i.e. a map of sets $Z \colon \Gamma_M(F) \longrightarrow \Gamma_M(VF)$ which assigns to every field configuration $\varphi \in \Gamma_M(F)$ a tangent vector $Z_\varphi \in T_\varphi\big(\Gamma_M(F)\big) = \Gamma_M(\varphi^* VF)$. Being a smooth map, the section also sends smooth $\mathbb{R}^k$-plots of field configurations, i.e. smoothly $\mathbb{R}^k$-parametrized sections $\varphi^k \colon \mathbb{R}^k \times M \to F$, to smooth $\mathbb{R}^k$-plots of tangent vectors, i.e. smoothly $\mathbb{R}^k$-parametrized sections $Z_{\varphi^k} \colon \mathbb{R}^k \times M \to VF$.
Vector fields on the field space $\mathcal{F}$ should be interpreted as infinitesimal smooth diffeomorphisms, in direct analogy with the finite-dimensional case. This would allow the symmetries of the field theory to be discussed within this setting. This can be done (cf. [GS25a, Ch. 2]), but it goes beyond the scope of these notes. Moreover, physically relevant smooth vector fields on field spaces are local vector fields (see Def. 3.1.24). We refer to [GS25a, Ch. 6] for an in-depth analysis of their role in infinitesimal symmetries and Noether theorems.
The fiber-wise linear structure (see Eq. (3.4)) of the smooth tangent bundle $T\mathcal{F}$ on a field space $\mathcal{F}$ allows us to define differential forms as fiber-wise linear and antisymmetric maps out of $T\mathcal{F}$.

Definition 3.1.7. The **set of differential m-forms** on $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as

$$\Omega^m(\mathcal{F}) \coloneqq \mathrm{Hom}^{\text{fib.lin.an.}}_{\mathrm{SmthSet}}\left(T^{\times m}\mathcal{F}, y(\mathbb{R})\right).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps with respect to the fiber-wise linear structure (see Eq. (3.4)) on the m-fold fiber product[9]

$$T^{\times m}\mathcal{F} \coloneqq T\mathcal{F} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} T\mathcal{F}$$

of the tangent bundle over $\mathcal{F}$.

[9] Here the product is the pullback (see Def. 1.1.12) of $T\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}^{T\mathcal{F}}} \mathcal{F} \xleftarrow{\pi_{\mathcal{F}}^{T\mathcal{F}}} T\mathcal{F}$.

The collection of differential forms of all degrees forms a graded $\mathbb{R}$-vector space (see Def. 1.3.3)

$$\Omega^\bullet(\mathcal{F}) \coloneqq \bigoplus_{m\in\mathbb{N}} \Omega^m(\mathcal{F}).$$

The latter has a well-defined notion of a wedge product $\wedge$ and contraction ι_Z along any vector field $Z \in \mathfrak{X}(\mathcal{F})$.[(10)] However, there is no obvious definition for a de Rham differential $\mathrm{d}_{\mathcal{F}} \colon \Omega^m(\mathcal{F}) \to \Omega^{m+1}(\mathcal{F})$. Indeed, if there were one, it would define a derivative on $C^\infty(\mathcal{F})$ through the Lie derivative, in contrast with Remark 3.1.4.

We want to solve this by noticing that, practically speaking, one often uses local forms and local fields[(11)] in a local Lagrangian field theory. That is, the variation of the action functional in a local theory only requires the existence of a bi-complex structure and Cartan calculus on the subset of local forms and local vector fields on $\mathcal{F} \times M$. Through the technology associated with the infinite jet bundle $J^\infty_M F$, which encodes locality in a globalized fashion, we build the complete Cartan calculus, with respect to local vector fields, on local forms in the sense of Definition 3.1.7.[(12)]

Infinite Jet Bundle as a Smooth Set

We provide a brief overview of finite jet bundles. We refer to [Blo24] for a more detailed discussion. Consequently, we introduce the infinite jet bundle as a smooth set exploiting the category LocProMfd of local pro-manifolds (cf. [GP17; DGV16; KS17]).

Definition 3.1.8. Let $\pi^F_M \colon F \to M$ be a smooth fiber bundle. Two local sections of it, defined on a neighborhood of m, have the same **k-jet** at m, denoted by $j^k_m\varphi = j^k_m\varphi'$, if they have the same value and partial derivatives up to k-th order at m.

This is a good definition, i.e. independent of the chosen chart, since if the derivatives agree in one chart, they agree in any due to the chain rule. Moreover, it is an equivalence relation on local sections in a neighborhood of m with k-jets $j^k_m\varphi = [\varphi]$ being the equivalence classes:

$$\varphi \sim \varphi' \in \Gamma(U,F) \Leftrightarrow \partial_I\varphi^a(m) = \partial_I\varphi'^a(m) \quad \forall 0 \leqslant |I| \leqslant k.$$

Here, I denotes a symmetric multi-index and a is the index of the coordinate chart $\{m^\mu, u^a\}$ in a trivialization of the fiber bundle $\pi^F_M \colon F \to M$.

We can now define the set of k-jets at m and the jet bundle as generalizations of the tangent space and tangent bundle. Indeed, first-order jets describe tangent planes, while higher-order jets correspond to higher-degree polynomial approximations of submanifolds through m.

Definition 3.1.9. The **set of k-jets at** m is defined as

$$J^k_m F \coloneqq \{j^k_m\varphi = [\varphi] \,|\, \forall \text{ open } U \ni m, \text{ and all } \varphi \in \Gamma(U,F)\}.$$

The **k-jet bundle** is given by

$$J^k_M F \coloneqq \bigcup_{m\in M} J^k_m F \in \mathrm{SmthMfd}$$

with induced[(13)] charts $\{m^\mu, \{u^a_I\}_{|I|\leqslant k}\} \coloneqq \{m^\mu, u^a, u^a_\mu, u^a_{\mu_1\mu_2}, \cdots, u^a_{\mu_1\ldots\mu_k}\}$. The extra coordinates are the partial derivatives of φ at m. In other words, the k-jet bundle represents k-jets of local sections of $F \to M$.

(10) Thanks to the fiber-wise linear structure of $T\mathcal{F}$ and graded structure of $\Omega^\bullet(\mathcal{F})$, they can be defined as usual (Def. 1.3.13).

(11) The term "local" gains a rigorous meaning in Definition 3.1.24 and Definition 3.1.26.

(12) One could also define de Rham forms abstractly in SmthSet, i.e. via a notion that naturally allows one to consider an m-form on an arbitrary smooth set. This requires the notion of moduli space of de Rham forms (cf. [GS25a][Def. 2.30]). This allows us to define the wedge product and differential naturally. However, there is no notion of contraction. This is solved again by moving to the infinite jet bundle and showing that the space of local forms on $y(J^\infty_M F)$ in the sense of Definition 3.1.7, for which we have a notion of Cartan calculus, is canonically identified with the smooth set-theoretical forms defined via the moduli space on it [GS25a][Lemma 4.15]. However, this is not yet proven in the most general case, and therefore this setting cannot yet be applied straightforwardly. The authors of [GS25a] suggest that the solution to this issue and a complete proof of this identification might be found once the category is enriched with infinitesimal structure via synthetic geometry technology. However, as for our purposes it is enough to work at the level of forms as in Definition 3.1.7, we do not delve into this in this thesis.

(13) The charts are induced by the above chart on the trivialization of the fiber bundle.

One can also see $J^k_m F$ as the fiber over m of $\pi^k_M \colon J^k F \to M$ (cf. [Blo24, Prop. 3.1.10]). There are two important maps that one can define on jet bundles.

Definition 3.1.10. Let $\pi^F_M \colon F \to M$ be a fiber bundle, $\Gamma_M(F)$ the set of its global sections and fix $k \in \mathbb{N}$. Then, we define the following maps:

1. The **k,l-forgetful map** for all $k > l \geqslant 0$ is the projection $\pi^k_l \colon J^k_M F \to J^l_M F$, $j^k_m \varphi \mapsto j^l_m \varphi$;
2. The **k-th jet prolongation map** is defined as $j^k \colon \Gamma_M(F) \to \Gamma_M(J^k F)$, $\varphi \mapsto j^k\varphi$ such that $j^k\varphi(m) \coloneqq j^k_m \varphi$.

Note that all the forgetful maps are surjective submersions, i.e. smooth surjective maps whose pushforward is an everywhere surjective linear map, whereas in general prolongation maps are not even surjective (cf. [Blo24, Ch. 3]). Moreover, we get the diagram of smooth manifolds

$$\to J^k_M F \to J^{k-1}_M F \to \cdots \to J^1_M F \to J^0_M F \cong F. \tag{3.6}$$

The arrows are the appropriate forgetful maps and therefore they are all surjective submersions.
To better work with jet bundles of finite but arbitrarily large order, we introduce the following definition.

Definition 3.1.11. Let $\pi^F_M \colon F \to M$ be a smooth fiber bundle. Two local sections of it φ and φ', defined on a neighborhood of m, have the same ∞-jet at m, denoted by $j^\infty_m \varphi = j^\infty_m \varphi'$, if they have the same k-jet at m for all $k \geqslant 0$.

Then, everything follows as before except for the smooth structure of the ∞-jet bundle $J^\infty_M F$. Indeed, the above definition can be given as a projective limit of the diagram (3.6) within Set, but the limit does not exist in SmthMfd. This can be seen by noticing that for every $k \geqslant 0$, the forgetful maps of sets $\pi^\infty_k \colon J^\infty F \to J^k F$, $j^\infty_m \varphi \mapsto j^k_m \varphi$, satisfying $\pi^k_{k-1} \circ \pi^\infty_k = \pi^\infty_{k-1}$, define the commutative diagram

$$\begin{tikzcd}
 & & J^\infty_M F \arrow[dl, "\pi^\infty_k"'] \arrow[d, "\pi^\infty_{k-1}"] \arrow[drrr, "\pi^\infty_0"] & & & \\
\arrow[r] & J^k_M F \arrow[r, "\pi^k_{k-1}"] & J^{k-1}_M F \arrow[r] & \cdots \arrow[r] & J^0_M F.
\end{tikzcd}$$

As can be checked in jet coordinates, any other cone over the diagram (3.6) induces a unique map to $J^\infty F$, which shows that $J^\infty F$ is the categorical limit of the sequence (3.6). However, the limit does not exist in SmthMfd, since it is necessarily infinite-dimensional. Therefore, projections and infinite jet prolongations at this level exist only set-theoretically.
To give a smooth structure to this limit, we therefore have to embed SmthMfd as a subcategory into an ambient category in which such limits exist compatibly with the set-theoretical limits. Moreover, morphisms out of $J^\infty_M F$ should descend to a finite jet bundle, i.e. they should factor through a classically smooth map out of a finite jet bundle in the sequence (3.6). While in [Blo24; Leó18] the ∞-jet bundle is studied within the category of pro-manifolds (see Rmk. 3.1.5), in our setting it is more natural to work in the category of locally pro-manifolds. The latter is a full subcategory of FrMfd where the limit exists as an infinite-dimensional paracompact manifold, as proven in [Sau89, Prop. 7.2.6] and [Tak79, Prop. 2.1]. The fact that the maps in diagram (3.6) are surjective submersions is essential for the existence of the limit within FrMfd.

Definition 3.1.12. The **infinite jet bundle** $J^\infty_M F$ is the paracompact and Hausdorff Fréchet manifold defined by the limit

$$J^\infty_M(F) \coloneqq \lim_k^{\mathrm{FrMfd}} \; J^k_M(F) \in \mathrm{FrMfd},$$

whose local model is $\mathbb{R}^\infty = \lim_k^{\mathrm{FrMfd}} \mathbb{R}^k \in \mathrm{FrMfd}$, with the following local coordinates
$\{x^\mu, \{u^a_I\}_{0 \leqslant |I|}\} \coloneqq \{x^\mu, u^a, u^a_{\mu_1}, u^a_{\mu_1\mu_2}, \ldots\}$.

Consequently, the forgetful maps defining the cone of the projective limit are smooth Fréchet maps. In particular, the limit lives in LocProMfd, a full subcategory of FrMfd (c.f. [GS25a, Sec. 3.1]).

Definition 3.1.13. We define the category of **locally pro-manifolds**, **LocProMfd** $\hookrightarrow$ FrMfd, to be the full subcategory of Fréchet manifolds consisting of projective limits of finite-dimensional manifolds.

Note that LocProMfd, being a full subcategory of FrMfd, is also fully faithfully embedded into SmthSet via the Yoneda embedding of FrMfd into SmthSet (see Prop. 3.1.2). Therefore, $J^\infty_M F \in \mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd} \xrightarrow{y} \mathrm{SmthSet}$ where the first arrow is an inclusion and the second is the Yoneda embedding (see Eq. (3.2)). Also, finite order jet bundles are objects of SmthMfd, and hence can be viewed in SmthSet through the Yoneda embedding (see Prop. 3.1.1). Then, smooth maps between jet bundles, whether of finite or infinite order, correspond to morphisms of sheaves. This, together with the smooth set characterizations of field space, vector bundles, and forms, shows that SmthSet satisfies requirement 1 of Section 2.1.
We have restricted ourselves to this subcategory because it is enough to describe the infinite jet bundle and to embed it into SmthSet, allowing at the same time to easily characterize smooth Fréchet maps into the infinite jet bundle and out of it.

Proposition 3.1.5. *Let $\Sigma \in \mathrm{SmthMfd}$ be a finite-dimensional manifold. This proposition characterize $\mathrm{Hom}_{\mathrm{FrMfd}}\big(J^\infty_M(F), \Sigma\big)$ and $\mathrm{Hom}_{\mathrm{FrMfd}}\big(\Sigma, J^\infty_M(F)\big)$, respectively in 1. and 2..*

1. *A map of sets $f\colon \Sigma \longrightarrow J^\infty_M F$ is smooth if and only if, for every $k \in \mathbb{N}$, the composite $\pi^\infty_k \circ f\colon \Sigma \to J^k_M F$ is smooth. Furthermore, smooth maps $f\colon \Sigma \to J^\infty_M F$ are in one-to-one correspondence with families of smooth maps $\{f_k\colon \Sigma \to J^k_M F\}_{k\in\mathbb{N}}$ such that for all $k_2 \geqslant k_1$ the following commutes:*

$$\begin{tikzcd} \Sigma \arrow[d, "f_{k_2}"'] \arrow[dr, "f_{k_1}"] & \\ J^{k_2}_M F \arrow[r, "\pi^{k_2}_{k_1}"] & J^{k_1}_M F. \end{tikzcd}$$

2. *Let $\pi^\infty_k\colon J^\infty_M F \to J^k_M F$ be the canonical projection for each $k \in \mathbb{N}$. A function of sets $f\colon J^\infty_M F \to \mathbb{R}$ is smooth if and only if, locally around every point $x \in J^\infty_M F$, it factors through some projection π^∞_k. Equivalently, for each $x \in J^\infty_M F$ there exists $k \in \mathbb{N}$, a neighborhood $U \subset J^k_M F$ of $\pi^\infty_k(x) \in J^k_M F$, and a smooth function of* SmthMfd *$\tilde f^k_U\colon J^k_M F|_U \to \mathbb{R}$ such that the following diagram commutes:*

$$\begin{tikzcd} J^\infty_M F|_{(\pi^\infty_k)^{-1}(U)} \arrow[d, "\pi^\infty_k"'] \arrow[dr, "f"] & \\ J^k_M F|_U \arrow[r, "\tilde f^k_U"] & \mathbb{R}. \end{tikzcd}$$

The result is readily extended if the target manifold $\mathbb{R}$ is replaced by $\mathbb{R}^n$, and consequently by any finite-dimensional manifold $\Sigma \in \mathrm{SmthMfd}$.

The first part follows directly from the universal cone property of the limit of the sequence (3.6), whereas the proof of the second is a bit trickier and can be found in [KS17, Prop. 2.29].
In particular, the result *1.* holds for all $\Sigma \in \mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd}$, allowing us to describe the smooth set incarnation of $J^\infty_M F$ via the embedding of Proposition 3.1.2:

$$y(J^\infty_M F) := \mathrm{Hom}_{\mathrm{FrMfd}}(-, J^\infty_M F)|_{\mathrm{CartSp}} \cong \lim_k^{\mathrm{SmthSet}} y(J^k_M F).$$

Here, the morphism in FrMfd is the one in Proposition 3.1.5 and the second equivalence comes from the fact that the limit is computed objectwise. Therefore, $\mathbb{R}^n$-plots of $y(J^\infty_M F)$ are

$$y(J^\infty_M F)(\mathbb{R}^n) \cong \{\{s^n_k\colon \mathbb{R}^n \to J^k_M F \,|\, \pi^k_{k-1} \circ s^n_k = s^n_{k-1}\}_{k\in\mathbb{N}}\}.$$

One can show (cf. [GS25a]) that the infinite jet prolongation is smooth and uniquely extends to a smooth map

$$\begin{aligned} j^\infty\colon \mathbf{\Gamma}_M(F) &\longrightarrow \mathbf{\Gamma}_M(J^\infty_M F) \\ \varphi^k &\longmapsto j^\infty \varphi^k. \end{aligned} \tag{3.7}$$

Finally, we can characterize smooth maps of locally pro-manifolds $\mathrm{Hom}_{\mathrm{LocProMfd}}(J^\infty_M F, G^\infty)$, with $G^\infty = \lim_j^{\mathrm{FrMfd}} G^j$, as follows. The proof of the following proposition can be found in [GS25a, Prop. 3.7].

Proposition 3.1.6. *Let $G^\infty = \lim_j^{\mathrm{FrMfd}} G^j$ be any locally pro-manifold.*

1. *A map of sets $f\colon J^\infty_M F \to G^\infty$ is smooth if and only if*

$$\mathrm{p}^\infty_j \circ f\colon J^\infty_M F \to G^\infty \xrightarrow{\mathrm{p}^\infty_j} G^j$$

is smooth for each $j \in \mathbb{N}$, and hence if each $\mathrm{p}^\infty_j \circ f$ factors locally around every $x \in J^\infty_M F$ through $\pi^\infty_k\colon J^\infty_M F \to J^k_M F$ for some $k \in \mathbb{N}$, where $\mathrm{p}^\infty_j\colon G^\infty \to G^j$ denotes the universal cone projections of G^∞.

2. *Furthermore, smooth maps $f\colon J^\infty_M F \to G^\infty$ are in one-to-one correspondence with compatible families of smooth maps $\{f_j\colon J^\infty_M F \to G^j \mid j \in \mathbb{N}\}$ such that for each pair $j_2 \geqslant j_1$ the diagram*

$$\begin{tikzcd}
J^\infty_M F \arrow[d, "f_{j_2}"'] \arrow[dr, "f_{j_1}"] & \\
G^{j_2} \arrow[r, "\mathrm{p}^{j_2}_{j_1}"'] & G^{j_1}
\end{tikzcd}$$

commutes, and hence, such that furthermore each f_j locally factors through $\pi^\infty_k\colon J^\infty_M F \to J^k_M F$ for some $k \in \mathbb{N}$.

Remark 3.1.5. In [Blo24; Leó18], the infinite jet bundle is studied within the category of pro-manifolds **ProMfd**. ProMfd can be informally thought of as an extension of SmthMfd, adding only well-behaved limits. Then, $J^\infty_M F$ is not thought of as a Fréchet manifold, but simply as a formal limit of finite-dimensional manifolds. Smooth functions on it are those that globally factor through some $J^k_M F$ (cf. [Leó18, Sec. 1.2]), i.e. the smooth functions $C^\infty_{\mathrm{glb}}(J^\infty_M F)$ on the formal limit are the union over $k \in \mathbb{N}$ of those on $J^k_M F$. Note that the latter is a subalgebra of the algebra of locally finite order functions that we characterized via Proposition 3.1.5:

$$C^\infty_{\mathrm{glb}}(J^\infty_M F) \hookrightarrow C^\infty(J^\infty_M F).$$

This somehow justifies (cf. [KS17, Sec. 2.2]) the name of the category LocProMfd, which is, however, really different from ProMfd, being a full subcategory of FrMfd.
Moreover, in [Blo24; Leó18] the smooth structure of the space of sections of smooth fiber bundles is described within the category of concrete smooth sets, i.e. DflSp. Therefore, to obtain a uniform description of the relevant field-theoretical objects, the two approaches must be combined in the category **ProDflSp** of pro-diffeological spaces.
This approach has the following three main drawbacks (cf. [GS25a, Rmk. 3.8]).

1. $C^\infty_{\mathrm{glb}}(J^\infty_M F)$ do not form a petit sheaf on the underlying topological space, in contrast to $C^\infty(J^\infty_M F)$.
2. Contrary to what we will show in the next sections, concrete sheaves do not naturally generalize to include fermionic fields or infinitesimal structure (cf. [GS25b; GS26; Gio25]).
3. Treating the relevant field-theoretical objects in different categorical footing results in the introduction of heavy categorical machinery that can be avoided within the more natural setting of SmthSet.

Tangent Bundle, Vector Fields and Cartan Connection

We rigorously define the tangent bundle, vector fields, and classical forms on the infinite jet bundle. We then show that the tangent bundle has a canonical smooth horizontal splitting that is essential for building the variational bi-complex (see Def. 3.1.20)).
The definition of the infinite jet tangent bundle is analogous to the definition of the infinite jet bundle in SmthSet. First, notice that the diagram (3.6) induces the diagram of finite-dimensional tangent bundles

$$\to T(J^k_M F) \xrightarrow{\mathrm{d}\pi^k_{k-1}} T(J^{k-1}_M F) \to \cdots \to T(J^1_M F) \xrightarrow{\mathrm{d}\pi^1_0} T(J^0_M F) \cong TF$$

with the maps being the pushforwards of the projections $\{\pi^k_{k-1}\colon J^k_M F \to J^{k-1}_M F\}_{k\in\mathbb{N}}$. The diagram is in SmthMfd and can be embedded in SmthSet by the Yoneda embedding (see Prop. 3.1.1).

Definition 3.1.14. The **smooth infinite jet tangent bundle** $T\big(y(J^\infty_M F)\big) \in \mathrm{SmthSet}$ is defined by

$$T\big(y(J^\infty_M F)\big) := \lim^{\mathrm{SmthSet}}_{k\in\mathbb{N}} y\big(T(J^k_M F)\big). \tag{3.8}$$

The latter is a concrete smooth set. Indeed, $T(y(J^\infty_M F))(*) := \lim^{\mathrm{Set}}_{k\in\mathbb{N}} T(J^k_M F)$ is represented by

$$\bigcup_{s\in J^\infty_M F} T_s(J^\infty_M F) := \bigcup_{s\in J^\infty_M F} \big\{\{X^k_s \in T_{\pi^\infty_k(s)}(J^k_M F) | \mathrm{d}\pi^k_{k-1} X^k_s = X^{k-1}_s\}_{k\in\mathbb{N}}\big\}.$$

That is, each point, or tangent vector, $X_s \in T_s(J^\infty_M F)$ at $s = j^\infty_p \varphi \in J^\infty_M F$ is represented by a family of tangent vectors $\{X^k_s \in T_{\pi^\infty_k(s)}(J^k_M F)\}_{k\in\mathbb{N}}$ on each finite order tangent bundle at $\pi^\infty_k(s) = \pi^\infty_k(j^\infty_p\varphi) = j^k_p\varphi \in J^k_M F$. Such a family is compatible along the pushforward projections, i.e. $\mathrm{d}\pi^k_{k-1} X^k_s = X^{k-1}_s$. In a local coordinate chart $\{x^\mu, \{u^a_I\}_{0\leqslant|I|}\}$ of $J^\infty_M F \in \mathrm{LocProMfd}$, we can think of this family as the formal sum

$$X_s = X^\mu\, \partial_{x^\mu}\big|_s \;+\; \sum_{|I|=0}^{\infty} Y^a_I\, \partial_{u^a_I}\big|_s \,.$$

Here $\{X^\mu, Y^a_I\} \subset \mathbb{R}$, with each X^k corresponding to the case where the sum is terminated at order $|I| = k$. From this definition one can recover the notion of tangent vectors as infinitesimal curves.

Proposition 3.1.7. *The set of tangent vectors $T\big(y(J^\infty_M F)\big)(*)$ is in bijection with equivalence classes of curves in $J^\infty_M F \in \mathrm{LocProMfd}$:*

$$T\big(y(J^\infty_M F)\big)(*) \cong \mathrm{Hom}_{\mathrm{FrMfd}}(\mathbb{R}, J^\infty_M F)/\sim_{O(t^1)} \;\cong\; y(J^\infty_M F)(\mathbb{R})/\sim_{O(t^1)}, \tag{3.9}$$

where the equivalence relation is agreement up to first-order derivatives at $0 \in \mathbb{R}$.

A proof can be found in [GS25a, Lem. 4.2].
Even if the latter approach is more intuitive, we would rather use Definition 3.1.14 since it allows us to consider not only points, but any $\mathbb{R}^n$-plot of the tangent bundle of $J^\infty_M F$

$$T\big(y(J^\infty_M F)\big)(\mathbb{R}^n) = \lim^{\mathrm{SmthSet}}_k\; y\big(T(J^k_M F)\big)(\mathbb{R}^n) := \lim^{\mathrm{Set}}_k \mathrm{Hom}_{\mathrm{SmthMfd}}\big(\mathbb{R}^n, T(J^k_M F)\big). \tag{3.10}$$

This is represented by

$$\bigcup_{s^n\in J^\infty_M F(\mathbb{R}^n)} T_{s^n}(J^\infty_M F) := \bigcup_{s^n\in J^\infty_M F(\mathbb{R}^n)} \Big\{\{X^k_{s^n} : \mathbb{R}^n \to T(J^k_M F) \mid \mathrm{d}\pi^k_{k-1}\circ X^k_{s^n} = X^{k-1}_{s^n}\}_{k\in\mathbb{N}}\Big\},$$

where $X^k_{s^n}(x) \in T_{\pi^\infty_k\circ s^n(x)}(J^k_M F)\ \forall x \in \mathbb{R}^n$. In local charts, we can write the formal sum

$$X_{s^n} = X^\mu_{s^n}\, \partial_{x^\mu}\big|_{s^n} + \sum_{|I|=0}^{\infty} Y^a_{I,s^n} \partial_{u^a_I}\big|_{s^n},$$

where now $\{X^\mu_{s^n}, Y^a_{I,s^n}\}_{0\leqslant|I|}$ are smooth functions on $\mathbb{R}^n$, with $X^k_{s^n}$ corresponding to the case where the sum is terminated at order $|I| = k$.
We now define fiberwise addition and $y(\mathbb{R})$-multiplication as the natural transformations

$$+ \colon T\big(y(J^\infty_M F)\big) \times_{y(J^\infty_M F)} T\big(y(J^\infty_M F)\big) \to T\big(y(J^\infty_M F)\big),\ \big(\{X^k_{s^n}\}_{k\in\mathbb{N}}, \{\tilde{X}^k_{s^n}\}_{k\in\mathbb{N}}\big) \mapsto \{X^k_{s^n} + \tilde{X}^k_{s^n}\}_{k\in\mathbb{N}} \tag{3.11}$$

$$\cdot \colon y(\mathbb{R}) \times T\big(y(J^\infty_M F)\big) \to T\big(y(J^\infty_M F)\big),\ (f_n, \{X^k_{s^n}\}_{k\in\mathbb{N}}) \mapsto \{f_n \cdot X^k_{s^n}\}_{k\in\mathbb{N}}$$

where $f_n \colon \mathbb{R}^n \to \mathbb{R}$ is a plot of $y(\mathbb{R})$. This is possible thanks to the $C^\infty(\mathbb{R}^n)$-linear structure on the fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $y(J^\infty_M F)$ that is induced by the linear structures on each $T(J^k_M F)$, natural in $\mathbb{R}^n$. To understand this, note that there is a canonical smooth bundle projection

$$\pi^{TJ^\infty_M F}_{J^\infty_M F} : T\big(y(J^\infty_M F)\big) \to y(J^\infty_M F).$$

We have dropped the Yoneda embedding y from the notation for readability. From now on, we do the same for any new object defined on $y(J^\infty_M F)$. Then, the above means that the $\mathbb{R}^n$-plots of $T(y(J^\infty_M F))$ over a single $\mathbb{R}^n$-plot of $y(J^\infty_M F)$ have a $C^\infty(\mathbb{R}^n)$-linear structure. Naturality ensures that the operations

of the module commute with pulling back along a smooth change of plots $f\colon \mathbb{R}^n \to \mathbb{R}^m$, allowing us to define the two above morphisms of smooth sets.
We now define vector fields on the infinite jet bundle, internally to SmthSet, as the geometrical smooth sections of its tangent bundle. Consequently, we show that this recovers the usual algebraic definition as derivatives on $C^\infty(J^\infty_M F)$.[14]

Definition 3.1.15. The **set of smooth vector fields** on the infinite jet bundle $y(J^\infty_M F)$ is defined as smooth sections of its tangent bundle:

$$\mathfrak{X}\big(J^\infty_M F\big) \coloneqq \Gamma_{y(J^\infty_M F)}\big(T(y(J^\infty_M F))\big) = \big\{X\colon y(J^\infty_M F) \to T(y(J^\infty_M F)) \;\big|\; \pi^{TJ^\infty_M F}_{J^\infty_M F} \circ X = \mathrm{id}_{y(J^\infty_M F)}\big\}.$$

By the limit property for $T\big(y(J^\infty_M F)\big)$, a vector field X corresponds to a compatible family $\{X^k\colon y\big(J^\infty_M F\big) \to y\big(T(J^k_M F)\big)\}_{k\in\mathbb{N}}$ such that the following commutes for all $k \in \mathbb{N}$:

$$\begin{tikzcd} & y\big(T(J^k_M F)\big) \arrow[d, "\mathrm{d}\pi^k_{k-1}"] \\ y(J^\infty_M F) \arrow[ur, "X^k"] \arrow[r, "X^{k-1}"'] & y\big(T(J^{k-1}_M F)\big). \end{tikzcd}$$

Since the embedding of LocProMfd into SmthSet is fully faithful, each map of such a family can be thought of as a smooth map in LocProMfd. The latter is, via Proposition 3.1.5, locally of finite order. In other words, it is, locally around each $s \in J^\infty_M F$, the pullback of some smooth function of a finite-order jet bundle via the appropriate projection. In particular, such a family $\{X^k\colon J^\infty_M F \to T(J^k_M F)\}_{k\in\mathbb{N}}$ may be represented in a local chart of $J^\infty_M F$ by the formal sum

$$X = X^\mu \partial_{x^\mu} + \sum_{|I|=0}^{\infty} Y^a_I\, \partial_{u^a_I}.$$

Here $\big\{X^\mu, \{Y^a_I\}_{0\leqslant |I|}\big\} \subset C^\infty(J^\infty_M F)$ is a family of locally defined smooth functions, with each X^k corresponding to the truncation of the sum at order $|I| = k$.
We denote the vector subspace of globally finite order vector fields,[15] i.e. those that correspond to a compatible family in which every map $X^k\colon J^\infty_M F \to T(J^k_M F)$ is globally of finite order, by

$$\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F) \;\subset\; \mathfrak{X}(J^\infty_M F).$$

Note that, $\mathfrak{X}(J^\infty_M F)$ has a $C^\infty(J^\infty_M F)$-module structure induced by the fiber-wise $y(\mathbb{R})$-linear structure on $T\big(y(J^\infty_M F)\big)$ through the composition with the operations in Equation (3.11). In local coordinates, this is represented as a formal sum of indices and multiplication by $C^\infty(J^\infty_M F)$. Similarly, $\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F)$ inherit the structure of a $C^\infty_{\mathrm{glb}}(J^\infty_M F)$-module.
The following result establishes a connection between our definition of a vector space and the more intuitive interpretation of vector fields as derivations on the jet bundle.

Proposition 3.1.8. *Vector fields on $J^\infty_M F$ are in 1-1 correspondence with derivations of the algebra of smooth functions:*

$$\mathfrak{X}(J^\infty_M F) \;\cong\; \mathrm{Der}\big(C^\infty(J^\infty_M F)\big).$$

For the proof, see [GS25a, Lem. 4.5].
The latter result restricts to the global case $\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F) = \mathrm{Der}\big(C^\infty_{\mathrm{glb}}(J^\infty_M F)\big)$. Notably, this point of view allows to define a Lie algebra structure on $\mathfrak{X}(J^\infty_M F)$ (cf. [GS25a]) with the usual bracket defined by

$$[X, \tilde{X}](f) \coloneqq X(\tilde{X}(f)) - \tilde{X}(X(f)) \tag{3.12}$$

for any $f \in C^\infty(J^\infty_M F)$ and $X, \tilde{X} \in \mathfrak{X}(J^\infty_M F)$.
Remarkably,[16] the infinite jet tangent bundle $T(y(J^\infty_M F)) \to y(J^\infty_M F)$ has a smooth canonical horizontal splitting.

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[14] Once the spaces are enriched with infinitesimal structure, this definition is naturally recovered as the synthetic tangent bundle of the infinite jet bundle (see Def. 3.2.4).
[15] These are the vector fields that usually appear in field-theoretic examples, but restricting to them is not enough for some applications.
[16] This is not true in general for infinite-dimensional manifolds, and even if the split always exists in the finite-dimensional case (cf. [GSM09, Thm. 1.1.12]) this is not canonical because it corresponds to an arbitrary choice of connection.

Remark 3.1.6. We recall the vertical tangent bundle and the splitting of an exact sequence of vector bundles, and heuristically show how the latter is related to a connection, namely the Cartan connection. Let $\pi^{TF}_F : TF \to F$ be the tangent bundle of a fiber bundle $\pi^F_M : F \to M$, and consider its vertical subbundle[17] $VF = \ker(\mathrm{d}\pi^F_M)$, where $\mathrm{d}\pi^F_M : TF \to TM$, consists of the vectors tangent to fibers of F. One can show that it fits in the following **short exact sequence** of vector bundles over F

$$0 \to VF \xrightarrow{\iota} TF \xrightarrow{\mathrm{d}\pi^F_M} (\pi^F_M)^*TM \to 0$$

where $(\pi^F_M)^*TM \cong F \times_M TM$. A short exact sequence here means that the arrows are bundle maps such that the image of each morphism is the kernel of the next morphism.
The sequence is said to **split** if there exist a bundle map $H : (\pi^F_M)^*TM \to TF$ such that $\mathrm{d}\pi^{\mathrm{F}}_{\mathrm{M}} \circ H = \mathrm{id}_{(\pi^F_M)^*TM}$ or equivalently $TF \cong \iota(VF) \oplus H((\pi^F_M)^*TM)$, where the sum has to be interpreted as direct sum compatible with the vector bundle structure (cf. Whitney sum [GSM09, Sec. 1.1.3]). The latter direct sum suggests identifying $(\pi^F_M)^*TM$ with the horizontal tangent subbundle HF, that is, the image of the lifting map H. In other words, the splitting is used to lift tangent vectors from M to horizontal vectors in TF, consistently with the bundle structure.[18] Such a lift can be described explicitly as

$$T_uF \ni X^H_u := \sigma(u)\big(X_{\pi^F_M(u)}\big) = H\big(u, X_{\pi^F_M(u)}\big) \quad \text{with} \quad X_{\pi^F_M(u)} \in T_{\pi^F_M(u)}M,$$

where the connection split $\sigma : F \to \mathrm{Hom}(TM, TF)$ assigns to each point $u \in F$ a bundle map in $\mathrm{Hom}(TM, TF)$.
This splitting is essential for comparing vertical field variations, which are useful for formalizing the action principle in field theories, separately from horizontal variations. It ensures that when one computes the variation of the action, the change of fields can be decomposed into genuine field transformation contributions (vertical) and variation solely due to a diffeomorphism of the base (horizontal).
The theorem [GSM09, Thm 1.1.12] states that every exact sequence of vector bundles is split in the case of finite-dimensional manifolds, even if the splitting is not canonical,[19] but this is not always true for the infinite-dimensional case.[20]
The choice of a complementary horizontal bundle defines an **Ehresmann connection** on the bundle. It can be shown that an Ehresmann connection uniquely determines a horizontal lift and therefore the splitting. That is, for a fiber bundle $\pi^F_M : F \to M$, a connection defines a splitting of the tangent bundle $TF \cong VF \oplus HF$, that is a vector bundle over F. VF denotes the vertical bundle, and HF is the horizontal distribution determined by the connection, called Cartan connection.
Such a connection is more general than a linear one, but it coincides with a linear connection, when considering a vector bundle $\pi^F_M : F \to M$, if the horizontal distribution is linear in the fibers, i.e. compatible with the vector space structure of each fiber. Nevertheless, the two concepts are analogous in the scope: determining the parallel transport along M. Presenting the relation between the two type of connections goes beyond the ambit of the project.[21] Still, it can be intuitively understood by defining the horizontal lift as follows, showing an analogy with parallel transport (cf. [Tec19]). Let $p \in M$ and $f \in F$ such that $\pi^F_M(f) = p$. Given a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$, we define a **lift** of γ through f as the curve $\tilde{\gamma}$, satisfying $\tilde{\gamma}(0) = f$ and $\pi^F_M(\tilde{\gamma}(t)) = \gamma(t)\ \forall t$. Then a lift is **horizontal** if every tangent to $\tilde{\gamma}$ lies in a fiber of HF, namely $\dot{\tilde{\gamma}}(t) \in HF_{\tilde{\gamma}(t)}$ for all $t \in \mathbb{R}$.
The splitting of the tangent bundle induces a splitting of the smooth vector fields into vertical or horizontal, depending on whether they belong to $\Gamma_F(VF)$ or $\Gamma_F(HF)$, respectively. That is a splitting of the $C^\infty(F)$-module[22] $\mathfrak{X}(F) \cong \mathfrak{X}_V(F) \oplus \mathfrak{X}_H(F)$.
The splitting of the tangent bundle induces also a splitting of differential forms into vertical and horizontal forms. The former are those forms on F that vanish when contracted with vectors tangent to the base M, i.e. when restricted to the horizontal subbundle. They are usually denoted by $\Omega^\bullet(F/M)$. The latter are those that vanish when restricted to the vertical subbundle. When the de Rham differential itself decomposes into horizontal and vertical parts, this structure endows the space of differential forms on F with the structure of a bicomplex.

(17) A subbundle of a given vector bundle is a submanifold of the total space with vector bundle structure over the same base.
(18) This is used to depict how fields change under a spacetime diffeomorphism described by fields in TM.
(19) The connection becomes canonical only after fixing additional geometric data, such as a metric.
(20) The properties of the infinite jet bundle ensure the existence of a canonical splitting of its tangent bundle.
(21) For an extensive treatment of connections on fiber bundles see [Tec19; GSM09].
(22) One can show that $C^\infty(F)$ forms a ring and that the action $(f \cdot X)(p) = f(p)\,X_p$, for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, $p \in M$, makes $\mathfrak{X}(M)$ into a $C^\infty(M)$-module. Equivalently for $\mathfrak{X}(F)$.

Once the vertical subbundle is introduced, we prove that it fits into a short exact sequence (see Eq. (3.14)) that has a canonical splitting induced by the Cartan connection (see Prop. 3.1.9).

To this end, first notice that the total projection $\pi^\infty_M \colon y(J^\infty_M F) \xrightarrow{\pi^\infty_0} y(F) \xrightarrow{\pi^F_M} y(M)$ induces the following smooth pushforward

$$\mathrm{d}\pi^\infty_M \colon T\big(y(J^\infty_M F)\big) \longrightarrow y(TM). \tag{3.13}$$

The action of this map on $\mathbb{R}^n$-plots is given by

$$\big\{ X^k_{s^n} \colon \mathbb{R}^n \to T(J^k_M F) \;\big|\; \mathrm{d}\pi^k_{k-1} \circ X^k_{s^n} = X^{k-1}_{s^n} \big\}_{k\in\mathbb{N}} \longmapsto \mathrm{d}\pi^k_M \circ X^k_{s^n} \in \; y(TM)(\mathbb{R}^n),$$

where $X^k_{s^n}$ is a representative in the representing set of the plots and $\mathrm{d}\pi^k_M \colon T(J^k_M F) \to TM$ is the pushforward of $\pi^F_M \circ \pi^k_0 =: \pi^k_M \colon J^k_M F \to M$. In local coordinates, we can concretely understand the action of the map on $X_s \in T\big(y(J^\infty_M F)\big)(*)$ at $s \in J^\infty_M F$ writing

$$X_s = X^\mu \, \partial_{x^\mu}\big|_s \; + \sum_{|I|=0}^{\infty} Y^a_I \, \partial_{u^a_I}\big|_s \longmapsto X^\mu \partial_{x^\mu}\big|_{\pi^\infty_M(s)}.$$

This, in turn, allows us to define the smooth vertical subbundle.

