

Batalin-Vilkovisky formalism in topological field theory

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Abstract

This thesis is comprised of 8 papers where we develop different aspects of perturbative topological field theory revolving around the Batalin-Vilkovisky framework for cohomological treatment of gauge symmetry in the perturbative path integral. In particular, we present interesting examples of the pushforward construction for solutions of the Batalin-Vilkovisky quantum master equation; we develop an extension of the Batalin-Vilkovisky formalism to topological field theories (in fact, to general gauge theories) on spacetime manifolds with boundary. We also develop an algebraic version of a special class of topological field theories – the AKSZ sigma models – based on differential graded Frobenius algebras, with the example of Chern-Simons theory studied in particular detail.

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Preface

Here we sum up some basic background material on topological field theories and on the Batalin-Vilkovisky formalism and give a brief account of the results of the papers included in the present Habilitationsschrift.

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1. INTRODUCTION

1.1. Topological quantum field theory: historical background. Topological quantum field theory was conceived in the works of A. S. Schwarz [43] and E. Witten [49] as a tool for constructing invariants of manifolds and knots. It arose as a spin-off of the quantum field theoretic idea of the integral of the exponential of a local (and diffeomorphism-invariant, in case of a topological theory) action functional over a space of local fields over the spacetime manifold – the functional integral (or “path integral” by analogy with Feynman’s path integrals representing evolution kernels in quantum mechanics). The path integral studied by Schwarz in [43] corresponds to the so-called “abelian BF theory” and produces the Ray-Singer torsion of a manifold, which is known, by comparison to Reidemeister torsion via the Cheeger-Müller theorem, to be invariant under simple homotopy equivalences. In [49], Witten studied the path integral over the space of connections in a principal bundle over a closed oriented 3-manifold with the action given by the integral of the Chern-Simons 3-form associated to the connection. Witten showed that the value of the path integral in this case is a diffeomorphism invariant of framed oriented closed 3-manifolds and presented a way to calculate this invariant through surgery. He also suggested a deformation of this path integral by the product of holonomies of the connection along components of a fixed link and showed that this deformation yields the Jones polynomial of the link evaluated at a root of unity (in the case of connections in principal $SU(2)$ -bundles and with holonomy evaluated in the fundamental representation).

Based on the axiomatic definition of conformal field theory by G. Segal [45], M. Atiyah [5] suggested an axiomatic definition of an $(n + 1)$ -dimensional topological quantum field theory (TQFT) as an association, mapping every closed n -manifold to a vector space (the “space of states”) and every $(n + 1)$ -manifold with boundary to a vector (the “partition function”) in the space of states associated to the boundary n -manifold. This association has to satisfy a set of axioms, in particular the multiplicativity axiom and the gluing axiom, expressing the partition function of an $(n + 1)$ -manifold with two diffeomorphic boundary components glued together in terms of the partition function for the original $(n + 1)$ -manifold. An immediate consequence of the axioms is that the partition function for a closed $(n + 1)$ -manifold is a number, invariant under diffeomorphisms of the manifold; using the gluing axiom this number can be recovered, once the manifold is cut into simpler pieces, from values of the TQFT on the pieces. In algebraic language, Atiyah’s topological quantum field theory is a functor of symmetric monoidal categories from the category of $(n + 1)$ -cobordisms to the category of vector spaces. Atiyah’s definition summarized the known properties of a collection of manifold invariants, e.g.

Floer-Gromov theory ($n = 1$) and Donaldson-Floer invariants ($n = 3$), as well as properties of the Chern-Simons path integral ($n = 2$) studied by Witten.

The idea of constructing topological quantum field theories via path integrals immediately runs into a number of problems:

- (I) Path integrals over spaces of local fields typically cannot be defined as measure theoretic integrals. Instead one can try to define the “perturbative” path integral by postulating the stationary phase approximation formula as the definition of the path integral, giving the latter as a sum over critical points of the action of sums of contributions of Feynman diagrams evaluated at a critical point (cf. e.g. [27], see also [39] for a modern exposition).
- (II) To apply, even formally, the stationary phase formula, one needs the critical points of the action to be isolated. Typically this is not the case with topological Lagrangians due to the presence of a local symmetry – the gauge symmetry. There are several ways out: Faddeev-Popov [25], BRST [12, 48] and Batalin-Vilkovisky [10, 11] constructions of gauge-fixing (in the chronological order, each a generalization of previous one). Generally they amount to extending the space of fields to a superspace and extending the action by additional terms depending on the newly introduced fields in such a way that the critical points of the new action are isolated and thus one can proceed with the stationary phase formula.
- (III) The gauge fixing procedure needed to define Feynman diagrams for the path integral typically destroys the manifest diffeomorphism invariance of the Lagrangian by introducing dependence on an additional geometric structure, e.g. a Riemannian metric on the spacetime manifold (chosen arbitrarily for the gauge fixing). One has to prove then that the resulting partition functions are independent on the chosen geometric structure. (In fact, the situation can be more subtle, e.g. Witten [49] discovered that the phase of the 1-loop Chern-Simons partition function is metric-dependent, though the metric dependence can be cancelled by a local counterterm, at the cost of introducing dependence on the framing.)
- (IV) Values of Feynman diagrams in the perturbative path integral contain integrals over configuration spaces of points on the spacetime manifold, with the integrand a product of propagators, singular on diagonals. One has to prove finiteness (convergence) of the Feynman diagrams, possibly after a suitable renormalization procedure.
- (V) Finally, one has to extend the path integral construction of TQFT partition functions to allow the spacetime manifold to have a boundary, in such a way that Atiyah’s gluing axiom is satisfied.

A purely perturbative treatment of the Chern-Simons path integral (as opposed to Witten’s treatment of [49], where certain properties of the path integral, e.g. extendability to an Atiyah’s TQFT on 3-manifolds with boundary and relation to 2-dimensional conformal field theory on the boundary, are assumed, instead of being derived from some definition) was given by S. Axelrod and I. M. Singer in [6, 7]. They constructed the perturbative Chern-Simons partition function on a *closed* oriented 3-manifold on the background of a fixed *acyclic* flat connection (a chosen critical point of the Chern-Simons action functional) as a formal power series in formal parameter \hbar , with finitely many Feynman graphs contributing to each order in \hbar . Feynman diagrams depend explicitly, via the propagator, on the

choice of an arbitrary Riemannian metric on the 3-manifold, necessary to do the Faddeev-Popov gauge fixing. The main results of Axelrod and Singer are:

- Finiteness of Feynman diagrams – Axelrod and Singer proved that the products of propagators involved lift to smooth differential forms on the Fulton-Macpherson compactification of the configuration space of points on the 3-manifold.
- Metric independence of the resulting partition function. More precisely, Axelrod and Singer proved (extending Witten’s 1-loop result to higher loop corrections) that there is metric dependence, but it can be cancelled by a local counterterm, proportional to the Chern-Simons action functional evaluated on the Levi-Civita connection, dependent on the choice of an arbitrary framing of the 3-manifold.

Another important example of topological quantum field theory constructed via path integral quantization is the 2-dimensional Poisson sigma model (originally considered in [29, 42]) which, in the case when the spacetime manifold is a disk, with certain boundary condition for fields imposed at the boundary circle, plays a crucial role in Kontsevich’s deformation quantization of Poisson manifolds [32, 17]. Here, in order to make sense of perturbation theory for the path integral, one is forced to employ the Batalin-Vilkovisky (BV) formalism [10], since the gauge symmetry is not given by a group action on the space of fields, but is instead a non-integrable distribution on the space of fields, and thus the less general Faddeev-Popov and BRST constructions do not work.

We should stress that neither quantum Chern-Simons theory, nor quantum Poisson sigma model are constructed by a perturbative path integral on manifolds with boundary, as an Atiyah’s TQFT (in Poisson sigma model on a disk one does indeed have a boundary, but fixing *one* boundary condition is not enough to be able to glue; one should instead allow a family of boundary conditions given by a Lagrangian foliation of the boundary phase space). In fact, the only example of perturbative Atiyah’s TQFT constructed so far seems to be the 1-dimensional Chern-Simons theory [4]. Papers [4, 21, 22, 3] are the first steps in the project of the author together with A. S. Cattaneo and N. Reshetikhin, aimed at constructing Atiyah’s TQFTs by path integral quantization of diffeomorphism-invariant Lagrangians (e.g. those provided by the AKSZ construction [1]) on manifolds with boundary. One outcome of the project would be the extension of Axelrod-Singer’s construction of the Chern-Simons path integral on closed 3-manifolds to a full Atiyah’s TQFT.

We should also mention that there are well known constructions of Atiyah’s TQFTs not coming from quantization (of a classical field theory) and not relying on perturbative path integrals. In particular, there are Turaev-Viro [47] and Reshetikhin-Turaev [40] constructions of 3-dimensional Atiyah’s TQFTs based on the representation theory of quantum groups.

In [1], Aleksandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) suggested a geometric construction of local diffeomorphism-invariant functionals on mapping spaces of the form $\text{Map}(T[1]\Sigma, \mathcal{M})$ where Σ is a manifold (the spacetime) and the target \mathcal{M} is a graded supermanifold endowed with differential graded symplectic structure of appropriate degree; $T[1]\Sigma$ stands for the tangent bundle to Σ with fiber coordinates prescribed degree 1. The AKSZ construction is from the start consistent with the Batalin-Vilkovisky formalism: the resulting action functionals automatically satisfy the classical master equation. As particular examples of the

AKSZ construction, one gets Batalin-Vilkovisky extensions of the Chern-Simons action, the Poisson sigma model, the BF theory, the Courant sigma model [41], Witten's A and B models and more.

1.2. What is done in the papers. We will give a more detailed account of the results in Section 4, here we just outline the general logic of the development.

- In [20] we construct a finite-dimensional algebraic model for Chern-Simons action on closed 3-manifolds, based on differential graded Frobenius algebras (in [15] this construction is extended to general AKSZ sigma models). We study the BV pushforward to cohomology in this model and the arising invariants of dg Frobenius algebras. Then we apply this construction in the case of standard Chern-Simons theory, obtaining a generalization of Axelrod-Singer's treatment of Chern-Simons path integral to non-acyclic flat background connections.
- In [13] we study the BV pushforward to zero modes for the Poisson sigma model on a closed surface. We prove that in genus 1 or for a regular unimodular target Poisson structure there are no quantum corrections in the effective action on zero modes. In genus 1 case, we perform the non-perturbative integration over zero-modes, yielding (under further assumptions on the target Poisson structure) the Euler characteristic of the target.
- In [4] we construct simplicial 1-dimensional Chern-Simons theory as Atiyah's TQFT on triangulated 1-cobordisms. Two definitions of the partition functions are given (and proven to coincide): a perturbative path integral definition and a non-perturbative one, expressing partition functions in terms of exponentials in Clifford algebra of the gauge Lie algebra. The partition functions are proven to satisfy the Batalin-Vilkovisky quantum master equation extended by boundary terms. The space of states associated to a point here is the spinor module, which arises as the geometric quantization of the parity-shifted gauge Lie algebra in operator formalism and as the space of functions on the parameter space of the chosen family of boundary conditions in the path integral formulation. It is an important observation that the cubic Dirac operator on the spinor module arising as the boundary term of the quantum master equation in this model is the geometric quantization of the target Hamiltonian in the AKSZ formulation of 1-dimensional Chern-Simons theory.
- In [21, 22] we extend the classical Batalin-Vilkovisky formalism to gauge theories on manifolds with boundary. In this formalism, the graded phase space (the space of boundary fields) associated to the boundary of the spacetime manifold carries a Batalin-Fradkin-Vilkovisky structure, i.e. is equipped with a degree 0 symplectic form and a degree 1 function (the BFV action) generating, as its Hamiltonian vector field, the cohomological vector field. This structure is in perfect accord with the quantum structure appearing in [4]. The work [21] is the first step in constructing perturbative Atiyah's TQFTs, e.g. starting from AKSZ Lagrangians.
- In [3] we apply the construction of [21] to Chern-Simons theory on a 3-manifold with boundary. We also incorporate the Wilson lines, possibly ending on the boundary, using the enhancement of AKSZ sigma models by observables suggested in [37]. Then we proceed with geometric quantization of the boundary BFV phase space. The cohomology in degree 0 of the

corresponding quantized BFV action is, at least in the case of a boundary surface of genus 0 or 1, the space of conformal blocks of the Wess-Zumino-Witten model.

Overall, the results group around the following subjects:

- (i) Effective BV actions for topological field theories (examples of pushforward construction for solutions of the master equation), [20, 13, 4, 37].
- (ii) Incorporation of spacetime boundary and Atiyah's gluing in the BV formalism, [4, 21, 22, 3].
- (iii) Algebraic finite-dimensional models for topological actions, [20, 15].
- (iv) Observables in AKSZ theories [37, 3].

1.3. Prospects. The results of the papers presented in this thesis (in particular, [4, 21, 22, 3]) can be regarded as steps towards the construction of a class of examples of functorial topological quantum field theories in the sense of Atiyah (and, more ambitiously, extended TQFTs in the sense of Baez-Dolan-Lurie) from perturbative path integral quantization of gauge- and diffeomorphism-invariant action functionals (in particular, AKSZ sigma models provide natural testing grounds for such a quantization) on manifolds with boundary or corners, in a way consistent with gluing. The natural framework for the treatment of gauge symmetry is provided by the Batalin-Vilkovisky formalism, extended to the setting with boundary/corners as in [21].

As a part of this program, Axelrod-Singer's perturbative treatment of Chern-Simons theory on closed 3-manifolds [6, 7] can be extended to 3-manifolds with boundary, constructing Chern-Simons theory as Atiyah's TQFT by the perturbative path integral and providing a mathematically transparent explanation for Witten's non-perturbative results of [49]. Another anticipated result is the proof of a long-standing conjecture that the $k \rightarrow \infty$ asymptotics of the Reshetikhin-Turaev invariant [40] coincides with the perturbative partition function in Chern-Simons theory (with Atiyah's canonical framing on the 3-manifold).

The output of the quantization program outlined above in algebraic topology would be in constructing the gluing-cutting formulae for invariants of manifolds and knots.

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2. PAPERS INCLUDED

Here we list the abstracts of the papers comprising the thesis.

1. [20] *Remarks on Chern-Simons invariants*, with A. S. Cattaneo.

The perturbative Chern-Simons theory is studied in a finite-dimensional version or assuming that the propagator satisfies certain properties (as is the case, e.g., with the propagator defined by Axelrod and Singer). It turns out that the effective BV action is a function on cohomology (with shifted degrees) that solves the quantum master equation and is defined modulo certain canonical transformations that can be characterized completely. Out of it one obtains invariants.

2. [15] *Finite-dimensional AKSZ-BV theories*, with F. Bonechi and M. Zabzine.

We describe a canonical reduction of AKSZ-BV theories to the cohomology of the source manifold. We get a finite-dimensional BV theory that describes the contribution of the zero modes to the full QFT. Integration can be defined and correlators can be computed. As an illustration of the general construction, we consider two-dimensional Poisson sigma model and three-dimensional Courant sigma model. When the source manifold is compact, the reduced theory is a generalization of the AKSZ construction where we take as source the cohomology ring. We present the possible generalizations of the AKSZ theory.

3. [4] *One-dimensional Chern-Simons theory*, with A. Alekseev.

We study a one-dimensional toy version of the Chern-Simons theory. We construct its simplicial version which comprises features of a low-energy effective gauge theory and of a topological quantum field theory in the sense of Atiyah.

4. [13] *The Poisson sigma model on closed surfaces*, with F. Bonechi and A. S. Cattaneo.

Using methods of formal geometry, the Poisson sigma model on a closed surface is studied in perturbation theory. The effective action, as a function on vacua, is shown to have no quantum corrections if the surface is a torus or if the Poisson structure is regular and unimodular (e.g., symplectic). In the case of a Kähler structure or of a trivial Poisson structure, the partition function on the torus is shown to be the Euler characteristic of the target; some evidence is given for this to happen more generally. The methods of formal geometry introduced in this paper might be applicable to other sigma models, at least of the AKSZ type.

5. [21] *Classical BV theories on manifolds with boundary*, with A. S. Cattaneo and N. Reshetikhin.

In this paper we extend the classical BV framework to gauge theories on spacetime manifolds with boundary. In particular, we connect the BV construction in the bulk with the BFV construction on the boundary and we develop its extension to strata of higher codimension in the case of manifolds with corners. We present several examples including electrodynamics, Yang-Mills theory and topological field theories coming from the AKSZ construction, in particular, the Chern-Simons theory, the BF theory, and the Poisson sigma model. This paper is the first step towards developing the perturbative quantization of such theories on manifolds with boundary in a way consistent with gluing.

6. [22] *Classical and quantum Lagrangian field theories with boundary*, with A. S. Cattaneo and N. Reshetikhin.

This note gives an introduction to Lagrangian field theories in the presence of boundaries. After an overview of the classical aspects, the cohomological formalisms to resolve singularities in the bulk and in the boundary theories (the BV and the BFV formalisms, respectively) are recalled. One of the goals here (and in [21]) is to show how the latter two formalisms can be put together in a consistent way, also in view of perturbative quantization.

7. [37] *A construction of observables for AKSZ sigma models*.

A construction of gauge-invariant observables is suggested for a class of topological field theories, the AKSZ sigma-models. The observables are associated to extensions of the target Q-manifold of the sigma model to a Q-bundle over it with additional Hamiltonian structure in fibers.

8. [3] *Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism*, with A. Alekseev and Y. Barmaz.

We consider the Chern-Simons theory with Wilson lines in 3D and in 1D in the BV-BFV formalism of Cattaneo-Mnev-Reshetikhin. In particular, we allow for Wilson lines to end on the boundary of the spacetime manifold. In the toy model of 1D Chern-Simons theory, the quantized BFV boundary action coincides with the Kostant cubic Dirac operator which plays an important role in representation theory. In the case of 3D Chern-Simons theory, the boundary action turns out to be the odd (degree 1) version of the BF model with source terms for the B field at the points where the Wilson lines meet the boundary. The boundary space of states arising as the cohomology of the quantized BFV action coincides with the space of conformal blocks of the corresponding WZW model.

3. BACKGROUND

3.1. Batalin-Vilkovisky formalism. The Batalin-Vilkovisky formalism [10, 11], sometimes also referred to as “symplectic BRST formalism”, was suggested as an enhancement of the BRST formalism [12, 48] for the gauge fixing of Lagrangian field theories with local symmetry, thus curing the problem of non-isolated critical points of the action obstructing the construction of the perturbative path integral. The Batalin-Vilkovisky formalism is applicable to a larger class of field theories than the BRST formalism and is the most general known method of gauge-fixing. Apart from being unavoidable in some examples (e.g. in case of the Poisson sigma model [32, 17] and in non-abelian BF theory in dimension ≥ 4 , cf. [23]), the BV formalism turned out to be a particularly elegant language in case of topological field theories, emphasizing the supergeometric origins (and permitting the superfield formulation) of diffeomorphism-invariant Lagrangians [1]. The geometric structure underlying the Batalin-Vilkovisky formalism was elucidated in the works of E. Witten [50], A. Schwarz [44], H. Khudaverdian [30], P. Ševera [46].

3.1.1. BV algebra. One can summarize the structure of the BV formalism as follows. First, one defines a *BV algebra* as a quadruple $\{\mathcal{A}^\bullet, \cdot, \{, \}, \Delta\}$ consisting of:

- A \mathbb{Z} -graded vector space \mathcal{A}^\bullet .¹
- A degree 0 multiplication $\cdot : \mathcal{A}^p \otimes \mathcal{A}^q \rightarrow \mathcal{A}^{p+q}$ satisfying supercommutativity $yx = (-1)^{pq}xy$ and associativity $(xy)z = x(yz)$ properties; one also assumes that the multiplication possesses a unit.
- A degree 1 bi-derivation, the “odd Poisson bracket”² $\{\bullet, \bullet\} : \mathcal{A}^p \otimes \mathcal{A}^q \rightarrow \mathcal{A}^{p+q+1}$ satisfying

$$\begin{aligned} \{y, x\} &= -(-1)^{(p+1)(q+1)}\{x, y\}, \\ \{xy, z\} &= x\{y, z\} + (-1)^{q(r+1)}\{x, z\}y, \\ \{x, \{y, z\}\} &= \{\{x, y\}, z\} + (-1)^{(p+1)(q+1)}\{y, \{x, z\}\} \end{aligned}$$

– graded skew-symmetry, derivation property and Jacobi identity. Here we assume that $x \in \mathcal{A}^p, y \in \mathcal{A}^q, z \in \mathcal{A}^r$ are homogeneous elements.

¹I.e. a collection of vector spaces \mathcal{A}^p indexed by $p \in \mathbb{Z}$.

²Also known as “anti-bracket”, “BV bracket” or “Gerstenhaber bracket”.

- A degree 1 linear operator $\Delta : \mathcal{A}^p \rightarrow \mathcal{A}^{p+1}$ (the “BV Laplacian”) squaring to zero

$$\Delta^2 = 0$$

satisfying the second order derivation property

$$\Delta(xyz) = \Delta(xy)z + (-1)^{qr} \Delta(xz)y + (-1)^p x \Delta(yz) - (\Delta x)yz - (-1)^p x(\Delta y)z - (-1)^{p+q} xy(\Delta z)$$

and the normalization property $\Delta(1) = 0$, and generating the odd Poisson bracket as the defect of the first order Leibnitz identity:

$$\Delta(xy) = (\Delta x)y + (-1)^p x(\Delta y) + (-1)^p \{x, y\}$$

(The set of axioms above is redundant: properties of $\{, \}$ follow from axioms for Δ ; excessive axioms are provided for exposition purposes.) The set of data $(\mathcal{A}^\bullet, \cdot, \{, \})$ comprises the so-called Gerstenhaber (or odd Poisson) algebra. Thus a BV algebra is in particular a Gerstenhaber algebra enriched by an additional structure – the BV Laplacian. Another remark is that, by forgetting the multiplication, a BV algebra $(\mathcal{A}^\bullet, \cdot, \{, \}, \Delta)$ induces a differential graded Lie algebra (dgLa) $(\mathcal{A}^{\bullet-1}, \{, \}, \Delta)$.

3.1.2. *Quantum and classical master equation.* Given a BV algebra $(\mathcal{A}^\bullet, \cdot, \{, \}, \Delta)$ one calls the equation

$$\{S, S\} = 0$$

for an element $S \in \mathcal{A}^0$ the *classical master equation* (CME).

The equation

$$\Delta e^{\frac{i}{\hbar} S} = 0$$

for an element $S \in \mathcal{A}^0[[\hbar]]$ in the ring of formal power series in a formal variable \hbar with coefficients in \mathcal{A}^0 is called the *quantum master equation* (QME). Equivalently, it can be written in the Maurer-Cartan form:

$$(1) \quad \frac{1}{2} \{S, S\} - i\hbar \Delta S = 0$$

Given a solution $S = S^{(0)} + \hbar S^{(1)} + \hbar^2 S^{(2)} + \dots$ of the quantum master equation with $S^{(k)} \in \mathcal{A}^0$, the zeroth term $S^{(0)}$ automatically satisfies the classical master equation and the higher order corrections satisfy the sequence of equations

$$\{S^{(0)}, S^{(1)}\} - i\Delta S^{(0)} = 0, \quad \{S^{(0)}, S^{(2)}\} + \frac{1}{2} \{S^{(1)}, S^{(1)}\} - i\Delta S^{(1)} = 0, \quad \dots$$

The obstructions to order-by-order extension of a solution $S^{(0)}$ of CME to a solution of $S = S^{(0)} + O(\hbar)$ of QME lie in cohomology of the coboundary operator $\{S^{(0)}, \bullet\} : \mathcal{A}^p \rightarrow \mathcal{A}^{p+1}$ in degree 1. (Hence if this obstruction space vanishes, all solutions of CME extend to solutions of QME.) In the special case $S = S^{(0)}$ – a solution of QME independent of \hbar – QME is equivalent to the pair of equations $\{S^{(0)}, S^{(0)}\} = 0$, $\Delta S^{(0)} = 0$ holding simultaneously.

For a solution $S^{(0)}$ of the classical master equation, the corresponding degree 1 derivation $Q = \{S^{(0)}, \bullet\} \in \text{Der}(\mathcal{A}^\bullet)_1$ (sometimes called the BRST operator) automatically satisfies the property $Q^2 = 0$.

If S, S' are two solutions of QME related by

$$e^{\frac{i}{\hbar} S'} = e^{\frac{i}{\hbar} S} + \Delta \left(e^{\frac{i}{\hbar} S} R \right)$$

one says that S and S' are *equivalent*, or related by a (*quantum*) *canonical BV transformation* with generator $R \in \mathcal{A}^{-1}[[\hbar]]$. An infinitesimal canonical transformation (i.e. with an infinitesimal generator) can be expressed as³

$$S' = S + \{S, R\} - i\hbar\Delta R + O(R^2)$$

3.1.3. *Examples of BV algebras.* The main source of examples of BV algebras is the following construction [44]: assume that \mathcal{F} is a finite-dimensional⁴ graded supermanifold endowed with a degree -1 symplectic structure $\omega \in \Omega^2(\mathcal{F})$ (the *BV 2-form*) and a volume element (Berezinian) μ . Then one constructs the BV algebra structure $\mathcal{A}^\bullet = C^\infty(\mathcal{F})$ with grading coming from the grading of \mathcal{F} ; the product is the pointwise product of functions, the odd Poisson bracket $\{, \}$ is induced from ω by $\{f, g\} = \check{f}g$ where the Hamiltonian vector field $\check{f} \in \mathfrak{X}(\mathcal{F})$ is defined by $\iota_{\check{f}}\omega = \delta f$.⁵ The BV Laplacian is constructed as

$$\Delta f = \frac{1}{2} \operatorname{div}_\mu \check{f}$$

where $\operatorname{div}_\mu : \mathfrak{X}(\mathcal{F}) \rightarrow C^\infty(\mathcal{F})$ is the divergence of a vector field associated to the chosen volume element μ by $\int_{\mathcal{F}} \mu v(g) = - \int_{\mathcal{F}} \mu (\operatorname{div}_\mu v) g$ for a fixed vector field v and any test function g . The requirement that $\Delta^2 = 0$ imposes a restriction on ω and μ . Following [44], we will call the set of data $(\mathcal{F}, \omega, \mu)$ (assuming that the induced BV Laplacian does square to zero) an *SP manifold*.

In this example, if $S^{(0)}$ is a solution of CME, the degree 1 Hamiltonian vector field $Q = \{S^{(0)}, \bullet\} \in \mathfrak{X}(\mathcal{F})_1$ satisfies $Q^2 = \frac{1}{2}[Q, Q] = 0$, i.e. it is (by definition) a cohomological vector field on \mathcal{F} .

A special example of the construction above arises when one takes M an (ordinary) manifold with a volume form $\nu \in \Omega^{\operatorname{top}}(M)$ and constructs the BV algebra $\mathcal{V}^{-\bullet}(M)$ – polyvector fields on M with reversed grading. The product is the wedge product of polyvectors, the odd Poisson bracket $\{, \} = [\cdot, \cdot]_{NS}$ is the Nijenhuis-Schouten bracket of polyvectors and the BV Laplacian is $\Delta = \operatorname{div}_\nu : \mathcal{V}^{-\bullet}(M) \rightarrow \mathcal{V}^{-\bullet-1}(M)$ – the divergence of a polyvector field with respect to ν , lowering the degree of a polyvector by 1. This example embeds into the construction above by setting $\mathcal{F} = T^*[-1]M$ – the cotangent bundle with shifted homological degree of the fiber coordinates, carrying a canonical degree -1 symplectic form ω (as a cotangent bundle) and endowed with the Berezinian $\mu = \nu^{\otimes 2}$ (we are using here the canonical isomorphism of line bundles over M , $\operatorname{Ber}(T^*[-1]M) \cong \operatorname{Det}(M)^{\otimes 2}$).

Another important special case is when \mathcal{F} is a graded vector space \mathcal{F}^\bullet with a translation-invariant symplectic form $\omega \in \Omega^2(\mathcal{F}^\bullet)$. Then there is a canonical choice of translation-invariant volume element μ , defined up to normalization. This induces a canonical BV algebra structure on $C^\infty(\mathcal{F}^\bullet)$.

³Note that modulo \hbar , this is a Hamiltonian transformation of S with generator R (hence the name “canonical transformation”).

⁴Finite-dimensionality is an important simplifying assumption that we will make to clarify the structure, despite the fact that it excludes spaces of fields of local field theories (which are typically Fréchet manifolds). Local field theories can be cast in BV formalism with an appropriate renormalization procedure, cf. e.g. the approaches of [24, 35, 36].

⁵We denote the de Rham differential on (forms on) \mathcal{F} by δ , since we want to reserve d for the de Rham differential on spacetime manifolds.

3.1.4. *BV integrals.* Let $(\mathcal{F}, \omega, \mu)$ be an *SP* manifold and $S \in C^\infty(\mathcal{F})_0$ (subscript denotes the degree of the function) a solution of the quantum master equation in the associated BV algebra. A *BV integral* is an expression of the form

$$Z_{S, \mathcal{L}} = \int_{\mathcal{L} \subset \mathcal{F}} e^{\frac{i}{\hbar} S} \sqrt{\mu}|_{\mathcal{L}}$$

where \mathcal{L} is a Lagrangian submanifold of \mathcal{F} with respect to the odd symplectic structure ω and $\sqrt{\mu}|_{\mathcal{L}}$ is the volume element on \mathcal{L} canonically induced from μ , cf. [44]. We assume that S and \mathcal{L} are such that the integral converges. The crucial properties of the BV integral (known as the BV-Stokes theorem, [10, 44]) are:

- (i) For two solutions S, S' of QME related by a canonical transformation and $\mathcal{L} \subset \mathcal{F}$ a fixed Lagrangian, one has $Z_{S, \mathcal{L}} = Z_{S', \mathcal{L}}$.
- (ii) For S a fixed solution of QME and $\mathcal{L}, \mathcal{L}' \subset \mathcal{F}$ two Lagrangians which can be connected by a smooth family of Lagrangians in \mathcal{F} (we will call, somewhat abusively, the corresponding equivalence relation on Lagrangians the *Lagrangian homotopy*), one has $Z_{S, \mathcal{L}} = Z_{S, \mathcal{L}'}$.

3.1.5. *Fiber BV integrals (pushforward of solutions of the quantum master equation).* Assume $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$ is a direct product of two *SP* manifolds, with $\omega = \omega' + \omega''$ and $\mu = \mu' \cdot \mu''$. The corresponding BV algebra is the (completed) tensor product $\mathcal{A} = \mathcal{A}' \hat{\otimes} \mathcal{A}''$ of the BV algebras for $\mathcal{F}', \mathcal{F}''$, with BV Laplacian on \mathcal{F} splitting as $\Delta = \Delta' + \Delta''$. Then, given a solution S of QME on \mathcal{F} can construct a solution of QME on \mathcal{F}' , $S' \in C^\infty(\mathcal{F}')_0[[\hbar]]$ by means of the *fiber BV integral* [33, 36, 37]:

$$(2) \quad e^{\frac{i}{\hbar} S'} = \int_{\mathcal{L}'' \subset \mathcal{F}''} e^{\frac{i}{\hbar} S} \sqrt{\mu''}|_{\mathcal{L}''}$$

Here \mathcal{L}'' is a Lagrangian submanifold of \mathcal{F}'' and the r.h.s. is to be understood as an integral over a Lagrangian submanifold of the fiber of the bundle $\mathcal{F}' \times \mathcal{F}'' \rightarrow \mathcal{F}'$ parameterized by a point of \mathcal{F}' . One calls S' an *effective BV action* induced from S on \mathcal{F}' . The main properties of this construction (cf. [36, 37] for details), generalizing the properties of the BV integrals (the BV-Stokes theorem) are:

- (i) The effective action S' as defined by (2) is indeed a solution of QME on \mathcal{F}' , provided that S is a solution of QME on \mathcal{F} .
- (ii) If \mathcal{L}'' and $\tilde{\mathcal{L}}''$ are two Lagrangian submanifolds of \mathcal{F}'' connected by a Lagrangian homotopy and S is a solution of QME on \mathcal{F} , then the two corresponding effective actions S', \tilde{S}' are related by a quantum BV canonical transformation. Moreover, if $\tilde{\mathcal{L}}''$ is a graph of the 1-form $\delta\Psi$ for $\Psi \in C^\infty(\mathcal{L}'')_{-1}$, then one can write the generator of the canonical transformation $R' \in C^\infty(\mathcal{F}')_{-1}[[\hbar]]$ explicitly in terms of Ψ :

$$(3) \quad R' = e^{-\frac{i}{\hbar} S'} \int_{\mathcal{L}''} \Psi e^{\frac{i}{\hbar} S} \sqrt{\mu''}|_{\mathcal{L}''}$$

- (iii) If S and \tilde{S} are two solutions of QME on \mathcal{F} related by a canonical transformation with generator R and the Lagrangian \mathcal{L}'' is fixed, then the two corresponding effective actions S', \tilde{S}' are related by a canonical transformation with generator given again by formula (3) where one substitutes $\Psi \mapsto R$.

Thus the fiber BV integral construction is a *pushforward*, mapping solutions of QME on \mathcal{F} modulo canonical transformations to solutions of QME on \mathcal{F}' modulo

canonical transformation. The pushforward depends on a choice of Lagrangian $\mathcal{L}'' \subset \mathcal{F}''$ modulo Lagrangian homotopy.

3.1.6. *The BV formalism and gauge-fixing.* The general abstract setup for a gauge system is a manifold F_{cl} – the “space of classical fields” (for a bosonic theory, this is an ordinary, non-super, manifold; we are assuming finite-dimensionality here to elucidate the geometric content of the formalism) endowed with a volume form μ_{cl} , a function S_{cl} and a distribution E – a subbundle

$$\begin{array}{ccc} E & \xrightarrow[\rho]{\hookrightarrow} & TF_{\text{cl}} \\ \downarrow & & \downarrow \\ F_{\text{cl}} & \xlongequal{\quad} & F_{\text{cl}} \end{array}$$

– the gauge symmetry. One requires that the following conditions hold:

- S_{cl} and μ_{cl} are E -invariant,
- distribution E is integrable on the critical locus of S_{cl} .

One is interested in the integral

$$(4) \quad Z = \int_{F_{\text{cl}}} e^{\frac{i}{\hbar} S_{\text{cl}}} \mu_{\text{cl}}$$

Due to E -invariance of the integrand, critical points are not isolated and the stationary phase formula is not applicable.

One solution of the problem is to extend the space of fields F_{cl} to the “space of BV fields” – an SP manifold

$$(5) \quad \mathcal{F} = T^*[-1](E[1])$$

where $E[1]$ is the total space of the vector bundle $E \rightarrow F_{\text{cl}}$ with fiber coordinates prescribed degree (“ghost number”) 1. The BV 2-form is the canonical symplectic structure on the cotangent bundle, the Berezinian is constructed as $(\mu_{\text{cl}} \cdot \mu_{\text{ghost}})^{\otimes 2}$ where μ_{ghost} is a fiber Berezinian on $E[1]$. If x^i are local coordinates on F_{cl} (“classical fields”) and c^a are fiber coordinates on $E[1]$ (“ghosts”), associated to a choice of basis $\{e_a\}$ in the fibers of $E \rightarrow F_{\text{cl}}$, then on \mathcal{F} one has coordinates c^a, x^i, x_i^+, c_a^+ (plus denotes the canonically conjugate coordinate on the cotangent fiber; the coordinates x_i^+, c_a^+ are called “anti-fields” and “anti-ghosts”, respectively) of degrees 1, 0, $-1, -2$ respectively. The BV 2-form is locally written as

$$\omega = \delta x_i^+ \wedge \delta x^i + \delta c_a^+ \wedge \delta c^a$$

Next, one constructs a solution of the classical master equation $S^{(0)} \in C^\infty(\mathcal{F})_0$ satisfying the properties

- The restriction of $S^{(0)}$ to $F_{\text{cl}} \subset \mathcal{F}$ is the original classical action S_{cl} .
- The cohomological vector field $Q = \{S^{(0)}, \bullet\} \in \mathfrak{X}(\mathcal{F})_1$ induces⁶ the distribution E on $F_{\text{cl}} \subset \mathcal{F}$.

⁶In the sense that, locally, $Q = \sum_a c^a \rho(e_a) +$ higher order monomials in fields of degree $\neq 0$. Here ρ is the inclusion of E as a subbundle of TF_{cl} (the “anchor map” in the terminology of Lie algebroids).

The construction of $S^{(0)}$ proceeds by adding corrections order by order in anti-fields, cf. [10] (see also [26]⁷). The next step is to extend $S^{(0)}$ to a solution of the quantum master equation $S = S^{(0)} + \hbar S^{(1)} + \hbar^2 S^{(2)} + \dots$ order by order in \hbar .

Then one regularizes the expression (4) to the BV integral

$$(6) \quad Z_{\text{BV}} = \int_{\mathcal{L} \subset \mathcal{F}} e^{\frac{i}{\hbar} S} \sqrt{\mu}|_{\mathcal{L}}$$

over a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$, which now plays the role of the gauge-fixing condition. If \mathcal{L} is such that the critical points of the integrand of (6) are isolated, one can proceed with the stationary phase formula and construct Feynman diagrams for the BV integral.

An important special case is when the gauge symmetry is given by a faithful action of a Lie group G on F_{cl} . Here E is a trivial bundle over F_{cl} with fiber the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ and the anchor map ρ is the differential of the group action. In this case, the BV action is, locally:

$$(7) \quad S = S_{\text{cl}}(x) + x_i^+ c^a v_a^i(x) + \frac{1}{2} f_{bc}^a c_a^+ c^b c^c$$

where $v_a = \rho(e_a)$ are the vector fields on F_{cl} representing the infinitesimal action of the chosen generators of \mathfrak{g} ; f_{bc}^a are the structure constants of \mathfrak{g} . The quantum master equation for S follows from: G -invariance of S_{cl} , the fact that $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(F_{\text{cl}})$ is a Lie algebra homomorphism, the Jacobi identity for structure constants \mathfrak{g} , G -invariance of the volume form μ_{cl} (or equivalently, divergence-free property of vector fields v_a), unimodularity of \mathfrak{g} (following from existence of a Haar measure on G). Choosing $\mathcal{L} \subset \mathcal{F} = T^*[-1](F_{\text{cl}} \oplus \mathfrak{g}[1])$ to be the conormal bundle to $\phi^{-1}(0) \oplus \mathfrak{g}[1] \subset F_{\text{cl}} \oplus \mathfrak{g}[1]$ with $\phi : F_{\text{cl}} \rightarrow \mathfrak{g}$ the gauge-fixing function (in the Faddeev-Popov sense), reduces the BV integral (6) to the Faddeev-Popov gauge-fixing of the integral (4). In this case the BV integral (6) coincides, up to a factor of the volume of G , with the original integral (4).

In the more general case, when the gauge symmetry does not come from a group action, one cannot directly compare (6) with (4), rather one should view the BV integral as a redefinition/regularization of (4).

An important generalization of the construction, see [11], is to allow the anchor map $\rho : E \rightarrow TF_{\text{cl}}$ to be non-injective and to introduce a resolution of the gauge symmetry – an exact⁸ sequence of vector bundles $0 \rightarrow E_k \rightarrow \dots \rightarrow E_1 = E \rightarrow TF_{\text{cl}}$. Then one constructs the space of BV fields as

$$(8) \quad \mathcal{F} = T^*[-1](E_1[1] \oplus \dots \oplus E_k[k])$$

The fiber coordinates of $E_i[i]$ are called “ i -th ghosts”. Examples of gauge systems where higher ghosts appear include the BF theory in dimension ≥ 4 [23] and the Courant sigma model [41].

⁷In the setting we outlined here, there may be obstructions for the construction of $S^{(0)}$ order by order in anti-fields as a solution of the classical master equation. Felder and Kazhdan [26] consider a different setup where the distribution E is not a part of initial data, but is recovered as the “maximal” symmetry of S . In this setting they prove existence and uniqueness (up to stable equivalence) of the solution of CME compatible with given initial data.

⁸In fact, in the context of local field theory, when each E_i is required to be the space of sections of some locally free sheaf over the spacetime manifold, one has to allow the resolution to have finite-dimensional cohomology.

In case when the gauge system has zero-modes, i.e. (set of critical points of $S_{\text{cl}})/E$ is not a discrete set of points, the procedure above does not lead directly to a perturbatively well-defined integral. Instead, one should use the construction of fiber BV integral (2) to pushforward the action S to zero-modes and then integrate over the zero-modes non-perturbatively (thus the problem of defining the path integral in this case becomes split into a perturbatively well-defined fiber BV integral, parameterized by zero-modes, and a finite-dimensional non-perturbative integral over zero-modes).

3.1.7. *The classical part of the BV formalism.* The classical part of the BV formalism, i.e. the structure modulo \hbar -corrections is as follows:

- The space of BV fields \mathcal{F} is a graded supermanifold with a degree -1 symplectic form ω . Functions on \mathcal{F} form a Gerstenhaber (odd Poisson) algebra $(C^\infty(\mathcal{F}), \cdot, \{, \})$.
- The BV action $S \in C^\infty(\mathcal{F})$ is a solution of CME: $\{S, S\} = 0$. The Hamiltonian vector field $Q = \{S, \bullet\}$ on \mathcal{F} is a cohomological vector field, $Q^2 = 0$.

3.1.8. *Remarks.*

- (i) Above we defined, following [44], the BV Laplacian Δ on an SP manifold \mathcal{F} as an operator acting on functions and depending on a choice of a Berezinian μ . An alternative viewpoint developed in [30, 46] is to define a *canonical* (independent on a choice of a Berezinian) BV Laplacian Δ_{can} acting on *semi-densities* on \mathcal{F} (which is just required to be an odd-symplectic supermanifold in this setting), i.e. on sections of $\text{Ber}(\mathcal{F})^{\otimes \frac{1}{2}}$. The drawback of this approach is that there is no natural BV algebra associated to Δ_{can} .
- (ii) The fiber BV integral can be viewed as a quantum version of the homological perturbation theory ([28]): the classical part of the induced BRST operator Q' on \mathcal{F}' can be recovered from the classical part of the original BRST operator Q on \mathcal{F} by means of homological perturbation theory, which corresponds to the tree-level approximation to the fiber BV integral (see [36, 20] for details).
- (iii) For \mathcal{F} a graded vector space with constant BV 2-form, a solution of the classical master equation is a generating function for a cyclic L_∞ algebra structure on $\mathcal{F}[-1]$, see [20].
- (iv) In much of the discussion in this section we were working on the premise that \mathcal{F} is a finite-dimensional supermanifold and BV integrals make sense as measure-theoretic integrals. In the context of local field theory this is no longer true and certain corrections have to be made to the discussion. In particular, the BV Laplacian is not defined on local functionals, thus the classical master equation makes sense right away, but QME has to be understood in a regularized/renormalized sense, see [24]. BV integrals over Lagrangian submanifolds $\mathcal{L} \subset \mathcal{F}$ now become perturbative path integrals and the Stokes' theorem (independence on deformations of \mathcal{L}) for them has to be verified via the definition through Feynman diagrams. In the cases of Chern-Simons theory on closed 3-manifolds and Poisson sigma model on a disk, this analysis was performed in [7, 32, 17]. The analysis showed that arguments proving the Stokes' theorem for measure-theoretic BV integrals translate, in the setup of path integrals, to cancellation of the contribution of part of the boundary (so-called principal boundary strata) of configuration spaces of points on the spacetime manifold to the variation of partition function. Cancellation

of the contribution of rest of the boundary (the hidden boundary strata) has to be checked separately and does not hold automatically. In case of Chern-Simons theory with Hodge-theoretic propagator [7], certain hidden strata do contribute and lead to the “gravitational anomaly” in Chern-Simons theory – dependence of the partition function on the choice of a Riemannian metric, which can be cancelled at the cost of introducing framing on the spacetime manifold [49, 7]. Different constructions of propagator in Chern-Simons theory were suggested by Kontsevich [31] and by Bott and Cattaneo [14]. In the first case, the propagator depends on a choice of framing from the start, but leads to hidden strata not contributing to the variation of the partition function. In the construction of [14], one constructs the propagator using a choice of metric connection on the 3-manifold and corrects for the contributions of the hidden strata using a framing (see e.g. [20] for an in-depth discussion).

- (v) The classical BV formalism of Section 3.1.7 makes sense in the context of local field theory on closed spacetime manifolds without any regularization, if one restricts to variation calculus type functionals on the space of fields.

3.2. Chern-Simons theory.

3.2.1. *Classical Chern-Simons theory on closed 3-manifolds.* For M a closed 3-manifold and G a compact simply-connected Lie group with Lie algebra \mathfrak{g} , one defines the space of classical fields of Chern-Simons theory to be the space of connections in the trivial G -bundle over M , $\text{Conn}(M) \simeq \mathfrak{g} \otimes \Omega^1(M)$. The action functional is

$$(9) \quad S_{\text{CS}}(A) = \int_M \text{tr}_{\mathfrak{g}} \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right)$$

where $A \in \text{Conn}(M)$ is a connection viewed as a 1-form on M with values in \mathfrak{g} , and we view \mathfrak{g} as a matrix Lie algebra using the fundamental representation of G .

The critical point equation (Euler-Lagrange equation) for the action (9) is the zero curvature equation for the connection:

$$F_A = dA + A \wedge A = 0$$

Gauge transformations are given by the action of $\text{Map}(M, G)$ on the connection. For $g : M \rightarrow G$, the transformation is:

$$(10) \quad A \mapsto A^g = g^{-1} A g + g^{-1} dg$$

Infinitesimal gauge transformations are given by

$$(11) \quad A \mapsto A + \epsilon(d\alpha + [A, \alpha]) + O(\epsilon^2)$$

for the generator $\alpha : M \rightarrow \mathfrak{g}$.

Gauge transformations preserve the zero curvature condition and, moreover, preserve the action S_{CS} , provided that the generator $g \in \text{Map}(M, G)$ is in the connected component of identity in the mapping space. For a general gauge transformation, one has (with conventional normalization of trace in (9)):

$$S_{\text{CS}}(A^g) = S_{\text{CS}}(A) + 4\pi^2 n$$

with n an integer, $n = \langle g^*[\theta], [M] \rangle$ where $[M]$ is the fundamental class of M and $\theta = \frac{1}{24\pi^2} \text{tr}(g^{-1} dg)^{\wedge 3} \in \Omega^3(G)$ is the Cartan integral 3-form on the Lie group G with $[\theta]$ its cohomology class.

The space of critical points of the action modulo gauge transformations is the moduli space of flat G -connections on M and is, generally, a finite-dimensional singular variety $\mathcal{M}(M, G) \simeq \text{Hom}(\pi_1(M), G)/G$.

3.2.2. *Chern-Simons theory in the Batalin-Vilkovisky formalism.* The (minimal) space of BV fields is, by construction (5):

$$\begin{aligned} \mathcal{F} &= T^*[-1](\mathfrak{g} \otimes \Omega^1(M) \oplus \mathfrak{g} \otimes \Omega^0(M)[1]) \simeq \\ &\simeq \mathfrak{g} \otimes \Omega^0(M)[1] \oplus \mathfrak{g} \otimes \Omega^1(M) \oplus \mathfrak{g} \otimes \Omega^2(M)[-1] \oplus \mathfrak{g} \otimes \Omega^3(M)[-2] \end{aligned}$$

where we identify g^* with \mathfrak{g} using the non-degenerate pairing $\text{tr}_{\mathfrak{g}} XY$. A point in \mathcal{F} is a quadruple (c, A, A^+, c^+) of a ghost, a connection, an anti-field and an anti-ghost, i.e., \mathfrak{g} -valued differential forms on M of degrees 0, 1, 2, 3 with ghost number 1, 0, -1 , -2 respectively. The BV 2-form is:

$$\omega = \int_M \text{tr} (\delta A \wedge \delta A^+ + \delta c \wedge \delta c^+)$$

and the BV action (7) is:

$$(12) \quad S_{\text{BV CS}} = \int_M \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A] + A^+ \wedge d_A c + \frac{1}{2} c^+ \wedge [c, c] \right)$$

where $d_A c = dc + [A, c]$ is the de Rham differential twisted by connection A . Lorentz gauge-fixing employed in [49, 6, 7] consists in choosing a Riemannian metric g on M and defining the gauge-fixing Lagrangian subspace $\mathcal{L} \subset \mathcal{F}$ as

$$(13) \quad \mathcal{L} = \{(c, A, A^+, c^+) \in \mathcal{F} \mid d^* A = 0, A^+ = d^* \bar{c}, c^+ = 0\}$$

Here $d^* = - * d *$ is the Hodge adjoint to the de Rham operator, and $\bar{c} \in \mathfrak{g} \otimes \Omega^3(M)[-1]$ is a new field introduced to parameterize the allowed values of the antifield A^+ on \mathcal{L} . Restriction of the BV action (12) to the Lagrangian \mathcal{L} yields

$$S_{\text{BV CS}}|_{\mathcal{L}} = \int_M \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A] + \bar{c} \wedge d^* d_A c \right)$$

the Faddeev-Popov action for Chern-Simons theory in Lorentz gauge.

3.2.3. *The perturbative Chern-Simons path integral on closed 3-manifolds.* Witten's idea in [49] was to use the action (9) to define diffeomorphism invariants of oriented 3-manifolds as

$$(14) \quad Z(M) = \int_{\text{Conn}(M)} \mathcal{D}A e^{\frac{i}{\hbar} S_{\text{CS}}(A)} \in \mathbb{C}$$

where $\int \mathcal{D}A$ should be understood as a single symbol for the path integral, which is to be made sense of. The integrand in (14) is manifestly independent on any geometric structure on M other than the orientation, so it is reasonable to expect $Z(M)$ to be a diffeomorphism invariant of M .

Under the simplifying assumption that the moduli space of flat connections $\mathcal{M}(M, G)$ is a discrete set of points (e.g. for M a rational homology sphere, for instance a lens space $L(p, q)$), the perturbative Chern-Simons path integral (14) splits into contributions of the points α of $\mathcal{M}(M, G)$:

$$Z(M) = \sum_{\alpha \in \mathcal{M}(M, G)} Z(M, A^{(\alpha)})$$

where $A^{(\alpha)}$ are some flat connections representing points $\alpha \in \mathcal{M}(M, G)$.

The contribution of a flat connection $A^{(\alpha)}$ to the perturbative partition function can be evaluated through the BV formalism as

$$(15) \quad Z(M, A^{(\alpha)}) = \int_{\mathcal{L}_{A^{(\alpha)}}} e^{\frac{i}{\hbar} S_{\text{BV CS}}}$$

where

$$\mathcal{L}_{A^{(\alpha)}} = \{(c, A^{(\alpha)} + a, A^+, c^+) \in \mathcal{F} \mid d_{A^{(\alpha)}}^* a = 0, A^+ = d_{A^{(\alpha)}}^* \bar{c}, c^+ = 0\}$$

– the construction (13) twisted by the flat connection $A^{(\alpha)}$. In (15), the right hand side is understood as the perturbative contribution of a single critical point, $c = a = A^+ = c^+ = 0$.

The analysis of the semiclassical approximation of the r.h.s. of (15) was done in [49] and extended to higher loop corrections in [6, 7] (without using the BV language explicitly), in case of an *acyclic* flat connection $A^{(\alpha)}$. The result is:

$$(16) \quad Z(M, A^{(\alpha)}, s) = e^{\frac{i}{\hbar} S_{\text{CS}}(A^{(\alpha)})} \tau(M, A^{(\alpha)}) e^{\frac{i\pi}{4} \eta(M, A^{(\alpha)}, g)} e^{ic(\hbar) S_{\text{grav}}(g, s)} \cdot \exp \left(\frac{i}{\hbar} \sum_{\text{connected 3-valent graphs } \Gamma} \frac{(i\hbar)^{l(\Gamma)}}{|\text{Aut}(\Gamma)|} \int_{\text{Conf}_{V(\Gamma)}(M)} \prod_{\text{edges } e \text{ of } \Gamma} \pi_{e_1 e_2}^* \eta \right)$$

Here the notations are as follows:

- $\tau(M, A^{(\alpha)})$ is the Ray-Singer torsion of M with respect to the local system defined by $A^{(\alpha)}$.
- $\eta(M, A^{(\alpha)}, g)$ is the Atiyah η -invariant of the elliptic operator $L_- = *d_{A^{(\alpha)}} + d_{A^{(\alpha)}}^* : \mathfrak{g} \otimes \Omega^{\text{odd}}(M) \rightarrow \mathfrak{g} \otimes \Omega^{\text{odd}}(M)$. It depends explicitly on the chosen Riemannian metric on M . By the Atiyah-Patodi-Singer theorem [49], the dependence of $\eta(M, A^{(\alpha)}, g)$ on the flat connection $A^{(\alpha)}$ can be spelled out explicitly: $\frac{\pi}{4}(\eta(M, A^{(\alpha)}, g) - \eta(M, 0, g)) = \frac{\hbar}{2\pi} S_{\text{CS}}(A^{(\alpha)})$ where h is the dual Coxeter number of the Lie algebra \mathfrak{g} .
- $V(\Gamma)$ and $l(\Gamma)$ are the number of vertices and the number of loops in the graph Γ ; $|\text{Aut}(\Gamma)|$ is the order of the automorphism group of Γ .
- $\text{Conf}_n(M)$ is the Fulton-Macpherson-Axelrod-Singer compactification of the configuration space of n -tuples of distinct points on M .
- $\eta \in \Omega^2(\text{Conf}_2(M))$ is the parametrix for the Hodge-theoretic inverse for the de Rham operator twisted by the flat connection, $\frac{d_{A^{(\alpha)}}^*}{[d_{A^{(\alpha)}}^*, d_{A^{(\alpha)}}^*]}$.
- $\pi_{ij} : \text{Conf}_n(M) \rightarrow \text{Conf}_2(M)$ is the projection by forgetting all points except the i -th and j -th.
- $S_{\text{grav}}(g, s)$ is the Chern-Simons invariant of the Levi-Civita connection on the spin bundle over M ; it depends on the trivialization of TM up to homotopy, i.e. on a choice of *framing* s of the 3-manifold M . $c(\hbar) \in \mathbb{C}[[\hbar]]$ is some universal power series.

The term $e^{ic(\hbar) S_{\text{grav}}(g, s)}$ in (16) is a local counterterm introduced by hand to cancel the dependence of the result on the choice of metric g on M , which comes in in 1-loop approximation because of Atiyah's η -invariant and in higher loops due to the contribution of certain hidden boundary strata of $\text{Conf}_n(M)$ (namely, those corresponding to collapse of all n points together) to the variation of $Z(M, A^{(\alpha)})$ with the variation of the metric. A theorem of Axelrod and Singer [7] asserts the existence of such a power series $c(\hbar)$ that the r.h.s. of (16) becomes metric

independent. However, this metric independence comes at the cost of introducing an explicit dependence on the choice of a framing s of M .

An important result of Axelrod and Singer implicit in (16) is the *finiteness* of Chern-Simons perturbation theory: in each order in \hbar in (16) only finitely many Feynman graphs Γ contribute, and their contributions are given by integrals of smooth differential forms (products of propagators) over compact manifolds $\text{Conf}_n(M)$.

3.2.4. The boundary structure of classical Chern-Simons theory. The phase space associated to a closed 2-dimensional surface Σ in Chern-Simons theory is the space $\Phi_\Sigma = \text{Conn}(\Sigma)$ of G -connections on Σ and the constraint subspace is the subspace of *flat* G -connections $C_\Sigma = \text{FlatConn}(\Sigma) \subset \text{Conn}(\Sigma)$. The symplectic structure on Φ_Σ is given by

$$(17) \quad \omega_\Sigma = \frac{1}{2} \int_\Sigma \text{tr } \delta A \wedge \delta A$$

The subspace of flat connections is coisotropic with respect to ω_Σ and the characteristic foliation is given by gauge transformations (11) on Σ .

One construction leading to this phase space is through the Hamiltonian formalism for Chern-Simons theory on a cylinder $\Sigma \times [0, 1]$ (as e.g. in [49]). The coisotropic C_Σ arises in this approach as the image of the Legendre transform of the axial density of Chern-Simons action.

The other construction (cf. e.g. [22]) is by regarding Σ as a component of the boundary of some 3-manifold M (not necessarily a cylinder) and then constructing Φ_Σ as the space of pullbacks of bulk fields to the boundary; in this approach the symplectic form ω_Σ is constructed as the differential of

$$(18) \quad \alpha_\Sigma = \frac{1}{2} \int_\Sigma \text{tr } A \wedge \delta A \in \Omega^1(\Phi_\Sigma)$$

– the 1-form arising from the boundary term of the variation of the action S_{CS} . The coisotropic C_Σ arises as the subspace of boundary fields which can be extended to a solution of the Euler-Lagrange equation in a neighborhood of Σ in M .

The space of leaves of the characteristic foliation of C_Σ (i.e. the symplectic reduction of C_Σ) constitutes the *reduced* phase space of Chern-Simons theory, $\Phi_\Sigma^{\text{red}} = \underline{C}_\Sigma = \mathcal{M}(\Sigma, G)$ – the moduli space of flat connections on Σ . The symplectic structure (17) induces the Atiyah-Bott symplectic structure $\underline{\omega}_\Sigma$ on the moduli space $\mathcal{M}(\Sigma, G)$ via symplectic reduction.

The trivial $U(1)$ -bundle L over Φ_Σ with connection ∇ defined by the 1-form $\frac{1}{\hbar} \alpha_\Sigma$ pushes forward to the reduction in case $\hbar = 2\pi/k$ with k a nonzero integer (the “level” of Chern-Simons theory) by identifying the $U(1)$ -fibers of L over gauge orbits in C_Σ (integrality condition for the level exactly ensures that the monodromy of ∇ in gauge orbits in C_Σ is trivial, thus the identification of fibers can be performed globally). The pushforward of L to the reduction is the Quillen $U(1)$ -bundle $\underline{L}_k = (\underline{L}_1)^{\otimes k}$ over the moduli space $\mathcal{M}(\Sigma, G)$. By construction, it is endowed with a connection $\underline{\nabla}_k$ of curvature $\frac{k}{2\pi} \underline{\omega}_\Sigma$.

If Σ is the boundary of a 3-manifold M , one has a restriction map between the moduli space of flat connections $\pi : \mathcal{M}(M, G) \rightarrow \mathcal{M}(\Sigma, G)$ with Lagrangian image – the “evolution relation” (cf. e.g. [21]), and the exponential of the Chern-Simons action restricted to flat connections on M defines a horizontal section of the pullback $U(1)$ -bundle $\pi^* \underline{L}_k$ over $\mathcal{M}(M, G)$ – the “Hamilton-Jacobi action”.

Another important remark on classical Chern-Simons theory on manifolds with boundary is that if $\partial M = \Sigma \neq \emptyset$, then the action (9) is not gauge-invariant. Instead one has

$$S_{\text{CS}}(A^g) = S_{\text{CS}}(A) + \frac{1}{2} \int_{\Sigma} \text{tr} A \wedge dg g^{-1} - \frac{1}{6} \int_M \text{tr} (g^{-1} dg)^{\wedge 3}$$

where a Wess-Zumino term makes an appearance. Under infinitesimal gauge transformations (11), the action transforms as

$$S_{\text{CS}} \rightarrow S_{\text{CS}} + \epsilon \frac{1}{2} \int_{\Sigma} \text{tr} A \wedge d\alpha + O(\epsilon^2)$$

3.2.5. The space of states. The space of states \mathcal{H}_{Σ} that Chern-Simons *quantum* theory associates to a surface Σ is the geometric quantization of the moduli space $\mathcal{M}(\Sigma, G)$ of flat connections on the surface, endowed with the Atiyah-Bott symplectic form ω_{Σ} and a prequantum Hermitian line bundle \underline{L}_k with unitary connection $\underline{\nabla}_k$ with $k \in \mathbb{Z} - \{0\}$ the fixed *level* of Chern-Simons theory. A complex polarization on $\mathcal{M}(\Sigma, G)$ required for the geometric quantization is inferred from an arbitrary choice of a complex structure J on Σ , which induces a splitting of 1-forms on Σ into (1, 0)- and (0, 1)-forms:

$$\mathfrak{g}_{\mathbb{C}} \otimes \Omega^1(\Sigma) = \mathfrak{g} \otimes \Omega^{1,0}(\Sigma) \oplus \mathfrak{g} \otimes \Omega^{0,1}(\Sigma)$$

where $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$ the complexified Lie algebra. Performing the Marsden-Weinstein reduction with the moment map $\mathfrak{g} \otimes \Omega^1(\Sigma) \rightarrow \mathfrak{g} \otimes \Omega^2(\Sigma)$, sending a connection to its curvature 2-form, one obtains as the symplectic quotient the moduli space $\mathcal{M}(\Sigma, G)$ endowed with a Kähler structure (dependent on the choice of complex structure J on Σ). The output of geometric quantization of $\mathcal{M}(\Sigma, G)$ with this structure (cf. [8] for details) is the space of holomorphic sections of \underline{L}_k .

Let us summarize some properties of \mathcal{H}_{Σ} .

- \mathcal{H}_{Σ} is a finite-dimensional Hilbert space (we are assuming that G is compact), with dimension depending on the level k . In particular, for $G = SU(2)$ the dimension is given by the Verlinde formula:

$$\dim \mathcal{H}_{\Sigma} = \left(\frac{k+2}{2} \right)^{g-1} \sum_{j=0}^k \frac{1}{\left(\sin \frac{\pi(j+1)}{k+2} \right)^{2g-2}}$$

where g is the genus of Σ .

- On the other hand, $\dim \mathcal{H}_{\Sigma}$ coincides with the Euler characteristic of the complex $\Omega^{0,\bullet}(\mathcal{M}(\Sigma, G), \underline{L}_k)$ with Dolbeault differential twisted by the connection $\underline{\nabla}_k$ and is given by the Riemann-Roch-Hirzebruch formula as

$$\dim \mathcal{H}_{\Sigma} = \int_{\mathcal{M}(\Sigma, G)} \text{Td}(\mathcal{M}) e^{k\omega}$$

with $\text{Td}(\mathcal{M})$ the Todd class of the tangent bundle of $\mathcal{M}(\Sigma, G)$. Thus $\dim \mathcal{H}_{\Sigma}$ is a polynomial in k of degree $\frac{1}{2} \dim \mathcal{M} = (g-1) \dim(G)$ for $g \geq 2$, with the leading coefficient given by the symplectic volume of the moduli space of flat connections. This volume was separately calculated in [51] as

$$\text{Vol}(\mathcal{M}(\Sigma, G)) = \frac{\#Z(G) \cdot \text{Vol}(G)^{2g-2}}{(2\pi)^{\dim \mathcal{M}}} \cdot \sum_R \frac{1}{(\dim R)^{2g-2}}$$

where the sum is over irreducible unitary representations of G and $Z(G)$ stands for the center of G .

- \mathcal{H}_Σ coincides with the space of conformal blocks of the Wess-Zumino-Witten model of conformal field theory on the Riemann surface (Σ, J) , with affine Lie algebra $\hat{\mathfrak{g}}$ acting on the (chiral) space of states for a circle $\mathcal{H}_{S^1}^{WZW}$.
- \mathcal{H}_Σ is endowed with a projective representation of the mapping class group of Σ . In the non-perturbative approach to Chern-Simons theory [49] this representation is used to relate Chern-Simons partition functions for 3-manifolds related by surgery.
- One can consider \mathcal{H}_Σ for a surface Σ with boundary (or punctures), which corresponds to the inclusion of Wilson line observables supported on a tangle in the 3-manifold in Chern-Simons theory. Then in addition, \mathcal{H}_Σ carries an action of the affine Lie algebra $\hat{\mathfrak{g}}$ for each boundary component of Σ .
- Varying the complex structure J on Σ , one obtains a Hermitian vector bundle over the moduli space of complex structures on Σ with fiber \mathcal{H}_Σ . This vector bundle is endowed with a canonical projective connection – the Hitchin’s connection (corresponding to the stress-energy tensor of the WZW model).

3.3. AKSZ construction. In this Section we give a brief review of the Aleksandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction of topological sigma models in the Batalin-Vilkovisky formalism. This account is taken from [37]; we refer the reader to the original paper [1] and the later expositions in [18], [41] for details.

3.3.1. Target data. Let \mathcal{M} be a degree n symplectic Q -manifold, i.e. a \mathbb{Z} -graded manifold endowed with a degree 1 vector field Q satisfying $Q^2 = 0$ (the cohomological vector field) and with a degree n symplectic form $\omega \in \Omega^2(\mathcal{M})_n$ which is compatible with Q , i.e. $L_Q\omega = 0$.

Assume⁹ that Q has a Hamiltonian function $\Theta \in C^\infty(\mathcal{M})_{n+1}$ with $\{\Theta, \bullet\}_\omega = Q$ satisfying

$$(19) \quad \{\Theta, \Theta\}_\omega = 0$$

Also assume that ω is exact, with $\alpha \in \Omega^1(\mathcal{M})_n$ a primitive.

We call the set of data $(\mathcal{M}, Q, \omega = \delta\alpha, \Theta)$ a *Hamiltonian Q -manifold of degree n* .

3.3.2. AKSZ sigma model. Fix a Hamiltonian Q -manifold \mathcal{M} of degree $n \geq -1$ and let Σ be an oriented closed manifold, $\dim \Sigma = n + 1$. Then one constructs the space of fields as the space of graded maps between graded manifolds from the degree-shifted tangent bundle $T[1]\Sigma$ to \mathcal{M} :

$$(20) \quad \mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathcal{M})$$

⁹In fact (cf. [41]), for $n \neq -1$ the symplectic property of the cohomological vector field ($L_Q\omega = 0$) implies existence and uniqueness of the Hamiltonian $\Theta = \frac{1}{n+1} \iota_E \iota_Q \omega$ where E is the Euler vector field. The Maurer-Cartan equation (19) follows from $Q^2 = 0$ for $n \neq -2$. Also, for $n \neq 0$, a closed form ω is automatically exact.

It is a Q -manifold with the cohomological vector field coming from the lifting of Q on the target and of the de Rham operator d_Σ on Σ (viewed as a cohomological vector field on $T[1]\Sigma$) to the mapping space:

$$(21) \quad Q_\Sigma = (d_\Sigma)^{\text{lifted}} + (Q)^{\text{lifted}} \in \mathfrak{X}(\mathcal{F}_\Sigma)_1$$

The transgression map. The following natural maps

$$(22) \quad \begin{array}{ccc} \mathcal{F}_\Sigma \times T[1]\Sigma & \xrightarrow{\text{ev}} & \mathcal{M} \\ p \downarrow & & \\ \mathcal{F}_\Sigma & & \end{array}$$

(where p is the projection to the first factor) allow us to define the transgression map

$$(23) \quad \tau_\Sigma = p_* \text{ev}^* : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{F}_\Sigma)$$

Here p_* is the fiber integration over $T[1]\Sigma$ (with the canonical integration measure). The map τ_Σ preserves the de Rham degree of a form, but changes the internal grading (the ghost number) by $-\dim \Sigma$.

If $u \in \mathfrak{X}(T[1]\Sigma)$, $v \in \mathfrak{X}(\mathcal{M})$ are any vector fields on the source and the target and $\psi \in \Omega^\bullet(\mathcal{M})$ is a form on the target, then for the Lie derivatives of the transgressed form along the lifted vector fields we have

$$(24) \quad L_{u^{\text{lifted}}} \tau_\Sigma(\psi) = 0$$

$$(25) \quad L_{v^{\text{lifted}}} \tau_\Sigma(\psi) = (-1)^{|v| \dim \Sigma} \tau_\Sigma(L_v \psi)$$

The integrated version of (24) is that for $\Phi : T[1]\Sigma \rightarrow T[1]\Sigma$ a diffeomorphism of the source, we have

$$(26) \quad (\Phi^*)^* \tau_\Sigma(\psi) = \tau_\Sigma(\psi)$$

where $\Phi^* : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_\Sigma$ is the lifting of Φ to the mapping space and $(\Phi^*)^* : \Omega^\bullet(\mathcal{F}_\Sigma) \rightarrow \Omega^\bullet(\mathcal{F}_\Sigma)$ is the pull-back by Φ^* .

The BV 2-form and the master action. One obtains the degree -1 symplectic form (the “BV 2-form”) on \mathcal{F}_Σ from the target by transgression¹⁰:

$$(27) \quad \Omega_\Sigma = (-1)^{\dim \Sigma} \tau_\Sigma(\omega) \in \Omega^2(\mathcal{F}_\Sigma)_{-1}$$

The Hamiltonian function for Q_Σ (the master action) is constructed as

$$(28) \quad S_\Sigma = \underbrace{\iota_{d_\Sigma^{\text{lifted}}} \tau_\Sigma(\alpha)}_{S_\Sigma^{\text{kin}}} + \underbrace{\tau_\Sigma(\Theta)}_{S_\Sigma^{\text{target}}} \in C^\infty(\mathcal{F}_\Sigma)_0$$

It automatically satisfies the classical master equation

$$\{S_\Sigma, S_\Sigma\}_{\Omega_\Sigma} = 0$$

One can summarize the construction above by saying that we have a degree -1 Hamiltonian Q -manifold structure on the mapping space (20): $(\mathcal{F}_\Sigma, Q_\Sigma, \Omega_\Sigma, S_\Sigma)$. The primitive 1-form for the BV 2-form can also be constructed by transgression as $\tau_\Sigma(\alpha)$.

¹⁰We introduce the sign $(-1)^{\dim \Sigma}$ in this definition to avoid signs in the formula for the action below. The reader may encounter different sign conventions for the AKSZ construction in the literature.

Note that exact shifts of the target 1-form, $\alpha \mapsto \alpha + \delta f$, with $f \in C^\infty(M)$ leave S_Σ unchanged.

Why AKSZ theory is topological. For $\phi \in \text{Diff}(\Sigma)$ a diffeomorphism of Σ , denote $\tilde{\phi} \in \text{Diff}(T[1]\Sigma)$ the tangent lift of ϕ to $T[1]\Sigma$. Then

$$(29) \quad (\tilde{\phi}^*)^* S_\Sigma = S_\Sigma, \quad (\tilde{\phi}^*)^* \Omega_\Sigma = \Omega_\Sigma$$

because of (26) and because $d_\Sigma \in \mathfrak{X}(T[1]\Sigma)$ commutes with $\tilde{\phi}$, since the latter is a tangent lift.

In coordinates. Let x^a be local homogeneous coordinates on the target \mathcal{M} , let u^μ be local coordinates on Σ and $\theta^\mu = du^\mu$ be the associated degree 1 fiber coordinates on $T[1]\Sigma$. Then locally an element of \mathcal{F}_Σ is parameterized by:

$$(30) \quad X^a(u, \theta) = \sum_{k=0}^{\dim \Sigma} \underbrace{\sum_{1 \leq \mu_1 < \dots < \mu_k \leq \dim \Sigma} X_{\mu_1 \dots \mu_k}^a(u) \theta^{\mu_1} \dots \theta^{\mu_k}}_{X_{(k)}^a(u, \theta)}$$

The coefficient functions $X_{\mu_1 \dots \mu_k}^a(u)$ are local coordinates of degree $|x^a| - k$ on the mapping space \mathcal{F}_Σ . The expression (30) is known as (the component of) the *superfield*, and it can be regarded as a generating function for the coordinates on the mapping space \mathcal{F}_Σ .

For any function $f \in C^\infty(\mathcal{M})$, we have

$$(31) \quad \begin{aligned} \text{ev}^* f &= f(X) = \\ &= f(X_{(0)}) + X_{\geq 1}^a \cdot (\partial_a f)(X_{(0)}) + \frac{1}{2} X_{\geq 1}^a X_{\geq 1}^b \cdot (\partial_b \partial_a f)(X_{(0)}) + \dots \\ &\in C^\infty(\mathcal{F}_\Sigma \times T[1]\Sigma) \end{aligned}$$

where we denote by $X_{\geq 1}^a = \sum_{k \geq 1} X_{(k)}^a$ the part of the superfield of positive de Rham degree with respect to Σ ; ev is the horizontal arrow in (22).

Let α and ω be locally given as $\alpha = \alpha_a(x) \delta x^a$ and $\omega = \frac{1}{2} \omega_{ab}(x) \delta x^a \wedge \delta x^b$. Then the BV 2-form and its primitive are:

$$\Omega_\Sigma = (-1)^{\dim \Sigma} \int_\Sigma \frac{1}{2} \omega_{ab}(X) \delta X^a \wedge \delta X^b, \quad \alpha_\Sigma = \int_\Sigma \alpha_a(X) \delta X^a$$

(Note that we use δ to denote the de Rham differential on the target \mathcal{M} and on the mapping space \mathcal{F}_Σ . We reserve symbol d for the de Rham differential on the source Σ .)

The master action is:

$$S_\Sigma(X) = \underbrace{\int_\Sigma \alpha_a(X) dX^a}_{S_\Sigma^{\text{kin}}} + \underbrace{\int_\Sigma \Theta(X)}_{S_\Sigma^{\text{target}}}$$

If the cohomological vector field on the target is locally written as $Q = Q^a(x) \frac{\partial}{\partial x^a}$ then the cohomological vector field (21) is determined by its action on the components of the superfield:

$$Q_\Sigma X^a = dX^a + Q^a(X)$$

The critical points of S_Σ are (with our sign conventions) Q -anti-morphisms between $T[1]\Sigma$ and \mathcal{M} , i.e. Q -morphisms between $(T[1]\Sigma, d)$ and $(\mathcal{M}, -Q)$.

3.3.3. *Examples.* Here we recall some of the standard examples of the AKSZ construction.

Chern-Simons theory [1]. Let \mathfrak{g} be a quadratic Lie algebra, i.e. a Lie algebra with a non-degenerate invariant pairing $\langle \cdot, \cdot \rangle$. Denote by $\psi : \mathfrak{g}[1] \rightarrow \mathfrak{g}$ the degree 1 \mathfrak{g} -valued coordinate on $\mathfrak{g}[1]$. We choose the target Hamiltonian Q -manifold of degree 2 as

$$(32) \quad \mathcal{M} = \mathfrak{g}[1], \quad Q = \left\langle \frac{1}{2}[\psi, \psi], \frac{\partial}{\partial \psi} \right\rangle, \\ \omega = \frac{1}{2}(\delta\psi, \delta\psi), \quad \alpha = \frac{1}{2}(\psi, \delta\psi), \quad \Theta = \frac{1}{6}(\psi, [\psi, \psi])$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between \mathfrak{g} and \mathfrak{g}^* . The associated AKSZ sigma model on a closed oriented 3-manifold Σ has the space of fields

$$\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathfrak{g}[1]) \cong \mathfrak{g}[1] \otimes \Omega^\bullet(\Sigma)$$

The superfield is

$$(33) \quad A = A_{(0)} + A_{(1)} + A_{(2)} + A_{(3)}$$

with $A_{(k)}$ a coordinate on \mathcal{F}_Σ with values in \mathfrak{g} -valued k -forms on Σ , with internal degree (ghost number) $1 - k$, for $k = 0, 1, 2, 3$. The BV 2-form is

$$\Omega_\Sigma = -\frac{1}{2} \int_\Sigma (\delta A, \delta A)$$

and the action is

$$S_\Sigma = \int_\Sigma \frac{1}{2}(A, dA) + \frac{1}{6}(A, [A, A])$$

This is the action of Chern-Simons theory in the Batalin-Vilkovisky formalism (12) written in terms of the superfield (33).

In case $\mathfrak{g} = \mathbb{R}$ with abelian Lie algebra structure, we have $Q = \Theta = 0$ on the target and

$$(34) \quad \mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)[1], \quad \Omega_\Sigma = -\frac{1}{2} \int_\Sigma \delta A \wedge \delta A, \quad S_\Sigma = \frac{1}{2} \int_\Sigma A \wedge dA$$

This is the *abelian* Chern-Simons theory in the BV formalism.

BF theory. For \mathfrak{g} a Lie algebra (not necessarily quadratic¹¹) and D a non-negative integer, we define the target Hamiltonian Q -manifold of degree $D - 1$ as

$$(35) \quad \mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[D - 2], \quad Q = \left\langle \frac{1}{2}[\psi, \psi], \frac{\partial}{\partial \psi} \right\rangle + \left\langle \text{ad}_\psi^* \xi, \frac{\partial}{\partial \xi} \right\rangle, \\ \omega = \langle \delta \xi, \delta \psi \rangle, \quad \alpha = \langle \xi, \delta \psi \rangle, \quad \Theta = \frac{1}{2} \langle \xi, [\psi, \psi] \rangle$$

Here $\psi : \mathfrak{g}[1] \rightarrow \mathfrak{g}$ is as before and $\xi : \mathfrak{g}^*[D - 2] \rightarrow \mathfrak{g}^*$ is the \mathfrak{g}^* -valued coordinate on $\mathfrak{g}^*[D - 2]$ of degree $D - 2$; ad^* is the coadjoint action of \mathfrak{g} on \mathfrak{g}^* .

The associated AKSZ sigma model on a closed oriented D -manifold Σ has the space of fields:

$$(36) \quad \mathcal{F}_\Sigma = \mathfrak{g}[1] \otimes \Omega^\bullet(\Sigma) \oplus \mathfrak{g}^*[D - 2] \otimes \Omega^\bullet(\Sigma)$$

¹¹However, for the consistency of quantization, in particular for the quantum master equation (1), one should require that \mathfrak{g} is unimodular.

The superfields associated to ψ and ξ are respectively

$$A = \sum_{k=0}^D A_{(k)}, \quad B = \sum_{k=0}^D B_{(k)}$$

with $A_{(k)}$ a \mathfrak{g} -valued k -form on Σ of internal degree $1-k$; $B_{(k)}$ is a \mathfrak{g}^* -valued k -form of internal degree $D-2-k$. The BV 2-form and the action are:

$$(37) \quad \Omega_\Sigma = (-1)^D \int_\Sigma \langle \delta B, \delta A \rangle, \quad S_\Sigma = \int_\Sigma \left\langle B, dA + \frac{1}{2}[A, A] \right\rangle$$

This is the BF theory in BV formalism.

In the abelian case, $\mathfrak{g} = \mathbb{R}$, we have $Q = \Theta = 0$ on the target and

$$(38) \quad \mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)[1] \oplus \Omega^\bullet(\Sigma)[D-2],$$

$$\Omega_\Sigma = (-1)^D \int_\Sigma \delta B \wedge \delta A, \quad S_\Sigma = \int_\Sigma B \wedge dA$$

The Poisson sigma model [18]. Let M be a manifold endowed with a Poisson bivector $\pi \in \Gamma(M, \wedge^2 TM)$. We construct the target Hamiltonian Q -manifold of degree 1 as

$$(39) \quad \mathcal{M} = T^*[1]M,$$

$$Q = \left\langle \pi(x), p \wedge \frac{\partial}{\partial x} \right\rangle_{\wedge^2 T_x M} + \frac{1}{2} \left\langle \frac{\partial}{\partial x} \pi(x), (p \wedge p) \otimes \frac{\partial}{\partial p} \right\rangle_{(\wedge^2 T \otimes T^*)_x M},$$

$$\omega = \langle \delta p, \delta x \rangle, \quad \alpha = \langle p, \delta x \rangle, \quad \Theta = \frac{1}{2} \langle \pi(x), p \wedge p \rangle_{\wedge^2 T_x M}$$

Here x and p stand for the local base and fiber coordinates on $T^*[1]M$ respectively. Note that all objects in (39) are globally well defined.

The corresponding AKSZ sigma model on an oriented closed surface Σ has the space of fields

$$\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, T^*[1]M)$$

with superfields

$$X = X_{(0)} + X_{(1)} + X_{(2)}, \quad \eta = \eta_{(0)} + \eta_{(1)} + \eta_{(2)}$$

associated to local coordinates x and p on the target, respectively. Here $X_{(k)}$ and $\eta_{(k)}$ are k -forms on Σ with internal degrees $-k$ and $1-k$ respectively. The BV 2-form and the action are:

$$\Omega_\Sigma = \int_\Sigma \langle \delta \eta, \delta X \rangle, \quad S_\Sigma = \int_\Sigma \langle \eta, dX \rangle + \frac{1}{2} \langle \pi(X), \eta \wedge \eta \rangle$$

3.4. Functorial topological field theory. In Atiyah's axiomatic approach [5], an $(n+1)$ -dimensional topological quantum field theory associates:

- To a closed oriented n -manifold Σ – a Hilbert space \mathcal{H}_Σ (the *space of states*).
- To an oriented $(n+1)$ -cobordism M from Σ_{in} to Σ_{out} (i.e. with a splitting of the boundary as $\partial M = \bar{\Sigma}_{\text{in}} \sqcup \Sigma_{\text{out}}$) – a linear map of vector spaces $Z_M : \mathcal{H}_{\Sigma_{\text{in}}} \rightarrow \mathcal{H}_{\Sigma_{\text{out}}}$ (the *partition function*).
- Orientation-preserving diffeomorphisms $\phi : \Sigma \rightarrow \Sigma'$ act on the spaces of states by unitary maps $\rho(\phi) : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma'}$. The orientation-reversing identity diffeomorphism $s_\Sigma : \Sigma \rightarrow \bar{\Sigma}$ (the bar denotes the opposite orientation) acts by a \mathbb{C} -anti-linear map $\sigma_\Sigma = \rho(s_\Sigma) : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\bar{\Sigma}}$.

One requires this association to satisfy the following axioms:

- (i) (Multiplicativity) $\mathcal{H}_{\Sigma \sqcup \Sigma'} = \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\Sigma'}$ for the spaces of states and

$$Z_{M \sqcup M'} = Z_M \otimes Z_{M'} : \mathcal{H}_{\Sigma_{\text{in}}} \otimes \mathcal{H}_{\Sigma'_{\text{in}}} \rightarrow \mathcal{H}_{\Sigma_{\text{out}}} \otimes \mathcal{H}_{\Sigma'_{\text{out}}}$$

for the partition functions.

- (ii) (Gluing) For two cobordisms $M_1 : \Sigma_1 \rightarrow \Sigma_2$ and $M_2 : \Sigma_2 \rightarrow \Sigma_3$, one can construct the glued cobordism $M_1 \cup_{\Sigma_2} M_2 : \Sigma_1 \rightarrow \Sigma_3$ and the corresponding partition function is the composition of partition functions for M_1 and M_2 as linear maps:

$$Z_{M_1 \cup_{\Sigma_2} M_2} = Z_{M_2} \circ Z_{M_1} : \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_3}$$

- (iii) (Normalization) $\mathcal{H}_{\emptyset} = \mathbb{C}$, $Z_{\Sigma \times [0,1]} = \text{id} : \mathcal{H}_{\Sigma} \rightarrow \mathcal{H}_{\Sigma}$.

- (iv) For $\phi : M \rightarrow M'$ a diffeomorphism, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_{\text{in}}} & \xrightarrow{Z_M} & \mathcal{H}_{\Sigma_{\text{out}}} \\ \rho(\phi|_{\Sigma_{\text{in}}}) \downarrow & & \downarrow \rho(\phi|_{\Sigma_{\text{out}}}) \\ \mathcal{H}_{\Sigma'_{\text{in}}} & \xrightarrow{Z_{M'}} & \mathcal{H}_{\Sigma'_{\text{out}}} \end{array}$$

(In particular, Z_M is invariant under diffeomorphisms of M relative to the boundary of M .)

- (v) (Symmetry) The natural diffeomorphism $\Sigma \sqcup \Sigma' \rightarrow \Sigma' \sqcup \Sigma$ is sent by ρ to the natural isomorphism $\mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\Sigma'} \rightarrow \mathcal{H}_{\Sigma'} \otimes \mathcal{H}_{\Sigma}$.
- (vi) The partition function for the cylinder $\Sigma \times [0, 1]$, regarded as a cobordism $\Sigma \times \bar{\Sigma} \rightarrow \emptyset$, together with the anti-linear map $(\sigma_{\Sigma})^{-1} : \mathcal{H}_{\bar{\Sigma}} \rightarrow \mathcal{H}_{\Sigma}$, yields the Hermitian inner product $\langle \cdot, \cdot \rangle : \mathcal{H}_{\Sigma} \times \mathcal{H}_{\Sigma} \rightarrow \mathbb{C}$.

The following are immediate consequences of the axioms:

- For M a closed $(n+1)$ -manifold, regarded as a cobordism $\emptyset \rightarrow \emptyset$, the partition function $Z_M \in \mathbb{C}$ is a number invariant under orientation-preserving diffeomorphisms of M .
- $Z_{\Sigma \times S^1} = \dim \mathcal{H}_{\Sigma} \in \mathbb{Z}$. In particular, this implies that spaces of states have to be finite-dimensional. More generally, for a *mapping torus* $\frac{\Sigma \times [0,1]}{\Sigma \times \{0\} \cong \Sigma \times \{1\}}$ with $\phi : \Sigma \rightarrow \Sigma$ a gluing diffeomorphism, the partition function is $Z = \text{Tr}_{\mathcal{H}_{\Sigma}} \rho(\phi)$.

A short way to formulate Atiyah's axioms is to say that a TQFT is a functor of symmetric monoidal categories from the category of oriented cobordisms (with composition given by gluing and monoidal structure given by disjoint unions) to the category of Hilbert spaces, and diffeomorphisms act by natural transformations of the functor (alternatively, one can view diffeomorphisms as 2-morphisms; then (\mathcal{H}, Z, ρ) becomes a functor of 2-categories).

In dimensions 1 and 2, Atiyah's TQFTs admit a straightforward complete classification. For $n+1=1$, a TQFT, up to equivalence, is specified by a non-negative integer – the dimension of the space of states for a point. For $n+1=2$, the classification of TQFTs is given by a result of Dijkgraaf by isomorphism classes of unital Frobenius algebras (in particular, the underlying vector space of the algebra is \mathcal{H}_{S^1} , multiplication is the partition function for a pair of pants regarded as a cobordism $S^1 \sqcup S^1 \rightarrow S^1$, the unit is the partition function for the disk viewed as

a cobordism $\emptyset \rightarrow S^1$ etc.) In dimension 3 and above the complete classification of Atiyah’s TQFTs is not known.

Baez and Dolan [9] suggested an enhancement of Atiyah’s notion of TQFT as a functor from the (∞, n) -extension of the cobordism category (which essentially allows gluing-cutting with higher codimension strata) – the “extended” TQFT. Lurie proved in [34] that a fully extended TQFT (i.e. extended down to strata of maximal codimension – points) is determined by its values on points, thus providing a complete classification result for fully extended TQFTs. It is not known however whether e.g. Chern-Simons theory can be embedded in a fully extended TQFT.

4. MAIN RESULTS

4.1. Paper “Remarks on Chern-Simons invariants” [20], with A. S. Cattaneo. Given a differential graded Frobenius algebra C^\bullet with invariant pairing of degree -3 , and a quadratic Lie algebra \mathfrak{g} , we construct the algebraic model for Chern-Simons theory in the framework of the Batalin-Vilkovisky formalism, with the space of BV fields $\mathcal{F} = \mathfrak{g} \otimes C^\bullet[1]$, with BV 2-form coming from the pairing on C^\bullet tensored with the ad-invariant pairing on \mathfrak{g} . Out of the structure constants of algebraic operations on C^\bullet and \mathfrak{g} , we construct the BV action $S \in C^\infty(\mathcal{F})_0$ – a cubic polynomial in fields solving the quantum master equation. In the case $C^\bullet = \Omega^\bullet(M)$, the de Rham algebra of M , the action S becomes the master action for Chern-Simons theory (12).

In case of a finite-dimensional dg Frobenius algebra C^\bullet , we analyze in detail the effective BV action induced on the cohomology $\mathfrak{g} \otimes H^\bullet(C^\bullet)[1]$ via the pushforward construction (2), and the canonical transformations induced by the deformations of the induction data (deformations of the gauge-fixing Lagrangian and of the choice of representatives of cohomology classes of $H^\bullet(C^\bullet)$ in C^\bullet). Then we analyze the invariants coming from the effective action on cohomology by considering it modulo canonical transformations coming from deformations of the induction data (taking a quotient over *all* canonical transformation, one would kill interesting invariants).

In the special case of $H^1(C^\bullet) = 0$, we prove that the effective action is of the form $W_{\text{prod}} + F(\hbar)$ where W_{prod} is a cubic polynomial on $\mathfrak{g} \otimes H^\bullet(C^\bullet)[1]$ corresponding to the Lie bracket on \mathfrak{g} -valued cohomology of C^\bullet , while $F(\hbar) \in \hbar^2\mathbb{C}[[\hbar]]$ is a formal power series, comprising the complete invariant of the effective action under admissible (in the sense above) canonical transformations.

In the case when $\dim H^1(C^\bullet) = 1$ and C^\bullet is a formal dg algebra, we prove that the 1-loop part of the effective action restricted to the Maurer-Cartan set of the graded Lie algebra $\mathfrak{g} \otimes H^\bullet(C^\bullet)$ is an invariant.

Then we extend the results developed for C^\bullet finite-dimensional, to the differential-geometric Chern-Simons theory with $C^\bullet = \Omega^\bullet(M)$, taking care of the translation of proofs valid for finite-dimensional BV integrals in the language of perturbative path integrals, with Feynman diagrams given by integrals over configuration spaces. We prove that one gets invariants of closed, oriented, framed 3-manifolds from the effective BV action for Chern-Simons theory induced on de Rham cohomology of the 3-manifold.

This treatment generalizes Axelrod-Singer’s treatment of perturbative Chern-Simons path integral to the case of non-acyclic flat background connections.

4.2. Paper “Finite-dimensional AKSZ-BV theories” [15], with F. Bonechi and M. Zabzine. In the framework of AKSZ sigma models, we consider the symplectic reduction of the coisotropic subspace of the mapping space $\text{Map}(T[1]\Sigma, \mathcal{M})$ given by the zero locus of Q_0 – the lifting of the de Rham differential on Σ to a cohomological vector field on the mapping space (the source term of the full AKSZ cohomological vector field (21)). The action of the sigma model is a reducible function with respect to this reduction. The reduced theory provides the leading term in the perturbation expansion for the zero-mode effective action for the AKSZ sigma model. The reduced mapping space is finite-dimensional and can be viewed as a mapping space of sheaves from the de Rham cohomology of Σ as a sheaf over a point to the structure sheaf of \mathcal{M} . We develop the corresponding generalization of the AKSZ construction, based on replacing the source by a differential graded Frobenius algebra. We also consider in detail two examples of the symplectic reduction to source zero-modes: the Poisson sigma model (on closed surfaces and on surfaces with boundary, with coisotropic boundary conditions introduced in [19]) and the Courant sigma-model.

4.3. Paper “One-dimensional Chern-Simons theory” [4], with A. Alekseev. We start by considering the “1-dimensional Chern-Simons theory” on a circle, a model in BV formalism with \mathbb{Z}_2 -graded space of BV fields $\mathcal{F} = \Pi\mathfrak{g} \otimes \Omega^0(S^1) \oplus \mathfrak{g} \otimes \Omega^1(S^1)$ (where Π is the parity-reversal operation and \mathfrak{g} a fixed quadratic Lie algebra) and the action $S = \int_{S^1} \langle \psi, d\psi + [A, \psi] \rangle$ with ψ and A the odd 0-form field and the even 1-form field, respectively. This theory arises from the AKSZ construction (Section 3.3) with the target supermanifold $\Pi\mathfrak{g}$ (same as in 3-dimensional Chern-Simons theory), but 1-dimensional source instead of a 3-dimensional one.

We calculate the BV pushforward to simplicial cochains of a polygon (triangulation of S^1), in the spirit of the work [35, 36], obtaining an explicit result. The resulting effective action on cochains depends on the simplicial fields non-locally.

The problem of splitting the result for the polygon into contributions of individual 1-simplices is solved by going to operator formalism and constructing 1-dimensional simplicial Chern-Simons theory as Atiyah’s topological quantum field theory on the monoidal category of triangulated 1-cobordisms with additional operations given by merging of neighboring 1-simplices (simplicial aggregations). The value of 1-dimensional Chern-Simons theory on a point (the space of states) is the spinor module and the value on a triangulated 1-cobordism (the partition function) is given in terms of Clifford exponentials. Concatenation of triangulated 1-cobordisms corresponds to the multiplication in the Clifford algebra $\text{Cl}(\mathfrak{g})$ and simplicial aggregations corresponds to certain finite-dimensional fiber BV integrals. Moreover, the resulting partition functions satisfy a version of the quantum master equation extended by boundary terms.

We prove that the value of 1-dimensional Chern-Simons theory, as constructed via Clifford exponentials, on a polygon (triangulated circle) is the same as calculated originally via perturbative path integral. Also, we prove that the partition function for a triangulated interval can be written as a path integral with boundary conditions given by a complex Lagrangian foliation of $\Pi\mathfrak{g}$ imposed on the restrictions of field ψ to the boundary points of the interval. In the usual way for the BV formalism, one also imposes the gauge-fixing for fields in the bulk by specifying a Lagrangian subspace in the bulk fields (with fixed boundary conditions).

4.4. Paper “The Poisson sigma model on closed surfaces” [13], with F. Bonechi and A. S. Cattaneo. We study the effective action of the Poisson sigma model on a closed 2-dimensional surface, regarding the Poisson structure as a perturbation, induced on the zero-modes of the unperturbed theory. The perturbation theory is developed around constant maps and we employ the tools of formal geometry of the target to glue the perturbative results into the global effective action.

In two important cases, the case of the surface being a torus and the case of a regular unimodular Poisson structure, we manage to prove vanishing of the quantum corrections in the effective action. In the case of a torus we employ a version of the axial gauge, while in the case of a regular unimodular Poisson structure we make use of the appropriate local normal form theorem.

In the case of a torus, with additional assumption that the target Poisson structure is Kähler, we are able to prove that the partition function (obtained from the effective action by evaluating the BV integral over the zero-mode space) is the Euler characteristic of the target. We also present an argument, using a regularization of the effective action based on the source supersymmetry, that the same result for the torus partition function holds for a broader class of Poisson structures.

Finally, in the case of a general surface and a general Poisson structure, we prove that perturbative effective actions constructed in formal neighborhoods of points of the target can indeed be glued, using quantum canonical transformations, into a global object.

4.5. Papers “Classical BV theories on manifolds with boundary” [21] and “Classical and quantum Lagrangian field theories with boundary” [22], with A. S. Cattaneo and N. Reshetikhin. We extend the Batalin-Vilkovisky formalism for gauge theories to spacetime manifolds with boundary (and corners) in a way consistent with gluing.

In this setup to a boundary component Σ of the spacetime one associates a phase space \mathcal{F}_Σ – a Hamiltonian Q -manifold with an exact degree 0 symplectic form $\omega_\Sigma = \delta\alpha_\Sigma$ (this set of data is known as a BFV manifold, for the Batalin-Fradkin-Vilkovisky cohomological resolution of coisotropic reductions in symplectic geometry). To the spacetime manifold M one associates a Q -manifold \mathcal{F}_M with a degree -1 symplectic form ω and an action function S which, instead of being a Hamiltonian for the cohomological vector field, satisfies the equation $\iota_Q\omega = dS + \pi^*\alpha_\partial$ where $\pi : \mathcal{F}_M \rightarrow \mathcal{F}_{\partial M}$ is a surjective submersion – the pullback of bulk fields to the boundary (in first-order theories; more generally $\mathcal{F}_{\partial M}$ is constructed as a symplectic reduction of the space of normal jets of bulk fields at the boundary and thus by construction comes with a surjective submersion π). Instead of the classical master equation in the bulk, one has the equation $\iota_Q\iota_Q\omega = \pi^*S_\partial$ where S_∂ is the BFV action on the boundary – the degree 1 Hamiltonian for Q_∂ .

A classical gauge theory in this setup is therefore a functor of symmetric monoidal categories from the category of spacetime cobordisms (endowed with the local geometric data appropriate to the model in question) to the *BFV category*. On the target side, the objects are the BFV phase spaces and the morphisms are the BV spaces of bulk fields. Composition is given by fiber products and monoidal structure is given by Cartesian products of BFV manifolds. This functorial picture admits

a higher category extension (where on the source side one essentially allows space-time manifolds with corners); we construct the appropriate extension of the target category.

All AKSZ sigma models fit naturally in the *BV-BFV formalism* which we develop. In particular, the boundary BFV phase space is also a mapping space with the same AKSZ target, $\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathcal{M})$, and the BFV action is again given by the formula (28), where one replaces the bulk by the boundary. We also analyze several other examples of non-topological gauge theories, in particular the electrodynamics and the pure Yang-Mills theory on Riemannian manifolds with boundary.

We also analyze the reduced picture where to a bulk manifold and to its boundary components one associates the “Euler-Lagrange moduli spaces” arising as a reduction of the zero locus of the cohomological vector field Q on the space of fields (or the boundary phase space) by the distribution induced by Q . Under a certain Hodge-theoretic assumption, which holds for a broad class of gauge theories, we prove a version of Lefschetz duality for the moduli spaces. In particular, for a bulk manifold M , denoting the bulk moduli space \mathcal{M}_M and the boundary moduli space $\mathcal{M}_{\partial M}$, we have a degree 0 symplectic structure on $\mathcal{M}_{\partial M}$, a degree 1 Poisson structure on \mathcal{M}_M and a map $\pi_* : \mathcal{M}_M \rightarrow \mathcal{M}_{\partial M}$ whose image is Lagrangian in $\mathcal{M}_{\partial M}$ and whose fibers are the symplectic leaves of the Poisson structure on \mathcal{M}_M . In the case of Chern-Simons theory, the Euler-Lagrange moduli spaces are certain natural graded enhancements of the moduli spaces of flat connections on the bulk 3-manifold and the boundary surface. In the simplest example, the abelian Chern-Simons theory with gauge group \mathbb{R} , the moduli spaces are given by the de Rham cohomology (in all degrees) of the bulk and the boundary.

Our results in the case of AKSZ sigma models were subsequently generalized in the context of derived symplectic geometry of mapping stacks in [16, 38].

The formalism developed in [21, 22] is a step towards perturbative quantization of gauge theories on manifolds with boundary consistent with the Atiyah-Segal axiomatic. The motivating quantum example, where the interaction of bulk BV and boundary BFV structures was first noticed, was constructed by the author and A. Alekseev in [4].

4.6. Paper “A construction of observables for AKSZ sigma models” [37].

We present a general construction of observables for BV theories based on the construction of BV pushforward. Namely, one extends a BV theory by auxiliary fields and extends the original BV action by a term depending on the original and auxiliary fields, so that the extended action still satisfies the master equation (we call this auxiliary data a *pre-observable*). One then produces an observable out of the pre-observable by performing the BV integral over auxiliary fields.

The construction is then specialized to the setting of AKSZ theories where it is shown that pre-observables can be constructed out of Hamiltonian Q -bundles over the AKSZ target. The corresponding observables are associated to embedded submanifolds of the source manifolds and their expectation values are expected to produce higher-dimensional generalized knot invariants (or more generally, cocycles on the spaces of embeddings).

As a special case, the construction produces the Alekseev-Faddeev-Shatashvili path integral representation [2] for the Wilson loop observable in Chern-Simons theory, associated to a knot in a 3-manifold. Another special case is the observable

in BF theory associated to codimension 2 “long knots” in \mathbb{R}^n , constructed by Cattaneo and Rossi [23].

4.7. Paper “Chern-Simons theory with Wilson lines and boundary in the BV-BFV formalism” [3], with A. Alekseev and Y. Barmaz. We analyze the Chern-Simons theory on 3-manifolds with boundary enriched by Wilson lines which are allowed to end on the boundary in the BV-BFV formalism developed in [21]. Wilson lines are represented by BV integrals over auxiliary fields supported on a tangle in the 3-manifold by specialization of the construction of [37]. We study the toy example of a 1-dimensional Chern-Simons theory of [4] with a Wilson line where the quantized BFV operator on the space of states turns out to be the Kostant’s cubic Dirac operator. In the case of 3-dimensional Chern-Simons theory with Wilson lines ending on the boundary, we construct the quantum BFV operator on the space of states. Its cohomology in degree zero yields the space of conformal blocks of the Wess-Zumino-Witten model on the boundary (or equivalently the geometric quantization of the moduli space of flat connections on the boundary [8]), at least in genera 0 and 1.

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REMARKS ON CHERN–SIMONS INVARIANTS

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ABSTRACT. The perturbative Chern–Simons theory is studied in a finite-dimensional version or assuming that the propagator satisfies certain properties (as is the case, e.g., with the propagator defined by Axelrod and Singer). It turns out that the effective BV action is a function on cohomology (with shifted degrees) that solves the quantum master equation and is defined modulo certain canonical transformations that can be characterized completely. Out of it one obtains invariants.

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1. INTRODUCTION

Since its proposal in Witten's paper [18], Chern–Simons theory has been a source of fruitful constructions for 3-manifold invariants. In the perturbative framework one would like to get the invariants from the Feynman diagrams of the theory. These may be shown to be finite, see [2]. However, as in every gauge theory, one has to fix a gauge and then one has to show that the result, the invariant, is independent of the gauge fixing. In the case when one works around an acyclic connection, this was proved in [2], but this assumption rules out the trivial connection. Gauge-fixing independence for perturbation theory around the trivial connection of a rational homology sphere was proved in [1] and, for more general definitions for the propagator, in [11] and in [3]. The flexibility in the choice of propagator allows one to show that the invariant is of finite type [13].

The case of general 3-manifolds was not treated in detail, even though the propagators described in [2, 11, 3] are defined in general. The main point is the presence of zero modes, namely—working around the trivial connection—elements of de Rham cohomology (with shifted degree) of the manifold tensor the given Lie algebra. Out of formal properties of the BV formalism, it is however clear [14, 15, 7, 6] what the invariant in the general case (of a compact manifold) should be: a solution of the quantum master equation on the space of zero modes modulo certain BV canonical transformations. This effective action has already been studied—but only modulo constants—in [7], which has been a source of inspiration for us.

In the first part of this note, we make this precise and mathematically rigorous working with a finite-dimensional version [17] of Chern–Simons theory, where the algebra of smooth functions on the manifold is replaced by an arbitrary finite-dimensional dg Frobenius algebra (of appropriate degrees). We are able to produce the solution to the quantum master equation on cohomology and to describe the BV canonical transformations that occur. Out of this we are able to describe the invariant in the case of cohomology concentrated in degree zero and three (the algebraic version of a rational homology sphere) and to extract invariants in case the first Betti number is one or, more generally, when the Frobenius algebra is formal.

In the second part we revert to the infinite-dimensional case and show that whatever we did in the finite-dimensional case actually goes through if the propagator satisfies certain properties. It is good news that the propagator introduced by Axelrod and Singer in [2] does indeed satisfy them. In particular, we get an invariant for framed 3-manifolds as described in Theorem 1 on page 29.

The problems with this scheme are that there is little flexibility in the choice of propagator and that the invariants are defined up to a universal constant that is very difficult to compute. (Notice that this constant is the same that appears anyway in the case of rational homology spheres in [2, 3]). For the general case, that is of a propagator as in [11] or [3], we are able to show that all properties but one can easily be achieved. We reduce the last property to Conjecture 1 on page 27 which we hope to be able to prove in a forthcoming paper.

During the preparation of this note, we have become aware of independent work by Iacovino [10] on the same topic.

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2. EFFECTIVE BV ACTION

Let¹ (\mathcal{F}, σ) be a finite-dimensional graded vector space endowed with an odd symplectic form $\sigma \in \Lambda^2 \mathcal{F}^*$ of degree -1, which means $\sigma(u, v) \neq 0 \Rightarrow |u| + |v| = 1$ for $u, v \in \mathcal{F}$. The space of polynomial functions $\text{Fun}(\mathcal{F}) := S^\bullet \mathcal{F}^*$ is a BV algebra with anti-bracket $\{\bullet, \bullet\}$ and BV Laplacian² Δ generated by the odd symplectic form σ .

In coordinates: let $\{u_i\}$ be a basis in \mathcal{F} and $\{x^i\}$ be the dual basis in \mathcal{F}^* . Let us denote $\sigma_{ij} = \sigma(u_i, u_j)$. Then

$$\begin{aligned} \sigma &= \sum_{i,j} (-1)^{\text{gh}(x^i)} \sigma_{ij} \delta x^i \wedge \delta x^j \\ \Delta f &= \frac{1}{2} \sum_{i,j} (-1)^{\text{gh}(x^i)} (\sigma^{-1})^{ij} \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial x^j} f \\ \{f, g\} &= f \left(\sum_{i,j} (\sigma^{-1})^{ij} \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial x^j} \right) g \end{aligned}$$

In our convention $\sigma_{ij} = -\sigma_{ji}$ and we call grading on $\text{Fun}(\mathcal{F})$ the “ghost number”: $\text{gh}(x^i) = -|u_i|$.

Suppose (\mathcal{F}', σ') is another odd symplectic vector space and $\iota : \mathcal{F}' \hookrightarrow \mathcal{F}$ is an embedding (injective linear map of degree 0) that agrees with the odd symplectic structure:

$$\sigma'(u', v') = \sigma(\iota(u'), \iota(v'))$$

for $u', v' \in \mathcal{F}'$. Then \mathcal{F} can be represented as

$$(1) \quad \mathcal{F} = \iota(\mathcal{F}') \oplus \mathcal{F}''$$

where $\mathcal{F}'' := \iota(\mathcal{F}')^\perp$ is the symplectic complement of the image of ι in \mathcal{F} with respect to σ . Hence the algebra of functions on \mathcal{F} factors

$$\text{Fun}(\mathcal{F}) \cong \text{Fun}(\mathcal{F}') \otimes \text{Fun}(\mathcal{F}'')$$

(the isomorphism depends on the embedding ι). Since (1) is an (orthogonal) decomposition of odd symplectic vector spaces, the BV Laplacian also splits:

$$(2) \quad \Delta = \Delta' + \Delta''$$

Here Δ' is the BV Laplacian on $\text{Fun}(\mathcal{F}')$, associated to the odd symplectic form σ' , and Δ'' is the BV Laplacian on \mathcal{F}'' associated to the restricted odd symplectic form $\sigma|_{\mathcal{F}''}$.

Let $S \in \text{Fun}(\mathcal{F})[[\hbar]]$ be a solution to the quantum master equation (QME):

$$\Delta e^{S/\hbar} = 0 \quad \Leftrightarrow \quad \frac{1}{2} \{S, S\} + \hbar \Delta S = 0$$

(the BV action on \mathcal{F}). Let also $\mathcal{L} \subset \mathcal{F}''$ be a Lagrangian subspace in \mathcal{F}'' . We define the effective (or “induced”) BV action $S' \in \text{Fun}(\mathcal{F}')[[\hbar]]$ by the fiber BV integral

$$(3) \quad e^{S'(x')/\hbar} = \int_{\mathcal{L}} e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}}$$

¹This Section is an adaptation of Section 4.2 from [16].

²In the general case, on an odd-symplectic graded manifold, the BV Laplacian Δ is constructed from the symplectic form and a consistent measure (the “ SP -structure”). But here we treat only the linear case, and an odd-symplectic graded vector space (\mathcal{F}, σ) is automatically an SP -manifold with Lebesgue measure (the constant Berezinian) $\mu_{\mathcal{F}}$.

where $\mu_{\mathcal{L}}$ is the Lebesgue measure on \mathcal{L} and we use the notation $x = \sum_i u_i x^i$ for the canonical element of $\mathcal{F} \otimes \text{Fun}(\mathcal{F})$, corresponding to the identity map $\mathcal{F} \rightarrow \mathcal{F}$, and analogously for $x' \in \mathcal{F}' \otimes \text{Fun}(\mathcal{F}')$, $x'' \in \mathcal{F}'' \otimes \text{Fun}(\mathcal{F}'')$

Proposition 1. *The Effective action S' defined by (3) satisfies the QME on \mathcal{F}' , i.e.*

$$\Delta' e^{S'/\hbar} = 0$$

Proof. This is a direct consequence of (2) and the BV-Stokes theorem (integrals over Lagrangian submanifolds of BV coboundaries vanish):

$$\begin{aligned} \Delta' e^{S'(x')/\hbar} &= \Delta' \int_{\mathcal{L}} e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}} = \int_{\mathcal{L}} (\Delta - \Delta'') e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}} = \\ &= \int_{\mathcal{L}} \Delta e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}} - \int_{\mathcal{L}} \Delta'' e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}} = 0 \end{aligned}$$

In the last line the first term vanishes due to QME for S , while the second term is zero due to the BV-Stokes theorem. \square

The BV integral (3) depends on a choice of embedding $\iota : \mathcal{F}' \hookrightarrow \mathcal{F}$ and Lagrangian subspace $\mathcal{L} \subset \mathcal{F}''$ (notice that $\mathcal{F}'' = \iota(\mathcal{F}')^\perp$ itself depends on ι). We call the pair (ι, \mathcal{L}) the “induction data” in this setting. We are interested in the dependence of the effective BV action S' on deformations of induction data.

Recall that a generic small Lagrangian deformation \mathcal{L}_Ψ of a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}''$ is given by a gauge fixing fermion $\Psi \in \text{Fun}(\mathcal{L})$ of ghost number -1. If (q^a, p_a) are Darboux coordinates on \mathcal{F}'' such that \mathcal{L} is given by $p = 0$, then \mathcal{L}_Ψ is

$$\mathcal{L}_\Psi = \left\{ (q, p) : p_a = -\frac{\partial}{\partial q^a} \Psi \right\} \subset \mathcal{F}''$$

In our case we are interested in linear Lagrangian subspaces and thus only allow quadratic gauge fixing fermions.

A general small deformation of induction data (ι, \mathcal{L}) from (\mathcal{F}, σ) to (\mathcal{F}', σ') can be written as

$$(4) \quad \iota \mapsto \iota + \delta\iota_\perp + \delta\iota_\parallel \quad , \quad \mathcal{L} \mapsto (\text{id}_{\mathcal{F}} - \iota \circ (\delta\iota_\perp)^T) \mathcal{L}_\Psi$$

where the “perpendicular part” of the deformation of the embedding $\delta\iota_\perp : \mathcal{F}' \rightarrow \mathcal{F}$ is a linear map of degree 0 satisfying $\delta\iota_\perp(\mathcal{F}') \subset \mathcal{F}''$, while the “parallel part” is of the form $\delta\iota_\parallel = \iota \circ \delta\phi_\parallel$ with $\delta\phi_\parallel : \mathcal{F}' \rightarrow \mathcal{F}'$ a linear map of degree 0 satisfying $(\delta\phi_\parallel)^T = -\delta\phi_\parallel$ (i.e. $\delta\phi_\parallel$ lies in the Lie algebra of the group of linear symplectomorphisms $\delta\phi_\parallel \in \text{sp}(\mathcal{F}', \sigma')$). We use superscript T to denote transposition w.r.t. symplectic structure. Thus it is reasonable to classify small deformations into the three following types:

- Type I: small Lagrangian deformations of Lagrangian subspace $\mathcal{L} \mapsto \mathcal{L}_\Psi$ leaving the embedding ι intact.
- Type II: small perpendicular deformations of the embedding $\iota \mapsto \iota + \delta\iota_\perp$ accompanied by an associated deformation³ of the Lagrangian subspace

$$(5) \quad \mathcal{L} \mapsto (\text{id}_{\mathcal{F}} - \iota \circ (\delta\iota_\perp)^T) \mathcal{L}$$

³Formula (5) is explained as follows: one can express the deformation of the embedding as $\iota + \delta\iota_\perp = (\text{id}_{\mathcal{F}} + \delta\Phi) \circ \iota$, where $\delta\Phi = \delta\iota_\perp \circ \iota^T - \iota \circ \delta\iota_\perp^T \in \text{sp}(\mathcal{F}, \sigma)$, so that $\text{id}_{\mathcal{F}} + \delta\Phi$ is an infinitesimal symplectomorphism of the space \mathcal{F} , accounting for the deformation of ι . Then (5) just means $\mathcal{L} \mapsto (\text{id}_{\mathcal{F}} + \delta\Phi) \circ \mathcal{L}$.

(this is necessary since here we deform the splitting (1) and \mathcal{L} is supposed to be a subspace of the deformed \mathcal{F}'').

- Type III: small parallel deformations of the embedding

$$\iota \mapsto \iota + \delta\iota_{\parallel} = \iota \circ (\text{id}_{\mathcal{F}'} + \delta\phi_{\parallel})$$

leaving \mathcal{L} intact.

A general small deformation (4) is the sum of deformations of types I, II, III.

We call the following transformation of the action

$$(6) \quad S \mapsto \tilde{S} = S + \{S, R\} + \hbar\Delta R$$

(regarded in first order in R) the infinitesimal canonical transformation of the action with (infinitesimal) generator $R \in \text{Fun}(\mathcal{F})[[\hbar]]$ of ghost number -1. Equivalently

$$(7) \quad e^{S/\hbar} \mapsto e^{\tilde{S}/\hbar} = e^{S/\hbar} + \Delta(e^{S/\hbar}R)$$

The transformed action also solves the QME (in first order in R). Infinitesimal canonical transformations generate the equivalence relation on solutions of QME.

Lemma 1. *If the action S is changed by an infinitesimal canonical transformation $S \mapsto S + \{S, R\} + \hbar\Delta R$, then the effective BV action is also changed by an infinitesimal canonical transformation*

$$(8) \quad S' \mapsto S' + \{S', R'\} + \hbar\Delta' R'$$

with generator $R' \in \text{Fun}(\mathcal{F}')[[\hbar]]$ given by fiber BV integral

$$(9) \quad R' = e^{-S'/\hbar} \int_{\mathcal{L}} e^{S/\hbar} R \mu_{\mathcal{L}}$$

Proof. This follows straightforwardly from (2), the BV-Stokes theorem and the exponential form of canonical transformations (7):

$$\begin{aligned} e^{S'/\hbar} \mapsto e^{\tilde{S}'/\hbar} &= \int_{\mathcal{L}} (e^{S/\hbar} + \Delta(e^{S/\hbar}R)) \mu_{\mathcal{L}} = \\ &= e^{S'/\hbar} + \Delta' \int_{\mathcal{L}} e^{S/\hbar} R \mu_{\mathcal{L}} + \underbrace{\int_{\mathcal{L}} \Delta''(e^{S/\hbar}R) \mu_{\mathcal{L}}}_0 = e^{S'/\hbar} + \Delta'(e^{S'/\hbar}R') \end{aligned}$$

with R' given by (9). □

Proposition 2. *Under general infinitesimal deformation of the induction data (ι, \mathcal{L}) as in (4) the effective BV action S' is transformed canonically (up to constant shift):*

$$(10) \quad S' \mapsto S' + \{S', R'\} + \hbar\Delta'(R' - R'_{III})$$

with generator

$$(11) \quad R' = \underbrace{\sigma'(x', \delta\phi_{\parallel} x')}_{R'_{III}} + e^{-S'/\hbar} \int_{\mathcal{L}} e^{S/\hbar} (\Psi + \sigma(x'', \delta\iota_{\perp} x')) \mu_{\mathcal{L}}$$

Proof. The deformation (4) can be represented in the form

$$\iota \mapsto (\text{id}_{\mathcal{F}} + \delta\Phi) \circ \iota \quad , \quad \mathcal{L} \mapsto (\text{id}_{\mathcal{F}} + \delta\Phi)\mathcal{L}$$

where $\delta\Phi \in \text{sp}(\mathcal{F}, \sigma)$ is the infinitesimal symplectomorphism given by

$$\delta\Phi = \{\bullet, \Psi\} + (\delta\iota_{\perp} \circ \iota^T - \iota \circ \delta\iota_{\perp}^T) + \delta\iota_{\parallel} \circ \iota^T$$

Here $\{\bullet, \Psi\} = \exp(\{\bullet, \Psi\}) - \text{id}_{\mathcal{F}}$ is understood as the (infinitesimal) flow generated by Hamiltonian vector field $\{\bullet, \Psi\}$ in unit time. The pull-back $(\text{id}_{\mathcal{F}} + \delta\Phi)^* : \text{Fun}(\mathcal{F}) \rightarrow \text{Fun}(\mathcal{F})$ acts on functions as canonical transformation

$$(12) \quad f \mapsto f + \{f, R\}$$

with generator given by

$$R = \frac{1}{2}\sigma(x, \delta\Phi x) = \underbrace{\Psi}_{R_I} + \underbrace{\sigma(x, (\delta\iota_{\perp} \circ \iota^T)x)}_{R_{II}} + \underbrace{\frac{1}{2}\sigma(x, (\delta\iota_{\parallel} \circ \iota^T)x)}_{R_{III}}$$

It is important to note that only the third term (the effect of type III deformation) contributes to ΔR :

$$\Delta R = \Delta' R = \Delta R_{III} = \frac{1}{2} \text{Str}_{\mathcal{F}'} \delta\phi_{\parallel}$$

($\text{Str}_{\mathcal{F}'}$ denotes supertrace over \mathcal{F}') and $\Delta'' R = 0$. The latter implies that $(\text{id}_{\mathcal{F}} + \delta\Phi)^* \mu_{\mathcal{L}} = \mu_{\mathcal{L}}$. Now we can compute the transformation of the effective BV action S' due to infinitesimal change of induction data:

$$(13) \quad e^{S'/\hbar} \mapsto e^{\tilde{S}'/\hbar} = \int_{\mathcal{L}} (\text{id}_{\mathcal{F}} + \delta\Phi)^* e^{S/\hbar} \mu_{\mathcal{L}} = \\ = e^{-\text{Str}_{\mathcal{F}'} \delta\phi_{\parallel}} \int_{\mathcal{L}} e^{(S + \{S, R\} + \hbar \Delta R)/\hbar} \mu_{\mathcal{L}} = e^{-\text{Str}_{\mathcal{F}'} \delta\phi_{\parallel}} e^{(S' + \{S', R'\} + \hbar \Delta' R)/\hbar}$$

Here we used Lemma 1 and the generator is given by (9):

$$R' = e^{-S'/\hbar} \int_{\mathcal{L}} e^{S/\hbar} R \mu_{\mathcal{L}} = e^{-S'/\hbar} \int_{\mathcal{L}} e^{S/\hbar} (\Psi + \sigma(x, (\delta\iota_{\perp} \circ \iota^T)x) + \frac{1}{2}\sigma(x, (\delta\iota_{\parallel} \circ \iota^T)x)) \mu_{\mathcal{L}}$$

which yields (11). At last note that $\frac{1}{2} \text{Str}_{\mathcal{F}'} \delta\phi_{\parallel} = \Delta' \frac{1}{2} \sigma'(x', \delta\phi_{\parallel} x') = \Delta' R'_{III}$ which explains the constant shift in (10). \square

Remark 1. If we were treating BV actions as log-half-densities (meaning that $e^{S/\hbar}$ is a half-density), we would write an honest canonical transformation (8) instead of (10), with no $-\hbar \Delta' R'_{III}$ shift. This is because the pull-back $(\text{id}_{\mathcal{F}} + \delta\Phi)^*$ would then be acting on S by transformation (6) instead of (12). But in practice one works with effective BV actions defined by a normalized BV integral: if the initial BV action is of the form $S = S_0 + S_{\text{int}}$ with S_0 quadratic in fields (the “free part” of BV action) and S_{int} the “interaction part”, one usually defines

$$e^{S'(x')/\hbar} = \frac{1}{N} \int_{\mathcal{L}} e^{S(\iota(x') + x'')/\hbar} \mu_{\mathcal{L}}$$

with the normalization factor $N = \int_{\mathcal{L}} e^{S_0(x'')/\hbar} \mu_{\mathcal{L}}$. The effective action defined via a normalized BV integral is indeed a function rather than log-half-density, and transforms according to (10) under change of induction data.

3. TOY MODEL FOR EFFECTIVE CHERN–SIMONS THEORY ON ZERO-MODES: EFFECTIVE BV ACTION ON COHOMOLOGY OF DG FROBENIUS ALGEBRAS

By a dg Frobenius algebra (\mathcal{C}, d, m, π) we mean a unital differential graded commutative algebra \mathcal{C} with differential $d : \mathcal{C}^{\bullet} \rightarrow \mathcal{C}^{\bullet+1}$ and (super-commutative, associative) product $m : S^2 \mathcal{C} \rightarrow \mathcal{C}$, endowed in addition with a non-degenerate pairing

$\pi : S^2\mathcal{C} \rightarrow \mathbb{R}$ of degree $-k$ (which means $\pi(a, b) \neq 0 \Rightarrow |a| + |b| = k$; and k is some fixed integer), satisfying the following consistency conditions:

$$(14) \quad \pi(da, b) + (-1)^{|a|}\pi(a, db) = 0$$

$$(15) \quad \pi(a, m(b, c)) = \pi(m(a, b), c)$$

for $a, b, c \in \mathcal{C}$.

By a dg Frobenius–Lie⁴ algebra (\mathcal{A}, d, l, π) we mean a differential graded Lie algebra \mathcal{A} with differential $d : \mathcal{A}^\bullet \rightarrow \mathcal{A}^{\bullet+1}$ and Lie bracket $l : \wedge^2\mathcal{A} \rightarrow \mathcal{A}$, endowed with a non-degenerate pairing $\pi : S^2\mathcal{A} \rightarrow \mathbb{R}$ of degree $-k$ satisfying the conditions

$$(16) \quad \pi(A, l(B, C)) = \pi(l(A, B), C)$$

$$(17) \quad \pi(dA, B) + (-1)^{|A|}\pi(A, dB) = 0$$

for $A, B, C \in \mathcal{A}$.

If \mathfrak{g} is an (ordinary) Lie algebra with non-degenerate ad-invariant inner product $\pi_{\mathfrak{g}} : S^2\mathfrak{g} \rightarrow \mathbb{R}$ (one calls such Lie algebras “quadratic”) and (\mathcal{C}, d, m, π) is a dg Frobenius algebra, then $(\mathfrak{g} \otimes \mathcal{C}, d, l, \pi_{\mathfrak{g} \otimes \mathcal{C}})$ is a dg Frobenius–Lie algebra. Here one defines $d(X \otimes a) := X \otimes da$, $l(X \otimes a, Y \otimes b) := [X, Y] \otimes m(a, b)$, $\pi_{\mathfrak{g} \otimes \mathcal{C}}(X \otimes a, Y \otimes b) := \pi_{\mathfrak{g}}(X, Y) \pi(a, b)$. We will usually write π instead of $\pi_{\mathfrak{g} \otimes \mathcal{C}}$.

Example 1. If M is a closed (compact, without boundary) orientable smooth manifold of dimension D , then de Rham algebra $\Omega^\bullet(M)$ is a dg Frobenius algebra with de Rham differential, wedge product and Poincaré pairing $\int_M \bullet \wedge \bullet$ of degree $-D$. If \mathfrak{g} is a finite-dimensional Lie algebra with invariant non-degenerate trace tr , then the algebra $\Omega^\bullet(M, \mathfrak{g}) = \mathfrak{g} \otimes \Omega^\bullet(M)$ of \mathfrak{g} -valued differential forms on M is a dg Frobenius–Lie algebra with pairing $\text{tr} \int_M \bullet \wedge \bullet$ of degree $-D$.

3.1. Abstract Chern–Simons action from a dg Frobenius algebra. Let (\mathcal{C}, d, m, π) be a finite dimensional non-negatively graded dg Frobenius algebra with pairing π of degree -3

$$\mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^1[-1] \oplus \mathcal{C}^2[-2] \oplus \mathcal{C}^3[-3]$$

We denote by $B_i = \dim H^i(\mathcal{C})$ the Betti numbers. Due to non-degeneracy of π , there is an isomorphism $\pi_{\mathfrak{b}} : \mathcal{C}^\bullet \cong (\mathcal{C}^{3-\bullet})^*$ (the Poincaré duality). Induced pairing on cohomology is also automatically non-degenerate, and so Poincaré duality descends to cohomology: $\pi_{\mathfrak{b}} : H^\bullet(\mathcal{C}) \cong (H^{3-\bullet}(\mathcal{C}))^*$. Hence $B_0 = B_3$, $B_1 = B_2$. We will suppose in addition that $B_0 = B_3 = 1$ (so that \mathcal{C} models the de Rham algebra of a connected manifold).

Let \mathfrak{g} be a finite dimensional quadratic Lie algebra of coefficients and let $(\mathfrak{g} \otimes \mathcal{C}, d, l, \pi)$ be the corresponding dg Frobenius–Lie structure on $\mathfrak{g} \otimes \mathcal{C}$. Then we can construct an odd symplectic space of BV fields

$$\mathcal{F} = \mathfrak{g} \otimes \mathcal{C}[1]$$

with odd symplectic structure of degree -1 given by

$$(18) \quad \sigma(sA, sB) = (-1)^{|A|}\pi(A, B)$$

Here $s : \mathfrak{g} \otimes \mathcal{C} \rightarrow \mathfrak{g} \otimes \mathcal{C}[1]$ is the suspension map. Let us also introduce the notation ω for the canonical element of $(\mathfrak{g} \otimes \mathcal{C}) \otimes \text{Fun}(\mathcal{F})$ corresponding to the desuspension

⁴Alternatively one could use the term “cyclic dg Lie algebra”.

map $s^{-1} : \mathcal{F} \rightarrow \mathfrak{g} \otimes \mathcal{C}$. If $\{e_I\}$ is a basis in \mathcal{C} and $\{T_a\}$ is an orthonormal basis in \mathfrak{g} , then we can write

$$\omega = \sum_{I,a} T_a e_I \omega^{Ia}$$

where $\{\omega^{Ia}\}$ are the corresponding coordinates on \mathcal{F} . By abuse of terminology we call ω the ‘‘BV field’’. Let us introduce notations for the structure constants: $\pi_{IJ} = \pi(e_I, e_J)$, $m_{IJK} = \pi(e_I, m(e_J, e_K))$, $f_{abc} = \pi_{\mathfrak{g}}(T_a, [T_b, T_c])$, $d_{IJ} = \pi(e_I, de_J)$. We will also use the shorthand notation for degrees $|I| = |e_I|$. In terms of π the BV Laplacian and the anti-bracket are

$$\begin{aligned} \Delta f &= \frac{1}{2} \sum_{I,J,a} (\pi^{-1})^{IJ} \frac{\partial}{\partial \omega^{Ia}} \frac{\partial}{\partial \omega^{Ja}} f \\ \{f, g\} &= f \left(\sum_{I,J,a} (-1)^{|I|+1} (\pi^{-1})^{IJ} \frac{\overleftarrow{\partial}}{\partial \omega^{Ia}} \frac{\overrightarrow{\partial}}{\partial \omega^{Ja}} \right) g \end{aligned}$$

Proposition 3. *The action $S \in \text{Fun}(\mathcal{F})$ defined as*

$$\begin{aligned} (19) \quad S &:= \frac{1}{2} \pi(\omega, d\omega) + \frac{1}{6} \pi(\omega, l(\omega, \omega)) = \\ &= \frac{1}{2} \sum_{I,J,a} (-1)^{|I|+1} d_{IJ} \omega^{Ia} \omega^{Ja} + \frac{1}{6} \sum_{I,J,K,a,b,c} (-1)^{|J|(|K|+1)} f_{abc} m_{IJK} \omega^{Ia} \omega^{Jb} \omega^{Kc} \end{aligned}$$

satisfies the QME with BV Laplacian defined by the odd symplectic structure (18) on \mathcal{F} .