Definition 3.1.16. The **smooth vertical subbundle** of $T\big(y(J^\infty_M F)\big)$ is defined as the equalizer of $\mathrm{d}\pi^\infty_M$ and the canonical zero map $0_M \colon T\big(y(J^\infty_M F)\big) \to y(TM)$:

$$VJ^\infty_M F := \mathrm{eq}\left(T\big(y(J^\infty_M F)\big) \underset{0_M}{\overset{\mathrm{d}\pi^\infty_M}{\rightrightarrows}} y(TM) \right).$$

$\mathbb{R}^n$-plots of the above are represented by compatible families

$$\left\{ X^k_{s_n} \colon \mathbb{R}^n \to T(J^k_M F) \;\middle|\; \mathrm{d}\pi^k_{k-1} \circ X^k_{s_n} = X^{k-1}_{s_n}, \; \mathrm{d}\pi^k_M \circ X^k_{s_n} = (\pi^\infty_M \circ s_n, 0) \right\}_{k\in\mathbb{N}}.$$

For instance, a vertical tangent vector at $s \in J^\infty_M F$, namely $X_s \in VJ^\infty_M F(*)$, is represented in a local coordinate chart by

$$X_s = 0 + \sum_{|I|=0}^{\infty} Y^a_I \, \partial_{u^a_I}\Big|_s.$$

Furthermore, note that smooth sections of the vertical subbundle $VJ^\infty_M F \to y(J^\infty_M F)$ (see Def. 3.1.15), define the subbundle of vertical vector fields on $J^\infty_M F$:

$$\mathfrak{X}_V(J^\infty_M F) := \Gamma_{y(J^\infty_M F)}\big(VJ^\infty_M F\big).$$

In a local chart, these are represented by

$$X = 0 + \sum_{|I|=0}^{\infty} Y^a_I \, \partial_{u^a_I} \quad \text{where} \quad Y^a_I \in C^\infty(J^\infty_M F).$$

It follows that $\mathfrak{X}_V(J^\infty_M F)$ is closed under the Lie bracket of $\mathfrak{X}(J^\infty_M F)$ (see Eq. (3.12)).
Finally, we write the following natural short exact sequence of smooth sets[(23)] over $y(J^\infty_M F)$

$$0_{J^\infty_M F} \longrightarrow VJ^\infty_M F \longrightarrow T\big(y(J^\infty_M F)\big) \to y\big(J^\infty_M F\big) \times_{y(M)} y(TM) \longrightarrow 0_{J^\infty_M F}. \tag{3.14}$$

This is a short exact sequence of $C^\infty(\mathbb{R}^n)$-modules of fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $y(J^\infty_M F)$.[(24)] The third map is naturally given on $\mathbb{R}^n$-plots by

$$\{X^k_{s^n} \colon \mathbb{R}^n \to T\big(J^k_M F\big) \,|\, \mathrm{d}\pi^k_{k-1} \circ X^k_{s^n} = X^{k-1}_{s^n}\} \longmapsto (s^n, \mathrm{d}\pi^k_M \circ X^k_{s^n}).$$

The crucial property of the infinite jet bundle is that the above sequence has a canonical splitting

$$H \colon y\big(J^\infty_M F\big) \times_{y(M)} y(TM) \longrightarrow T\big(y(J^\infty_M F)\big).$$

(23) The fibered product is in SmthSet, with $\mathbb{R}^n$-plots being pairs of plots that project to the same plot in M. The coordinate charts take the form $\{x^\mu, \dot{x}^\mu, u^a, u^a_\mu, u^a_{\mu_1\mu_2}, \cdots\}$, with $\{\dot{x}^\mu\}$ denoting the fiber coordinates on TM.

(24) More concretely, exactness is checked plotwise: for each plot s^n of $J^\infty_M F$, the $\mathbb{R}^n$-plots of the smooth sets in the sequence above it, i.e. in the fiber of s^n, form a short exact sequence of $C^\infty(\mathbb{R}^n)$-modules.

That is, there exists a canonical connection on $y(J^\infty_M F)$, usually referred to as the Cartan connection.[(25)] Before stating and proving Proposition 3.1.9, let us work out the description of the splitting at the point set level.
Let $j^k_p\varphi \in J^k_M F$ and choose a representative local section $\tilde{\varphi}\colon U \subset M \to F$ such that $j^k\tilde{\varphi}(p) = j^k_p\varphi$ (see Def. 3.1.10). The induced pushforward map

$$\mathrm{d}\big(j^k\tilde{\varphi}\big)_p\colon\ T_pM \longrightarrow T_{j^k_p\varphi}(J^k_M F)$$

is given in local coordinates by the usual Cartan horizontal lift

$$\begin{aligned} X^\mu \partial_{x^\mu}\big|_p \longmapsto & X^\mu\bigg(\partial_{x^\mu}\big|_{j^k_p\varphi} + \sum_{|I|=0}^{k} \partial_{x^\mu}\partial_{x^I}\tilde{\varphi}^a(p)\cdot \partial_{u^a_I}\big|_{j^k_p\varphi}\bigg) \\ &= X^\mu\bigg(\partial_{x^\mu}\big|_{j^k_p\varphi} + \sum_{|I|=0}^{k} u^a_{I+\mu}(J^{k+1}_p\tilde{\varphi})\cdot \partial_{u^a_I}\big|_{j^k_p\varphi}\bigg). \end{aligned}$$

Here $\{x^\mu\}$ and $\{x^\mu, \{u^a_I\}_{|I|\leqslant k}\}$ are the local coordinates of M and $J^k_M F$ respectively. The expression depicts how a vector over the base manifold M is horizontally lifted to a vector over the infinite jet bundle, compatibly with respect to the structure of the latter. Note that, such a map depends smoothly only on the $(k+1)$-jet at $p \in M$ of the chosen representative section $\tilde{\varphi}$. Therefore the assignment defines, for each $k \in \mathbb{N}$, a smooth map of bundles over $J^k_M F$

$$H^k\colon J^{k+1}_M F \times_M TM \longrightarrow T(J^k_M F),\ \ \big(j^{k+1}_p\varphi, X_p\big) \longmapsto \mathrm{d}\big(j^k\varphi\big)_p(X_p). \tag{3.15}$$

Here $\varphi\colon U \subset M \to F$ on the right-hand side is any representative local section with $j^{k+1}\varphi(p)$ equal to the given jet. This fits inside the following commutative diagram:

$$\begin{tikzcd} J^{k+1}_M F \times_M TM \arrow[r, "H^k"] \arrow[d, "\pi^{k+1}_k \times \mathrm{id}"'] & T(J^k_M F) \arrow[d, "d\pi^k_{k-1}"] \\ J^k_M F \times_M TM \arrow[r, "H^{k-1}"] & T(J^{k-1}_F). \end{tikzcd} \tag{3.16}$$

Notably, there is an injective point-set map

$$J^\infty_M F \times_M TM \to T\big(y(J^\infty_M F)\big)(*),\quad \big(j^\infty_p\varphi, X_p\big) \mapsto \mathrm{d}\big(j^\infty\varphi\big)_p(X_p) \coloneqq \big\{\mathrm{d}\big(j^k\varphi\big)_p(X_p)\big\}_{k\in\mathbb{N}} \tag{3.17}$$

which splits the $*$-plot sequence of Equation (3.14). Again, $\varphi\colon U \subset M \to F$ is a representative local section of $j^\infty_p\varphi$. In the coordinate representation of points in $T_{j^\infty_p\varphi}J^\infty_M F$, the map takes the form

$$\Big(j^\infty_p\varphi, X^\mu\partial_{x^\mu}\big|_p\Big) \longmapsto X^\mu\bigg(\partial_{x^\mu}\big|_{j^\infty_p\varphi} + \sum_{|I|=0}^{\infty} u^a_{I+\mu}(j^\infty_p\varphi)\cdot\partial_{u^a_I}\big|_{j^\infty_p\varphi}\bigg),$$

where and $\{x^\mu, \{u^a_I\}_{0\leqslant|I|}\}$ are the local coordinates of $J^\infty_M F$. More generally, the map of Equation (3.17) extends to a smooth splitting of the corresponding smooth sets. Note that such splitting is not global on arbitrary jet bundles, but is canonical on $J^\infty_M F$ due to its properties.[(26)]

Proposition 3.1.9. *The family of smooth bundle maps* $\{H^k\colon J^{k+1}_M F \times_M TM \to T(J^k_M F)\}_{k\in\mathbb{N}}$ *determines a map, called* ***Cartan connection****, of smooth sets*

$$H\colon y(J^\infty_M F) \times_{y(M)} y(TM) \longrightarrow T\big(y(J^\infty_M F)\big), \tag{3.18}$$

which splits the corresponding exact sequence in Equation (3.14).
Moreover, the split is canonical, i.e. there is a canonical isomorphism of smooth sets over $y(J^\infty_M F)$

$$VJ^\infty_M F \times_{y(J^\infty_M F)} \big(y(J^\infty_M F) \times_{y(M)} y(TM)\big) \xrightarrow{\sim} T\big(y(J^\infty_M F)\big).$$

(25) The Cartan connection is the Ehresmann connection that chooses the complementary horizontal subbundle as $HJ^\infty_M F$. This is canonical for the infinite jet bundle, but it is not when considering a finite order jet bundle. In any case, it is possible to equivalently describe $HJ^k_M F$, with $k \in \mathbb{N}\cup\{\infty\}$, in terms of Cartan distribution (see ft. (27), p. 67) or associated foliation. However, the Cartan distribution has desirable properties only on the infinite jet bundle, and not on the finite order ones (cf. [Leó18; Blo24]).
(26) In particular, this is due to the contact structure of $J^\infty_M F$, i.e. the one forms annihilating the Cartan distribution.

*The induced splitting is $T\big(y(J^\infty_M F)\big) \cong VJ^\infty_M F \oplus HJ^\infty_M F$, where the plots of the smooth subbundle $HJ^\infty_M F$, called **Cartan distribution**,*[27] *are given by the image of the Cartan connection*[28] (see Eq. (3.18)).

Proof. We prove only the first part of the proposition, since the second comes as a corollary. The proof is a variation of Proposition 3.1.6. By the limit property of the infinite jet tangent bundle (see Eq. (3.8)) and the fully faithful embedding of LocProMfd into SmthSet, we compute

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F)\times_{y(M)} y(TM), T\big(y(J^\infty_M F)\big)\big) \cong \\
&\cong \mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F\times_M TM), \lim_k^{\mathrm{SmthSet}} y(T(J^k_M F))\big) \cong \\
&\cong \lim_k^{\mathrm{Set}} \mathrm{Hom}_{\mathrm{SmthSet}}\big(y(J^\infty_M F\times_M TM), y(TJ^k_M F)\big) \cong \\
&\cong \lim_k^{\mathrm{Set}} \mathrm{Hom}_{\mathrm{FrMfd}}\big(J^\infty_M F\times_M TM, TJ^k_M F\big).
\end{aligned}$$

In particular, in the second line we used the fact that the Yoneda embedding preserves limits. In the third line we applied the property of the hom-functor (see Prop. 1.1.1). Hence a smooth map $f\colon y(J^\infty_M F\times_M TM) \to T\big(y(J^\infty_M F)\big)$ corresponds to a family of smooth Fréchet maps $\{f^k\colon J^\infty_M F\times_M TM \to T(J^k_M F)\}_{k\in\mathbb{N}}$ such that $\mathrm{d}\pi^k_{k-1}\circ f^k = f^{k-1}$, and vice versa.
By Proposition 3.1.5, the set-theoretic maps

$$\big(\pi^\infty_{k+1}\times\mathrm{id}\big)^* H^k\colon J^\infty_M F\times_M TM \longrightarrow J^{k+1}_M F\times_M TM \longrightarrow T(J^k F)$$

are smooth Fréchet maps for each $k\in\mathbb{N}$, being the pullback of globally[29] finite order maps. Furthermore, by the commutativity of diagram (3.16), they satisfy

$$\begin{aligned}
\mathrm{d}\pi^k_{k-1}\circ(\pi^\infty_{k+1}\times\mathrm{id})^* H^k &= \mathrm{d}\pi^k_{k-1}\circ H^k\circ(\pi^\infty_{k+1}\times\mathrm{id}) = H^{k-1}\circ\big(\pi^{k+1}_k\times\mathrm{id}\big)\circ(\pi^\infty_{k+1}\times\mathrm{id}) \\
&= H^{k-1}\circ\big(\pi^\infty_k\times\mathrm{id}\big) = (\pi^\infty_k\times\mathrm{id})^* H^{k-1}.
\end{aligned}$$

Thus the family $\{(\pi^\infty_{k+1}\times\mathrm{id})^* H^k\colon J^\infty_M F\times_M TM \longrightarrow T(J^k_M F)\}_{k\in\mathbb{N}}$ uniquely corresponds to a map $H\colon y(J^\infty_M F\times_M TM)\to T\big(y(J^\infty_M F)\big)$. The underlying point set map is that of Equation (3.17), and since this splits the $*$-plot sequence, it follows that the $\mathbb{R}^n$-plot sequences split too. ∎

In a local coordinate chart for $J^\infty_M F$, the explicit action on $\mathbb{R}^n$-plots may be seen as

$$\Big(s^n, X^\mu\partial_{x^\mu}\big|_{\pi^\infty_M\circ s^n}\Big) \longmapsto X^\mu\Big(\partial_{x^\mu}\big|_{s^n} + \sum_{|I|=0}^{\infty} u^a_{I+\mu}\circ s^n\cdot\partial_{u^a_I}\big|_{s^n}\Big).$$

Here $s^n\colon\mathbb{R}^n\to J^\infty_M F$ is an $\mathbb{R}^k$-plot of the infinite jet bundle, and $\{X^\mu\colon\mathbb{R}^n\to M\}$ are the components of the corresponding plot in TM.
The second part of the above proposition implies the splitting, in the sense of $C^\infty(J^\infty_M F)$-modules, of smooth vector fields $\mathfrak{X}(J^\infty_M F)\cong\mathfrak{X}_V(J^\infty_M F)\oplus\mathfrak{X}_H(J^\infty_M F)$, where the two components denote smooth sections of the corresponding bundles (see Def. 3.1.15). Hence, any vector field X on $y(J^\infty_M F)$ decomposes uniquely as $X = X_V + X_H$, where X_V and X_H are sections of the vertical and horizontal smooth subbundles, respectively. The latter can be represented in local coordinates as

$$X_V = \sum_{|I|=0}^{\infty}\Big(Y^a_I - X^\mu\cdot u^a_{I+\mu}\Big)\cdot\frac{\partial}{\partial u^a_I} \quad\text{and}\quad X_H = X^\mu\Big(\partial_{x^\mu} + \sum_{|I|=0}^{\infty} u^a_{I+\mu}\partial_{u^a_I}\Big).$$

In particular, if $\tilde{X}^\mu\partial_{x^\mu}\in\Gamma_M(TM)$, its horizontal lift is $(\pi^\infty_M)^*\tilde{X}(\partial_{x^\mu}+\sum_{|I|=0}^\infty u^a_{I+\mu}\partial_{u^a_I})\in\mathfrak{X}_H(J^\infty_M F)$. It is helpful to denote the local basis for horizontal vector fields on $y(J^\infty_M F)$, i.e. the horizontal lifts of the local coordinate vector fields $\{\partial_{x^\mu}\}$ on M, by

$$D_\mu \coloneqq \partial_{x^\mu} + \sum_{|I|\geqslant 0} u^a_{I+\mu}\partial_{u^a_I}. \tag{3.19}$$

[27] A distribution is a smooth assignment of a vector subspace of the tangent space to each point of the manifold $m\mapsto\Delta_m\subseteq T_mM$ (see ft. (25), p. 66).
[28] The direct sum here is the fibered product $VJ^\infty_M F\times_{y(J^\infty_M F)}\big(y(J^\infty_M F)\times_{y(M)} y(TM)\big)$ computed in SmthSet, with the linear structure being fiberwise for each $\mathbb{R}^n$-plot over the induced plot of the base $y(J^\infty_M F)$, i.e. the direct sum of $C^\infty(\mathbb{R}^n)$-modules of fiber $\mathbb{R}^n$-plots over each $\mathbb{R}^n$-plot of $y(J^\infty_M F)$.
[29] In this particular case, the maps are globally of finite order, but this more generally works for locally finite order maps.

If $f\colon J^\infty_M F \to \mathbb{R}$ is a smooth function and $\varphi \in \Gamma_M(F)$ is a smooth section, then $f \circ j^\infty\varphi \in C^\infty(M)$. The vector fields $\{D_\mu\}$ encode the action of $\{\frac{\partial}{\partial x^\mu}\}$ on $f \circ j^\infty\varphi$ via the chain rule, that is:

$$\begin{aligned} D_\mu(f)\circ j^\infty\varphi &= \Big(\partial_{x^\mu} f + \sum_{|I|=0}^{\infty} u^a_{I+\mu}\partial_{u^a_I} f\Big)\circ j^\infty\varphi \\ &= \partial_{x^\mu} f\circ j^\infty\varphi + \sum_{|I|=0}^{\infty} \partial_{x^\mu}\partial_{x^I}\varphi^a \cdot \big(\partial_{u^a_I} f\circ j^\infty\varphi\big) = \partial_{x^\mu}\big(f\circ j^\infty\varphi\big). \end{aligned}$$

By construction, the vertical vector fields $\mathfrak{X}_V(J^\infty_M F)$ are closed under the Lie bracket defined in Equation (3.12). Crucially, the Cartan connection is flat. That is, the horizontal vector fields are closed under the Lie bracket

$$[X^1_H, X^2_H] \in \mathfrak{X}_H(J^\infty_M F), \quad \forall\, X^1_H, X^2_H \in \mathfrak{X}_H(J^\infty_M F), \tag{3.20}$$

as can be checked in local coordinates. This means that the Cartan distribution has zero curvature and that fields over the infinite jet bundle behave well when parallelly transported along a local path of the base manifold M, i.e. the transport is locally independent of the path. This greatly simplifies the horizontal lifting. Note that the splitting descends to the subspace of global finite order vector fields $\mathfrak{X}_{\mathrm{glb}}(J^\infty_M F) \cong \mathfrak{X}_{\mathrm{glb},V}(J^\infty_M F) \oplus \mathfrak{X}_{\mathrm{glb},H}(J^\infty_M F)$, with the same local representation formulas and properties.

Differential Forms and Variational Bi-complex

We now classically define differential forms on the infinite jet bundle as smooth antisymmetric $y(\mathbb{R})$-linear maps $T\big(y(J^\infty_M F)\big) \to y(\mathbb{R})$, exploiting the fiber-wise linear structure of $T(y(J^\infty_M F))$. Finally, we define the Cartan calculus and the relative variational bi-complex.

Definition 3.1.17. The set of **differential m-forms** on the infinite jet bundle $y(J^\infty_M F)$ is defined by

$$\Omega^m(J^\infty_M F) \coloneqq \mathrm{Hom}^{\text{fib.lin.an.}}_{\mathrm{SmthSet}}\Big(T^{\times m}(J^\infty_M F),\, y(\mathbb{R})\Big).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps, with respect to the fiber-wise linear structure induced by the operations in Equation (3.11), on the m-fold fiber product

$$T^{\times m}(J^\infty_M F) \;\coloneqq\; T(y(J^\infty_M F)) \times_{y(J^\infty_M F)} \cdots \times_{y(J^\infty_M F)} T(y(J^\infty_M F))$$

of the infinite jet tangent bundle over the infinite jet bundle.

Concretely, the above m-fold fiber product is the smooth set with $\mathbb{R}^n$-plots given by m-tuples of plots covering the same plot in $y(J^\infty_M F)$. Explicitly, the plots are

$$T^{\times m}(J^\infty_M F)(\mathbb{R}^n) \;=\; \big\{(X^1_{s^n}, \cdots, X^m_{s^n}) \in T(y(J^\infty_M F))(\mathbb{R}^n) \times \cdots \times T(y(J^\infty_M F))(\mathbb{R}^n)\big\},$$

where $p\circ X^1_{s^n} = \cdots = p \circ X^m_{s^n} = s^n \in y(J^\infty_M F)(\mathbb{R}^n)$, with $X^i_{s^n} \in T(y(J^\infty_M F))(\mathbb{R}^n)$ (see Eq. (3.10)). Note that sections of such m-fold fibered product are m-tuples of vector fields (see Def. 3.1.15). Accordingly, any $\omega \in \Omega^m(J^\infty_M F)$ defines, via pre-composition, a map of $C^\infty(J^\infty_M F)$-modules

$$\begin{aligned} \omega\colon\; \mathfrak{X}(J^\infty_M F)\times\cdots\times\mathfrak{X}(J^\infty_M F) &\longrightarrow C^\infty(J^\infty_M F) \\ \big((X^1,\dots,X^m)\colon y(J^\infty_M F)\to T^{\times m}(J^\infty_M F)\big) &\longmapsto \big(\omega\circ(X^1,\dots,X^m)\colon y(J^\infty_M F)\to y(\mathbb{R})\big). \end{aligned}$$

On the right-hand side, we identify $C^\infty(J^\infty_M F) = \mathrm{Hom}_{\mathrm{FrMfd}}(J^\infty_M F, \mathbb{R})$ since the Yoneda embedding is fully faithful.

Consider now $\omega \in \Omega^1(J^\infty_M F)$. In local coordinates around $s \in J^\infty_M F$, we can write

$$X^\mu\partial_{x^\mu} + \sum_{|I|=0}^{\infty} Y^a_I\,\partial_{u^a_I} \mapsto X^\mu\cdot\omega\big(\partial_{x^\mu}\big) + \sum_{|I|=0}^{\infty} Y^a_I\cdot\omega\big(\partial_{u^a_I}\big).$$

The right-hand side sum, to be a well-defined smooth function on $J^\infty_M F$ for all vector fields X, must terminate for some finite $|I| = k_s \in \mathbb{N}$, i.e. $\omega(\partial_{u^a_I}) = 0$ for $|I| \geqslant k_s$. By further restricting to a smaller neighborhood $U_s \subset J^\infty_M F$ around s, without loss of generality, we write

$$\omega = \omega_\mu \mathrm{d}x^\mu + \sum_{|I|=0}^{k_s} \omega^I_a \mathrm{d}u^a_i$$

where $\{\omega_\mu, \{\omega_a^I\}_{|I|\leqslant k_s}\} \subset C^\infty(J^\infty_M F)$ are locally defined and of finite order k_s. Extending this reasoning to the general case, we represent $\omega \in \Omega^m(J^\infty_M F)$, in local coordinates in a neighborhood U_s around a point $s \in J^\infty_M F$, by

$$\omega = \sum_{p+q=m} \sum_{I_1,\dots,I_q=0}^{k_s} \frac{\omega^{I_1\cdots I_q}_{\mu_1\cdots\mu_p\ a_1\cdots a_q}}{p!q!} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}u^{a_1}_{I_1} \wedge \cdots \wedge \mathrm{d}u^{a_q}_{I_q}.$$

Here the sum terminates at some finite order k_s, the coefficients are locally defined functions on $J^{k_s}_M F$, and the wedge is the usual exterior product.(30) That is, the map $\omega\colon\ \mathfrak{X}(J^\infty_M F) \times \cdots \times \mathfrak{X}(J^\infty_M F) \longrightarrow C^\infty(J^\infty_M F)$ is locally, for any $s' \in U_s \subset J^\infty_M F$, necessarily of the form

$$\omega\big(X^1, \cdots, X^m\big)(s') = \omega_{k_s}\big(\pi^\infty_{k_s}(s')\big)\Big(\mathrm{d}\pi^\infty_{k_s}\big(X_1(s')\big), \cdots, \mathrm{d}\pi^\infty_{k_s}\big(X_m(s')\big)\Big) \tag{3.21}$$

for some local form $\omega_{k_s} \in \Omega^m(J^{k_s}_M F)$ and $k_s \in \mathbb{N}$. Therefore, we can interpret m-forms on $y(J^\infty_M F)$ as being locally the pullback of finite order forms, and represent them by compatible families of locally defined m-forms on finite order jet bundles

$$\big\{\omega_{k_s} \in \Omega^m\big(U_{\pi^\infty_{k_s}(s)}\big) \big| U_{\pi^\infty_{k_s}(s)} \subset J^{k_s}_M F\big\}_{s \in J^\infty_M F}.$$

Compatible here means that for any two $s, s' \in J^\infty_M F$ with $k_{s'} \geqslant k_s$ such that(31) $(\pi^{k_{s'}}_{k_s})^{-1}\big(U_{k_s}\big) \cap U_{k_{s'}} \neq \varnothing \subset J^{k_{s'}}_M F$, we have $(\pi^{k_{s'}}_{k_s})^*\omega_{k_s} = \omega_{k_{s'}}$. Equation (3.21) determines the form of the corresponding map out of the tangent bundle (see Def. 3.1.17) on $*$-plots

$$\omega\colon T^{\times m}(J^\infty_M F)(*) \to \mathbb{R},\ (X^1_s, \cdots, X^m_s) \mapsto \omega_{k_s}\big(\pi^\infty_{k_s}(s)\big)\big(\mathrm{d}\pi^\infty_{k_s}(X^1_s), \cdots, \mathrm{d}\pi^\infty_{k_s}(X^m_s)\big),$$

for some locally defined form $\omega_{k_s} \in \Omega^m\big(J^{k_s}_M F\big)$ around $\pi^\infty_{k_s}(s) \in J^{k_s}_M F$. The action on any $\mathbb{R}^n$-plot follows immediately by substituting the $*$-plots denoted by s with $\mathbb{R}^n$-plots s^n.
Note that the globally finite order m-forms, i.e. those determined as a pullback of a single globally defined m-form $\omega_k \in \Omega^m(J^k_M F)$, fit in the above description. They form a vector subspace that we denote by $\Omega^m_{\mathrm{glb}}(J^\infty_M F) \subset \Omega^m(J^\infty_M F)$ in analogy to the case of smooth functions (or 0-forms).(32)

Remark 3.1.7. Equivalently, one could define the above within LocProMfd and consequently embed it in SmthSet. Recall that the local pro-manifold $J^\infty_M F$ is paracompact and has partitions of unity (cf. [Tak79]). By extending tangent vectors to vector fields, via a partition of unity, it follows that any $C^\infty(J^\infty_M F)$-linear map $\mathfrak{X}(J^\infty_M F) \times \cdots \times \mathfrak{X}(J^\infty_M F) \to C^\infty(J^\infty_M F)$ defines a fiberwise linear smooth map $T^{\times m}(J^\infty_M F) \to \mathbb{R}$. Thus, there is in fact a bijection

$$\begin{aligned}\Omega^m(J^\infty_M F) &= \mathrm{Hom}^{\text{fib.lin.an.}}_{\mathrm{SmthSet}}\big(T^{\times m}(J^\infty_M F), \mathbb{R}\big) &(3.22)\\ &\cong \mathrm{Hom}^{\text{antis.}}_{C^\infty(J^\infty_M F)\text{-Mod}}\big(\mathfrak{X}(J^\infty_M F) \times \cdots \times \mathfrak{X}(J^\infty_M F),\, C^\infty(J^\infty_M F)\big), &(3.23)\end{aligned}$$

where "antis." denotes the fact that the morphisms are antisymmetric.

Note that the wedge product can be defined as for Definition 3.1.7. Furthermore, it is possible to algebraically define the Cartan calculus(33) on $\Omega^\bullet(J^\infty_M F) := \bigoplus_{m\in\mathbb{N}} \Omega^m(J^\infty_M F)$.

Definition 3.1.18. Let $\omega \in \Omega^m(J^\infty_M F)$, $\omega' \in \Omega^{m'}(J^\infty_M F)$ (see Def. 3.1.17) for some m, $m' \in \mathbb{N}$, $f \in C^\infty(J^\infty_M F)$ and $X \in \mathfrak{X}(J^\infty_M F)$.

1. The **contraction** is defined as $\iota_X \omega := \omega(X, -, \cdots, -) \in \Omega^{m-1}(J^\infty_M F)$.

(30) In this case it is the usual wedge product since in the local coordinates we are considering forms locally of finite order, i.e. over the smooth manifold $J^{k_s}_M F$.
(31) $(\pi^{k_{s'}}_{k_s})^{-1}$ is the preimage of $\big(U_{k_s}\big)$ via the projection.
(32) Both $\Omega^m_{\mathrm{glb}}(J^\infty_M F)$ and $\Omega^m(J^\infty_M F)$ have a natural $C^\infty(J^\infty_M F)$-module structure, but only the latter also defines a petit sheaf on the topological space $J^\infty_M F \in$ LocProMfd.
(33) The Cartan calculus also naturally descends to $\Omega^\bullet_{\mathrm{glb}}(J^\infty_M F)$.

2. The **de Rham differential** $\mathrm{d}\colon \Omega^m(J^\infty_M F) \to \Omega^{m+1}(J^\infty_M F)$ is given by

$$\mathrm{d}\omega(X^0, \cdots, X^m) = \sum_{i=0}^{m} (-1)^i X^i\Big(\omega(X^0, \cdots, \widehat{X^i}, \cdots, X^m)\Big) + \sum_{i<j} (-1)^{i+j}\omega\big([X^i, X^j], X^0, \cdots, \widehat{X^i}, \cdots, \widehat{X^j}, \cdots, X^m\big). \tag{3.24}$$

   It follows that $\mathrm{d}^2 = 0$, and so $\big(\Omega^\bullet(J^\infty_M F), \mathrm{d}\big)$ defines a **cochain complex**.(34)

3. The **Lie derivative** of a function $f \in C^\infty(J^\infty_M F)$ along X is defined by $\mathbb{L}_X(f) \coloneqq X(f) \in C^\infty(J^\infty_M F)$. It extends to m-forms by $\mathbb{L}_X \coloneqq [\iota_X, \mathrm{d}]$, that is:

$$\mathbb{L}_X\colon\; \Omega^m(J^\infty_M F) \to \Omega^m(J^\infty_M F),\;\; \omega \mapsto \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X\omega).$$

The maps d, $\mathbb{L}_X$ and ι_X take the usual coordinate form around any point $s \in J^\infty_M F$, whereby ω and X are locally of finite order. One can show that the de Rham differential, the contraction by a vector field, and the Lie derivative with respect to a vector field are graded derivations of degree $+1$, -1, and 0, respectively. The usual Cartan calculus identities (see Thm. 1.1) between them also follow, since they hold locally around every point $s \in J^\infty_M F$ for the corresponding finite order representatives.
The splitting of the tangent bundle (see Prop. 3.1.9), induces a splitting on the set of forms $\Omega^\bullet(J^\infty_M F)$, that is a decomposition(35) on m-forms, for all $m \in \mathbb{N}$,

$$\Omega^m(J^\infty_M F) \cong \bigoplus_{p+q=m} \Omega^{p,q}(J^\infty_M F)$$

with $\omega \in \Omega^{p,q}(J^\infty_M F)$ if and only if

$$\omega(X^1_V, \ldots, X^p_V, -, \ldots, -) = 0 \quad \text{and} \quad \omega(-, \ldots, -, X^1_H, \ldots, X^q_H) = 0$$

for any vertical $\{X^1_V, \ldots, X^p_V\} \subset \mathfrak{X}_V(J^\infty_M F)$ and $\{X^1_H, \ldots, X^p_H\} \subset \mathfrak{X}_H(J^\infty_M F)$.
Locally around $s \in J^\infty_M F$, ω can be written as

$$\omega = \sum_{I_1,\ldots,I_q=0}^{k_s} \frac{\omega^{I_1\ldots I_q}_{\mu_1\ldots\mu_p a_1\ldots a_q}}{p!q!} \mathrm{d}_H x^{\mu_1} \wedge \ldots \wedge \mathrm{d}_H x^{\mu_p} \wedge \mathrm{d}_V u^{a_1}_{I_1} \wedge \ldots \wedge \mathrm{d}_V u^{\mu_q}_{I_q}$$

where the horizontal and vertical differentials are defined as follows.

Definition 3.1.19. The horizontal and vertical (p,q)-differentials

$$\mathrm{d}^{p,q}_H\colon \Omega^{p,q}(J^\infty_M F) \to \Omega^{p+1,q}(J^\infty_M F) \quad \text{and} \quad \mathrm{d}^{p,q}_V\colon \Omega^{p,q}(J^\infty_M F) \to \Omega^{p,q+1}(J^\infty_M F)$$

are defined by

$$\mathrm{d}^{p,q}_H \coloneqq \mathrm{pr}_{p+1,q} \circ \mathrm{d}|_{\Omega^{p,q}} \quad \text{and} \quad \mathrm{d}^{p,q}_V \coloneqq \mathrm{pr}_{p,q+1} \circ \mathrm{d}|_{\Omega^{p,q}}$$

where $\mathrm{pr}_{p,q}\colon \Omega^{p+q}(J^\infty_M F) \to \Omega^{p,q}(J^\infty_M F)$ denotes the subspace projection.

The involutive properties of the vertical and horizontal subbundles (see Eq. 3.20) imply that the de Rham differential decomposes as

$$\mathrm{d}|_{\Omega^{p,q}} = \mathrm{d}^{p,q}_H + \mathrm{d}^{p,q}_V. \tag{3.25}$$

Indeed, by projecting the de Rham differential onto each of the subspaces, the decomposition on functions is exactly

$$\mathrm{d}|_{C^\infty(J^\infty_M F)} =\colon \mathrm{d}^{0,0}_H + \mathrm{d}^{0,0}_V\colon C^\infty(J^\infty_M F) \to \Omega^1_H(J^\infty_M F) \oplus \Omega^1_V(J^\infty_M F),$$

where in local coordinates

$$\mathrm{d}_H(f) = \mathrm{d}x^\mu D_\mu(f) \quad \text{and} \quad \mathrm{d}_V(f) = \sum_{|I|=0}^{k_s} \big(\mathrm{d}u^a_I - u^a_{I+\mu}\mathrm{d}x^\mu\big)\, \partial_{u^a_I} f. \tag{3.26}$$

(34) It turns out that the global and local cohomologies are isomorphic, since the relative cochains are quasi-isomorphic (cf. [Tak79] and [GS25a, Prop. 4.14]).
(35) To show this, one first shows it for $m = 1$, where the uniqueness of the decomposition is due to $C^\infty(J^\infty_M F)$-linearity (cf. [GS25a][Def. 5.1]). Then it follows for $m \geqslant 2$ by $\Omega^m(J^\infty_M F) \cong \bigwedge^m \Omega^1(J^\infty_M F) \cong \bigwedge^m \big(\Omega^1_H(J^\infty_M F) \oplus \Omega^1_V(J^\infty_M F)\big)$.

A priori, the discussion is different on $(1,0)$-forms. Indeed, $\mathrm{d}|_{\Omega^{1,0}}$ could create 2-forms and therefore decompose as

$$\mathrm{d}|_{\Omega^{1,0}} = \mathrm{d}_H^{1,0} + \mathrm{d}_V^{1,0} + \mathrm{pr}_{0,2} \circ \mathrm{d}|_{\Omega^{1,0}}.$$

Then, to get the correct decomposition, we need to show that given any form $\omega_H \in \Omega^{1,0}(J^\infty_M F) = \Omega^1_H(J^\infty_M F)$, $\mathrm{d}\omega_H$ has zero $(0,2)$-component. This amounts to show that $\mathrm{d}\omega_H(X_V^1, X_V^2) = 0$ for all $\omega_H \in \Omega^{1,0}(J^\infty_M F) = \Omega^1_H(J^\infty_M F)$, and $X_V^1, X_V^2 \in \mathfrak{X}_V(J^\infty_M F)$. It is immediate to see that

$$\mathrm{d}\omega_H(X_V^1, X_V^2) = X_V^1(\omega_H(X_V^2)) - X_V^2(\omega_H(X_V^1)) - \omega_H([X_V^1, X_V^2]) = 0$$

by horizontality of ω_H and the involutive property of the vertical subbundle. A similar argument shows that $\mathrm{d}|_{\Omega^{0,1}} = \mathrm{d}_H^{0,1} + \mathrm{d}_V^{0,1}$. Since the bigraded algebra $\Omega^{\bullet,\bullet}(J^\infty_M F) := \bigoplus_{m\in\mathbb{N}} \bigoplus_{p+q=m} \Omega^{p,q}(J^\infty_M F)$ is generated by $C^\infty(J^\infty_M F)$, $\Omega^1_H(J^\infty_M F)$, and $\Omega^1_V(J^\infty_M F)$, it follows that the decomposition Equation (3.25) holds. Then, omitting the reference to (p,q), $0 = \mathrm{d}^2 = \mathrm{d}_H^2 + \mathrm{d}_V^2 + (\mathrm{d}_H\mathrm{d}_V + \mathrm{d}_V\mathrm{d}_H)$ implies

$$\mathrm{d}_H^2 = 0, \quad \mathrm{d}_V^2 = 0, \quad \text{and} \quad \mathrm{d}_H\mathrm{d}_V = -\mathrm{d}_V\mathrm{d}_H.$$

This is the case since each of the terms maps into different subspaces of the bigraded algebra $\Omega^{\bullet,\bullet}(J^\infty_M F)$. Then, the action on local coordinates and local basis of horizontal and vertical 1-forms is given by

$$\begin{aligned}
&\mathrm{d}_H(x^\mu) = \mathrm{d}x^\mu, \quad \mathrm{d}_V(x^\mu) = 0, \quad \mathrm{d}_H(u_I^a) = \mathrm{d}x^\mu u^a_{I+\mu}, \quad \mathrm{d}_V(u^a_I) = \mathrm{d}u^a_I - u^a_{I+\mu}\mathrm{d}x^\mu \\
&\mathrm{d}_H(\mathrm{d}x^\mu) = 0, \quad \mathrm{d}_V(\mathrm{d}x^\mu) = 0, \quad \mathrm{d}_H(\mathrm{d}_V u^a_I) = \mathrm{d}x^\mu \wedge \mathrm{d}_V u^a_{I+\mu}, \quad \mathrm{d}_V(\mathrm{d}_V u^a_I) = 0
\end{aligned} \tag{3.27}$$

and hence on any (p,q)-form, by derivation properties of the differentials and their action on functions (see Eq. (3.26)).
Note that any smooth bundle map $J^\infty_M F \to \bigwedge^p T^*M$ over M is equivalently a $(p,0)$-form on the infinite jet bundle.

Definition 3.1.20. The **variational bi-complex** of a fiber bundle $F \to M$ is the bi-complex

$$(\Omega^{\bullet,\bullet}(J^\infty_M F), \mathrm{d}_H, \mathrm{d}_V),$$

graphically depicted as

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & & & \uparrow{\scriptstyle \mathrm{d}_V} \\
\Omega^{0,2}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{1,2}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{2,2}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{d,2}(J^\infty_M F) \\
\uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & & & \uparrow{\scriptstyle \mathrm{d}_V} \\
\Omega^{0,1}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{1,1}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{2,1}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{d,1}(J^\infty_M F) \\
\uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & \uparrow{\scriptstyle \mathrm{d}_V} & & & & \uparrow{\scriptstyle \mathrm{d}_V} \\
\Omega^{0}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{1,0}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \Omega^{2,0}(J^\infty_M F) & \xrightarrow{\mathrm{d}_H} & \cdots & \xrightarrow{\mathrm{d}_H} & \Omega^{d,0}(J^\infty_M F).
\end{array}$$

A full description of the properties of the variational bicomplex can be found in [Sau89; Tak79; Leó18], and the discussion naturally fits into SmthSet (cf. [GS25a, Ch. 5]). We here only report an important result: Takens' acyclicity theorem.

Proposition 3.1.10. *The horizontal rows $(\Omega^{\bullet,q\geqslant 1}(J^\infty_M F), d_H)$ are exact, apart from degrees (d,q). Similarly the vertical columns $(\Omega^{p,\bullet}(J^\infty_M F), d_V)$, apart from $(p, q \leqslant f)$, where f is the dimension of the fiber $(\dim F = f + d)$.*

Local Cartan Calculus and Variational Bicomplex and Transgression

We introduce local vector fields and forms to demonstrate how the infinite jet bundle technology can be utilized to globally discuss locality, i.e. without referring to a specific local coordinate chart, and to define the local Cartan calculus on $\mathcal{F} \times M$. Such a description is crucial for the formal implementation of the action principle in classical local field theories. This can also be implemented in other settings, for instance within ProDflSp (cf. [Blo24]). However, SmthSet affords a more natural treatment of all field-theoretic objects within a single category (see Rmk. 3.1.5). In particular, the local Lagrangian density and local vector fields are morphisms in SmthSet, i.e. natural transformations between sheaves, that factor through the infinite jet bundle. Since the integration over M can also be viewed as a morphism of sheaves, the action functional itself defines a smooth map in SmthSet. Crucially, local vector fields form a Lie algebra with an appropriate bracket. Finally, local forms are morphisms in SmthSet obtained as pullbacks, along an appropriate smooth map, of differential forms on the infinite jet bundle; these, too, are natural transformations. Restricting to these local objects, one can define the local Cartan calculus and local variational bi-complex on $M \times \mathcal{F}$ and further transgress it to $\mathcal{F}$. For a more detailed treatment, we refer to [GS25a, Ch. 3, 5, 6 and Ch. 7].

Definition 3.1.21. A **local Lagrangian density** is a map of smooth sections $\mathcal{L}\colon \Gamma_M(F) \to \Omega^d(M)$, $\varphi \mapsto \mathsf{L} \circ j^\infty\varphi$ where $j^\infty\colon \Gamma_M(F) \to \Gamma_M(J^\infty_M F)$ is the set-theoretic jet prolongation, d is the dimension of M, and L is a smooth bundle map

$$\begin{tikzcd} J^\infty_M F \arrow[rr, "\mathsf{L}"] \arrow[dr, "\pi^\infty_M"'] & & \bigwedge^d T^*M. \arrow[dl] \\ & M & \end{tikzcd}$$

The Lagrangian form is local if it locally factors through finite order jet bundles. This definition does indeed reflect the formulas written in the physics literature (see Def. 2.1.2). Locally, the value of the local Lagrangian density $\mathcal{L}(\varphi)$ on a field $\varphi \in \Gamma_M(F)$ may be represented by

$$\mathcal{L}(\varphi) = \mathsf{L} \circ j^\infty\varphi = \mathsf{L}\big(x^\mu, \varphi^a, \{\partial_I \varphi^a\}_{|I|\leqslant k}\big) = \bar{\mathsf{L}}\big(x^\mu, \varphi^a, \{\partial_I \varphi^a\}_{|I|\leqslant k}\big) \cdot \mathrm{d}x^1 \cdots \mathrm{d}x^d,$$

for some smooth function $\bar{\mathsf{L}} \in C^\infty(J^\infty_M F)$. One can show that it canonically extends to a smooth map of smooth sets $\mathcal{L}\colon\ \mathbf{\Gamma}_M(F) \longrightarrow \mathbf{\Gamma}_M(\bigwedge^d T^*M)$, where the latter is the smooth set whose plots can be interpreted as $\mathbb{R}^k$-parametrized d-forms on M (cf. [GS25a, Lem. 3.11]).
Then, we define a bosonic smooth, local Lagrangian field theory to be a pair $(\mathcal{F}, \mathcal{L})$.
$\mathcal{F} := \mathbf{\Gamma}_M(F) \in$ SmthSet is the smooth field space of sections of a finite–dimensional fiber bundle F over the spacetime M, and $\mathcal{L} = \mathsf{L} \circ j^\infty$ is the smooth Lagrangian density defined by a smooth bundle map $\mathsf{L}\colon J^\infty_M F \to \bigwedge^d T^*M$ or, equivalently, by $\mathsf{L} \in \Omega^{d,0}(J^\infty_M F)$. Note that the majority of fundamental physical theories are described by Lagrangians that factor globally through a finite-degree jet bundle. However, generic algebraic operations on fixed order Lagrangians result in objects that necessarily factor through higher jet bundles, and so it is natural not to fix an order in the jet bundle and consider the infinite limit instead.
Integration defines another natural map of smooth sets. More precisely, if the base spacetime M is closed[(36)] and oriented, then integration along M defines a smooth map

$$\int_M \colon \mathbf{\Gamma}_M\big(\bigwedge^d T^*M\big) \longrightarrow y(\mathbb{R}). \tag{3.28}$$

Here, the integration is plot-wise along M while keeping the $\mathbb{R}^k$-dependence fixed. Explicitly, for any $\omega_{\mathbb{R}^k} \in \Omega^d(M)(\mathbb{R}^k)$, the value of $\int_M \omega_{\mathbb{R}^k} \in y(\mathbb{R})(\mathbb{R}^k)$ is given by

$$\left(\int_M \omega_{\mathbb{R}^k}\right)(x) := \int_M \iota_x^* \omega_{\mathbb{R}^k}$$

where $\iota_x^*\colon M \cong \{x\} \times M \hookrightarrow \mathbb{R}^k \times M$. The composition of a smooth Lagrangian $\mathcal{L}$ with the smooth integration map defines the action functional as a map of smooth sets

$$S = \int_M \circ \mathcal{L}\colon \mathbf{\Gamma}_M(F) \longrightarrow y(\mathbb{R}). \tag{3.29}$$

Therefore, SmthSet also satisfies requirement 2 in Section 2.1. S is a local functional in the following sense.

(36) For a non-compact manifold, we can restrict to closed submanifolds $\Sigma_d \subset M$. If $\partial M \neq \varnothing$, or $\partial \Sigma_d \neq \varnothing$, we should also take care of the boundary. A way of dealing with this at the level of the critical locus can be found in [GS25a][Def. 5.37 & Rmk. 5.38]. To apply the Cartan calculus on the boundary and, by means of it, formulate BV-BFV, one should first understand how the infinite jet bundle behaves in the presence of a boundary. This is an interesting question that, however, we do not delve into as it goes beyond the scope of the thesis.

Definition 3.1.22. The **algebra of smooth local real-valued functionals** $C^\infty_{\mathrm{loc}}(\mathcal{F}) \hookrightarrow C^\infty(\mathcal{F})$ on the field space $\mathcal{F}$ is the minimal subalgebra generated by integrating $\mathcal{P} \coloneqq \mathsf{P} \circ j^\infty \colon \mathcal{F} \to \mathbf{\Gamma}_M(\bigwedge^p T^*M)$, with $\mathsf{P} \in \Omega^{p,0}(J^\infty_M F)$, over compact oriented submanifolds $\Sigma^p \subset M$ of dimension p.(37)

The appropriate notion of a symmetry of a Lagrangian field theory is the following.

Definition 3.1.23.

1. Consider a Lagrangian field theory $(\mathcal{F}, \mathcal{L})$ and a local diffeomorphism

$$\mathcal{D} \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{F}) \coloneqq \{\mathrm{Hom}_{\mathrm{SmthSet}}(\mathcal{F},\mathcal{F}) \ni \mathcal{D} \coloneqq \mathsf{D} \circ j^\infty \colon \mathcal{F} \to \mathcal{F} | \exists \mathcal{D}^{-1} \coloneqq \mathsf{D}^{-1} \circ j^\infty \colon \mathcal{F} \to \mathcal{F}\},$$

   where $\mathsf{D}, \mathsf{D}^{-1}$ are bundle maps(38) $J^\infty_M F \to F$ over M. $\mathcal{D}$ is a **local symmetry** of $(\mathcal{F}, \mathcal{L})$ if there exist a map $\mathcal{K} \coloneqq \mathsf{K} \circ j^\infty \colon \mathcal{F} \to \mathbf{\Gamma}_M(\bigwedge^{d-1} T^*M)$, for some bundle map $\mathsf{K} \colon J^\infty_M F \to \bigwedge^{d-1} T^*M$, such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{L} + \mathrm{d}_M \mathcal{K}"'] & & F. \arrow[dl, "\mathcal{L}"] \\ & M & \end{tikzcd}$$

2. Consider a local Lagrangian field theory $(\mathcal{F}, \mathcal{L})$ and a local diffeomorphism $\mathrm{Diff}_{\mathrm{loc}}(\mathcal{F}) \ni \mathcal{D} \coloneqq \mathsf{D} \circ j^\infty(-) \circ (\mathrm{id}, f^{-1})$ induced by the bundle map $\mathsf{D} \colon J^\infty_M F \to F$ covering $\mathrm{Diff}(M) \ni f \colon M \to M$, i.e. such that the following commutes

$$\begin{tikzcd} J^\infty_M F \arrow[r, "\mathsf{D}"] \arrow[d, "\pi^\infty_M"'] & F \arrow[d, "\pi^F_M"] \\ M \arrow[r, "f"] & M. \end{tikzcd}$$

   $\mathcal{D}$ is a **spacetime covariant symmetry** for $(\mathcal{F}, \mathcal{L})$ if there exists $\mathcal{K} \coloneqq \mathsf{K} \circ j^\infty \colon \mathcal{F} \to \mathbf{\Gamma}_M(\bigwedge^{d-1} T^*M)$, for some bundle map $\mathsf{K} \colon J^\infty_M F \to \bigwedge^{d-1} T^*M$, such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{D}"] \arrow[d, "\mathcal{L} + \mathrm{d}_M \mathcal{K}"'] & F \arrow[d, "\mathcal{L}"] \\ \mathbf{\Gamma}_M(\bigwedge^{d-1} T^*M) \arrow[r, "f^*"] & \mathbf{\Gamma}_M(\bigwedge^{d-1} T^*M). \end{tikzcd}$$

Remark 3.1.8. This discussion applies verbatim for Yang–Mills, Chern-–Simons, and BF only if the set-theoretic field space is taken to be $\mathcal{F} = \Omega^1(M, \mathfrak{g}) = \Gamma_M(T^*M \times_M \mathfrak{g})$, that means considering connections on the trivial G-bundle $M \times G \to M$. Generally, however, the field space consists of connections $A \in \mathrm{Conn}(P)$ on an arbitrary principal G-bundle over M, which are not canonically the set of sections of a fiber bundle over M. With some restriction, one can choose a reference connection A_0 to yield a bijection $\mathrm{Conn}(P) \cong_{A_0} \Gamma_M(\mathrm{ad}(P))$ where $\mathrm{ad}(P) = P \times_G \mathfrak{g}$ is the adjoint bundle. The jet bundle formalism then applies to the corresponding smooth set of the right-hand side, therefore all diffeomorphisms and symmetries defined necessarily preserve A_0. Namely, such an approach treats the perturbation theory around a fixed connection A_0. A nonperturbative global approach to this requires a further natural generalization of smooth sets (cf. [GS25a][ft. 31, p. 32 & Rmk. 6.37]).

To properly treat the infinitesimal version of local symmetry and state Noether theorems(39), one needs the notion of local vector fields. We now introduce them as they are relevant to build the local variational bicomplex, but the full discussion on symmetries is out of the scope of this thesis and can be found in [GS25a][Ch. 6].

(37) If the submanifolds is non compact, we extend the definition by integrating against bump functions.
(38) One can show that bundle maps out of the infinite jet bundle naturally define a differential operator that extend to a smooth map (cf. [GS25a, Lemma 3.15]).
(39) [GS25a, Prop. 6.17] shows that local vector fields do capture the infinitesimal version of spacetime covariant symmetries, thus justifying the focus solely on local vector fields.

Definition 3.1.24. A smooth vector field $Z \in \mathfrak{X}(\mathcal{F}) = \Gamma_{\mathcal{F}}(T\mathcal{F})$ on the field space is **local** if it is given by $Z = \mathsf{Z} \circ j^\infty$ for some smooth bundle map, called **evolutionary vector field** $\mathsf{Z} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$,

$$\begin{tikzcd} J^\infty_M F \arrow[rr, "\mathsf{Z}"] \arrow[dr, "\pi^\infty_0"'] & & VF \arrow[dl, "\pi^{TF}_F"] \\ & F & \end{tikzcd}$$

over the total space F of the field bundle $F \to M$. The subset of local vector fields is denoted by $\mathfrak{X}_{\mathrm{loc}}(F) \subset \mathfrak{X}(F)$.

For each $\varphi \in \Gamma_M(F)$, the value of the tangent vector Z_φ depends only locally on φ via its jet $j^\infty_x \varphi$ at each $x \in M$. By Proposition 3.1.5, evolutionary vector fields are locally represented by finite sums

$$\mathsf{Z} = \mathsf{Z}^a \, \partial_{u^a} \,, \tag{3.30}$$

where $\{\mathsf{Z}^a\} \subset C^\infty(J^\infty_M F)$ are locally defined smooth functions on the infinite jet bundle, and $\{\partial_{u^a}\}$ is the local coordinate basis for vertical tangent vectors on F. This allows us to make sense of the field theoretical way of writing vector fields on the space of fields $\Gamma_M(F) \ni \varphi$

$$Z(\varphi) = Z^a(\varphi) \cdot \partial_{\varphi^a} = \mathsf{Z}^a(\varphi, \{\partial_I \varphi\}_{|I| \leqslant k}) \cdot \partial_{\varphi^a}$$

Strictly speaking, evolutionary vector fields are not vector fields, but they can be uniquely prolonged to a Lie subalgebra of $VJ^\infty_M F$ (cf. [GS25a][Prop. 6.4 and Cor. 6.6]).

Proposition 3.1.11. *For any $\mathsf{Z} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$, there exists a unique prolonged vertical vector field $\mathrm{pr}\mathsf{Z} \in \mathfrak{X}_V(J^\infty_M F)$ such that $\mathrm{d}\pi^\infty_0 \circ \mathrm{pr}\mathsf{Z} = Z$ and $[\iota_{\mathrm{pr}\mathsf{Z}}, \mathrm{d}_H] = 0$. Vice versa, any vertical vector field $Z \in \mathfrak{X}_V(J^\infty_M F)$ such that $[\iota_Z, \mathrm{d}_H] = 0$ defines an evolutionary vector field $\mathsf{Z} := \mathrm{d}\pi^\infty_0 \circ Z$. The two processes are inverse to each other. Furthermore,*

1. $\mathbb{L}_{\mathrm{pr}\mathsf{Z}} = [\iota_{\mathrm{pr}\mathsf{Z}}, \mathrm{d}_V]$ *satisfies* $\mathbb{L}_{\mathrm{pr}\mathsf{Z}} \mathrm{d}_{V/H} = \mathrm{d}_{V/H} \mathbb{L}_{\mathrm{pr}\mathsf{Z}}$.
2. $[\mathrm{pr}\mathsf{Z}_1, \mathrm{pr}\mathsf{Z}_2] = \mathrm{pr}\mathsf{Z}_3$ *where* $\mathfrak{X}_{\mathrm{ev}}(J^\infty_M F) \ni \mathsf{Z}_3 = \big(D_I \mathsf{Z}^b_1 \cdot \partial_{u^b_I} \mathsf{Z}^a_2 - D_I \mathsf{Z}^b_2 \cdot \partial_{u^b_I} \mathsf{Z}^a_1\big) \cdot \partial_{u^a}$ *in local coordinates.*

The condition $[\iota_{\mathrm{pr}\mathsf{Z}}, \mathrm{d}_H] = 0$ implies that, in local coordinates, we can write

$$\mathrm{pr}\mathsf{Z} = \sum_{|I|=0}^{\infty} D_I(\mathsf{Z}^a) \partial_{u^a_I}, \quad \text{where } D_I = D^{I_1}_{\mu_1} \circ \cdots \circ D^{I_d}_{\mu_d} \text{ with } D^{I_k}_{\mu_k} = \overbrace{D_{\mu_k} \circ \cdots \circ D_{\mu_k}}^{I_k \text{ times}} \tag{3.31}$$

for every multi-index $I = (I_1, \ldots, I_d)$ (see Eq. (3.19)). Crucially, this allows us to define the Lie algebra of local vector fields via the bracket

$$[Z_1, Z_2] = [\mathsf{Z}_1 \circ j^\infty, \mathsf{Z}_1 \circ j^\infty] := \mathsf{Z}_3 \circ j^\infty \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F}) \tag{3.32}$$

where $[\mathrm{pr}\mathsf{Z}_1, \mathrm{pr}\mathsf{Z}_2] = \mathrm{pr}\mathsf{Z}_3$. Finally, a local vector field $Z = \mathsf{Z} \circ j^\infty \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$ acts on a local functional $\int_{\Sigma^p} \mathcal{P} = \int_{\Sigma^p} \mathsf{P} \circ j^\infty \in C^\infty_{\mathrm{loc}}(\mathcal{F})$, where $\mathsf{P} \in \Omega^{p,0}(J^\infty_M F)$, (see Def. 3.1.22) as

$$Z(\mathcal{P}_{\Sigma^p}) := \int_{\Sigma^p} \mathbb{L}_{\mathrm{pr}\mathsf{Z}}(\mathsf{P}) \circ j^\infty \in C^\infty_{\mathrm{loc}}(\mathcal{F}). \tag{3.33}$$

In local coordinates such that $\mathsf{P} = \sum_{1 \leqslant \mu_1, \ldots, \mu_p \leqslant d} \frac{1}{p!} \mathsf{P}_{\mu_1 \cdots \mu_p}(\varphi, \{\partial_I \varphi\}_{|I| \leqslant k}) \cdot \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$, we can write $\mathbb{L}_{\mathrm{pr}\mathsf{Z}}(\mathsf{P}) = \sum_{1 \leqslant \mu_1, \ldots, \mu_p \leqslant d} \mathrm{pr}\mathsf{Z}\big(\frac{\mathsf{P}_{\mu_1 \cdots \mu_p}}{p!}\big) \cdot \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$.

To define local forms, we first consider $\mathcal{F} \times y(M) \in \mathrm{SmthSet}$ and define the tangent bundle on it to be $T(\mathcal{F} \times y(M)) := T(\mathcal{F}) \times y(T(M)) \in \mathrm{SmthSet}$. For readability, we drop the Yoneda embedding y for M and TM until the end of this section, since it is clear from the context whether they should be interpreted in SmthMfd or SmthSet. Then, the by definition splitting can be thought of as horizontal splitting along $\mathrm{pr}_2 \colon \mathcal{F} \times M \to M$, i.e.

$$V(\mathcal{F} \times M) \times_{\mathcal{F} \times M} H(\mathcal{F} \times M) := \big(T\mathcal{F} \times M\big) \times_{\mathcal{F} \times M} \big(\mathcal{F} \times TM\big) \;\cong\; T\mathcal{F} \times TM. \tag{3.34}$$

By definition, differential m-forms on the spacetime manifold M, the field space $\mathcal{F}$ and the infinite jet bundle $y(J^\infty_M F)$ are given by fiber-wise linear bundle maps out of their tangent bundle in SmthSet (see Def. 3.1.7). We therefore use this definition for differential forms on $\mathcal{F} \times M$ as well. As usual, the collection of differential forms of all degrees forms a graded $\mathbb{R}$-vector space

$$\Omega^\bullet(\mathcal{F} \times M) \coloneqq \bigoplus_{m \in \mathbb{N}} \Omega^m(\mathcal{F} \times M).$$

At this point, the development of Cartan calculus on $\mathcal{F} \times M$ reaches the same problems that we encounter for $\mathcal{F}$. Namely, sticking to this definition of forms, we lack of a notion of differential. Moreover, even though there is a natural notion of horizontal/vertical 1-forms on $\mathcal{F} \times M$ as the ones that vanish respectively on the vertical/horizontal subbundle, it is not obvious that every 1-form ω on $\mathcal{F} \times M$ splits accordingly. Similarly for any m-form, creating a non-trivial obstacle toward building the variational bicomplex (cf. [GS25a][Rmk. 7.2]). The resolution is that local classical field theory only requires the existence of the Cartan calculus and the corresponding bicomplex structure on the classical forms over $y(J^\infty_M F)$ (see Def. 3.1.18 and Def. 3.1.20). This structure can then be pulled back along ev^∞ (see Eq. (3.1.25)) to the subset of local forms on $\mathcal{F} \times M$ (see Def. 3.1.26). Then, since field theories are local, the usual field-theoretical manipulations may be concisely expressed via this local bicomplex and the related Cartan calculus.
More explicitly, consider the smooth evaluation map

$$\mathrm{ev}\colon \mathbf{\Gamma}_M(J^\infty_M F) \times M \longrightarrow y(J^\infty_M F), \;\; (\tilde{\varphi}^k, p^k) \longmapsto \tilde{\varphi}^k \circ (\mathrm{id}_{\mathbb{R}^k}, p^k) \in y(J^\infty_M F)(\mathbb{R}^k)$$

where φ^k and p^k are $\mathbb{R}^k$-plots of $\mathbf{\Gamma}_M(J^\infty_M F)$ and $y(M)$ respectively. Precomposing along the smooth jet prolongation $j^\infty\colon \mathcal{F} \to \mathbf{\Gamma}_M(J^\infty_M F)$, we may define the smooth prolongated evaluation map with values in $y(J^\infty_M F)$.

Definition 3.1.25. The **prolongated evaluation map** $\mathrm{ev}^\infty\colon \mathcal{F} \times M \to y(J^\infty_M F)$ is defined as the composition of maps of smooth sets

$$\mathrm{ev}^\infty\colon \mathcal{F} \times M \xrightarrow{(j^\infty, \mathrm{id}_M)} \mathbf{\Gamma}_M(J^\infty_M F) \times M \xrightarrow{\mathrm{ev}} y(J^\infty_M F). \tag{3.35}$$

At the level of $*$–plots this is given by $\mathrm{ev}^\infty(\varphi, p) = j^\infty\varphi(p) \in y(J^\infty_M F)(*)$, and similarly for higher plots.

Differentiating the latter, we can define the pushforward map along ev^∞ on $*$–plots as

$$\begin{aligned} \mathrm{dev}^\infty\colon T\mathcal{F} \times TM &\to VJ^\infty_M F \oplus HJ^\infty_M F \cong T\big(y(J^\infty_M F)\big) \\ (Z_\varphi, X_p) &\mapsto j^\infty Z_\varphi(p) + \big(\mathrm{d}j^\infty\varphi\big)_p X_p, \end{aligned} \tag{3.36}$$

and similarly for higher plots. In local compatible coordinates around p, we can write it as

$$\left(Z^a_\varphi \cdot \partial_{u^a}, X^\mu \cdot \partial_{x^\mu}\big|_p\right) \mapsto \sum_{|I|=0}^{\infty} \partial_{x^I}(Z^a_\varphi)(p) \cdot \partial_{u^a_I}\big|_{j^\infty_p\varphi} + X^\mu\left(\partial_{x^\mu}\big|_{j^\infty_p\varphi} + \sum_{|I|=0}^{\infty} \partial_{x^{I+\mu}}(\varphi^a)(p) \cdot \partial_{u^a_I}\big|_{j^\infty_p\varphi}\right).$$

By construction, the pushforward dev^∞ respects the natural splitting of the tangent bundles $T\mathcal{F} \times TM$ from Equation (3.34), and that of $T\big(y(J^\infty_M F)\big)$ from Proposition 3.1.9, as bundles over M:

$$\begin{tikzcd} V(\mathcal{F} \times M) \times_{\mathcal{F}\times M} H(\mathcal{F} \times M) \arrow[rr, "\mathrm{def}^\infty"] \arrow[dr] & & VJ^\infty_M F \times_{J^\infty_M F} HJ^\infty_M F. \arrow[dl] \\ & M & \end{tikzcd} \tag{3.37}$$

Thanks to this, we can define the bicomplex of local forms as follows.

Definition 3.1.26. The **bicomplex of local forms** on $\mathcal{F} \times M$ is defined as the image of the variational bicomplex on $y(J^\infty_M F)$ under the pullback along $\mathrm{ev}^\infty\colon \mathcal{F} \times M \to y(J^\infty_M F)$. That is,

$$\Omega^{\bullet,\bullet}_{\mathrm{loc}}(\mathcal{F} \times M) \coloneqq \mathrm{Im}\big((\mathrm{ev}^\infty)^*\colon \Omega^{\bullet,\bullet}(J^\infty_M F) \to \Omega^\bullet(\mathcal{F} \times M)\big),$$

equipped with the horizontal and vertical differentials

$$\mathrm{d}_M((\mathrm{ev}^\infty)^*\omega) \coloneqq (\mathrm{ev}^\infty)^*(\mathrm{d}_H\omega) \quad \text{and} \quad \mathrm{d}_{\mathcal{F}}((\mathrm{ev}^\infty)^*\omega) \coloneqq (\mathrm{ev}^\infty)^*(\mathrm{d}_V\omega).$$

By definition, the local bicomplex has a module structure over $C^\infty_{\mathrm{loc}}(\mathcal{F}\times M)\cong\Omega^{0,0}_{\mathrm{loc}}(\mathcal{F}\times M)$. More explicitly, the local pullback form $(\mathrm{ev}^\infty)^*\omega\in\Omega^{p,q}_{loc}(\mathcal{F}\times M)$ of a differential form $\omega\in\Omega^{p,q}(J^\infty_M F)$ is defined by the composition

$$(\mathrm{ev}^\infty)^*\omega\colon\ T\mathcal{F}\times TM\xrightarrow{\mathrm{dev}^\infty}T\big(y(J^\infty_M F)\big)\xrightarrow{\omega}\mathbb{R},$$

and, using the Einstein summation convention, takes the local form

$$\sum_{I_1,\dots,I_p=0}\frac{\omega^{I_1\cdots I_q}_{\mu_1\cdots\mu_p a_1\cdots a_q}(x,\{\partial_J\varphi\}_{|J|\leqslant k})}{p!q!}\cdot\mathrm{d}_M x^{\mu_1}\wedge\cdots\wedge\mathrm{d}_M x^{\mu_p}\wedge\mathrm{d}_{\mathcal{F}}(\partial_{I_1}\varphi^{a_1})\wedge\cdots\wedge\mathrm{d}_{\mathcal{F}}(\partial_{I_q}\varphi^{a_q}).\quad(3.38)$$

Here, for all $f\in C^\infty(J^\infty_M F)$, the differentials of $\bar f\equiv(\mathrm{ev}^\infty)^*f$ are locally given by

$$\mathrm{d}_M(\bar f)=\mathrm{d}x^\mu\cdot\partial_{x^\mu}\big(f(x,\{\partial_J\varphi\}_{|J|\leqslant k})\big),\quad \mathrm{d}_{\mathcal{F}}(\bar f)=\sum_{|I|=0}^{\infty}\mathrm{d}_{\mathcal{F}}(\partial_I\varphi^a)\cdot\partial_{\partial_I\varphi^a}\big(f(x,\{\partial_J\varphi\}_{|J|\leqslant k})\big).$$

Crucially, the local variational bicomplex admits a full local Cartan calculus (cf. [GS25a][pp. 98-100]) upon restriction to local vector fields $(\mathfrak{X}_{\mathrm{loc}}(\mathcal{F}),[\ ,\ ])$ (see Def. 3.1.24 and Eq. (3.32)).

Proposition 3.1.12. *Consider any local form $((\mathrm{ev}^\infty)^*\omega)\in\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)$ and any local vector field $Z=\mathsf{Z}\circ j^\infty\in\mathfrak{X}_{\mathrm{loc}}(\mathcal{F})\hookrightarrow\mathfrak{X}(\mathcal{F}\times M)$ viewed as a vector field $(\varphi,p)\mapsto(Z(\varphi),0_p)$ on $\mathcal{F}\times M$ with a constant vanishing horizontal component. Then,*

1. *the contraction by Z satisfies $\iota_Z((\mathrm{ev}^\infty)^*\omega))=(\mathrm{ev}^\infty)^*(\iota_{\mathrm{pr}\mathsf{Z}}\omega)$ and it is a graded derivation $\iota_Z\colon\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)\to\Omega^{p,q-1}_{\mathrm{loc}}(\mathcal{F}\times M)$;*
2. *The horizontal and vertical differentials are graded derivations $\mathrm{d}_M\colon\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)\to\Omega^{p+1,q}_{\mathrm{loc}}(\mathcal{F}\times M)$ and $\mathrm{d}_{\mathcal{F}}\colon\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)\to\Omega^{p,q+1}_{\mathrm{loc}}(\mathcal{F}\times M)$;*
3. *The Lie derivative along Z is a graded derivation $\mathbb{L}_Z:=[\iota_Z,\mathrm{d}]\colon\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)\to\Omega^{p,q+1}_{\mathrm{loc}}(\mathcal{F}\times M)$[40] satisfying $\mathbb{L}_Z((\mathrm{ev}^\infty)^*\omega))=(\mathrm{ev}^\infty)^*(\mathbb{L}_{\mathrm{pr}\mathsf{Z}})$ and $\mathbb{L}_Z\mathrm{d}_{M/\mathcal{F}}=\mathrm{d}_{M/\mathcal{F}}\mathbb{L}_Z$;*
4. *All the rest of the commutation relations between these derivations still hold* (see Prop. 1.1).

The proof of the above statements can be found in (cf. [GS25a][pp. 98-100]). In particular, it is based on Definition 3.1.18, the pullback properties, and Proposition 3.1.11.
We now introduce the transgression map, defined as the composition of pullback along the prolongated evaluation map and integration over a closed submanifold of M. We also introduce a suitable notion of weakly symplectic form on $\mathcal{F}\times M$, which induces a weakly symplectic structure on $\mathcal{F}$ via transgression.

Definition 3.1.27. Let $\Sigma^p\subset M$ be a closed[41] submanifold. The **transgression** $\tau_{\Sigma^p}\colon\Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)\to\Omega^q(\mathcal{F})$ of local (p,q)-forms on $\mathcal{F}\times M$ to q-forms on $\mathcal{F}$ is defined by

$$\begin{aligned}\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\colon (T\mathcal{F})^{\times q}&\longrightarrow y(\mathbb{R})\\ (Z^1_\varphi,\dots,Z^q_\varphi)&\longmapsto\int_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)_\varphi(Z^1_\varphi,\dots,Z^q_\varphi)\end{aligned}$$

and similarly for higher plots. We denote $\Omega^\bullet_{\mathrm{loc},\Sigma^p}(\mathcal{F})\hookrightarrow\Omega^\bullet(\mathcal{F})$ the subalgebra of forms generated by transgressed local forms along Σ^p and by $\Omega^\bullet_{\mathrm{loc}}(\mathcal{F})$ the one generated by transgressed local forms along arbitrary closed submanifolds. Note that, for $p=0$ this corresponds to local functionals of Definition 3.1.22.

The following result gives us a notion of de Rham differential on $\mathcal{F}$ when restricted to local vector fields and local transgressed forms. The caveat is that the latter should be interpreted as maps of local vector fields $(\mathfrak{X}_{\mathrm{loc}}(\mathcal{F}))^{\times q}\to C^\infty_{\mathrm{loc}}(\mathcal{F})$, and not as bundle maps $(T\mathcal{F})^{\times q}\to y(\mathbb{R})$ (cf. [GS25a][Rmk. 7.39]). Note that for our application, this is enough.

[40] We could have used $\mathbb{L}'_Z:=[\iota_Z,\mathrm{d}_M+\mathrm{d}_{\mathcal{F}}]=[\iota_Z,\mathrm{d}_M]+[\iota_Z,\mathrm{d}_{\mathcal{F}}]$, but this would be redundant since $[\iota_Z,\mathrm{d}_M]=0$, as can be seen in local coordinates.

[41] For non-closed manifolds, one has to generalize this definition. For example, see [GS25a][Rmk. 7.35] for non-compact submanifolds.

Proposition 3.1.13. *The vertical differential* $\mathrm{d}_{\mathcal{F}}\colon \Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M) \to \Omega^{p,q+1}_{\mathrm{loc}}(\mathcal{F}\times M)$ *transgresses to a well-defined map*

$$\begin{aligned} \mathrm{d}_{\mathcal{F}}\colon \Omega^{q}_{\mathrm{loc},\Sigma^p}(\mathcal{F}) &\longrightarrow \Omega^{q+1}_{\mathrm{loc},\Sigma^p}(\mathcal{F}) \\ \tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega\big) &\longmapsto \mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) \coloneqq \tau_{\Sigma^p}\big(\mathrm{d}_{\mathcal{F}}(\mathrm{ev}^\infty)^*\omega\big) \end{aligned} \tag{3.39}$$

that extends to a well-defined derivation $\Omega^{\bullet}_{\mathrm{loc}}(\mathcal{F}) \to \Omega^{\bullet+1}_{\mathrm{loc}}(\mathcal{F})$.

Proof. There might exist $(\mathrm{ev}^\infty)^*\omega \neq (\mathrm{ev}^\infty)^*\omega' \in \Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)$ with $\tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega\big) = \tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega'\big)$. So, for Equation (3.39) to be well defined, we need $\mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) = \mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega')\big)$. We prove that this is the case by induction on q.
For $q=0$, consider $(\mathrm{ev}^\infty)^*\omega \neq (\mathrm{ev}^\infty)^*\omega' \in \Omega^{p,0}_{\mathrm{loc}}(\mathcal{F}\times M)$ with $\tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega\big) = \tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega'\big)$. Then, for any 1-parameter family of fields φ_t we have $\int_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)(\varphi_t) = \int_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega')(\varphi_t) \in C^\infty(\mathbb{R})$. The derivative evaluated at $t=0$ reads

$$\int_{\Sigma^p} \mathrm{d}_{\mathcal{F}}((\mathrm{ev}^\infty)^*\omega)_{\varphi_0}(\partial_t\varphi_t|_{t=0}) = \int_{\Sigma^p} \mathrm{d}_{\mathcal{F}}((\mathrm{ev}^\infty)^*\omega')_{\varphi_0}(\partial_t\varphi_t|_{t=0}) \in C^\infty(\mathbb{R})$$

and thanks to Proposition 3.1.4 the result follows for $q=0$.
For a generic q, assuming the case $q-1$ to hold, we proceed as follows. Consider $Z_1, Z_2 \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$ and $(\mathrm{ev}^\infty)^*\omega \neq (\mathrm{ev}^\infty)^*\omega' \in \Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)$ with $\tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega\big) = \tau_{\Sigma^p}\big((\mathrm{ev}^\infty)^*\omega'\big)$. By definition of contraction along Z_1, we can write

$$\tau_{\Sigma^p}\big(\iota_{Z_1}(\mathrm{ev}^\infty)^*\omega\big) = \tau_{\Sigma^p}\big(\iota_{Z_1}(\mathrm{ev}^\infty)^*\omega'\big).$$

By the $q-1$ case, we get $\mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}(\iota_{Z_1}(\mathrm{ev}^\infty)^*\omega)\big) = \mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}(\iota_{Z_1}(\mathrm{ev}^\infty)^*\omega')\big)$. Further contracting the latter equality along Z_2 and recalling that $\mathbb{L}_{Z_2} = [\iota_{Z_2}, \mathrm{d}_{\mathcal{F}}]$, we get

$$\int_{\Sigma^p} \mathbb{L}_{Z_2}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega) = \int_{\Sigma^p} \iota_{Z_2}\mathrm{d}_{\mathcal{F}}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega) = \int_{\Sigma^p} \iota_{Z_2}\mathrm{d}_{\mathcal{F}}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega') = \int_{\Sigma^p} \mathbb{L}_{Z_2}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega').$$

If we invert the role of Z_1 and Z_2, the result is the same with $Z_1 \leftrightarrow Z_2$. Then, combining the above and using the local Cartan calculus (see Prop. 3.1.12), we get the following

$$\begin{aligned} \int_{\Sigma^p} \mathrm{d}_{\mathcal{F}}((\mathrm{ev}^\infty)^*\omega)(Z_1,Z_2) &= \int_{\Sigma^p} \mathbb{L}_{Z_1}\iota_{Z_2}((\mathrm{ev}^\infty)^*\omega) - \mathbb{L}_{Z_2}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega) - \iota_{[Z_1,Z_2]}((\mathrm{ev}^\infty)^*\omega) = \\ &= \int_{\Sigma^p} \mathbb{L}_{Z_1}\iota_{Z_2}((\mathrm{ev}^\infty)^*\omega') - \mathbb{L}_{Z_2}\iota_{Z_1}((\mathrm{ev}^\infty)^*\omega') - \iota_{[Z_1,Z_2]}((\mathrm{ev}^\infty)^*\omega') = \int_{\Sigma^p} \mathrm{d}_{\mathcal{F}}((\mathrm{ev}^\infty)^*\omega')(Z_1,Z_2). \end{aligned}$$

By generality of $Z_1, Z_2 \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$, we obtain $\mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) = \mathrm{d}_{\mathcal{F}}\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega')\big)$. ∎

Note that, for any local vector field $Z \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$ and local form $(\mathrm{ev}^\infty)^*\omega \in \Omega^{p,q}_{\mathrm{loc}}(\mathcal{F}\times M)$ the following holds

$$\begin{aligned} \iota_Z\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) &= \tau_{\Sigma^p}\big(\iota_Z((\mathrm{ev}^\infty)^*\omega)\big) \\ \mathbb{L}_Z\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) &\coloneqq [\iota_Z, \mathrm{d}_{\mathcal{F}}]\big(\tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)\big) = \tau_{\Sigma^p}\big(\mathbb{L}_Z((\mathrm{ev}^\infty)^*\omega)\big) \end{aligned} \tag{3.40}$$

In particular, the first holds by Definition 3.1.27 and Proposition 3.1.12 and the second follows from the first and Proposition 3.1.13. Finally, the following definition gives us the notion of symplectic and Poisson structure on $\mathcal{F}$.

Definition 3.1.28. Consider a closed submanifold $\Sigma^p \subset M$. A **(weakly) symplectic local form** on $\mathcal{F}$ is given by $\Omega_{\Sigma^p} \coloneqq \tau_{\Sigma^p}((\mathrm{ev}^\infty)^*\omega)$ where $\Omega^{p,2}_{\mathrm{loc}}(\mathcal{F}\times M) \ni (\mathrm{ev}^\infty)^*\omega \coloneqq \mathrm{d}_{\mathcal{F}}\alpha + (\mathrm{d}_M\text{-exact terms})$ is vertically closed and such that $\Omega_{\Sigma^p}|_{\mathfrak{X}_{\mathrm{loc}}(\mathcal{F})}\colon \mathfrak{X}_{\mathrm{loc}}(\mathcal{F}) \to \Omega^1_{\mathrm{loc}}(\mathcal{F})$ is (injective) bijective. For any local vector field $Z \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$, a local functional $\mathcal{H}_{Z,\Sigma^p} \coloneqq \tau_{\Sigma^p} h_Z = \int_{\Sigma^p} h_Z \in C^\infty_{\mathrm{loc}}(\mathcal{F})$, where $h_Z \in \Omega^{p,0}_{\mathrm{loc}}(\mathcal{F}\times M)$, is the **Hamiltonian functional**[(42)] for Z with respect to Ω_{Σ^p} if $\iota_Z\Omega_{\Sigma^p} = \mathrm{d}_{\mathcal{F}}\mathcal{H}_{Z,\Sigma^p}$. The (weakly) symplectic structure defines a **Poisson structure**[(43)] (cf. [GS25a][Def. 7.47 & Lemma 7.48]) on Hamiltonian functionals (for a fixed Σ^p) with bracket defined by

$$\{\mathcal{H}_{Z_1,\Sigma^p}, \mathcal{H}_{Z_2,\Sigma^p},\}_{\Omega_{\Sigma^p}} \coloneqq \iota_{Z_1}\iota_{Z_2}\Omega_{\Sigma^p}.$$

[(42)] Strictly speaking, we should talk about Hamiltonian pairs $(\mathcal{H}_{Z,\Sigma^p}, Z) \in C^\infty_{\mathrm{loc}}(\mathcal{F}) \times \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})$ as for every Z there is a family of local functionals that are Hamiltonian for Z parametrized by the freedom of adding any $(p,0)$ form to the integrand whose vertical differential vanishes up to horizontally exact terms. We omit this and think of the Hamiltonian $\mathcal{H}_{Z,\Sigma^p}$ as a representative of such a family.
[(43)] The product of Hamiltonian functionals is Hamiltonian, but not necessarily for a local vector field (cf. [GS25a][Lemma 7.46]).

All the structure in the above definition can be understood at the level of $\mathcal{F} \times M$ up to horizontally exact terms (cf. [GS25a][Def. 7.10, Def. 7.28, Def. 7.29 & Lemma 7.31]). As this is not strictly relevant for us, we omit this discussion.

In summary, SmthSet provides field spaces and local jet geometry; LocProMfd gives a workable infinite jet bundle; the variational bicomplex provides local Cartan calculus.

3.2 Thickened Smooth Sets

In this section, we recall the definition of the category ThSmthSet of infinitesimally thickened smooth sets (cf. [GS25b]). This setting subsumes all previous constructions. In particular, it recovers the ad hoc tangent bundles arising in traditional field-theoretic considerations via the synthetic tangent bundle construction.

Definition of Thickened Smooth Sets

Consider the category of **associative commutative unital real algebras** $\mathbf{CAlg}_{\mathbb{R}}$. The objects are real unital associative commutative ($a \cdot b = b \cdot a$) algebras, and the morphisms are structure-preserving maps, i.e. unital associative commutative algebras' morphisms.

Definition 3.2.1. The category **ThCartSp** of **thickened Cartesian spaces** is defined as

$$\begin{aligned} \mathrm{ThCartSp} &\hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^k \times \mathbb{D} &\mapsto C^\infty(\mathbb{R}^k) \otimes \mathcal{O}(\mathbb{D}) \equiv \mathcal{O}(\mathbb{R}^k \times \mathbb{D}). \end{aligned}$$

Here, $\mathcal{O}(\mathbb{D}) := \mathbb{R} \oplus V$, with V a finite-dimensional vector space containing only nilpotent elements, is the Weil algebra associated to the **infinitesimal point** $\mathbb{D}$.[(44)] Morphisms in ThCartSp are equivalently described as morphisms between the corresponding opposite unital commutative associative algebras

$$\mathrm{Hom}_{\mathrm{ThCartSp}}(\mathbb{R}^k \times \mathbb{D}, \mathbb{R}^{k'} \times \mathbb{D}') \cong \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}(\mathcal{O}(\mathbb{R}^{k'} \times \mathbb{D}'), \mathcal{O}(\mathbb{R}^k \times \mathbb{D})).$$

The latter definition can be understood in light of the following proposition. Intuitively, one can fully characterize a smooth manifold by considering the algebra of smooth functions on it. This is known as Milnor's Exercise (cf. [KMS93, Ch. 35]) and a generalized proof for manifolds with corners can be found in [GS25b, Prop. 4.13].

Proposition 3.2.1. *The functor*

$$\begin{aligned} C^\infty(-) \colon \mathrm{SmthMfd} &\hookrightarrow \mathrm{CAlg}_{\mathbb{R}} \\ M &\mapsto C^\infty(M) \end{aligned}$$

sending a finite-dimensional smooth second countable and Hausdorff manifold to its function algebra is fully faithful, in that for any pair $N, M \in \mathrm{SmthMfd}$ the smooth functions $f \colon N \to M$ biject onto the algebra homomorphisms $f^* \colon C^\infty(M) \mapsto C^\infty(N)$. Hence also the composite $C^\infty(-) \colon \mathrm{CartSp} \hookrightarrow \mathrm{SmthMfd} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}$ is fully faithful.