Proof. Indeed, let us check the CME:

$$\begin{aligned} \frac{1}{2} \{S, S\} &= \\ \frac{1}{8} \{ \pi(\omega, d\omega), \pi(\omega, d\omega) \} &+ \frac{1}{12} \{ \pi(\omega, d\omega), \pi(\omega, l(\omega, \omega)) \} + \frac{1}{72} \{ \pi(\omega, l(\omega, \omega)), \pi(\omega, l(\omega, \omega)) \} \\ &= -\frac{1}{2} \pi(\omega, d^2\omega) - \frac{1}{2} \pi(\omega, dl(\omega, \omega)) - \frac{1}{8} \pi(\omega, l(\omega, l(\omega, \omega))) = 0 \end{aligned}$$

The first term vanishes due to $d^2 = 0$, the second — due to the Leibniz identity for $\mathfrak{g} \otimes \mathcal{C}$, since property (16) implies

$$\frac{1}{2} \pi(\omega, dl(\omega, \omega)) = \frac{1}{6} \pi(\omega, dl(\omega, \omega) - l(d\omega, \omega) + l(\omega, d\omega)),$$

and the third term is zero due to the Jacobi identity for $\mathfrak{g} \otimes \mathcal{C}$. Next, check the quantum part of the QME:

$$\hbar \Delta S = -\hbar \frac{1}{2} \text{Str}_{\mathfrak{g} \otimes \mathcal{C}} d - \hbar \frac{1}{2} \text{Str}_{\mathfrak{g} \otimes \mathcal{C}} l(\omega, \bullet) = 0$$

Here the first term vanishes since d raises degree and the second term vanishes due to unimodularity of Lie algebra \mathfrak{g} . \square

The BV action (19) can be viewed as an abstract model (or toy model, since \mathcal{C} is finite dimensional) for Chern–Simons theory on a connected closed orientable 3-manifold. We associate such a model to any finite dimensional non-negatively graded dg Frobenius algebra \mathcal{C} with pairing of degree -3 and $B_0 = B_3 = 1$, and arbitrary finite dimensional quadratic Lie algebra of coefficients \mathfrak{g} . We are interested

in the effective BV action for (19) induced on cohomology $\mathcal{F}' = H^\bullet(\mathcal{C}, \mathfrak{g})[1]$. We will now specialize the general induction procedure sketched in Section 2 to this case.

3.2. Effective action on cohomology. Let $\iota : H^\bullet(\mathcal{C}) \hookrightarrow \mathcal{C}^\bullet$ be an embedding of cohomology into \mathcal{C} . Note that ι is not just an arbitrary chain map between two fixed complexes, but is also subject to condition $\iota([a]) = a + d(\dots)$ for any cocycle $a \in \mathcal{C}$. This implies in particular that the only allowed deformations of ι are of the form $\iota \mapsto \iota + d \delta I$ where $\delta I : H^\bullet(\mathcal{C}) \rightarrow \mathcal{C}^{\bullet-1}$ is an arbitrary degree -1 linear map. This is indeed a type II deformation (in the terminology of Section 2), while type III deformations are prohibited in this setting.

Let also $K : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet-1}$ be a symmetric chain homotopy retracting \mathcal{C}^\bullet to $H^\bullet(\mathcal{C})$, that is a degree -1 linear map satisfying

$$(20) \quad dK + Kd = \text{id}_{\mathcal{C}} - \mathcal{P}'$$

$$(21) \quad \pi(Ka, b) + (-1)^{|a|} \pi(a, Kb) = 0$$

$$(22) \quad K \circ \iota = 0$$

where $\mathcal{P}' = \iota \circ \iota^T : \mathcal{C} \rightarrow \mathcal{C}$ is the orthogonal (w.r.t. π) projection to the representatives of cohomology in \mathcal{C} . We require the additional property

$$(23) \quad K^2 = 0$$

Remark 2 (cf. [9]). An arbitrary linear map $K_0 : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet-1}$ satisfying just (20) can be transformed into a chain homotopy K with all the properties (20,21,22,23) via a chain of transformations $K_0 \mapsto K_1 \mapsto K_2 \mapsto K_3 = K$ where

$$(24) \quad \begin{aligned} K_1 &= \frac{1}{2}(K_0 - K_0^T) \\ K_2 &= (\text{id}_{\mathcal{C}} - \mathcal{P}') K_1 (\text{id}_{\mathcal{C}} - \mathcal{P}') \end{aligned}$$

$$(25) \quad K_3 = K_2 d K_2$$

Having ι and K we can define a Hodge decomposition for \mathcal{C} into representatives of cohomology, d -exact part and K -exact part:

$$(26) \quad \mathcal{C} = \text{im}(\iota) \oplus \underbrace{\mathcal{C}_{d\text{-ex}}}_{\text{im}(d)} \oplus \underbrace{\mathcal{C}_{K\text{-ex}}}_{\text{im}(K)}$$

Properties (21), (22), (23) and skew-symmetry of differential (14) imply the orthogonality properties for Hodge decomposition (26):

$$\text{im}(\iota)^\perp = \mathcal{C}_{d\text{-ex}} \oplus \mathcal{C}_{K\text{-ex}}, \quad (\mathcal{C}_{d\text{-ex}})^\perp = \text{im}(\iota) \oplus \mathcal{C}_{K\text{-ex}}, \quad (\mathcal{C}_{K\text{-ex}})^\perp = \text{im}(\iota) \oplus \mathcal{C}_{K\text{-ex}}$$

In terms of Hodge decomposition (26) the splitting (1) of the space of BV fields $\mathcal{F} = \mathfrak{g} \otimes \mathcal{C}[1]$ is given by

$$\underbrace{\mathfrak{g} \otimes \mathcal{C}[1]}_{\mathcal{F}} = \underbrace{\iota(H^\bullet(\mathcal{C}, \mathfrak{g})[1])}_{\iota(\mathcal{F}')} \oplus \underbrace{\mathfrak{g} \otimes \mathcal{C}_{d\text{-ex}}[1] \oplus \mathfrak{g} \otimes \mathcal{C}_{K\text{-ex}}[1]}_{\mathcal{F}''}$$

and we choose the Lagrangian subspace

$$(27) \quad \mathcal{L}_K = \mathfrak{g} \otimes \mathcal{C}_{K\text{-ex}}[1] \subset \mathcal{F}''$$

We define the “effective BV action on cohomology” (or “on zero-modes”) $W \in \text{Fun}(\mathcal{F}')[[\hbar]]$ for an abstract Chern–Simons action (19) by a normalized fiber BV

integral

$$(28) \quad e^{W(\alpha)/\hbar} = \frac{1}{N} \int_{\mathcal{L}_K} e^{S(\iota(\alpha)+\omega'')/\hbar} \mu_{\mathcal{L}_K}$$

where

$$(29) \quad N = \int_{\mathcal{L}_K} e^{S_0(\omega'')/\hbar} \mu_{\mathcal{L}_K}$$

is the normalization factor and

$$S_0(\omega'') = \frac{1}{2} \pi(\omega'', d\omega'')$$

is the free part of the action S . To lighten somewhat the notation, we denoted the effective action by W instead of S' and the BV field associated to $\mathcal{F}' = H^\bullet(\mathcal{C}, \mathfrak{g})[1]$ by α instead of ω' . Let $\{e_p\}$ be a basis of the cohomology $H^\bullet(\mathcal{C})$. Then $\alpha = \sum_{a,p} T_a e_p \alpha^{pa}$ where α^{pa} are coordinates on \mathcal{F}' with ghost numbers $\text{gh}(\alpha^{pa}) = 1 - |e_p|$. We have the following decomposition of $S(\iota(\alpha) + \omega'')$:

$$(30) \quad S(\iota(\alpha) + \omega'') = \underbrace{\frac{1}{6} \pi(\iota(\alpha), l(\iota(\alpha), \iota(\alpha)))}_{W_{\text{prod}}(\alpha)} + \underbrace{\frac{1}{2} \pi(\omega'', d\omega'')}_{S_0(\omega'')} + \underbrace{\frac{1}{2} \pi(\omega'', l(\iota(\alpha), \iota(\alpha))) + \frac{1}{2} \pi(\iota(\alpha), l(\omega'', \omega'')) + \frac{1}{6} \pi(\omega'', l(\omega'', \omega''))}_{S_{\text{int}}(\alpha, \omega'')}$$

The perturbation expansion for (28) is obtained in a standard way and can be written as

$$(31) \quad W(\alpha) = W_{\text{prod}}(\alpha) + \hbar \log \left(e^{-\hbar \frac{1}{2} \pi^{-1}(\frac{\partial}{\partial \omega''}, K \frac{\partial}{\partial \omega''})|_{\omega''=0}} \circ e^{S_{\text{int}}(\alpha, \omega'')/\hbar} \right) = \sum_{l=0}^{\infty} \hbar^l \sum_{n=0}^{\infty} \sum_{\Gamma \in G_{l,n}} \frac{1}{|\text{Aut}(\Gamma)|} W_{\Gamma}(\alpha)$$

Where $G_{l,n}$ denotes the set of connected non-oriented Feynman graphs with vertices of valence 1 and 3 (we would like to understand them as trivalent graphs with “leaves” allowed, i.e. external edges), with l loops and n leaves. The contribution $W_{\Gamma}(\alpha)$ of each Feynman graph $\Gamma \in G_{l,n}$ is a homogeneous polynomial of degree n in $\{\alpha^{pa}\}$ and of ghost number 0 obtained by decorating each leaf of Γ by $\iota_p^I \alpha^{pa}$, each trivalent vertex by $f_{abc} m_{IJK}$ and each (internal) edge by $\delta^{ab} K^{IJ}$, and taking contraction of all indices, corresponding to incidence of vertices and edges in Γ . One should also take into account signs for contributions, which can be obtained from the exponential formula for perturbation series (31). The cubic term

$$(32) \quad W_{\text{prod}}(\alpha) = \frac{1}{6} \pi(\iota(\alpha), l(\iota(\alpha), \iota(\alpha))) = \frac{1}{6} \sum_{a,b,c} \sum_{p,q,r} (-1)^{|e_q|(|e_r|+1)} f_{abc} \mu_{pqr} \alpha^{pa} \alpha^{qb} \alpha^{rc}$$

is the contribution of the simplest Feynman diagram $\Gamma_{0,3}$, the only element of $G_{0,3}$. Here $\mu_{pqr} = \pi(\iota(e_p), m(\iota(e_q), \iota(e_r)))$ are structure constants of the induced associative product on $H^\bullet(\mathcal{C})$ (hence the notation W_{prod}).

Remark 3. The perturbative expansion (31) is related to homological perturbation theory (HPT) in the following way. Denote by $c_{l,n}(\alpha) = n! \sum_{\Gamma \in G_{l,n}} \frac{1}{|\text{Aut}(\Gamma)|} W_{\Gamma}(\alpha)$ the total contribution of Feynman graphs with l loops and n leaves to the effective action (31), with additional factor $n!$. Then each

$$c_{l,n} \in S^n(\mathcal{F}^{l*}) \cong \text{Hom}(\wedge^n(H^{\bullet}(\mathcal{C}, \mathfrak{g})), \mathbb{R})$$

can be understood as a (super-)anti-symmetric n -ary operation on cohomology $H^{\bullet}(\mathcal{C}, \mathfrak{g})$, taking values in numbers. Now suppose $l_n \in \text{Hom}(\wedge^n(H^{\bullet}(\mathcal{C}, \mathfrak{g})), H^{\bullet}(\mathcal{C}, \mathfrak{g}))$ are the L_{∞} operations on cohomology, induced from dg Lie algebra $\mathfrak{g} \otimes \mathcal{C}$ (by means of HPT). Then it is easy to see that $c_{0,n+1}(\alpha_0, \dots, \alpha_n) = \pi'(\alpha_0, l_n(\alpha_1, \dots, \alpha_n))$ and trees for HPT (the Lie version of trees from [12], cf. also [9]) are obtained from Feynman trees for $W(\alpha)$ by assigning one leaf as a root and inserting the inverse of pairing⁵ $(\pi')^{-1}$ there, or vice versa: Feynman trees are obtained from trees of HPT by reverting the root⁶ with π' and forgetting the orientation of edges (cf. Section 7.2.1 of [16]). Thus we can loosely say that the BV integral (28) defines a sort of “loop enhancement” of HPT for a cyclic dg Lie algebra $\mathfrak{g} \otimes \mathcal{C}$. Also, in this language (due to A. Losev), using the BV-Stokes theorem to prove that the effective action W satisfies the quantum master equation can be viewed as the loop-enhanced version of using the HPT machinery to prove the system of quadratic relations (homotopy Jacobi identities) on induced L_{∞} operations l_n .

Let us introduce a Darboux basis in $H^{\bullet}(\mathcal{C})$. Namely, let $e_{(0)} = [1]$ be the basis vector in $H^0(\mathcal{C})$, the cohomology class of unit $1 \in \mathcal{C}^0$ (recall that we assume $B_0 = \dim H^0(\mathcal{C}) = 1$) and let $e_{(3)}$ be the basis vector in $H^3(\mathcal{C})$, satisfying $\pi'(e_{(0)}, e_{(3)}) = 1$ (i.e. $e_{(3)}$ is represented by some top-degree element $v = \iota(e_{(3)}) \in \mathcal{C}^3$, normalized by the condition $\pi(1, v) = 1$). Let also $\{e_{(1)i}\}$ be some basis in $H^1(\mathcal{C})$ and $\{e_{(2)}^i\}$ the dual basis in $H^2(\mathcal{C})$, so that $\pi'(e_{(1)i}, e_{(2)}^j) = \delta_i^j$. The BV field α is then represented as

$$\alpha = \sum_a e_{(0)} T_a \alpha_{(0)}^a + \sum_{a,i} e_{(1)i} T_a \alpha_{(1)}^{ia} + \sum_{a,i} e_{(2)}^i T_a \alpha_{(2)i}^a + \sum_a e_{(3)} T_a \alpha_{(3)}^a$$

In Darboux coordinates $\{\alpha_0^a, \alpha_{(1)}^{ia}, \alpha_{(2)i}^a, \alpha_{(3)}^a\}$ the BV Laplacian on \mathcal{F}' is

$$\Delta' = \sum_a \frac{\partial}{\partial \alpha_{(0)}^a} \frac{\partial}{\partial \alpha_{(3)}^a} + \sum_{a,i} \frac{\partial}{\partial \alpha_{(1)}^{ia}} \frac{\partial}{\partial \alpha_{(2)i}^a}$$

It is also convenient to introduce \mathfrak{g} -valued coordinates on \mathcal{F}' :

$$\alpha_{(0)} = \sum_a T_a \alpha_{(0)}^a, \quad \alpha_{(1)}^i = \sum_a T_a \alpha_{(1)}^{ia}, \quad \alpha_{(2)i} = \sum_a T_a \alpha_{(2)i}^a, \quad \alpha_{(3)} = \sum_a T_a \alpha_{(3)}^a$$

⁵We use notation $\pi' = \pi(\iota(\bullet), \iota(\bullet)) : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathbb{R}$ for the induced pairing on cohomology $H^{\bullet}(\mathcal{C})$. By a slight abuse of notation, we also use π' to denote the pairing on $H^{\bullet}(\mathcal{C}, \mathfrak{g})$ induced from $\pi_{\mathfrak{g} \otimes \mathcal{C}}$.

⁶This means the following: let T be a binary rooted tree with n leaves, oriented towards the root; let \bar{T} be the non-oriented (and non-rooted) tree with $n+1$ leaves, obtained from T by forgetting the orientation (and treating the root as additional leaf). Then the weight $W_{\bar{T}}(\alpha)$ of \bar{T} as a Feynman graph and the contribution l_T of tree T (without the symmetry factor) to the induced L_{∞} operation l_n are related by $W_{\bar{T}}(\alpha) = \pi'(\alpha, l_T(\underbrace{\alpha, \dots, \alpha}_n))$. Pictorially this is represented by inserting a bivalent vertex (associated to the operation $\pi'(\bullet, \bullet)$) at the root of T . Both edges incident to this vertex are incoming, thus we say that the root becomes reverted.

The ghost numbers of $\alpha_{(0)}, \alpha_{(1)}^i, \alpha_{(2)i}, \alpha_{(3)}$ are 1, 0, -1, -2 respectively. In terms of this Darboux basis, the trivial part (32) of the effective action is

$$\begin{aligned} W_{\text{prod}}(\alpha) &= \\ &= \frac{1}{2} \sum_{a,b,c} f_{abc} \alpha_{(0)}^a \alpha_{(0)}^b \alpha_{(3)}^c - \sum_{a,b,c} \sum_i f_{abc} \alpha_{(0)}^a \alpha_{(1)}^{ib} \alpha_{(2)i}^c + \frac{1}{6} \sum_{a,b,c} \sum_{i,j,k} f_{abc} \mu_{ijk} \alpha_{(1)}^{ia} \alpha_{(1)}^{jb} \alpha_{(1)}^{kc} = \\ &= \underbrace{\frac{1}{2} \pi_{\mathfrak{g}}(\alpha_{(3)}, [\alpha_{(0)}, \alpha_{(0)}])}_{W_{\text{prod}}^{003}} - \underbrace{\sum_i \pi_{\mathfrak{g}}(\alpha_{(0)}, [\alpha_{(1)}^i, \alpha_{(2)i}])}_{W_{\text{prod}}^{012}} + \underbrace{\frac{1}{6} \sum_{i,j,k} \mu_{ijk} \pi_{\mathfrak{g}}(\alpha_{(1)}^i, [\alpha_{(1)}^j, \alpha_{(1)}^k])}_{W_{\text{prod}}^{111}} \end{aligned}$$

Here μ_{ijk} is the totally antisymmetric tensor of structure constants of multiplication of 1-cohomologies: $\mu_{ijk} = \pi(\iota(e_{(1)i}), m(\iota(e_{(1)j}), \iota(e_{(1)k})))$.

Proposition 4. *The effective BV action W , induced from the abstract Chern–Simons action (19) on $\mathcal{F}' = H^\bullet(\mathcal{C}, \mathfrak{g})[1]$ has the form*

$$(33) \quad W(\alpha) = W_{\text{prod}}(\alpha) + F(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1}; \hbar)$$

where $F \in (\text{Fun}(\mathfrak{g}^{B_1})[[\hbar]])^{\mathfrak{g}}$ is some \hbar -dependent function on $H^1(\mathcal{C}, \mathfrak{g})[1] \cong \mathfrak{g}^{B_1}$, invariant under the diagonal adjoint action of \mathfrak{g} , i.e.

$$(34) \quad F(\alpha_{(1)}^1 + [X, \alpha_{(1)}^1], \dots, \alpha_{(1)}^{B_1} + [X, \alpha_{(1)}^{B_1}]; \hbar) = F(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1}; \hbar) \pmod{X^2}$$

at the first order in $X \in \mathfrak{g}$.

Proof. Ansatz (33) follows from the observation that the values of individual Feynman graphs $\Gamma \neq \Gamma_{0,3}$ in (31) depend only on the 1-cohomology: $W_\Gamma = W_\Gamma(\{\alpha_{(1)}^i\})$. The argument is as follows: suppose not all leaves of Γ are decorated with insertions of 1-cohomology. Then, since $\text{gh}(W_\Gamma) = 0$, there is at least one leaf decorated with insertion of 0-cohomology. Since $\iota(H^0(\mathcal{C})) = \mathbb{R} \cdot 1$, the value of Γ will contain one of the expressions $K^2(\dots)$, $K(\iota(\dots))$. Hence Feynman diagrams with insertion of cohomology of degree $\neq 1$ vanish due to (22), (23). This proves (33).

By construction, W has to satisfy the QME (Proposition 1). The QME for an action satisfying ansatz (33) is equivalent to ad-invariance of F (34):

$$\begin{aligned} \frac{1}{2} \{W, W\}' + \hbar \Delta' W &= \{W_{\text{prod}}^{012}, F\}' = \sum_i \sum_{a,b,c} f_{abc} \alpha_{(0)}^a \alpha_{(1)}^{ib} \frac{\partial}{\partial \alpha_{(1)}^{ic}} F = \\ &= \sum_i \left\langle [\alpha_{(0)}, \alpha_{(1)}^i], \frac{\partial}{\partial \alpha_{(1)}^i} \right\rangle_{\mathfrak{g}} F \end{aligned}$$

where $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ denotes the canonical pairing between \mathfrak{g} and \mathfrak{g}^* .

Another explanation of (34) is the following: if \mathfrak{g} is the Lie algebra of the Lie group \mathcal{G} , then the original abstract Chern–Simons action (19) is invariant under the adjoint action of \mathcal{G} , i.e. $\omega \mapsto g\omega g^{-1}$ for $g \in \mathcal{G}$. The embedding ι and the choice of the Lagrangian subspace $\mathcal{L} \subset \mathcal{F}''$ are also compatible with this symmetry. Hence $W(\alpha)$ is also invariant under the adjoint action of \mathcal{G} : $\alpha \mapsto g\alpha g^{-1}$, and (34) is the infinitesimal form of this symmetry. \square

Remark 4. There is another argument for ansatz (33) that can be formulated on the level of BV integral (28) itself, rather than on the level of Feynman diagrams.

Namely, the restriction of $S_{\text{int}}(\alpha, \omega'')$ (we refer to decomposition (30)) to the Lagrangian subspace \mathcal{L}_K does not depend on $\alpha_{(0)}$. This means that the only term depending on $\alpha_{(0)}$ in (30) is the trivial one $W_{\text{prod}}(\alpha)$, constant on \mathcal{L}_K . Hence the non-trivial part of the effective action $W(\alpha) - W_{\text{prod}}(\alpha)$ does not depend on $\alpha_{(0)}$. Since it also has to be of ghost number zero, it can only depend on $\alpha_{(1)}$.

3.3. Dependence of the effective action on cohomology on induction data.

The effective action $W(\alpha)$ depends on induction data (ι, K) , and we are interested in describing how $W(\alpha)$ changes due to the deformation of (ι, K) .

In the terminology of Section 2, the type I deformations of induction data are of the form $\iota \mapsto \iota, K \mapsto K + [d, \delta\kappa]$ where $\delta\kappa : \mathcal{C}_{d-ex}^\bullet \rightarrow \mathcal{C}_{K-ex}^{\bullet-2}$ is a skew-symmetric linear map of degree -2. The corresponding deformation of the Lagrangian subspace \mathcal{L}_K is described by the gauge fixing fermion $\Psi = \frac{1}{2}\pi(\omega, d \delta\kappa d \omega)$. Type II deformations change embedding as $\iota \mapsto \iota + d \delta I$ where $\delta I : H^\bullet(\mathcal{C}) \rightarrow \mathcal{C}_{K-ex}^{\bullet-1}$, and change chain homotopy in a minimal way (so as not to spoil properties (22), (20)): $K \mapsto K + \iota \delta I^T - \delta I \iota^T$. Type III transformations are forbidden in this setting, as discussed above (we have a canonical surjective map from $\ker d \subset \mathcal{C}$ to $H^\bullet(\mathcal{C})$ that sends the cocycle α to its cohomology class $[\alpha]$).

Due to Proposition 2, an infinitesimal deformation of (ι, K) induces an infinitesimal canonical transformation of $W(\alpha)$

$$W \mapsto W + \{W, R'\}' + \hbar \Delta' R'$$

with generator given by the fiber BV integral

(35)

$$R'(\alpha) = e^{-W(\alpha)/\hbar} \int_{\mathcal{L}_K} e^{S(\iota(\alpha) + \omega'')/\hbar} \left(\frac{1}{2}\pi(\omega'', d \delta\kappa d\omega'') + \pi(\omega'', d \delta I \alpha) \right) \mu_{\mathcal{L}_K}$$

This integral is evaluated perturbatively in analogy with (31):

(36)

$$R'(\alpha) =$$

$$e^{-W(\alpha)/\hbar} \cdot e^{-\hbar \frac{1}{2}\pi^{-1}(\frac{\partial}{\partial \omega''}, K \frac{\partial}{\partial \omega''})} \Big|_{\omega''=0} \circ \left(e^{S_{\text{int}}(\alpha, \omega'')/\hbar} \cdot \left(\frac{1}{2}\pi(\omega'', d \delta\kappa d\omega'') + \pi(\omega'', d \delta I \alpha) \right) \right) = \sum_{l=0}^{\infty} \hbar^l \sum_{n=0}^{\infty} \sum_{\Gamma^M \in G_{l,n}^M} \frac{1}{|\text{Aut}(\Gamma^M)|} R'_{\Gamma^M}(\alpha)$$

Here the superscript M stands for ‘‘marked edge’’, $G_{l,n}^M$ is the set of connected non-oriented trivalent graphs with l loops and n leaves and either one leaf or one internal edge marked. Values of Feynman graphs $R'_{\Gamma^M}(\alpha)$ are now homogeneous polynomials of degree n and ghost number -1 on \mathcal{F}^l , obtained by the same Feynman rules as for $W_\Gamma(\alpha)$, supplemented with a Feynman rule for marked edge: we decorate the marked leaf with $\delta I_p^I \alpha^{pa}$ and marked internal edge with $\delta^{ab} \delta \kappa^{IJ}$.

Proposition 5. *The generator of the the infinitesimal canonical transformation induced on the effective BV action $W(\alpha)$ by the infinitesimal change of induction data (ι, K) has the following form:*

(37)

$$R'(\alpha) = \sum_{a,i} \alpha_{(2)i}^a G^{ia}(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1}; \hbar)$$

where $G = \sum_{a,i} G^{ia} \frac{\partial}{\partial \alpha_{(1)}^a} \in (\text{Vect}(\mathfrak{g}^{B_1})[[\hbar]])^{\mathfrak{g}}$ is some \hbar -dependent vector field on $H^1(\mathcal{C}, \mathfrak{g})[1] \cong \mathfrak{g}^{B_1}$, equivariant under the diagonal adjoint action of \mathfrak{g} , i.e.

$$(38) \quad G^i(\alpha_{(1)}^1 + [X, \alpha_{(1)}^1], \dots, \alpha_{(1)}^{B_1} + [X, \alpha_{(1)}^{B_1}]; \hbar) = [X, G^i(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1}; \hbar)] \pmod{X^2}$$

at first order in $X \in \mathfrak{g}$, for all $i = 1, \dots, B_1$ (and we set $G^i := \sum_a T_a G^{ia}$). The canonical transformation with generator (37) in terms of ansatz (33) is

$$(39) \quad F \mapsto F + G \circ (W_{\text{prod}}^{111} + F) + \hbar \text{div}(G)$$

Proof. The argument for ansatz (37) is pretty much the same as for (33): the value $R'_{\Gamma^M}(\alpha)$ of each Feynman graph $\Gamma^M \in G_{l,n}^M$ is linear in $\alpha_{(2)}$ and does not depend on $\alpha_{(0)}, \alpha_{(3)}$ for the following reason: Unless we decorate one leaf of Γ^M by $\alpha_{(2)}$ and all other leaves by $\alpha_{(1)}$, some leaf has to be decorated by $\alpha_{(0)}$ (since the total ghost number of $R'_{\Gamma^M}(\alpha)$ has to be -1). Then the contribution of this decoration of Γ^M vanishes due to $\iota(H^0(\mathcal{C})) = \mathbb{R} \cdot 1$ and the vanishing of the expressions $\delta I(e_{(0)})$, K^2 , $K \delta \kappa$, $\delta \kappa K$, $K \iota$, $\delta \kappa \iota$, one of which necessarily appears as contribution of a neighborhood of the place of insertion of 0-cohomology on the Feynman graph. This proves ansatz (37).

The equivariance of G (38) is equivalent to the fact that a canonical transformation with generator (37) preserves ansatz (33) for $W(\alpha)$. Indeed, if G were not equivariant, the term $\{W_{\text{prod}}^{012}, R'\}$ would produce $\alpha_{(0)}$ -dependence for the canonically transformed effective action. The other explanation is that the equivariance of G is due to the invariance of R' under the adjoint action of the group \mathcal{G} , which is due to the fact that a deformation of (ι, K) is trivial in \mathfrak{g} -coefficients and hence consistent with the adjoint \mathcal{G} -action.

Rewriting the canonical transformation of the effective action (33) with generator (37) as (39) is straightforward. \square

Remark 5. Analogously to Proposition 4, one can prove ansatz (37) on the level of BV integral (35) instead of using Feynman diagrams. Namely, expressions $S(\iota(\alpha) + \omega'') - W_{\text{prod}}(\alpha)$, $W(\alpha) - W_{\text{prod}}(\alpha)$ and the expression in parentheses in (35) all do not depend on $\alpha_{(0)}$. Hence R' does not depend on $\alpha_{(0)}$. But it also has to be of ghost number -1, which can only be achieved if it is of form (37).

3.4. Invariants. We are interested in describing the effective action $W(\alpha)$ on cohomology modulo changes of induction data (ι, K) . Due to Propositions 4, 5, we can give a complete solution (i.e. describe the complete invariant) in case $B_1 = 0$, and find some partial solution (i.e. describe some, probably incomplete, invariant) for the case of a formal Frobenius algebra \mathcal{C} , meaning that we can find representatives for cohomology $\iota : H^\bullet(\mathcal{C}) \hookrightarrow \mathcal{C}$ closed under multiplication. In particular, in case $B_1 = 1$ the algebra \mathcal{C} is necessarily formal.

Proposition 6. *If $B_1 = 0$, the effective action on cohomology is*

$$(40) \quad W(\alpha) = W_{\text{prod}}^{003}(\alpha) + F(\hbar)$$

where $W_{\text{prod}}^{003}(\alpha) = \frac{1}{2} \pi_{\mathfrak{g}}(\alpha_{(3)}, [\alpha_{(0)}, \alpha_{(0)}])$ and $F(\hbar)$ is an \hbar -dependent constant, invariant under deformations of induction data (ι, K) .

Proof. Ansatz (40) is a restriction of (33) to the case $B_1 = 0$. Due to (37) and $B_1 = 0$, the generator of induced canonical transformation necessarily vanishes $R' = 0$. Hence $F(\hbar)$ is invariant under deformation of (ι, K) . \square

So $F(\hbar)$ is the complete invariant of $W(\alpha)$ for the $B_1 = 0$ case (which is an abstract model for Chern–Simons theory on a rational homology sphere) and is given by

$$(41) \quad F(\hbar) = \hbar \log \left(e^{-\hbar \frac{1}{2} \pi^{-1} \left(\frac{\partial}{\partial \omega^I}, K \frac{\partial}{\partial \omega^I} \right) \Big|_{\omega''=0}} \circ e^{\frac{1}{\hbar} \frac{1}{6} \pi(\omega'', l(\omega'', \omega''))} \right) = \\ = \sum_{l=2}^{\infty} \hbar^l \sum_{\Gamma^{\text{vac}} \in G_{l,0}} \frac{1}{|\text{Aut}(\Gamma^{\text{vac}})|} F_{\Gamma^{\text{vac}}}$$

Where we sum over trivalent connected non-oriented graphs without leaves Γ^{vac} (the “vacuum loops”). The contribution of a Feynman graph $F_{\Gamma^{\text{vac}}} \in \mathbb{R}$ is a number, computed by the same Feynman rules as for (31), just without the insertions of α .

Example 2 (Chevalley–Eilenberg complex of $\mathfrak{su}(2)$). We obtain an interesting example of abstract Chern–Simons theory with $B_1 = 0$ if we choose

$$\mathcal{C} = S^\bullet(\mathfrak{su}(2)^*[-1]) = \mathbb{R} \oplus \mathfrak{su}(2)^*[-1] \oplus (\wedge^2 \mathfrak{su}(2)^*)[-2] \oplus (\wedge^3 \mathfrak{su}(2)^*)[-3]$$

— the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{su}(2)$. This \mathcal{C} is naturally a dg Frobenius algebra with super-commutative wedge product, Chevalley–Eilenberg differential

$$d: \quad e_1 \mapsto e_2 e_3, \quad e_2 \mapsto e_3 e_1, \quad e_3 \mapsto e_1 e_2$$

and pairing

$$\pi(\mathbf{1}, e_1 e_2 e_3) = \pi(e_1, e_2 e_3) = \pi(e_2, e_3 e_1) = \pi(e_3, e_1 e_2) = 1$$

Here $\{e_1, e_2, e_3\}$ is the basis in $\mathfrak{su}(2)^*$, dual to the basis $\{-\frac{i}{2}\sigma_1, -\frac{i}{2}\sigma_2, -\frac{i}{2}\sigma_3\}$ in $\mathfrak{su}(2)$, where $\{\sigma_i\}$ are the Pauli matrices; $\mathbf{1}$ is the unit in $\mathcal{C}^0 = \mathbb{R}$. This \mathcal{C} can be understood as the algebra of left-invariant differential forms on the Lie group $SU(2) \sim S^3$ which is indeed quasi-isomorphic (as a dg algebra) to the whole de Rham algebra of the sphere S^3 ; thus the abstract Chern–Simons theory associated to this \mathcal{C} is in a sense a toy model for true Chern–Simons theory on S^3 . The Hodge decomposition (26) for \mathcal{C} is unique:

$$\mathcal{C} = \underbrace{\mathbb{R} \oplus (\wedge^3 \mathfrak{su}(2)^*)[-3]}_{H^\bullet(\mathcal{C})} \oplus \underbrace{\mathfrak{su}(2)^*[-1]}_{\mathcal{C}_{K-ex}} \oplus \underbrace{(\wedge^2 \mathfrak{su}(2)^*)[-2]}_{\mathcal{C}_{d-ex}}$$

and the BV field ω is

$$\omega = \underbrace{\alpha_{(0)} \mathbf{1} + \alpha_{(3)} e_1 e_2 e_3}_{\alpha} + \underbrace{\omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3}_{\omega''_{K-ex}} + \underbrace{\omega^{23} e_2 e_3 + \omega^{31} e_3 e_1 + \omega^{12} e_1 e_2}_{\omega''_{d-ex}}$$

Here $\alpha_{(0)}, \alpha_{(3)}$ are \mathfrak{g} -valued coordinates on $\mathcal{F}' = \mathfrak{g} \otimes H^\bullet(\mathcal{C})[1]$ and $\{\omega^I\}, \{\omega^{IJ}\}$ are \mathfrak{g} -valued coordinates on $\mathcal{F}'' \subset \mathfrak{g} \otimes \mathcal{C}[1]$. The Lagrangian subspace (27) is $\mathcal{L} = \mathfrak{g} \otimes \mathfrak{su}(2)^*$. The effective action on cohomology, as defined by the integral (28), satisfies the ansatz (40) with $F(\hbar)$ given by

$$(42) \quad F(\hbar) = \hbar \log \frac{\int_{\mathfrak{g} \otimes \mathfrak{su}(2)^*} e^{\frac{1}{\hbar} \left(\frac{1}{2} \sum_{I=1}^3 \pi_{\mathfrak{g}}(\omega^I, \omega^I) + \pi_{\mathfrak{g}}(\omega^1, [\omega^2, \omega^3]) \right)} d\omega^1 d\omega^2 d\omega^3}{\int_{\mathfrak{g} \otimes \mathfrak{su}(2)^*} e^{\frac{1}{\hbar} \cdot \frac{1}{2} \sum_{I=1}^3 \pi_{\mathfrak{g}}(\omega^I, \omega^I)} d\omega^1 d\omega^2 d\omega^3}$$

The perturbative expansion (41) now reads

$$(43) \quad F(\hbar) = \hbar \log \left(e^{-\hbar \frac{1}{2} \sum_{I=1}^3 \pi_{\mathfrak{g}}^{-1} \left(\frac{\partial}{\partial \omega^I}, \frac{\partial}{\partial \omega^I} \right) \Big|_{\omega^I=0}} \circ e^{\frac{1}{\hbar} \pi_{\mathfrak{g}}(\omega^1, [\omega^2, \omega^3])} \right) =$$

$$= \sum_{l=2}^{\infty} (-1)^{l+1} \hbar^l \sum_{\Gamma^{\text{vac}} \in G_{l,0}} \frac{1}{|\text{Aut}(\Gamma^{\text{vac}})|} L_{\Gamma^{\text{vac}}}^{\mathfrak{su}(2)} \cdot L_{\Gamma^{\text{vac}}}^{\mathfrak{g}}$$

Here $L_{\Gamma^{\text{vac}}}^{\mathfrak{g}}$ denotes the ‘‘Lie algebra graph’’ (or the ‘‘Jacobi graph’’), i.e. the number⁷ obtained by decorating vertices of Γ^{vac} with the structure constants f_{abc} of the Lie algebra \mathfrak{g} and taking contraction of indices over edges of Γ^{vac} . In particular, for $L_{\Gamma^{\text{vac}}}^{\mathfrak{su}(2)}$ vertices are decorated with structure constants of $\mathfrak{su}(2)$ — the Levi-Civita symbol⁸ $\epsilon_{IJK} \in \{\pm 1, 0\}$. First terms in the series (43) are:

$$F(\hbar) = -\hbar^2 \frac{1}{12} \cdot 6 \cdot f_{abc} f_{abc} + \hbar^3 \left(\frac{1}{16} \cdot 12 \cdot f_{acd} f_{bcd} f_{aef} f_{bef} + \frac{1}{24} \cdot 6 \cdot f_{abc} f_{ade} f_{bef} f_{cfd} \right) + \dots$$

In particular, for $\mathfrak{g} = \mathfrak{su}(N)$ we have

$$F(\hbar) = -\hbar^2 \frac{1}{2} (N^3 - N) + \hbar^3 \frac{7}{8} (N^4 - N^2) - \hbar^4 \frac{23}{8} (N^5 - N^3) + \dots$$

As a side note, $L_{\Gamma^{\text{vac}}}^{\mathfrak{su}(2)}$ can be interpreted combinatorially as the number of ways to decorate edges of Γ^{vac} with 3 colors, such that in each vertex edges of all 3 colors meet (these decorations should be counted with signes, determined by the cyclic order of colors in each vertex). Thus for $\mathfrak{g} = \mathfrak{su}(2)$ the invariant $F(\hbar)$ is given by

$$F(\hbar) = \sum_{l=2}^{\infty} (-1)^{l+1} \hbar^l \sum_{\Gamma^{\text{vac}} \in G_{l,0}} \frac{1}{|\text{Aut}(\Gamma^{\text{vac}})|} (L_{\Gamma^{\text{vac}}}^{\mathfrak{su}(2)})^2$$

and it can be viewed as a generating function for certain interesting combinatorial quantities (and on the other hand, there is an ‘‘explicit’’ integral formula (42) for $F(\hbar)$ in terms of a 9-dimensional Airy-type integral).

Proposition 7. *Suppose \mathcal{C} is formal and the embedding $\iota : H^\bullet(\mathcal{C}) \hookrightarrow \mathcal{C}$ is an algebra homomorphism, then:*

- *The tree part of the effective action on cohomology contains only the trivial W_{prod} term, i.e.*

$$(44) \quad W(\alpha) = W_{\text{prod}}(\alpha) + \hbar F^{(1)}(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1}) + \sum_{l=2}^{\infty} \hbar^l F^{(l)}(\alpha_{(1)}^1, \dots, \alpha_{(1)}^{B_1})$$

where $F^{(l)} \in (\text{Fun}(\mathfrak{g}^{B_1}))^{\mathfrak{g}}$ for $l = 1, 2, \dots$

- *The 1-loop part of effective action can be written as*

$$(45) \quad F^{(1)}(\alpha_{(1)}) = \frac{1}{2} \text{Str}_{\mathfrak{g} \otimes \mathcal{C}} \log (1 + K \circ l(\iota(\alpha_{(1)}), \bullet))$$

⁷Strictly speaking, one also has to choose a cyclic ordering of half-edges for each vertex of Γ^{vac} , and this choice affects the total sign of $L_{\Gamma^{\text{vac}}}^{\mathfrak{g}}$. But this ambiguity is cancelled in (43) since the factors $L_{\Gamma^{\text{vac}}}^{\mathfrak{g}}$ and $L_{\Gamma^{\text{vac}}}^{\mathfrak{su}(2)}$ change their signs simultaneously if we change the cyclic ordering of half-edges in any vertex of Γ^{vac} .

⁸We assume that the inner product on $\mathfrak{su}(2)$ is normalized as $\pi_{\mathfrak{su}(2)}(x, y) = -2 \text{tr}(xy)$ (in the fundamental representation of $\mathfrak{su}(2)$), so that the structure constants for the orthonormal basis are really ϵ_{IJK} .

- *Restriction of the 1-loop part of the effective action $F^{(1)}|_{\text{MC}}$ to the Maurer-Cartan set is invariant under deformations of (ι, K) (preserving the homomorphism condition for ι). Here $\text{MC} \subset H^1(\mathcal{C}, \mathfrak{g})[1]$ is given by*

$$\sum_{j,k} \mu_{ijk}[\alpha_{(1)}^j, \alpha_{(1)}^k] = 0.$$

Proof. The fact that ι is a homomorphism implies

$$(46) \quad K \circ l(\iota(\alpha), \iota(\alpha)) = 0$$

This means that all Feynman diagrams for $W(\alpha)$ that contain a vertex adjacent to two leaves and one internal edge, vanish. In particular, all tree diagrams except $\Gamma_{0,3}$ vanish. Together with Proposition 4 this implies (44). Also (46) implies that among 1-loop diagrams only “wheels” survive, and they are summed up to form (45) in standard way.

The next observation is that the tree part R^0 of the generator of the canonical transformation (36) induced by changing (ι, K) vanishes. This is a consequence of properties (46), $\delta\kappa \circ l(\iota(\alpha), \iota(\alpha)) = 0$ and $\pi(\delta I(\alpha), l(\iota(\alpha), \iota(\alpha))) = 0$ (which all follow from the fact that ι is a homomorphism). In terms of the vector field G (37), we have $G = \sum_{l=1}^{\infty} \hbar^l G^{(l)}$ and the l -loop part of effective action is transformed according to (39) as

$$F^{(l)} \mapsto F^{(l)} + G^{(l)} \circ W_{\text{prod}}^{111} + \sum_{i=1}^{l-1} G^{(l-i)} \circ F^{(i)} + \text{div } G^{(l-1)}$$

In particular, the 1-loop part is transformed as

$$F^{(1)} \mapsto F^{(1)} + G^{(1)} \circ W_{\text{prod}}^{111}$$

Since the Maurer-Cartan set is precisely the locus of stationary points of W_{prod}^{111} , the restriction $F^{(1)}|_{\text{MC}}$ is invariant. This finishes the proof. \square

Special case of formal \mathcal{C} is the case $B_1 = 1$. Here the Maurer-Cartan set is $\text{MC} = H^1(\mathcal{C}, \mathfrak{g})[1] \cong \mathfrak{g}$, so the 1-loop part of effective action $F^{(1)} \in \text{Fun}(\mathfrak{g})^{\mathfrak{g}}$ is invariant (without any restriction).

3.5. Comments on relaxing the condition $K^2 = 0$ for chain homotopy. Let us introduce the notation

$$(47) \quad \text{Ind}_{\iota, K}(\bar{S})(\alpha) := \hbar \log \left(e^{-\hbar \frac{1}{2} \pi^{-1} \left(\frac{\partial}{\partial \omega''}, K \frac{\partial}{\partial \omega''} \right) |_{\omega''=0}} \circ e^{(\bar{S}(\iota(\alpha) + \omega'') - S_0(\omega''))/\hbar} \right) \in \text{Fun}(\mathcal{F}')[[\hbar]]$$

for the “effective action” for some (not necessarily abstract Chern–Simons) action $\bar{S} \in \text{Fun}(\mathcal{F})[[\hbar]]$, defined by perturbation series, rather than by BV integral itself. Expression (47) is indeed the perturbation series generated by the BV integral

$$(48) \quad e^{\text{Ind}_{\iota, K}(\bar{S})(\alpha)/\hbar} = \frac{1}{N} \int_{\mathcal{L}_K} e^{\bar{S}(\iota(\alpha) + \omega'')/\hbar} \mu_{\mathcal{L}_K}$$

with normalization N as before (29). In particular for abstract Chern–Simons action $\bar{S} = S$ we recover the definition of W : $\text{Ind}_{\iota, K}(S)(\alpha) = W(\alpha)$.

The important observation with which we are concerned here is that definition (47) makes sense for a chain homotopy K not necessarily satisfying property $K^2 = 0$ (we assume that the other properties we demanded (20,21,22) hold), while the

definition via BV integral (48) is less general and uses essentially $K^2 = 0$ property for construction of Lagrangian subspace \mathcal{L}_K . To avoid confusion we will denote by \hat{K} the chain homotopy without property (23) and reserve notation K for the “honest” chain homotopy with property (23). We will call the effective action defined via (47) with \hat{K} as chain homotopy the “relaxed” effective action, while for an honest chain homotopy K we call the effective action (defined equivalently by (47) or by BV integral (48)) “strict”.

Proposition 8. *Let $\hat{K} : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet-1}$ be a chain homotopy satisfying properties (20,21,22), but with $\hat{K}^2 \neq 0$, and let $K = \hat{K}d\hat{K}$ be the construction (25) applied to \hat{K} (i.e. K satisfies all the properties (20,21,22,23)). Then the relaxed effective action $\text{Ind}_{l,\hat{K}}(S)$ is equivalent (i.e. connected by a canonical transformation) to the strict effective action $\text{Ind}_{l,K}(S)$.*

Proof. The first observation is that since $K = \hat{K}d\hat{K}$, we can write the relaxed chain homotopy as

$$(49) \quad \hat{K} = K + d\Lambda d$$

where $\Lambda = \hat{K}^3 : \mathcal{C}^3 \rightarrow \mathcal{C}^0$. In fact, formula (49), with arbitrary skew-symmetric, degree -3 linear map $\Lambda : \mathcal{C}^3 \rightarrow \mathcal{C}^0$, gives a general (finite) deformation of honest chain homotopy K , preserving properties (20,21,22), but violating (23) and satisfying in addition $K = \hat{K}d\hat{K}$. In other words, deformation (49) is the inverse of projection (25).

Second, we interpret the Feynman diagram decomposition for $\text{Ind}_{l,\hat{K}}$ with propagator \hat{K} given by (49) as sum over graphs with edges decorated either by K or by $d\Lambda d$, and then raise the Feynman subgraphs with edges decorated only by $d\Lambda d$ into action. I.e. we obtain

$$(50) \quad \text{Ind}_{l,\hat{K}}(S) = \text{Ind}_{l,K}(S + \Phi_\Lambda)$$

where

$$(51) \quad \begin{aligned} \Phi_\Lambda = & \frac{1}{8}\pi(l(\omega, \omega), d\Lambda d l(\omega, \omega)) + \frac{1}{8}\pi(l(\omega, \omega), d\Lambda d l(\omega, d\Lambda d l(\omega, \omega))) + \\ & + \frac{1}{8}\pi(l(\omega, \omega), d\Lambda d l(\omega, d\Lambda d l(\omega, d\Lambda d l(\omega, \omega)))) + \dots + \\ & + \hbar \frac{1}{2} \cdot \frac{1}{2} \text{Str}_{\mathcal{F}} d\Lambda d l(\omega, d\Lambda d l(\omega, \bullet)) + \hbar \frac{1}{2} \cdot \frac{1}{3} \text{Str}_{\mathcal{F}} d\Lambda d l(\omega, d\Lambda d l(\omega, d\Lambda d l(\omega, \bullet))) + \dots \end{aligned}$$

Only the simplest trees (“branches”) and only the simplest one-loop diagrams (“wheels”) contribute to Φ_Λ , because any diagram with 3 incident internal edges decorated by $d\Lambda d$ automatically vanishes due to

$$\pi(d\Lambda d(\dots), l(d\Lambda d(\dots), d\Lambda d(\dots))) = 0$$

(which is implied by Leibniz identity in $\mathfrak{g} \otimes \mathcal{C}$, by $d^2 = 0$ and skew-symmetry of d).

Third, we notice that there is a canonical transformation from S to $S + \Phi_\Lambda$. A convenient way to describe a finite canonical transformation is to present a “homotopy”

$$S_\Lambda(t, dt) = S_\Lambda(t) + dt \cdot R_\Lambda(t) \in \text{Fun}(\mathcal{F})[[\hbar]] \otimes \Omega^\bullet([0, 1])$$

— a differential form on interval $[0, 1]$ with values in functions on \mathcal{F} ($t \in [0, 1]$ is a coordinate on the interval), satisfying the QME on the extended space $\mathcal{F} \oplus$

$T[1]([0, 1])$:

$$(52) \quad (d_t + \hbar\Delta)S_\Lambda(t, dt) + \frac{1}{2}\{S_\Lambda(t, dt), S_\Lambda(t, dt)\} = 0$$

(where $d_t = dt \frac{\partial}{\partial t}$ is the de Rham differential on the interval), and satisfying boundary conditions

$$S_\Lambda(t)|_{t=0} = S, \quad S_\Lambda(t)|_{t=1} = S + \Phi_\Lambda$$

The extended QME is equivalent to the fact that $S_\Lambda(t)$ is a solution to the QME on \mathcal{F} for any given $t \in [0, 1]$ plus the fact that $S_\Lambda(t + \delta t)$ is obtained from $S_\Lambda(t)$ by the infinitesimal canonical transformation with generator $\delta t \cdot R_\Lambda(t)$. In our case it is a straightforward exercise to present the desired homotopy between S and $S + \Phi_\Lambda$:

$$S_\Lambda(t, dt) = S + \Phi_{t\Lambda} + dt \frac{1}{t} R_{t\Lambda}$$

where $\Phi_{t\Lambda}$ is defined by (51) with Λ rescaled by t , and R_Λ is given by

$$R_\Lambda = \frac{1}{4}\pi(d\Lambda d\omega, l(\omega, \omega)) + \frac{1}{4}\pi(d\Lambda d\omega, l(\omega, d\Lambda d l(\omega, \omega))) + \\ + \frac{1}{4}\pi(d\Lambda d\omega, l(\omega, d\Lambda d l(\omega, d\Lambda d l(\omega, \omega)))) + \dots$$

We showed that S and $S + \Phi_\Lambda$ are connected by a homotopy. Due to Proposition 1 and Lemma 1 the effective actions $\text{Ind}_{\iota, K}(S)$ and $\text{Ind}_{\iota, K}(S + \Phi_\Lambda)$ are also connected by a homotopy defined by

$$(53) \quad e^{W_\Lambda(t, dt)(\alpha)/\hbar} = \frac{1}{N} \int_{\mathcal{L}_K} e^{S_\Lambda(t, dt)(\iota(\alpha) + \omega'')/\hbar} \mu_{\mathcal{L}_K}$$

Together with (50) this finishes the proof. \square

Remark 6. An immediate consequence of Proposition 8 is that the relaxed effective action $\text{Ind}_{\iota, \hat{K}}(S)$ satisfies the QME on \mathcal{F}' .

Remark 7. Expression (51) suggests that it is itself the value of a certain Gaussian integral. Namely, the restriction of $S + \Phi_\Lambda$ to the subspace $\iota(\mathcal{F}') \oplus \mathcal{L}_K \subset \mathcal{F}$ can be written as a fiber integral over fibers $\mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]$ of vector bundle $\iota(\mathcal{F}') \oplus \mathcal{L}_K \oplus \mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1] \rightarrow \iota(\mathcal{F}') \oplus \mathcal{L}_K$ (which is a sub-bundle of $\mathcal{F} \rightarrow \iota(\mathcal{F}') \oplus \mathcal{L}_K$):

$$e^{(S + \Phi_\Lambda)/\hbar} \Big|_{\iota(\mathcal{F}') \oplus \mathcal{L}_K} = \int_{\mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]} e^{(S + \frac{1}{2}\pi(\omega, K\Lambda^{-1}K\omega))/\hbar} \mu_{\mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]}$$

(by $\mu(\dots)$ we always mean the Lebesgue measure on the vector space). Here we assume for simplicity that $\Lambda : \mathcal{C}^3 \rightarrow \mathcal{C}^0$ is an isomorphism, and we denote $\Lambda^{-1} : \mathcal{C}^0 \rightarrow \mathcal{C}^3$ its inverse. Now we can write the relaxed effective action as

$$(54) \quad e^{\text{Ind}_{\iota, \hat{K}}(S)(\alpha)/\hbar} = \int_{\mathcal{L}_K \oplus \mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]} e^{S(\iota(\alpha) + \omega'')/\hbar} e^{\frac{1}{2\hbar}\pi(\omega'', K\Lambda^{-1}K\omega'')} \mu_{\mathcal{L}_K \oplus \mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]}$$

where we integrate over the coisotropic subspace $\mathcal{L}_K \oplus \mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1] \subset \mathcal{F}''$ instead of just the Lagrangian subspace $\mathcal{L}_K \subset \mathcal{F}''$, with measure $e^{\frac{1}{2\hbar}\pi(\omega, K\Lambda^{-1}K\omega)} \mu_{\mathcal{L}_K \oplus \mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]}$ that is constant in direction of \mathcal{L}_K and is Gaussian in direction of $\mathfrak{g} \otimes \mathcal{C}_{d-ex}^1[1]$. So expression (54) gives an elegant interpretation of the relaxed effective action via a “thick” fiber BV integral (over a Gaussian-smeared Lagrangian subspace). In the case of Λ of general rank (not necessarily an isomorphism), we should replace \mathcal{C}_{d-ex}^1

by $\text{im}(d\Lambda) \subset C_{d-ee}^1$ in this discussion, which leads to a thick fiber BV integral over a smaller coisotropic $\mathcal{L}_K \oplus \mathfrak{g} \otimes \text{im}(d\Lambda)[1] \subset \mathcal{F}''$.

3.5.1. Invariants from the relaxed effective action. We are interested in describing the invariants of the relaxed effective action $\text{Ind}_{\iota, \hat{K}}(S)$ modulo deformations of the “relaxed induction data” (ι, \hat{K}) . Since we have a general description (49) for a non-strict chain homotopy, a general (infinitesimal) deformation of the relaxed induction data (ι, \hat{K}) can be described as a deformation of the underlying strict induction data (ι, K) studied in the beginning of Section 3.3, plus a deformation of Λ . Now we can restate some weak version of Proposition 6 for the case of relaxed induction, where we make a special choice for the Lie algebra of coefficients: $\mathfrak{g} = \mathfrak{su}(2)$ (we generalize this in the Remark afterwards). We are able to recover only the two-loop part of the complete invariant $F(\hbar)$ of the strict effective action in this discussion.

Proposition 9. *If $B_1 = 0$ and $\mathfrak{g} = \mathfrak{su}(2)$, the relaxed effective action on cohomology has the form*

$$(55) \quad \text{Ind}_{\iota, \hat{K}}(S)(\alpha) = A(\hbar) \cdot W_{\text{prod}}^{003}(\alpha) + B(\hbar)$$

where $W_{\text{prod}}^{003} = \frac{1}{2} \sum_{a,b,c} \epsilon_{abc} \alpha_{(0)}^a \alpha_{(0)}^b \alpha_{(3)}^c$, and $A(\hbar) = 1 + \hbar A^{(1)} + \hbar^2 A^{(2)} + \dots$, $B(\hbar) = \hbar^2 B^{(2)} + \hbar^3 B^{(3)} + \dots$ are some \hbar -dependent constants. The number

$$(56) \quad F^{(2)} = 3A^{(1)} + B^{(2)}$$

is invariant under deformations of the relaxed induction data (ι, \hat{K}) .

Proof. Ansatz (55) follows from the following argument. Since the relaxed effective action $\hat{W} = \text{Ind}_{\iota, \hat{K}}(S) \in \text{Fun}(\mathfrak{g}[1] \oplus \mathfrak{g}[-2])[[\hbar]]$ is a function of ghost number zero and $\mathfrak{g} = \mathfrak{su}(2)$ is 3-dimensional (and hence an at most cubic dependence on $\alpha_{(0)}$ is possible), it has to be of the form

$$\hat{W}(\alpha) = \frac{1}{2} \sum_{a,b,c} A_{abc}(\hbar) \alpha_{(0)}^a \alpha_{(0)}^b \alpha_{(3)}^c + B(\hbar)$$

Since \hat{W} also inherits the invariance under adjoint action of $\mathcal{G} = SU(2)$: $\alpha \mapsto g\alpha g^{-1}$ for $g \in \mathcal{G}$ from the ad-invariance of the original abstract Chern–Simons action S , the tensor $A_{abc}(\hbar)$ has to be of the form $A_{abc}(\hbar) = A(\hbar)\epsilon_{abc}$. This proves (55). The fact that the series for $A(\hbar)$ starts as $A(\hbar) = 1 + O(\hbar)$ is due to the vanishing of all tree diagrams $\Gamma \neq \Gamma_{0,3}$ which follows from formality of \mathcal{C} (it is automatic for the $B_1 = 0$ case) and $\hat{K} \circ \iota = 0$. The series for $B(\hbar)$ starts with an $O(\hbar^2)$ -term just because $G_{l,0}$ is empty for $n = 0, 1$.

Next, we know from Proposition 8 that there is a homotopy $W_\Lambda(t, dt)(\alpha)$ connecting the strict effective action $W = \text{Ind}_{\iota, K}(S) = W_{\text{prod}}^{003}(\alpha) + F(\hbar)$ (ansatz (40)) and the relaxed one $\hat{W} = \text{Ind}_{\iota, \hat{K}}(S)$. The general ansatz for $W_\Lambda(t, dt)$, taking into account that $\mathfrak{g} = \mathfrak{su}(2)$ and that construction (53) is compatible with ad-invariance $\alpha \mapsto g\alpha g^{-1}$, is the following:

$$(57) \quad W_\Lambda(t, dt)(\alpha) = A(\hbar; t)W_{\text{prod}}^{003}(\alpha) + B(\hbar; t) + dt \cdot \left(C(\hbar; t) \frac{1}{12} \sum_{a,b,c,d,e} \epsilon_{abc} \delta_{de} \alpha_{(0)}^a \alpha_{(0)}^b \alpha_{(0)}^c \alpha_{(3)}^d \alpha_{(3)}^e + D(\hbar; t) \sum_{a,b} \delta_{ab} \alpha_{(0)}^a \alpha_{(3)}^b \right)$$

where A, B, C, D are some functions of \hbar and the homotopy parameter $t \in [0, 1]$. The boundary conditions are:

$$A(\hbar; 0) = 1, \quad A(\hbar; 1) = A(\hbar), \quad B(\hbar; 0) = F(\hbar), \quad B(\hbar; 1) = B(\hbar)$$

The extended QME (52) for homotopy $W_\Lambda(t, dt)$ is equivalent to the system

$$(58) \quad \frac{\partial}{\partial t} A(\hbar; t) = \hbar C(\hbar; t) - A(\hbar; t) \cdot D(\hbar; t)$$

$$(59) \quad \frac{\partial}{\partial t} B(\hbar; t) = 3\hbar D(\hbar; t)$$

Next, the argument of vanishing of non-trivial trees (due to formality of \mathcal{C} , $K \circ \iota = 0$ and $d \circ \iota = 0$) applies again to construction (53). Hence, we have

$$A(\hbar; t) = 1 + \hbar A^{(1)}(t) + O(\hbar^2), \quad C(\hbar; t) = \hbar C^{(1)}(t) + O(\hbar^2), \quad D(\hbar; t) = \hbar D^{(1)}(t) + O(\hbar^2)$$

And due to $G_{0,0} = G_{1,0} = \emptyset$, we again have $B(\hbar; t) = \hbar^2 B^{(2)}(t) + O(\hbar^3)$. Equation (58) in order $O(\hbar)$ together with (59) in order $O(\hbar^2)$ yield

$$\frac{\partial}{\partial t} A^{(1)}(t) = -D^{(1)}(t), \quad \frac{\partial}{\partial t} B^{(2)}(t) = 3D^{(1)}(t)$$

Hence the expression $3A^{(1)}(t) + B^{(2)}(t)$ does not depend on t . For $t = 0$ it is the two-loop part of the invariant $F(\hbar)$ from (40), while for $t = 1$ it is the right hand side of (56). This concludes the proof. \square

Remark 8 (Generalization). The generalization of Proposition 9 to an arbitrary (quadratic) \mathfrak{g} is straightforward. We no longer have ansatz (55) for \hat{W} , since there might be more ad-invariant functions on $\mathfrak{g}[1] \oplus \mathfrak{g}[-2]$, but we still can write

$$\hat{W}(\alpha) = (1 + \hbar A^{(1)}) W_{\text{prod}}^{003}(\alpha) + \hbar^2 B^{(2)} + O(\hbar^3 + \hbar^2(\alpha_{(0)})^2 \alpha_{(3)} + \hbar(\alpha_{(0)})^4 (\alpha_{(3)})^2)$$

(we could prescribe weight 1 to \hbar and weight $1/3$ to $\alpha_{(0)}$ and $\alpha_{(3)}$, then we write explicitly terms with weight ≤ 2). The fact that $O(\hbar \alpha_{(0)} \alpha_{(0)} \alpha_{(3)})$ -contribution is proportional to W_{prod}^{003} (in principle there could be some other invariant tensor of rank 3) is explained by the fact that it is given by a single Feynman diagram $\Gamma \in G_{1,3}$ — the wheel with 3 leaves — and the Lie algebra part of this diagram is described by contraction $\sum_{d,e,f} f_{ade} f_{bef} f_{cfd} \propto f_{abc}$. Following the proof of Proposition 9, for the homotopy we have

$$\begin{aligned} W_\Lambda(t, dt)(\alpha) = & \\ & (1 + \hbar A^{(1)}(t)) W_{\text{prod}}^{003}(\alpha) + \hbar^2 B^{(2)}(t) + O(\hbar^3 + \hbar^2(\alpha_{(0)})^2 \alpha_{(3)} + \hbar(\alpha_{(0)})^4 (\alpha_{(3)})^2) + \\ & + dt \cdot \left(\hbar D^{(1)}(t) \sum_a \alpha_{(0)}^a \alpha_{(3)}^a + O(\hbar^2 \alpha_{(0)} \alpha_{(3)} + \hbar(\alpha_{(0)})^3 (\alpha_{(3)})^2) \right) \end{aligned}$$

The extended QME at order $O(dt \hbar (\alpha_{(0)})^2 \alpha_{(3)} + dt \hbar^2)$ yields

$$\frac{\partial}{\partial t} A^{(1)}(t) = -D^{(1)}(t), \quad \frac{\partial}{\partial t} B^{(2)}(t) = \dim \mathfrak{g} \cdot D^{(1)}(t)$$

Hence the two-loop part of the invariant $F(\hbar)$ is expressed in terms of coefficients of the relaxed effective action \hat{W} as

$$(60) \quad F^{(2)} = \dim \mathfrak{g} \cdot A^{(1)} + B^{(2)}$$

The other case discussed in Section 3.4, the case of formal \mathcal{C} with general B_1 , is translated straightforwardly into the setting of relaxed effective actions.

Proposition 10. *Suppose \mathcal{C} is formal and $\iota : H^\bullet(\mathcal{C}) \hookrightarrow \mathcal{C}$ is an algebra homomorphism. Then the relaxed effective action has the form*

$$(61) \quad \text{Ind}_{\iota, \hat{K}}(S)(\alpha) = W_{\text{prod}}(\alpha) + \hbar \hat{F}^{(1)}(\alpha) + O(\hbar^2)$$

where the one-loop part can be expressed as a super-trace:

$$(62) \quad \hat{F}^{(1)}(\alpha) = \frac{1}{2} \text{Str}_{\mathfrak{g} \otimes \mathcal{C}} \log \left(1 + \hat{K} \circ l(\iota(\alpha), \bullet) \right)$$

The restriction of the one-loop relaxed effective action to the Maurer-Cartan set $\hat{F}^{(1)}|_{\text{MC}}$ is invariant under deformations of (ι, \hat{K}) , preserving the homomorphism property of ι .

Proof. Analogously to the case of a strict chain homotopy K (Proposition 7), formality of \mathcal{C} together with $\hat{K} \circ \iota = 0$ imply vanishing of non-trivial trees (hence the ansatz (61)) and that the only possibly non-vanishing one-loop diagrams are wheels (hence the super-trace formula (62)).

Second, we know that there is a homotopy $W_\Lambda(t, dt)(\alpha)$ connecting the strict effective action $W(\alpha) = W_{\text{prod}}(\alpha) + \hbar F^{(1)}(\alpha_{(1)}) + \dots$ to the relaxed one $\hat{W}(\alpha) = W_{\text{prod}}(\alpha) + \hbar \hat{F}^{(1)}(\alpha) + \dots$:

$$W_\Lambda(t, dt)(\alpha) = \left(W_{\text{prod}}(\alpha) + \hbar F^{(1)}(t)(\alpha) + O(\hbar^2) \right) + dt \cdot \left(\hbar R'_\Lambda(t)(\alpha) + O(\hbar^2) \right)$$

(here we again exploit the vanishing of trees implied by construction (53), formality of \mathcal{C} and $K \circ \iota = d \circ \iota = 0$). The extended QME for the homotopy at order $O(dt \cdot \hbar)$ is

$$\frac{\partial}{\partial t} F^{(1)}(t)(\alpha) = \{W_{\text{prod}}(\alpha), R'_\Lambda(t)(\alpha)\}'$$

Hence the restriction $F^{(1)}(t)|_{\overline{\text{MC}}}$ does not depend on t , where

$$\overline{\text{MC}} = \{\alpha \mid l(\iota(\alpha), \iota(\alpha)) = 0\} \subset \mathcal{F}'$$

is the set of critical points of W_{prod} (the “non-homogeneous Maurer-Cartan set”). As $\text{MC} \subset \overline{\text{MC}}$, the restriction $F^{(1)}(t)|_{\text{MC}}$ also does not depend on t , hence

$$\hat{F}^{(1)}|_{\text{MC}} = F^{(1)}|_{\text{MC}}$$

As the right hand side is invariant (Proposition 7), so is the left hand side. This concludes the proof. \square

Obviously, going from the restriction to $\overline{\text{MC}}$ to the restriction to MC , we do not lose any invariant information, since $F^{(1)}$ depends only on $\alpha_{(1)}$ and not on other components of α , which implies $\hat{F}^{(1)}|_{\overline{\text{MC}}} = \hat{F}^{(1)}|_{\text{MC}}$. Note also that in Proposition 9 we managed to reconstruct only the two-loop part of the invariant $F(\hbar)$ (Proposition 6) in the relaxed setting. On the other hand, we can completely reconstruct the invariant $F^{(1)}|_{\text{MC}}$ of Proposition 7 in the relaxed setting.

3.6. Examples of dg Frobenius algebras. In this Section we will provide some examples of non-negatively graded dg Frobenius algebras with pairing of degree -3 and zeroth Betti number $B_0 = 1$, i.e. algebras suitable for constructing abstract Chern–Simons actions.

Example 1: minimal dg Frobenius algebra. Let V be a vector space and $\mu \in \wedge^3 V^*$ an arbitrary exterior 3-form on V . Then we construct the dg Frobenius algebra \mathcal{C} from this data as

$$\mathcal{C} := \mathbb{R} \cdot 1 \oplus V[-1] \oplus V^*[-2] \oplus \mathbb{R} \cdot v$$

where v is a degree 3 element. The pairing π is defined to be the canonical pairing between $V[-1]$ and $V^*[-2]$, and also we set $\pi(1, v) := 1$. We define the product m as

$$m(1, x) = x, \quad m(x_{(1)}, y_{(1)}) = \langle \mu, \bar{x}_{(1)} \wedge \bar{y}_{(1)} \rangle, \quad m(x_{(1)}, z_{(2)}) = \langle \bar{x}_{(1)}, \bar{z}_{(2)} \rangle$$

where $x \in \mathcal{C}$ (element of arbitrary degree), $x_{(1)}, y_{(1)} \in V[-1]$, $z_{(2)} \in V^*[-2]$ and bar means shifting an element to degree zero, e.g. $\bar{x}_{(1)} = sx_{(1)} \in V$, $\bar{z}_{(2)} = s^2 z_{(2)} \in V^*$; $\langle \bullet, \bullet \rangle$ denotes the canonical pairing. The differential d is set to zero. This construction gives the most general minimal (i.e. with zero differential) dg Frobenius algebra (non-negatively graded, with pairing of degree -3 and $B_0 = 1$). Abstract Chern–Simons action associated to a minimal algebra is a purely cubic polynomial in the fields, with coefficients being the components of μ . Inducing the effective action on cohomology is the identity operation, since here $\mathcal{C} = H^\bullet(\mathcal{C})$.

Example 2: differential concentrated in degree $\mathcal{C}^1 \rightarrow \mathcal{C}^2$. Let V be a vector space, $\mu \in \wedge^3 V^*$ be some 3-form on V and $\delta : S^2 V \rightarrow \mathbb{R}$ some symmetric pairing on V (not necessarily non-degenerate). We define \mathcal{C} , the pairing π and product m as in Example 1, but now we construct the differential $d : V[-1] \rightarrow V^*[-2]$ from δ as $x_{(1)} \mapsto s^{-2} \delta(\bar{x}_{(1)}, \bullet)$. The two other components of the differential $\mathbb{R} \cdot 1 \rightarrow V[-1]$ and $V^*[-2] \rightarrow \mathbb{R} \cdot v$ are set to zero as before. This construction gives the general dg Frobenius algebra with differential concentrated in degree $\mathcal{C}^1[-1] \rightarrow \mathcal{C}^2[-2]$ only.

Here the Hodge decomposition $\mathcal{C} = \iota(H^\bullet(\mathcal{C})) \oplus \mathcal{C}_{d-ex} \oplus \mathcal{C}_{K-ex}$ is defined by a choice of projection $p : V \rightarrow \ker \delta$ or equivalently by choosing a complement V'' of $\ker \delta \subset V$ in V (here we understand δ as the self-dual map $V \rightarrow V^*$). We set

$$H^\bullet(\mathcal{C}) = \mathbb{R} \cdot 1 \oplus (\ker \delta)[-1] \oplus (\ker \delta)^*[-2] \oplus \mathbb{R} \cdot v, \\ \mathcal{C}_{d-ex} = (\text{im } \delta)[-2], \quad \mathcal{C}_{K-ex} = V''[-1]$$

The embedding $\iota : H^\bullet(\mathcal{C}) \hookrightarrow \mathcal{C}$ is canonical in degrees 0,1,3 and given by $p^* : (\ker \delta)^* \hookrightarrow V^*$ in degree 2. The (non-vanishing part of the) chain homotopy K is the inverse map for the isomorphism $d : V''[-1] \rightarrow (\text{im } \delta)[-2]$. So the induction data (ι, K) is completely determined by the choice of p . This means in particular that only the deformations of induction data of type II are possible here. Also there are no relaxed chain homotopies \hat{K} (other than the strict one described above), which is obvious from (49). Despite these simplifications, the effective action on cohomology for such \mathcal{C} is in general non-trivial. A particular example here is the case $\mathcal{C} = S^\bullet(\mathfrak{su}(2)^*[-1])$ discussed in section 3.4.

Example 3: “doubled” commutative dga. Let $\mathcal{V} = V^0 \oplus V^1[-1]$ be a unital commutative associative dg algebra, concentrated in degrees 0 and 1, with differential $d_{\mathcal{V}}$ and multiplication $m_{\mathcal{V}}$, and satisfying $\dim H^0(\mathcal{V}) = 1$. Then we set

$$\mathcal{C} := \mathcal{V} \oplus \mathcal{V}^*[-3] = V^0 \oplus V^1[-1] \oplus (V^1)^*[-2] \oplus (V^0)^*[-3]$$

with pairing π generated by the canonical pairing between \mathcal{V} and \mathcal{V}^* . The component of differential $V^0 \rightarrow V^1[-1]$ is given by $d_{\mathcal{V}}$, the component $(V^1)^*[-2] \rightarrow (V^0)^*[-3]$ — by the dual map $(d_{\mathcal{V}})^*$, and the component $V^1[-1] \rightarrow (V^1)^*[-2]$ is

set to zero. Multiplication for elements of \mathcal{C} of degrees 0 and 1 is given by $m_{\mathcal{V}}$ and is extended to other degrees by the cyclicity property (15):

$$m(x_{(0,1)}, y_{(0,1)}) := m_{\mathcal{V}}(x_{(0,1)}, y_{(0,1)}), \quad m(x_{(0,1)}, z_{(2,3)}) := \langle m_{\mathcal{V}}(\bullet, x_{(0,1)}), z_{(2,3)} \rangle$$

for $x_{(0,1)}, y_{(0,1)} \in \mathcal{V}$ and $z_{(2,3)} \in \mathcal{V}^*[-3]$. In particular, the product of elements of degree 1 in \mathcal{C} is zero.

Hodge decomposition for \mathcal{C} is fixed by choosing an embedding $\iota_{\mathcal{V}} : H^{\bullet}(\mathcal{V}) \hookrightarrow \mathcal{V}$ (it is canonical in degree zero since $H^0(\mathcal{V}) = \mathbb{R} \cdot 1$, but non-canonical in degree one) and a retraction $r_{\mathcal{V}} : \mathcal{V} \rightarrow H^{\bullet}(\mathcal{V})$. Equivalently, we choose a splitting of \mathcal{V} into representatives of cohomology and the complement $\mathcal{V} = \iota_{\mathcal{V}}(H^{\bullet}(\mathcal{V})) \oplus \mathcal{V}''$, i.e. $V^0 = \mathbb{R} \cdot 1 \oplus V^{0''}$, $V^1 = \iota_{\mathcal{V}}(H^1(\mathcal{V})) \oplus V^{1''}$. Then the Hodge decomposition for \mathcal{C} is $\mathcal{C} = \iota(H^{\bullet}(\mathcal{C})) \oplus \mathcal{C}_{d-ex} \oplus \mathcal{C}_{K-ex}$ with

$$\begin{aligned} H^{\bullet}(\mathcal{C}) &= \mathbb{R} \oplus H^1(\mathcal{V})[-1] \oplus (H^1(\mathcal{V}))^*[-2] \oplus \mathbb{R} \cdot v, \\ \mathcal{C}_{d-ex} &= V^{1''}[-1] \oplus (V^{0''})^*[-3], \quad \mathcal{C}_{K-ex} = V^{0''} \oplus (V^{1''})^*[-2] \end{aligned}$$

Here $v \in (V^0)^*[-3]$ is the element defined by the component of retraction $r_{\mathcal{V}} : V^0 \rightarrow \mathbb{R} \cdot 1$. The embedding $\iota : H^{\bullet}(\mathcal{C}) \rightarrow \mathcal{C}$ is given by $\iota_{\mathcal{V}}$ in degrees 0,1 and by $(r_{\mathcal{V}})^*$ in degrees 2,3. As in the Example 2, only type II deformations of the induction data (ι, K) are possible here. However, one can introduce the relaxed chain homotopy (49) here with $\Lambda : (V^{0''})^*[-3] \rightarrow V^{0''}$.

The BV integral (28) is Gaussian in this example since the cyclic product $\pi(\bullet, m(\bullet, \bullet))$ vanishes on $(\mathcal{C}_{K-ex})^{\otimes 3}$ (for trivial degree reasons). Hence, the only Feynman graphs contributing to $W(\alpha)$ are wheels and the trivial tree $\Gamma_{0,3}$ (“branches” could also contribute, but they vanish since the component of the multiplication $\iota(H^1(\mathcal{C})) \otimes \iota(H^1(\mathcal{C})) \rightarrow \mathcal{C}^2$ vanishes and the non-trivial part of $W(\alpha)$ depends only on $\alpha_{(1)}$ due to Proposition 4). The effective action can be written as

$$W(\alpha) = W_{\text{prod}}^{003}(\alpha) + W_{\text{prod}}^{012}(\alpha) + 2 \cdot \frac{1}{2} \hbar \text{tr}_{\mathfrak{g} \otimes V^0} \log(1 + K \circ l(\iota(\alpha_{(1)}), \bullet))$$

(the factor 2 accounts for the contribution of the trace over $\mathfrak{g} \otimes (V^{1''})^*$ — the other half of the Lagrangian subspace \mathcal{L}_K).

4. THREE-MANIFOLD INVARIANTS

We now wish to extend the results of the previous Sections to the Frobenius algebra $\Omega^{\bullet}(M)$ of differential forms on a smooth compact 3-manifold M (with de Rham differential, wedge product and integration pairing). This will provide us with the effective action (around the trivial connection) of Chern–Simons theory [18]. The invariant of this Frobenius algebra will also constitute an invariant of 3-manifolds modulo diffeomorphisms.

The discussion of the previous Sections, however, does not go through automatically since $\Omega^{\bullet}(M)$ is infinite dimensional. As in previous works [2, 11, 3] the way out is to restrict oneself to a special class of chain homotopies K for which the finite dimensional arguments are simply replaced by the application of Stokes’ theorem. However, at some point of the construction we have to choose a framing and hence we get invariants of framed 3-manifolds.

4.1. Induction data and the propagator. As in the finite-dimensional case we fix an embedding $\iota: H^\bullet(M) \hookrightarrow \Omega^\bullet(M)$. By $\chi \in \Omega^3(M \times M)$ we will denote the representative of the Poincaré dual of the diagonal Δ determined by this embedding. Namely, if $\{1, \{\alpha_i, \beta^i\}_{i=1, \dots, B_1}, v\}$ is a basis of $\iota(H^\bullet(M))$ (with 1 the constant function, $\alpha_i \in \Omega^1$, $\beta^i \in \Omega^2$, $v \in \Omega^3$ and $\int_M \alpha_i \beta^j = \delta_i^j$, $\int_M v = 1$), then

$$\chi = v_2 - \alpha_{i,1} \beta_2^i + \beta_1^i \alpha_{i,2} - v_1,$$

where we have used Einstein's convention over repeated indices and for any form $\gamma \in \Omega^\bullet(M)$ we write γ_i for $\pi_i^* \gamma$ with π_1 and π_2 the two projections $M \times M \rightarrow M$.

The chain homotopy K is assumed to be determined by a smooth integral kernel $\hat{\eta}$. Namely, let $C_2^0(M) := M \times M \setminus \Delta = \{(x_1, x_2) \in M \times M : x_1 \neq x_2\}$ be the open configuration space of two points in M and let $C_2(M)$ be its Fulton–MacPherson–Axelrod–Singer (FMAS) compactification [8, 2] obtained by replacing the diagonal Δ with its unit normal bundle. Let $\pi_{1,2}$ be the extensions to the compactification of the projection maps $\pi_i(x_1, x_2) = x_i$. Then $\hat{\eta}$ is a smooth 2-form on $C_2(M)$ and the chain homotopy K is defined by

$$K\alpha = -\pi_{2*}(\hat{\eta} \pi_1^* \alpha), \quad \alpha \in \Omega^\bullet(M),$$

where a lower $*$ denotes integration along the fiber.

For K to be a symmetric chain homotopy satisfying (20), $\hat{\eta}$ must satisfy the following properties (see [3, 5] for more details):

- P1:** $d\hat{\eta} = \pi^* \chi$, with π the extension to $C_2(M)$ of the inclusion of $C_2^0(M)$ into $M \times M$.
- P2:** $\int_{\partial C_2(M)} \hat{\eta} = -1$.
- P3:** $T^* \hat{\eta} = -\hat{\eta}$, where T is the extension to $C_2(M)$ of the involution $(x_1, x_2) \mapsto (x_2, x_1)$ of $C_2^0(M)$.

Observe that $\partial C_2(M)$ is canonically diffeomorphic to the sphere tangent bundle STM of M (STM is the quotient of TM by the action $(x, v) \mapsto (x, \lambda v)$, $\lambda \in \mathbb{R}_*^+$). Condition P2 may then be refined to

- P2':** $\iota_\partial^* \hat{\eta} = -\eta$, where ι_∂ is the inclusion map $STM = \partial C_2(M) \hookrightarrow C_2(M)$ and η is a given odd global angular form on STM .

Recall that a global angular form on a sphere bundle $S \rightarrow M$ is a differential form on the total space S whose restriction to each fiber generates its cohomology and whose differential is minus the pullback of a representative of the Euler class (in our case zero). These two properties are consistent with the restriction to the boundary of P1 and P2. Odd here means $T^* \eta = -\eta$ where T is the antipodal map on each fiber, and this is compatible with the restrictions to the boundary of P3. Global angular forms always exist.

Remark 9 (Kontsevich [11]). Since STM is trivial for every 3-manifold, there is a simple choice for η only depending on a choice of framing or, equivalently, a choice of trivialization $f: STM \xrightarrow{\sim} M \times S^2$. One simply sets $\eta = f^* \omega$ where ω is the normalized $SO(3)$ -invariant volume form on S^2 (tensor 1 $\in \Omega^0(M)$).

Remark 10 ([3]). It is also possible to construct an odd global angular form depending on the choice of a connection but not on a framing. Namely, realize STM as the S^2 -bundle associated to the frame bundle $F(M)$:

$$STM = F(M) \times_{SO(3)} S^2$$

(we reduce the structure group of the frame bundle to $SO(3)$ by picking a Riemannian metric). By choosing a metric connection θ (e.g., the Levi-Civita connection), one defines $\bar{\eta} := \omega + d(\theta_i x^i)/(2\pi)$; here the x^i s, $i = 1, 2, 3$, are the homogeneous coordinates on S^2 , while the θ_i s are the coefficients of the connection in the basis $\{\xi_i\}$ of $\mathfrak{so}(3)$ given by $(\xi_i)_{jk} = \epsilon_{ijk}$. It is then easy to show that $\bar{\eta}$ is a global angular form on $F(M) \times S^2$ and that it is basic. Hence it defines a global angular form η on STM .

Lemma 2 ([3]). *For any given odd global angular form η on STM , there exists a propagator $\hat{\eta} \in \Omega^2(C_2(M))$ satisfying P1, P2', P3.*

Proof. The complete proof in the case of a rational homology sphere is contained in [3]. (The general case, whose proof is a straightforward generalization of the previous case, is spelled out in [5].) For completeness, we briefly recall the idea of the proof. One chooses a tubular neighborhood U of $\partial C_2(M)$ and a subneighborhood V . One picks a compactly supported function ρ which is constant and equal to -1 on V and is even under the action of T . Let p be the projection $U \rightarrow \partial C_2(M)$. The differential of the form $\rho p^* \eta$ is a representative of the Thom class of the normal bundle of Δ . The form $\rho p^* \eta$ may be extended by zero to the whole of $C_2(M)$ and its differential is then a representative of the Poincaré dual of Δ . Thus, its difference from the given representative χ will be exact and actually **regular** on the diagonal (see Appendix B for the definition). Namely, $d\rho p^* \eta = \pi^*(\chi + d\alpha)$, $\alpha \in \Omega^3(M \times M)$. Since both $\rho p^* \eta$ and χ are odd under the action of T , one may also choose α to be odd. Finally, one sets $\hat{\eta} := \rho p^* \eta - \pi^* \alpha$. \square

4.2. The improved propagator. In the previous Sections it was also important to assume the conditions $K \circ \iota = 0$ and $K^2 = 0$. In terms of the propagator $\hat{\eta}$ they correspond to additional conditions that are easily expressed by using the

Definition 1 (Compactification of configuration spaces). The FMAS compactification $C_n(M)$ of the configuration space $C_n^0(M) := \{(x_1, \dots, x_n) \in M^n : x_i \neq x_j \forall i \neq j\}$ is obtained by taking the closure

$$C_n(M) := \overline{C_n^0(M)} \subset M^n \times \prod_{S \subset \{1, \dots, n\}: |S| \geq 2} \text{Bl}(M^S, \Delta_S),$$

where $\text{Bl}(M^S, \Delta_S)$ denotes the differential-geometric blowup obtained by replacing the principal diagonal Δ_S in M^S with its unit normal bundle $N(\Delta_S)/\mathbb{R}_*^+$. See [2, 4].

We recall that the $C_n(M)$ are smooth manifolds with corners.

Notation 1. Given a differential form γ on M , we write $\gamma_i := \pi_i^* \gamma$ where π_i is the extension to $C_n(M)$ of the projection $C_n^0(M) \rightarrow M$, $(x_1, \dots, x_n) \mapsto x_i$. We also set $\hat{\eta}_{ij} := \pi_{ij}^* \hat{\eta}$, where π_{ij} is the extension to the compactifications of the projection $C_n^0(M) \rightarrow C_2^0(M)$, $(x_1, \dots, x_n) \mapsto (x_i, x_j)$. Given a product of such forms on a compactified configuration space and a set of indices i, j, \dots , we denote by $\int_{i, j, \dots}$ the fiber integral over the points x_i, x_j, \dots . This notation also takes care of orientation. Namely, if i, j, \dots are not ordered, the integral carries the sign of the permutation to order them.

We are finally ready to list the two simple properties corresponding to $K \circ \iota = 0$ and $K^2 = 0$:

P4: $\int_2 \hat{\eta}_{12} \gamma_2 = 0$ for all $\gamma \in \iota(H^\bullet(M))$.

P5: $\int_2 \hat{\eta}_{12} \hat{\eta}_{23} = 0$.

Lemma 3. *For any given odd global angular form η on STM, there exists a propagator $\hat{\eta} \in \Omega^2(C_2(M))$ satisfying P1, P2', P3, P4.*

Proof. The idea is to apply transformation (24) to a propagator as the one constructed in Lemma 2. This must however be reformulated in terms of integral kernels. Namely, let $\hat{\eta}$ satisfy P1, P2', P3. Set

$$\begin{aligned} \lambda_{12} := & \int_3 \eta_{13} v_3 - \int_3 \eta_{23} v_3 + \int_3 \eta_{13} \alpha_{3,i} \beta_2^i - \beta_1^i \int_3 \eta_{23} \alpha_{3,i} + \int_3 \eta_{13} \beta_3^i \alpha_{2,i} + \\ & + \alpha_{1,i} \int_3 \eta_{23} \beta_3^i - \alpha_{1,i} \int_{3,4} \beta_3^i \eta_{34} \beta_4^j \alpha_{2,j} - \beta_1^i \int_{3,4} \alpha_{3,i} \eta_{34} v_4 - \int_{3,4} v_3 \eta_{34} \alpha_{4,i} \beta_2^i. \end{aligned}$$

By construction the new propagator $\hat{\eta} - \lambda$ satisfies P3 and P4. It is not difficult to check that λ is closed, so P1 is still satisfied. Finally observe that the integrals on one argument simply produce a form on M , while the integrals on two arguments simply produce a number. As a consequence, λ is (the pullback by π of) a form on $M \times M$. As it is also T -odd, its restriction to the boundary vanishes. Thus, P2' still holds. \square

4.3. On property P5.

Lemma 4. *For any given odd global angular form η on STM, there exists a propagator $\hat{\eta} \in \Omega^2(C_2(M))$ satisfying P1, P2, P3, P4, P5.*

Proof. We apply transformation (25) in the equivalent form

$$K_3 = K_2 + [d, dK_2^3]$$

to the propagator $\hat{\eta}$ constructed in Lemma 3. In terms of integral kernels, the new propagator is $\hat{\eta} + \gamma$ with $\gamma_{12} := dd_1 f_{12}$ and

$$f_{12} := \int_{3,4} \hat{\eta}_{13} \hat{\eta}_{34} \hat{\eta}_{42}.$$

Properties P1 and P2 are obviously satisfied as we have changed the propagator by an exact term. As for property P3, observe that equivalently γ may be written as $d_2 d_1 f_{12}$ and that f is even under the action of T . Finally property P4 is easily checked by integration. \square

It would be very useful to prove the following

Conjecture 1. *The propagator constructed in Lemma 4 also satisfies property P2'.*

Observe that since γ is T -odd, it would suffice to show that it is regular on the diagonal (i.e., a pullback from $M \times M$). For this it would suffice to show that f or at least df has this property. By its definition f looks rather regular, but at the moment we have no complete proof of this fact.

Remark 11 (The Riemannian propagator). The physicists' treatment of Chern–Simons theory would simply be to choose a Riemannian metric g and use it to impose the Lorentz gauge-fixing. Out of this one gets the propagator $d^* \circ G$, where G is the Green function for $\Delta + \mathcal{P}'$, where Δ is the Laplace operator and \mathcal{P}' is the projection to harmonic forms in the Hodge decomposition. The integral kernel of this propagator is a smooth two-form $\hat{\eta}$ on $C_2(M)$ satisfying properties P1, P2', P3,

P4, P5 with odd angular form on the boundary of the type described in Remark 10. In this case, the metric connection is actually the Levi-Civita connection for the chosen Riemannian metric. See [2] and [3, Remark 3.6].

4.4. The construction of the invariant by a framed propagator. We now fix the boundary value of the propagator as in Remark 9 for a given choice f of framing. We also choose an embedding ι of cohomology and denote by $P_{\iota,f}$ the space of propagators satisfying properties P1,P2',P3. Observe that, by Lemma 3, $P_{\iota,f}$ is not empty. For $\hat{\eta} \in P_{\iota,f}$ we define $\text{Ind}_{\iota,\hat{\eta}} \in \text{Fun}(\mathcal{F}')[[\hbar]]$, $\mathcal{F}' = H^\bullet(M, \mathfrak{g})[1] = H^\bullet(M)[1] \otimes \mathfrak{g}$, analogously to $\text{Ind}_{\iota,K}$ as at the beginning of subsection 3.5 by the following obvious changes of notations:

- (1) Every chain homotopy K is replaced by a propagator $\hat{\eta}$;
- (2) every vertex is replaced by a point in the compactified configuration space over which we eventually integrate.

Signs may be taken care of by choosing an ordering of vertices and of half edges at each vertex (see, e.g., [3]).

All computations in Section 3 go through as they are simply replaced by Stokes theorem:

$$d \int_{C_n(M)} = (-1)^n \int_{C_n(M)} d + \int_{\partial C_n(M)}.$$

Among the codimension-one boundary components of $C_n(M)$ we distinguish between principal and hidden faces: the former correspond to the collapse of exactly two points, the latter to the collapse of more than two points. Principal faces contribute by the same combinatorics as in Section 3, whereas hidden faces do not contribute by our choice of propagators because of Kontsevich's vanishing Lemmata [11]. As a result we conclude that $\text{Ind}_{\iota,\hat{\eta}}$ satisfies the quantum master equation for every choice of induction data as above. For more details, we refer to Appendix A.

By the same reasoning and by the arguments of subsection 3.5, we may prove that $\text{Ind}_{\iota,\hat{\eta}}$ is canonically equivalent to a strict effective action. This allows us to recover at least the two-loop part of the complete invariant of a rational homology sphere as discussed in subsection 3.5.1.

Remark 12. If Conjecture 1 were true, the space $P'_{\iota,f}$ of propagators satisfying properties P1,P2',P3,P4,P5 would not be empty. We could then repeat the above construction using $P'_{\iota,f}$ instead of $P_{\iota,f}$ and get a strict effective action directly. The discussions of subsections 3.2, 3.3 and 3.4 would then go through and, in particular, Propositions 4, 5, 6, 7 would hold.

Remark 13. In [3] an invariant for framed rational homology spheres was introduced. The boundary condition for the propagator was different (see next subsection), but this is immaterial for the present discussion. Namely, choose a propagator in $P_{\iota,f}$ and define $\tilde{\eta}_{123} := \hat{\eta}_{12} + \hat{\eta}_{23} + \hat{\eta}_{31}$. If M is a rational homology sphere, $\tilde{\eta}_{123}$ is closed. Now take the graphs appearing in the constant part of the strict effective action and reinterpret them as follows:

- (1) Each vertex is replaced by a point in the compactified configuration space;
- (2) an extra point x_0 is added on which one puts the representative $v \in \iota(H^3(M))$ with $\int v = 1$;
- (3) each chain homotopy is replaced by $\tilde{\eta}$ (more precisely, the chain homotopy between vertices i and j is replaced by $\tilde{\eta}_{ij0}$).

It is now possible to show that this produces an invariant of (M, f) . This is a different way of getting the invariant corresponding to the constant part of the strict effective action for a choice of propagator not necessarily satisfying property P5. We do not have a direct proof that this invariant is the same. The indirect proof consists of showing that both invariants are finite type with the same normalizations along the lines of [13].

If Conjecture 1 were true, then it would immediately follow that this invariant is exactly the constant part of the strict effective action. In fact, for a propagator as in the Conjecture, it is not difficult to show that only the term $\hat{\eta}_{ij}$ in each $\tilde{\eta}_{ij0}$ would contribute (and the integration over x_0 would then decouple). Since the induced action is constant on $P_{\iota, f}$, by restriction to $P'_{\iota, f} \subset P_{\iota, f}$ we would prove the claim.

4.5. The unframed propagator. Instead of using Kontsevich’s propagator, one may proceed as in [3] and define the propagator by choosing the global angular form on $\partial C_2(M)$ as in Remark 10. Recall that in this case no choice of framing is required. On the other hand, one needs to specify a Riemannian metric g and a metric connection θ . We denote by $P_{\iota, g, \theta}$ the space of propagators corresponding to these choices.

We proceed exactly as in the previous subsection to define the effective action (see Appendix A for more details). In particular we want to check independence on the induction data; so we choose a path in $P_{\iota, g, \theta}$ and consider the effective action W as a function on the shifted cohomology tensor the differential forms on $[0, 1]$ and check whether the extended QME $(d + \hbar\Delta)W + \{W, W\}/2 = 0$ holds. The only difference with respect to the previous subsection is that there is an extra set of boundary components of the configuration spaces that may appear: namely, the most degenerate faces corresponding to the collapse of all points. These faces may be treated exactly as in [2, 3] and one shows that their contribution is a multiple of the first Pontryagin form $-\text{tr } F_\theta^2/(8\pi^2)$, where F_θ is the curvature of θ . The important point is that the coefficient depends only on the graph involved but not on the 3-manifold M . As a result the effective action might not satisfy the extended quantum master equation. However, one may easily compensate for this by adding to it the integral over M of the Chern–Simons 3-form of the connection θ pulled back from $F(M)$ to M by choosing a section f (i.e., a framing). The framing now appears because of this correction but is not present in the propagator.

The main disadvantage of this approach is that one does not know how to compute the universal coefficients. (It is known that the coefficients vanish for graphs with an odd number of loops, while for the graph with two loops one may compute the coefficient explicitly and see that it is not zero.) The advantage is that $P_{\iota, g, \theta}$ contains a subspace of propagators satisfying also property P5: These are the integral kernels constructed in [2], see Remark 11. With these choices, and the addition of the frame-dependent constant as in the previous paragraph, one gets an induced effective action satisfying all properties stated in subsections 3.2, 3.3 and 3.4. More precisely, let

$$\text{CS}(M, \theta, f) := -\frac{1}{8\pi^2} \int_M f^* \text{Tr} \left(\theta d\theta + \frac{2}{3} \theta^3 \right)$$

be the Chern–Simons integral for a connection θ on the frame bundle $F(M)$ of M and a framing f (regarded here as a section of $F(M)$).

Theorem 1. *Let M be a compact 3-manifold and \mathfrak{g} a quadratic Lie algebra. Then*

- (1) For every choice of Riemannian metric g on M , the effective action W constructed using the Riemannian propagator of Remark 11 is a function on $H^\bullet(M, \mathfrak{g})[1]$, solves the quantum master equation and has the properties described in Proposition 4. In addition it has the form given in (40) in case $B_1(M) = 0$ and in (44) in case M is formal.
- (2) There is a universal element $\phi \in \hbar^2 \mathbb{R}[[\hbar^2]]$, depending only on the choice of Lie algebra \mathfrak{g} , such that the modified effective action

$$\widetilde{W}(M, g, f) := W(M, g) + \phi \text{CS}(M, \theta_g, f),$$

where θ_g is the Levi-Civita connection for g , solves the QME and is independent of g modulo canonical transformations as in Proposition 5. In particular we get invariants for the framed 3-manifold (M, f) as in Propositions 6 and 7.

Remark 14. The leading contribution to ϕ may be explicitly computed and yields

$$\phi = C_2(\mathfrak{g})\hbar^2/48 + O(\hbar^4)$$

with $C_2(\mathfrak{g}) = f_{abc}f_{abc}$, where the f_{abc} are the structure constants of \mathfrak{g} in an orthonormal basis. It is not known whether there are nonvanishing higher order corrections.

APPENDIX A. THE CHERN–SIMONS MANIFOLD INVARIANT

In this Appendix we give more details on the construction outlined in subsection 4.5. To a graph Γ with $|\Gamma|$ vertices we associate an element $\omega_\Gamma \in \Omega^\bullet(C_{|\Gamma|}(M)) \otimes \text{Fun}(\mathcal{F}')$ as follows:⁹

- to each edge we associate the pullback of a propagator by the corresponding projection from $C_{|\Gamma|}(M)$ to $C_2(M)$;
- to each leaf we associate the pullback (by the corresponding projection from $C_{|\Gamma|}(M)$ to M) of $\sum z_\mu^\alpha \gamma^\mu e_a$, where $\{\gamma^\mu\}$ is the chosen basis of $H^\bullet(M)$, $\{e_a\}$ is an orthonormal basis of \mathfrak{g} , and the $\{z_\mu^\alpha\}$ s are the corresponding coordinate functions on \mathcal{F}' ;
- on each vertex we put a structure constant in the orthonormal basis chosen above.

Then we take the wedge product of the differential forms and sum over Lie algebra indices for each edge. The result does not depend on the choice of orthonormal basis for \mathfrak{g} but depends on a choice of numbering of the vertices and of orientation of the edges. If we however make the same choice also to orient $C_{|\Gamma|}(M)$, then

$$\int_{C_{|\Gamma|}(M)} \omega_\Gamma \in \text{Fun}(\mathcal{F}')$$

is well defined. We define Z (the exponential of W) to be the sum of $\omega_\Gamma/|\text{aut } \Gamma|$ over all trivalent graphs, where $|\text{aut } \Gamma|$ is the order or the group of automorphisms of Γ . Proving the QME for W is equivalent to proving $\Delta Z = 0$.

The main observation is that Property P1 of the propagator and the same combinatorics as in the toy model imply that ΔZ is obtained by replacing the ω_Γ s by $d\omega_\Gamma$ s one by one. We then use Stokes theorem. The contributions of principal faces (i.e., boundary faces of configuration spaces corresponding to the collapse

⁹Here and in the rest of the Appendix, the symbol \otimes is understood as the completed tensor product: i.e., the space of functions on the Cartesian product of the corresponding supermanifolds.

of two vertices) sum up to zero thanks to the Lie algebra contributions (this is also the same combinatorics as in the toy model). Hidden faces (i.e., the other boundary faces) may in principle contribute. Let γ be the subgraph corresponding to a hidden face (i.e., the vertices of γ are those that collapse and its edges are the edges between such vertices). By simple dimensional reasons, the hidden faces corresponding to γ vanishes if γ has a univalent vertex; if γ has a bivalent vertex, its contribution also vanishes by Kontsevich’s lemma thanks to Property P3. Since we only consider trivalent graphs, we are left with contributions coming from the collapse of *all* vertices of a connected component of Γ s with *no leaves*. These are the graphs that contribute to the constant part of Z and thus of W . The latter contributions also vanish by a simple dimensional argument.

More generally, to keep track of the choices involved in the propagator, we consider a one parameter family of choices with parameter $t \in I := [0, 1]$ and show $\Delta' \tilde{Z} = 0$ with $\Delta' := \Delta + dt \frac{d}{dt}$ and $\tilde{Z} \in Fun(\mathcal{F}') \otimes \Omega^\bullet(I)$ constructed as follows. Let $\hat{\eta}$ be a one-parameter family of propagators regarded as an element of $\Omega^2(C_2(M) \times I)$ related, at every $t \in I$, by Property P1 to the one parameter family $\{\gamma^\mu\}$ of bases of $\Omega^\bullet(M)$. Let $\hat{\eta} \in \Omega^2(C_2(M) \times I)$ and $\dot{\gamma}^\mu \in \Omega^\bullet(M \times I)$ be their t -derivatives. We may assume $\int_M \gamma^\mu \dot{\gamma}^\nu = 0, \forall \mu, \nu$, for more general choices may be compensated by a linear transformation of $H^\bullet(M)$. Observe that by Property P2', the restriction of $\hat{\eta}$ to the boundary is fixed, so the restriction of $\dot{\eta}$ vanishes. Actually, by construction $\dot{\eta}$ vanishes in a whole neighborhood of the boundary, so it is a regular form. Let

$$\lambda_{13} := \frac{\int_2 \hat{\eta}_{12} \dot{\eta}_{23} - \dot{\eta}_{12} \hat{\eta}_{23}}{2},$$

which is regular by Lemma 5 in Appendix B. Also set $\xi_1^\mu := (\int \hat{\eta}_{12} \dot{\gamma}_2^\mu)$. We define

$$\begin{aligned} \tilde{\eta} &:= \hat{\eta} - \lambda dt, \\ \tilde{\gamma}^\mu &:= \gamma^\mu + \xi^\mu dt, \\ \tilde{\chi} &:= g_{\mu\nu} \tilde{\gamma}^\mu \tilde{\gamma}^\nu = \chi + O(dt), \end{aligned}$$

where $g_{\mu\nu}$ is the metric on $H^\bullet(M)$ in the given basis. A simple computation then shows $D \tilde{\eta} = \tilde{\chi}$ and $D \tilde{\gamma}^\mu = 0$ with $D := d + dt \frac{d}{dt}$. Finally, we define \tilde{Z} as above by using $\tilde{\eta}$ and $\tilde{\gamma}$ instead of $\hat{\eta}$ and γ , respectively. We now observe that applying Δ' to \tilde{Z} is the same as applying D to the propagators. Reasoning as above by Stokes theorem, we see that the only possible nonvanishing contributions come from hidden faces corresponding to the collapse of all vertices of a connected component with *no leaves*.¹⁰ These contributions also vanish if one uses a framed propagator, whereas the choice of an unframed propagator yields some constant times the integral of the Pontryagin form on M as a one-form on I , see [3].

By writing $\tilde{Z} = Z + \zeta dt$, we may see that the equivalences are produced by ζ . These are very particular kinds of BV equivalences as ζ consists of graphs decorated by propagators and generators of cohomology classes with the exception of one edge that is decorated by λ or one leaf that is decorated by ξ .

If Property P5 holds, the computation of \tilde{Z} simplifies drastically. By construction we obtain $\int_2 \hat{\eta}_{12} \lambda_{23} = \int_2 \lambda_{12} \hat{\eta}_{23} = 0$ and $\int_2 \eta_{12} \xi_2^\mu = 0$. As a result, whenever a vertex is decorated by $1 \in \Omega^0(M)$ the corresponding integral vanishes. Thus, \tilde{Z} , as a function on $H^\bullet(M, \mathfrak{g})[1]$ is independent of the coordinates in degree 1 apart

¹⁰Observe that, since λ is regular, only η will appear in the boundary computations.

from the trivial classical term. Since Z is of degree zero, it will only depend on the coordinates of degree zero. Since ζ is of degree -1 , it will be linear in the coordinates of degree -1 with coefficients depending on the coordinates of degree zero.

APPENDIX B. REGULAR FORMS

Let $\varpi: C_n(M) \rightarrow M^n$ be the extension to the compactification of the inclusion $C_n(M)^0 \rightarrow M^n$. We call a form on $C_n(M)$ regular if it is a pullback by ϖ . Recall the maps π_i s and $\pi_{i,j}$ s defined in Notation 1 on page 26.

Lemma 5. *Let α and β be differential forms on $C_2(M)$ and let at least one of them be regular. Then their convolution $\alpha * \beta := \int_2 \alpha_{12} \beta_{23} := \pi_{13,*}(\pi_{12}^* \alpha \pi_{23}^* \beta)$ is regular.*

Proof. Suppose that, e.g., α is regular; i.e., $\alpha = \varpi^* \alpha'$, $\alpha' \in \Omega^\bullet(M \times M)$. Define

$$\gamma = (\text{pr}_1 \times \pi_2)_* ((\text{pr}_1 \times \pi_1)^* \alpha' \text{pr}_2^* \beta) \in \Omega^\bullet(M \times M),$$

where pr_1 and pr_2 are the projections from $M \times C_2(M)$ to the two factors. It then follows that $\alpha * \beta = \varpi^* \gamma$. \square

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Finite dimensional AKSZ-BV theories

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Abstract

We describe a canonical reduction of AKSZ-BV theories to the cohomology of the source manifold. We get a finite dimensional BV theory that describes the contribution of the zero modes to the full QFT. Integration can be defined and correlators can be computed. As an illustration of the general construction we consider two dimensional Poisson sigma model and three dimensional Courant sigma model. When the source manifold is compact, the reduced theory is a generalization of the AKSZ construction where we take as source the cohomology ring. We present the possible generalizations of the AKSZ theory.

1 Introduction

The Batalin-Vilkovisky (BV) formalism [2] is widely regarded as the most powerful and general approach to the quantization of gauge theories. The idea is to extend the space of fields of the original gauge theory by auxiliary fields (ghosts, antighosts, Lagrangian multipliers etc.) and their conjugate antifields in such way that the total field-antifield space is equipped with two canonical structures: an odd Poisson bracket (antibracket) and an odd second order differential operator Δ (BV Laplacian). The original gauge action should be extended to master action defined on the total field-antifield space in such way that the master action satisfies the master equation (the equation involving antibracket and BV Laplacian). The path integral is defined as an appropriate integral over half of the field-antifield space. The master action can be computed within the homological perturbation theory which can be very difficult to carry out in general. In [1] Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) proposed a way to construct solutions of the BV classical master equation directly, without any reference to a classical action with a set of gauge symmetries. Their approach (AKSZ method) uses mapping spaces of supermanifolds equipped with the additional structures. The AKSZ method was applied in [1] to the Chern-Simons theory, the Witten A- and B-models. Furthermore the AKSZ approach was applied to two-dimensional Poisson sigma model in [7] and to three-dimensional Courant sigma model in [17], [24]. Moreover the higher dimensional case of open p-branes were discussed in [22, 15, 16].

In the present work we propose the reduction within the AKSZ framework to a finite dimensional BV theory which governs the zero mode contribution and is responsible for the semiclassical approximation in the full theory. These finite dimensional BV theories offer interesting perspective on some of the standard geometry. The proposed reduction naturally suggests a generalization of AKSZ framework.

The paper is organized as follows. In Section 2 we sketch the essentials of the BV formalism. Section 3 reviews the AKSZ formalism and discusses its reduction in general terms. The rest of the paper is devoted to the consideration of the examples of this reduction and to the discussion of finite dimensional BV theories arising as a result of the reduction. Section 4 treats the Poisson sigma model defined over a closed Riemann surface of genus g . Section 5 considers the Courant sigma model, a three dimensional topological sigma model. Section 6 considers the more involved setup of the Poisson sigma model on the Riemann surface with a boundary. In Section 7 we discuss the generalization of the AKSZ construction. In Section 8 we give a summary and discuss the possible developments. In Appendix A we calculate the Berezinian measure for the reduced BV manifold of the Courant sigma model. In Appendix B we collect some facts on the relative

cohomology of the manifolds with boundaries.

2 Summary of BV formalism

In this Section we recall the basic notions of BV formalism and fix the notation. For further details the reader may consult the following reviews [9, 12].

The BV algebra can be defined in many different but equivalent ways. In particular, a Gerstenhaber algebra (odd Poisson algebra) $(\mathcal{A}, \{ , \})$ together with an odd \mathbb{R} -linear map

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A},$$

which squares to zero $\Delta^2 = 0$ and generates the bracket $\{ , \}$ as

$$\{f, g\} = (-1)^{|f|} \Delta(fg) + (-1)^{|f|+1} (\Delta f)g - f(\Delta g),$$

is called a BV-algebra. Δ is called odd Laplace operator (odd Laplacian). Quite often such odd Poisson bracket is called antibracket.

The canonical example of BV algebra is given by the space of functions on $W \oplus \Pi W^*$, where W is a superspace, W^* is its dual and Π stands for the reversed parity functor. $W \oplus \Pi W^*$ is equipped with an odd non-degenerate pairing. Let y^a be the coordinates on W (the fields) and y_a^+ be the corresponding coordinates on ΠW^* (the antifields). We denote the parity of y^a as $(-1)^{|y^a|}$ and that of y_a^+ as $(-1)^{|y_a^+|} = (-1)^{|y^a|+1}$. Then the odd Laplacian is defined as follows

$$\Delta = (-1)^{|y_a|} \frac{\partial}{\partial y_a^+} \frac{\partial}{\partial y^a}. \quad (2.1)$$

It generates the canonical antibracket on $C^\infty(W \oplus \Pi W^*)$

$$\{f, g\} = (-1)^{|y^a|} \frac{\overleftarrow{\partial} f}{\partial y_a^+} \frac{\overrightarrow{\partial} g}{\partial y^a} + (-1)^{|y^a|} \frac{\overleftarrow{\partial} f}{\partial y^a} \frac{\overrightarrow{\partial} g}{\partial y_a^+}, \quad (2.2)$$

where we use the notation $\overrightarrow{\partial}_v f = \partial_v f$ and $\overleftarrow{\partial}_v f = (-1)^{|v||f|} \partial_v f$. Indeed the bracket (2.2) is non degenerate and defines the canonical odd symplectic structure on $W \oplus \Pi W^*$.

A Lagrangian submanifold $\mathcal{L} \subset W \oplus \Pi W^*$ is an isotropic supermanifold of maximal dimension. The volume form $dy^1 \dots dy^n dy_1^+ \dots dy_n^+$ induces a well defined volume form on \mathcal{L} . Thus the integral

$$\int_{\mathcal{L}} f, \quad f \in C^\infty(W \oplus \Pi W^*) \quad (2.3)$$

is defined for any \mathcal{L} . If $\Delta f = 0$, then then the integral (2.3) depends only on the homology class of \mathcal{L} . Moreover the integral (2.3) is zero for any Lagrangian \mathcal{L} if $f = \Delta g$.

The canonical example $W \oplus \Pi W^*$ can be generalized to the cotangent bundle $T^*[-1]\mathcal{M}$ of any graded manifold \mathcal{M} [25]. By a graded manifold we mean a sheaf of freely generated \mathbb{Z} -graded commutative algebras over a smooth manifold [26]. As a cotangent bundle, $T^*[-1]\mathcal{M}$ is naturally equipped with an odd Poisson bracket that makes $C^\infty(T^*[-1]\mathcal{M})$ to a Gerstenhaber algebra. The idea is that locally one can map $T^*[-1]\mathcal{M}$ to $W \oplus \Pi W^*$, define the bracket on coordinates with (2.2) and then glue the patches in a consistent manner.

Now in order to define the odd Laplacian Δ we need an integration over $T^*[-1]\mathcal{M}$. Namely, the choice of a volume form v on \mathcal{M} produces the corresponding volume¹ form μ_v on $T^*[-1]\mathcal{M}$. The divergence operator is defined as a map from the vector fields on $T^*[-1]\mathcal{M}$ to $C^\infty(T^*[-1]\mathcal{M})$ through the following integral relation

$$\int_{T^*[-1]\mathcal{M}} X(f) \mu_v = - \int_{T^*[-1]\mathcal{M}} \operatorname{div}_{\mu_v} X f \mu_v, \quad \forall f \in C^\infty(T^*[-1]\mathcal{M}), \quad (2.4)$$

with X being a vector field. As one can easily check, for any function f and vector field X the divergence satisfies

$$\operatorname{div}_{\mu_v}(fX) = f \operatorname{div}_{\mu_v}(X) + (-1)^{|f||X|} X(f). \quad (2.5)$$

Now the odd Laplacian of $f \in C^\infty(T^*[-1]\mathcal{M})$ is defined through the divergence of the corresponding Hamiltonian vector field as

$$\Delta_v f = \frac{(-1)^{|f|}}{2} \operatorname{div}_{\mu_v} X_f, \quad \{f, g\} = X_f(g). \quad (2.6)$$

Indeed one can check that thanks to (2.5) Δ_v generates the bracket and $\Delta_v^2 = 0$. Thus $C^\infty(T^*[-1]\mathcal{M})$ is a BV-algebra as defined above, see [18] for the explicit calculations. If the volume form is written in terms of an even density ρ_v as

$$\mu_v = \rho_v dy^1 \cdots dy^n dy_1^+ \cdots dy_n^+,$$

then the Laplacian can be written as

$$\Delta_v = (-1)^{|y_a|} \frac{\partial}{\partial y_a^+} \frac{\partial}{\partial y^a} + \frac{1}{2} \{\log \rho_v, -\}. \quad (2.7)$$

This Laplacian squares to zero since we take a specific Berezinian measure ρ_v which originates from the measure on \mathcal{M} . Thus $T^*[-1]\mathcal{M}$ is equipped with the SP-structure in the

¹We require that the measure is well-defined functional on the space of compactly supported functions and locally it is Berezinian measure.

Schwarz's terminology [25] (also see [19] for a nice review of the related issues) and in what follows we refer to them simply as BV-manifolds.

There exists a canonical way (up to a sign) of restricting a volume form μ_v on $T^*[-1]\mathcal{M}$ to a volume form on a Lagrangian submanifold \mathcal{L} . We denote such restriction as $\sqrt{\mu_v}$ and consider the integrals of the form

$$\int_{\mathcal{L}} \sqrt{\mu_v} f, \quad f \in C^\infty(T^*[-1]\mathcal{M}). \quad (2.8)$$

Again if f is Δ -closed then the integral depends only on homology class of \mathcal{L} and if f is Δ -exact then the integral is zero. In particular we are interested in the situation when the integrands in (2.8) are of the form

$$\int_{\mathcal{L}} \sqrt{\mu_v} \Psi e^S \equiv \langle \Psi \rangle, \quad (2.9)$$

where we assume naturally that $\Delta_v(\Psi e^S) = 0$. If $\Psi = 1$ then we get the following relation

$$\Delta_v(e^S) = 0 \iff \Delta_v S + \frac{1}{2}\{S, S\} = 0, \quad (2.10)$$

which is known as the *quantum master equation*. In the general case we have

$$\Delta_v(\Psi e^S) = 0 \iff \Delta_{(v,S)}\Psi = \Delta_v\Psi + \{S, \Psi\} = 0, \quad (2.11)$$

where we refer to $\Delta_{(v,S)}$ as the quantum Laplacian. In the derivation of (2.11) we have used the quantum master equation (2.10). A function S that satisfies the quantum master equation is called a quantum BV action and Ψ satisfying (2.11) is a quantum observable. Indeed the quantum observables are elements of the cohomology $H(\Delta_{(v,S)})$; by the above construction it is clear that S defines the isomorphism

$$H^\bullet(\Delta_v) \approx H^\bullet(\Delta_{(v,S)}). \quad (2.12)$$

The integral (2.9) defines a trace ($H^\bullet(\Delta_{(v,S)}) \rightarrow \mathbb{C}$) on this cohomology.

If we change S to S/\hbar , we see that in the classical limit ($\hbar \rightarrow 0$) S must satisfy the classical master equation $\{S, S\} = 0$ and the classical observables Ψ are such that $\delta_{BV}\Psi \equiv \{S, \Psi\} = 0$. Due to the classical master equation the vector field δ_{BV} squares to zero and defines the cohomology $H(\delta_{BV})$ of classical observables.

If \mathcal{M} is a finite dimensional manifold then everything is well-defined. In the following Sections we provide several finite dimensional BV manifolds equipped with a solution of the classical (quantum) master equation. However in field theory one deals with \mathcal{M} being infinite dimensional. In fact, \mathcal{M} is usually the space of the physical fields, ghosts and

Lagrange multipliers, that is infinite dimensional. This set of fields is then extended by adding antifields such that together they form $T^*[-1]\mathcal{M}$, where an odd Poisson bracket is well-defined on large enough class of functions, as described above. However there is no well-defined measure on \mathcal{M} and thus there is no well-defined odd Laplace operators. In physics literature, the naive Laplacian of the form (2.2) is used. Moreover the field theory suffers from the problems with renormalization which can be resolved within the perturbative setup.

3 AKSZ formalism and its reduction

In this Section we review the AKSZ construction [1] of the solutions of the classical master equation. Here we will follow the presentation given in [24] and we use the language of graded manifolds. Relevant definitions are postponed to Section 7. For further details the reader may consult [26].

The AKSZ solution of the classical master equation is defined starting from the following data:

The source: A graded manifold \mathcal{N} endowed with a cohomological vector field D and a measure $\int_{\mathcal{N}} \mu$ of degree $-n - 1$ for some positive integer n . In what follows the source will be $\mathcal{N} = T[1]\mathcal{N}_0$, for any smooth manifold \mathcal{N}_0 of dimension $n + 1$, with $D = d$ the de Rham differential over \mathcal{N}_0 and the canonical coordinate measure.

The target: A graded symplectic manifold (\mathcal{M}, ω) with $\deg(\omega) = n$ and an homological vector field Q preserving ω . We require that Q is Hamiltonian, *i.e.* it exists $\Theta \in C_{n+1}(\mathcal{M})$ (functions of degree $n + 1$) such that $Q = \{\Theta, -\}$. Therefore Θ satisfies the following Maurer-Cartan equation

$$\{\Theta, \Theta\} = 0 .$$

Since graded manifolds are ringed spaces, we consider the space $\text{Maps}(\mathcal{N}, \mathcal{M})$ to be the space of morphisms of ringed spaces (see Definition 9). We choose a set of coordinates $X^A = \{x^\mu; \xi^m\}$ on the target \mathcal{M} , where $\{x^\mu\}$ are the coordinates for an open $U \subset \mathcal{M}_0$ and $\{\xi^m\}$ are the coordinates in the formal directions. We also choose the coordinates $\{u^\alpha; \theta^a\}$ on the source \mathcal{N} correspondently. Let $(\phi, \Phi) \in \text{Maps}(\mathcal{N}, \mathcal{M})$, where $\phi : \mathcal{N}_0 \rightarrow \mathcal{M}_0$. The superfield Φ is defined as an expansion over the formal coordinates of \mathcal{N} for $\phi^{-1}(U)$

$$\Phi^A = \Phi_0^A(u) + \Phi_a^A(u)\theta^a + \Phi_{a_1 a_2}^A(u)\theta^{a_1}\theta^{a_2} + \dots , \quad (3.13)$$

such that $\Phi_0^\mu = x^\mu \circ \phi$. The symplectic form ω of degree n on \mathcal{M} can be written in the Darboux coordinates $\omega = dX^A \omega_{AB} dX^B$. Using this form we define the symplectic form of

degree -1 on $\text{Maps}(\mathcal{N}, \mathcal{M})$ as

$$\Omega = \int_{\mathcal{N}} \mu \delta\Phi^A \omega_{AB} \delta\Phi^B . \quad (3.14)$$

Thus the space of maps $\text{Maps}(\mathcal{N}, \mathcal{M})$ is naturally equipped with the odd Poisson bracket $\{ , \}$. Since the space $\text{Maps}(\mathcal{N}, \mathcal{M})$ is infinite dimensional we cannot define the BV Laplacian properly. We can only talk about the naive Laplacian adapted to the local field-antifield splitting, given by the formula (2.1). However on $\text{Maps}(\mathcal{N}, \mathcal{M})$ we can discuss the solutions of the classical master equation. The AKSZ action then reads

$$S_{BV}[\Phi] = S_{kin}[\Phi] + S_{int}[\Phi] = \int_{\mathcal{N}} \mu \left(\frac{1}{2} \Phi^A \omega_{AB} D\Phi^B + (-1)^{n+1} \Phi^*(\Theta) \right) . \quad (3.15)$$

and solves the classical master equation $\{S_{BV}, S_{BV}\} = 0$ with respect to the bracket defined by the symplectic structure (3.14). We denote with δ_{BV} the Hamiltonian vector field for S_{BV} , which is homological as a consequence of classical master equation. The action (3.15) is invariant under all orientation preserving diffeomorphisms of \mathcal{N}_0 and thus defines a topological field theory. The solutions of the classical field equations of (3.15) are graded differentiable maps $(\mathcal{N}, D) \rightarrow (\mathcal{M}, Q)$, i.e. maps which commute with the homological vector fields.

The homological vector field Q on \mathcal{M} defines a complex on $C^\infty(\mathcal{M})$ whose cohomology we denote with $H_Q(\mathcal{M})$. Take $f \in C^\infty(\mathcal{M})$ and expand $\Phi^* f$ in the formal variables on \mathcal{N} , i.e.

$$\Phi^* f = O^{(0)}(f) + O_a^{(1)}(f)\theta^a + O_{a_1 a_2}^{(2)}(f)\theta^{a_1}\theta^{a_2} + \dots .$$

We compute

$$\delta_{BV}(\Phi^* f) = \{S_{BV}, \Phi^* f\} = D\Phi^* f + \Phi^* Qf ,$$

so that, if $Qf = 0$ and μ_k is a D -invariant linear functional on $C_k(\mathcal{N})$, i.e. a representative of a homology class of \mathcal{N}_0 , then $\mu_k(O^{(k)}(f))$ is δ_{BV} -closed, i.e. it is a classical observable. Therefore $H_Q(\mathcal{M})$ naturally defines a set of classical observables in the theory.

Like in the smooth case, the symplectic reduction of the odd symplectic manifold $\text{Maps}(\mathcal{N}, \mathcal{M})$ is specified by the choice of a coisotropic submanifold. The reduced space naturally inherits the odd symplectic structure and any function on $\text{Maps}(\mathcal{N}, \mathcal{M})$ descends to the quotient provided it is invariant once restricted to the coisotropic submanifold. In this paper we consider the symplectic reduction of $\text{Maps}(\mathcal{N}, \mathcal{M})$ to the cohomology $H_D(\mathcal{N}) = H_{dR}(\mathcal{N}_0)$ of the source. We consider the constraint

$$\Lambda = \int_{\mathcal{N}} \mu \Lambda_A D\Phi^A ,$$

which defines the coisotropic submanifold $D\Phi = 0$ of $\text{Maps}(\mathcal{N}, \mathcal{M})$ (the constrained surface). The corresponding infinitesimal gauge transformations

$$\delta_\Lambda \Phi^A = D\Lambda^A \quad (3.16)$$

identify two configurations which differ by a D -exact term. If \mathcal{N}_0 has a boundary then boundary conditions must be discussed. We leave the case with boundary to Section 6 and for now we assume that \mathcal{N}_0 does not have boundary. The reduced odd symplectic manifold is finite dimensional and can be globally described as the space of maps from the cohomology of \mathcal{N}_0 to \mathcal{M} . More precisely, we can consider a generalized AKSZ construction where the source is the cohomology of \mathcal{N}_0 seen as a sheaf $X_{\mathcal{N}_0}$ of graded commutative algebras over a point, equipped with the zero homological vector field and the integration naturally induced on cohomology. The source is not anymore a graded manifold since the cohomology ring is not freely generated in general. However the space of maps $\text{Maps}(X_{\mathcal{N}_0}, \mathcal{M})$ is still defined; this simply corresponds to interpret $\{\theta^a\}$ in the superfield (3.13) as the generators of the cohomology ring $H_{dR}(\mathcal{N}_0)$. The simplest case is when \mathcal{N}_0 has cohomology ring concentrated in degree zero and top degree. In this case the reduced theory will be simply $T^*[-1]\mathcal{M}$.

For any $f \in C^\infty(\mathcal{M})$, $\Phi^* f - (\Phi + D\Lambda)^* f$ is D -exact. Thus any observable $\mu_k(O^{(k)}(f))$ in the full theory is invariant under the gauge transformations (3.16) and defines a function on the reduced odd symplectic manifold. In particular, the BV-action (3.15) $S_{BV} = \int_{\mathcal{N}} \mu \Phi^*(\Theta)$ defines a solution of the classical master equation that coincides with the AKSZ action on $\text{Maps}(X_{\mathcal{N}_0}, \mathcal{M})$.

It is crucial that the reduced odd symplectic manifold is finite dimensional so that integration and BV Laplacian can be well defined. Thus we can define properly the quantum master equation and discuss the possible obstructions to satisfy it. The rest of the paper consists of a detailed account of these reduced BV-theories in several examples.

4 Two dimensional case: Poisson sigma model

Let us consider the AKSZ construction described in the previous section with source $\mathcal{N} = T[1]\Sigma_g$, where Σ_g is the two dimensional compact surface of genus g . Then the target \mathcal{M} is a symplectic graded manifold of degree 1 with a homological vector field Q preserving the symplectic structure. It must be necessarily of the form $\mathcal{M} = T^*[1]M$ where M is a Poisson manifold with Poisson tensor α . By choosing coordinates $\{x^\mu, \beta_\mu\}$, the Hamiltonian for Q is $\Theta = \alpha^{\mu\nu} \beta_\mu \beta_\nu \in C_2(\mathcal{M})$. We have that $C^\infty(\mathcal{M}) = C^\infty(\Lambda TM)$ and the cohomology $H_Q(M)$ of the complex $(C^\infty(\mathcal{M}), Q)$ coincides with the usual Lichnerowicz-Poisson cohomology $H_{LP}(M; \alpha)$. This is the case studied in [6] as a BV quantization of

the Poisson Sigma Model. We will first review the relevant results from [6] and [7] and then study the reduction procedure to the finite dimensional BV -manifold outlined in the previous section. In the case of $\Sigma_0 = \mathbb{S}^2$ the reduction to the finite dimensional theory has been studied in [3].

The Poisson Sigma Model covers many interesting geometrical structures. For example, if A is a Lie algebroid then the dual vector bundle A^* regarded as a manifold is naturally equipped with a Poisson structure. This is an example of a non-compact Poisson manifold where some issues related to the integration require extra care. We assume that for non-compact case the integration is defined as functional over the space of functions with compact support (or with exponential decay along the non-compact directions) and thus formally all our considerations are equally applicable both for compact and non-compact cases.

4.1 BV action

Let $\{u^\alpha\}$ be coordinates on Σ_g and $\{x^\mu\}$ on M . The superfields read as

$$\begin{aligned} \mathbf{X}^\mu &= X^\mu + \theta^\alpha \eta_\alpha^{+\mu} - \frac{1}{2} \theta^\alpha \theta^\beta \beta_{\alpha\beta}^{+\mu} , \\ \boldsymbol{\eta}_\mu &= \beta_\mu + \theta^\alpha \eta_{\alpha\mu} + \frac{1}{2} \theta^\alpha \theta^\beta X_{\alpha\beta\mu}^+ , \end{aligned}$$

with $\{\theta^\alpha\}$ being the odd coordinates on $T[1]\Sigma_g$. In the BV language, the components of ghost number 0, X and η are the classical fields, β is a ghost with the ghost number 1, while η^+ , β^+ and X^+ are antifields of ghost number -1 , -2 and -1 respectively. The space of maps $\text{Map}(T[1]\Sigma_g, T^*[1]M)$ can be seen as $T^*[-1]\mathcal{M}$, where \mathcal{M} is the infinite dimensional manifold corresponding to the fields (X, η, β) .