The category of ThCartSp is a site when endowed with the Grothendieck pretopology given by the trivially extended differentiably good open covers $\{U_i \xrightarrow{\iota_i} \mathbb{R}^k\}_{i\in I}$[(45)]

$$\{U_i \times \mathbb{D} \xrightarrow{\iota_i \times \mathrm{id}_{\mathbb{D}}} \mathbb{R}^k \times \mathbb{D}\}_{i \in I}. \tag{3.41}$$

(44) We stress that infinitesimally thickened points are taken to be the formal duals of arbitrary Weil algebras (also known as finite-dimensional real Artinian local algebras), i.e. finite-dimensional commutative local $\mathbb{R}$-algebras with residue field $\mathbb{R}$ and nilpotent maximal ideal. The infinitesimal disks $D^m_{(l)}$, whose defining algebras form a subclass of the Weil algebras, are used primarily for convenience of exposition, since any infinitesimal point embeds into an infinitesimal disk (see point 1. of Rmk. 3.2.1).

(45) That is, a collection $\{U_i \subset \mathbb{R}^k\}_{i\in I}$ of open subsets of $\mathbb{R}^k$ such that $\mathbb{R}^k = \bigcup_{i\in I} U_i$ and such that all U_i, together with all their inhabited finite intersections, are diffeomorphic to an open ball.

To check this, notice that isomorphisms are coverings,[46] coverings are stable under pullbacks[47] and transitivity holds trivially. The chosen Grothendieck topology is generated only by covers in the smooth even component, while the infinitesimal direction is carried along unchanged. This means that all the presheaves in the infinitesimal component are also sheaves. One needs to check the locality and gluing conditions only in the smooth component: a presheaf $\mathcal{G}\colon \mathrm{ThCartSp}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf if and only if $\mathcal{G}(\mathbb{R}^k \times \mathbb{D})$ is a sheaf in the petit sense when restricted along $\mathbb{R}^k \hookrightarrow \mathbb{R}^k \times \mathbb{D}$ for any $\mathbb{R}^k \times \mathbb{D} \in \mathrm{ThCartSp}$.

Remark 3.2.1.

1. One can work with **infinitesimal m-disks of order l** instead, since any infinitesimal point embeds into an infinitesimal disk (cf. [GS25b, Lemma 2.8 & Rmk. 2.9]). These are similarly defined by

$$\mathcal{O}(\mathbb{D}^m(l)) := \mathbb{R}[\epsilon^1, \dots \epsilon^m]/(\epsilon^1, \dots, \epsilon^m)^{l+1} \cong C^\infty(\mathbb{R}^m)/(x^1, \dots, x^m)^{l+1},$$

where the last canonical identification follows from Hadamard's Lemma (cf. [GS25b, Lemma 2.6]). Every infinitesimal disk $\mathbb{D}^m(l)$ also embeds into the corresponding m-dimensional Cartesian space $\mathbb{R}^m$, around its origin. More explicitly, we can understand $\iota_0\colon \mathbb{D}^m(l) \to \mathbb{R}^m$ in the formal dual sense of the algebra projection map defined by the l-order Taylor expansion

$$\begin{aligned} &C^\infty(M) \to \mathcal{O}(\mathbb{D}^m(l)) \\ &\quad f \mapsto f(0) + \epsilon^i \partial_i f(0) + \frac{1}{2}\epsilon^i\epsilon^j \partial_i\partial_j f(0) + \dots + \frac{1}{l!}\sum_{i_1,\dots,i_l=1}^{m} \epsilon^{i_1}\cdots\epsilon^{i_l}\partial_{i_1}\cdots\partial_{i_l} f(0). \end{aligned}$$

2. The archetypical example of infinitesimal space is $\mathbb{D}^1(1) := \mathbb{R}[\epsilon]/\epsilon^2$, which may be thought of as the smooth functions on the first order infinitesimal neighborhood $\mathbb{D}^1(1) \hookrightarrow \mathbb{R}$ of the origin $0 \in \mathbb{R}$. $\epsilon \in \mathcal{O}(\mathbb{D}^1(1))$ can be interpreted as a coordinate function on $\mathbb{D}^1(1)$ that squares to zero since all the points in it are infinitesimally close to the origin. Crucially, for any smooth manifold M, potentially with corners, there is a canonical bijection (cf. [GS25b, Lemma 2.2])

$$TM \cong_{\mathrm{Set}} \mathrm{Hom}_{\mathrm{Calg}_{\mathbb{R}}}(C^\infty(M), \mathcal{O}(\mathbb{D}^1(1))).$$

That is, the formal dual map $\mathbb{D}^1(1) \to M$ is a first order infinitesimal curve segment in M. For the case of manifolds with corners, this notion rightfully substitutes the intuitive idea of considering tangent vectors as equivalence classes of smooth curves, allowing us to describe also vectors that point outside the boundary (cf. [GS25b, Rmk. 2.3]).

Then, the idea is to capture both the infinitesimal and smooth structure by probing (see discussion before Def. 3.1.1) the generalized spaces with ThCartSp instead of only with CartSp.

Definition 3.2.2. The category of **thickened smooth sets**

$$\mathbf{ThSmthSet} := \mathrm{Sh}(\mathrm{ThCartSp})$$

is the sheaf category over the site ThCartSp w.r.t. the Grothendieck topology induced by the pretopology defined by Equation (3.41). The morphisms are natural transformations between sheaves.

The Yoneda Lemma applies as the ThCartSp is subcanonical (cf. [GS25b, ft. 4, p. 9]).

Proposition 3.2.2. *For $\mathcal{G} \in$ ThSmthSet and $\mathbb{R}^k \times \mathbb{D} \in$ ThCartSp, there is a natural bijection between the $\mathbb{R}^k \times \mathbb{D}$-plots of $\mathcal{G}$ and the smooth maps of ThSmthSet from $y(\mathbb{R}^k \times \mathbb{D})$ to $\mathcal{G}$:*

$$\mathcal{G}(\mathbb{R}^k \times \mathbb{D}) \equiv \mathrm{Plots}(\mathbb{R}^k \times \mathbb{D}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{ThSmthSet}}(y(\mathbb{R}^k \times \mathbb{D}), G).$$

Here, $y(\mathbb{R}^k \times \mathbb{D}) := \mathrm{Hom}_{\mathrm{ThCartSp}}(-, \mathbb{R}^k \times \mathbb{D})$ is the image via Yoneda embedding of $\mathbb{R}^k \times \mathbb{D}$ into the category ThSmthSet*. Choosing $\mathcal{G} = y(\mathbb{R}^{\tilde{k}} \times \tilde{\mathbb{D}}) \in$ ThCartSp for an arbitrary $\mathbb{R}^{\tilde{k}} \times \tilde{\mathbb{D}}$, the above immediately implies that the embedding functor*

$$y\colon \mathrm{ThCartSp} \hookrightarrow \mathrm{ThSmthSet}, \ \mathbb{R}^k \times \mathbb{D} \mapsto y(\mathbb{R}^k \times \mathbb{D}) := \mathrm{Hom}_{\mathrm{ThCartSp}}(-, \mathbb{R}^k \times \mathbb{D})$$

is fully faithful. Hence, any results and constructions on ThCartSp *may equivalently be phrased in terms of their thickened smooth set incarnation and vice versa.*

[46] For thickened points one can restrict to identity maps to describe isomorphisms.
[47] To see this, one can dually compute the pushout in $\mathrm{CAlg}_{\mathbb{R}}$, that is, a tensor product of algebras.

Remark 3.2.2. If one defines the category of ThSmthMfd[(48)] following Definition 3.2.1, the above result still works with $y(M \times \mathbb{D}) \coloneqq \mathrm{Hom}_{\mathrm{ThSmthMfd}}(-, M \times \mathbb{D})|_{\mathrm{ThCartSp}}$ essentially by the same arguments of [GS25b, Ex. 2.13].[(49)] This means that ThSmthMfd are representable.[(50)]

Note that all the properties of categories of sheaves discussed in Chapter 1 apply. This means that all limits and colimits exist in ThSmthSet and they are computed pointwise (see Prop. 1.2.1).

Remark 3.2.3. SmthSet is fully faithfully embedded in ThSmthSet. Indeed, SmthSet$\hookrightarrow$ ThSmthSet, $G \mapsto \tilde{G}$ with $\tilde{G}(\mathbb{R}^k \times \mathbb{D}) \coloneqq G(\mathbb{R}^k)$ given by precomposing with $\iota_1 \colon \mathrm{CartSp} \hookrightarrow \mathrm{ThCartSp}$, $\mathbb{R}^k \mapsto \mathbb{R}^k \times \mathbb{D}^0(0)$. The ι_1 has right adjoints given by the appropriate forgetful functor. Indeed, this is due to the fact that the only even-nilpotent element in $C^\infty(\mathbb{R}^k)$ is the zero function (cf. [GS25b, Eq. (8) & following discussion]).

Vector Fields and Differential Forms on the Field Space

As for any (pre-)sheaf category (cf. [MM12, Prop. 1, p. 97]), ThSmthSet has an internal hom functor

$$[-,-] \colon \mathrm{ThSmthSet}^{\mathrm{op}} \times \mathrm{ThSmthSet} \to \mathrm{ThSmthSet},$$

defined as follows for all $\mathcal{G}, \mathcal{H} \in \mathrm{ThSmthSet}$

$$[\mathcal{G}, \mathcal{H}](\mathbb{R}^{k|q} \times \mathbb{D}) \coloneqq \mathrm{Hom}_{\mathrm{ThSmthSet}}(y(\mathbb{R}^{k|q} \times \mathbb{D}) \times \mathcal{G}, \mathcal{H}). \tag{3.42}$$

The exponential property for ThSmthSet

$$\mathrm{Hom}_{\mathrm{ThSmthSet}}(\mathcal{K}, [\mathcal{G}, \mathcal{H}]) = \mathrm{Hom}_{\mathrm{ThSmthSet}}(\mathcal{K} \times \mathcal{G}, \mathcal{H}) \tag{3.43}$$

follows from the Yoneda lemma (see Prop. 3.2.2) and from the fact that any sheaf is a colimit of a representable sheaves $\mathcal{X} \cong \mathrm{colim}_i^{\mathrm{ThSmthSet}} y(\mathbb{R}^{k_i} \times \mathbb{D}_i)$ (see Prop. 1.1.1 and Prop. 1.2.1).

In field theoretic terms, the mapping space $[y(M), y(N)]$ describes the smooth and infinitesimal structure of the off-shell space of fields for a $\sigma-$model with domain M and target N. For more general field spaces consisting of sections of a fiber bundle, we introduce the following object.[(51)]

Definition 3.2.3. Consider fiber bundle

$$\pi^F_M \colon F \to M$$

over a closed oriented d-manifold M. The **thickened smooth space of off-shell fields** on M is defined by

$$\mathcal{F}(\mathbb{R}^k \times \mathbb{D}) \equiv \mathbf{\Gamma}_M(F)(\mathbb{R}^k \times \mathbb{D}) \coloneqq \{\varphi^{k,\epsilon} \colon \mathbb{R}^k \times \mathbb{D} \times M \to F | \pi^F_M \circ \varphi^{k,\epsilon} = \pi_M\} \tag{3.44}$$

where $\pi_M \colon \mathbb{R}^k \times \mathbb{D} \times M \to M$ can be understood as a map of ThSmthMfd (see Rmk. 3.2.2). Explicitly, these $(\mathbb{R}^k \times \mathbb{D})$-plots, or $(\mathbb{R}^k \times \mathbb{D})$-parametrized sections of $F \to M$,[(52)] consist of formal dual of algebra maps $(\varphi^{k,\epsilon})^* \colon \mathcal{O}(F) \to \mathcal{O}(\mathbb{R}^k \times \mathbb{D} \times M)$ such that the following commutes

$$\begin{tikzcd}
 & \mathcal{O}(F) \arrow[dl, "(\varphi^{k,\epsilon})^*"'] \\
\mathcal{O}(\mathbb{R}^k \times \mathbb{D} \times M) & C^\infty(M). \arrow[l, "(\pi_M)^*"] \arrow[u, "(\pi^F_M)^*"']
\end{tikzcd}$$

[(48)] There is a more general definition of thickened manifolds which are only (petit) locally isomorphic to the product form $\mathbb{R}^k \times \mathbb{D}$. It is possible to show that sheaves over these also yield an equivalent topos (cf. [GS25b, ft. 5, p. 9]).
[(49)] In more detail, the pullback along the inclusion $\iota \colon \mathrm{ThCartSp} \to \mathrm{ThSmthMfd}$ defines an equivalence of categories $\iota^* \colon \mathrm{Sh}(\mathrm{ThSmthMfd}) \to \mathrm{Sh}(\mathrm{ThCartSp})$ essentially because any thickened smooth manifold $M \times \mathbb{D}$ ($\dim M = m$) is always covered by a (trivially extended) differentiably good open cover $\{U_i \times \mathbb{D} \hookrightarrow M \times \mathbb{D}\}_{i \in I}$ of thickened Cartesian spaces. It follows that the composition $\mathrm{ThSmthMfd} \xrightarrow{y} \mathrm{Sh}(\mathrm{ThSmthMfd}) \xrightarrow{\iota^*} \mathrm{Sh}(\mathrm{ThCartSp})$ is also fully faithful.
[(50)] An analogous argument allows to see also manifolds with corners and therefore also manifolds with boundaries, as "representables" (cf. [GS25b, Prop. 4.13 & Prop. 4.14]).
[(51)] This can be generalized to field spaces of connections of a non-trivial principal bundle (cf. [GSM09; FF03]).
[(52)] This interpretation holds only for concrete thickened smooth sets.

Remark 3.2.4. An arbitrary bundle $F \to M$ might not have global sections, so Definition 3.2.3 might be null. However, noticing that the assignment of local smooth sections $U \mapsto \Gamma_U(F)$, $U \in M$ open, defines a petit sheaf on M with values in ThSmthSet, one might instead consider the functor

$$\begin{aligned}\mathbf{\Gamma}_{(-)}(F)\colon M_{\mathrm{cl}}^{\mathrm{op}} \times \mathrm{ThCartSp}^{\mathrm{op}} &\longrightarrow \mathrm{Set}\\ (U \times \mathbb{R}^k \times \mathbb{D}) &\longmapsto \mathbf{\Gamma}_U(F)(\mathbb{R}^k \times \mathbb{D}).\end{aligned}$$

Since all statements and definitions regarding the thickened smooth set of fields $\mathcal{F} = \mathbf{\Gamma}_M(F)$ functorially apply for each such open set U, we can continue to work with Definition 3.2.3.

Following the intuition of Remark 3.2.1 (point 2.), one can prove that $y(TM) \cong [y(\mathbb{D}^1(1)), y(M)] \in$ ThSmthSet (cf. [GS25b, Prop. 2.22]). This suggests the following definition.

Definition 3.2.4. The **synthetic tangent bundle** $T\mathcal{G}$ of $\mathcal{G} \in$ ThSmthSet is defined as the mapping thickened smooth set

$$T\mathcal{G} := [\mathbb{D}^1(1), \mathcal{G}] \in \mathrm{ThSmthSet},$$

where we omit the Yoneda embedding y for $\mathbb{D}(1)^1$ for readability.

The unique embedding map $\iota_0\colon * \hookrightarrow \mathbb{D}^1(1)$ defines, by precomposition, the projection

$$T\mathcal{G} := [\mathbb{D}^1(1), \mathcal{G}] \xrightarrow{(-)\circ\iota_0} [*, \mathcal{G}] \cong \mathcal{G}.$$

In the case of $M \in$ SmthMfd, the latter is the usual bundle projection.
Furthermore, the construction above forms a functor

$$T := [\mathbb{D}^1(1), -]\colon \mathrm{ThSmthSet} \to \mathrm{ThSmthSet},$$

with smooth maps $f\colon \mathcal{G} \to \mathcal{H}$ mapped to $Tf\colon T\mathcal{G} \to T\mathcal{H}, v^{k,\epsilon} \mapsto f \circ v^{k,\epsilon}$. The latter can be interpreted as the pushforward of any map f. Indeed, in the case of $\mathcal{G} = y(M)$, $\mathcal{H} = y(N)$, one can check that $Tf \equiv \mathrm{d}f\colon y(TM) \to y(TN)$.
Crucially, by its definition and by internal hom property (see Eq. (3.43)), it has a left-adjoint functor[53]

$$(\mathbb{D}^1(1) \times -) \dashv T(-). \tag{3.45}$$

Remark 3.2.5. Note that in full generality $T\mathcal{G} := [\mathbb{D}^1(1), \mathcal{G}]$ does not have a fiber-wise $\mathbb{R}$-linear structure (cf. [GS25b, Rmk. 2.25]), that is fiberwise addition and $y(\mathbb{R})$-multiplication

$$+\colon T\mathcal{G} \times_{\mathcal{G}} T\mathcal{G} \to T\mathcal{G} \qquad \text{and} \qquad \cdot\colon y(\mathbb{R}) \times T\mathcal{G} \to T\mathcal{G}. \tag{3.46}$$

In the field-theoretically relevant cases, this structure is guaranteed by the $\mathbb{R}$-linearity of the targets, as it is evident in what follows.

Proposition 3.2.3. *The synthetic tangent bundle $T\mathcal{F} = [\mathbb{D}^1(1), \mathcal{F}]$ of a thickened smooth field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is canonically isomorphic to $\mathbf{\Gamma}_M(VF) \in$ ThSmthSet, where the vertical bundle $VF \xrightarrow{\pi_F^{TF}} F \xrightarrow{\pi_M^F} M$ is defined by* (see ft. (6) p. 32)

$$VF := \mathrm{eq}\Big(TF \underset{0_M}{\overset{\mathrm{d}\pi_M^F}{\rightrightarrows}} TM\Big).$$

For a proof, see [GS25b][Lemma 2.28].
Composing the projection $\pi_F^{TF}\colon VF \to F$ with $T\mathcal{F}(\mathbb{R}^k \times \mathbb{D}) \ni Z_{\varphi^{k,\epsilon}}$ yields $(\mathbb{R}^{k,\epsilon})$-parametrized sections of $F \to M$, $\varphi^{k,\epsilon} := \pi_F^{TF} \circ Z_{\varphi^{k,\epsilon}}$. That is, any $Z_{\varphi^{k,\epsilon}}$ covers a unique plot of fields $\varphi^{k,\epsilon}$, which realizes the projection

$$\pi_{\mathcal{F}}^{T\mathcal{F}}\colon T\mathcal{F} \cong \mathbf{\Gamma}_M(VF) \xrightarrow{(\pi_F^{TF})_*} \mathbf{\Gamma}_M(F) \cong \mathcal{F}.$$

Furthermore, plots of a tangent bundle can be represented via paths of plots (cf. [GS25b][Lemma 2.29] extending Thm. 3.1.4) and path derivations depend only on tangent vectors (cf. [GS25b][Lemma 2.32]). This latter result means that any tangent vector and vector field, defined as a section of the bundle $\pi_{\mathcal{F}}^{T\mathcal{F}}\colon T\mathcal{F} \to \mathcal{F}$ (see diagram (3.5)), defines a unique real-valued derivation (see Rmk. 3.1.4). Forms are defined as in SmthSet.

(53) One should also use the monoidal structure of ThSmthSet via the direct product.

Definition 3.2.5. The **set of differential m-forms** on $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as

$$\Omega^m(\mathcal{F}) \coloneqq \mathrm{Hom}^{\text{fib.lin.an.}}_{\mathrm{ThSmthSet}}\left(T^{\times m}\mathcal{F}, y(\mathbb{R})\right).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps with respect to the fiber-wise linear structure (see Eq. (3.46)) on the m-fold fiber product[(54)]

$$T^{\times m}\mathcal{F} \coloneqq T\mathcal{F} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} T\mathcal{F}$$

of the tangent bundle over $\mathcal{F}$.

The collection of differential forms of all degrees forms a graded $\mathbb{R}$-vector space (see Def. 1.3.3)

$$\Omega^\bullet(\mathcal{F}) \coloneqq \bigoplus_{m\in\mathbb{N}} \Omega^m(\mathcal{F}).$$

The latter has a well-defined notion of a wedge product $\wedge$ and contraction ι_Z along any vector field $Z \in \mathfrak{X}(\mathcal{F})$.[(55)] Moreover, there is a good notion of Lie derivative along $Z \in \mathfrak{X}(\mathcal{F})$ given by the unique derivation on $C^\infty(\mathcal{F})$ associated with the vector field Z. However, there is no obvious definition for a de Rham differential $\mathrm{d}_{\mathcal{F}}\colon \Omega^m \to \Omega^{m+1}$. This is because, given the Lie derivative, Cartan's magic formula alone does not uniquely determine the de Rham differential. Indeed, there remain infinitely many possible choices, since the formula cannot distinguish between two candidate differentials whose difference commutes with the contraction. We overcome this issue by employing the geometry of the infinite jet bundle as done in SmthSet.

Infinite Jet Bundle Technology and Local Lagrangian Field Theory

Using Definition 3.1.9 and Remark 3.2.2, one can compute the defining limit (see Def. 3.1.12) of the infinite jet bundle within ThSmthSet[(56)]

$$J^\infty_M F \coloneqq \lim_k^{\mathrm{ThSmthSet}} y(J^k_M F) \in \mathrm{ThSmthSet}. \tag{3.47}$$

By construction, the limit comes with thickened smooth projection maps $\pi^\infty_k\colon J^\infty_M F \to y(J^k_M F)$. The synthetic tangent bundle recovers the ad hoc definition of tangent bundle of the infinite jet bundle (see Def. 3.1.14).

Proposition 3.2.4. *The tangent bundle $TJ^\infty_M F \coloneqq [\mathbb{D}^1(1), J^\infty_M F]$ is isomorphic to $\lim_k^{\mathrm{ThSmthSet}} y(TJ^k_M F)$.*

Proof. This can be computed plotwise using that $y(TJ^k_M F) \cong [\mathbb{D}^1(1), y(J^k_M F)]$ (cf. [GS25b, Prop. 2.22]) and that $T \coloneqq [\mathbb{D}^1(1), -]$ has a left adjoint and hence preserves limits (see Thm. 1.2). Indeed, recalling the definition of mapping space (see Def. 3.42), the exponential property for ThSmthSet (see Eq. (3.43)) and that limits commute with the hom functor (see Prop. 1.2), we get

$$\begin{aligned}[]
[\mathbb{D}^1(1), y(J^\infty_M F)](\mathbb{R}^k \times \mathbb{D}) &\cong \lim_k^{\mathrm{Set}} \mathrm{Hom}(\mathbb{R}^k \times \mathbb{D} \times \mathbb{D}^1(1), y(J^k_M F)) \cong \lim_k^{\mathrm{Set}} \mathrm{Hom}(\mathbb{R}^k \times \mathbb{D}, y(TJ^k_M F)) \cong \\
&\cong \mathrm{Hom}(\mathbb{R}^k \times \mathbb{D}, \lim_k^{\mathrm{ThSmthSet}} y(TJ^k_M F)) \cong \lim_k^{\mathrm{ThSmthSet}} y(TJ^k_M F)(\mathbb{R}^k \times \mathbb{D}),
\end{aligned}$$

where $\mathrm{Hom} \equiv \mathrm{Hom}_{\mathrm{ThSmthSet}}$. ■

Similarly, one can show that the corresponding limit of the pushforward projection maps $\mathrm{d}\pi^\infty_k\colon TJ^\infty_M F \to y(TJ^k_M F)$ coincides with the synthetic pushforwards of π^∞_k. Moreover, Proposition 3.1.7 and Proposition 3.1.8 are corollaries of [GS25b][Lemma 2.32]). Note that the limit characterization of both $J^\infty_M F$ and $TJ^\infty_M F$ means that their plots may be formally represented and manipulated precisely as in the purely smooth setting, essentially by carrying along the dependence on nilpotent elements corresponding to

(54) Here the product is the pullback (see Def. 1.1.12) of $T\mathcal{F} \xrightarrow{\pi^{T\mathcal{F}}_{\mathcal{F}}} \mathcal{F} \xleftarrow{\pi^{T\mathcal{F}}_{\mathcal{F}}} T\mathcal{F}$.

(55) Thanks to the fiber-wise linear structure of $T\mathcal{F}$ and graded structure of $\Omega^\bullet(\mathcal{F})$, they can be defined as usual (see Def. 1.3.13).

(56) In [GS25b][Sec. 2.4], the synthetic infinitesimal neighborhood is introduced to fully describe the infinite jet bundle within ThSmthSet in a chart-reference-free manner (cf. [GS25b, App. 4.3]). This is not necessary if one works locally and with finite-dimensional fiber bundles, but it is necessary for a rigorous global approach to perturbative field theory (cf. [GS25b, pp. 27-28]). This can also be done for a submanifold of M to rigorously treat initial conditions of PDEs.

the infinitesimal thickening of the Cartesian probes. This can be done in local coordinates by dually defining the action on the algebra generators (cf. [GS25b][pp. 22-23]). Finally, notice that $TJ^\infty_M F$ has a fiber-wise and plot-wise $\mathbb{R}$-linear structure as in SmthSet (see Eq. (3.11)). The latter allows us to define m-differential forms as in SmthSet by

$$\Omega^m(J^\infty_M F) \coloneqq \mathrm{Hom}^{\text{fib.lin.an.}}_{\text{ThSmthSet}}\Big(T^{\times m}(J^\infty_M F),\, y(\mathbb{R})\Big)$$

That is, m-forms are thickened smooth real-valued, fiber-wise linear antisymmetric maps on the m-fold fiber product of the tangent bundle. The discussion on the Cartan calculus follows verbatim the one given for SmthSet (see Def. 3.1.18), thanks to the following remark.

Remark 3.2.6. This discussion in SmthSet is heavily based on the point-set characterization of forms as those of locally finite order, which in turn follows from Propositions 3.1.5 and 3.1.6. In particular, these include those of globally finite order, which are the only ones appearing in examples from field theory. The fact that we can do the same in ThSmthSet is not trivial. Indeed, there exists a fully faithful functor $\iota_!\colon \mathrm{SmthSet} \to \mathrm{ThSmthSet}$, which hence fully faithfully embeds also FrMfd, and LocProMfd, into ThSmthSet and so produces a potentially different version of the thickened infinite jet bundle $\iota_!(\lim^{\mathrm{SmthSet}}_k J^k_M F)$. In [Gio+26], it is proven that such passage from FrMfd to ThSmthSet preserves the projective limits that define infinite jet bundles. Therefore, $\iota_!(\lim^{\mathrm{SmthSet}}_k J^k_M F)$ coincides with Equation (3.47) and smooth functions on the jet bundle defined in ThSmthSet. Similarly, differential forms on it are also fully characterized as being exhausted by the point-set locally finite order functions and differential forms on $J^\infty_M F$ (cf. [GS25b][Rmk. 3.5]). This also applies to each component of the family characterizing a vector field on the infinite jet bundle (see Def. 3.1.15).

All the discussion in Section 3.1 on the horizontal splitting depends only on universal properties, so it applies verbatim in ThSmthSet (cf. [GS25b][Sec. 3.1]). Consequently, the characterization of the variational bicomplex also follows as in Definition 3.1.20.
Similarly, all the discussion about Local Lagrangian field theories, local Cartan calculus and local Variational bicomplex follows the path outlined in Section 3.1, provided that the prolongation map (see Eq. (3.7)), the prolongated evaluation map (see Def. 3.1.25), and the integration map (see Eq. (3.28)) are well-defined maps of ThSmthSet. In particular, the synthetic pushforward of the first two should yield the ad-hoc definitions of their pushforward (see Eq. 3.15 and Eq. (3.36)).

Proposition 3.2.5. *The following statements hold.*

1. *The infinite jet prolongation $j^\infty\colon \mathbf{\Gamma}_M(F)\times M \to \mathbf{\Gamma}_M(J^\infty_M F)$ exists as a map in ThSmthSet and its synthetic pushforward $Tj^\infty\colon T\mathbf{\Gamma}_M(F) \to T\mathbf{\Gamma}_M(J^\infty_M F)$ is given by $j^\infty_{VF}\colon \mathbf{\Gamma}_M(VF) \to \mathbf{\Gamma}_M(J^\infty_M VF)$.*

2. *The prolongated evaluation $\mathrm{ev}^\infty\colon \mathbf{\Gamma}_M(F) \to J^\infty_M F$ exists as a map in ThSmthSet and its synthetic pushforward $T\mathrm{ev}^\infty\colon T\mathbf{\Gamma}_M(F) \to TJ^\infty_M F$ is given by* Equation (3.36).

3. *For any closed oriented submanifold $\Sigma^p \subset M$, the integration defines a map of thickened smooth sets*
$$\int_{\Sigma^p}\colon \mathbf{\Gamma}\big(\textstyle\bigwedge^p T^*M\big) \to y(\mathbb{R}),$$
that is defined by plot-wise SmthSet *integration* (see Eq. (3.28))*, carrying along the finite-polynomial dependence on nilpotent coordinates.*

The proof of the three above statements can be found respectively in [GS25b][Lemma 3.1, Lemma 3.8, Lemma 3.10 & Prop. 3.11], [GS25b][Prop. 3.12 & Cor. 3.13] and in [GS25b][Sec. 3.3]. Note that the prolongation and the prolonged evaluation are essentially the same as in SmthSet once one considers higher plots not only in $\mathbb{R}^k$, but also in $\mathbb{D}$, which can be dually defined by their action on generators of the proper algebras. Finally, note that the discussion on local symmetries (cf. [GS25a, Ch. 6]) and on the critical locus (cf. [GS25a, Ch. 5]) are subsumed within ThSmthSet, with the advantage that the latter gives a more natural description of the ad hoc definitions used in SmthSet (cf. [GS25b, Ch. 3.3]).

In summary, the thickened category ThSmthSet provides synthetic tangent bundles and naturally recovers all constructions available in SmthSet.

3.3 Super Thickened Smooth Sets

Following the hinted approach to the super version of ThSmthSet presented in [GS25b; Gio25; GSS24], we generalize ThSmthSet to sThSmthSet by considering the sheaf category over the site sThCartSp. The latter is given as the opposite category to an appropriate full subcategory of $\mathbb{Z}_2$-CAlg$^{\mathrm{op}}$, using the fact that we have the following fully faithful embeddings

$$\mathrm{CartSp} \hookrightarrow \mathrm{ThCartSp} \hookrightarrow \mathrm{CAlg}^{\mathrm{op}} \hookrightarrow \mathbb{Z}_2\mathrm{CAlg}^{\mathrm{op}}.$$

What follows is a tentative exposition of the constructions in sThSmthSet that we need for our application, all of which should be fully explained and proved in [GS26].
For clarity, we emphasize the following points:

- Proposition 3.3.1 is conjectural and relies on the assumption that the site sThCartSp (see Def. 3.3.1) is subcanonical, which is needed for the purpose of the thesis. A similar caveat applies to Remark 3.3.1 and Remark 3.3.2;
- The exponential property for sThSmthSet has a sketched proof (see Eq. (3.50)), which likewise depends on the subcanonicality assumption;
- Example 3.3.2 is taken from [Gio25][Ex. 3.6];
- Proposition 3.3.2 is proven by closely following the proof of the analogous result for ThSmthSet (cf. [GS25b][Lemma 2.28]);
- The entire discussion concerning graded infinite jet bundle technology and local geometric structures, leading to the development of the graded local Cartan calculus and the graded local variational bicomplex, is treated as a working assumption for the purposes of this thesis;
- Under the above assumptions and the hypothesis listed in Definition 3.4.1, a local BV system is shown to transgress to the BV theory described by Equation (3.58). Remark 3.4.2 and Remark 3.4.3 discuss the limitations of this framework and indicate possible outlooks;
- As an illustrative example, the Ansatz for (non-)abelian BF theory in dimension $d = 4$ is shown to define a local BV system in the sense of Definition 3.4.1.

The Category sThSmthSet

Consider the category of $\mathbb{Z}_2$ **commutative real algebras** $\mathbb{Z}_2\mathbf{CAlg}_{\mathbb{R}}$. The objects are real $\mathbb{Z}_2$-graded unital associative commutative ($a\cdot b = (-1)^{|a||b|} b\cdot a$) algebras, and the morphisms are structure-preserving maps, i.e. grading-preserving unital commutative algebras' morphisms. Following the intuition given by Milnor's exercise, we define sThCartSp as follows.

Definition 3.3.1. The category of **super thickened Cartesian spaces** is given by

$$\begin{gathered}\mathrm{sThCartSp} \hookrightarrow \mathbb{Z}_2\mathrm{CAlg}^{\mathrm{op}}_{\mathbb{R}} \\ \mathbb{R}^{k|q} \times \mathbb{D} \equiv \mathbb{R}^k \times \mathbb{R}^{0|q} \times \mathbb{D} \mapsto C^\infty(\mathbb{R}^k) \otimes \mathcal{O}(\mathbb{R}^{0|q}) \otimes \mathcal{O}(\mathbb{D}) \equiv \mathcal{O}(\mathbb{R}^{k|q} \times \mathbb{D})\end{gathered}$$

where $\mathcal{O}(\mathbb{R}^{0|q}) \coloneqq \mathbb{R}[\theta_1, \dots, \theta_q]$ is the Grassmann algebra associated to the **superpoint** $\mathbb{R}^{0|q}$ and $\mathcal{O}(\mathbb{D}) \coloneqq \mathbb{R} \oplus V$ as in Definition 3.2.1. Morphisms in sThCartSp are equivalently described as a morphism between the corresponding opposite $\mathbb{Z}_2$-graded commutative algebras

$$\mathrm{Hom}_{\mathrm{sThCartSp}}(\mathbb{R}^{k|q} \times \mathbb{D}, \mathbb{R}^{k'|q'} \times \mathbb{D}') \cong \mathrm{Hom}_{\mathbb{Z}_2\mathrm{CAlg}_{\mathbb{R}}}(\mathcal{O}(\mathbb{R}^{k'|q'} \times \mathbb{D}'), \mathcal{O}(\mathbb{R}^{k|q} \times \mathbb{D})).$$

The category of sThCartSp is a site when endowed with the Grothendieck pretopology given by the trivially extended differentiably good open covers $\{U_i \xrightarrow{\iota_i} \mathbb{R}^k\}_{i\in I}$

$$\{U_i \times \mathbb{R}^{0|q} \times \mathbb{D} \xrightarrow{\iota_i \times \mathrm{id}_{\mathbb{R}^{0|q}} \times \mathrm{id}_{\mathbb{D}}} \mathbb{R}^k \times \mathbb{R}^{0|q} \times \mathbb{D}\}_{i\in I}. \tag{3.48}$$

To check this, notice that isomorphisms are coverings,[(57)] coverings are stable under pullbacks[(58)] and transitivity holds trivially. The chosen Grothendieck topology is generated only by covers in the smooth even component, while the odd and infinitesimal directions are carried along unchanged. This means that all the presheaves in those components are also sheaves. One needs to check the locality and gluing conditions only in the smooth component: a presheaf $\mathcal{G}\colon \mathrm{sThCartSp}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf if and only if $\mathcal{G}(\mathbb{R}^{k|q}\times\mathbb{D})$ is a sheaf in the petit sense when restricted along $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k|q}\times\mathbb{D}$ for any $\mathbb{R}^{k|q}\times\mathbb{D} \in \mathrm{sThCartSp}$.

Definition 3.3.2. The category of **super thickened smooth sets**

$$\mathbf{sThSmthSet} := \mathrm{Sh}(\mathrm{sThCartSp})$$

is the sheaf category over the site sThCartSp w.r.t. the Grothendieck topology induced by the pretopology defined by Equation (3.48). The morphisms are natural transformations between sheaves.

The intuition is that, enlarging the site CartSp both with infinitesimal and super structure, we are able to probe simultaneously the smooth, infinitesimal and super structure of the generalized space, the second one giving us all the necessary tangent bundle constructions for free via synthetic geometry (see Sec. 3.2).[(59)]. By the Yoneda lemma for sThSmthSet, sSmthMfd are representable in sThSmthSet.

Proposition 3.3.1. *For $\mathcal{G} \in \mathrm{sThSmthSet}$ and $\mathbb{R}^{k|q}\times\mathbb{D} \in \mathrm{sThCartSp}$, there is a natural bijection between the $\mathbb{R}^{k|q}\times\mathbb{D}$-plots of $\mathcal{G}$ and the smooth maps of super thickened smooth sets from $y(\mathbb{R}^{k|q}\times\mathbb{D})$ to $\mathcal{G}$:*

$$\mathcal{G}(\mathbb{R}^{k|q}\times\mathbb{D}) \equiv \mathrm{Plots}(\mathbb{R}^{k|q}\times\mathbb{D},\mathcal{G}) \cong \mathrm{Hom}_{\mathbf{sThSmthSet}}(y(\mathbb{R}^{k|q}\times\mathbb{D}),G).$$

Here, $y(\mathbb{R}^{k|q}\times\mathbb{D}) := \mathrm{Hom}_{\mathbf{sThCartSp}}(-,\mathbb{R}^{k|q}\times\mathbb{D})$ is the image via Yoneda embedding of $\mathbb{R}^{k|q}\times\mathbb{D}$ into the category sThSmthSet.
Choosing $\mathcal{G} = y(\mathbb{R}^{\tilde{k}|\tilde{q}}\times\tilde{\mathbb{D}}) \in \mathrm{sThCartSp}$ for an arbitrary $\mathbb{R}^{\tilde{k}|\tilde{q}}\times\tilde{\mathbb{D}}$, the above immediately implies that the embedding functor

$$y\colon \mathrm{sThCartSp} \hookrightarrow \mathrm{sThSmthSet},\ \mathbb{R}^{k|q}\times\mathbb{D} \mapsto y(\mathbb{R}^{k|q}\times\mathbb{D}) := \mathrm{Hom}_{\mathbf{sThCartSp}}(-,\mathbb{R}^{k|q}\times\mathbb{D})$$

is fully faithful. Hence, any results and constructions on sThCartSp *may equivalently be phrased in terms of their super thickened smooth set incarnation and vice versa.*

A proof of the statement should appear in [GS26] and should follow from the fact that sThCartSp should be subcanonical as ThCartSp since the Grothendieck topology on these two sites is morally the same (cf. [GS25b, ft. 4, p. 9]).

Remark 3.3.1. If one defines the category of sThSmthMfd following Def. 3.3.1, the above result still works with $y(\mathcal{M}\times\mathbb{D}) := \mathrm{Hom}_{\mathbf{sThSmthMfd}}(-,\mathcal{M}\times\mathbb{D})|_{\mathbf{sThCartSp}}$ essentially by the same arguments of [GS25b, Ex. 2.13] adapted to the case of super manifolds ($\mathcal{M} \in \mathrm{sSmthMfd}$).[(60)] This means that sThSmthMfd are representable.

By Definition 3.3.2, all the properties of categories of sheaves discussed in Chapter 1 apply. Note that this means that all limits and colimits exist in sThSmthSet and they are computed pointwise (Prop. 1.2.1).

Remark 3.3.2. SmthSet, ThSmthSet, and sSmthSet[(61)] should be fully faithfully embedded in sThSmthSet. Furthermore, each embedding functor should have right adjoint given by the appropriate forgetful functor. The embeddings and the adjunctions should follow as in the case of ThSmthSet (see Rmk. 3.2.3 and [GS25b, Eq. (8) & following discussion]).

(57) For super points one can restrict to identity maps (c.f. [Sac08, Prop. 2.4]), and similarly for thickened points.

(58) To see this, one can dually compute the pushout in $\mathbb{Z}_2\mathrm{CAlg}_{\mathbb{R}}$, that is, a tensor product of algebras.

(59) To have non-trivial fermionic fields, one usually adds ad hoc odd auxiliary coordinates. Then, to display a fermionic Lagrangian, or any related observable functional, of polynomial order $q \in \mathbb{N}$ as a non-trivial map requires the introduction of q auxiliary odd coordinates (cf. [Gio25, Ch. 3]). In sThSmthSet, this is taken care of automatically by odd-probes plots (cf. [GSS24, Rmk. 2.51]). Indeed, the auxiliary coordinates are nothing but the content of odd plots of the corresponding sheaf, that is, they correspond to a certain $\mathbb{R}^{0|q}$-parametrization. It follows that the symbol used for fermionic fields implicitly refers to an arbitrary $\mathbb{R}^{p|q}$-plot of the super thickened smooth field space. Formulas, like a Lagrangian, made out of these are implicitly functorial under maps in the probe site, that is, they may naturally be interpreted as natural transformations within the category sThSmthSet.

(60) In more detail, the pullback along the inclusion $\iota\colon \mathrm{sThCartSp} \to \mathrm{sThSmthMfd}$ defines an equivalence of categories $\iota^*\colon \mathrm{Sh}(\mathrm{sThSmthMfd}) \to \mathrm{Sh}(\mathrm{sThCartSp})$ essentially because any super thickened smooth manifold $\mathcal{M}\times\mathbb{D}$ ($\dim\mathcal{M} = (m,n)$) is always covered by a (trivially extended) differentiably good open cover $\{U_i \times \mathbb{R}^{0|n}\times\mathbb{D} \hookrightarrow \mathcal{M}\times\mathbb{D}\}_{i\in I}$ of super thickened Cartesian spaces. It follows that the composition $\mathrm{sThSmthMfd} \xrightarrow{y} \mathrm{Sh}(\mathrm{sThSmthMfd}) \xrightarrow{i^*} \mathrm{Sh}(\mathrm{sThCartSp})$ is also fully faithful.

(61) sSmthSet is the sheaf category over the site sCartSp w.r.t. the Grothendieck topology induced by the opportunely restricted pretopology defined by Equation (3.48).

Mapping Spaces, Synthetic Super Tangent Bundle and Differential Forms

As for any (pre-)sheaf category (cf. [MM12, Prop. 1, p. 97]), sThSmthSet has an internal hom functor

$$[-,-]\colon \mathrm{sThSmthSet}^{\mathrm{op}} \times \mathrm{sThSmthSet} \to \mathrm{sThSmthSet},$$

defined as follows for all $\mathcal{G}, \mathcal{H} \in \mathrm{sThSmthSet}$

$$[\mathcal{G},\mathcal{H}](\mathbb{R}^{k|q} \times \mathbb{D}) \coloneqq \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(y(\mathbb{R}^{k|q} \times \mathbb{D}) \times \mathcal{G}, \mathcal{H}\big). \tag{3.49}$$

The exponential property for sThSmthSet follows from the Yoneda lemma (see Prop. 3.3.1) and from the fact that, if the site is subcanonical, any sheaf can be written as a colimit of representable sheaves $\mathcal{X} \cong \mathrm{colim}_i^{\mathrm{sThSmthSet}} y(\mathbb{R}^{k_i|q_i} \times \mathbb{D}_i)$ (see Prop. 1.2.1):

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{sThSmthSet}}(\mathcal{X}, [\mathcal{G},\mathcal{H}]) &\cong \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(\mathrm{colim}_i^{\mathrm{sThSmthSet}} y(\mathbb{R}^{k_i|q_i} \times \mathbb{D}_i), [\mathcal{G},\mathcal{H}]\big)\\
&\cong \lim_i^{\mathrm{sThSmthSet}} \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(y(\mathbb{R}^{k_i|q_i} \times \mathbb{D}_i), [\mathcal{G},\mathcal{H}]\big)\\
&\cong \lim_i^{\mathrm{sThSmthSet}} \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(y(\mathbb{R}^{k_i|q_i} \times \mathbb{D}_i) \times \mathcal{G},\mathcal{H}\big)\\
&\cong \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(\mathrm{colim}_i^{\mathrm{sThSmthSet}} y(\mathbb{R}^{k_i|q_i} \times \mathbb{D}_i) \times \mathcal{G},\mathcal{H}\big)\\
&\cong \mathrm{Hom}_{\mathrm{sThSmthSet}}(\mathcal{X} \times \mathcal{G},\mathcal{H}).
\end{aligned} \tag{3.50}$$

In the second and second-to-last equalities, we used the fact that the internal hom functor contravariantly sends colimits in the first variable to limits (see *2.* of Prop. 1.1.1), and in the third equality we used the plotwise definition of the internal Hom-functor for sThSmthSet (see Eq. (3.49)).

Example 3.3.1. The mapping space between $\mathcal{M},\mathcal{N} \in \mathrm{sSmthMfd}$, is naturally encoded by the internal hom functor

$$\begin{aligned}
[y(\mathcal{M}), y(\mathcal{N})](\mathbb{R}^{k|q} \times \mathbb{D}) &\coloneqq \mathrm{Hom}_{\mathrm{sThSmthSet}}\big(y(\mathbb{R}^{k|q} \times \mathbb{D} \times \mathcal{M}), y(\mathcal{N})\big)\\
&\cong \mathrm{Hom}_{\mathbb{Z}_2\mathrm{Calg}_{\mathbb{R}}}\big(\mathcal{O}(\mathcal{N}), \mathcal{O}(\mathbb{R}^{k|q} \times \mathbb{D} \times \mathcal{M})\big).
\end{aligned}$$

Notice that $*$-plots are morphisms in Grassmann-graded algebras

$$[y(\mathcal{M}), y(\mathcal{N})](*) \cong \mathrm{Hom}_{\mathbb{Z}_2\mathrm{Calg}_{\mathbb{R}}}\big(\mathcal{O}(\mathcal{N}), \mathcal{O}(\mathcal{M})\big) \cong \mathrm{Hom}_{\mathrm{sSmthMfd}}(\mathcal{M},\mathcal{N}) \cong \mathrm{Mor}(\mathcal{M},\mathcal{N}).$$

However, the exponential property (see Eq. (3.50)) suggests that higher plots of this object describe $\mathrm{Map}(\mathcal{M},\mathcal{N})$ (see Def. 2.3.1), i.e. the space of grading non-preserving maps from $\mathcal{M}$ to $\mathcal{N}$. Therefore, the sThSmthSet above encodes, by probing it with different sThCartSp, both morphisms and grading non-preserving maps.

To describe both bosonic (even) and fermionic (odd) fields simultaneously, we introduce the following.[62]

Definition 3.3.3. Let M be a closed oriented smooth d-manifold. Consider the composite bundle

$$F \xrightarrow{\pi^F_E} E \xrightarrow{\pi^E_M} M$$

where $F \to E$ is an odd vector bundle and $E \to M$ is a fiber bundle. The full composite bundle $\pi^F_M\colon F \to M$ encodes both bosonic and fermionic fields simultaneously. The **thickened smooth super space of off-shell fields** on M is defined by

$$\mathcal{F}(\mathbb{R}^{k|q} \times \mathbb{D}) \equiv \mathbf{\Gamma}_M(F)(\mathbb{R}^{k|q} \times \mathbb{D}) \coloneqq \{\varphi^{k|q,\epsilon}\colon \mathbb{R}^{k|q} \times \mathbb{D} \times M \to F \,|\, \pi^F_M \circ \varphi^{k|q,\epsilon} = \pi_M\} \tag{3.51}$$

where $\pi_M\colon \mathbb{R}^{k|q} \times \mathbb{D} \times M \to M$ can be understood as a map of sThSmthMfd (see Rmk. 3.3.1). Explicitly, these $(\mathbb{R}^{k|q} \times \mathbb{D})$-plots, or $(\mathbb{R}^{k|q} \times \mathbb{D})$-parametrized sections of the bundle $\pi^F_M\colon F \to M$,[63] consist of formal dual of algebra maps $(\varphi^{k|q,\epsilon})^*\colon \mathcal{O}(F) \to \mathcal{O}(\mathbb{R}^{k|q} \times \mathbb{D} \times M)$ such that the following commutes

$$\begin{tikzcd}
 & \mathcal{O}(F) \arrow[dl, "(\varphi^{k|q,\epsilon})^*"'] \\
\mathcal{O}(\mathbb{R}^{k|q} \times \mathbb{D} \times M) & C^\infty(M). \arrow[l, "(\pi_M)^*"] \arrow[u, "(\pi^F_M)^*"']
\end{tikzcd}$$

[62] This can be generalized to field spaces of connections of a non-trivial principal bundle (cf. [GSM09; FF03]).
[63] This interpretation holds only for concrete thickened smooth super sets.

Remark 3.3.3.

1. The above can be equivalently defined as the pullback in sThSmthSet

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[y(M), y(F)]} \arrow[d, "(\pi^F_M)_*"] \\ \mathrm{id}_M \cong * \arrow[r, hook] & {[y(M), y(M)],} \end{tikzcd} \tag{3.52}$$

   where $(\pi^F_M)_*$ denotes post-composition and the bottom embedding is defined probe-wise by sending the unique $(\mathbb{R}^{k|q} \times \mathbb{D})$-plot of the point $* \cong \mathrm{id}_M$ to the projection-plot $[y(M), y(M)](\mathbb{R}^{k|q} \times \mathbb{D}) \ni \pi_M \colon \mathbb{R}^{k|q} \times \mathbb{D} \times M \to M$ for each $\mathbb{R}^{k|q} \times \mathbb{D}$. Remembering that limits in sheaf categories are computed objectwise, it is easy to compute the above diagram plotwise in Set and recover the set of plot-parametrized sections as in Equation (3.51).

2. An arbitrary bundle $\pi^F_M \colon F \to M$ might not have global sections, so Definition 3.3.3 might be null. However, noticing that the assignment of local smooth sections $U \mapsto \Gamma_U(F)$, $U \in M$ open, defines a petit sheaf on M with values in sThSmthSet, one might instead consider the functor

$$\begin{aligned} \mathbf{\Gamma}_{(-)}(F) \colon M^{\mathrm{op}}_{\mathrm{cl}} \times \mathrm{sThCartSp}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ (U \times \mathbb{R}^{k|q} \times \mathbb{D}) &\longmapsto \mathbf{\Gamma}_U(F)(\mathbb{R}^{k|q} \times \mathbb{D}). \end{aligned}$$

   Since all statements and definitions regarding the thickened smooth super set of fields $\mathcal{F} = \mathbf{\Gamma}_M(F)$ functorially apply for each such open set U, we can continue to work with Definition 3.3.3.

Example 3.3.2. Consider the simple case (cf. [Gio25]) where the composite bundle is given by the pullback[(64)] of a separate odd vector bundle $V_{\mathrm{odd}} \to M$ over M

$$F = (\pi^E_M)^* V_{\mathrm{odd}}.$$

It follows that[(65)]

$$\mathcal{F}(*) \cong \Gamma_M(F).$$

Forgetting for a moment about the infinitesimal structure, one can show that

$$\begin{aligned} \mathcal{F}(\mathbb{R}^{0|1}) &\cong \Gamma_M(F) \times (\Gamma_M(V_{\mathrm{odd}}) \cdot \theta) \\ \mathcal{F}(\mathbb{R}^{1|1}) &\cong \Gamma_{\mathbb{R}\times M}(\pi^*_M F) \times (\Gamma_{\mathbb{R}\times M}(\pi^*_M V_{\mathrm{odd}}) \cdot \theta) \end{aligned}$$

where θ is the coordinate in $\mathbb{R}^{0|1}$ and we used that the following diagram commutes

$$\begin{tikzcd} & \pi^*_M Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ \mathbb{R} \times M \arrow[ur] \arrow[r, "\mathrm{id}_{\mathbb{R}\times M}"] & \mathbb{R} \times M \arrow[r, "\pi_M"] & M, \end{tikzcd}$$

both for $Y = F$ or $F = V_{\mathrm{odd}}$. The arrow $Y \to M$ is the projection of the considered bundle. In particular, the $\mathbb{R}^{1|1}$-plots are pairs of smoothly $\mathbb{R}$-parametrized families of a bosonic field configuration and a fermionic field configuration (of linear order).[(66)] This is the basic notion for implementing the variational principle in bosonic and (first-order) fermionic Lagrangian field theories.[(67)]. Finally, the $\mathbb{D}$-plots encode the infinitesimal structure of such generalized space (cf. [GS25b]).

A full treatment of the field-theoretic framework within sThSmthSet lies beyond the scope of this thesis. Nevertheless, the guiding idea behind such an extension can already be illustrated in the following discussion. Since we require a notion of Cartan calculus on the space of fields, we outline how such a

(64) This is not the case when considering a gravitational theory with coupled fermionic spinorial fields, as the latter depends on the choice of metric and hence the fully general composite description is necessary (cf. [Gio25; GSS24]).
(65) This is actually the case for any general composite bundle: the infinitesimal and fermionic structures are encoded only in higher infinitesimal and odd plots of the thickened smooth super space of fields.
(66) See ft. (59) at p. 85.
(67) For a Lagrangian of polynomial order k in the fermionic field, we would need $\mathbb{R}^{1|k}$-plots.

structure may arise by relying on two observations. First, the category sThSmthSet behaves in many respects similarly to ThSmthSet, since the underlying Grothendieck topology is essentially the same. Second, provided that the grading is handled appropriately, the definitions from super differential geometry that we employ are straightforward generalizations of the corresponding notions in ordinary differential geometry. For these reasons, it is reasonable to expect that the material presented in [GS26] will extend the constructions of [GS25a; GS25b] in a direct manner.

Definition 3.3.4. The **synthetic tangent bundle** $T\mathcal{G}$ of $\mathcal{G} \in \mathrm{sThSmthSet}$ is defined as the mapping thickened smooth super set

$$T\mathcal{G} \coloneqq [\mathbb{D}^1(1), \mathcal{G}] \in \mathrm{sThSmthSet},$$

where we omit the Yoneda embedding y for $\mathbb{D}(1)^1$ for readability.

The unique embedding map $\iota_0 \colon * \hookrightarrow \mathbb{D}^1(1)$ defines, by precomposition, the projection

$$T\mathcal{G} \coloneqq [\mathbb{D}^1(1), \mathcal{G}] \xrightarrow{(-)\circ\iota_0} [*, \mathcal{G}] \cong \mathcal{G}.$$

In the case of $\mathcal{M} \in \mathrm{sSmthMfd}$, the latter is the usual bundle projection.
Furthermore, the construction above forms a functor

$$T \coloneqq [\mathbb{D}^1(1), -] \colon \mathrm{sThSmthSet} \to \mathrm{sThSmthSet},$$

with smooth maps $f \colon \mathcal{G} \to \mathcal{H}$ mapped to $Tf \colon T\mathcal{G} \to T\mathcal{H}, v^{k|q,\epsilon} \mapsto f \circ v^{k|q,\epsilon}$. The latter can be interpreted as the pushforward of any map f. Indeed, in the case of $\mathcal{G} = y(\mathcal{M})$, $\mathcal{H} = y(\mathcal{N})$, one can check that $Tf \equiv \mathrm{d}f \colon y(T\mathcal{M}) \to y(T\mathcal{N})$.
Crucially, by its definition and by internal hom property (see Eq. (3.50)), it has a left-adjoint functor[(68)]

$$(\mathbb{D}^1(1) \times -) \dashv T(-). \tag{3.53}$$

Remark 3.3.4. Note that in full generality $T\mathcal{G} \coloneqq [\mathbb{D}^1(1), \mathcal{G}]$ does not have a fiber-wise $\mathbb{R}$-linear structure, that is fiberwise addition and $y(\mathbb{R})$-multiplication

$$+ \colon T\mathcal{G} \times_{\mathcal{G}} T\mathcal{G} \to T\mathcal{G} \qquad \text{and} \qquad \cdot \colon y(\mathbb{R}) \times T\mathcal{G} \to T\mathcal{G}. \tag{3.54}$$

In the field-theoretically relevant cases, this structure is guaranteed by the $\mathbb{R}$-linearity of the targets, as it is evident in what follows.

Proposition 3.3.2. *The synthetic tangent bundle $T\mathcal{F} \coloneqq [\mathbb{D}^1(1), \mathcal{F}]$ of a thickened smooth super field space $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is canonically isomorphic to $\mathbf{\Gamma}_M(VF) \in \mathrm{sThSmthSet}$, where the vertical bundle $VF \xrightarrow{\pi^{TF}_F} F \xrightarrow{\pi^F_M} M$ is defined by*

$$VF \coloneqq \mathrm{eq}\Big(TF \overset{\mathrm{d}\pi^F_M}{\underset{0_M}{\rightrightarrows}} TM\Big).$$

Proof. In the proof, we omit the Yoneda embedding y for $\mathcal{M}$, $\mathcal{N}$, $T\mathcal{N}$, M, TM, F and TF. First, we prove that $T[\mathcal{M}, \mathcal{N}] \cong [\mathcal{M}, T\mathcal{N}]$ by the internal hom property and the symmetric monoidal structure of sThSmthSet via the direct product.

$$T[\mathcal{M},\mathcal{N}] \coloneqq [\mathbb{D}^1(1), [\mathcal{M},\mathcal{N}]] \cong [\mathbb{D}^1(1) \times \mathcal{M}, \mathcal{N}] \cong [\mathcal{N} \times \mathbb{D}^1(1), \mathcal{N}] \cong [\mathcal{M}, [\mathbb{D}^1(1), \mathcal{N}]] \cong [\mathcal{M}, T\mathcal{N}].$$

Note that in this case the fiberwise $y(\mathbb{R})$-linear structure in $T[\mathcal{M},\mathcal{N}]$ is inherited from the $\mathbb{R}$-linear structure of $T\mathcal{N}$ (cf. [GS25b, Prop. 2.27]).
Secondly, we apply the synthetic tangent functor to diagram (3.52) and obtain, using the computation above,

$$\begin{tikzcd} T\mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[M, TF]} \arrow[d, "(T\pi^F_M)_*"] \\ T(\mathrm{id}_M) \cong * \arrow[r, hook] & {[M, TM]}, \end{tikzcd}$$

(68) One should also use monoidal structure of sThSmthSet via the direct product.

where the bottom embedding is identified with the embedding defined probe-wise by sending the unique $(\mathbb{R}^{k|q} \times \mathbb{D})$-plot of the point $*$ to the zero plot $[M, TM](\mathbb{R}^{k|q} \times \mathbb{D}) \ni (\mathbb{R}^{k|q} \times \mathbb{D} \times M \xrightarrow{\pi_M} M \xrightarrow{0_M} TM)$, for each $\mathbb{R}^{k|q} \times \mathbb{D} \in \mathrm{sThCartSp}$. This means that $T\mathcal{F}$ plots are given by

$$T\mathcal{F}(\mathbb{R}^{k|q} \times \mathbb{D}) = \{Z_{\varphi^{k|q,\epsilon}} \in [M, TF](\mathbb{R}^{k|q} \times \mathbb{D}) \mid T\pi^F_M \circ Z_{\varphi^{k|q,\epsilon}} = 0_M \circ \pi_M\}.$$

These maps are exactly the maps factoring through the vertical subbundle $VF \hookrightarrow TF$ (cf. [GS25b, Prop. 2.28]), that is, $(\mathbb{R}^{k|q} \times \mathbb{D})$-plots of $\mathbf{\Gamma}_M(VF)$. Note that the $y(\mathbb{R})$-linear structure in $T\mathcal{F}$ is inherited from the $\mathbb{R}$-linear structure of VF. ■

Composing the projection $\pi^{TF}_F \colon VF \to F$ with $Z_{\varphi^{k|q,\epsilon}}$ yields $(\mathbb{R}^{k|q,\epsilon})$–parametrized sections of $F \to M$, $\varphi^{k|q,\epsilon} \coloneqq \pi^{TF}_F \circ Z_{\varphi^{k|q,\epsilon}}$. That is, any $Z_{\varphi^{k|q,\epsilon}}$ covers a unique plot of fields $\varphi^{k|q,\epsilon}$, which realizes the projection

$$\pi^{T\mathcal{F}}_{\mathcal{F}} \colon T\mathcal{F} \cong \mathbf{\Gamma}_M(VF) \xrightarrow{(\pi^{TF}_F)_*} \mathbf{\Gamma}_M(F) \cong \mathcal{F}.$$

Vector fields are sections of this tangent bundle as in Diagram (3.5). As in ThSmthSet, vector fields should be in one-to-one relation with derivatives on the algebra of smooth functions, connecting this definition with Definition 1.3.20. Forms are defined in the usual fashion, once one takes into account the $\mathbb{Z}_2$-graded structure (cf. [GSS24]).

Definition 3.3.5. The **set of differential m-forms** on $\mathcal{F} = \mathbf{\Gamma}_M(F)$ is defined as

$$\Omega^m(\mathcal{F}) \coloneqq \mathrm{Hom}^{\mathrm{fib.lin.an.}}_{\mathrm{sThSmthSet}}\left(T^{\times m}\mathcal{F}, y(\mathbb{R}^{1|1})\right).$$

That is, m-forms are smooth real-valued, fiber-wise linear antisymmetric maps with respect to the fiber-wise linear structure (see Eq. (3.54)) on the m-fold fiber product(69)

$$T^{\times m}\mathcal{F} \coloneqq T\mathcal{F} \times_{\mathcal{F}} \cdots \times_{\mathcal{F}} T\mathcal{F}$$

of the tangent bundle over $\mathcal{F}$.

Homogeneous form of degree 0 and 1 (see Def. 1.3.20) are given by maps into $y(\mathbb{R})$ and $y(\mathbb{R}^{0|1})$, respectively (cf. [GSS24][Ex. 2.36]). The total degree of an homogeneous m-form ω of degree $k = 0, 1$, i.e. $\omega \in \Omega^m(\mathcal{F})_k$ is $|\omega| = m + k$. The collection of differential forms of all (cohomological) degrees forms a graded $\mathbb{R}$-vector space (see Def. 1.3.3)

$$\Omega^\bullet(\mathcal{F}) \coloneqq \bigoplus_{m\in\mathbb{N}} \Omega^m(\mathcal{F}).$$

The latter has a well-defined notion of a wedge product $\wedge$ and contraction ι_Z along any vector field $Z \in \mathfrak{X}(\mathcal{F})$.(70) However, there is no obvious definition for a de Rham differential $\mathrm{d}_{\mathcal{F}} \colon \Omega^m \to \Omega^{m+1}$ as in the case of ThSmthSet. We can resolve this via infinite jet bundle technologies.

Super Infinite Jet Bundle Technology and Local Lagrangian Field Theory

For $F \xrightarrow{\pi^F_E} E \xrightarrow{\pi^E_M} M$ as in Definition 3.3.3, we can define the super infinite jet bundle $J^\infty_M F$ following the construction in [GSM09][Ch. 3].
The idea is to define the infinite jet bundle algebraically, starting from the construction of its algebra of forms. Via Batchelor theorem (see Ch. 1), any odd vector bundle bundle $F \to E$ can be described as the $\mathbb{Z}_2$-graded manifold $\Pi\bar{F}$ modeled over the classical vector bundle $\pi^{\bar{F}}_E \colon \bar{F} \to E$. That is, $\Pi\bar{F} \coloneqq (E, \mathcal{O}_{\Pi\bar{F}})$ with $\mathcal{O}_{\Pi\bar{F}}(E) = \Gamma_E(\bigwedge \bar{F}^*)$. The latter has local coordinates $\{x^\mu, y^A, \theta^i\}$ where the first two are even and only the last is odd. Alternatively, we do not distinguish E and F-fibers coordinates and write $\{x^\mu, u^a\}$. Second, notice that the k-jet manifold $J^k_M\bar{F}$ is a vector bundle $J^k_M\bar{F} \to J^k_M E$ over the k-jet manifold $J^k_M E$ (see Def. 3.1.9). Define the $\mathbb{Z}_2$-graded manifold $\Pi J^r_M\bar{F} \coloneqq (J^r_M E, \mathcal{O}_{\Pi J^r_M\bar{F}})$ with $\mathcal{O}_{\Pi J^r_M\bar{F}} =$

(69) Here the product is the pullback (see Def. 1.1.12) of $T\mathcal{F} \xrightarrow{\pi^{T\mathcal{F}}_{\mathcal{F}}} \mathcal{F} \xleftarrow{\pi^{T\mathcal{F}}_{\mathcal{F}}} T\mathcal{F}$.
(70) Thanks to the fiber-wise linear structure of $T\mathcal{F}$ and graded structure of $\Omega^\bullet(\mathcal{F})$, they can be defined as usual (see Def. 1.3.13). Note that the wedge product commutation relation is governed by the total degree, while in [GSS24] the Deligne and Freed convention is used (see ft. (22), p. 14).

$\Gamma_{J^r_M E}\big(\bigwedge(J^r_M \bar{F})^*\big)$ and local coordinates $\{x^\mu, \{y^A_I, \theta^i_I\}_{|I|\leqslant r}\} = \{x^\mu, \{u^a\}_{|I|\leqslant r}\}$. We can associate the $\mathbb{Z}_2$-graded Lie algebra of derivations $\mathrm{Der}\big(\mathcal{O}_{\Pi J^r_M \bar{F}}\big)$ and the algebra of $\mathbb{Z}$-graded forms (see Def. 1.3.20)

$$\Omega^\bullet(\Pi J^r_M \bar{F}) = \Big(\big(\mathrm{Der}(\mathcal{O}_{\Pi J^r_M \bar{F}})\big)^*\Big)^{\wedge \bullet}. \tag{3.55}$$

The surjection $\pi^{r+1}_r \colon J^{r+1}_M E \to J^r_M E$ yields dually a morphism of the corresponding graded algebras $\Omega^\bullet(\Pi J^r_M \bar{F}) \to \Omega^\bullet(\Pi J^{r+1}_M \bar{F})$. Taking the projective limit of this system, we get the algebra of forms on the would-be infinite jet bundle that we denote by $\Omega^\bullet(\Pi J^\infty_M \bar{F})$. The latter is $C^\infty(E)$-algebra locally generated by $\{\theta^i_I, \mathrm{d}x^\mu, \mathrm{d}u^a_I - u^a_{I+\mu}\mathrm{d}x^\mu\}$. It is possible to recover from it the sheaf of graded functions defining the graded infinite jet bundle $J^\infty_M F \coloneqq (J^\infty_M E, \mathcal{O}_{J^\infty_M F})$. The splitting of the infinite jet bundle tangent space is shown at the level of this algebra, as it decomposes into horizontal and vertical forms. The latter is the notion of graded variational bicomplex (see Def. 3.1.20). All the properties of the horizontal and vertical differential remain the same (see Eqs. (3.26) and (3.27)). Instead of building the algebra of forms as above, to cast this in sThSmthSet we can take the projective limit of the algebras $\Gamma_{J^r_M E}\big(\bigwedge(J^r_M \bar{F})^*\big)$ in $\mathbb{Z}_2\mathrm{CAlg}_{\mathbb{R}}$ and dually use it to define the infinite graded smooth jet bundle (cf. [Gio25; GS26]).
The discussion of infinite jet bundle technology and local structure should then follow by universal properties as in (Th)SmthSet (see Prop. 3.2.5).
Developing all this goes far beyond the scope of the thesis and it should appear in [GS26]. This section only provides a justification for the following section, which is based on this future work by Sati and Giotopoulos.

In summary, the super-thickened category sThSmthSet is proposed as the setting that combines smoothness, infinitesimals, and parity.

3.4 Local BV System via Super Thickened Smooth Set

The motivation behind this thesis is to find a categorical setting in which the field-theoretic objects relevant to the BV formalism appear on equal footing. In this section, we show how sThSmthSet provides this setting when working locally. In particular, the existence of the local Cartan calculus allows us to mimic the finite-dimensional case (see Sec. 2.2) and make rigorous sense of the expressions in Section 2.3, where we assumed this calculus to exist. Since the category does not provide a notion of measure on the space of fields, which is necessary to mimic the finite-dimensional quantum BV formalism,[(71)] we focus only on the classical case. For ease of notation, we henceforth suppress the Yoneda embedding y whenever it is implicitly understood from the context.

Grading

The classical BV formalism relies on symplectic $\mathbb{Z}$-graded geometry and the existence of the infinite-dimensional Cartan calculus (see Sec. 2.2). Note that, the nature of the anticommuting structure needed to describe ghost fields and antifields is intrinsically different from the anticommuting nature of fermionic fields, which we do not consider. However, if one works locally, the results related to the BV formalism (see for example Prop. 1.4.2 and Prop. 1.4.3) only need the compatible[(72)] $\mathbb{Z}_2$ underlying structure, the full Cartan calculus, and the existence of an even vector field that tracks the ghost $\mathbb{Z}$-grading.[(73)]
Therefore, working within sThSmthSet should be enough. Indeed, in this category, we have an intrinsic notion of superstructure and a good notion of local Cartan calculus on the infinite-dimensional space of fields. Note that restricting to the local setting is not an issue, as all the relevant field theories are local. Finally, we can postulate the existence of a local even vector field $\mathbb{E}$ (see Def. 3.1.24) on the space of fields that keeps track of the ghost $\mathbb{Z}$-grading.
In particular, let $\mathcal{F}$ be the super thickened smooth space of fields for some composite bundle $\pi^F_M \colon F \to M$ (see Def. 3.3.3) over a closed oriented smooth d-manifold M. Let $\{x^\mu, u^a\}$ and $\{x^\mu, \{u^a_I\}_{|I|\geqslant 0}\}$ be

(71) Note that there exist different approaches to the quantum BV formalism (cf. [CMS25]).
(72) Compatibility implies that the Koszul sign in the graded commutation rule is actually governed by the underlying $\mathbb{Z}_2$ structure. That is, what actually governs the signs is the $\mathbb{Z}$-grading modulo 2.
(73) Strictly speaking, we can think of a BV system just as an odd symplectic supermanifold with an even function satisfying the master equation (cf. [Sch93]). The $\mathbb{Z}$-grading facilitates bookkeeping and becomes important when considering theories with boundaries and corners (cf. [CMR14; CMR18]).

local coordinates on F and $J^\infty_M F$ (see discussion around Eq. (3.55)). Note that, since the base is even, $\deg_{\mathbb{Z}_2}(u^a) = \deg_{\mathbb{Z}_2}(u^a_I)$. Then we define[(74)]

$$\mathbb{E} \coloneqq \mathsf{E} \circ j^\infty \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})_0, \quad \text{where } \mathsf{E} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F) \text{ s.t. } \mathsf{E}^\mu = \mathsf{E}(x^\mu) = 0 \text{ and } \mathsf{E}^a = \mathsf{E}(u^a) = \mathrm{gh}(u^a)u^a.$$

Importantly, $\mathrm{gh}(u^a) \bmod 2 = \deg_{\mathbb{Z}_2}(u^a)$. Note that, the prolongation of the latter (see Eq. (3.35)) is given by $\mathrm{pr}\mathsf{E}(u^a_I) = \mathrm{gh}(u^a)u^a_I$ as one would expect, since the derivatives along the base bring zero degree. To see it, we used that $D_\mu(\mathsf{E}^a) = (\partial_{x^\mu} + \sum_{|I|\geqslant 0} u^a_{I+\mu}\partial_{u^a_I})\mathsf{E}^a = u^a_\mu \mathrm{gh}(u^a)$ (see Eq. (3.19)) and concluded by induction. While $\mathbb{L}_{\mathrm{pr}\mathsf{E}}$ gives the ghost $\mathbb{Z}$-grading on $J^\infty_M F$, $\mathbb{L}_{\mathbb{E}}$ reads the ghost $\mathbb{Z}$-grading of local objects on $\mathcal{F} \times M$ and $\mathcal{F}$. Importantly, these gradings coincide (see Prop. 3.1.11, Prop. 3.1.12 and Eq. (3.40)[(75)]).

Remark 3.4.1. From now on, we consider the ghost $\mathbb{Z}$-grading defined in this way, keeping in mind that, at the level of sThSmthSet, we only have an underlying $\mathbb{Z}_2$-grading. The latter is given by the ghost $\mathbb{Z}$-grading modulo 2. Also, the definitions and results we take from Section 3.1 have to be modified with the correct Koszul sign as done in Chapter 1 and Chapter 2. This is always the case from now on: we explicitly state it only when strictly necessary to understand the computations.

To work with the total degree, we should define two derivations $\mathbb{D}_H, \mathbb{D}_V \in \mathrm{Der}(\mathfrak{X}(\Omega^{\bullet,\bullet}(J^\infty_M F))$ that count horizontal and vertical cohomological degree (see Eq. (1.15)), and consider $\hat{\mathbb{D}} \coloneqq \mathbb{L}_{\mathrm{pr}\mathsf{E}} + \mathbb{D}_H + \mathbb{D}_V$. Since the pullback along ev^∞ preserves the cohomological bigrading (see Diagram (3.37)), it has zero total degree and there exist compatible derivations $\mathbb{D}_M, \mathbb{D}_{\mathcal{F}} \in \mathrm{Der}(\Omega^{\bullet;\bullet}_{\mathrm{loc}}(\mathcal{F} \times M))$. By abuse of notation, we denote the total grading derivation on $\mathcal{F} \times M$ by $\hat{\mathbb{D}} \coloneqq \mathbb{L}_{\mathbb{E}} + \mathbb{D}_M + \mathbb{D}_{\mathcal{F}}$. Finally, note that transgressing to $\mathcal{F}$ along τ_M (see Def. 3.1.27) kills the component $\mathbb{D}_M$ and preserves the other. Therefore, τ_M has total degree $-d$. We also denote by $\hat{\mathbb{D}} = \mathbb{L}_{\mathbb{E}} + \mathbb{D}_{\mathcal{F}}$ the total grading after the transgression. The total grading defined like this is denoted by $|\cdot|$

Local BV System

The idea is to define a local BV system at the level of the super infinite jet bundle. This system, once pulled back along ev^∞ and transgressed to $\mathcal{F}$, should yield a local classical BV theory on $\mathcal{F}$ (see Def. 2.2.4). BV results should then follow by the existence of Cartan calculus.

Definition 3.4.1. Let us consider the following:

1. M is a closed oriented smooth d-manifold. As it has no boundary, horizontally exact terms vanish after integration and therefore along the transgression τ_M;
2. $\pi^F_M \colon F \to E \to M$ is a globally defined composite fiber bundle (see Def. 3.3.3). Assume the existence of a local Cartan calculus given within sThSmthSet introducing the graded infinite jet bundle $J^\infty_M F$. Furthermore, assume compatibility of the ghost $\mathbb{Z}$-grading given by $\mathbb{L}_{\mathbb{E}}$ with the intrinsic $\mathbb{Z}_2$-grading (see Rmk. 3.4.1);
3. Globally defined local finite-order data on the graded infinite jet bundle encoded by:
   a. An evolutionary vector field $\mathsf{Q} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$ of total degree $\hat{\mathbb{D}}(Q) = 1Q$ such that $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = 0$ (see Prop. 3.1.11 with extra Koszul sign $(-1)^{|\mathsf{Z}_1||\mathsf{Z}_2|}$ in the local expression as in Eq. (1.13));
   b. A local Lagrangian $\mathsf{L} \in \Omega^{d,0}_{\mathrm{loc}}(J^\infty_M F)_0$ given by a local $(d,0)$-form of total degree $\hat{\mathbb{D}}\mathsf{L} = d\mathsf{L}$;
   c. A local $(d,2)$-form $\omega \in \Omega^{d,2}_{\mathrm{loc}}(J^\infty_M F)_{-1}$ of total degree $\hat{\mathbb{D}}\omega = (-1+d+2)\omega$ locally of the form
   $$\omega = (\mathrm{d_H}x)^d \omega_{AB}\mathrm{d_V}\psi^A \mathrm{d_V}\psi^B$$
   with ω_{AB} invertible. This ensures the weak non-degeneracy of the local symplectic form after the transgression τ_M.

[(74)]The fact that evolutionary vector fields are zero on the base might be an obstruction to treating in a similar way the general case where the base is a supermanifold (see Rmk. 3.4.3), since in this case the Euler vector would have non-zero horizontal components. In this case, working with a derivation, as for the cohomological degree, might be the solution.

[(75)]The latter holds when restricting to local transgressed forms interpreted as maps of local vector fields $(\mathfrak{X}_{\mathrm{loc}}(\mathcal{F}))^{\times q} \to C^\infty_{\mathrm{loc}}(\mathcal{F})$, and not as bundle maps $(T\mathcal{F})^{\times q} \to y(\mathbb{R})$ (cf. [GS25a][Rmk. 7.39]). Note that for our application, this is enough.

The tuple $(J^\infty_M F, \omega, \mathsf{L}, \mathsf{Q})$ is a **local BV system** (cf. [Gri23]) if, up to horizontally exact terms, it satisfies the following:

$$\iota_{\mathrm{pr}\mathsf{Q}}\omega - \mathrm{d}_V \mathsf{L} = 0 \tag{3.56}$$

$$\frac{1}{2}\iota_{\mathrm{pr}\mathsf{Q}}\iota_{\mathrm{pr}\mathsf{Q}}\omega = 0. \tag{3.57}$$

Consider the transgression $\tau_M\colon \Omega^{d,q}_{\mathrm{loc}}(\mathcal{F}\times M) \to \Omega^q(\mathcal{F})$ of local (d,q)-forms on $\mathcal{F}\times M$ to q-forms on $\mathcal{F}$ (see Def. 3.1.27). Under the hypotheses listed in Definition 3.4.1, the local BV system $(J^\infty_M F, \omega, \mathsf{L}, \mathsf{Q})$ transgresses to the following BV theory:

$$(\mathcal{F} \coloneqq \mathbf{\Gamma}_M(F),\ \hat{\mathbb{D}},\ \Omega \coloneqq (-1)^d \tau_M((\mathrm{ev}^\infty)^*\omega),\ S \coloneqq \tau_M((\mathrm{ev}^\infty)^*\mathsf{L},\ Q \coloneqq \mathsf{Q}\circ j^\infty). \tag{3.58}$$

The derivation $\hat{\mathbb{D}}$ defines the total grading as in Remark 3.4.1. Recall the conventions in Subsection 2.0.1. Since the transgression sends top forms to real numbers, it has total degree $|\tau_M| = -d$. By definition, $\Omega \in \Omega^2_{\mathrm{loc}}(\mathcal{F})_{-1}$ has total degree $\hat{\mathbb{D}}\Omega = 1\Omega$, $S \in C^\infty_{\mathrm{loc}}(\mathcal{F})_0$ has zero total degree, and $Q \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})_1$ has total degree $\hat{\mathbb{D}}Q = 1Q$. In what follows, we use the local Cartan calculus and the transgressed one (see Prop. 3.1.12, Prop. 3.1.13 and Eq. (3.40)). The transgressed de Rham differential comes with the extra Koszul sign $(-1)^{-d}$ given by exchanging $\mathrm{d}_\mathcal{F}$ ($|\mathrm{d}_\mathcal{F}| = 1$) with τ_M ($|\tau_M| = -d$)). The extra sign $(-1)^d$ is added to Ω to cancel the Koszul sign coming from the transgressed Cartan calculus in Equation (3.59). The tuple (3.58) is a classical BV theory in sThSmthSet, indeed

1. The space of fields $\mathcal{F} \in$ sThSmthSet has a smooth superstructure compatible (modulo 2) with the ghost $\mathbb{Z}$-grading given by $\mathbb{L}_\mathbb{E}$ in $\hat{\mathbb{D}}$.
2. Given the local expression of ω and invertibility of ω_{AB}, $\Omega \in \Omega^2_{\mathrm{loc}}(\mathcal{F})_{-1}$ can be shown to be odd-weakly symplectic as a map $(\mathfrak{X}_{\mathrm{loc}}(\mathcal{F}))^{\times 2} \to C^\infty_{\mathrm{loc}}(\mathcal{F})_{-1}$.
3. $Q \in \mathfrak{X}_{\mathrm{loc}}(\mathcal{F})_1$ is cohomological by $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = 0$ and Equation (3.32). Furthermore, it is the Hamiltonian vector field of S by Equation (3.56) (see Def. 3.1.28). Indeed, recalling the discussion around Remark 3.4.1 and that $|\iota_Q| = |Q| - 1 = 0$, we get

$$\begin{aligned}\iota_Q\Omega &= (-1)^d\tau_M(\iota_Q((\mathrm{ev})^*\omega)) = (-1)^d\tau_M((\mathrm{ev}^\infty)^*(\iota_{\mathrm{pr}\mathsf{Q}}\omega)) \overset{(3.56)}{=} (-1)^d\tau_M((\mathrm{ev}^\infty)^*(\mathrm{d}_V\mathsf{L})) = \\ &= (-1)^d\tau_M(\mathrm{d}_\mathcal{F}(\mathrm{ev}^\infty)^*\mathsf{L}) = (-1)^{d-d}\mathrm{d}_\mathcal{F}\tau_M((\mathrm{ev}^\infty)^*\mathsf{L}) = \mathrm{d}_\mathcal{F}S,\end{aligned} \tag{3.59}$$

where the potential horizontally exact terms do not contribute by Stokes theorem and $\partial M = \varnothing$.

4. The BV action $S \in C^\infty_{\mathrm{loc}}(\mathcal{F})_0$ satisfies the CME by Equation (3.57) and Definition 3.1.28 (modified with the appropriate Koszul sign as in Def. 1.4.8). Indeed, recalling that $|\iota_Q| = |Q| - 1 = 0$, we get

$$\{S,S\}_\Omega = (-1)^{|Q|-1}\frac{1}{2}\iota_Q\iota_Q\Omega = \frac{1}{2}(-1)^d\tau_M(\iota_Q\iota_Q((\mathrm{ev})^*\omega)) = (-1)^d\tau_M\left((\mathrm{ev}^\infty)^*\left(\frac{1}{2}\iota_{\mathrm{pr}\mathsf{Q}}\iota_{\mathrm{pr}\mathsf{Q}}\omega\right)\right) = 0,$$

where the potential horizontally exact terms do not contribute by Stokes theorem and $\partial M = \varnothing$.

BV results follow from the existence of Cartan calculus. For example, under opportune regularity conditions, it should be possible to characterize the Euler locus of the classical theory via H^0_Q and confront it with the characterization as a pullback within sThSmthSet (cf. [GS25a][Ch. 5]).

Remark 3.4.2. In general, the description of the local BV theory above is heavily based on the bundle structure defining the space of fields. The assumption to work with gauge theories described by a trivial principal G-bundle (see Rmk. 2.1.1) allows us to describe the fields of BV extension of the theory as sections of a composite bundle $F \to M$.[76] Thanks to this caveat, we can apply all the technology developed in the previous sections of this chapter and describe gauge symmetries as natural transformations $\mathcal{F} \to \mathcal{F}$ that preserve the action (see Def. 3.1.23). This means that for more general theories, we can only work locally. To formulate a categorical description of gauge theories and the BV formalism, one should enrich sThSmthSet with higher structure, considering a category of sheaves of groupoids (cf. [GS25a][Rmk. 6.37 & Ch. 8]). A derived approach to the global description of BV/AKSZ theory can be found in (cf. [Ste24]). Along these lines, a possibility might be considering a derived extension of the category of SmthSet, using the universal property of derived manifolds and their relation with dg-manifolds (cf. [CS19; Car23]).