If we change coordinates $y^i = y^i(x)$, the superfields transform as

$$\mathbf{Y}^i = y^i(\mathbf{X}) , \quad \boldsymbol{\eta}_i = \frac{\partial x^\mu}{\partial y^i}(\mathbf{X}) \boldsymbol{\eta}_\mu . \quad (4.17)$$

For later utility we add the explicit component content of (4.17)

$$\begin{aligned} \eta_\alpha^{+i} &= \frac{\partial y^i}{\partial x^\mu} \eta_\alpha^{+\mu} , \quad \beta_{\alpha\beta}^{+i} = \frac{\partial y^i}{\partial x^\mu} \beta_{\alpha\beta}^{+\mu} + \frac{\partial^2 y^i}{\partial x^\mu \partial x^\nu} \eta_\alpha^{+\mu} \eta_\beta^{+\nu} , \quad \beta_i = \frac{\partial x^\mu}{\partial y^i} \beta_\mu , \\ \eta_i &= \frac{\partial x^\mu}{\partial y^i} \eta_\mu + \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} \eta^{+\nu} \beta_\mu , \\ X_{i\alpha\beta}^+ &= \frac{\partial x^\mu}{\partial y^i} X_{\mu\alpha\beta}^+ - 2 \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} \eta_\alpha^{+\nu} \eta_{\mu\beta} - \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} \beta_{\alpha\beta}^{+\nu} \beta_\mu - \frac{\partial^2}{\partial x^\nu \partial x^\rho} \frac{\partial x^\mu}{\partial y^i} \eta_\alpha^{+\nu} \eta_\beta^{+\rho} \beta_\mu . \end{aligned} \quad (4.18)$$

The AKSZ action defined in (3.15) reads

$$S_{BV} = \int d^2\theta d^2u \left(\boldsymbol{\eta}_\mu DX^\mu + \frac{1}{2} \alpha^{\mu\nu}(X) \boldsymbol{\eta}_\mu \boldsymbol{\eta}_\nu \right), \quad (4.19)$$

where $D = \theta^\alpha \partial_\alpha$. The odd symplectic structure is

$$\omega = \int_{\Sigma_g} (\delta X \wedge \delta X^+ + \delta \boldsymbol{\eta} \wedge \delta \boldsymbol{\eta}^+ + \delta \beta \wedge \delta \beta^+) . \quad (4.20)$$

The action (4.19) satisfies both classical and naive quantum master equations [6]. The corresponding BV operator δ_{BV} acts on the superfields as follows

$$\delta_{BV} X^\mu = DX^\mu + \alpha^{\mu\nu}(X) \boldsymbol{\eta}_\nu, \quad (4.21)$$

$$\delta_{BV} \boldsymbol{\eta}_\mu = D\boldsymbol{\eta}_\mu + \frac{1}{2} \partial_\mu \alpha^{\nu\rho}(X) \boldsymbol{\eta}_\nu \boldsymbol{\eta}_\rho. \quad (4.22)$$

The local and non-local classical observables are labelled by the Lichnerowicz- Poisson cohomology [3]. In components the AKSZ-BV action (4.19) has the form

$$\begin{aligned} S_{BV} = \int_{\Sigma_g} & \boldsymbol{\eta}_\mu \wedge dX^\mu + \frac{1}{2} \alpha^{\mu\nu}(X) \boldsymbol{\eta}_\mu \wedge \boldsymbol{\eta}_\nu + X_\mu^+ \alpha^{\mu\nu}(X) \beta_\nu - \boldsymbol{\eta}^{+\mu} \wedge (d\beta_\mu + \partial_\mu \alpha^{\rho\nu}(X) \boldsymbol{\eta}_\rho \beta_\nu) - \\ & - \frac{1}{2} \beta^{+\mu} \partial_\mu \alpha^{\rho\nu}(X) \beta_\rho \beta_\nu - \frac{1}{4} \boldsymbol{\eta}^{+\mu} \wedge \boldsymbol{\eta}^{+\nu} \partial_\mu \partial_\nu \alpha^{\rho\sigma}(X) \beta_\rho \beta_\sigma. \end{aligned} \quad (4.23)$$

4.2 The reduced BV-AKSZ theory

In this subsection we describe the reduction of the BV manifold $\text{Maps}(T[1]\Sigma_g, T^*[1]M)$ down to "the constant configurations" as described in Section 3. We obtain a solution of the classical master equation on the finite dimensional BV manifold. Namely we define the reduction with respect to the following constraints

$$\Lambda = \int_{T[1]\Sigma} \Lambda_\mu DX^\mu, \quad T = \int_{T[1]\Sigma} T^\mu D\boldsymbol{\eta}_\mu, \quad (4.24)$$

with $\Lambda_\mu(u, \theta) = \Lambda_\mu^{(0)} + \Lambda_{\mu\alpha}^{(1)} \theta^\alpha + \Lambda_{\mu\alpha\beta}^{(2)} \theta^\alpha \theta^\beta$ and $T^\mu(u, \theta) = T_{(0)}^\mu + T_{(1)\alpha}^\mu \theta^\alpha + T_{(2)\alpha\beta}^\mu \theta^\alpha \theta^\beta$. We assign $|\Lambda_\mu| = 0$ and $|T^\mu| = -1$. After a short computation one gets

$$\delta_{\Lambda, T} X^\mu = \{T, X^\mu\} = DT^\mu, \quad \delta_{\Lambda, T} \boldsymbol{\eta}_\nu = \{\Lambda, \boldsymbol{\eta}_\nu\} = D\Lambda_\nu, \quad (4.25)$$

or in components

$$\delta_{\Lambda, T} X^\mu = 0, \quad \delta_{\Lambda, T} \boldsymbol{\eta}^{+\mu} = dT_{(0)}^\mu, \quad \delta_{\Lambda, T} \beta^{+\mu} = dT_{(1)}^\mu$$

$$\delta_{\Lambda,T}\beta_\mu = 0, \quad \delta_{\Lambda,T}\eta_\mu = d\Lambda_\mu^{(0)}, \quad \delta_{\Lambda,T}X_\mu^+ = d\Lambda_\mu^{(1)}. \quad (4.26)$$

We see that the reduction with respect to the constraints $D\mathbf{X} = D\boldsymbol{\eta} = 0$ consists in identifying $(\mathbf{X}, \boldsymbol{\eta})$ with $(\mathbf{X} + DT, \boldsymbol{\eta} + D\Lambda)$ and then in going to the cohomology of Σ_g .

In order to define the reduced variables, let us choose a basis for the homology $H_\bullet(\Sigma_g)$. Let $u_0 \in \Sigma_g$ be a generator in degree zero and the whole surface Σ_g be the generator in degree two and consider the canonical basis $\{c_I, c^J\}$ in $H_1(\Sigma_g)$, in degree one, with the following intersection numbers

$$\#(c_I, c_J) = \#(c^I, c^J) = 0, \quad \#(c_I, c^J) = -\#(c^J, c_I) = \delta_I^J.$$

The Poincaré dual basis $\{e^J, e_I\}$ in $H^1(\Sigma_g)$ is defined as

$$\int_{c_I} e^J = \int_{c^J} e_I = \delta_I^J, \quad \int_{c_I} e_J = \int_{c^I} e^J = 0,$$

and satisfies

$$\int_{\Sigma_g} e^J \wedge e_I = \delta_I^J.$$

Let us introduce the reduced coordinates obtained from the zero form fields

$$x^\mu = X^\mu(u_0), \quad b_\mu = \beta_\mu(u_0),$$

from the one form fields

$$\eta_I = \int_{c_I} \eta, \quad \eta^J = \int_{c^J} \eta, \quad \eta_I^+ = \int_{c_I} \eta^+, \quad \eta^{+J} = \int_{c^J} \eta^+.$$

and finally from the two forms fields

$$x_\mu^+ = \int_{\Sigma_g} X_\mu^+, \quad b^{+\mu} = \int_{\Sigma_g} \beta^{+\mu}.$$

The global structure, *i.e.* the change of coordinates under the change $y^i = y^i(x)$ can be obtained from (4.18). One explicitly gets

$$\begin{aligned} \eta_I^{+i} &= \frac{\partial y^i}{\partial x^\mu} \eta_I^{+\mu}, \quad \eta^{+iI} = \frac{\partial y^i}{\partial x^\mu} \eta^{+\mu I}, \quad b_i = \frac{\partial x^\mu}{\partial y^i} b_\mu, \quad b^{+i} = \frac{\partial y^i}{\partial x^\mu} b^{+\mu} + \frac{\partial^2 y^i}{\partial x^\mu \partial x^\nu} \eta_I^{+\mu} \eta^{+\nu I}, \\ \eta_{Ii} &= \frac{\partial x^\mu}{\partial y^i} \eta_{I\mu} + \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} \eta_I^{+\nu} b_\mu, \quad \eta_i^I = \frac{\partial x^\mu}{\partial y^i} \eta_\mu^I + \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} \eta^{+I\nu} b_\mu, \\ x_i^+ &= \frac{\partial x^\mu}{\partial y^i} x_\mu^+ - \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} (\eta_I^{+\nu} \eta_\mu^I - \eta^{+I\nu} \eta_{I\mu}) - \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^i} b^{+\nu} b_\mu - \frac{\partial^2}{\partial x^\nu \partial x^\rho} \frac{\partial x^\mu}{\partial y^i} \eta_I^{+\nu} \eta^{+\rho I} b_\mu. \end{aligned} \quad (4.27)$$

All the BV structure goes to the quotient which is a finite dimensional BV manifold. The odd symplectic structure (4.20) reads

$$\omega = dx^\mu dx_\mu^+ + d\eta_{I\mu} d\eta^{+I\mu} - d\eta_\mu^I d\eta_I^{+\mu} + db_\mu db^{+\mu} . \quad (4.28)$$

Moreover, the BV action S_{BV} defined in (4.23) when restricted to the constraint surface depends only on the reduced variables, *i.e.* it is a pull-back of a function on the reduced manifold. We use the same notation S_{BV} for the reduced action. The reduced BV action is computed from (4.23) as

$$\begin{aligned} S_{BV} = & \alpha^{\mu\nu}(x)\eta_{I\mu}\eta_\nu^I + x_\mu^+ \alpha^{\mu\nu}(x)b_\nu - \eta_I^{+\mu} \partial_\mu \alpha^{\rho\nu}(x)\eta_\rho^I b_\nu + \\ & + \eta^{+I\mu} \partial_\mu \alpha^{\rho\nu}(x)\eta_{I\rho} b_\nu - \frac{1}{2} b^{+\mu} \partial_\mu \alpha^{\rho\nu}(x)b_\rho b_\nu - \frac{1}{2} \eta_I^{+\mu} \eta^{+I\nu} \partial_\mu \partial_\nu \alpha^{\rho\sigma}(x)b_\rho b_\sigma , \end{aligned} \quad (4.29)$$

which obviously satisfies the classical master equation.

The reduced variables can be assembled in the superfields $\underline{\Phi} = (\mathbf{x}^\mu, \mathbf{e}_\mu)$

$$\mathbf{x}^\mu = x^\mu + e^J \eta_J^{+\mu} + e_J \eta^{+J\mu} - s b^{+\mu} , \quad \mathbf{e}_\nu = b_\nu + e_I \eta_\nu^I + e^I \eta_{I\nu} + s x_\nu^+ , \quad (4.30)$$

where s is the generator of $H_{dR}^2(\Sigma_g)$ normalized to $\int_\Sigma s = 1$. The ring structure for $g > 0$ is defined by $e^J \wedge e_I = \delta_I^J s$ and by $s^2 = 0$ for $g = 0$. One can check that the transformations of coordinates (4.18) can be deduced from the transformations of superfields

$$\mathbf{y}^i = y^i(\mathbf{x}) , \quad \mathbf{e}_i = \frac{\partial x^\mu}{\partial y^i}(\mathbf{x}) \mathbf{e}_\mu .$$

The action (4.29) can be written as

$$S_{BV} = \int ds \alpha^{\mu\nu}(\mathbf{x}) \mathbf{e}_\mu \mathbf{e}_\nu , \quad (4.31)$$

where $\int ds$ is the induced integral on $H_{dR}^\bullet(\Sigma_g)$. We easily see that this is an AKSZ formulation of the reduced theory, where we take as ‘‘coordinates’’ on the source the generators of the ring $H_{dR}(\Sigma_g)$. Compared with the discussion in Section 3, here coordinates $\{e_J, e^J\}$ on the source for $g \neq 1$ are not free but simply generate the commutative graded algebra $H_{dR}(\Sigma_g)$. We may regard it as the sheaf X_{Σ_g} obtained by putting over a point the commutative graded algebra $H_{dR}(\Sigma_g)$ such that that the reduced manifold is the space $\text{Maps}(X_{\Sigma_g}, T^*[1]M)$ of morphisms between X_{Σ_g} and $T^*[1]M$. The case $g = 1$ is special since the cohomology ring is freely generated and we have a true graded manifold $X_{\Sigma_1} = \mathbb{R}^2[1]$. We will comment more on this in Section 8.

Remark 1 For $g = 0$, $\text{Maps}(X_{\mathbb{S}^2}, T^*[1]M)$ can be equivalently described as $T^*[-1]T^*[1]M$, see [3]. For $g > 0$ the BV manifold can be described as

$$\text{Maps}(X_{\Sigma_g}, T^*[1]M) = T^*[-1](\text{Maps}(X_L, T^*[1]M)) ,$$

where X_L is the subsheaf over a point generated by $(1, e_I)$. The total dimension of the BV manifold is given by the following formula

$$\dim \text{Maps}(X_{\Sigma_g}, T^*[1]M) = 4(g + 1) \dim M .$$

Since the reduced BV-manifold is finite dimensional we can introduce an integral. Let $\Omega = \rho_\Omega dx^1 \dots dx^n$ be a volume form on M . It defines the generator of the Schouten bracket D_Ω on $\Lambda TM = C^\infty(T^*[1]M)$. Moreover it defines a Berezinian integration on the reduced BV manifold $\text{Maps}(X_{\Sigma_g}, T^*[1]M)$ as

$$\mu_\Omega = \rho_\Omega^{2(2-2g)} Dz ,$$

where $Dz = dx \dots dx^+ \dots db \dots db^+ \dots d\eta \dots d\eta^+$ is the coordinate volume form. Since under the change of coordinates $\tilde{z} = \tilde{z}(z)$ the coordinate volume form transforms as

$$D\tilde{z} = \text{Ber} \frac{\partial \tilde{z}}{\partial z} Dz , \quad \text{Ber} \begin{pmatrix} I_{00} & I_{01} \\ I_{10} & I_{11} \end{pmatrix} = \frac{\det(I_{00} - I_{01} I_{11}^{-1} I_{10})}{\det I_{11}} \quad (4.32)$$

it is a tedious but straightforward computation to put (4.27) in (4.32) and check that μ_Ω is well defined.

The corresponding generator of the BV-bracket is

$$\Delta_\Omega = \frac{\partial}{\partial x_\mu^+} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial b^{+\mu}} \frac{\partial}{\partial b_\mu} - \frac{\partial}{\partial \eta_I^{+\mu}} \frac{\partial}{\partial \eta_\mu^I} + \frac{\partial}{\partial \eta^{+I\mu}} \frac{\partial}{\partial \eta_{I\mu}} + (2 - 2g) \{ \log \rho_\Omega, - \} . \quad (4.33)$$

We then compute

$$\Delta_\Omega S_{BV} = (g - 1)(D_\Omega \alpha)^\mu b_\mu = (g - 1) \chi_\Omega^\mu b_\mu ,$$

where $\chi_\Omega = D_\Omega \alpha$ is the modular vector field that satisfies $d_{LP} \chi_\Omega = 0$. The class $[\chi_\Omega] \in H_{LP}(M; \alpha)$ is called *modular class* and it is independent from the choice of the volume form Ω . The modular class represents an obstruction for S_{BV} to satisfy the quantum master equation for any genus $g \neq 1$. This extends to any genus the construction in [3]; the result has been already observed in [10]. In order to understand from the point of view of QFT this dependence on the genus, we recall the renormalization of the PSM on the disk in [6]; in fact in the expansion on Feynman graphs, the modular vector field appeared as a factor of the divergent graphs with self insertions. Their renormalization

was obtained by introducing a non vanishing vector field, which in the compact case is possible only for $g = 1$.

Any $w = w^{\mu_1 \dots \mu_k} \beta_{\mu_1} \dots \beta_{\mu_k}$ representing a class in Lichnerowicz-Poisson cohomology defines the chain of observables $\Phi^* w = w^{\mu_1 \dots \mu_k}(X) \boldsymbol{\eta}_{\mu_1} \dots \boldsymbol{\eta}_{\mu_k} = O^{(0)}(w) + O_\alpha^{(1)}(w) \theta^\alpha + \dots$ of the full theory. They are invariant with respect to gauge transformations (4.25) and correspond to the observables $\underline{\Phi}^* w = w^{\mu_1 \dots \mu_k}(x) \mathbf{e}_{\mu_1} \dots \mathbf{e}_{\mu_k} = O^{(0)}(w) + O_I^{(1)}(w) e^I + \dots$ of the reduced AKSZ theory. By using (4.33) we compute the following relation

$$\Delta_\Omega \underline{\Phi}^* w = 2s(1-g) \underline{\Phi}^* D_\Omega w .$$

We conclude that, given a unimodular Poisson structure α , for any genus $g \neq 1$ the finite dimensional BV theory maps any class $[w] \in H_{LP}(M; \alpha)$ represented by a divergenceless vector field w to the class represented by $e^{S_{BV}} \underline{\Phi}^* w$ in the cohomology of the complex $(C^\infty(\text{Maps}(X_{\Sigma_g}, T^*[1]M)), \Delta_\Omega)$. For $g = 1$ the hypothesis of unimodularity is not needed and such invariants are defined for any class in Lichnerowicz Poisson cohomology. The main point is now to characterize these invariants in terms of more canonical cohomologies and mainly understand when they are not trivial. Here we simply sketch some obvious considerations and leave a more systematic analysis for further investigations.

One has to compute the correlators of such observables by choosing a gauge fixing, *i.e.* a lagrangian submanifold \mathcal{L} . There is always one canonical choice which consists simply in putting all antifields equal to zero, *i.e.* $x^+ = \eta^+ = b^+ = 0$. The case $g = 0$ has been studied in [3] and we refer to it for the result. The case $g > 0$ is badly defined due to fibrewise integration along η in the degenerate directions of the Poisson tensor.

If one allows on the target M a complex structure and introduces complex coordinates $\{z^a, z^{\bar{a}}\}$, then one can define a gauge fixing by putting $x^+ = b^+ = 0$ and $\eta_{Ia} = \eta_{\bar{a}}^I = \eta^{+I\bar{a}} = \eta_I^{+a} = 0$. By looking at (4.18) one can easily check that this is invariant under holomorphic transformation of variables. In the symplectic case the computation of the partition function in this gauge fixing gives the Euler number of M ; due to degeneracy of the Poisson tensor it is not clear how to give a meaning to this integral in the general Poisson case.

5 Three dimensional case: Courant sigma model

In this Section we consider the reduction of three dimensional topological theory associated to any Courant algebroid, the Courant sigma model. We follow closely the presentation given in [24] (see also for the related discussion [17, 15, 16]).

We follow notations of Section 4 of [24]. Let $E \rightarrow M$ be a vector bundle equipped with a fiberwise nondegenerate symmetric inner product \langle, \rangle , of arbitrary signature. A *Courant*

algebroid structure on (E, \langle, \rangle) is a bilinear operation \circ on sections of E and a bundle map (the *anchor*) $a : E \rightarrow TM$ satisfying the following properties

- i) $s \circ (s_1 \circ s_2) = (s \circ s_1) \circ s_2 + s_1 \circ (s \circ s_2)$;
- ii) $a(s_1 \circ s_2) = [a(s_1), a(s_2)]$;
- iii) $s_1 \circ (f s_2) = f(s_1 \circ s_2) + (a(s_1)(f))s_2$;
- iv) $\langle s, s_1 \circ s_2 + s_2 \circ s_1 \rangle = a(s)(\langle s_1, s_2 \rangle)$;
- v) $a(e)(\langle s_1, s_2 \rangle) = \langle s \circ s_1, s_2 \rangle + \langle s_1, s \circ s_2 \rangle$,

for $s, s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

We can associate to these data a symplectic graded manifold (\mathcal{M}, ω) with $\deg \omega = 2$. Locally we introduce a set of coordinates $\{x^\mu\}$ on M and the local basis $\{e_a\}$ of sections of E such that $\langle e_a, e_b \rangle = g_{ab}$, with constant g_{ab} . Then the symplectic non negatively graded manifold (\mathcal{M}, ω) is described in terms of the local coordinates $\{x^\mu, p_\mu, \xi^a\}$, with $\deg(x) = 0$, $\deg(p) = 2$ and $\deg(\xi) = 1$ with the symplectic form of degree 2

$$\omega = dp_\mu dx^\mu + \frac{1}{2} d\xi^a g_{ab} d\xi^b .$$

Globally \mathcal{M} can be interpreted as the symplectic submanifold of $T^*[2]E[1]$ defined by $\theta_a = \frac{1}{2} g_{ab} \xi^b$ with θ being a momentum for ξ . As shown in [23] the Hamiltonian function of degree 3

$$\Theta = \xi^a P_a^\mu p_\mu - \frac{1}{6} T_{abc} \xi^a \xi^b \xi^c , \quad (5.34)$$

where the structure constants $T_{abc} = \langle e_a \circ e_b, e_c \rangle$ and the anchor $a(e_a) = P_a^\mu \partial_\mu$ defines on (E, \langle, \rangle) a *Courant algebroid* structure if and only if $\{\Theta, \Theta\} = 0$. Due to the AKSZ logic reviewed in Section 3 one can define a three dimensional theory whose space of fields is $\text{Maps}(T[1]\Sigma, \mathcal{M})$ with Σ being any three dimensional manifold. Introducing the superfields (X^μ, P_μ, ξ^a) corresponding to the local description of \mathcal{M} the Courant sigma model is defined by the following BV action

$$S_{BV} = \int d^3\theta d^3u \left(P_\mu D X^\mu + \frac{1}{2} \xi^a g_{ab} D \xi^b - \xi^a P_a^\mu P_\mu + \frac{1}{6} T_{abc} \xi^a \xi^b \xi^c \right) . \quad (5.35)$$

The AKSZ construction guarantees that (5.35) is a solution of the classical master equation associated to any three dimensional manifold Σ and the Courant algebroid E . The reader may consult [24] for the explicit expression of this *BV* action in components of the superfield.

Here we are interested in the finite dimensional BV theory obtained through the symplectic reduction of the AKSZ theory described above. The whole construction is done in complete analogy with the two dimensional case, see subsection 4.2. Thus we skip the details of the actual reduction and present the final answer. Introduce the basis in $H_{dR}(\Sigma)$ such that $\{e_I\}$ is a basis in $H_{dR}^1(\Sigma)$, e^I is the dual basis in $H_{dR}^2(\Sigma)$, 1 basis element in $H_{dR}^0(\Sigma)$ and s is a basis element in $H_{dR}^3(\Sigma)$. With this choice the natural ring structure on $H_{dR}(\Sigma)$ is given

$$e_I \wedge e^J = \delta_J^I s, \quad e_I \wedge e_J = f_{IJK} e^K$$

and also it follows that

$$e_I \wedge e_J \wedge e_K = f_{IJK} s.$$

The constants f_{IJK} have the interpretation as intersection numbers of one cycles. Define the sheaf X_Σ by putting over a point the commutative graded algebra $H_{dR}(\Sigma)$. The space of maps (*i.e.* morphisms of ringed spaces) $\text{Maps}(X_\Sigma, \mathcal{M})$ is equipped with an odd symplectic structure. We can introduce the superfields expanded in $(1, e_I, e^I, s)$

$$\begin{aligned} \mathbf{X}^\mu &= x^\mu + F^{+\mu I} e_I + \alpha_I^{+\mu} e^I + \gamma^{+\mu} s, \\ \boldsymbol{\xi}^a &= \beta^a + A^{aI} e_I + g^{ab} A_{bI}^+ e^I + g^{ab} \beta_b^+ s, \\ \mathbf{P}_\mu &= \gamma_\mu + \alpha_\mu^I e_I + F_{\mu I} e^I + X_\mu^+ s, \end{aligned} \quad (5.36)$$

which correspond to the local description of element from $\text{Maps}(X_\Sigma, \mathcal{M})$. The integration is naturally defined by the relation $\int ds s = 1$ and all other integrals are zero (modulo ring relations). The odd symplectic structure is

$$\omega = \int ds \left(\delta \mathbf{X}^\mu \delta \mathbf{P}_\mu + \frac{1}{2} \delta \boldsymbol{\xi}^a g_{ab} \delta \boldsymbol{\xi}^b \right). \quad (5.37)$$

Upon the reduction the action (5.35) gives rise to the following BV action

$$\begin{aligned} S_{BV} &= -A^{aI} P_a^\nu(x) F_{\mu I} + \frac{1}{6} T_{abc}(x) f_{IJK} A^{aI} A^{bJ} A^{cK} - \beta^a P_a^\mu(x) X_\mu^+ \\ &+ (-g^{ac} P_a^\mu(x) \alpha_\mu^I - \beta^a A^{bI} T_{abr} g^{rc}) A_{cI}^+ \\ &+ (-\beta^a \partial_\mu P_a^\nu F_{\nu I} + f_{JKI} A^{aJ} \partial_\mu P_a^\nu \alpha_\nu^K + \frac{1}{2} f_{JKI} \partial_\mu T_{abc} \beta^a A^{bJ} A^{cK}) F^{+\mu I} \\ &+ (-\beta^a \partial_\mu P_a^\nu \alpha_\nu^I - A^{aI} \partial_\mu P_a^\nu \gamma_\nu + \frac{1}{2} \partial_\mu T_{abc} \beta^a \beta^b A^{cI}) \alpha_I^{+\mu} \\ &+ (\frac{1}{2} T_{abr} g^{rc} \beta^a \beta^b - g^{ac} P_a^\mu \gamma_\mu) \beta_c^+ + (\frac{1}{6} \partial_\mu T_{abc} \beta^a \beta^b \beta^c - \beta^a \partial_\mu P_a^\nu \gamma_\nu) \gamma^{+\mu} \\ &- (\frac{1}{2} \partial_\mu T_{abr} g^{rc} \beta^a \beta^b - g^{ac} \partial_\mu P_a^\nu \gamma_\nu) F^{+\mu I} A_{cI}^+ + \frac{1}{2} (\beta^a \partial_\mu \partial_\nu P_a^\rho \alpha_\rho^K + A^{aK} \partial_\mu \partial_\nu P_a^\rho \gamma_\rho \\ &- \frac{1}{2} \partial_\mu \partial_\nu T_{abc} \beta^a \beta^b A^{cK}) F^{+\mu I} F^{+\nu J} f_{KIJ} - (\beta^a \partial_\mu \partial_\nu P_a^\rho \gamma_\rho - \frac{1}{6} \partial_\mu \partial_\nu T_{abc} \beta^a \beta^b \beta^c) F^{+\mu I} \alpha_I^{+\nu} \\ &- \frac{1}{6} (-\beta^a \partial_\mu \partial_\nu \partial_\rho P_a^\sigma \gamma_\sigma + \frac{1}{6} \partial_\mu \partial_\nu \partial_\rho T_{abc} \beta^a \beta^b \beta^c) F^{+\mu I} F^{+\nu J} F^{+\rho K} f_{IJK}, \end{aligned} \quad (5.38)$$

hich automatically satisfies the classical master equation if E is equipped with the structure of a Courant algebroid.

On the given BV-manifold $\text{Maps}(X_\Sigma, \mathcal{M})$ the Berezinian integration can be defined as

$$\mu = g^{1-b_1} Dz , \quad (5.39)$$

where $b_1 = \dim H_{dR}^1(\Sigma)$, Dz is the canonical volume form with respect to the coordinates introduced in (5.36) and $g = \det(g^{ab})$. The Berezinian integration is canonically defined (see Appendix A for the details). Thus the corresponding generator (odd Laplacian) of the BV bracket coincides with the naive one

$$\Delta_\mu = \frac{\partial}{\partial x_\mu^+} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial \beta_a^+} \frac{\partial}{\partial \beta^a} + \frac{\partial}{\partial A^{aI}} \frac{\partial}{\partial A_{aI}^+} + \frac{\partial}{\partial \gamma^{+\mu}} \frac{\partial}{\partial \gamma_\mu} - \frac{\partial}{\partial \alpha_I^{+\mu}} \frac{\partial}{\partial \alpha_\mu^I} - \frac{\partial}{\partial F^{+\mu I}} \frac{\partial}{\partial F_{\mu I}} \quad (5.40)$$

Using the explicit expression (5.38) and the axioms of Courant algebroid we easily check that

$$\Delta_\mu S_{BV} = 0 ,$$

i.e. S_{BV} satisfies the quantum master equation.

As an example we can consider the case when the source manifold Σ is a three sphere \mathbb{S}^3 . In this case the reduced BV manifold $\text{Maps}(X_\Sigma, \mathcal{M})$ corresponds to $T^*[-1]\mathcal{M}$ since $H_{dR}(\mathbb{S}^3)$ has only elements of degree 0 and 3. To describe the reduced BV theory for \mathbb{S}^3 we have to set fields A^{aI} , α_μ^a , $F_{\mu I}$ together with their antifields A_{bI}^+ , $\alpha_I^{+\mu}$, $F^{+\mu I}$ to zero in (5.36), (5.37), (5.38) and remove them from the measure (5.39) and Laplacian (5.40). It is easy to describe the observables and the correlators in this theory. The Hamiltonian (5.34) defines an homological vector field $Q = \{\Theta, -\}$. The complex $(C^\infty(\mathcal{M})_{pol}, Q)$ of functions polynomial in p is called the *standard complex* and we denote its cohomology with $H_Q(\mathcal{M})_{pol}$. However in what follows we need to consider the complex $(C^\infty(\mathcal{M})_{exp}, Q)$ of functions with exponential decay in p directions. We have to use this complex in order to make sense of the integrals. For $f \in C^\infty(\mathcal{M})_{exp}$, $Q(f) = 0$ the corresponding observable is

$$\Phi^*(f) = O^{(0)}(f) + sO^{(3)}(f) ,$$

which satisfies by construction

$$\delta_{BV}(\Phi^*(f)) = \{S_{BV}, \Phi^*(f)\} = 0 ,$$

where $\Phi \in \text{Maps}(X_{\mathbb{S}^3}, \mathcal{M})$. For $T^*[-1]\mathcal{M}$ we choose \mathcal{M} as a Lagrangian submanifold defined by setting all antifields to zero, *i.e.* $X^+ = \gamma^+ = \beta^+ = 0$. Thus on \mathcal{M} we have that $\Phi^*(f) = O^{(0)}(f) = f$ and the berezinian measure on \mathcal{M} reads

$$\sqrt{\mu} = \sqrt{g} dx d\gamma d\beta .$$

The correlator is defined as integral over the Lagrangian submanifold

$$\langle \Phi^*(f) \rangle = \int_{\mathcal{M}} \sqrt{\mu} f . \quad (5.41)$$

To make this integral well-defined we have to assume that f decays fast enough along non-compact directions. By construction the correlator (5.41) satisfies the version of the Stokes theorem with respect to BV-differential and odd Laplacian.

Example 2 Consider E to be a Lie algebra with invariant metric interpreted as a vector bundle over a point. In this case the Courant sigma model is just standard Chern-Simons theory. The reduced finite dimensional theory is BV version of the matrix theory. The standard complex is just ΛE^* equipped with the Chevalley-Eilenberg differential. The integral defined in (5.41) is different from zero only on top forms.

Example 3 Consider $E = TM + T^*M$ with the canonical inner product and the Dorfman bracket

$$[X_1 \oplus \omega_1, X_2 \oplus \omega_2] = [X_1, X_2] \oplus L_{X_1} \omega_2 - \iota_{X_2} d\omega_1 .$$

The coordinates on the fiber now split $\xi \rightarrow \{\psi^\mu, \theta_\mu\}$ and the corresponding graded manifold is described as the even symplectic manifold $\mathcal{M} = T^*[2]T[1]M$. The Hamiltonian (5.34) reads $\Theta = \psi^\mu p_\mu$ giving rise to the homological vector field

$$Q = \psi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \theta_\mu} .$$

As shown in [23], Q commutes with

$$\epsilon = p_\mu \frac{\partial}{\partial p_\mu} + \theta_\mu \frac{\partial}{\partial \theta_\mu} .$$

The complex of polynomial functions in p ($C^\infty(\mathcal{M})_{pol}, Q$) then decomposes according to the ϵ -degree, *i.e.* $C^\infty(\mathcal{M})_{pol} = \bigoplus_{k \geq 0} C^\infty(\mathcal{M})_{(k)}$, with the subcomplex of degree zero is $(C^\infty(\mathcal{M})_{(0)}, Q) = (\Omega(M), d)$ being the de Rham complex for M . Moreover, since $\epsilon = Q\iota + \iota Q$ with $\iota = \theta_\mu \frac{\partial}{\partial p_\mu}$ the cohomology $H_Q(\mathcal{M})_{pol}$ is concentrated in degree zero and isomorphic to the de Rham cohomology $H_{dR}(M)$. However for the BV theory we need the complex $(C^\infty(\mathcal{M})_{exp}, Q)$ of functions with the exponential decay in p . It is not clear how the corresponding cohomology is related to $H_{dR}(M)$. Otherwise the correlators can be defined in the way we described above.

6 Two dimensional case with boundary

In this section we discuss the reduction of PSM on the surface with a boundary. Take a surface $\Sigma_{g,n}$, of genus g and n boundary components. We consider the boundary conditions for PSM that have been introduced in [8]. Let $\partial_i \Sigma_{g,n}$ the i -th boundary component of $\Sigma_{g,n}$ and let C_i a coisotropic submanifold of the Poisson manifold (M, α) . We recall that a submanifold C of M is coisotropic if $\alpha(N^*C) \subset TC$, where N^*C is the conormal bundle of C . We assume that the superfields $(\mathbf{X}, \boldsymbol{\eta})$ restrict to $\text{Maps}(T[1]\partial_i \Sigma, N^*[1]C_i)$ for any $i = 1, \dots, n$. More general boundary conditions have been introduced in [4], where it is required that the rank of $\alpha(N_x^*C) + T_x C \subset T_x M$ is constant as x varies over C .

We apply the same construction as in the closed case, *i.e.* we perform the reduction with respect to the constraints defined in (4.24). Since the bracket between the constraints has now a boundary contribution

$$\{\Lambda, T\} = \int_{T[1]\partial\Sigma} \Lambda_\mu DT^\mu ,$$

we require that

$$\Lambda|_{\partial_i \Sigma_{g,n}} \in \Gamma(\Lambda T^* \partial_i \Sigma_{g,n} \otimes X^* N^* C_i) , \quad T|_{\partial_i \Sigma_{g,n}} \in \Gamma(\Lambda T^* \partial_i \Sigma_{g,n} \otimes X^* T C_i) . \quad (6.42)$$

in order to have a consistent reduction. Due to the presence of the boundary conditions for fields and constraints, cohomologies of $\Sigma_{g,n}$ relative to boundary components will appear in the description of the reduced BV-manifolds. In Appendix B we collect the relevant facts about relative (co)homology.

6.1 The case with one boundary component

Consider the case of a surface $\Sigma_{g,1}$ with one boundary component with the boundary condition corresponding to a coisotropic submanifold C . Let us introduce a set of coordinates of M adapted to C with $\{x^a\}$ tangent to C and $\{x^n\}$ normal. Coisotropy of C is then simply expressed by the condition $\alpha^{mn} = 0$.

The gauge transformations (4.25) defined by the constraints together with boundary conditions (6.42) imply that the reduced BV manifold is described by the following variables

$$\mathbf{X}^a \in H_{dR}(\Sigma_{g,1}) , \quad \mathbf{X}^n \in H_{dR}(\Sigma_{g,1}, \partial \Sigma_{g,1}) , \quad \boldsymbol{\eta}_a \in H_{dR}(\Sigma_{g,1}, \partial \Sigma_{g,1}) , \quad \boldsymbol{\eta}_n \in H_{dR}(\Sigma_{g,1}) .$$

The covariant meaning of the above statements and the gluing data of the reduced BV-manifold are better understood once that we introduce the reduced variables.

We recall from Appendix B that $H^2(\Sigma_{g,1}) = H^0(\Sigma_{g,1}, \partial\Sigma_{g,1}) = 0$. In order to define the reduced coordinates, let us choose a set of representatives defining a basis of the relevant homologies. Let $u_0 \in \Sigma_{g,1}$ be a representative generating $H_0(\Sigma_{g,1})$, $\{c^I\}$ for $H_1(\Sigma_{g,1})$, $\{g^{I'}\}$ for $H_1(\Sigma_{g,1}, \partial\Sigma_{g,1})$ and the whole surface $\Sigma_{g,1}$ for $H_2(\Sigma_{g,1}, \partial\Sigma_{g,1})$. Let $\{c_I\}$ and $\{g_{I'}\}$ be the dual basis for $H_{dR}^1(\Sigma_{g,1})$ and $H_{dR}^1(\Sigma_{g,1}, \partial\Sigma_{g,1})$ respectively. Remark that beyond the natural pairings between homology and cohomology one can pair $H_1(\Sigma_{g,1})$ with $H_{dR}^1(\Sigma_{g,1}, \partial\Sigma_{g,1})$. The matrix

$$\lambda_{I'}^I = \int_{c^I} g_{I'}$$

describes the natural homomorphism $H_{dR}^1(\Sigma_{g,1}, \partial\Sigma_{g,1}) \rightarrow H_{dR}^1(\Sigma_{g,1})$ as $g_{I'} \rightarrow \lambda_{I'}^I c_I$. Finally we define the following matrices

$$B_{I'J'}^{(g,1)} = \int_{\Sigma_{g,1}} g_{I'} \wedge g_{J'} , \quad A_{II'}^{(g,1)} = \int_{\Sigma_{g,1}} c_I \wedge g_{I'} ,$$

where $A^{(g,1)}$ is non degenerate matrix.

Then we define

$$\begin{aligned} x^a &= X^a(u_0) , & b_n &= \beta_n(u_0) , \\ \eta_a^{I'} &= \int_{g^{I'}} \eta_a , & \eta_n^I &= \int_{c^I} \eta_n , & \eta^{+aI} &= \int_{c^I} \eta^{+a} , & \eta^{+nI'} &= \int_{g^{I'}} \eta^{+n} \\ x_a^+ &= \int_{\Sigma} X_a^+ , & b^{+n} &= \int_{\Sigma} \beta^{+n} . \end{aligned}$$

The reduced odd symplectic form reads

$$\omega = dx^a dx_a^+ + d\eta_a^{I'} d\eta^{+aI} A_{II'}^{(g,1)} + d\eta_n^I d\eta^{+nI'} A_{II'}^{(g,1)} + db_n db^{+n} . \quad (6.43)$$

If we take another adapted system of coordinates $\{y^\alpha = y^\alpha(x^a, x^n), y^\nu = y^\nu(x^a, x^n)\}$ with y^α being the coordinates along C and y^ν transverse to C , then the law for the corresponding transformation of the reduced coordinates can be derived from (4.18). We will not explicitly write it here; we simply remark that beyond the matrix $(\partial y / \partial x)$, it involves all matrices $A_{II'}^{(g,1)}$, $B_{I'J'}^{(g,1)}$ and $\lambda_{I'}^J$ introduced above.

We give here a non canonical description of the reduced BV manifold that depends on the choice of a tubular neighborhood for C . We recall that a *tubular neighborhood* for C is an embedding $j : NC \rightarrow M$, where NC is the normal bundle of C inside M , such that $j(C) = C$. Let $X_{\Sigma_{g,1}}$ be the sheaf obtained by putting over a point the commutative graded algebra $H_{dR}(\Sigma_{g,1})$.

Proposition 4 For any choice of a tubular neighborhood of C inside M , there exists an isomorphism between the reduced BV-manifold and $T^*[-1](\text{Maps}(X_{\Sigma_{g,1}}, N^*[1]C))$. Such isomorphism is canonical in the case $g = 0$, i.e. $\Sigma = D^1$, when the reduced BV manifold is $T^*[-1]N^*[1]C$.

Proof. The superfields for the graded manifold $\text{Maps}(X_{\Sigma_{g,1}}, N^*[1]C)$ are

$$x^a = x^a + c_I \eta^{+Ia}, \quad e_n = b_n + c_I \eta_n^I.$$

One can easily check that the degree of the momenta are correct. Let us choose the tubular neighborhood of C . A trivialization of NC with (t_n^ν) as transition functions of NC , defines the atlas of adapted coordinates $\{x^a, x^n\}$, where $\{x^a\}$ are coordinates of C and $\{x^n\}$ are coordinates on the fibre. The change of coordinates is given by

$$y^\alpha = y^\alpha(x^a), \quad y^\nu = t_n^\nu(x^a)x^n.$$

It is tedious but straightforward to check that the rules of change of variables defined in $\text{Maps}(X_{\Sigma_{g,1}}, N^*[1]C)$ together with those of momenta are the same that we get from (4.18). \square

The reduced BV action reads as

$$\begin{aligned} S_{BV} = & x_a^+ \alpha^{an} b_n - \frac{1}{2} b^{+m} \partial_m \alpha^{np} b_n b_p + \frac{1}{2} \alpha^{ab} \eta_a^{I'} \eta_b^{J'} B_{I'J'}^{(g,1)} + \alpha^{am} \eta_a^{I'} \eta_m^J A_{I'J}^{(g,1)} + \\ & \partial_b \alpha^{na} \eta^{+bI} \eta_a^{J'} b_n A_{IJ'}^{(g,1)} + \partial_m \alpha^{na} \eta^{+mI'} \eta_a^{J'} b_n B_{I'J'}^{(g,1)} - \partial_m \alpha^{np} \eta^{+mI'} \eta_p^J b_n A_{JJ'}^{(g,1)} + \\ & - \frac{1}{4} \partial_p \partial_q \alpha^{mn} b_m b_n \eta^{+pI'} \eta^{+qJ'} B_{I'J'}^{(g,1)} - \frac{1}{2} \partial_a \partial_p \alpha^{mn} b_m b_n \eta^{+aI} \eta^{pI'} A_{I'J'}^{(g,1)}, \end{aligned}$$

which automatically satisfies the classical master equation.

Remark 5 Apparently, there is no natural description of these solutions of BV equation as AKSZ-actions.

Recall that $\dim H_{dR}^1(\Sigma_{g,n}) = \dim H_{dR}^1(\Sigma_{g,n}, \partial\Sigma_{g,n}) = 2g + n - 1$. Any choice of volume form $\Omega = \rho_\Omega dx^a dx^n$ on M defines the berezinian

$$\mu_\Omega = \rho_\Omega^{2(1-2g)} dx^a dx_a^+ db_n db^{+n} d\eta d\eta^+$$

and the BV-generator

$$\Delta_\Omega = \frac{\partial}{\partial x_a^+} \frac{\partial}{\partial x^a} - \frac{\partial}{\partial b^{+n}} \frac{\partial}{\partial b_n} + A^{(g,1)I'J} \left(\frac{\partial}{\partial \eta_a^{I'}} \frac{\partial}{\partial \eta^{+aJ}} - \frac{\partial}{\partial \eta_m^J} \frac{\partial}{\partial \eta^{+mI'}} \right) + (1-2g) \{ \log \rho_\Omega, - \}.$$

We easily compute

$$\Delta_\Omega S_{BV} = (1-2g)(\partial_a \alpha^{an} + \partial_m \alpha^{mn} + \partial_a \log \rho_\Omega \alpha^{an}) b_n = (1-2g) \chi_{\Omega, N^*C}^n b_n,$$

where $\chi_{\Omega, N^*C} \in NC = T_C M / TC$ is a representative of the modular class of the Lie algebroid N^*C . We conclude that the solution of the classical master equation S_{BV} satisfies the quantum master equation if and only if N^*C is unimodular.

7 Abstract AKSZ models from dg Frobenius algebras

Inspired by the previous consideration we suggest the extension of the AKSZ idea. The AKSZ construction admits the following straightforward generalization: loosely, one can use any differential graded (“dg”) Frobenius algebra (or a sheaf of dg algebras such that the global sections carry the structure of the Frobenius algebra) instead of differential forms on the worldsheet. In this section we sketch two versions of the construction — vector space version and sheaf version.

7.1 Vector space version

The source: A unital dg Frobenius algebra \mathcal{C} , i.e. a \mathbb{Z} -graded vector space² $\mathcal{C} = \mathcal{C}^0 \oplus \dots \oplus \mathcal{C}^{n+1}$ endowed with (super-)commutative associative multiplication $m : S^2\mathcal{C} \rightarrow \mathcal{C}$ of degree 0, differential $D : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet+1}$ of degree 1 and non-degenerate symmetric pairing $\Pi : S^2\mathcal{C} \rightarrow \mathbb{R}$ of degree $-n - 1$, satisfying the following axioms:

- Degree properties:

$$\begin{aligned} |m(u, v)| &= |u| + |v| \\ |Du| &= |u| + 1 \\ \Pi(u, v) \neq 0 &\text{ implies } |u| + |v| = n + 1 \end{aligned}$$

- Symmetry properties:

$$\begin{aligned} m(u, v) &= (-1)^{|u|\cdot|v|} m(v, u) \\ \Pi(u, v) &= (-1)^{|u|\cdot|v|} \Pi(v, u) \end{aligned}$$

- Poincaré, Leibniz and associativity identities for differential and multiplication:

$$\begin{aligned} D^2 &= 0 \\ Dm(u, v) &= m(Du, v) + (-1)^{|u|} m(u, Dv) \\ m(m(u, v), w) &= m(u, m(v, w)) \end{aligned}$$

- Multiplication is cyclic w.r.t. the pairing:

$$\Pi(u, m(v, w)) = \Pi(m(u, v), w)$$

²Our convention is that *elements* of \mathcal{C}^i have degree i (not coordinates on \mathcal{C}^i).

- Differential is skew-symmetric w.r.t. the pairing:

$$\Pi(Du, v) + (-1)^{|u|}\Pi(u, Dv) = 0$$

- Pairing Π is non-degenerate, i.e. induces an isomorphism

$$\mathcal{C}^\bullet \xrightarrow{\sim} (\mathcal{C}^{n+1-\bullet})^*$$

Here we assume that $u, v, w \in \mathcal{C}$ are homogeneous elements and $|\dots|$ denotes the degree of an element. Also, we denote the unit of \mathcal{C} by $\mathbf{1} \in \mathcal{C}^0$.

Equivalently, one can describe \mathcal{C} as a unital (super-)commutative dg algebra with trace $\text{Tr} : \mathcal{C}^{n+1} \rightarrow \mathbb{R}$ (and extended by zero on lower degree components of \mathcal{C}) satisfying

$$\text{Tr}(Du) = 0$$

and such that the pairing $\text{Tr}(m(\bullet, \bullet))$ is non-degenerate. The trace is constructed from the pairing as $\text{Tr}(u) = \Pi(\mathbf{1}, u)$ (and vice versa, pairing can be constructed from the trace as $\Pi(u, v) = \text{Tr}(m(u, v))$).

The target: A \mathbb{Z} -graded vector space \mathcal{W} endowed with a (constant) symplectic form $\omega \in S^2(\mathcal{W}[1])^*$ of degree n and a function $\Theta \in S^\bullet(\mathcal{W}^*)$ of degree $n+1$, satisfying $\{\Theta, \Theta\} = 0$.

The space of BV fields of abstract AKSZ model is defined in this setting to be the \mathbb{Z} -graded vector space

$$\mathcal{F} = \mathcal{C} \otimes \mathcal{W} . \tag{7.44}$$

Let $\{e_\zeta\}$ be a basis in \mathcal{C} and $\{\tau_A\}$ be a basis in \mathcal{W} (we denote the corresponding coordinates on \mathcal{W} by $\{X^A\}$). Then $\{e_\zeta \otimes \tau_A\}$ is the basis in \mathcal{F} and we denote the corresponding coordinates on \mathcal{F} by $\{\Phi^{A\zeta}\}$. The degree (ghost number) of $\Phi^{A\zeta}$ is $-|e_\zeta| + |X^A|$. If the symplectic form ω on \mathcal{W} is $\omega = dX^A \omega_{AB} dX^B$, then the degree -1 symplectic form Ω on \mathcal{F} is defined as

$$\Omega = \text{Tr}(\delta\Phi^A \omega_{AB} \delta\Phi^B) = (-1)^{(|A|+1)\cdot|\zeta|+n+1} \delta\Phi^{A\zeta} \Pi(e_\zeta, e_\eta) \omega_{AB} \delta\Phi^{B\eta} ,$$

where $\Phi^A := \Phi^{A\zeta} e_\zeta$ (and we use multiplication m under trace implicitly) and we use the obvious shorthand notation $|A| = |X^A|$, $|\zeta| = |e_\zeta|$. The abstract AKSZ action is

$$S = S_{kin} + S_{int} = \underbrace{\frac{1}{2} \text{Tr}(\Phi^A \omega_{AB} D\Phi^B)}_{=(-1)^{|A|\cdot(|\zeta|+1)} \frac{1}{2} \Phi^{A\zeta} \Pi(e_\zeta, D e_\eta) \omega_{AB} \Phi^{B\eta}} + (-1)^{n+1} \text{Tr}(\Phi^*(\Theta)) , \tag{7.45}$$

where $\Phi^* : S^\bullet(\mathcal{W}^*) \rightarrow \mathcal{C}$ is the ring homomorphism induced by the field $\Phi \in \mathcal{C} \otimes \mathcal{W} \cong \text{Hom}(\mathcal{W}^*, \mathcal{C})$ (i.e. we first interpret Φ as a map of graded vector spaces from \mathcal{W}^* to \mathcal{C} and

then extend it as a ring homomorphism, according to free multiplication in $S^\bullet(\mathcal{W}^*)$ and multiplication m in \mathcal{C}). In coordinates: if

$$\Theta = \Theta_A X^A + \frac{1}{2} \Theta_{AB} X^A X^B + \dots \quad (7.46)$$

then

$$S_{int} = (-1)^{n+1} \Theta_A \text{Tr}(e_\zeta) \Phi^{A\zeta} + (-1)^{(|A|+1) \cdot |\zeta| + n+1} \frac{1}{2} \Theta_{AB} \text{Tr}(m(e_\zeta, e_\eta)) \Phi^{A\zeta} \Phi^{B\eta} + \dots \quad (7.47)$$

We claim that the action (7.45) satisfies classical master equation $\{S, S\} = 0$ w.r.t. the anti-bracket on \mathcal{F} associated to the odd symplectic form Ω .

Remark 6 The coordinate-free version of the construction above is as follows. We have a ring homomorphism $\text{ev}^* : S^\bullet(\mathcal{W}^*) \rightarrow S^\bullet(\mathcal{F}^*) \otimes \mathcal{C}$ defined on generators as the canonical map $\mathcal{W}^* \rightarrow \mathcal{F}^* \otimes \mathcal{C}$ associated to the identity $\mathcal{F} \rightarrow \mathcal{C} \otimes \mathcal{W} = \mathcal{F}$. (Notation ev^* should remind of the pull-back by evaluation map $\text{ev} : \text{Maps}(\mathcal{N}, \mathcal{M}) \times \mathcal{N} \rightarrow \mathcal{M}$ in usual AKSZ construction which goes as $\text{ev}^* : C^\infty(\mathcal{M}) \rightarrow C^\infty(\text{Maps}(\mathcal{N}, \mathcal{M})) \otimes C^\infty(\mathcal{N})$.) The interaction part of action (7.45) is then

$$S_{int} = (\text{id} \otimes \text{Tr}) \circ \text{ev}^*(\Theta) .$$

We can also formally extend ev^* to differential forms on \mathcal{W} as a homomorphism of dg algebras $\text{ev}^* : \Omega^\bullet(\mathcal{W}) \rightarrow \Omega^\bullet(\mathcal{F}) \otimes \mathcal{C}$ (here \mathcal{C} is treated as an algebra with zero de Rham differential). Then the odd symplectic form on \mathcal{F} is given by

$$\Omega = (\text{id} \otimes \text{Tr}) \circ \text{ev}^*(\omega) .$$

The kinetic part of action (7.45) is defined as the Hamiltonian function for the cohomological vector field on \mathcal{F} , induced by the differential $D : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet+1}$.

Example 7 (*AKSZ with target a vector space.*) Usual AKSZ models on the space $\text{Maps}(\mathcal{N}, \mathcal{M})$, in the case when \mathcal{M} is a graded vector space with constant symplectic form, can be interpreted as abstract AKSZ with $\mathcal{W} = \mathcal{M}$ and $\mathcal{C} = C^\infty(\mathcal{N})$.

Example 8 (*abstract Chern-Simons.*) Taking arbitrary \mathcal{C} with $n = 2$ (i.e. concentrated in degrees 0,1,2,3 and with pairing of degree -3), taking $\mathcal{W} = \mathfrak{g}[1]$ for a quadratic Lie algebra \mathfrak{g} with invariant pairing $\pi_{\mathfrak{g}}$ and setting

$$\omega = dX^A \pi_{\mathfrak{g}}(\tau_A, \tau_B) dX^B \quad , \quad \Theta = \frac{1}{6} \pi_{\mathfrak{g}}(\tau_A, [\tau_B, \tau_C]) X^A X^B X^C$$

we obtain abstract Chern-Simons in the sense of [11].

In general, in this way (i.e. by allowing \mathcal{C} to be an arbitrary dg Frobenius algebra with pairing of appropriate degree, instead of demanding that it is of form $C^\infty(\mathcal{N})$) we can construct abstract versions of AKSZ models with vector space targets.

7.2 Sheaf version

We recall first some basic definition that can be found in [13]. A ringed space is a couple (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is sheaf of rings on X , see [13]. We denote with $\mathcal{O}_X(U)$ the local sections on the open $U \subset X$. A graded manifold \mathcal{M} is a ringed space $(\mathcal{M}_0, \mathcal{M})$ such that $\mathcal{M}(U)$ is locally isomorphic to $C^\infty(U) \otimes S(V^*)$, for some open $U \subset \mathcal{M}_0$ and graded vector space V .

Definition 9 *A morphism of ringed spaces from (X, \mathcal{O}_X) to (Y, \mathcal{O}_Y) is a couple (ϕ, Φ) , where $\phi : X \rightarrow Y$ and $\Phi : \mathcal{O}_Y \rightarrow \phi_*\mathcal{O}_X$ is a morphism of sheaves on Y .*

The pushforward sheaf is defined as the sheaf on Y with local sections $\phi_*\mathcal{O}_X(U) = \mathcal{O}_X(\phi^{-1}U)$, for any open $U \subset Y$. The morphism (ϕ, Φ) assigns to any open $U \subset Y$ a ring morphism $\Phi^*(U) : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\phi^{-1}U)$.

The abstract AKSZ construction depends on the following data.

The source: A sheaf \mathcal{J} of dg supercommutative algebras over some closed manifold \mathcal{N}_0 , such that locally, for some open $U \subset \mathcal{N}_0$, we have

$$\mathcal{J}(U) \cong \Omega^\bullet(U) \otimes \mathcal{C} , \quad (7.48)$$

where $\Omega^\bullet(U)$ is the algebra of differential forms on U , \mathcal{C} is some fixed finite dimensional unital dg Frobenius algebra with differential D , product m and pairing Π of degree $-n_{\mathcal{C}}$ (with Tr the corresponding trace). We denote the unit of \mathcal{C} by $\mathbf{1}$ and impose that \mathcal{C} is equipped with the splitting

$$\mathcal{C} = \mathbb{R} \cdot \mathbf{1} \oplus \bar{\mathcal{C}} , \quad (7.49)$$

where $\bar{\mathcal{C}}$ is an ideal. The ring $\Omega^\bullet(U) \otimes \mathcal{C} = C^\infty(U) \oplus \dots$ inherits the splitting and the restriction homomorphisms must respect the splitting. Moreover we require the existence of a morphism of sheaves of complexes

$$\text{Tr}_J : \mathcal{J} \rightarrow \Omega^\bullet ,$$

where Ω^\bullet is sheaf of differential forms. This morphism has a degree $-n_{\mathcal{C}}$ and locally has the form $\text{id}_U \otimes \text{Tr}$. Therefore the set of the global sections of \mathcal{J} is equipped with the structure of dg Frobenius algebra with the trace given by the composition of Tr_J and integration of differential forms over \mathcal{N}_0 and with total differential $d_{\mathcal{N}_0} + D$.

The target: A \mathbb{Z} -graded manifold \mathcal{M} with body \mathcal{M}_0 , equipped with symplectic form $\omega \in \Omega^2(\mathcal{M})$ of degree n and a function $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$, satisfying $\{\Theta, \Theta\} = 0$. Here $n := \dim \mathcal{N}_0 + n_{\mathcal{C}} - 1$.

We define the space of BV fields $\mathcal{F} = \text{Maps}(\mathcal{J}, \mathcal{M})$ as the space of morphisms of ringed spaces from $(\mathcal{N}_0, \mathcal{J})$ to $(\mathcal{M}_0, \mathcal{M})$. Let $(\phi, \Phi) \in \mathcal{F}$ and let $U \subset \mathcal{M}_0$ be a coordinate neighborhood such that $\mathcal{M}(U) \sim C^\infty(U) \otimes S(V^*)$. Let $\{e_\zeta\}$ be some basis in \mathcal{C} with $e_0 = \mathbf{1}$ and $e_\zeta \in \bar{\mathcal{C}}$ for $\zeta \neq 0$; let $\{X^A\} = \{x^\mu; \xi^m\}$ be local coordinates on \mathcal{M} (i.e. $\{x^\mu\}$ are coordinates on $U \subset \mathcal{M}_0$ and $\{\xi^m\}$ are coordinates on V), and let $\{u^a\}$ be local coordinates on \mathcal{N}_0 . Then the BV field $(\phi, \Phi) \in \mathcal{F}$ is locally described by the *superfields* that are the values of $\Phi^*(U)$ on the generators $\{X^A\}$ of $\mathcal{M}(U)$:

$$\Phi^*(U) : X^A \mapsto \Phi_0^{A\zeta}(u)e_\zeta + \Phi_a^{A\zeta}(u)\theta^a e_\zeta + \Phi_{a_1 a_2}^{A\zeta}(u)\theta^{a_1}\theta^{a_2} e_\zeta + \dots = \Phi^{A\zeta}(u, \theta)e_\zeta . \quad (7.50)$$

Here $\theta^a := d_{\mathcal{N}_0} u^a$ are the odd generators of $\Omega^\bullet(\phi^{-1}(U))$. The splitting condition (7.49) allows to define the coefficients of 1 that define a local map $\phi_U : \phi^{-1}(U) \rightarrow U$ as $x^\mu \circ \phi_U(u) = \Phi_0^{\mu 0}(u)$. By using a standard argument we see that $\phi_U(u) = \phi(u)$, i.e. we recover the global map ϕ .

Let us denote the symplectic form as $\omega = dX^A \omega_{AB} dX^B$; we construct the degree -1 symplectic form Ω on \mathcal{F} as

$$\begin{aligned} \Omega &= \int_{\mathcal{N}_0} \text{Tr} (\delta\Phi^A \omega_{AB} \delta\Phi^B) = \\ &= (-1)^{(|A|+\dim\mathcal{N}_0+1)\cdot|\zeta|+(n+1)n_{\mathcal{C}}} \int_{\mathcal{N}_0} \delta\Phi^{A\zeta} \Pi(e_\zeta, e_\eta) \omega_{AB} \delta\Phi^{B\eta} , \end{aligned} \quad (7.51)$$

where $\Phi^A = \Phi^*(X^A)$ is the right hand side of (7.50) and the expression under trace in the first line implicitly uses the multiplication m in \mathcal{C} . The abstract AKSZ action is

$$S[\Phi] = \underbrace{\frac{1}{2} \int_{\mathcal{N}_0} \text{Tr} (\Phi^A \omega_{AB} (d_{\mathcal{N}_0} + D)\Phi^B)}_{S_{kin}} + (-1)^{n+1} \underbrace{\int_{\mathcal{N}_0} \text{Tr}(\Phi^*(\Theta))}_{S_{int}} . \quad (7.52)$$

The kinetic term may be also written as

$$\begin{aligned} S_{kin} &= (-1)^{(|A|+\dim\mathcal{N}_0)\cdot|\zeta|+n_{\mathcal{C}} \dim\mathcal{N}_0} \cdot \int_{\mathcal{N}_0} \left((-1)^{|A|+\dim\mathcal{N}_0} \frac{1}{2} \Phi^{A\zeta} \Pi(e_\zeta, D e_\eta) \omega_{AB} \Phi^{B\eta} + \right. \\ &\quad \left. + (-1)^{|\zeta|} \frac{1}{2} \Phi^{A\zeta} \Pi(e_\zeta, e_\eta) \omega_{AB} d_{\mathcal{N}_0} \Phi^{B\eta} \right) . \end{aligned} \quad (7.53)$$

If Θ is expanded in local coordinates on \mathcal{M} as (7.46), then the expansion analogous to (7.47) in this setting is

$$S_{int} = (-1)^{n+1+n_{\mathcal{C}}} \int_{\mathcal{N}_0} \Theta_A \text{Tr}(e_\zeta) \Phi^{A\zeta} + (-1)^{(|A|+\dim\mathcal{N}_0+1)\cdot|\zeta|} \frac{1}{2} \Theta_{AB} \text{Tr}(m(e_\zeta, e_\eta)) \Phi^{A\zeta} \Phi^{B\eta} + \dots$$

We again claim that $\{S, S\} = 0$.

Example 10 Taking the trivial dg Frobenius algebra $\mathcal{C} = \mathbb{R}$, we obtain standard AKSZ models on $\text{Maps}(T[1]\mathcal{N}_0, \mathcal{M})$.

Example 11 Taking $\mathcal{N}_0 = pt$, specifying some finite dimensional dg Frobenius algebra \mathcal{C} with splitting (7.49) and a finite-dimensional target $(\mathcal{M}, \omega, \Theta)$, we obtain a finite-dimensional abstract AKSZ model. A particular class of such sources is provided by the de Rham cohomology of connected closed orientable manifolds $\mathcal{C} = H_{dR}^\bullet(\Sigma)$, viewed as dg Frobenius algebras with $D = 0$ and Π associated to Poincaré duality. Since Σ is connected, splitting (7.49) is automatic: $\mathcal{C} = \mathbb{R} \cdot \mathbf{1} \oplus H_{dR}^{\geq 1}(\Sigma)$. For these models $S_{kin} = 0$, and the AKSZ action is just the pull-back of the function Θ on the target. These are the examples that we studied in (4.31) when Σ is two-dimensional and in (5.38) when Σ is three-dimensional.

Example 12 (*Source given by fiber cohomology of a fiber bundle.*) Suppose E is a fiber bundle over \mathcal{N}_0 with typical fiber F (closed connected orientable manifold), endowed with a flat connection ∇_E . Then we define fiber differential $d_{\text{fib}} : \Omega^\bullet(E) \rightarrow \Omega^{\bullet+1}(E)$ as follows: for 0-forms $f \in C^\infty(E)$ we define d_{fib} by the property $i_v(d_{\text{fib}}f) := v_\perp(f)$ (where i_v is the convolution with arbitrary vector field $v \in \text{Vect}(E)$ and the projection to fiber $v \mapsto v_\perp$ is the projection to second term in the splitting of tangent bundle $TE = T_{\parallel}E \oplus T_{\perp}E$ defined by the connection ∇_E); then we extend d_{fib} to all forms on E by Leibniz rule and property $d_E d_{\text{fib}} + d_{\text{fib}} d_E = 0$ (where d_E is the de Rham differential on E). Flatness of ∇_E implies that $d_{\text{fib}}^2 = 0$. Then we construct sheaf \mathcal{J} over \mathcal{N}_0 as the cohomology of d_{fib} :

$$\Gamma(\mathcal{N}_0, \mathcal{J}) = H_{d_{\text{fib}}}^\bullet(\Omega^\bullet(E)) .$$

Locally \mathcal{J} splits as (7.48) with $\mathcal{C} = H^\bullet(F)$ (de Rham cohomology of the fiber).

In particular we can take a trivial fiber bundle $E = \mathcal{N}_0 \times F$ with canonical flat connection ∇_E . Then d_{fib} is just the de Rham differential along fiber $d_{\text{fib}} = \text{id}_{\mathcal{N}_0} \otimes d_F$ and the fiber cohomology sheaf \mathcal{J} splits globally:

$$\Gamma(\mathcal{N}_0, \mathcal{J}) = \Omega^\bullet(\mathcal{N}_0) \otimes H^\bullet(F) .$$

Abstract AKSZ models with source \mathcal{J} and some target $(\mathcal{M}, \omega, \Theta)$ may in some cases arise as a partial reduction of usual AKSZ models on $\text{Maps}(T[1]E, \mathcal{M})$. The simplest example here is: E is the 2-torus, viewed as a trivial bundle over circle $\mathcal{N}_0 = \mathbb{S}^1$ with fiber a circle $F = \mathbb{S}^1$, and \mathcal{M} is a Poisson manifold. The corresponding abstract AKSZ model is a partial reduction of Poisson sigma model on torus.

We hope to explore this set of examples in more detail in a future publication.

Remark 13 Global sections $\Gamma(\mathcal{N}_0, \mathcal{J})$ of the sheaf \mathcal{J} themselves form a dg Frobenius algebra with differential $D_{\mathcal{J}} = d_{\mathcal{N}_0} + D$ (where $d_{\mathcal{N}_0}$ is the de Rham differential on \mathcal{N}_0), multiplication $m_{\mathcal{J}}$ coming from wedge product of forms and multiplication m on \mathcal{C} , and with the pairing $\Pi_{\mathcal{J}}(\chi, \psi) = \int_{\mathcal{N}_0} \text{Tr}(m(\chi, \psi))$. When the target is a graded vector space, we can apply the vector space construction of Subsection 7.1 with $\Gamma(\mathcal{N}_0, \mathcal{J})$ as source. We easily verify that the two constructions coincide. Remark that requirement (7.49) is not needed in the vector space version, so that more general Frobenius algebras are allowed in the vector space version.

8 Conclusions

In this paper we studied a canonical reduction of the AKSZ-BV field theory to a finite dimensional BV theory which governs the semi-classical approximation. As illustration of the general construction, we discussed the two dimensional Poisson sigma model and the three dimensional Courant sigma model.

Our main perspective has been the odd symplectic reduction of the infinite dimensional manifold of fields. It is important to remark that one can look at the reduced action S_{BV} as the leading contribution in the effective BV action which controls the low energy fields (we can call them either "constant" maps or zero modes). It is convenient to consider the idea of effective BV theories suggested by Losev [20] (see [21], [5] and [10] for further developments). In fact, given any embedding of the cohomology in the space of forms of the source, one can look at the reduced variables as "infrared" degrees of freedom of the full theory. The effective action is then defined by integrating over the "ultraviolet" degrees of freedom and, in the perturbative approach, is a series in \hbar and in the hamiltonian function Θ of the target. From this point of view, the reduced BV manifolds that we studied in this paper are the spaces of the infrared degrees of freedom and the action S_{BV} is the lowest order in the expansion of the effective action. In principle one can calculate the corrections by applying Feynman diagrams techniques.

From the examples considered in this paper, it is natural to consider the generalization of the AKSZ construction [1]. In Section 4.2 we observed that the reduced BV theory can be described in terms of "supermaps" to the target graded manifold. The novelty is that the formal variables of the source manifold have to satisfy some constraints and thus they cannot be anymore considered as the coordinates of a graded manifold. This is the generalization of the AKSZ construction that we introduced in Section 7. The examples

that appeared in this paper have sources that are commutative graded algebras seen as sheaves over a point and so are zero dimensional TFT's. One can consider generalized AKSZ theories in any dimension, as we described for instance in the Example (12), that will be the object of future study.

Finally, one can consider more general type of BV reductions, not necessary to the constant map configurations. Moreover many ideas presented here can be applied to a wider setup than simply AKSZ-BV theories. For example, it could be interesting to study the reduction of the two dimensional BV theories described in [27, 28].

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A Computation of Berezinian (5.39)

We show here that the volume form introduced in (5.39) is globally defined on $\text{Map}(X_\Sigma, \mathcal{M})$. The coordinates $z = (\mathbf{X}^\mu, \boldsymbol{\xi}_a, \mathbf{P}_\mu)$ defined as coefficients of the superfields (5.36) depend on the choice of coordinates $\{x^\mu\}$ on M and of a trivialization $\{e_a\}$ of E . If we change to coordinates $\{y^i = y^i(x)\}$ on M and to trivialization $\{e_\alpha = t_\alpha^a(x)e_a\}$, the coordinates on $\text{Map}(X_\Sigma, \mathcal{M})$ change to $\tilde{z} = (\mathbf{X}^i, \boldsymbol{\xi}_\alpha, \mathbf{P}_i)$ accordingly as

$$\mathbf{X}^i = y^i(\mathbf{X}), \quad \boldsymbol{\xi}^\alpha = t_\alpha^a(\mathbf{X})\boldsymbol{\xi}^a, \quad \mathbf{P}_i = \frac{\partial x^\mu}{\partial y^i}(\mathbf{X})\mathbf{P}_\mu + \frac{1}{2}\boldsymbol{\xi}^a\boldsymbol{\xi}^b\frac{\partial t_a^\alpha}{\partial y^i}(\mathbf{X})g_{\alpha\beta}t_b^\beta(\mathbf{X}). \quad (\text{A.54})$$

The quadratic term in the transformation of \mathbf{P}_i can be removed by introducing a connection on the vector bundle E . In fact, the coordinate $\mathbf{P}_\mu^\Gamma = \mathbf{P}_\mu + \frac{1}{2}\Gamma_{\mu a}^b\boldsymbol{\xi}^a g_{bc}\boldsymbol{\xi}^c$ transforms as a tensor

$$\mathbf{P}_i^\Gamma = \frac{\partial x^\mu}{\partial y^i}(\mathbf{X})\mathbf{P}_\mu^\Gamma.$$

It can be easily checked that the Berezinian of the transformation from $\{\mathbf{X}^\mu, \boldsymbol{\xi}^a, \mathbf{P}_\mu\}$ to $\{\mathbf{X}^\mu, \boldsymbol{\xi}^a, \mathbf{P}_\mu^\Gamma\}$ is one so that the coordinate volume forms are the same. Equivalently, in

order to compute the Berezinian of the transformation $\tilde{z}(z)$ we are allowed to ignore the quadratic term in the transformation of \mathbf{P} in (A.54).

The final result is that the berezinian of the transformation matrix $I = \left(\frac{\partial \tilde{z}}{\partial z}\right)$ is

$$\text{Ber} I = \text{Ber} \begin{pmatrix} I_{00} & I_{01} \\ I_{10} & I_{11} \end{pmatrix} = (\det t_a^\alpha)^{2(1-b_1)}.$$

We first compute it with respect to the transformation $y^i = y^i(x)$ with fixed trivialization. Let us order the relevant coordinates $z = \{z_0, z_1\}$, where z_0 are the even ones and z_1 are the odd ones, as $z_0 = (x^\mu, \gamma_\mu, F_{\mu I}, \alpha_I^{+\mu})$ and $z_1 = (\alpha_\mu^I, F^{+\mu I}, \gamma^{+\mu}, x_\mu^+)$. It is important to see where the zeros are located in I , so that at the end only few matrix elements (the diagonal ones) enter the result. By inspection of the degree in (A.54) we can easily write

$$I_{00} = \frac{\partial \tilde{z}_0}{\partial z_0} = \begin{pmatrix} \frac{\partial y^i}{\partial x^\mu} & 0 & 0 & 0 \\ * & \frac{\partial \gamma_i}{\partial \gamma_\mu} & 0 & 0 \\ & * & \frac{\partial F_{iI}}{\partial F_{\mu J}} & * \\ & & 0 & \frac{\partial \alpha_I^{+i}}{\partial \alpha_J^{+\mu}} \end{pmatrix} \quad I_{01} = \frac{\partial \tilde{z}_0}{\partial z_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & * & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$I_{11} = \frac{\partial \tilde{z}_1}{\partial z_1} = \begin{pmatrix} \frac{\partial \alpha_i^I}{\partial \alpha_\mu^J} & * & 0 & 0 \\ 0 & \frac{\partial F^{+iI}}{\partial F^{+\mu J}} & 0 & 0 \\ & * & \frac{\partial \gamma^{+i}}{\partial \gamma^{+\mu}} & 0 \\ & & * & \frac{\partial x_{+i}}{\partial x_{+\mu}} \end{pmatrix} \quad I_{10} = \frac{\partial \tilde{z}_1}{\partial z_0} = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & \end{pmatrix},$$

where the block structure is easily understood. It is then easy to compute that

$$I_{01} I_{11}^{-1} I_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & * & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

so that

$$\text{Ber} I = \det(I_{00} - I_{01} I_{11}^{-1} I_{10}) / \det I_{11} = \det I_{00} / \det I_{11} = 1.$$

Consider now the change of trivialization $e_\alpha = t_\alpha^a(x) e_a$, without changing coordinates $\{x^\mu\}$. Let us order the relevant coordinates as follows: the even ones are $z_0 = (x^\mu, \alpha_I^{+\mu}, A^{aI}, \beta_a^+)$ and the odd ones are $z_1 = (F^{+\mu I}, \gamma^{+\mu}, \beta^a, A_{aI}^+)$. Since we are ignoring the quadratic terms in (A.54) the coefficients of \mathbf{P}_μ do not appear. Then we compute

$$I_{00} = \begin{pmatrix} \delta_\nu^\mu & 0 & 0 & 0 \\ 0 & \delta_J^I \delta_\nu^\mu & 0 & 0 \\ & * & \frac{\partial A^{aI}}{\partial A^{aJ}} & 0 \\ & & * & \frac{\partial \beta_a^+}{\partial \beta_a^+} \end{pmatrix} \quad I_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & * & * & 0 \\ & & * & * \end{pmatrix}$$

$$I_{11} = \begin{pmatrix} \delta_\nu^\mu \delta_J^I & 0 & 0 & 0 \\ 0 & \delta_\mu^\nu & 0 & 0 \\ * & \frac{\partial \beta^\alpha}{\partial \beta^a} & 0 & \\ * & * & \frac{\partial A_{\alpha I}^+}{\partial A_{a I}^+} & \end{pmatrix} \quad I_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & \\ * & * & 0 & \end{pmatrix}.$$

We then compute that

$$I_{01} I_{11}^{-1} I_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & \\ * & * & 0 & \end{pmatrix}$$

and finally we get

$$\text{Ber} I = \det(I_{00} - I_{01} I_{11}^{-1} I_{10}) / \det I_{11} = \det I_{00} / \det I_{11} = (\det t_a^\alpha)^{2(b_1-1)}.$$

B Relative (co)homology

We recall in this appendix basic facts about relative (co)homology and Lefschetz duality for manifolds with boundary [14].

Let Σ be a smooth manifold of dimension d with boundary $\partial\Sigma$. The relative k -chains with real coefficients are defined as $C_k(\Sigma, \partial\Sigma) = C_k(\Sigma) / C_k(\partial\Sigma)$, *i.e.* the chains in Σ modulo the chains in $\partial\Sigma$. We will always work with real coefficient and we will omit it in the notation. The usual boundary ∂ goes to the quotient and defines the relative homology $H_k(\Sigma, \partial\Sigma)$. An alternative description of chains is obtained by defining $C'_k(\Sigma, \partial\Sigma) = \{(c_k, \sigma_{k-1}), c_k \in C_k(\Sigma), \sigma_{k-1} \in C_{k-1}(\partial\Sigma)\}$ with boundary $\partial(c_k, \sigma_{k-1}) = (\partial c_k + (-)^k \sigma_{k-1}, \partial \sigma_{k-1})$. It is easy to check that the map $(c_k, \sigma_{k-1}) \in C'_k(\Sigma, \partial\Sigma) \rightarrow \underline{c}_k \in C_k(\Sigma, \partial\Sigma)$ is a quasisomorphism.

The exact sequence $0 \rightarrow C_k(\partial\Sigma) \rightarrow C_k(\Sigma) \rightarrow C_k(\Sigma, \partial\Sigma) \rightarrow 0$ gives rise to the long exact sequence in homology

$$\dots \rightarrow H_k(\partial\Sigma) \rightarrow H_k(\Sigma) \rightarrow H_k(\Sigma, \partial\Sigma) \rightarrow H_{k-1}(\partial\Sigma) \rightarrow \dots, \quad (\text{B.1})$$

where the last map sends $[\underline{c}] \in H_k(\Sigma, \partial\Sigma) \rightarrow [\partial c] \in H_{k-1}(\partial\Sigma)$, for some $c \in C_k(\Sigma)$.

The complex of relative cochains $C^\bullet(\Sigma, \partial\Sigma)$ can be described as the restriction of de Rham complex to those forms ω whose restriction $\omega|_{\partial\Sigma}$ to the boundary is zero. We denote the relative cohomology as $H_{dR}(\Sigma, \partial\Sigma)$. By the universal coefficient theorem we have that $H_{dR}^\bullet(\Sigma) = H(\Sigma)_*$. The alternative description for k -relative cochains is $C'^k(\Sigma, \partial\Sigma) = \Omega^k \Sigma \oplus \Omega^{k-1} \partial\Sigma$, with differential $d(\omega_k, \nu_{k-1}) = (d\omega_k, d\nu_{k-1} - (-)^k \omega_k|_{\partial\Sigma})$. The map $\omega_k \in$

$C^k(\Sigma, \partial\Sigma) \rightarrow (\omega_k, 0) \in C^k(\Sigma, \partial\Sigma)$ is a quasisomorphism. The pairing is then defined as

$$\langle \omega_k, \underline{c_k} \rangle = \int_{c_k} \omega_k .$$

or alternatively as

$$\langle (\omega_k, \nu_{k-1}), (c_k, \sigma_{k-1}) \rangle = \int_{c_k} \omega_k + \int_{\sigma_{k-1}} \nu_{k-1} .$$

The above notion of relative (co)homology makes sense for any subspace of Σ , in particular we can consider (union of) components of $\partial\Sigma$. If $\partial\Sigma = \cup_{i \in I} \partial_i \Sigma$ and $\partial_J \Sigma = \cup_{i \in J} \partial_i \Sigma$, for $J \subset I$, we will consider the relative homology $H(\Sigma, \partial_J \Sigma)$ and cohomology $H_{dR}(\Sigma, \partial_J \Sigma)$. Let $I = I_1 \cup I_2$, with $I_1 \cap I_2 = \emptyset$; the choice of the fundamental class $[\Sigma] \in H_d(\Sigma, \partial\Sigma)$ determines the following isomorphism (for a proof see Theorem 3.43 of [14])

$$H_k(\Sigma, \partial_{I_1} \Sigma) = H_{d-k}^{dR}(\Sigma, \partial_{I_2} \Sigma) .$$

In particular, the case $I_1 = \emptyset$ or $I_2 = \emptyset$ is known as *Lefschetz duality*,

$$H_{dR}^k(\Sigma, \partial\Sigma) \sim H_{d-k}(\Sigma) , \quad H_{dR}^k(\Sigma) \sim H_{d-k}(\Sigma, \partial\Sigma) .$$

Let us describe more explicitly the case $d = 2$ and consider a compact surface $\Sigma_{g,n}$ of genus g and n boundary components.

Since $H_0(\Sigma_{g,n}, \partial\Sigma_{g,n}) = H_2(\Sigma_{g,n}) = 0$, by Lefschetz duality we get that $H_{dR}^2(\Sigma_{g,n}) = H_{dR}^0(\Sigma_{g,n}, \partial\Sigma_{g,n}) = 0$. The Lefschetz duality in degree one can be seen as the non degeneracy of the pairing $H_{dR}^1(\Sigma_{g,n}) \otimes H_{dR}^1(\Sigma_{g,n}, \partial\Sigma_{g,n}) \rightarrow \mathbb{R}$, $(a, b) \rightarrow \int_{\Sigma_{g,n}} a \wedge b$. Equivalently, for any basis $\{c_I\}$ for $H_{dR}^1(\Sigma_{g,n})$ and $\{g_{I'}\}$ for $H_{dR}^1(\Sigma_{g,n}, \partial\Sigma_{g,n})$ the matrix $A_{II'}^{(g,n)} = \int_{\Sigma_{g,n}} c_I \wedge g_{I'}$ is non degenerate. Let us denote with $A^{(g,n)I'J}$ the inverse matrix. It will be useful even the (possibly degenerate) matrix $B_{I'J'}^{(g,n)} = \int_{\Sigma_{g,n}} g_{I'} \wedge g_{J'}$.

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ONE-DIMENSIONAL CHERN-SIMONS THEORY

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ABSTRACT. We study a one-dimensional toy version of the Chern-Simons theory. We construct its simplicial version which comprises features of a low-energy effective gauge theory and of a topological quantum field theory in the sense of Atiyah.

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1. INTRODUCTION

We begin (see section 2) by considering a one-dimensional version of the Chern-Simons theory on a circle. This is a gauge theory in the Batalin-Vilkovisky formalism defined by the action

$$(1) \quad \frac{1}{2} \int (\psi, d\psi) + (\psi, [A, \psi]),$$

where the field ψ is an odd function on the circle with values in a quadratic Lie algebra \mathfrak{g} , and the field A is an even 1-form with values in \mathfrak{g} . We address the problem of constructing an effective BV action induced on a triangulation of the circle.

This problem is interesting by itself since it is related to discretization of differential geometry. Indeed, the action (1) can be viewed as a generating function for natural operations on differential forms on the circle: the de Rham differential, the wedge product and the integral over the circle (more precisely, it is a generating function for a unimodular cyclic DGLA structure on \mathfrak{g} -valued forms, cf. [7]). In this language, the effective action on a triangulation is a generating function for some discretized (homotopy) version of this structure induced on cochains of the triangulations (viewed as discrete differential forms).

Another motivation for studying the effective action for (1) is that it might give a new insight for constructing a discrete version of the 3-dimensional Chern-Simons theory. Such a discrete Chern-Simons theory would allow to compute invariants of 3-manifolds as finite-dimensional integrals, and it would be compatible with the gauge symmetry (*i.e.* it would satisfy the Batalin-Vilkovisky quantum master equation).

The effective action for the one-dimensional Chern-Simons theory on a triangulated circle turns out to be given by an explicit but somewhat bizarre formula (42). It immediately raises a number of questions. For instance, the result is expected to satisfy the quantum master equation (QME) and to be compatible with simplicial aggregations (merging several 1-simplices of the triangulation). How can we check this directly? Another desire is to represent the result (42) in a “simplicially-local” form.

It turns out that answers to these questions come from the following construction. We give a new definition of the one-dimensional simplicial Chern-Simons theory in the “operator formalism”, *i.e.* in the language of Clifford algebras $Cl(\mathfrak{g})$ (section 3.2.1). The partition function for a simplicial complex is an element of $Cl(\mathfrak{g})^{\otimes i}$ (where i is half the number of boundary points of the simplicial complex), and it given by a product of local $Cl(\mathfrak{g})$ -valued expressions (61) for 1-simplices. In particular, for a triangulated circle the partition function takes values in numbers (and it also depends on simplicial “bulk fields”). In section 4.4.2, we establish the equivalence between the operator formalism and the path integral formalism of section 2. Consistency with simplicial aggregations is checked straightforwardly in the operator language (section 3.2.2). The partition function for an interval can be

shown to satisfy equation (68):

$$\hbar \frac{\partial}{\partial \bar{\psi}^a} \frac{\partial}{\partial A^a} Z_{\mathcal{I}} + \frac{1}{\hbar} \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, Z_{\mathcal{I}} \right]_{Cl(\mathfrak{g})} = 0$$

which is interpreted as a version of the quantum master equation adjusted for the presence of the boundary. This immediately implies the QME with boundary contributions for arbitrary one-dimensional simplicial complex (73) and the usual QME for the triangulated circle (44).

To formulate one-dimensional simplicial Chern-Simons theory in the spirit of Atiyah's axioms of TQFT (section 4.2), we choose a complex polarization of \mathfrak{g} :

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bar{\mathfrak{h}}$$

(which can always be introduced if \mathfrak{g} is even-dimensional; however, by introducing a complex polarization we break the $O(\mathfrak{g})$ -symmetry of the original problem). The Clifford algebra $Cl(\mathfrak{g})$ is isomorphic to the matrix (super-)algebra $\text{End}(\wedge^{\bullet} \mathfrak{h}) = \text{End}(\text{Fun}(\Pi \mathfrak{h}))$. Therefore, the space of states associated to a point in the one-dimensional Chern-Simons theory is $\mathcal{H}_{pt} = \text{Fun}(\Pi \mathfrak{h})$ — the super vector space of polynomials in $(\dim \mathfrak{g})/2$ odd variables. The super-space \mathcal{H}_{pt} is endowed with an odd third-order differential operator δ . One-dimensional “cobordisms” are now equipped with triangulations. To a triangulated cobordism Θ we associate the “space of bulk fields” $\mathcal{F}_{\Theta}^{\text{bulk}}$, equipped with the BV Laplacian $\Delta_{\Theta}^{\text{bulk}}$. The partition function for a triangulated cobordism satisfies the quantum master equation (80). In addition to the operations of gluing and disjoint union (which are standard in Atiyah's picture), simplicial aggregations are allowed for triangulated cobordisms. The original continuum theory can be thought of as the simplicial theory in the limit of dense triangulation.

Matrix elements of the partition function for a triangulated cobordism can be written as path integrals for the one-dimensional Chern-Simons with BV gauge fixing in the bulk and holomorphic-antiholomorphic boundary conditions (section 4.4.2). This brings us back to the formalism of effective BV actions. The action for an interval is given by a Gaussian integral and it is easy to compute it explicitly (115).

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1.2. Authorship. The idea of looking at the one-dimensional Chern-Simons theory and some parts of section 3 (operator formalism approach) are joint work of both authors (the idea to use operator formalism for the one-dimensional Chern-Simons theory was suggested by A. A.). Other parts of the paper are due to P. M.

2. SIMPLICIAL CHERN-SIMONS THEORY ON THE CIRCLE

In this section we study the Chern-Simons theory on the circle in Batalin-Vilkovisky (BV) formalism and construct an effective BV action induced on cochains of a triangulation. Much of this discussion is inspired by [7] and [14]. In particular,

the reader is referred to sections 2 and 3.2 of [7] for details of the effective BV action construction.

2.1. Continuum theory on the circle: fields, BV structure, action. Let \mathfrak{g} be a quadratic Lie algebra with Lie bracket $[\cdot, \cdot]$ and non-degenerate ad-invariant pairing (\cdot, \cdot) . We will denote by $\{T^a\}$ an orthonormal basis in \mathfrak{g} and by $f^{abc} = (T^a, [T^b, T^c])$ the structure constants in this basis. We will also use the Einstein summation convention for the Lie algebra indices.

The Chern-Simons theory on a 3-manifold M can be constructed as an AKSZ sigma model [1] with the space of fields

$$\mathcal{F} = \text{Maps}(\Pi TM, \Pi \mathfrak{g}) = \Pi \mathfrak{g} \otimes \Omega^\bullet(M).$$

That is, \mathcal{F} is the space of maps of super-manifolds from the parity-shifted tangent bundle of M to the parity-shifted Lie algebra. Equivalently, this is the space of differential forms on M with values in $\Pi \mathfrak{g}$. From the canonical integration measure on ΠTM and the even symplectic structure $\omega_{\Pi \mathfrak{g}} = \frac{1}{2} \delta X^a \wedge \delta X^a$ on $\Pi \mathfrak{g}$ (we denote by $\{X^a\}$ the set of odd coordinates on $\Pi \mathfrak{g}$ associated to the orthonormal basis $\{T^a\}$ on \mathfrak{g}) one constructs an odd symplectic form (the ‘‘BV 2-form’’) on \mathcal{F} :

$$(2) \quad \omega = \frac{1}{2} \int_M (\delta \alpha, \delta \alpha).$$

Here the superfield α is the canonical odd map (the parity-shifted identity operator)

$$(3) \quad \alpha : \mathcal{F} \rightarrow \mathfrak{g} \otimes \Omega^\bullet(M)$$

which can be viewed as the generating function for coordinates on \mathcal{F} with values in \mathfrak{g} -valued differential forms on M . By splitting α into components according to the degrees of differential forms we obtain

$$\alpha = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}$$

where $A^{(p)}$ takes values in \mathfrak{g} -valued p -forms.¹ Since α is totally odd, $A^{(0)}$ and $A^{(2)}$ are intrinsically odd, $A^{(1)}$ and $A^{(3)}$ are intrinsically even (the intrinsic parity is the total parity minus the de Rham degree modulo 2). The Chern-Simons action is built of the 1-form $\frac{1}{2} X^a \wedge \delta X^a$ on $\Pi \mathfrak{g}$ (which is a primitive for $\omega_{\Pi \mathfrak{g}}$) and the odd function on $\Pi \mathfrak{g}$

$$(4) \quad \theta = \frac{1}{6} f^{abc} X^a X^b X^c$$

which satisfies $\{\theta, \theta\}_{\Pi \mathfrak{g}} = 0$. The action is given by formula,

$$(5) \quad S = \int_M \frac{1}{2} (\alpha, d\alpha) + \frac{1}{6} (\alpha, [\alpha, \alpha]).$$

By the general construction [1], S satisfies the classical master equation

$$\{S, S\} = 0$$

where $\{\cdot, \cdot\}$ is the BV anti-bracket on functions on \mathcal{F} defined by the odd symplectic form ω .

¹In the BV formalism, $A^{(1)}$ is the ‘‘classical field’’, $A^{(0)}$ is the ‘‘ghost’’; $A^{(2)}$ and $A^{(3)}$ are the ‘‘anti-fields’’ for $A^{(1)}$ and $A^{(0)}$, respectively.

We would like to define the one-dimensional Chern-Simons theory on the circle \mathcal{S}^1 by substituting $M = \mathcal{S}^1$ into the construction described above. Then, the space of fields becomes

$$\mathcal{F} = \text{Maps}(\Pi T\mathcal{S}^1, \Pi\mathfrak{g}) = \Pi\mathfrak{g} \otimes \Omega^0(\mathcal{S}^1) \oplus \Pi\mathfrak{g} \otimes \Omega^1(\mathcal{S}^1)$$

The superfield α can now be written as

$$(6) \quad \alpha = \psi + A,$$

where the component $\psi = T^a\psi^a(\tau)$ takes values in \mathfrak{g} -valued functions on the circle and is intrinsically odd (τ is the coordinate on \mathcal{S}^1); and the component $A = d\tau T^a A^a(\tau)$ takes values in \mathfrak{g} -valued 1-forms on the circle and is intrinsically even. Thus, $\{\psi^a(\tau), A^a(\tau)\}$ are odd and even coordinates on \mathcal{F} , respectively. The space \mathcal{F} is equipped with an odd symplectic structure (2):

$$(7) \quad \omega = \int_{\mathcal{S}^1} (\delta\psi, \delta A),$$

defining the anti-bracket $\{\bullet, \bullet\} : \text{Fun}(\mathcal{F}) \times \text{Fun}(\mathcal{F}) \rightarrow \text{Fun}(\mathcal{F})$,

$$\{f, g\} = \int_{\mathcal{S}^1} d\tau f \left(\frac{\overleftarrow{\delta}}{\delta\psi^a(\tau)} \frac{\overrightarrow{\delta}}{\delta A^a(\tau)} - \frac{\overleftarrow{\delta}}{\delta A^a(\tau)} \frac{\overrightarrow{\delta}}{\delta\psi^a(\tau)} \right) g$$

and the BV Laplacian $\Delta : \text{Fun}(\mathcal{F}) \rightarrow \text{Fun}(\mathcal{F})$

$$\Delta f = \int_{\mathcal{S}^1} d\tau \frac{\delta}{\delta\psi^a(\tau)} \frac{\delta}{\delta A^a(\tau)} f.$$

Note that the operator Δ is ill-defined on local functionals.

The action (5) can be written in terms of components (6) of the superfield as

$$(8) \quad S = \frac{1}{2} \int_{\mathcal{S}^1} ((\psi, d\psi) + (\psi, [A, \psi])).$$

Here d is the de Rham differential on \mathcal{S}^1 . By the general AKSZ construction², the action S satisfies the classical master equation

$$\{S, S\} = 0.$$

Naïvely, one could also say that the unimodularity of \mathfrak{g} implies unimodularity of $\mathfrak{g} \otimes \Omega^\bullet(\mathcal{S}^1)$, and therefore the quantum master equation is fulfilled

$$\frac{1}{2}\{S, S\} + \hbar\Delta S = 0.$$

However, ΔS is ill-defined in continuum theory.

Remark 1. The \mathbb{Z}_2 -grading on the space of fields of the Chern-Simons theory on a 3-manifold can be promoted to a \mathbb{Z} -grading (by setting $\mathcal{F} = \text{Maps}(T[1]M, \mathfrak{g}[1])$) in such a way that the odd symplectic form Ω attains grade³ -1 (so that the anti-bracket has degree $+1$) and the action S is in degree zero. However, this does not apply to the one-dimensional Chern-Simons theory which is essentially \mathbb{Z}_2 -graded: there is no consistent \mathbb{Z} -grading on the space of fields.

2.2. Effective action on the cohomology of the circle.

²Or, in the algebraic language, due to relations (Leibniz identity, Jacobi identity, cyclicity of differential, cyclicity of Lie bracket) in the cyclic dg Lie algebra $\mathfrak{g} \otimes \Omega^\bullet(\mathcal{S}^1)$, cf. [7].

³Grade is defined as the total degree minus the de Rham degree of a differential form.

2.2.1. *Harmonic gauge.* Let's split the space of differential forms on the circle into constant 0- and 1-forms and those with vanishing integral⁴: $\Omega^\bullet(\mathcal{S}^1) = \Omega'^\bullet(\mathcal{S}^1) \oplus \Omega''^\bullet(\mathcal{S}^1)$ where

$$(9) \quad \Omega'^\bullet(\mathcal{S}^1) = \{f + d\tau g \mid f, g \in \mathbb{R}\},$$

$$(10) \quad \Omega''^\bullet(\mathcal{S}^1) = \{f''(\tau) + d\tau g''(\tau) \mid \int_{\mathcal{S}^1} d\tau f''(\tau) = 0, \int_{\mathcal{S}^1} d\tau g''(\tau) = 0\}.$$

It induces the splitting for fields into infrared and ultraviolet parts $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$, where

$$\begin{aligned} \mathcal{F}' &= \{\psi_0 + d\tau A_0 \mid \psi_0 \in \Pi\mathfrak{g}, A_0 \in \mathfrak{g}\}, \\ \mathcal{F}'' &= \{\psi'' + A'' \mid \int_{\mathcal{S}^1} d\tau \psi''(\tau) = 0, \int_{\mathcal{S}^1} A'' = 0\}. \end{aligned}$$

This splitting respects both the BV 2-form and the de Rham differential. We define the Lagrangian subspace $\mathcal{L} \subset \mathcal{F}''$ as

$$(11) \quad \mathcal{L} = \{\psi'' + A'' \mid A'' = 0\}.$$

2.2.2. *Effective action on cohomology.*⁵ We define the effective action W on \mathcal{F}' by the fiber BV integral

$$\begin{aligned} (12) \quad e^{\frac{1}{\hbar}W(\psi_0, A_0, \hbar)} &= \int_{\mathcal{L}} e^{\frac{1}{\hbar}S(\psi_0 + \psi'', d\tau A_0 + A'')} = \\ &= \int \mathcal{D}\psi'' e^{\frac{1}{2\hbar} \int_{\mathcal{S}^1} (\psi_0 + \psi'', (d + d\tau \text{ad}_{A_0})(\psi_0 + \psi''))}. \end{aligned}$$

The Lagrangian subspace $\mathcal{L} \subset \mathcal{F}''$ is uniquely⁶ fixed by the requirement that the “free” part of the action $\frac{1}{2} \int_{\mathcal{S}^1} (\psi, d\psi)$ be non-degenerate when restricted to \mathcal{L} .

Integral (12) is Gaussian, and it yields the following result.

Proposition 1. *The effective BV action W of the one-dimensional Chern-Simons theory is given by*

$$(13) \quad e^{\frac{1}{\hbar}W(\psi_0, A_0, \hbar)} = \det_{\mathfrak{g}}^{1/2} \left(\frac{\sinh \frac{\text{ad}_{A_0}}{2}}{\frac{\text{ad}_{A_0}}{2}} \right) \cdot e^{-\frac{1}{2\hbar}(\psi_0, \text{ad}_{A_0} \psi_0)}.$$

A simple form of the 0-loop part is due to the fact that multiplication by constant 1-forms respects the splitting of forms into infrared and ultraviolet parts (9,10).⁷ The functional determinant is easily computed *e.g.* by using the exponential basis $\{e^{2\pi i k \tau}\}$ for 0-forms on the circle.

⁴Properties of being constant for a 1-form and being of integral zero for a 0-form are non-covariant. This is not a problem as choosing a gauge always relies on introducing some additional structure. In the case of harmonic gauge, this extra structure is the parametrization of the circle.

⁵More precisely, we consider the effective action as a function on $\Pi\mathfrak{g} \otimes H^\bullet(\mathcal{S}^1)$, *i.e.* on the parity-shifted de Rham cohomology of the circle (which we represented by harmonic forms) with coefficients in \mathfrak{g} .

⁶This a special property of the one-dimensional theory related to the fact that the de Rham operator $d : \Omega''^0(\mathcal{S}^1) \rightarrow \Omega''^1(\mathcal{S}^1)$ is an isomorphism, and there is unique chain homotopy $K = d^{-1} : \Omega''^1(\mathcal{S}^1) \rightarrow \Omega''^0(\mathcal{S}^1)$.

⁷Higher order terms in the 0-loop effective action would correspond to Massey operations on the de Rham cohomology (cf. [7], [14]). In the case of the circle, Massey operations vanish.

The effective action (13) satisfies the quantum master equation

$$\frac{\partial}{\partial \psi_0^a} \frac{\partial}{\partial A_0^a} e^{\frac{1}{\hbar} W(\psi_0, A_0, \hbar)} = 0.$$

The classical master equation is implied by the Jacobi identity and by cyclicity property of the Lie bracket on \mathfrak{g} ; the quantum part of master equation follows from the fact that the one-loop part of W is manifestly ad-invariant.

2.3. Simplicial Chern-Simons action on circle. Let's assume that the circle \mathcal{S}^1 is glued of n intervals $\mathcal{I}_1 = [\mathbf{p}_1, \mathbf{p}_2], \dots, \mathcal{I}_n = [\mathbf{p}_n, \mathbf{p}_1]$ (where $\mathbf{p}_1, \dots, \mathbf{p}_n$ is a cyclically ordered collection of points on the circle), and that each interval \mathcal{I}_k is equipped with a coordinate function $\tau : \mathcal{I}_k \rightarrow [0, 1]$. We will denote a point of \mathcal{I}_k with coordinate τ by (k, τ) . We will assume n to be odd (otherwise, our gauge will be inconsistent, see remark 2). We denote this “triangulation” of the circle by Ξ_n .

2.3.1. Cyclic Whitney gauge. Splitting of fields into infrared and ultraviolet parts is defined by splitting for differential forms on the circle $\Omega^\bullet(\mathcal{S}^1) = \Omega_{\Xi_n}^{\bullet}(\mathcal{S}^1) \oplus \Omega''_{\Xi_n}(\mathcal{S}^1)$, where we split 0-forms into continuous piecewise-linear ones and those with vanishing integrals over each \mathcal{I}_k , and we split 1-forms into piecewise-constant ones and the orthogonal complement of piecewise-linear 0-forms:

$$(14) \quad \Omega_{\Xi_n}^{\bullet}(\mathcal{S}^1) = \{f' + d\tau g' \mid f'|_{\mathcal{I}_k} = (1 - \tau)f_k + \tau f_{k+1}, g'|_{\mathcal{I}_k} = g_k \forall k\}$$

$$\Omega''_{\Xi_n}(\mathcal{S}^1) = \{f'' + d\tau g'' \mid \int_{\mathcal{I}_k} d\tau f'' = 0, \int_{\mathcal{I}_k} \tau d\tau g'' + \int_{\mathcal{I}_{k+1}} (1 - \tau) d\tau g'' = 0 \forall k\}$$

where $f_k, g_k \in \mathbb{R}$ are numbers. Thus, $\Omega_{\Xi_n}^{\bullet}(\mathcal{S}^1) \cong \mathbb{R}^{n|n}$ (as a super-space). As a cochain complex, $\Omega_{\Xi_n}^{\bullet}(\mathcal{S}^1)$ is isomorphic to $C^\bullet(\Xi_n)$, the cochain complex of the simplicial complex Ξ_n . As in section 2.2.1, this splitting agrees with the de Rham differential, and the associated splitting for fields $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ agrees with the BV 2-form.

We call splitting (14) the “cyclic Whitney gauge”, because our representatives Ω^\bullet for cell cochains of triangulation are exactly the Whitney forms [16] for Ξ_n . The word “cyclic” indicates that Ω''^\bullet is constructed as an orthogonal complement of Ω^\bullet with respect to the Poincaré pairing $\int_{\mathcal{S}^1} \bullet \wedge \bullet$.

Coordinates on \mathcal{F}' are given by values of ψ' at the vertices of triangulation:

$$(15) \quad \psi_k = \psi'(\mathbf{p}_k) \in \Pi\mathfrak{g},$$

and by integrals of A over intervals:

$$(16) \quad A_k = \int_{\mathcal{I}_k} A' \in \mathfrak{g}.$$

The BV 2-form on \mathcal{F}' is given by

$$(17) \quad \omega' = \sum_{k=1}^n \frac{\delta\psi_k^a + \delta\psi_{k+1}^a}{2} \wedge \delta A_k^a.$$

We will denote $\mathcal{F}' = \mathcal{F}_{\Xi_n}$ to emphasize its dependence on n . We have $\mathcal{F}_{\Xi_n} \cong \Pi\mathfrak{g} \otimes C^\bullet(\Xi_n)$.

Remark 2. The requirement that n be odd is needed since for n even the piecewise-linear 0-form

$$f(k, \tau) = (-1)^k (\tau - 1/2)$$

belongs to both the infrared and ultraviolet subspaces.

As in section 2.2.1, we define the Lagrangian subspace $\mathcal{L} \subset \mathcal{F}''$ by setting the 1-form part of the ultraviolet field to zero (11).

2.3.2. *Chain homotopy, dressed chain homotopy.* Let us define the infrared projector $\mathcal{P}' : \Omega^\bullet(\mathcal{S}^1) \rightarrow \Omega_{\Xi_n}^\bullet(\mathcal{S}^1)$ by formula

$$(18) \quad f + d\tau \cdot g \mapsto \\ \mapsto \sum_{k=1}^n \left(\int_{\mathcal{I}_k} d\tau' f - (1-2\tau) \cdot \left(\int_{\mathcal{I}_{k+1}} d\tau' f - \int_{\mathcal{I}_{k+2}} d\tau' f + \cdots + \int_{\mathcal{I}_{k-2}} d\tau' f - \int_{\mathcal{I}_{k-1}} d\tau' f \right) \right) \theta_{\mathcal{I}_k} + \\ + d\tau \sum_{k=1}^n \left(\int_{\mathcal{I}_k} d\tau' g + \int_{\mathcal{I}_{k+1}} (1-2\tau') d\tau' g - \int_{\mathcal{I}_{k+2}} (1-2\tau') d\tau' g + \cdots - \int_{\mathcal{I}_{k-1}} (1-2\tau') d\tau' g \right) \theta_{\mathcal{I}_k},$$

where $\theta_{\mathcal{I}_k}$ is the function on the circle with value 1 on \mathcal{I}_k and zero elsewhere.

The chain homotopy $\kappa : \Omega^1(\mathcal{S}^1) \rightarrow \Omega^0(\mathcal{S}^1)$ is uniquely defined by the properties

$$(19) \quad d\kappa + \kappa d = \text{id} - \mathcal{P}',$$

$$(20) \quad \mathcal{P}'\kappa = 0,$$

$$(21) \quad \kappa\mathcal{P}' = 0.$$

Lemma 1. *The operator κ defined by relations (19), (20), (21) acts on the 1-form $d\tau \cdot g \in \Omega^1(\mathcal{S}^1)$ by⁸*

$$(22) \quad \kappa(d\tau \cdot g)(k, \tau) = \sum_{k'=1}^n \int_{\mathcal{I}_{k'}} d\tau' \kappa((k, \tau), (k', \tau')) g(k', \tau'),$$

where the integral kernel is given by

$$(23) \quad \kappa((k, \tau), (k', \tau')) = \begin{cases} \theta(\tau - \tau') - \frac{1}{2} - \tau + \tau' & \text{if } k = k' \\ (-1)^{k-k'} 2(\frac{1}{2} - \tau)(\frac{1}{2} - \tau') & \text{if } k' < k < k' + n \end{cases}$$

and θ is the unit step function. In addition, the kernel has the anti-symmetry property:

$$\kappa((k', \tau'), (k, \tau)) = -\kappa((k, \tau), (k', \tau'))$$

To obtain formula (23), one observes that relations (19) and (20) imply the differential equation

$$(24) \quad \frac{\partial}{\partial \tau} \kappa((k, \tau), (k', \tau')) = \delta_{k,k'} \delta(\tau - \tau') + C_k(k', \tau')$$

subject to conditions

$$(25) \quad \int_0^1 d\tau \kappa((k, \tau), (k', \tau')) = 0, \quad \kappa((k, 1), (k', \tau')) = \kappa((k+1, 0), (k', \tau')) \quad \forall k.$$

Here $C_k(k', \tau')$ are some functions independent of τ . Solving (24) together with (25) immediately yields (23). This proves the uniqueness property. In order to prove existence, one checks that (23) satisfies (19), (20), (21).

⁸Recall that (k, τ) denotes a point on the circle which belongs to the interval \mathcal{I}_k and which has a local coordinate τ . Hence, the integral kernel here is actually a function on $\mathcal{S}^1 \times \mathcal{S}^1$.

We will also need the “dressed” chain homotopy (*i.e.* dressed by the connection) $\kappa_{A'} : \mathfrak{g} \otimes \Omega^1(\mathcal{S}^1) \rightarrow \mathfrak{g} \otimes \Omega^0(\mathcal{S}^1)$, where $A' = \sum_{k=1}^n A_k \theta_{\mathcal{I}_k}$ is a piecewise-constant \mathfrak{g} -valued 1-form on the circle. The operator $\kappa_{A'}$ is uniquely defined by the properties

$$(26) \quad (\text{id} - \mathcal{P}') d_{A'} \kappa_{A'} + \kappa_{A'} d_{A'} (\text{id} - \mathcal{P}') = \text{id} - \mathcal{P}',$$

$$(27) \quad \mathcal{P}' \kappa_{A'} = 0,$$

$$(28) \quad \kappa_{A'} \mathcal{P}' = 0,$$

where $d_{A'} = d + \text{ad}_{A'}$. One can summarize these properties by saying that $\kappa_{A'} : \Omega_{\Xi_n}^{\prime 1}(\mathcal{S}^1) \rightarrow \Omega_{\Xi_n}^{\prime 0}(\mathcal{S}^1)$ is the inverse of the operator $(\text{id} - \mathcal{P}') d_{A'} (\text{id} - \mathcal{P}') : \Omega_{\Xi_n}^{\prime 0}(\mathcal{S}^1) \rightarrow \Omega_{\Xi_n}^{\prime 1}(\mathcal{S}^1)$. The dressed chain homotopy can either be expressed perturbatively as a series

$$\kappa_{A'} = \kappa - \kappa \text{ad}_{A'} \kappa + \kappa \text{ad}_{A'} \kappa \text{ad}_{A'} \kappa - \dots,$$

or it can be computed explicitly by solving the differential equation (26).

To present an explicit formula for the integral kernel of $\kappa_{A'}$ (defined as in (22)), we have to first introduce some notation. Let $F(\mathcal{A}, \tau)$ be an $\text{End}(\mathfrak{g})$ -valued function on the interval $[0, 1]$ depending on an anti-symmetric matrix $\mathcal{A} \in \mathfrak{so}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$ and defined by the following properties

$$(29) \quad F(\mathcal{A}, \bullet) \in \text{Span}_{\text{End}(\mathfrak{g})}(1, e^{-\mathcal{A}\tau}),$$

$$\int_0^1 d\tau F(\mathcal{A}, \tau) = 0,$$

$$F(\mathcal{A}, 0) = 1.$$

Property (29) is equivalently stated as $(d + \mathcal{A})F(\mathcal{A}, \bullet) = \text{const}$ (this constant depends on \mathcal{A}). It is convenient to introduce notation

$$R(\mathcal{A}) = -F(\mathcal{A}, 1) \in \text{End}(\mathfrak{g}).$$

More explicitly, we have

$$(30) \quad F(\mathcal{A}, \tau) = \frac{\frac{1}{2} (\coth \frac{\mathcal{A}}{2} + 1) e^{-\mathcal{A}\tau} - \mathcal{A}^{-1}}{\frac{1}{2} (\coth \frac{\mathcal{A}}{2} + 1) - \mathcal{A}^{-1}},$$

$$(31) \quad R(\mathcal{A}) = -\frac{\mathcal{A}^{-1} + \frac{1}{2} - \frac{1}{2} \coth \frac{\mathcal{A}}{2}}{\mathcal{A}^{-1} - \frac{1}{2} - \frac{1}{2} \coth \frac{\mathcal{A}}{2}}.$$

We will use notation

$$(32) \quad \mu_k(A') = R(\text{ad}_{A_{k-1}})R(\text{ad}_{A_{k-2}}) \cdots R(\text{ad}_{A_{k+1}})R(\text{ad}_{A_k}) \in \text{End}(\mathfrak{g}).$$

The following reflection properties

$$F(-\mathcal{A}, \tau) = -\frac{F(\mathcal{A}, 1 - \tau)}{R(\mathcal{A})}, \quad R(-\mathcal{A}) = R(\mathcal{A})^{-1}, \quad \mu_k(A')^T = \mu_k(A')^{-1}$$

mean that $R(\mathcal{A})$ and $\mu_k(A')$ take values in orthogonal matrices $O(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$.

We can now present the result for the integral kernel of the dressed chain homotopy $\kappa_{A'}$:

Lemma 2. *The operator $\kappa_{A'}$ defined by relations (26), (27), (28) acts on a \mathfrak{g} -valued 1-form $d\tau \cdot \beta \in \mathfrak{g} \otimes \Omega^1(\mathcal{S}^1)$ by formula*

$$(33) \quad \kappa_{A'}(d\tau \cdot \beta)(k, \tau) = \sum_{k'=1}^n \int_{\mathcal{I}_{k'}} d\tau' \cdot \kappa_{A'}((k, \tau), (k', \tau')) \circ \beta(k', \tau'),$$

where the integral kernel is given by

$$(34) \quad \kappa_{A'}((k, \tau), (k', \tau')) = \begin{cases} \left(\theta(\tau - \tau') - \frac{1}{2} + \frac{1}{2} \coth \frac{\text{ad}_{A_k}}{2} \right) e^{(\tau' - \tau) \text{ad}_{A_k}} - (\text{ad}_{A_k})^{-1} + \\ \quad + F(\text{ad}_{A_k}, \tau) \left(\frac{1}{1 + \mu_k(A')} - \frac{1}{1 + R(\text{ad}_{A_k})} \right) F(-\text{ad}_{A_k}, \tau') & \text{if } k = k', \\ (-1)^{k-k'} F(\text{ad}_{A_k}, \tau) R(\text{ad}_{A_{k-1}}) \cdots R(\text{ad}_{A_{k'}}) \frac{1}{1 + \mu_{k'}(A')} F(-\text{ad}_{A_{k'}}, \tau') & \text{if } k' < k < k' + n \end{cases}$$

This integral kernel is anti-symmetric:

$$\kappa_{A'}((k', \tau'), (k, \tau))^T = -\kappa_{A'}((k, \tau), (k', \tau'))$$

To obtain (34) we proceed as in the derivation of (23). The differential equation implied by (26) is as follows:

$$(35) \quad \left(\frac{\partial}{\partial \tau} + \text{ad}_{A_k} \right) \kappa_{A'}((k, \tau), (k', \tau')) = \delta_{k,k'} \delta(\tau - \tau') + C_k(k', \tau').$$

Conditions (25) are still fulfilled. Solving (35) with these conditions imposed yields formula (34).

2.3.3. *Simplicial action.* Simplicial Chern-Simons action S_{Ξ_n} for the triangulation Ξ_n of circle is defined by the fiber BV integral

$$(36) \quad e^{\frac{1}{\hbar} S_{\Xi_n}(\psi', A', \hbar)} = \int_{\mathcal{L}} e^{\frac{1}{\hbar} S(\psi' + \psi'', A' + A'')}.$$

As in the case of induction to cohomology, one puts $A''|_{\mathcal{L}} = 0$, and the integral becomes Gaussian:

$$(37) \quad e^{\frac{1}{\hbar} S_{\Xi_n}(\psi', A', \hbar)} = \int \mathcal{D}\psi'' e^{\frac{1}{2\hbar} \int_{S^1} (\psi' + \psi'', d_{A'}(\psi' + \psi''))}.$$

Expanding the integrand as

$$(\psi', d_{A'}\psi') + (\psi'', d\psi'') + (\psi'', \text{ad}_{A'}\psi'') + 2(\psi'', \text{ad}_{A'}\psi')$$

one can consider the second term as the “free” part of the action and the third and fourth terms as a perturbation. In this way, we arrive at the following Feynman diagram expansion for S_{Ξ_n} :

$$(38) \quad \begin{aligned} S_{\Xi_n} = & \underbrace{\frac{1}{2}(\psi', d\psi') + \frac{1}{2}(\psi', \text{ad}_{A'}\psi') - \frac{1}{2}(\psi', \text{ad}_{A'}\kappa \text{ad}_{A'}\psi') + \frac{1}{2}(\psi', \text{ad}_{A'}\kappa \text{ad}_{A'}\kappa \text{ad}_{A'}\psi') - \cdots}_{S_{\Xi_n}^0} + \\ & \underbrace{\hbar \frac{1}{2 \cdot 2} \text{tr}_{\mathfrak{g} \otimes \Omega^0(S^1)} (\kappa \text{ad}_{A'}\kappa \text{ad}_{A'}) - \hbar \frac{1}{2 \cdot 3} \text{tr}_{\mathfrak{g} \otimes \Omega^0(S^1)} (\kappa \text{ad}_{A'}\kappa \text{ad}_{A'}\kappa \text{ad}_{A'}) + \cdots}_{\hbar S_{\Xi_n}^1} \end{aligned}$$

Here the first line is the sum of “tree” diagrams and the second line is the sum of “wheel diagrams”.

The tree part of S_{Ξ_n} can be expressed in terms of the dressed chain homotopy,

$$(39) \quad S_{\Xi_n}^0 = \frac{1}{2} \int_{S^1} (\psi', d\psi') + \frac{1}{2} \int_{S^1} (\psi', \text{ad}_{A'}\psi') - \frac{1}{2} \int_{S^1} (\psi', \text{ad}_{A'}\kappa \text{ad}_{A'}\psi').$$

For the 1-loop part, we first write

$$S_{\Xi_n}^1 = \frac{1}{2} \operatorname{tr}_{\mathfrak{g} \otimes \Omega^0(\mathcal{S}^1)} \log(1 + \kappa \operatorname{ad}_{A'}).$$

Then, by using the general formula

$$\frac{\partial}{\partial s} \operatorname{tr} \log M_s = \operatorname{tr} \left(M_s^{-1} \frac{\partial M_s}{\partial s} \right)$$

one obtains

$$S_{\Xi_n}^1 = \int_0^1 ds \frac{\partial}{\partial s} \left(\frac{1}{2} \operatorname{tr}_{\mathfrak{g} \otimes \Omega^0(\mathcal{S}^1)} \log(1 + \kappa \operatorname{ad}_{sA'}) \right) = \frac{1}{2} \int_0^1 ds \operatorname{tr}_{\mathfrak{g} \otimes \Omega^0(\mathcal{S}^1)} \underbrace{(1 + \kappa \operatorname{ad}_{sA'})^{-1} \kappa \operatorname{ad}_{A'}}_{\kappa_{sA'}}.$$

One can evaluate the functional trace in the integrand in the coordinate representation (*i.e.* in the basis of delta-functions) by expressing it in terms of the integral kernel (34) restricted to the diagonal $\mathcal{S}^1 \subset \mathcal{S}^1 \times \mathcal{S}^1$:

$$(40) \quad S_{\Xi_n}^1 = \int_0^1 ds \frac{1}{2} \sum_{k=1}^n \int_0^1 d\tau \operatorname{tr}_{\mathfrak{g}} (\kappa_{sA'}((k, \tau), (k, \tau)) \operatorname{ad}_{A_k}).$$

There is an ambiguity in this expression since the integral kernel (34) is discontinuous on the diagonal. We regularize this ambiguity by using the convention

$$(41) \quad \theta(0) = \frac{1}{2}$$

Observe that the value assigned to $\theta(0)$ does not really matter: changing the convention for $\theta(0)$ changes $S_{\Xi_n}^1$ by $\propto \operatorname{tr}_{\mathfrak{g}} \operatorname{ad}_{A'} = 0$. It is interesting that the integral over the auxiliary parameter s in (40) can be computed explicitly. By putting together (39) and (40) and substituting (34) we obtain the following explicit result for the simplicial Chern-Simons action on the triangulated circle.

Theorem 1. *Simplicial Chern-Simons action on the triangulated circle is given by*

$$(42) \quad \begin{aligned} S_{\Xi_n} &= \\ &= -\frac{1}{2} \sum_{k=1}^n \left((\psi_k, \psi_{k+1}) + \frac{1}{3} (\psi_k, \operatorname{ad}_{A_k} \psi_k) + \frac{1}{3} (\psi_{k+1}, \operatorname{ad}_{A_k} \psi_{k+1}) + \frac{1}{3} (\psi_k, \operatorname{ad}_{A_k} \psi_{k+1}) \right) + \\ &+ \frac{1}{2} \sum_{k=1}^n (\psi_{k+1} - \psi_k, \left(\frac{1 - R(\operatorname{ad}_{A_k})}{2} \left(\frac{1}{1 + \mu_k(A')} - \frac{1}{1 + R(\operatorname{ad}_{A_k})} \right) \frac{1 - R(\operatorname{ad}_{A_k})}{2R(\operatorname{ad}_{A_k})} + \right. \\ &\quad \left. + (\operatorname{ad}_{A_k})^{-1} + \frac{1}{12} \operatorname{ad}_{A_k} - \frac{1}{2} \coth \frac{\operatorname{ad}_{A_k}}{2} \right) \circ (\psi_{k+1} - \psi_k)) + \\ &+ \frac{1}{2} \sum_{k'=1}^n \sum_{k=k'+1}^{k'+n-1} (-1)^{k-k'} (\psi_{k+1} - \psi_k, \frac{1 - R(\operatorname{ad}_{A_k})}{2} R(\operatorname{ad}_{A_{k-1}}) \cdots R(\operatorname{ad}_{A_{k'}}) \cdot \\ &\quad \cdot \frac{1}{1 + \mu_{k'}(A')} \cdot \frac{1 - R(\operatorname{ad}_{A_{k'}})}{2R(\operatorname{ad}_{A_{k'}})} \circ (\psi_{k'+1} - \psi_{k'})) + \\ &+ \hbar \frac{1}{2} \operatorname{tr}_{\mathfrak{g}} \log \left((1 + \mu_{\bullet}(A')) \prod_{k=1}^n \left(\frac{1}{1 + R(\operatorname{ad}_{A_k})} \cdot \frac{\sinh \frac{\operatorname{ad}_{A_k}}{2}}{\frac{\operatorname{ad}_{A_k}}{2}} \right) \right). \end{aligned}$$

In the last term (the 1-loop contribution) $\mu_\bullet(A')$ stands for $\mu_l(A')$ for arbitrary l . For different l , these matrices differ by conjugation. Hence, the expression $\det_{\mathfrak{g}}(1 + \mu_\bullet(A'))$ is well defined.

2.3.4. Remarks.

Remark 3. In deriving formula (42) we were sloppy about additive constants (we did not pay attention to normalization of the measure in the functional integral (37)). We chose an ad hoc normalization

$$S_{\Xi_n}(0, 0, \hbar) = -\hbar \frac{n-1}{2} \dim \mathfrak{g} \log 2$$

which turns out to be consistent with the operator formalism (see section 3.1).

Remark 4. Setting $n = 1$ in (42) yields the effective action on cohomology (13).

Remark 5. By expanding (42) in a power series with respect to A' we get back the perturbative expansion (38),

$$(43) \quad S_{\Xi_n} = \\ = -\frac{1}{2} \sum_{k=1}^n (\psi_k, \psi_{k+1}) - \frac{1}{2} \sum_{k=1}^n \left(\frac{1}{3} (\psi_k, \text{ad}_{A_k} \psi_k) + \frac{1}{3} (\psi_{k+1}, \text{ad}_{A_k} \psi_{k+1}) + \frac{1}{3} (\psi_k, \text{ad}_{A_k} \psi_{k+1}) \right) + \\ + \frac{1}{2} \sum_{k'=1}^n \sum_{k=k'+1}^{k'+n-1} \frac{1}{72} (-1)^{k-k'} (\psi_{k+1} - \psi_k, \text{ad}_{A_k} \text{ad}_{A_{k'}} (\psi_{k'+1} - \psi_{k'})) - \\ - \hbar \frac{n-1}{2} \dim \mathfrak{g} \log 2 + \hbar \frac{1}{2 \cdot 2} \left(\sum_{k=1}^n \frac{1}{12} \text{tr}_{\mathfrak{g}} (\text{ad}_{A_k})^2 + \sum_{k'=1}^n \sum_{k=k'+1}^{k'+n-1} \frac{1}{36} \text{tr}_{\mathfrak{g}} (\text{ad}_{A_k} \text{ad}_{A_{k'}}) \right) + \mathcal{O}((A')^3).$$

Remark 6. Naïvely, at large n the simplicial action S_{Ξ_n} can be viewed as a lattice approximation to the continuum action (8). For ψ and A fixed, we have

$$S_{\Xi_n} \left(\{ \psi_k = \psi(k/n) \}, \{ A_k = \int_{k/n}^{(k+1)/n} A \} \right) \longrightarrow S(\psi, A) + \mathcal{O}(1/n)$$

when n tends to infinity. The point is that rather than being just an approximation the simplicial theory $(\mathcal{F}_{\Xi_n}, S_{\Xi_n})$ is exactly equivalent to the continuum theory (\mathcal{F}, S) for any finite n .

Remark 7. The simplicial theory is constructed by the fiber BV integral from the continuum theory. Hence, we expect the simplicial action to satisfy the quantum master equation

$$(44) \quad \Delta_{\Xi_n} e^{\frac{1}{\hbar} S_{\Xi_n}} = 0$$

with BV Laplacian associated to the BV 2-form (17),

$$(45) \quad \Delta_{\Xi_n} = \sum_{k=1}^n \frac{\partial}{\partial \tilde{\psi}_k^a} \frac{\partial}{\partial A_k^a},$$

where

$$(46) \quad \tilde{\psi}_k = \frac{\psi_k + \psi_{k+1}}{2}.$$

However, the BV Laplacian in the continuum theory is ill-defined. Therefore, the quantum master equation (44) is not automatic, and it must be checked independently. It is easy to do it in low degrees in A' by using the expansion (43). We will prove the quantum master equation via the operator approach in section 3.2.3.

Remark 8. Another property we expect from the simplicial theory is its compatibility with simplicial aggregations $\Xi_{n+2} \rightarrow \Xi_n$. We will prove this property in section 3.2.2.

Remark 9. The gauge choice (14) is actually rigid up to diffeomorphisms of intervals \mathcal{I}_k . More exactly, if we require compatibility of the splitting $\Omega = \Omega' \oplus \Omega''$ with the de Rham differential and with the pairing $\int_{\mathcal{S}^1} \bullet \wedge \bullet$ (i.e. so that Ω' and Ω'' be subcomplexes of Ω , and Ω'' be orthogonal to Ω'), then the splitting is completely determined by the images of basis 1-cochains of Ξ_n in $\Omega^1(\mathcal{S}^1)$. If in addition, we require that the image of each basis 1-cochain $e_k^{(1)}$ be supported exactly on the respective interval $\mathcal{I}_k \subset \mathcal{S}^1$ (a kind of simplicial locality property), we obtain the splitting (14) up to diffeomorphisms of intervals \mathcal{I}_k taking the representatives of basis 1-cochains $e_k^{(1)}$ to constant forms $\theta_{\mathcal{I}_k} d\tau$. It is important to note that we are implicitly assuming that the splitting for fields is induced by the splitting for real-valued differential forms. If we drop this assumption and allow splittings of $\mathfrak{g} \otimes \Omega$ which are not obtained from a splitting of Ω by tensoring with \mathfrak{g} , we can introduce other gauges.

Remark 10. The embedding of cell cochains of Ξ_n to the space of differential forms in (14) is the same as in the 1-dimensional simplicial BF theory [14]: 0-cochains are represented by continuous piecewise-linear functions, and 1-cochains are represented by piecewise-constant 1-forms. However, the projection (18) is very different. In fact, it is non-local: for 0-forms instead of evaluation at vertices (as in simplicial BF) we have a sum of integrals over all intervals \mathcal{I}_k with certain signs. And for 1-forms, instead of integration over one interval we have an integral over the whole circle with a certain piecewise-linear integral kernel. Thus, in the splitting $\Omega = \Omega' \oplus \Omega''$ the infrared part is like in the simplicial BF theory, but the ultraviolet part is different.

Remark 11. The action (42) can be viewed as a generating function of a certain infinity-structure on $\mathfrak{g} \otimes C^\bullet(\Xi_n)$. In more detail, this is a structure of a loop-enhanced (or “quantum”, or “unimodular”) cyclic L_∞ algebra with the structure maps (operations) $c_k^{(l)} : \wedge^k(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathbb{R}$ related to the action (42) by

$$S_{\Xi_n} = \sum_{l=0}^1 \sum_{k=2}^{\infty} \frac{\hbar^l}{k!} c_k^{(l)}(\underbrace{\psi' + A', \dots, \psi' + A'}_k)$$

where the superfield $\psi' + A'$ is understood as the parity-shifted identity map $\mathcal{F}_{\Xi_n} \rightarrow \mathfrak{g} \otimes C^\bullet(\Xi_n)$ (as in (3)). The quantum master equation (44) generates a family of structure equations on operations $c_k^{(l)}$. In particular, $c_k^{(0)}$ satisfy the structure equations of the usual (nonunimodular) cyclic L_∞ algebra. This algebraic structure on \mathfrak{g} -valued cochains of Ξ_n can be viewed as a homotopy transfer of the unimodular cyclic DGLA structure on $\mathfrak{g} \otimes \Omega^\bullet(\mathcal{S}^1)$. Only the first two cyclic operations, $c_2^{(0)} : \wedge^2(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathbb{R}$ and $c_3^{(0)} : \wedge^3(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathbb{R}$ are simplicially-local. All the other operations are non-local. We can also use the pairing on $\mathfrak{g} \otimes C^\bullet(\Xi_n)$ (induced by the pairing $\int_{\mathcal{S}^1}(\bullet, \bullet)$ on \mathfrak{g} -valued differential forms)

to invert one input in cyclic operations. This gives an oriented (non-cyclic) version of the unimodular L_∞ structure (see e.g. [10]) on $\mathfrak{g} \otimes C^\bullet(\Xi_n)$ with structure maps $l_k^{(0)} : \wedge^k(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathfrak{g} \otimes C^\bullet(\Xi_n)$ and $l_k^{(1)} : \wedge^k(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathbb{R}$ related to the cyclic operations $c_k^{(l)}$ by the following formula

$$\begin{aligned} c_{k+1}^{(0)}(\underbrace{\psi' + A', \dots, \psi' + A'}_{k+1}) &= \left(\psi' + A', l_k^{(0)}(\underbrace{\psi' + A', \dots, \psi' + A'}_k) \right), \quad k \geq 1 \\ c_k^{(1)} &= l_k^{(1)}, \quad k \geq 2 \end{aligned}$$

In this unimodular L_∞ algebra, only the differential $l_1^{(0)}$ is local while all other operations (including the binary bracket $l_2^{(0)} : \wedge^2(\mathfrak{g} \otimes C^\bullet(\Xi_n)) \rightarrow \mathfrak{g} \otimes C^\bullet(\Xi_n)$) are non-local — unlike in the simplicial BF theory [14] where all higher operations are simplicially-local.

Remark 12. The dressed chain homotopy $\kappa_{A'}$ can be used to construct the non-linear map $U : \mathcal{F}_{\Xi_n} \rightarrow \mathcal{F}$,

$$U : \psi' + A' \mapsto \psi' + A' - \kappa_{A'} \text{ad}_{A'} \psi'.$$

It sends the infrared field $\psi' + A'$ to the conditional extremum of the continuum action S restricted to $\{\psi' + A'\} \oplus \mathcal{L}$. The tree part of the simplicial action can be expressed in terms of U ,

$$\begin{aligned} S_{\Xi_n}^0 &= S(U(\psi' + A')) = \\ &= \int_{\mathcal{S}^1} \frac{1}{2} (U(\psi' + A'), dU(\psi' + A')) + \frac{1}{6} (U(\psi' + A'), [U(\psi' + A'), U(\psi' + A')]) \end{aligned}$$

In the language of infinity algebras, U is an L_∞ morphism intertwining the DGLA structure on $\mathfrak{g} \otimes \Omega^\bullet(\mathcal{S}^1)$ and the L_∞ structure on $\mathfrak{g} \otimes C^\bullet(\Xi_n)$.