[76] In particular, the gauge fields, which are 1-form connections, can be described as forms on the base taking values into the Lie algebra of G. For the more general case of a non-trivial G-bundle, when working locally, we can always reduce to the triviality hypothesis by working around a trivial connection (see Rmk. 3.1.8).

Before passing to the example of BF theory, let us make the following remark regarding the limitations of our framework and a possible outlook.

Remark 3.4.3. As discussed in Section 2.3, the AKSZ formalism allows to automatically generate BV theories on the mapping space $\mathrm{Map}(T[1]M,\mathcal{N})$, when the target is equipped with some supergeometric data. We would like to find a categorical description of it within sThSmthSet. For simplicity, assume $\mathcal{N}$ to be a superspace. Already at this level, some problems arise when trying to use a general perspective as in Section 2.3.
Indeed, although the mapping space is well understood in sThSmthSet (see Def. 3.3.1), the infinite jet bundle technology developed above strictly relies on a bundle structure that a general mapping space does not have. A possible solution to this problem, which we present as an outlook, is the following.
If it were possible to extend Definition 3.3.3 to the case in which the base is a finite-dimensional supermanifold $\mathcal{M}$[77] (see Def. 1.3.18) and show that all the results that lead us to the local Cartan calculus above also hold in this extension, we would have a usable notion of Cartan calculus on the mapping space.[78] Indeed, notice that in this case the mapping space of Example 3.3.1 would be isomorphic in sThSmthSet to the thickened smooth superspace of sections of the trivial bundle $\mathcal{M}\times\mathcal{N}\xrightarrow{\mathrm{pr}_1}\mathcal{M}$. Indeed, we would have that plotwise

$$\begin{aligned}\mathbf{\Gamma}_{\mathcal{M}}(\mathcal{M}\times\mathcal{N})(\mathbb{R}^{k|q}\times\mathbb{D}) &\coloneqq \{\varphi^{k|q,\epsilon}\colon \mathbb{R}^{k|q}\times\mathbb{D}\times\mathcal{M}\to\mathcal{M}\times\mathcal{N}\,|\,\mathrm{pr}_1\circ\varphi^{k|q,\epsilon}=\pi_{\mathcal{M}}\}\\ &\cong \{\tilde{\varphi}^{k|q,\epsilon}\colon \mathbb{R}^{k|q}\times\mathbb{D}\times\mathcal{M}\to\mathcal{N}\}\cong[y(\mathcal{M}),y(\mathcal{N})](\mathbb{R}^{k|q}\times\mathbb{D}),\end{aligned}$$

where $\pi_M\colon\mathbb{R}^{k|q}\times\mathbb{D}\times M\to M$. Then, the local Cartan calculus on the first might induce one on the second. Note that the definition of infinite supergraded jet bundle we use (see around Eq. (3.55)) relies on the base being even. More general notions of infinite jet bundles are described in [GK19; Vys23] from a local and global perspective, respectively. This would allow us to mimic the proof of Theorem 2.1 within sThSmthSet (following the Diagram (3.60)), giving a systematic way to build a local BV theory (see Eq. (3.58)). In this case, locality would follow directly from the definition of $\mathbb{T}$ and the fact that the supergeometric data we start with are defined on a finite-dimensional superspace $\mathcal{N}$.
Alternatively, since the target is assumed to be a superspace, we can make sense of the mapping space by $\mathrm{Map}(T[1]M,\mathcal{N})\cong\Omega^\bullet(M,\mathcal{N})$ (see Eq. (2.22)). Thanks to this, we can describe the fields of the BV extension as sections of a composite bundle $F\to E\to M$. This forces us to work with forms over M rather than with functions over $T[1]M$ (see Rmk. 1.3.7). The drawback of this approach is that there is no canonical map from $J^\infty_M F$ (or from $M\times\mathbf{\Gamma}_M(F)$ as in Diagram (3.61)) to $\mathcal{N}$ along which consistently pull back the supergeometric data of the AKSZ theory. Therefore, we cannot mimic the proof of Theorem 2.1. What we can do is start from a good Ansatz for a local BV system on $J^\infty_M F$ (see Def. 3.4.1), show that it is well defined and build the BV theory as in Equation (3.58). The Ansatz can be obtained by grading considerations and by working formally as in Section 2.3.

Diagram with
$\mathcal{F}\cong\mathbf{\Gamma}_{T[1]M}(T[1]M\times\mathcal{N})$:

$$\begin{tikzcd} T[1]M\times\mathcal{F} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"'] & \mathcal{N} \\ \mathcal{F} & \end{tikzcd}\tag{3.60}$$

Diagram with
$\mathcal{F}\cong\mathbf{\Gamma}_M(\bigwedge^\bullet T^*M\otimes(M\times\mathcal{N}))\equiv\mathbf{\Gamma}_M(F)$:

$$\begin{tikzcd} M\times\mathcal{F} \arrow[r, "\mathrm{ev}^\infty"] \arrow[d, "\tau_M"'] & J^\infty_M F \arrow[r, dashed, "?"] & \mathcal{N} \\ \mathcal{F} & & \end{tikzcd}\tag{3.61}$$

Abelian BF Theory in $d=4$

From now on, we use Einstein summation convention. Moreover, recall that $\mathrm{d}_H=\mathrm{d}x^\nu\partial_{x^\nu}$ and $\mathrm{d}_H\omega=\mathrm{d}x^\nu\frac{1}{k!}\partial_{[\nu}\omega_{\mu_1,\dots,\mu_k]}\mathrm{d}x^{\mu_1}\cdots\mathrm{d}x^{\mu_k}$.[79] The classical abelian[80] BF theory in $d=4$ (see Sec. 2.3) can be understood within SmthSet. Let M be a closed oriented smooth 4-manifold. The space of fields of the

[77] Actually, showing this for a base of the form $T[1]M$ should be enough for the purposes of AKSZ theory.
[78] The integral map then would be defined as in Equation (3.28), using super integration theory (see Eq. (1.17)) instead of classical integration theory.
[79] Where we omitted the wedge product for readability and we use the convention for which the anticommutation of the indices has a factor of $(k+1)!$ at denominator.
[80] Since we are in the abelian case, $\mathfrak{g}=\mathbb{R}$. We can drop the dual $\mathfrak{g}^*=\mathbb{R}^*$, as we can interpret any dual element as a multiplication by an element in $\mathbb{R}$. Implicitly, we are choosing a basis of $\mathbb{R}$.

bundle $F_{\text{cl}} := T^*M \oplus \bigwedge^2 T^*M \ni \{A_{\mu_1}, B_{\mu_1\mu_2}/2!\}$ is the smooth set (see Def. 3.1.4)

$$\mathcal{F}_{\text{cl}} := \mathbf{\Gamma}_M\Big(T^*M \oplus \bigwedge^2 T^*M\Big),$$

with the original fields $(A, B) \in \Omega^1(M) \oplus \Omega^2(M) \cong \mathcal{F}_{\text{cl}}(*)$. Considering the infinite jet bundle $J^\infty_M F_{\text{cl}} \ni \{x^\mu, \{A_{\mu_1|I}, B_{\mu_1\mu_2|I}/2!\}_{|I|\geqslant 0}\}$, we can write the classical lagrangian as $\mathcal{L}_{\text{cl}} = \mathsf{L}_{\text{cl}} \circ j^\infty$ (see Def. 3.1.21). Here L_{cl} is the $(d, 0)$-form on $J^\infty_M F_{\text{cl}}$ locally given by

$$\mathsf{L}\bigg(x^\mu, \Big\{A_{\mu_1|I}, \frac{B_{\mu_1\mu_2|I}}{2!}\Big\}_{|I|\leqslant 1}\bigg) = \bigg(\frac{B_{\mu_1\mu_2}}{2!}\mathrm{d}_H x^{\mu_1} \wedge \mathrm{d}_H x^{\mu_2}\bigg) \wedge \Big(\mathrm{d}_H x^{\mu_3}\partial_{[\mu_3}A_{\mu_4]} \wedge \mathrm{d}_H x^{\mu_4}\Big) = B \wedge \mathrm{d}_H A.$$

Notice that it globally factors through the 1-jet. Then, working within SmthSet, one can recover the BF action as a natural transformation of smooth sets (see Eq. (3.29)) and describe the smooth structure of the Euler–Lagrange locus (cf. [GS25a][Ch. 5]). Also, the gauge symmetries (see Eq. (2.25)) can be understood via point 1. of Definition 3.1.23. In particular, the first symmetry in Equation (2.25) can be extended to a smooth map $\mathcal{D}_1 \colon \mathcal{F}_{\text{cl}} \to \mathcal{F}_{\text{cl}}$ factoring through the 0-jet bundle such that $\mathcal{L} \circ \mathcal{D}_1 = \mathcal{L}$. The second one, at the level of $*$-plots, acts on the Lagrangian by $\mathcal{L}_{\text{cl}} \mapsto \mathcal{L}_{\text{cl}} + \mathrm{d}_M(t \wedge F_A)$. The latter extends to a smooth map $\mathcal{K} \colon \mathcal{F}_{\text{cl}} \to \mathbf{\Gamma}_M(\bigwedge^3 T^*M)$ factoring through the 1-jet such that $\mathcal{L}_{\text{cl}} \circ \mathcal{D}_2 = \mathcal{L}_{\text{cl}} + \mathrm{d}_M\mathcal{K}$. Here, $\mathcal{D}_2 \colon \mathcal{F}_{\text{cl}} \to \mathcal{F}_{\text{cl}}$ is the extension of the second symmetry, which also factors through the 0-jet. The latter discussion applies similarly to the higher translations in the tower (see Eq. (2.26)), which, for $d = 4$, terminates at $\mathcal{G}_2$. Alternatively, we can build the BV extension of this theory, following the AKSZ construction, as the latter intrinsically describes the gauge symmetries of the original theory. To do so, odd fields arise as ghost fields and antifields, as discussed in Chapter 2.

Following Section 2.3, consider the AKSZ theory $(M, \mathcal{N}, Q_{\mathcal{N}}, \omega_{\mathcal{N}}, \alpha_{\mathcal{N}}, \Theta_{\mathcal{N}})$ where the source $M \in$ SmthMfd is as above and the target is $\mathcal{N} := \mathbb{R}[1] \oplus \mathbb{R}[2]$ with coordinate functions $\psi \in C^\infty(\mathcal{N})_1 \cong \mathrm{Mor}(\mathcal{N}, \mathbb{R}^{0|1})$ and $\xi \in C^\infty(\mathcal{N})_2 \cong \mathrm{Mor}(\mathcal{N}, \mathbb{R}^{1|0})$. We can think of the target as a super vector space $\mathcal{N} \cong \mathbb{R}^{1|1} \in$ sCartSp considering $\mathbb{Z}$-grading modulo 2. As it is important to keep track of the ghost $\mathbb{Z}$-grading, we postulate (see Rmk. 3.4.1) the existence of an even vector field $\mathbb{E}_{\mathcal{N}} \in \mathfrak{X}(\mathcal{N})_0$ and an even derivation $\mathbb{D}_{\mathcal{N}} \in \mathrm{Der}_{\mathbb{R}}(\Omega^\bullet(\mathcal{N}))$ (see discussion after Def. 1.3.21) and define $\hat{\mathbb{D}}_{\mathcal{N}} := \mathbb{L}_{\mathbb{E}_{\mathcal{N}}} + \mathbb{D}_{\mathcal{N}}$ such that

$$\hat{\mathbb{D}}_{\mathcal{N}}\psi = \mathbb{L}_{\mathbb{E}_{\mathcal{N}}}\psi = 1\psi \quad \text{abd} \quad \hat{\mathbb{D}}_{\mathcal{N}}\xi = \mathbb{L}_{\mathbb{E}_{\mathcal{N}}}\xi = 2\xi.$$

The latter keeps track of the ghost $\mathbb{Z}$-grading, which coincides modulo 2 with the underlying $\mathbb{Z}_2$-structure of $\mathcal{N}$, and the cohomological degree. Since we are in the abelian case ($f^c_{ab} = 0$), the cohomological vector field and the Hamiltonian on the target are null. Finally, the symplectic form is given by

$$\omega_{\mathcal{N}} := \mathrm{d}_{\mathcal{N}}\xi \wedge \mathrm{d}_{\mathcal{N}}\psi = \mathrm{d}_{\mathcal{N}}\big(\xi\mathrm{d}_{\mathcal{N}}\psi\big) =: \mathrm{d}_{\mathcal{N}}\alpha_{\mathcal{N}} \in \Omega^2(\mathcal{N})_{d-1},$$

where $\alpha_{\mathcal{N}} \in \Omega^1(\mathcal{N})_{d-1}$. Note that $\hat{\mathbb{D}}_{\mathcal{N}}\omega_{\mathcal{N}} = 5\omega_{\mathcal{N}}$ and $\hat{\mathbb{D}}_{\mathcal{N}}\alpha_{\mathcal{N}} = 4\alpha_{\mathcal{N}}$. This AKSZ theory can be fully described within sSmthMfd and is therefore naturally encoded in sThSmthSet. We want to build the BV extension of this theory within sThSmthSet. Consider the following composite bundle over M (see Diagram (3.61))

$$F := \bigoplus_{k=0}\Big(\bigwedge^k T^*M \otimes (M \times \mathcal{N})\Big)$$

with the following local coordinates.

	x^μ	c	A_{μ_1}	$\frac{1}{2!}B^+_{\mu_1\mu_2}$	$\frac{1}{3!}\tau^+_{1\mu_1\mu_2\mu_3}$	$\frac{1}{4!}\tau^+_{2\mu_1\mu_2\mu_3\mu_4}$	τ_2	$\tau_{1\mu_1}$	$\frac{1}{2!}B_{\mu_1\mu_2}$	$\frac{1}{3!}A^+_{\mu_1\mu_2\mu_3}$	$\frac{1}{4!}c^+_{\mu_1\mu_2\mu_3\mu_4}$
gh	0	1	0	-1	-2	-3	2	1	0	-1	-2

where the ghost $\mathbb{Z}$-grading is defined as above. That is, the total grading is given by $\hat{\mathbb{D}}(x^\mu) = (0+0+0)x^\mu$, $\hat{\mathbb{D}}(c) = (1+0+0)c$, $\hat{\mathbb{D}}(A_{\mu_1}\mathrm{d}_H x^{\mu_1}) = (0+1+0)A$ and so on. The relation between $\hat{\mathbb{D}}$ and $\hat{\mathbb{D}}_{\mathcal{N}}$ is evident if one notice that the first five fiber components are form components with values in $\mathbb{R}[1]$ ($\hat{\mathbb{D}}\varphi = 1\varphi$ with $\varphi = c, A, B^+, \tau_1^+, \tau_2^+$) and the last five as form components with values in $\mathbb{R}[2]$ ($\hat{\mathbb{D}}\varphi = 2\varphi$ with $\varphi = \tau_2, \tau_1, B, A^+, c^+$).

Then, the super thickened smooth space of fields is defined as in Definition 3.3.3.

Remark 3.4.4. Note that, we are considering a $\mathbb{Z}_2$-graded vector bundle $F \to M$ (see Def. 1.3.18). That is, the supermanifold locally looking like $M \times K$. Here, K is some supervector space and M can be seen as a purely even supermanifold. Supposing $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$, we can write

$$F = \Big(\big(\bigwedge^{\bullet} T^*M \otimes (M \times \mathcal{N}_0)\big) \oplus \big(\bigwedge^{\bullet} T^*M \otimes (M \times \mathcal{N}_1)\big) \Big) \to \underbrace{\big(\bigwedge^{\bullet} T^*M \otimes (M \times \mathcal{N}_0)\big)}_{E} \to M$$

to reconduct explicitly to the composite bundle of Definition 3.3.3 (cf. [GSM09][Notation 4.1.1]). This is possible by noticing that F is naturally isomorphic to the pullback along π^E_M of the odd graded bundle $\big(\bigwedge^{\bullet} T^*M \otimes (M \times \mathcal{N}_1)\big) \to M$.

The AKSZ mapping space corresponds to the set of $*$-plots of the latter

$$\mathcal{F}(*) = \Omega^{\bullet}(M)[1] \oplus \Omega^{\bullet}(M)[2].$$

where the gradings are given by $\hat{\mathbb{D}}$. We can define the super infinite jet bundle $J^\infty_M F$ with local coordinates

$$\Big\{ x^\mu, \{c_I, A_{\mu_1|I}, \tfrac{1}{2!}B^+_{\mu_1\mu_2|I}, \tfrac{1}{3!}\tau^+_{1\mu_1\mu_2\mu_3|I}, \tfrac{1}{4!}\tau^+_{2\mu_1\mu_2\mu_3\mu_4|I}, \tau_{2I}, \tau_{1\mu_1|I}, \tfrac{1}{2!}B_{\mu_1\mu_2|I}, \tfrac{1}{3!}A^+_{\mu_1\mu_2\mu_3|I}, \tfrac{1}{4!}c^+_{\mu_1\mu_2\mu_3\mu_4|I}\}_{|I|\geqslant 0} \Big\}$$

where the grading is as above for every I. Indeed, $\partial_I \coloneqq \partial^{I_1}_{x^{\mu_1}} \cdots \partial^{I_4}_{x^{\mu_4}}$, with $I = I_1 + \cdots + I_4$, has zero degree since $\mathbb{L}_{\mathrm{prE}} x^\mu = 0$. The local BV system is given in local coordinates by the following Ansatz(81)

$$\begin{aligned}
\overset{4}{\omega} &\coloneqq \ \mathrm{d}_V A \wedge \mathrm{d}_V A^+ + \mathrm{d}_V B \wedge \mathrm{d}_V B^+ + \mathrm{d}_V c \wedge \mathrm{d}_V c^+ + \mathrm{d}_V \tau_1 \wedge \mathrm{d}_V \tau_1^+ + \mathrm{d}_V \tau_2 \wedge \mathrm{d}_V \tau_2^+, \\
\overset{4}{\alpha} &\coloneqq \ \mathrm{d}_V A \wedge A^+ + B \wedge \mathrm{d}_V B^+ + \mathrm{d}_V c \wedge c^+ + \tau_1 \wedge \mathrm{d}_V \tau_1^+ + \tau_2 \wedge \mathrm{d}_V \tau_2^+, \\
\overset{4}{\mathsf{L}} &\coloneqq B \wedge \mathrm{d}_H A + A^+ \wedge \mathrm{d}_H c + B^+ \wedge \mathrm{d}_H \tau_1 + \tau_1^+ \wedge \mathrm{d}_H \tau_2,
\end{aligned}$$

with $\hat{\mathbb{D}}\overset{4}{\omega} = (-1+4+2)\overset{4}{\omega}$, $\hat{\mathbb{D}}\overset{4}{\alpha} = (-1+4+1)\overset{4}{\alpha}$ and $\hat{\mathbb{D}}\overset{4}{\mathsf{L}} = (0+4+0)\overset{4}{\mathsf{L}}$. Finally, we locally define $\mathsf{Q} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$ by

$$\begin{array}{llll}
\mathsf{Q}(c) = 0 & \mathsf{Q}(A_{\mu_1}) = -c_{\mu_1} & \mathsf{Q}\left(\dfrac{B^+_{\mu_1\mu_2}}{2!}\right) = A_{[\mu_2|\mu_1]} & \mathsf{Q}\left(\dfrac{\tau^+_{1\mu_1\mu_2\mu_3}}{3!}\right) = -\dfrac{B^+_{[\mu_2\mu_3|\mu_1]}}{2!} \\
\mathsf{Q}(\tau_2) = 0 & \mathsf{Q}(\tau_{1\mu_1}) = \tau_{2\mu_1} & \mathsf{Q}\left(\dfrac{B_{\mu_1\mu_2}}{2!}\right) = -\tau_{1[\mu_2|\mu_1]} & \mathsf{Q}\left(\dfrac{A^+_{\mu_1\mu_2\mu_3}}{3!}\right) = \dfrac{B_{[\mu_2\mu_3|\mu_1]}}{2!}
\end{array}$$

$$\mathsf{Q}\left(\frac{\tau^+_{2\mu_1\mu_2\mu_3\mu_4}}{4!}\right) = \frac{\tau^+_{1[\mu_2\mu_3\mu_4|\mu_1]}}{3!} \qquad \mathsf{Q}\left(\frac{c^+_{\mu_1\mu_2\mu_3\mu_4}}{4!}\right) = -\frac{A^+_{[\mu_2\mu_3\mu_4|\mu_1]}}{3!}$$

Note that $\overset{4}{\omega} \in \Omega^{4,2}_{\mathrm{glb}}(J^\infty_M F)_{-1}$, $\overset{4}{\alpha} \in \Omega^{4,1}_{\mathrm{glb}}(J^\infty_M F)_{-1}$ and $\overset{4}{\mathsf{L}} \in \Omega^{4,0}_{\mathrm{glb}}(J^\infty_M F)_0$ depend globally only on the 0-jet or 1-jet. Furthermore, the components in the fibers coordinates form an invertible supermatrix and $\mathrm{d}_V \overset{4}{\alpha} = \overset{4}{\omega}$. The evolutionary vector field $\mathsf{Q} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$ only depends on the 1-jet and it can be checked that $\hat{\mathbb{D}}\mathsf{Q} = (1+0+0)\mathsf{Q}$ and $\hat{\mathbb{D}}\mathrm{pr}\mathsf{Q} = (1+0+0)\mathrm{pr}\mathsf{Q}$.(82) To prove that this is indeed a local BV system, it remains to show that $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = 0$, Equation (3.56) and Equation (3.57). First, we show $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = 0$. By modifying the local expression of $[\mathrm{pr}\mathsf{Z}_1, \mathrm{pr}\mathsf{Z}_2] = \mathrm{pr}\mathsf{Z}_3$ in Proposition 3.1.11 with the right Koszul rule, we get

$$\mathsf{Z}_3 = D_I \mathsf{Z}^a_1 \cdot \partial_{u^a_I} \mathsf{Z}^b_2 \partial_{u^b} - (-1)^{|\mathsf{Z}_1||\mathsf{Z}_2|} D_I \mathsf{Z}^a_2 \cdot \partial_{u^a_I} \mathsf{Z}^b_1 \partial_{u^b}.$$

From now on, we write $\partial_\bullet = \frac{\partial}{\partial \bullet}$ for readability. Using that $|\mathsf{Q}| = 1$ and that it only depends on the first jet, we can write

$$\mathsf{Z}_3 = 2 D_I \mathsf{Q}^a \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a_I} \frac{\partial}{\partial u^b} = 2\left[\left(\mathsf{Q}^a \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a}\right) + \left(\frac{\partial \mathsf{Q}^a}{\partial x^\mu} \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a_\mu}\right) + \left(u^\gamma_{\nu\mu} \frac{\partial \mathsf{Q}^a}{\partial u^\gamma_\nu} \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a_\mu}\right)\right] \frac{\partial}{\partial u^b}. \tag{3.62}$$

(81) We could start from an Ansatz on F and pull it back to the infinite jet bundle along $\pi^\infty_0 \colon J^\infty_M F \to F$ to get $\overset{4}{\omega}, \overset{4}{\alpha}, \overset{4}{\mathsf{L}}$ (see [Gri23]).

(82) Comparing the components of Q with the abelian ($f^c_{ab} = 0$) case of Equation (2.30), one can interpret the integration over M in terms of the action of the local vector field $Q = \mathsf{Q} \circ j^\infty$ on local coordinate functionals (see Eq. (3.33)). More concretely, local functionals with values in $\mathbb{R}$ need to combine fields so that the total ghost $\mathbb{Z}$-grading is zero. For BF theory, this means building top forms that then need to be integrated to give a real value.

Noticing that $\partial_{u^a}\mathsf{Q} = 0$, we can expand the second and third term to obtain

$$
\begin{aligned}
\mathsf{Z}_3 =&2\Bigg[\Bigg(0 - \frac{\partial \mathsf{c}_\nu}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial A_{\nu|\mu}} + \frac{\partial A_{[\mu_1|\nu]}}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial B^+_{\nu\mu_1|\mu}/2!} - \frac{\partial B^+_{[\mu_1\mu_2|\nu]}/2!}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial \tau^+_{1\nu\mu_1\mu_2|\mu}/3!} + 0 + \\
&\quad + 0 + \frac{\partial \tau_{2\nu}}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial \tau_{1\nu|\mu}} - \frac{\partial \tau_{1[\mu_1|\nu]}}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial B_{\nu\mu_1|\mu}/2!} + \frac{\partial B_{[\mu_1\mu_2|\nu]}/2!}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial A^+_{\nu\mu_1\mu_2|\mu}/3!} + 0\Bigg) + \\
&+\Bigg(c_{\nu\mu}\frac{\partial \mathsf{Q}(A_{\mu_1'})}{\partial c_\nu}\cdot\frac{\partial \mathsf{Q}^b}{\partial A_{\mu_1'|\mu}} + A_{\mu_1|\nu\mu}\frac{\partial \mathsf{Q}(B^+_{\mu_1'\mu_2'}/2!)}{\partial A_{\mu_1|\nu}}\cdot\frac{\partial \mathsf{Q}^b}{\partial B^+_{\mu_1'\mu_2'|\mu}/2!} + \\
&\quad + \frac{B^+_{\mu_1\mu_2|\nu\mu}}{2!}\frac{\partial \mathsf{Q}(\tau^+_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial B^+_{\mu_1\mu_2|\nu}/2!}\cdot\frac{\partial \mathsf{Q}^b}{\partial \tau^+_{1\mu_1'\mu_2'\mu_3'|\mu}/3!} + 0 + 0 + \tau_{2\nu\mu}\frac{\partial \mathsf{Q}(\tau_{1\mu_1'})}{\partial \tau_{2\nu}}\cdot\frac{\partial \mathsf{Q}^b}{\partial \tau_{1\mu_1'|\mu}} + \\
&\quad + \tau_{1\mu_1|\nu\mu}\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2'}/2!)}{\partial \tau_{1\mu_1|\nu}}\cdot\frac{\partial \mathsf{Q}^b}{\partial B_{\mu_1'\mu_2'|\mu}/2!} + \frac{B_{\mu_1\mu_2|\nu\mu}}{2!}\frac{\partial \mathsf{Q}(A^+_{\mu_1'\mu_2'\mu_3'}/3!)}{\partial B_{\mu_1\mu_2|\nu}/2!}\cdot\frac{\partial \mathsf{Q}^b}{\partial A^+_{\mu_1'\mu_2'\mu_3'|\mu}/3!} + 0 + 0\Bigg)\Bigg]\frac{\partial}{\partial u^b} = \\
=&2\Bigg[-2c_{\nu\mu}\frac{\partial}{\partial B^+_{\mu\nu}/2!} - 2A_{\mu_1|\nu\mu}\frac{\partial}{\partial \tau^+_{1\mu\nu\mu_1}/3!} - 2\frac{B^+_{\mu_1\mu_2|\nu\mu}}{2!}\frac{\partial}{\partial \tau^+_{2\mu\nu\mu_1\mu_2}/4!} + \\
&\qquad -2\tau_{2\nu\mu}\frac{\partial}{\partial B_{\mu\nu}/2!} - 2\tau_{1\mu_1|\nu\mu}\frac{\partial}{\partial A^+_{1\mu\nu\mu_1}/3!} - 2\frac{B_{\mu_1\mu_2|\nu\mu}}{2!}\frac{\partial}{\partial c^+_{\mu\nu\mu_1\mu_2}/4!}\Bigg] = 0.
\end{aligned}
$$

In the first equality we used the component-wise definition of Q and the fact that it does not depend on $\tau^+_{2\mu_1\mu_2\mu_3\mu_4}$ or $c^+_{\mu_1\mu_2\mu_3\mu_4}$. In the second equality, we antisymmetrized, computed the derivatives, and contracted the resulting Kronecker delta. In the last equality, we used the fact that the inner product of symmetric and antisymmetric tensors gives a null contribution.
Let us check Equation (3.56). Recall that the degree 1 differentials satisfy $\mathrm{d}_V^2 = \mathrm{d}_H^2 = \mathrm{d}_V(\mathrm{d}_H x^\mu) = 0$ and that $\mathrm{d}_V\mathrm{d}_H = -\mathrm{d}_H\mathrm{d}_V$. Furthermore, $|\iota_{\mathrm{pr}\,\mathsf{Q}}| = 1 - 1 = 0$. Then,

$$
\begin{aligned}
\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{4}{\omega} =& \underline{\mathrm{d}_H c \wedge \mathrm{d}_V A^+} + \mathrm{d}_V A \wedge \mathrm{d}_H B + \underline{\mathrm{d}_H \tau_1 \wedge \mathrm{d}_V B^+} + \underline{\mathrm{d}_V B \wedge \mathrm{d}_H A} + 0 + \mathrm{d}_V c \wedge \mathrm{d}_H A^+ + \\
&+ \underline{\mathrm{d}_H \tau_2 \wedge \mathrm{d}_V \tau_1^+} + \mathrm{d}_V \tau_1 \wedge \mathrm{d}_H B^+ + 0 + \mathrm{d}_V \tau_2 \wedge \mathrm{d}_H \tau_1^+ + \\
&+ \underline{B \wedge \mathrm{d}_V(\mathrm{d}_H A)} - B \wedge \mathrm{d}_V(\mathrm{d}_H A) + \underline{A^+ \wedge \mathrm{d}_V(\mathrm{d}_H c)} - A^+ \wedge \mathrm{d}_V(\mathrm{d}_H c) + \\
&- \underline{B^+ \wedge \mathrm{d}_V(\mathrm{d}_H \tau_1)} + B^+ \wedge \mathrm{d}_V(\mathrm{d}_H \tau_1) - \underline{\tau_1^+ \wedge \mathrm{d}_V(\mathrm{d}_H \tau_2)} + \tau_1^+ \wedge \mathrm{d}_V(\mathrm{d}_H \tau_2) = \\
=&\mathrm{d}_V\Big[\underline{B \wedge \mathrm{d}_H A} + \underline{A^+ \wedge \mathrm{d}_H c} + \underline{B^+ \wedge \mathrm{d}_H \tau_1} + \underline{\tau_1^+ \wedge \mathrm{d}_H \tau_2}\Big] + \\
&+ \mathrm{d}_H\Big[\mathrm{d}_V A \wedge B + \mathrm{d}_V c \wedge A^+ - \mathrm{d}_V \tau_1 \wedge B^+ - \mathrm{d}_V \tau_2 \wedge \tau_1^+\Big] = \mathrm{d}_V\overset{4}{\mathsf{L}} + \mathrm{d}_H\overset{3}{\chi},
\end{aligned}
$$

where $\overset{3}{\chi} \in \Omega^{3,1}_{\mathrm{glb}}(J^\infty_M F)_0$. In the first equality we computed the contraction using that $\mathrm{d}_V u^b(\partial_{u^a}) = \delta^b_a$ (first two lines) and summed and subtracted the extra terms (second two lines). The second equality can be checked backward by explicitly computing it. Let us now check the Equation (3.57).

$$
\begin{aligned}
\frac{1}{2}\iota_{\mathrm{pr}\,\mathsf{Q}}\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{4}{\omega} =& \frac{1}{2}\iota_{\mathrm{pr}\,\mathsf{Q}}\Big[\mathrm{d}_H c \wedge \mathrm{d}_V A^+ + \mathrm{d}_V A \wedge \mathrm{d}_H B + \mathrm{d}_H \tau_1 \wedge \mathrm{d}_V B^+ + \mathrm{d}_V B \wedge \mathrm{d}_H A + 0 + \\
&+ \mathrm{d}_V c \wedge \mathrm{d}_H A^+ + \mathrm{d}_H \tau_2 \wedge \mathrm{d}_V \tau_1^+ + \mathrm{d}_V \tau_1 \wedge \mathrm{d}_H B^+ + 0 + \mathrm{d}_V \tau_2 \wedge \mathrm{d}_H \tau_1^+\Big] = \\
=&\Big[\mathrm{d}_H c \wedge \mathrm{d}_H B + \mathrm{d}_H \tau_1 \wedge \mathrm{d}_H A + 0 + \mathrm{d}_H \tau_2 \wedge \mathrm{d}_H B^+\Big] = \mathrm{d}_H\overset{3}{\mathsf{L}},
\end{aligned}
$$

for some $\overset{3}{\mathsf{L}} \in \Omega^{3,0}_{\mathrm{glb}}(J^\infty_M F)_1$.

Remark 3.4.5. This is compatible with a full lax presentation (cf. [Gri23; SS25; MSW19]). The first step consists of checking that $\mathrm{d}_V\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{d}{\omega} = \mathrm{d}_H\overset{d-1}{\omega}$ and that $\mathrm{d}_V\overset{d-1}{\chi} = -\overset{d-1}{\omega}$, with $\overset{d-1}{\omega} \in \Omega^{d-1,2}_{\mathrm{glb}}(J^\infty_M F)_0$. Then the above arguments can be iterated, potentially only locally, to check that $\iota_{\mathrm{pr}\mathsf{Q}}\overset{d-1}{\omega} = \mathrm{d}_V\overset{d-1}{\mathsf{L}} + \mathrm{d}_H\overset{d-2}{\chi}$ for some $\overset{d-2}{\chi} \in \Omega^{d-2,1}_{\mathrm{glb}}(J^\infty_M F)_1$ and so on, finally arriving to the full lax presentation. This is the structure one should consider to treat the case with boundaries, corners, and higher co-dimensions structures [MSW19].

In our case, it is straightforward to see that $\mathrm{d}_V \overset{3}{\chi} = -\overset{3}{\omega}$ for $\overset{3}{\omega}$ satisfying

$$\begin{aligned}
\mathrm{d}_V \iota_{\mathrm{pr}\,\mathsf{Q}} \overset{4}{\omega} =& \mathrm{d}_V \Big[\mathrm{d}_H c \wedge \mathrm{d}_V A^+ + \mathrm{d}_V A \wedge \mathrm{d}_H B + \mathrm{d}_H \tau_1 \wedge \mathrm{d}_V B^+ + \mathrm{d}_V B \wedge \mathrm{d}_H A + 0 + \\
& + \mathrm{d}_V c \wedge \mathrm{d}_H A^+ + \mathrm{d}_H \tau_2 \wedge \mathrm{d}_V \tau_1^+ + \mathrm{d}_V \tau_1 \wedge \mathrm{d}_H B^+ + 0 + \mathrm{d}_V \tau_2 \wedge \mathrm{d}_H \tau_1^+ \Big] = \\
=& - \mathrm{d}_H(\mathrm{d}_V c) \wedge \mathrm{d}_V A^+ - \mathrm{d}_V A \wedge \mathrm{d}_H(\mathrm{d}_V B) - \mathrm{d}_H(\mathrm{d}_V \tau_1) \wedge \mathrm{d}_V B^+ + \mathrm{d}_V B \wedge \mathrm{d}_H(\mathrm{d}_V A) + \\
& - \mathrm{d}_V c \wedge \mathrm{d}_H(\mathrm{d}_V A^+) - \mathrm{d}_H(\mathrm{d}_V \tau_2) \wedge \mathrm{d}_V \tau_1^+ + \mathrm{d}_V \tau_1 \wedge \mathrm{d}_H(\mathrm{d}_V B^+) + \mathrm{d}_V \tau_2 \wedge \mathrm{d}_H(\mathrm{d}_V \tau_1^+) = \\
=& \mathrm{d}_H \Big[- \mathrm{d}_V c \wedge \mathrm{d}_V A^+ - \mathrm{d}_V A \wedge \mathrm{d}_V B - \mathrm{d}_V \tau_1 \wedge \mathrm{d}_V B^+ - \mathrm{d}_V \tau_2 \wedge \mathrm{d}_V \tau_1^+ \Big] = \mathrm{d}_H \overset{3}{\omega}.
\end{aligned}$$

This follows from direct computations, $|\mathrm{d}_V| = 1$, and $\mathrm{d}_V \mathrm{d}_H = -\mathrm{d}_H \mathrm{d}_V$.

Finally, transgressing this structure on the infinite jet bundle to the space of fields via the map

$$\mathbb{T} = \int_M \circ (ev^\infty)^* \colon \Omega^{4,\bullet}_{\mathrm{glb}}(J^\infty_M F) \longrightarrow \Omega^\bullet_{\mathrm{glb}}(\mathcal{F}),$$

we obtain the local BV structure on $\mathcal{F}$ (see Eq. (3.58)). Note that the transgression changes the total degree by -4, as the top-horizontal component, which contributes with a degree of 4 at the level of $J^\infty_M F$, is sent to a real number of degree 0. Explicitly, $\mathfrak{X}_{\mathrm{loc}}(\mathcal{F})_1 \ni Q = \mathsf{Q} \circ j^\infty$ and[83] L

$$\Omega \coloneqq \mathbb{T}\overset{n}{\omega} = \int_M \mathrm{d}_{\mathcal{F}} A \wedge \mathrm{d}_{\mathcal{F}} A^+ + \mathrm{d}_{\mathcal{F}} B \wedge \mathrm{d}_{\mathcal{F}} B^+ + \mathrm{d}_{\mathcal{F}} c \wedge \mathrm{d}_{\mathcal{F}} c^+ + \mathrm{d}_{\mathcal{F}} \tau_1 \wedge \mathrm{d}_{\mathcal{F}} \tau_1^+ + \mathrm{d}_{\mathcal{F}} \tau_2 \wedge \mathrm{d}_{\mathcal{F}} \tau_2^+ \in \Omega^2_{\mathrm{loc}}(\mathcal{F})_{-1}$$

$$S \coloneqq \mathbb{T}\overset{n}{\mathsf{L}} = \int_M B \wedge \mathrm{d}_M A + A^+ \wedge \mathrm{d}_M c + B^+ \wedge \mathrm{d}_M \tau_1 + \tau_1^+ \wedge \mathrm{d}_M \tau_2 \in C^\infty_{\mathrm{loc}}(\mathcal{F})_0$$

The form is closed because it is exact, with primitive

$$\alpha \coloneqq \mathbb{T}\overset{n}{\alpha} = \int_M \mathrm{d}_{\mathcal{F}} A \wedge A^+ + B \wedge \mathrm{d}_{\mathcal{F}} B^+ + \mathrm{d}_{\mathcal{F}} c \wedge c^+ + \tau_1 \wedge \mathrm{d}_{\mathcal{F}} \tau_1^+ + \tau_2 \wedge \mathrm{d}_{\mathcal{F}} \tau_2^+ \in \Omega^1_{\mathrm{loc}}(\mathcal{F})_{-1}.$$

Conditions (3.57) and (3.56) are satisfied by construction as above. That is

$$\delta S = \iota_Q \Omega \quad \text{and} \quad \{S, S\}_\Omega = 0.$$

The latter is the rigorous understanding of the formal manipulations we did for the (abelian) BF theory in Section 2.3 (see Rmk. 2.3.1 and Eqs. (2.28), (2.29) and (2.30)). We now discuss the non-abelian case.

Non-abelian BF Theory in $d = 4$

As for the abelian case, the classical non-abelian BF theory in $d = 4$ (see Sec. 2.3) can be understood within SmthSet. Let M be a closed oriented smooth 4-manifold and G a finite-dimensional compact Lie group with Lie algebra $\mathfrak{g}$ (with basis $\{T_a\}$). The space of fields of the bundle $F_{\mathrm{cl}} \coloneqq (T^*M \otimes (M \times \mathfrak{g})) \oplus (\bigwedge^2 T^*M \otimes (M \times \mathfrak{g}^*)) \ni \{A^a_{\mu_1}, B^a_{\mu_1\mu_2}/2!\}$ is the smooth set (see Def. 3.1.4)

$$\mathcal{F}_{\mathrm{cl}} \coloneqq \mathbf{\Gamma}_M \Big((T^*M \otimes (M \times \mathfrak{g})) \oplus \Big(\bigwedge^2 T^*M \otimes (M \times \mathfrak{g}^*) \Big) \Big),$$

with the original fields $(A, B) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^2(M, \mathfrak{g}^*) \cong \mathcal{F}_{\mathrm{cl}}(*)$. Considering the infinite jet bundle $J^\infty_M F_{\mathrm{cl}}$ with local coordinates $\{x^\mu, \{A^a_{\mu_1|I}, B_{\mu_1\mu_2|I,a}/2!\}_{|I|\geqslant 0}\}$, we can write the classical Lagrangian as $\mathcal{L}_{\mathrm{cl}} = \mathsf{L}_{\mathrm{cl}} \circ j^\infty$ (see Def. 3.1.21). Here L_{cl} is the $(d, 0)$-form on $J^\infty_M F_{\mathrm{cl}}$ locally given by

$$\begin{aligned}
\mathsf{L}\Big(x^\mu, \{A^a_{\mu_1|I}, \tfrac{B_{\mu_1\mu_2,a|I}}{2!}\}_{|I|\leqslant 1}\Big) =& \Big(\frac{B_{\mu_1\mu_2,a}}{2!} \mathrm{d}_H x^{\mu_1} \wedge \mathrm{d}_H x^{\mu_2} \Big) \wedge \Big(\mathrm{d}_H x^{\mu_3} \wedge \partial_{[\mu_3} A^a_{\mu_4]} \wedge \mathrm{d}_H x^{\mu_4} + \\
& + \frac{1}{2} f^a_{bc} A^b_{\mu_3} \wedge \mathrm{d}_H x^{\mu_3} \wedge A^c_{\mu_4} \wedge \mathrm{d}_H x^{\mu_4} \Big) = B_a \wedge \Big(\mathrm{d}_H A^a + \frac{1}{2} [A \,\overset{\wedge}{,}\, A]^a \Big) = \langle B \,\overset{\wedge}{,}\, F_A \rangle.
\end{aligned}$$

Notice that it globally factors through the 1-jet. Then, working within SmthSet, one can recover the BF action as a natural transformation of smooth sets (see Eq. (3.29)) and describe the smooth structure

[83] Pulling back along the infinite evaluation locally means substituting $\mathrm{d}_{V/H}$ with $\mathrm{d}_{\mathcal{F}/M}$.

of the Euler–Lagrange locus (cf. [GS25a][Ch. 5]). Also, the gauge symmetries (see Eq. (2.25)) can be understood via point 1. of Definition 3.1.23, as discussed for the abelian case. We can build the BV extension of this theory, following the AKSZ construction, as the latter intrinsically describes the gauge symmetries of the original theory. To do so, odd fields arise as ghost fields and antifields, as discussed in Chapter 2.