Remark 13. There are two natural systems of \mathfrak{g} -valued coordinates on the space of simplicial BV fields $\mathcal{F}_{\Xi_n} : \{\psi_k, A_k\}$ and $\{\tilde{\psi}_k, A_k\}$ (where variables $\tilde{\psi}_k$ are defined by (46)). The first coordinate system is associated to the realization of the space of fields through the cochain complex of a triangulation: $\mathcal{F}_{\Xi_n} = \Pi\mathfrak{g} \otimes C^\bullet(\Xi_n)$. The second coordinate system is associated to the realization through 1-chains and 1-cochains: $\mathcal{F}_{\Xi_n} \cong \mathfrak{g} \otimes C_1(\Xi_n) \oplus \Pi\mathfrak{g} \otimes C^1(\Xi_n)$ (or instead of 1-chains of Ξ_n one can talk of 0-cochains of the dual cell decomposition Ξ_n^\vee : $\mathcal{F}_{\Xi_n} \cong \Pi\mathfrak{g} \otimes C^0(\Xi_n^\vee) \oplus \Pi\mathfrak{g} \otimes C^1(\Xi_n)$). The convenience of the first coordinate system is that the abelian part of the simplicial action (42) is local in variables (ψ_k, A_k) . The convenience of the second coordinate system $\{\tilde{\psi}_k, A_k\}$ is that the BV Laplacian becomes diagonal (45).

3. APPROACH THROUGH OPERATOR FORMALISM

In this section, our strategy is to give a new definition of the one-dimensional Chern-Simons theory. It will be inspired by the definition of section 2.1, but we will be able to consider our theory on an interval and to define a concatenation (gluing) procedure. We will check that the results obtained by the new approach are consistent with those of section 2.

For the rest of the paper, we will assume that $\dim \mathfrak{g} = 2m$ is even. This is important for theorem 2 (the correspondence between the operator and path integral

formalism): its proof relies on recovering the path integral by using the fundamental representation of Clifford algebra $Cl(\mathfrak{g})$ (section 4.1) which is simpler for $\dim \mathfrak{g} = 2m$.

3.1. One-dimensional Chern-Simons theory in operator formalism.

3.1.1. *First approximation.* We would like to define the one-dimensional Chern-Simons theory (8) as a quantum mechanics where components of the quantized odd field $\{\hat{\psi}^a\}$ are subject to the anti-commutation relations

$$(47) \quad \hat{\psi}^a \hat{\psi}^b + \hat{\psi}^b \hat{\psi}^a = \hbar \delta^{ab},$$

i.e. $\{\hat{\psi}^a\}$ are generators of the Clifford algebra $Cl(\mathfrak{g})$. The even 1-form (connection) field $A = d\tau T^a A^a(\tau)$ is non-dynamical, and it is treated as a classical background. The evolution operator for the theory on the interval is defined as a path-ordered exponential of A in the spin representation:

$$(48) \quad U_{\mathcal{I}}(A) = \overleftarrow{P} \exp \left(-\frac{1}{2\hbar} \int_{\mathcal{I}} d\tau f^{abc} \hat{\psi}^a A^b(\tau) \hat{\psi}^c \right) \in Cl(\mathfrak{g})$$

Here, the intuition is as follows: the term $\frac{1}{2} \int (\psi, d\psi)$ in the action (8) generates the canonical anti-commutation relations (47) while the term $\frac{1}{2} \int (\psi, [A, \psi])$ generates the time-dependent quantum Hamiltonian $\hat{H}(\tau) = -\frac{1}{2} f^{abc} \hat{\psi}^a A^b(\tau) \hat{\psi}^c$ which appears in (48). The evolution operator for the concatenation of intervals $\mathcal{I}_1 = [\mathfrak{p}_1, \mathfrak{p}_2]$, $\mathcal{I}_2 = [\mathfrak{p}_2, \mathfrak{p}_3]$ with connections A_1, A_2 is naturally given by the product of the corresponding evolution operators in the Clifford algebra:

$$U_{[\mathfrak{p}_1, \mathfrak{p}_3]}(A_1 \theta_{[\mathfrak{p}_1, \mathfrak{p}_2]} + A_2 \theta_{[\mathfrak{p}_2, \mathfrak{p}_3]}) = U_{[\mathfrak{p}_2, \mathfrak{p}_3]}(A_2) \cdot U_{[\mathfrak{p}_1, \mathfrak{p}_2]}(A_1).$$

As in section 2.3.2, $\theta_{[\mathfrak{p}_k, \mathfrak{p}_{k+1}]} = \theta_{\mathcal{I}_k}$ denotes the function taking value 1 on the interval \mathcal{I}_k and zero everywhere else. The partition function for a circle is given by

$$Z_{S^1}(A) = \text{Str}_{Cl(\mathfrak{g})} \overleftarrow{P} \exp \left(-\frac{1}{2\hbar} \int_{\mathcal{I}} d\tau f^{abc} \hat{\psi}^a A^b(\tau) \hat{\psi}^c \right) \in \mathbb{R},$$

where $\text{Str}_{Cl(\mathfrak{g})}$ is the super-trace on $Cl(\mathfrak{g})$ defined as

$$(49) \quad \text{Str}_{Cl(\mathfrak{g})} : \hat{a} \mapsto (i\hbar)^m \cdot \left(\text{Coefficient of } \hat{\psi}^1 \dots \hat{\psi}^{2m} \text{ in } \hat{a} \right).$$

3.1.2. *Imposing the cyclic Whitney gauge.* Our next task is to model in operator formalism the fiber BV integral (36) for the theory on circle. We will look for the analogue of the cyclic Whitney gauge introduced in section 2.3.1. For the connection A , imposing the gauge just amounts to saying that A is now infrared, *i.e.* a piecewise-constant connection $A' = \sum_{k=1}^n d\tau A_k \theta_{\mathcal{I}_k}$ with $A_k \in \mathfrak{g}$. For ψ , we would like to restrict the integration to ψ 's with given integrals (average values) $\tilde{\psi}_k = \frac{\psi_k + \psi_{k+1}}{2}$ over intervals \mathcal{I}_k . So, we are interested in the integral

$$(50) \quad e^{\frac{1}{\hbar} S_{\Xi_n}} = \int \mathcal{D}\psi e^{\frac{1}{2\hbar} \int_{S^1} ((\psi, d\psi) + (\psi, \text{ad}_{A'} \psi))} \prod_{k=1}^n \delta \left(\int_{\mathcal{I}_k} d\tau \psi - \tilde{\psi}_k \right) = \\ = \int \prod_{k=1}^n D\lambda_k e^{-\sum_{k=1}^n (\lambda_k, \tilde{\psi}_k)} \underbrace{\int \mathcal{D}\psi e^{\frac{1}{2\hbar} \int_{S^1} ((\psi, d\psi) + (\psi, \text{ad}_{A'} \psi)) + \sum_{k=1}^n (\lambda_k, \psi)}}_{Z_{\Xi_n}(\lambda', A')}.$$

Here we got rid of δ -functions at the cost of introducing odd auxiliary variables $\lambda_k = T^a \lambda_k^a \in \Pi\mathfrak{g}$. One can organize them into an odd piecewise-constant function on the circle $\lambda' = \sum_{k=1}^n \lambda_k \theta_{\mathcal{I}_k}$ which plays the role of a source for the field ψ . The entity $Z_{\Xi_n}(\lambda', A')$ that appeared in the integrand can be written in the operator formalism as

$$(51) \quad Z_{\Xi_n}(\lambda', A') = \text{Str}_{Cl(\mathfrak{g})} \overleftarrow{\prod}_{k=1}^n \exp \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A_k^b \hat{\psi}^c + \lambda_k^a \hat{\psi}^a \right).$$

Then, the partition function of the one-dimensional Chern-Simons theory on the circle (in the Whitney gauge) is given by the odd Fourier transform of (51):

$$(52) \quad Z_{\Xi_n}(\psi', A') = (i\hbar)^{-nm} \int \prod_{k=1}^n D\lambda_k e^{-\sum_{k=1}^n \lambda_k^a \tilde{\psi}_k^a} Z_{\Xi_n}(\lambda', A').$$

Remark 14. To be precise with signs, we should introduce an ordering convention for the Berezin measure in (52). We set

$$\prod_{k=1}^n D\lambda_k = \overrightarrow{\prod}_{k=1}^n \overrightarrow{\prod}_{a=1}^{2m} D\lambda_k^a.$$

Theorem 2. *For n odd and $\dim \mathfrak{g} = 2m$ even, one has*

$$(53) \quad Z_{\Xi_n}(\psi', A') = e^{\frac{1}{\hbar} S_{\Xi_n}(\psi', A')},$$

where the right hand side is given by (42) and the left hand side is defined by (51, 52).

We will prove this theorem in section 4.4.2 by constructing a path integral representation for Z_{Ξ_n} . But first (see sections 3.1.3, 3.1.4) we will perform some direct tests of formula (53).

Remark 15. Note that we can define the right hand side of (53) only for n odd while the definition of left hand side makes sense for both even and odd n . A simple computation shows that for n even the partition function $Z_{\Xi_n}(\psi', A')$ vanishes at $\psi' = 0, A' = 0$. This agrees with the observation that for n even the Whitney gauge does not apply (see section 2).

Remark 16. We chose the normalization for the super-trace in Clifford algebra (49) and for Z_{Ξ_n} (52) in a way consistent with the path integral formalism (see (76), (77)).

3.1.3. *Consistency check: the effective action on cohomology.*⁹ The first test of the correspondence (53) is the case of $n = 1$. Let us first compute the following expression in the Clifford algebra with two generators Cl_2 :

$$(54) \quad \varphi(\tilde{\psi}^1, \tilde{\psi}^2, a) = (i\hbar)^{-1} \int D\lambda^1 D\lambda^2 e^{-\lambda^1 \tilde{\psi}^1 - \lambda^2 \tilde{\psi}^2} \text{Str}_{Cl_2} \exp \left(-\frac{1}{\hbar} \hat{\psi}^1 a \hat{\psi}^2 + \lambda^1 \hat{\psi}^1 + \lambda^2 \hat{\psi}^2 \right),$$

where $a \in \mathbb{R}$ is a number. For the exponential under the super-trace we have

$$(55) \quad \exp \left(-\frac{1}{\hbar} \hat{\psi}^1 a \hat{\psi}^2 + \lambda^1 \hat{\psi}^1 + \lambda^2 \hat{\psi}^2 \right) = \left(-\frac{2}{\hbar} \sin(a/2) - \frac{\sin(a/2)}{a/2} \lambda^1 \lambda^2 \right) \hat{\psi}^1 \hat{\psi}^2 +$$

⁹For reader's convenience, we present explicit calculations in the Clifford algebra here and in subsequent sections; the general reference for the Clifford calculus is [4].

$$+ \frac{\sin(a/2)}{a/2} (\lambda^1 \hat{\psi}^1 + \lambda^2 \hat{\psi}^2) + \left(\cos(a/2) + \hbar \frac{1}{a} \left(\cos(a/2) - \frac{\sin(a/2)}{a/2} \right) \lambda^1 \lambda^2 \right).$$

In this expression, only the first term contributes to the super-trace in (54), and we obtain

$$\begin{aligned} \varphi(\tilde{\psi}^1, \tilde{\psi}^2, a) &= \int D\lambda^1 D\lambda^2 e^{-\lambda^1 \tilde{\psi}^1 - \lambda^2 \tilde{\psi}^2} \left(-\frac{2}{\hbar} \sin(a/2) - \frac{\sin(a/2)}{a/2} \lambda^1 \lambda^2 \right) = \\ &= \frac{\sin(a/2)}{a/2} - \frac{2}{\hbar} \sin(a/2) \tilde{\psi}^1 \tilde{\psi}^2 = \frac{\sin(a/2)}{a/2} e^{-\frac{1}{\hbar} \tilde{\psi}^1 a \tilde{\psi}^2}. \end{aligned}$$

Next let us consider an anti-symmetric block-diagonal matrix with blocks 2×2

$$(56) \quad \mathcal{A} = \begin{pmatrix} 0 & a^1 & & & & \\ -a^1 & 0 & & & & \\ & & 0 & a^2 & & \\ & & -a^2 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & a^m \\ & & & & & -a^m & 0 \end{pmatrix} \in \mathfrak{so}(\mathfrak{g}) \subset \text{End}(\mathfrak{g}).$$

Then, we have

$$(57) \quad (i\hbar)^{-m} \int \prod_a D\lambda^a e^{-\lambda^a \tilde{\psi}^a} \text{Str}_{Cl(\mathfrak{g})} \exp \left(-\frac{1}{2\hbar} \hat{\psi}^a \mathcal{A}^{ab} \hat{\psi}^b + \lambda^a \hat{\psi}^a \right) = \\ = \prod_{p=1}^m \varphi(\tilde{\psi}^{2p-1}, \tilde{\psi}^{2p}, a^p) = \det_g^{1/2} \left(\frac{\sinh \frac{A}{2}}{\frac{A}{2}} \right) \cdot e^{-\frac{1}{2\hbar} \tilde{\psi}^a \mathcal{A}^{ab} \tilde{\psi}^b}.$$

Since both the left hand side and the right hand side of (57) are $SO(\mathfrak{g})$ -invariant, the equality actually holds for all anti-symmetric matrices \mathcal{A} , and in particular for $\mathcal{A} = \text{ad}_{A_0}$, where $A_0 \in \mathfrak{g}$. Thus, we have shown that

$$Z_{\Xi_1}(\tilde{\psi}, A_0) = \det_g^{1/2} \left(\frac{\sinh \frac{\text{ad}_{A_0}}{2}}{\frac{\text{ad}_{A_0}}{2}} \right) \cdot e^{-\frac{1}{2\hbar} (\tilde{\psi}, \text{ad}_{A_0} \tilde{\psi})}$$

The right hand side coincides with (13), so we have checked the correspondence (53) in the case of $n = 1$.

3.1.4. *Consistency check: the case of mutually commuting $\{A_k\}$ and $\psi' = 0$.* Now we would like to perform a direct check of the correspondence (53) at the point $\psi' = 0$ (i.e. neglecting the tree part of the simplicial action) and assuming that $[A_k, A_{k'}] = 0$ for all $k, k' = 1, \dots, n$. That is, we will check that

$$(58) \quad (i\hbar)^{-nm} \int \prod_{k=1}^n D\lambda_k \text{Str}_{Cl(\mathfrak{g})} \overleftarrow{\prod}_{k=1}^n \exp \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A_k^b \hat{\psi}^c + \lambda_k^a \hat{\psi}^a \right) = \\ = \det_{\mathfrak{g}}^{1/2} \left(\frac{1 + \prod_{k=1}^n R(\text{ad}_{A_k})}{\prod_{k=1}^n (1 + R(\text{ad}_{A_k}))} \cdot \prod_{k=1}^n \frac{\sinh \frac{\text{ad}_{A_k}}{2}}{\frac{\text{ad}_{A_k}}{2}} \right).$$

We use the idea of section 3.1.3 to reduce (58) to a computation in Cl_2 . Since all A_k mutually commute, we can choose an orthonormal basis in \mathfrak{g} , such that the

matrices ad_{A_k} simultaneously assume the standard form (56). Then, both sides of (58) factorize into contributions of 2×2 blocks, and it suffices to check the identity

$$(59) \quad (i\hbar)^{-n} \int \prod_{k=1}^n (D\lambda_k^1 D\lambda_k^2) \text{Str}_{Cl_2} \overleftarrow{\prod}_{k=1}^n \exp \left(-\frac{1}{\hbar} \hat{\psi}^1 a_k \hat{\psi}^2 + \lambda_k^1 \hat{\psi}^1 + \lambda_k^2 \hat{\psi}^2 \right) = \\ = \det^{1/2} \left(\frac{1 + \prod_{k=1}^n R(i\sigma_2 a_k)}{\prod_{k=1}^n (1 + R(i\sigma_2 a_k))} \cdot \prod_{k=1}^n \frac{\sinh \frac{i\sigma_2 a_k}{2}}{\frac{i\sigma_2 a_k}{2}} \right).$$

Here $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix, and $i\sigma_2 a_k = \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}$. To evaluate the left hand side of (59), we use the result (55) for the Clifford exponential:

$$\text{l.h.s of (59)} = (i\hbar)^{-n} \text{Str}_{Cl_2} \overleftarrow{\prod}_{k=1}^n \left(\frac{\sin(a_k/2)}{a_k/2} \hat{\psi}^1 \hat{\psi}^2 + \hbar \frac{1}{a_k} \left(\frac{\sin(a_k/2)}{a_k/2} - \cos(a_k/2) \right) \right)$$

An easy way to evaluate this expression is to use the matrix representation $Cl_2 \rightarrow \text{End}(\mathbb{C}^{1|1})$ which maps

$$\hat{\psi}^1 \mapsto \left(\frac{\hbar}{2} \right)^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\psi}^2 \mapsto \left(\frac{\hbar}{2} \right)^{1/2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{Str}_{Cl_2} \mapsto \text{Str}_{\text{End}(\mathbb{C}^{1|1})},$$

and

$$\text{Str}_{\text{End}(\mathbb{C}^{1|1})} : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \alpha - \delta$$

is the standard super-trace on matrices of the size $(1|1) \times (1|1)$. Using this representation, we obtain

$$\text{l.h.s of (59)} = \\ i^{-n} \text{Str}_{\text{End}(\mathbb{C}^{1|1})} \overleftarrow{\prod}_{k=1}^n \left(\frac{\frac{1}{a_k} \left(\frac{\sin(a_k/2)}{a_k/2} - \cos(a_k/2) \right) + \frac{i}{2} \frac{\sin(a_k/2)}{a_k/2}}{0} \middle| \begin{array}{c} 0 \\ \frac{1}{a_k} \left(\frac{\sin(a_k/2)}{a_k/2} - \cos(a_k/2) \right) - \frac{i}{2} \frac{\sin(a_k/2)}{a_k/2} \end{array} \right) \\ = i^{-n} \prod_{k=1}^n \frac{\sin(a_k/2)}{a_k/2} \cdot 2i \text{Im} \prod_{k=1}^n \left(\frac{i}{2} + \frac{1}{a_k} - \frac{1}{2} \cot(a_k/2) \right) = \\ = 2 \prod_{k=1}^n \frac{\sin(a_k/2)}{a_k/2} \cdot \text{Re} \prod_{k=1}^n \frac{1}{1 + R(ia_k)}.$$

In the last line we used the assumption that n is odd.

To evaluate the right hand side of (59), first observe that the matrix

$$\frac{1 + \prod_{k=1}^n R(i\sigma_2 a_k)}{\prod_{k=1}^n (1 + R(i\sigma_2 a_k))} = \frac{1}{\prod_{k=1}^n (1 + R(i\sigma_2 a_k))} + \frac{1}{\prod_{k=1}^n (1 + R(-i\sigma_2 a_k))}$$

is constructed from matrices $i\sigma_2 a_k = \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}$ and is therefore of form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ (*i.e.* belongs to $\text{Span}_{\mathbb{R}}(1, i\sigma_2)$), and that it is symmetric. Hence, it is actually a multiple of the identity matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, and the determinant may be expressed as

$$\det^{1/2} \left(\frac{1 + \prod_{k=1}^n R(i\sigma_2 a_k)}{\prod_{k=1}^n (1 + R(i\sigma_2 a_k))} \right) = \frac{1}{2} \text{tr} \left(\frac{1 + \prod_{k=1}^n R(i\sigma_2 a_k)}{\prod_{k=1}^n (1 + R(i\sigma_2 a_k))} \right) =$$

$$= \prod_{k=1}^n \frac{1}{1 + R(ia_k)} + \prod_{k=1}^n \frac{1}{1 + R(-ia_k)} = 2 \operatorname{Re} \prod_{k=1}^n \frac{1}{1 + R(ia_k)}$$

Now it is obvious that (59) is verified, and this implies (58).

3.2. One-dimensional Chern-Simons with boundary.

3.2.1. *One-dimensional simplicial Chern-Simons in the operator formalism. Concatenations.* There is a natural construction of the one-dimensional simplicial Chern-Simons theory in the operator formalism: to a one-dimensional oriented simplicial complex Θ (a collection of $i(\Theta)$ triangulated intervals and $c(\Theta)$ circles) it associates a partition function

$$(60) \quad Z_{\Theta} \in \operatorname{Fun}(\underbrace{\mathfrak{g} \otimes C_1(\Theta) \oplus \Pi\mathfrak{g} \otimes C^1(\Theta)}_{\mathcal{F}_{\Theta}^{\text{bulk}}}) \otimes Cl(\mathfrak{g})^{\otimes i(\Theta)},$$

where we think of variables $A_k \in \mathfrak{g}$ as coordinates on \mathfrak{g} -valued simplicial 1-cochains of Θ with parity shifted, and we think of variables $\tilde{\psi}_k \in \Pi\mathfrak{g}$ as coordinates on \mathfrak{g} -valued simplicial 1-chains of Θ . The partition function Z_{Θ} is defined by the following properties:

- For a single interval with a standard triangulation, we have

$$(61) \quad Z_{\mathcal{I}} = (i\hbar)^{-m} \int \prod_{a=1}^{2m} D\lambda^a e^{-\lambda^a \tilde{\psi}^a} \exp\left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a\right).$$

- For a disjoint union of Θ_1 and Θ_2 ,

$$Z_{\Theta_1 \sqcup \Theta_2} = Z_{\Theta_1} \otimes Z_{\Theta_2}.$$

- For a concatenation $\Theta_1 \cup \Theta_2$ with intersection $\Theta_1 \cap \Theta_2 = \mathfrak{p}$ being a single point embedded as a boundary point of positive orientation $\mathfrak{p}_1 \in \Theta_1$ belonging to the i -th triangulated interval of Θ_1 , and embedded as a boundary point of negative orientation $\mathfrak{p}_2 \in \Theta_2$ belonging to the j -th triangulated interval of Θ_2 , we have:

$$(62) \quad Z_{\Theta_1 \cup \Theta_2} = \mathfrak{m}(Z_{\Theta_2} \otimes Z_{\Theta_1}).$$

Here $\mathfrak{m} : Cl(\mathfrak{g}) \otimes Cl(\mathfrak{g}) \rightarrow Cl(\mathfrak{g})$ is the Clifford algebra multiplication, and \mathfrak{m} in (62) acts on Clifford algebras associated to the j -th triangulated interval of Θ_2 and the i -th triangulated interval of Θ_1 .

- If the simplicial complex Θ' is obtained from Θ by closing the i -th triangulated interval into a circle, we have:

$$(63) \quad Z_{\Theta'} = \operatorname{Str}_{Cl(\mathfrak{g})} Z_{\Theta},$$

where the super-trace is taken over the Clifford algebra associated to the i -th interval of Θ .

Obviously, this construction gives (52) for a triangulated circle $\Theta = \Xi_n$, and due to the correspondence (53) it is consistent with the results of section 2.

Remark 17. We may regard $Z_{\mathcal{I}}$ as a contribution of an open interval. Then, the concatenation (62) is understood as taking a disjoint union of open triangulated intervals and then gluing them together at the point \mathfrak{p} . Likewise, (63) is understood as gluing together of two end-points of the same interval. Thus, a contribution

of point is either the (1,2)-tensor¹⁰ \mathfrak{m} on the Clifford algebra or the (0,1)-tensor $\text{Str}_{Cl(\mathfrak{g})}$, depending on whether we are gluing together two different or the same connected component.

Remark 18. Once we come to a description of the one-dimensional Chern-Simons theory in terms of Atiyah-Segal's axioms and specify the vector space associated to a point (the space of states), properties (62) and (63) become a single sewing axiom (see section 4.2 below).

3.2.2. *Simplicial aggregations.* Let Θ' be the interval $[\mathfrak{p}_1, \mathfrak{p}_3]$ with the standard triangulation, and let Θ be the subdivision of Θ' with three 0-simplices $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ and two 1-simplices $[\mathfrak{p}_1, \mathfrak{p}_2], [\mathfrak{p}_2, \mathfrak{p}_3]$. The aggregation morphism r^ξ acts on simplicial chains as

$$\begin{aligned} r_{C_\bullet}^\xi : C_\bullet(\Theta) &\rightarrow C_\bullet(\Theta') \\ \alpha_{\mathfrak{p}_1} e^{\mathfrak{p}_1} + \alpha_{\mathfrak{p}_2} e^{\mathfrak{p}_2} + \alpha_{\mathfrak{p}_3} e^{\mathfrak{p}_3} + \alpha_{\mathfrak{p}_1 \mathfrak{p}_2} e^{\mathfrak{p}_1 \mathfrak{p}_2} + \alpha_{\mathfrak{p}_2 \mathfrak{p}_3} e^{\mathfrak{p}_2 \mathfrak{p}_3} &\mapsto \\ \mapsto (\alpha_{\mathfrak{p}_1} + (1 - \xi)\alpha_{\mathfrak{p}_2}) e'^{\mathfrak{p}_1} + (\alpha_{\mathfrak{p}_3} + \xi\alpha_{\mathfrak{p}_2}) e'^{\mathfrak{p}_3} + (\xi\alpha_{\mathfrak{p}_1 \mathfrak{p}_2} + (1 - \xi)\alpha_{\mathfrak{p}_2 \mathfrak{p}_3}) e'^{\mathfrak{p}_1 \mathfrak{p}_3}, \end{aligned}$$

and on simplicial cochains as

$$\begin{aligned} r_{C^\bullet}^\xi : C^\bullet(\Theta) &\rightarrow C^\bullet(\Theta') \\ \alpha^{\mathfrak{p}_1} e_{\mathfrak{p}_1} + \alpha^{\mathfrak{p}_2} e_{\mathfrak{p}_2} + \alpha^{\mathfrak{p}_3} e_{\mathfrak{p}_3} + \alpha^{\mathfrak{p}_1 \mathfrak{p}_2} e_{\mathfrak{p}_1 \mathfrak{p}_2} + \alpha^{\mathfrak{p}_2 \mathfrak{p}_3} e_{\mathfrak{p}_2 \mathfrak{p}_3} &\mapsto \alpha^{\mathfrak{p}_1} e'_{\mathfrak{p}_1} + \alpha^{\mathfrak{p}_3} e'_{\mathfrak{p}_3} + (\alpha^{\mathfrak{p}_1 \mathfrak{p}_2} + \alpha^{\mathfrak{p}_2 \mathfrak{p}_3}) e'_{\mathfrak{p}_1 \mathfrak{p}_3}. \end{aligned}$$

Here $0 < \xi < 1$ is a parameter determining the relative weight of intervals $[\mathfrak{p}_1, \mathfrak{p}_2]$ and $[\mathfrak{p}_2, \mathfrak{p}_3]$ inside $[\mathfrak{p}_1, \mathfrak{p}_3]$ (the symmetric choice corresponds to $\xi = 1/2$). Basis chains and cochains corresponding a simplex σ are denoted by e^σ and e_σ , respectively. We use primes to distinguish the basis of $C_\bullet(\Theta'), C^\bullet(\Theta')$; $\alpha_{\dots}, \alpha^{\dots}$ are numerical coefficients.

One can also introduce the subdivision morphism which is dual to the aggregation morphism:

$$i_{C_\bullet}^\xi = (r_{C^\bullet}^\xi)^* : C_\bullet(\Theta') \rightarrow C_\bullet(\Theta), \quad i_{C^\bullet}^\xi = (r_{C_\bullet}^\xi)^* : C^\bullet(\Theta') \rightarrow C^\bullet(\Theta).$$

More explicitly,

$$\begin{aligned} i_{C_\bullet}^\xi : \alpha'_{\mathfrak{p}_1} e'^{\mathfrak{p}_1} + \alpha'_{\mathfrak{p}_3} e'^{\mathfrak{p}_3} + \alpha'_{\mathfrak{p}_1 \mathfrak{p}_3} e'^{\mathfrak{p}_1 \mathfrak{p}_3} &\mapsto \alpha'_{\mathfrak{p}_1} e^{\mathfrak{p}_1} + \alpha'_{\mathfrak{p}_3} e^{\mathfrak{p}_3} + \alpha'_{\mathfrak{p}_1 \mathfrak{p}_3} (e^{\mathfrak{p}_1 \mathfrak{p}_2} + e^{\mathfrak{p}_2 \mathfrak{p}_3}), \\ i_{C^\bullet}^\xi : \alpha'^{\mathfrak{p}_1} e'_{\mathfrak{p}_1} + \alpha'^{\mathfrak{p}_3} e'_{\mathfrak{p}_3} + \alpha'^{\mathfrak{p}_1 \mathfrak{p}_3} e'_{\mathfrak{p}_1 \mathfrak{p}_3} &\mapsto \\ \mapsto \alpha'^{\mathfrak{p}_1} e_{\mathfrak{p}_1} + ((1 - \xi)\alpha'^{\mathfrak{p}_1} + \xi\alpha'^{\mathfrak{p}_3}) e_{\mathfrak{p}_2} + \alpha'^{\mathfrak{p}_3} e_{\mathfrak{p}_3} + \alpha'^{\mathfrak{p}_1 \mathfrak{p}_3} (\xi e_{\mathfrak{p}_1 \mathfrak{p}_2} + (1 - \xi)e_{\mathfrak{p}_2 \mathfrak{p}_3}). \end{aligned}$$

Aggregation and subdivision morphisms are chain maps. Moreover, they are quasi-isomorphisms.

These maps induce an embedding $i_{\Theta, \Theta'}^\xi : \mathcal{F}_{\Theta'}^{\text{bulk}} \hookrightarrow \mathcal{F}_\Theta^{\text{bulk}}$ and projection $r_{\Theta, \Theta'}^\xi : \mathcal{F}_\Theta^{\text{bulk}} \rightarrow \mathcal{F}_{\Theta'}^{\text{bulk}}$ for the spaces of “bulk fields”

$$\mathcal{F}_\Theta^{\text{bulk}} = \mathfrak{g} \otimes C_1(\Theta) \oplus \Pi \mathfrak{g} \otimes C^1(\Theta), \quad \mathcal{F}_{\Theta'}^{\text{bulk}} = \mathfrak{g} \otimes C_1(\Theta') \oplus \Pi \mathfrak{g} \otimes C^1(\Theta').$$

That is, we have a splitting

$$(64) \quad \mathcal{F}_\Theta^{\text{bulk}} = i_{\Theta, \Theta'}^\xi(\mathcal{F}_{\Theta'}^{\text{bulk}}) \oplus \mathcal{F}_{\Theta, \Theta'}^{\prime \text{bulk}, \xi}.$$

In more detail, we split the bulk fields as

$$\tilde{\psi}_1 = \tilde{\psi}' + (1 - \xi)\tilde{\psi}'', \quad \tilde{\psi}_2 = \tilde{\psi}' - \xi\tilde{\psi}'', \quad A_1 = \xi A' - A'', \quad A_2 = (1 - \xi)A' + A''.$$

¹⁰We mean the rank of the tensor: one time covariant, two times contravariant.

The maps $i_{\Theta, \Theta'}^\xi$ and $r_{\Theta, \Theta'}^\xi$ are dual to each other with respect to the odd pairing on $\mathcal{F}_{\Theta}^{\text{bulk}}$ and $\mathcal{F}_{\Theta'}^{\text{bulk}}$ (thus, (64) is an orthogonal decomposition). We denote by $\mathcal{L}_{\Theta, \Theta'}^\xi \subset \mathcal{F}_{\Theta, \Theta'}^{\text{bulk}, \xi}$ the Lagrangian subspace defined by setting to zero the 1-cochain part of the ultraviolet field A'' .

We define the action of the aggregation map $\Theta \rightarrow \Theta'$ on the partition function of the one-dimensional simplicial Chern-Simons theory by the fiber BV integral associated to the splitting (64):

$$(65) \quad (r_{\Theta, \Theta'}^\xi)_*(Z_\Theta) = \int_{\mathcal{L}_{\Theta, \Theta'}^\xi} Z_\Theta \in \text{Fun}(\mathcal{F}_{\Theta'}^{\text{bulk}}) \otimes CI(\mathfrak{g}).$$

It is easy to give a direct check of the following statement.

Lemma 3.

$$(66) \quad (r_{\Theta, \Theta'}^\xi)_*(Z_\Theta) = Z_{\Theta'}$$

Proof. Indeed, by definition (65) we have

$$\begin{aligned} & (r_{\Theta, \Theta'}^\xi)_*(Z_\Theta) = \\ &= (i\hbar)^m \int \prod_{a=1}^{\overleftarrow{2m}} D\tilde{\psi}''^a Z_{[\mathfrak{p}_2, \mathfrak{p}_3]}(\tilde{\psi}' - \xi\tilde{\psi}'', (1-\xi)A') \cdot Z_{[\mathfrak{p}_1, \mathfrak{p}_2]}(\tilde{\psi}' + (1-\xi)\tilde{\psi}'', \xi A') = \\ &= (i\hbar)^{-m} \int \prod_{a=1}^{\overleftarrow{2m}} D\tilde{\psi}''^a \prod_{a=1}^{\overrightarrow{2m}} D\lambda_2^a \prod_{a=1}^{\overrightarrow{2m}} D\lambda_1^a e^{-\lambda_1^a(\tilde{\psi}'^a + (1-\xi)\tilde{\psi}''^a) - \lambda_2^a(\tilde{\psi}'^a - \xi\tilde{\psi}''^a)} \\ & \quad \cdot \exp\left(-\frac{1-\xi}{2\hbar} f^{abc} \hat{\psi}^a A'^b \hat{\psi}^c + \lambda_2^a \hat{\psi}^a\right) \cdot \exp\left(-\frac{\xi}{2\hbar} f^{abc} \hat{\psi}^a A'^b \hat{\psi}^c + \lambda_1^a \hat{\psi}^a\right). \end{aligned}$$

Here we made a change of coordinates $(\lambda_1, \lambda_2) \rightarrow (\lambda = \lambda_1 + \lambda_2, \nu = (1-\xi)\lambda_1 - \xi\lambda_2)$. The integral over $\tilde{\psi}''$ produces the delta-function $\delta(\nu)$; by integrating over ν we obtain

$$\begin{aligned} & (r_{\Theta, \Theta'}^\xi)_*(Z_\Theta) = (i\hbar)^{-m} \int \prod_{a=1}^{\overrightarrow{2m}} D\lambda^a e^{-\lambda^a \tilde{\psi}'^a} \\ & \cdot \exp\left(-\frac{1-\xi}{2\hbar} f^{abc} \hat{\psi}^a A'^b \hat{\psi}^c + (1-\xi)\lambda^a \hat{\psi}^a\right) \cdot \exp\left(-\frac{\xi}{2\hbar} f^{abc} \hat{\psi}^a A'^b \hat{\psi}^c + \xi\lambda^a \hat{\psi}^a\right) = \\ &= (i\hbar)^{-m} \int \prod_{a=1}^{\overrightarrow{2m}} D\lambda^a e^{-\lambda^a \tilde{\psi}'^a} \cdot \exp\left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A'^b \hat{\psi}^c + \lambda^a \hat{\psi}^a\right) = Z_{\Theta'}. \end{aligned}$$

□

Remark 19. We normalize the measure on $\mathcal{L}_{\Theta, \Theta'}^\xi$ in such a way that relation (66) holds with no additional factors.

Up to now, we only discussed the elementary aggregation which takes an interval subdivided into two smaller intervals and into an interval with the standard triangulation (that is, one removes the middle point \mathfrak{p}_2 and merges intervals $[\mathfrak{p}_1, \mathfrak{p}_2]$ and $[\mathfrak{p}_2, \mathfrak{p}_3]$). A general simplicial aggregation for a one-dimensional simplicial complex is a sequence of elementary aggregations made at each step on an incident pair of

intervals. In particular, there are many simplicial aggregations for triangulated circles: $\Xi_n \rightarrow \Xi_{n'}$ with $n > n'$.

The following is an immediate consequence of (66):

Proposition 2. *For a general simplicial aggregation*

$$r = r_{\Theta_{l-1}, \Theta'}^{\xi_l} \circ \cdots \circ r_{\Theta_1, \Theta_2}^{\xi_2} \circ r_{\Theta, \Theta_1}^{\xi_1} : \Theta \rightarrow \Theta'$$

(where Θ is an arbitrary one-dimensional simplicial complex and Θ' is some aggregation of Θ), one has

$$(67) \quad r_*(Z_\Theta) = Z_{\Theta'}.$$

The compatibility with aggregations is an important property expected from a simplicial theory. In particular, (53) implies that the simplicial action for a circle S_{Ξ_n} given by (42) is compatible with simplicial aggregations $\Xi_n \rightarrow \Xi_{n'}$ for n, n' odd.

3.2.3. Quantum master equation.

Lemma 4.¹¹ *The partition function for an interval (61) satisfies the following differential equation:*

$$(68) \quad \hbar \frac{\partial}{\partial \tilde{\psi}^a} \frac{\partial}{\partial A^a} Z_{\mathcal{I}} + \frac{1}{\hbar} \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, Z_{\mathcal{I}} \right]_{Cl(\mathfrak{g})} = 0,$$

where $[\cdot, \cdot]_{Cl(\mathfrak{g})}$ denotes the super-commutator on $Cl(\mathfrak{g})$.

Proof. We will check (68) using variables (λ, A) . We have,

$$(69) \quad \left(\hbar \lambda^a \frac{\partial}{\partial A^a} + \frac{1}{\hbar} \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \bullet \right]_{Cl(\mathfrak{g})} \right) \exp \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right) = 0.$$

After the Fourier transform from the variable λ to the variable $\tilde{\psi}$, this expression becomes (68). Observe that

$$(70) \quad \hbar \lambda^a \frac{\partial}{\partial A^a} \exp \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right) = \\ = \int_0^1 d\tau e^{\tau \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right)} \cdot \frac{1}{2} f^{abc} \hat{\psi}^a \lambda^b \hat{\psi}^c \cdot e^{(1-\tau) \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right)}.$$

Next, compute commutators in the Clifford algebra,

$$(71) \quad \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \lambda^{a'} \hat{\psi}^{a'} \right]_{Cl(\mathfrak{g})} = -\frac{\hbar}{2} f^{abc} \hat{\psi}^a \lambda^b \hat{\psi}^c,$$

and

$$(72) \quad \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \frac{1}{2} f^{a'b'c'} \hat{\psi}^{a'} A^{b'} \hat{\psi}^{c'} \right]_{Cl(\mathfrak{g})} = \\ = \frac{1}{12} f^{abc} f^{a'b'c'} A^{b'} ([\hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \hat{\psi}^{a'}] \hat{\psi}^{c'} - \hat{\psi}^{a'} [\hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \hat{\psi}^{c'}]) = \\ = \frac{\hbar}{4} f^{abc} f^{a'b'c'} A^{b'} (\delta^{ca'} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^{c'} - \delta^{cc'} \hat{\psi}^{a'} \hat{\psi}^a \hat{\psi}^b) = \frac{\hbar}{4} f^{abc} f^{cb'c'} A^{b'} (\hat{\psi}^a \hat{\psi}^b \hat{\psi}^{c'} + \hat{\psi}^{c'} \hat{\psi}^a \hat{\psi}^b) =$$

¹¹In a different setting equation (68) appeared in [2].

$$\begin{aligned}
&= \frac{\hbar}{6} f^{abc} f^{cb'c'} A^{b'} (\hat{\psi}^a \hat{\psi}^b \hat{\psi}^{c'} + \hat{\psi}^{c'} \hat{\psi}^a \hat{\psi}^b + \hat{\psi}^b \hat{\psi}^{c'} \hat{\psi}^a) + \\
&\quad \underbrace{\hspace{10em}}_{=0 \text{ by Jacobi identity}} \\
&+ \frac{\hbar}{12} f^{abc} f^{cb'c'} A^{b'} (\hat{\psi}^a \hat{\psi}^b \hat{\psi}^{c'} + \hat{\psi}^{c'} \hat{\psi}^a \hat{\psi}^b - 2\hat{\psi}^b \hat{\psi}^{c'} \hat{\psi}^a) = \\
&= \frac{\hbar}{24} f^{abc} f^{cb'c'} A^{b'} ((\hat{\psi}^a \hat{\psi}^b \hat{\psi}^{c'} + \hat{\psi}^{c'} \hat{\psi}^a \hat{\psi}^b - 2\hat{\psi}^b \hat{\psi}^{c'} \hat{\psi}^a) - (\hat{\psi}^b \hat{\psi}^a \hat{\psi}^{c'} + \hat{\psi}^{c'} \hat{\psi}^b \hat{\psi}^a - 2\hat{\psi}^a \hat{\psi}^{c'} \hat{\psi}^b)) = \\
&= \frac{\hbar}{24} f^{abc} f^{cb'c'} A^{b'} (\hat{\psi}^a [\hat{\psi}^b, \hat{\psi}^{c'}] + [\hat{\psi}^{c'}, \hat{\psi}^a] \hat{\psi}^b - \hat{\psi}^b [\hat{\psi}^{c'}, \hat{\psi}^a] - [\hat{\psi}^b, \hat{\psi}^{c'}] \hat{\psi}^a) = 0.
\end{aligned}$$

For brevity, we are omitting the subscript in $[\cdot, \cdot]_{Cl(\mathfrak{g})}$ in computations. Identities (71,72) imply

$$\begin{aligned}
&\frac{1}{\hbar} \left[\frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c, \exp \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right) \right]_{Cl(\mathfrak{g})} = \\
&= - \int_0^1 d\tau e^{\tau \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right)} \cdot \frac{1}{2} f^{abc} \hat{\psi}^a \lambda^b \hat{\psi}^c \cdot e^{(1-\tau) \left(-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a \right)}.
\end{aligned}$$

Together with (70), this implies (69) which finishes the proof of (68). \square

Let us denote¹² by

$$\hat{\theta} := \frac{1}{6} f^{abc} \hat{\psi}^a \hat{\psi}^b \hat{\psi}^c$$

the Clifford element in (68).

Proposition 3. *The partition function for any one-dimensional simplicial complex Θ satisfies the differential equation*

$$(73) \quad (\hbar \Delta_{\Theta}^{\text{bulk}} + \frac{1}{\hbar} \delta_{\Theta}) Z_{\Theta} = 0,$$

where

$$\Delta_{\Theta}^{\text{bulk}} = \sum_k \frac{\partial}{\partial \tilde{\psi}_k^a} \frac{\partial}{\partial A_k^a}$$

(the sum goes over all 1-simplices of Θ), and

$$\delta_{\Theta} = \sum_{j=1}^{i(\Theta)} \left[\hat{\theta}^{(j)}, \bullet \right]_{Cl(\mathfrak{g})}.$$

Here the sum goes over connected components of Θ that are triangulated intervals; $\hat{\theta}^{(j)}$ denotes $\hat{\theta}$ as an element of the j -th copy of $Cl(\mathfrak{g})$.

Proof. Equation (73) follows from (68). The compatibility with disjoint unions is obvious as $\Delta_{\Theta_1 \sqcup \Theta_2}^{\text{bulk}} = \Delta_{\Theta_1}^{\text{bulk}} + \Delta_{\Theta_2}^{\text{bulk}}$, and $\delta_{\Theta_1 \sqcup \Theta_2} = \delta_{\Theta_1} + \delta_{\Theta_2}$. For concatenations, it suffices to check the case of two triangulated intervals Θ_1 and Θ_2 :

$$\begin{aligned}
&(\hbar \Delta_{\Theta_1 \cup \Theta_2}^{\text{bulk}} + \hbar^{-1} \delta_{\Theta_1 \cup \Theta_2}) Z_{\Theta_1 \cup \Theta_2} = \left(\hbar \Delta_{\Theta_1}^{\text{bulk}} + \hbar \Delta_{\Theta_2}^{\text{bulk}} + \hbar^{-1} \left[\hat{\theta}, \bullet \right]_{Cl(\mathfrak{g})} \right) \circ (Z_{\Theta_2} \cdot Z_{\Theta_1}) = \\
&= Z_{\Theta_2} \cdot \left(\left(\hbar \Delta_{\Theta_1}^{\text{bulk}} + \hbar^{-1} \left[\hat{\theta}, \bullet \right]_{Cl(\mathfrak{g})} \right) \circ Z_{\Theta_1} \right) + \left(\left(\hbar \Delta_{\Theta_2}^{\text{bulk}} + \hbar^{-1} \left[\hat{\theta}, \bullet \right]_{Cl(\mathfrak{g})} \right) \circ Z_{\Theta_2} \right) \cdot Z_{\Theta_1} = 0.
\end{aligned}$$

¹²The notation stems from the fact that this is a quantization of the Maurer-Cartan element $\theta \in \text{Fun}(\Pi\mathfrak{g})$ (4).

The compatibility with closure of a triangulated interval into a triangulated circle follows from $\text{Str}_{Cl(\mathfrak{g})}[\hat{\theta}, Z_\Theta]_{Cl(\mathfrak{g})} = 0$. \square

Remark 20. We understand (73) as a kind of quantum master equation with the boundary term $\delta_\Theta Z_\Theta$. It is tempting to think of the operator $\Delta_\Theta^{\text{bulk}} + \delta_\Theta$ appearing in (73) as a new BV Laplacian adjusted for the presence of the boundary.

Corollary 1. *In the case of a triangulated circle $\Theta = \Xi_n$, the partition function satisfies the usual (non modified) quantum master equation*

$$\Delta_{\Xi_n} Z_{\Xi_n} = 0.$$

4. BACK TO PATH INTEGRAL

Sections 4.1 and 4.4 mostly go along the lines of the standard derivation of the path integral representation for quantum mechanics, see [8].

4.1. Representation of $Cl(\mathfrak{g})$, complex polarization of \mathfrak{g} . The Clifford algebra $Cl(\mathfrak{g})$ admits a representation ρ on the space of polynomials of m odd variables $\text{Fun}(\mathbb{C}^{0|m}) \cong \mathbb{C}[\eta^1, \dots, \eta^m]$. This representation is defined on generators of $Cl(\mathfrak{g})$ as

$$(74) \quad \rho : \hat{\psi}^a \mapsto \begin{cases} \frac{1}{\sqrt{2}} \left(\eta^p + \hbar \frac{\partial}{\partial \eta^p} \right) & \text{if } a = 2p - 1, \\ \frac{i}{\sqrt{2}} \left(\eta^p - \hbar \frac{\partial}{\partial \eta^p} \right) & \text{if } a = 2p. \end{cases}$$

In fact, $\rho : Cl(\mathfrak{g}) \rightarrow \text{End}(\text{Fun}(\mathbb{C}^{0|m})) \cong \text{End}(\mathbb{C}^{2^{m-1}|2^{m-1}})$ is an isomorphism of super-algebras. There is a natural identification

$$\phi : \text{End}(\text{Fun}(\mathbb{C}^{0|m})) \xrightarrow{\sim} \text{Fun}(\mathbb{C}^{0|m} \oplus \mathbb{C}^{0|m}) \cong \mathbb{C}[\eta^1, \dots, \eta^m, \bar{\eta}^1, \dots, \bar{\eta}^m].$$

This is not an algebra morphism with respect to the standard algebra structure on polynomials. Instead, it takes the product of endomorphisms into the convolution; see [3], [8]. We are interested in the composition $\Phi = \phi \circ \rho : Cl(\mathfrak{g}) \rightarrow \text{Fun}(\mathbb{C}^{0|m} \oplus \mathbb{C}^{0|m})$ which maps

$$(75) \quad \Phi : \begin{cases} \hat{1} & \mapsto e^{\frac{1}{\hbar} \sum_q \eta^q \bar{\eta}^q} \\ \hat{\psi}^{2p-1} & \mapsto \frac{1}{\sqrt{2}} (\eta^p + \bar{\eta}^p) e^{\frac{1}{\hbar} \sum_q \eta^q \bar{\eta}^q} \\ \hat{\psi}^{2p} & \mapsto \frac{i}{\sqrt{2}} (\eta^p - \bar{\eta}^p) e^{\frac{1}{\hbar} \sum_q \eta^q \bar{\eta}^q} \end{cases}$$

and sends the product in $Cl(\mathfrak{g})$ into the convolution

$$(76) \quad \begin{aligned} \Phi(\hat{\alpha} \cdot \hat{\beta})(\eta_2, \bar{\eta}_1) &= \\ &= \hbar^m \int \prod_p (D\eta_1^p D\bar{\eta}_2^p) \Phi(\hat{\alpha})(\eta_2, \bar{\eta}_2) \cdot e^{\frac{1}{\hbar} \sum_q \bar{\eta}_2^q \eta_1^q} \cdot \Phi(\hat{\beta})(\eta_1, \bar{\eta}_1). \end{aligned}$$

Formula (76) is a key point in reconstructing the path integral from the operator formalism. Another useful identity is as follows,

$$(77) \quad \text{Str}_{Cl(\mathfrak{g})}(\hat{\alpha}) = \hbar^m \int \prod_p (D\eta^p D\bar{\eta}^p) e^{\frac{1}{\hbar} \sum_q \bar{\eta}^q \eta^q} \cdot \Phi(\hat{\alpha})(\eta, \bar{\eta}).$$

More generally, a representation of type (74) is associated to a choice of a linear complex structure J on \mathfrak{g} compatible with the pairing:

$$J : \mathfrak{g} \rightarrow \mathfrak{g}, \quad J^2 = -\text{id}_{\mathfrak{g}}, \quad (Ja, b) = -(a, Jb) \text{ for } a, b \in \mathfrak{g}.$$

It induces the splitting of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$ into “holomorphic” and “anti-holomorphic” subspaces:

$$(78) \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bar{\mathfrak{h}},$$

where J acts on $\mathfrak{h}, \bar{\mathfrak{h}}$ by multiplication by $+i$ and $-i$, respectively. (Note that \mathfrak{h} and $\bar{\mathfrak{h}}$ are complex subspaces of $\mathfrak{g}_{\mathbb{C}}$ with respect to the standard complex structure; bar in $\bar{\mathfrak{h}}$ does not mean conjugation.) The subspaces \mathfrak{h} and $\bar{\mathfrak{h}}$ are Lagrangian with respect to the pairing (\cdot, \cdot) . The complex Lagrangian polarization (78) induces a polarization for the parity-reversed Lie algebra

$$(79) \quad \Pi\mathfrak{g} = \Pi\mathfrak{h} \oplus \Pi\bar{\mathfrak{h}}.$$

We denote coordinates on $\Pi\mathfrak{h}$ by η^1, \dots, η^m and coordinates on $\Pi\bar{\mathfrak{h}}$ by $\bar{\eta}^1, \dots, \bar{\eta}^m$. The representation $\rho : Cl(\mathfrak{g}) = \widehat{\text{Fun}(\Pi\mathfrak{g})} \rightarrow \text{End}(\text{Fun}(\Pi\mathfrak{h}))$ sends quantized holomorphic coordinates to multiplication operators and quantized anti-holomorphic coordinates to partial derivatives:

$$\rho : \begin{cases} \hat{\eta}^p \mapsto \eta^p \cdot, \\ \hat{\bar{\eta}}^p \mapsto \hbar \frac{\partial}{\partial \bar{\eta}^p}. \end{cases}$$

Morphism (75) from $Cl(\mathfrak{g})$ to the convolution algebra $\text{Fun}(\Pi\mathfrak{h} \oplus \Pi\bar{\mathfrak{h}})$ is given on generators by

$$\Phi : \begin{cases} \hat{\eta}^p \mapsto \eta^p e^{\frac{1}{\hbar} \sum_a \eta^a \bar{\eta}^a}, \\ \hat{\bar{\eta}}^p \mapsto \bar{\eta}^p e^{\frac{1}{\hbar} \sum_a \eta^a \bar{\eta}^a}, \end{cases}$$

and it extends to the other elements of $Cl(\mathfrak{g})$ by the convolution formula (76).

We will use notation $\pi, \bar{\pi}$ for projections from $\Pi\mathfrak{g}$ to $\Pi\mathfrak{h}$ and $\Pi\bar{\mathfrak{h}}$, respectively. We denote by $\iota, \bar{\iota}$ embeddings of $\Pi\mathfrak{h}$ and $\Pi\bar{\mathfrak{h}}$ into $\Pi\mathfrak{g}$.

4.2. One-dimensional Chern-Simons in terms of Atiyah-Segal’s axioms.

To a point with positive orientation we associate the vector super-space (the space of states)

$$\mathcal{H}_{pt^+} = \text{Fun}(\Pi\mathfrak{h}) \cong \mathbb{C}[\eta^1, \dots, \eta^m],$$

and to a point with negative orientation – the dual space

$$\mathcal{H}_{pt^-} = (\mathcal{H}_{pt^+})^* = \text{Fun}(\Pi\bar{\mathfrak{h}}) \cong \mathbb{C}[\bar{\eta}^1, \dots, \bar{\eta}^m].$$

To an interval $\mathcal{I} = [\mathfrak{p}_1, \mathfrak{p}_2]$ we associate the partition function

$$Z_{\mathcal{I}}^{\rho} := \rho(Z_{\mathcal{I}}) \in \text{Fun}(\Pi\mathfrak{g} \oplus \mathfrak{g}) \otimes \underbrace{\mathcal{H}_{\mathfrak{p}_2^+} \otimes \mathcal{H}_{\mathfrak{p}_1^-}}_{\cong \text{End}(\mathcal{H}_{pt^+})}$$

given by formula (61) in representation ρ (see equation (74)). In general, to a one-dimensional simplicial complex Θ we associate the partition function (60) of section 3.2.1 taken in representation ρ :

$$Z_{\Theta}^{\rho} := \rho^{\otimes i(\Theta)} \circ Z_{\Theta} \in \text{Fun}(\mathcal{F}_{\Theta}^{\text{bulk}}) \otimes (\mathcal{H}_{pt^+} \otimes \mathcal{H}_{pt^-})^{\otimes i(\Theta)}.$$

The one-dimensional Chern-Simons theory features three types of operations:

- To a disjoint union $\Theta_1 \sqcup \Theta_2$ corresponds the tensor product for partition functions.
- To a sewing of boundary points \mathfrak{p}_1^- and \mathfrak{p}_2^+ in a simplicial complex Θ corresponds the convolution of spaces of states $\mathcal{H}_{\mathfrak{p}_2^+}$ and $\mathcal{H}_{\mathfrak{p}_1^-}$.

- To a simplicial aggregation $r : \Theta \rightarrow \Theta'$ corresponds a fiber BV integral r_* which reduces the space of bulk fields from $\mathcal{F}_{\Theta}^{\text{bulk}}$ to $\mathcal{F}_{\Theta'}^{\text{bulk}}$.

In addition, \mathcal{H}_{pt^+} is equipped with an odd third-order differential operator

$$\delta^\rho = \rho(\hat{\theta}) : \mathcal{H}_{pt^+} \rightarrow \mathcal{H}_{pt^+},$$

and \mathcal{H}_{pt^-} is equipped with minus its dual $-(\delta^\rho)^* : \mathcal{H}_{pt^-} \rightarrow \mathcal{H}_{pt^-}$. The partition function Z_{Θ}^ρ satisfies the quantum master equation

$$(80) \quad (\hbar \Delta_{\Theta}^{\text{bulk}} + \hbar^{-1} \delta_{\Theta}^\rho) Z_{\Theta}^\rho = 0,$$

where the “boundary BV operator” δ_{Θ}^ρ is the sum over boundary points of Θ of operators δ^ρ or $-(\delta^\rho)^*$ acting on the corresponding \mathcal{H}_{pt} (depending on whether the orientation of pt is positive or negative).

Remark 21. δ^ρ is “almost” a coboundary operator: its square is proportional to identity:

$$(81) \quad (\delta^\rho)^2 = -\frac{\hbar^3}{48} f^{abc} f^{abc} \cdot \text{id}_{\mathcal{H}_{pt^+}}$$

(see [11]). This implies that the boundary BV operator for an interval

$$\delta_{\mathcal{I}}^\rho = \delta^\rho \otimes \text{id}_{\mathcal{H}_{pt^-}} - \text{id}_{\mathcal{H}_{pt^+}} \otimes (\delta^\rho)^* : \underbrace{\mathcal{H}_{pt^+} \otimes \mathcal{H}_{pt^-}}_{\cong \text{End}(\mathcal{H}_{pt^+})} \rightarrow \underbrace{\mathcal{H}_{pt^+} \otimes \mathcal{H}_{pt^-}}_{\cong \text{End}(\mathcal{H}_{pt^+})}$$

squares to zero

$$(\delta_{\mathcal{I}}^\rho)^2 = 0$$

Cases when δ^ρ squares to zero (*i.e.* when $f^{abc} f^{abc} = 0$) are quite interesting as then the reduced space of states for a point $\mathcal{H}_{pt^+}^{\text{red}}$ emerges (see remarks 25, 34, 35 below).

Remark 22. The space of states \mathcal{H}_{pt^+} can be viewed as a geometric quantization of the classical phase super-space $\Pi\mathfrak{g}$ (viewed as an odd Kähler manifold). The operator δ^ρ is the quantization of the Maurer-Cartan element θ (4); the operator $\hbar^{-1} \delta_{\mathcal{I}}^\rho$ is the quantization of the Hamiltonian vector field $\{\theta, \bullet\}$ on $\Pi\mathfrak{g}$.

Remark 23. Topological quantum mechanics (TQM) in the sense of A. Losev [12] assigns to an interval a manifold $Geom$ (the “space of geometric data”) and to a point — a vector superspace \mathcal{H} endowed with an odd coboundary operator Q . The evolution operator U for an interval is a differential form on $Geom$ with values in $\text{End}(\mathcal{H})$ and has to satisfy the “homotopy topologicity” equation

$$(82) \quad (d + \text{ad}_Q) U = 0,$$

where d is the de Rham operator on $Geom$. A standard class of examples of TQMs comes from choosing $Geom = \mathbb{R}_{>0}$ (with coordinate $t > 0$) and setting

$$(83) \quad U(t, dt) = e^{[Q, G]t + dt G} = e^{(d + \text{ad}_Q) \circ (tG)}$$

where G is an odd operator on \mathcal{H} . For instance, for the Hodge TQM [12] on a Riemannian manifold M , one sets $\mathcal{H} = \Omega^\bullet(M)$, $Q = d_M$ and $G = d_M^*$ — the Hodge operator on forms on M . For the Morse TQM [17], [9], one takes the same \mathcal{H} and Q , but now $G = \iota_v$ is the substitution of the gradient vector field. The one-dimensional Chern-Simons theory on an interval can be viewed as a TQM: here $Geom = \mathfrak{g}$ (with

coordinates A^a), $\mathcal{H} = \text{Fun}(\Pi\mathfrak{h})$, $Q = \hbar^{-1}\delta^\rho$. The odd Fourier transform in variable $\tilde{\psi}$ of the partition function for an interval (61) is

$$(84) \quad U(A, \lambda) = \rho \left(e^{-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \lambda^a \hat{\psi}^a} \right) = e^{(d + \hbar^{-1} \text{ad}_{\delta^\rho}) \circ \hbar^{-1} \rho(A^a \hat{\psi}^a)}$$

where $d = \hbar \lambda^a \frac{\partial}{\partial A^a}$ is the de Rham operator on $Geom$. Note that the expression (84) is similar to (83) where we make a substitution $tG \mapsto \hbar^{-1} \rho(A^a \hat{\psi}^a)$. The quantum master equation (68) is equivalent to

$$(85) \quad (d + \hbar^{-1} \text{ad}_{\delta^\rho}) U(A, \lambda) = 0$$

which is exactly the ‘‘homotopy topologicity’’ equation (82). The peculiarity of the one-dimensional Chern-Simons theory viewed as a TQM is that δ^ρ is not necessarily a coboundary operator on \mathcal{H} .

4.3. Integrating out the bulk fields. In section 3.2.2, we discussed simplicial aggregations which reduce the space of bulk fields of the 1-dimensional Chern-Simons theory $\mathcal{F}_\Theta^{\text{bulk}} \rightarrow \mathcal{F}_{\Theta'}^{\text{bulk}}$ according to combinatorial moves applied to the triangulation $\Theta \rightarrow \Theta'$. It is interesting to consider the ‘‘ultimate aggregation’’ — integrating out the bulk fields completely. This procedure should yield the partition function in the sense of Atiyah-Segal (*i.e.* without bulk fields). We will denote it by Z° .

For an interval, we have

$$(86) \quad (i\hbar)^m \int D\tilde{\psi} Z_{\mathcal{I}}(\tilde{\psi}, A) = e^{-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c}.$$

We view (86) as a BV integral over the Lagrangian subspace

$$(87) \quad \mathcal{L}_A = \{\tilde{\psi} + A \mid \tilde{\psi} \text{ is free, } A \text{ fixed}\} \subset \mathcal{F}_{\mathcal{I}}^{\text{bulk}}.$$

This subspace depends on the value of A , and integral (86) also depends on A . However, this dependence is $\text{ad}_{\hat{\theta}}$ -exact:

$$(88) \quad e^{-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a (A + \delta A)^b \hat{\psi}^c} - e^{-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c} = \frac{1}{\hbar} [\hat{\theta}, e^{-\frac{1}{2\hbar} f^{abc} \hat{\psi}^a A^b \hat{\psi}^c + \frac{1}{\hbar} \delta A^a \hat{\psi}^a}]_{Cl(\mathfrak{g})} + \mathcal{O}((\delta A)^2)$$

(this can be checked analogously to the proof of lemma 4). Therefore, we should understand the partition function $Z_{\mathcal{I}}^\circ$ as an element of cohomology of the operator $\text{ad}_{\hat{\theta}}$ (the fact that (86) is $\text{ad}_{\hat{\theta}}$ -closed is an immediate consequence of the quantum master equation (68)). More exactly, $Z_{\mathcal{I}}^\circ$ is the class of Clifford unit $\hat{1}$ in $\text{ad}_{\hat{\theta}}$ -cohomology:

$$(89) \quad Z_{\mathcal{I}}^\circ = [\hat{1}] \in H_{\text{ad}_{\hat{\theta}}}(Cl(\mathfrak{g}))$$

Equivalently, in terms of representation ρ , we have

$$(90) \quad \rho(Z_{\mathcal{I}}^\circ) = [\text{id}_{\mathcal{H}_{pt+}}] \in H_{\delta_{\mathcal{I}}^\rho}(\text{End}(\mathcal{H}_{pt+})).$$

Remark 24. If the contraction of structure constants $f^{abc} f^{abc}$ for \mathfrak{g} is nonzero, the cohomology class (89), (90) vanishes since

$$\hat{1} = -\frac{24\hbar^{-3}}{f^{abc} f^{abc}} \text{ad}_{\hat{\theta}} \hat{\theta}$$

In fact, whole cohomology group $H_{\text{ad}_{\hat{\theta}}} \cong H_{\delta_{\mathcal{I}}^\rho}$ vanishes since very $\text{ad}_{\hat{\theta}}$ -cocycle $\hat{\alpha} \in Cl(\mathfrak{g})$ is automatically exact:

$$\hat{\alpha} = -\frac{24\hbar^{-3}}{f^{abc}f^{abc}} \text{ad}_{\hat{\theta}}(\hat{\theta} \cdot \hat{\alpha}).$$

Remark 25. If $f^{abc}f^{abc} = 0$, we can define the reduced space of states for a point as the δ^ρ -cohomology:

$$\mathcal{H}_{pt^+}^{\text{red}} := H_{\delta^\rho}(\mathcal{H}_{pt^+}), \quad \mathcal{H}_{pt^-}^{\text{red}} := H_{-(\delta^\rho)^*}(\mathcal{H}_{pt^-}) = (\mathcal{H}_{pt^+}^{\text{red}})^*$$

By Künneth formula, we have

$$(91) \quad H_{\delta_{\mathcal{I}}^\rho}(\text{End}(\mathcal{H}_{pt^+})) \cong \mathcal{H}_{pt^+}^{\text{red}} \otimes \mathcal{H}_{pt^+}^{\text{red}}.$$

The partition function (90) is then represented by the identity operator

$$\rho(Z_{\mathcal{I}}^\circ) : \mathcal{H}_{pt^+}^{\text{red}} \xrightarrow{\text{id}} \mathcal{H}_{pt^+}^{\text{red}}.$$

For a circle, we can obtain the partition function $Z_{S^1}^\circ$ (which is just a number) either as a Clifford super-trace of (86) or as a BV integral of the effective action (13) over the Lagrangian subspace (87). Either way, we have

$$(92) \quad Z_{S^1}^\circ = 0$$

due to non-saturation of fermionic modes either in Clifford super-trace or in the Berezin integral over $\tilde{\psi}$.

4.4. From operator formalism to path integral.

4.4.1. *Abelian one-dimensional Chern-Simons theory.* The one-dimensional abelian Chern-Simons theory associates to an interval \mathcal{I} the unit of $Cl(\mathfrak{g})$ (here \mathfrak{g} can be viewed as a Euclidean vector space; the Lie algebra structure is irrelevant). The path integral arises upon applying the map (75) to this trivial partition function:

$$(93) \quad \Phi(\hat{\mathbf{1}})(\eta_{out}, \bar{\eta}_{in}) = \Phi(\underbrace{\hat{\mathbf{1}} \cdot \hat{\mathbf{1}} \cdots \hat{\mathbf{1}}}_N)(\eta_{out}, \bar{\eta}_{in}) = \\ = \int \left(\prod_{k=1}^{N-1} \hbar^m D\eta_k D\bar{\eta}_{k+1} \right) \cdot \\ \cdot \exp \frac{1}{\hbar} (\langle \eta_{out}, \bar{\eta}_N \rangle + \langle \bar{\eta}_N, \eta_{N-1} \rangle + \langle \eta_{N-1}, \bar{\eta}_{N-1} \rangle + \cdots + \langle \eta_2, \bar{\eta}_2 \rangle + \langle \bar{\eta}_2, \eta_1 \rangle + \langle \eta_1, \bar{\eta}_{in} \rangle) = \\ = \int \left(\prod_{k=1}^{N-1} \hbar^m D\eta_k D\bar{\eta}_{k+1} \right) \cdot \exp \frac{1}{\hbar} \left(\langle \eta_1, \bar{\eta}_{in} \rangle + \sum_{k=2}^N \langle \eta_k - \eta_{k-1}, \bar{\eta}_k \rangle \right).$$

For convenience, we set $\bar{\eta}_1 := \bar{\eta}_{in}$, $\eta_N := \eta_{out}$ and introduced a notation

$$\langle \eta, \bar{\eta} \rangle := \sum_q \eta^q \bar{\eta}^q.$$

In the exponential of (93), N terms of type $\langle \eta_k, \bar{\eta}_k \rangle$ correspond to Clifford units, and $N - 1$ terms of type $\langle \bar{\eta}_{k+1}, \eta_k \rangle$ correspond to convolutions kernels as in (76); the symbol $D\eta_k D\bar{\eta}_{k+1}$ is defined as $\prod_p (D\eta_k^p D\bar{\eta}_{k+1}^p)$. Expression (93) corresponds to triangulating an interval by N smaller intervals; the terms in the exponential correspond to 0- and 1-simplices of this triangulation. In the limit $N \rightarrow \infty$, one formally writes (93) as a path integral over paths $\eta(\tau)$, $\bar{\eta}(\tau)$ with η at the right end-point and $\bar{\eta}$ at left end-point of \mathcal{I} fixed by the boundary conditions.

Lemma 5.

$$(94) \quad \Phi(\hat{1})(\eta_{out}, \bar{\eta}_{in}) = \int_{\bar{\eta}(0)=\bar{\eta}_{in}, \eta(1)=\eta_{out}} \mathcal{D}\eta \mathcal{D}\bar{\eta} \cdot \exp \frac{1}{\hbar} \left(\langle \eta(0), \bar{\eta}(0) \rangle + \int_{\mathcal{I}} \langle d\eta, \bar{\eta} \rangle \right)$$

A perturbative computation of the path integral (94) is trivial: the integral is given by the contribution of the critical point

$$\eta(\tau) = \eta_{out}, \quad \bar{\eta}(\tau) = \bar{\eta}_{in} \quad \text{for all } \tau \in [0, 1]$$

and yields

$$\Phi(\hat{1})(\eta_{out}, \bar{\eta}_{in}) = e^{\frac{1}{\hbar} \langle \eta_{out}, \bar{\eta}_{in} \rangle}.$$

It is instructive to write the integral (94) in terms of the field ψ instead of fields $\eta, \bar{\eta}$. For simplicity, we first choose a complex polarization (as in (74))

$$(95) \quad \begin{cases} \psi^{2p-1} &= \frac{1}{\sqrt{2}}(\eta^p + \bar{\eta}^p) \\ \psi^{2p} &= \frac{i}{\sqrt{2}}(\eta^p - \bar{\eta}^p) \end{cases} \Leftrightarrow \begin{cases} \eta^p &= \frac{1}{\sqrt{2}}(\psi^{2p-1} - i\psi^{2p}) \\ \bar{\eta}^p &= \frac{1}{\sqrt{2}}(\psi^{2p-1} + i\psi^{2p}) \end{cases}$$

(this corresponds to the complex structure J on \mathfrak{g} which assigns ψ^{2p-1} as “real” coordinates and ψ^{2p} as “imaginary” coordinates on $\Pi\mathfrak{g}$). We have,

$$\begin{aligned} \sum_p \eta_k^p \bar{\eta}_k^p &= \sum_p i\psi_k^{2p-1} \psi_k^{2p}, \\ \sum_p \bar{\eta}_{k+1}^p \eta_k^p &= \sum_p \frac{\psi_{k+1}^{2p-1} \psi_k^{2p-1} + \psi_{k+1}^{2p} \psi_k^{2p}}{2} - \sum_p i \frac{\psi_{k+1}^{2p-1} \psi_k^{2p} - \psi_{k+1}^{2p} \psi_k^{2p-1}}{2}. \end{aligned}$$

Substituting these expressions into the integral representation (93) for $\Phi(1)$, we obtain

$$(96) \quad \begin{aligned} \Phi(\hat{1})(\eta_{out}, \bar{\eta}_{in}) &= \int \left(\hbar^{m/2} D\eta_1 \right) \left(\prod_{k=2}^{N-1} (i\hbar)^m D\psi_k \right) \left(\hbar^{m/2} D\bar{\eta}_N \right) \cdot \\ &\cdot \exp \frac{1}{\hbar} \left(\sum_{k=1}^{N-1} \frac{1}{2} (\psi_{k+1}, \psi_k) + \sum_p \sum_{k=1}^{N-1} \frac{i}{2} (\psi_{k+1}^{2p-1} - \psi_k^{2p-1})(\psi_{k+1}^{2p} - \psi_k^{2p}) + \right. \\ &\quad \left. + \sum_p \frac{i}{2} \psi_1^{2p-1} \psi_1^{2p} + \sum_p \frac{i}{2} \psi_N^{2p-1} \psi_N^{2p} \right). \end{aligned}$$

Here $D\psi_k := \overleftarrow{\prod}_a D\psi_k^a$ is the Berezin measure on $\Pi\mathfrak{g}$; the variable ψ_1 is constructed by formulae (95) from the integration variable η_1 and the boundary value $\bar{\eta}_1 := \bar{\eta}_{in}$, and ψ_N is constructed from the integration variable $\bar{\eta}_N$ and the boundary value $\eta_N := \eta_{out}$.

Remark 26. We think of integral (93) as corresponding to cutting the interval $\mathcal{I} = [\mathfrak{p}_{in}, \mathfrak{p}_{out}]$ into N intervals $[\mathfrak{p}_{in}, \mathfrak{p}_2] \cup [\mathfrak{p}_2, \mathfrak{p}_3] \cup \dots \cup [\mathfrak{p}_N, \mathfrak{p}_{out}]$. Integration variables $\eta_k, \bar{\eta}_{k+1}$ are associated to the point \mathfrak{p}_{k+1} (more specifically, to the right end of interval $[\mathfrak{p}_k, \mathfrak{p}_{k+1}]$ and to the left end of the interval $[\mathfrak{p}_{k+1}, \mathfrak{p}_{k+2}]$, respectively); the boundary value $\bar{\eta}_{in}$ corresponds to the point \mathfrak{p}_{in} , the boundary value η_{out} — to the point \mathfrak{p}_{out} . However, variables ψ_k are linear combinations of η_k and $\bar{\eta}_k$. Thus, they are not associated to any single point, but rather to a pair of neighboring points $(\mathfrak{p}_k, \mathfrak{p}_{k+1})$.

Again, we formally write the limit $N \rightarrow \infty$ of the integral (96) as a path integral over paths $\psi : \mathcal{I} \rightarrow \Pi\mathfrak{g}$ with a fixed anti-holomorphic projection of ψ at the right end-point of the interval and a fixed holomorphic projection at the left end-point:

Lemma 6. *The path integral expression for the partition function of the abelian Chern-Simons theory on an interval is given by*

$$(97) \quad \Phi(\hat{1})(\eta_{out}, \bar{\eta}_{in}) = \int_{\bar{\pi}(\psi(0))=\bar{\eta}_{in}, \pi(\psi(1))=\eta_{out}} \mathcal{D}\psi \cdot \exp \frac{1}{\hbar} \left(\sum_p \frac{i}{2} \psi^{2p-1}(0) \psi^{2p}(0) + \int_{\mathcal{I}} \frac{1}{2} (\psi, d\psi) + \sum_p \frac{i}{2} \psi^{2p-1}(1) \psi^{2p}(1) \right).$$

The exact meaning of the conditional measure on paths in (97) is the formal $N \rightarrow \infty$ limit of the measure in (96). The second term in the exponential in (96) does not contribute to the limit $N \rightarrow \infty$: once we assume that ψ_k are values of a differentiable path $\psi(\tau)$ at times $\tau = k/N$, the contribution of this term becomes of order $\mathcal{O}(1/N)$.

For a general complex structure J on \mathfrak{g} , path integral (97) becomes

$$\begin{aligned} & \Phi(\hat{1})(\eta_{out}, \bar{\eta}_{in}) = \\ & = \int_{\bar{\pi}(\psi(0))=\bar{\eta}_{in}, \pi(\psi(1))=\eta_{out}} \mathcal{D}\psi \cdot \exp \frac{1}{\hbar} \left(\frac{i}{4} (\psi(0), J\psi(0)) + \int_{\mathcal{I}} \frac{1}{2} (\psi, d\psi) + \frac{i}{4} (\psi(1), J\psi(1)) \right). \end{aligned}$$

4.4.2. *Path integral for the non-abelian one-dimensional Chern-Simons theory in the cyclic Whitney gauge. End of proof of theorem 2.* To obtain a path integral representation for the partition function of the one-dimensional Chern-Simons theory on an interval (61) we use the same strategy as in section 4.4.1: we cut the interval into N smaller intervals and then apply the map Φ (75). The new point here is that for small intervals we have to use the ‘‘heat kernel’’ approximation which gives an exact result only in the limit $N \rightarrow \infty$.

Applying Φ to $Z_{\mathcal{I}}$ (61), we have

$$(98) \quad \begin{aligned} \Phi(Z_{\mathcal{I}})(\eta_{out}, \bar{\eta}_{in}) &= \int (i\hbar)^{-m} D\lambda \cdot e^{-(\lambda, \bar{\psi})} \cdot \Phi \left(\exp \left(-\frac{1}{2\hbar} (\hat{\psi}, [A, \hat{\psi}]) + (\lambda, \hat{\psi}) \right) \right) (\eta_{out}, \bar{\eta}_{in}) = \\ &= \int (i\hbar)^{-m} D\lambda \cdot e^{-(\lambda, \bar{\psi})} \cdot \Phi \left(\left(\exp \left(-\frac{1}{2\hbar N} (\hat{\psi}, [A, \hat{\psi}]) + \frac{1}{N} (\lambda, \hat{\psi}) \right) \right)^N \right) (\eta_{out}, \bar{\eta}_{in}) = \\ &= \int (i\hbar)^{-m} D\lambda \cdot e^{-(\lambda, \bar{\psi})} \int \left(\prod_{k=1}^{N-1} \hbar^m D\eta_k D\bar{\eta}_{k+1} \right) \cdot \\ & \quad \cdot \Phi \left(\exp \left(-\frac{1}{2\hbar N} (\hat{\psi}, [A, \hat{\psi}]) + \frac{1}{N} (\lambda, \hat{\psi}) \right) \right) (\eta_{out}, \bar{\eta}_N) \cdot e^{\frac{1}{\hbar} \langle \bar{\eta}_N, \eta_{N-1} \rangle} \dots \\ & \quad \dots e^{\frac{1}{\hbar} \langle \bar{\eta}_2, \eta_1 \rangle} \cdot \Phi \left(\exp \left(-\frac{1}{2\hbar N} (\hat{\psi}, [A, \hat{\psi}]) + \frac{1}{N} (\lambda, \hat{\psi}) \right) \right) (\eta_1, \bar{\eta}_{in}). \end{aligned}$$

Next, we need to evaluate the partition function for a small interval in the limit $N \rightarrow \infty$:

$$(99) \quad \Phi \left(\exp \left(-\frac{1}{2\hbar N} (\hat{\psi}, [A, \hat{\psi}]) + \frac{1}{N} (\lambda, \hat{\psi}) \right) \right) (\eta, \bar{\eta}) =$$

$$\begin{aligned}
&= \Phi \left(\hat{1} - \frac{1}{2\hbar N} (\hat{\psi}, [A, \hat{\psi}]) + \frac{1}{N} (\lambda, \hat{\psi}) + \mathcal{O} \left(\frac{1}{N^2} \right) \right) (\eta, \bar{\eta}) = \\
&= e^{\frac{1}{\hbar} \langle \eta, \bar{\eta} \rangle} \left(1 - \frac{1}{2\hbar N} (\psi, [A, \psi]) + \frac{1}{N} (\lambda, \psi) + \frac{i}{4N} \text{tr} (J \cdot \text{ad}_A) + \mathcal{O} \left(\frac{1}{N^2} \right) \right).
\end{aligned}$$

Here ψ is a linear combination of $\eta, \bar{\eta}$, prescribed by the choice of a complex structure J (e.g. (95)); the term with a trace appeared due to the following identity:

$$\Phi(\hat{\psi}^a \hat{\psi}^b)(\eta, \bar{\eta}) = e^{\frac{1}{\hbar} \langle \eta, \bar{\eta} \rangle} \left(\psi^a \psi^b + \frac{\hbar}{2} \delta^{ab} + \frac{i\hbar}{2} J^{ab} \right).$$

Here the third term generates the trace term in (99). Substituting the ‘‘heat kernel’’ asymptotics (99) into (98), we get

$$\begin{aligned}
(100) \quad \Phi(Z_{\mathcal{I}})(\eta_{out}, \bar{\eta}_{in}) &= \\
&= \int \left(\prod_{k=1}^{N-1} \hbar^m D\eta_k D\bar{\eta}_{k+1} \right) \int (i\hbar)^{-m} D\lambda \cdot e^{(\lambda, \frac{1}{N} \sum_{k=1}^N \psi_k - \bar{\psi})} \cdot \\
&\cdot e^{\frac{i}{4} \text{tr}(J \cdot \text{ad}_A)} \cdot e^{\frac{1}{\hbar} (\langle \eta_{out}, \bar{\eta}_N \rangle + \langle \bar{\eta}_N, \eta_{N-1} \rangle + \dots + \langle \bar{\eta}_2, \eta_1 \rangle + \langle \eta_1, \bar{\eta}_{in} \rangle)} \cdot e^{-\frac{1}{2\hbar N} \sum_{k=1}^N (\psi_k, [A, \psi_k])} + \mathcal{O} \left(\frac{1}{N} \right).
\end{aligned}$$

Taking the limit $N \rightarrow \infty$, we obtain the following.

Proposition 4. *The Chern-Simons partition function for an interval is given by the path integral:*

$$\begin{aligned}
(101) \quad \Phi(Z_{\mathcal{I}})(\eta_{out}, \bar{\eta}_{in}) &= e^{\frac{i}{4} \text{tr}(J \cdot \text{ad}_A)} \int_{\bar{\pi}(\psi(0))=\bar{\eta}_{in}, \pi(\psi(1))=\eta_{out}, \int_{\mathcal{I}} d\tau \psi(\tau)=\bar{\psi}} \mathcal{D}\psi \cdot \\
&\cdot \exp \frac{1}{\hbar} \left(\int_{\mathcal{I}} \frac{1}{2} (\psi, (d + d\tau \cdot \text{ad}_A)\psi) + \frac{i}{4} (\psi(0), J\psi(0)) + \frac{i}{4} (\psi(1), J\psi(1)) \right)
\end{aligned}$$

The conditional measure on paths $\psi(\tau)$ with a fixed holomorphic projection at $\tau = 1$, a fixed anti-holomorphic projection at $\tau = 0$ and with a fixed integral over τ in (101) is the $N \rightarrow \infty$ limit of the measure in (100).

Applying concatenation formulae (76), (77) to (101), we obtain a path integral representation of the Chern-Simons partition function for one-dimensional simplicial complexes:

Corollary 2. *For a triangulated interval $\Theta = [\mathfrak{p}_{in}, \mathfrak{p}_2] \cup [\mathfrak{p}_2, \mathfrak{p}_3] \cup \dots \cup [\mathfrak{p}_n, \mathfrak{p}_{out}]$ we have*

$$\begin{aligned}
(102) \quad \Phi(Z_{\Theta})(\eta_{out}, \bar{\eta}_{in}) &= e^{\sum_{k=1}^n \frac{i}{4} \text{tr}(J \cdot \text{ad}_{A_k})} \int_{\bar{\pi}(\psi(\mathfrak{p}_{in}))=\bar{\eta}_{in}, \pi(\psi(\mathfrak{p}_{out}))=\eta_{out}, \int_{\mathfrak{p}_k}^{\mathfrak{p}_{k+1}} d\tau \psi(\tau)=\bar{\psi}_k} \mathcal{D}\psi \cdot \\
&\cdot \exp \frac{1}{\hbar} \left(\int_{\mathcal{I}} \frac{1}{2} (\psi, (d + d\tau \cdot \text{ad}_A)\psi) + \frac{i}{4} (\psi(\mathfrak{p}_{in}), J\psi(\mathfrak{p}_{in})) + \frac{i}{4} (\psi(\mathfrak{p}_{out}), J\psi(\mathfrak{p}_{out})) \right).
\end{aligned}$$

For a triangulated circle $\Xi_n = [\mathfrak{p}_1, \mathfrak{p}_2] \cup \dots \cup [\mathfrak{p}_{n-1}, \mathfrak{p}_n] \cup [\mathfrak{p}_n, \mathfrak{p}_1]$, we obtain:

$$(103) \quad Z_{\Xi_n} = e^{\sum_{k=1}^n \frac{i}{4} \text{tr}(J \cdot \text{ad}_{A_k})} \int_{\int_{\mathfrak{p}_k}^{\mathfrak{p}_{k+1}} d\tau \psi(\tau)=\bar{\psi}_k} \mathcal{D}\psi \cdot e^{\frac{1}{2\hbar} \int_{\mathcal{I}} (\psi, (d + d\tau \cdot \text{ad}_A)\psi)}.$$

Remark 27. Expression (103) returns us to the “naïve” path-integral (37) for the simplicial Chern-Simons on a circle, up to a somewhat puzzling factor $e^{\sum_k \frac{i}{4} \text{tr}(J \cdot \text{ad}_{A_k})}$. The explanation is as follows: the path integral in (103) was obtained from the path integral with boundaries (102) by concatenation formula (77). Hence, it is secretly using the normal ordering prescribed by the choice of a complex structure J on \mathfrak{g} (which dictates the regularization for the one-loop determinant in (103)). This implicit dependence on J is exactly cancelled by the factor $e^{\sum_k \frac{i}{4} \text{tr}(J \cdot \text{ad}_{A_k})}$ (indeed, we know that the left hand side of (103) is defined in terms of the Clifford algebra $Cl(\mathfrak{g})$ and therefore cannot possibly depend on J). For the naïve path integral (37), we implicitly assumed the symmetric normal ordering by making a regularization (41) in our computation of the one-loop determinant (the important point is that $\theta(0)$ is a number, and not a matrix).

Path integral representation (103) returns us to perturbative computation of section 2.3 and thus finishes the proof of theorem 2.

Remark 28. It is easy to compute (101) in the case of $A = 0$:

$$(104) \quad \Phi(Z_{\mathcal{I}}|_{A=0})(\eta_{out}, \bar{\eta}_{in}) = 2^{-m} e^{\frac{1}{\hbar} (\langle \eta_{out}, \bar{\eta}_{in} \rangle - 2(\eta_{out} - \bar{\eta}, \bar{\eta}_{in} - \bar{\eta}))},$$

where $\bar{\eta}, \tilde{\eta}$ are holomorphic and anti-holomorphic components of the bulk field $\tilde{\psi}$. We can also write (104) as

$$\Phi(Z_{\mathcal{I}}|_{A=0})(\eta_{out}, \bar{\eta}_{in}) = 2^{-m} e^{\frac{1}{\hbar} (\frac{i}{2}(\psi_{bd}, J\psi_{bd}) - i(\psi_{bd} - \tilde{\psi}, J(\psi_{bd} - \tilde{\psi}))},$$

where $\psi_{bd} = \iota(\eta_{out}) + \bar{\iota}(\bar{\eta}_{in})$ is the linear combination of boundary fields $\bar{\eta}_{out}, \bar{\eta}_{in}$.

4.5. Simplicial action on an interval.

Proposition 5. *The path integral for the Chern-Simons partition function on an interval (101) is given by*

$$(105) \quad \Phi(Z_{\mathcal{I}}(\tilde{\psi}, A))(\eta_{out}, \bar{\eta}_{in}) = \det_{\mathfrak{g}}^{1/2} \left(\frac{\sinh \frac{\text{ad}_A}{2}}{\frac{\text{ad}_A}{2}} \right) \cdot \det_{\mathfrak{g}}^{-1/2} M(\text{ad}_A) \cdot \exp \frac{1}{\hbar} \left(\langle \eta_{out}, \bar{\eta}_{in} \rangle - \frac{1}{2}(\tilde{\psi}, \text{ad}_A \tilde{\psi}) + \frac{1}{2} \begin{pmatrix} \tilde{\eta} - \eta_{out} & \bar{\eta} - \bar{\eta}_{in} \\ \tilde{\eta} - \bar{\eta}_{in} & \eta_{out} \end{pmatrix} \cdot M(\text{ad}_A) \cdot \begin{pmatrix} \tilde{\eta} - \eta_{out} \\ \bar{\eta} - \bar{\eta}_{in} \end{pmatrix} \right),$$

where the bilinear form $M(\text{ad}_A)$ in basis $(\eta, \bar{\eta})$ is represented by the block matrix

$$(106) \quad M(\text{ad}_A) = \begin{pmatrix} R_{-+} R_{++}^{-1} & -1 - R_{--} + R_{-+} R_{++}^{-1} R_{+-} \\ 1 + R_{++}^{-1} & R_{++}^{-1} R_{+-} \end{pmatrix}.$$

Here symbols $R_{\pm\pm}$ stand for blocks of $R(\text{ad}_A)$ (defined by formula (31)) in the basis $(\eta, \bar{\eta})$:

$$(107) \quad R(\text{ad}_A) = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}.$$

Proof. The path integral (101) is Gaussian with a critical point $\psi^{cr}(\tau)$ being the solution of

$$(108) \quad (d + d\tau \text{ad}_A)\psi^{cr} = \text{const}$$

subject to conditions

$$(109) \quad \int d\tau \psi^{cr}(\tau) = \tilde{\psi},$$

$$(110) \quad \pi(\psi^{cr}(1)) = \eta_{out},$$

$$(111) \quad \bar{\pi}(\psi^{cr}(0)) = \bar{\eta}_{in}.$$

Equation (108) together with (109) gives

$$(112) \quad \psi^{cr}(\tau) = \tilde{\psi} + F(\text{ad}_A, \tau)(\psi^{cr}(0) - \tilde{\psi}),$$

where $F(\text{ad}_A, \tau)$ is defined by (30). The value of the action (together with boundary terms) for this path is

$$(113) \quad S(\psi^{cr}) + \text{boundary terms} = \\ = -\frac{1}{2}(\tilde{\psi}, \text{ad}_A \tilde{\psi}) - \frac{1}{2}(\tilde{\psi}, \psi^{cr}(1) - \psi^{cr}(0)) + \frac{1}{2}\langle \eta^{cr}(0), \bar{\eta}_{in} \rangle + \frac{1}{2}\langle \eta_{out}, \bar{\eta}^{cr}(1) \rangle. \\ \underbrace{\hspace{10em}}_{\frac{1}{2} \int (\psi^{cr}, d_A \psi^{cr})}$$

Boundary values of ψ^{cr} can be found as follows: (112) implies that

$$\psi^{cr}(1) = (1 + R(\text{ad}_A))\tilde{\psi} - R(\text{ad}_A)\psi^{cr}(0).$$

Solving equation this together with (110) and (111) in coordinates $(\eta, \bar{\eta})$, we obtain

$$(114) \quad \begin{pmatrix} \eta_{out} \\ \bar{\eta}^{cr}(1) \end{pmatrix} = \begin{pmatrix} 1 + R_{++} & R_{+-} \\ R_{-+} & 1 + R_{--} \end{pmatrix} \begin{pmatrix} \tilde{\eta} \\ \bar{\tilde{\eta}} \end{pmatrix} - \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix} \begin{pmatrix} \eta^{cr}(0) \\ \bar{\eta}_{in} \end{pmatrix}.$$

Here $\eta^{cr}(0)$ and $\bar{\eta}^{cr}(1)$ are the unknowns. Solving (114), we get

$$\begin{aligned} \eta^{cr}(0) &= \eta_{out} + (1 + R_{++}^{-1})(\tilde{\eta} - \eta_{out}) + R_{++}^{-1}R_{+-}(\bar{\tilde{\eta}} - \bar{\eta}_{in}), \\ \bar{\eta}^{cr}(1) &= \bar{\eta}_{in} - R_{-+}R_{++}^{-1}(\tilde{\eta} - \eta_{out}) - (-1 - R_{--} + R_{-+}R_{++}^{-1}R_{+-})(\bar{\tilde{\eta}} - \bar{\eta}_{in}). \end{aligned}$$

By substituting into (113), we obtain the exponential in (105).

The pre-exponential in (105) can be derived as follows. We know that it is a function of A only (since it is a square root of a functional determinant of the operator¹³ d_A acting on functions with vanishing integral, vanishing holomorphic part at $\tau = 1$ and vanishing anti-holomorphic part at $\tau = 0$); let us denote it by $G(A)$. Closing the interval into a circle (by using (77)), we find

$$\int \hbar^m D\eta_{out} D\bar{\eta}_{in} e^{\frac{1}{\hbar}\langle \bar{\eta}_{in}, \eta_{out} \rangle} \cdot \Phi(Z_{\mathcal{I}}(\tilde{\psi}, A))(\eta_{out}, \bar{\eta}_{in}) = \det_{\mathfrak{g}}^{1/2} M(\text{ad}_A) \cdot G(A) \cdot e^{-\frac{1}{2\hbar}(\tilde{\psi}, \text{ad}_A \tilde{\psi})}.$$

Comparing with the known result for a circle (13), we obtain the pre-exponential in (105). \square

Remark 29. In this computation, we neglected the factor of $e^{\frac{i}{4}\text{tr}(J \cdot \text{ad}_A)}$. If we did not, it would anyway be cancelled by the pre-exponential obtained by comparison with (13) (and this result is obtained by an explicit computation in operator formalism in section 3.1.3).

We can define the simplicial Chern-Simons action for the interval $S_{\mathcal{I}}(\tilde{\psi}, A; \eta_{out}, \bar{\eta}_{in})$ by

$$e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta_{out}, \bar{\eta}_{in})} = \Phi(Z_{\mathcal{I}}(\tilde{\psi}, A))(\eta_{out}, \bar{\eta}_{in}),$$

or explicitly

$$(115) \quad S_{\mathcal{I}}(\tilde{\psi}, A; \eta_{out}, \bar{\eta}_{in}) =$$

¹³More exactly, the determinant of a matrix of the bilinear form $f(\bullet, d_A \bullet)$.

$$\begin{aligned}
&= \langle \eta_{out}, \bar{\eta}_{in} \rangle - \frac{1}{2} \langle \tilde{\psi}, \text{ad}_A \tilde{\psi} \rangle + \frac{1}{2} \left(\tilde{\eta} - \eta_{out} \quad \bar{\eta} - \bar{\eta}_{in} \right) \cdot M(\text{ad}_A) \cdot \begin{pmatrix} \tilde{\eta} - \eta_{out} \\ \bar{\eta} - \bar{\eta}_{in} \end{pmatrix} + \\
&\quad + \frac{\hbar}{2} \text{tr}_{\mathfrak{g}} \log \left(\frac{\sinh \frac{\text{ad}_A}{2}}{\frac{\text{ad}_A}{2}} \right) - \frac{\hbar}{2} \text{tr}_{\mathfrak{g}} \log M(\text{ad}_A).
\end{aligned}$$

Remark 30. Expansion of (115) as a power series in A starts as

$$\begin{aligned}
(116) \quad S_{\mathcal{I}}(\tilde{\psi}, A; \eta_{out}, \bar{\eta}_{in}) &= \\
&= \frac{i}{2} (\psi_{bd}, J\psi_{bd}) - i(\psi_{bd} - \tilde{\psi}, J(\psi_{bd} - \tilde{\psi})) - \frac{1}{2} \langle \tilde{\psi}, \text{ad}_A \tilde{\psi} \rangle - \frac{1}{6} (\psi_{bd} - \tilde{\psi}, J \text{ad}_A J(\psi_{bd} - \tilde{\psi})) + \mathcal{O}(A^2) - \\
&\quad - \hbar m \log 2 + \hbar \frac{i}{12} \text{tr}(J \text{ad}_A) + \mathcal{O}(\hbar A^2),
\end{aligned}$$

where $\psi_{bd} = \iota(\eta_{out}) + \bar{\iota}(\bar{\eta}_{in})$.

We can obtain the action for a simplicial complex Θ by gluing actions (115) for individual intervals using concatenation formulae (76), (77). E.g. for a triangulated interval $\Theta = [\mathfrak{p}_{in}, \mathfrak{p}_2] \cup [\mathfrak{p}_2, \mathfrak{p}_3] \cup \dots \cup [\mathfrak{p}_n, \mathfrak{p}_{out}]$ we have

$$\begin{aligned}
(117) \quad e^{\frac{1}{\hbar} S_{\Theta}(\tilde{\psi}_1, A_1, \dots, \tilde{\psi}_n, A_n; \eta_{out}, \bar{\eta}_{in})} &= \int \prod_{k=1}^{n-1} (\hbar^m D\eta_k D\bar{\eta}_{k+1}) \cdot \\
&\cdot \exp \frac{1}{\hbar} \left(S_{\mathcal{I}}(\tilde{\psi}_n, A_n; \eta_{out}, \bar{\eta}_n) + \langle \bar{\eta}_n, \eta_{n-1} \rangle + \dots + \langle \bar{\eta}_2, \eta_1 \rangle + S_{\mathcal{I}}(\tilde{\psi}_1, A_1; \eta_1, \bar{\eta}_{in}) \right).
\end{aligned}$$

For a triangulated circle $\Xi_n = [\mathfrak{p}_1, \mathfrak{p}_2] \cup \dots \cup [\mathfrak{p}_{n-1}, \mathfrak{p}_n] \cup [\mathfrak{p}_n, \mathfrak{p}_1]$, we have

$$(118) \quad e^{\frac{1}{\hbar} S_{\Xi_n}(\tilde{\psi}_1, A_1, \dots, \tilde{\psi}_n, A_n)} = \int \prod_{k=1}^n (\hbar^m D\eta_k D\bar{\eta}_{k+1}) \cdot \exp \frac{1}{\hbar} \sum_{k=1}^n \left(S_{\mathcal{I}}(\tilde{\psi}_k, A_k; \eta_k, \bar{\eta}_k) + \langle \bar{\eta}_{k+1}, \eta_k \rangle \right).$$

Remark 31. Looking at formulae (117), (118), it is tempting to identify $\langle \bar{\eta}_{k+1}, \eta_k \rangle$ as a simplicial action for the point \mathfrak{p}_{k+1} .

Remark 32. Formula (118) explains how the simplicially non-local expression (42) is produced from a simplicially local expression (the sum of contributions of individual intervals — the integrand in (118)). The key is integration over boundary fields $\{\eta_k, \bar{\eta}_k\}$.

Let us introduce the notation $f_{\pm\pm\pm}^{pqr}$ for structure constants¹⁴ of \mathfrak{g} in the basis $(\eta, \bar{\eta})$:

$$(119) \quad \theta = \frac{1}{6} f^{abc} \psi^a \psi^b \psi^c = \frac{1}{6} f_{+++}^{pqr} \eta^p \eta^q \eta^r + \frac{1}{2} f_{++-}^{pqr} \eta^p \eta^q \bar{\eta}^r + \frac{1}{2} f_{+-}^{pqr} \eta^p \bar{\eta}^q \bar{\eta}^r + \frac{1}{6} f_{---}^{pqr} \bar{\eta}^p \bar{\eta}^q \bar{\eta}^r.$$

(this is the same θ as in (4) rewritten in holomorphic-antiholomorphic coordinates on $\Pi\mathfrak{g}$). Then, the quantum master equation (68) for the action (115) has the following form:

$$(120) \quad \hbar \frac{\partial}{\partial \tilde{\psi}^a} \frac{\partial}{\partial A^a} e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} + \frac{1}{\hbar} \left(\frac{1}{6} f_{+++}^{pqr} \eta^p \eta^q \eta^r + \frac{\hbar}{2} f_{++-}^{pqr} \eta^p \eta^q \frac{\partial}{\partial \eta^r} - \frac{\hbar}{2} f_{+-}^{pqr} \eta^p + \dots \right)$$

¹⁴Here we mean the structure constants of the cyclic operation $(\bullet, [\bullet, \bullet]) : \wedge^3 \mathfrak{g} \rightarrow \mathbb{R}$.

$$\begin{aligned}
& + \frac{\hbar^2}{2} f_{+--}^{pqr} \eta^p \frac{\partial}{\partial \eta^q} \frac{\partial}{\partial \eta^r} - \frac{\hbar^2}{2} f_{+--}^{ppr} \frac{\partial}{\partial \eta^r} + \frac{\hbar^3}{6} f_{+--}^{pqr} \frac{\partial}{\partial \eta^p} \frac{\partial}{\partial \eta^q} \frac{\partial}{\partial \eta^r} \Big) e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} - \\
& - e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} \frac{1}{\hbar} \left(\frac{\hbar^3}{6} f_{+++}^{pqr} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^q} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^r} + \frac{\hbar^2}{2} f_{+++}^{pqr} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^q} \bar{\eta}^r - \frac{\hbar^2}{2} f_{+++}^{pqq} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} + \right. \\
& \quad \left. + \frac{\hbar}{2} f_{+--}^{pqr} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} \bar{\eta}^q \bar{\eta}^r - \frac{\hbar}{2} f_{+--}^{ppr} \bar{\eta}^r + \frac{\hbar^3}{6} f_{+--}^{pqr} \bar{\eta}^p \bar{\eta}^q \bar{\eta}^r \right) = 0.
\end{aligned}$$

Remark 33. The one-dimensional BF theory is a special case of the one-dimensional Chern-Simons theory where the complex polarization (78) is compatible with the Lie algebra structure on \mathfrak{g} . In more detail, let \mathfrak{h} in (78) be a Lie subalgebra, and let the Lie algebra structure on \mathfrak{g} be given by a semidirect product of \mathfrak{h} with its coadjoint module $\bar{\mathfrak{h}}$:

$$\mathfrak{g} = \mathfrak{h} \ltimes \bar{\mathfrak{h}}.$$

In this case, formula (115) for $S_{\mathcal{I}}$ simplifies: the block R_{+-} in (107) vanishes, and the matrix $M(\text{ad}_A)$ (106) becomes

$$M(\text{ad}_A) = \begin{pmatrix} R_{-+} + R_{++}^{-1} & -1 - R_{--} \\ 1 + R_{++}^{-1} & 0 \end{pmatrix}, \quad \det_{\mathfrak{g}}^{1/2} M(\text{ad}_A) = \det_{\mathfrak{h}}(1 + R_{++}^{-1}).$$

In (119), only the second term on the right hand side survives:

$$\theta = \frac{1}{2} F_{pq}^r \eta^p \eta^q \bar{\eta}^r.$$

(Here F_{pq}^r are the structure constants of \mathfrak{h} ; we distinguish between upper and lower indices to emphasize that we do not assume that \mathfrak{h} comes with a pairing). So, the quantum master equation (120) is simplified:

$$\begin{aligned}
(121) \quad & \hbar \frac{\partial}{\partial \tilde{\psi}^a} \frac{\partial}{\partial A^a} e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} + \frac{1}{\hbar} \left(\frac{\hbar}{2} F_{pq}^r \eta^p \eta^q \frac{\partial}{\partial \eta^r} - \frac{\hbar}{2} F_{pq}^q \eta^p \right) e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} - \\
& - e^{\frac{1}{\hbar} S_{\mathcal{I}}(\tilde{\psi}, A; \eta, \bar{\eta})} \frac{1}{\hbar} \left(\frac{\hbar^2}{2} F_{pq}^r \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^q} \bar{\eta}^r - \frac{\hbar^2}{2} F_{pq}^q \frac{\overleftarrow{\partial}}{\partial \bar{\eta}^p} \right) = 0.
\end{aligned}$$

(If in addition \mathfrak{h} is unimodular, the last terms in brackets vanish.) Note that the result for BF theory that we obtain from (115) cannot be directly compared to the result in [14] as the choice of gauge fixing is very different¹⁵.

Remark 34. Another interesting point about the BF case is that $f^{abc} f^{abc} = 0$. Hence, the operator $\delta^\rho : \mathcal{H}_{pt+} \rightarrow \mathcal{H}_{pt+}$ becomes a coboundary operator. If we assume in addition that \mathfrak{h} is unimodular, then $(\mathcal{H}_{pt+}, \delta^\rho)$ can be identified with the Chevalley-Eilenberg complex of Lie algebra \mathfrak{h} . Thus, the reduced space of states associated to a point (see remark 25) is the Chevalley-Eilenberg cohomology of \mathfrak{h} :

$$\mathcal{H}_{pt+}^{\text{red}} \cong H_{CE}(\mathfrak{h}).$$

¹⁵Indeed, here we fix the field A to be constant on the interval, and we fix the integral $\tilde{\psi}$ of field ψ over the interval, and the the holomorphic and anti-holomorphic projections of ψ at the right and left end-points of the interval. The gauge used in [14] fixes $\pi(A)$ to be constant, $\bar{\pi}(A)$ to be a sum of delta-functions at the ends of the interval; and it fixes the values $\pi(\psi)$ at the ends of the interval and the integral for $\bar{\pi}(\psi)$. The latter gauge choice features better simplicial locality properties, but is only \mathfrak{h} -equivariant.

Therefore, the cohomology space

$$(122) \quad H_{\text{ad}_{\mathfrak{g}}}(\mathcal{C}l(\mathfrak{g})) \cong H_{\delta_T^p}(\text{End}(\mathcal{H}_{pt+})) \cong H_{CE}(\mathfrak{h}) \otimes (H_{CE}(\mathfrak{h}))^*$$

becomes non-trivial. In this case, the partition function Z_T° can be understood as an identity operator acting on the Chevalley-Eilenberg cohomology $H_{CE}(\mathfrak{h})$.

Remark 35. One can also view the one-dimensional version of the BF theory with cosmological term [6] as a special case of the one-dimensional Chern-Simons theory for $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$, where \mathfrak{h} is itself a quadratic Lie algebra, and the Lie algebra structure on \mathfrak{g} is given by

$$(123) \quad \theta = \frac{1}{2} F^{pqr} \eta^p \eta^q \bar{\eta}^r + \kappa \frac{1}{6} F^{pqr} \bar{\eta}^p \bar{\eta}^q \bar{\eta}^r.$$

Here F^{pqr} are the structure constants of \mathfrak{h} (in an orthonormal basis) and the parameter κ is the ‘‘cosmological constant’’. For Lie algebra \mathfrak{g} , we automatically have $f^{abc} f^{abc} = 0$, and remark 25 applies in this case.

Let us denote \mathfrak{g} with Lie algebra structure defined by (123) by $\mathfrak{g}_{BF,\kappa}$. Then, one-dimensional Chern-Simons theories with Lie algebras $\mathfrak{g}_{BF,\kappa}$ and \mathfrak{h} are related, similarly to the 3-dimensional case [6]. In particular, for continuum action on the circle we have

$$(124) \quad S_{\mathfrak{g}_{BF,\kappa}}(\underbrace{\iota(\eta) + \bar{\iota}(\bar{\eta})}_{\psi}, \underbrace{\iota(\mathbf{A}) + \bar{\iota}(\bar{\mathbf{A}})}_A) = \\ = \frac{1}{2\kappa} (S_{\mathfrak{h}}(\eta + \kappa \bar{\eta}, \mathbf{A} + \kappa \bar{\mathbf{A}}) - S_{\mathfrak{h}}(\eta - \kappa \bar{\eta}, \mathbf{A} - \kappa \bar{\mathbf{A}}))$$

where ι and $\bar{\iota}$ denote the embeddings of \mathfrak{h} , \mathfrak{h}^* into $\mathfrak{g}_{BF,\kappa}$ and on the right hand side we implicitly use the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ given by the pairing on \mathfrak{h} . Relation (124) implies the following relation for partition functions for triangulated circle Ξ_n for Lie algebras $\mathfrak{g}_{BF,\kappa}$ and \mathfrak{h} :

$$(125) \quad Z_{\mathfrak{g}_{BF,\kappa}, \Xi_n}(\{\iota(\eta_k) + \bar{\iota}(\bar{\eta}_k)\}, \{\iota(\mathbf{A}_k) + \bar{\iota}(\bar{\mathbf{A}}_k)\}; \hbar) = \\ = Z_{\mathfrak{h}, \Xi_n}(\{\eta_k + \kappa \bar{\eta}_k\}, \{\mathbf{A}_k + \kappa \bar{\mathbf{A}}_k\}; 2\kappa \hbar) \cdot Z_{\mathfrak{h}, \Xi_n}(\{\eta_k - \kappa \bar{\eta}_k\}, \{\mathbf{A}_k - \kappa \bar{\mathbf{A}}_k\}; -2\kappa \hbar).$$

Remark 36. Another special case of a one-dimensional Chern-Simons theory can be constructed from a Lie bialgebra \mathfrak{h} . Here we set $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ with the canonical pairing and with Lie algebra structure on \mathfrak{g} defined by

$$\theta = \frac{1}{2} F_{pq}^r \eta^p \eta^q \bar{\eta}_r + \frac{1}{2} G_p^{qr} \eta^p \bar{\eta}_q \bar{\eta}_r.$$

Here F_{pq}^r and G_p^{qr} are structure constants of the Lie bracket and co-bracket on \mathfrak{h} . This is a one-dimensional version of the Lie bialgebra BF theory, cf. [13] (the underlying unimodular Lie bialgebra for continuum theory on circle is $\mathfrak{h} \otimes \Omega^{\bullet}(\mathcal{S}^1)$). It does not seem to enjoy any particular simplifications with respect to the general case other than having a canonical complex polarization on \mathfrak{g} .

Remark 37. The odd third-order differential operator in variables $\tilde{\psi}, A, \eta_{out}, \bar{\eta}_{in}$ that appears in (120) endows the algebra of functions $\text{Fun}(\Pi\mathfrak{g} \oplus \mathfrak{g} \oplus \Pi\mathfrak{h} \oplus \Pi\bar{\mathfrak{h}})$ with a structure of homotopy BV algebra in the sense of Tamarkin-Tsygan [15]. In general, the same applies to $\text{Fun}(\mathcal{F}_{\Theta}^{\text{bulk}} \oplus (\Pi\mathfrak{h})^{\times i(\Theta)} \oplus (\Pi\bar{\mathfrak{h}})^{\times i(\Theta)})$ for any 1-dimensional simplicial complex Θ . If Θ has no boundary, this homotopy BV structure is strict.

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THE POISSON SIGMA MODEL ON CLOSED SURFACES

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ABSTRACT. Using methods of formal geometry, the Poisson sigma model on a closed surface is studied in perturbation theory. The effective action, as a function on vacua, is shown to have no quantum corrections if the surface is a torus or if the Poisson structure is regular and unimodular (e.g., symplectic). In the case of a Kähler structure or of a trivial Poisson structure, the partition function on the torus is shown to be the Euler characteristic of the target; some evidence is given for this to happen more generally. The methods of formal geometry introduced in this paper might be applicable to other sigma models, at least of the AKSZ type.

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1. INTRODUCTION

In this note we study the Poisson sigma model [17, 20] with worldsheet a connected, closed surface Σ . To do so we treat the Poisson structure on the target manifold M as a perturbation and expand around the vacua (a.k.a. zero modes) of the unperturbed action.¹ As a critical point of the latter in particular contains a constant map, we have first to localize around its image $x \in M$. To glue perturbations around different points x , we use formal geometry [16]. Our first result is that the perturbative effective action (as a function on the moduli space of vacua for the unperturbed theory) has no quantum corrections if Σ is the torus or if the Poisson structure is regular and unimodular (e.g., symplectic). In the former case, under the further assumption that the Poisson structure is Kähler, we can also perform the integration over vacua and show that the partition function is the Euler characteristic of M . For a general Poisson structure we can use worldsheet supersymmetry to regularize the effective action² and study it like in [22]; this argument is however a bit formal unless some extra conditions on the Poisson structure are assumed.

Notice that on the torus we need not assume unimodularity. For other genera, the requirement of unimodularity was first remarked in [6] where the leading term of the effective action on the sphere was also computed.

The techniques presented in this note, in particular the way of using formal geometry to get a global effective action, should be applicable to other field theories, in particular of the AKSZ type [1]. The techniques of subsection 4.3 and of subsections 5.2 and 5.3 should also extend to higher dimensional AKSZ theories in which the source manifold is a Cartesian product with a torus.

The torus case may also be understood as follows. Recall that the BV (Batalin–Vilkovisky) action for the Poisson sigma model can be given in terms of the AKSZ construction [10]. It is a function on the infinite dimensional graded manifold $\text{Map}(T[1]\Sigma, T^*[1]M)$. On a cylinder $\Sigma = S^1 \times I$, the partition function should be interpreted as an operator on the Hilbert space associated to the boundary S^1 . As the theory is topological, this operator is the identity and the partition

¹What we compute is then $\langle e^{\frac{i}{\hbar} S_\pi} \rangle_0$ where S_π is the interaction part depending on the target Poisson structure π and $\langle \rangle_0$ denotes the expectation value for the Poisson sigma model with zero Poisson structure.

²We prove that the regularized effective action does not depend on the regularization as long as one is present. However, in principle this is not the same theory as the non regularized one.

function on the torus is just its supertrace. Now, in the case of trivial Poisson structure, the BFV (Batalin–Fradkin–Vilkovisky) reduced phase space associated to the boundary is the graded symplectic manifold $T^*T^*[1]M = T^*T[-1]M$. If we choose the vertical polarization in the second presentation, the Hilbert space will be $C^\infty(T[-1]M)$, i.e., the de Rham complex with opposite grading. It is then to be expected that the partition function on the torus should be the Euler characteristic of M . In the perturbative computation, however, the final result is usually of the form $0 \cdot \infty$, but in the Kähler case we get an unambiguous answer. We might then think of a Kähler structure on M , if it exists, as a regularization of the Poisson sigma model with trivial Poisson structure.³ Notice that, if such structures exist, they are open dense in the space of all Poisson structures on M . Another regularization, which produces the same result, consists in adding the Hamiltonian functions of the supersymmetry generators for the effective action. Formally, this can even be done before integrating over fluctuations around vacua.

Finally, notice that apart from the cases mentioned above we do expect the effective action to have quantum corrections. Moreover, the naively computed effective action in formal coordinates might happen not to be global. We show however that it is always possible to find a quantum canonical transformation which makes it into the Taylor expansion of a global effective action. Its class modulo quantum canonical transformations is then the well-defined object associated to the theory.

Section 2 is a crash course in formal geometry (essentially following [11, §2]). In Section 3, we develop the construction of [11, §6] to define the effective action in formal coordinates. Next using the results of Section 4, we show that, in the two special cases mentioned above, the effective action has no quantum correction and is the expression in formal coordinates of a global effective action. In Section 5, we study the effective action for the case of the torus and perform the computation of the partition function. Finally, in Section 6 we study the globalization of the effective action in general.

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³That is, we regularize $\text{STr}_{C^\infty(T[-1]M)} \text{id} = \langle 1 \rangle_0$ as

$$\langle 1 \rangle_0 := \lim_{\epsilon \rightarrow 0} \langle e^{\frac{i\epsilon}{\hbar} S_\pi} \rangle_0.$$

We then show that, in the Kähler case, $\langle e^{\frac{i\epsilon}{\hbar} S_\pi} \rangle_0$ is independent of ϵ and equal to the Euler characteristic of M .

2. FORMAL LOCAL COORDINATES

We shortly review the notion of formal local coordinates following the simple introduction of [11, §2] (for more on formal geometry see [2, 7, 16]).

A generalized exponential map for a manifold M is just a smooth map $\phi: U \rightarrow M$, where U is some open neighborhood of the zero section of M in TM , $(x, y \in U_x) \mapsto \phi_x(y)$, satisfying $\phi_x(0) = x$ and $d_y\phi_x(0) = \text{id} \forall x \in M$. As an example, one may take the exponential map of a connection.

If f is a smooth function on M , then the function $\phi^*f \in C^\infty(U)$ satisfies $d(\phi^*f) = df \circ d\phi$. Denoting by d_x (d_y) the horizontal (vertical) part of the differential, we then get $d_x(\phi^*f) = df \circ d_x\phi$ and $d_y(\phi^*f) = df \circ d_y\phi$. Because of the assumptions on ϕ , there is an open neighborhood $U' \subset U$ of the zero section of M in TM on which $d_y\phi$ is invertible. As a consequence, on U' we have the formula

$$(2.1) \quad d_x(\phi^*f) = d_y(\phi^*f) \circ (d_y\phi)^{-1} \circ d_x\phi.$$

Notice that, for each x , ϕ_x^*f is a smooth function on U_x . By $T\phi_x^*f \in \widehat{ST}_x^*M$ we then denote its Taylor expansion in the $y \in U_x$ -variables around $y = 0$.⁴ In doing this, we associate to $f \in C^\infty(M)$ a section $T\phi^*f$ of \widehat{ST}^*M over M . We may now reinterpret (2.1) as a condition on the section $T\phi^*f$ simply taking Taylor expansions w.r.t. y on both sides. Notice that in the definition of $T\phi^*f$ and in the resulting condition only the Taylor coefficients of ϕ appears. We are thus let to considering two generalized exponential maps as equivalent if all their partial derivatives in the vertical directions, for each point of the base M , coincide at the zero section. We call formal exponential map an equivalence class of generalized exponential maps.

If ϕ is a formal exponential map, then $T\phi^*f \in \Gamma(\widehat{ST}^*M)$ is constructed as above just by picking any generalized exponential map in the given equivalence class. Choosing local coordinates $\{x^i\}$ on the base and $\{y^i\}$ on the fiber, we have explicit expressions

$$(2.2) \quad \phi_x^i(y) = x^i + y^i + \frac{1}{2}\phi_{x,jk}^i y^j y^k + \frac{1}{3!}\phi_{x,jkl}^i y^j y^k y^l + \dots,$$

and the class of ϕ is simply given by the collection of coefficients $\phi_{x,\bullet}$. One can easily see that the coefficients $\phi_{x,jk}^i$ of the quadratic term transform as the components of a connection. We will refer to this as

⁴Here \widehat{S} denotes the formal completion of the symmetric algebra.

the connection in ϕ . Also explicitly we may compute

$$(2.3) \quad T\phi_x^* f = f(x) + y^i \partial_i f(x) + \frac{1}{2} y^j y^k (\partial_j \partial_k f(x) + \phi_{x,jk}^i \partial_i f(x)) + \cdots .$$

Above we have proved that sections of \widehat{ST}^*M of the form $T\phi^* f$ satisfy (the Taylor expansion of) equation (2.1). One can easily prove that the converse is also true. In fact, one has even more. We may think of the Taylor expansion of the r.h.s. as an operator acting on the section $T\phi^* f$. Actually, for every section σ of \widehat{ST}^*M one can define a section $R(\sigma)$ of $T^*M \otimes \widehat{ST}^*M$ by taking the Taylor expansion of $-\mathrm{d}_y \sigma \circ (\mathrm{d}_y \phi)^{-1} \circ \mathrm{d}_x \phi$. Notice that R is $C^\infty(M)$ -linear. As a consequence we have a connection⁵

$$(X, \sigma) \in \Gamma(TM) \otimes \Gamma(\widehat{ST}^*M) \mapsto i_X R(\sigma) \in \Gamma(\widehat{ST}^*M)$$

on \widehat{ST}^*M . One can check that this connection is flat. We can also regard R as a one-form on M taking values in the bundle $\mathrm{End}(\widehat{ST}^*M)$. Also notice that \widehat{ST}^*M is a bundle of algebras and that R acts as a derivation; so we can regard R as a one-form on M taking values in the bundle $\mathrm{Der}(\widehat{ST}^*M)$, which is tantamount to saying the bundle of formal vertical vector fields $\widehat{\mathfrak{X}}(TM) := TM \otimes \widehat{ST}^*M$. Notice that the flatness of the connection may be expressed as the MC (Maurer–Cartan) equation

$$(2.4) \quad \mathrm{d}_x R + \frac{1}{2} [R, R] = 0,$$

where $[,]$ is the Lie bracket of vector fields. Finally, equation (2.1) may now be expressed by saying that $\mathrm{d}\sigma + R(\sigma) = 0$ if σ is of the form $T\phi^* f$ for some f . Below we will see that also the converse is true.

We first extend this connection to a differential D on the complex of \widehat{ST}^*M -valued differential forms $\Gamma(\Lambda^\bullet T^*M \otimes \widehat{ST}^*M)$.⁶ The main result is that the cohomology of D is concentrated in degree zero and $H_D^0 = T\phi^* C^\infty(M)$. This can be easily seen working in local coordinates again:

$$R(\sigma)_i = \frac{\partial \sigma}{\partial y^k} \left(\left(\frac{\partial \phi}{\partial y} \right)^{-1} \right)_j^k \frac{\partial \phi^j}{\partial x^i}.$$

⁵This is the Grothendieck connection in the presentation given by the choice of the formal exponential map ϕ .

⁶Since $\Gamma(\Lambda^\bullet T^*M \otimes \widehat{ST}^*M)$ is the algebra of functions on the formal graded manifold $\mathcal{M} := T[1]M \oplus T[0]M$, the differential D gives \mathcal{M} the structure of a differential graded manifold. In particular since D vanishes on the body, we may linearize at each $x \in M$ and get an L_∞ -algebra structure on $T_x M[1] \oplus T_x M \oplus T_x M$.

Using (2.2) we get $R = \delta + R'$ with $\delta = -dx^i \frac{\partial}{\partial y^i}$ and R' a one-form on the base taking value in the vector fields vanishing at $y = 0$. Hence we have $D = \delta + D'$ with

$$D' = dx^i \frac{\partial}{\partial x^i} + R'.$$

Notice that δ is itself a differential and that it decreases the polynomial degree in y , whereas the operator D' does not decrease this degree. The fundamental remark is that the cohomology of δ consists of zero forms constant in y . This is easily shown by introducing $\delta^* := y^i \iota_{\frac{\partial}{\partial x^i}}$ and observing that $(\delta\delta^* + \delta^*\delta)\sigma = k\sigma$ if σ is an r -form of degree s in y and $r + s = k$. By cohomological perturbation theory the cohomology of D is isomorphic to the cohomology of δ , which is what we wanted to prove.

Finally, observe that, if σ is a D -closed section, we can immediately recover the function f for which $\sigma = T\phi^*f$ simply by setting $y = 0$, $f(x) = \sigma_x(0)$, as follows from (2.3).

We can now extend the whole story to other natural objects. Let $\mathcal{V}(M)$ denote the multivector fields on M (i.e., sections of ΛTM), $\Omega(M)$ the differential forms, $\mathcal{W}^j(M) := \Gamma(S^j TM)$ and $\mathcal{O}^j(M) := \Gamma(S^j T^*M)$. We use similar symbols for formal vertical vector fields $\widehat{\mathcal{V}}(TM) := \Gamma(\Lambda TM \otimes \widehat{S}T^*M)$ and formal vertical differential forms $\widehat{\Omega}(TM) := \Gamma(\Lambda T^*M \otimes \widehat{S}T^*M)$. We have injective maps

$$T\phi^* := T(\phi_*)^{-1}: \mathcal{V}(M) \rightarrow \widehat{\mathcal{V}}(TM), \quad T\phi^*: \Omega(M) \rightarrow \widehat{\Omega}(TM).$$

Similarly, we set $\widehat{\mathcal{W}}^j(TM) := \Gamma(S^j TM \otimes \widehat{S}T^*M)$ and $\widehat{\mathcal{O}}^j(TM) := \Gamma(S^j T^*M \otimes \widehat{S}T^*M)$ and get

$$T\phi^* := T(\phi_*)^{-1}: \mathcal{W}^j(M) \rightarrow \widehat{\mathcal{W}}^j(TM), \quad T\phi^*: \mathcal{O}^j(M) \rightarrow \widehat{\mathcal{O}}^j(TM).$$

We can now let R naturally act on $\widehat{\mathcal{V}}(TM)$, $\widehat{\Omega}(TM)$, $\widehat{\mathcal{W}}^j(TM)$ and $\widehat{\mathcal{O}}^j(TM)$ by Lie derivative and hence get a differential D on the corresponding complexes of differential forms. Notice that D respects the Gerstenhaber algebra structure (by the vertical Schouten–Nijenhuis bracket) of $\widehat{\mathcal{V}}(TM)$ and the differential complex structure (by the vertical differential) of $\widehat{\Omega}(TM)$, so that these structures are induced in cohomology. By the same argument as above, we get that all these cohomologies are concentrated in degree zero with $H_D^0(\widehat{\mathcal{V}}(TM)) = T\phi^*\mathcal{V}(M)$, $H_D^0(\widehat{\Omega}(TM)) = T\phi^*\Omega(M)$, $H_D^0(\widehat{\mathcal{W}}^j(TM)) = T\phi^*\mathcal{W}^j(M)$, and $H_D^0(\widehat{\mathcal{O}}^j(TM)) = T\phi^*\mathcal{O}^j(M)$. Notice in particular that a section is

in the image of $T\phi^*$ if and only if

$$d_x\sigma + L_R\sigma = 0.$$

In order to recover, in local coordinates, the global object corresponding to a solution to the above equation, we should only observe that by assumption $d_y\phi_x(0) = \text{id}$, so that it is enough to evaluate the components of σ at $y = 0$ and to replace formally each dy^i by dx^i and each $\frac{\partial}{\partial y^i}$ by $\frac{\partial}{\partial x^i}$. More explicitly, if $\sigma_x(y) = \sigma_{x;i_1,\dots,i_n}(y) dy^{i_1} \cdots dy^{i_n}$ is equal to $T\phi^*\omega$, then

$$\omega(x) = \sigma_{x;i_1,\dots,i_n}(0) dx^{i_1} \cdots dx^{i_n}.$$

If on the other hand, $\sigma_x(y) = \sigma_x^{i_1,\dots,i_n}(y) \frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_n}}$ is equal to $T\phi^*(Y)$, then

$$Y(x) = \sigma_x^{i_1,\dots,i_n}(0) \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}}.$$

One can immediately extend these results to direct sums of the vector bundles above. Notice that cohomology also commutes with direct limits. This implies that the cohomology of $\prod_j \widehat{\mathcal{W}}^j(TM)$ is also concentrated in degree zero and coincides with $T\phi^* \prod_j \mathcal{W}^j(M)$. Now we have that $\prod_j \mathcal{W}^j(M) = \Gamma(\widehat{STM})$ whereas $\prod_j \widehat{\mathcal{W}}^j(TM) = \Gamma(\widehat{S}(TM \otimes T^*M))$. Similarly, we see that the cohomology with values in $\widehat{S}(T^*M \otimes T^*M)$ is concentrated in degree zero and coincides with $T\phi^*\Gamma(\widehat{ST^*M})$. To summarize:

$$H_D^\bullet(\Gamma(\widehat{S}(TM \otimes T^*M))) = H_D^0(\Gamma(\widehat{S}(TM \otimes T^*M))) = T\phi^*\Gamma(\widehat{STM})$$

and

$$H_D^\bullet(\Gamma(\widehat{S}(T^*M \otimes T^*M))) = H_D^0(\Gamma(\widehat{S}(T^*M \otimes T^*M))) = T\phi^*\Gamma(\widehat{ST^*M}).$$

2.1. Gauge transformations. We now wish to consider the effects of changing the choice of formal exponential map. Namely, let ϕ be a family of formal exponential maps depending on a parameter t belonging to an open interval I . We may associate to this family a formal exponential map ψ for the manifold $M \times I$ by $\psi(x, t, y, \tau) := ((\phi)_{x,t}(y), t + \tau)$, where τ denotes the tangent variable to t . We want to define the associated connection \widetilde{R} : on a section $\widetilde{\sigma}$ of $\widehat{ST^*}(M \times I)$ we have, by definition,

$$\widetilde{R}(\widetilde{\sigma}) = -(d_y\widetilde{\sigma}, d_\tau\widetilde{\sigma}) \circ \begin{pmatrix} (d_y\phi)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} d_x\phi & \dot{\phi} \\ 0 & 1 \end{pmatrix}.$$

So we can write $\widetilde{R} = R + C dt + T$ with R defined as before (but now t -dependent),

$$C(\widetilde{\sigma}) = -d_y\widetilde{\sigma} \circ (d_y\phi)^{-1} \circ \dot{\phi}$$

and $T = -dt \frac{\partial}{\partial \tau}$. We now spell out the MC equation for \tilde{R} observing that $d_x T = d_t T = 0$ and that T commutes with both R and C . The $(2, 0)$ -form component over $M \times I$ yields again the MC equation for R , whereas the $(1, 1)$ -component reads

$$\dot{R} = d_x C + [R, C].$$

Hence, under a change of formal exponential map, R changes by a gauge transformation with generator the section C of $\widehat{\mathfrak{X}}(TM)$.

Finally, if σ is a section in the image of $T\phi^*$, then by a simple computation one gets

$$\dot{\sigma} = -L_C \sigma,$$

which can be interpreted as the associated gauge transformation for sections.

3. PSM IN FORMAL COORDINATES

The Poisson sigma model (PSM) [17, 20] is a topological field theory with source a two-manifold Σ and target a Poisson manifold M . Before getting to the BV action for the PSM, we fix the notations and introduce the AKSZ formalism [1] (for a gentle introduction to it, especially suited to the PSM, see [10]). Let $\text{Map}(T[1]\Sigma, T^*[1]M)$ be the infinite dimensional graded manifold of maps from $T[1]\Sigma$ to $T^*[1]M$. It fibers over $\text{Map}(T[1]\Sigma, M)$. We denote by \mathbf{X} a ‘‘point’’ of $\text{Map}(T[1]\Sigma, M)$ and by $\boldsymbol{\eta}$ a ‘‘point’’ of the fiber. In local target coordinates, the super fields \mathbf{X} and $\boldsymbol{\eta}$ have simple expressions:

$$\begin{aligned} \mathbf{X}^i &= X^i + \eta_+^i + \beta_+^i, \\ \boldsymbol{\eta}_i &= \beta_i + \eta_i + X_i^+, \end{aligned}$$

where we have ordered the terms in increasing order of form degree on Σ . The ghost number is 0 for X and η , 1 for β , -1 for η_+ and X^+ , and -2 for β_+ . As unperturbed BV action one considers

$$S_0 := \int_{\Sigma} \boldsymbol{\eta}_i d\mathbf{X}^i.$$

Notice that it satisfies the classical master equation (CME) $(S_0, S_0) = 0$ if Σ has no boundary or if appropriate boundary conditions are taken (which we assume throughout). Here $(,)$ is the BV bracket corresponding to the odd symplectic structure on the space of fields for which the superfield $\boldsymbol{\eta}$ is the momentum conjugate to the superfield \mathbf{X} . Formally one may also assume $\Delta S_0 = 0$ where Δ is the BV operator, so S_0 satisfies also the quantum master equation (QME).

To perturb this action, we pick a multivector field Y on M . We may regard it as a function on $T^*[1]M$. We then define S_Y as the integral over $T[1]\Sigma$ of the pullback of Y by the evaluation map

$$\text{ev}: T[1]\Sigma \times \text{Map}(T[1]\Sigma, T^*[1]M) \rightarrow T^*[1]M.$$

Explicitly, for a k -vector field Y , we have

$$S_Y = \frac{1}{k!} \int_{\Sigma} Y^{i_1, \dots, i_k}(\mathbf{X}) \boldsymbol{\eta}_{i_1} \dots \boldsymbol{\eta}_{i_k}.$$

This construction has several interesting properties. First, $(S_0, S_Y) = 0$ (for $\partial\Sigma = \emptyset$ or with appropriate boundary conditions). Second, for any two multivector fields Y and Y' , we have $(S_Y, S_{Y'}) = S_{[Y, Y']}$. The BV action for the PSM with target Poisson structure π is recovered as $S = S_0 + S_{\pi}$. Notice that by the above mentioned properties it satisfies the CME. As for the quantum master equation, we refer to [12], where it is shown that one can assume $\Delta S_{\pi} = 0$ if the Euler characteristic of Σ is zero or if π is unimodular. In the latter case, one picks a volume form v on M such that $\text{div}_v \pi = 0$ and defines Δ according to it.

We consider S_{π} as a perturbation, so we expand the functional integral around the critical points of S_0 . They consist of closed superfields. In particular, the component X of \mathbf{X} will be a constant map, say with image $x \in M$. Fluctuations will explore only a formal neighborhood of x in M , so as in [11, §6], it makes sense to make the change of variables

$$\mathbf{X} = \phi_x(\mathbf{A}), \quad \boldsymbol{\eta} = d\phi_x(\mathbf{A})^{*, -1} \mathbf{B},$$

where ϕ is a formal exponential map and the new superfields (\mathbf{A}, \mathbf{B}) are in $\text{Map}(T[1]\Sigma, T^*[1]T_x M)$. Notice that this change of variables

$$\phi_x: \text{Map}(T[1]\Sigma, T^*[1]T_x M) \rightarrow \text{Map}(T[1]\Sigma, T^*[1]M)$$

is a local symplectomorphism and that

$$T\phi_x^* S_0 = \int_{\Sigma} \mathbf{B}_i d\mathbf{A}^i.$$

The moduli space of vacua (i.e., the space of critical points modulo gauge transformations) is now $\mathcal{H}_x := H^{\bullet}(\Sigma) \otimes T_x M \oplus H^{\bullet}(\Sigma) \otimes T_x^* M[1]$. (Here $H^{\bullet}(\Sigma)$ is regarded as a graded vector space with its natural grading.) We should regard $\mathcal{H} = \bigcup_x \mathcal{H}_x$ as a vector bundle over M , but for the moment we concentrate on a single x . Later on we will also consider the remaining integration over vacua and, in particular, over M (which actually shows up as the space of constant maps $\Sigma \rightarrow M$); see Section 5.

We may repeat the AKSZ construction on $\text{Map}(T[1]\Sigma, T^*[1]T_x M)$. In particular, if Y is a function of degree k on $T^*[1]T_x M$ (i.e., a formal vertical k -vector field), we may construct a functional

$$S_Y = \frac{1}{k!} \int_{\Sigma} Y^{i_1, \dots, i_k}(\mathbf{A}) \mathbf{B}_{i_1} \dots \mathbf{B}_{i_k}.$$

In particular, we have

$$T\phi_x^* S_{\pi} = S_{T\phi_x^* \pi}.$$

As a result, we have a solution $S_x := T\phi_x^* S$ of the QME and may compute its partition function Z_x (as a function on \mathcal{H}_x) upon integrating over a Lagrangian submanifold \mathcal{L} of a complement of \mathcal{H}_x in $\text{Map}(T[1]\Sigma, T^*[1]T_x M)$:

$$Z_x := \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_x}.$$

Notice [19, 12] that there is an induced BV operator Δ on \mathcal{H}_x and that Z_x satisfies $\Delta Z_x = 0$. Moreover, upon changing the gauge fixing \mathcal{L} , Z_x changes by a Δ -exact term. We wish to compare the class of Z_x with the globally defined partition function morally obtained by integrating in $\text{Map}(T[1]\Sigma, T^*[1]M)$. For this we have to understand the collection $\{Z_x\}_{x \in M}$ as a section $\widehat{Z}: x \mapsto Z_x$ of $\widehat{S}\mathcal{H}^*$ (we hide the dependency on \hbar here) and compute how it changes over M . Using all properties above and setting $\widehat{S}: x \mapsto S_x$, we get⁷

$$d_x \widehat{Z} = \frac{i}{\hbar} \int_{\mathcal{L}} e^{\frac{i}{\hbar} \widehat{S}} S_{d_x T\phi^* \pi} = -\frac{i}{\hbar} \int_{\mathcal{L}} e^{\frac{i}{\hbar} \widehat{S}} (S_R, \widehat{S}).$$

Notice that this may also be rewritten as

$$d_x \widehat{Z} = -\Delta \int_{\mathcal{L}} e^{\frac{i}{\hbar} \widehat{S}} S_R$$

if we assume $\Delta S_R = 0$. This is correct if Σ has zero Euler characteristic or if $\text{div}_{T\phi^* v} R = 0$. From the equation $d_x T\phi^* v + L_R T\phi^* v = 0$, we see that the latter condition is satisfied if and only if $d_x T\phi^* v = 0$. Given a volume form v , it is always possible to find a formal exponential map ϕ satisfying this condition; actually, one can even get $T\phi_x^* v = dy^1 \dots dy^d \forall x$.

We can collect the above identities nicely if we define⁸

$$\widetilde{S} := \widehat{S} + S_R.$$

⁷The choice of \mathcal{L} might be different for different x s, but for simplicity we assume it not to be the case.

⁸For $\pi = 0$, \widetilde{S} is also the BV action for the BF_{∞} -theory [19] with target the L_{∞} algebra of footnote 6. See also [14].

Notice that \tilde{S} is of total degree zero (the term S_R has ghost number minus one but is a one-form on M) and satisfies the modified CME

$$d_x \tilde{S} + \frac{1}{2}(\tilde{S}, \tilde{S}) = 0$$

and by assumption also $\Delta \tilde{S} = 0$ (so it satisfies a modified QME as well). We then define

$$(3.1) \quad \tilde{Z} := \int_{\mathcal{C}} e^{\frac{i}{\hbar} \tilde{S}}$$

as a nonhomogeneous differential form on M taking values in \mathcal{H} . It satisfies

$$d_x \tilde{Z} - i\hbar \Delta \tilde{Z} = 0.$$

Remark 3.1. We are now also in a position to understand the change of \tilde{Z} under a change of the formal exponential map. Using the results of subsection 2.1, we immediately see that

$$\tilde{Z} = (d_x - i\hbar \Delta) \int_{\mathcal{C}} e^{\frac{i}{\hbar} \tilde{S}} \frac{i}{\hbar} S_C,$$

assuming $\Delta S_C = 0$. The assumption is verified if Σ has zero Euler characteristic or if we let ϕ vary only in the class of formal exponential maps that make $T\phi^*v$ constant. Notice that the space of such formal exponential maps is connected. Therefore, the class of \tilde{Z} under these transformations is independent of all choices needed to compute it.

Finally, we consider the effective action \tilde{S}_{eff} defined by the identity $\tilde{Z} = e^{\frac{i}{\hbar} \tilde{S}_{\text{eff}}}$. It is a differential form taking values in $\hat{S}\mathcal{H}^*[[\hbar]]$ and satisfies the modified QME

$$(3.2) \quad d_x \tilde{S}_{\text{eff}} + \frac{1}{2}(\tilde{S}_{\text{eff}}, \tilde{S}_{\text{eff}}) - i\hbar \Delta \tilde{S}_{\text{eff}} = 0.$$

This equation formally follows from the properties of BV integrals. In the case when Σ has no boundary, it may be proved directly by considering the expansion of \tilde{S}_{eff} in Feynman diagrams and applying the usual Stokes theorem techniques on integrals over configuration spaces, see [18, 13]. If Σ has a boundary, additional terms corresponding to several points collapsing to the boundary together may appear and spoil (3.2). From now on we therefore assume that Σ is closed.

Equation (3.2) contains both information on the QME satisfied by the zero-form component $\tilde{S}_{\text{eff}}^{(0)}$ and on its global properties. The equations are in general mixed: we do not simply get a flat connection with respect to which $\tilde{S}_{\text{eff}}^{(0)}$ is covariantly constant. However, it is possible to find a modified quantum BV canonical transformation that produces

a flat connection with respect to which the zero form part of the effective action is horizontal and hence global; we postpone this discussion to Section 6. In the remaining of this Section, we concentrate on two special cases where the general theory is not needed.

The first special case is when Σ is a torus. In Section 4, see Lemma 4.4, we will show that, in an appropriate gauge, there are no quantum corrections, so

$$\tilde{S}_{\text{eff}} = \tilde{S}_{\text{eff}}^{(0)} + \tilde{S}_{\text{eff}}^{(1)},$$

where $\tilde{S}_{\text{eff}}^{(0)}$ is the zero-form obtained by restricting $S_{T\phi^*\pi}$ to vacua and $\tilde{S}_{\text{eff}}^{(1)}$ is the one-form obtained by restricting S_R to vacua. One can explicitly check, see Section 5, that $\Delta\tilde{S}_{\text{eff}}^{(0)} = \Delta\tilde{S}_{\text{eff}}^{(1)} = 0$. Hence the modified QME now simply yields the CME for $\tilde{S}_{\text{eff}}^{(0)}$,

$$(\tilde{S}_{\text{eff}}^{(0)}, \tilde{S}_{\text{eff}}^{(0)}) = 0,$$

the flatness condition for $\tilde{S}_{\text{eff}}^{(1)}$,

$$d_x \tilde{S}_{\text{eff}}^{(1)} + \frac{1}{2}(\tilde{S}_{\text{eff}}^{(1)}, \tilde{S}_{\text{eff}}^{(1)}) = 0,$$

and the fact that $\tilde{S}_{\text{eff}}^{(0)}$ is covariantly constant,

$$d_x \tilde{S}_{\text{eff}}^{(0)} + (\tilde{S}_{\text{eff}}^{(1)}, \tilde{S}_{\text{eff}}^{(0)}) = 0.$$

Now notice that $(\tilde{S}_{\text{eff}}^{(1)}, \cdot)$ is just the natural action of R on the sections of \mathcal{H} . Hence we can conclude that $\tilde{S}_{\text{eff}}^{(0)}$ is just $T\phi^*(S|_{\text{vacua}})$.

The second special case is when π is regular and unimodular (and Σ is any two-manifold). Also in Section 4 we will show that, upon choosing an appropriate formal exponential map, $S_{\text{eff}}^{(0)}$ and $S_{\text{eff}}^{(1)}$ have no quantum corrections. Therefore, we may use the same reasoning as above and conclude that $\tilde{S}_{\text{eff}}^{(0)}$ is just $T\phi^*(S|_{\text{vacua}})$.

A final important remark is that in the two cases above the effective action depends polynomially on all vacua but, possibly, for those related to the X field; therefore, $S_{\text{eff}}^{(0)}$ is a section of $S\tilde{\mathcal{H}}^* \otimes \widehat{S}T^*M$, where $\tilde{\mathcal{H}}_x = H^{>0}(\Sigma) \otimes T_x M \oplus H^\bullet(\Sigma) \otimes T_x^* M[1]$ is the moduli space of vacua excluding those for X . The corresponding global effective action $S|_{\text{vacua}}$ will then be a section of $S\tilde{\mathcal{H}}^*$, i.e., a function on the vector bundle $\tilde{\mathcal{H}}$ (polynomial in the fibers). Notice that this vector bundle is diffeomorphic, by choosing a connection (e.g., the one contained in the choice of ϕ), to the natural global definition of the moduli space of vacua as presented, e.g., in [5].

4. SOME COMPUTATIONS OF THE EFFECTIVE ACTION

In this Section we discuss the perturbative computation for the effective action and show that it has no quantum corrections in two important cases.

4.1. Factorization of Feynman graphs. Consider the effective action \tilde{S}_{eff} defined in (3.1) by the identity $\tilde{Z} = e^{\frac{i}{\hbar}\tilde{S}_{\text{eff}}}$. Here the Lagrangian subspace \mathcal{L} in the complement of \mathcal{H}_x inside the space of fields

$$(4.1) \quad \mathcal{F}_x = \text{Map}(T[1]\Sigma, T^*[1]T_x M) \cong \Omega^\bullet(\Sigma) \otimes (T_x M \oplus T_x^*[1]M)$$

accounts for the gauge fixing. Let \mathcal{L} have the factorized form

$$(4.2) \quad \mathcal{L} = \mathcal{L}_K \otimes (T_x M \oplus T_x^*[1]M)$$

where $\mathcal{L}_K \subset \Omega^\bullet(\Sigma)$ is defined as

$$\mathcal{L}_K = \ker \mathcal{P} \cap \ker K$$

with \mathcal{P} the projector from differential forms on Σ to (the chosen representatives of) de Rham cohomology of Σ and

$$K : \Omega^\bullet(\Sigma) \rightarrow \Omega^{\bullet-1}(\Sigma)$$

a linear operator satisfying

$$(4.3) \quad dK + Kd = \text{id} - \mathcal{P}, \quad \mathcal{P}K = K\mathcal{P} = 0, \quad K^T = K, \quad K^2 = 0$$

(i.e., K is the chain homotopy between identity and the projection to cohomology, also known as a parametrix). The transpose is w.r.t. the Poincaré pairing on forms $\int_\Sigma \bullet \wedge \bullet$. We assume the operator K (which now determines the gauge fixing) to be an integral operator with a distributional integral kernel $\omega \in \Omega^1(\Sigma \times \Sigma)$ – the propagator. An explicit construction may be done along the same lines as in [8, 9]. Let us introduce a basis $\{\chi_\alpha\}$ in the cohomology space $H^\bullet(\Sigma)$; denote the matrix of the Poincaré pairing by $\Pi_{\alpha\beta} = \int_\Sigma \chi_\alpha \wedge \chi_\beta$. In terms of ω , properties (4.3) read:

- (1) $d\omega = \delta_{\text{diag}} - \sum_{\alpha,\beta} (\Pi^{-1})^{\alpha\beta} \chi_\alpha \otimes \chi_\beta$, where δ_{diag} is the delta-form supported on the diagonal of $\Sigma \times \Sigma$;
- (2) $\int_{\Sigma_{(1)}} \omega \pi_1^* \chi_\alpha = \int_{\Sigma_{(2)}} \omega \pi_2^* \chi_\alpha = 0$, $\forall \alpha$, where $\Sigma_{(1)}$ and $\Sigma_{(2)}$ denote the two factors of $\Sigma \times \Sigma$, and π_1 and π_2 are the two projections from $\Sigma \times \Sigma$ to its factors;
- (3) $t^* \omega = \omega$, where $t : \Sigma \times \Sigma \rightarrow \Sigma \times \Sigma$ is the map swapping the two copies of Σ ;
- (4) $\int_{\Sigma_{(2)}} \pi_{12}^* \omega \pi_{23}^* \omega = 0$, where $\Sigma_{(2)}$ denotes the middle factor in $\Sigma \times \Sigma \times \Sigma$, whereas π_{12} and π_{23} are the projections from $\Sigma \times \Sigma \times \Sigma$ to the first two and last two factors, respectively.

Notice that the restriction of ω to the configuration space $C_2^0(\Sigma) := \{(u, v) \in \Sigma : u \neq v\}$ is smooth and it extends to the Fulton–MacPherson–Axelrod–Singer compactification as a smooth form. In [13] it is shown how to implement the property $\mathcal{PK} = K\mathcal{P} = 0$ on the propagator. Once this is done, the propagator will also satisfy the property $K^2 = 0$. This is proved exactly as in [12, Lemma 10].

The perturbation expansion for the effective action (3.1) has the form

$$(4.4) \quad S^{\text{eff}}(\mathbf{A}_{\text{z.m.}}, \mathbf{B}_{\text{z.m.}}; \hbar) = \sum_{\Gamma} \frac{(i\hbar)^{l(\Gamma)}}{|\text{Aut}(\Gamma)|} W_{\Gamma}^{\text{target}}(\mathbf{A}_{\text{z.m.}}, \mathbf{B}_{\text{z.m.}}) \cdot W_{\Gamma}^{\text{source}}$$

where the sum is over connected oriented graphs Γ with leaves⁹ decorated by basis cohomology classes $\{\chi_{\alpha}\}$; $l(\Gamma)$ stands for the number of loops, $|\text{Aut}(\Gamma)|$ is the number of graph automorphisms; $\{\mathbf{A}_{\text{z.m.}}, \mathbf{B}_{\text{z.m.}}\} = \{\mathbf{A}^{\alpha i}, \mathbf{B}_i^{\alpha}\}$ are the coordinates on the moduli space of vacua \mathcal{H}_x .

The “target part” $W_{\Gamma}^{\text{target}}$ of the contribution of a graph to \tilde{S}_{eff} is a homogeneous polynomial function on \mathcal{H}_x of degree equal to the number of leaves, computed using the following set of rules:

- (1) to an incoming leaf of Γ decorated by χ_{α} one associates $\mathbf{A}^{\alpha i}$
- (2) to an outgoing leaf decorated by χ_{α} one associates \mathbf{B}_i^{α}
- (3) to a vertex with m inputs and n outputs one associates the expression

$$\partial_{i_1} \dots \partial_{i_m} Y^{j_1 \dots j_n}$$

– the m -th derivative of n -vector¹⁰ contribution to the action S .

- (4) for every edge contract the dummy Latin indices for the two constituent half-edges.

The result of contraction is a polynomial function on \mathcal{H}_x .

The “source” (or “de Rham”) part $W_{\Gamma}^{\text{source}}$ is a number defined as

$$(4.5) \quad W_{\Gamma}^{\text{source}} = \int_{\Sigma^{\times V(\Gamma)}} \left(\prod_{\text{edges } (h_{\text{in}}, h_{\text{out}})} \pi_{v(h_{\text{in}}, v(h_{\text{out}}))}^* \omega \right) \cdot \left(\prod_{\text{leaves } l} \pi_{v(l)}^* \chi_{\alpha_l} \right)$$

⁹A leaf for us is a loose half-edge, i.e., one not connected to another half-edge to form an edge.

¹⁰In the standard setup for the Poisson sigma model, only the perturbation by Poisson bivector field $Y = \pi^{ij} \partial_i \wedge \partial_j$ is present in the action, hence all vertices have to have exactly two outputs, otherwise the graph does not contribute. In the present case, we also have the vector field R .

where $V(\Gamma)$ is the number of vertices, $\pi_v : \Sigma^{\times V(\Gamma)} \rightarrow \Sigma$ is the projection to v -th copy of Σ , $\pi_{u,v} : \Sigma^{\times V(\Gamma)} \rightarrow \Sigma \times \Sigma$ is the projection to u -th and v -th copies of Σ ; $v(h)$ is the vertex incident to the half-edge h . Notice that all these integrals converge. The usual way to show this is to observe that the integrals are actually defined on configuration spaces (i.e., the complements of all diagonals in the Cartesian products of copies of Σ) and that the propagators ω extend to their compactifications.

Remark 4.1. The factorization into source- and target contributions for Feynman diagrams in the expansion (4.4) is due to the factorization of the space of fields (4.1) and to the fact that our ansatz for the gauge fixing (4.2) is compatible with this factorization.

Remark 4.2. The orientation of Γ is irrelevant for the source parts W_Γ^{source} .

4.2. Regular Poisson structures. If π is nondegenerate, it is always possible to find a formal exponential map ϕ such that $T\phi^*\pi$ is constant (in the y variables). One simply has to go to formal Darboux coordinates. Notice, moreover, that $\text{div}_v \pi = 0$ if for v one chooses v to be the Liouville volume form $\omega^k/k!$, $k = \dim M/2$. It then follows that $T\phi^*v$ is also constant and that $\text{div}_{T\phi^*v} R = 0$. A slight generalization occurs when π is regular (i.e., its kernel has constant rank) and unimodular (notice that this is not guaranteed if π is degenerate [21]). After choosing v such that $\text{div}_v \pi = 0$, it is again possible to find a formal exponential map ϕ such that $T\phi^*\pi$ and $T\phi^*v$ are both constant and hence $\text{div}_{T\phi^*v} R = 0$.

In the perturbative expansion, we may thus assume that we have a bivalent vertex, corresponding to $T\phi^*\pi$, with no incoming arrows. If one of the outgoing arrows is replaced by a vacuum mode (i.e., a cohomology class), the result is zero by the property $\mathcal{P}K = K\mathcal{P} = 0$, otherwise it is zero by the property $K^2 = 0$. As a result, every graph containing a $T\phi^*\pi$ -vertex will vanish, apart from the one with both outgoing arrows evaluated on vacua. As a consequence $S_{\text{eff}}^{(0)}$ and $S_{\text{eff}}^{(1)}$ have no quantum corrections.

4.3. Axial gauge on the torus $\Sigma = \mathbb{T}^2 := S^1 \times S^1$. In the case of a torus, differential forms have a bigrading with respect to the two circles. One may choose the axial gauge¹¹ by setting the superfields to vanish if they have nonzero degree with respect to the first circle. Naïvely this implies the propagator to be the product of a propagator for the de

¹¹The axial gauge for topological field theories was originally proposed in the context of Chern–Simons theory in [15].

Rham differential on the first circle and the identity operator on the second circle (just plug in the gauge fixed fields into in the unperturbed action to realize this). This argument however does not take vacua into account nor the fact that the axial gauge fixing does not fix all the gauge freedom. In fact, one can prove that the propagator in the axial gauge has one additional term, see (4.7) below.

To start with a rigorous construction of the propagator, observe that differential forms on a circle admit the Hodge decomposition

$$\Omega^\bullet(S^1) = \underbrace{\Omega_{\text{Harm}}^\bullet(S^1)}_{\text{Span}(1, d\tau)} \oplus \underbrace{\tilde{\Omega}^0(S^1)}_{\{f(\tau) \mid \int_{S^1} f(\tau) d\tau = 0\}} \oplus \underbrace{\tilde{\Omega}^1(S^1)}_{\{g(\tau) d\tau \mid \int_{S^1} g(\tau) d\tau = 0\}}$$

(In our convention the coordinate τ on the circle runs from 0 to 1). The associated chain homotopy operator is

$$K_{S^1} : g(\tau) d\tau \mapsto \int_{S^1} \omega_{S^1}(\tau, \tau') g(\tau') d\tau'$$

with the integral kernel

$$\omega_{S^1}(\tau, \tau') = \theta(\tau - \tau') - \tau + \tau' - \frac{1}{2}$$

Projection to harmonic forms on the circle (representatives of cohomology) is

$$\mathcal{P}_{S^1} : f(\tau) + g(\tau) d\tau \mapsto \int_{S^1} (d\tau' - d\tau) \wedge (f(\tau') + g(\tau') d\tau')$$

For the torus we may decompose the de Rham complex in the following way:

$$\begin{aligned} (4.6) \quad \Omega^\bullet(S^1 \times S^1) &= \Omega^\bullet(S^1) \hat{\otimes} \Omega^\bullet(S^1) = \\ &= \underbrace{\Omega_{\text{Harm}}^\bullet(S^1) \otimes \Omega_{\text{Harm}}^\bullet(S^1)}_{\cong H^\bullet(S^1 \times S^1)} \oplus \underbrace{\tilde{\Omega}^0(S^1) \hat{\otimes} \Omega^\bullet \oplus \Omega_{\text{Harm}}^\bullet(S^1) \otimes \tilde{\Omega}^0(S^1)}_{\mathcal{L}_K} \oplus \\ &\quad \oplus \tilde{\Omega}^1(S^1) \hat{\otimes} \Omega^\bullet \oplus \Omega_{\text{Harm}}^\bullet(S^1) \otimes \tilde{\Omega}^1(S^1) \end{aligned}$$

The associated chain homotopy operator is

$$(4.7) \quad K = \underbrace{K_{S^1} \otimes \text{id}_{S^1}}_{K_I} + \underbrace{\mathcal{P}_{S^1} \otimes K_{S^1}}_{K_{II}} : \Omega^\bullet(S^1 \times S^1) \rightarrow \Omega^{\bullet-1}(S^1 \times S^1)$$

Its integral kernel (the propagator) is

$$(4.8) \quad \omega = \underbrace{\left(\theta(\sigma - \sigma') - \sigma + \sigma' - \frac{1}{2} \right) \cdot \delta(\tau - \tau') \cdot (d\tau' - d\tau)}_{\omega_I} + \underbrace{(d\sigma' - d\sigma) \cdot \left(\theta(\tau - \tau') - \tau + \tau' - \frac{1}{2} \right)}_{\omega_{II}}$$

where we denote by $\sigma, \tau \in \mathbb{R}/\mathbb{Z}$ the coordinates on the first and the second circles, respectively.

Remark 4.3. The chain homotopy (4.7) arises from the composition of two quasi-isomorphisms:

$$\begin{aligned} \Omega^\bullet(S^1) \hat{\otimes} \Omega^\bullet(S^1) & \quad \equiv \quad \Omega_{\text{Harm}}^\bullet(S^1) \otimes \Omega^\bullet(S^1) \oplus \tilde{\Omega}^0(S^1) \hat{\otimes} \Omega^\bullet(S^1) \oplus \tilde{\Omega}^1(S^1) \hat{\otimes} \Omega^\bullet(S^1) \\ & \quad \downarrow \\ \Omega_{\text{Harm}}^\bullet(S^1) \otimes \Omega^\bullet(S^1) & \quad \equiv \quad \Omega_{\text{Harm}}^\bullet \otimes \Omega_{\text{Harm}}^\bullet \oplus \Omega_{\text{Harm}}^\bullet(S^1) \otimes \tilde{\Omega}^0(S^1) \oplus \Omega_{\text{Harm}}^\bullet(S^1) \otimes \tilde{\Omega}^1(S^1) \\ & \quad \downarrow \\ \Omega_{\text{Harm}}^\bullet(S^1) \otimes \Omega_{\text{Harm}}^\bullet(S^1) & \end{aligned}$$

i.e., we first contract the first circle to cohomology, then the second one.

4.4. Vanishing of quantum corrections.

Lemma 4.4. *For the Poisson sigma model in the axial gauge on torus, the source parts W_Γ^{source} vanish for all connected graphs Γ except for trees with one vertex (“corollas”).*

Proof. Let us introduce the basis in cohomology of the torus:

$$\chi_{(0,0)} = 1, \quad \chi_{1,0} = d\sigma, \quad \chi_{(0,1)} = d\tau, \quad \chi_{(1,1)} = d\sigma \wedge d\tau$$

Define a decoration c of Γ as an assignment of bidegree

$$c(h) \in \{(0,0), (1,0), (0,1), (1,1)\}$$

to each half-edge h of Γ (so that on leaves the bidegree coincides with the prescribed leaf decoration α) together with an assignment of an index $c(e) \in \{I, II\}$ to each edge e . Define the source part for a decorated graph Γ as

$$(4.9) \quad W_{\Gamma,c}^{\text{source}} = \int_{\Sigma \times V(\Gamma)} \left(\prod_{\text{edges } e=(h_{\text{in}}, h_{\text{out}})} \pi_{v(h_{\text{in}}), v(h_{\text{out}})}^* \omega_{c(e)}|_c \right) \cdot \left(\prod_{\text{leaves } l} \pi_{v(l)}^* \chi_{\alpha_l} \right)$$

where the $\omega|_c$ symbol means the component of the propagator (as an element of $\Omega^\bullet(S^1 \times S^1) \hat{\otimes} \Omega^\bullet(S^1 \times S^1)$) of de Rham bidegrees $c(h_{in}), c(h_{out})$ where h_{in}, h_{out} are the constituent half-edges of the edge; $\omega_{c(e)}$ is one of the two pieces of propagator, ω_I or ω_{II} , as defined in (4.8). Then we have

$$W_\Gamma^{\text{source}} = \sum_{\text{decorations } c} W_{\Gamma,c}^{\text{source}}$$

The source part $W_{\Gamma,c}^{\text{source}}$ vanishes automatically unless the following conditions are satisfied simultaneously:

- (i) At every vertex there is exactly one incident half-edge decorated by $(1, \bullet)$, all others are $(0, \bullet)$.
- (ii) At every vertex there is exactly one incident half-edge decorated by $(\bullet, 1)$, all others are $(\bullet, 0)$. (This half-edge may be the same as in (i)).
- (iii) Compatibility between edge decorations and half-edge decorations: for any edge $e = (h_1, h_2)$ we have

$$\begin{aligned} (c(e) = I) &\implies \begin{cases} c(h_1) = (0, 0), c(h_2) = (0, 1) \\ c(h_1) = (0, 1), c(h_2) = (0, 0) \end{cases} & \text{or} \\ (c(e) = II) &\implies \begin{cases} c(h_1) = (0, 0), c(h_2) = (1, 0) \\ c(h_1) = (1, 0), c(h_2) = (0, 0) \end{cases} & \text{or} \end{aligned}$$

- (iv) Number of edges decorated as I adjacent to any given vertex should be different from one.
- (v) If a vertex has no adjacent I -edges, then the number of adjacent II -edges should be different from one.

Requirements (i,ii) follow directly from degree counting in (4.9); (iii) follows from the formula for propagator (4.8); (iv,v) follow from the property $K_{S^1} \mathcal{P}_{S^1} = 0$ and from the fact that harmonic forms on a circle are closed under wedge multiplication.

Fix some decoration c of Γ satisfying (i–v). Consider the subgraph Γ_I of Γ obtained by deleting all II -edges in Γ ; Γ_I may be disconnected. Let $\Gamma_I = \sqcup_a \Gamma_I^a$ where Γ_I^a are the connected components of Γ_I . Due to (ii), the number of vertices V_I^a of Γ_I^a is equal to the number of $(0, 1)$ -half-edges in Γ_I^a which is in turn greater or equal to the number of edges E_I^a due to (iii). Hence the Euler characteristic of Γ_I^a non-negative: $V_I^a - E_I^a \geq 0$. Therefore Γ_I^a is either a tree or a 1-loop graph. Next, property (iv) shows that Γ_I^a has to be a wheel graph, with arbitrary number of leaves attached at vertices, or a corolla. On the other hand,

if Γ_I contains a wheel then the corresponding source part vanishes:

$$(4.10) \quad \begin{aligned} W_{\Gamma,c}^{\text{source}} &= \int_{(S^1 \times S^1)^{\times V}} (d\tau_1 - d\tau_2) \delta(\tau_2 - \tau_1) \wedge \cdots \wedge (d\tau_n - d\tau_1) \delta(\tau_1 - \tau_n) \wedge F = \\ &= \int_{(S^1 \times S^1)^{\times V}} \underbrace{(d\tau_1 \wedge d\tau_2 \wedge \cdots \wedge d\tau_n + (-1)^n d\tau_2 \wedge \cdots \wedge d\tau_n \wedge d\tau_1)}_{=0} \wedge \\ &\quad \wedge \delta(\tau_2 - \tau_1) \cdots \delta(\tau_1 - \tau_n) \wedge F = 0 \end{aligned}$$

where n is the length of the wheel and $F \in \Omega^\bullet((S^1 \times S^1)^{\times V})$ is some differential form.

Remark 4.5. Argument (4.10) has the fault that the integrand is singular and the result is $0 \cdot \delta(0)$. This can be remedied by regularizing the propagator ω , e.g., by changing $\delta(\tau - \tau')$ in (4.8) to a smeared delta-function. Notice that the source parts of all diagrams except corollas still vanish exactly: in this vanishing argument the chain homotopy equation is never used; we only use the de Rham bigrading properties, $PK = 0$ and the fact that harmonic forms on a circle are closed under multiplication.

Thus we have shown that $W_{\Gamma,c}^{\text{source}}$ vanishes unless Γ_I is a collection of corollas (i.e. there are no I -edges).

Now fix a decoration c satisfying (i-v) with $c(e) = II$ for all edges. Repeating the Euler characteristic argument as above (using properties (i,iii)), we show that Γ has to be either a tree or a 1-loop graph and using property (v) we show that it has to be either a wheel or a corolla. If it is a wheel then

$$(4.11) \quad \begin{aligned} W_{\Gamma,c}^{\text{source}} &= \int_{(S^1 \times S^1)^{\times V}} (d\sigma_1 - d\sigma_2) \wedge \cdots \wedge (d\sigma_V - d\sigma_1) \wedge F = \\ &= \int_{(S^1 \times S^1)^{\times V}} \underbrace{(d\sigma_1 \wedge \cdots \wedge d\sigma_V + (-1)^V d\sigma_2 \wedge \cdots \wedge d\sigma_V \wedge d\sigma_1)}_{=0} \wedge F = \\ &= 0 \end{aligned}$$

Therefore $W_{\Gamma,c}^{\text{source}}$ vanishes for any decoration c unless Γ is a corolla. This concludes the proof of the Lemma. \square

An immediate consequence of the Lemma is that the effective action S_x^{eff} is just the restriction of the action S_x to vacua: there are no quantum corrections.

5. THE PARTITION FUNCTION ON THE TORUS

Let S_{eff} be the global effective action on the moduli space of vacua for the torus \mathbb{T}^2 . In Lemma 4.4, we have shown that it has no quantum corrections. The moduli space of vacua can be viewed as $\text{Map}(\mathbb{R}^2[1], T^*[1]M)$ and in [5] it has been remarked that the action restricted to vacua is the AKSZ action for this mapping space. In local coordinates the superfields are

$$(5.1) \quad x^\mu = x^\mu + e^1 \eta_1^{+\mu} + e^2 \eta_2^{+\mu} - s b^{+\mu}, \quad \mathbf{e}_\nu = b_\nu + e^1 \eta_{\nu 1} + e^2 \eta_{\nu 2} + s x_\nu^+,$$

where $s = e^1 e^2$ is the generator of $H_{dR}^2(\mathbb{T}^2)$ normalized to $\int_{\mathbb{R}^2[1]} ds s = 1$.

If π is the Poisson bivector field on M , then

$$(5.2) \quad S_{\text{eff}} = \frac{1}{2} \int_{\mathbb{R}^2[1]} ds \pi^{\mu\nu}(x) \mathbf{e}_\mu \mathbf{e}_\nu.$$

There exists a canonical Berezinian given by the coordinate volume form

$$(5.3) \quad \nu = dx \cdots dx^+ \cdots db \cdots db^+ \cdots d\eta_i \cdots d\eta_i^+ \cdots.$$

If we denote with Δ the corresponding Laplacian, the AKSZ action satisfies

$$\Delta e^{\frac{i}{\hbar} S_{\text{eff}}} = 0$$

and defines a class in Δ -cohomology.

5.1. Kähler gauge fixing and Euler class. Now let π be symplectic such that π^{-1} is the Kähler form of the hermitian structure (J, g) . In the complex coordinates $\{x^i\}$ of M we have $\pi^{i\bar{j}} = ig^{i\bar{j}}$. Let us fix a complex structure on \mathbb{T}^2 defined by $z = \theta^1 + \tau\theta^2$, for $\tau = \tau_1 + i\tau_2$ and $\tau_2 > 0$. Let

$$\eta_z^{+\mu} = (\eta_2^{+\mu} - \bar{\tau}\eta_1^{+\mu})/2i\tau_2, \quad \eta_{z\mu} = (\eta_{2\mu} - \bar{\tau}\eta_{1\mu})/2i\tau_2.$$

Let $\mathcal{L}_{\varepsilon, \tau}$ be the following Lagrangian submanifold of $\text{Map}(\mathbb{R}^2[1], T^*[1]M)$:

$$(5.4) \quad \eta_z^{+i} = \eta_{\bar{z}}^{+\bar{i}} = \eta_{zi} = \eta_{\bar{z}\bar{i}} = x^+ = b^+ = 0.$$

Let us define

$$p_k = \eta_{\bar{z}k} + \Gamma_{ki}^j \eta_{\bar{z}}^{+i} b_j,$$

where Γ are the Christoffel symbols of the Levi-Civita connection. All fiber coordinates b, p, η^+ transform tensorially with respect to a transformation of coordinates on M so that $\mathcal{L}_{\varepsilon, \tau} = (T^*[1] + T^*M + T[-1])M$. After a straightforward computation we get

$$S_{\text{eff}} = \tau_2 \left(R_{s\bar{i}k}^j g^{s\bar{r}} \eta_{\bar{z}}^{+k} \eta_z^{+\bar{l}} b_j b_{\bar{r}} + g^{i\bar{j}} p_i p_{\bar{j}} \right)$$

$$= \frac{1}{2} \tau_2 (R^{\mu\lambda} b_\mu b_\lambda + g^{\mu\nu} p_\mu p_\nu) .$$

The induced Berezinian on $\mathcal{L}_{\varepsilon, \tau}$ reads

$$\sqrt{\nu} = \frac{dx^\mu \cdots db_\mu \cdots dp_\mu \cdots d\eta^{+\mu} \cdots}{(2\pi)^m},$$

with $m = \dim M$.¹² If we perform the fiberwise integration with respect to the fibration $\mathcal{L}_{\varepsilon, \tau} \rightarrow T[-1]M$, we get

$$\begin{aligned} \frac{1}{(2\pi)^m} \int dp_\mu \cdots db_\mu \cdots e^{\frac{i}{\hbar} S_{\text{eff}}} &= \\ &= \frac{1}{(2\pi)^m} \frac{1}{(i\hbar)^{\frac{m}{2}}} \int db_\mu \cdots \frac{1}{\det(\tau_2 g^{\mu\nu})^{1/2}} e^{\frac{i}{2\hbar} \tau_2 R^{\mu\nu} b_\mu b_\nu} = \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}} \int db'_\mu \cdots \frac{1}{\det(g^{\mu\nu})^{1/2}} e^{-\frac{1}{2} R^{\mu\nu} b'_\mu b'_\nu} = \\ &= \frac{1}{(2\pi)^{\frac{m}{2}}} \sqrt{g} \text{Pf}(R) \in C^\infty(T[-1]M) = \Omega M, \end{aligned}$$

which, by the Chern–Gauss–Bonnet theorem, is a representative of the Euler class. Notice that \hbar and τ_2 disappear in the final formula. (The main reason for this is that scaling the b and p variables by the same factor preserves the Berezinian since the former are odd and the latter are even variables.) Finally, we can integrate over M getting

$$Z = \chi(M),$$

the Euler characteristic of M . (Actually, by the argument in the Introduction that the partition function should be the Euler characteristic, we might in reverse think of this result as one more physical proof of the Chern–Gauss–Bonnet theorem, in the case of Kähler manifolds.)

Remark 5.1. In this Section we have assumed the existence of a Kähler structure on M . We expect the above results to hold if we just use an almost Kähler structure, but computations become much more involved.

¹²We choose here the standard convention that the measure for a pair of even conjugate coordinates p, q is $dp dq / (2\pi i \hbar)$, whereas the measure for a pair of odd conjugate coordinates b, η^+ is $i \hbar db d\eta^+$. This is consistent with the standard normalization

$$\int e^{\frac{i}{\hbar} pq} \frac{dp dq}{2\pi i \hbar} = \int e^{\frac{i}{\hbar} b \eta^+} i \hbar db d\eta^+ = 1.$$

Remark 5.2. Another possible gauge fixing consists in setting all $+$ variables to zero. The effective action then reduces to $\pi^{\mu\nu}(x)\eta_{\mu 1}\eta_{\nu 2}$ and is independent of the b variables. If M is compact, the integrals over the η and x variables is finite (and proportional to the symplectic volume of M); because of the b -integration, the partition function then vanishes. If M is not compact, the partition function is ambiguous and of the form $0 \cdot \infty$. This gauge fixing is then in general not equivalent to the Kähler one used above. From the considerations in the Introduction, the Kähler gauge fixing is the one compatible with the Hamiltonian interpretation of the theory.

5.2. Regularized effective action. We now show that the symmetries of S_{eff} induce a regularization which allows one to compute the partition function for every Poisson structure and to show that, independently of the Poisson structure, one gets the Euler characteristic of the target.¹³

The main remark is that the effective action and the symplectic form are invariant under the action of the Lie algebra of divergenceless vector fields of $\mathbb{R}^2[1]$ on the moduli space of vacua. This Lie algebra is spanned by the vector fields $\frac{\partial}{\partial e^1}$, $\frac{\partial}{\partial e^2}$, $e^2 \frac{\partial}{\partial e^1}$, $e^1 \frac{\partial}{\partial e^2}$ and $e^1 \frac{\partial}{\partial e^1} - e^2 \frac{\partial}{\partial e^2}$. The fifth vector field is generated by the previous ones and we are not going to need it in the following. We lift the first four vector fields first to $\text{Map}(\mathbb{R}^2[1], M)$ and next to its cotangent bundle shifted by one. We will denote the resulting vector fields by δ_1 , δ_2 , K_1 and K_2 , respectively. Since they have degree -1 or 0 and the symplectic form has degree -1 , they are also automatically Hamiltonian with uniquely defined Hamiltonian functions τ_1 and τ_2 (of degree -2), and ρ_1 and ρ_2 (of degree -1). The Lie algebra relations translate into the Poisson bracket relations

$$(5.5) \quad (\tau_2, \rho_1) = \tau_1, \quad (\tau_1, \rho_2) = \tau_2, \quad (\tau_1, \rho_1) = 0, \quad (\tau_2, \rho_2) = 0.$$

Also notice that we have

$$(5.6) \quad K_1 \circ \delta_1 = K_2 \circ \delta_2 = 0,$$

which implies that ρ_i Poisson commutes with every δ_i -exact function. Finally, since we started with divergenceless vector fields, we get

$$(5.7) \quad \Delta\tau_1 = \Delta\tau_2 = \Delta\rho_1 = \Delta\rho_2 = 0.$$

Remark 5.3. Even though we do not need the explicit form of these vector fields and their Hamiltonian functions, we give them for completeness of our presentation. From the defining formulae $\delta_i \mathbf{x} = \frac{\partial \mathbf{x}}{\partial e^i}$,

¹³We thank T. Johnson-Freyd for pointing out this approach.

$\delta_i \mathbf{e} = \frac{\partial \mathbf{e}}{\partial e^i} K_1 \mathbf{x} = e^2 \frac{\partial \mathbf{x}}{\partial e^1}$, $K_1 \mathbf{e} = e^2 \frac{\partial \mathbf{e}}{\partial e^1}$, $K_2 \mathbf{x} = e^1 \frac{\partial \mathbf{x}}{\partial e^2}$ and $K_2 \mathbf{e} = e^1 \frac{\partial \mathbf{e}}{\partial e^2}$, we get

$$\begin{aligned}\delta_1 &= \eta_1^{+\mu} \frac{\partial}{\partial x^\mu} + b^{+\mu} \frac{\partial}{\partial \eta_2^{+\mu}} + \eta_{1\mu} \frac{\partial}{\partial b_\mu} - x_\mu^+ \frac{\partial}{\partial \eta_{2\mu}}, \\ \delta_2 &= \eta_2^{+\mu} \frac{\partial}{\partial x^\mu} - b^{+\mu} \frac{\partial}{\partial \eta_1^{+\mu}} + \eta_{2\mu} \frac{\partial}{\partial b_\mu} + x_\mu^+ \frac{\partial}{\partial \eta_{1\mu}}, \\ K_1 &= \eta_{1\mu} \frac{\partial}{\partial \eta_{2\mu}} + \eta_1^{+\mu} \frac{\partial}{\partial \eta_2^{+\mu}}, \\ K_2 &= \eta_{2\mu} \frac{\partial}{\partial \eta_{1\mu}} + \eta_2^{+\mu} \frac{\partial}{\partial \eta_1^{+\mu}}.\end{aligned}$$

The corresponding Hamiltonian functions, with respect to the symplectic structure

$$\omega = \int de^2 de^1 \delta x^\mu \delta e_\mu = \delta x^\mu \delta x_\mu^+ + \delta \eta_1^{+\mu} \delta \eta_{2\mu} - \delta \eta_2^{+\mu} \delta \eta_{1\mu} - \delta b^{+\mu} \delta b_\mu,$$

are given by

$$\begin{aligned}\tau_1 &= x_\mu^+ \eta_1^{+\mu} - \eta_{1\mu} b^{+\mu}, \\ \tau_2 &= x_\mu^+ \eta_2^{+\mu} - \eta_{2\mu} b^{+\mu}, \\ \rho_1 &= -\eta_{1\mu} \eta_1^{+\mu}, \\ \rho_2 &= \eta_{2\mu} \eta_2^{+\mu}.\end{aligned}$$

We now turn back to the effective action. It turns out that it is not only δ_1 - and δ_2 -closed, but actually exact:

$$S_{\text{eff}} = \delta_2 \delta_1 \sigma, \quad \sigma := \frac{1}{2} \pi^{\mu\nu}(x) b_\mu b_\nu.$$

From all the above it follows that S_{eff} Poisson commutes not only with τ_1 and τ_2 , but also with ρ_1 and ρ_2 . Notice that the Jacobi identity for π implies $(S_{\text{eff}}, \sigma) = 0$.

Now consider the regularized effective action

$$S_{\text{eff}}^{\epsilon, t_1, t_2} := \epsilon S_{\text{eff}} - i\hbar(t_1 \tau_1 + t_2 \tau_2),$$

which satisfies the QME for all ϵ, t_1, t_2 . By all the above it follows that

$$\begin{aligned}\frac{\partial}{\partial t_1} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} &= \Delta \left(\frac{1}{t_2} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} \rho_1 \right), \\ \frac{\partial}{\partial t_2} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} &= \Delta \left(\frac{1}{t_1} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} \rho_2 \right),\end{aligned}$$

which means that, as long as the parameters t_1 and t_2 are different from zero, the regularized effective action is independent of them up to quantum canonical transformations. We also have

$$\frac{\partial}{\partial \epsilon} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} = \Delta \left(\frac{i}{\hbar t_2} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} \delta_1 \sigma \right) = -\Delta \left(\frac{i}{\hbar t_1} e^{\frac{i}{\hbar} S_{\text{eff}}^{\epsilon, t_1, t_2}} \delta_2 \sigma \right),$$

which implies that, as long as one of the parameters t_1 and t_2 is different from zero, the regularized effective action is independent of ϵ up to quantum canonical transformations. This in particular means that the partition function is independent of ϵ and that, in order to compute it, we may simply set ϵ to zero.

To perform the final computation we further deform the regularized effective action by adding one more irrelevant term. Namely, let G be a function on $\text{Map}(\mathbb{R}^2[1], M)$. Then

$$S_{\text{eff}}^{0, t_1, t_2, G} := -i\hbar(t_1\tau_1 + t_2\tau_2 + \delta_2\delta_1G)$$

satisfies the QME for all t_1, t_2, G . Moreover, if we take a path $G(t)$ of such functions, we get

$$\frac{\partial}{\partial t} e^{\frac{i}{\hbar} S_{\text{eff}}^{0, t_1, t_2, G(t)}} = \Delta \left(\frac{e^{\frac{i}{\hbar} S_{\text{eff}}^{0, t_1, t_2, G(t)}} \delta_1 \dot{G}(t)}{t_2} \right) = -\Delta \left(\frac{e^{\frac{i}{\hbar} S_{\text{eff}}^{0, t_1, t_2, G(t)}} \delta_2 \dot{G}(t)}{t_1} \right),$$

which means that, as long as one of the two parameters t_1 and t_2 is different from zero, adding the new term is irrelevant up to quantum canonical transformations. We are now ready to compute the partition function. Namely, we choose $\mathcal{L} := \text{Map}(\mathbb{R}^2[1], M)$ as the Lagrangian submanifold of $\text{Map}(\mathbb{R}^2[1], T^*[1]M)$ over which we integrate. Since $\tau_1|_{\mathcal{L}} = \tau_2|_{\mathcal{L}} = 0$, we get

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_{\text{eff}}^{0, t_1, t_2, G}} = \int_{\text{Map}(\mathbb{R}^2[1], M)} e^{\delta_2 \delta_1 G}$$

and we already know that the last integral is independent of G . We only have to make sure that G is chosen in such a way that the integral is well defined (choosing $G = 0$, e.g., would lead to $\infty \cdot 0$). A good choice is $G := g_{\mu\nu}(x)\eta_1^{+\mu}\eta_2^{+\nu}$ where $g^{\mu\nu}$ is a Riemannian metric on target. An explicit computation [3, 4] then shows that $Z = \chi(M)$.

Remark 5.4. Switching ϵ to zero first and then turning on the regularizing term in G is a bit formal since we pass through the solution to the QME where both terms are absent. This solution has a singular integral (of the type $0 \cdot \infty$) on \mathcal{L} . In order to find a non formal regularization it is necessary to have additional structure on the Poisson manifold. For instance let us look for G such that $S_{\text{eff}}^{\epsilon, t_1, t_2, G}$ satisfies the

QME for any ϵ , preserving the property that the variation of G produces a quantum canonical transformation. Indeed, let us assume G as above but let g be possibly degenerate. If $(S_{\text{eff}}, G) = 0$ then $S_{\text{eff}}^{\epsilon, t_1, t_2, G}$ satisfies the QME and the change of G is a quantum canonical transformation. This property is equivalent to require that $\pi \circ g = 0$ and $L_V g = 0$ for every vector field V tangent to the symplectic leaves. In special cases we may find such a g and in addition a Kähler structure on the leaves, compatible with the symplectic structure, such that a gauge fixing given by a mixture of what we discussed in this subsection and the Kähler one is available. We plan to investigate the geometrical conditions needed for this gauge fixing in the future.

5.3. Regularization on the space of fields. The argument of the previous subsection may formally be lifted to the space of fields to show that the regularized action is actually independent, up to quantum canonical transformations, of the Poisson structure. Let s^1 and s^2 denote the coordinates on the two S^1 factors of the torus \mathbb{T}^2 , and let e^1 and e^2 denote the corresponding fiber coordinates on $T[1]\mathbb{T}^2$. We now denote by δ_1, δ_2, K_1 and K_2 the lifts of the vector fields $\frac{\partial}{\partial e^1}, \frac{\partial}{\partial e^2}, e^2 \frac{\partial}{\partial e^1}$ and $e^1 \frac{\partial}{\partial e^2}$ to the space of fields $\mathcal{F} = \text{Map}(T[1]\mathbb{T}^2, T^*[1]M)$. We denote by τ_1, τ_2, ρ_1 and ρ_2 their Hamiltonian functions. They satisfy (5.5) and (5.6), and formally also (5.7).

Remark 5.5. For completeness, we give explicit expressions also in this case, even if we are not going to need them. If we write

$$\begin{aligned} \mathbf{X} &= X + \eta_1^+ e^1 + \eta_2^+ e^2 + \beta^+ e^1 e^2, \\ \boldsymbol{\eta} &= \beta + \eta_1 e^1 + \eta_2 e^2 + X^+ e^1 e^2, \end{aligned}$$

we then have

$$\begin{aligned} \delta_1 X &= -\eta_1^+, & \delta_1 \eta_2^+ &= \beta^+, & \delta_1 \beta &= \eta_1, & \delta_1 \eta_2 &= -X^+, \\ \delta_2 X &= -\eta_2^+, & \delta_2 \eta_1^+ &= -\beta^+, & \delta_2 \beta &= \eta_2, & \delta_2 \eta_1 &= X^+, \end{aligned}$$

and

$$\begin{aligned} K_1 \eta_2^+ &= \eta_1^+, & K_2 \eta_1^+ &= \eta_2^+, \\ K_1 \eta_2 &= \eta_1, & K_2 \eta_1 &= \eta_2. \end{aligned}$$

With respect to the symplectic form

$$\Omega = \int_{\mathcal{F}} \delta X \delta \boldsymbol{\eta} = \int_{\mathbb{T}^2} (\delta X^\mu \delta X_\mu^+ - \delta \eta_1^{+\mu} \delta \eta_{2\mu} + \delta \eta_2^{+\mu} \delta \eta_{1\mu} + \delta \beta^{+\mu} \delta \beta_\mu) ds^1 ds^2,$$

the corresponding Hamiltonian functions are

$$\begin{aligned}\tau_1 &= \int_{\mathbb{T}^2} (-\eta_1^{+\mu} X_\mu^+ + \beta^{+\mu} \eta_{1\mu}) ds^1 ds^2, \\ \tau_2 &= \int_{\mathbb{T}^2} (-\eta_2^{+\mu} X_\mu^+ + \beta^{+\mu} \eta_{2\mu}) ds^1 ds^2, \\ \rho_1 &= \int_{\mathbb{T}^2} \eta_1^{+\mu} \eta_{1\mu} ds^1 ds^2, \\ \rho_2 &= - \int_{\mathbb{T}^2} \eta_2^{+\mu} \eta_{2\mu} ds^1 ds^2.\end{aligned}$$

Notice that despite their non covariant look the above formulae are actually globally well defined.

The action $S = S_0 + S_\pi$ is δ_1 - and δ_2 -closed; it turns out that the interaction part S_π is actually exact:

$$S_\pi = \delta_2 \delta_1 \sigma_\pi, \quad \sigma_\pi := \int_{\mathbb{T}^2} \frac{1}{2} \pi^{\mu\nu}(X) \beta_\mu \beta_\nu ds^1 ds^2.$$

From all the above it follows that S Poisson commutes not only with τ_1 and τ_2 , but also with ρ_1 and ρ_2 . Notice that the Jacobi identity for π implies $(S, \sigma_\pi) = 0$.

Now consider the regularized action

$$S^{\epsilon, t_1, t_2} := S_0 + \epsilon S_\pi - i\hbar(t_1 \tau_1 + t_2 \tau_2),$$

which satisfies the CME and formally also the QME for all ϵ, t_1, t_2 . By all the above it follows that, formally,

$$\begin{aligned}\frac{\partial}{\partial t_1} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} &= \Delta \left(\frac{1}{t_2} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} \rho_1 \right), \\ \frac{\partial}{\partial t_2} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} &= \Delta \left(\frac{1}{t_1} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} \rho_2 \right),\end{aligned}$$

which means that, as long as the parameters t_1 and t_2 are different from zero, the regularized action is independent of them up to quantum canonical transformations. We also have, again formally,

$$\frac{\partial}{\partial \epsilon} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} = \Delta \left(\frac{i}{\hbar t_2} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} \delta_1 \sigma \right) = -\Delta \left(\frac{i}{\hbar t_1} e^{\frac{i}{\hbar} S^{\epsilon, t_1, t_2}} \delta_2 \sigma \right),$$

which implies that, as long as one of the parameters t_1 and t_2 is different from zero, the regularized action is independent of ϵ up to quantum canonical transformations. This in particular means that the partition function is independent of ϵ and that, in order to compute it, we may simply set ϵ to zero. It is now easy to see that, for a reasonable choice of propagators, the effective action for S^{0, t_1, t_2} is simply the restriction

to vacua, that is the the regularized effective action $S_{\text{eff}}^{0,t_1,t_2}$ considered in the previous subsection.

6. GLOBALIZATION OF THE EFFECTIVE ACTION

We now go back to the problem of globalizing $\tilde{S}_{\text{eff}}^{(0)}$ in the general case. Recall that $\tilde{S}_{\text{eff}} \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathcal{H}^*[[\hbar]])$ satisfies the modified QME (3.2). We write $\tilde{S}_{\text{eff}} = \sum_{i=0}^m \tilde{S}_{\text{eff}}^{(i)}$, where $\tilde{S}_{\text{eff}}^{(i)}$ is the i -form component and $m = \dim M$. In form degree zero, we have

$$\frac{1}{2}(\tilde{S}_{\text{eff}}^{(0)}, \tilde{S}_{\text{eff}}^{(0)}) - i\hbar\Delta\tilde{S}_{\text{eff}}^{(0)} = 0,$$

which is the usual QME.

The modified QME is preserved under modified quantum canonical transformations. Namely, $T \in \Gamma(\Lambda T^*M \otimes \widehat{S}\mathcal{H}^*[[\hbar]])$ of total degree -1 generates the infinitesimal transformation

$$\delta\tilde{S}_{\text{eff}} = d_x T + (\tilde{S}_{\text{eff}}, T) - i\hbar\Delta T$$

which preserves the modified QME at the infinitesimal level. Notice that, setting $T = \sum_{i=0}^m T^{(i)}$, we get in form degree zero

$$\delta\tilde{S}_{\text{eff}}^{(0)} = (\tilde{S}_{\text{eff}}^{(0)}, T^{(0)}) - i\hbar\Delta T^{(0)},$$

which is a usual infinitesimal quantum canonical transformation. The goal of this Section is to prove the following

Theorem 6.1. *There is a quantum canonical transformation starting at order 1 in \hbar that makes the form degree zero part $\tilde{S}_{\text{eff}}^{(0)}$ of the effective action closed with respect to the induced Grothendieck differential $D = d_x + (S_R|_{\text{vacua}}, \cdot)$ on $\Gamma(\Lambda T^*M \otimes \widehat{S}\mathcal{H}^*[[\hbar]])$, where $S_R|_{\text{vacua}}$ denotes the evaluation of S_R on vacua.*

This will ensure that the so obtained effective action, call it $\check{S}_{\text{eff}}^{(0)}$, is the image under $T\phi^*$ of a global effective action S_{eff} . Since $\check{S}_{\text{eff}}^{(0)} \in \Gamma(\widehat{S}\mathcal{H}^*[[\hbar]])$, it follows from the discussion just before subsection 2.1 that S_{eff} is a section of $\widehat{S}\tilde{\mathcal{H}}^*[[\hbar]]$,¹⁴ i.e., a formal power series in \hbar of functions on $\tilde{\mathcal{H}}$ (formal in the fiber coordinates). Again, we may identify $\tilde{\mathcal{H}}$ with the canonical global moduli space of vacua by using a connection (e.g., the one in ϕ). By Remark 3.1, we conclude that the class of S_{eff} under quantum canonical transformations is a well-defined object independent of all choices.

¹⁴Recall that $\tilde{\mathcal{H}}_x = H^{>0}(\Sigma) \otimes T_x M \oplus H^\bullet(\Sigma) \otimes T_x^* M[1]$.

6.1. Proof of Theorem 6.1. We start with a simple observation:

Lemma 6.2. *Write $\widetilde{S}_{\text{eff}}^{(i)} = \sum_{k=0}^{\infty} \hbar^k S_k^{(i)}$. If the propagator satisfies the properties in (4.3), then $S_0^{(i)} = 0 \forall i > 1$, whereas $S_0^{(0)}$ and $S_0^{(1)}$ are obtained by the evaluation on vacua of \widehat{S} and S_R , respectively.*

Proof. The terms for $k = 0$ correspond to trees in the expansion in Feynman diagrams; so, using the notations of Section 4, what we have to prove is that the source part $W_{\Gamma}^{\text{source}}$ vanishes for any tree Γ containing more than one vertex.

This is checked by the following degree counting argument. Consider a tree Γ containing more than one vertex. Let V_k be the number of vertices in Γ of internal valence (i.e., not counting the leaves) equal to $k \geq 1$. Then the total number of vertices is

$$V = \sum_{k \geq 1} V_k,$$

the number of internal edges is

$$(6.1) \quad E = \frac{1}{2} \sum_{k \geq 1} k V_k,$$

and the Euler characteristic of Γ is

$$(6.2) \quad 1 = V - E = \sum_{k \geq 1} \frac{2-k}{2} V_k.$$

Next, the source part $W_{\Gamma}^{\text{source}}$ vanishes automatically due to $K\mathcal{P} = \mathcal{P}K = 0$ and $K^2 = 0$, unless the following two properties hold for the decoration of leaves by cohomology classes $\chi_{\alpha} \in H^{\bullet}(\Sigma)$:

- (i) At every vertex of internal valence 1 there are at least two incident leaves decorated by cohomology classes of non-zero degree. (Otherwise $W_{\Gamma}^{\text{source}}$ vanishes due to $K\mathcal{P} = 0$.)
- (ii) At every vertex of internal valence 2 there is at least one incident leaf decorated by a cohomology class of non-zero degree. (Otherwise $W_{\Gamma}^{\text{source}}$ vanishes due to $K^2 = 0$.)

This gives a lower bound $E + 2V_1 + V_2$ for the form degree of the integrand in (4.5); since it should coincide with the dimension of the space $\Sigma^{\times V}$ it is integrated against, we have the inequality

$$(6.3) \quad E + 2V_1 + V_2 \leq 2V.$$

By (6.1) this is equivalent to

$$\frac{1}{2} V_1 + \sum_{k \geq 3} \frac{k-4}{2} V_k \leq 0.$$

Subtracting (6.2) from this inequality, we get

$$\sum_{k \geq 3} (k-3)V_k \leq -1$$

which is a contradiction. Thus it is impossible to find a decoration of leaves of Γ satisfying properties (i) and (ii) simultaneously. Therefore, W_Γ^{source} vanishes for any decoration. \square

We now set $S^{(1)'} = \tilde{S}_{\text{eff}}^{(1)} - S_0^{(1)}$ and $S^{(i)'} = \tilde{S}_{\text{eff}}^{(i)}$ for $i > 1$.

Lemma 6.3. *There is a modified quantum canonical transformation starting at order 1 in \hbar after which all $S^{(i)'}$ for $i \geq 1$ vanish.*

Proof. We work by induction on the order of \hbar . At order zero the statement holds by Lemma 6.2. Assume that $S_r^{(i)'} = 0 \forall i \geq 1$ and $\forall r < k$. Then the modified quantum master equation yields the identities

$$DS_k^{(i)'} + \Omega S_k^{(i+1)'} = 0, \quad \forall i \geq 1,$$

with $D := d_x + (S_0^{(1)}, \cdot)$ and $\Omega := (S_0^{(0)}, \cdot)$. We already know that $D^2 = \Omega^2 = 0$ and that D and Ω commute. If $T = \sum_{r=k}^{\infty} \hbar^r T_r$ is a generator starting at the order k , we then have the infinitesimal transformations

$$\delta S_k^{(i)'} = DT_k^{(i-1)} + \Omega T_k^{(i)}, \quad \forall i \geq 1.$$

By dimensional reasons $DS^{(m)'} = 0$, with $m = \dim M$, and since the D -cohomology is concentrated in degree zero, we can find a $\tau \in \Gamma(\Lambda T^*M \otimes \widehat{\mathcal{H}}^*)$ such that $S^{(m)'} = D\tau$. We now consider the transformation with generator $T = -\hbar^k \tau$. Hence we get $\delta S_k^{(m)'} = -D\tau$, $\delta S_k^{(m-1)'} = -\Omega\tau$ and $\delta S_k^{(i)'} = 0$ for $i < m-1$. Integrating this transformation up to time 1, we make $S_k^{(m)'}$ vanish; as a result the new $S_k^{(m-1)'}$ will be D -closed. We may then proceed like this until we make all the $S^{(i)'}$ vanish. This proves our claim.

Notice that these transformations may change the $S_r^{(i)'}$ for $r > k$. Moreover, the generator used to kill $S_k^{(1)'}$ will act on $S_r^{(0)}$ for $r \geq k$ by a quantum canonical transformation. \square

This completes the proof of Theorem 6.1.

As a final remark, observe that in the case when π is regular and unimodular we start with $S^{(1)'} = 0$, so we have two different but equivalent ways of getting the global action. One consists in taking the original $\tilde{S}_{\text{eff}}^{(0)}$, the other in applying the method described in this Section since nothing guarantees that the remaining $S^{(i)'}$ vanish at the start. After applying the method we get another effective action $\check{S}_{\text{eff}}^{(0)}$ that simply

differs from $\tilde{S}_{\text{eff}}^{(0)}$ by a quantum canonical transformation and is also the image of $T\phi^*$ of a global action which we denote by \check{S} . Eventually, the two global effective actions S and \check{S} simply differ by a quantum canonical transformation starting at order \hbar .

7. CONCLUSIONS AND PERSPECTIVES

In this paper we have studied the effective action of the Poisson sigma model on a closed surface Σ , where the Poisson structure π on the target M is treated perturbatively and, for consistency, has to be assumed to be unimodular unless Σ is a torus. We have shown how to obtain a global effective action S_{eff} as an \hbar -dependent function on the moduli space of vacua of the theory with zero Poisson structure, around which we are perturbing. Because of the freedom in the choice of gauge fixing and the details of globalization, S_{eff} is, as usual, only well-defined up to quantum canonical transformations. By a reasonable choice of the class of allowed gauge fixings—namely, those for which the propagator enjoys properties (4.3)—we make sure that the order zero $S_{\text{eff},0}$ of the effective action is fixed and equal to the evaluation on vacua of the Poisson-dependent part S_π of the action; moreover, the remaining quantum canonical transformations will start at order 1.

In the cases when Σ is a torus or π is regular and unimodular, we have shown that S_{eff} has (a representative with) no quantum corrections. In the particular case when Σ is a torus, π is nondegenerate and there is a compatible complex structure, we can use the latter to gauge-fix the remaining integration over vacua: the final result is that, as expected from the Hamiltonian formulation and from comparison with the A-model, the partition function is the Euler characteristic of the target. An alternative approach that produces the same result consists in regularizing the effective action by adding the Hamiltonian functions of supersymmetry generators. In general, the effective action modulo quantum canonical transformations is an invariant of the Poisson structure.

Recall that each order in \hbar of S_{eff} is actually a section of a vector bundle $\mathcal{Z}_g := \widehat{S}\widetilde{\mathcal{H}}^*$ over the target M whose structure is fixed by the genus g of the source Σ . These sections are just tensors of a particular sort. The lowest order in the quantum master equation for S_{eff} implies that $S_{\text{eff},0}$ solves the classical master equation, i.e., that it defines a differential on $\Gamma(\mathcal{Z}_g)$, which we call the genus g Poisson complex. Since $S_{\text{eff},0}$ is also Δ -closed, the lowest nonvanishing quantum contribution

to S_{eff} is a cocycle in the genus g Poisson complex¹⁵ and defines an invariant of the Poisson structure, which might be possible to compute in concrete examples.

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¹⁵If this happens at order 1, which can be easily seen not to be the case for $g \leq 1$, this class is also what is left after modding out by quantum canonical transformations at this order.

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CLASSICAL BV THEORIES ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. In this paper we extend the classical BV framework to gauge theories on spacetime manifolds with boundary. In particular, we connect the BV construction in the bulk with the BFV construction on the boundary and we develop its extension to strata of higher codimension in the case of manifolds with corners. We present several examples including electrodynamics, Yang-Mills theory and topological field theories coming from the AKSZ construction, in particular, the Chern-Simons theory, the BF theory, and the Poisson sigma model. This paper is the first step towards developing the perturbative quantization of such theories on manifolds with boundary in a way consistent with gluing.

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1. INTRODUCTION

One of the key features in quantum field theory is locality. Physically, it is based on the concept of an ideal point-like particle. Mathematically, it means that the action in the corresponding classical field theory should be a local functional.

In quantum field theory locality is expected to correspond to the gluing property of the partition function for manifolds with boundary. For a quantum field theory on Minkowski cylinders this implies the usual composition law for evolution operators. In topological and conformal field theories the locality as a gluing property of partition functions was discussed in [4, 51, 56].

In this paper we formulate the classical Batalin-Vilkovisky (BV) framework for theories on manifolds with boundary. This combines the properties of the standard BV [8] and the Batalin-Fradkin-Vilkovisky (BFV) [9] frameworks. Recall that standard BV theories are odd symplectic extensions of classical gauge theories by ghosts, anti-fields and antighosts. The gauge symmetry appears as a cohomological Hamiltonian vector field whose Hamiltonian function is an extension of the classical action. The BFV theories give a similar cohomological description of the gauge symmetries on the boundary. Another relation between BV and BFV theories is described in [30].

This paper should be considered as a first step towards the perturbative quantization of classical BV theories on manifolds with boundary, and possibly with corners.

In case of topological field theories the ultimate goal is to construct perturbative topological invariants of manifolds with boundary which satisfy the cutting-gluing principle. All known classical topological field theories are gauge theories. Examples are the BF theory, the Chern-Simons theory, and the Poisson sigma model (see Sections 5 and 7 for definitions and details).

Recall some basic facts about BV theories. These are classical field theories with \mathbb{Z} -graded spaces of fields which appear naturally in quantization of gauge theories. The BV construction is a generalization of the Faddeev-Popov [25] and BRST [10] methods. The BV construction is particularly important in cases when the gauge symmetry is given by a non-integrable distribution on the space of fields or when the gauge symmetry is reducible with stabilizers of varying rank. Examples of such theories include the Poisson sigma model and the non-abelian BF theory when the dimension of spacetime is greater than 3. Examples of theories where the gauge symmetry is an integrable distribution include Yang-Mills and Chern-Simons. Even in cases when the distribution is integrable, BV formalism is useful because it is compatible with Wilsonian renormalization of Feynman diagrams, see e.g. [23].

Topological field theories such as Chern-Simons, BF theories and the Poisson sigma model are special for several reasons. First, in the perturbative quantization of such theories Feynman diagrams have no ultraviolet divergencies. Another reason is that the natural choice of gauge condition, the Lorenz gauge, is induced by a choice of metric on the spacetime manifold. Thus the gauge independence of the Feynman diagram expansion in such theories implies that the partition function is metric independent and therefore is a topological invariant.

All these three examples, Chern-Simons, BF and the Poisson sigma model, fit into the general AKSZ construction [1]. The AKSZ construction also gives other examples of topological field theories such as the Rozansky-Witten model and others [21].

There is a consensus that perturbative quantization of the classical Chern-Simons theory gives the same asymptotical expansions as the combinatorial topological field theory based on quantized universal enveloping algebras at roots of unity [45], or, equivalently, on the modular category corresponding to the Wess-Zumino-Witten conformal field theory [56, 42] with the first semiclassical computations involving torsion made in [56]. However this conjecture is still open despite a number of important results in this direction, see for example [47, 3].

One of the reasons why the conjecture is still open is that for manifolds with boundary the perturbative quantization of Chern-Simons theory has not been developed yet. On the other hand, for closed manifolds the perturbation theory involving Feynman diagrams was developed in [32, 27, 7] and in [5, 35, 13]. For the latest development see [19]. Closing this gap and developing the perturbative quantization of Chern-Simons theory for manifolds with boundary is one of the main motivations for the project started in this paper.

Having laid the framework of classical BV-BFV theory, in a forthcoming work we plan to discuss the formal perturbative quantization of BV theories on manifolds with boundary. The main difference with respect to the formal perturbative quantization for closed spacetime manifolds is that in addition to a choice of gauge fixing in the bulk, one has to fix a polarization in the space of boundary fields. Given such choices, the partition function is a function on the space of leaves of boundary polarization. Its value on a leaf is the path integral (understood as a series of

Feynman diagrams) over fluctuations lying in the gauge fixing submanifold with boundary values being in a fixed leaf of this polarization. The resulting partition function is a state in the space of boundary states determined by the polarization. We expect that the equation (10) will be replaced by a deformed quantum master equation for this state, as it happens in the example of 1d Chern-Simons theory [2]. Gluing along a boundary component will be simply given by pairing the partition functions in the space of boundary states.

A classical BV-BFV theory is a functor from the spacetime category to the BFV category. One should expect a similar description in the quantum case. Classically it extends to higher categories, and we anticipate that quantum counterparts of such extensions are higher category versions of topological quantum field theories.

In this paper spacetime manifolds are always compact oriented, possibly with boundary and corners.

In this paper we will use freely the notion of a \mathbb{Z} -graded manifold. See [11, 20], and the Appendix B for details. To simplify the terminology we will call \mathbb{Z} -graded manifolds simply graded manifolds. By a point in a graded manifold we understand the application of the functor of points.

This paper is organized in the following manner. In section 2 we recall the basic structure of classical BV theory for closed spacetime manifolds.

Section 3 is the central part of the paper. Here we formulate the BV-BFV framework for gauge theories. In this section we propose the analog of the Classical Master Equation for spacetime manifolds with boundary. We define the relevant moduli spaces and address the problem of composing them when we glue two spacetime manifolds together. We also show that first order BV theories naturally extend to lower dimensional strata.

In section 4 we introduce the notion of BFV category. We also define the classical BV theory as a functor from the spacetime category to the BFV category.

In section 5 we describe several examples of BV-BFV extensions of classical gauge theories: electrodynamics, Yang-Mills theory, scalar field and the abelian BF theory. The latter is in fact topological and is an example of the AKSZ construction.

The AKSZ construction which provides a class of examples of topological BV-BFV theories is described in section 6. This construction is determined by the choice of a target manifold, which should be a Hamiltonian differential graded manifold. An AKSZ theory extends to strata of all codimensions.

Further examples of BV-BFV extensions of AKSZ theories are described in section 7. We start with abelian and non-abelian Chern-Simons theories, then we describe the non-abelian BF theory and the Poisson sigma model.

In Appendix A we review some useful facts about coisotropic submanifolds and reduction. In Appendix B we recall some basic facts on graded manifolds. A brief discussion of smooth points on moduli spaces associated to differential graded manifolds is given in Appendix C. Elements of Cartan calculus for local forms and functionals on mapping spaces are developed in Appendix D and applications to AKSZ theories are presented.

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2. CLASSICAL BV THEORIES FOR CLOSED SPACETIMES

2.1. **Non-reduced theory.** When the spacetime manifold N is closed, the space of fields \mathcal{F}_N in a BV theory is a graded manifold with a symplectic form ω_N of degree -1 and the action functional S_N of degree zero. Usually fields are sections of or connections on a fiber bundle on N . Some basic facts on graded manifolds are summarized in Appendix B. The degree in the grading on the space of fields is usually called the ghost number. Thus, $gh(\omega_N) = -1$ and $gh(S_N) = 0$. The space of fields is usually infinite dimensional. The BV data should satisfy the following property: the Poisson bracket of the action functional with itself is zero [8, 9] (see also [49, 52]). This property is known as the classical master equation (CME). The action functional generates the Hamiltonian vector field Q_N of degree 1. The CME implies that the Lie derivative of this vector field squares to zero.

Most interesting for applications both in physics and in geometry and topology are local theories. The notion of locality is more transparent for theories on spacetime manifolds with boundary, see section 3. For closed manifolds locality implies natural isomorphisms $\mathcal{F}_{N_1 \sqcup N_2} \simeq \mathcal{F}_{N_1} \times \mathcal{F}_{N_2}$ and the additivity of the symplectic form and of the action: $\omega_{N_1 \sqcup N_2} = \omega_{N_1} + \omega_{N_2}$, $S_{N_1 \sqcup N_2} = S_{N_1} + S_{N_2}$.

We will use the following definition of classical BV theory for closed spacetime manifolds.

Definition 2.1. *For a closed spacetime N a BV theory [8] on the space of fields \mathcal{F}_N is a triple (ω_N, Q_N, S_N) where ω_N is a symplectic form on \mathcal{F}_N with $gh = -1$, Q_N is a vector field on \mathcal{F}_N with $gh = 1$, and S_N is a function on \mathcal{F}_N with $gh = 0$, subject to*

$$\iota_{Q_N} \omega_N = (-1)^n \delta S_N, \quad L_{Q_N}^2 = 0$$

where L_Q is the Lie derivative for the vector field Q .

These two equations imply the *classical master equation*

$$\{S_N, S_N\} = 0,$$

which can also be written as

$$L_{Q_N} S_N = 0$$

The first equation in the definition means that the vector field Q_N is Hamiltonian and therefore it preserves the symplectic form:

$$L_{Q_N} \omega_N = 0,$$

Critical points of the action functional S_N are solutions to Euler-Lagrange (EL) equations

$$(1) \quad \delta S_N = 0,$$

Because the space of fields is an odd-symplectic manifold, solutions to the EL equations are also zeroes of the Hamiltonian vector field generated by S_N , i.e. of the cohomological vector field Q_N . Denote the space of solutions to Euler-Lagrange equations (1) by \mathcal{EL}_N .

Because vector field Q_N is Hamiltonian, and Poisson brackets of local functionals are defined, we can write $L_{Q_N}F = \{S_N, F\}$ for the action of the vector field Q_N on the functional F . Denote by V_F the vector field generated by F , then $L_{V_F}G = \{F, G\}$. It is clear that vector fields $[Q_N, V_F]$ annihilate the functional S_N :

$$L_{[Q_N, V_F]}S_N = \{\{S_N, F\}, S_N\} = 0$$

This follows from CME and from the Jacobi identity. Thus, any such vector field can be considered as a local (off-shell) gauge symmetry of S_N .

From now on we will consider only first order BV theories. In such theories the action functional is local and it is linear in derivatives of fields. It is well known that any local theory can be reformulated as a first order theory by adding corresponding ‘‘momenta’’ variables in fields.

2.2. Other degrees. So far we have assumed the symplectic form ω_N to have degree -1 . But everything can be generalized to the case when the degree is some integer k (we fix however the degree of the cohomological vector field Q_N to be 1). In this setting the action S_N will have degree $k + 1$ [46]. Only a few remarks are in order

- (1) Unless $k = 0$, ω_N is automatically exact (so for $k = 0$ we need this extra assumption). In the following, we will be interested in specifying a primitive 1-form of the same degree k .
- (2) Unless $k = -1$, the condition $L_{Q_N}\omega = 0$ implies that Q_N is Hamiltonian and that the Hamiltonian function S_N is uniquely determined.
- (3) Unless $k = -2$, the condition $Q_N^2 = 0$ implies $\{S_N, S_N\} = 0$.

We will mainly be interested in the cases $k = -1$ (BV manifolds) and $k = 0$ (BFV manifolds). For extended BV theories, we will also be interested in k positive. We are not aware of any application of the cases $k < -1$.

2.3. The Q -reduction. The vector field Q_N vanishes on the subspace $\mathcal{EL}_N \subset \mathcal{F}_N$. It also defines the Lie subalgebra \widetilde{Vect}_Q of the Lie algebra of vector fields on \mathcal{F}_N generated by vector fields which are Lie brackets with Q_N , i.e. by vector fields of the form $[V, Q_N]$. This Lie subalgebra in general does not determine a distribution on \mathcal{F}_N . However, the restriction of \widetilde{Vect}_Q to \mathcal{EL}_N defines an integrable distribution $Vect_Q$.

Definition 2.2. *The Q -reduction \mathcal{EL}_N/Q of \mathcal{EL}_N is the space of leaves of the distribution $Vect_Q$.*

Assume that X is a zero of Q_N such that the leaf of $Vect_Q$ through X is a smooth point in \mathcal{EL}_N/Q . Define the linear operator $\hat{Q}_X : T_X\mathcal{F}_N \rightarrow T_X\mathcal{F}_N$ as the

linearization of Q_N at X . If we write Q_N in local coordinates as $Q_N = \sum_a q^a(X)\partial_a$, the operator \hat{Q}_X acts on the basis ∂_a in $T_X\mathcal{F}_N$ as

$$\hat{Q}_X\partial_a = \sum_b \partial_a q^b(X)\partial_b$$

Because Q_N squares to zero we have $\hat{Q}_X^2 = 0$. By definition $T_X\mathcal{E}\mathcal{L}_N = \ker(\hat{Q}_X)$. Also observe that $\text{Vect}_Q|_X = \text{Im}(\hat{Q}_X) \subset T_X\mathcal{E}\mathcal{L}_N$. From now on for geometric considerations we will focus only on smooth points as defined in the Appendix C.

Proposition 2.3. *For a smooth point $[X] \in \mathcal{E}\mathcal{L}_N/Q$, there is a natural linear isomorphism*

$$T_{[X]}\mathcal{E}\mathcal{L}_N/Q = \ker(\hat{Q}_X)/\text{Im}(\hat{Q}_X)$$

2.4. The symplectic reduction. The following Proposition says that if the subspace $\mathcal{E}\mathcal{L}_N$ is a submanifold then it is a coisotropic submanifold.

Proposition 2.4. *If $X \in \mathcal{E}\mathcal{L}_N$ is a smooth point, then $T_X\mathcal{E}\mathcal{L}_N \subset T_X\mathcal{F}_N$ is a coisotropic subspace.*

Proof. The invariance of ω_N with respect to the vector field Q_N means that $L_{Q_N}\omega_N = 0$. If X is a zero of Q_N this implies $\omega_N(\hat{Q}_X\xi, \eta) = \omega_N(\xi, \hat{Q}_X\eta)$.

These properties of \hat{Q}_X imply $\omega_N(\hat{Q}_X\xi, \hat{Q}_X\eta) = \omega_N(\xi, \hat{Q}_X^2\eta) = 0$. Therefore $\text{Im}(\hat{Q}_X)$ is an isotropic subspace and therefore its symplectic orthogonal $\text{Im}(\hat{Q}_X)^\perp$ is coisotropic.

On the other hand

$$\begin{aligned} (2) \quad \text{Im}(\hat{Q}_X)^\perp &= \{\xi \mid \omega_N(\xi, \hat{Q}_X\eta) = 0, \text{ for any } \eta\} \\ &= \{\xi \mid \omega_N(\hat{Q}_X\xi, \eta) = 0, \text{ for any } \eta\} = \ker(\hat{Q}_X) \end{aligned}$$

This proves that $T_X\mathcal{E}\mathcal{L}_N = \ker(\hat{Q}_X)$ is a coisotropic subspace. Because we used only nondegeneracy of the form ω_N on $T_X\mathcal{F}$, the proof works both in finite-dimensional and in the infinite-dimensional case. \square

Recall the definition of the symplectic reduction of a coisotropic submanifold of a symplectic manifold. The symplectic reduction \underline{C} of the coisotropic submanifold C in the symplectic manifold S is the space of leaves of the characteristic foliation of C . The characteristic foliation of C is spanned by Hamiltonian vector fields generated by the vanishing ideal I_C (the ideal in the commutative algebra of functions on S generated by functions vanishing on C). When \underline{C} is a smooth manifold, it is a symplectic manifold. Otherwise only at smooth points the tangent space to \underline{C} has a natural symplectic form. If $[X]$ is such a smooth point, the tangent space $T_{[X]}\underline{C}$ is naturally isomorphic to $T_X C / (T_X C)^\perp$.

Remark 2.5. *An alternative proof of the Proposition 2.4 is algebraic. Observe that the vanishing ideal for $\mathcal{E}\mathcal{L}_N$ is generated by functionals of the form $\{S_N, T\}$ where $\{.,.\}$ is the Poisson bracket for the form ω_N and T is a local functional. The simple calculations shows that*

$$\{\{S, U\}, \{S, T\}\} = \{S, \{U, \{S, T\}\}\}$$

The bracket $\{U, \{S, T\}\}$ is a local functional if U and T are. Therefore the vanishing ideal is a Poisson algebra and therefore $\mathcal{E}\mathcal{L}_N$ is coisotropic in the algebraic sense.

In Proposition 2.4 we proved that $Im(\hat{Q}_X)^\perp = \ker(\hat{Q}_X)$. This implies $Im(\hat{Q}_X) \subset \ker(\hat{Q}_X)^\perp$, which means that the \hat{Q} -distribution on \mathcal{EL}_N is contained in the characteristic distribution. In the finite dimensional case the inclusion becomes an equality.

Assumption 2.6. *In classical field theory we set*

$$(3) \quad Im(\hat{Q}_X) = \ker(\hat{Q}_X)^\perp,$$

as an assumption.

This corresponds to certain ellipticity condition for Q_N . In all our examples this condition follows from the usual Hodge-de Rham decomposition.

Remark 2.7. *As a consequence of condition (3) the characteristic foliation of \mathcal{EL}_N is the same as the foliation by $Vect_Q$, and in particular, if $[X] \in \underline{\mathcal{EL}}_N$ is a smooth point, then*

$$(4) \quad T_{[X]}\underline{\mathcal{EL}}_N = \ker(\hat{Q}_X)/Im(\hat{Q}_X)$$

In other words, locally the symplectic reduction of \mathcal{EL}_N coincides with its Q -reduction: $\underline{\mathcal{EL}}_N = \mathcal{EL}_N/Q$.

We will call $\underline{\mathcal{EL}}_N = \mathcal{EL}_N/Q$ the \mathcal{EL} -moduli space for N . If the cohomology spaces of \hat{Q}_X are finite dimensional at a generic point X , the \mathcal{EL} -moduli space is finite dimensional, but possibly singular.

Remark 2.8. *With the appropriate assumptions the ring of functions on the reduced space \mathcal{EL}_N/Q is isomorphic to the cohomology space of the ring of functions on \mathcal{F}_N with the differential L_{Q_N} (the Lie derivative with respect to Q_N). To be more specific, one should expect that the smooth locus of \mathcal{EL}/Q is isomorphic to the smooth locus of $Spec(H_Q(Fun(\mathcal{F}_N)))$ with the appropriate definition of $Spec$ and of the ring of functionals $Fun(\mathcal{F}_N)$.*

Because $\delta S_N = 0$ on the subspace $\mathcal{EL}_N \subset \mathcal{F}_N$ we have the following Proposition.

Proposition 2.9. *The action is constant on connected components of \mathcal{EL}_N .*

3. BV THEORIES FOR SPACETIME MANIFOLDS WITH BOUNDARY

3.1. Non-reduced theory.

3.1.1. *Non-reduced BV-BFV theory.* In this section, unless otherwise specified, N is an n -dimensional spacetime manifold, with ∂N being its $n - 1$ dimensional boundary. The BV theory for spacetime manifolds with boundary consists of two parts.

First, in the bulk we have the same data as for BV theory, that is: the space of fields \mathcal{F}_N which is a graded (the degree is called ghost number) symplectic manifold with the symplectic form ω_N with $gh = -1$, the cohomological vector field Q_N with $gh = 1$ and the action functional S_N of ghost number zero.

Second, on the boundary we have the BFV data [9] which consist of: the space of boundary fields $\mathcal{F}_{\partial N}$ which is a graded (usually infinite dimensional) symplectic manifold with the symplectic form $\omega_{\partial N}$ which is exact $\omega_{\partial N} = (-1)^{n-1}\delta\alpha_{\partial N}$ with $gh = 0$, and a vector field $Q_{\partial N}$ with $gh = 1$. The sign in $(-1)^{n-1}$ is here because it appears naturally in AKSZ examples of topological gauge theories. The BFV data satisfy the following conditions:

$$(5) \quad Q_{\partial N}^2 = 0, \quad L_{Q_{\partial N}}\omega_{\partial N} = 0$$

These conditions imply that $Q_{\partial N}$ is Hamiltonian vector field [46] with ghost number 1, with the Hamiltonian $S_{\partial N}$, which is by definition the BFV action. In Appendix B we recall this construction.

A BV theory on a spacetime with boundary can be regarded as a pairing of the BV data with the BFV data through the natural mapping $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ which is the restriction of fields to the boundary.

We will say that BV theory in the bulk and BFV theory on the boundary agree and form the BV-BFV theory on the manifold with boundary if the restriction mapping is a surjective submersion and if the BV data in the bulk and BFV data on the boundary satisfy the following conditions

$$(6) \quad Q_N^2 = 0, \quad \delta\pi(Q_N) = Q_{\partial N}, \quad L_{Q_{\partial N}}\omega_{\partial N} = 0$$

$$(7) \quad \iota_{Q_N}\omega_N = (-1)^n\delta S_N + \pi^*(\alpha_{\partial N}),$$

The sign is purely conventional, but it fits well with the natural definition of BV data for AKSZ theories.

Note that first two equations imply that

$$(8) \quad Q_{\partial N}^2 = 0$$

which means BFV axioms are satisfied. The equation (7) implies

$$(9) \quad L_{Q_N}\omega_N = (-1)^n\pi^*(\omega_{\partial N})$$

which means, in particular, that in BV-BFV theory the symplectic form is no longer Q -invariant.

The bulk action satisfies an important identity which describes how the action changes with respect to gauge transformations.

Proposition 3.1. *The following identity holds in any BV-BFV theory*

$$(10) \quad L_{Q_N}S_N = (-1)^{\dim(N)}\pi^*(2S_{\partial N} - \iota_{Q_{\partial N}}\alpha_{\partial N})$$

Remark 3.2. *We can remove the signs $(-1)^n$ by redefining vector fields $Q_N, Q_{\partial N}$ and the 1-form $\alpha_{\partial N}$. Set $Q = (-1)^n Q_N, Q_{\partial} = (-1)^n Q_{\partial N}$ and $\alpha_{\partial} = (-1)^{n-1}\alpha_{\partial N}$. Then $\delta\pi(Q) = Q_{\partial}, \iota_Q\omega_N = S_N - \pi^*(\alpha_{\partial})$ and $L_Q\omega_N = \pi^*(\omega_{\partial N})$.*

Proof. By definition of the Lie derivative

$$L_{Q_N} = \iota_{Q_N}\delta - \delta\iota_{Q_N}$$

and the same formula for the Lie derivative holds for $Q_{\partial N}$. The sign is minus in this formula because ι_Q is an even operation. We will prove the Proposition by applying L_{Q_N} to the equation:

$$\iota_{Q_N}\omega_N = (-1)^n\delta S_N + \pi^*(\alpha_{\partial N})$$

By definition of L_{Q_N} and because of $Q_N^2 = 0$ we have $[L_{Q_N}, \iota_{Q_N}] = L_{Q_N}\iota_{Q_N} - \iota_{Q_N}L_{Q_N} = 0$ and we already know that

$$L_{Q_N}\omega_N = -\pi^*\delta\alpha_{\partial N} = (-1)^{n+1}\pi^*\omega_{\partial N},$$

Therefore we have identities:

$$(11) \quad L_{Q_N}\iota_{Q_N}\omega_N = \iota_{Q_N}L_{Q_N}\omega_N = -\pi^*\iota_{Q_{\partial N}}\delta\alpha_{\partial N},$$

Now we can apply the Lie derivative of Q_N to the classical master equation:

$$(12) \quad L_{Q_N} \iota_{Q_N} \omega_N = (-1)^n L_{Q_N} \delta S_N + L_{Q_N} \pi^* \alpha_{\partial N} = -(-1)^n \delta \iota_{Q_N} \delta S_N + \pi^* (L_{Q_{\partial N}} \alpha_{paN}) = \\ -(-1)^n \delta L_{Q_N} S + \pi^* (\iota_{Q_{\partial N}} \delta \alpha_{\partial N} - \delta \iota_{Q_{\partial N}} \alpha_{\partial N})$$

Here, in the last line, we used the formula for the Lie derivative $L_{Q_{\partial N}}$. Using (11) and the identity $\delta S_{\partial N} = \iota_{Q_{\partial N}} \omega_{\partial N}$ we arrive at:

$$-\pi^* (\delta S_{\partial N}) = -(-1)^n \delta L_{Q_N} S_N + \pi^* (\delta S_{\partial N}) - \pi^* (\delta \iota_{Q_{\partial N}} \alpha_{\partial N})$$

This is equivalent to

$$\delta((-1)^n L_{Q_N} S_N - \pi^* (2S_{\partial N} - \iota_{Q_{\partial N}} \alpha_{\partial N})) = 0$$

We have δ (function of degree 1) = 0, which means the function must vanish. This proves the Proposition. \square

Remark 3.3. *In the definition of BV-BFV theory we made an assumption that ω_{∂} is exact. This assumption can be removed. Two examples, a charged particle in an electro-magnetic field, and the WZW model suggest that a more natural condition is to require that there is a line bundle \mathfrak{L}_{∂} over $\mathcal{F}_{\partial N}$. The symplectic form is $\omega_{\partial N} = (-1)^{n-1} \delta \alpha_{\partial N}$ where $\alpha_{\partial N}$ is a connection on \mathfrak{L}_{∂} . The condition*

$$L_{Q_N} \omega_N = (-1)^n \pi^* \omega_{\partial N}$$

implies that the connection $(-1)^n \frac{i}{\hbar} (-\iota_{Q_N} \omega_N + \pi^ \alpha_{\partial N})$ on the line bundle $\pi^*(\mathfrak{L}_{\partial})$ over \mathcal{F}_N is flat. In this case the action functional S_N should be regarded as a horizontal section $s_N = \exp(\frac{iS_N}{\hbar})$ of $\pi^*(\mathfrak{L}_{\partial})$ and the equation (7) becomes*

$$(\delta + (-1)^n \frac{i}{\hbar} (-\iota_{Q_N} \omega_N + \pi^* \alpha_{\partial N})) s_N = 0$$

3.1.2. *Digression: gauge invariance.* Now let us discuss the gauge invariance of BV-BFV theory. When $\partial N \neq \emptyset$, Poisson bracket of local functionals is not defined. In this case we will call the vector field V Hamiltonian if there exists a local functional F such that

$$\iota_V \omega_N = \delta F$$

Recall that a vector field V is called *projectable* if for all $X \in \mathcal{F}_N$

$$\delta \pi_X (V_X) = v_{\pi(X)}$$

for some vector field v on $\mathcal{F}_{\partial N}$. The gauge invariance of the BV-BFV action can be formulated as the following statement:

Proposition 3.4. *If V is a projectable Hamiltonian vector field, then the vector field $[Q_N, V]$ preserves S_N up to a pullback from the boundary:*

$$L_{[Q_N, V]} S_N = (-1)^{\dim N} \pi^* (\iota_V L_{Q_{\partial N}} \alpha_{\partial N} + \iota_{Q_{\partial N}} L_V \alpha_{\partial N})$$

Indeed, we have:

$$(13) \quad L_{[Q_N, V]} S_N = L_{Q_N} L_V S + L_V L_{Q_N} S_N = \\ \iota_{Q_N} \delta \iota_V \delta S_N + (-1)^{\dim N} L_V \pi^* (2S_{\partial N} - \iota_{Q_{\partial N}} \alpha_{\partial N}) = \\ = (-1)^{\dim N} (\iota_{Q_{\partial N}} \delta \iota_{Q_N} \iota_V \omega_N + \pi^* (-\iota_{Q_{\partial N}} \delta \iota_V \alpha_{\partial N} + \\ 2\iota_V \delta S_{\partial N} - \iota_V \delta \iota_{Q_{\partial N}} \alpha_{\partial N})) = (-1)^{\dim N} \pi^* (\iota_V L_{Q_{\partial N}} \alpha_{\partial N} + \iota_{Q_{\partial N}} L_V \alpha_{\partial N})$$

Here we used identities $\delta S_N = (-1)^{\dim N}(\iota_{Q_N}\omega_N - \pi^*\alpha_{\partial N})$, $\iota_V\omega_N = \delta F$, $L_{Q_N}L_{Q_N}F = \frac{1}{2}L_{[Q_N, Q_N]}F = 0$, $\delta S_{\partial N} = (-1)^{\dim N-1}\iota_{Q_{\partial N}}\omega_{\partial N} = \iota_{Q_{\partial N}}\omega_{\partial N} = \iota_{Q_{\partial N}}\delta\alpha_{\partial N}$, and F is the generating function for V and v is the projection of V to the boundary.

As a corollary we also have the formula

$$\delta L_{[Q_N, V]}S = -(-1)^{\dim N}\pi^*(L_{[Q_{\partial N}, v]}\alpha_{\partial N})$$

3.1.3. *The boundary structure from the bulk.* Let us show that a first order BV theory of spacetime manifolds (on the bulk) induces BFV theory on the boundary. Denote the pullback of fields to the boundary by $\mathcal{F}_N|_{\partial N}$ (we say pullback because fields are usually either sections of a fiber bundle, or connections). The differential of the action can be written as the bulk part plus the boundary contribution from the use of Stokes theorem:

$$\delta S_N[X] = \int_N A \wedge \delta X - (-1)^n \pi^* \tilde{\alpha}_{\partial N}[X]$$

where $\tilde{\alpha}_{\partial N}$ is a one-form on the space $\mathcal{F}_N|_{\partial N}$ and A defines the Euler-Lagrange equation. The kernel of $\delta\tilde{\alpha}_{\partial N}$ forms a distribution on the space $\mathcal{F}_N|_{\partial N}$. Denote by $\mathcal{F}_{\partial N}$ the space of leaves of this distribution. We have a natural projection $\pi: \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$.

Denote by $\alpha_{\partial N}$ the form on $\mathcal{F}_{\partial N}$ corresponding to the form $\tilde{\alpha}_{\partial N}$ on $\mathcal{F}_N|_{\partial N}^1$.

Taking into account that for closed manifolds the vector field Q_N is Hamiltonian, generated by the classical action functional, we conclude that the bulk term in the differential of the action functional is $(-1)^n \iota_{Q_N}\omega_N$. Thus, we have equation (7).

Observe that Q_N is projectable to $\mathcal{F}_{\partial N}$ denote its projection by $Q_{\partial N} = \delta\pi(Q_N)$. For any form θ on boundary fields we will have $L_{Q_N}\pi^*(\theta) = \pi^*(L_{Q_{\partial N}}\theta)$. Equations $Q_N^2 = 0$ and (9) imply that $L_{Q_N}^2\omega_N = \pi^*(L_{Q_{\partial N}}\omega_{\partial N}) = 0$. Therefore $L_{Q_{\partial N}}\omega_{\partial N} = 0$.

It is clear that the BFV action induced by a first order BV theory is also of first order.

3.1.4. *Some properties of non-reduced BV-BFV theories.* Associated with BV and BFV data we have the following important subspaces in the space of bulk fields and in the spaces of boundary fields:

- The space $\mathcal{EL}_N \subset \mathcal{F}_N$ of zeroes of the vector field Q_N .
- The space $\mathcal{EL}_{\partial N} \subset \mathcal{F}_{\partial N}$ of zeroes of the vector field $Q_{\partial N}$.
- The space $\mathcal{L}_N = \pi(\mathcal{EL}_N) \subset \mathcal{F}_{\partial N}$ of boundary values of solutions of Euler-Lagrange equations. It is clear that $\mathcal{L}_N \subset \mathcal{EL}_{\partial N} \subset \mathcal{EL}_N$.

Proposition 3.5. *a) The subspace $\mathcal{EL}_{\partial N}$ is locally coisotropic in $\mathcal{F}_{\partial N}$ (its tangent space at a smooth point is coisotropic in the tangent space to the space of fields at this point).*

b) \mathcal{EL}_N is still locally coisotropic when $\partial N \neq \emptyset$.

The proof of a) is identical to the proof of Proposition 2.4 for closed manifolds. The same proof carries through for b) by requiring η to vanish on the boundary. The algebraic proof outlined after Proposition 2.4 also works, the only difference is that one should take local functionals with test functions vanishing on the boundary. An example of such functional is $\int_N \alpha_a \wedge X^a$ where X^a are coordinate fields and α_a are test functions which vanish on ∂N .

¹Strictly speaking the form $\tilde{\alpha}_{\partial N}$ becomes a connection on a line bundle over $\mathcal{F}_{\partial N}$ with the curvature $\omega_{\partial N}$. However in our examples this line bundle is trivial. This is why for the time being we will assume this and will regard $\alpha_{\partial N}$ as a 1-form.

Proposition 3.6. *The subspace $\mathcal{L}_N \subset \mathcal{EL}_{\partial N} \subset \mathcal{F}_{\partial N}$ is locally isotropic.*

Proof. Assume X is a smooth point of \mathcal{EL}_N . The equation (7) implies that

$$(14) \quad \omega_X(\hat{Q}_X \xi, \eta) - \omega_X(\xi, \hat{Q}_X \eta) = \tilde{\omega}_{\pi(X)}(\delta\pi(\xi), \delta\pi(\eta))$$

where ξ, η are tangent vectors to \mathcal{F}_N at the point X . If ξ and η are tangent to $\mathcal{EL}_N \subset \mathcal{F}_N$ then $\hat{Q}_X \xi = \hat{Q}_X \eta = 0$. Therefore $\tilde{\omega}_{\pi(X)}(\delta\pi(\xi), \delta\pi(\eta)) = 0$ which means that $\delta\pi(T_X \mathcal{EL}_N) = T_{\pi(X)} \mathcal{L}_N$ is an isotropic subspace in $T_{\pi(X)} \mathcal{F}_{\partial N}$. \square

We proved that \mathcal{L}_N is always isotropic. We will prove in Section 3.4.3 (Corollary 3.21) that under a natural regularity assumption (see Definition 3.18), \mathcal{L}_N is in fact Lagrangian.

Another natural property of a BV-BFV classical field theory is locality. If two spacetime manifolds N_1 and N_2 have common boundary Σ (say $\Sigma \subset \partial N_1$ and it is identified by an orientation preserving diffeomorphism with part of the boundary of N_2) then

$$\begin{aligned} \mathcal{F}_{N_1 \cup_{\Sigma} N_2} &= \mathcal{F}_{N_1} \times_{\mathcal{F}_{\Sigma}} \mathcal{F}_{N_2} \\ S_{N_1 \cup_{\Sigma} N_2} &= S_{N_1} + S_{N_2} \end{aligned}$$

3.2. The reduction of the boundary BFV theory. The boundary manifold ∂N is closed. This is why both the Q -reduction and the symplectic reduction for the boundary BFV theory work as they do for the BV theory on closed spacetime manifolds. The additional structure in the BV-BFV theory is the reduction of the submanifold $\mathcal{L}_N \subset \mathcal{EL}_{\partial N}$ and the reduction of the fibers of the fiber bundle $\pi : \mathcal{EL}_N \rightarrow \mathcal{L}_N$.

3.2.1. The Q -reduction. The following Proposition is an immediate corollary of the identity $\delta\pi \circ Q_N = Q_{\partial N} \circ \delta\pi$.

Proposition 3.7. *The distribution $\text{Vect}_{Q_{\partial}}$ on $\mathcal{F}_{\partial N}$ generated by Lie brackets of vector fields with $Q_{\partial N}$ is parallel to \mathcal{L}_N .*

Definition 3.8. *Define Q -reductions of \mathcal{L}_N and of $\mathcal{EL}_{\partial N}$ as the space of leaves of $\text{Vect}_{Q_{\partial}}$ through \mathcal{L}_N and $\mathcal{EL}_{\partial N}$ respectively.*

In general these spaces are singular. Because our goal here is to develop the set up for the perturbative quantization in a vicinity of a generic classical solution, we will focus on smooth points of $\mathcal{EL}_{\partial N}$ and \mathcal{L}_N and tangent spaces at such points. See Appendix C below for the discussion of smooth points.

3.2.2. The symplectic reduction. Let $\underline{\mathcal{EL}}_{\partial N}$ be the symplectic reduction of $\mathcal{EL}_{\partial N}$. We make the following assumption, which is the analogue of (3) for the boundary fields:

Assumption 3.9. *We assume that for any $l \in \mathcal{EL}_{\partial N}$,*

$$(15) \quad \ker(\hat{Q}_l)^\perp = \text{Im}(\hat{Q}_l)$$

in $T_l \mathcal{F}_{\partial N}$, where \hat{Q}_l is the linearization of $Q_{\partial N}$ at l .

The following Proposition is the counterpart of the Remark 2.7.

Proposition 3.10. *Under the assumption (15), locally, the symplectic reduction of $\mathcal{EL}_{\partial N}$ is equal to its Q -reduction.*

The difference from Remark 2.7 is just in the shift of gradings.

Denote by $\underline{\mathcal{L}}_N \subset \underline{\mathcal{E}\mathcal{L}}_{\partial N}$ the image of \mathcal{L}_N with respect to this reduction. That is $\underline{\mathcal{L}}_N$ is the space of leaves of the characteristic foliation of $\mathcal{E}\mathcal{L}_{\partial N}$ through \mathcal{L}_N . This Proposition implies that locally the reduction $\underline{\mathcal{L}}_N$ of \mathcal{L}_N is equal to its Q -reduction \mathcal{L}_N/Q .

Under the regularity assumption (see Definition 3.18, Proposition 3.20), $\underline{\mathcal{L}}_N \subset \underline{\mathcal{E}\mathcal{L}}_{\partial N}$ is a Lagrangian submanifold.

3.3. The reduction of the bulk BV theory. The meaning of the Q -reduction is passing from fields to gauge classes of fields. This is why it is natural to reduce not the whole space of solutions to Euler-Lagrange equations but a subspace with fixed boundary conditions. Points of this reduced space correspond to gauge classes of fields with boundary values in a given boundary gauge class.

The space $\mathcal{E}\mathcal{L}_N$ is naturally fibered over $\mathcal{L}_N = \pi(\mathcal{E}\mathcal{L}_N)$ where $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ is the restriction mapping.

For $l \in \mathcal{L}_N$ denote by $[l]$ the leaf of $Vect_{Q_{\partial N}}$ through l , i.e. the image of l in the Q -reduced space \mathcal{L}_N/Q . Denote by $\mathcal{E}\mathcal{L}(N, [l])$ the fiber of π over $[l]$,

$$\mathcal{E}\mathcal{L}(N, [l]) = \pi^{-1}([l]) \cap \mathcal{E}\mathcal{L}_N$$

Because the restriction mapping π intertwines vector fields Q_N and $Q_{\partial N}$, the vector field Q_N is parallel to $\mathcal{E}\mathcal{L}(N, [l])$. It also induces the projection of reduced spaces $p : \mathcal{E}\mathcal{L}_N/Q \rightarrow \mathcal{L}_N/Q$ where $\mathcal{E}\mathcal{L}_N/Q$ is the space of leaves of $Vect_Q$ in $\mathcal{E}\mathcal{L}_N$.

Definition 3.11. *The reduced space $\mathcal{E}\mathcal{L}(N, [l])/Q$ is the space of leaves of $Vect_Q$ on $\mathcal{E}\mathcal{L}(N, [l])$.*

Because the projection π intertwines vector fields Q_N and $Q_{\partial N}$ the reduced space $\mathcal{E}\mathcal{L}(N, [l])/Q$ is the fiber of p over $[l]$: $\mathcal{E}\mathcal{L}(N, [l])/Q = p^{-1}([l])$.

Assume that the image of $X \in \mathcal{E}\mathcal{L}_N$ in $\mathcal{E}\mathcal{L}(N, [\pi(X)])/Q$ is a smooth point. We have the natural mappings of tangent spaces:

$$(16) \quad \begin{array}{ccc} T_{[X]}\mathcal{E}\mathcal{L}_N/Q & \xleftarrow{\iota} & T_{[X]}\mathcal{E}\mathcal{L}(N, [l])/Q \\ \delta p \downarrow & & \\ T_{[l]}\mathcal{L}_N/Q \subset T_{[l]}\mathcal{E}\mathcal{L}_{\partial N}/Q & & \end{array}$$

where $l = \pi(X)$, and the horizontal mapping is the natural inclusion of fibers, $Im(\iota) = \ker(\delta p)$. We know that $T_{[X]}\mathcal{E}\mathcal{L}_N/Q \simeq H(T_X\mathcal{F}_N; \hat{Q}_X)$, and that $T_{[l]}\mathcal{E}\mathcal{L}(\partial N)/Q \simeq H(T_l\mathcal{F}_{\partial N}; \hat{Q}_l)^2$.

We also know that the short exact sequence

$$\begin{array}{ccc} T_X\mathcal{F}_N & \xleftarrow{i} & \ker(\delta\pi)_X \\ (\delta\pi)_X \downarrow & & \\ T_l\mathcal{F}_{\partial N} & & \end{array}$$

gives the long exact sequence

$$\cdots \rightarrow H_{\hat{Q}_X}^\bullet(\ker(\delta\pi)_X) \xrightarrow{[i]} H_{\hat{Q}_X}^\bullet(T_X\mathcal{F}_N) \xrightarrow{[\delta\pi_X]} H_{\hat{Q}_l}^\bullet(T_l\mathcal{F}_{\partial N}) \xrightarrow{\phi} H_{\hat{Q}_X}^{\bullet+1}(\ker(\delta\pi)_X) \rightarrow \cdots$$

²Recall that \hat{Q}_X is the linearization of Q_N at $X \in \mathcal{E}\mathcal{L}_N$ and \hat{Q}_l is the linearization of $Q_{\partial N}$ at $l = \pi(X) \in \mathcal{E}\mathcal{L}_{\partial N}$.

on the \hat{Q} -cohomology spaces.

Truncating this long exact sequence we obtain the short exact sequence

(17)

$$\begin{array}{ccc} H_{\hat{Q}_X}(T_X \mathcal{F}_N) & \xleftarrow{[i]} & H_{\hat{Q}_X}(\ker(\delta\pi)_X)/\phi(H_{\hat{Q}_l}(T_l \mathcal{F}_{\partial N})) \\ \downarrow [\delta\pi_X] & & \end{array}$$

$$[\delta\pi_X](H_{\hat{Q}_X}(T_X \mathcal{F}_N)) \subset H_{\hat{Q}_l}(T_l \mathcal{F}_{\partial N})$$

Because we have natural isomorphisms $H_{\hat{Q}_X}(T_X \mathcal{F}_N) \cong T_{[X]}\mathcal{E}\mathcal{L}_N/Q$, $H_{\hat{Q}_l}(T_l \mathcal{F}_{\partial N}) \cong T_{[l]}\mathcal{E}\mathcal{L}_{\partial N}/Q$ and $[\delta\pi_X](H_{\hat{Q}_X}(T_X \mathcal{F}_N)) = T_{[l]}\mathcal{L}_N/Q \subset T_{[l]}\mathcal{E}\mathcal{L}_{\partial N}/Q$ we can identify $T_{[X]}\mathcal{E}\mathcal{L}(N, [l])/Q$ with the fiber of $[\delta\pi_X]$. We proved the following:

Proposition 3.12. *If $[\pi(X)] \in \mathcal{E}\mathcal{L}_N/Q$ is a smooth point, then*

$$T_{[X]}\mathcal{E}\mathcal{L}(N, [l])/Q \cong H_{\hat{Q}_X}(\ker(\delta\pi_X))/\phi(H_{\hat{Q}_l}(T_l \mathcal{F}_{\partial N}))$$

Here is another, equivalent characterization of this space. By definition

$$T_X \mathcal{E}\mathcal{L}(N, [l]) = \{\xi \in \ker(\hat{Q}_X) \mid \delta\pi_X(\xi) \in \text{Im}(\hat{Q}_l)\}$$

On the other hand $(\text{Vect}_Q)_X = \text{Im}(\hat{Q}_X)$. Therefore the tangent space to the reduced space is the quotient

$$T_X \mathcal{E}\mathcal{L}(N, [l])/Q = \{\xi \in \ker(\hat{Q}_X) \mid \delta\pi_X(\xi) \in \text{Im}(\hat{Q}_l)\}/\text{Im}(\hat{Q}_X)$$

This implies the following.

Proposition 3.13. *We have*

$$T_X \mathcal{E}\mathcal{L}(N, [l])/Q \cong \{\xi \in \ker(\hat{Q}_X) \mid \delta\pi_X(\xi) = 0\}/\{\xi \in \text{Im}(\hat{Q}_X) \mid \delta\pi_X(\xi) = 0\}$$

3.4. Symplectic $\mathcal{E}\mathcal{L}$ -moduli spaces. In previous sections we introduced $\mathcal{E}\mathcal{L}$ -moduli space as the Q -reduction of the space of solutions to Euler-Lagrange equations. In case when $\partial N = \emptyset$ we proved that they carry a symplectic structure coming from symplectic reduction, but this is no longer true in the case with boundary.

In this subsection, for the case with boundary we will introduce a different reduction which is symplectic (see subsection 3.4.3), fibers over the Q -reduction and has simple gluing properties. We call it the symplectic $\mathcal{E}\mathcal{L}$ -moduli space.

Denote the bulk and boundary $\mathcal{E}\mathcal{L}$ -moduli spaces by

$$\mathcal{M}_N = \mathcal{E}\mathcal{L}_N/Q_N, \quad \mathcal{M}_{\partial N} = \mathcal{E}\mathcal{L}_{\partial N}/Q_{\partial N}$$

When N is fixed, we will usually suppress the subscript N for brevity.

3.4.1. The main construction. Denote by $\widetilde{\text{Vect}}_Q^{\text{rel}}$ the space of vector fields of the form $[Q_N, v]$ where v is a *vertical* vector field, i.e. a vector field tangent to the fibers of π . This space is also a Lie subalgebra of $\widetilde{\text{Vect}}_Q^{\text{rel}}$. The restriction of $\widetilde{\text{Vect}}_Q^{\text{rel}}$ to $\mathcal{E}\mathcal{L}$ gives a distribution which we denote by $\text{Vect}_Q^{\text{rel}}$.

Define the *symplectic $\mathcal{E}\mathcal{L}$ -moduli space* $\mathcal{M}^{\text{symp}}$ as the space of leaves of $\text{Vect}_Q^{\text{rel}}$ on $\mathcal{E}\mathcal{L}$:

$$\mathcal{M}^{\text{symp}} = \mathcal{E}\mathcal{L}/\text{Vect}_Q^{\text{rel}}$$

It is clear that

$$T_{[X]}\mathcal{M}^{\text{symp}} = \ker(\hat{Q}_X)/\hat{Q}_X(\ker(\delta\pi_X))$$

The restriction to the boundary $\pi : \mathcal{F} \rightarrow \mathcal{F}_\partial$ induces the projection $\pi_* : \mathcal{M}^{\text{symp}} \rightarrow \mathcal{EL}_\partial$.

The distribution Vect_Q on \mathcal{EL} induces the distribution b on $\mathcal{M}^{\text{symp}}$ which projects to the distribution Vect_{Q_∂} on \mathcal{EL}_∂ by $\delta\pi_*$. It is easy to see that b is involutive and that

$$b_{[X]} \simeq \text{Im}(\hat{Q}_X)/\hat{Q}_X(\ker(\delta\pi_X))$$

where $[X]$ is the leaf of $\text{Vect}_Q^{\text{rel}}$ through X . The image of this projection can be thought of as a quotient distribution $\text{Vect}_Q/\text{Vect}_Q^{\text{rel}}$.

Notice that there is a natural isomorphism $T_X\mathcal{F}/\ker\delta\pi_X \simeq T_l\mathcal{F}_\partial$ where $l = \pi(X)$. We also have a natural degree 1 surjective map

$$\beta : T\mathcal{F}_\partial \rightarrow b, \quad \beta_X : T_l\mathcal{F}_\partial \rightarrow b_{[X]}$$

defined as follows: for $\xi \in T_X\mathcal{F}$ and $\xi + \ker(\delta\pi)$ being identified with an element in $T_l\mathcal{F}_\partial$, we set

$$\beta_X(\xi + \ker(\delta\pi)) = \hat{Q}_X(\xi) + \hat{Q}_X(\ker(\delta\pi))$$

The following statement follows immediately from $\hat{Q}_X^2 = 0$.

Lemma 3.14. *β vanishes on $\text{Im}(\hat{Q}_l)$.*

The usual \mathcal{EL} -moduli space, i.e. space of leaves of Vect_Q on \mathcal{EL} , is naturally isomorphic to the space of leaves of b on $\mathcal{M}^{\text{symp}}$. This follows immediately from the definition of $\mathcal{M}^{\text{symp}}$.

Finally, from the definitions above, it is clear that the following diagram is commutative:

$$(18) \quad \begin{array}{ccccc} b_{[X]} & \xlongequal{\quad} & b_{[X]} & \xrightarrow{\subset} & T_{[X]}\mathcal{M}^{\text{symp}} \\ \beta_{[X]} \uparrow & & (\delta\pi_*)_{[X]} \downarrow & & (\delta\pi_*)_{[X]} \downarrow \\ T_l[-1]\mathcal{F}_\partial & \xrightarrow{\hat{Q}_l} & (\text{Vect}_{Q_\partial})_l & \xrightarrow{\subset} & T_l\mathcal{EL}_\partial \end{array}$$

3.4.2. *More on tangent spaces.* Define the vertical component $T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}}$ of the tangent space $T_{[X]}\mathcal{M}^{\text{symp}}$ as

$$T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} = \ker((\delta\pi_*)_X) \subset T_{[X]}\mathcal{M}^{\text{symp}}$$

and denote the quotient map by $\chi : T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \rightarrow T_{[X]^Q}\mathcal{M}$ where \mathcal{M} is the usual \mathcal{EL} -moduli space for N , i.e. the space of leaves of Vect_Q on \mathcal{EL} , and $[X]^Q$ is the leaf of Vect_Q through X .

The projection $\pi : \mathcal{F} \rightarrow \mathcal{F}_\partial$ restricted to \mathcal{EL} induces the natural projection $\pi_*^Q : \mathcal{M} \rightarrow \mathcal{M}_\partial$. Recall that the fiber of π_*^Q over $[l] \in \mathcal{M}_\partial$ is the space $\mathcal{EL}(N, [l])/Q$ discussed in Section 3.3; the image of π_*^Q is the Q -reduction of the space $\mathcal{L} \subset \mathcal{EL}$ discussed in Section 3.2. Denote by $\psi = \delta\pi_*^Q$ the corresponding mapping of tangent bundles, $\psi_{[X]^Q} : T_{[X]^Q}\mathcal{M} \rightarrow T_{[l]}\mathcal{M}_\partial$.

The restriction of the mapping $\beta : T\mathcal{F}_\partial \rightarrow b$ defined earlier to $\ker(\hat{Q}_l)$ vanishes on $\text{Im}(\hat{Q}_l)$ and therefore induces a linear mapping $(\beta_*)_{[X]} : T_l[-1]\mathcal{M}_\partial \rightarrow T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}}$.

We have a sequence of linear maps:

$$(19) \quad \cdots \rightarrow T_{[l]}[-1]\mathcal{M}_\partial \xrightarrow{\beta_*} T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \xrightarrow{\chi} T_{[X]^Q}\mathcal{M} \xrightarrow{\psi} T_{[l]}\mathcal{M}_\partial \rightarrow \cdots$$

Proposition 3.15. *This is an exact sequence.*

Proof. Sequence (19) can be written as

$$\cdots \rightarrow H_{\hat{Q}_l}^{\bullet-1}(T_l\mathcal{F}_\partial) \rightarrow H_{\hat{Q}_X}^{\bullet}(T_X^{\text{vert}}\mathcal{F}) \rightarrow H_{\hat{Q}_X}^{\bullet}(T_X\mathcal{F}) \rightarrow H_{\hat{Q}_l}^{\bullet}(T_l\mathcal{F}_\partial) \rightarrow \cdots$$

which is induced from the short exact sequence in the non-reduced picture

$$0 \rightarrow T_X^{\text{vert}}\mathcal{F} \rightarrow T_X\mathcal{F} \rightarrow T_l\mathcal{F}_\partial \rightarrow 0$$

by passing to \hat{Q} -cohomology and so is exact by snake lemma. \square

3.4.3. Symplectic structure, Lefschetz duality. Denote by \hat{Q}_X^{vert} the restriction of \hat{Q}_X to $T_X^{\text{vert}}\mathcal{F}$.

We know (cf. the proof of Proposition 3.5) that in $T_X\mathcal{F}$ we have

$$\text{Im}(\hat{Q}_X^{\text{vert}})^\perp = \ker(\hat{Q}_X), \quad \text{Im}(\hat{Q}_X)^\perp = \ker(\hat{Q}_X^{\text{vert}})$$

Assumption 3.16. *We will assume that also the opposite holds:*

$$(20) \quad \ker(\hat{Q}_X)^\perp = \text{Im}(\hat{Q}_X^{\text{vert}}), \quad \ker(\hat{Q}_X^{\text{vert}})^\perp = \text{Im}(\hat{Q}_X)$$

This is a stronger version of assumption (3); in our examples it follows from Hodge-Morrey decomposition theorem for manifolds with boundary [14]. It immediately implies that $\text{Vect}_Q^{\text{rel}}$ coincides with the characteristic distribution on $\mathcal{E}\mathcal{L}$, thus we have the following.

Proposition 3.17. *Assuming (20), $\mathcal{M}^{\text{symp}}$ is the symplectic reduction of $\mathcal{E}\mathcal{L}$ and carries a degree -1 symplectic structure $\underline{\omega}$ coming from \mathcal{F} .*

Definition 3.18. *We call a BV-BFV theory **regular** if the assumption (20) holds for any $X \in \mathcal{E}\mathcal{L}$, together with the assumption (15) for any $l \in \mathcal{E}\mathcal{L}_\partial$.*

Symplectic structure ω on \mathcal{F} induces a bilinear pairing $H_{\hat{Q}_X^{\text{vert}}} \otimes H_{\hat{Q}_X} [1] \rightarrow \mathbb{R}$ or equivalently $T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \otimes T_{[X]Q} [1]\mathcal{M} \rightarrow \mathbb{R}$ which is well-defined due to (14) and non-degenerate due to (20). Together with the symplectic structure on $\mathcal{M}_\partial = \mathcal{E}\mathcal{L}_\partial$ it gives the Lefschetz duality for the long exact sequence (19), i.e. a non-degenerate pairing on $T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \oplus T_{[X]Q} [1]\mathcal{M} \oplus T_{[l]}\mathcal{M}_\partial$:

$$(21) \quad \langle \bullet, \bullet \rangle : \begin{cases} T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \otimes T_{[X]Q} [1]\mathcal{M} & \rightarrow \mathbb{R} \\ T_{[X]Q} [1]\mathcal{M} \otimes T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} & \rightarrow \mathbb{R} \\ T_{[l]}\mathcal{M}_\partial \otimes T_{[l]}\mathcal{M}_\partial & \rightarrow \mathbb{R} \end{cases}$$

Observe that (14) implies that the map χ is self-adjoint with respect to $\langle \bullet, \bullet \rangle$, and ψ and β_* are adjoint to each other, therefore Lefschetz duality can be stated as an injective chain map between the complex (19) and the dual one:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & T_{[l]}\mathcal{M}_\partial & \xrightarrow{\beta_*} & T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} & \xrightarrow{\chi} & T_{[X]Q}\mathcal{M} & \xrightarrow{\psi} & T_{[l]}\mathcal{M}_\partial & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & T_{[l]}^*\mathcal{M}_\partial & \xrightarrow{\psi^*} & T_{[X]Q}^*[-1]\mathcal{M} & \xrightarrow{\chi^*} & (T_{[X]}^{\text{vert}})^*[-1]\mathcal{M}^{\text{symp}} & \xrightarrow{(\beta_*)^*} & T_{[l]}^*\mathcal{M}_\partial & \longrightarrow & \cdots \end{array}$$

In case of a topological field theory, the complex (19) consists of finite dimensional vector spaces and vertical arrows in the diagram above become isomorphisms.

The form $\underline{\omega}$ induces presymplectic structures $\omega^{\text{symp,vert}}$ on the fibers of $\pi_* : \mathcal{M}^{\text{symp}} \rightarrow \mathcal{E}\mathcal{L}_\partial$. The form $\omega^{\text{symp,vert}}$ can be written in terms of Lefschetz duality as:

$$\omega_{[X]}^{\text{symp,vert}} = \langle \bullet, \chi(\bullet) \rangle : \quad T_{[X]}^{\text{vert}}\mathcal{M}^{\text{symp}} \otimes T_{[X]}^{\text{vert}} [1]\mathcal{M}^{\text{symp}} \rightarrow \mathbb{R}$$

Proposition 3.19. *Fibers of $\pi_*^Q : \mathcal{M} \rightarrow \mathcal{M}_\partial$ carry a natural degree -1 symplectic structure coming from \mathcal{F} .*

Proof. By non-degeneracy of (21), the kernel of $\omega^{\text{symp,vert}}$ is exactly the kernel of χ . Thus $\omega^{\text{symp,vert}}$ induces a non-degenerate degree -1 symplectic structure on $\text{Im}(\chi) = \ker(\psi) \subset T_{[X]^Q} \mathcal{M}$ and hence on fibers of $\pi_*^Q : \mathcal{M} \rightarrow \mathcal{M}_\partial$. \square

Thus, the Q -reduced fibers $\mathcal{EL}(N, [l])/Q$ have natural symplectic structure. We call these reduced fibers *the moduli spaces of vacua*.

Proposition 3.20. *The image of $\pi_*^Q : \mathcal{M} \rightarrow \mathcal{M}_\partial$ is locally Lagrangian in \mathcal{M}_∂ .*

Proof. Indeed, for a smooth point $[l] = [\pi(X)] \in \mathcal{M}_\partial$, tangent space to the image of π_*^Q is $\text{Im}(\psi : H_{\hat{Q}_X}^\bullet(T_X \mathcal{F}) \rightarrow H_{\hat{Q}_l}^\bullet(T_l \mathcal{F}_\partial))$. Lagrangianity is proven as follows:

$$\begin{aligned} \text{Im}(\psi)^\perp &= \{a \in H_{\hat{Q}_l}(T_l \mathcal{F}_\partial) \mid \langle a, \psi(b) \rangle = 0 \forall b \in H_{\hat{Q}_X}(T_X \mathcal{F})\} \\ &= \{a \in H_{\hat{Q}_l} \mid \langle \beta_*(a), b \rangle = 0 \forall b \in H_{\hat{Q}_X}\} \\ &= \ker(\beta_*) \\ &= \text{Im}(\psi) \end{aligned}$$

where in the third line we used the non-degeneracy of Lefschetz pairing. \square

The following is a corollary of the above, using also Propositions A.1 and 3.7:

Corollary 3.21. *The space $\mathcal{L} = \pi(\mathcal{EL}) \subset \mathcal{F}_\partial$ is locally Lagrangian.*

3.4.4. *Digression: fibers of π_*^Q via symplectic reduction.* Let $\Lambda \subset \mathcal{F}_\partial$ be a Lagrangian submanifold which is transversal to $\mathcal{L} = \pi(\mathcal{EL})$. Assume that Λ intersects \mathcal{L} at the point l . The subspace $\pi^{-1}(\Lambda) \subset \mathcal{F}$ is the space of fields with boundary values on Λ . We will assume that the subspace $\pi^{-1}(\Lambda)$ is symplectic.

Let us prove that the subspace $\pi^{-1}(\Lambda) \cap \mathcal{EL}$ is locally coisotropic in $\pi^{-1}(\Lambda)$. For this we need to prove that at a smooth point X the tangent space $\mathcal{S}_X = T_X \pi^{-1}(\Lambda) \cap \mathcal{EL}$ is a coisotropic subspace in $T_X \mathcal{F}$.

Because Lagrangian subspaces \mathcal{L} and Λ are transversal, the intersection of their tangent spaces is trivial, $T_X \Lambda \cap T_X \mathcal{L} = \{0\}$, and therefore for each $\xi \in T_X \pi^{-1}(\Lambda) \cap \mathcal{EL}$ we have $\delta\pi_X(\xi) = 0$. Thus

$$\mathcal{S}_X = \{\xi \in \ker(\hat{Q}_X) \mid \delta\pi(\xi) = 0\}$$

Denote

$$I_X = \{\xi \in \text{Im}(\hat{Q}_X) \mid \delta\pi(\xi) = 0\}$$

Observe that $I_X \subset \mathcal{S}_X$ and that

$$(22) \quad \mathcal{S}_X / I_X \cong T_X \mathcal{EL}(N, [l]) / Q$$

by Proposition 3.13.