Following (see Sec. 2.3), consider the AKSZ theory $(M, \mathcal{N}, Q_{\mathcal{N}}, \omega_{\mathcal{N}}, \alpha_{\mathcal{N}}, \Theta_{\mathcal{N}})$ where the source $M \in$ SmthMfd is as above and the target is $\mathcal{N} := \mathfrak{g}[1] \oplus \mathfrak{g}^*[2]$ with coordinate functions $\psi \in C^\infty(\mathcal{N}, \mathfrak{g})_1$ and $\xi \in C^\infty(\mathcal{N}, \mathfrak{g}^*)_2$. As it is important to keep track of the ghost $\mathbb{Z}$-grading, we postulate (see Rmk. 3.4.1) the existence of an even vector field $\mathbb{E}_{\mathcal{N}} \in \mathfrak{X}(\mathcal{N})_0$ and an even derivation $\mathbb{D}_{\mathcal{N}} \in \mathrm{Der}_{\mathbb{R}}(\Omega^\bullet(\mathcal{N}))$ and define $\hat{\mathbb{D}}_{\mathcal{N}} := \mathbb{L}_{\mathbb{E}_{\mathcal{N}}} + \mathbb{D}_{\mathcal{N}}$ such that

$$\hat{\mathbb{D}}_{\mathcal{N}}\psi = \mathbb{L}_{\mathbb{E}_{\mathcal{N}}}\psi = 1\psi \quad \text{abd} \quad \hat{\mathbb{D}}_{\mathcal{N}}\xi = \mathbb{L}_{\mathbb{E}_{\mathcal{N}}}\xi = 2\xi.$$

The latter keeps track of the ghost $\mathbb{Z}$-grading, which coincides modulo 2 with the underlying $\mathbb{Z}_2$-structure of $\mathcal{N}$, and the cohomological degree. Finally, the symplectic form, the cohomological vector field, and the Hamiltonian are given by (see Sec. 2.3)

$$\begin{aligned}
\omega_{\mathcal{N}} &:= \langle \mathrm{d}_{\mathcal{N}}\xi \mathbin{\overset{\wedge}{,}} \mathrm{d}_{\mathcal{N}}\psi \rangle = \sum_a \mathrm{d}_{\mathcal{N}}\xi_a \wedge \mathrm{d}_{\mathcal{N}}\psi^a = \mathrm{d}_{\mathcal{N}} \sum_a \xi_a \mathrm{d}_{\mathcal{N}}\psi^a = \mathrm{d}_{\mathcal{N}} \underbrace{\langle \xi, \mathrm{d}_{\mathcal{N}}\psi \rangle}_{\alpha_{\mathcal{N}} \in \Omega^1(\mathcal{N})_3} \in \Omega^2(\mathcal{N})_3; \\
Q_{\mathcal{N}} &:= \frac{1}{2}\langle [\psi, \psi], \partial_\psi \rangle + \langle \mathrm{ad}^*_\psi(\xi), \partial_\xi \rangle = \sum_{a,b,c} \frac{1}{2} f^a_{bc}\psi^b\psi^c \partial_{\psi^a} - \sum_{a,b,c} f^a_{b,c}\psi^b \xi_a \partial_{\xi_c} \in \mathfrak{X}(\mathcal{N})_1 \\
\Theta_{\mathcal{N}} &:= \{Q_{\mathcal{N}}, -\} = \frac{1}{2}\langle \xi, [\psi, \psi] \rangle = \sum_{a,b,c} \frac{1}{2} f^a_{bc}\xi_a\psi^b\psi^c \in C^\infty(\mathcal{N})_4.
\end{aligned}$$

Note that $\hat{\mathbb{D}}_{\mathcal{N}}\omega_{\mathcal{N}} = 5\omega_{\mathcal{N}}$, $\hat{\mathbb{D}}_{\mathcal{N}}\alpha_{\mathcal{N}} = 4\alpha_{\mathcal{N}}$, $\hat{\mathbb{D}}_{\mathcal{N}}Q_{\mathcal{N}} = 1Q_{\mathcal{N}}$ and $\hat{\mathbb{D}}_{\mathcal{N}}\Theta_{\mathcal{N}} = 4\Theta_{\mathcal{N}}$. This AKSZ theory can be fully described within sSmthMfd and is therefore naturally encoded in sThSmthSet. We want to build the BV extension of this theory within sThSmthSet. Consider the following composite (see Rmk. 3.4.4) bundle over M (see Diagram (3.61))

$$F := \bigoplus_{k=0} \Big(\bigwedge^k T^*M \otimes (M \times \mathcal{N}) \Big)$$

with the following local coordinates.

	x^μ	c^a	$A^a_{\mu_1}$	$\frac{1}{2!}B^{+a}_{\mu_1\mu_2}$	$\frac{1}{3!}\tau^{+a}_{1\mu_1\mu_2\mu_3}$	$\frac{1}{4!}\tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}$	τ_{2a}	$\tau_{1\mu_1,a}$	$\frac{1}{2!}B_{\mu_1\mu_2,a}$	$\frac{1}{3!}A^+_{\mu_1\mu_2\mu_3,a}$	$\frac{1}{4!}c^+_{\mu_1\mu_2\mu_3\mu_4,a}$
gh	0	1	0	-1	-2	-3	2	1	0	-1	-2

where the ghost $\mathbb{Z}$-grading is defined as above. That is, the total grading is given by $\hat{\mathbb{D}}(x^\mu) = (0+0+0)x^\mu$, $\hat{\mathbb{D}}(c^a) = (1+0+0)c^a$, $\hat{\mathbb{D}}(A^a_{\mu_1}\mathrm{d}_H x^{\mu_1}) = (0+1+0)A^a$ and so on. The relation between $\hat{\mathbb{D}}$ and $\hat{\mathbb{D}}_{\mathcal{N}}$ is evident if one notice that the first five fiber components are form components with values in $\mathfrak{g}[1]$ ($\hat{\mathbb{D}}\varphi^a = 1\varphi^a$ with $\varphi^a = c^a, A^a, B^{+a}, \tau_1^{+a}, \tau_2^{+a}$) and the last five as form components with values in $\mathfrak{g}^*[2]$ ($\hat{\mathbb{D}}\bar{\varphi}_a = 2\bar{\varphi}_a$ with $\bar{\varphi}_a = \tau_{2a}, \tau_{1,a}, B_a, A^+_a, c^+_a$).

Then, the super thickened smooth space of fields is defined as in Definition 3.3.3. The AKSZ mapping space corresponds to the set of $*$-plots of $\mathcal{F}(*) = \Omega^\bullet(M, \mathfrak{g})[1] \oplus \Omega^\bullet(M, \mathfrak{g}^*)[2]$, where the gradings are given by $\hat{\mathbb{D}}$. We can define the super infinite jet bundle $J^\infty_M F$ with local coordinates

$$\Big\{ x^\mu, \{c^a_I, A^a_{\mu_1|I}, \frac{1}{2!}B^{+a}_{\mu_1\mu_2|I}, \frac{1}{3!}\tau^{+a}_{1\mu_1\mu_2\mu_3|I}, \frac{1}{4!}\tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4|I}, \tau_{2I,a}, \tau_{1\mu_1|I,a}, \frac{1}{2!}B_{\mu_1\mu_2|I,a}, \frac{1}{3!}A^+_{\mu_1\mu_2\mu_3|I,a}, \frac{1}{4!}c^+_{\mu_1\mu_2\mu_3\mu_4|I,a}\} \Big\}$$

where the grading is as above for every I (see analogous discussion in the abelian case). The local BV system is given in local coordinates by the following Ansatz[84]

$$\begin{aligned}
\overset{4}{\omega} &:= \langle \mathrm{d}_V A \mathbin{\overset{\wedge}{,}} \mathrm{d}_V A^+ \rangle + \langle \mathrm{d}_V B \mathbin{\overset{\wedge}{,}} \mathrm{d}_V B^+ \rangle + \langle \mathrm{d}_V c \mathbin{\overset{\wedge}{,}} \mathrm{d}_V c^+ \rangle + \langle \mathrm{d}_V \tau_1 \mathbin{\overset{\wedge}{,}} \mathrm{d}_V \tau_1^+ \rangle + \langle \mathrm{d}_V \tau_2 \mathbin{\overset{\wedge}{,}} \mathrm{d}_V \tau_2^+ \rangle; \\
\overset{4}{\alpha} &:= \langle \mathrm{d}_V A \mathbin{\overset{\wedge}{,}} A^+ \rangle + \langle B \mathbin{\overset{\wedge}{,}} \mathrm{d}_V B^+ \rangle + \langle \mathrm{d}_V c \mathbin{\overset{\wedge}{,}} c^+ \rangle + \langle \tau_1 \mathbin{\overset{\wedge}{,}} \mathrm{d}_V \tau_1^+ \rangle + \langle \tau_2 \mathbin{\overset{\wedge}{,}} \mathrm{d}_V \tau_2^+ \rangle;
\end{aligned}$$

(84) We could start from an Ansatz on F and pull it back to the infinite jet bundle along $\pi^\infty_0 \colon J^\infty_M F \to F$ to get $\overset{4}{\omega}, \overset{4}{\alpha}, \overset{4}{\mathsf{L}}$ (see [Gri23]).

$$\begin{aligned}\overset{4}{\mathsf{L}} \coloneqq &\langle B \,\hat{,}\, F_A\rangle + \langle A^+ \,\hat{,}\, \mathrm{d}_A c\rangle + \langle B^+ \,\hat{,}\, \mathrm{ad}^*_c(B) + \mathrm{d}_A \tau_1\rangle + \langle \tau_1^+ \,\hat{,}\, \mathrm{ad}^*_c(\tau_1) + \mathrm{d}_A\tau_2\rangle + \frac{1}{2}\langle c^+ \,\hat{,}\, [c \,\hat{,}\, c]\rangle + \\ &+ \langle \tau_2^+ \,\hat{,}\, \mathrm{ad}^*_c(\tau_2)\rangle + \frac{1}{2}\langle \tau_2 \,\hat{,}\, [B^+ \,\hat{,}\, B^+]\rangle.\end{aligned}$$

Using the notation introduced above, recall $\langle \varphi \,\hat{,}\, \bar{\varphi}\rangle = \varphi^a \wedge \varphi_a$ and $\langle \bar{\varphi} \,\hat{,}\, \varphi\rangle = \bar{\varphi}_a \wedge \varphi^a$, $\mathrm{d}_A\varphi = \mathrm{d}_H\varphi + \mathrm{ad}_A\varphi$ and $\mathrm{d}_A\bar{\varphi} = \mathrm{d}_H\bar{\varphi} + \mathrm{ad}^*_A\bar{\varphi}$, $\mathrm{ad}_\varphi(\varphi') = [\varphi \,\hat{,}\, \varphi'] = (f^c_{ab}\varphi^a \wedge \varphi'^b)T_c$ and $\mathrm{ad}_\varphi(\bar{\varphi}) = (-f^c_{ab}\varphi^a \wedge \bar{\varphi}_c)T^b$, with $\{T^a\}$ a basis of $\mathfrak{g}^*$. Note that, $\hat{\mathbb{D}}\overset{4}{\omega} = (-1+4+2)\overset{4}{\omega}$, $\hat{\mathbb{D}}\overset{4}{\alpha} = (-1+4+1)\overset{4}{\alpha}$ and $\hat{\mathbb{D}}\overset{4}{\mathsf{L}} = (0+4+0)\overset{4}{\mathsf{L}}$.
Finally, we locally define $\mathsf{Q} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$ by

$$\begin{aligned}
\mathsf{Q}(c^a) =&0 + \frac{1}{2} f^a_{bc} c^b c^c\\
\mathsf{Q}(A^a_{\mu_1}) =& - c^a_{\mu_1} - f^a_{bc} A^b_{\mu_1} c^c\\
\mathsf{Q}\left(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\right) =& A^a_{[\mu_2|\mu_1]} + \frac{1}{2} f^a_{bc} A^b_{\mu_1} A^c_{\mu_2} + f^a_{bc} c^b \frac{B^{+c}_{\mu_1\mu_2}}{2!}\\
\mathsf{Q}\left(\frac{\tau^{+a}_{1\mu_1\mu_2\mu_3}}{3!}\right) =& -\frac{B^{+a}_{[\mu_2\mu_3|\mu_1]}}{2!} - f^a_{bc} A^b_{\mu_1} \frac{B^{+c}_{\mu_2\mu_3}}{2!} + f^a_{bc} c^b \frac{\tau^{+c}_{1\mu_1\mu_2\mu_3}}{3!}\\
\mathsf{Q}\left(\frac{\tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}}{4!}\right) =& \frac{\tau^{+a}_{1[\mu_2\mu_3\mu_4|\mu_1]}}{3!} + f^a_{bc} A^b_{\mu_1} \frac{\tau^{+c}_{1\mu_2\mu_3\mu_4}}{3!} + f^a_{bc} c^b \frac{\tau^{+c}_{2\mu_1\mu_2\mu_3\mu_4}}{4!} + \frac{1}{2} f^a_{bc} \frac{B^{+b}_{\mu_1\mu_2}}{2!}\frac{B^{+c}_{\mu_3\mu_4}}{2!}\\
\mathsf{Q}(\tau_{2a}) =& 0 - f^c_{ba} c^b \tau_{2c}\\
\mathsf{Q}(\tau_{1\mu_1,a}) =& \tau_{2\mu_1,a} - f^c_{ba} c^b \tau_{1\mu_1,c} - f^c_{ba} A^b_{\mu_1}\tau_{2c}\\
\mathsf{Q}\left(\frac{B_{\mu_1\mu_2,a}}{2!}\right) =& -\tau_{1[\mu_2|\mu_1],a} - f^c_{ba} c^b \frac{B_{\mu_1\mu_2,c}}{2!} + f^c_{ba} A^b_{\mu_1}\tau_{1\mu_2,c} - f^c_{ba}\frac{B^{+b}_{\mu_1\mu_2}}{2!}\tau_{2c}\\
\mathsf{Q}\left(\frac{A^+_{\mu_1\mu_2\mu_3,a}}{3!}\right) =& \frac{B_{1[\mu_2\mu_3|\mu_1],a}}{2!} - f^c_{ba} c^b \frac{A^+_{\mu_1\mu_2\mu_3,c}}{3!} - f^c_{ba} A^b_{\mu_1}\frac{B_{\mu_2\mu_3,c}}{2!} - f^c_{ba}\frac{B^{+b}_{\mu_1\mu_2}}{2!}\tau_{1\mu_3,c} - f^c_{ba}\frac{\tau^{+b}_{1\mu_1\mu_2\mu_3}}{3!}\tau_{2c} +\\
\mathsf{Q}\left(\frac{c^+_{\mu_1\mu_2\mu_3\mu_4,a}}{4!}\right) =& -\frac{A^+_{[\mu_2\mu_3\mu_4|\mu_1],a}}{3!} - f^c_{ba} c^b \frac{c^+_{\mu_1\mu_2\mu_3\mu_4,c}}{4!} + f^c_{ba} A^b_{\mu_1}\frac{A^+_{\mu_2\mu_3\mu_4,c}}{3!} - f^c_{ba}\frac{B^{+b}_{\mu_1\mu_2}}{2!}\frac{B_{\mu_3\mu_4,c}}{2!} +\\
&+ f^c_{ba}\frac{\tau^{+b}_{1\mu_1\mu_2\mu_3}}{3!}\tau_{1\mu_4,c} - f^c_{ba}\frac{\tau^{+b}_{2\mu_1\mu_2\mu_3\mu_4}}{4!}\tau_{2c}.
\end{aligned}$$

Note that $\overset{4}{\omega} \in \Omega^{4,2}_{\mathrm{glb}}(J^\infty_M F)_{-1}$, $\overset{4}{\alpha} \in \Omega^{4,1}_{\mathrm{glb}}(J^\infty_M F)_{-1}$ and $\overset{4}{\mathsf{L}} \in \Omega^{4,0}_{\mathrm{glb}}(J^\infty_M F)_0$ and they depend globally only on the 0-jet or 1-jet. Furthermore, the components in the fibers coordinates form an invertible supermatrix and $\mathrm{d}_V\overset{4}{\alpha} = \overset{4}{\omega}$. The vector field $\mathsf{Q} \in \mathfrak{X}_{\mathrm{ev}}(J^\infty_M F)$ only depends on the 1-jet and it is such that $\hat{\mathbb{D}}\mathsf{Q} = (1+0+0)\mathsf{Q}$ and $\hat{\mathbb{D}}\mathrm{pr}\mathsf{Q} = (1+0+0)\mathrm{pr}\mathsf{Q}$.[(85)] It remains to show that $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = 0$, Equation (3.56) and Equation (3.57). First, we show $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}] = \mathrm{pr}\mathsf{Z}_3 = 0$. As for the abelian case (see Eq. 3.62), we can write

$$\mathsf{Z}_3 = 2D_I\mathsf{Q}^a \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a_I}\frac{\partial}{\partial u^b} = 2\left[\left(\mathsf{Q}^a \cdot \frac{\partial \mathsf{Q}^b}{\partial u^a}\right) + \left(\frac{\partial \mathsf{Q}^a}{\partial x^\mu}\cdot\frac{\partial \mathsf{Q}^b}{\partial u^a_\mu}\right) + \left(u^\gamma_{\nu\mu}\frac{\partial \mathsf{Q}^a}{\partial u^\gamma_\nu}\cdot\frac{\partial \mathsf{Q}^b}{\partial u^a_\mu}\right)\right]\frac{\partial}{\partial u^b}. \tag{3.63}$$

The third component goes away with a calculation similar to the abelian case. We now expand the first and second contributions componentwise. The more relevant manipulations are done using Jacobi identity (Jac.) $f^e_{ba}f^a_{cd} + f^e_{ca}f^a_{db} + f^e_{da}f^a_{bc} = 0$ and antisymmetrization with respect to some indices, for example (ant. (a,b,c)). We also use the redefinition of summed indices, for example $(a \leftrightarrow b)$, and $f^c_{ab} = -f^c_{ba}$. In general, we use the total degree $|\cdot|$ $(|\partial_\bullet| = -|\bullet|)$. Finally, all the terms involving a second-order partial derivative (from the second term in Eq. (3.63)) vanish independently as in the abelian case (Sym − Ant). The components coming from $\mathrm{d}^{\mathrm{lift}}_M$ are already antisymmetrized.

$$\boxed{c^{a'}} \to \mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(c^{a'})}{\partial c^a}\frac{\partial}{\partial c^{a'}} = \frac{1}{2}f^{a'}_{ac'}f^a_{bc}c^b c^c c^{c'}\frac{\partial}{\partial c^{a'}} \overset{\substack{\text{ant.}\\ b,c,c'}}{=} -\frac{1}{6}(f^{a'}_{c'a}f^a_{bc} + f^{a'}_{ba}f^a_{cc'} + f^{a'}_{ca}f^a_{c'b})c^b c^c c^{c'}\frac{\partial}{\partial c^{a'}} \overset{\text{Jac.}}{=} 0$$

$$\begin{aligned}\boxed{A^{a'}_{\mu'_1}} \to &\left(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(A^{a'}_{\mu'_1})}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(A^{a'}_{\mu'_1})}{\partial A^a_{\mu_1}} + \frac{\partial \mathsf{Q}(c^a)}{\partial x^\mu}\frac{\partial \mathsf{Q}(A^{a'}_{\mu'_1})}{\partial c^a_\mu}\right)\frac{\partial}{\partial A^{a'}_{\mu'_1}} = \Bigg(\underbrace{-\frac{1}{2}f^{a'}_{b'a}f^a_{bc}c^b c^c A^{b'}_{\mu'_1}}_{\text{Jac.+ant. }(b,c)} +\\ &+ f^{a'}_{ac'}c^a_{\mu_1}c^{c'} + f^{a'}_{ac'}f^a_{bc}c^c c^{c'}A^b_{\mu'_1} - f^{a'}_{bc}c^b_{\mu'_1}c^c\Bigg)\frac{\partial}{\partial A^{a'}_{\mu'_1}} = 0\end{aligned}$$

[(85)]Comparing the components of Q with the Equation (2.30), one can interpret the integration over M in terms of the action of the local vector field $Q = \mathsf{Q}\circ j^\infty$ on local coordinate functionals (see Eq. (3.33)). More concretely, local functionals with values in $\mathbb{R}$ need to combine fields so that the total ghost $\mathbb{Z}$-grading is zero. For BF theory, this means building top forms that then need to be integrated to give a real value.

$$\boxed{\frac{B^{+a'}_{\mu_1'\mu_2'}}{2!}} \to \Big(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(B^{+a'}_{\mu_1'\mu_2'}/2!)}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(B^{+a'}_{\mu_1'\mu_2'}/2!)}{\partial A^a_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(B^{+a'}_{\mu_1'\mu_2'}/2!)}{\partial B^{+a}_{\mu_1\mu_2}/2!} +$$

$$+ \frac{\partial \mathsf{Q}(A^a_{\mu_1})}{\partial x^\mu}\frac{\partial \mathsf{Q}(B^{+a'}_{\mu_1'\mu_2'}/2!)}{\partial A^a_{\mu_1|\mu}}\Big)\frac{\partial}{\partial B^{+a'}_{\mu_1'\mu_2'}/2!} = \Big(\underbrace{-\frac{1}{2} f^{a'}_{c'a} f^a_{bc} c^b c^c \frac{B^{+c'}_{\mu_1'\mu_2'}}{2!}}_{\text{Jac.+ant. }(b,c)} - f^{a'}_{ac'} c^a_{\mu_1'} A^{c'}_{\mu_2'} +$$

$$+ f^{a'}_{c'a} f^a_{bc} c^c A^b_{\mu_1'} A^{c'}_{\mu_2'} - f^{a'}_{b'a} A^a_{\mu_2'|\mu_1'} c^{b'} - \underbrace{\frac{1}{2} f^{a'}_{b'a} f^a_{bc} c^{b'} A^b_{\mu_1'} A^c_{\mu_2'}}_{\text{Jac.+ant. }(b,c)} + f^{a'}_{b'a} f^a_{bc} c^b c^{b'} \frac{B^{+c}_{\mu_1'\mu_2'}}{2!} - \underbrace{c^{a'}_{\mu_2'\mu_1'}}_{\text{Sym}-\text{Ant}} +$$

$$- f^{a'}_{bc} A^b_{\mu_2'|\mu_1'} c^c - f^{a'}_{bc} A^b_{\mu_2'} c^c_{\mu_1'} \Big)\frac{\partial}{\partial B^{+a'}_{\mu_1'\mu_2'}/2!} = 0$$

$$\boxed{\frac{\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}}{3!}} \to \Big(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial A^a_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial B^{+a}_{\mu_1\mu_2}/2!} +$$

$$+ \mathsf{Q}\Big(\frac{\tau^{+a}_{1\mu_1\mu_2\mu_3}}{3!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial \tau^{+a}_{1\mu_1\mu_2\mu_3}/3!} + \frac{\partial \mathsf{Q}(B^{+a}_{\mu_1\mu_2}/2!)}{\partial x^\mu}\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial B^{+a}_{\mu_1\mu_2|\mu}/2!}\Big)\frac{\partial}{\partial \tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!} =$$

$$= \Big(\underbrace{-\frac{1}{2} f^{a'}_{c'a} f^a_{bc} c^b c^c \frac{\tau^{+c'}_{1\mu_1'\mu_2'\mu_3'}}{3!}}_{\text{Jac.+ant. }(b,c)} + f^{a'}_{ac'} c^a_{\mu_1'} \frac{B^{+c'}_{\mu_2'\mu_3'}}{2!} - f^{a'}_{c'a} f^a_{bc} A^b_{\mu_1'} c^c \frac{B^{+c'}_{\mu_2'\mu_3'}}{2!} - \underbrace{f^{a'}_{b'a} A^a_{\mu_3'|\mu_2'} A^{b'}_{\mu_1'}}_{\mu_2'\leftrightarrow\mu_3'\leftrightarrow\mu_1'} +$$

$$- \underbrace{\frac{1}{2} f^{a'}_{b'a} f^a_{bc} A^b_{\mu_2'} A^c_{\mu_3'} A^{b'}_{\mu_1'}}_{\text{ant. }(b,c,b')+\text{Jac.}} - f^{a'}_{b'a} f^a_{bc} A^{b'}_{\mu_1'} c^b \frac{B^{+c}_{\mu_2'\mu_3'}}{2!} - f^{a'}_{b'a} \frac{B^{+a}_{\mu_2'\mu_3'|\mu_1'}}{2!} c^{b'} + f^{a'}_{b'a} f^a_{bc} A^b_{\mu_1'} c^{b'} \frac{B^{+c}_{\mu_2'\mu_3'}}{2!} +$$

$$+ f^{a'}_{b'a} f^a_{bc} c^b c^{b'} \frac{\tau^{+c}_{1\mu_1'\mu_2'\mu_3'}}{3!} - \underbrace{A^{a'}_{\mu_3'|\mu_2'\mu_1'}}_{\text{Sym}-\text{Ant}} - f^{a'}_{bc} A^b_{\mu_2'|\mu_1'} A^c_{\mu_3'} - f^{a'}_{bc} c^b_{\mu_1'} \frac{B^{+c}_{\mu_2'\mu_3'}}{2!} - f^{a'}_{bc} c^b \frac{B^{+c}_{\mu_2'\mu_3'|\mu_1'}}{2!} \Big)\frac{\partial}{\partial \tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!} \overset{\text{Jac.}}{=} 0$$

$$\boxed{\frac{\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}}{4!}} \to \Big(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial A^a_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial B^{+a}_{\mu_1\mu_2}/2!} +$$

$$+ \mathsf{Q}\Big(\frac{\tau^{+a}_{1\mu_1\mu_2\mu_3}}{3!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial \tau^{+a}_{1\mu_1\mu_2\mu_3}/3!} + \mathsf{Q}\Big(\frac{\tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}}{4!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial \tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}/4!} +$$

$$+ \frac{\partial \mathsf{Q}(\tau^{+a}_{1\mu_1\mu_2\mu_3}/3!)}{\partial x^\mu}\frac{\partial \mathsf{Q}(\tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!)}{\partial \tau^{+a}_{1\mu_1\mu_2\mu_3|\mu}/3!}\Big)\frac{\partial}{\partial \tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!} = \Big(\underbrace{-\frac{1}{2} f^{a'}_{c'a} f^a_{bc} c^b c^c \frac{\tau^{+c'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}}{4!}}_{\text{Jac.+ant. }(b,c)} - f^{a'}_{ac'} c^a_{\mu_1'} \frac{\tau^{+c'}_{1\mu_2'\mu_3'\mu_4'}}{3!} +$$

$$+ f^{a'}_{c'a} f^a_{bc} A^b_{\mu_1'} c^c \frac{\tau^{+c'}_{1\mu_2'\mu_3'\mu_4'}}{3!} + f^{a'}_{ac'} A_{\mu_2\mu_1'} \frac{B^{+c'}_{\mu_3'\mu_4'}}{2!} - \underbrace{\frac{1}{2} f^{a'}_{c'a} f^a_{bc} A^b_{\mu_1'} A^c_{\mu_2'} \frac{B^{+c'}_{\mu_3'\mu_4'}}{2!}}_{\text{Jac.+ant. }(b,c)} - f^{a'}_{c'a} f^a_{bc} c^b \frac{B^{+c}_{\mu_1'\mu_2'}}{2!} \frac{B^{+c'}_{\mu_3'\mu_4'}}{2!} +$$

$$- f^{a'}_{b'a} A^{b'}_{\mu_1'} \frac{B^{+a}_{\mu_3'\mu_4'|\mu_2'}}{2!} - f^{a'}_{b'a} f^a_{bc} A^{b'}_{\mu_1'} A^b_{\mu_2'} \frac{B^{+c}_{\mu_3'\mu_4'}}{2!} + f^{a'}_{b'a} f^a_{bc} A^{b'}_{\mu_1'} c^b \frac{\tau^{+c}_{1\mu_2'\mu_3'\mu_4'}}{3!} - f^{a'}_{b'a} \frac{\tau^{+a}_{1\mu_2'\mu_3'\mu_4'|\mu_1'}}{3!} c^{b'} +$$

$$- f^{a'}_{b'a} f^a_{bc} A^b_{\mu_1'} c^{b'} \frac{\tau^{+c}_{1\mu_2'\mu_3'\mu_4'}}{3!} + f^{a'}_{b'a} f^a_{bc} c^b c^{b'} \frac{\tau^{+c}_{2\mu_1'\mu_2'\mu_3'\mu_4'}}{4!} - \underbrace{\frac{1}{2} f^{a'}_{b'a} f^a_{bc} \frac{B^{+b}_{\mu_1'\mu_2'}}{2} \frac{B^{+c}_{\mu_3'\mu_4'}}{2!} c^{b'}}_{\text{Jac.+ant. }(b,c)} - \underbrace{\frac{B^{+a'}_{\mu_3'\mu_4'|\mu_2'\mu_1'}}{2!}}_{\text{Sym}-\text{Ant}} +$$

$$- f^{a'}_{bc} A^b_{\mu_2'|\mu_1'} \frac{B^{+c}_{\mu_3'\mu_4'}}{2!} - \underbrace{f^{a'}_{bc} A^b_{\mu_2'} \frac{B^{+c}_{\mu_3'\mu_4'|\mu_1'}}{2!}}_{\mu_1'\leftrightarrow\mu_2'} + f^{a'}_{bc} c^b_{\mu_1'} \frac{\tau^{+c}_{1\mu_2'\mu_3'\mu_4'}}{3!} + f^{a'}_{bc} c^b \frac{\tau^{+c}_{1\mu_2'\mu_3'\mu_4'|\mu_1'}}{3!} \Big)\frac{\partial}{\partial \tau^{+a'}_{2\mu_1'\mu_2'\mu_3'\mu_4'}/4!} \overset{\text{Jac.}}{=} 0$$

$$\boxed{\tau_{2a'}} \to \Big(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(\tau_{2a'})}{\partial c^a} + \mathsf{Q}(\tau_{2a})\frac{\partial \mathsf{Q}(\tau_{2a'})}{\partial \tau_{2a}}\Big)\frac{\partial}{\partial \tau_{2a'}} = \Big(\underbrace{\frac{1}{2} f^{c'}_{a'a} f^a_{bc} c^b c^c \tau_{2c'}}_{\text{Jac.+ant. }(b,c)} + f^c_{ba} f^a_{b'a'} c^b c^{b'} \tau_{2c} \Big)\frac{\partial}{\partial \tau_{2a'}} = 0$$

$$\boxed{\tau_{1\mu_1',a'}} \to \Big(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(\tau_{1\mu_1',a'})}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(\tau_{1\mu_1',a'})}{\partial A^a_{\mu_1}} + \mathsf{Q}(\tau_{2a})\frac{\partial \mathsf{Q}(\tau_{1\mu_1',a'})}{\partial \tau_{2a}} + \mathsf{Q}(\tau_{1\mu_1,a})\frac{\partial \mathsf{Q}(\tau_{1\mu_1',a'})}{\partial \tau_{1\mu_1,a}} +$$

$$
+\frac{\partial \mathsf{Q}(\tau_{2a})}{\partial x^{\mu}}\frac{\partial \mathsf{Q}(\tau_{1\mu_1',a'})}{\partial \tau_{2\mu,a}}\Big)\frac{\partial}{\partial \tau_{1\mu_1',a'}} = \Big(\underbrace{\frac{1}{2}f^{c'}_{a'a}f^{a}_{bc}c^{b}c^{c}\tau_{1\mu_1',c'}}_{\text{Jac.+ant. }(b,c)} + f^{c'}_{aa'}c^{a}_{\mu_1'}\tau_{2c'} - f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}c^{c}\tau_{2c'} +
$$

$$
+ f^{c}_{ba}f^{a}_{b'a'}A^{b'}_{\mu_1'}c^{b}\tau_{2c} + f^{a}_{b'a'}\tau_{2\mu_1',a}c^{b'} + f^{c}_{ba}f^{a}_{b'a'}c^{b}c^{b'}\tau_{1\mu_1',c} - f^{c}_{ba}f^{a}_{b'a'}A^{b}_{\mu_1'}c^{b'}\tau_{2c} - f^{c}_{ba'}c^{b}_{\mu_1'}\tau_{2c} +
$$

$$
- f^{c}_{ba'}c^{b}\tau_{2\mu_1',c}\Big)\frac{\partial}{\partial \tau_{1\mu_1',a'}} \overset{\text{Jac.}}{=} 0
$$

$$
\boxed{\frac{B_{\mu_1'\mu_2',a'}}{2!}} \to \Big(\mathsf{Q}(c^{a})\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial c^{a}} + \mathsf{Q}(A^{a}_{\mu_1})\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial A^{a}_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial B^{+a}_{\mu_1\mu_2}/2!} +
$$

$$
+ \mathsf{Q}(\tau_{2a})\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial \tau_{2a}} + \mathsf{Q}(\tau_{1\mu_1,a})\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial \tau_{1\mu_1,a}} + \mathsf{Q}\Big(\frac{B_{\mu_1\mu_2,a}}{2!}\Big)\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial B_{\mu_1\mu_2,a}/2!} +
$$

$$
+ \frac{\partial \mathsf{Q}(\tau_{1\mu_1,a})}{\partial x^{\mu}}\frac{\partial \mathsf{Q}(B_{\mu_1'\mu_2',a'}/2!)}{\partial \tau_{1\mu_1|\mu,a}}\Big)\frac{\partial}{\partial_{\mu_1'\mu_2',a'}/2!} = \Big(\underbrace{\frac{1}{2}f^{c'}_{a'a}f^{a}_{bc}c^{b}c^{c}\frac{B_{\mu_1'\mu_2',c'}}{2!}}_{\text{Jac.+ant. }(b,c)} - f^{c'}_{aa'}c^{a}_{\mu_1'}\tau_{1\mu_2',c'} +
$$

$$
+ f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}c^{c}\tau_{1\mu_2',c'} - f^{c'}_{aa'}A^{a}_{\mu_2'|\mu_1'}\tau_{2c'} + \underbrace{\frac{1}{2}f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}A^{c}_{\mu_2'}\tau_{2c'}}_{\text{Jac.+ant. }(b,c)} + f^{c'}_{a'a}f^{a}_{bc}c^{b}\frac{B^{+c}_{\mu_1'\mu_2'}}{2!}\tau_{2c'} +
$$

$$
+ f^{c}_{ba}f^{a}_{b'a'}c^{b}\frac{B^{+b'}_{\mu_1'\mu_2'}}{2!}\tau_{2c} + f^{a}_{b'a'}\tau_{2\mu_2',a}A^{b'}_{\mu_1'} - f^{c}_{ba}f^{a}_{b'a'}A^{b'}_{\mu_1'}c^{b}\tau_{1\mu_2'c} - f^{c}_{ba}f^{a}_{b'a'}A^{b}_{\mu_2'}A^{b'}_{\mu_1'}c^{b}\tau_{2c} +
$$

$$
- f^{a}_{b'a'}c^{b'}\tau_{1\mu_2'|\mu_1',a} + f^{c}_{ba}f^{a}_{b'a'}c^{b}c^{b'}\frac{B_{\mu_1'\mu_2',c}}{2!} + f^{c}_{ba}f^{a}_{b'a'}A^{b}_{\mu_1'}c^{b'}\tau_{1\mu_2',c} - f^{c}_{ba}f^{a}_{b'a'}c^{b'}\frac{B^{+b}_{\mu_1'\mu_2'}}{2!}\tau_{2c} +
$$

$$
- \underbrace{\tau_{1\mu_2'\mu_1',a'}}_{\text{Sym}-\text{Ant}} + f^{c}_{ba'}c^{b}_{\mu_1'}\tau_{1\mu_2,c} + f^{c}_{ba'}c^{b}\tau_{1\mu_2'|\mu_1',c} + f^{c}_{ba'}A^{b}_{\mu_2'|\mu_1'}\tau_{2c} + f^{c}_{ba'}A^{b}_{\mu_2'}\tau_{2\mu_1'c}\Big)\frac{\partial}{\partial B_{\mu_1\mu_2,a}/2!} \overset{\text{Jac.+Jac.}}{=} 0
$$

$$
\boxed{\frac{A^{+}_{\mu_1'\mu_2'\mu_3',a'}}{3!}} \to \Big(\mathsf{Q}(c^{a})\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial c^{a}} + \mathsf{Q}(A^{a}_{\mu_1})\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial A^{a}_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!}{\partial B^{+a}_{\mu_1\mu_2}/2!} +
$$

$$
+ \mathsf{Q}\Big(\frac{\tau^{+a}_{1\mu_1\mu_2\mu_3}}{3!}\Big)\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial \tau^{+a}_{1\mu_1\mu_2\mu_3}/3!} + \mathsf{Q}(\tau_{2a})\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial \tau_{2a}} + \mathsf{Q}(\tau_{1\mu_1,a})\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial \tau_{1\mu_1,a}} +
$$

$$
+ \mathsf{Q}\Big(\frac{B_{\mu_1\mu_2,a}}{2!}\Big)\frac{\partial \mathsf{Q}(\tau^{+a'}_{1\mu_1'\mu_2'\mu_3'}/3!)}{\partial B_{\mu_1\mu_2,a}/2!} + \mathsf{Q}\Big(\frac{A^{+}_{\mu_1\mu_2\mu_3,a}}{3!}\Big)\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial A^{+}_{\mu_1\mu_2\mu_3,a}/3!} +
$$

$$
+ \frac{\partial \mathsf{Q}(B_{\mu_1\mu_2,a}/2!)}{\partial x^{\mu}}\frac{\partial \mathsf{Q}(A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!)}{\partial B_{\mu_1\mu_2|\mu,a}/2!}\Big)\frac{\partial}{\partial A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!} = \Big(\underbrace{\frac{1}{2}f^{c'}_{a'a}f^{a}_{bc}c^{b}c^{c}\frac{A^{+}_{\mu_1'\mu_2'\mu_3',c'}}{3!}}_{\text{Jac.+ant. }(b,c)} - f^{c'}_{aa'}c^{a}_{\mu_1'}\frac{B_{\mu_2'\mu_3',c'}}{2!} +
$$

$$
- f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}c^{c}\frac{B_{\mu_2'\mu_3',c'}}{2!} + f^{c'}_{aa'}A^{a}_{\mu_2'|\mu_1'}\tau_{1\mu_3',c'} + \underbrace{\frac{1}{2}f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}A^{c}_{\mu_2'}\tau_{1\mu_3',c'}}_{\text{Jac.+ant. }(b,c)} + f^{c'}_{a'a}f^{a}_{bc}c^{b}\frac{B^{+c}_{\mu_1'\mu_2'}}{2!}\tau_{1\mu_3',c'} +
$$

$$
- f^{c'}_{a'a}\frac{B^{+a}_{\mu_2'\mu_3'|\mu_1'}}{2!}\tau_{2c'} - f^{c'}_{a'a}f^{a}_{bc}A^{b}_{\mu_1'}\frac{B^{+c}_{\mu_2'\mu_3'}}{2!}\tau_{2c'} + f^{c'}_{a'a}f^{a}_{bc}c^{b}\frac{\tau^{+c}_{1\mu_1'\mu_2'\mu_3'}}{3!}\tau_{2c'} + f^{c}_{ba}f^{a}_{b'a'}c^{b}\frac{\tau^{+b'}_{1\mu_1'\mu_2'\mu_3'}}{3!}\tau_{2c} +
$$