Lemma 3.22. *For a smooth point X , the symplectic orthogonal subspace to I_X in $T_X \pi^{-1}(\Lambda)$ is \mathcal{S}_X .*

Proof. The symplectic orthogonal of I_X in $T_X \pi^{-1}(\Lambda)$ is:

$$I_X^\perp = \{\eta \in T_X \pi^{-1}(\Lambda) \mid \omega_N(\eta, \xi) = 0, \text{ for any } \xi \in I_X\}$$

That is for any $\xi = \hat{Q}_X \lambda$ such that $\delta\pi(\lambda) \in \ker(\hat{Q}_\partial)$. The orthogonality of ξ and η is equivalent to

$$\omega(\hat{Q}_X \eta, \lambda) + \omega_\partial(\delta\pi(\eta), \delta\pi(\lambda)) = 0$$

for any λ such that $\delta\pi(\lambda) \in \ker(\hat{Q}_l)$. Because the first term should vanish for all λ , we have $\eta \in \ker(\hat{Q}_X)$. Thus the last term should vanish separately for any such λ . This implies that $\delta\pi(\eta)$ is symplectic orthogonal to $\ker(\hat{Q}_l)$. By condition (15) this implies that $\delta\pi(\eta) \in \text{Im}(\hat{Q}_l)$. Because $\text{Im}(\hat{Q}_l) \subset T_l\mathcal{L}$ and because \mathcal{L} and Λ are transversal, we have $\delta\pi(\eta) = 0$. Therefore $I_X^\perp = \mathcal{S}_X$. \square

Corollary 3.23. *The subspace $\mathcal{S}_X \subset T_X\pi^{-1}(\Lambda)$ is coisotropic. Thus $\pi^{-1}(\Lambda) \cap \mathcal{E}\mathcal{L}$ is locally coisotropic in $\pi^{-1}(\Lambda)$.*

Lemma 3.24. *The symplectic orthogonal subspace to \mathcal{S}_X in $T_X\pi^{-1}(\Lambda)$ is I_X .*

Proof. By Lemma 3.22, $I_X^\perp = \mathcal{S}_X$, which implies $I_X \subseteq \mathcal{S}_X^\perp$. On the other hand, due to (22) and to Proposition 3.19, \mathcal{S}_X/I_X is symplectic and thus I_X has to coincide with \mathcal{S}_X^\perp . \square

The following is the immediate corollary of the Lemma above.

Proposition 3.25. *The symplectic reduction of \mathcal{S}_X is*

$$\underline{\mathcal{S}}_X = \mathcal{S}_X/I_X$$

Comparing with the Q -reduction of $T_X\mathcal{E}\mathcal{L}(N, [l])$ we see that two reductions are naturally isomorphic:

$$\underline{T_X\pi^{-1}(\Lambda) \cap \mathcal{E}\mathcal{L}} = T_X\mathcal{E}\mathcal{L}(N, [l])/Q$$

Thus the fiber of $\pi_^Q : \mathcal{M} \rightarrow \mathcal{M}_\partial$ over $[l]$ coincides with the symplectic reduction of $\pi^{-1}(\Lambda) \cap \mathcal{E}\mathcal{L}$ in $\pi^{-1}(\Lambda)$.*

3.4.5. *Example: symplectic $\mathcal{E}\mathcal{L}$ -moduli space for the abelian Chern-Simons theory.* In the abelian Chern-Simons theory, see section 7.1 for details, the following occurs:

- The symplectic $\mathcal{E}\mathcal{L}$ -moduli space is

$$\begin{aligned} \mathcal{M}^{\text{symp}} &= \mathcal{E}\mathcal{L}/\text{Vect}_Q^{\text{rel}} = \\ &= \{A \in \Omega^{\bullet+1}(N) \mid dA = 0\} / (A \sim A + d\alpha \quad \text{for any } \alpha \in \Omega^\bullet(N) \text{ s.t. } \alpha|_{\partial N} = 0) \end{aligned}$$

- The restriction map $\pi_* : \mathcal{M}^{\text{symp}} \rightarrow \mathcal{E}\mathcal{L}_\partial = \Omega_{\text{closed}}^{\bullet+1}(\partial N)$ sends the class $[A] \in \mathcal{M}^{\text{symp}}$ to $A|_{\partial N} \in \mathcal{E}\mathcal{L}_\partial$ (which is well defined).
- Fibers of π_* are isomorphic to the relative cohomology $H^{\bullet+1}(N, \partial N)$.
- The map $\beta_{[A]} : T_{A|_{\partial N}}\mathcal{F}_{\partial N} \rightarrow T_{[A]}\mathcal{M}^{\text{symp}}$ sends $\alpha_\partial \in \Omega^\bullet(\partial N)$ to $[d\tilde{\alpha}]$ for arbitrary $\tilde{\alpha} \in \Omega^\bullet(N)$ such that $\tilde{\alpha}|_{\partial N} = \alpha_\partial$.
- Taking the quotient of $\mathcal{M}^{\text{symp}}$ by the distribution $b = \text{im}(\beta)$ gives the usual $\mathcal{E}\mathcal{L}$ -moduli space

$$\mathcal{M}^{\text{symp}}/b = \mathcal{M} = H^{\bullet+1}(N)$$

which is the absolute de Rham cohomology of N .

- Exact sequence (19) is the usual long exact sequence of relative cohomology

$$\dots \rightarrow H^\bullet(\partial N) \rightarrow H^{\bullet+1}(N, \partial N) \rightarrow H^{\bullet+1}(N) \rightarrow H^{\bullet+1}(\partial N) \rightarrow \dots$$

- The symplectic form ω on $\Omega^\bullet(N)$ descends to

$$\underline{\omega}([A], [B]) = \int_N A \wedge B$$

on $\mathcal{M}^{\text{symp}}$. Lefschetz duality (21) is the usual Lefschetz duality between absolute and relative cohomology, plus Poincaré duality for the cohomology of the boundary.

3.5. The gluing (cutting) of symplectic \mathcal{EL} -moduli spaces.

3.5.1. *The non-reduced case.* Assume that a manifold N is cut into two pieces N_1, N_2 along a codimension 1 submanifold N_2^∂ such that $\partial N_1 = N_1^\partial \sqcup N_2^\partial$ and $\partial N_2 = N_2^\partial \sqcup N_3^\partial$. We have natural commutative diagram:

$$(23) \quad \begin{array}{ccc} \mathcal{F}_N & \longrightarrow & \mathcal{F}_{N_1} \\ \downarrow & & \downarrow \\ \mathcal{F}_{N_2} & \longrightarrow & \mathcal{F}_{N_2^\partial} \end{array}$$

The space of fields \mathcal{F}_N is the subspace in the fiber product of spaces \mathcal{F}_{N_1} and \mathcal{F}_{N_2} over $\mathcal{F}_{N_2^\partial}$ consisting of fields which are smooth at N_∂ .

The Euler-Lagrange space for N is the fiber product of Euler-Lagrange spaces for N_1 and for N_2 :

$$\mathcal{EL}_N = \mathcal{EL}_{N_1} \times_{\mathcal{EL}_{N_2^\partial}} \mathcal{EL}_{N_2}$$

3.5.2. *The gluing.* The gluing for symplectic \mathcal{EL} -moduli spaces goes as follows. Let N_1, N_2 be two spacetime manifolds with boundaries $\partial N_1 = N_1^\partial \sqcup N_2^\partial$ and $\partial N_2 = (N_2^\partial)' \sqcup N_3^\partial$ respectively. Assume that N_2^∂ diffeomorphic to $(N_2^\partial)'$, and denote by N the result of gluing N_1 and N_2 along the common boundary component:

$$N = (N_1 \sqcup N_2) / (N_2^\partial \sim (N_2^\partial)')$$

The space $\mathcal{M}_N^{\text{symp}}$ can be constructed intrinsically in terms of $\mathcal{M}_{N_1}^{\text{symp}}$ and $\mathcal{M}_{N_2}^{\text{symp}}$ as follows:

(i) First consider the fiber product

$$(24) \quad \widetilde{\mathcal{M}} = \mathcal{M}_{N_1}^{\text{symp}} \times_{\mathcal{EL}_{N_2^\partial}} \mathcal{M}_{N_2}^{\text{symp}}$$

For any point $\tilde{X} \in \widetilde{\mathcal{M}}$, we have a map

$$(25) \quad \tilde{\beta}_{\tilde{X}} : T_{\pi_1(\tilde{X})} \mathcal{F}_{N_1^\partial} \times T_{\pi_2(\tilde{X})} \mathcal{F}_{N_2^\partial} \times T_{\pi_3(\tilde{X})} \mathcal{F}_{N_3^\partial} \rightarrow T_{\tilde{X}} \widetilde{\mathcal{M}}$$

induced from the two maps

$$\begin{aligned} (\beta_1 \times \beta_2)_{[X_1]} : T_{\pi_1(X_1)} \mathcal{F}_{N_1^\partial} \times T_{\pi_2(X_1)} \mathcal{F}_{N_2^\partial} &\rightarrow T_{[X_1]} \mathcal{M}_{N_1}^{\text{symp}} \\ (\beta_{2'} \times \beta_3)_{[X_2]} : T_{\pi_{2'}(X_2)} \mathcal{F}_{(N_2^\partial)'} \times T_{\pi_3(X_2)} \mathcal{F}_{N_3^\partial} &\rightarrow T_{[X_2]} \mathcal{M}_{N_2}^{\text{symp}} \end{aligned}$$

(ii) The space $\mathcal{M}_N^{\text{symp}}$ now can be identified with the leaf space of the distribution $\tilde{\beta}(0 \times T\mathcal{F}_{N_2^\partial} \times 0)$ on $\widetilde{\mathcal{M}}$:

$$\mathcal{M}_N^{\text{symp}} = \widetilde{\mathcal{M}} / \tilde{\beta}(0 \times T\mathcal{F}_{N_2^\partial} \times 0)$$

It inherits the quotient distribution

$$b_N = b_N^1 \times b_N^3 = \text{im}(\tilde{\beta}) / \tilde{\beta}(0 \times T\mathcal{F}_{N_2^\partial} \times 0)$$

parameterized by $T\mathcal{F}_{N_1^\partial} \times T\mathcal{F}_{N_3^\partial}$.

The important point here is that the gluing of symplectic \mathcal{EL} -moduli spaces is done in intrinsic terms of the symplectically reduced picture, i.e. in terms of the ingredients of diagram (18): $(\mathcal{M}^{\text{symp}}, \mathcal{F}_\partial, Q_\partial, \pi_*, \beta)$.

3.5.3. *Gluing tangent spaces.* The construction of gluing of symplectic \mathcal{EL} -moduli spaces described above implies the following Mayer-Vietoris type long exact sequence for $T_{[X]}\mathcal{M}^{\text{symp}}$:

$$(26) \quad \cdots \rightarrow T_{[\pi_2(X)]}[-1]\mathcal{M}_{N_2^\partial} \xrightarrow{(\beta_2 - \beta_{2'})_*} T_{[X]}\mathcal{M}_N^{\text{symp}} \rightarrow \\ \rightarrow T_{[X_1]}\mathcal{M}_{N_1}^{\text{symp}, Q_2} \oplus T_{[X_2]}\mathcal{M}_{N_2}^{\text{symp}, Q_{2'}} \rightarrow T_{[\pi_2(X)]}\mathcal{M}_{N_2^\partial} \rightarrow \cdots$$

Here $\mathcal{M}_{N_1}^{\text{symp}, Q_2}$ is the quotient of $\mathcal{M}_{N_1}^{\text{symp}}$ by the distribution b_2 . Similarly, $\mathcal{M}_{N_2}^{\text{symp}, Q_{2'}}$ is the quotient of $\mathcal{M}_{N_2}^{\text{symp}}$ by the distribution $b_{2'}$.³

We have a similar long exact sequence for the tangent space to the usual \mathcal{EL} -moduli space:

$$(27) \quad \cdots \rightarrow T_{[\pi_2(X)]}[-1]\mathcal{M}_{N_2^\partial} \rightarrow T_{[X]}\mathcal{M}_N \rightarrow T_{[X_1]}\mathcal{M}_{N_1} \oplus T_{[X_2]}\mathcal{M}_{N_2} \rightarrow T_{[\pi_2(X)]}\mathcal{M}_{N_2^\partial} \rightarrow \cdots$$

Here $[X]$ is the space of leaves of Vect_Q through X .

3.5.4. *Example: gluing in abelian Chern-Simons theory.* Here we will give an example of gluing symplectic \mathcal{EL} -moduli spaces. Let $N = N_1 \cup_{N_2^\partial} N_2$ as before. Then

$$\widetilde{\mathcal{M}} = \frac{\{A \in \Omega^{\bullet+1}(N) \mid dA = 0\}}{A \sim A + d\alpha \quad \text{for any } \alpha \in \Omega^\bullet(N) \text{ s.t. } \alpha|_{N_1^\partial} = \alpha|_{N_2^\partial} = \alpha|_{N_3^\partial} = 0}$$

Mapping (25) acts as

$$\tilde{\beta}_{[A]} : (\alpha_1^\partial, \alpha_2^\partial, \alpha_3^\partial) \mapsto [d\tilde{\alpha}]$$

Here $(\alpha_1^\partial, \alpha_2^\partial, \alpha_3^\partial) \in T_{A|_{N_1^\partial}}\mathcal{F}_{N_1^\partial} \times T_{A|_{N_2^\partial}}\mathcal{F}_{N_2^\partial} \times T_{A|_{N_3^\partial}}\mathcal{F}_{N_3^\partial}$ and $\tilde{\alpha} \in \Omega^\bullet(N)$ is any form such that $\tilde{\alpha}|_{N_k^\partial} = \alpha_k^\partial$ for $k = 1, 2, 3$. The class $[d\tilde{\alpha}] \in T_{[A]}\widetilde{\mathcal{M}}$ does not depend on the choice of $\tilde{\alpha}$. To pass from $\widetilde{\mathcal{M}}$ to $\mathcal{M}_N^{\text{symp}}$ we should mod out differentials of forms on the interface N_2^∂ extended to N . Now we can write the symplectic \mathcal{EL} -moduli space as the quotient

$$\mathcal{M}_N^{\text{symp}} = \widetilde{\mathcal{M}} / \tilde{\beta}(0 \times \Omega^\bullet(N_2^\partial) \times 0) \simeq \\ \simeq \frac{\{A \in \Omega^{\bullet+1}(N) \mid dA = 0\}}{A \sim A + d\alpha \quad \text{for any } \alpha \in \Omega^\bullet(N) \text{ s.t. } \alpha|_{N_1^\partial} = \alpha|_{N_3^\partial} = 0}$$

It is equipped with two commuting distributions b_N^1, b_N^3 parameterized by $\Omega^\bullet(N_1^\partial), \Omega^\bullet(N_3^\partial)$ respectively.

Mayer-Vietoris sequence (26) becomes the following:

$$\cdots \rightarrow H^\bullet(N_2^\partial) \rightarrow H^{\bullet+1}(N, N_1^\partial \sqcup N_3^\partial) \rightarrow \\ \rightarrow H^{\bullet+1}(N_1, N_1^\partial) \oplus H^{\bullet+1}(N_2, N_3^\partial) \rightarrow H^{\bullet+1}(N_2^\partial) \rightarrow \cdots$$

³An equivalent description: $\mathcal{M}_{N_1}^{\text{symp}, Q_2} = \mathcal{EL}_{N_1} / \text{Vect}_Q^{\text{rel}1}$ where $\text{Vect}_Q^{\text{rel}1}$ is the distribution generated by vector fields of the form $[Q_{N_1}, v]$ with v tangent to the fibers of π_1 . Likewise, $\mathcal{M}_{N_2}^{\text{symp}, Q_{2'}} = \mathcal{EL}_{N_2} / \text{Vect}_Q^{\text{rel}3}$.

whereas the version for usual \mathcal{EL} -moduli spaces (27) reads

$$\dots \rightarrow H^\bullet(N_2^\partial) \rightarrow H^{\bullet+1}(N) \rightarrow H^{\bullet+1}(N_1) \oplus H^{\bullet+1}(N_2) \rightarrow H^{\bullet+1}(N_2^\partial) \rightarrow \dots$$

3.6. Higher codimensions. The BV-BFV theory is the extension of the BV theory to manifolds with boundary. In a similar way the BV theory extends to higher codimension submanifolds.

Definition 3.26. *The following collection of data is called a k -extended BV theory in dimension n .*

For each $i = 0, \dots, k - 1$, a k -extended BV theory assigns to every $(n - i)$ -dimensional manifold N_i with boundary N_{i+1} :

- (1) a space of fields \mathcal{F}_{N_i} which is a graded manifold with exact symplectic form $\omega_{N_i} = (-1)^{n-i} \delta \alpha_{N_i}$ with $gh(\omega_{N_i}) = gh(\alpha_{N_i}) = i - 1$, and with cohomological vector field Q_{N_i} ,
- (2) a projection $\pi_i : \mathcal{F}_{N_i} \rightarrow \mathcal{F}_{N_{i+1}}$,
- (3) an action functional S_{N_i} on \mathcal{F}_{N_i} with $gh(S_{N_i}) = i$.

These data should satisfy the following axioms:

- $\delta \pi_{i-1}(Q_{N_{i-1}}) = Q_{N_i}$,
- $\iota_{Q_{N_{i-1}}} \omega_{N_{i-1}} = (-1)^{n-i+1} \delta S_{N_{i-1}} + \pi_i^*(\alpha_{N_i})$

We call k -extension of a given BV theory in dimension n a k -extended theory with the original data for $i = 0$.

Definition 3.27. *A BV field theory has length k if k is the maximal number such that in its k -extension for $i = 1, \dots, k$, all $\pi_{i-1}(\mathcal{EL}_{N_{i-1}}) = \mathcal{L}_{N_{i-1}} \subset \mathcal{EL}_{N_i} \subset \mathcal{F}_{N_i}$ are Lagrangian and all Q -reduced fibers are finite dimensional. Here $\mathcal{EL}_{N_i} \subset \mathcal{F}_{N_i}$ is the set of zeroes of the vector field Q_{N_i} . If k is equal to n , we say that the theory is **maximally extended**.*

Usually BV field theories have length 1. For example, scalar field theory in dimension greater than one or Yang-Mills theory in dimension greater than two. We never consider theories of length zero. In the rest of the paper we will show that scalar field theory in dimension 1, Yang-Mills theory in dimension 2 and all AKSZ theories are maximally extended (this includes BF , Chern-Simons theories and the Poisson sigma model). Notice that scalar theory in dimension one is just quantum mechanics and the Yang-Mills theory in dimension 2 is known to be almost topological [57] (meaning that it only depends on the topology of the spacetime manifold and on its volume).

Remark 3.28 (Extended BV theories on manifolds with corners). *A k -extended BV theory in dimension n naturally leads to an associated theory on n -dimensional manifolds with corners up to codimension k . Namely, to such a manifold N we associate the data for $N_0 = N$ and N_1 the union of the codimension one⁴ strata in ∂N . To each such stratum $N' \subset \partial N$ we associate the data for $N_1 = N'$ and N_2 the union of the codimension one strata in $\partial N'$ (notice that this is the union of only some codimension two strata in ∂N). To each such stratum $N'' \subset \partial N'$ we associate the data for $N_2 = N''$ and N_3 the union of the codimension one strata in $\partial N''$ (notice that this is the union of only some codimension two strata in $\partial N'$ and in turn of some codimension three strata in ∂N), and so on.*

⁴Here and in the following, the codimension of a boundary stratum is computed in terms of the bulk manifold.

3.7. Boundary conditions. In the current paper we insist on having free boundary conditions. An alternative approach, first explored in [18, 43], fixes boundary conditions in a way compatible with the BFV structure on the boundary. This corresponds to choosing a Lagrangian submanifold \mathcal{L} of the BFV space of boundary fields to which the boundary cohomological vector field is tangent and on which the boundary 1-form vanishes. Equivalently, the boundary action should vanish on \mathcal{L} . We call such Lagrangian submanifolds *adapted*.

At first sight, it would also seem natural to impose \mathcal{L} to be transversal to the Lagrangian submanifold L_M . However, this condition is too restrictive and rules out a lot of interesting boundary conditions. Instead, a better assumption would be that the intersection of \mathcal{L} with L_N should be finite dimensional after reduction.

As shown in [18], in the case of AKSZ theories, one can obtain an adapted Lagrangian submanifold \mathcal{L} by choosing an adapted Lagrangian submanifold L_i of the target manifold \mathcal{M} for each boundary component $\partial_i N$ of the spacetime N . The submanifold $\mathcal{L} = \prod_i \text{Map}(T[1]\partial_i N, L_i)$ in this case is an adapted Lagrangian in $\mathcal{F}_{\partial N}$. Notice that usually one does not require the decomposition $\partial M = \cup_i \partial_i M$ to be disjoint, rather the pairwise intersections are assumed to be of lower dimension. The adapted Lagrangian submanifolds of the target are usually referred to as *branes*.

In the case when the target is $T^*[1]P$, with P a Poisson manifold, one can easily show that a brane is necessarily of the form $N^*[1]C$, where C is a coisotropic submanifold of P and N^*C denotes its conormal bundle. When the target is a differential graded symplectic manifold (with symplectic form of degree 2) associated to a Courant algebroid over a manifold N , then a brane covering N is the same as a Dirac structure.

An intermediate type of boundary condition consists in splitting the boundary into two components and in choosing an adapted Lagrangian submanifold of the BFV space of fields on the first component while keeping free boundary conditions for the second component. The analysis performed in this paper still holds for boundary fields of the second component. This mixed approach might have several applications. For example [15], the study of the Poisson Sigma Model on the disk whose boundary is split into an even number of ordered intervals I_i with boundary conditions on I_{2s} determined by the $C = P \forall s$ leads to the construction of what is known as the relative symplectic groupoid integrating P .

3.8. $gh = 0$ part of the BV-BFV theory.

3.8.1. Non reduced theory. The $gh = 0$ part of the BV-BFV theory is a first order classical field theory with the space of fields $F_N = \mathcal{F}_N^{(0)}$ (the $gh = 0$ part of \mathcal{F}_N), the classical action $S_N^{(0)} = S_N|_{\mathcal{F}_N^{(0)}}$.

The $gh = 0$ part of the space of boundary fields $F_{\partial N} = \mathcal{F}_{\partial N}^{(0)}$ is a symplectic manifold with the symplectic form $\omega_{\partial N}^{(0)} = \delta\alpha_{\partial N}^{(0)}$ which is the $gh = 0$ part of the symplectic form $\omega_{\partial N}$ on boundary BFV fields. The one-form $\delta\alpha_{\partial N}^{(0)}$ is the $gh = 0$ part of the form $\alpha_{\partial N}$ and it determined by the $gh = 0$ part of the classical action.

Gauge transformations are $gh = 0$ part of the BV-BFV gauge transformations generated by vector fields $[V, Q]$ and they are Hamiltonian on the boundary.

The $gh = 0$ part of the coisotropic submanifold $\mathcal{EL}_{\partial N} \subset \mathcal{F}_N$ is a coisotropic submanifold $C_{\partial N} = \mathcal{EL}_{\partial N}^{(0)} \subset F_N$. It consists of boundary fields which can be

extended to a solution to Euler-Lagrange equations in the vicinity of the boundary in N .

In regular BV theories the space of solutions to the Euler-Lagrange equations $EL_N = \mathcal{EL}_N^{(0)}$ projects to the Lagrangian subspace $L_N = \mathcal{L}_N^{(0)} = \pi(EL_N) \subset C_{\partial N} \subset F_{\partial N}$.

3.8.2. The reduction. The classical moduli space EL_N/G_N is the $gh = 0$ part of the \mathcal{EL} -moduli space \mathcal{EL}_N/Q_N and maps to the reduced phase space $C_{\partial N}/G_{\partial N} = \underline{C}_{\partial N}$, which is in turn the $gh = 0$ part of the boundary \mathcal{EL} -moduli space $\underline{\mathcal{EL}}_{\partial N}$. Here \underline{G}_N denotes the distribution on EL_N induced by Vect_{Q_N} , likewise for $\underline{G}_{\partial N}$; also $\underline{G}_{\partial N}$ can be seen as the coisotropic distribution on $C_{\partial N}$.

In the regular case, the image of

$$(28) \quad \pi_* : EL_N/G_N \rightarrow \underline{C}_{\partial N}$$

is the reduced Lagrangian $L_N/G_{\partial N} = \underline{L}_N$.

Remark 3.29. *In regular case, there is also the following relation between the smooth loci (cf. Appendix C) of moduli spaces:*

$$(29) \quad \begin{array}{ccc} T_{\text{vert}}^*[-1](EL_N/G_N)^{\text{smooth}} & \xrightarrow{\subseteq} & (\mathcal{EL}_N/Q_N)^{\text{smooth}} \\ \pi_* \downarrow & & \pi_* \downarrow \\ (C_{\partial N})^{\text{smooth}} & \xrightarrow{\subseteq} & (\underline{\mathcal{EL}}_{\partial N})^{\text{smooth}} \end{array}$$

Here T_{vert}^* denotes the dual of the vertical tangent bundle of the fibration (28). Horizontal arrows in (29) may be strict inclusions (e.g. abelian Chern-Simons theory where \mathcal{EL} -moduli spaces contain additional smooth pieces: $H^0(N)$ and $H^3(N)$, cf. section 7.1.3), or may be equalities (e.g. non-abelian Chern-Simons theory with a simple gauge group G).

4. THE BFV CATEGORY

4.1. Spacetime categories. Recall that an n -dimensional spacetime category is the category of n -dimensional cobordisms which may have additional structure (smooth, Riemannian etc.). See [4, 51, 54] for the discussion of various examples.

In most general terms *objects* of a d -dimensional spacetime category are $(d - 1)$ -dimensional manifolds (space manifolds). In specific examples of spacetime categories, space manifolds are equipped with a structure (orientation, symplectic structure, Riemannian metric, etc.).

A *morphism* between two space manifolds Σ_1 and Σ_2 is a d -dimensional manifold M , possibly with a structure (orientation, symplectic, Riemannian metric, etc.), together with the identification of $\Sigma_1 \sqcup \overline{\Sigma_2}$ with the boundary of M . Here $\overline{\Sigma}$ is the manifold Σ with reversed orientation.

Composition of morphisms is the gluing along the common boundary. Here are examples of spacetime categories.

The d -dimensional topological category. Objects are smooth, compact, oriented $(d - 1)$ -dimensional manifolds. A morphism between Σ_1 and Σ_2 is a d -dimensional smooth compact oriented manifold with $\partial M = \Sigma_1 \sqcup \overline{\Sigma_2}$. The orientation on M should agree with the orientations of Σ_i in a natural way. The composition consists of gluing two morphisms along the common boundary.

The d -dimensional Riemannian category. Objects are oriented $(d - 1)$ Riemannian manifolds with collars. Morphisms between two objects N_1 and N_2 are oriented d -dimensional Riemannian manifolds M , such that $\partial M = N_1 \sqcup \overline{N_2}$. The orientation on all three manifolds should naturally agree, and the metric on M agrees with the metric on N_1 and N_2 on a collar of the boundary. The composition is the gluing of such Riemannian cobordisms. For details see [54].

The d -dimensional metrized cell complexes. Objects are $(d-1)$ -dimensional oriented metrized cell complexes (edges have length, 2-cells have area, etc.). A morphism between two such complexes C_1 and C_2 is an oriented metrized d -dimensional cell complex C together with two embeddings of metrized cell complexes $i : C_1 \hookrightarrow C$, $j : \overline{C_2} \hookrightarrow C$ where i is orientation reversing and j is orientation preserving. The composition is the gluing of such triples along the common $(d-1)$ -dimensional subcomplex.

This is the underlying category for all lattice models in statistical mechanics.

The Pseudo-Riemannian category The difference between this category and the Riemannian category is that morphisms are pseudo-Riemannian with the signature $(d - 1, 1)$. This is the most interesting category for physics. When $d = 4$ it represents the spacetime structure of our universe.

4.2. The BFV category. The category BFV has the following objects, morphisms and compositions of morphisms.

Objects of BFV are triples (\mathcal{F}, α, Q) where \mathcal{F} is an exact graded symplectic manifold with the symplectic form $\omega = d\alpha$ with ghost number 0, and Q is a cohomological vector field (i.e. its Lie derivative squares to zero) with ghost number 1. The symplectic form should be preserved by Q :

$$L_Q \omega = 0$$

Let \mathcal{EL} be the space of zeroes of the vector field Q . The restriction of the Lie subalgebra $Vect_Q = [Vect(\mathcal{F}), Q] \subset Vect(\mathcal{F})$ defines an involutive⁵ distribution on \mathcal{EL} .

Morphisms between $(\mathcal{F}_1, \alpha_1, Q_1)$ and $(\mathcal{F}_2, \alpha_2, Q_2)$ are differential graded manifolds \mathcal{F} with symplectic form $\omega^{\mathcal{F}}$ with ghost number -1 , with cohomological vector field $Q^{\mathcal{F}}$ with ghost number 1, with the function $S^{\mathcal{F}}$ (action function) on \mathcal{F} with $gh(S^{\mathcal{F}}) = 0$, and with two projection mappings $\pi_i : \mathcal{F} \rightarrow \mathcal{F}_i$. These data should satisfy the following conditions:

- Projections π_i are mappings of differential graded manifolds, i.e. $\delta\pi_i(Q^{\mathcal{F}}) = Q_i$.
- The following identity should hold

$$\iota_{Q^{\mathcal{F}}} \omega^{\mathcal{F}} = (-1)^n \delta S^{\mathcal{F}} - \pi_1^*(\alpha_1) + \pi_2^*(\alpha_2),$$

- Let $\mathcal{EL}^{\mathcal{F}}$ be the zero-locus of the vector field $Q^{\mathcal{F}}$, then $\mathcal{L}^{\mathcal{F}} = (\pi_1 \times \pi_2)(\mathcal{EL}^{\mathcal{F}}) \subset \overline{\mathcal{F}_1} \times \mathcal{F}_2$ should be Lagrangian. Here $\overline{\mathcal{F}_1}$ is the dg manifold \mathcal{F}_1 with symplectic form $-\omega_1$.

⁵We will always assume that this distribution is actually integrable, which is not automatic in the infinite dimensional case.

- For each Lagrangian submanifold \mathcal{L} which is generic, relative to $\mathcal{L}^{\mathcal{F}}$ (the intersection is transversal), the preimage $(\pi_1 \times \pi_2)^{-1}(\mathcal{L}) \subset \mathcal{F}$ should be symplectic.

Composition of morphisms Let $\mathcal{F} : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ and $\mathcal{F}' : \mathcal{F}_2 \rightarrow \mathcal{F}_3$ be two morphisms in BFV. The composition $\mathcal{F}' \cdot \mathcal{F}$ is the fiber product $\mathcal{F} \times_{\mathcal{F}_2} \mathcal{F}' = \{(x, x') \in \mathcal{F} \times \mathcal{F}' \mid \pi_2(x) = \pi_2(x')\}$. It is a submanifold in $\mathcal{F}' \times \mathcal{F}$.

The symplectic form on $\mathcal{F}' \cdot \mathcal{F}$ is the pullback of the symplectic form on $\mathcal{F}' \times \mathcal{F}$. The vector field $Q + Q'$ on $\mathcal{F}' \times \mathcal{F}$ is tangent to $\mathcal{F}' \cdot \mathcal{F} \subset \mathcal{F}' \times \mathcal{F}$ and it induced the vector field Q^{comp} on the fiber product. The action function is additive: $S^{\mathcal{F}' \cdot \mathcal{F}}(x, x') = S^{\mathcal{F}}(x) + S^{\mathcal{F}'}(x')$.

Let f be a mapping which assigns to each object \mathcal{F} of BFV the functional $f^{\mathcal{F}}$ on \mathcal{F} . Define the mapping $F_f : \text{BFV} \rightarrow \text{BFV}$ as follows. It acts trivially on \mathcal{F} . It acts on one forms α_{Σ} as

$$\alpha^{\mathcal{F}} \rightarrow \alpha^{\mathcal{F}} + \delta f^{\mathcal{F}}$$

and does not change $Q^{\mathcal{F}}$. On the morphism $(\mathcal{F}, \omega^{\mathcal{F}}, Q^{\mathcal{F}}, S^{\mathcal{F}}) : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ this mapping acts it as follows. It acts trivially on $\omega^{\mathcal{F}}$, on $Q^{\mathcal{F}}$, and on \mathcal{F} while on the action $S^{\mathcal{F}}$ it acts as

$$S^{\mathcal{F}} \rightarrow S^{\mathcal{F}} + \pi_1^*(f^{\mathcal{F}_1}) - \pi_2^*(f^{\mathcal{F}_2})$$

It is easy to see that F is a covariant endofunctor for BFV.

Remark 4.1. *The notion of BV-BFV category introduced above has a natural generalization where the objects are quadruples $(\mathcal{F}, \mathcal{L}, \alpha, Q)$. Here \mathcal{F} is a \mathbb{Z} -graded manifold, \mathcal{L} is a line bundle over it, α is a connection on \mathcal{L} and Q is a cohomological degree 1 vector field on \mathcal{F} .*

A morphism between $(\mathcal{F}_1, \mathcal{L}_1, \alpha_1, Q_1)$ and $(\mathcal{F}_2, \mathcal{L}_2, \alpha_2, Q_2)$ is a \mathbb{Z} -graded manifold with the same data as before but instead of $S^{\mathcal{F}}$ being a function on \mathcal{F} we have a section of the line bundle $\mathcal{L}^{\mathcal{F}} = \pi^(\mathcal{L}_1 \times \mathcal{L}_2)$ over \mathcal{F} which is horizontal for the flat connection $(-1)^{n-1} \frac{i}{\hbar} (\iota_{Q^{\mathcal{F}}} \omega^{\mathcal{F}} + \pi_1^*(\alpha_1) - \pi_2^*(\alpha_2))$ on $\mathcal{L}^{\mathcal{F}}$, cf. Remark 3.3.*

4.3. The BV functor. Now classical BV-BFV theories can be regarded as functors from spacetime categories to BFV category.

Fix a spacetime category. A classical BV-BFV field theory for a spacetime from this category defines the covariant functor, the *BV functor*, from the spacetime category to the BFV category.

On objects: The BV functor assigns the space of fields \mathcal{F}_{Σ} to the object Σ of the spacetime category with the BFV data α_{Σ} and Q_{Σ} .

On morphisms: The BV functor assigns the space of fields \mathcal{F}_N for a morphism $N : \Sigma_1 \rightarrow \Sigma_2$ with the BV data ω_N, Q_N, S_N , and mappings $\pi_i : \mathcal{F}_N \rightarrow \mathcal{F}_{\Sigma_i}$.

The properties of the BV-BFV field theories guarantee that this mapping is a covariant functor.

Remark 4.2. *The BFV category is 1-category. The corresponding BV functor is a functor from 1-category of cobordisms to the BFV category. This can be extended to higher categories. The natural target structure is a k -extended BFV category. It is a k -category. The k -extended classical BV field theory is the k -functor from the k -category of k -cobordisms to the k -extended BFV category, similar to k -extended topological quantum field theories [6, 39]. We will discuss extended theories in another publication.*

5. EXAMPLES OF BV-BFV THEORIES

5.1. Electrodynamics. Here we will consider the BV-extended classical Euclidean electrodynamics in the trivial $U(1)$ -bundle. Spacetime manifolds in Euclidean electrodynamics are smooth oriented n -dimensional Riemannian manifolds. By $*$: $\Omega^i(N) \rightarrow \Omega^{n-i}(N)$ we denote the Hodge operation induced by the metric on N .

5.1.1. The BV-BFV structure for classical electrodynamics. The space of fields in the BV-extended classical electrodynamics on the spacetime manifold N is $T^*[-1]E_N$. Here $E_N = \Omega^1(N) \oplus \Omega^{n-2}(N) \oplus \Omega^0(N)[1]$ where the first summand is the space of connections A in the trivial $U(1)$ bundle over N , the second summand is the space of fields B , the Hamiltonian counterpart (“momentum”) of A , the third summand is the space of ghost fields. The total space of fields in BV classical electrodynamics is

$$\mathcal{F}_N = \Omega^1(N) \oplus \Omega^{n-2}(N) \oplus \Omega^0(N)[1] \oplus \Omega^{n-1}(N)[-1] \oplus \Omega^2(N)[-1] \oplus \Omega^n(N)[-2]$$

We will use notations $A, B, c, A^\dagger, B^\dagger, c^\dagger$ for the fields from corresponding summands⁶. Here we regard $\Omega^k(N)$ as a vector space concentrated in degree zero.

The BV-symplectic form on the space of fields is the canonical symplectic form on $T^*[-1]E_N$:

$$\omega_N = \int_N (\delta A \wedge \delta A^\dagger + \delta B \wedge \delta B^\dagger + \delta c \wedge \delta c^\dagger)$$

It has degree -1 .

The BV-extended action of the classical electrodynamics on N has degree zero:

$$S_N = \int_N \left(B \wedge F(A) + \frac{1}{2} B \wedge *B + A^\dagger \wedge dc \right),$$

where $F(A) = dA$ is the curvature of the connection A . The vector field Q_N is

$$Q_N = \int_N \left(dc \wedge \frac{\delta}{\delta A} + dB \wedge \frac{\delta}{\delta A^\dagger} + (*B + dA) \wedge \frac{\delta}{\delta B^\dagger} + dA^\dagger \wedge \frac{\delta}{\delta c^\dagger} \right)$$

and it has $gh = 1$.

It acts on coordinate fields as

$$Q_N A = dc, \quad Q_N A^\dagger = dB, \quad Q_N B^\dagger = *B + dA, \quad Q_N c^\dagger = dA^\dagger$$

On other coordinate fields Q_N acts trivially. Here and below we are using the same notation for the vector field Q_N and for its Lie derivative.

The boundary BFV theory has the space of fields

$$\mathcal{F}_{\partial N} = \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N) \oplus \Omega^0(\partial N)[1] \oplus \Omega^{n-1}(\partial N)[-1]$$

We will denote corresponding fields by A, B, c, A^\dagger respectively. The projection $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ acts as

$$\pi(A) = i^*(A), \quad \pi(B) = i^*(B), \quad \pi(c) = i^*(c), \quad \pi(A^\dagger) = i^*(A^\dagger), \quad \pi(B^\dagger) = 0, \quad \pi(c^\dagger) = 0$$

The boundary symplectic form is the differential of the form

$$\alpha_{\partial N} = \int_{\partial N} (B \wedge \delta A + A^\dagger \wedge \delta c)$$

$$\omega_{\partial N} = \delta \alpha_{\partial N} = \int_{\partial N} (\delta B \wedge \delta A + \delta A^\dagger \wedge \delta c)$$

⁶Here we discuss the *minimal* BV extension of Hamiltonian classical electrodynamics.

The boundary vector field $Q_{\partial N} = \delta\pi Q_N$ is

$$(30) \quad Q_{\partial N} = \int_{\partial N} \left(dB \wedge \frac{\delta}{\delta A^\dagger} + dc \wedge \frac{\delta}{\delta A} \right)$$

The boundary action is

$$S_{\partial N} = \int_{\partial N} c \wedge dB$$

Proposition 5.1. *The data described above satisfy the BV-BFV axioms.*

Proof. The only non-trivial computation is to check the classical master equation (7) when $\partial N \neq \emptyset$. Contracting the vector field Q_N with the symplectic form ω_N we obtain

$$\iota_{Q_N} \omega_N = \int_N (dc \wedge \delta A^\dagger + \delta A \wedge dB + \delta B \wedge (*B + dA) + dA^\dagger \wedge \delta c)$$

The differential of the action is easy to compute:

$$\delta S_N = \int_N (\delta B \wedge dA + B \wedge d\delta A + \delta B \wedge *B + \delta A^\dagger \wedge dc + A^\dagger \wedge d\delta c)$$

Comparing these two formulae and using the Stokes formula we obtain the classical master equation . \square

Remark 5.2. *The connection field A in electrodynamics is called the vector potential. When $n = 4$ and $N = [t_1, t_2] \times M$ is equipped with Minkowsky metric choose a basis e_0, e_1, e_2, e_3 in the tangent space where e_0 is a time direction. On EL_N the components B_{0i} , $i = 1, 2, 3$ give the magnetic field on M and the components B_{ij} give the electric field.*

5.1.2. *The Q -reduction of \mathcal{EL}_N .* The Euler-Lagrange equations in the bulk are:

$$dB = 0, \quad B = -*dA, \quad dA^\dagger = 0, \quad dc = 0$$

Note that this implies the usual Maxwell's equation for the vector potential $d*dA = 0$.

This defines the subspace $\mathcal{EL}_N \subset \mathcal{F}_N$:

$$(31) \quad \mathcal{EL}_N = \{(A, B) \in \Omega^1(N) \oplus \Omega^{n-2}(N) \mid d*dA = 0, B = -*dA\} \oplus \\ \oplus \Omega_{closed}^0(N)[1] \oplus \Omega_{closed}^{n-1}(N)[-1] \oplus \Omega^2(N)[-1] \oplus \Omega^n(N)[-2]$$

The summands correspond to fields $A, B, c, A^\dagger, B^\dagger, c^\dagger$ respectively.

Proposition 5.3. *The Q -reduced space of solutions to the Euler-Lagrange equations is*

$$(32) \quad \mathcal{EL}_N/Q \simeq \Omega_{Maxw}^1(N)/\Omega_{exact}^1(N) \oplus H^0(N)[1] \oplus H^{n-1}(N)[-1] \oplus H^n(N)[-2]$$

Here $\Omega_{Maxw}^1(N)$ is the space of solutions to Maxwell's equations, i.e. 1-forms A , such that $d*dA = 0$. The summands represent quotient spaces in fields $A, c, A^\dagger, c^\dagger$ respectively.

Proof. Let us first find the Q -reduced tangent space to \mathcal{EL}_N :

$$T_X \mathcal{EL}_N/Q = \ker(\hat{Q}_X)/\text{Im}(\hat{Q}_X)$$

Because $\mathcal{E}\mathcal{L}_N$ is a vector space its tangent space, which is isomorphic to $\ker(\hat{Q}_X)$, is given by (31). The image of \hat{Q}_X is easy to compute:

$$(33) \quad \text{Im}(\hat{Q}_X) = \Omega_{exact}^1(N) \oplus \{(A^\dagger, B^\dagger) \in \Omega^{n-1}(N)[-1] \oplus \Omega^2(N)[-1] \mid A^\dagger = d\beta, B^\dagger = *\beta + \text{exact}, \beta \in \Omega^{n-2}(N)\} \oplus \Omega_{exact}^n(N)[-2]$$

Here the components correspond to fields $A, A^\dagger, B^\dagger, c^\dagger$ respectively. Let us prove now that

$$(34) \quad \{(A^\dagger, B^\dagger) \in \Omega^{n-1}(N) \oplus \Omega^2(N) \mid A^\dagger = d\beta, B^\dagger = *\beta + \text{exact}, \beta \in \Omega^{n-2}(N)\} = \Omega_{exact}^{n-1}(N) \oplus \Omega^2(N)$$

By the Hodge-Morrey decomposition (see for example [14] and references therein) we can write $\Omega(N) = \Omega_{coclosed}(N) + \Omega_{exact}(N)$. Using this decomposition, for an exact $(n-1)$ -form $A^\dagger = d\gamma$ and an arbitrary 2-form B^\dagger we can write $B^\dagger - *\gamma = *\theta + d\eta$ with θ closed. Then $A^\dagger = d\beta, B^\dagger = *\beta + d\eta$ for $\beta = \gamma + \theta$ and therefore the r.h.s. of (34) is the subspace of l.h.s. and since to opposite inclusion is obvious we proved (34).

Now we can write

$$\text{Im}(\hat{Q}_X) = \Omega_{exact}^1(N) \oplus \Omega_{exact}^{n-1}(N)[-1] \oplus \Omega^2(N)[-1] \oplus \Omega_{exact}^n(N)[-2]$$

Together with the formula (31) this proves the Proposition. \square

Proposition 5.4. *When $\partial N = \emptyset$*

$$\mathcal{E}\mathcal{L}_N/Q = H^1(N) \oplus H^0(N)[1] \oplus H^{n-1}(N)[-1] \oplus H^n(N)[-2]$$

The summands correspond to fields $A, c, A^\dagger, c^\dagger$ respectively.

Proof. A 1-form A satisfies the Maxwell's equation $d * dA = 0$ if and only if $dA \in \ker(d^*)$. The metric on N gives the Hodge decomposition:

$$\Omega(N) = H(N) \oplus \Omega_{d-exact}(N) \oplus \Omega_{d^*-exact}(N)$$

where $H(N)$ are harmonic forms representing cohomology classes of N . It is clear from this decomposition that the first summand in (32) is $H^1(N)$. \square

When N does not have a boundary the space $\mathcal{E}\mathcal{L}_N/Q$ described above has a natural symplectic structure given by the Poincaré pairing between $H^1(N)$ and $H^{n-1}(N)$ and between $H^0(N)$ and $H^n(N)$.

5.1.3. *The reduction of boundary structures.* Recall that the space of boundary fields is

$$\mathcal{F}_{\partial N} = \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N) \oplus \Omega^0(\partial N)[1] \oplus \Omega^{n-1}(\partial N)[-1]$$

Where the summands correspond to pullbacks of fields A, B, c, A^\dagger respectively. The Euler-Lagrange equations on the boundary (equations for zeroes of the vector field $Q_{\partial N}$) are

$$dB = 0, \quad dc = 0$$

Thus, the space of solutions to boundary Euler-Lagrange equations is

$$\mathcal{E}\mathcal{L}_{\partial N} = \Omega^1(\partial N) \oplus \Omega_{closed}^{n-2}(\partial N) \oplus \Omega_{closed}^0(\partial N)[1] \oplus \Omega^{n-1}(\partial N)[-1]$$

Because it is a vector space, it is isomorphic to its tangent space at every point.

Similarly to the discussion for the bulk, for $l \in \mathcal{EL}_{\partial N}$ we get

$$(35) \quad \text{Im}(\hat{Q}_l) = \Omega_{exact}^1(\partial N) \oplus \Omega_{exact}^{n-1}(\partial N)[-1] \subset T_l \mathcal{EL}_{\partial N}$$

Taking the quotient space $\ker(\hat{Q}_{\partial N})/\text{Im}(\hat{Q}_{\partial N})$ and identifying the tangent space with the space itself we prove the following.

Proposition 5.5. *The Q -reduced space of boundary fields is*

$$\mathcal{EL}(\partial N)/Q = \Omega^1(\partial N)/\{exact\} \oplus \Omega_{closed}^{n-2}(\partial N) \oplus H^0(\partial N)[1] \oplus H^{n-1}(\partial N)[-1]$$

Here summands correspond to fields A, B, c, A^\dagger respectively.

This space is clearly infinite-dimensional. It coincides with the symplectic reduction of $\mathcal{EL}_{\partial N}$ with symplectic structure given by the natural pairing between the first and the second and between the third and the fourth summands.

Proposition 5.6. *BV-extended classical electrodynamics is a regular theory (in the sense of Definition 3.18).*

Proof. By direct calculation, we have

$$(36) \quad \ker(\hat{Q}_X) = \underbrace{\{(a, b) \in \Omega^1 \oplus \Omega^{n-2} \mid b = - * da, db = 0\}}_{\cong \Omega_{cl}^1 \oplus \Omega_{coex, cl}^{n-2}} \oplus \Omega_{cl}^0[1] \oplus \Omega_{cl}^{n-1}[-1] \oplus \Omega^2[-1] \oplus \Omega^n[-2],$$

$$(37) \quad \text{Im}(\hat{Q}_X) = \Omega_{ex}^1 \oplus \Omega_{ex}^{n-1}[-1] \oplus \Omega^2[-1] \oplus \Omega_{ex}^n[-2],$$

$$(38) \quad \ker(\hat{Q}_X^{\text{vert}}) = \Omega_{cl, D}^1 \oplus \Omega_{cl, D}^0[1] \oplus \Omega_{cl, D}^{n-1}[-1] \oplus \Omega^2[-1] \oplus \Omega^n[-2],$$

$$(39) \quad \text{Im}(\hat{Q}_X^{\text{vert}}) = \Omega_{ex, D}^1 \oplus \Omega_{ex, D}^n[-2] \oplus \underbrace{\{(db, da + *b) \in \Omega^{n-1} \oplus \Omega^2 \mid a \in \Omega_D^1, b \in \Omega_D^{n-2}\}}_{\cong \Omega_{ex, D}^{n-1} \oplus (\Omega_{ex, D}^2 \oplus \Omega_{cocl, N}^2)}[-1]$$

Here we use a shorthand notation with $ex, cl, coex, cocl, D, N$ standing for exact, closed, coexact, coclosed, Dirichlet, Neumann respectively (the last two indicating the imposed boundary condition; $\Omega_{ex, D}$ means exact, with the *primitive* being subject to Dirichlet condition). Hodge-Morrey decomposition theorem implies that subspaces (36) and (39) are mutually orthogonal in $T_X \mathcal{F}_N$, and (38) and (37) are mutually orthogonal too. Thus the assumption (20) holds.

Also,

$$\ker(\hat{Q}_l) = \Omega_{\partial}^1 \oplus \Omega_{cl, \partial}^{n-2} \oplus \Omega_{cl, \partial}^0[1] \oplus \Omega_{\partial}^{n-1}[-1]$$

and

$$\text{Im}(\hat{Q}_l) = \Omega_{ex, \partial}^1 \oplus \Omega_{ex, \partial}^{n-1}[-1]$$

(where ∂ stands for forms on the boundary) are mutually orthogonal in $T_l \mathcal{F}_{\partial N}$ due to Hodge decomposition on ∂N , thus the assumption (15) also holds. Therefore electrodynamics is a regular theory. \square

5.1.4. *The Lagrangian subspace \mathcal{L}_N and its reduction.* Now let us describe the evolution relation $\mathcal{L}_N = \pi(\mathcal{EL}_N) \subset \mathcal{F}_{\partial N}$ and its reduction. Due to regularity, \mathcal{L}_N

is Lagrangian. This subspace consists of pullbacks of fields A, B, c, A^\dagger satisfying Euler-Lagrange equation in N :

$$(40) \quad \mathcal{L}_N = \{(i^*(A), i^*(- * dA)) \in \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N) \mid d * dA = 0, A \in \Omega^1(N)\} \\ \oplus i^*(\Omega_{closed}^0(N))[1] \oplus i^*(\Omega_{closed}^{n-1}(N))[-1]$$

The Q -reduction of the Lagrangian subspace \mathcal{L}_N , or equivalently, its symplectic reduction is:

$$(41) \quad \mathcal{L}_N/Q = \{(i^*(A), i^*(- * dA)) \in \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N) \mid \\ d * dA = 0, A \in \Omega^1(N)\} / \{\text{exact } A\} \oplus \tilde{H}^0(\partial N)[1] \oplus \tilde{H}^{n-1}(\partial N)[-1]$$

Here $\tilde{H}^i(\partial N)$ are cohomology classes of ∂N which are pullbacks of cohomology classes on N .

5.1.5. *Gauge classes of solutions to Euler-Lagrange equations with fixed gauge classes of boundary values.* It is clear that the tangent space $T_X \mathcal{E}\mathcal{L}(N, [l])$ depends neither on X nor on l and is

$$(42) \quad \{(A, - * dA) \in \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N) \mid d * dA = 0, i^*(A) \text{ exact}, A \in \Omega^1(N)\} \oplus \\ \oplus \Omega_{closed}^0(N, \partial N)[1] \oplus \{A^\dagger \in \Omega_{closed}^{n-1}(N)[-1] \mid i^*(A^\dagger) \text{ exact}\} \oplus \\ \oplus \Omega^2(N)[-1] \oplus \Omega^n(N)[-2]$$

For the quotient space we have:

$$(43) \quad T_X \mathcal{E}\mathcal{L}(N, [l]) / \text{Im}(\hat{Q}_X) = \\ = \frac{H^1(N, \partial N)}{H^0(\partial N)} \oplus \frac{H^{n-1}(N, \partial N)}{H^{n-2}(\partial N)}[-1] \oplus H^0(N, \partial N)[1] \oplus H^n(N)[-2]$$

It is the Q -reduction of the tangent space $T_X \mathcal{E}\mathcal{L}(N, [l])$. It is a finite dimensional symplectic space. Since we are in a linear case, $\mathcal{E}\mathcal{L}(N, [l])/Q$ is isomorphic to (43).

5.1.6. *Codimension 2 BV structure.* Here we will describe the extension of the BV electrodynamics to codimension 2 strata. Let Σ be an $(n-1)$ -dimensional manifold with the boundary $\partial\Sigma$. The boundary $\partial\Sigma$ is closed and the space of boundary fields is

$$\mathcal{F}_{\partial\Sigma} = \Omega^{n-2}(\partial\Sigma) \oplus \Omega^0(\partial\Sigma)[1]$$

where the summands correspond to pullbacks of fields B and c from Σ to the boundary. We will denote the pullbacks of B and c by the same letters.

The restriction mapping $\pi : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_{\partial\Sigma}$ acts as $\pi(A) = \pi(A^\dagger) = 0$ and $\pi(B) = B$, $\pi(c) = c$.

The symplectic structure on this space is exact

$$\omega_{\partial\Sigma} = \int_{\partial\Sigma} \delta B \wedge \delta c = \delta \alpha_{\partial\Sigma}$$

where

$$\alpha_{\partial\Sigma} = \int_{\partial\Sigma} \delta B \wedge c$$

The action and the vector field Q on $\mathcal{F}_{\partial\Sigma}$ are trivial:

$$S_{\partial\Sigma} = 0, \quad Q_{\partial\Sigma} = 0$$

Proposition 5.7. *The action $S_\Sigma = \int_\Sigma c \wedge dB$ satisfies the equation (7).*

The proof is a straightforward computation.

It is clear that

$$\mathcal{E}\mathcal{L}_{\partial\Sigma} = \mathcal{F}_{\partial\Sigma},$$

The evolution relation $\mathcal{L}_\Sigma = \pi(\mathcal{E}\mathcal{L}_\Sigma)$

$$\mathcal{L}_\Sigma = i^*(\Omega_{closed}^{n-2}(\Sigma)) \oplus i^*(\Omega_{closed}^0(\Sigma))[1]$$

When there is only one connected component of the boundary

$$\mathcal{L}_\Sigma = \Omega_{exact}^{n-2}(\partial\Sigma) \oplus \Omega_{closed}^0(\partial\Sigma)[1]$$

Because $Q_{\partial\Sigma} = 0$ the reduction is trivial: the reduced structures are the same as non-reduced ones. The reduced fiber over $l = (B, c)$ is

$$\mathcal{E}\mathcal{L}(\Sigma, [l])/Q = \Omega^1(\Sigma)/\{exact\} \oplus \Omega_{closed}^{n-2}(\Sigma) \oplus H^0(\Sigma, \partial\Sigma)[1] \oplus H^{n-1}(\Sigma)[-1]$$

This space is infinite dimensional for $n > 2$ and it is finite dimensional when $n = 2$.

Remark 5.8. *The reduced space \mathcal{L}_Σ/Q and fibers of $\pi : \mathcal{E}\mathcal{L}_\Sigma/Q \rightarrow \mathcal{L}_\Sigma/Q$ are infinite dimensional when $n > 2$. When $n = 2$ they are finite dimensional. This corresponds to two dimensional electrodynamics which is an almost topological field theory [57].*

5.2. Yang-Mills theory. As in the classical Euclidean electrodynamics spacetime manifolds in the Yang-Mills theory are oriented compact smooth, possibly with boundary (and with corners), Riemannian manifolds. Let \mathfrak{g} be the Lie algebra of a finite dimensional simply connected Lie group G with \mathfrak{g} -invariant scalar product. To simplify notations we assume that \mathfrak{g} is a matrix algebra and that the scalar product is given by the trace: $\langle a, b \rangle = tr(ab)$.

5.2.1. The non-reduced theory. In the first order formulation of Yang-Mills theory fields are connections A in a principal G -bundle P over N and $(n-2)$ -forms B with coefficients in the associated adjoint bundle. For simplicity, we assume that the principal bundle is trivial and consider connections as 1-forms with coefficients in \mathfrak{g} and B fields as $(n-2)$ -forms with coefficients in \mathfrak{g} . The ghost fields c are 0-forms with coefficients in \mathfrak{g} . The BV extension includes anti-fields $A^\dagger, B^\dagger, c^\dagger$. The total space of BV extended Yang-Mills theory is⁷

$$\begin{aligned} \mathcal{F}_N = & \underbrace{\mathfrak{g} \otimes \Omega^1(N)}_A \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-2}(N)}_B \oplus \underbrace{\mathfrak{g} \otimes \Omega^0(N)[1]}_c \oplus \\ & \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-1}(N)[-1]}_{A^\dagger} \oplus \underbrace{\mathfrak{g} \otimes \Omega^2(N)[-1]}_{B^\dagger} \oplus \underbrace{\mathfrak{g} \otimes \Omega^n(N)[-2]}_{c^\dagger} \end{aligned}$$

This is a graded infinite dimensional vector space with the symplectic form

$$\omega_N = \int_N tr (\delta A \wedge \delta A^\dagger + \delta B \wedge \delta B^\dagger + \delta c \wedge \delta c^\dagger)$$

The action functional is

$$S_N = \int_N tr \left(B \wedge F_A + \frac{1}{2} B \wedge *B + A^\dagger \wedge d_A c + B^\dagger \wedge [B, c] + \frac{1}{2} c^\dagger \wedge [c, c] \right)$$

⁷Here, as in the case of classical electrodynamics we only discuss the minimal BV extension. When $n = 4$ the BV extension of Yang-Mills theory can also be presented in a different way using the decomposition of 2-forms into self-dual and anti-self-dual parts, see [16] and [22].

where $F_A = dA + \frac{1}{2}[A, A]$, and the cohomological vector field is

$$Q_N = \int_N \text{tr} \left(d_{AC} \wedge \frac{\delta}{\delta A} + [B, c] \wedge \frac{\delta}{\delta B} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} + (d_A B + [A^\dagger, c]) \wedge \frac{\delta}{\delta A^\dagger} + (F_A + *B + [B^\dagger, c]) \wedge \frac{\delta}{\delta B^\dagger} + (d_A A^\dagger + [B, B^\dagger] + [c, c^\dagger]) \wedge \frac{\delta}{\delta c^\dagger} \right)$$

The boundary structure. We will denote the pullback to the boundary of forms A, B, A^\dagger, c by the same letters. The space of boundary fields is the quotient space of the pullback of \mathcal{F}_N to the boundary over the kernel of the form $\delta\tilde{\alpha}_{\partial N}$, as it is explained in section 3.1:

$$\begin{aligned} \mathcal{F}_{\partial N} = & \underbrace{\mathfrak{g} \otimes \Omega^1(\partial N)[1]}_{A, \text{gh}=0} \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-2}(\partial N)[n-2]}_{B, \text{gh}=0} \oplus \\ & \oplus \underbrace{\mathfrak{g} \otimes \Omega^0(\partial N)[1]}_{c, \text{gh}=1} \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-1}(\partial N)[n-2]}_{A^\dagger, \text{gh}=-1} \end{aligned}$$

The structure of an exact symplectic manifold on $\mathcal{F}_{\partial N}$ is given by

$$\begin{aligned} \alpha_{\partial N} &= \int_{\partial N} \text{tr} (B \wedge \delta A + A^\dagger \wedge \delta c), \\ \omega_{\partial N} &= \int_{\partial N} \text{tr} (\delta B \wedge \delta A + \delta A^\dagger \wedge \delta c) \end{aligned}$$

The vector field $Q_{\partial N}$ and the action $S_{\partial N}$ for the boundary BFV theory are

$$\begin{aligned} Q_{\partial N} &= \int_{\partial N} \text{tr} \left(d_{AC} \wedge \frac{\delta}{\delta A} + [B, c] \wedge \frac{\delta}{\delta B} + (d_A B + [A^\dagger, c]) \wedge \frac{\delta}{\delta A^\dagger} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} \right), \\ S_{\partial N} &= \int_{\partial N} \text{tr} \left(B \wedge d_{AC} + \frac{1}{2}A^\dagger \wedge [c, c] \right) \end{aligned}$$

The codimension 2 structure. Let Σ be a stratum of codimension 2. The BV-BFV theory on N and on ∂N induce the following data associated on Σ (see section 3.1. The space of fields:

$$\mathcal{F}_\Sigma = \underbrace{\mathfrak{g} \otimes \Omega^{n-2}(\Sigma)[n-2]}_{B, \text{gh}=0} \oplus \underbrace{\mathfrak{g} \otimes \Omega^0(\Sigma)[1]}_{c, \text{gh}=1}$$

Here we denoted the pullback of B and of c to Σ by the same letters. The rest of the BFV data, the exact symplectic form, the vector field Q and the action S (which can be obtained from Q by the Roytenberg's construction) are:

$$\begin{aligned} \alpha_\Sigma &= \int_\Sigma \text{tr}(B \wedge \delta c), \\ \omega_\Sigma &= \int_\Sigma \text{tr}(\delta B \wedge \delta c), \\ Q_\Sigma &= \int_\Sigma \text{tr} \left([B, c] \wedge \frac{\delta}{\delta B} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} \right), \\ S_\Sigma &= \int_\Sigma \text{tr} \left(\frac{1}{2}B \wedge [c, c] \right) \end{aligned}$$

5.2.2. *The non-reduced $gh = 0$ part.*

The bulk. The fields A and its Hamiltonian counterpart remain, so the space of fields is

$$F_N = \underbrace{\mathfrak{g} \otimes \Omega^1(\partial N)}_A \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-2}(\partial N)}_B$$

The classical action is

$$S_N^{cl} = \int_N \text{tr} \left(B \wedge F_A + \frac{1}{2} B \wedge *B \right)$$

Its critical points are solutions to Euler-Lagrange equations

$$d_A B = 0, \quad F_A + *B = 0$$

or, equivalently

$$d_A *F_A = 0, \quad B = -*F_A$$

Infinitesimal gauge transformations are infinitesimal automorphisms of the trivial G -bundle over N , i.e. elements of the Lie algebra $\text{Map}(N, \mathfrak{g})$. They act as

$$A \mapsto A + d_A \alpha, \quad B \mapsto B + [B, \alpha]$$

where $\alpha \in \Omega^0(N, \mathfrak{g})$.

The boundary. Boundary fields in the Yang-Mills theory are pullbacks of fields A and B . We will denote them by the same letter:

$$F_{\partial N} = \underbrace{\mathfrak{g} \otimes \Omega^1(\partial N)}_A \oplus \underbrace{\mathfrak{g} \otimes \Omega^{n-2}(\partial N)}_B$$

The exact symplectic structure on $\mathcal{F}_{\partial N}$ is given by

$$\begin{aligned} \alpha_{\partial N}^{cl} &= \int_{\partial N} \text{tr}(B \wedge \delta A) \\ \omega_{\partial N}^{cl} &= \int_{\partial N} \text{tr}(\delta B \wedge \delta A) = \delta \alpha_{\partial N}^{cl} \end{aligned}$$

Euler-Lagrange subspace, $gh = 0$ part of $\mathcal{EL}_{\partial N}$, is defined by the constraint

$$d_A B = 0$$

Boundary gauge transformations are:

$$A \mapsto A + d_A \alpha, \quad B \mapsto B + [B, \alpha]$$

The action of the Lie algebra of gauge transformations is Hamiltonian with the moment map $\mu : F_{\partial N} \rightarrow \mathfrak{g} \otimes \Omega^{n-1}(\partial N)$:

$$(A, B) \mapsto d_A B$$

Codimension 2 part. The $gh = 0$ part of the codimension 2 structure is given by the pullback of B to the boundary:

$$F_{\Sigma} = \mathfrak{g} \otimes \Omega^{n-2}(\Sigma)$$

with no constraints and the gauge transformations given by

$$B \mapsto B + [B, \alpha]$$

where $\alpha \in \Omega^0(\Sigma, \mathfrak{g})$.

5.2.3. *The reduction in the $gh = 0$ part.* The space EL_N is naturally isomorphic to the space of solutions to the Yang-Mills equation $d_A * F(A) = 0$.

The classical moduli space is naturally isomorphic to the space of gauge classes of solutions to the YM equation:

$$EL_N/G_N = \{(A, B) \mid A \in \mathfrak{g} \otimes \Omega^1(N), d_A * F_A = 0, B = -*F_A\} / \{(A, B) \sim (A + d_A \alpha, B + [B, \alpha])\}$$

The reduced phase space is naturally isomorphic to the cotangent bundle of the space of gauge classes of all connections:

$$EL_{\partial N}/G_{\partial N} = \{(A, B) \mid A \in \mathfrak{g} \otimes \Omega^1(\partial N), B \in \mathfrak{g} \otimes \Omega^{n-2}(\partial N), d_A B = 0\} / \{(A, B) \sim (A + d_A \alpha, B + [B, \alpha])\}$$

The $gh = 0$ part of the \mathcal{EL} -moduli space for a codimension 2 stratum is simply

$$EL_{\Sigma}/G_{\Sigma} \cong (\mathfrak{g} \otimes \Omega^{n-2}(\Sigma))/G$$

where Σ is $(n-2)$ -dimensional and the quotient is by the adjoint action of G .

The restriction map

$$\pi_* : EL_N/G_N \rightarrow EL_{\partial N}/G_{\partial N}$$

is the pullback of the connection A and of the form B to the boundary. The fibers are finite-dimensional. One way to see this is by homological perturbation theory around electrodynamics.

The reduced phase space $EL_{\partial N}/G_{\partial N}$ is infinite-dimensional for $n \geq 3$. The case $n = 2$ is special as for $n = 2$ we have

$$EL_{\partial N}/G_{\partial N} \cong T^* \mathcal{M}_{\partial N}^G$$

where $\mathcal{M}_{\partial N}^G$ denotes the moduli space of flat G -connections on ∂N which is finite-dimensional.

Under the regularity assumption, the image of π_* is Lagrangian. Smooth loci of \mathcal{EL} -moduli spaces for the bulk and the boundary are described by diagram (29) with horizontal arrows being equalities (we assume the group G to be simple for this to hold).

5.3. Scalar field.

5.3.1. *The non-reduced picture.* For the massless free n -dimensional scalar field the bulk BV data is:

$$\begin{aligned} \mathcal{F}_N &= \underbrace{\Omega^0(N)}_{\phi} \oplus \underbrace{\Omega^{n-1}(N)}_p \oplus \underbrace{\Omega^n(N)[-1]}_{\phi^\dagger} \oplus \underbrace{\Omega^1(N)[-1]}_{p^\dagger} \\ \omega_N &= \int_N (\delta\phi \wedge \delta\phi^\dagger + \delta p \wedge \delta p^\dagger) \\ S_N &= \int_N (p \wedge d\phi + \frac{1}{2} p \wedge *p) \\ Q_N &= \int_N \left(dp \wedge \frac{\delta}{\delta\phi^\dagger} + (d\phi + *p) \wedge \frac{\delta}{\delta p^\dagger} \right) \end{aligned}$$

The boundary BFV data in this theory is:

$$\begin{aligned}\mathcal{F}_{\partial N} &= \underbrace{\Omega^0(\partial N)}_{\phi} \oplus \underbrace{\Omega^{n-1}(\partial N)}_p \\ \alpha_{\partial N} &= \int_{\partial N} p \wedge \delta\phi \\ \omega_{\partial N} &= \delta\alpha_{\partial N} = \int_{\partial N} \delta p \wedge \delta\phi \\ Q_{\partial N} &= 0 \\ S_{\partial N} &= 0\end{aligned}$$

It is easy to derive them from the BV data in the bulk as it is explained in section 3.1. It is clear that the theory has length one.

For the Euler-Lagrange space we have

$$(44) \quad \mathcal{EL}_N = \{(\phi, p, \phi^\dagger, p^\dagger) \mid dp = 0, p = - * d\phi\}$$

At the boundary $\mathcal{EL}_{\partial N} = \mathcal{F}_{\partial N}$.

5.3.2. *The reduction.* Similarly to the case of electrodynamics we have,

$$\ker(\hat{Q})/Im(\hat{Q}) \simeq \Omega_{Harmon}^0(N) \oplus H^n(N)[-1]$$

where $\Omega_{Harmon}^0(N)$ are harmonic zero-forms on N ; the first summand correspond to the field ϕ and the second to ϕ^\dagger . Because of the linearity, the moduli space \mathcal{EL}_N/Q_N is given by the same formula.

Remark 5.9. *If $\partial N = \emptyset$ then $\mathcal{EL}_N/Q_N \simeq H^0(N) \oplus H^n(N)[-1]$ is a finite-dimensional odd-symplectic space.*

Because $Q_{\partial N} = 0$, the reduced boundary Euler-Lagrange space is the space of boundary fields:

$$(45) \quad \mathcal{EL}_{\partial N}/Q_{\partial N} = \mathcal{F}_{\partial N} = \Omega^0(\partial N) \oplus \Omega^{n-1}(\partial N)$$

The reduced evolution relation is the same as the non-reduced one:

$$(46) \quad \begin{aligned}\underline{\mathcal{L}}_N &= \\ &= \{(\phi, p) \in \mathcal{F}_{\partial N} \mid p = -(*d\tilde{\phi})|_{\partial N} \text{ for } \tilde{\phi} \text{ a harmonic continuation of } \phi \text{ into } N\} \simeq \\ &\simeq \Omega^0(\partial N)\end{aligned}$$

Because the harmonic continuation $\tilde{\phi}$ exists and is unique, the subspace $\underline{\mathcal{L}}_N$ is Lagrangian.

The restriction (the pullback to the boundary) mapping is surjective over $\underline{\mathcal{L}}_N$

$$(47) \quad \begin{array}{c} \mathcal{EL}_N/Q_N \\ \downarrow \\ \underline{\mathcal{L}}_N \subset \mathcal{EL}_{\partial N}/Q_{\partial N} \end{array}$$

Its fibers are finite-dimensional odd-symplectic spaces canonically isomorphic to

$$(48) \quad H^0(N, \partial N) \oplus H^n(N)[-1] \simeq T^*[-1]\mathbb{R}^k$$

Where $k \geq 0$ is the number of closed connected components of N .

5.3.3. *Massive scalar field.* The BV extension of the massive scalar field is similar to the massless scalar field. The space of fields and the symplectic structure are the same. The action and the vector field Q_N are

$$(49) \quad S_N = \int_N \left(p \wedge d\phi + \frac{1}{2} p \wedge *p - \frac{m^2}{2} \phi \wedge *\phi \right)$$

$$(50) \quad Q_N = \int_N \left((dp - m^2 * \phi) \wedge \frac{\partial}{\partial \phi^\dagger} + (d\phi + *p) \wedge \frac{\partial}{\partial p^\dagger} \right)$$

The boundary data are unchanged. The reduced bulk Euler-Lagrange space is

$$(51) \quad \mathcal{EL}_N / Q_N \simeq \Omega_{Klein-Gordon}^0(N)$$

Here $\Omega_{Klein-Gordon}^0(N)$ is the space of functions satisfying the equation $\Delta\phi - m^2\phi = 0$. It fibers over $\underline{\mathcal{L}} \simeq \Omega^0(\partial N)$ with zero fibers. That is, when $m \neq 0$ the fibers of the \mathcal{EL} -moduli spaces over given boundary values are trivial.

It is also easy to construct BV extensions of scalar fields interacting with the Yang-Mills theory, and to generalize it to Dirac and Majorana fermions.

5.4. Abelian BF theory.

5.4.1. *The BV-BFV structure of BF theory.* Here we will focus on the BV-BFV extension of the classical abelian BF gauge theory. Let us start with the description of the corresponding BV theory.

The space of fields in this theory is:

$$\mathcal{F}_N = \Omega^\bullet(N)[1] \oplus \Omega^\bullet(N)[n-2]$$

For coordinate fields we will write $\mathcal{A} \in \Omega^\bullet(N)[1]$ and $\mathcal{B} \in \Omega^\bullet(N)[n-2]$. Here $\Omega^\bullet(N)$ is regarded as a \mathbb{Z} -graded vector space with $\Omega^k(N)$ being its component of degree $-k$.

Remark 5.10. *The BV-BF theory is an example of the AKSZ theory (cf. section 6) with the target manifold $T^*[1]\mathbb{R}$ and with $\Theta = 0$.*

Remark 5.11. *Dimensions $n = 2, 3$ are special. When $n = 2$ the theory is equivalent to the topological sector of electrodynamics. When $n = 3$ it is equivalent to two copies of abelian Chern-Simons theories.*

The BV symplectic form

$$\omega_N = \int_N \delta\mathcal{A} \wedge \delta\mathcal{B}$$

The total degree (ghost number) is $gh(\omega_N) = -1$.

The action functional is:

$$S_N = \int_N \mathcal{B} \wedge d\mathcal{A}$$

The vector field Q_N is

$$Q_N = \int_N \left(d\mathcal{A} \wedge \frac{\delta}{\delta\mathcal{A}} + d\mathcal{B} \wedge \frac{\delta}{\delta\mathcal{B}} \right)$$

On “coordinate functions” it acts as

$$(52) \quad Q_N \mathcal{A} = d\mathcal{A}, \quad Q_N \mathcal{B} = d\mathcal{B}$$

The space \mathcal{EL}_N of solutions to Euler-Lagrange equations, i.e. the space of zeroes of the vector field Q_N , is the space of closed forms:

$$\mathcal{EL}_N = \Omega_{\text{closed}}^\bullet(N)[1] \oplus \Omega_{\text{closed}}^\bullet(N)[n-2]$$

Clearly this space is coisotropic in \mathcal{F}_N .

The space of boundary fields is:

$$\mathcal{F}_{\partial N} = \Omega^\bullet(\partial N)[1] \oplus \Omega^\bullet(\partial N)[n-2]$$

and the natural restriction map $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ is the pullback of forms to the boundary.

The space of boundary fields is a symplectic space with the exact symplectic form of degree 0:

$$\omega_{\partial N} = \delta\alpha_{\partial N}, \quad \alpha_{\partial N} = \int_{\partial N} \mathcal{A} \wedge \delta\mathcal{B}$$

The boundary cohomological vector field $Q_{\partial N} = \delta\pi Q_N$ is

$$Q_{\partial N}\mathcal{A} = d\mathcal{A}, \quad Q_{\partial N}\mathcal{B} = d\mathcal{B}$$

The Euler-Lagrange subspace $\mathcal{EL}_{\partial N} \subset \mathcal{F}_{\partial N}$ is the subspace of closed forms on ∂N .

Proposition 5.12. *The data described above is an example of the BV-BFV theory.*

The proof is a straightforward computation which proves identities from section 3.1.

The evolution relation $\mathcal{L}_N \subset \mathcal{EL}_{\partial N}$ consists of closed forms on ∂N which extend to closed forms on N .

Proposition 5.13. *The subspace $\mathcal{L}_N \subset \mathcal{EL}_{\partial N} \subset \mathcal{F}_{\partial N}$ is Lagrangian.*

To prove this Proposition we need the following lemma.

Lemma 5.14. *Denote by $\iota_* : H_\bullet(\partial N) \rightarrow H_\bullet(N)$ the pushforward by $\iota : \partial N \hookrightarrow N$ in homology. The subspace $\ker(\iota_*)$ is isotropic in $H_\bullet(\partial N)$ with respect to the bilinear form given by the intersection pairing.*

Proof. Let U and V be two cycles in N relative to the boundary, with $\dim(U) + \dim(V) = \dim(N) + 1$, and let $u = \partial U$, $v = \partial V$ be their boundaries in ∂N . By general position argument we can assume that $U \cap V$ is a one dimensional chain. This chain gives a cobordism between $u \cap v$ and the empty set. Hence the sum of coefficients in $u \cap v$, which is the intersection pairing, vanishes. \square

Proof of Proposition 5.13. By the natural identification of $H_\bullet(\partial N)^*$ with $H^\bullet(\partial N)$, the annihilator of $\ker(\iota_*)$ is $\text{Im}(\iota^*)$. Therefore, by Lemma 5.14, the subspace $\text{Im}(\iota^*)$ is coisotropic in $H^\bullet(\partial N)$.

Because $\mathcal{L}_N \subset \mathcal{EL}_N$ is isotropic by Proposition 3.6, the reduced space \mathcal{L}_N is also isotropic. Since we just proved that it is coisotropic, we conclude that it is Lagrangian. By Proposition A.1 the preimage of a Lagrangian subspace with respect to the symplectic reduction is Lagrangian if it contains the kernel of the presymplectic form. In our case this kernel consists of exact forms on ∂N , and exact forms are clearly in \mathcal{L}_N . \square

Proposition 5.15. *Abelian BF theory is regular.*

Proof. We have

$$(53) \quad \ker(\hat{Q}_X) = \Omega_{closed}^\bullet(N)[1] \oplus \Omega_{closed}^\bullet(N)[n-2],$$

$$(54) \quad \text{Im}(\hat{Q}_X) = \Omega_{exact}^\bullet(N)[1] \oplus \Omega_{exact}^\bullet(N)[n-2],$$

$$(55) \quad \ker(\hat{Q}_X^{\text{vert}}) = \Omega_{closed}^\bullet(N, \partial N)[1] \oplus \Omega_{closed}^\bullet(N, \partial N)[n-2],$$

$$(56) \quad \text{Im}(\hat{Q}_X^{\text{vert}}) = d(\Omega^\bullet(N, \partial N))[0] \oplus d(\Omega^\bullet(N, \partial N))[n-3],$$

$$(57) \quad \ker(\hat{Q}_l) = \Omega_{closed}^\bullet(\partial N)[1] \oplus \Omega_{closed}^\bullet(\partial N)[n-2],$$

$$(58) \quad \text{Im}(\hat{Q}_l) = \Omega_{exact}^\bullet(\partial N)[1] \oplus \Omega_{exact}^\bullet(\partial N)[n-1]$$

Due to Hodge-Morrey decomposition for forms on N , pairs of subspaces (53), (56) and (55), (54) are mutually orthogonal. Due to Hodge decomposition on ∂N , subspaces (57), (58) are also mutually orthogonal. Thus the theory is regular. \square

5.4.2. *The reduction of boundary structures.* From section 2.4 we know that the Q -reduced Euler-Lagrange space coincides with its symplectic reduction and is

$$\mathcal{EL}_{\partial N}/Q = \underline{\mathcal{EL}}_{\partial N} = H^\bullet(\partial N)[1] \oplus H^\bullet(\partial N)[n-2]$$

The reduced space \mathcal{L}_N/Q is the space of cohomology classes of closed forms on ∂N which continue to closed forms on N , and we already proved that it is Lagrangian.

5.4.3. *The reduction of of bulk fields.* The Q -reduced space of solutions to the EL equations is:

$$\mathcal{EL}_N/Q = H^\bullet(N)[1] \oplus H^\bullet(N)[n-2]$$

To compute the space $\mathcal{EL}(N, [l])/Q$, for $l \in \mathcal{L}_N$ consider the natural exact sequence:

$$0 \rightarrow \Omega^\bullet(N, \partial N) \rightarrow \Omega^\bullet(N) \xrightarrow{\iota^*} \Omega^\bullet(\partial N) \rightarrow 0$$

where ι^* is the pullback to the boundary corresponding to the inclusion mapping $\iota : \partial N \hookrightarrow N$, and $\Omega^\bullet(N, \partial N)$ are forms vanishing on the boundary (their pullback to the boundary is zero). It induces the standard long exact sequence for cohomology spaces:

$$\begin{aligned} \dots &\rightarrow H^\bullet(N, \partial N) \xrightarrow{\chi} H^\bullet(N) \xrightarrow{\iota^*} H^\bullet(\partial N) \xrightarrow{\beta} \\ &\rightarrow H^{\bullet+1}(N, \partial N) \xrightarrow{\chi} H^{\bullet+1}(N) \xrightarrow{\iota^*} H^{\bullet+1}(\partial N) \xrightarrow{\beta} \dots \end{aligned}$$

For $l \in \mathcal{L}_N$ the Q -reduction of the space $\mathcal{EL}(N, [l]) = \pi^{-1}([l]) \cap \mathcal{EL}_N$ is the Q -reduced space $\mathcal{EL}(N, [l])/Q = (\pi^{-1}([l]) \cap \mathcal{EL}_N)/Q$

$$(59) \quad \begin{aligned} \mathcal{EL}(N, [l])/Q &= \text{Im}(\chi)[1] \oplus \text{Im}(\chi)[n-2] \simeq \\ &\simeq H^\bullet(N, \partial N)/\text{Im}(\beta)[1] \oplus H^\bullet(N, \partial N)/\text{Im}(\beta)[n-2] \end{aligned}$$

Recall that $[l]$ is the leaf of Vect_Q through l . Due to regularity, (59) is symplectic, with the symplectic structure coming from the Lefschetz duality between $H^\bullet(N)$ and $H^\bullet(N, \partial N)$.

In terms of the long exact sequence of the pair $(N, \partial N)$, the reduced space \mathcal{L}_N is:

$$\mathcal{L}_N/Q = \text{Im}(\iota^*)[1] \oplus \text{Im}(\iota^*)[n-2] \subset H^\bullet(\partial N)[1] \oplus H^\bullet(\partial N)[n-2]$$

Remark 5.16. *The symplectic reduction of \mathcal{EL}_N is infinite dimensional and, as a vector space,*

$$\underline{\mathcal{EL}}_N = H^\bullet(N, \partial N)[1] \oplus H^\bullet(N, \partial N)[n-2] \oplus \Omega_{closed}^\bullet(\partial N)[1] \oplus \Omega_{closed}^\bullet(\partial N)[n-2]$$

Indeed, it is easy to check that the symplectic orthogonal subspace $\mathcal{EL}_N^{\perp, symp}$ to \mathcal{EL}_N is

$$\Omega_{exact}(N, \partial N)[1] \oplus \Omega_{exact}(N, \partial N)[n-2]$$

where $\Omega_{exact}(N, \partial N)$ is the space of exact forms with the pullback to the boundary being zero. The symplectic reduction is the quotient space $\mathcal{EL}_N / \mathcal{EL}_N^{\perp, symp}$. It is the symplectic \mathcal{EL} -moduli space discussed in section 3.4 and comes with residual gauge symmetry data, which in particular allow for a simple gluing formula.

5.4.4. *BV extensions to strata of codimension k .* The BF theory can be maximally extended. On an $n-k$ dimensional stratum N_k the space of fields is

$$\mathcal{F}_{N_k} = \Omega^\bullet(N_k)[1] \oplus \Omega^\bullet(N_k)[n-2]$$

The symplectic form is

$$\omega_{N_k} = \int_{N_k} \delta \mathcal{A} \wedge \delta \mathcal{B}$$

The cohomological vector field Q_{N_k} is

$$Q_{N_k} = \int_{N_k} \left(d\mathcal{A} \wedge \frac{\delta}{\delta \mathcal{A}} + d\mathcal{B} \wedge \frac{\delta}{\delta \mathcal{B}} \right)$$

The action is

$$S_{N_k} = \int_{N_k} \mathcal{B} \wedge d\mathcal{A}$$

Formulae for α, Q, S are structurally the same for all k . This is a general feature of AKSZ theories of which abelian BF theory is an example.

5.4.5. *The $gh = 0$ part of the abelian BF theory.* In this section we will consider the restriction of the abelian BF theory to the fields with $gh = 0$.

The $gh = 0$ part of the space of fields is $F_N = \Omega^1(N) \oplus \Omega^{n-2}(N)$. The action is the restriction of the BV action to the $gh = 0$ sector:

$$S_N = \int_N B \wedge dA$$

where $A \in \Omega^1(N)$ and $B \in \Omega^{n-2}(N)$.

Euler-Lagrange equations are:

$$dA = 0, \quad dB = 0,$$

Solutions are closed forms

$$EL_N = \Omega_{closed}^1(N) \oplus \Omega_{closed}^{n-2}(N)$$

This is the degree zero part of the space \mathcal{EL}_N in the BF -BV theory.

The gauge group $G_N = \Omega^0(N) \oplus \Omega^{n-3}(N)$ acts on fields as

$$A \mapsto A + d\alpha, \quad B \mapsto B + d\beta$$

Vector fields generated by these transformations are degree zero parts of vector fields $Vect_Q$. The action of the abelian BF theory is invariant with respect to these transformations.

The set of gauge classes of solutions is the degree zero part of the Q -reduced Euler-Lagrange space:

$$EL_N/G_N = H^1(N) \oplus H^{n-2}(N)$$

Non-reduced boundary theory. The boundary fields are:

$$F_{\partial N} = \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N)$$

This is a symplectic manifold with the symplectic form induced by the intersection pairing:

$$\omega_{\partial N} = \int_N \delta A \wedge \delta B$$

This form is restriction of the symplectic form in the BF theory to the $gh = 0$ subspace in the space of fields.

The boundary Euler-Lagrange subspace consists of closed 1- and $(n-2)$ -forms on ∂N :

$$EL_{\partial N} = \Omega_{closed}^1(\partial N) \oplus \Omega_{closed}^{n-2}(\partial N)$$

It is a coisotropic subspace of $F_{\partial N}$.

The evolution relation $L_N = \pi(EL_N)$ is the space of closed 1- and $(n-2)$ -forms on ∂N which continue to closed forms on N . As follows from Lemma 5.14, L_N is a Lagrangian subspace in $EL_{\partial N}$.

Reduced boundary theory. Boundary gauge group $G_{\partial N} = \Omega^0(\partial N) \oplus \Omega^{n-3}(\partial N)$ is the pullback of the group of gauge transformations on N to the boundary.

The symplectically reduced Euler-Lagrange subspace $\underline{EL}_{\partial N}$ is the set of leaves of the characteristic foliation of $EL_{\partial N}$ (the foliation of $EL_{\partial N}$ by Hamiltonian vector fields generated by the ideal $I_{EL_{\partial N}}$ of functions on $F_{\partial N}$ vanishing on $EL_{\partial N}$).

Remark 5.17. *Because the action of the gauge group is Hamiltonian with*

$$H_\alpha = \int_{\partial N} \alpha \wedge dB, \quad H_\beta = \int_{\partial N} \beta \wedge dA$$

The reduced space $\underline{EL}_{\partial N}$ is also the result of Hamiltonian reduction: $\overline{EL}_{\partial N} = J^{-1}(0)/G_{\partial N}$, where $J : F_{\partial N} \rightarrow [\Omega^0(\partial N) \oplus \Omega^{n-3}(\partial N)]^$ is the moment map $(A, B) \mapsto (H_\alpha, H_\beta)$. It is also clear that $EL_{\partial N} = J^{-1}(0)$.*

The Q -reduced Euler-Lagrange space

$$EL_{\partial N}/G_{\partial N} = H^1(\partial N) \oplus H^{n-2}(\partial N)$$

has the natural symplectic structure given by the Poincaré pairing.

The reduced evolution relation $L_N/G_{\partial N}$ consists of cohomology classes of closed forms on ∂N which continue to closed forms on N . In other words, this is the image of the restriction mapping:

$$H^1(N) \oplus H^{n-2}(N) \xrightarrow{\iota^* \oplus \iota^*} H^1(\partial N) \oplus H^{n-2}(\partial N)$$

where $\iota : \partial N \hookrightarrow N$ is the inclusion of the boundary mapping. As follows from the Proposition 5.13, it is a Lagrangian subspace.

The kernel of the restriction map is isomorphic to

$$H^1(N)/Im(\beta_0) \oplus H^{n-2}(N)/Im(\beta_{n-3})$$

Here $\beta_i : H^i(\partial N) \rightarrow H^{i+1}(N, \partial N)$ is the mapping in the long exact sequence of the pair $(N, \partial N)$.

6. THE AKSZ CONSTRUCTION OF CLASSICAL TOPOLOGICAL GAUGE THEORIES

In this section we will recall the construction of classical topological field theories for closed manifolds known as the AKSZ construction [1] and we will extend this construction to manifolds with boundary. We will show that this extension gives an example of a BV theory for spacetimes with boundary. The AKSZ construction generalizes the BV extension of the BF gauge theory.

6.1. The target manifold.

6.1.1. *Hamiltonian dg manifolds.* The AKSZ construction requires the choice of a Hamiltonian differential graded manifold as the target space.

Recall that a *differential graded (dg) manifold* is a pair (\mathcal{M}, Q) where \mathcal{M} is a graded manifold and Q is a cohomological vector field of degree one. A vector field is cohomological if its Lie derivative squares to zero.

A dg manifold is a *dg symplectic manifold* of degree m if it has a symplectic form ω of degree m which is Q -invariant (i.e. $L_Q\omega = 0$ where L_Q is the Lie derivative with respect to Q). We denote the degree by *deg*.

Definition 6.1. *A dg symplectic manifold (\mathcal{M}, ω, Q) of degree m is Hamiltonian if there exists an element $\Theta \in Fun(\mathcal{M})$ with $deg(\Theta) = m + 1$ such that*

$$(60) \quad \{\Theta, \Theta\} = 0$$

and Q is the Hamiltonian vector field of Θ .

Remark 6.2. *A graded symplectic manifold is always exact when $deg(\omega) \neq 0$: $\omega = d\alpha$. If $deg(\omega) = 0$ we will require it to be exact. A dg symplectic manifold is automatically Hamiltonian when $deg(\omega) \neq -1, -2$. See section 2.2 and [46].*

Notice that a Hamiltonian dg manifold is actually defined by the symplectic form and by the function Θ satisfying (60). The vector field Q is defined as the Hamiltonian vector field of Θ and acts of functions as on functions on \mathcal{M} as:

$$Qf = \{\Theta, f\}$$

In local coordinates

$$Qf = \sum_{ab} \Theta \overleftarrow{\frac{\partial}{\partial x^a}} \omega^{ab} \overrightarrow{\frac{\partial}{\partial x^b}} f$$

For polynomial functions f we have

$$\frac{df}{dt}(x + t\epsilon) = \sum_a \epsilon^a \overrightarrow{\frac{\partial}{\partial x^a}} f = \sum_a f \overleftarrow{\frac{\partial}{\partial x^a}} \epsilon^a$$

were $deg(t) = 0$.

The condition (60) implies

$$Q^2 = 0$$

Because $deg(\omega) = m$, we have $deg(\omega^{-1}) = -m$ and because $deg(\Theta) = m + 1$, we have $deg(Q) = 1$.

6.1.2. *Examples of Hamiltonian dg manifolds.* Here we will give few examples of Hamiltonian dg manifolds.

Example 1. Let $m = 2$, \mathfrak{g} be a finite dimensional Lie algebra with an invariant inner product, and x^a be coordinates on \mathfrak{g} in an orthonormal linear basis e_a ⁸. Choose $\mathcal{M} = \mathfrak{g}[1]$ with $\deg(x^a) = 1$ and define

$$\omega = \frac{1}{2} \sum_a dx^a \wedge dx^a$$

and

$$\Theta = \frac{1}{6} \sum_{abc} f_{abc} x^a x^b x^c$$

where f_{abc} are structure constants of \mathfrak{g} in the basis e_a . Clearly $\deg(\omega) = 2$, and $\deg(\Theta) = 3$, which agrees with $m = 2$.

The differential Q in this example is given by

$$Q = \frac{1}{2} \sum_{abc} f_{ab}^c x^a x^b \frac{\partial}{\partial x^c}$$

which corresponds to the Chevalley-Eilenberg differential for the Lie algebra \mathfrak{g} .

This example describes the target space for the Chern-Simons theory in the BV formalism.

Example 2 Let n be any integer and \mathfrak{g} be a finite dimensional Lie algebra. Define

$$\mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[n-2] = T^*[n-1](\mathfrak{g}[1])$$

Let x^a be coordinates on \mathfrak{g} , and p_a be corresponding coordinates on the dual space \mathfrak{g}^* , $gh(x^a) = 1$, $gh(p_a) = n-2$. Define

$$\omega = \sum_a dp_a \wedge dx^a, \quad \alpha = \sum_a p_a dx^a$$

$$\Theta = \frac{1}{2} \sum_{abc} f_{bc}^a p_a x^b x^c$$

It is clear that $\deg(\omega) = n-1$ and $\deg(\Theta) = n$.

This is the target space for BF models. When $n = 2$ this space is the same as in the next example (Poisson sigma model) for $M = g^*$ with the Kirillov-Kostant Poisson structure.

Example 3 Let $m = 1$ and $\mathcal{M} = T^*[1]M$, where M is a finite dimensional Poisson manifold with the Poisson tensor π . Let x^i be local coordinates on M , and p_i be corresponding coordinates on the cotangent space T_x^*M . The grading is such that $\deg(x^i) = 0$ and $\deg(p_i) = 1$

Define

$$\omega = \sum_i dp_i \wedge dx^i, \quad \alpha = \sum_i p_i dx^i$$

and

$$\Theta = \frac{1}{2} \sum_{ij} \pi^{ij}(x) p_i p_j$$

Condition (60) is equivalent to the fact that π is Poisson. Clearly $\deg(\omega) = 1$, and $\deg(\Theta) = 2$ which agrees with $m = 1$.

This example describes the target space in the Poisson sigma model.

⁸ \mathfrak{g} can be also a Lie superalgebra

Example 4 When $m = 2$ and the Lie algebra has an invariant bilinear form, the Example 2 can be modified by adding a cubic term in p to Θ . For this we identify \mathfrak{g} and \mathfrak{g}^* using the Killing form and assume x^a, p_a are coordinates in an orthonormal basis. The new potential is

$$\Theta = \frac{1}{2} \sum_{abc} f_{bc}^a p_a x^b x^c \pm \frac{1}{6} f^{abc} p_a p_b p_c$$

This Hamiltonian dg manifold is isomorphic to two copies of the Hamiltonian dg manifold from the first example.

Example 5 When $m = 3$ and \mathfrak{g} has an invariant bilinear form, we can add a quadratic term to Θ . The new potential is

$$\Theta = \frac{1}{2} \sum_{abc} f_{bc}^a p_a x^b x^c + \frac{1}{2} \sum_a p_a^2$$

6.2. The space of fields. Fix a Hamiltonian dg manifold $(\mathcal{M}, \omega, \Theta)$ of degree $n - 1$. The classical n -dimensional AKSZ field theory with the target manifold \mathcal{M} on the spacetime manifold N has $\mathcal{F}_N = \text{Map}(T[1]N, \mathcal{M})$ as the space of fields⁹. We assume that the spacetime is a compact oriented smooth manifold. See Appendix B for basic facts on graded manifolds and their mapping spaces. As in the previous sections we will say that the grading on the space of fields is given by ghost numbers gh .

Let f be a smooth function of fixed degree on \mathcal{M} and $X \in \mathcal{F}_N$, then the composition $X_f = f \circ X$ is a smooth function on $T[1]N$. Let ξ^i be local coordinates on the fiber $T_u[1]N$, $u \in N$. Then X_f at (ξ, u) can be written as

$$X_f(\xi, u) = \sum_{k=0}^n X_f(u)_{i_1, \dots, i_k} \xi^{i_1} \dots \xi^{i_k}$$

with $gh(X_f(u)_{i_1, \dots, i_k}) = deg(f) - k$.

We have natural identification $\phi : C^\infty(T[1]N) \simeq \Omega(N)$. In coordinates this mapping brings the function X_f to the form

$$\phi(X_f)(u) = \sum_{k=0}^n X_f(u)_{i_1, \dots, i_k} du^{i_1} \wedge \dots \wedge du^{i_k}$$

Let $x^a, a = 1, \dots, n$ be homogeneous local coordinates on \mathcal{M} . Denote by X^a the composition of the field $X \in \mathcal{F}_N$ and the coordinate function x^a . Component fields X^a can be regarded as forms $\phi(X^a)(u)$ on $N, a = 1, \dots, n$:

$$(61) \quad \phi(X^a)(u) = \sum_{k=0}^d X^a(u)_{i_1, \dots, i_k} du^{i_1} \wedge \dots \wedge du^{i_k}$$

We will denote the component of degree k by $X_k^a(u)$. We will also call forms $X^a(u)$ coordinate fields, or superfields, when it will not cause a confusion.

As it follows from the definition of $X^a(u)_{i_1, \dots, i_k}$ the ghost number of this field is $gh(X_k^a) = deg(x^a) - k$.

⁹The AKSZ construction does not have to have $T[1]N$ as the source graded manifold. However we will consider only these cases, as they seem to be more important in field theory

6.3. The non-reduced AKSZ theory for spacetime manifolds with boundary.

6.3.1. *The AKSZ theory in terms of coordinate fields.* The AKSZ action consists of “kinetic” and “interaction” parts:

$$S_N[X] = S_N^{kin}[X] + S_N^{int}[X]$$

Let x^a be local coordinates on \mathcal{M} and $\alpha(x) = \sum_a \alpha_a(x) dx^a$. In local coordinates the “kinetic” part of the AKSZ action is

$$S_N^{kin}[X] = \int_N \sum_a \alpha_a(X(u)) \wedge dX^a(u)$$

Here and below $X^a(u)$ are coordinate components of fields as in (61). It is easy to see that $gh(S_N^{kin}) = 0$.

The “interaction” part of the AKSZ action is the functional

$$S_N^{int}[X] = \int_N \Theta(X)$$

where Θ is the potential function for the target manifold. Because $deg(\Theta) = n$, the action has the zero grading, i.e. $gh(S_N^{int}) = 0$. Here and below the expression $\int_N \Theta(X)$ for a homogeneous polynomial $\Theta(x) = \sum_{\{a\}} \Theta_{\{a\}} x^{a_1} \dots x^{a_k}$ of degree k on \mathcal{M} means $\sum_{\{a\}} \Theta_{\{a\}} \int_N X^{a_1} \wedge \dots \wedge X^{a_k}$.

Assume that in local coordinates $\omega_{\mathcal{M}} = \frac{1}{2} \sum_{ab} \omega_{ab} dx^a \wedge dx^b$, then

$$\omega_N = \frac{1}{2} \int_N \sum_{ab} \omega_{ab}(X(u)) \wedge \delta X^a(u) \wedge \delta X^b(u)$$

where $\delta X^a(x)$ is the de Rham differential on the space of fields.

In local coordinates, the action of Q_N on local functionals is:

$$Q_N F = \int_N \left(dX^a + \omega^{ab}(X) \wedge \frac{\partial \Theta}{\partial x^b}(X) \right) \wedge \frac{\delta F}{\delta X^a}$$

Here ω^{ab} are components of the Poisson bivector field corresponding to the symplectic form ω . Because $deg(\omega) = n - 1$ and $deg(\Theta) = n$, we have $gh(Q_N) = 1$.

It acts on coordinate fields as:

$$(62) \quad Q_N X^a = dX^a + \omega^{ab}(X) \wedge \frac{\partial \Theta}{\partial x^b}(X)$$

The variation of S_N :

$$\delta S_N[X] = \int_N (\delta X^a(u) \wedge \partial_a \alpha_b(X(u)) \wedge dX^b(u) + \alpha_a(X(u)) \wedge \delta dX^a(u) + \delta \Theta(X))$$

After integration by parts this expression becomes

$$\int_{\partial N} \alpha_a(X(u)) \wedge \delta X^a(u) + \int_N \left(\omega_{ab}(X(u)) \wedge dX^a(u) + \frac{\partial \Theta}{\partial X^b}(X(u)) \right) \wedge \delta X^b(u)$$

In this section we assume that $\partial N = \emptyset$, which means the first term is absent. The Euler-Lagrange equations are:

$$(63) \quad dX^a(u) + \sum_b \omega^{ab}(X(u)) \wedge \frac{\partial \Theta}{\partial x^a}(X(u)) = 0$$

The same formulae hold for the boundary BFV action $S_{\partial N}$, for the boundary symplectic form $\omega_{\partial N}$, and for the boundary cohomological vector field $Q_{\partial N}$. One

should simply substitute ∂N instead of N . The only difference is the degree change. Because $\dim(\partial N) = n - 1$, $gh(S_{\partial N}) = 1$, $gh(\omega_{\partial N}) = -1$ and $gh(Q_{\partial N}) = -1$ as in the bulk N .

The BV and BFV data described above have the right grading and give an example of a BV-BFV theory:

Proposition 6.3. *The cohomological vector field Q_N is Hamiltonian, up to a boundary term:*

$$(64) \quad \iota_{Q_N} \omega_N = (-1)^{\dim(N)} \delta S_N + \pi^* \alpha_{\partial N}$$

We defer the proof to Appendix D.

Remark 6.4. *When N has a non-empty boundary, the AKSZ action depends of the choice of the form α , not only on its cohomology class as in case of closed manifolds. Let \tilde{S}_N be the action corresponding to $\tilde{\alpha} = \alpha + df$, then*

$$\tilde{S}_N[X] = S_N[X] + \int_N df(X) = S_N + \int_{\partial N} f(X)$$

Any AKSZ theory can be maximally extended. On an $n - k$ dimensional stratum Σ the space of fields is the space of maps $\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathcal{M})$. The symplectic form ω_Σ , the form α_Σ , the action functional S_Σ and the vector field Q_Σ are all given by the same formulae as above. The difference from the n -dimensional stratum is only in the grading:

$$gh(\omega_\Sigma) = k - 1, \quad gh(Q_\Sigma) = 1, \quad gh(S_\Sigma) = k$$

7. EXAMPLES OF AKSZ THEORIES

7.1. Abelian Chern-Simons theory. The target space for this AKSZ theory is $\mathbb{R}[1]$ with symplectic structure $\omega = da \wedge da$, where a is the coordinate on $\mathbb{R}[1]$ and $\Theta = 0$.

7.1.1. The bulk BV theory. The space of fields on the 3-dimensional spacetime manifold N is :

$$\mathcal{F}_N = \Omega^\bullet(N)[1]$$

The fields corresponding to forms of degree 0, 1, 2, 3 will be denoted by $c, A, A^\dagger, c^\dagger$ respectively. The ghost numbers are 1, 0, -1, -2. We will write $\mathcal{A} = c + A + A^\dagger + c^\dagger$ for the BV superfield.

The symplectic form, the vector field Q , and the classical action are:

$$\begin{aligned} \omega_N &= \frac{1}{2} \int_N \delta \mathcal{A} \wedge \delta \mathcal{A} = \int_N (\delta c \wedge \delta c^\dagger + \delta A \wedge \delta A^\dagger) \\ Q_N &= \int_N d\mathcal{A} \wedge \frac{\delta}{\delta \mathcal{A}} = \int_N (dc \wedge \frac{\delta}{\delta A} + dA \wedge \frac{\delta}{\delta A^\dagger} + dA^\dagger \wedge \frac{\delta}{\delta c^\dagger}) \\ S_N &= \frac{1}{2} \int_N \mathcal{A} \wedge d\mathcal{A} = \frac{1}{2} \int_N (A \wedge dA + A^\dagger \wedge dc + c \wedge dA^\dagger) \end{aligned}$$

The Euler-Lagrange space is

$$\mathcal{E}\mathcal{L}_N = \Omega_{closed}^\bullet(N)[1]$$

7.1.2. *The boundary BFV theory.* Boundary fields are pullbacks of the bulk fields to the boundary.

$$\mathcal{F}_{\partial N} = \Omega^\bullet(N)[1]$$

We will use the same notation for pullbacks as for bulk fields. This means 0, 1, 2 forms will be denoted by c, A, A^\dagger respectively. They have ghost numbers 1, 0, -1 .

The one form $\alpha_{\partial N}$, the symplectic structure, the vector field Q and the action for the boundary BFV theory are:

$$\begin{aligned}\alpha_{\partial N} &= \frac{1}{2} \int_{\partial N} \mathcal{A} \wedge \delta \mathcal{A} = \frac{1}{2} \int_{\partial N} (A \wedge \delta A + c \wedge \delta A^\dagger + A^\dagger \wedge \delta c) \\ \omega_{\partial N} &= \frac{1}{2} \int_{\partial N} \delta \mathcal{A} \wedge \delta \mathcal{A} = \int_{\partial N} \left(\frac{1}{2} \delta A \wedge \delta A + \delta c \wedge \delta A^\dagger \right) \\ Q_{\partial N} &= \int_{\partial N} d\mathcal{A} \wedge \frac{\delta}{\delta \mathcal{A}} = \int_{\partial N} \left(dc \wedge \frac{\delta}{\delta A} + dA \wedge \frac{\delta}{\delta A^\dagger} \right) \\ S_{\partial N} &= \frac{1}{2} \int_{\partial N} \mathcal{A} \wedge \delta \mathcal{A} = \int_{\partial N} c \wedge dA\end{aligned}$$

The boundary Euler-Lagrange space is

$$\mathcal{E}\mathcal{L}_{\partial N} = \Omega_{closed}^\bullet(\partial N)[1]$$

The evolution relation $\mathcal{L}_N \subset \mathcal{E}\mathcal{L}_{\partial N}$ is the subspace of forms in $\Omega_{closed}^\bullet(\partial N)[1]$ which continue to closed forms on N .

Abelian Chern-Simons theory is regular, which is proven similarly to Proposition 5.15.

7.1.3. *Reduced BV-BFV theory.* Reduced bulk and boundary Euler-Lagrange spaces are

$$\mathcal{E}\mathcal{L}_N/Q \simeq H^\bullet(N)[1], \quad \mathcal{E}\mathcal{L}_{\partial N}/Q \simeq H^\bullet(\partial N)[1]$$

respectively. The symplectic form on $\mathcal{E}\mathcal{L}_{\partial N}/Q$ is given by the Poincaré duality.

The pullback mapping $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ induces the mapping of reduced spaces:

$$\pi_* : \mathcal{E}\mathcal{L}_N/Q \rightarrow \mathcal{E}\mathcal{L}_{\partial N}/Q$$

The reduced evolution relation $\mathcal{L}_N/Q = \text{Im}(\pi_*) \subset \mathcal{E}\mathcal{L}_{\partial N}/Q$ is a Lagrangian subspace.

Because π_* is a linear mapping, its fiber over any point of $\mathcal{E}\mathcal{L}_{\partial N}/Q$ is simply $\ker(\pi_*)$. This space

$$\ker(\pi_*) \simeq H^\bullet(N, \partial N)/H^{\bullet-1}(\partial N)[1]$$

has a natural symplectic structure of degree -1 coming from the Lefschetz duality.

7.1.4. *The $gh = 0$ part of the theory.* The $gh = 0$ part of the space of fields is $F_N = \Omega^1(N)$. The $gh = 0$ part of Vect_Q gives the gauge action of $\Omega^0(N)$ on the space of fields: $A \rightarrow A + d\beta$ where $A \in \mathcal{F}_N$ and $\beta \in \Omega^0(N)$. The abelian Chern-Simons action

$$S_N^{cl} = \frac{1}{2} \int_N A \wedge dA$$

is gauge invariant when $\partial N = \emptyset$. When N has non-empty boundary the gauge transformation generated by β changes the action by $\int_{\partial N} \beta \wedge A$.

The space EL_N is the space of closed 1-forms on N . The space of boundary fields $F_{\partial N} = \Omega^1(\partial N)$ is exact symplectic with

$$\alpha_{\partial N} = \frac{1}{2} \int_{\partial N} A \wedge \delta A, \quad \omega_{\partial N} = \delta \alpha_{\partial N} = \frac{1}{2} \int_{\partial N} \delta A \wedge \delta A$$

The $gh = 0$ part of the boundary Euler-Lagrange space, the space $C_{\partial N}$, is the space of closed 1-forms on ∂N . It is clearly coisotropic. The gauge action is Hamiltonian with the momentum map $\mu = d : \Omega^1(\partial N) \rightarrow \Omega^2(\partial N)$, where we consider $\Omega^2(N)$ as the dual space to the abelian Lie algebra of 0-forms.

The $gh = 0$ part of the moduli space \mathcal{EL}_N/Q is the space of gauge orbits EL_N/G_N , and we have the natural isomorphism $EL_N/G_N \simeq H^1(N)$. The $gh = 0$ part of the moduli space $\mathcal{EL}_{\partial N}/Q$ is isomorphic to the space $C_{\partial N}/G_{\partial N}$ of gauge orbits through $C_{\partial N}$, or equivalently, since the action is Hamiltonian, this space is the symplectic reduction of $C_{\partial N}$. It is clear that we have the natural isomorphism $C_{\partial N}/G_{\partial N} \simeq H^1(\partial N)$. This space is symplectic with the symplectic structure given by the Poincaré duality.

The pullback to the boundary $\pi : F_N \rightarrow F_{\partial N}$ induces the restriction mapping $\pi_* : H^1(N) \rightarrow H^1(\partial N)$. The subspace

$$L_N = \text{Im}(\pi_*) \subset H^1(\partial N)$$

is Lagrangian. The fiber over any point of $C_{\partial N}/G_{\partial N}$ is $\ker(\pi_*)$.

7.2. Non-abelian Chern-Simons theory. In this case the target manifold is constructed from a Lie algebra \mathfrak{g} with an invariant scalar product (for example a simple Lie algebra). The target manifold is described in details in section 6.1.2. We assume that the Lie algebra and the corresponding simply connected Lie group are matrix groups and will write $\text{tr}(ab)$ for the Killing form evaluated on two Lie algebra elements.

7.2.1. The bulk BV theory. Fields are graded connections in a principal G -bundle over the spacetime N . We assume the bundle is trivial, so the space of fields is

$$\mathcal{F}_N = \Omega^\bullet(N, \mathfrak{g})[1]$$

Here $\Omega^\bullet(N, \mathfrak{g})[1] = \Omega^\bullet(N) \otimes \mathfrak{g}[1]$. The fields corresponding to forms of degree 0, 1, 2, 3 will be denoted by $c, A, A^\dagger, c^\dagger$ respectively. The ghost numbers are 1, 0, $-1, -2$. We will write $\mathcal{A} = c + A + A^\dagger + c^\dagger$ for the BV superfield.

The symplectic form, the vector field Q , and the classical action are:

$$\omega_N = \frac{1}{2} \int_N \text{tr}(\delta \mathcal{A} \wedge \delta \mathcal{A}) = \int_N \text{tr}(\delta c \wedge \delta c^\dagger + \delta A \wedge \delta A^\dagger),$$

$$\begin{aligned} (65) \quad Q_N &= \int_N \text{tr} \left((d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]) \wedge \frac{\delta}{\delta \mathcal{A}} \right) = \\ &= \int_N \text{tr} \left(d_{AC} \wedge \frac{\delta}{\delta A} + (F(A) + [c, A^\dagger]) \wedge \frac{\delta}{\delta A^\dagger} + (d_A A^\dagger + [c, c^\dagger]) \wedge \frac{\delta}{\delta c^\dagger} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} \right), \end{aligned}$$

$$\begin{aligned} (66) \quad S_N &= \int_N \text{tr} \left(\frac{1}{2} \mathcal{A} \wedge d\mathcal{A} + \frac{1}{6} \mathcal{A} \wedge [\mathcal{A}, \mathcal{A}] \right) = \\ &= \int_N \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A] + \frac{1}{2} A^\dagger \wedge d_A c + \frac{1}{2} c \wedge d_A A^\dagger + \frac{1}{2} c^\dagger \wedge [c, c] \right) \end{aligned}$$

The Euler-Lagrange space is the space of flat graded connections

$$\mathcal{EL}_N = \{\mathcal{A} \in \mathcal{F}_N \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0\}$$

In coordinate fields the Euler-Lagrange space consists of $c, A, A^\dagger, c^\dagger$ which satisfy

$$[c, c] = 0, \quad d_A c = 0, \quad F(A) + [c, A^\dagger] = 0, \quad d_A A^\dagger + [c, c^\dagger] = 0$$

The tangent space at $\mathcal{A} \in \mathcal{EL}_N$ is kernel of $\hat{Q}_{\mathcal{A}}$. It consists of elements $(\gamma, \alpha, \alpha^\dagger, \gamma^\dagger) \in \Omega^\bullet(N)[1]$ such that

$$(67) \quad \begin{aligned} [\gamma, c] = 0, \quad d_A \gamma + [\alpha, c] = 0, \quad d_A \alpha + [c, \alpha^\dagger] + [\gamma, A^\dagger] = 0, \\ d_A \alpha^\dagger + [c, \gamma^\dagger] + [A^\dagger, \alpha] + [c^\dagger, \gamma] = 0, \end{aligned}$$

The point \mathcal{A} belongs to the $gh = 0$ part of \mathcal{EL}_N if and only if $c = c^\dagger = A^\dagger = 0$ and $F(A) = 0$. The $gh = 0$ part of \mathcal{EL}_N is the space flat connections. The tangent space to \mathcal{EL}_N at such point is naturally isomorphic to

$$(68) \quad \Omega_{d_A\text{-closed}}^\bullet(N, \mathfrak{g})$$

where forms are closed with respect to the differential $d_A = d + [A, \cdot]$.

In this case the reduced tangent space is

$$\ker(\hat{Q}_{\mathcal{A}})/\text{Im}(\hat{Q}_{\mathcal{A}}) \simeq H_{d_A}^\bullet(N, \mathfrak{g})$$

7.2.2. *The boundary BFV theory.* Boundary fields in the non-abelian Chern-Simons theory are pullbacks of the bulk fields to the boundary.

$$\mathcal{F}_{\partial N} = \Omega^\bullet(N, \mathfrak{g})[1]$$

We will use the same notation for pullbacks as for bulk fields. This means 0, 1, 2 forms will be denoted by c, A, A^\dagger respectively. They have ghost numbers 1, 0, -1.

The one form $\alpha_{\partial N}$, the symplectic structure, the vector field Q and the action for the boundary BFV theory are:

$$\alpha_{\partial N} = \frac{1}{2} \int_{\partial N} \text{tr}(\mathcal{A} \wedge \delta \mathcal{A}) = \frac{1}{2} \int_{\partial N} \text{tr}(A \wedge \delta A + c \wedge \delta A^\dagger + A^\dagger \wedge \delta c),$$

$$\omega_{\partial N} = \frac{1}{2} \int_{\partial N} \text{tr}(\delta \mathcal{A} \wedge \delta \mathcal{A}) = \int_{\partial N} \text{tr} \left(\frac{1}{2} \delta A \wedge \delta A + \delta c \wedge \delta A^\dagger \right),$$

$$\begin{aligned} Q_{\partial N} &= \int_{\partial N} \text{tr} \left(\left(dA + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) \wedge \frac{\delta}{\delta \mathcal{A}} \right) = \\ &= \int_{\partial N} \text{tr} \left(d_A c \wedge \frac{\delta}{\delta A} + (F(A) + [c, A^\dagger]) \wedge \frac{\delta}{\delta A^\dagger} + \frac{1}{2}[c, c] \wedge \frac{\delta}{\delta c} \right), \end{aligned}$$

$$S_{\partial N} = \int_{\partial N} \text{tr} \left(\frac{1}{2} \mathcal{A} \wedge d\mathcal{A} + \frac{1}{6} \mathcal{A} \wedge [\mathcal{A}, \mathcal{A}] \right) = \int_{\partial N} \text{tr} \left(c \wedge F(A) + \frac{1}{2}[c, c] \wedge A^\dagger \right)$$

The boundary Euler-Lagrange space is the space of graded flat \mathfrak{g} -connections on ∂N :

$$\mathcal{EL}_{\partial N} = \{\mathcal{A} \in \mathcal{F}_{\partial N} \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0\}$$

The evolution relation $\mathcal{L}_N \subset \mathcal{EL}_{\partial N}$ is the subspace of graded flat \mathfrak{g} -connections on $G \times \partial N$ which continue to flat graded \mathfrak{g} -connections on $G \times N$.

The flatness of the graded connection \mathcal{A} means its graded components satisfy

$$[c, c] = 0, \quad d_A c = 0, \quad F(A) + [c, A^\dagger] = 0,$$

The tangent space $T_{\mathcal{A}}\mathcal{E}\mathcal{L}_{\partial N} \subset T_{\mathcal{A}}\mathcal{F}_{\partial N}$ is the kernel of $\hat{Q}_{\mathcal{A}}$:

$$(69) \quad \ker(\hat{Q}_{\mathcal{A}}) = \{(\gamma, \alpha, \alpha^\dagger) \mid [\gamma, c] = 0, \quad d_A\gamma + [\alpha, c] = 0, \quad d_A\alpha + [c, \alpha^\dagger] + [\gamma, A^\dagger] = 0\}$$

In $gh = 0$ part of $\mathcal{E}\mathcal{L}_{\partial N}$ we have $c = A^\dagger = 0$ and $F(A) = 0$

$$(70) \quad \ker(\hat{Q}_{\mathcal{A}}) = \{(\gamma, \alpha, \alpha^\dagger) \mid d_A\gamma = 0, \quad d_A\alpha = 0\} \simeq \Omega_{d_A\text{-closed}}^\bullet(\partial N, \mathfrak{g})[1]$$

$$Im(\hat{Q}_{\mathcal{A}}) = \{(0, d_A\tilde{\gamma}, d_A\tilde{\alpha})\}$$

Thus, in this case the reduced tangent space is

$$\ker(\hat{Q}_{\mathcal{A}})/Im(\hat{Q}_{\mathcal{A}}) \simeq H_{d_A}^\bullet(\partial N, \mathfrak{g})[1]$$

The smooth locus of the Euler-Lagrange space $\mathcal{E}\mathcal{L}_{\partial N}$ is the vector bundle over the space of flat connections on ∂N with fibers $\oplus_{i \neq 1} \Omega_{d_A\text{-closed}}^i(\partial N)$. The smooth locus of the boundary $\mathcal{E}\mathcal{L}$ -moduli space for the non-abelian Chern-Simons theory is isomorphic to a vector bundle over the smooth locus of the representation variety $\text{Hom}(\pi_1(\partial N), G)/G$ with fiber $H_{d_A}^0(\partial N, \mathfrak{g}) \oplus H_{d_A}^2(\partial N, \mathfrak{g})$ with the symplectic structure naturally extending the Atiyah-Bott symplectic form on the representation variety.

Regularity for non-abelian Chern-Simons theory can be proven using homological perturbation theory around the abelian Chern-Simons theory. This argument extends to general AKSZ theories.

7.2.3. Reduced BV-BFV theory. Reduced Euler-Lagrange spaces for the bulk and for the boundary are the spaces of leaves of the foliation $Vect_Q$ on $\mathcal{E}\mathcal{L}_N$ and $\mathcal{E}\mathcal{L}_{\partial N}$ respectively. We will write

$$\mathcal{E}\mathcal{L}_N/Q_N = \{\mathcal{A} \in \mathcal{F}_N \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0\} / \{\mathcal{A} \mapsto \mathcal{A} + d\lambda + [\mathcal{A}, \lambda] \mid \lambda \in \Omega^\bullet(N)\}$$

and similarly for ∂N . These notations indicate that leaves of $Vect_Q$ should be considered as gauge orbits on $\mathcal{E}\mathcal{L}_N$.

The projection $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ (the pullback to the boundary) defines the mapping

$$\pi_* : \mathcal{E}\mathcal{L}_N/Q_N \rightarrow \mathcal{E}\mathcal{L}_{\partial N}/Q_{\partial N}$$

Its image $\mathcal{L}_N/Q \subset \mathcal{E}\mathcal{L}_{\partial N}/Q$ is Lagrangian.

Tangent spaces to $\mathcal{E}\mathcal{L}_N$ and $\mathcal{E}\mathcal{L}_{\partial N}$ are kernels of corresponding operators \hat{Q} and are described in components in (67), (69). The image of \hat{Q} is also easy to compute. For the bulk we have

$$Im(\hat{Q}_{\mathcal{A}}) = \{([\tilde{\gamma}, c], d_A\tilde{\gamma} + [\tilde{\alpha}, c], d_A\tilde{\alpha} + [c, \tilde{\alpha}^\dagger] + [\tilde{\gamma}, A^\dagger], d_A\tilde{\alpha}^\dagger + [c, \gamma^\dagger] + [A^\dagger, \tilde{\alpha}] + [c^\dagger, \gamma]\}$$

for the boundary:

$$Im(\hat{Q}_{\mathcal{A}}) = \{([\tilde{\gamma}, c], d_A\tilde{\gamma} + [\tilde{\alpha}, c], d_A\tilde{\alpha} + [c, \tilde{\alpha}^\dagger] + [\tilde{\gamma}, A^\dagger])\}$$

The reduced spaces are quotient spaces $\ker(\hat{Q})/Im(\hat{Q})$. In other words these are cohomology spaces of the cochain complexes $\Omega^\bullet(N)[1]$ and $\Omega^\bullet(\partial N)[1]$ respectively with respect to the differential $d_A\omega = [c, \omega] + d_A\omega + [A^\dagger, \omega]$.

When \mathcal{A} is of degree zero, i.e. $c = c^\dagger = A^\dagger = 0$ and $F(A) = 0$ we have

$$T_{\mathcal{A}}\mathcal{E}\mathcal{L}_N/Q_N = H_{d_A}^\bullet(N, \mathfrak{g})[1], \quad T_{\mathcal{A}}\mathcal{E}\mathcal{L}_{\partial N}/Q_{\partial N} = H_{d_A}^\bullet(\partial N, \mathfrak{g})[1]$$

7.2.4. *The $gh = 0$ part of the theory.* The non-reduced $gh = 0$ part of the theory has the space of fields $F_N = \Omega^1(N, \mathfrak{g})$. In case of a general principal G -bundle, this is the space of connections. The classical action is the Chern-Simons functional:

$$S_N^{cl} = \int_N \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A, A] \right)$$

Its critical points are flat connections: $F(A) = dA + \frac{1}{2}[A, A] = 0$. The \mathfrak{g} -valued zero-forms act by infinitesimal gauge transformations $A \mapsto A + d_A \alpha$, $\alpha \in \omega^0(N, \mathfrak{g})$.

The $gh = 0$ part of the space of boundary fields is $F_{\partial N} = \Omega^1(N, \mathfrak{g})$. This is an exact symplectic space with

$$\alpha_{\partial N} = \frac{1}{2} \int_{\partial N} \text{tr}(A \wedge \delta A), \quad \omega_{\partial N} = \delta \alpha_{\partial N} = \frac{1}{2} \int_{\partial N} \text{tr}(\delta A \wedge \delta A)$$

The degree zero part $C_{\partial N}$ of the boundary Euler-Lagrange space is the space of flat connections in a trivial G -bundle over ∂N . It is coisotropic in the space of all connections.

The gauge group $G_{\partial N} = \text{Map}(\partial N, G)$ acts on $F_{\partial N}$ by Hamiltonian transformations. Infinitesimally, the action is $A \rightarrow A + d_A \alpha$. The momentum map $\mu : \Omega^1(N, \mathfrak{g}) \rightarrow \Omega^2(N, \mathfrak{g})$ is the curvature. The symplectic reduction of $C_{\partial N}$ coincides with the Hamiltonian reduction. The reduced space $C_{\partial N}/G_{\partial N}$ is the moduli space of flat G -connections on the trivial G -bundle over ∂N .

The reduced $gh = 0$ part of the Euler-Lagrange space is the moduli space of flat connections in the trivial G -bundle over N . The mapping $\pi_* : \mathcal{M}_N^G \rightarrow \mathcal{M}_{\partial N}^G$ is the natural restriction mapping of representations of $\pi_1(N)$ to representations of $\pi_1(\partial N) \subset \pi_1(N)$. The image of this mapping is the Lagrangian subvariety in $\mathcal{M}_{\partial N}^G$, the fibers are those representations of $\pi_1(N)$ which restrict trivially to $\pi_1(\partial N)$. For detailed exposition of the global aspects of the classical Chern-Simons theory see [26].

7.3. **Non-abelian BF theory.** The target space for the non-abelian BF theory is described in section 6.1.2.

7.3.1. *The bulk BV theory.* The space of fields in the BV-extended non-abelian BF theory is

$$\mathcal{F}_N = \Omega^\bullet(N, \mathfrak{g})[1] \oplus \Omega^\bullet(N, \mathfrak{g})[n-2]$$

$$\begin{aligned} \omega_N &= \int_N \text{tr}(\delta \mathcal{B} \wedge \delta \mathcal{A}) \\ Q_N &= \int_N \text{tr} \left(\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) \wedge \frac{\delta}{\delta \mathcal{A}} + d\mathcal{A} \mathcal{B} \wedge \frac{\delta}{\delta \mathcal{B}} \right) \\ S_N &= \int_N \text{tr} \left(\mathcal{B} \wedge \left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) \right) \end{aligned}$$

The Euler-Lagrange space:

$$\mathcal{EL}_N = \{(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_N \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0, \quad d\mathcal{A} \mathcal{B} = 0\}$$

As in the Chern-Simons theory we are interested in the smooth locus of the Euler-Lagrange space which is, in this case, a vector bundle over the space of flat connections on $G \times N$ with fiber $(\oplus_{i \neq 1} \Omega_{d_A\text{-closed}}^i(N, \mathfrak{g})[1]) \oplus \Omega_{d_A\text{-closed}}^\bullet(N, \mathfrak{g})[n-2]$ over a flat connection A .

7.3.2. *The boundary BFV theory.* The space of boundary fields is

$$\mathcal{F}_{\partial N} = \Omega^\bullet(\partial N, \mathfrak{g})[1] \oplus \Omega^\bullet(\partial N, \mathfrak{g})[n-2]$$

The BFV structure on it is given by the AKSZ construction:

$$\alpha_{\partial N} = \int_{\partial N} \text{tr}(\mathcal{B} \wedge \delta \mathcal{A}), \quad \omega_{\partial N} = \delta \alpha_{\partial N} = \int_{\partial N} \text{tr}(\delta \mathcal{A} \wedge \delta \mathcal{B})$$

$$Q_{\partial N} = \int_{\partial N} \text{tr} \left(\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) \wedge \frac{\delta}{\delta \mathcal{A}} + d_{\mathcal{A}}\mathcal{B} \wedge \frac{\delta}{\delta \mathcal{B}} \right)$$

$$S_{\partial N} = \int_{\partial N} \text{tr} \left(\mathcal{B} \wedge \left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) \right)$$

The boundary Euler-Lagrange space is

$$\mathcal{EL}_{\partial N} = \{(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_{\partial N} \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0, \quad d_{\mathcal{A}}\mathcal{B} = 0\}$$

7.3.3. *The $gh = 0$ part of the theory.* The degree zero gauge theory has the space of fields $F_N = \Omega^1(N, \mathfrak{g}) \oplus \Omega^{n-2}(N, \mathfrak{g})$ we will denote 1-forms A and $(n-2)$ -forms B . The action is

$$S_N^{cl} = \int_N \text{tr}(B \wedge F(A))$$

Euler-Lagrange equations are

$$F(A) = 0, \quad d_A B = 0$$

Infinitesimal gauge transformations act as

$$A \rightarrow A + d_A \mu, \quad B \rightarrow B + [B, \mu] + d_A \lambda$$

The space of boundary fields is the pullback of bulk fields. The restriction mapping is the pullback. The symplectic structure on the space of bulk fields is exact with $\omega_N = \delta \alpha_N$ where

$$\alpha_N = \int_{\partial N} \text{tr}(B \wedge \delta A), \quad \omega_N = \int_{\partial N} \text{tr}(\delta A \wedge \delta B)$$

The coisotropic submanifold $C_{\partial N}$ is the $gh = 0$ part of $\mathcal{EL}_{\partial N}$. The Lagrangian submanifold $L_N = \pi(EL_N)$ consists of pairs (A, B) where A is a flat connection on ∂N which continues to a flat connection on N and $B \in \Omega^{n-2}(\partial N, \mathfrak{g})$ is horizontal with respect to the flat connection A and extends to a horizontal $(n-2)$ -form on N .

The reduced space EL_N/G_N is a vector bundle over the moduli space of flat connections on the trivial G -bundle on N with fiber $H_A^{n-2}(N, \mathfrak{g})$ over a gauge class $[A]$.

The moduli space $EL_{\partial N}/G_{\partial N}$ has the same structure and can be identified with $T^*M_{\partial N}^G$.

7.4. **$BF + B^2$ theory.** In this theory the space is 4-dimensional. The target space is the same as the one for the BF theory.

7.4.1. *The bulk BV theory.* The space of fields is the same as in the 4-dimensional BV-extended BF-theory with the same symplectic structure.

The action and the vector field Q_N are:

$$Q_N = tr \int_N \left(\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] + \mathcal{B} \right) \wedge \frac{\delta}{\delta \mathcal{A}} + d_{\mathcal{A}}\mathcal{B} \wedge \frac{\delta}{\delta \mathcal{A}} \right)$$

$$S_N = tr \int_N \left(\mathcal{B} \wedge \left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) + \frac{1}{2}\mathcal{B} \wedge \mathcal{B} \right)$$

The Euler-Lagrange space:

$$\mathcal{E}\mathcal{L}_N = \{(\mathcal{A}, \mathcal{B}) \in \mathcal{F}_N \mid d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] + \mathcal{B} = 0, d_{\mathcal{A}}\mathcal{B} = 0\} \simeq \Omega^\bullet(N, \mathfrak{g})[1]$$

Indeed, the first condition $F(\mathcal{A}) + \mathcal{B} = 0$ gives no restriction on \mathcal{A} , and the second condition $d_{\mathcal{A}}\mathcal{B} = 0$ follows from the first one and from the Bianchi identity.

7.4.2. *The boundary BFV theory.* The space of boundary fields is the same as for the BF theory with the same symplectic structure of degree 0.

The boundary vector field $Q_{\partial N}$ and the boundary action are

$$Q_{\partial N} = tr \int_{\partial N} \left(\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] + \mathcal{B} \right) \wedge \frac{\delta}{\delta \mathcal{A}} + d_{\mathcal{A}}\mathcal{B} \wedge \frac{\delta}{\delta \mathcal{A}} \right)$$

$$S_{\partial N} = tr \int_{\partial N} \left(\mathcal{B} \wedge \left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \right) + \frac{1}{2}\mathcal{B} \wedge \mathcal{B} \right)$$

The boundary Euler-Lagrange space:

$$\mathcal{E}\mathcal{L}_{\partial N} \simeq \Omega^\bullet(\partial N)[1]$$

7.4.3. *The $gh = 0$ part of the theory.* The space of fields is the same as the space of fields of the non-abelian BF theory in four dimensions. The action is

$$S_N^{cl} = tr \int_N \left(B \wedge F(A) + \frac{1}{2}B \wedge B \right)$$

Euler-Lagrange equations are

$$B + F(A) = 0, \quad d_A B = 0$$

The second equation follows from the first and from the Bianchi identity. Infinitesimal gauge transformations are

$$A \mapsto A + d_A \alpha - \beta, \quad B \mapsto B + [B, \alpha] + d_A \beta$$

The reduced theory is trivial:

$$\mathcal{E}\mathcal{L}_N/Q = \{0\}, \quad \mathcal{E}\mathcal{L}_{\partial N}/Q = \{0\}$$

so the $gh = 0$ part of it also trivial.

7.5. **The Poisson sigma model.** The target space for the Poisson sigma model is described in section 6.1.2. In this section M is a Poisson manifold with the Poisson tensor $\pi(x)$. In local coordinates $\pi(x) = \sum_{ab} \pi^{ab}(x) \partial_a \partial_b$.

7.5.1. *The bulk BV theory.* In case of the Poisson sigma model the spacetime is two dimensional. The space of fields in the models is $\mathcal{F}_N = \text{Map}(T[1]N, T^*[1]M)$. We will use coordinate field $\tilde{X}^a = X^a + \eta^{+,a} + \beta^{+,a}$ for coordinates q^a on M and $\tilde{\eta}_a = \beta_a + \eta_a + X_a^\dagger$ for coordinates p_a in the cotangent directions. Components of fields \tilde{X}^a and $\tilde{\eta}_a$ are forms of degree 0, 1, 2 respectively. They have ghost numbers $gh(X) = 0$, $gh(\eta^\dagger) = -1$, $gh(\beta^\dagger) = -2$, $gh(\beta) = 1$, $gh(\eta) = 0$, $gh(X^\dagger) = -1$. The symplectic form and the action functional are given by the AKSZ construction:

$$\omega_N = \int_N \sum_a \delta \tilde{\eta}_a \wedge \delta \tilde{X}^a = \int_N \sum_a (\delta X^a \wedge \delta X_a^\dagger + \delta \eta^{+,a} \wedge \delta \eta_a + \delta \beta^{+,a} \wedge X_a^\dagger)$$

$$S_N = \int_N \sum_a \left(\tilde{\eta}_a \wedge d\tilde{X}^a + \frac{1}{2} \int_N \sum_{ab} \pi^{ab}(\tilde{X}) \wedge \tilde{\eta}_a \wedge \tilde{\eta}_b \right)$$

In field components this action is

$$(71) \quad S_N = \int_N \left(\eta_a \wedge dX^a - \eta^{+,a} \wedge \beta_a + \frac{1}{2} \pi^{ab}(X) \wedge \eta_a \wedge \eta_b + \pi^{ab}(X) \wedge \beta_a \wedge X_b^\dagger + \right. \\ \left. \eta^{+,a} \wedge \partial_a \pi^{bc}(X) \wedge \beta_b \wedge \eta_c + \frac{1}{2} \beta^{+,c} \wedge \partial_c \pi^{ab}(X) \wedge \beta_b \wedge \beta_c + \frac{1}{4} \eta^{+,c} \wedge \eta^{+,d} \wedge \partial_c \partial_d \pi^{ab}(X) \wedge \beta_a \wedge \beta_b \right)$$

The vector field Q is

$$Q_N = \int_N \left((d\tilde{X}^a + \pi^{ab}(\tilde{X}) \wedge \tilde{\eta}_a \wedge \tilde{\eta}_b) \wedge \frac{\delta}{\delta \tilde{X}^a} + (d\tilde{\eta}_a + \frac{1}{2} \partial_a \pi^{bc}(\tilde{X}) \wedge \tilde{\eta}_b \wedge \tilde{\eta}_c) \wedge \frac{\delta}{\delta \tilde{\eta}_a} \right)$$

The Euler-Lagrange space \mathcal{EL}_N is the space of zeroes of Q (the space of critical points of S_N), or the space of solutions to

$$d\tilde{X}^a + \pi^{ab}(\tilde{X}) \wedge \tilde{\eta}_a \wedge \tilde{\eta}_b = 0, \quad d\tilde{\eta}_a + \frac{1}{2} \partial_a \pi^{bc}(\tilde{X}) \wedge \tilde{\eta}_b \wedge \tilde{\eta}_c$$

7.5.2. *The boundary BFV theory.* We will use the same notations for coordinate fields in the space of boundary fields $\mathcal{F}_{\partial N} = \text{Map}(T[1]\partial N, T^*[1]M)$. In terms of this coordinate fields the BFV data for the Poisson sigma model are:

$$\alpha_{\partial N} = \int_{\partial N} \tilde{\eta}_a \wedge \delta \tilde{X}^a$$

$$\omega_{\partial N} = \int_{\partial N} \delta \tilde{\eta}_a \wedge \delta \tilde{X}^a$$

$$Q_{\partial N} = \int_{\partial N} \left((d\tilde{X}^a + \pi^{ab}(\tilde{X}) \wedge \tilde{\eta}_a \wedge \tilde{\eta}_b) \wedge \frac{\delta}{\delta \tilde{X}^a} + (d\tilde{\eta}_a + \frac{1}{2} \partial_a \pi^{bc}(\tilde{X}) \wedge \tilde{\eta}_b \wedge \tilde{\eta}_c) \wedge \frac{\delta}{\delta \tilde{\eta}_a} \right)$$

$$S_{\partial N} = \int_{\partial N} \left(\sum_a \tilde{\eta}_a \wedge d\tilde{X}^a + \frac{1}{2} \sum_{ab} \pi^{ab}(\tilde{X}) \wedge \tilde{\eta}_a \wedge \tilde{\eta}_b \right)$$

APPENDIX A. COISOTROPIC SUBMANIFOLDS AND REDUCTION

In this paper we use the definition of Lagrangian submanifolds as submanifolds which are both isotropic and coisotropic. This is because many of the symplectic spaces we work with are infinite dimensional. In finite dimensional symplectic manifolds, Lagrangian submanifolds have half-dimension of the total manifold. In the infinite dimensional case this property becomes meaningless.

Let W be a presymplectic space (possibly infinite-dimensional) and $K \subset W$ be the kernel of its presymplectic form. Then $W' = W/K$ is symplectic. Denote by $p : W \rightarrow W'$ the natural projection map. For a subspace $L \subset W$ define $L' \subset W'$ as $L' = p(L) = L/L \cap K$.

It is clear that $p^{-1}(L) = L + K$, and it is easy to see that $p^{-1}((L')^\perp) = L^\perp$. Here A^\perp is the orthogonal space with respect to the symplectic (presymplectic) form.

Proposition A.1. *The subspaces L and L' have the following properties:*

- (1) *L is isotropic if and only if L' is isotropic.*
- (2) *If L is Lagrangian, then L' is also Lagrangian.*
- (3) *If L' is Lagrangian and $K \subset L$, then L is Lagrangian.*

Proof. For $\xi, \eta \in W$, denote by $[\xi], [\eta] \in W'$ the equivalence classes of these elements. By definition of the symplectic structure on W' :

$$([\xi], [\eta])_{W'} = (\xi, \eta)$$

The first statement is obvious from this definition.

Now assume that L is coisotropic, i.e. if $(\xi, \eta) = 0$ for any $\eta \in L$, then $\xi \in L$. This implies that also $(\xi + \kappa_1, \eta + \kappa_2) = 0$ for each $\kappa_i \in K$. But this means that if $([\xi], [\eta])_{W'} = 0$ for each $[\eta] \in L'$ then $[\xi] \in L'$. This means L' contains its symplectic orthogonal and therefore is coisotropic. We proved the second statement.

To prove the last statement, we have to prove that if L' is coisotropic and $K \subset L$ then L is coisotropic. Then the first statement implies the third one.

Assume L' is coisotropic, i.e. that if $([\xi], [\eta])_{W'} = 0$ for each $[\eta] \in L'$ then $[\xi] \in L'$. But this means that if $(\xi + \kappa_1, \eta + \kappa_2) = 0$ for each $\eta \in L$ and $\kappa_1, \kappa_2 \in K$ then $\xi \in L + K$. If $K \subset L$, this means that if $(\xi, \eta) = 0$ for each $\eta \in L$ then $\xi \in L$. That is that $L \subset W$ is coisotropic. □

An important particular case is when W is a coisotropic subspace in a bigger symplectic space. In this case $K = W^\perp$. The space W' is the Hamiltonian reduction of the space W .

Remark A.2. *If $L \subset V$, the space V is symplectic and $W \subset V$ is coisotropic but we do not assume that $L \subset W$, then the first statement holds when $L' = p(L \cap W)$ but the second and the third statements do not hold in general unless V is finite dimensional.*

Now, let W be a presymplectic manifold and $K \subset TW$ be the integrable distribution which is the kernel of the presymplectic form. Assume the space of leaves of K is a smooth manifold W' . Let $L \subset W$ be a submanifold, and L' be the space of leaves of K which passes through L . Assume it is also smooth. Then Proposition A.1 holds but the condition in the last statement should be replaced with $K|_L \subset TL$.

APPENDIX B. SOME FACTS ON GRADED MANIFOLDS

B.1. Graded manifolds. Recall that a smooth super manifold M with body M_e is a sheaf of super algebras over M_e locally isomorphic to the tensor product of the algebra of smooth functions on the body with an exterior algebra. Namely, there is an atlas $\{(U_\alpha, \phi_\alpha)\}$ of M_e with super algebra isomorphisms

$$\Phi_\alpha: M|_{U_\alpha} \rightarrow C^\infty(\phi_\alpha(U_\alpha)) \otimes \bigwedge V^* =: \mathcal{A}_\alpha$$

for a fixed vector space V . The coordinate map ϕ_α s take values in some given vector space W . For infinite dimensional supermanifolds one needs more structure: W is assumed to be a Banach or Fréchet space and, if V is infinite dimensional, the tensor product has to be completed (the dual has also to be defined properly).

If $V = \oplus_{k \in \mathbb{Z}} V_k$ and $W = \oplus_{k \in \mathbb{Z}} W_k$ are \mathbb{Z} -graded vector spaces (it is safer to assume they have only finitely many nontrivial, but possibly infinite dimensional, summands), then \mathcal{A}_α gets additional structure as it contains the \mathbb{Z} -graded subalgebra of polynomial functions where by definition linear functions on W_k or V_k have degree $-k$. To extend the notion of grading to nonpolynomial functions, we introduce the local graded Euler vector field E as follows: Pick graded bases (i.e., bases adapted to the decompositions) $\{x^i\}$ and $\{y^i\}$ of W^* and V^* , respectively; then define

$$E := \sum_i |x^i| x^i \frac{\partial}{\partial x^i} + \sum_i |y^i| y^i \frac{\partial}{\partial y^i},$$

where $|x^i| := -k$ for $x^i \in (W_k)^*$ and $|y^i| := -k$ for $y^i \in (V_k)^*$. Notice that this definition is independent of the choice of graded bases. We then say that a function f is of degree k if it satisfies $E(f) = kf$.

If it is possible to choose an atlas of M as above such that all transition functions are compatible with the local graded Euler vector fields on charts,¹⁰ then we say that M is a graded manifold. Notice that a graded manifold is a super manifold with additional structure; namely, that of a globally well-defined graded Euler vector field, which we will keep denoting by E . A global function is said to have degree k if it satisfies

$$E(f) = kf.$$

One may let E act on vector fields and differential forms by Lie derivative and define degree accordingly. Namely, a vector field X has degree k if satisfies $[E, X] = kX$ and a differential form α has degree k if $L_E \alpha := i_E d\alpha + di_E \alpha = k\alpha$. Notice that the graded Euler vector field has degree zero. We recall from [46] some useful facts whose proof is a straightforward computation:

- (1) Let ω be a closed form of degree $m \neq 0$. Then $\omega = d\theta$ with $\theta = \frac{1}{m} \iota_E \omega$.
- (2) Let X be a vector field of degree l and ω a closed X -invariant form of degree m with $m + l \neq 0$. Then $\iota_X \omega = dS$ with $S = \frac{1}{m+l} \iota_E \iota_X \omega$.

In particular, this implies that symplectic forms of degree different from zero are automatically exact and that a symplectic cohomological vector field is automatically Hamiltonian if the degree of the symplectic form is different from -1 .

Definition B.1. *Morphisms of graded manifolds are morphisms of the underlying supermanifolds that respect the graded Euler vector fields.*

¹⁰That is, for any two charts U_α and U_β we have $E_\beta(\Phi_\beta \circ \Phi_\alpha^{-1})^* f = (\Phi_\beta \circ \Phi_\alpha^{-1})^* E_\alpha f$ for all $f \in \mathcal{A}_\alpha$.

Remark B.2. *A different definition of graded manifolds commonly used in the literature is that of a sheaf M of graded algebras over the degree zero body M_0 locally isomorphic to the tensor product of smooth functions on the degree zero body with the graded symmetric algebra of a graded vector space. In this setting functions (and transition functions) are polynomial in the coordinates of degree different from zero. To distinguish graded manifolds defined this way from the ones defined above we will call them polygraded manifolds. Notice that a polygraded manifold is not a super manifold with additional structure. On the other hand, if a graded manifold happens to have an atlas for which all transition functions are polynomial in the coordinates of degree different from zero, then it can be given the structure of a polygraded manifold simply by restricting the sheaf. In all the examples discussed in this paper we could work with polygraded manifolds, but this would be problematic for functional integral quantization where one exponentiates the action and wants to consider integration.*

Remark B.3. *A common notion in the literature is that of an N -manifold. This is a graded manifold with no coordinates of negative degree. They are commonly used as targets of the AKSZ construction. Notice that by degree reasons all transition functions are polynomial in the coordinates of degree different from zero.*

Remark B.4. *A special class of (poly)graded manifolds are those in which the \mathbb{Z} and \mathbb{Z}_2 gradings agree. This means that $W_{2k+1} = 0 = V_{2k}$ for all k or, equivalently, that for all k homogenous sections of degree $2k$ ($2k+1$) are even (odd) with respect to the original super algebra grading. These graded manifolds occur in the BV formalism whenever no fermionic physical fields are present. In this paper we will restrict to this case throughout.*

B.2. Mapping spaces. Let M and N be finite dimensional super manifolds. Then the set of morphisms $\text{Mor}(M, N)$ can naturally be given the structure of a (usually infinite dimensional) manifold (non super) as follows: If the target is a super vector space Z , then $\text{Mor}(M, Z)$ is the (usually infinite dimensional) vector space

$$(72) \quad (C^\infty(M) \otimes Z)_e = C^\infty(M)_e \otimes Z_e \oplus C^\infty(M)_o \otimes Z_o$$

For the general case, one applies this construction to local charts of the target to define local charts of the manifold of morphisms.

It is important to extend this construction to define the mapping space $\text{Map}(M, N)$ as a (usually infinite dimensional) super manifold with body $\text{Mor}(M, N)$. Again this is done in terms of local charts, so it is enough to define $\text{Map}(M, Z)$. This is just the super space $C^\infty(M) \otimes Z$ (of which (72) is the even part). The main property of the mapping space is

$$\text{Mor}(X \times M, N) = \text{Mor}(X, \text{Map}(M, N)) \quad \forall X.$$

Vector fields on the source and on the target can naturally be lifted to the mapping space. If X_M (Y_N) is a vector field on M (N), we denote by \check{X}_M (\hat{Y}_N) its lift. Notice that the map $Y_N \mapsto \hat{Y}_N$ is a morphism of Lie algebras, whereas the map $X_M \mapsto \check{X}_M$ is an antimorphism.

If M and N are graded manifolds, with graded Euler vector fields E_M and E_N , then one can give $\text{Map}(M, N)$ the structure of a (usually infinite dimensional) graded manifold with graded Euler vector field

$$E_{\text{Map}(M, N)} = \hat{E}_N - \check{E}_M.$$

The set of morphisms from M to N in the category of graded manifolds can then be regarded as the degree zero submanifold of $\text{Map}(M, N)$.

Remark B.5. *In the category of polygraded manifolds, one can also define the mapping space as a polygraded manifold. In this case the local data are given by $\text{Map}(M, Z)$ for Z a graded vector space. Here $\text{Map}(M, Z)$ is defined as the graded tensor product*

$$C^\infty(M) \otimes Z = \bigoplus_k (C^\infty(M) \otimes Z)_k$$

with

$$(C^\infty(M) \otimes Z)_k = \bigoplus_j C^\infty(M)_j \otimes Z_{k-j}.$$

APPENDIX C. ON SMOOTH POINTS OF THE MODULI SPACE \mathcal{EL}/Q

Here we will discuss the notion of smooth points in the \mathcal{EL} -moduli spaces which we use throughout the paper. We will use notations: \mathcal{F} for the space of BV fields, F for its $\text{gh} = 0$ part, \mathcal{EL} for BV Euler-Lagrange space and EL for its $\text{gh} = 0$ part. We assume \mathcal{F} is given together with the cohomological vector field Q . Denote by $Vect_Q$ the Lie subalgebra of the Lie algebra of vector fields on \mathcal{F} formed by Lie brackets with Q and by G the distribution on the body F of \mathcal{F} induced by the span of $Vect_Q$. The distribution G on F should be regarded as infinitesimal gauge transformations.

For a $x \in F$ the Taylor expansion of a vector field Q in the formal neighborhood of x is:

$$Q_x^{\text{formal}} = Q_x^{(0)} + Q_x^{(1)} + Q_x^{(2)} + \dots$$

where each

$$Q_x^{(k)} \in \text{Coder}(\hat{S}T_x\mathcal{F})$$

is the extension of a k -linear map $\hat{Q}_x^{(k)} : S^k(T_x\mathcal{F}) \rightarrow T_x\mathcal{F}$ to a coderivation of $\hat{S}T_x\mathcal{F}$ by co-Leibniz identity. The linear maps $\hat{Q}_x^{(k)}$ arise from the Taylor expansion of Q at x . By the natural inclusion $\text{Coder}(\hat{S}T_x\mathcal{F}) \hookrightarrow \text{Der}(\hat{S}T_x^*\mathcal{F})$, Q_x^{formal} acts on $\hat{S}T_x^*\mathcal{F}$ as a derivation.

If $x \in EL \subset F$, then $\hat{Q}_x^{(0)} = 0$ and Q_x^{formal} endows $T_x[-1]\mathcal{F}$ with the structure of L_∞ algebra with differential $\hat{Q}_x^{(1)}$ and higher polylinear operations $\hat{Q}_x^{(k)}$, $k \geq 2$.

We define the formal neighborhood of $[x] \in EL/G$ in \mathcal{EL}/Q as

$$(73) \quad U_{[x]}^{\text{formal}}(\mathcal{EL}/Q) := \text{Spec}(H_{Q_x^{\text{formal}}}(\hat{S}T_x^*\mathcal{F}))$$

It can be regarded as the ‘‘BV’’ (or ‘‘stable’’) version of the Maurer-Cartan set for the L_∞ structure on $T_x[-1]\mathcal{F}$. When operations $\hat{Q}_x^{(k)}$ for $k \geq 2$ are identically zero, the spectrum (73) is

$$H_{\hat{Q}_x^{(1)}}(T_x\mathcal{F})$$

which means that the formal neighborhood of $[x]$ in \mathcal{EL}/Q is the graded vector space $H_{\hat{Q}_x^{(1)}}(T_x\mathcal{F})$.

When operations $\hat{Q}_x^{(k)}$ do not vanish for $k \geq 2$ they induce operations $\hat{Q}'_x^{(k)}$ on the cohomology space $V_x = H_{\hat{Q}_x^{(1)}}(T_x\mathcal{F})$. These operations endow the graded space $V_x[-1]$ with the structure of minimal L_∞ algebra. In other words the formal series

$$Q'_x = Q'_x^{(0)} + Q'_x^{(1)} + Q'_x^{(2)} + \dots \in \text{Coder}(\hat{S}(V_x))$$

is a (formal) cohomological vector field on V_x . Note that first two terms vanish by our assumptions.

This construction is known as homological perturbation theory [33] (see also [31] for a more concise exposition), more specifically, as the homotopy transfer of L_∞ algebras [37, 38]. By the same homological perturbation theory the formal neighborhood of $[x] \in EL/G \subset \mathcal{EL}/Q$ is

$$U_{[x]}^{\text{formal}}(\mathcal{EL}/Q) = \text{Spec}(H_{Q'_x}(\hat{S}(V_x)^*))$$

This is a singular variety unless Q'_x vanishes.

This is why we define a *smooth point* $[x] \in EL/G$ as a point for which all induced L_∞ operations $\hat{Q}_x^{(k)}$ vanish. The set of such points we will call the smooth locus of the body of the \mathcal{EL} -moduli space and will denote it $(EL/G)^{\text{smooth}}$.

The smooth locus in the \mathcal{EL} -moduli space is a graded vector bundle over $(EL/G)^{\text{smooth}}$ with fiber $H_{\hat{Q}_x^{(1)}}^{\neq 0}(T_x\mathcal{F})$ over $[x]$. Here $H^{\neq 0}$ means that we do not count the cohomology in ghost number 0 to avoid double counting of geometric ($gh = 0$) directions for the tangent space.

APPENDIX D. CARTAN CALCULUS OF LOCAL DIFFERENTIAL FORMS

D.1. Transgression map from forms on the target to forms on the mapping space. Recall that the space of fields in the AKSZ theory is $\mathcal{F}_N = \text{Map}(T[1]N, \mathcal{M})$ with \mathcal{M} a Hamiltonian dg manifold.

Consider natural projections

$$\begin{array}{ccc} \mathcal{F}_N \times T[1]N & \xrightarrow{ev} & \mathcal{M} \\ p \downarrow & & \\ \mathcal{F}_N & & \end{array}$$

Here $ev(f, x) = f(x)$ and $p(f, x) = f$. The pullback ev^* of a form \mathcal{M} gives a form on $\mathcal{F}_N \times T[1]N$. Let Ω be a form on $\mathcal{F}_N \times T[1]N$ of type $(k, 0)$. The pushforward p_* of a form on $\mathcal{F}_N \times T[1]N$ gives a form on \mathcal{F}_N .

$$p_*(\Omega)(X) = \int_{T[1]N} i_X^*(\Omega)$$

Here i_X is the composition of the natural isomorphism of $T[1]N$ with the fiber $(X, T[1]N) = p^{-1}(X)$ and the embedding of this fiber into $\mathcal{F}_N \times T[1]N$. The integration over $T[1]N$ is defined as usual taking into account natural isomorphism $\phi : C^\infty(T[1]N) \simeq \Omega^\bullet(N)$. If $f \in C^\infty(T[1]N)$

$$\int_{T[1]N} f \stackrel{\text{def}}{=} \int_N \phi(f)$$

Having in mind this formula we will write for p_*

$$p_*(\Omega)(X) = \int_N \Omega$$

The transgression map is defined as

$$T_N := p_*ev^* : \quad \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{F}_N)$$

Given $\alpha \in \Omega^\bullet(\mathcal{M})$ which is $\alpha = \sum_{\{a\}} \alpha_{a_1, \dots, a_k} dx^{a_1} \dots dx^{a_k}$ in local coordinates, its transgression is

$$(74) \quad (T_N \alpha)(X) = \int_N \sum_{a_1, \dots, a_k} \alpha_{a_1, \dots, a_k}(X(u)) \delta X^{a_1}(u) \dots \delta X^{a_k}(u)$$

where $X^a(u)$ are coordinate fields and $\delta X^a(u)$ are their de Rham differentials (on \mathcal{F}_N).

Define the space of *ultra local forms* $\Omega_{uloc}^\bullet(\mathcal{F}_N)$ as the image of the transgression map.

D.2. De Rham and lifted vector fields. Recall that the de Rham differential for N can be regarded as a vector field D_N on $T[1]N$ of degree 1. The identification of forms on N with functions on $T[1]N$ identifies the action of the de Rham differential on forms with the action of the vector field D_N on functions on $T[1]N$. In local coordinates $\{u^i\}$ on N and $\xi^i = du^i$ on $T_u[1]N$ the Lie derivative of a function along the de Rham vector field on $T[1]N$ is $D_N f = \sum_i \xi^i \frac{\partial f}{\partial u^i}$.

The *de Rham vector field* \check{D}_N is defined as the lift of de Rham vector field $\xi^i \frac{\partial}{\partial u^i}$ on N to the mapping space \mathcal{F}_N .

Indeed, the tangent space $T_X \mathcal{F}_N$ is the space of mappings $Y : T[1]N \rightarrow T\mathcal{M}$ such that $(\xi, u) \mapsto Y(u, \xi) \in T_{X(u, \xi)} \mathcal{M}$. A vector field V on \mathcal{F}_N is a section of the tangent bundle, i.e. it assigns a vector $V_X : \mathcal{F}_N \rightarrow T_X \mathcal{F}_N$ to each $X \in \mathcal{F}_N$.

For de Rham vector field \check{D}_N on \mathcal{F}_N the vector $(\check{D}_N)_X \in T_X \mathcal{F}_N$ is the mapping

$$(\check{D}_N)_X : T[1]N \xrightarrow{D_N} T(T[1]N) \xrightarrow{dX} T\mathcal{M}$$

Here $dX : T(T[1]N) \rightarrow T\mathcal{M}$ is the differential of X . In local coordinates:

$$(\check{D}_N)_X(\xi, u) = \sum_a \sum_{\{i\}, j} \frac{\partial X^a_{i_1, \dots, i_k}}{\partial u^j}(u) \xi^j \xi^{i_1} \dots \xi^{i_k} \frac{\partial}{\partial x^a}$$

The de Rham vector field on \mathcal{F}_N has ghost number +1.

Another important class of vector fields on \mathcal{F}_N comes from lifting vector fields on \mathcal{M} to the mapping space \mathcal{F}_N . If $v : \mathcal{M} \rightarrow T\mathcal{M}$ is a vector field on \mathcal{M} , its lifting is the vector field \hat{v} on \mathcal{F}_N with $\hat{v}_X \in T_X \mathcal{F}_N$ given by the mapping

$$T[1]N \xrightarrow{X} \mathcal{M} \xrightarrow{v} T\mathcal{M}$$

D.3. Local differential forms. *Local differential forms*¹¹ are defined as substitutions of several copies of \check{D}_N into an ultra-local form:

$$(75) \quad T_N^k \chi := (\iota_{\check{D}_N})^k T_N \chi, \quad k \geq 0$$

The space of forms $\Omega^\bullet(\mathcal{M})$ is naturally bi-graded with (de Rham degree of forms on \mathcal{M} , \mathbb{Z} -grading on \mathcal{M}). The space $\Omega^\bullet(\mathcal{F}_N)$ is also naturally bi-graded with (de Rham degree of forms on \mathcal{F}_N , ghost number). Transgression mappings $T_N^k : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{F}_N)$ are bi-graded with

$$\text{deg}(T_N^k) = (-k, -\dim(N) + k)$$

Also note that if $\chi \in \Omega^n(\mathcal{M})$ and $k > n$, by degree reasons we have $T_N^k \chi = 0$.

Let $\{x^a\}$ be local coordinates on \mathcal{M} and $X^a(u)$ be corresponding coordinate fields. If $\alpha = \sum_{\{a\}} \alpha_{a_1, \dots, a_k} dx^{a_1} \wedge \dots \wedge dx^{a_k}$ is a differential form on \mathcal{M} written in

¹¹Note that, by our definition, local 0-forms are special cases of local functionals.

local coordinates, the corresponding local forms on \mathcal{F}_N can be written in coordinate fields as

$$T_N^l(\alpha) = \int_N \sum_{\{a\}=\{a_I\} \sqcup \{a_{II}\}} \alpha_{\{a_I\} \sqcup \{a_{II}\}}(X(u)) \wedge dX^{\{a_I\}}(u) \wedge \delta X^{\{a_{II}\}}(u)$$

Here the sum is taken over partitions of the set $\{a\}$ into two subsets, so that $\{a\}$ is a shuffle of $\{a_I\}$ and $\{a_{II}\}$. We denote $dX^{\{a\}} = dX^{a_1} \wedge \cdots \wedge dX^{a_k}$.

We will use the notation $\Omega_{loc}^\bullet(\mathcal{F}_N)$ for the space of such forms.

D.4. The Cartan calculus. Let us denote $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ the restriction map (pullback by the natural inclusion $\partial N \hookrightarrow N$) and let

$$(76) \quad \pi^* : \quad \Omega^\bullet(\mathcal{F}_{\partial N}) \rightarrow \Omega^\bullet(\mathcal{F}_N)$$

be the pullback by π for differential forms on the space of fields.

Objects we introduced satisfy the following properties:

(i) De Rham differential δ on \mathcal{F}_N acts on local forms by

$$(77) \quad \delta(T_N^k \chi) = (-1)^{\dim(N)} (T_N^k d\chi - k \cdot \pi^*(T_{\partial N}^{k-1} \chi))$$

where d is the de Rham differential on target.

(ii) ‘‘Total derivative property’’, which is a special case of (77) for $k = \deg \chi + 1$:

$$(78) \quad \chi \in \Omega^n(\mathcal{M}) \quad \Rightarrow \quad T_N^{n+1}(d\chi) = (n+1) \cdot \pi^*(T_{\partial N}^n \chi)$$

(iii) Substitution of \check{D}_N into a local form:

$$(79) \quad \iota_{\check{D}_N}(T_N^k \chi) = T_N^{k+1} \chi$$

by definition (75).

(iv) Substitution of a lifted vector field into a local form:

$$(80) \quad \iota_{\check{v}}(T_N^k \chi) = (-1)^{(\text{gh}(v)+1) \cdot \dim(N)} T_N^k(\iota_v \chi)$$

for any vector field $v \in \text{Vect}(\mathcal{M})$ of ghost number $\text{gh}(v)$ on target.

(v) Rules for commuting the substitution of a vector field on \mathcal{F}_N with the map π^* (76):

$$(81) \quad \iota_{\check{D}_N} \pi^* = \pi^* \iota_{\check{D}_{\partial N}}$$

$$(82) \quad \iota_{\check{v}} \pi^* = \pi^* \iota_{\check{v}}$$

Here \check{v} is the lifting of the vector field v on \mathcal{M} to a vector field on $\mathcal{F}_{\partial N}$. This is a manifestation of the fact that vector fields $\check{D}_N, \check{v} \in \text{Vect}(\mathcal{F}_N)$ are projectable by the restriction map $\pi : \mathcal{F}_N \rightarrow \mathcal{F}_{\partial N}$ with projections being $\check{D}_{\partial N}, \check{v} \in \text{Vect}(\mathcal{F}_{\partial N})$ respectively.

(vi) The Lie derivative of a local form along \check{D}_N :

$$(83) \quad L_{\check{D}_N}(T_N^k \chi) = (-1)^{\dim(N)} \pi^*(T_{\partial N}^k \chi)$$

We define the Lie derivative by Cartan formula $L_V := [\iota_V, \delta] = \iota_V \circ \delta + (-1)^{\text{gh}(V)} \delta \circ \iota_V$ for any vector field $V \in \text{Vect}(\mathcal{F}_N)$. With this definition the formula (83) comes as a consequence of (77), (79), (81).

(vii) Lie derivative of a local form along a lifted vector field:

$$(84) \quad L_{\check{v}}(T_N^k \chi) = (-1)^{\text{gh}(v) \cdot \dim(N)} T_N^k(L_v \chi)$$

(this is a consequence of (77), (80), (82)).

(viii) Commutators of vector fields:

$$(85) \quad [\check{D}_N, \check{D}_N] = 0$$

This is because $d^2 = 0$ on the source N . Also

$$(86) \quad [\check{D}_N, \hat{v}] = 0$$

and

$$(87) \quad [\hat{u}, \hat{v}] = \widehat{[u, v]}$$

D.5. Applications to AKSZ theories.

D.5.1. *Invariant definition of AKSZ theories.* Recall that the target manifold in an n -dimensional AKSZ theory is a Hamiltonian dg manifold of degree $n - 1$. In other words, it is symplectic with an exact symplectic form $\omega = d\alpha \in \Omega^2(\mathcal{M})$ and $\deg(\omega) = n - 1$ and with a potential function Θ of degree n such that $\{\Theta, \Theta\} = 0$. The potential function generates the cohomological vector field Q of degree 1:

$$\iota_Q \omega = d\Theta$$

By definition of Q , the symplectic form ω is Q -invariant, i.e.

$$L_Q \omega = 0$$

where $L_Q \eta$ is the Lie derivative of η . The condition $\{\Theta, \Theta\} = 0$ implies that

$$L_Q^2 = 0$$

which we will write as $Q^2 = 0$.

The AKSZ action is the following local functional on the space $\text{Map}(T[1]N \rightarrow \mathcal{M})$

$$S_N = S_N^{kin} + S_N^{int}$$

where

$$\begin{aligned} S_N^{kin} &= T_N^1 \alpha \\ S_N^{int} &= T_N^0 \Theta \end{aligned}$$

are kinetic and interaction parts of the action respectively. Here T_N^k are the transgression maps defined above.

The BV cohomological vector field Q_N in this theory is defined as

$$Q_N = Q_N^{kin} + Q_N^{int} \in \text{Vect}(\mathcal{F}_N)$$

with

$$\begin{aligned} Q_N^{kin} &= \check{D}_N \\ Q_N^{int} &= \hat{Q} \end{aligned}$$

Here \hat{Q} is the lift of the vector field Q on \mathcal{M} to a vector field on \mathcal{F}_N , see subsection D.2.

The BV 2-form and its primitive 1-form:

$$\begin{aligned} \omega_N &= T_N^0 \omega \\ \alpha_N &= T_N^0 \alpha \end{aligned}$$

Note that with our conventions we have an extra sign: $\omega_N = (-1)^{\dim(N)} \delta \alpha_N$.

These formulae applied to the boundary ∂N of N also define the BFV action $S_{\partial N}$, the cohomological vector field $Q_{\partial N}$, BFV 2-form $\omega_{\partial N}$ and its primitive 1-form $\alpha_{\partial N}$.

D.5.2. *Proof of Proposition 6.3.* Using the definition of Q_N and the Cartan calculus we can compute the contraction of kinetic and interaction parts of Q_N with ω_N :

$$\iota_{Q_N^{kin}}\omega_N = \iota_{\tilde{D}_N}(T_N^0\omega) = T_N^1\omega = T_N^1\delta\alpha$$

The rule (77) from Cartan calculus on local forms implies that this is equal to

$$(88) \quad (-1)^{\dim(N)}\delta T_N^1\alpha + \pi^*(T_{\partial N}^0\alpha) = (-1)^{\dim(N)}\delta S_N^{kin} + \pi^*\alpha_{\partial N}$$

Contracting Q_N^{int} with ω_N we obtain

$$\iota_{Q_N^{int}}\omega_N = \iota_{\hat{Q}}(T_N^0\omega) = T_N^0(\iota_Q\omega)$$

Here in the last equality we used (80). Because Q is a Hamiltonian vector field generated by Θ , $\iota_Q\omega = d\Theta$ and for the above expression we obtain

$$(89) \quad T_N^0(d\Theta) = (-1)^{\dim(N)}\delta(T_N^0\Theta) = (-1)^{\dim(N)}\delta S_N^{int}$$

Here the first equality follows from (77).

Equations (88), (89) together give (64).

D.5.3. *Further applications to the AKSZ formalism.* Here we will reprove the Proposition 3.1 using the Cartan calculus.

Proposition D.1. *The following identity holds:*

$$(90) \quad L_{Q_N}S_N = (-1)^{\dim(N)}\pi^*(2S_{\partial N} - \iota_{Q_{\partial N}}\alpha_{\partial N})$$

Proof. The proof is computational:

$$(91) \quad \begin{aligned} L_{Q_N^{kin}}S_N^{kin} &= L_{\tilde{D}_N}T_N^1\alpha = (-1)^{\dim(N)}\pi^*(T_{\partial N}^1\alpha) = \\ &= (-1)^{\dim(N)}\pi^*(\iota_{Q_{\partial N}^{kin}}\alpha_{\partial N}) = (-1)^{\dim(N)}\pi^*(2S_{\partial N}^{kin} - \iota_{Q_{\partial N}^{kin}}\alpha_{\partial N}) \end{aligned}$$

Here we used $\iota_{Q_{\partial N}^{kin}}\alpha_{\partial N} = S_{\partial N}^{kin}$.

$$(92) \quad \begin{aligned} L_{Q_N^{int}}S_N^{int} &= L_{\tilde{D}_N}(T_N^0\Theta) = (-1)^{\dim(N)}\pi^*(T_{\partial N}^0\Theta) = \\ &= (-1)^{\dim(N)}\pi^*(S_{\partial N}^{int}) \end{aligned}$$

$$(93) \quad \begin{aligned} L_{Q_N^{int}}S_N^{kin} &= L_{\hat{Q}}(T_N^1\alpha) = (-1)^{\dim(N)}T_N^1(L_Q\alpha) \quad \underbrace{\quad}_{\text{Cartan formula}} \\ &= (-1)^{\dim(N)}T_N^1(\iota_Q d\alpha - d\iota_Q\alpha) = (-1)^{\dim(N)}T_N^1d(\Theta - \iota_Q\alpha) \end{aligned}$$

Here we used the Cartan formula and the exactness of the symplectic form on the target, $\omega = d\alpha$. Now applying (78) we obtain

$$(94) \quad = (-1)^{\dim(N)}\pi^*T_{\partial N}^0(\Theta - \iota_Q\alpha) = (-1)^{\dim(N)}\pi^*(S_{\partial N}^{int} - \iota_{Q_{\partial N}^{int}}\alpha_{\partial N})$$

$$(95) \quad L_{Q_N^{int}}S_N^{int} = L_{\hat{Q}}T_N^0\Theta = (-1)^{\dim(N)}T_N^0(L_Q\Theta) = 0$$

Collecting (91)–(95), we obtain (90). □

The following is a corollary of Proposition 6.3 but here we give an independent proof.

Proposition D.2. *The following holds:*

$$(96) \quad L_{Q_N} \omega_N = (-1)^{\dim(N)} \pi^* \omega_{\partial N}$$

Proof. Indeed:

$$(97) \quad L_{Q_N^{kin}} \omega_N = L_{\tilde{D}_N} (T_N^0 \omega) = (-1)^{\dim(N)} \pi^* \underbrace{T_{\partial N}^0 \omega}_{\omega_{\partial N}}$$

where we used the rule (83). Also,

$$(98) \quad L_{Q_N^{int}} \omega_N = L_{\hat{Q}} (T_N^0 \omega) = (-1)^{\dim(N)} T_N^0 (L_Q \omega) = 0$$

where we used the rule (84) and the fact that the target cohomological vector field is symplectic. Putting (97) and (98) together we get (96). \square

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CLASSICAL AND QUANTUM LAGRANGIAN FIELD THEORIES WITH BOUNDARY

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ABSTRACT. This note gives an introduction to Lagrangian field theories in the presence of boundaries. After an overview of the classical aspects, the cohomological formalisms to resolve singularities in the bulk and in the boundary theories (the BV and the BFV formalisms, respectively) are recalled. One of the goals here (and in [7]) is to show how the latter two formalisms can be put together in a consistent way, also in view of perturbative quantization.

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1. INTRODUCTION

Lagrangian field theories are used to describe physical models. Their quantization is somehow expected to satisfy Segal's [25] axioms (which generalize in higher dimensions our understanding of quantum mechanics). Roughly speaking they say

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that a d -dimensional quantum field theory should provide a functor from the category of d -dimensional cobordisms (with possible extra structures) to the category of vector spaces and linear maps. This program has been completed in conformal field theories. Atiyah proposed a stricter version of these axioms in the case of topological field theories, which was first realized by the Reshetikhin–Turaev invariant [20] which may be thought of as some nonperturbative quantization of Chern–Simons theory [28, 11]. Another example is implicitly present in an old work by Migdal [19].

It would be interesting to understand how this picture emerges from the perturbative functional integral quantization of Lagrangian fields theories. An important application would be the construction of perturbative quantum field theories on manifolds out of topologically simple or geometrically small pieces, where the computations might be more tractable. Even in the case of topological field theories this would produce new insight (and might possibly lead to a full understanding of the relation between the perturbative expansion of Chern–Simons theory and the asymptotics of the Reshetikhin–Turaev invariant).

The first step in developing this program, performed in [7], consists in developing the analogous pictures in the classical formalism and in the BV formalism [5] which is the main tool for the perturbative quantization of theories with symmetries. In principle, this already yields the possibility of constructing moduli spaces of solutions to variational problems out of computations on topologically simple or geometrically small pieces.

This note reviews some results of [7], see Section 5, with a didactical introduction through classical Lagrangian field theory given in Sections 2 and 3, and with the BFV formalism [4], being outlined in Section 4. Notice that Section 5 is self contained, so the hasty reader who does not need a motivation or an introduction might well jump directly there.

As a final remark, notice that in this paper every manifold is assumed to be compact, though possibly with boundary.

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2. LAGRANGIAN FIELD THEORY I: OVERVIEW

We start reviewing classical Lagrangian mechanics. This is usually defined by specifying a Lagrangian function L on the tangent bundle TN of some manifold N . The action $S_{[t_0, t_1]}$ corresponding to an interval $[t_0, t_1]$ is a function on the path space $N^{[t_0, t_1]}$ defined by

$$(2.1) \quad S_{[t_0, t_1]}[x] = \int_{t_0}^{t_1} L(\dot{x}(t), x(t)) dt.$$

The Euler–Lagrange (EL) equations describe the critical points of the action. As this requires integration by parts, one usually puts appropriate boundary conditions for the boundary terms to vanish, e.g., one fixes the initial and final values $x(t_0)$ and $x(t_1)$. In the sequel we will want to avoid this.

2.1. Symplectic formulation. The typical example is Newtonian mechanics on a Riemannian manifold. In this case N is a Riemannian manifold and the Lagrangian is $L(q, v) = \frac{1}{2}m\|v\|^2 - V(q)$, where $\| \cdot \|$ is the norm induced by a metric on N , V is a function on N and m is a parameter (usually assumed to be strictly positive). The EL equations in this case are the Newton equations with force given by $-\nabla V$. They admit, locally, a unique solution if initial position and velocity are specified. We denote by $C = TN$ the space of such initial conditions. Here C stands here for Cauchy, but we will see later it can also stand for coisotropic. Notice the peculiar coincidence that C is the same as the space on which L is defined. This will not be the case in further examples.

One usually reformulates the problem in symplectic terms using the Legendre mapping $\phi_L: TN \rightarrow T^*N$, $(v, q) \mapsto (p(v, q), q)$, with $p_i = \frac{\partial L}{\partial v^i}$. The Newton equations of motions now become first order and their solution yields the symplectic flow Φ_{H_L} with respect to the canonical symplectic form ω_{can} on T^*M and Hamiltonian function H_L given by the Legendre transform of L : $H_L(p, q) = p_i v^i(p, q) - L(v(p, q), q)$. Here we have used the inverse of the Legendre mapping $\phi_L^{-1}: T^*N \rightarrow TN$, $(p, q) \mapsto (v(p, q), q)$, where $v(p, q)$ is the inverse to $p(v, q)$. Since in the following we will also be interested in degenerate Lagrangians, for which the Legendre mapping is not a local diffeomorphism, we now recall how to reformulate things without going to the cotangent bundle. The first simple fact is that

$$(2.2) \quad \alpha = \frac{\partial L}{\partial v^i} dq^i$$

is a well-defined one-form on $C = TN$ (it would be more precise to write α_L instead of α to stress the dependency on L , but we regard L as given for the whole discussion). Moreover, $\omega := d\alpha$ is non degenerate precisely when the Lagrangian is regular, i.e. when the Legendre mapping is a local diffeomorphism. In this case, one can also easily show that $\omega = \phi_L^* \omega_{\text{can}}$. We can now formulate the Hamiltonian evolution directly on C . For later considerations, however, it is better to consider the graph of the Hamiltonian flow instead of the flow itself. Borrowing notations from the cotangent bundle, we then consider $L_{[t_0, t_1]} = (\phi_L^{-1} \times \phi_L^{-1})(\text{graph}(\Phi_{H_L}|_{t_0}^{t_1}))$. As a Hamiltonian flow is a symplectomorphism and as the graph of a symplectomorphism is a Lagrangian submanifold of the Cartesian product with reversed sign of the symplectic form on the first factor, we have that $L_{[t_0, t_1]}$ is Lagrangian submanifold in $\bar{C} \times C$. The fact that it comes from the graph of a flow yields the property

$$(2.3) \quad L_{[t_1, t_2]} \circ L_{[t_0, t_1]} = L_{[t_0, t_2]}, \quad \lim_{t_1 \rightarrow t_0} L_{[t_0, t_1]} = \text{graph of the Id map}.$$

The limit has to be understood by putting an appropriate topology on the space of submanifolds of C .

The crucial point now is that $L_{[t_0, t_1]}$ may be defined directly without making reference to the Hamiltonian flow. Let

$$\pi_{[t_0, t_1]}: \begin{array}{ccc} N^{[t_0, t_1]} & \rightarrow & C \times C \\ \{x(t)\} & \mapsto & ((\dot{x}(t_0), x(t_0)), (\dot{x}(t_1), x(t_1))) \end{array}$$

and let $EL_{[t_0, t_1]} \subset N^{[t_0, t_1]}$ be the space of solutions to the EL equations. Then we simply have

$$(2.4) \quad L_{[t_0, t_1]} = \pi_{[t_0, t_1]}(EL_{[t_0, t_1]}).$$

This is the fundamental equation we are going to use. Notice that (2.3) immediately follow from this new definition (the limiting property follows from the fact that C is the space of initial conditions that guarantee local existence and uniqueness).

To insure that $L_{[t_0, t_1]}$ is Lagrangian we have to make some further observations. First we observe that, if we do not impose boundary conditions, the variational calculus yields

$$(2.5) \quad \delta S_{[t_0, t_1]} = \text{EL}_{[t_0, t_1]} + \pi_{[t_0, t_1]}^* \alpha,$$

where $\text{EL}_{[t_0, t_1]}$ is the term containing the EL equations that we reinterpret as a one-form on $N^{[t_0, t_1]}$. Notice that we also interpret the variation symbol δ as the de Rham differential on $\Omega^\bullet(N^{[t_0, t_1]})$. The appearance of the same α here and in (2.2) is crucial. The equation yields in fact, after differentiation, $\pi_{[t_0, t_1]}^* \omega = -\delta \text{EL}_{[t_0, t_1]}$, which in turn implies that the restriction of ω to $L_{[t_0, t_1]}$ vanishes (observe here that the space of solutions to the EL equations on $M^{[t_0, t_1]}$ is the zero locus of $\text{EL}_{[t_0, t_1]}$ and that $\pi_{[t_0, t_1]}$ is a surjective submersion). This amounts to saying that $L_{[t_0, t_1]}$ is isotropic.

In addition we know that a unique solution is given, locally, once we specify initial conditions in TN or, equivalently, if we specify initial and final position for a short enough time interval $[t_0, t_1]$. Hence, if we fix initial and final positions Q_0 and Q_1 and let $L_{Q_0, Q_1} := \{(v, q, v', q') \in C \times C \mid q = Q_0, q' = Q_1\}$, we get that, for t_1 sufficiently close to t_0 , $L_{Q_0, Q_1} \cap L_{[t_0, t_1]}$ consists of one point; by dimension counting, assuming the intersection is transversal, this implies that $L_{[t_0, t_1]}$ has half dimension than C and so is Lagrangian¹. Finally, if t_0 and t_1 are not close, we can decompose the interval into short ones on which we can use the previous argument and recover $L_{[t_0, t_1]}$ as the composition of the canonical relations corresponding to the subintervals. Hence $L_{[t_0, t_1]}$ is also a canonical relation.

After understanding this, we can also think of more general boundary conditions by replacing L_{Q_0, Q_1} with another submanifold L of $\overline{C} \times C$ on which α vanishes. The latter condition ensures that the variational problem has no boundary contributions. It also implies that L is isotropic. In order to have, generically, intersection points of L with $L_{[t_0, t_1]}$, one has to require L to have maximal dimension, and hence to be Lagrangian². Now, the intersection $L \cap L_{[t_0, t_1]}$ can be considered as the space of solutions to the EL equations. Notice however that this intersection might as well be empty or contain (infinitely) many points, though generically, it will be a discrete set.

2.2. A degenerate example: geodesics on the Euclidean plane. We now consider the non-regular Lagrangian $L(v, q) := \|v\|$, where $\|\cdot\|$ is the Euclidean norm and $v, q \in T\mathbb{R}^2$. The action is still given by (2.1) which we now define only on the space $N_0^{[t_0, t_1]}$ of immersed paths (i.e., we impose the condition $\dot{\gamma}(t) \neq 0 \forall t \in [t_0, t_1]$). The EL equations have as solutions parameterized segments of straight lines in \mathbb{R}^2 . By analogy with Newtonian mechanics the Cauchy data include initial position and initial velocity, but these data do not give uniqueness: they define one end of the segment and its slope but do not define a parametrization of the

¹Here is another argument. For a short interval, the initial and final positions specify a unique solution. Hence initial and final positions determine initial and final velocities, which implies the L is a graph, hence Lagrangian.

²If L is Lagrangian but α does not vanish on it, we can modify the action by adding boundary terms such that the modified one-form α is vanishing on L

segment uniquely. Nevertheless, let us consider what will happen if we take the “wrong” space of Cauchy data $C = T\mathbb{R}^2 \setminus \{\text{zero section}\}$. We defer to Remark 2.3 the discussion of the “true” space of Cauchy data.

Using polar coordinates for the velocities, we can write $C = \mathbb{R}^2 \times S^1 \times \mathbb{R}_{>0}$ with coordinates the position \mathbf{q} , the normalized velocity \mathbf{v} (i.e. $\|\mathbf{v}\| = 1$) and the speed ρ . Then (2.2) yields $\alpha = \mathbf{v} \cdot d\mathbf{q}$, so that $\omega = d\mathbf{v} \cdot d\mathbf{q}$ is clearly degenerate (as it does not have a $d\rho$ component). On the other hand, (2.4) (with π the evaluation of position and velocity at the initial and at the final point) yields

$$L_{[t_0, t_1]} = \{(\mathbf{q}_0, \mathbf{v}, \rho_0), (\mathbf{q}_1, \mathbf{v}, \rho_1)\} | (\mathbf{q}_1 - \mathbf{q}_0) \text{ parallel to } \mathbf{v}\}$$

which is clearly not a graph but can easily be checked to be Lagrangian in $\overline{C} \times C$ (see Appendix A for the definition of Lagrangianity in the case of a degenerate two-form).

In this example one can easily get rid of the degeneracy of ω by taking the quotient by its kernel which is the span of the vector fields $\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{q}}$ and $\frac{\partial}{\partial \rho}$ (geometrically, these vector fields represent the space directions parallel to the velocity and the rescalings of the velocity). The quotient turns out to be TS^1 with symplectic structure given by pullback of the canonical one on T^*S^1 by the induced metric on S^1 . Notice that the base S^1 here is the space of normalized velocities, whereas the tangent fiber can be thought of as the space direction orthogonal to the given velocity. One can also project $L_{[t_0, t_1]}$ down to the quotient $TS^1 \times TS^1$. The result is just the graph of the identity map on TS^1 . This is a consequence of the fact that the action is invariant under reparametrization. Notice that this is an example of topological theory: the action does not depend of the metric on $[t_0, t_1]$.

In this example we passed to the quotient space which appeared to be smooth. However that in general reduction may produce very singular quotients, so passing directly to the reduced space had better be avoided. Instead, as we will see, it is better to use the BV-BFV approach.

Remark 2.1. Notice that the one-form $\alpha = \mathbf{v} \cdot d\mathbf{q}$ is not horizontal with respect to the kernel of ω , so it cannot be reduced to TS^1 . On the other hand, we may regard α as a connection one-form on the trivial line bundle on C . We can then reduce this line bundle to a line bundle over TS^1 and reduce α as a connection.

Also notice that evaluating the action on a solution yields a well-defined function S_{HJ} (the Hamilton–Jacobi action) on $L_{[t_0, t_1]}$ which is just the length of the path. Again, S_{HJ} cannot be reduced to a well-defined function on $\overline{TS^1} \times TS^1$, but $\exp \frac{i}{\hbar} S_{HJ}$ can be reduced to a section of the reduced line bundle.

Remark 2.2. If one considers the same example but now with the Minkowski metric, also in higher dimensions, and considers only timelike velocities (i.e., with $ds^2 = dt^2 - d\mathbf{x}^2$, one assumes $\|v\| > 0$), the reduction yields TH , where H is the upper hyperboloid $v_0^2 - \mathbf{v}^2 = 1$, and the symplectic structure is obtained by pullback of the canonical one on the cotangent bundle by the hyperbolic metric on H (which is induced by the Minkowski metric). The reduced Lagrangian is again just the graph of the identity.

Remark 2.3. We mentioned above that $T\mathbb{R}^2 \setminus \{\text{zero section}\}$ is not the true space of initial conditions because giving initial position and velocity does not select a unique parametrized segment. In order to obtain uniqueness in a formal neighborhood of the initial point we can enlarge C by setting $\widetilde{C} = T\mathbb{R}^2 \setminus \{\text{zero section}\} \times \mathbb{R}^\infty$, where

the coordinates on \mathbb{R}^∞ are all higher derivatives of the path. In other words here we work with the space of jets. Define the one form $\tilde{\alpha}$ as the pull-back of α with respect to the natural projection $\tilde{C} \rightarrow C$. It is clear the \mathbb{R}^∞ factor is in the kernel of $\tilde{\alpha}$ and of $\tilde{\omega} = \omega$. This is the reason why this factor can be completely neglected just as we neglected the space $\mathbb{R} > 0$ of speeds in the discussion above. The reduction of \tilde{C} is the same as that of C . Boundary values (2.4) of solutions to the Euler-Lagrange equations define a Lagrangian subspace in $(\tilde{C} \times \tilde{C})$. Its reduction is again the graph of the identity.

2.3. Example: Free scalar field theory. We now describe an example of a field theory in dimension d . The space time in such theory is a Riemannian d -manifold (M, g_M) . The space of fields is \mathbb{R}^M (functions on M) and the action is

$$S_{(M, g_M)}[\phi] = \frac{1}{2} \int_M g_M^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, dvol_{g_M} = \frac{1}{2} \int_M (d\phi, d\phi) dvol_{g_M}$$

where $\phi \in \mathbb{R}^M$. Solutions to the EL equations are harmonic functions on M .

In order to understand the boundary structure for an arbitrary space time manifold M consider first a thin neighborhood of its boundary $\Sigma = \partial M$. That is consider a short cylinder $\Sigma \times [0, \epsilon]$ where (Σ, g_Σ) is a $(d-1)$ -dimensional Riemannian manifold and $g_M = g_\Sigma + ds^2$, $s \in [0, \epsilon]$. A unique solution to the EL equation is obtained if one specifies the values of ϕ and of its normal derivative on $\Sigma \times \{0\}$. This gives the natural space of Cauchy data associated to Σ , $C_\Sigma = \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma$. Similarly to (2.2) the boundary term in the variation of $S_{\Sigma \times [0, \epsilon]}$ gives the one-form

$$\alpha_{(\Sigma, g_\Sigma)} = \int_\Sigma \chi \delta\phi \, dvol_{g_\Sigma}$$

with $(\chi, \phi) \in C_\Sigma$. Here χ should be thought of as the restriction to $\Sigma \times \{0\}$ of the normal derivative of the bulk field ϕ . Notice that $\omega_{(\Sigma, g_\Sigma)} = \delta\alpha_{(\Sigma, g_\Sigma)}$ is (weakly) nondegenerate.

For a general Riemannian d -manifold M with boundary ∂M , we have the surjective submersion $\pi_M: \mathbb{R}^M \rightarrow C_{\partial M}$ obtained by evaluating the field ϕ and its normal derivative on the boundary. Formula (2.5) still holds (with a little change in notation):

$$\delta S_{(M, g_M)} = EL_{(M, g_M)} + \pi_M^* \alpha_{(\partial M, g_M|_{\partial M})}.$$

Moreover, in the spirit of (2.4), define $L_{(M, g_M)} \subset C_{\partial M}$ as $\pi_M(EL_{(M, g_M)})$.

It is easy to see that $L_{(M, g_M)}$ is Lagrangian. Indeed, the Dirichlet problem for ϕ has unique solution on M . Thus, if $\phi \in EL_M$, its boundary values define the the normal derivative of ϕ at the boundary. This map from the Dirichlet data to Neumann data is known as the Dirichlet-to-Neumann mapping. Thus, the submanifold $L_M \in C_{\partial M}$ is the graph of the Dirichlet-to-Neumann mapping $\mathbb{R}^{\partial M} \rightarrow \mathbb{R}^{\partial M}$.

Notice that one may distinguish the connected components of ∂M into incoming and outgoing: $\partial M = \partial_{\text{in}} M \sqcup \partial_{\text{out}} M$. Denoting $\partial_{\text{in}} M$ with opposite orientation by $\partial_{\text{in}} M^{\text{opp}}$, we may then view $L_{(M, g_M)} \subset \overline{C_{\partial_{\text{in}} M^{\text{opp}}} \times C_{\partial_{\text{out}} M}}$ as a canonical relation from $C_{\partial_{\text{in}} M^{\text{opp}}}$ to $C_{\partial_{\text{out}} M}$.

2.4. Conclusions. From these examples we see that the Hamiltonian framework for non-regular Lagrangians has to be replaced by its weaker version. However, certain important patterns remain. We have seen in these examples that that *i*) in all cases we were able to derive a, possibly degenerate, two-form on the space

of initial conditions associated to the boundary of the space time and *ii*) we were able to assign to the bulk of the space time a Lagrangian/isotropic submanifold (not necessarily a graph) in such spaces. In the next Section we will see that this a quite general fact.

3. LAGRANGIAN FIELD THEORY II (AFTER V. FOCK)

In an unpublished account [10], V. Fock has considered the general structure of Lagrangian field theories on manifolds with boundary. We give here our recapitulation of this account. (To different levels of generality, this structure has been rediscovered many times, see e.g. [13, 24] and references therein.)

Notice that this Section is rather about a philosophical account leading to a concrete construction than a precise mathematical formulation. For simplicity, we also assume that all the "spaces" occurring in the following are actually (possibly infinite dimensional) manifolds.

A Lagrangian field theory is specified by

- (1) fixing the dimension d of the source manifolds;
- (2) fixing a class of d -manifolds, possibly with extra structure, such as a metric in the example of subsection 2.3;
- (3) associating a space of fields F_M (functions, maps to a fixed target manifold, sections of bundles, connections, ...) to every d -manifold M in the class;
- (4) defining a density, the Lagrangian L , of the fields and finitely many of their derivatives.³

The action functional associated to a manifold M , as a function on F_M , is then given by $S_M = \int_M L$. The variation of the action, neglecting the boundary terms, yields the EL equations. We denote by $EL_M \subset F_M$ the space of solutions to the EL equations on a given manifold M .

Let now Σ be a $(d-1)$ -manifold. We extend it to a d -manifold $M := \Sigma \times [0, \epsilon]$ (taking care of the possible additional structure). The variational calculus on this particular M produces two new pieces of data:

- (1) The space C_Σ of Cauchy data consisting on the information on the fields (and their derivatives) that one has to specify on Σ so that there is a unique solution to EL equations on $\Sigma \times [0, \epsilon]$ for ϵ small enough (possibly, one might have to work with a formal neighborhood of 0 like in the example of subsection 2.2).
- (2) A one-form α_Σ on C_Σ arising from the $\Sigma \times \{0\}$ -boundary contribution to the variation of $S_{\Sigma \times [0, \epsilon]}$.

One can see that the Lagrangian is regular iff $\omega_\Sigma := d\alpha_\Sigma$ is non degenerate.

Using these data, one can further develop the induced structure. Namely, for every M in the class we now have a surjective submersion

$$\pi_M: F_M \rightarrow C_{\partial M}$$

and the variation of the action leads to the fundamental equation

$$(3.1) \quad \delta S_M = EL_M + \pi_M^* \alpha_{\partial M}.$$

Again we define $L_M := \pi_M(EL_M) \subset C_{\partial M}$. It follows from (3.1) that L_M is isotropic (i.e., the restriction of $\omega_{\partial M}$ vanishes). In most examples L_M is actually Lagrangian.

³For a precise definition, see, e.g., [9]

Remark 3.1. From now on for simplicity we are going to assume that all F_M and C_Σ can be given a manifold structure and that EL_M and L_M are smooth submanifolds and, apart from the following counterexample, we are always going to assume that L_M is Lagrangian.

Example 3.2 (L_M is not Lagrangian). Consider a one-dimensional example with target \mathbb{R}^2 , space of fields $F_{[t_0, t_1]} = (\mathbb{R}^2)^{[t_0, t_1]}$, with Lagrangian $L \in C^\infty(T\mathbb{R}^2)$ given by $L(v_x, v_y, x, y) = \frac{1}{2}y v_x^2$ and with the action $S_{[t_0, t_1]} = \int_{t_0}^{t_1} \frac{1}{2}y(t) \dot{x}(t)^2 dt$. This example is related to the one given in [16](section 1.2.2) where it is used to disprove a related conjecture by Dirac. Note also that this is a 1-dimensional version of the Polyakov string action.

The EL equations are $\dot{x}^2 = 0$ and $\frac{d}{dt}(y \dot{x}) = 0$, the latter being trivially implied from the first. The x -component of a solution is then completely determined by its initial value, whereas the y -component is completely free. To get formal uniqueness of solutions to the EL equations define $C = \mathbb{R} \times \mathbb{R}^\infty$. Here the second factor contains the information about y and all its derivatives at the initial time. The variation of the action is $\int_{t_0}^{t_1} (\frac{1}{2}\delta y \dot{x}^2 + y \dot{x} \delta \dot{x}) dt$. However the boundary term here is absent because we have to assume $\dot{x} = 0$ on the boundary in order to have a solution and therefore $\alpha = 0$. The projection $\pi_{[t_0, t_1]}: F_{[t_0, t_1]} \rightarrow C \times C$ is then simply given by

$$\pi(x(\cdot), y(\cdot)) = (x(t_0), y(t_0), \dot{y}(t_0), \ddot{y}(t_0), \dots, x(t_1), y(t_1), \dot{y}(t_1), \ddot{y}(t_1), \dots)$$

so that $L_{[t_0, t_1]} = \{(x, y_0, y_1, y_2, \dots, x, \tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \dots), x, y_i, \tilde{y}_i \in \mathbb{R} \forall i\}$. Now $L_{[t_0, t_1]}$ is obviously isotropic since $\omega = 0$. On the other hand, since $\omega = 0$, $L_{[t_0, t_1]}^\perp = C \neq L$. Hence $L_{[t_0, t_1]}$ is not Lagrangian.

As in subsection 2.3, we can decide to split the boundary of a given d -manifold M into incoming and outgoing boundary components, $\partial M = \partial M_{\text{in}} \sqcup \partial M_{\text{out}}$, and regard L_M as a canonical relation from $C_{\partial M_{\text{in}}^{\text{opp}}}$ to $C_{\partial M_{\text{out}}}$, which we will call the evolution relation since it generalizes the evolution flow.

Suppose we cut a manifold M along a submanifold Σ into two manifolds M_1 and M_2 in such a way that $\partial_{\text{in}} M \subset \partial M_1$ and $\partial_{\text{out}} M \subset \partial M_2$. Then we set $\partial_{\text{in}} M_1 = \partial_{\text{in}} M$, $\partial_{\text{out}} M_1 = \Sigma$, $\partial_{\text{in}} M_2 = \Sigma^{\text{opp}}$ and $\partial_{\text{out}} M_2 = \partial_{\text{out}} M$. We then have

$$L_M = L_{M_2} \circ L_{M_1}$$

since a solution on M corresponds to solutions on M_1 and M_2 that match on Σ . This composition of canonical relations replaces the usual composition of flows.

In particular, in the case of a cylinder $\Sigma \times [t_0, t_1]$ we have

$$L_{\Sigma \times [t_0, t_1]} = L_{\Sigma \times [t, t_1]} \circ L_{\Sigma \times [t_0, t]}$$

for all $t \in (t_0, t_1)$. On the other hand, in general we cannot expect $L_\Sigma := \lim_{t_1 \rightarrow t_0} L_{\Sigma \times [t_0, t_1]}$ to be the graph of the identity on C_Σ as this will happen only in regular theories. We call gauge theories those for which L_Σ , which can be seen to be an equivalence relation, is not the graph of the identity.

Remark 3.3 (Evolution correspondences). Recall that the Cauchy space C_Σ determines uniqueness only locally (or just formally, like in example 3.2 or in remark 2.3) on cylinders. Therefore, we may miss some important information by looking only at L_M . The information about nonuniqueness is contained in the fibers of $\pi_M: EL_M \rightarrow L_M$. Notice that by cutting M along some Σ as above we get $EL_M = EL_{M_2} \times_{C_\Sigma} EL_{M_1}$, so we may also interpret EL_M as a canonical correspondence, the evolution correspondence.

It is tempting to think in terms of gluing manifolds along boundary components instead of cutting them (even though this might require some extra pieces of data, like collars, or to work up to homotopy as in [18]). From this perspective we could think of Lagrangian field theories as inducing a functor from the cobordism category (of manifolds with appropriate structure) to the extended presymplectic category. We may think of this as the classical version of the Segal–Atiyah [25, 3] axioms for quantum field theory.

Remark 3.4. Recall that a closed two-form on a manifold P is called presymplectic if its kernel is a subbundle of TP (in finitely many dimensions this is equivalent to requiring ω to have constant rank). There is no reason why the two-forms ω_Σ we obtained above should be presymplectic, but this is a fundamental requirement for making sense of the rest of this program. This requirement puts some constraints on the theories one can write down.

Remark 3.5. In the above description the two-form is always exact. On the other hand, physical examples with non exact symplectic forms abound. One source for them is reduction, see the next subsection, others arise from dropping the restrictive condition that the action is a (well defined) function. More generally, one should think of the action S , or rather of the Gibbs weight $\exp \frac{i}{\hbar} S$, as the section of a line bundle over F_M . In this more general setting, α_Σ/\hbar is no longer a one-form on C_Σ , but a connection one-form on a line bundle.

A simple example where this occurs is that of a charged particle moving in a magnetic field on a manifold N . The action contains the term $\int_{t_0}^{t_1} A_i(x(t)) \dot{x}^i(t) dt$, where $A = A_i dx^i$ is the vector potential regarded as a one-form. This term is also equal to $\int_\gamma A$, where γ is the image of the path. If we make a gauge transformation, the action then changes by boundary terms. Such action is not a function on the space of paths when A is a connection on some nontrivial line bundle E over N . Using the evaluation map at the endpoints, we can pullback this line bundle to $F_{[t_0, t_1]}$. Namely, we define $E_{[t_0, t_1]} = \text{ev}_{t_0}^* E^* \otimes \text{ev}_{t_1}^* E$. We can then see the Gibbs weight as a section of $(E_{[t_0, t_1]})^{\otimes k}$ (where $k = 1/\hbar$ is an integer). The boundary one-form α has the term A as a contribution from the magnetic term in the action and therefore α/\hbar is defined only as a connection on $p^* E^{\otimes k}$, where p is the projection from $C = TN$ to N . The symplectic form ω is the canonical one for a particle on N plus the curvature of A .

Another example is the WZW model, as discussed in [12] [14]. In this paper, for simplicity, we will assume that the action is defined as a function.

3.1. Reduction. If the two-form ω_Σ is degenerate, one may perform reduction by its kernel. If the leaf space \underline{C}_Σ is smooth it inherits a symplectic structure $\underline{\omega}_\Sigma$. We may also project an evolution relation L_M to the reduction and denote this by \underline{L}_M . If it is a smooth submanifold of \underline{C}_Σ , it is automatically isotropic. Actually, in all the examples at hand it is Lagrangian. (In example 3.2, this is trivial since the reduction of C is zero-dimensional.) Finally, notice that at the reduced level $\lim_{t_1 \rightarrow t_0} \underline{L}_{\Sigma \times [t_0, t_1]}$ is the graph of the identity on \underline{C}_Σ . In TFTs, this is so even without taking the limit.

Remark 3.6. In general we cannot expect α_Σ to be horizontal with respect to the kernel of ω_Σ , even though this happens in most examples discussed in this paper (with the notable exception of subsection 2.2). If this is not the case, we should

regard α_Σ as a connection one-form on the trivial line bundle $E_\Sigma := C_\Sigma \times \mathbb{C}$. Since, by definition, the restriction of this line bundle to each leaf is flat, we may reduce to a line bundle with connection $(\underline{E}_\Sigma, \underline{\alpha}_\Sigma)$ if the holonomy of α_Σ is trivial on each leaf. Equivalently, we may think of α_Σ as a contact form on the total space of E_Σ . Its reduction, if smooth, will be a contact manifold \underline{E}_Σ . Under the same conditions as above it will be the total space of a line bundle with connection over \underline{C}_Σ .

Remark 3.7. In general the reduced space \underline{C}_Σ is singular and we want to avoid reduction. We will see in the next Sections how to give a good cohomological replacement for it. However, some partial reduction is very often possible and useful. See for example Remark 2.3. We will see several other examples in the following.

3.2. Axiomatization. By the above discussion we see that a Lagrangian field theory in d dimensions induces the following "categorical" description:

- The source category is a category of cobordisms: objects are $(d - 1)$ -manifolds and morphisms are d -manifolds with boundary. Depending on the theory there might be restriction or additional data (e.g., a metric). Composition of morphisms is given by gluing along boundary components; one way to make sense of this consists in putting a choice of collar of the boundary in the additional data.
- The target "category" has (usually infinite-dimensional) presymplectic manifolds as its objects and correspondences with Lagrangian (or just isotropic) image as morphisms.

A few comments are in order.

- (1) What is actually important is not really gluing manifolds along common boundary, but cutting manifolds along submanifolds. This structure is more relevant than the categorical structure and much less problematic.
- (2) In the case of a regular field theory, the dynamics of the problem on the space time M may be recovered by choosing boundary conditions—viz., the choice of a submanifold L of the symplectic manifold $C_{\partial M}$ —and take the fiber of the evolution correspondence EL_M over the intersection points between L and the evolution relation L_M as the space of solutions for these boundary conditions. We might require for a field theory to be good that these fibers should be generically finite dimensional; we call elements of these fibers the (classical) vacua of the theory.

In order for the variational problem to be well-defined, we have to avoid boundary terms and, as a consequence, require L to be such that the restriction of $\alpha_{\partial M}$ to it vanishes. This in particular implies that L must be isotropic. In order to have, generically, solutions, we should also require L to be maximally isotropic, i.e., Lagrangian.

- (3) In a non regular theory, boundary conditions are also given by the choice of a Lagrangian submanifold L on which the one-form α vanishes. In the reduced theory, one considers the intersection between the reduction of L and that of the evolution relation L_M and the fibers over them. In addition, one also has to consider a reduction of these fibers. This is an additional piece of data (not contained in the Lagrangian function defining the field theory). A more refined definition of the target category would then require endowing the evolution correspondences EL_M with an integrable distribution—the

gauge symmetries of the theory—with the consistency requirement that its image in the evolution relation L_M should coincide with the restriction of the characteristic distribution of the presymplectic manifold.

- (4) The above setting might be too rigid as the one-form $\alpha_{\partial M}$ might not restrict to zero on the Lagrangian submanifold L one wishes to consider. To add more flexibility, one can allow changing $\alpha_{\partial M}$ by an exact term df . By consistency with (3.1) we see that in the Lagrangian field theory we started with we have to change the action by $\pi_M^* f$. To preserve locality we might want f itself to be a local functional. By using Stokes Theorem, we may also write this as a bulk term. The original Lagrangian is changed by a total derivative (which is hence invisible on a manifold without boundary).
- (5) In the target “category,” we might also work in the more refined setting where objects are endowed with a line bundle with connection whose curvature is the presymplectic form (a prequantization bundle). The shift of $\alpha_{\partial M}$ by an exact form in the previous comment should now be replaced by a gauge transformation for the connection one-form. In addition, we may take care of the Hamilton–Jacobi action as a covariantly closed section of the pullback of the flat line bundle from the evolution relation to the evolution correspondence. If the fibers of the correspondence over the relation are connected, this defines a section of the flat line bundle over the relation. In many relevant examples in addition the line bundle over the presymplectic manifold is trivial; in these cases, the presymplectic form is exact and the Hamilton–Jacobi action is a function.
- (6) It might also make sense to allow for singular presymplectic manifolds or for singular relations/correspondences.

3.3. Perturbative quantization. The perturbative functional integral may be extended in the presence of boundary.⁴ Assume first that the theory is regular. For simplicity, assume that the symplectic manifold $C_{\partial M}$ is endowed with a Lagrangian foliation along which $\alpha_{\partial M}$ vanishes and with a smooth leaf space $B_{\partial M}$.⁵ Denote by $p_{\partial M}$ the projection $C_{\partial M} \rightarrow B_{\partial M}$. We then define the boundary vector space $H_{\partial M}$ as the space of functions on $B_{\partial M}$ and the state ψ_M associated to the bulk M as

$$\psi_M(\phi) = \int_{\Phi \in \pi_M^{-1}(p_{\partial M}^{-1}(\phi))} e^{\frac{i}{\hbar} S_M(\Phi)} [D\Phi].$$

The integral is defined by the formal saddle point approximation around critical points. As explained in the previous subsection, we may allow for some finite-dimensional degeneracy. In this case, we should think of ψ_M as a function on the total space of the bundle of vacua over $B_{\partial M}$. If this makes sense, one could also eventually perform the remaining finite-dimensional fiber integration.

In non-regular theories we have two (related) problems. The first is that on the boundary space of fields we only have a presymplectic structure. The second is that the critical points are degenerate (with infinite-dimensional fibers). To approach this problem we have to require as an additional piece of data the choice of gauge symmetries. The idea is that the situation reduces to the regular one if we mod

⁴What we call here perturbative perhaps should be called semiclassical. Strictly speaking the perturbative expansion would be taking a formal power series expansion in coupling constants of the action.

⁵A more general setting would require a discussion of geometric quantization.

out by gauge symmetries in the bulk and by the characteristic foliation on the boundary. However, this usually leads to singular spaces and, even when it is not the case, one should make sense of the functional integral on the quotient. The way out is to replace reduction by its cohomological version. When $\partial M = \emptyset$ this goes under the name of BV formalisms [5] and it is known as BFV formalism [4] in the case of boundary reduction by the characteristic foliation. The goal of this note (and of [7]) is to show how the two formalisms fit together in a consistent way.

3.4. An alternative approach. Instead of introducing the space C_Σ of Cauchy data directly, one can “derive” it from the following construction which is somehow more natural and better fitted to the BFV formalism which will be discussed in Section 4. The space of Cauchy data obtained in this way may not coincide with the one introduced previously, but the two constructions agree after reduction.

The main idea is to associate to a $(d - 1)$ -manifold Σ the space \tilde{F}_Σ of germs of fields at $\Sigma \times \{0\}$ on $\Sigma \times [0, \epsilon]$. We will call it the space of preboundary fields. The boundary term in the variational calculus yields, as above, a one-form $\tilde{\alpha}_\Sigma$ on \tilde{F}_Σ and the fundamental equation (3.1) now reads

$$(3.2) \quad \delta S_M = EL_M + \tilde{\pi}_M^* \tilde{\alpha}_{\partial M},$$

where $\tilde{\pi}_M$ is the natural surjective submersion from F_M to $\tilde{F}_{\partial M}$.

We then introduce $\tilde{\omega}_\Sigma := d\tilde{\alpha}_\Sigma$. This two-form will have a huge kernel but is assumed to be presymplectic. We denote by $(F_\Sigma^\partial, \omega_\Sigma^\partial)$ the reduced space, which we will call the space of boundary fields. For the rest of the discussion, we are going to assume F_Σ^∂ is a smooth manifold. In general, we do not require $\tilde{\alpha}_\Sigma$ to be basic. If the trivial line bundle with connection $\tilde{\alpha}_\Sigma$ on \tilde{F}_Σ may be reduced to a smooth line bundle on F_Σ^∂ , we will denote by α_Σ^∂ the induced connection one-form.

For simplicity, we are now going to assume that $\tilde{\alpha}_\Sigma$ is indeed horizontal, so α_Σ^∂ is a one-form on F_Σ^∂ , and leave the general case to the reader. If we denote by π_M the composition of $\tilde{\pi}_M$ with the natural projection from $\tilde{F}_{\partial M}$ to $F_{\partial M}^\partial$, we get

$$\delta S_M = EL_M + \pi_M^* \alpha_{\partial M}^\partial.$$

Out of it we get that $L_M := \pi_M(EL_M)$ is isotropic. We finally define the (new version of the) space of Cauchy data C_Σ as the space of points of F_Σ^∂ that can be completed to a pair belonging to $L_{\Sigma \times [0, \epsilon]}$ for some ϵ . Notice that we think of F_Σ^∂ as a relation from the one-point manifold to F_Σ^∂ and of $L_{\Sigma \times [0, \epsilon]}$ as a relation from F_Σ^∂ to itself, we may write

$$C_\Sigma = \bigcup_{\epsilon \in (0, +\infty)} L_{\Sigma \times [0, \epsilon]} \circ F_\Sigma^\partial.$$

Notice that F_Σ^∂ is coisotropic in itself. If $\forall \epsilon$ we assume $L_{\Sigma \times [0, \epsilon]}$ also to be so—and hence to be Lagrangian—, then each composition is coisotropic (up to some infinite-dimensional subtleties), and so will be the union. However, it may happen that C_Σ is coisotropic even if the L s are not Lagrangian.

3.4.1. Examples.

Example 3.8. Consider a nondegenerate Lagrangian function on TN as at the beginning of Section 2. The space \tilde{F}_{pt} is just the infinite jet bundle over N . The one-form is given in (2.2). The kernel of the corresponding two-form consists of all

jets higher than the first, so $F_{pt}^\partial = TN$. Since every point in it can be completed to a pair in $L_{[0,\epsilon]}$, for ϵ small enough, we recover $C_{pt} = TN$.

Example 3.9. In the case discussed in subsection 2.2, \tilde{F}_{pt} is the open submanifold in the infinite jet bundle over \mathbb{R}^2 obtained by requiring the first jet to be different from zero. The reduced space F_{pt}^∂ is TS^1 with symplectic form obtained by pullback from the cotangent bundle using the metric and L_I is the graph of the identity for every interval I ; hence, $C_{pt} = F_{pt}^\partial = TS^1$. Notice that in this example the space of Cauchy data given by this construction is different from the previous one, though their reductions are obviously the same. Moreover, in this example the one-form $\tilde{\alpha}_{pt}$ is not basic. The induced one-form connection α_{pt}^∂ is the one discussed in remark 2.1.

Example 3.10. We now work out the new description of example 3.2. Here \tilde{F}_{pt} is the infinite jet bundle over \mathbb{R}^2 and $\tilde{\alpha}_{pt} = yv_x dx$. The kernel of the two-form is given by all jets higher than the first for x , by all jets higher than the zero jet for y , and by $X := y\frac{\partial}{\partial y} - v_x\frac{\partial}{\partial v_x}$. So in this case the form is not presymplectic. To solve this problem we assume $(v_x, y) \neq (0, 0)$. This means that the original space of fields F_I has to be defined as paths in \mathbb{R}^2 that can hit the x -axis only with non zero x -velocity. The reduction is then $F_{pt}^\partial = \mathbb{R}^2$. If we denote by (p, q) its coordinates we have $\alpha_{pt}^\partial = pdq$. Moreover, $\pi_{[t_0, t_1]}(x(\cdot), y(\cdot)) = (y(t_0)\dot{x}(t_0), x(t_0), y(t_1)\dot{x}(t_1), x(t_1))$. Since EL_I consists of paths that are constant in the x -direction, we get $L_{[t_0, t_1]} = \{(0, q, 0, q), q \in \mathbb{R}\}$ which is clearly not Lagrangian. On the other hand, we have $C_{pt} = \{(0, q), q \in \mathbb{R}\}$ which is coisotropic.

Example 3.11 (Electrodynamics). We now discuss the case of electrodynamics (we leave to the reader the generalization to nonabelian Yang–Mills theory). The space of fields on a manifold M is the space \mathcal{A}_M of connection one-forms for a fixed line bundle over M . The action is $S_M(A) = (dA, dA) = \int_M dA \wedge *dA$, where (\cdot, \cdot) is the Hodge pairing of forms for a fixed metric on M and $*$ is the Hodge $*$ -operation. The EL equations are $d^*dA = 0$, where d^* is the formal adjoint of d with respect to the Hodge pairing.

For simplicity of exposition, we now reformulate electrodynamics in the first-order formalism (and leave to the reader its study in the usual second-order formalism). Namely, we enlarge the space of fields to $F_M := \mathcal{A}_M \times \Omega^{d-2}(M)$ and extend the action to

$$S_M(A, B) = \int_M B \wedge dA + \frac{1}{2}B \wedge *B$$

The EL equations are $dA + *B = 0$ and $dB = 0$. Hence, B is completely determined by A and then A must satisfy $d^*dA = 0$.

On the space of preboundary fields \tilde{F}_Σ we have the one-form $\tilde{\alpha}_\Sigma = \int_\Sigma B \wedge \delta A$. It then follows that the space of boundary fields is $F_\Sigma^\partial = \mathcal{A}_\Sigma \times \Omega^{d-2}(\Sigma)$ with one-form $\alpha_\Sigma^\partial = \int_\Sigma B \delta A$ and symplectic form $\omega_\Sigma^\partial = d\alpha_\Sigma^\partial$.

Let us now consider $L_{\Sigma \times [0, \epsilon]}$. The equation $dB = 0$ restricts to the boundary, so it has to be satisfied by a field in $L_{\Sigma \times [0, \epsilon]}$. We will call the direction along $[0, \epsilon]$ vertical. The evolution will impose some other conditions, but we claim that $C_\Sigma = \mathcal{A}_\Sigma \times \Omega_{closed}^{d-2}(\Sigma)$, namely that no other conditions have to be imposed on the first boundary component. The reason is that A on the first boundary can always be extended to a solution. In the axial gauge (i.e., when we require that the vertical component of A vanishes), the solution is unique once we specify the

vertical derivative of A at the first boundary. But this first derivative may be chosen, actually uniquely, so as to yield the given closed B on the boundary.

Notice that B —despite the notation—is the electric field on the boundary (or, better, the $(d - 1)$ -form corresponding to the electric vector field using the metric) and the equation $dB = 0$ is just the Gauss law (i.e., divergence of electric field equal to zero). The characteristic distribution on C_Σ consists of just the gauge transformations on A .

Example 3.12 (Abelian BF theories). We may consider the “topological” limit of the first-order formulation of electrodynamics, i.e., drop the term with the Hodge $*$ operator. This way we get the action $S_M = \int_M B \wedge dA$ on the same space of fields $F_M = \mathcal{A}_M \times \Omega^{d-2}(M)$. This theory is called abelian BF theory.

The space of boundary fields is the same as for electrodynamics. What changes are the Lagrangian submanifolds L_M . Since the EL equations are just $dA = 0$ and $dB = 0$, we see that L_M consists closed A and B on the boundary that can be extended to closed A and B in the bulk. As a result $C_\Sigma = \mathcal{A}_\Sigma^{flat} \times \Omega_{closed}^{d-2}(\Sigma)$. The characteristic distribution consists of gauge transformations for A and shifts of B by exact forms.

4. THE BFV FORMALISM

In this Section we address the problem of reformulating the reduction of a presymplectic manifold C cohomologically.

If we work in the setting of subsection 3.4, our presymplectic submanifold is actually given as a coisotropic submanifold of a symplectic manifold. Otherwise, we first recall that Gotay [15] proved that every presymplectic manifold (C, ω_C) may be embedded into a symplectic manifold (F, ω_F) as a coisotropic submanifold such that ω_C is the restriction of ω_F to C . Moreover, such an embedding is unique up to neighborhood equivalence. The existence part is simply proven by taking $F = D^*$, where D is the kernel of ω_C , and $\omega_F = p^*\omega_C + \sigma^*\omega_{can}$, where p is the projection $D^* \rightarrow C$, ω_{can} is the canonical symplectic form on T^*C and σ is a splitting of $T^*C \rightarrow D^*$. Notice that ω_{can} is exact; so an exact presymplectic manifold can be coisotropically embedded into an exact symplectic manifold⁶.

We are then led to consider the problem of how to describe symplectic reduction of a coisotropic submanifold cohomologically. This goes under the name of BFV formalism [4].

We start with the finite dimensional setting. Locally, a coisotropic submanifold C of F can be described as the common zero locus of some differentially independent functions ϕ_i on F . The characteristic foliation is then the span of the Hamiltonian vector fields X_i of the ϕ_i s. The space of functions on the quotient \underline{C} , if it is smooth, is the same as $(C^\infty(F)/\langle \phi_1, \phi_2, \dots \rangle)^{(X_1, X_2, \dots)}$, where $\langle \phi_1, \phi_2, \dots \rangle$ denotes the ideal generated by the ϕ_i s and the exponent denotes taking the subalgebra invariant under all the X_i s. The goal is to describe this space (actually this Poisson algebra) as the zeroth cohomology of a complex (actually the differential graded Poisson algebra of functions on a graded symplectic manifold).

To do this we add to F new odd coordinates b_i of degree -1 and define a vector field Q on the supermanifold so obtained by imposing $Q(f) = 0$ for any

⁶ In this note we will not focus on subtleties of this statement in the infinite dimensional setting.

$f \in C^\infty(F)$ and $Q(b_i) = \phi_i$. The cohomology is concentrated in degree zero and yields $C^\infty(F)/\langle \phi_1, \phi_2, \dots \rangle$. To select the invariant part, we add more odd coordinates c^i of degree +1 and extend Q to the supermanifold so obtained \mathcal{F} by requiring $Q(b_i) = \phi_i$, $Q(c^i) = 0$ and $Q(f) = c^i X_i(f)$. This Q is a Hamiltonian vector field on \mathcal{F} with respect to the symplectic form $\omega_F + db_i dc^i$ and Hamiltonian function $c^i \phi_i$; yet, in general it is not a differential on $C^\infty(\mathcal{F})$. However, using cohomological perturbation theory one can prove [26] that the Hamiltonian function may be deformed in such a way this occurs. The construction may also be globalized [6, 17, 22]. To summarize, we have the

Theorem 4.1. *Let C be a coisotropic submanifold of a finite-dimensional symplectic manifold F . Then one can embed F as the body of a supermanifold \mathcal{F} with an additional \mathbb{Z} -grading endowed with an even symplectic form $\omega_{\mathcal{F}}$ of degree zero and an even function S of degree +1 such that its Hamiltonian vector field Q squares to zero and its cohomology in degree zero is isomorphic as a Poisson algebra to the algebra of functions on C that are invariant under its characteristic distribution. This construction is unique up to symplectomorphisms of \mathcal{F} if one requires it to be minimal (in terms of the newly added coordinates).*

In the case of field theory, the analogous result—with the additional condition that S and $\omega_{\mathcal{F}}$ are local—was proved long ago by [4] (in the description above the index i is now replaced by a worldsheet coordinate and the sum over i by an integral). Notice however that, in order to get S as a local functional, one often has to add extra fields of degree greater than +1 (and consequently extra fields of degree less than −1). In any case, the final result is what we will call a BFV manifold.

Definition 4.2. A BFV manifold is a triple (F, ω, Q) where F is a supermanifold with additional \mathbb{Z} -grading, ω is an even symplectic form of degree zero, and Q is an odd symplectic vector field of degree +1 satisfying $[Q, Q] = 0$.

Remark 4.3. Recall that Q symplectic means $L_Q \omega = 0$. On the other hand the \mathbb{Z} -grading amounts to the existence of an even vector field E of degree zero (the graded Euler vector field) such that the grading on functions, forms and vector fields is given by the eigenvalues of the Lie derivative L_E . We then have $L_E \omega = 0$ and $[E, Q] = Q$. This then implies that Q is automatically Hamiltonian, $\iota_Q \omega = dS$, with $S = \iota_E \iota_Q \omega$ (this remark is due to Roytenberg [21]). Notice that the condition $[Q, Q] = 0$ implies the “classical master equation” (CME) in the BFV formalism:

$$\{S, S\} = 0,$$

where $\{ , \}$ denotes the Poisson bracket induced by ω_F .

The coisotropic submanifold C can also be recovered geometrically. Namely, one defines \mathcal{EL} as the zero locus of Q . More precisely (when EL is singular as is often the case), one considers the ideal $I_{\mathcal{EL}}$ generated by functions of the form $\{S, f\}$ with $f \in C^\infty(\mathcal{F})$. This ideal is clearly a Lie subalgebra (with respect to the Poisson bracket) thanks to the CME. This amounts to saying that \mathcal{EL} is a coisotropic submanifold. The original C is just its body.

Remark 4.4 (Quantization). The (geometric) quantization of F is in this setting replaced by a quantization of \mathcal{F} with a compatible quantization of S . Namely, one has to produce a graded vector space $\mathcal{H}_{\mathcal{F}}$ quantizing $(\mathcal{F}, \omega_{\mathcal{F}})$ together with an

odd operator Ω of degree 1 quantizing S and satisfying $\Omega^2 = 0$. Notice that the CME is the classical counterpart of the last equality and there might be obstruction (“anomalies”) in finding such an Ω . If everything works, however, one can consider the cohomology of Ω . Its degree zero component may be thought of the quantization of the reduction of C .

4.1. BFV as a boundary theory. In Section 3 we saw that a d -dimensional Lagrangian field theory associates a space C_Σ with a closed (often exact) two-form ω_Σ to a $(d - 1)$ -dimensional manifold Σ . It was part of the assumptions that C_Σ is a manifold and that ω_Σ is presymplectic. Following the description above, we now associate to Σ a BFV manifold $(\mathcal{F}_\Sigma^\partial, \omega_\Sigma^\partial, Q_\Sigma^\partial)$. (The upper symbol ∂ is a reminder that this is the boundary construction as in the following we will have a similar construction, with similar notations, for the bulk.) If we work in the settings of subsection 3.4, then we take F_Σ^∂ to be the degree zero part of $\mathcal{F}_\Sigma^\partial$.

An important remark is that other cohomology groups may turn out to be non-trivial. As an example, consider first-order electrodynamics as in example 3.11. Recall that the space of boundary fields on Σ is $\mathcal{A}_\Sigma \times \Omega^{d-2}(\Sigma)$ whereas $C_\Sigma = \mathcal{A}_\Sigma \times \Omega_{cl}^{d-2}(\Sigma)$. To implement the BFV construction we add odd fields $c \in \Omega^0(\Sigma)$ of degree +1 and $b \in \Omega^{d-1}(\Sigma)$ of degree -1 and consider the BFV action $\mathcal{S}_\Sigma = \int_\Sigma cdB$. The Hamiltonian vector field Q acts trivially on c and B . On the other hand $Qb = dB$ and $QA = dc$. So the BFV cohomology yields functions on $H^0(\Sigma)[1] \times \mathcal{A}_\Sigma^{flat}/gauge \times \Omega_{cl}^{d-2}(\Sigma) \times H^{d-1}(\Sigma)[-1]$. The extra factors, in degree 0 and $d - 1$, express the stacky nature of the reduction and become even more important in the nonabelian Yang–Mills case.

5. THE BV FORMALISM FOR MANIFOLDS WITH BOUNDARY

The BV formalism [5] deals with the degeneracy problem for an action in the bulk. In the BV case, as in Sections 2 and 3, we have a d -dimensional Lagrangian field theory, i.e. the assignment of a space of fields F_M and an action $S_M = \int_M L$ over F_M to a d -manifold M . But in addition we have a distribution $D_M \subset TF_M$ on F_M which describe the “symmetries”. This distribution does not have to be given by an action of a Lie group. It is involutive and of finite codimension when restricted to EL_M . The construction aims at cohomologically resolving the quotient of EL_M by the symmetries. Let us sketch the last point assuming at the beginning that our manifold M has no boundary. The BV construction proceeds by

- (1) first extending the space of fields F_M to the supermanifold $D_M[1] \subset T[1]F_M$ (i.e., one assigns degree 1 and the according Grassmann parity to coordinates on fibers of D_M),
- (2) extending the action S_M to a new local functional \mathcal{S}_M on $\mathcal{F}_M := T^*[-1]D_M[1]$ which has the two following properties:
 - (a) It satisfies the classical master equation (CME) $\{\mathcal{S}_M, \mathcal{S}_M\} = 0$, where $\{ , \}$ is the (degree +1) Poisson bracket associated to the canonical symplectic form (of degree -1) on \mathcal{F}_M , and
 - (b) the restriction of D_M to EL_M is the same as the restriction of the characteristic distribution of the coisotropic submanifold \mathcal{EL}_M of critical points of \mathcal{S}_M to its degree zero part.

The solution can be found by cohomological perturbation theory. In order to preserve locality, it is often necessary to extend the above procedure by allowing dependent symmetries and resolving their relations by adding new variables of degree 2 (ghosts for ghosts) and so on. The final result is anyway a supermanifold with odd symplectic form of degree -1 and a solution of the CME.

Remark 5.1. The CME is also the starting point for making sense of the integral of $e^{\frac{1}{\hbar}S_M}$ over F_M . In the saddle point approximation, one expands around critical points, i.e., points of EL_M . If the action is degenerate—namely, its Hessian at a critical point is degenerate—one cannot even begin the perturbative expansion. However, if one quotients by a distribution as above, one saves the game (or at least reduces the problem to a residual finite dimensional integration). This quotient might be very singular; also notice that in general situations the distribution is not even involutive outside of EL_M ; and even if everything worked out properly, it might be difficult to define the perturbative functional integral on the quotient, which might have a much more involved manifold structure. The way out is to extend S_M to a (possibly \hbar -dependent) solution \tilde{S}_M of the quantum master equation (QME) on \mathcal{F}_M . Namely, one picks a Berezinian ρ (formally, since we are working in an infinite dimensional context) on \mathcal{F}_M and defines the BV Laplacian Δ by $\Delta f = \frac{1}{2} \operatorname{div}_\rho X_f$, where X_f is the Hamiltonian vector field of a function f and div_ρ is the divergence operator with respect to ρ . One requires ρ to restrict to the original measure on F_M and to be compatible with the symplectic structure: namely, one requires $\Delta^2 = 0$. The QME then reads $\frac{1}{2}\{\tilde{S}_M, \tilde{S}_M\} - i\hbar\Delta\tilde{S}_M = 0$. The limit of \tilde{S}_M for $\hbar \rightarrow 0$ solves the CME and is taken to be S_M . One actually starts with S_M and tries to extend it to a formal power series in \hbar that solves the QME if there are no obstructions (“anomalies”). A consequence of the QME is that the integral of $e^{\frac{1}{\hbar}S_M}$ on a Lagrangian submanifold is invariant under deformations of the Lagrangian submanifold. One then replaces the originally ill-defined integral over F_M by the integral over a deformation of the Lagrangian submanifold $D_M[1]$ where it is well-defined. We refer to [23] for a good introduction.

Definition 5.2. A BV manifold is a triple given by a supermanifold with additional \mathbb{Z} -grading, an odd symplectic form of degree -1 and a function of degree 0 that satisfies the CME, i.e. Poisson commutes with itself.

We may then formulate the result of the BV construction in d -dimensional Lagrangian field theory as the assignment of a BV manifold $(\mathcal{F}_M, \omega_M, S_M)$ to a d -manifold M . Notice that as a consequence of the CME the Hamiltonian vector field Q_M of S_M ,

$$(5.1) \quad \iota_{Q_M} \omega_M = \delta S_M,$$

is cohomological, i.e., it satisfies $[Q_M, Q_M] = 0$.

5.1. The case with boundary. Now let us allow M to have a nonempty boundary. Since the BV construction is local it still assigns to M a quadruple $(\mathcal{F}_M, \omega_M, S_M, Q_M)$. It is still true that ω_M is symplectic and that Q_M is cohomological. On the other hand, S_M is no longer its Hamiltonian. The problem is that (5.1) involves integration by parts. We may overcome this problem working as in the previous Sections (in particular, subsection 3.4).

Namely, we define the space $\tilde{\mathcal{F}}_\Sigma$ of preboundary fields on a $(d-1)$ -manifold Σ as the germs at $\Sigma \times \{0\}$ of $\mathcal{F}_{\Sigma \times [0, \epsilon]}$. Integration by parts in the computation of

$\delta\mathcal{S}_{\Sigma \times [0, \epsilon]}$ yields a one-form $\tilde{\alpha}_\Sigma$ on $\tilde{\mathcal{F}}_\Sigma$. We denote by $\tilde{\omega}_\Sigma$ its differential—which we assume to be presymplectic—and by $(\mathcal{F}_\Sigma^\partial, \omega_\Sigma^\partial)$ its reduction. We also assume that $\tilde{\alpha}_\Sigma$ reduces to a connection one-form α_Σ^∂ on $\mathcal{F}_\Sigma^\partial$. In most examples, α_Σ^∂ will be an actual one-form.

If we take care of boundary terms, instead of (5.1) we now get

$$(5.2) \quad \iota_{Q_M} \omega_M = \delta\mathcal{S}_M + \tilde{\pi}_M^* \tilde{\alpha}_{\partial M},$$

where $\tilde{\pi}_M$ is the natural surjective submersion from \mathcal{F}_M to $\tilde{\mathcal{F}}_{\partial M}$. If we denote by π_M the composition of $\tilde{\pi}_M$ with the natural surjective submersion from $\tilde{\mathcal{F}}_{\partial M}$ to $\mathcal{F}_{\partial M}^\partial$, we finally get the fundamental equation of the BV theory for manifolds with boundary [7]:

$$(5.3) \quad \iota_{Q_M} \omega_M = \delta\mathcal{S}_M + \pi_M^* \alpha_{\partial M}^\partial,$$

To complete the description of the theory with boundary, we still have to study Q_M . The first obvious remark is that it is $\tilde{\pi}_M$ -projectable. More precisely, for every Σ , there is a uniquely defined vector field \tilde{Q}_Σ (automatically cohomological) on $\tilde{\mathcal{F}}_\Sigma$ such that for every M the vector field Q_M projects to $\tilde{Q}_{\partial M}$: namely, $\tilde{Q}_{\partial M}(\phi) = d_{\hat{\phi}} \tilde{\pi}_M(Q_M(\hat{\phi}))$, $\forall \phi \in \tilde{\mathcal{F}}_{\partial M}$ and $\forall \hat{\phi} \in \tilde{\pi}_M^{-1}(\phi)$.

Let us now differentiate (5.2). Using the fact that ω_M is closed, we get $L_{Q_M} \omega_M = \tilde{\pi}_M^* \tilde{\omega}_{\partial M}$ (which by the way proves that Q_M is not even symplectic). We now apply L_{Q_M} to this equation. Using the fact Q_M is cohomological and projectable, we get $\tilde{\pi}_M^* L_{\tilde{Q}_{\partial M}} \tilde{\omega}_{\partial M} = 0$. Since $\tilde{\pi}_M$ is a surjective submersion, we conclude that $L_{\tilde{Q}_{\partial M}} \tilde{\omega}_{\partial M} = 0$.

Actually, this proves that, for every Σ , $\tilde{\omega}_\Sigma$ is \tilde{Q}_Σ -invariant. This implies that \tilde{Q}_Σ is projectable to the reduction. To show this, we have just to check that $[\tilde{Q}_\Sigma, X]$ belongs to the kernel of $\tilde{\omega}_\Sigma$ for every X in the kernel. This follows from the identities $\iota_{[\tilde{Q}_\Sigma, X]} \tilde{\omega}_\Sigma = [L_{\tilde{Q}_\Sigma}, \iota_X] \tilde{\omega}_\Sigma = 0$.

We conclude that, for every Σ , there is a uniquely defined vector field Q_Σ^∂ on $\mathcal{F}_\Sigma^\partial$ (automatically cohomological and symplectic) to which \tilde{Q}_Σ projects. This has two fundamental consequences:

- (1) To each $(d-1)$ -dimensional manifold Σ , we now associate a BFV manifold $(\mathcal{F}_\Sigma^\partial, \omega_\Sigma^\partial, Q_\Sigma^\partial)$, see definition 4.2.
- (2) For each d -manifold M , Q_M π_M -projects to Q_M^∂ .

These two final observations together with (5.3) constitute the framework of the BV formalism extended to manifolds with boundaries [7], which we call the BV-BFV formalism.

Definition 5.3. We define a BV-BFV manifold over a given exact BFV manifold $(\mathcal{F}^\partial, \omega^\partial = d\alpha^\partial, Q^\partial)$ as a quintuple $(\mathcal{F}, \omega, \mathcal{S}, Q, \pi)$ where \mathcal{F} is a supermanifold with additional \mathbb{Z} -grading, ω is an odd symplectic form of degree -1 , \mathcal{S} is an even function of degree 0, Q is a cohomological vector field and $\pi: \mathcal{F} \rightarrow \mathcal{F}^\partial$ is a surjective submersion such that

- (1) $\iota_Q \omega = d\mathcal{S} + \pi^* \alpha^\partial$,
- (2) $Q^\partial = d\pi Q$.

The definition may be extended to BV-BFV manifolds over a BFV manifold with connection α^∂ . This requires introducing a line bundle over \mathcal{F}^∂ and viewing $\exp(\frac{i}{\hbar} \mathcal{S})$ as a section of the pulled-back bundle.

Remark 5.4 (Axiomatization). We may now reformulate Lagrangian field theories axiomatically as a "functor" from some "cobordisms category" to a category where the objects are BFV manifold with connection and the morphisms are BV-BFV manifolds over the Cartesian product of the objects.

Remark 5.5. Notice that using this method the BFV construction associated to the boundaries is obtained from the BV construction in the bulk and does not have to be done independently. Also recall that, by general principles [21], Q_Σ^∂ is Hamiltonian with a uniquely defined odd Hamiltonian function S_Σ^∂ of degree +1. This yields as a consequence the following generalization of the CME

$$(5.4) \quad Q_M(\mathcal{S}_M) = \pi_M^*(2\mathcal{S}_{\partial M}^\partial - \iota_{Q_{\partial M}^\partial} \alpha_{\partial M}^\partial),$$

which can be proved as follows. First differentiate (5.3) to obtain $L_{Q_M} \omega_M = \pi_M^* \omega_{\partial M}^\partial$. Then apply ι_{Q_M} to (5.3) and use the obtained equation and the fact that Q_M is cohomological and projects to $Q_{\partial M}^\partial$ to obtain the differential of (5.4). Then observe that the differential of a function of degree 1 vanishes if and only if the function itself vanishes (we have no constants in degree +1).

Example 5.6 (First-order electrodynamics). We return to example 3.11 (first-order YM is explained in details in [7]; we leave the usual second-order formulation as an exercise to the reader). Since we want to implement gauge transformations for A , we define $D_M[1]$ by adding the "ghost field" $c \in \Omega^0(M)$ which is odd and of degree +1. Gauge transformations are given by the vector field dc in the A -direction (here d denotes the de Rham differential on M). The BV space of fields $\mathcal{F}_M = T^*[-1]D_M[1]$ is then

$$\Omega^0(M)[1] \times \mathcal{A}_M \times \Omega^2(M)[-1] \times \Omega^{d-2}(M) \times \Omega^{d-1}(M)[-1] \times \Omega^d(M)[-2].$$

We add a superscript $+$ to denote the canonically conjugate coordinates (a.k.a. antifields) to the fields: namely, $B^+ \in \Omega^2(M)[-1]$, $A^+ \in \Omega^{d-1}(M)[-1]$, $c^+ \in \Omega^d(M)[-2]$. The BV action is just

$$\mathcal{S}_M = \int_M B \wedge dA + \frac{1}{2} B \wedge *B + A^+ \wedge dc.$$

The cohomological vector field Q_M acts as follows (we omit the terms where the action is zero):

$$Q_M A = dc, \quad Q_M A^+ = dB, \quad Q_M B^+ = *B + dA, \quad Q_M c^+ = dA^+.$$

On the space of preboundary fields on a $(d-1)$ -manifold Σ , we get $\tilde{\omega}_\Sigma = \int_\Sigma B \wedge \delta A + A^+ \delta c$. We immediately see that the kernel of $\tilde{\omega}_\Sigma$ consists of all jets of B^+ and c^+ and all jets higher than the zeroth of A, B, A^+, c . Moreover, $\tilde{\omega}_\Sigma$ is also basic. We then get

$$\mathcal{F}_\Sigma^\partial = \Omega^0(\Sigma)[1] \times \mathcal{A}_\Sigma \times \Omega^{d-2}(\Sigma) \times \Omega^{d-1}(\Sigma)[-1].$$

Projecting the cohomological vector field, we get the cohomological vector field Q_Σ^∂ , which acts by $Q^\partial A^+ = dB$ and $Q^\partial A = dc$ and has Hamiltonian function $S_\Sigma^\partial = \int_\Sigma c dB$.

5.2. EL correspondences. We now define the space \mathcal{EL}_M as the space of zeros of Q_M and $\mathcal{L}_{\partial M}$ as its image under π_M . This generalizes the classical story of evolution correspondences and evolution relations. Notice that (5.3) implies that $\mathcal{L}_{\partial M}$ is isotropic, and we are going to assume that it is actually Lagrangian. There are two problems to be tackled though: the first is that \mathcal{EL}_M is not smooth in general, the second is that we are usually interested in reduction, which is even more singular in general.

One way to avoid the first problem is by working with an algebraic description only, but we will often pretend that we are dealing with smooth manifolds. Namely, instead of \mathcal{EL}_M we consider its vanishing ideal $I_{\mathcal{EL}_M}$, i.e., the ideal generated by functions of the form $Q_M f$, with $f \in C^\infty(\mathcal{F}_M)$. This ideal is a Lie algebra with respect to the Poisson bracket, which amounts to saying that \mathcal{EL}_M is coisotropic. If M has no boundary, this is obvious since Q_M is symplectic and squares to zero. If M has a boundary, this is still true since, to generate it, it is enough to consider functions f that “vanish near the boundary” (namely, functions in $\pi_{M,U}^* \mathcal{F}_U$ where U is compact in the interior of M and $\pi_{M,U}$ is the restriction map). The characteristic distribution \mathcal{D}_M is generated by the Hamiltonian vector fields of functions of the form Qf . If f is as above, we have $Qf = \{S, f\}$ and hence the characteristic distribution is generated by vector fields of the form $[Q, X]$ where X is a Hamiltonian vector field vanishing near the boundary (we assume here that components of Q on EL are differentiably independent). The reduction $\underline{\mathcal{EL}}_M$ of \mathcal{EL}_M by \mathcal{D}_M carries again a symplectic form of degree -1 (if it is singular, this has to make sense of; e.g., by considering the open smooth locus or using the language of derived algebraic geometry [27]).

If M has a boundary, it makes sense to consider another reduction, namely by the distribution $\mathcal{D}_M^Q \supset \mathcal{D}_M$ generated by vector fields of the form $[Q, X]$ where X is Hamiltonian (but with no vanishing condition). More precisely, observe that, since Q_M projects to $Q_{\partial M}^\partial$, we have that $\mathcal{L}_{\partial M}$ is contained in $\mathcal{EL}_{\partial M}^\partial$, the space of zeros of $Q_{\partial M}^\partial$, which is also coisotropic. Hence its characteristic distribution, generated by vector fields of the form $[Q_{\partial M}^\partial, X]$ with X Hamiltonian, is tangent to $\mathcal{L}_{\partial M}$. Now let ℓ be a point in $\mathcal{L}_{\partial M}$ and let $[\ell]$ denote its orbit. Then \mathcal{D}_M^Q restricts to $\pi_M^{-1}([\ell]) \cap \mathcal{EL}_M$ and we denote by $\underline{\mathcal{E}}_{[\ell]}$ its quotient. The union of the $\underline{\mathcal{E}}_{[\ell]}$ s over $[\ell] \in \mathcal{L}_{\partial M}$ is the quotient $\underline{\mathcal{EL}}_{M,Q}$ of \mathcal{EL}_M by \mathcal{D}_M^Q , which is by itself a quotient of the symplectic reduction $\underline{\mathcal{EL}}_M$.

In [7] it is shown that each fiber $\underline{\mathcal{E}}_{[\ell]}$ carries a symplectic form of degree -1 . This follows from a different but equivalent description of this quotient. Namely, pick a Lagrangian submanifold \mathcal{L} of $\mathcal{F}_{\partial M}^\partial$ transversal to $\mathcal{L}_{\partial M}$ at ℓ . Then one shows that $\pi_M^{-1}(\ell) \cap \mathcal{EL}_M$ is coisotropic in $\pi_M^{-1}(\mathcal{L})$ and that its reduction is $\underline{\mathcal{E}}_{[\ell]}$.

We call $\underline{\mathcal{E}}_{[\ell]}$ the moduli space of vacua at $[\ell]$ and we assume it to be finite dimensional (if this is not the case, it means that we have not considered enough symmetries).

Notice that there is in principle a second (usually coisotropic) submanifold \mathcal{C}'_Σ of \mathcal{F}_Σ . Namely, the elements of \mathcal{F}_Σ are zeroes of Q'_Σ which can be extended as zeroes of $Q_{\Sigma \times [0, \epsilon]}$ for some $\epsilon > 0$. It is meaningful to require $\mathcal{C}'_\Sigma = \mathcal{C}_\Sigma$. Otherwise, it again means that we have not taken enough symmetries into account.

Remark 5.7 (Axiomatization). If reduction were always nice, we could get the following induced axiomatization of a d -dimensional Lagrangian field theory in the

BV-BFV formalism. To a $(d - 1)$ -manifold Σ we associate a symplectic supermanifold $\mathcal{E}\mathcal{L}_{\partial M}^\partial$ and to a d -manifold M we associate the “evolution correspondence” $\mathcal{E}\mathcal{L}_{M_Q} \rightarrow \mathcal{E}\mathcal{L}_\Sigma^\partial$ which has a Lagrangian image and whose fibers are finite dimensional symplectic manifolds in degree -1 . If we cut a manifold M along a submanifold Σ , we may try to recover $\mathcal{E}\mathcal{L}_M$ out of the composition of the evolution correspondences, and some more data, for the two halves. This problem started to be addressed in [7].

Example 5.8 (Electrodynamics—continued). In [7], to which we refer for details, it is shown that, in the case of first-order electrodynamics, for any ℓ we have

$$\mathcal{E}_{[\ell]} \simeq H^1(M, \partial M) \oplus H^{n-1}(M)[-1] \oplus H^0(M, \partial M)[1] \oplus H^n(M)[-2],$$

which is indeed finite-dimensional. (Here $H^*(M, \partial M)$ denotes cohomology relative to the boundary.)

5.3. Extended theories. The construction in subsection 5.1 may be applied iteratively to go to lower and lower dimension. Namely, there we have obtained a BFV structure $(\mathcal{F}_\Sigma^\partial, \omega_\Sigma^\partial, Q_\Sigma^\partial)$, to which we canonically associate a function $\mathcal{S}_\Sigma^\partial$, for every $(d - 1)$ -dimensional manifold Σ without boundary; yet, since the construction is local, we can use these data on a $(d - 1)$ -dimensional manifold Σ with boundary. Again what is not going to work is the condition that $\mathcal{S}_\Sigma^\partial$ is the Hamiltonian function of Q_Σ^∂ . We correct this equation using the induced one-form on the space of preboundary fields on $\partial\Sigma$ and reduce by the kernel of the two form. Since $\mathcal{S}_\Sigma^\partial$ has degree 1, this will also be the degree of the induced symplectic form.

As a result, to a $(d - 2)$ -manifold γ we associate a triple $(\mathcal{F}_\gamma^{\partial\partial}, \omega_\gamma^{\partial\partial} = d\alpha_\gamma^{\partial\partial}, Q_\gamma^{\partial\partial})$, where $\omega_\gamma^{\partial\partial}$ is an odd symplectic form of degree $+1$ and $Q_\gamma^{\partial\partial}$ is a cohomological, symplectic vector field (hence automatically Hamiltonian with a uniquely defined even Hamiltonian function $\mathcal{S}_\gamma^{\partial\partial}$ of degree $+2$). To a $(d - 1)$ -manifold Σ with boundary we now associate a quintuple $(\mathcal{F}_\Sigma^\partial, \omega_\Sigma^\partial, Q_\Sigma^\partial, \mathcal{S}_\Sigma^\partial, \pi_\Sigma^\partial)$, where ω_Σ^∂ is an even symplectic form of degree 0, Q_Σ^∂ is a cohomological vector field, $\mathcal{S}_\Sigma^\partial$ is an odd function of degree $+1$ and $\pi_\Sigma^\partial: \mathcal{F}_\Sigma^\partial \rightarrow \mathcal{F}_{\partial\Sigma}^{\partial\partial}$ is a surjective submersion such that

- (1) $\iota_{Q_\Sigma^\partial} \omega_\Sigma^\partial = d\mathcal{S}_\Sigma^\partial + (\pi_\Sigma^\partial)^* \alpha_{\partial\Sigma}^{\partial\partial}$,
- (2) $Q_{\partial\Sigma}^{\partial\partial} = d\pi_{Q_\Sigma^\partial}$.

We can now consider the zero locus $\mathcal{E}\mathcal{L}_{\partial\Sigma}^{\partial\partial}$ of $Q_{\partial\Sigma}^{\partial\partial}$, which is coisotropic and contains $L_\Sigma^\partial := \pi_\Sigma^\partial(\text{EL}_\Sigma^\partial)$. Repeating the same analysis as above, we conclude that $\mathcal{E}\mathcal{L}_{\Sigma_Q}^\partial \rightarrow \mathcal{E}\mathcal{L}_{\partial\Sigma}^{\partial\partial}$ has fibers with a symplectic structure in degree zero. Notice that in general these fibers will not be finite dimensional (it would be too restrictive to ask for that).

The construction may now be iterated to $(d - 2)$ -manifolds with boundaries. Every time the degree of the symplectic form and of the action increase by 1. However, it probably makes sense to continue this construction only as long as the $\mathcal{E}^{\partial\partial\dots\partial}_{[\ell]}$ spaces are finite dimensional.

Typically, at some point we get $\mathcal{S}^{\partial\partial\dots\partial} = 0$, so that $\mathcal{E}^{\partial\partial\dots\partial}_{[\ell]}$ is the whole space of fields in the bulk over a point on the boundary and this will usually be infinite dimensional.

On the other hand, in topological field theories of the AKSZ type [2], this construction can be iterated down to dimension 0 always with finite dimensional

$\underline{\mathcal{E}}^{\partial\partial\dots\partial}_{[l]}$ fibers; see [7] for details. Hence we can speak of fully extended field theories. These looks like the BV-BFV version of [18].

Example 5.9 (Electrodynamics—continued). In example 5.6, we got the BFV structure for first-order electrodynamics. Applying the above reasoning, we first consider the space of preboundary fields with one-form $\tilde{\alpha}_\gamma^{\partial\partial} = \int_\gamma B \delta c$. The kernel of its differential consists of all jets for A and A^+ and all jets higher than the zeroth for B and c , and $\tilde{\alpha}_\gamma^{\partial\partial}$ is basic. Hence we get

$$\mathcal{F}_\gamma^{\partial\partial} = \Omega^0(\gamma)[1] \times \Omega^{d-2}(\gamma).$$

One can also easily realize that $Q_\gamma^{\partial\partial} = 0$. Moreover, one can also compute, see [7],

$$\underline{\mathcal{E}}\mathcal{L}_{\Sigma_Q}^{\partial} = \Omega^1(\Sigma)/\Omega^1(\Sigma)_{\text{exact}} \oplus \Omega_{\text{closed}}^{d-2}(\Sigma, \partial\Sigma) \oplus H^0(\Sigma, \partial\Sigma)[1] \oplus H^{d-1}(\Sigma)[-1],$$

which is infinite dimensional for $d > 2$. If $d = 2$, this space is finite dimensional, so it makes sense to extend the theory down to codimension two. This is another way of observing that two-dimensional electrodynamics is almost topological (this holds also for nonabelian Yang Mills theories).

5.4. Perturbative quantization. We may finally present the generalization of the formalism discussed in subsection 3.3 to the case of degenerate Lagrangians in the BV-BFV formalism. For simplicity, we assume that the boundary one-form $\alpha_{\partial M}^{\partial}$ is globally well-defined and that $\mathcal{F}_{\partial M}^{\partial}$ is endowed with a Lagrangian foliation on which $\alpha_{\partial M}^{\partial}$ vanishes and which has a smooth leaf space $\mathcal{B}_{\partial M}$. The space of functions on $\mathcal{B}_{\partial M}$ defines the boundary graded vector space $\mathcal{H}_{\partial M}$. Let $p_{\partial M}$ be the projection $\mathcal{F}_{\partial M}^{\partial} \rightarrow \mathcal{B}_{\partial M}$. To produce a state ψ_M associated to the bulk M we first have to choose an embedding of $\underline{\mathcal{E}}\mathcal{L}_{[p_{\partial M}^{-1}(\phi) \cap \mathcal{L}_M]}$ into $\pi_M^{-1}(p_{\partial M}^{-1}(\phi))$ and a tubular neighborhood thereof. Then we have to pick a Lagrangian submanifold \mathcal{L}_ϕ in the fiber of this tubular neighborhood. Finally,

$$\psi_M(\phi) = \int_{\mathcal{L}_\phi} e^{\frac{i}{\hbar} S_M(\Phi)} [D\Phi].$$

Notice that $\psi_M(\phi)$ is also a function on the moduli space of vacua $\underline{\mathcal{E}}\mathcal{L}_{[p_{\partial M}^{-1}(\phi) \cap \mathcal{L}_M]}$. As already observed, each of these spaces carries a symplectic structure of degree -1 and is by assumption finite dimensional. The integral has to be computed perturbatively. One can then define a BV operator Δ on the moduli spaces of vacua and (if the theory is not anomalous) a coboundary operator Ω on $\mathcal{H}_{\partial M}$. By general BV arguments we expect that ψ_M satisfies the following generalization of the QME

$$(5.5) \quad \hbar^2 \Delta \psi_M + \Omega \psi_M = 0,$$

whose classical limit should correspond to (5.4).

An example where this kind of quantization has been performed and a solution to (5.5) has been explicitly obtained is described in [1]. Other examples are currently being studied [8].

APPENDIX A. SOME USEFUL FACTS

A relation from a set X to a set Y is just a subset of $X \times Y$. If R_1 is a relation from X to Y and R_2 is a relation from Y to Z , the composition $R_2 \circ R_1$ from X to

Z is defined as

$$R_2 \circ R_1 = \{(x, z) \in X \times Z : \exists y \in Y (x, y) \in R_1, (y, z) \in R_2\}.$$

The composition is associative. If $\phi: X \rightarrow Y$ and $\psi: Y \rightarrow Z$ are maps, then $\text{graph}(\psi) \circ \text{graph}(\phi) = \text{graph}(\psi \circ \phi)$.

If X and Y are symplectic manifolds, a relation from X to Y is called canonical if it is a Lagrangian submanifold. A map $\phi: X \rightarrow Y$ is a symplectomorphism iff $\text{graph}(\phi)$ is a canonical relation. The composition of two canonical relations in general is not a submanifold. On the other hand, being Lagrangian is preserved if X and Y are finite dimensional; otherwise one can only ensure being isotropic in general. A composition of isotropic relations is again isotropic.

In this paper, we often work with presymplectic and weakly symplectic forms. Recall that a closed two-form ω is presymplectic if it has constant rank and is weakly symplectic if it defines an injective linear map from the tangent to the cotangent bundle (in finitely many dimensions, this implies that the form is also symplectic).

The notion of Lagrangian submanifold naturally extends to presymplectic and weakly symplectic manifolds. A submanifold L of (M, ω) is called Lagrangian, if $T_x L^\perp = T_x L \forall x \in L$. Here

$$T_x L^\perp := \{v \in T_x M : \omega_x(v, w) = 0 \forall w \in T_x L\},$$

which makes sense also if ω_x is degenerate. Similarly, $L \subset M$ is coisotropic if $TL^\perp \subset TL$ and it is isotropic when $TL \subset TL^\perp$. Here \perp means orthogonal subbundle with respect to the two-form ω .

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A CONSTRUCTION OF OBSERVABLES FOR AKSZ SIGMA MODELS

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ABSTRACT. A construction of gauge-invariant observables is suggested for a class of topological field theories, the AKSZ sigma-models. The observables are associated to extensions of the target Q -manifold of the sigma model to a Q -bundle over it with additional Hamiltonian structure in fibers.

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1. INTRODUCTION

The interest in observables in topological field theories is largely due to the applications in algebraic topology, where expectation values of observables are known to yield (at least in some examples) invariants of knots and links under ambient isotopy or, more generally, cohomology classes of spaces of embeddings. The seminal example here was given in the work of Witten [16], where the expectation value of the Wilson loop observable in Chern-Simons theory with gauge group $SU(2)$, associated to a knot in the 3-sphere, was found out to yield the Jones polynomial of the knot.

Given a diffeomorphism-invariant action S_Σ of a topological field theory on a manifold Σ with space of fields F_Σ , one is particularly interested in observables $O \in C^\infty(F_\Sigma)$ associated to embedded submanifolds $i : \gamma \hookrightarrow \Sigma$ which depend on fields only via their pull-back from Σ to γ :

$$(1) \quad O_{\gamma,i}(X) = O(i^*X)$$

where X is the field. Such observables are automatically invariant with respect to diffeomorphisms $\phi : \Sigma \rightarrow \Sigma$ in the sense that

$$(2) \quad O_{\gamma,\phi \circ i}((\phi^{-1})^*X) = O_{\gamma,i}(X)$$

where $(\phi^{-1})^*X$ is the pull-back of the field by ϕ^{-1} .

On a formal level, the expectation value of the observable

$$(3) \quad \langle O_{\gamma,i} \rangle = \int_{F_\Sigma} \mathcal{D}X O_{\gamma,i}(X) e^{iS_\Sigma(X)}$$

is diffeomorphism-invariant

$$(4) \quad \langle O_{\gamma,\phi \circ i} \rangle = \langle O_{\gamma,i} \rangle$$

due to (2), diffeomorphism-invariance of the action S_Σ and of the path integral measure $\mathcal{D}X$. This means in particular that the expectation value is an invariant of the embedding $i : \gamma \hookrightarrow \Sigma$ under ambient isotopy.

Since topological field theories possess gauge symmetry, one also requires that the observable is gauge-invariant, so that in the path integral (3) one could pass to integration over gauge equivalence classes of fields.

In this paper we employ Batalin-Vilkovisky (BV) formalism to treat systems with gauge symmetry (cf. e.g. [15] for details). In particular the space of fields F_Σ becomes extended to an odd-symplectic Q -manifold \mathcal{F}_Σ with the action S_Σ extended to a function on \mathcal{F}_Σ , satisfying the master equation¹ $\{S_\Sigma, S_\Sigma\} = 0$ and generating the cohomological vector field Q_Σ as its Hamiltonian vector field. In this formalism the gauge-invariance of the observable is expressed as

$$(5) \quad Q_\Sigma O = 0$$

and the path integral (3) for the expectation value is traded for an integral over a Lagrangian submanifold \mathcal{L} in \mathcal{F}_Σ , with \mathcal{L} playing the role of the gauge-fixing condition.

One idea how to construct an observable for a gauge theory in BV formalism is to consider an extension of the BV theory $(\mathcal{F}_\Sigma, S_\Sigma)$ to a BV theory on a larger space $(\mathcal{F}_\Sigma \times \mathcal{F}^{\text{aux}}, S_\Sigma + S^{\text{aux}})$ with \mathcal{F}^{aux} the space of ‘‘auxiliary fields’’ and S^{aux}

¹In this outline we do not make explicit distinction between classical and quantum master equation. We write the classical one which also coincides with the quantum one if $\Delta_\Sigma S = 0$ for Δ_Σ the BV Laplacian. Likewise, we write the classical gauge-invariance condition on the observable $Q_\Sigma O = 0$ which coincides with the quantum one if $\Delta_\Sigma O = 0$, cf. section 4 for details.

the action for auxiliary fields Y which is also allowed to depend on the “ambient” fields $X \in \mathcal{F}_\Sigma$. Then one uses the BV push-forward construction

$$(6) \quad O(X) = \int_{\mathcal{L}^{\text{aux}} \subset \mathcal{F}^{\text{aux}}} \mathcal{D}Y e^{iS^{\text{aux}}(X,Y)}$$

(propositions 1, 3 in section 4, cf. also [13]) to integrate out the auxiliary fields and produce an observable for the ambient theory $(\mathcal{F}_\Sigma, S_\Sigma)$. In this paper we call such extensions by auxiliary fields “pre-observables” (to be more precise, we use a slightly better definition with equation (40) on S^{aux} instead of the master equation on $S_\Sigma + S^{\text{aux}}$, which still suffices to produce an observable, cf. definition 6 and remark 5).

In this paper we consider observables obtained by the construction outlined above for topological sigma models coming from AKSZ construction [1] (cf. a brief reminder in section 2), where \mathcal{F}_Σ is the mapping space from the tangent bundle of Σ with shifted parity of fibers to a target Q -manifold \mathcal{M} with additional Hamiltonian structure (cf. definition 1). In this setting we give a construction of pre-observables (section 5), which associates a pre-observable for the AKSZ sigma model to an embedded submanifold $i : \gamma \hookrightarrow \Sigma$ and an extension of the target \mathcal{M} to a Q -bundle over \mathcal{M} with additional Hamiltonian structure in fibers (the “Hamiltonian Q -bundle”, cf. section 3). For the ambient and auxiliary actions to have degree zero, degrees of Hamiltonian structures in the base and the fiber of the target Q -bundle have to match² dimensions of Σ and γ respectively.