$$
+ f^{a}_{b'a'}\tau_{2\mu_3',a}\frac{B^{+b'}_{\mu_1'\mu_2'}}{2} + f^{c}_{ba}f^{a}_{b'a'}c^{b}\frac{B^{+b'}_{\mu_1'\mu_2'}}{2!}\tau_{1\mu_3',c} - f^{c}_{ba}f^{a}_{b'a'}A^{b}_{\mu_3'}\frac{B^{+b'}_{\mu_1'\mu_2'}}{2!}\tau_{2c} + f^{a}_{b'a'}\tau_{1\mu_3'|\mu_2',a}A^{b'}_{\mu_1'} +
$$

$$
+ f^{c}_{ba}f^{a}_{b'a'}A^{b'}_{\mu_1'}c^{b}\frac{B_{\mu_2'\mu_3',c}}{2!} - f^{c}_{ba}f^{a}_{b'a'}A^{b'}_{\mu_1'}A^{b}_{\mu_2'}\tau_{1\mu_3',c} + f^{c}_{ba}f^{a}_{b'a'}A^{b'}_{\mu_1'}\frac{B^{+b}_{\mu_2'\mu_3'}}{2!}\tau_{2c} + f^{a}_{b'a'}\frac{B_{\mu_2'\mu_3'|\mu_1',a}}{2!}c^{b'} +
$$

$$
+ f^{c}_{ba}f^{a}_{b'a'}c^{b}c^{b'}\frac{A^{+}_{\mu_1'\mu_2'\mu_3',c}}{3!} - f^{c}_{ba}f^{a}_{b'a'}A^{b}_{\mu_1'}c^{b'}\frac{B_{\mu_2'\mu_3',c}}{2!} - f^{c}_{ba}f^{a}_{b'a'}c^{b'}\frac{B^{+b}_{\mu_1'\mu_2'}}{2!}\tau_{1\mu_3',c} - f^{c}_{ba}f^{a}_{b'a'}c^{b'}\frac{\tau^{+b}_{1\mu_1'\mu_2'\mu_3'}}{2!}\tau_{2c} +
$$

$$
- \underbrace{\tau_{1\mu_3'|\mu_2'\mu_1',a'}}_{\text{Sym}-\text{Ant}} - f^{c}_{ba'}c^{b}_{\mu_1'}\frac{B_{\mu_2'\mu_3',c}}{2!} - f^{c}_{ba'}c^{b}\frac{B_{\mu_2'\mu_3'|\mu_1',c}}{2!} + f^{c}_{ba'}A^{b}_{\mu_2'|\mu_1'}\tau_{1\mu_3',c} + \underbrace{f^{c}_{ba'}A^{b}_{\mu_2'}\tau_{1\mu_3'|\mu_1'c}}_{\mu_1'\leftrightarrow\mu_2'} +
$$

$$
- f^{c}_{ba'}\frac{B^{+b}_{\mu_2'\mu_3'|\mu_1'}}{2!}\tau_{2c} - \underbrace{f^{c}_{ba'}\frac{B^{+b}_{\mu_2'\mu_3'}}{2!}\tau_{2\mu_1',c}}_{\mu_1'\leftrightarrow\mu_3'\leftrightarrow\mu_2'}\Big)\frac{\partial}{\partial A^{+}_{\mu_1'\mu_2'\mu_3',a'}/3!} \overset{\text{Jac.+Jac.}}{\underset{\text{Jac.+Jac.}}{=}} 0
$$

$$
\begin{aligned}
&\boxed{\frac{c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}}{4!}} \to \Bigg(\mathsf{Q}(c^a)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial c^a} + \mathsf{Q}(A^a_{\mu_1})\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial A^a_{\mu_1}} + \mathsf{Q}\Big(\frac{B^{+a}_{\mu_1\mu_2}}{2!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial B^{+a}_{\mu_1\mu_2}/2!}+\\
&+\mathsf{Q}\Big(\frac{\tau^{+a}_{1\mu_1\mu_2\mu_3}}{3!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial \tau^{+a}_{1\mu_1\mu_2\mu_3}/3!} + \mathsf{Q}\Big(\frac{\tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}}{4!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial \tau^{+a}_{2\mu_1\mu_2\mu_3\mu_4}/4!} + \mathsf{Q}(\tau_{2a})\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial \tau_{2a}}+\\
&+\mathsf{Q}(\tau_{1\mu_1,a})\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial \tau_{1\mu_1,a}} + \mathsf{Q}\Big(\frac{B_{\mu_1\mu_2,a}}{2!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial B_{\mu_1\mu_2,a}/2!} + \mathsf{Q}\Big(\frac{A^+_{\mu_1\mu_2\mu_3,a}}{3!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial A^+_{\mu_1\mu_2\mu_3,a}/3!}+\\
&+\mathsf{Q}\Big(\frac{c^+_{\mu_1\mu_2\mu_3\mu_4,a}}{4!}\Big)\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial c^+_{\mu_1\mu_2\mu_3\mu_4,a}/4!} + \frac{\partial \mathsf{Q}(A^+_{\mu_1\mu_2\mu_3,a}/3!)}{\partial x^\mu}\frac{\partial \mathsf{Q}(c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!)}{\partial A^+_{\mu_1\mu_2\mu_3|\mu,a}/3!}\Bigg)\frac{\partial}{\partial c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,a'}/4!} =\\
&= \Bigg(\underbrace{\frac{1}{2}f^{c'}_{a'a}f^a_{bc}c^bc^c\frac{c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,c'}}{4!}}_{\text{Jac.+ant. }(b,c)} - f^{c'}_{aa'}c^a_{\mu'_1}\frac{A^+_{\mu'_2\mu'_3\mu'_4,c'}}{3!} + f^{c'}_{a'a}f^a_{bc}A^b_{\mu'_1}c^c\frac{A^+_{\mu'_2\mu'_3\mu'_4,c'}}{3!} + f^{c'}_{aa'}A^a_{\mu'_2|\mu'_1}\frac{B_{\mu'_3\mu'_4,c'}}{2!}+\\
&+\underbrace{\frac{1}{2}f^{c'}_{a'a}f^a_{bc}A^b_{\mu'_1}A^c_{\mu'_2}B_{\mu'_3\mu'_4,c'}}_{\text{Jac.+ant. }(b,c)} + f^{c'}_{a'a}f^a_{bc}c^b\frac{B^{+c}_{\mu'_1\mu'_2}}{2!}\frac{B_{\mu'_3\mu'_4,c'}}{2!} - f^{c'}_{a'a}\frac{B^{+a}_{\mu'_2\mu'_3|\mu'_1}}{2!}\tau_{1\mu'_4,c'} + f^{c'}_{a'a}f^a_{bc}A^b_{\mu'_1}\frac{B^{+c}_{\mu'_2\mu'_3}}{2!}\tau_{1\mu'_4,c'}+\\
&- f^{c'}_{a'a}f^a_{bc}c^b\frac{\tau^{+c}_{1\mu'_1\mu'_2\mu'_3}}{3!}\tau_{1\mu'_4,c'} + f^{c'}_{a'a}\frac{\tau^{+a}_{1\mu'_2\mu'_3\mu'_4|\mu'_1}}{3!}\tau_{2c'} + f^{c'}_{a'a}f^a_{bc}A^b_{\mu'_1}\frac{\tau^{+c}_{1\mu'_2\mu'_3\mu'_4}}{3!}\tau_{2c'} + f^{c'}_{a'a}f^a_{bc}c^b\frac{\tau^{+c}_{2\mu'_1\mu'_2\mu'_3\mu'_4}}{4!}\tau_{2c'}+\\
&+\underbrace{\frac{1}{2}f^{c'}_{a'a}f^a_{bc}\frac{B^{+b}_{\mu'_1\mu'_2}}{2!}\frac{B^{+c}_{\mu'_3\mu'_4}}{2!}\tau_{2c'}}_{\text{Jac.+ant. }(b,c)} + f^c_{ba}f^a_{b'a'}c^b\frac{\tau^{+b'}_{2\mu'_1\mu'_2\mu'_3\mu'_4}}{4!}\tau_{2c} + \underbrace{f^a_{b'a'}\tau_{2\mu'_4,a}\frac{\tau^{+b'}_{1\mu'_1\mu'_2\mu'_3}}{3!}}_{\mu'_1\leftrightarrow\mu'_2\leftrightarrow\mu'_3\leftrightarrow\mu'_4} - f^c_{ba}f^a_{b'a'}c^b\frac{\tau^{+b'}_{1\mu'_1\mu'_2\mu'_3}}{3!}\tau_{1\mu'_4,c}+\\
&-\underbrace{f^c_{ba}f^a_{b'a'}A^b_{\mu'_4}\frac{\tau^{+b'}_{1\mu'_1\mu'_2\mu'_3}}{3!}\tau_{2c}}_{\mu'_1\leftrightarrow\mu'_2\leftrightarrow\mu'_3\leftrightarrow\mu'_4} + f^a_{b'a'}\tau_{1\mu'_4|\mu'_3,a}\frac{B^{+b'}_{\mu'_1\mu'_2}}{2} + f^c_{ba}f^a_{b'a'}c^b\frac{B^{+b'}_{\mu'_1\mu'_2}}{2!}\frac{B_{\mu'_3\mu'_4,c}}{2!} + \underbrace{f^c_{ba}f^a_{b'a'}A^b_{\mu'_3}\frac{B^{+b'}_{\mu'_1\mu'_2}}{2!}\tau_{1\mu'_4,c}}_{\mu'_1\leftrightarrow\mu'_2\leftrightarrow\mu'_3}+\\
&- f^c_{ba}f^a_{b'a'}\frac{B^{+b'}_{\mu'_1\mu'_2}}{2!}\frac{B^{+b}_{\mu'_3\mu'_4}}{2!}\tau_{2c} + f^a_{b'a'}\frac{B_{\mu'_3\mu'_4|\mu'_2,a}}{2!}A^{b'}_{\mu'_1} - f^c_{ba}f^a_{b'a'}A^{b'}_{\mu'_1}c^b\frac{A^+_{\mu'_2\mu'_3\mu'_4,c}}{3!} - f^c_{ba}f^a_{b'a'}A^{b'}_{\mu'_1}A^b_{\mu'_2}\frac{B_{\mu'_3\mu'_4,c}}{2!}+\\
&- f^c_{ba}f^a_{b'a'}A^{b'}_{\mu'_1}\frac{B^{+b}_{\mu'_2\mu'_3}}{2!}\tau_{1\mu'_4,c} - f^c_{ba}f^a_{b'a'}A^{b'}_{\mu'_1}\frac{\tau^{+b}_{1\mu'_2\mu'_3\mu'_4}}{3!}\tau_{2c} - f^a_{b'a'}c^{b'}\frac{A^+_{\mu'_2\mu'_3\mu'_4|\mu'_1,a}}{3!} + f^c_{ba}f^a_{b'a'}c^bc^{b'}\frac{c^+_{\mu'_1\mu'_2\mu'_3\mu'_4,c}}{4!}+\\
&+ f^c_{ba}f^a_{b'a'}A^b_{\mu'_1}c^{b'}\frac{A^+_{\mu'_2\mu'_3\mu'_4,c}}{3!} - f^c_{ba}f^a_{b'a'}c^{b'}\frac{B^{+b}_{\mu'_1\mu'_2}}{2!}\frac{B_{\mu'_3\mu'_4,c}}{2!} + f^c_{ba}f^a_{b'a'}c^{b'}\frac{\tau^{+b}_{1\mu'_1\mu'_2\mu'_3}}{3!}\tau_{1\mu'_4,c} - f^c_{ba}f^a_{b'a'}c^{b'}\frac{\tau^{+b}_{2\mu'_1\mu'_2\mu'_3\mu'_4}}{4!}\tau_{2c}+\\
&-\underbrace{\frac{B_{\mu'_3\mu'_4|\mu'_2\mu'_1,a'}}{2!}}_{\text{Sym}-\text{Ant}} + f^c_{ba'}c^b_{\mu'_1}\frac{A^+_{\mu'_2\mu'_3\mu'_4,c}}{3!} + f^c_{ba'}c^b\frac{A^+_{\mu'_2\mu'_3\mu'_4|\mu'_1,c}}{3!} + f^c_{ba'}A^b_{\mu'_2|\mu'_1}\frac{B_{\mu'_3\mu'_4,c}}{2!} + \underbrace{f^c_{ba'}A^b_{\mu'_2}\frac{B_{\mu'_3\mu'_4|\mu'_1c}}{2!}}_{\mu'_1\leftrightarrow\mu'_2}+\\
&+ f^c_{ba'}\frac{B^{+b}_{\mu'_2\mu'_3|\mu'_1}}{2!}\tau_{1\mu'_4,c} + \underbrace{f^c_{ba'}\frac{B^{+b}_{\mu'_2\mu'_3}}{2!}\tau_{1\mu'_4|\mu'_1,c}}_{\mu'_1\leftrightarrow\mu'_3\leftrightarrow\mu'_2} + f^c_{ba'}\frac{\tau^{+b}_{1\mu'_2\mu'_3\mu'_4|\mu'_1}}{3!}\tau_{2c} + f^c_{ba'}\frac{\tau^{+b}_{1\mu'_2\mu'_3\mu'_4}}{3!}\tau_{2\mu'_1,c}\Bigg)\frac{\partial}{\partial A^+_{\mu'_1\mu'_2\mu'_3,a'}/3!} = 0.
\end{aligned}
$$

In the latter equality we used Jac. + Jac. + Jac. + Jac. + Jac. + Jac..

Let us check Equation (3.56). Recall that the degree 1 differentials satisfy $\mathrm{d}_V^2 = \mathrm{d}_H^2 = \mathrm{d}_V(\mathrm{d}_H x^\mu) = 0$ and that $\mathrm{d}_V\mathrm{d}_H = -\mathrm{d}_H\mathrm{d}_V$. Furthermore, $|\iota_{\mathrm{pr}\,\mathsf{Q}}| = 1-1 = 0$. Then,

$$
\begin{aligned}
\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{4}{\omega} - \mathrm{d}_V\overset{4}{\mathsf{L}} &= \iota_{\mathrm{pr}\,\mathsf{Q}}\Big[\langle\, \mathrm{d}_V A \,\overset{\wedge}{,}\, \mathrm{d}_V A^+\rangle + \langle \mathrm{d}_V B \,\overset{\wedge}{,}\, \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V c \,\overset{\wedge}{,}\, \mathrm{d}_V c^+\rangle + \langle \mathrm{d}_V \tau_1 \,\overset{\wedge}{,}\, \mathrm{d}_V \tau_1^+\rangle + \langle \mathrm{d}_V \tau_2 \,\overset{\wedge}{,}\, \mathrm{d}_V \tau_2^+\rangle\Big]+\\
&- \mathrm{d}_V\Big[\langle B \,\overset{\wedge}{,}\, F_A\rangle + \langle A^+ \,\overset{\wedge}{,}\, \mathrm{d}_A c\rangle + \langle B^+ \,\overset{\wedge}{,}\, \mathrm{ad}^*_c(B) + \mathrm{d}_A\tau_1\rangle + \langle \tau_1^+ \,\overset{\wedge}{,}\, \mathrm{ad}^*_c(\tau_1) + \mathrm{d}_A\tau_2\rangle + \frac{1}{2}\langle c^+ \,\overset{\wedge}{,}\, [c \,\overset{\wedge}{,}\, c]\rangle+\\
&+ \langle \tau_2^+ \,\overset{\wedge}{,}\, \mathrm{ad}^*_c(\tau_2)\rangle + \frac{1}{2}\langle \tau_2 \,\overset{\wedge}{,}\, [B^+ \,\overset{\wedge}{,}\, B^+]\rangle\Big] = \Big[\langle\, \mathrm{d}_A c \,\overset{\wedge}{,}\, \mathrm{d}_V A^+\rangle + \langle\, \mathrm{d}_V A \,\overset{\wedge}{,}\, \mathrm{d}_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle+\\
&+ \langle \mathrm{d}_A\tau_1 + \mathrm{ad}^*_c B + \mathrm{ad}^*_{B^+}\tau_2 \,\overset{\wedge}{,}\, \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V B \,\overset{\wedge}{,}\, F_A + [c \,\overset{\wedge}{,}\, B^+]\rangle + \frac{1}{2}\langle [c \,\overset{\wedge}{,}\, c] \,\overset{\wedge}{,}\, \mathrm{d}_V c^+\rangle+\\
&+ \langle \mathrm{d}_V c \,\overset{\wedge}{,}\, \mathrm{d}_A A^+ + \mathrm{ad}^*_c c^+ + \mathrm{ad}^*_{B^+}B + \mathrm{ad}^*_{\tau_1^+}\tau_1 + \mathrm{ad}^*_{\tau_2^+}\tau_2\rangle + \langle \mathrm{d}_A\tau_2 + \mathrm{ad}^*_c\tau_1 \,\overset{\wedge}{,}\, \mathrm{d}_V\tau_1^+\rangle+\\
&+ \langle \mathrm{d}_V\tau_1 \,\overset{\wedge}{,}\, \mathrm{d}_A B^+ + [c \,\overset{\wedge}{,}\, \tau_1^+]\rangle + \langle \mathrm{ad}^*_c\tau_2 \,\overset{\wedge}{,}\, \mathrm{d}_V\tau_2^+\rangle + \langle \mathrm{d}_V\tau_2 \,\overset{\wedge}{,}\, \mathrm{d}_A\tau_1^+ + [c \,\overset{\wedge}{,}\, \tau_2^+] + \frac{1}{2}[B^+ \,\overset{\wedge}{,}\, B^+]\rangle\Big]+
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-\langle \mathrm{d}_V B \curlywedge F_A\rangle - \langle B \curlywedge \mathrm{d}_V F_A\rangle - \langle \mathrm{d}_V A^+ \curlywedge \mathrm{d}_A c\rangle - \langle A^+ \curlywedge [\mathrm{d}_V A \curlywedge c] - \mathrm{d}_A(\mathrm{d}_V c)\rangle - \langle \mathrm{d}_V B^+ \curlywedge \mathrm{ad}^*_c(B) + \mathrm{d}_A\tau_1\rangle + \\
&+\langle B^+ \curlywedge \mathrm{ad}^*_{\mathrm{d}_V c}(B) - \mathrm{ad}^*_c(\mathrm{d}_V B) + \mathrm{ad}^*_{\mathrm{d}_V A}(\tau_1) - \mathrm{d}_A(\mathrm{d}_V\tau_1)\rangle - \langle \mathrm{d}_V\tau_1^+ \curlywedge \mathrm{ad}^*_c(\tau_1) + \mathrm{d}_A\tau_2\rangle + \\
&+\langle \tau_1^+ \curlywedge \mathrm{ad}^*_{\mathrm{d}_V c}(\tau_1) - \mathrm{ad}^*_c(\mathrm{d}_V\tau_1) + \mathrm{ad}^*_{\mathrm{d}_V A}(\tau_2) - \mathrm{d}_A\mathrm{d}_V\tau_2\rangle - \frac{1}{2}\langle \mathrm{d}_V c^+ \curlywedge [c \curlywedge c]\rangle - \frac{1}{2}\langle c^+ \curlywedge \mathrm{d}_V([c \curlywedge c])\rangle + \\
&-\langle \mathrm{d}_V\tau_2^+ \curlywedge \mathrm{ad}^*_c(\tau_2)\rangle + \langle \tau_2^+ \curlywedge \mathrm{ad}^*_{\mathrm{d}_V c}(\tau_2) - \mathrm{ad}^*_c(\mathrm{d}_V\tau_2)\rangle - \frac{1}{2}\langle \mathrm{d}_V\tau_2 \curlywedge [B^+ \curlywedge B^+]\rangle + \frac{1}{2}\langle \tau_2 \curlywedge \mathrm{d}_V([B^+ \curlywedge B^+])\rangle\Big] = \\
&= \mathrm{d}_H\Big[\langle \mathrm{d}_V A \curlywedge B\rangle + \langle \mathrm{d}_V c \curlywedge A^+\rangle - \langle \mathrm{d}_V\tau_1 \curlywedge B^+\rangle - \langle \mathrm{d}_V\tau_2 \curlywedge \tau_1^+\rangle\Big] = \mathrm{d}_H \overset{3}{\chi}
\end{aligned}
$$

where $\overset{3}{\chi} \in \Omega^{3,1}_{\mathrm{glb}}(J^\infty_M F)_0$. In the first equality we computed the contraction using that $\mathrm{d}_V u^b(\partial_{u^a}) = \delta^b_a$ and the derivative d_V. Expanding the above in components, the second equality can be checked with manipulations similar to those used for the components of $[\mathrm{pr}\mathsf{Q}, \mathrm{pr}\mathsf{Q}]$.
Let us now check the Equation (3.57).

$$
\begin{aligned}
\frac{1}{2}\iota_{\mathrm{pr}\,\mathsf{Q}}\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{4}{\omega} =& \frac{1}{2}\iota_{\mathrm{pr}\,\mathsf{Q}}\Big[\langle\, \mathrm{d}_A c \curlywedge \mathrm{d}_V A^+\rangle + \langle\, \mathrm{d}_V A \curlywedge \mathrm{d}_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle + \\
&+\langle \mathrm{d}_A\tau_1 + \mathrm{ad}^*_c B + \mathrm{ad}^*_{B^+}\tau_2 \curlywedge \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V B \curlywedge F_A + [c \curlywedge B^+]\rangle + \frac{1}{2}\langle [c \curlywedge c] \curlywedge \mathrm{d}_V c^+\rangle + \\
&+\langle \mathrm{d}_V c \curlywedge \mathrm{d}_A A^+ + \mathrm{ad}^*_c c^+ + \mathrm{ad}^*_{B^+} B + \mathrm{ad}^*_{\tau_1^+}\tau_1 + \mathrm{ad}^*_{\tau_2^+}\tau_2\rangle + \langle \mathrm{d}_A\tau_2 + \mathrm{ad}^*_c\tau_1 \curlywedge \mathrm{d}_V\tau_1^+\rangle + \\
&+\langle \mathrm{d}_V\tau_1 \curlywedge \mathrm{d}_A B^+ + [c \curlywedge \tau_1^+]\rangle + \langle \mathrm{ad}^*_c\tau_2 \curlywedge \mathrm{d}_V\tau_2^+\rangle + \langle \mathrm{d}_V\tau_2 \curlywedge \mathrm{d}_A\tau_1^+ + [c \curlywedge \tau_2^+] + \frac{1}{2}[B^+ \curlywedge B^+]\rangle\Big] = \\
&= \Big[\langle\, \mathrm{d}_A c \curlywedge \mathrm{d}_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle + \langle \mathrm{d}_A\tau_1 + \mathrm{ad}^*_c B + \mathrm{ad}^*_{B^+}\tau_2 \curlywedge F_A + [c \curlywedge B^+]\rangle + \\
&+\frac{1}{2}\langle [c \curlywedge c] \curlywedge \mathrm{d}_A A^+ + \mathrm{ad}^*_c c^+ + \mathrm{ad}^*_{B^+} B + \mathrm{ad}^*_{\tau_1^+}\tau_1 + \mathrm{ad}^*_{\tau_2^+}\tau_2\rangle + \\
&+\langle \mathrm{d}_A\tau_2 + \mathrm{ad}^*_c\tau_1 \curlywedge \mathrm{d}_A B^+ + [c \curlywedge \tau_1^+]\rangle + \langle \mathrm{ad}^*_c\tau_2 \curlywedge \mathrm{d}_A\tau_1^+ + [c, \tau_2^+] + \frac{1}{2}[B^+ \curlywedge B^+]\rangle\Big] = \\
&= \Big[\langle\, \mathrm{d}_H c \curlywedge \mathrm{d}_H B + \mathrm{ad}^*_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle + \langle [A \curlywedge c] \curlywedge \mathrm{d}_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle + \\
&+\langle \mathrm{d}_H\tau_1 \curlywedge \mathrm{d}_H A + \frac{1}{2}[A \curlywedge A] + [c \curlywedge B^+]\rangle + \langle \mathrm{ad}^*_A\tau_1 \curlywedge F_A + [c \curlywedge B^+]\rangle + \langle \mathrm{ad}^*_c B \curlywedge F_A + [c \curlywedge B^+]\rangle + \\
&+\langle \mathrm{ad}^*_{B^+}\tau_2 \curlywedge F_A + [c \curlywedge B^+]\rangle + \frac{1}{2}\langle [c \curlywedge c] \curlywedge \mathrm{d}_A A^+ + \underbrace{\mathrm{ad}^*_c c^+}_{\text{ant.+Jac.}} + \underbrace{\mathrm{ad}^*_{B^+} B}_{\text{Jac.+ant.}} + \underbrace{\mathrm{ad}^*_{\tau_1^+}\tau_1}_{\text{Jac.+ant.}} + \underbrace{\mathrm{ad}^*_{\tau_2^+}\tau_2}_{\text{Jac.+ant}}\rangle + \\
&+\langle \mathrm{d}_H\tau_2 \curlywedge \mathrm{d}_H B^+ + [A \curlywedge B^+] + [c \curlywedge \tau_1^+]\rangle + \langle \mathrm{ad}^*_A\tau_2 \curlywedge \mathrm{d}_A B^+ + [c \curlywedge \tau_1^+]\rangle + \langle \mathrm{ad}^*_c\tau_1 \curlywedge \mathrm{d}_A B^+ + [c \curlywedge \tau_1^+]\rangle \\
&+\langle \mathrm{ad}^*_c\tau_2 \curlywedge \mathrm{d}_A\tau_1^+ + [c \curlywedge \tau_2^+] + \underbrace{\frac{1}{2}[B^+ \curlywedge B^+]\rangle}_{\text{Jac.+ant.}}\Big] = \Big(\langle \mathrm{d}_H c \curlywedge \mathrm{d}_H B\rangle + \langle \mathrm{d}_H\tau_1 \curlywedge \mathrm{d}_H A\rangle + \langle \mathrm{d}_H\tau_2 \curlywedge \mathrm{d}_H B^+\rangle + \\
&-\mathrm{d}_H(\langle \mathrm{ad}^*_c\tau_2 \curlywedge \tau_1^+\rangle) - \mathrm{d}_H(\langle \mathrm{ad}^*_A\tau_2 \curlywedge B^+\rangle) + \mathrm{d}_H\Big(\frac{1}{2}\langle [A \curlywedge A] \curlywedge \tau_1\rangle\Big) + \mathrm{d}_H\Big(\frac{1}{2}\langle [c \curlywedge c] \curlywedge A^+\rangle\Big) + \\
&+\mathrm{d}_H(\langle [c \curlywedge A] \curlywedge B\rangle + \mathrm{d}_H(\langle \tau_1 \curlywedge [c \curlywedge B^+]\rangle\Big) = \mathrm{d}_H\overset{3}{\mathsf{L}},
\end{aligned}
$$

for some $\overset{3}{\mathsf{L}} \in \Omega^{3,0}_{\mathrm{glb}}(J^\infty_M F)_1$. In the first and second equality, we computed the first and second contractions. In the third, we expanded the result. The final equality can be checked in components with usual manipulations. Note that the first three terms of $\mathrm{d}_H\overset{3}{\mathsf{L}}$ are the ones that we also encounter in the abelian case.

Remark 3.4.6. As in Remark 3.4.5, it is possible to see that $\mathrm{d}_V\overset{3}{\chi} = -\overset{3}{\omega}$ for $\overset{3}{\omega}$ satisfying

$$
\begin{aligned}
\mathrm{d}_V\iota_{\mathrm{pr}\,\mathsf{Q}}\overset{4}{\omega} =& \mathrm{d}_V\Big[\langle\, \mathrm{d}_A c \curlywedge \mathrm{d}_V A^+\rangle + \langle\, \mathrm{d}_V A \curlywedge \mathrm{d}_A B + \mathrm{ad}^*_c A^+ + \mathrm{ad}^*_{B^+}\tau_1 + \mathrm{ad}^*_{\tau_1^+}\tau_2\rangle + \\
&+\langle \mathrm{d}_A\tau_1 + \mathrm{ad}^*_c B + \mathrm{ad}^*_{B^+}\tau_2 \curlywedge \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V B \curlywedge F_A + [c \curlywedge B^+]\rangle + \frac{1}{2}\langle [c \curlywedge c] \curlywedge \mathrm{d}_V c^+\rangle + \\
&+\langle \mathrm{d}_V c \curlywedge \mathrm{d}_A A^+ + \mathrm{ad}^*_c c^+ + \mathrm{ad}^*_{B^+} B + \mathrm{ad}^*_{\tau_1^+}\tau_1 + \mathrm{ad}^*_{\tau_2^+}\tau_2\rangle + \langle \mathrm{d}_A\tau_2 + \mathrm{ad}^*_c\tau_1 \curlywedge \mathrm{d}_V\tau_1^+\rangle + \\
&+\langle \mathrm{d}_V\tau_1 \curlywedge \mathrm{d}_A B^+ + [c \curlywedge \tau_1^+]\rangle + \langle \mathrm{ad}^*_c\tau_2 \curlywedge \mathrm{d}_V\tau_2^+\rangle + \langle \mathrm{d}_V\tau_2 \curlywedge \mathrm{d}_A\tau_1^+ + [c \curlywedge \tau_2^+] + \frac{1}{2}[B^+ \curlywedge B^+]\rangle\Big] =
\end{aligned}
$$

$$
\begin{aligned}
=&\Big[\langle \mathrm{d}_V(\mathrm{d}_A c) \,\dot{\wedge}\, \mathrm{d}_V A^+\rangle + \langle\, \mathrm{d}_V A \,\dot{\wedge}\, \mathrm{d}_V(\mathrm{d}_A B) + \mathrm{d}_V(\mathrm{ad}^*_c A^+) + \mathrm{d}_V(\mathrm{ad}^*_{B^+}\tau_1) + \mathrm{d}_V(\mathrm{ad}^*_{\tau_1^+}\tau_2)\rangle +\\
&+ \langle \mathrm{d}_V(\mathrm{d}_A \tau_1) \,\dot{\wedge}\, \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V(\mathrm{ad}^*_c B) \,\dot{\wedge}\, \mathrm{d}_V B^+\rangle + \langle \mathrm{d}_V(\mathrm{ad}^*_{B^+}\tau_2) \,\dot{\wedge}\, \mathrm{d}_V B^+\rangle +\\
&- \langle \mathrm{d}_V B \,\dot{\wedge}\, \mathrm{d}_V(F_A) + \mathrm{d}_V([c \,\dot{\wedge}\, B^+])\rangle + \frac{1}{2}\langle \mathrm{d}_V([c \,\dot{\wedge}\, c]) \,\dot{\wedge}\, \mathrm{d}_V c^+\rangle +\\
&+ \langle \mathrm{d}_V c \,\dot{\wedge}\, \mathrm{d}_V(\mathrm{d}_A A^+) + \mathrm{d}_V(\mathrm{ad}^*_c c^+) + \mathrm{d}_V(\mathrm{ad}^*_{B^+} B) + \mathrm{d}_V(\mathrm{ad}^*_{\tau_1^+}\tau_1) + \mathrm{d}_V(\mathrm{ad}^*_{\tau_2^+}\tau_2)\rangle +\\
&+ \langle \mathrm{d}_V(\mathrm{d}_A \tau_2) \,\dot{\wedge}\, \mathrm{d}_V \tau_1^+\rangle + \langle \mathrm{d}_V(\mathrm{ad}^*_c \tau_1) \,\dot{\wedge}\, \mathrm{d}_V \tau_1^+\rangle - \langle \mathrm{d}_V \tau_1 \,\dot{\wedge}\, \mathrm{d}_V(\mathrm{d}_A B^+) + \mathrm{d}_V([c \,\dot{\wedge}\, \tau_1^+])\rangle +\\
&+ \langle \mathrm{d}_V(\mathrm{ad}^*_c \tau_2) \,\dot{\wedge}\, \mathrm{d}_V \tau_2^+\rangle - \langle \mathrm{d}_V \tau_2 \,\dot{\wedge}\, \mathrm{d}_V(\mathrm{d}_A \tau_1^+) + \mathrm{d}_V([c \,\dot{\wedge}\, \tau_2^+]) + \mathrm{d}_V\Big(\frac{1}{2}[B^+ \,\dot{\wedge}\, B^+]\Big)\rangle\Big] =\\
=&\mathrm{d}_H\Big[- \langle \mathrm{d}_V c \,\dot{\wedge}\, \mathrm{d}_V A^+\rangle - \langle \mathrm{d}_V A \,\dot{\wedge}\, \mathrm{d}_V B\rangle - \langle \mathrm{d}_V \tau_1 \,\dot{\wedge}\, \mathrm{d}_V B^+\rangle - \langle \mathrm{d}_V \tau_2 \,\dot{\wedge}\, \mathrm{d}_V \tau_1^+\rangle\Big] = \mathrm{d}_H \overset{3}{\omega}.
\end{aligned}
$$

This follows from direct computations, $|\mathrm{d}_V| = 1$, and $\mathrm{d}_V \mathrm{d}_H = -\mathrm{d}_H \mathrm{d}_V$.

Finally, transgressing this structure on the infinite jet bundle to the space of fields via the map

$$
\mathbb{T} = \int_M \circ (ev^\infty)^* : \Omega^{4,\bullet}_{\mathrm{glb}}(J^\infty_M F) \longrightarrow \Omega^\bullet_{\mathrm{glb}}(\mathcal{F}),
$$

we obtain the local BV structure on $\mathcal{F}$ (see Eq. (3.58)). Note that the transgression changes the total degree by -4, as the top-horizontal component, which contributes with a degree of 4 at the level of $J^\infty_M F$, is sent to a real number of degree 0. Explicitly, $\mathfrak{X}_{\mathrm{loc}}(\mathcal{F})_1 \ni Q = \mathsf{Q} \circ j^\infty$ and[86]

$$
\begin{aligned}
\Omega &:= \mathbb{T}\overset{n}{\omega} = \int_M \langle \mathrm{d}_{\mathcal{F}} A \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} A^+\rangle + \langle \mathrm{d}_{\mathcal{F}} B \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} B^+\rangle + \langle \mathrm{d}_{\mathcal{F}} c \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} c^+\rangle + \langle \mathrm{d}_{\mathcal{F}} \tau_1 \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} \tau_1^+\rangle + \langle \mathrm{d}_{\mathcal{F}} \tau_2 \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} \tau_2^+\rangle;\\
S &:= \mathbb{T}\overset{n}{\mathsf{L}} = \int_M \langle B \,\dot{\wedge}\, F_A\rangle + \langle A^+ \,\dot{\wedge}\, \mathrm{d}_A c\rangle + \langle B^+ \,\dot{\wedge}\, \mathrm{ad}^*_c(B) + \mathrm{d}_A \tau_1\rangle + \langle \tau_1^+ \,\dot{\wedge}\, \mathrm{ad}^*_c(\tau_1) + \mathrm{d}_A \tau_2\rangle +\\
&\qquad + \frac{1}{2}\langle c^+ \,\dot{\wedge}\, [c \,\dot{\wedge}\, c]\rangle + \langle \tau_2^+ \,\dot{\wedge}\, \mathrm{ad}^*_c(\tau_2)\rangle + \frac{1}{2}\langle \tau_2 \,\dot{\wedge}\, [B^+ \,\dot{\wedge}\, B^+]\rangle \in C^\infty_{\mathrm{loc}}(\mathcal{F})_0.
\end{aligned}
$$

Note that, $\Omega \in \Omega^2_{\mathrm{loc}}(\mathcal{F})_{-1}$ and it is closed because it is exact, with primitive

$$
\alpha := \mathbb{T}\overset{n}{\alpha} = \int_M \langle\, \mathrm{d}_{\mathcal{F}} A \,\dot{\wedge}\, A^+\rangle + \langle B \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} B^+\rangle + \langle \mathrm{d}_{\mathcal{F}} c \,\dot{\wedge}\, c^+\rangle + \langle \tau_1 \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} \tau_1^+\rangle + \langle \tau_2 \,\dot{\wedge}\, \mathrm{d}_{\mathcal{F}} \tau_2^+\rangle \in \Omega^1_{\mathrm{loc}}(\mathcal{F})_{-1}.
$$

Conditions (3.57) and (3.56) are satisfied by construction as above. That is

$$
\delta S = \iota_Q \Omega \quad \text{and} \quad \{S, S\}_\Omega = 0.
$$

The latter is the rigorous understanding of the formal manipulations we did for BF theory in Section 2.3 (see Rmk. 2.3.1 and Eqs. (2.28), (2.29) and (2.30)). In particular, now the Cartan calculus is well defined.

Remark 3.4.7. Notice that the resulting structure is local and globally of finite order. That is, at the level of infinite jet bundles, the forms and vector fields we consider always factor globally through a finite order jet bundle of fixed degree. This is usually the case for physically relevant theories. In SmthSet, the subalgebra of globally of local order differential forms on the infinite jet bundle is canonically identified with a subalgebra of de Rham forms on the infinite jet bundle (cf. [GS25a][Lemma 4.15]). The latter is a categorical notion of forms defined as natural transformations into the smooth moduli space of de Rham forms (cf. [GS25a][Def. 2.30] and see ft. (12) at p. 58). Working within this subalgebra would provide a cleaner categorical approach to the full Cartan calculus (cf. [GS25a][Def. 2.33]), with the contraction induced by the contraction on globally finite-order forms. If adapted to the smooth and thickened structure, this could be a direction to investigate, as it would provide a nice functorial approach for locally describing classical BV theories.

In summary, under specific assumptions, sThSmthSet locally allows for a categorical approach to the BV formalism via the notion of local BV systems.

[86] Pulling back along the infinite evaluation locally means substituting $\mathrm{d}_{V/H}$ with $\mathrm{d}_{\mathcal{F}/M}$.

Conclusions

This thesis studies the BV formalism (cf. [Mne19]), the categories SmthSet and ThSmthSet (cf. [GS25a; GS25b]), and explains how the super extension of the latter (cf. [Gio25; GS26]) naturally encodes the local classical BV formalism (cf. [Gri23]). This yields a rigorous understanding of the Cartan calculus on the infinite-dimensional BV space of fields.
As an example, we present the AKSZ construction of the BV extension of classical BF theory in $d = 4$. Unfortunately, both a complete treatment of the AKSZ formalism (see Rmk. 3.4.3) and a global formulation of BV theories (see Rmk. 3.4.2) require further extensions of sThSmthSet.

Three gauge-fixing methods are introduced in the finite-dimensional case to address the degeneracy arising in the perturbative path integral quantization of gauge theories (see Sec. 2.2): the Faddeev–Popov method, the BRST formalism, and the BV formalism. Being the most general, the BV formalism subsumes the other two. Although the latter is well understood in the finite-dimensional case, the discussion becomes more delicate when the space of fields is infinite-dimensional (cf. [CMS25]). In particular, one lacks the notion of Cartan calculus and of a measure on the space of fields, essential for the classical and quantum versions of the BV formalism. Although the measure-theoretic issues are not resolved within sThSmthSet, the Cartan calculus is rigorously understood upon restriction to the local setting.
To establish this result, we show that SmthSet naturally incorporates the action principle and many fundamental field-theoretic constructions, with a focus on bosonic Lagrangian field theories (see Sec. 3.1). The strength of this framework lies in its ability to perform the standard operations of finite-dimensional differential geometry pointwise on ordinary manifolds, thereby extending them to smooth sets. In particular, through the use of the infinite jet bundle, it allows one to formalize the local Cartan calculus on the field space when restricted to well-behaved de Rham forms (see Prop. 3.1.13). Its thickening ThSmthSet is introduced to recover ad hoc tangent bundle constructions via the synthetic tangent bundle (see Sec. 3.2). Following [Gio25; GSS24], the super extension sThSmthSet is briefly outlined (see Sec. 3.3). This is needed to describe the odd nature of ghost fields. sThSmthSet should be fully presented in [GS26]. Finally, this setting is used to rigorously formulate the local BV formalism. As an example, the local BV extension of the BF theory in $d = 4$ is explicitly given (see Sec. 3.4).

We emphasize that the approach we use is purely local, meaning that the gauge structure is described via a trivial G-bundle (see Rmk. 2.1.1). This allows one to describe gauge fields, i.e. 1-form connections, as $\mathfrak{g}$-valued 1-forms over the base manifold M and to describe the BV space of fields within Definition 3.3.3. To formulate a global categorical description of gauge theories and the BV formalism, one should enrich sThSmthSet with higher structure, considering a category of sheaves of groupoids (cf. [GS25a][Rmk. 6.37 & Ch. 8]). A derived approach to the global description of BV/AKSZ theory can be found in (cf. [Ste24]). Along these lines, a possible direction is to consider a derived extension of SmthSet, using the universal property of derived manifolds and their relation to dg-manifolds (cf. [CS19; Car23]).
Furthermore, the general AKSZ formalism is not yet described in this setting. Indeed, although the mapping space is well understood in sThSmthSet (see Def. 3.3.1), the category still does not provide it with the full Cartan calculus. This difficulty can be circumvented by assuming that the target is a graded vector space (see Eq. (2.22)), but this means that the AKSZ formal construction can only be used as an Ansatz for the local BV system, which then has to be checked to be well defined (see Sec. 3.4). A general description of the AKSZ formalism could be gained once the notion of graded infinite jet bundle is extended to also allow a graded base (see Rmk. 3.4.3 and [Vys23]).
Finally, while this framework is powerful and promising, it does not yet provide a straightforward way to define the Cartan calculus universally on forms defined via the de Rham moduli space (see Rmk. 3.4.7), which would give a cleaner universal description of local BV systems. These questions remain open and motivate further research in this direction.

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