Having a pre-observable for an AKSZ sigma model, we construct the corresponding observable by the BV push-forward (6). By construction, this observable is gauge-invariant in the sense of (5) and satisfies (1), and so formally produces expectation values which are invariant under ambient isotopy. Of course, the problem with this definition of the observable is that (6) is generally a path integral. In section 6 we consider several simple cases when this integral can be made sense of and its expected properties can be rigorously checked:

- case when the fiber in the target is a point (which gives rise to observables given by an exponential of an integral of a local expression),
- case of one-dimensional observables (when the path integral becomes a path integral of quantum mechanical type and can be regularized via geometric quantization, which gives rise to certain generalization of Wilson loops),
- case when the integral (6) is Gaussian which gives rise to a class of observables which we call “torsion-like” for their similarity to Ray-Singer torsion.

In section 7 we give explicit examples of observables falling within one of the three classes above. In particular, we recover the usual Wilson loop observable in Chern-Simons theory, with the corresponding path integral expression being the Alekseev-Faddeev-Shatashvili path integral formula for the Wilson loop [2]. We also recover the Cattaneo-Rossi “Wilson surface” observable for BF theory [9].

In this paper we concentrate only on the construction of observables, we are not calculating their expectation values.

1.1. Logic of the construction.

- (i) For an AKSZ sigma model on manifold Σ with target \mathcal{M} , find a Hamiltonian Q -bundle \mathcal{E} over \mathcal{M} . Then for every embedded submanifold $i : \gamma \hookrightarrow \Sigma$ of fixed dimension matching the degree of Hamiltonian structure in fibers of \mathcal{E} , we construct a pre-observable.

²More precisely, degrees of Hamiltonians (not of symplectic structures) have to match the dimensions.

- (ii) Take the fiber BV integral (6) over the auxiliary fields of the pre-observable constructed in step (i) to obtain an observable for the AKSZ sigma model.

Unlike step (i), step (ii) is not canonical, in the sense that it is a quantization problem: the path integral has to be made sense of (and the gauge-invariance of the result has to be checked) which can be done for certain classes of target Q -bundles, but it is not clear whether it is possible to do in greater generality³.

1.2. Plan of the paper. The logical organization of the exposition is as follows.

- Sections 2, 3.1, bits of 4 concerning BV observables — background reminders.
- Section 5 — step (i) of the main construction.
- Section 3 — auxiliary construction for step (i) of the main construction.
- Section 4 — motivation for step (ii) of the main construction.
- Section 6 — examples for step (ii) of the main construction.
- Section 7 — fully explicit examples.

Section 2 is a short reminder of the AKSZ construction of topological sigma models in Batalin-Vilkovisky formalism. We also recall how some well-known topological field theories fit into the construction: Chern-Simons theory, BF theory, Poisson sigma model.

In section 3 we first briefly recall the standard notion of a Q -bundle and then define a “Hamiltonian” Q -bundle, preparing the grounds for the construction of pre-observables in section 5. We start with the definition 2 of a trivial Hamiltonian Q -bundle (“trivial” here means trivial as a fiber bundle; the cohomological vector field is *not* required to be a sum of a cohomological vector field on the base and another one on the fiber). All our examples are of this type, but for the completeness of the exposition, we also give a slightly more general definition 3, which does not explicitly rely on the total space being a direct product. However in a local trivialization definition 3 boils down to definition 2.

In section 4 we recall the standard notions of classical and quantum observables in BV formalism and introduce the notion of a pre-observable, which comes in three modifications:

- (i) Classical pre-observable, definition 6: essentially, an extension of the action of the ambient classical BV theory to a solution of classical master equation on the space of fields extended by auxiliary fields. We give a technically more convenient definition with equation (40) required instead of the classical master equation on extended space of fields (cf. remark 5 for the relation between the two).
- (ii) Semi-quantum pre-observable, definition 7, suited for integrating out auxiliary fields to obtain an observable for the ambient theory (proposition 1).
- (iii) Quantum pre-observable for a quantum ambient BV theory (i.e. one with the action satisfying the quantum master equation), definition 9: this is also an extension of the ambient theory by auxiliary fields, plus an extension of the action to a solution of quantum master equation on the extended space. From a quantum pre-observable one can induce a quantum observable for the ambient theory, by integrating out auxiliary fields (proposition 3).

³One can indeed try to define the path integral perturbatively, as sum of Feynman diagrams represented by integrals over compactified configuration spaces of tuples of points on γ . However, for the formal argument of proposition 1 that the result is gauge-invariant to become rigorous, one has to prove that the hidden boundary strata of the configuration spaces appearing in the calculation of $Q_{\Sigma}O$ (cf. the proof of proposition 6) do not contribute, which is not true in general case.

In this section in the discussion of quantization we always work with spaces of fields as with finite dimensional spaces and the integrals are over finite-dimensional super-manifolds. In the context of local field theory, the BV push-forward becomes a path integral, so the proofs given in section 4 stop to work and the propositions become conjectures that have to be proven by more delicate means (cf. the proofs of propositions 5, 6 in section 6).

Section 5 is the logical core of the paper. Here we give the construction of a classical pre-observable for an AKSZ theory out of an extension of the target to a Hamiltonian Q -bundle.

In section 6 we treat several classes of situations when the BV push-forward yielding an observable out of a pre-observable constructed in section 5 can be performed rigorously: case of target fiber being a point, case of 1-dimensional observables (via geometric quantization), case when the BV push-forward is given by a Gaussian integral.

In section 7 we specialize the constructions of section 6 to present several explicit examples, including the Wilson loop together with its path integral representation known from [2], Cattaneo-Rossi codimension 2 observable for BF theory, torsion observables in Chern-Simons and BF theories, etc.

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TERMINOLOGY, NOTATIONS

Terminology.

- Q -manifold (or bundle) = differential graded manifold (or bundle).
- Ghost number = internal degree (to distinguish from de Rham degree of a differential form) = \mathbb{Z} -grading on functions on \mathbb{Z} -graded manifolds.
- Observable = gauge-invariant functional on the space of fields (cf. section 4 for definitions).
- Expectation value = correlator.

Conventions. We set the Planck's constant $\hbar = 1$ (cf. remark 6 on how to re-introduce \hbar).

Notations. We use $\mathcal{M}, \mathcal{N}, \mathcal{E}, \mathcal{F}$ etc. for \mathbb{Z} -graded manifolds, M, N etc. for ordinary (non-graded) manifolds; Σ always denotes the source (spacetime) manifold of the sigma model, γ is typically a submanifold of Σ on which the observable is supported. We use \mathcal{L} for a Lagrangian submanifold of a degree -1 symplectic graded manifold.

We denote by $\mathfrak{X}(\mathcal{M})$ the Lie algebra of vector fields on \mathcal{M} . For a fiber bundle $\pi : \mathcal{E} \rightarrow \mathcal{M}$, we denote by $\mathfrak{X}^{\text{vert}}(\mathcal{E})$ the Lie algebra of vertical vector fields on the total space \mathcal{E} , i.e. the space of sections $\Gamma(\mathcal{E}, T^{\text{vert}}\mathcal{E})$ of the vertical distribution on \mathcal{E} , $T^{\text{vert}}\mathcal{E} = \ker(d\pi) \subset T\mathcal{E}$.

We denote the degree (the ghost number) of functions/differential forms/vector fields on a graded manifold by $|\dots|$.

2. AKSZ REMINDER

What follows is a very short reminder of the AKSZ construction of topological sigma models in Batalin-Vilkovisky formalism, to fix the terminology and notation. We refer the reader to the original paper [1] and the later expositions in [7], [14] for details.

2.1. Target data. Let \mathcal{M} be a degree n symplectic Q -manifold, i.e. a \mathbb{Z} -graded manifold endowed with a degree 1 vector field Q satisfying $Q^2 = 0$ (the cohomological vector field) and with a degree⁴ n symplectic form $\omega \in \Omega^2(\mathcal{M})$ which is compatible with Q , i.e. $L_Q\omega = 0$.

Assume⁵ that Q has a Hamiltonian function $\Theta \in C^\infty(\mathcal{M})$ with $|\Theta| = n + 1$, $\{\Theta, \bullet\}_\omega = Q$ and satisfying

$$(7) \quad \{\Theta, \Theta\}_\omega = 0$$

Also assume that ω is exact, with $\alpha \in \Omega^1(\mathcal{M})$ a primitive.

Definition 1. We call the set of data $(\mathcal{M}, Q, \omega = \delta\alpha, \Theta)$ a *Hamiltonian Q -manifold of degree n* .

2.2. The AKSZ sigma model. Fix a Hamiltonian Q -manifold \mathcal{M} of degree $n \geq -1$ and let Σ be an oriented closed manifold, $\dim \Sigma = n + 1$. Then one constructs the space of fields as the space of graded maps between graded manifolds from the degree-shifted tangent bundle $T[1]\Sigma$ to \mathcal{M} :

$$(8) \quad \mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathcal{M})$$

It is a Q -manifold with the cohomological vector field coming from the lifting of Q on the target and of the de Rham operator d_Σ on Σ (viewed as a cohomological vector field on $T[1]\Sigma$) to the mapping space:

$$(9) \quad Q_\Sigma = (d_\Sigma)^{\text{lifted}} + (Q)^{\text{lifted}} \in \mathfrak{X}(\mathcal{F}_\Sigma)$$

Transgression map. The following natural maps

$$(10) \quad \begin{array}{ccc} \mathcal{F}_\Sigma \times T[1]\Sigma & \xrightarrow{\text{ev}} & \mathcal{M} \\ p \downarrow & & \\ \mathcal{F}_\Sigma & & \end{array}$$

(where p is the projection to the first factor) allow us to define the transgression map

$$(11) \quad \tau_\Sigma = p_* \text{ev}^* : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{F}_\Sigma)$$

Here p_* is the fiber integration over $T[1]\Sigma$ (with the canonical integration measure). Map τ_Σ preserves the de Rham degree of a form, but changes the internal grading (the ghost number) by $-\dim \Sigma$.

If $u \in \mathfrak{X}(T[1]\Sigma)$, $v \in \mathfrak{X}(\mathcal{M})$ are any vector fields on the source and the target and $\psi \in \Omega^\bullet(\mathcal{M})$ is a form on the target, then for the Lie derivatives of the transgressed form along the lifted vector fields we have

$$(12) \quad L_{u^{\text{lifted}}} \tau_\Sigma(\psi) = 0$$

$$(13) \quad L_{v^{\text{lifted}}} \tau_\Sigma(\psi) = (-1)^{|v| \dim \Sigma} \tau_\Sigma(L_v \psi)$$

The finite version of (12) is that for $\Phi : T[1]\Sigma \rightarrow T[1]\Sigma$ a diffeomorphism of the source, we have

$$(14) \quad (\Phi^*)^* \tau_\Sigma(\psi) = \tau_\Sigma(\psi)$$

where $\Phi^* : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_\Sigma$ is the lifting of Φ to the mapping space and $(\Phi^*)^* : \Omega^\bullet(\mathcal{F}_\Sigma) \rightarrow \Omega^\bullet(\mathcal{F}_\Sigma)$ is the pull-back by Φ^* .

⁴By “degree” here we mean the internal \mathbb{Z} -grading coming from the grading on \mathcal{M} .

⁵In fact (cf. [14]), for $n \neq -1$ the symplectic property of the cohomological vector field ($L_Q\omega = 0$) implies existence and uniqueness of the Hamiltonian $\Theta = \frac{1}{n+1} \iota_E \iota_Q \omega$ where E is the Euler vector field. Maurer-Cartan equation (7) follows from $Q^2 = 0$ for $n \neq -2$. Also, for $n \neq 0$, a closed form ω is automatically exact.

The BV 2-form and the master action. One obtains the degree -1 symplectic form (the “BV 2-form”) on \mathcal{F}_Σ from the target by transgression⁶:

$$(15) \quad \Omega_\Sigma = (-1)^{\dim \Sigma} \tau_\Sigma(\omega) \in \Omega^2(\mathcal{F}_\Sigma)$$

The Hamiltonian function for Q_Σ (the master action) is constructed as

$$(16) \quad S_\Sigma = \underbrace{\iota_{d_\Sigma^{\text{lifted}}} \tau_\Sigma(\alpha)}_{S_\Sigma^{\text{kin}}} + \underbrace{\tau_\Sigma(\Theta)}_{S_\Sigma^{\text{target}}} \in C^\infty(\mathcal{F}_\Sigma)$$

It automatically satisfies the classical master equation

$$\{S_\Sigma, S_\Sigma\}_{\Omega_\Sigma} = 0$$

One can summarize the construction above by saying that we have a degree -1 Hamiltonian Q -manifold structure on the mapping space (8): $(\mathcal{F}_\Sigma, Q_\Sigma, \Omega_\Sigma, S_\Sigma)$. The primitive 1-form for the BV 2-form can also be constructed by transgression as $\tau_\Sigma(\alpha)$.

Note that exact shifts of the target 1-form, $\alpha \mapsto \alpha + \delta f$, with $f \in C^\infty(M)$ leave S_Σ unchanged.

Why AKSZ theory is topological. For $\phi \in \text{Diff}(\Sigma)$ a diffeomorphism of Σ , denote $\tilde{\phi} \in \text{Diff}(T[1]\Sigma)$ the tangent lift of ϕ to $T[1]\Sigma$. Then

$$(17) \quad (\tilde{\phi}^*)^* S_\Sigma = S_\Sigma, \quad (\tilde{\phi}^*)^* \Omega_\Sigma = \Omega_\Sigma$$

because of (14) and because $d_\Sigma \in \mathfrak{X}(T[1]\Sigma)$ commutes with $\tilde{\phi}$, since the latter is a tangent lift.

In coordinates. Let x^a be local homogeneous coordinates on the target \mathcal{M} , let u^μ be local coordinates on Σ and $\theta^\mu = du^\mu$ be the associated degree 1 fiber coordinates on $T[1]\Sigma$. Then locally an element of \mathcal{F}_Σ is parameterized by

$$(18) \quad X^a(u, \theta) = \sum_{k=0}^{\dim \Sigma} \underbrace{\sum_{1 \leq \mu_1 < \dots < \mu_k \leq \dim \Sigma} X_{\mu_1 \dots \mu_k}^a(u) \theta^{\mu_1} \dots \theta^{\mu_k}}_{X_{(k)}^a(u, \theta)}$$

Coefficient functions $X_{\mu_1 \dots \mu_k}^a(u)$ are local coordinates of degree $|x^a| - k$ on the mapping space \mathcal{F}_Σ . Expression (18) is known as (the component of) the *superfield*, and it can be regarded as a generating function for the coordinates on the mapping space \mathcal{F}_Σ .

For any function $f \in C^\infty(\mathcal{M})$, we have

$$(19) \quad \begin{aligned} \text{ev}^* f &= f(X) = \\ &= f(X_{(0)}) + X_{\geq 1}^a \cdot (\partial_a f)(X_{(0)}) + \frac{1}{2} X_{\geq 1}^a X_{\geq 1}^b \cdot (\partial_b \partial_a f)(X_{(0)}) + \dots \\ &\in C^\infty(\mathcal{F}_\Sigma \times T[1]\Sigma) \end{aligned}$$

where we denoted $X_{\geq 1}^a = \sum_{k \geq 1} X_{(k)}^a$ the part of the superfield of positive de Rham degree with respect to Σ ; ev is the horizontal arrow in (10).

Let α and ω be locally given as $\alpha = \alpha_a(x) \delta x^a$ and $\omega = \frac{1}{2} \omega_{ab}(x) \delta x^a \wedge \delta x^b$. Then the BV 2-form and its primitive are:

$$\Omega_\Sigma = (-1)^{\dim \Sigma} \int_\Sigma \frac{1}{2} \omega_{ab}(X) \delta X^a \wedge \delta X^b, \quad \alpha_\Sigma = \int_\Sigma \alpha_a(X) \delta X^a$$

⁶We introduce the sign $(-1)^{\dim \Sigma}$ in this definition to avoid signs in the formula for the action below. The reader may encounter different sign conventions for the AKSZ construction in the literature.

(Note that we use δ to denote the de Rham differential on the target \mathcal{M} and on the mapping space \mathcal{F}_Σ . We reserve symbol d for the de Rham differential on the source Σ .)

The master action is:

$$S_\Sigma(X) = \underbrace{\int_\Sigma \alpha_a(X) dX^a}_{S_\Sigma^{\text{kin}}} + \underbrace{\int_\Sigma \Theta(X)}_{S_\Sigma^{\text{target}}}$$

If the cohomological vector field on the target is locally written as $Q = Q^a(x) \frac{\partial}{\partial x^a}$ then the cohomological vector field (9) is determined by its action on the components of the superfield:

$$Q_\Sigma X^a = dX^a + Q^a(X)$$

Critical points of S_Σ are (with our sign conventions) Q -anti-morphisms between $T[1]\Sigma$ and \mathcal{M} , i.e. Q -morphisms between $(T[1]\Sigma, d)$ and $(\mathcal{M}, -Q)$.

2.3. Examples. Here we recall some of the standard examples of the AKSZ construction.

Chern-Simons theory [1]. Let \mathfrak{g} be a quadratic Lie algebra, i.e. a Lie algebra with a non-degenerate invariant pairing $\langle \cdot, \cdot \rangle$. Denote by $\psi : \mathfrak{g}[1] \rightarrow \mathfrak{g}$ the degree 1 \mathfrak{g} -valued coordinate on $\mathfrak{g}[1]$. We choose the target Hamiltonian Q -manifold of degree 2 as

$$(20) \quad \mathcal{M} = \mathfrak{g}[1], \quad Q = \left\langle \frac{1}{2}[\psi, \psi], \frac{\partial}{\partial \psi} \right\rangle, \\ \omega = \frac{1}{2}(\delta\psi, \delta\psi), \quad \alpha = \frac{1}{2}(\psi, \delta\psi), \quad \Theta = \frac{1}{6}(\psi, [\psi, \psi])$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing⁷ between \mathfrak{g} and \mathfrak{g}^* . The associated AKSZ sigma model on a closed oriented 3-manifold Σ has the space of fields

$$\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathfrak{g}[1]) \cong \mathfrak{g}[1] \otimes \Omega^\bullet(\Sigma)$$

The superfield is

$$(21) \quad A = A_{(0)} + A_{(1)} + A_{(2)} + A_{(3)}$$

with $A_{(k)}$ a coordinate on \mathcal{F}_Σ with values in \mathfrak{g} -valued k -forms on Σ , with internal degree (ghost number) $1 - k$, for $k = 0, 1, 2, 3$. The BV 2-form is

$$\Omega_\Sigma = -\frac{1}{2} \int_\Sigma (\delta A, \delta A)$$

and the action is

$$S_\Sigma = \int_\Sigma \frac{1}{2}(A, dA) + \frac{1}{6}(A, [A, A])$$

This is the action of Chern-Simons theory in Batalin-Vilkovisky formalism.

In case $\mathfrak{g} = \mathbb{R}$ with abelian Lie algebra structure, we have $Q = \Theta = 0$ on the target and

$$(22) \quad \mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)[1], \quad \Omega_\Sigma = -\frac{1}{2} \int_\Sigma \delta A \wedge \delta A, \quad S_\Sigma = \frac{1}{2} \int_\Sigma A \wedge dA$$

This is the *abelian* Chern-Simons theory in BV formalism.

BF theory. For \mathfrak{g} a Lie algebra (not necessarily quadratic⁸) and D a non-negative integer, we define the target Hamiltonian Q -manifold of degree $D - 1$ as

⁷We will generally be using notation $\langle \cdot, \cdot \rangle$ for the canonical pairing $V \otimes V^* \rightarrow \mathbb{R}$ between a vector space V and its dual; sometimes we will indicate the respective V as a subscript: $\langle \cdot, \cdot \rangle_V$.

⁸However, for the consistency of quantization, in particular for the quantum master equation (48), one has to require that \mathfrak{g} is unimodular.

$$(23) \quad \mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[D - 2], \quad Q = \left\langle \frac{1}{2}[\psi, \psi], \frac{\partial}{\partial \psi} \right\rangle + \left\langle \text{ad}^*_\psi \xi, \frac{\partial}{\partial \xi} \right\rangle,$$

$$\omega = \langle \delta \xi, \delta \psi \rangle, \quad \alpha = \langle \xi, \delta \psi \rangle, \quad \Theta = \frac{1}{2} \langle \xi, [\psi, \psi] \rangle$$

Here $\psi : \mathfrak{g}[1] \rightarrow \mathfrak{g}$ is as before and $\xi : \mathfrak{g}^*[D - 2] \rightarrow \mathfrak{g}^*$ is the \mathfrak{g}^* -valued coordinate on $\mathfrak{g}^*[D - 2]$ of degree $D - 2$; ad^* is the coadjoint action of \mathfrak{g} on \mathfrak{g}^* .

The associated AKSZ sigma model on a closed oriented D -manifold Σ has the space of fields

$$(24) \quad \mathcal{F}_\Sigma = \mathfrak{g}[1] \otimes \Omega^\bullet(\Sigma) \oplus \mathfrak{g}^*[D - 2] \otimes \Omega^\bullet(\Sigma)$$

The superfields associated to ψ and ξ are respectively

$$A = \sum_{k=0}^D A_{(k)}, \quad B = \sum_{k=0}^D B_{(k)}$$

with $A_{(k)}$ a \mathfrak{g} -valued k -form on Σ of internal degree $1 - k$; $B_{(k)}$ is a \mathfrak{g}^* -valued k -form of internal degree $D - 2 - k$. The BV 2-form and the action are:

$$(25) \quad \Omega_\Sigma = (-1)^D \int_\Sigma \langle \delta B, \delta A \rangle, \quad S_\Sigma = \int_\Sigma \left\langle B, dA + \frac{1}{2}[A, A] \right\rangle$$

This is the BF theory in BV formalism.

In abelian case, $\mathfrak{g} = \mathbb{R}$, we have $Q = \Theta = 0$ on the target and

$$(26) \quad \mathcal{F}_\Sigma = \Omega^\bullet(\Sigma)[1] \oplus \Omega^\bullet(\Sigma)[D - 2],$$

$$\Omega_\Sigma = (-1)^D \int_\Sigma \delta B \wedge \delta A, \quad S_\Sigma = \int_\Sigma B \wedge dA$$

Poisson sigma model [7]. Let M be a manifold endowed with a Poisson bivector $\pi \in \Gamma(M, \wedge^2 TM)$. We construct the target Hamiltonian Q -manifold of degree 1 as

$$(27) \quad \mathcal{M} = T^*[1]M,$$

$$Q = \left\langle \pi(x), p \wedge \frac{\partial}{\partial x} \right\rangle_{\wedge^2 T_x M} + \frac{1}{2} \left\langle \frac{\partial}{\partial x} \pi(x), (p \wedge p) \otimes \frac{\partial}{\partial p} \right\rangle_{(\wedge^2 T \otimes T^*)_x M},$$

$$\omega = \langle \delta p, \delta x \rangle, \quad \alpha = \langle p, \delta x \rangle, \quad \Theta = \frac{1}{2} \langle \pi(x), p \wedge p \rangle_{\wedge^2 T_x M}$$

Here x and p stand for the local base and fiber coordinates on $T^*[1]M$ respectively. Note that all objects in (27) are globally well defined.

The corresponding AKSZ sigma model on an oriented closed surface Σ has the space of fields

$$\mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, T^*[1]M)$$

with superfields

$$X = X_{(0)} + X_{(1)} + X_{(2)}, \quad \eta = \eta_{(0)} + \eta_{(1)} + \eta_{(2)}$$

associated to local coordinates x and p on the target, respectively. Here $X_{(k)}$ and $\eta_{(k)}$ are k -forms on Σ with internal degrees $-k$ and $1 - k$ respectively. The BV 2-form and the action are:

$$\Omega_\Sigma = \int_\Sigma \langle \delta \eta, \delta X \rangle, \quad S_\Sigma = \int_\Sigma \langle \eta, dX \rangle + \frac{1}{2} \langle \pi(X), \eta \wedge \eta \rangle$$

3. HAMILTONIAN Q -BUNDLES

In this section we briefly recall the standard notion of a Q -bundle and then introduce ‘‘Hamiltonian Q -bundles’’, an auxiliary notion necessary for our construction of observables.

3.1. Q -bundles reminder. Recall that a “ Q -bundle” [11] is a fiber bundle in the category of Q -manifolds.

In particular, a *trivial* Q -bundle is trivial bundle of graded manifolds

$$\pi : \underbrace{\mathcal{M} \times \mathcal{N}}_{\mathcal{E}} \rightarrow \mathcal{M}$$

where the base \mathcal{M} is endowed with a cohomological vector field Q and the total space is endowed with a cohomological vector field Q^{tot} in such a way that π is a Q -morphism, i.e. $d\pi(Q^{\text{tot}}) = Q$. This implies the following ansatz for Q^{tot} :

$$(28) \quad Q^{\text{tot}} = Q + \mathcal{A}$$

where $\mathcal{A} \in \mathfrak{X}^{\text{vert}}(\mathcal{E}) \cong \mathfrak{X}(\mathcal{N}) \hat{\otimes} C^\infty(\mathcal{M})$ is the vertical part of Q^{tot} . Cohomological property for Q^{tot} is equivalent to

$$(29) \quad Q\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0$$

plus the cohomological property for Q . Note that in the first term on the l.h.s. of (29) we are thinking of Q as acting on $C^\infty(\mathcal{M})$ part of \mathcal{A} . Equivalently, we can lift Q to a horizontal vector field on \mathcal{E} and write the first term as $[Q, \mathcal{A}]$.

3.2. Trivial Hamiltonian Q -bundles.

Definition 2. We define a trivial Hamiltonian Q -bundle of degree $n' \in \mathbb{Z}$ as the following collection of data.

- (i) A trivial Q -bundle

$$\pi : \mathcal{E} = \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M}$$

with $Q^{\text{tot}} = Q + \mathcal{A}$ as in (28).

- (ii) The fiber \mathcal{N} is endowed with a degree n' exact symplectic form $\omega' = \delta\alpha'$ and a degree $(n' + 1)$ Hamiltonian $\Theta' \in C^\infty(\mathcal{E})$ satisfying $\mathcal{A} = \{\Theta', \bullet\}_{\omega'}$ and

$$(30) \quad Q\Theta' + \frac{1}{2}\{\Theta', \Theta'\}_{\omega'} = 0$$

Remark 1. (Roytenberg) Symplectic structure ω' is automatically exact for $n' \neq 0$, since $\omega' = \delta\left(\frac{1}{n'}\iota_E\omega'\right)$ where E is the Euler vector field on \mathcal{N} .

Remark 2. If $n' = 0$, then ω' is not automatically exact. In this situation, exactness condition may be relaxed⁹ to the Bohr-Sommerfeld condition that $\omega'/2\pi$ defines an integral cohomology class in $H^2(\mathcal{N})$. Then the role of primitive 1-form α' is taken by a Hermitian line bundle L' over \mathcal{N} equipped with a $U(1)$ -connection ∇' of curvature ω' .

Remark 3. For a Hamiltonian Q -manifold of degree $n \neq -2$, equation $\{\Theta, \Theta\} = 0$ follows from the fact that Θ is the Hamiltonian for a cohomological vector field (Jacobi identity implies $\frac{1}{2}\{\Theta, \Theta\}, \bullet\} = Q^2(\bullet) = 0$ which implies that $\{\Theta, \Theta\}$ is a constant of degree $n + 2$ and thus vanishes unless $n = -2$).

Observe that this argument fails for the fiber \mathcal{N} : applying the same reasoning to derive the equation (30) from (29), we obtain that l.h.s. of (30) is a pull-back of some function on the base of degree $n' + 2$, which does not imply that it is zero.

⁹Cf. remark 8 for the motivation: primitive α' (or connection ∇' in the relaxed version) will be needed in the construction of section 5 to define the kinetic part of the auxiliary action.

3.3. General Hamiltonian Q -bundles. The assumption of triviality of the bundle \mathcal{E} , used in the definition 2, can be relaxed as follows.

Definition 3. A Hamiltonian Q -bundle is the following set of data:

- (i) A Q -bundle $\pi : \mathcal{E} \rightarrow \mathcal{M}$.
- (ii) A degree n' exact pre-symplectic form $\omega' = \delta\alpha'$ on the total space \mathcal{E} , with the property that the distribution $\ker \omega' \subset T\mathcal{E}$ is transversal to the vertical distribution $T^{\text{vert}}\mathcal{E}$; thus $\ker \omega'$ defines a flat Ehresmann connection $\nabla_{\omega'}$ on \mathcal{E} .
- (iii) A Hamiltonian function $\Theta' \in C^\infty(\mathcal{E})$ satisfying

$$\iota_{Q^{\text{tot}}}\omega' = \delta^{\text{vert}}\Theta'$$

where $\delta^{\text{vert}} = (i^{\text{vert}})^* \circ \delta$ is the vertical part of de Rham differential on \mathcal{E} ; we denoted $i^{\text{vert}} : T^{\text{vert}}\mathcal{E} \hookrightarrow T\mathcal{E}$ the natural inclusion. We also require that

$$(Q^{\text{hor}} + \frac{1}{2}Q^{\text{vert}})\Theta' = 0$$

where the splitting $Q^{\text{tot}} = Q^{\text{hor}} + Q^{\text{vert}}$ of the cohomological vector field on \mathcal{E} into the horizontal and the vertical part is defined by connection $\nabla_{\omega'}$.

Remark 4. Note that one can introduce a local trivialization of \mathcal{E} consistent with the flat connection $\nabla_{\omega'}$ (i.e. where the horizontal distribution $\ker \omega' \subset T\mathcal{E}$ is defined by the direct product structure on $\mathcal{E}|_U$ coming from the local trivialization, where $U \subset \mathcal{M}$ is a trivializing neighborhood). In such a trivialization, we recover back the definition 2. In this sense, definition 3 does not bring in a drastic increase of generality.

3.4. Examples.

- (i) Let \mathfrak{g} be a Lie algebra and let $\rho : \mathfrak{g} \rightarrow \mathfrak{so}(R) \subset \text{End}(R)$ be a representation of \mathfrak{g} on by anti-symmetric matrices on a Euclidean vector space $R, (\cdot, \cdot)$. Then for any $k \in \mathbb{Z}$ we can define a degree $4k + 2$ trivial Hamiltonian Q -bundle with base $\mathcal{M} = \mathfrak{g}[1]$ and fiber $\mathcal{N} = R[2k + 1]$. Let ψ be the degree 1 \mathfrak{g} -valued coordinate on $\mathfrak{g}[1]$ and let x be the degree $2k + 1$ R -valued coordinate on $R[2k + 1]$. Then we set

$$(31) \quad Q = \frac{1}{2} \left\langle [\psi, \psi], \frac{\partial}{\partial \psi} \right\rangle$$

– the Chevalley-Eilenberg differential, and

$$(32) \quad \mathcal{A} = \left\langle \rho(\psi)x, \frac{\partial}{\partial x} \right\rangle, \quad \omega' = \frac{1}{2}(\delta x, \delta x),$$

$$\alpha' = \frac{1}{2}(x, \delta x), \quad \Theta' = \frac{1}{2}(x, \rho(\psi)x)$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between a vector space and its dual. It is a straightforward check that this comprises the data of a trivial Hamiltonian Q -bundle of degree $4k + 2$, according to definition 2.

As a variation of this example, we may require that the pairing (\cdot, \cdot) is anti-symmetric (so that R is a symplectic space instead of Euclidean) and that $\rho : \mathfrak{g} \rightarrow \mathfrak{sp}(R) \subset \text{End}(R)$ is a symplectic representation. Then formulae (31,32) define a Hamiltonian Q -bundle structure of degree $4k$ on $\mathfrak{g}[1] \times R[2k]$ (note that here we choose an even degree shift for the fiber).

- (ii) Example above generalizes straightforwardly to the setting of L_∞ algebras and L_∞ modules. Namely, let \mathfrak{g} be an L_∞ algebra with operations $\{l_j : \wedge^j \mathfrak{g} \rightarrow \mathfrak{g}\}_{j \geq 1}$. Also, let R be graded vector space with graded symmetric pairing (\cdot, \cdot) of degree q and with the structure of an L_∞ module over \mathfrak{g} with

the module operations $\{\rho_j : \wedge^j \mathfrak{g} \otimes R \rightarrow R\}_{j \geq 0}$, where we additionally require that operations $(\bullet, \rho_j(\cdots; \bullet))$ are graded anti-symmetric w.r.t. the two entries in the module. For ψ a \mathfrak{g} -valued degree 1 coordinate on $\mathfrak{g}[1]$ and x a R -valued degree $2k + 1$ coordinate on $R[2k + 1]$, we construct a trivial Hamiltonian Q -bundle with $\mathcal{M} = \mathfrak{g}[1]$, $\mathcal{N} = R[2k + 1]$ and the data are:

$$(33) \quad Q = \sum_{j \geq 1} \frac{1}{j!} \left\langle l_j(\psi, \dots, \psi), \frac{\partial}{\partial \psi} \right\rangle$$

for the base and

$$(34) \quad \mathcal{A} = \sum_{j \geq 0} \frac{1}{j!} \left\langle \rho_j(\psi, \dots, \psi; x), \frac{\partial}{\partial x} \right\rangle, \quad \omega' = \frac{1}{2}(\delta x, \delta x),$$

$$\alpha' = \frac{1}{2}(x, \delta x), \quad \Theta' = \sum_{j \geq 0} \frac{1}{2j!}(x, \rho_j(\psi, \dots, \psi; x))$$

This comprises the data of a trivial Hamiltonian Q -bundle of degree $4k + 2 + q$.

- (iii) Example (ii) admits the following variation. If \mathfrak{h} is an L_∞ algebra with $I \subset \mathfrak{h}$ an L_∞ ideal¹⁰, then there is a natural Q -bundle

$$(35) \quad \pi : \mathfrak{h}[1] \rightarrow (\mathfrak{h}/I)[1]$$

with bundle projection being the quotient map and with the fiber I . If in addition we fix a section of the quotient map $\mathfrak{h} \rightarrow \mathfrak{h}/I$, i.e. a splitting of the short exact sequence

$$(36) \quad I \rightarrow \mathfrak{h} \rightarrow \mathfrak{h}/I$$

and if I carries a degree q cyclic¹¹ inner product, then (35) becomes a Hamiltonian Q -bundle of degree $2 + q$. Explicitly, the structure on the base $\mathcal{M} = (\mathfrak{h}/I)[1]$ is again given by (33) with $\{l_j\}$ the quotient L_∞ operations on \mathfrak{h}/I , ψ the degree 1 (\mathfrak{h}/I) -valued coordinate on $(\mathfrak{h}/I)[1]$. The structure on the fiber $\mathcal{N} = I[1]$ is:

$$(37) \quad \mathcal{A} = \sum_{j \geq 0, k \geq 0, j+k \geq 1} \frac{1}{j!k!} \left\langle P_I \lambda_{j+k}(\underbrace{\psi, \dots, \psi}_j, \underbrace{x, \dots, x}_k), \frac{\partial}{\partial x} \right\rangle,$$

$$\omega' = \frac{1}{2}(\delta x, \delta x), \quad \alpha' = \frac{1}{2}(x, \delta x),$$

$$\Theta' = \sum_{j \geq 0, k \geq 0, j+k \geq 1} \frac{1}{j!(k+1)!} (x, P_I \lambda_{j+k}(\underbrace{\psi, \dots, \psi}_j, \underbrace{x, \dots, x}_k))$$

where $P_I : \mathfrak{h} \rightarrow I$ is the projection to the ideal fixed by the choice of splitting of (36), $\{\lambda_j\}$ are the L_∞ operations on \mathfrak{h} , x is the degree 1 I -valued coordinate on $I[1]$ and $(,)$ is the cyclic pairing on I .

- (iv) Let \mathfrak{g} be a Lie algebra and suppose there is a Hamiltonian action of \mathfrak{g} on an exact symplectic manifold $(M, \omega_M = \delta \alpha_M)$ with equivariant moment map $\mu : M \rightarrow \mathfrak{g}^*$. Then we set $\mathcal{M} = \mathfrak{g}[1]$ as in (i), with the structure (31). For the fiber, we set $\mathcal{N} = M$ and

$$(38) \quad \mathcal{A} = \{\langle \psi, \mu \rangle, \bullet\}, \quad \omega' = \omega_M, \quad \alpha' = \alpha_M, \quad \Theta' = \langle \psi, \mu \rangle$$

¹⁰Recall that a subspace $I \subset \mathfrak{h}$ is called an L_∞ ideal if the L_∞ operations on \mathfrak{h} take values in I if at least one argument is in I .

¹¹By cyclic property for the pairing on the ideal we mean that extending the pairing on I to \mathfrak{h} by zero on \mathfrak{h}/I , we obtain a degenerate cyclic pairing for the L_∞ algebra \mathfrak{h} .

This is the data of a trivial Hamiltonian Q -bundle of degree 0 on $\mathfrak{g}[1] \times M$. Exactness condition for ω_M can be relaxed to the Bohr-Sommerfeld integrality condition, cf. remark 2.

(v) For any Q -manifold (\mathcal{M}, Q) , given a function $\theta \in C^\infty(\mathcal{M})$ satisfying

$$Q\theta = 0, \quad |\theta| = p$$

we can construct a trivial Hamiltonian Q -bundle over \mathcal{M} of degree $p - 1$ with fiber \mathcal{N} a point, $\mathcal{A} = 0$, $\omega' = 0$, $\Theta' = \theta$ (and we formally prescribe degree $p - 1$ to ω').

4. BV PRE-OBSERVABLES AND OBSERVABLES

In this section we recall some basic structures of BV formalism: classical and quantum BV theory (cf. [15]), observables in classical and quantum setting. We also introduce the notion of a *pre-observable*¹²: an extension of a BV theory by auxiliary fields endowed with an odd-symplectic structure and an action (which is also dependent on the fields of the ambient theory). Given a pre-observable, one can construct an observable by integrating out the auxiliary fields (propositions 1, 3). Pre-observables come in three versions: purely classical, *semi-quantum* – suited for integrating out auxiliary fields to produce a classical observable for the ambient theory, and *quantum* – suited for integrating out auxiliary fields to produce a quantum observable for the ambient theory.

Throughout this section in our treatment of quantum objects we are assuming that the spaces of fields are finite-dimensional. This is almost never the case in the interesting examples. In the setting of local quantum field theory, when the spaces of fields are infinite-dimensional, measures on fields are problematic to define; however the BV Laplacians and the integrals over auxiliary fields (which are perturbative path integrals now) can be made sense of with an appropriate regularization/renormalization procedure (one systematic approach is via Wilson's renormalization flow, cf. [10]). Proofs of propositions 1, 3 cannot be taken for granted in the infinite-dimensional setting, and should be redone within the framework of perturbative path integrals (cf. e.g. the proof of proposition 6 in section 6.3).

In view of this, our treatment of quantum objects in this section should be viewed as a simplified discussion, to motivate the interest in pre-observables and argue that given a pre-observable one can expect a path integral expression for an observable.

4.1. Pre-observables and observables for a classical ambient BV theory.

Definition 4. We will call a (*classical*) *BV theory* a Hamiltonian Q -manifold of degree -1 , i.e. a quadruple $(\mathcal{F}, Q, \Omega, S)$ consisting of a Q -manifold (\mathcal{F}, Q) (the *space of fields* with the *BRST operator*), the degree -1 symplectic structure (the *BV 2-form*) Ω and the degree 0 action S satisfying $\{S, \bullet\}_\Omega = Q$ and the *classical master equation*:

$$(39) \quad \{S, S\}_\Omega = 0$$

Definition 5. A (*classical*) *observable* for a classical BV theory $(\mathcal{F}, Q, \Omega, S)$ is a degree 0 function¹³ $O \in C^\infty(\mathcal{F})$ satisfying

$$Q(O) = 0$$

¹²This is not a part of the standard BV lore; the term is sometimes used in a totally different sense in the literature.

¹³We will be assuming degree 0 condition for observables by default, but sometimes it is interesting to relax it. In such cases we will indicate the degree explicitly.

Observables O and O' are called *equivalent* if $O' - O = Q(\Psi)$ for some $\Psi \in C^\infty(\mathcal{F})$, i.e. O' and O give the same class in Q -cohomology.

Definition 6. For a classical BV theory $(\mathcal{F}, Q, \Omega, S)$ (which we will call the *ambient theory*), a *pre-observable* is a degree -1 Hamiltonian Q -bundle over \mathcal{F} . We will call the fiber the *space of auxiliary fields* \mathcal{F}^{aux} , which comes with its own degree -1 symplectic structure Ω^{aux} and the *action for auxiliary fields* $S^{\text{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}^{\text{aux}})$ satisfying

$$(40) \quad QS^{\text{aux}} + \frac{1}{2}\{S^{\text{aux}}, S^{\text{aux}}\}_{\Omega^{\text{aux}}} = 0$$

Remark 5. Note that if in addition to (40) we have the property

$$\{S^{\text{aux}}, S^{\text{aux}}\}_\Omega = 0$$

then the pre-observable gives a new classical BV theory

$$(\mathcal{F} \times \mathcal{F}^{\text{aux}}, Q + \{S^{\text{aux}}, \bullet\}_{\Omega + \Omega^{\text{aux}}}, \Omega + \Omega^{\text{aux}}, S + S^{\text{aux}})$$

Note also that this cohomological vector field on $\mathcal{F} \times \mathcal{F}^{\text{aux}}$ differs from the one arising from the Q -bundle structure by the term $\{S^{\text{aux}}, \bullet\}_\Omega$ and is in general not projectable to \mathcal{F} .

In the case when \mathcal{F}^{aux} is finite-dimensional, it makes sense to introduce the following notion.

Definition 7. For a classical BV-theory $(\mathcal{F}, Q, \Omega, S)$, a *semi-quantum pre-observable* is a quadruple $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$ consisting of a \mathbb{Z} -graded manifold \mathcal{F}^{aux} and a degree -1 symplectic form Ω^{aux} on it. Further,

- \mathcal{F}^{aux} is endowed with a volume element μ^{aux} compatible with Ω^{aux} in the sense that the associated BV Laplacian on $C^\infty(\mathcal{F}^{\text{aux}})$,

$$\Delta^{\text{aux}} : f \mapsto \frac{1}{2} \operatorname{div}_{\mu^{\text{aux}}} \{f, \bullet\}_{\Omega^{\text{aux}}}$$

satisfies

$$(\Delta^{\text{aux}})^2 = 0$$

- Degree 0 action $S^{\text{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}^{\text{aux}})$ satisfies

$$(41) \quad QS^{\text{aux}} + \frac{1}{2}\{S^{\text{aux}}, S^{\text{aux}}\}_{\Omega^{\text{aux}}} - i\Delta^{\text{aux}}S^{\text{aux}} = 0$$

or, equivalently,

$$(42) \quad (Q - i\Delta^{\text{aux}}) e^{iS^{\text{aux}}} = 0$$

We call two semi-quantum pre-observables $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$ and $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, \tilde{S}^{\text{aux}})$ *equivalent*, if there exists a degree -1 function $R^{\text{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}^{\text{aux}})$ such that

$$(43) \quad e^{i\tilde{S}^{\text{aux}}} - e^{iS^{\text{aux}}} = (Q - i\Delta^{\text{aux}}) \left(e^{iS^{\text{aux}}} R^{\text{aux}} \right)$$

Infinitesimally, this equivalence condition can be written as

$$(44) \quad \tilde{S}^{\text{aux}} - S^{\text{aux}} = QR^{\text{aux}} + \{S^{\text{aux}}, R^{\text{aux}}\}_{\Omega^{\text{aux}}} - i\Delta^{\text{aux}}R^{\text{aux}} + \mathcal{O}(R^2)$$

In particular, a classical pre-observable can be promoted to a semi-quantum pre-observable if there exists a volume element μ^{aux} on \mathcal{F}^{aux} compatible with Ω^{aux} , such that in addition to (40) we have $\Delta^{\text{aux}}S^{\text{aux}} = 0$.

Given a semi-quantum pre-observable, one can construct an observable for the ambient classical BV theory using fiber BV integrals [13] as follows.

Proposition 1. *Given a semi-quantum pre-observable $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$, set*

$$(45) \quad O_{\mathcal{L}} = \int_{\mathcal{L} \subset \mathcal{F}^{\text{aux}}} e^{iS^{\text{aux}}} \sqrt{\mu^{\text{aux}}} |_{\mathcal{L}} \in C^\infty(\mathcal{F})$$

for $\mathcal{L} \subset \mathcal{F}^{\text{aux}}$ a Lagrangian submanifold. Then:

- (i) $O_{\mathcal{L}}$ is an observable, i.e. $Q(O_{\mathcal{L}}) = 0$.
- (ii) If Lagrangian submanifolds \mathcal{L} and \mathcal{L}' can be connected by a Lagrangian homotopy, then the difference $O_{\mathcal{L}'} - O_{\mathcal{L}}$ is Q -exact, i.e. observables $O_{\mathcal{L}}$ and $O_{\mathcal{L}'}$ are equivalent.
- (iii) Given two equivalent semi-quantum pre-observables S^{aux} and \tilde{S}^{aux} , the associated observables $O_{\mathcal{L}}$ and $\tilde{O}_{\mathcal{L}}$ are equivalent.

Proof. Applying Q to the integral (45), we obtain

$$(46) \quad Q(O_{\mathcal{L}}) = Q \int_{\mathcal{L}} e^{iS^{\text{aux}}} = \int_{\mathcal{L}} (Q - i\Delta^{\text{aux}}) e^{iS^{\text{aux}}} = 0$$

where we used (42) and the Stokes' theorem for BV integrals [15], that the BV integral of a Δ -coboundary vanishes. We suppress the measure in the short-hand notation for integrals here. This proves (i).

Now let $\{\mathcal{L}_t\}_{t \in [0,1]}$ be a smooth family of Lagrangian submanifolds of $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}})$ connecting $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{L}' = \mathcal{L}_1$. An infinitesimal deformation of a Lagrangian submanifold \mathcal{L}_t is given by a graph of a Hamiltonian vector field¹⁴

$$(47) \quad \frac{d}{dt} \mathcal{L}_t = \text{graph}(\{\bullet, \Psi_t\}_{\Omega^{\text{aux}}}) \in \Gamma(\mathcal{L}_t, N\mathcal{L}_t)$$

viewed as a section of the normal bundle of \mathcal{L}_t , with $\Psi_t \in C^\infty(\mathcal{L}_t)$, $|\Psi_t| = -1$ the generator. Hence,

$$\begin{aligned} \frac{d}{dt} O_{\mathcal{L}_t} &= \frac{d}{dt} \int_{\mathcal{L}_t} e^{iS^{\text{aux}}} |_{\mathcal{L}_t} \\ &= \int_{\mathcal{L}_t} \{e^{iS^{\text{aux}}}, \Psi_t\}_{\Omega^{\text{aux}}} + e^{iS^{\text{aux}}} \cdot \Delta^{\text{aux}}(\Psi_t) \\ &= \int_{\mathcal{L}_t} \Delta^{\text{aux}}(e^{iS^{\text{aux}}} \Psi_t) - \Delta^{\text{aux}}(e^{iS^{\text{aux}}}) \cdot \Psi_t \\ &= \int_{\mathcal{L}_t} \Delta^{\text{aux}}(e^{iS^{\text{aux}}} \Psi_t) + iQ(e^{iS^{\text{aux}}}) \cdot \Psi_t \\ &= Q\left(i \int_{\mathcal{L}_t} e^{iS^{\text{aux}}} \Psi_t\right) \end{aligned}$$

The term $\Delta^{\text{aux}}(\Psi_t)$ in the second line comes from the transformation of measure $\sqrt{\mu^{\text{aux}}} |_{\mathcal{L}_t}$ by the infinitesimal shift of Lagrangian (47). This calculation implies that

$$O_{\mathcal{L}'} - O_{\mathcal{L}} = Q\left(\int_0^1 dt i \int_{\mathcal{L}_t} e^{iS^{\text{aux}}} \Psi_t\right)$$

and thus proves (ii).

Item (iii) follows immediately from the construction (45), our definition of equivalence (43) and the Stokes' theorem for BV integrals:

$$\tilde{O}_{\mathcal{L}} - O_{\mathcal{L}} = \int_{\mathcal{L}} e^{i\tilde{S}^{\text{aux}}} - e^{iS^{\text{aux}}}$$

¹⁴A general infinitesimal Lagrangian deformation is given by $\text{graph}(\iota_{(\Omega^{\text{aux}})^{-1}\chi_t})$ for any degree -1 closed 1-form $\chi_t \in \Omega^1(\mathcal{L}_t)$. Since in non-zero degree a closed form is automatically *exact* (cf. remark 1), we have (47) with the generator $\Psi_t = \iota_E \chi_t$.

$$= \int_{\mathcal{L}} (Q - i\Delta^{\text{aux}}) \left(e^{iS^{\text{aux}}} R^{\text{aux}} \right) = Q \left(\int_{\mathcal{L}} e^{iS^{\text{aux}}} R^{\text{aux}} \right)$$

□

Here we are implicitly assuming convergence of the integral (45), and for the proof of (ii) the integral over \mathcal{L}_t should converge for every $t \in [0, 1]$.

The following construction allows one to produce new semi-quantum pre-observables out of old ones by partially integrating out the auxiliary fields.

Proposition 2. *Given a semi-quantum pre-observable $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$, assume that \mathcal{F}^{aux} is given as a product¹⁵, $\mathcal{F}^{\text{aux}} = \mathcal{F}_z^{\text{aux}} \times \tilde{\mathcal{F}}^{\text{aux}}$, so that Ω^{aux} and μ^{aux} split:*

$$\Omega^{\text{aux}} = \Omega_z^{\text{aux}} + \tilde{\Omega}^{\text{aux}}, \quad \mu^{\text{aux}} = \mu_z^{\text{aux}} \times \tilde{\mu}^{\text{aux}}$$

Define $S_z^{\text{aux}} \in C^\infty(\mathcal{F} \times \mathcal{F}_z^{\text{aux}})$ by

$$e^{iS_z^{\text{aux}}} = \int_{\tilde{\mathcal{L}} \subset \tilde{\mathcal{F}}^{\text{aux}}} e^{iS^{\text{aux}}} \sqrt{\tilde{\mu}^{\text{aux}}} |_{\mathcal{L}}$$

with $\tilde{\mathcal{L}}$ a Lagrangian submanifold of $(\tilde{\mathcal{F}}^{\text{aux}}, \tilde{\Omega}^{\text{aux}})$. Then

- (i) $(\mathcal{F}_z^{\text{aux}}, \Omega_z^{\text{aux}}, \mu_z^{\text{aux}}, S_z^{\text{aux}})$ is a semi-quantum pre-observable for the same ambient classical BV theory.
- (ii) The observable for the ambient theory induced from S_z^{aux} using (45) with Lagrangian $\mathcal{L}_z \subset \mathcal{F}_z^{\text{aux}}$ is equivalent to the one induced directly from S^{aux} using Lagrangian $\mathcal{L} \subset \mathcal{F}^{\text{aux}}$, provided that \mathcal{L} can be connected with $\mathcal{L}_z \times \tilde{\mathcal{L}}$ by a Lagrangian homotopy in \mathcal{F}^{aux} .

Proof. Similarly to (46), we have

$$(Q - i\Delta_z^{\text{aux}}) e^{iS_z^{\text{aux}}} = \int_{\tilde{\mathcal{L}}} \underbrace{(Q - i\Delta_z^{\text{aux}} - i\tilde{\Delta}^{\text{aux}})}_{-i\Delta^{\text{aux}}} e^{iS^{\text{aux}}} = 0$$

which proves (i). Item (ii) is an immediate corollary of (ii) of proposition 1. □

4.2. Pre-observables and observables for a quantum ambient BV theory.

In the case when the space of fields of the ambient theory \mathcal{F} is finite-dimensional, it makes sense to introduce the following notions.

Definition 8 (cf. [15]). A *quantum BV theory* is a quadruple $(\mathcal{F}, \Omega, \mu, S)$ where (\mathcal{F}, Ω) is a \mathbb{Z} -graded manifold with a degree -1 symplectic form Ω ; μ is a volume element on \mathcal{F} , compatible with Ω as in definition 7, i.e. the BV Laplacian $\Delta : f \mapsto \frac{1}{2} \text{div}_\mu \{f, \bullet\}_\Omega$ satisfies $\Delta^2 = 0$; the degree 0 action $S \in C^\infty(\mathcal{F})$ is required to satisfy the *quantum master equation*

$$(48) \quad \frac{1}{2} \{S, S\}_\Omega - i\Delta S = 0$$

or, equivalently,

$$\Delta e^{iS} = 0$$

For a quantum BV theory it is convenient to introduce a degree 1 second order differential operator

$$\delta_{\text{BV}} = \{S, \bullet\}_\Omega - i\Delta = e^{-iS} (-i\Delta) e^{iS}$$

which satisfies $(\delta_{\text{BV}})^2 = 0$.

¹⁵Subscript “z” refers to auxiliary zero-modes.

Definition 9. A *quantum observable*¹⁶ for a quantum BV theory is a degree 0 function $O \in C^\infty(\mathcal{F})$ satisfying

$$\delta_{\text{BV}}O = 0$$

which is equivalent to

$$\Delta(O e^{iS}) = 0$$

Observables O and O' are called *equivalent* if $O' - O = \delta_{\text{BV}}(\Psi)$ for some $\Psi \in C^\infty(\mathcal{F})$, i.e. O' and O give the same class in δ_{BV} -cohomology.

Given a quantum observable, one can define its *correlator* as

$$(49) \quad \langle O \rangle_{\mathcal{L}} = \frac{\int_{\mathcal{L} \subset \mathcal{F}} O e^{iS} \sqrt{\mu}|_{\mathcal{L}}}{\int_{\mathcal{L} \subset \mathcal{F}} e^{iS} \sqrt{\mu}|_{\mathcal{L}}} \in \mathbb{C}$$

where \mathcal{L} is a Lagrangian submanifold of (\mathcal{F}, Ω) and we are assuming convergence. By the Stokes' theorem for BV integrals, this expression does not change with Lagrangian homotopy of \mathcal{L} : $\langle O \rangle_{\mathcal{L}} = \langle O \rangle_{\mathcal{L}'}$, and also correlators of equivalent observables coincide: $\langle O \rangle_{\mathcal{L}} = \langle O' \rangle_{\mathcal{L}}$, cf. [15].

Finally, if both \mathcal{F} and \mathcal{F}^{aux} are finite-dimensional, the following definition makes sense.

Definition 10. A *quantum pre-observable* for a quantum BV theory is a semi-quantum pre-observable $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$ where instead of equation (41) we require that $S + S^{\text{aux}}$ satisfies the quantum master equation on $\mathcal{F} \times \mathcal{F}^{\text{aux}}$:

$$(\Delta + \Delta^{\text{aux}}) e^{i(S+S^{\text{aux}})} = 0$$

The analog of proposition 1 in the setting of quantum BV theory is as follows.

Proposition 3. Given a quantum pre-observable $(\mathcal{F}^{\text{aux}}, \Omega^{\text{aux}}, \mu^{\text{aux}}, S^{\text{aux}})$, define again $O_{\mathcal{L}}$ as

$$O_{\mathcal{L}} = \int_{\mathcal{L} \subset \mathcal{F}^{\text{aux}}} e^{iS^{\text{aux}}} \sqrt{\mu^{\text{aux}}}|_{\mathcal{L}} \in C^\infty(\mathcal{F})$$

for $\mathcal{L} \subset \mathcal{F}^{\text{aux}}$ a Lagrangian submanifold. Then:

- (i) $O_{\mathcal{L}}$ is a quantum observable, i.e. $\delta_{\text{BV}}O_{\mathcal{L}} = 0$.
- (ii) If Lagrangian submanifolds \mathcal{L} and \mathcal{L}' can be connected by a Lagrangian homotopy, then the difference $O_{\mathcal{L}'} - O_{\mathcal{L}}$ is δ_{BV} -exact.

Proof. The proof is obtained from the proof of proposition 1 by replacing Q by $-i\Delta, S^{\text{aux}}$ by $S + S^{\text{aux}}$ and O by $O e^{iS}$ everywhere. \square

Similarly, proposition 2 holds in the context of quantum pre-observables.

Remark 6. It is customary for perturbative quantization to introduce a formal parameter (the ‘‘Planck’s constant’’) \hbar by making a rescaling

$$(50) \quad \Omega \rightarrow \hbar^{-1}\Omega, \quad S \rightarrow \hbar^{-1}S$$

(which means $\Omega^{\text{old}} = \hbar^{-1}\Omega^{\text{new}}$ etc.) where the new action is a formal power series in \hbar , $S \in C^\infty(\mathcal{F})[[\hbar]]$. Hamiltonian vector field $\{S, \bullet\}_{\Omega}$ does not rescale and the

¹⁶Here we define a quantum observable in the framework of Lagrangian field theory in BV formalism, as something that can be averaged over the space of fields to yield a correlator (49). In other contexts one has quite different definitions of quantum observables: e.g. in the framework of Atiyah’s topological quantum field theory [12], an observable is a pair consisting of a submanifold $\gamma \subset \Sigma$ of the spacetime manifold and a vector \hat{O}_γ in the space of states associated to the boundary of a tubular neighborhood of γ in Σ . The passage from the first picture to the second one goes through path integral quantization of the ambient topological theory on the tubular neighborhood (as on a manifold with boundary) with the insertion of observable.

BV Laplacian rescales as $\Delta \rightarrow \hbar\Delta$. For the structure on auxiliary fields, one does the same:

$$(51) \quad \Omega^{\text{aux}} \rightarrow \hbar^{-1}\Omega^{\text{aux}}, \quad S^{\text{aux}} \rightarrow \hbar^{-1}S^{\text{aux}}$$

With these redefinitions the equation (41) on semi-quantum pre-observables becomes

$$QS^{\text{aux}} + \frac{1}{2}\{S^{\text{aux}}, S^{\text{aux}}\}_{\Omega^{\text{aux}}} - i\hbar\Delta^{\text{aux}}S^{\text{aux}} = 0$$

and can be solved by obstruction theory order by order in \hbar for $S^{\text{aux}} = S_{(0)}^{\text{aux}} + \hbar S_{(1)}^{\text{aux}} + \hbar^2 S_{(2)}^{\text{aux}} + \dots$ starting from $S_{(0)}^{\text{aux}}$ a solution of (40). Likewise, the BV push-forward formula (45) becomes

$$O = \int_{\mathcal{L} \subset \mathcal{F}^{\text{aux}}} e^{\frac{i}{\hbar}S^{\text{aux}}} \sqrt{\mu^{\text{aux}}}|_{\mathcal{L}} \in C^\infty(\mathcal{F})[[\hbar]]$$

and can be evaluated by the stationary phase formula.

Note that one can also introduce two independent Planck's constants $\hbar, \hbar^{\text{aux}}$ for the ambient (50) and auxiliary (51) theory respectively.

5. AKSZ PRE-OBSERVABLE ASSOCIATED TO A HAMILTONIAN Q -BUNDLE

Let $(\mathcal{M}, Q, \omega = \delta\alpha, \Theta)$ be a Hamiltonian Q -manifold and Σ a closed oriented spacetime manifold, $\dim \Sigma = |\omega| + 1$. Then we have the AKSZ theory with

$$(52) \quad \mathcal{F}_\Sigma = \text{Map}(T[1]\Sigma, \mathcal{M}), \quad Q_\Sigma = (d_\Sigma)^{\text{lifted}} + Q^{\text{lifted}}, \\ \Omega_\Sigma = (-1)^{\dim \Sigma} \tau_\Sigma(\omega), \quad S_\Sigma = \iota_{d_\Sigma^{\text{lifted}}} \tau_\Sigma(\alpha) + \tau_\Sigma(\Theta)$$

as in section 2.

Given a trivial Hamiltonian Q -bundle over \mathcal{M} with fiber data $(\mathcal{N}, \mathcal{A}, \omega' = \delta\alpha', \Theta')$, a closed oriented manifold γ of dimension $\dim \gamma = |\omega'| + 1$ and a smooth map¹⁷ $i : \gamma \rightarrow \Sigma$, we can construct a pre-observable for the AKSZ theory (52) with target \mathcal{M} as follows. Set

$$(53) \quad \mathcal{F}_\gamma = \text{Map}(T[1]\gamma, \mathcal{N}), \quad \mathcal{A}_\gamma = (d_\gamma)^{\text{lifted}} + p^*\mathcal{A}^{\text{lifted}}, \\ \Omega_\gamma = (-1)^{\dim \gamma} \tau_\gamma(\omega'), \quad S_\gamma = \iota_{(d_\gamma)^{\text{lifted}}} \tau_\gamma(\alpha') + p^*\tau_\gamma^{\text{tot}}(\Theta')$$

Here the notations are as follows.

- $\tau_\gamma : \Omega^\bullet(\mathcal{N}) \rightarrow \Omega^\bullet(\mathcal{F}_\gamma)$ and $\tau_\gamma^{\text{tot}} : \Omega^\bullet(\mathcal{E}) \rightarrow \Omega^\bullet(\text{Map}(T[1]\gamma, \mathcal{E}))$ are the transgression maps, defined as in (11); $\mathcal{E} = \mathcal{M} \times \mathcal{N}$ is the total space of the target Hamiltonian Q -bundle.
- $\text{Map } p = i^* : \mathcal{F}_\Sigma \rightarrow \text{Map}(T[1]\gamma, \mathcal{M})$ is the pull-back of ambient fields by $i : \gamma \rightarrow \Sigma$ and $p^* : C^\infty(\text{Map}(T[1]\gamma, \mathcal{M})) \rightarrow C^\infty(\mathcal{F}_\Sigma)$ is the pull-back by p . By the trivial extension to auxiliary fields, $p = p \times \text{id}_{\mathcal{F}_\gamma}$ also maps $\mathcal{F}_\Sigma \times \mathcal{F}_\gamma$ to $\text{Map}(T[1]\gamma, \mathcal{E})$.
- $(d_\gamma)^{\text{lifted}} \in \mathfrak{X}(\mathcal{F}_\gamma) \subset \mathfrak{X}^{\text{vert}}(\mathcal{F}_\Sigma \times \mathcal{F}_\gamma)$ is the lifting of the de Rham differential on γ to a vector field on the mapping space $\text{Map}(T[1]\gamma, \mathcal{N})$.
- $\mathcal{A}^{\text{lifted}}$ is the lifting of the vertical vector field $\mathcal{A} \in \mathfrak{X}^{\text{vert}}(\mathcal{E})$ to a vertical vector field on the mapping space $\text{Map}(T[1]\gamma, \mathcal{E})$, so that $p^*\mathcal{A}^{\text{lifted}}$ becomes a vertical vector field on $\mathcal{F}_\Sigma \times \mathcal{F}_\gamma$.

Proposition 4. *The data (53) defines a classical pre-observable for the AKSZ theory (52), in particular*

$$(54) \quad Q_\Sigma S_\gamma + \frac{1}{2}\{S_\gamma, S_\gamma\}_{\Omega_\gamma} = 0$$

¹⁷By default, we assume that i is an embedding, cf. remark 10 below.

Proof. First let us check that \mathcal{A}_γ as defined in (53) is the Hamiltonian vector field for S_γ . Introduce the notations for the source (kinetic) and target parts of \mathcal{A}_γ and S_γ :

$$\mathcal{A}_\gamma = \underbrace{(d_\gamma)^{\text{lifted}}}_{\mathcal{A}_\gamma^{\text{kin}}} + \underbrace{p^* \mathcal{A}^{\text{lifted}}}_{\mathcal{A}_\gamma^{\text{target}}}, \quad S_\gamma = \underbrace{\iota_{(d_\gamma)^{\text{lifted}}} \tau_\gamma(\alpha')}_{S_\gamma^{\text{kin}}} + \underbrace{p^* \tau_\gamma^{\text{tot}}(\Theta')}_{S_\gamma^{\text{target}}}$$

Then we have

$$(55) \quad \begin{aligned} \iota_{\mathcal{A}_\gamma^{\text{kin}}} \Omega_\gamma &= \iota_{(d_\gamma)^{\text{lifted}}} (-1)^{\dim \gamma} \tau_\gamma(\omega') = \iota_{(d_\gamma)^{\text{lifted}}} \delta \tau_\gamma(\alpha') = \\ &= \underbrace{L_{(d_\gamma)^{\text{lifted}}} \tau_\gamma(\alpha') + \delta \iota_{(d_\gamma)^{\text{lifted}}} \tau_\gamma(\alpha')}_{=0 \text{ by (12)}} = \delta S_\gamma^{\text{kin}} = \delta^{\text{vert}} S_\gamma^{\text{kin}} \end{aligned}$$

and

$$(56) \quad \begin{aligned} \iota_{\mathcal{A}_\gamma^{\text{target}}} \Omega_\gamma &= p^* \iota_{\mathcal{A}^{\text{lifted}}} (-1)^{\dim \gamma} \tau_\gamma^{\text{tot}}(\omega') = \\ &= (-1)^{\dim \gamma} p^* \tau_\gamma^{\text{tot}}(\underbrace{\iota_{\mathcal{A}} \omega'}_{=\delta^{\text{vert}} \Theta'}) = \delta^{\text{vert}} p^* \tau_\gamma^{\text{tot}}(\Theta') = \delta^{\text{vert}} S_\gamma^{\text{target}} \end{aligned}$$

Here δ^{vert} stands for the de Rham differential in the fiber direction, as in section 3.3. Collecting (55,56), we get

$$\{S_\gamma, \bullet\}_{\Omega_\gamma} = \mathcal{A}_\gamma$$

Next, let us prove (54). Note that

$$Q_\Sigma S_\gamma^{\text{kin}} = 0$$

since S_γ^{kin} does not depend on the ambient fields (i.e. is a function on $\mathcal{F}_\Sigma \times \mathcal{F}_\gamma$, constant in the direction of \mathcal{F}_Σ); also

$$Q_\Sigma^{\text{kin}} S_\gamma^{\text{target}} = p^*(Q_\Sigma^{\text{kin}}|_\gamma \tau_\gamma^{\text{tot}}(\Theta')) = 0$$

as in (12), where $Q_\Sigma^{\text{kin}}|_\gamma$ is the lifting of d_γ to $\text{Map}(T[1]\gamma, \mathcal{M})$, extended to a horizontal vector field on $\text{Map}(T[1]\gamma, \mathcal{E})$. Thus

$$(57) \quad \begin{aligned} Q_\Sigma S_\gamma + \frac{1}{2} \{S_\gamma, S_\gamma\}_{\Omega_\gamma} &= \\ &= Q_\Sigma^{\text{target}} S_\gamma^{\text{target}} + \frac{1}{2} \{S_\gamma^{\text{kin}}, S_\gamma^{\text{kin}}\}_{\Omega_\gamma} + \{S_\gamma^{\text{kin}}, S_\gamma^{\text{target}}\} + \frac{1}{2} \{S_\gamma^{\text{target}}, S_\gamma^{\text{target}}\}_{\Omega_\gamma} \\ &= \frac{1}{2} \{S_\gamma^{\text{kin}}, S_\gamma^{\text{kin}}\}_{\Omega_\gamma} + \underbrace{\mathcal{A}_\gamma^{\text{kin}} S_\gamma^{\text{target}}}_{0 \text{ by (12)}} + (Q_\Sigma^{\text{target}} + \frac{1}{2} \mathcal{A}_\gamma^{\text{target}}) S_\gamma^{\text{target}} \end{aligned}$$

Here the first term on the r.h.s. has $((d_\gamma)^{\text{lifted}})^2 = 0$ as its Hamiltonian vector field on \mathcal{F}_γ , thus it is a degree 1 constant function on \mathcal{F}_γ , and hence vanishes. Therefore, continuing the calculation (57) and using (13), we have

$$(58) \quad \begin{aligned} Q_\Sigma S_\gamma + \frac{1}{2} \{S_\gamma, S_\gamma\}_{\Omega_\gamma} &= (-1)^{\dim \gamma} p^* \tau_\gamma^{\text{tot}}(Q\Theta' + \frac{1}{2} \mathcal{A}\Theta') = \\ &= (-1)^{\dim \gamma} p^* \tau_\gamma^{\text{tot}}(Q\Theta' + \frac{1}{2} \{\Theta', \Theta'\}_{\omega'}) = 0 \end{aligned}$$

since the target is a Hamiltonian Q -bundle and equation (30) holds there. This finishes the proof. \square

In coordinates. Alongside with the superfield (18) for the ambient theory on Σ , we now have a superfield for the auxiliary theory on γ :

$$(59) \quad Y^i(v^1, \dots, v^{\dim \gamma}, \xi^1, \dots, \xi^{\dim \gamma}) =$$

$$= \sum_{k=1}^{\dim \gamma} \underbrace{\sum_{1 \leq \nu_1 < \dots < \nu_k \leq \dim \gamma} Y_{\nu_1 \dots \nu_k}^i(v^1, \dots, v^{\dim \gamma}) \xi^{\nu_1} \dots \xi^{\nu_k}}_{Y_{(k)}^i(v, \xi)}$$

associated to local homogeneous coordinates y^i on \mathcal{N} . Here $v^1, \dots, v^{\dim \gamma}$ are local coordinates on γ and ξ^ν stands for dv^ν , $\nu = 1, \dots, \dim \gamma$.

Let α' and ω' be locally given on \mathcal{N} as $\alpha' = \alpha'_i(y) \delta y^i$, $\omega' = \frac{1}{2} \omega'_{ij}(y) \delta y^i \wedge \delta y^j$. Then the BV 2-form for auxiliary fields is:

$$\Omega_\gamma = (-1)^{\dim \gamma} \int_\gamma \frac{1}{2} \omega'_{ij}(Y) \delta Y^i \wedge \delta Y^j$$

Its primitive is:

$$\alpha_\gamma = \int_\gamma \alpha'_i(Y) \delta Y^i$$

The action for auxiliary fields is:

$$S_\gamma = \int_\gamma \alpha'_i(Y) dY^i + \int_\gamma \Theta'(i^* X, Y)$$

Vertical part of the cohomological vector field \mathcal{A}_γ acts on the auxiliary superfield by

$$\mathcal{A}_\gamma Y^i = dY^i + \mathcal{A}^i(i^* X, Y)$$

if on the target we have $\mathcal{A} = \mathcal{A}^i(x, y) \frac{\partial}{\partial y^i}$.

Remark 7. In the case when $i(\gamma) \subset \Sigma$ is an embedded submanifold of positive codimension, the expression

$$(60) \quad \{S_\gamma, S_\gamma\}_{\Omega_\Sigma}$$

is ill-defined, because it formally contains $\int_\Sigma (\delta_\gamma)^2 \dots$ where δ_γ is the delta function supported on $i(\gamma) \subset \Sigma$. On the other hand, if we use adapted local coordinates $u^1, \dots, u^{\dim \Sigma}$ on Σ in which $i(\gamma)$ is given by $u^{\dim \gamma + 1} = \dots = u^{\dim \Sigma} = 0$, the auxiliary action S_γ only depends on the components of the ambient superfield $X_{\mu_1 \dots \mu_k}^a$ with $\mu_1, \dots, \mu_k \leq \dim \gamma$, whereas the BV 2-form Ω_Σ couples the components $X_{\mu_1 \dots \mu_k}^a$ and $X_{\nu_1 \dots \nu_l}^b$ if and only if $\{\mu_1, \dots, \mu_k\} \sqcup \{\nu_1, \dots, \nu_l\} = \{1, \dots, \dim \Sigma\}$ (for the sake of this argument we choose local Darboux coordinates on \mathcal{M} in which $\omega_{ab}(x)$ does not depend on x). So, formally (60) is $\infty \cdot 0$, and we naively regularize it to zero. Thus, by remark 5, we have (in the sense of our regularization) the classical master equation on $\mathcal{F}_\Sigma \times \mathcal{F}_\gamma$:

$$(61) \quad \{S_\Sigma + S_\gamma, S_\Sigma + S_\gamma\}_{\Omega_\Sigma + \Omega_\gamma} = 0$$

In the case when $i : \gamma \rightarrow \Sigma$ is a diffeomorphism, we have

$$\{S_\gamma, S_\gamma\}_{\Omega_\Sigma} = \{S_\gamma^{\text{target}}, S_\gamma^{\text{target}}\}_{\Omega_\Sigma} = (-1)^{\dim \Sigma} p^* \tau_\gamma^{\text{tot}}(\{\Theta', \Theta'\}_\omega)$$

Thus, (61) holds if and only if on the target we have

$$(62) \quad \{\Theta', \Theta'\}_\omega = 0$$

in addition to (30).

Remark 8. If $\dim \gamma = 1$, ω' is not automatically exact. The interesting case is when ω' is not exact but satisfies Bohr-Sommerfeld integrality condition, see remark 2. We assume that γ is a circle; the discussion extends to a disjoint union of circles trivially. The kinetic part of the action for auxiliary fields S_γ^{kin} cannot be defined as in (53), because the primitive α' does not exist globally on \mathcal{N} . Instead one can define

$$(63) \quad e^{i S_\gamma^{\text{kin}}(Y)} := \text{Hol}_{Y_{(0)}^* \nabla'}(\gamma) \in U(1)$$

– the holonomy around γ of the pull-back of the connection ∇' (cf. remark 2) by $Y_{(0)} : \gamma \rightarrow \mathcal{N}$. Here the l.h.s. should be viewed as one symbol. Target part of the action is defined as before, so we have

$$e^{iS_\gamma(X,Y)} = e^{i\int_\gamma \Theta'(i^*X,Y)} \text{Hol}_{Y_{(0)}}^* \nabla'(\gamma)$$

Otherwise, using Stokes' theorem one can rewrite the kinetic part of the auxiliary action as

$$(64) \quad S_\gamma^{\text{kin}}(Y) = \int_D \tilde{Y}_{(0)}^* \omega'$$

for D a disc bounded by γ and $\tilde{Y}_{(0)}$ an arbitrary extension of $Y_{(0)}$ from γ to D . Due to integrality of $\omega'/2\pi$, arbitrariness of $\tilde{Y}_{(0)}$ results in (64) being defined modulo multiples of 2π :

$$S_\gamma^{\text{kin}}(Y) \in \mathbb{R}/2\pi\mathbb{Z}$$

From this point of view, the l.h.s. of (63) is well-defined.

It is easy to see that the proof of proposition 4 still works in this case (e.g. because one can find a primitive of ω' in the tubular neighborhood of the loop $Y_{(0)}(\gamma) \subset \mathcal{N}$).

Remark 9. Pre-observable (53) is invariant w.r.t. $\text{Diff}(\gamma)$ (reparametrizations) and $\text{Diff}(\Sigma)$ (ambient diffeomorphisms) in the following sense. Let us denote explicitly the dependence of S_γ on $i : \gamma \rightarrow \Sigma$ (through $p = i^*$ in (53)) as $S_\gamma(X, Y; i)$. Then for an ambient diffeomorphism $\Phi \in \text{Diff}(\Sigma)$ and a reparametrization $\phi \in \text{Diff}(\gamma)$ we have

$$(65) \quad S_\gamma(X, Y; \Phi \circ i \circ \phi) = S_\gamma(\Phi^*X, (\phi^{-1})^*Y; i)$$

which can be seen directly from the definition (53).

6. FROM PRE-OBSERVABLES TO OBSERVABLES

The general idea of passage from pre-observables constructed in section 5 to observables for the underlying AKSZ theory is by means of integrating out the auxiliary fields as in proposition 1, i.e.

$$(66) \quad O_\gamma = \int_{\mathcal{L} \subset \mathcal{F}_\gamma} e^{iS_\gamma} \in C^\infty(\mathcal{F}_\Sigma)$$

where \mathcal{L} is a Lagrangian submanifold of $(\mathcal{F}_\gamma, \Omega_\gamma)$. When we want to emphasize the dependence of O_γ on the map $i : \gamma \rightarrow \Sigma$, we will write $O_{\gamma,i}$.

Assuming that we can make sense of the path integral (66), the observable O_γ is expected to have the following properties:

- (i) O_γ depends on the fields of the ambient AKSZ theory only via pull-back by $i : \gamma \rightarrow \Sigma$.
- (ii) Gauge invariance: $Q_\Sigma O_\gamma = 0$ (which is, indeed, our definition of an observable).
- (iii) The class of O_γ in Q_Σ -cohomology is independent of deformations of the gauge fixing \mathcal{L} (as a Lagrangian submanifold of \mathcal{F}_γ).
- (iv) The class of O_γ in Q_Σ -cohomology is invariant under isotopy of γ (reparametrizations of γ homotopic to identity): for $\phi \in \text{Diff}_0(\gamma)$, we have

$$O_{\gamma,i \circ \phi} = O_{\gamma,i} + Q_\Sigma(\dots)$$

- (v) Invariance under ambient diffeomorphisms: for $\Phi \in \text{Diff}(\Sigma)$, we have

$$O_{\gamma, \Phi \circ i}(X) = O_{\gamma,i}(\Phi^*X)$$

(vi) Invariance of the correlator $\langle O_\gamma \rangle$ under ambient isotopy: for $\Phi \in \text{Diff}_0(\Sigma)$, we have

$$\langle O_{\gamma, \Phi \circ i} \rangle = \langle O_{\gamma, i} \rangle$$

Here (i) follows from our construction of the pre-observable (53) and (v) follows from (i). Properties (ii) and (iii) are expected to hold in view of the proposition 1 for finite-dimensional fiber BV integrals. Property (iv) formally follows from (65), (iii) and $\text{Diff}(\gamma)$ -invariance of the measure on \mathcal{F}_γ :

$$\begin{aligned} (67) \quad O_{\gamma, i \circ \phi} &= \int_{\mathcal{L} \subset \mathcal{F}_\gamma} \mathcal{D}Y e^{iS_\gamma(X, Y; i \circ \phi)} = \int_{\mathcal{L}} \mathcal{D}Y e^{iS_\gamma(X, (\phi^{-1})^* Y; i)} \\ &= \int_{(\phi^{-1})^* \mathcal{L}} (\phi^{-1})_* (\mathcal{D}Y) e^{iS_\gamma(X, Y; i)} = \int_{(\phi^{-1})^* \mathcal{L}} \mathcal{D}Y e^{iS_\gamma(X, Y; i)} \\ &= \int_{\mathcal{L}} \mathcal{D}Y e^{iS_\gamma(X, Y; i)} + Q_\Sigma(\dots) = O_{\gamma, i} + Q_\Sigma(\dots) \end{aligned}$$

Here $\int_{\mathcal{L}} \mathcal{D}Y \dots$ should be understood as $\int_{\mathcal{L}} \sqrt{|\mu|} \mathcal{L} \dots$ where μ is the path integral measure on \mathcal{F}_γ . Note that we do use the fact that ϕ is an isotopy rather than a general diffeomorphism, since otherwise Lagrangians \mathcal{L} and $(\phi^{-1})^* \mathcal{L}$ are not guaranteed to be homotopic. Property (vi) formally follows from (v), $\text{Diff}(\Sigma)$ -invariance of the AKSZ action (17) and of the path integral measure on \mathcal{F}_Σ , and from the BV-Stokes' theorem:

$$\begin{aligned} (68) \quad \langle O_{\gamma, \Phi \circ i} \rangle &= \int_{\mathcal{L} \subset \mathcal{F}_\Sigma} \mathcal{D}X O_{\gamma, \Phi \circ i}(X) e^{iS_\Sigma(X)} = \int_{\mathcal{L}} \mathcal{D}X O_{\gamma, i}(\Phi^* X) e^{iS_\Sigma(X)} \\ &= \int_{\Phi^* \mathcal{L}} (\Phi^*)_* (\mathcal{D}X) O_{\gamma, i}(X) e^{iS_\Sigma((\Phi^{-1})^* X)} = \int_{\Phi^* \mathcal{L}} \mathcal{D}X O_{\gamma, i}(X) e^{iS_\Sigma(X)} \\ &= \int_{\mathcal{L}} \mathcal{D}X O_{\gamma, i}(X) e^{iS_\Sigma(X)} = \langle O_{\gamma, i} \rangle \end{aligned}$$

Remark 10. For our construction of observable O_γ it is not a priori necessary to impose any restrictions on the map $i : \gamma \rightarrow \Sigma$. However, if we want to make sense of the correlator $\langle O_\gamma \rangle$ via perturbation theory for the path integral (which is outside of the scope of this paper), we have to require that i is an embedding. The technical reason is that if $i : \gamma \rightarrow \Sigma$ is not an embedding, it does not lift to a map between compactified configuration spaces of pairs of points on γ and Σ , $\text{Conf}_2(\gamma) \rightarrow \text{Conf}_2(\Sigma)$, which prevents one from defining the pull-back of the propagator of the ambient theory to γ , which in turn leads to Feynman diagrams for the correlator being ill-defined.

We will not try to give meaning to the path integral (66) in the most general situation here, but rather will discuss several special cases.

6.1. Case $\mathcal{N} = \text{point}$. In the case when \mathcal{N} is a point the target Hamiltonian Q -bundle is described in example (v) in section 3.4: $\mathcal{A} = 0$, $\omega' = 0$, $\Theta' = \theta \in C^\infty(\mathcal{M})$ — a Q -cocycle of degree p for the target \mathcal{M} . The associated pre-observable (53) for a p -dimensional closed manifold γ and a map $i : \gamma \rightarrow \Sigma$ is:

$$\mathcal{F}_\gamma = \text{point}, \quad \mathcal{A}_\gamma = 0, \quad \Omega_\gamma = 0, \quad S_\gamma(X) = \int_\gamma \theta(i^* X)$$

The passage to the observable (66) is trivial in this setting, since there are no auxiliary fields:

$$(69) \quad O_\gamma = e^{i \int_\gamma \theta(i^* X)}$$

This observable obviously satisfies properties (i-v) above.

6.2. Case of 1-dimensional observables. Let $i : \gamma = S^1 \rightarrow \Sigma$ be a circle mapped into Σ and suppose we have a Hamiltonian Q -bundle of degree 0 over the AKSZ target $(\mathcal{M}, Q, \omega = \delta\alpha, \Theta)$ with fiber data $(\mathcal{N}, \mathcal{A}, \omega' = \delta\alpha', \Theta')$. We will assume for simplicity that \mathcal{N} is concentrated in degree 0, i.e. (\mathcal{N}, ω') is an ordinary (non-graded) symplectic manifold.

In local coordinates, the auxiliary superfield (59) is:

$$Y^i(u, du) = Y_{(0)}^i(u) + Y_{(1)}^i(u) du$$

where u is the coordinate on $\gamma = S^1$ and index i corresponds to local coordinates on \mathcal{N} . For the pre-observable (53) we have

$$(70) \quad \mathcal{F}_\gamma = \text{Map}(T[1]\gamma, \mathcal{N}) \\ = \left\{ (Y_{(0)}, Y_{(1)}) \mid Y_{(0)} : \gamma \rightarrow \mathcal{N}, Y_{(1)} \in \Gamma(\gamma, T^*\gamma \otimes Y_{(0)}^* T\mathcal{N})[-1] \right\}, \\ \Omega_\gamma = - \oint_\gamma \omega'_{ij}(Y_{(0)}) \delta Y_{(0)}^i \wedge \delta Y_{(1)}^j + \frac{1}{2} Y_{(1)}^k \partial_k \omega'_{ij}(Y_{(0)}) \delta Y_{(0)}^i \wedge \delta Y_{(0)}^j, \\ S_\gamma = \oint_\gamma \alpha'_i(Y_{(0)}) dY_{(0)}^i + \Theta'(i^* X, Y)$$

One natural choice of Lagrangian in (66) is to set

$$(71) \quad \mathcal{L} = \text{Map}(\gamma, \mathcal{N}) \subset \text{Map}(T[1]\gamma, \mathcal{N})$$

I.e. the 1-form component $Y_{(1)}$ of the auxiliary superfield vanishes on \mathcal{L} . With this choice (66) becomes

$$(72) \quad O_\gamma = \int_{\text{Map}(\gamma, \mathcal{N})} \mathcal{D}Y_{(0)} e^{i \oint_\gamma \alpha'_i(Y_{(0)}) dY_{(0)}^i + \Theta'(i^* X, Y_{(0)})} \\ = \int_{\text{Map}(\gamma, \mathcal{N})} \mathcal{D}Y_{(0)} e^{i \oint_\gamma \alpha'_i(Y_{(0)}) dY_{(0)}^i + i^* X_{(1)}^a \partial_a \Theta'(i^* X_{(0)}, Y_{(0)})}$$

Note that with this gauge fixing, one formally has a strict version of property (iv), namely that O_γ is invariant under isotopy of γ (instead of only the class of O_γ in Q_Σ -cohomology being invariant). This follows from invariance of \mathcal{L} under $\text{Diff}(\gamma)$, together with $\text{Diff}(\gamma)$ -invariance of the path integral measure $\mathcal{D}Y_{(0)}$.

The quantum mechanical path integral (72) can be understood in the Hamiltonian formalism, using the fiber geometric quantization of the target Hamiltonian Q -bundle, as follows.

Proposition 5. *Assume that the symplectic manifold (\mathcal{N}, ω') can be geometrically quantized to a complex vector space of states \mathcal{H} and the Hamiltonian $\Theta' \in C^\infty(\mathcal{M} \times \mathcal{N})$ can be quantized (viewing coordinates on \mathcal{M} as external parameters) to an operator-valued function on \mathcal{M} , $\hat{\Theta}' \in C^\infty(\mathcal{M}) \otimes \text{End}(\mathcal{H})$ satisfying*

$$(73) \quad Q\hat{\Theta}' + i(\hat{\Theta}')^2 = 0$$

(the quantum counterpart of the equation (30)). Then define O_γ as the trace of a path ordered exponential¹⁸

$$(74) \quad O_\gamma = \text{tr}_{\mathcal{H}} \mathcal{P} \exp \left(i \oint_\gamma \hat{\Theta}'(i^* X) \right)$$

Then

$$Q_\Sigma O_\gamma = 0$$

i.e. O_γ is an observable.

¹⁸To define the path ordered exponential, we need to choose a starting point $p \in \gamma$, but the dependence on p is cancelled by taking the trace.

Proof. Let $j : [0, 1] \rightarrow \Sigma$ be a path in Σ parameterized by $t \in [0, 1]$. Let us denote $\psi = \hat{\Theta}'(j^* X) \in \Omega^\bullet([0, 1]) \hat{\otimes} C^\infty(\mathcal{F}_\Sigma) \otimes \text{End}(\mathcal{H})$, also let $\psi_{(0)}(t)$ and $\psi_{(1)}(t, dt)$ be the 0- and 1-form components of ψ . Consider the path ordered exponential

$$(75) \quad W_j = \mathcal{P} \exp \left(i \int_0^1 \psi_{(1)} \right) \\ = \lim_{N \rightarrow \infty} \overleftarrow{\prod}_{0 \leq k < N} \left(\text{id}_{\mathcal{H}} + i \iota_{\frac{1}{N} \partial_t} \psi_{(1)}(k/N, dt) \right) \in C^\infty(\mathcal{F}_\Sigma) \otimes \text{End}(\mathcal{H})$$

Then

$$(76) \quad Q_\Sigma W_j = -i \int_0^1 \mathcal{P} \exp \left(i \int_t^1 \psi_{(1)} \right) \cdot Q_\Sigma \psi(t, dt) \cdot \mathcal{P} \exp \left(i \int_0^t \psi_{(1)} \right) \\ = -i \int_0^1 \mathcal{P} \exp \left(i \int_t^1 \psi_{(1)} \right) \cdot \left(dt \frac{\partial}{\partial t} \psi_{(0)}(t) - i [\psi_{(0)}(t), \psi_{(1)}(t, dt)] \right) \cdot \mathcal{P} \exp \left(i \int_0^t \psi_{(1)} \right) \\ = -i \lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \overleftarrow{\prod}_{l < k < N} \left(\text{id}_{\mathcal{H}} + i \iota_{\frac{1}{N} \partial_t} \psi_{(1)} \left(\frac{k}{N}, dt \right) \right) \cdot \\ \cdot \left(\psi_{(0)} \left(\frac{l+1}{N} \right) \left(\text{id}_{\mathcal{H}} + i \iota_{\frac{1}{N} \partial_t} \psi_{(1)} \left(\frac{l}{N}, dt \right) \right) - \left(\text{id}_{\mathcal{H}} + i \iota_{\frac{1}{N} \partial_t} \psi_{(1)} \left(\frac{l}{N}, dt \right) \right) \psi_{(0)} \left(\frac{l}{N} \right) \right) \\ \cdot \overleftarrow{\prod}_{0 \leq k < l} \left(\text{id}_{\mathcal{H}} + i \iota_{\frac{1}{N} \partial_t} \psi_{(1)} \left(\frac{k}{N}, dt \right) \right) \\ = -i (\psi_{(0)}(1) \cdot W_j - W_j \cdot \psi_{(0)}(0))$$

where we used that due to (73), we have $Q_\Sigma \psi = d\psi - \frac{i}{2}[\psi, \psi]$. For $j = i : S^1 \rightarrow \Sigma$, we have $j(0) = j(1)$ and applying (76), we get

$$Q_\Sigma O_\gamma = \text{tr}_{\mathcal{H}} Q_\Sigma W_j = -i \text{tr}_{\mathcal{H}} [\hat{\Theta}'(X_{(0)}(j(0))), W_j] = 0$$

□

Note that the construction of proposition 5 does not require ω' to be exact, it is enough to require that ω' satisfies Bohr-Sommerfeld integrality condition.

6.3. Torsion-like observables. Now let again $i : \gamma \rightarrow \Sigma$ be an oriented closed manifold of arbitrary dimension mapped into Σ and suppose that \mathcal{N} is a finite-dimensional graded vector space, ω' is a graded-antisymmetric pairing on \mathcal{N} of degree $\dim \Sigma - 1$ and Θ' is quadratic in \mathcal{N} directions:

$$\Theta'(x, y) = \frac{1}{2} \omega'(y, \theta(x)y)$$

with $\theta \in C^\infty(\mathcal{M}) \otimes \text{End}(\mathcal{N})$ of degree 1. Equation (30) is then equivalent to

$$(77) \quad Q\theta - \theta^2 = 0$$

We also assume that

$$(78) \quad \text{Str}_{\mathcal{N}} \theta(x) = 0$$

where $\text{Str}_{\mathcal{N}}$ stands for the super-trace over \mathcal{N} .

One way to make sense of the expression (66) is to choose a Riemannian metric on γ . Then we have the Hodge decomposition for differential forms on γ with values in \mathcal{N}

$$(79) \quad \underbrace{\mathcal{N} \otimes \Omega^\bullet(\gamma)}_{\mathcal{F}_\gamma} = \underbrace{\mathcal{N} \otimes \Omega_{\text{harm}}^\bullet(\gamma)}_{\mathcal{F}_\gamma^z} \oplus \underbrace{\mathcal{N} \otimes \Omega_{\text{exact}}^\bullet(\gamma)}_{\mathcal{F}_\gamma} \oplus \overbrace{\mathcal{N} \otimes \Omega_{\text{co-exact}}^\bullet(\gamma)}^{\tilde{\mathcal{L}}}$$

which can be used to first take the BV integral over the complement of harmonic forms, as in proposition 2, and then take the BV integral over the zero-modes (harmonic forms):

$$(80) \quad O_\gamma = \int_{\mathcal{L}_z \subset \mathcal{F}_\gamma^z} \mathcal{D}Y_z \int_{\tilde{\mathcal{L}} \subset \tilde{\mathcal{F}}_\gamma} \mathcal{D}\tilde{Y} e^{\frac{i}{2} \int_\gamma \omega'(Y, dY + \theta(i^*X)Y)}$$

where $Y = Y_z + \tilde{Y}$ is the splitting of the auxiliary superfield into the harmonic part and the part in the complement; \mathcal{F}_γ^z comes with the degree -1 symplectic structure Ω_γ^z induced from ω' and the Poincaré pairing on cohomology of γ ; $\mathcal{L}_z \subset \mathcal{F}_\gamma^z$ is a Lagrangian vector subspace w.r.t. Ω_γ^z .

The internal integral in (80) is a path integral and can be made sense of using perturbation theory¹⁹:

$$(81) \quad O_\gamma^z(X, Y_z) = \exp\left(\frac{i}{2} \int_\gamma \omega'(Y_z, \theta(i^*X)) (\text{id} + G \theta(i^*X))^{-1} Y_z + \text{Str}_{\mathcal{N}} \sum_{k \geq 2} \frac{(-1)^k}{2k} \int_{\text{Conf}_k(\gamma)} \overleftarrow{\prod}_{1 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta(i^*X)\right)$$

The notations here are:

- $\text{Conf}_k(\gamma)$ is the Fulton-Macpherson-Axelrod-Singer compactification of the configuration space of k points on γ , cf. [5].
- $G = d^*(\Delta + \mathcal{P}_{\text{harm}})^{-1} : \Omega^\bullet(\gamma) \rightarrow \Omega^{\bullet-1}(\gamma)$ is the Hodge-theoretic inverse of d ; its integral kernel extends to a smooth $(\dim \gamma - 1)$ -form η on $\text{Conf}_2(\gamma)$ (the *Hodge propagator*), cf. [5]. By $\mathcal{P}_{\text{harm}}$ we denote the orthogonal projection to harmonic forms, using the Hodge inner product.
- $\pi_{j+1, j} : \text{Conf}_k \rightarrow \text{Conf}_2$ is the map associated to forgetting all points except points j and $j + 1$; also $\pi_j : \text{Conf}_k \rightarrow \text{Conf}_1 = \gamma$ is forgetting all points except the point j . By convention, $\pi_{k+1, k} = \pi_{1, k}$.
- In the first term in the exponential in (81), $\theta(i^*X)$ is understood as a multiplication operator on $\Omega^\bullet(\gamma)$ and an endomorphism of \mathcal{N} , depending on ambient fields. Expression $(\text{id} + G \theta(i^*X))^{-1}$ is understood as $\sum_{k=0}^\infty (-1)^k (G \theta(i^*X))^k$.

The BV integral over zero-modes in (80) then is:

$$(82) \quad O_\gamma = \int_{\mathcal{L}_z \subset \mathcal{F}_\gamma^z} e^{\frac{i}{2} \int_\gamma \omega'(Y_z, \theta(i^*X)) (\text{id} + G \theta(i^*X))^{-1} Y_z} \sqrt{\mu_z|_{\mathcal{L}_z}} \cdot \exp\left(\text{Str}_{\mathcal{N}} \sum_{k \geq 2} \frac{(-1)^k}{2k} \int_{\text{Conf}_k(\gamma)} \overleftarrow{\prod}_{1 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta(i^*X)\right)$$

where μ_z is some fixed translation-invariant volume element on \mathcal{F}_z .

Remark 11. The requirement that the integral over zero-modes in (82) should converge imposes the following restriction on \mathcal{L}_z . If $\{Y_{\mathcal{L}_z}^I\}$ are coordinates on \mathcal{L}_z and if we write

$$\int_\gamma \omega'(Y_z, \theta(i^*X)) (\text{id} + G \theta(i^*X))^{-1} Y_z = Y_{\mathcal{L}_z}^I H_{IJ}(i^*X) Y_{\mathcal{L}_z}^J$$

then we require the even-even block (i.e. the block pairing coordinates $Y_{\mathcal{L}_z}^I$ of even degree with coordinates $Y_{\mathcal{L}_z}^J$ of even degree) of the matrix (H_{IJ}) to be invertible. One universal way to satisfy this condition is to choose $\mathcal{L}_z \subset \mathcal{F}_\gamma^z$ by setting all even

¹⁹We are working modulo constants here. We are also implicitly assuming convergence of the sum over k in (81).

coordinates on \mathcal{F}_γ^z to zero. With this choice, the integral over \mathcal{L}_z in (82) is (up to normalization) the Pfaffian of (H_{IJ}) .

Proposition 6. (i) Expression (82) is an observable, i.e.

$$Q_\Sigma O_\gamma = 0$$

(ii) Dependence of O_γ on deformations of Riemannian metric on γ and on deformations of \mathcal{L}_z is Q_Σ -exact.

Proof. Define $S_\gamma^z \in C^\infty(\mathcal{F}_\Sigma \times \mathcal{F}_\gamma^z)$ by (81) and $O_\gamma^z = e^{iS_\gamma^z}$:

$$(83) \quad S_\gamma^z = \frac{1}{2} \int_\gamma \omega'(Y_z, \theta (\text{id} + G \theta)^{-1} Y_z) - \\ - i \text{Str}_\mathcal{N} \sum_{k \geq 2} \frac{(-1)^k}{2k} \int_{\text{Conf}_k(\gamma)} \overleftarrow{\prod}_{1 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta$$

where θ actually means $\theta(i^* X)$.

First let us check that S_γ^z is a semi-quantum pre-observable, i.e. satisfies the equation (41):

$$(84) \quad Q_\Sigma S_\gamma^z + \frac{1}{2} \{S_\gamma^z, S_\gamma^z\}_{\Omega_\gamma^z} - i \Delta_z S_\gamma^z = 0$$

Indeed, we have

$$(85) \quad \frac{1}{2} \{S_\gamma^z, S_\gamma^z\}_{\Omega_\gamma^z} = \frac{1}{2} \int_\gamma \omega'(Y_z, (\text{id} + \theta G)^{-1} \theta \mathcal{P}_{\text{harm}} \theta (\text{id} + G \theta)^{-1} Y_z)$$

$$(86) \quad -i \Delta_z S_\gamma^z = -\frac{i}{2} \text{Str}_{\mathcal{F}_\gamma^z} (\mathcal{P}_{\text{harm}} \theta (\text{id} + G \theta)^{-1})$$

$$(87) \quad Q_\Sigma S_\gamma^z = -\frac{1}{2} \int_\gamma \omega'(Y_z, (\text{id} + \theta G)^{-1} \underbrace{(d\theta + \theta^2)}_{Q_\Sigma \theta(i^* X)} (\text{id} + G \theta)^{-1} Y_z) - \\ + \frac{i}{2} \text{Str}_\mathcal{N} \sum_{k \geq 2} (-1)^k \int_{\text{Conf}_k(\gamma)} \left(\overleftarrow{\prod}_{2 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta \right) \cdot \pi_{2,1}^* \eta \cdot \pi_1^* (d\theta + \theta^2)$$

Using Stokes' theorem, (87) can be written as

$$(88) \quad Q_\Sigma S_\gamma^z = \frac{1}{2} \int_\gamma \omega'(Y_z, (\text{id} + \theta G)^{-1} \theta ([d, G] - \text{id}) \theta (\text{id} + G \theta)^{-1} Y_z) + \\ + \frac{i}{2} \text{Str}_\mathcal{N} \sum_{k \geq 2} (-1)^{k+\dim \gamma} \int_{\text{Conf}_k(\gamma)} \left(\overleftarrow{\prod}_{2 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta \right) \cdot \pi_{2,1}^* d\eta \cdot \pi_1^* \theta + \\ + \frac{i}{2} \text{Str}_\mathcal{N} \sum_{k \geq 2} (-1)^k \int_{\text{Conf}_k(\gamma)} \left(\overleftarrow{\prod}_{2 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta \right) \cdot \pi_{2,1}^* \eta \cdot \pi_1^* \theta^2 - \\ - \frac{i}{2} \text{Str}_\mathcal{N} \sum_{k \geq 2} \frac{(-1)^{k+k \dim \gamma}}{k} \int_{\partial \text{Conf}_k(\gamma)} \overleftarrow{\prod}_{1 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \theta$$

Two last terms here cancel due to the contribution of principal boundary strata of $\text{Conf}_k(\gamma)$, where points j and $j+1$ collapse (and we use the property of the propagator that the fiber integral of η over fibers of $\partial \text{Conf}_2(\gamma) \rightarrow \gamma$ is the constant function $-(-1)^{\dim \gamma} \mathbf{1} \in C^\infty(\gamma)$). Hidden boundary strata (i.e. those corresponding to the collapse of ≥ 3 points) do not contribute due to the fact that the collapsed subgraph of the “wheel” graph either has a vertex of valence 2, and then the integral

vanishes by (anti-) symmetry of η on $\partial\text{Conf}_2(\gamma)$ w.r.t. the antipodal involution, or the collapsed subgraph has an isolated vertex, and then the integral vanishes for degree reasons, cf. [6].

Since $[d, G] = \text{id} - \mathcal{P}_{\text{harm}}$ and since $d\eta = -(-1)^{\dim \gamma} \mathbf{P}_{\text{harm}}$ where $\mathbf{P}_{\text{harm}} \in \Omega^{\dim \gamma}(\gamma \times \gamma)$ is the integral kernel of $\mathcal{P}_{\text{harm}}$, collecting (85,86,88) we get the semi-quantum master equation (84).

Next, knowing that (84) holds, we infer by proposition 1 that $O_\gamma = \int_{\mathcal{L}_z \subset \mathcal{F}_\gamma^z} e^{iS_\gamma^z}$ given by (82) satisfies $Q_\Sigma O_\gamma = 0$. Thus we proved (i).

To prove (ii) we first note that deformations of \mathcal{L}_z induce Q_Σ -exact shifts in O_γ by proposition 1.

Next we check the dependence of S_γ^z on metric on γ . Let us now be more careful with the space of zero-modes and define it as $\mathcal{F}_\gamma^z = \mathcal{N} \otimes H^\bullet(\gamma)$ – the \mathcal{N} -valued de Rham cohomology of γ ; it comes with a metric-dependent embedding of cohomology into differential forms $i : H^\bullet(\gamma) \hookrightarrow \Omega^\bullet(\gamma)$ which realizes cohomology classes by harmonic forms. In view of this redefinition we should now write $i(Y_z)$ instead of Y_z in (83).

Now assume that we have a path in the space of Riemannian metrics on γ , $g_t \in \text{Met}(\gamma)$ with $t \in [0, 1]$ the parameter. Then we have the t -dependent Hodge-theoretic inverse G_t for de Rham operator, the t -dependent propagator η_t and the t -dependent embedding of cohomology into differential forms i_t with exact time-derivatives:

$$\frac{d}{dt} G_t = [d, H_t], \quad \frac{\partial}{\partial t} \eta_t = d\zeta_t, \quad \frac{\partial}{\partial t} i_t = dj_t$$

for some linear operator $H_t : \Omega^\bullet(\gamma) \rightarrow \Omega^{\bullet-2}(\gamma)$ with smooth integral kernel $\zeta_t \in \Omega^{\dim \gamma - 2}(\gamma \times \gamma)$ and some $j_t : H^\bullet(\gamma) \rightarrow \Omega^{\bullet-1}(\gamma)$; H_t , ζ_t and j_t depend on g_t and $\frac{\partial}{\partial t} g_t$. Then one can introduce the “extended” versions of G , η and i :

$$\begin{aligned} \tilde{G} = G_t + dt H_t &\in \Omega^\bullet([0, 1]) \hat{\otimes} \text{End}(\Omega^\bullet(\gamma)) \\ \tilde{\eta} = \eta_t + dt \zeta_t &\in \Omega^{\dim \Sigma - 1}([0, 1] \times \text{Conf}_2(\gamma)) \\ \tilde{i} = i_t + dt j_t &\in \Omega^\bullet([0, 1]) \hat{\otimes} \text{Hom}(H^\bullet(\gamma), \Omega^\bullet(\gamma)) \end{aligned}$$

They satisfy respectively

$$\begin{aligned} \left(dt \frac{\partial}{\partial t} + [d_\gamma, \bullet] \right) \tilde{G} &= \text{id} - \mathcal{P}_{\text{harm}} \\ \left(dt \frac{\partial}{\partial t} + d_{\text{Conf}_2(\gamma)} \right) \tilde{\eta} &= -(-1)^{\dim \gamma} \mathbf{P}_{\text{harm}} \\ \left(dt \frac{\partial}{\partial t} + d_\gamma \right) \tilde{i} &= 0 \end{aligned}$$

Next, we define the extended version of S_γ^z , $\tilde{S}_\gamma^z \in \Omega^\bullet([0, 1]) \hat{\otimes} C^\infty(\mathcal{F}_\Sigma \times \mathcal{F}_\gamma^z)$ by substituting \tilde{G} instead of G , $\tilde{\eta}$ instead of η and $\tilde{i}(Y_z)$ instead of Y_z into (83). Repeating our proof of (84) in the extended case we find that

$$\left(dt \frac{\partial}{\partial t} + Q_\Sigma \right) \tilde{S}_\gamma^z + \frac{1}{2} \{ \tilde{S}_\gamma^z, \tilde{S}_\gamma^z \}_{\Omega_\gamma^z} - i \Delta_z \tilde{S}_\gamma^z = 0$$

which, when we decompose \tilde{S}_γ^z into the 0-form and 1-form parts w.r.t. t , $\tilde{S}_\gamma^z = S_\gamma^z(t) + dt R_\gamma^z(t)$, is equivalent to semi-quantum master equation (84) for $S_\gamma^z(t)$ plus the equation

$$(89) \quad \frac{\partial}{\partial t} S_\gamma^z(t) = Q_\Sigma R_\gamma^z(t) + \{ S_\gamma^z(t), R_\gamma^z(t) \}_{\Omega_\gamma^z} - i \Delta_z R_\gamma^z(t)$$

Therefore (cf. (44)) semi-quantum pre-observables $S_\gamma^z(t)$ are equivalent for all values of the parameter t and so by item (iii) of proposition 1, dependence of the induced

observable $O_\gamma(t) = \int_{\mathcal{L}_z \subset \mathcal{F}_\gamma^z} e^{iS_\gamma^z(t)}$ on t is Q_Σ -exact. This finishes the proof of (ii). \square

7. EXAMPLES OF OBSERVABLES

Here we will illustrate the construction of AKSZ pre-observables/observables of sections 5, 6 by several explicit examples.

7.1. **Observables with \mathcal{N} a point.** First we give some examples of the situation described in section 6.1.

- For abelian Chern-Simons theory (22) we may take $\theta = c\psi \in C^\infty(\mathbb{R}[1])$ with $c \in \mathbb{R}$ any constant. This yields by (69) a 1-dimensional observable for $i : \gamma = S^1 \hookrightarrow \Sigma$

$$O_\gamma = e^{ic \oint_\gamma i^* A_{(1)}}$$

— the abelian Wilson loop; parameter c can be viewed as the weight of a one-dimensional representation of the gauge group \mathbb{R} . Note that this is also an observable for the abelian BF theory (26).

- For D -dimensional abelian BF theory, take $\theta = c\xi$ with $c \in \mathbb{R}$ a parameter. The associated observable for a closed oriented submanifold $i : \gamma \hookrightarrow \Sigma$ of codimension 2 is

$$(90) \quad O_\gamma = e^{ic \int_\gamma i^* B_{(D-2)}}$$

- For 2-dimensional BF theory assume that \mathfrak{g} is quadratic so that we can identify \mathfrak{g}^* with \mathfrak{g} and let $\rho : \mathfrak{g} \rightarrow \text{End}(R)$ be a representation. Set $\theta = -i \log \text{tr}_R f(\rho(\xi))$ for $f \in C^\infty(\mathbb{R})$ a function. The corresponding 0-dimensional observable for $\gamma \in \Sigma$ a point is

$$(91) \quad O_\gamma = \text{tr}_R f(\rho(B_{(0)}|_\gamma))$$

- For 2-dimensional abelian BF theory take $\theta = f(\xi)\psi$ for $f \in C^\infty(\mathbb{R})$. The corresponding 1-dimensional observable for $i : \gamma = S^1 \hookrightarrow \Sigma$ is

$$(92) \quad O_\gamma = e^{i \oint_\gamma i^*(A \cdot f(B))}$$

Note that for f non-constant, O_γ depends non-trivially on components of the superfields of ghost number $\neq 0$, namely on $A_{(0)}$ and $B_{(1)}$:

$$O_\gamma = e^{i \oint_\gamma i^*(A_{(1)} \cdot f(B_{(0)}) + A_{(0)} \cdot B_{(1)} \cdot f'(B_{(0)}))}$$

- For Poisson sigma model and $\theta = f \in C^\infty(M)$ a Casimir of the Poisson structure (i.e. $[\pi, f] = 0$, with $[\cdot, \cdot]$ the Schouten-Nijenhuis bracket on polyvector fields on M), we get a 0-dimensional observable for γ a point in Σ :

$$(93) \quad O_\gamma = e^{i \int_{X_{(0)}|_\gamma} f}$$

- For Poisson sigma model and $v \in \mathfrak{X}(M)$ a Poisson vector field (i.e. $[\pi, v] = 0$), set $\theta = \langle p, v(x) \rangle$. The associated 1-dimensional observable is:

$$(94) \quad O_\gamma = e^{i \oint_\gamma i^* \langle \eta, v(X) \rangle}$$

Note that if $\tilde{v} = v + [\pi, f]$ – another Poisson vector field giving the same class in Poisson cohomology of M as v , then the observable \tilde{O}_γ constructed from \tilde{v} by (94) is equivalent to O_γ : $\langle p, \tilde{v}(x) \rangle - \langle p, v(x) \rangle = Qf(x)$ on the target implies $\oint_\gamma i^* \langle \eta, \tilde{v}(X) \rangle - \oint_\gamma i^* \langle \eta, v(X) \rangle = -Q_\Sigma \oint_\gamma i^* f(X)$ on the mapping space \mathcal{F}_Σ . This implies in turn

$$\tilde{O}_\gamma = O_\gamma + Q_\Sigma \left(-i \int_0^1 dt e^{i \oint_\gamma i^* \langle \eta, (1-t)v(X) + t\tilde{v}(X) \rangle} \oint_\gamma i^* f(X) \right)$$

Thus (94) defines a map from Poisson cohomology of M in degree one to Q_Σ -cohomology in degree zero.

Note that the observable (91) is a special case of (93) when we take M to be \mathfrak{g}^* with Kirillov-Kostant Poisson structure. Also, (92) is a special case of (94) for $M = \mathbb{R}$ with $\pi = 0$.

7.2. Wilson loop observable in Chern-Simons theory. Here we will combine the discussion of section 6.2 with the example (iv) of section 3.4.

Let \mathfrak{g} be a quadratic Lie algebra and let (M, ω_M) be a symplectic manifold with a Hamiltonian action of \mathfrak{g} with equivariant moment map $\mu : M \rightarrow \mathfrak{g}^*$. Further, we assume that ω_M satisfies Bohr-Sommerfeld integrality condition and is the curvature of a $U(1)$ -connection ∇ on a line bundle L over M .

Then we have a degree 0 Hamiltonian Q -bundle structure on $\mathfrak{g}[1] \times M \rightarrow \mathfrak{g}[1]$, with base structure given by (20) and fiber structure (38), with the correction that now we relaxed the exactness condition on ω_M . Applying the construction of section 5 to this target data we get the following 1-dimensional pre-observable for Chern-Simons theory:

$$(95) \quad \mathcal{F}_\gamma = \text{Map}(T[1]\gamma, M) \\ = \{(Y_{(0)}, Y_{(1)}) \mid Y_{(0)} : \gamma \rightarrow M, Y_{(1)} \in \Gamma(\gamma, T^*\gamma \otimes Y_{(0)}^* TM)[-1]\}, \\ \Omega_\gamma = - \oint_\gamma \omega_M(Y) = - \oint_\gamma \omega_{ij}(Y_{(0)}) \delta Y_{(0)}^i \wedge \delta Y_{(1)}^j + \frac{1}{2} Y_{(1)}^k \partial_k \omega_{ij}(Y_{(0)}) \delta Y_{(0)}^i \wedge \delta Y_{(0)}^j, \\ e^{iS_\gamma} = \text{Hol}_{Y_{(0)}^* \nabla}(\gamma) e^{i \oint_\gamma \langle i^* A, \mu(Y) \rangle} = \text{Hol}_{Y_{(0)}^* \nabla}(\gamma) e^{i \oint_\gamma \langle i^* A_{(1)}, \mu(Y_{(0)}) \rangle + \langle i^* A_{(0)}, Y_{(1)}^i \partial_i \mu(Y_{(0)}) \rangle}$$

where $i : \gamma = S^1 \hookrightarrow \Sigma$ is a circle embedded into 3-manifold Σ , $A = \sum_{k=0}^3 A_{(k)}$ is the superfield of the ambient Chern-Simons theory, $Y = Y_{(0)} + Y_{(1)}$ is the auxiliary superfield (21), with components $Y_{(0)}, Y_{(1)}$ having internal degrees 0 and -1 respectively; indices i, j, k refer to local coordinates on M .

To construct an observable out of the pre-observable (95), we impose the gauge fixing (71) which consists in setting $Y_{(1)} = 0$. The path integral (72) reads

$$(96) \quad O_\gamma = \int_{\text{Map}(\gamma, M)} \mathcal{D}Y_{(0)} \text{Hol}_{Y_{(0)}^* \nabla}(\gamma) e^{i \oint_\gamma \langle i^* A_{(1)}, \mu(Y_{(0)}) \rangle}$$

Assume that there exists a Lagrangian polarization P of (M, ω_M) such that the geometric quantization of $(M, \omega, L, \nabla, P)$ yields a space of states \mathcal{H} and components $\mu_a \in C^\infty(M)$ of the moment map are quantized into operators $\hat{\mu}_a \in \text{End}(\mathcal{H})$ such that the map $\hat{\mu} : \mathfrak{g} \rightarrow \text{End}(\mathcal{H})$ sending generator $T_a \in \mathfrak{g}$ into $\hat{\mu}_a$ is a Lie algebra homomorphism (i.e. $\hat{\mu}$ is a representation of \mathfrak{g} on \mathcal{H}). Then

$$\hat{\Theta}' = -i \langle \psi, \hat{\mu} \rangle \in C^\infty(\mathfrak{g}[1]) \otimes \text{End}(\mathcal{H})$$

(where we view $\hat{\mu}$ as an element of $\mathfrak{g}^* \otimes \text{End}(\mathcal{H})$) is the quantization of $\Theta' = \langle \psi, \mu(x) \rangle$ satisfying (73). Then the construction of proposition 5 produces out of the pre-observable (95) the usual Wilson loop of the connection component $A_{(1)}$ of the ambient superfield in representation $\hat{\mu}$ along γ :

$$(97) \quad O_\gamma = \text{tr}_{\mathcal{H}} \mathcal{P} \exp \oint_\gamma \hat{\mu}(i^* A_{(1)})$$

which is indeed a gauge invariant observable (i.e. $Q_\Sigma O_\gamma = 0$) for Chern-Simons theory on Σ .

Expression (97) can be viewed as a regularization of the path integral (96).

Coadjoint orbit case. In case when (M, ω_M) is an integral coadjoint orbit of a compact Lie group G such that $\mathfrak{g} = \text{Lie}(G)$, with ω_M the Kirillov symplectic structure and $\mu : M \hookrightarrow \mathfrak{g}^*$ the embedding of M into \mathfrak{g}^* (which corresponds to

the coadjoint Hamiltonian action of G on M), formula (96) becomes the Alekseev-Faddeev-Shatashvili path integral formula for the Wilson loop [2]. The respective $\hat{\mu}$ is the irreducible representation of G associated by Kirillov’s orbit method to the coadjoint orbit M , and (97) is just the Wilson loop in this irreducible representation.

7.3. “Wilson loops” in Poisson sigma model. Let (M, π) be a Poisson manifold, $(N, \omega_N = \delta\alpha_N)$ an exact symplectic manifold and $F : N \rightarrow \mathfrak{X}(M)$ a map²⁰ with the property

$$(98) \quad \frac{1}{2}\{F, F\}_{\omega_N} + [\pi, F] = 0$$

where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket on polyvector fields on M ; expression $\frac{1}{2}\{F, F\}$ is a map from N to bivectors on M , given in local coordinates as $\frac{1}{2}(\omega^{-1}(y))^{ab} \frac{\partial}{\partial y^a} F^i(x, y) \frac{\partial}{\partial y^b} F^j(x, y) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ where $\{x^i\}$ are local coordinates on M and $\{y^a\}$ are local coordinates on N . Then we have a degree 0 Hamiltonian Q -bundle structure on $T^*[1]M \times N \rightarrow T^*[1]M$ with base structure (27) and fiber structure

$$(99) \quad \mathcal{N} = N, \quad \mathcal{A} = \langle p, \{F, \bullet\}_{\omega_N} \rangle, \\ \omega' = \omega_N, \quad \alpha' = \alpha_N, \quad \Theta' = \langle p, F \rangle \in C^\infty(T^*[1]M \times N)$$

The corresponding 1-dimensional pre-observable for Poisson sigma model associated to (M, π) is given by $\mathcal{F}_\gamma, \Omega_\gamma$ as in (95), changing M to N , and the auxiliary action is

$$S_\gamma = \oint_\gamma \alpha_a(Y) dY^a + \langle i^* \eta, F(i^* X, Y) \rangle$$

The corresponding observable is formally given (in the gauge (71)) by the path integral (72):

$$(100) \quad O_\gamma = \int_{\text{Map}(\gamma, N)} \mathcal{D}Y_{(0)} e^{i \oint_\gamma \alpha_a(Y_{(0)}) dY_{(0)}^a + \langle i^* \eta, F(i^* X, Y_{(0)}) \rangle}$$

Assume that (N, ω_N) can be geometrically quantized to a space of states \mathcal{H} and F is quantized to an operator-valued vector field $\hat{F} \in \text{End}(\mathcal{H}) \hat{\otimes} \mathfrak{X}(M)$ satisfying

$$(101) \quad [\pi, \hat{F}] + i\hat{F} \wedge \hat{F} = 0$$

Then (74) gives

$$(102) \quad O_\gamma = \text{tr}_{\mathcal{H}} \mathcal{P} \exp \left(i \oint_\gamma \langle i^* \eta, \hat{F}(i^* X) \rangle \right)$$

which can be regarded as the evaluation of the path integral (100) via Hamiltonian formalism. By proposition 5, O_γ is indeed an observable for the Poisson sigma model.

Remark 12. Note that in case $N = \text{point}$ we get back the observable (94). Also, in case $M = \mathfrak{g}^*$ with Kirillov-Kostant Poisson structure and requiring that F takes values in constant vector fields on F , equation (98) becomes the equation on equivariant moment map and the corresponding observable (102) is the usual Wilson loop (97) for 2-dimensional BF theory.

²⁰Alternatively, one can view F as a vertical vector field on the trivial fiber bundle $N \times M \rightarrow N$.

7.4. Torsion observables in Chern-Simons theory. Fix $n' \in \{-1, 0, 1, 2\}$ and $m \in \mathbb{Z}$ (only the parity of m will matter for our discussion). Let again \mathfrak{g} be a quadratic Lie algebra and let $\rho : \mathfrak{g} \rightarrow \text{End}(R)$ be a representation on a vector space R with values in traceless matrices (cf. (78)). Then we have a degree n' Hamiltonian Q -bundle structure on $\mathfrak{g}[1] \oplus R[m] \oplus R^*[n' - m] \rightarrow \mathfrak{g}[1]$ with base structure (20) and fiber structure

$$(103) \quad \mathcal{N} = R[m] \oplus R^*[n' - m], \quad \mathcal{A} = \left\langle \rho(\psi)q, \frac{\partial}{\partial q} \right\rangle + \left\langle \rho^*(\psi)p, \frac{\partial}{\partial p} \right\rangle,$$

$$\omega' = \langle \delta p, \delta q \rangle, \quad \alpha' = \langle p, \delta q \rangle, \quad \Theta' = \langle p, \rho(\psi)q \rangle$$

where $\rho^* : \mathfrak{g} \rightarrow \text{End}(R^*)$ is the representation dual to ρ ; q is the R -valued coordinate of degree m on $R[m]$ and p is the R^* -valued coordinate of degree $n' - m$ on $R^*[n' - m]$.

The corresponding pre-observable for Chern-Simons theory on a closed oriented 3-manifold Σ , for $i : \gamma \hookrightarrow \Sigma$ an embedded closed oriented $(n' + 1)$ -manifold, reads:

$$(104) \quad \mathcal{F}_\gamma = R[m] \otimes \Omega^\bullet(\gamma) \oplus R^*[n' - m] \otimes \Omega^\bullet(\gamma), \quad \Omega_\gamma = \int_\gamma \langle \delta \mathbf{p}, \delta \mathbf{q} \rangle,$$

$$S_\gamma = \int_\gamma \langle \mathbf{p}, d\mathbf{q} \rangle + \langle \mathbf{p}, \rho(i^* A)\mathbf{q} \rangle$$

where $\mathbf{q} = \sum_{k=0}^{n'+1} \mathbf{q}_{(k)}$ and $\mathbf{p} = \sum_{k=0}^{n'+1} \mathbf{p}_{(k)}$ are the auxiliary superfields corresponding to q and p ; component $\mathbf{q}_{(k)}$ is a R -valued k -form on γ with internal degree $m - k$ and $\mathbf{p}_{(k)}$ is a R^* -valued k -form on γ of internal degree $n' - m - k$.

Pre-observable (104) can be pushed forward to zero-modes (in the sense of proposition 2) as in section 6.3:

$$(105) \quad \mathcal{F}_\gamma^z = R[m] \otimes H^\bullet(\gamma) \oplus R^*[n' - m] \otimes H^\bullet(\gamma), \quad \Omega_\gamma^z = \int_\gamma \langle \delta \mathbf{p}_z, \delta \mathbf{q}_z \rangle,$$

$$S_\gamma^z = \int_\gamma \langle \mathbf{p}_z, \rho(i^* A)(\text{id} + G \rho(i^* A))^{-1} \mathbf{q}_z \rangle - i \log \text{tor}(\gamma, i^* A, \rho)$$

where we introduced the notation

$$(106) \quad \text{tor}(\gamma, i^* A, \rho) = \exp \text{tr}_R \sum_{k \geq 2} \frac{(-1)^k}{k} \int_{\text{Conf}_k(\gamma)} \overleftarrow{\prod}_{1 \leq j \leq k} \pi_{j+1, j}^* \eta \cdot \pi_j^* \rho(i^* A)$$

Notations G, η, π are the same as in section 6.3. We use harmonic representatives for zero-modes $\mathbf{q}_z, \mathbf{p}_z$ in \mathcal{F}_γ .

Taking the BV integral over zero-modes in (105) we obtain the observable for Chern-Simons theory on Σ :

$$(107) \quad O_\gamma = \int_{\mathcal{L}_z \subset \mathcal{F}_\gamma^z} \sqrt{\mu_z} |_{\mathcal{L}_z} e^{i \int_\gamma \langle \mathbf{p}_z, \rho(i^* A)(\text{id} + G \rho(i^* A))^{-1} \mathbf{q}_z \rangle} \cdot \text{tor}(\gamma, i^* A, \rho)$$

with $\mathcal{L}_z \subset \mathcal{F}_\gamma^z$ a Lagrangian subspace and μ_z a translation invariant volume element on \mathcal{F}_γ^z . Proposition 6 implies that O_γ is Q_Σ -closed and that deformations of gauge-fixing induce Q_Σ -exact shifts in O_γ .

In case $n' = 0, m = 1$, with $\gamma = S^1$, (107) can be evaluated explicitly²¹:

$$(108) \quad O_\gamma = \int_{R[1] \oplus R^*[-1]} d\mathbf{q}_{(0)}^z d\mathbf{p}_{(0)}^z e^{i \oint_\gamma \langle \mathbf{p}_{(0)}^z, \rho(\bar{A})\mathbf{q}_{(0)}^z \rangle} \cdot \det_R \left(\frac{\sinh \rho(\bar{A}/2)}{\rho(\bar{A}/2)} \right)$$

$$= \det_R \left(\rho(W)^{1/2} - \rho(W)^{-1/2} \right) = \det_R(\rho(W) - \text{id}_R)$$

²¹One trick to simplify the calculation is to use gauge-invariance of O_γ to pick the constant representative of the gauge equivalence class of connection $i^* A$.

where $W = \mathcal{P} \exp \oint_{\gamma} i^* A_{(1)}$ is the holonomy of the ambient connection around γ (since this requires the choice of a starting point on γ , one can think of W as a group element defined modulo conjugation). We also denote $\bar{A} = \log W$.

In (108) we used the gauge-fixing Lagrangian (71). In particular, the gauge-fixing for the integral over zero-modes is

$$\underbrace{(R[1] \oplus R^*[-1]) \otimes H^0(\gamma)}_{\mathcal{L}_z} \subset \underbrace{(R[1] \oplus R^*[-1]) \otimes H^\bullet(\gamma)}_{\mathcal{F}_\gamma^z}$$

Remark 13. Observable (108) is only interesting (not identically zero) for representations ρ such that $\det \rho \neq 0$, i.e. $\rho(x)$ does not have zero eigenvalue for x in an open dense subset of \mathfrak{g} . In particular, (108) is identically zero for $\rho = \text{ad}$ the adjoint representation.

Remark 14. Observable (108) has the property that it only depends on $i^* A_{(1)}$ and not on the other components of the ambient field. Also, it is the only representative of its class in Q_Σ -cohomology which depends only on $i^* A$, for degree reasons (since $C^\infty(\mathcal{F}_\Sigma|_\gamma)$ is non-negatively graded; we denote $\mathcal{F}_\Sigma|_\gamma = \text{Map}(T[1]\gamma, \mathcal{M})$ the space of pull-backs of ambient fields to γ).

In case $n' > 0$, O_γ defined by (107) generally depends on $i^* A_{(k)}$ for $0 \leq k \leq n' + 1$. Also, there is no canonical representative of the class of O_γ in Q_Σ -cohomology and no canonical choice for the gauge-fixing Lagrangian $\mathcal{L} \subset \mathcal{F}_\gamma$.

For $n' = -1$, convergence requirement for the integral over zero-modes (107) forces (cf. remark 11) $\mathcal{L}_z = (R[m] \oplus R^*[-1 - m])^{\text{odd}}$ which yields $O_\gamma = 0$.

For $n' = 2$, for γ a connected closed oriented 3-manifold with non-zero first Betti number, we may take $m = 1$ and choose the gauge-fixing for zero-modes as

$$(109) \quad \mathcal{L}_z = (R[1] \oplus R^*[1]) \otimes (H^0(\gamma) \oplus \underbrace{\lambda^\perp}_{\subset H^1(\gamma)} \oplus \underbrace{\lambda}_{\subset H^2(\gamma)})$$

where λ is any line in $H^1(\gamma)$ and λ^\perp is its orthogonal complement in $H^2(\gamma)$ w.r.t. the Poincaré duality.

If the condition $\det \rho \neq 0$ as in remark 13 holds, O_γ defined with zero-mode gauge-fixing (109) is well-defined on an open subset of $\mathcal{F}_\Sigma|_\gamma$ and is not identically zero.

In case $\dim H^1(\gamma) = 1$, Lagrangian (109) is the same as in remark 11, i.e. $\mathcal{L}_z = (\mathcal{F}_\gamma^z)^{\text{odd}}$. In case $\dim H^1(\gamma) > 1$, the Lagrangian $(\mathcal{F}_\gamma^z)^{\text{odd}}$ yields O_γ which is identically zero.

In case of γ a rational homology sphere, $H^1(\gamma) = 0$, construction (109) is not applicable. However, one can take

$$\mathcal{L}_z = (R[1] \oplus R^*[1]) \otimes H^0(\gamma)$$

which produces an observable O_γ of degree $-2 \dim R$.

Remark 15. In case $n' = 2$ and $\gamma = \Sigma$, $i = \text{id}$, property (62) does not generally hold, so (104) is a pre-observable in the sense of definition 6, but $S_\Sigma + S_\gamma$ does not satisfy the classical master equation (61) on the space $\mathcal{F}_\Sigma \times \mathcal{F}_\gamma$. On the level of observable O_γ this means that $Q_\Sigma O_\gamma = 0$, but $S_\Sigma - i \log O_\gamma$ does not satisfy the classical master equation on \mathcal{F}_Σ .

In case $n' = 1$, for γ a connected closed oriented surface of genus $g > 0$, we may take $m = 0$ and set

$$\mathcal{L}_z = R \otimes (H^0(\gamma) \oplus \lambda^\perp) \oplus R^*[1] \otimes (H^0(\gamma) \oplus \lambda)$$

where λ is again any line $H^1(\gamma)$ and λ^\perp is its orthogonal in $H^1(\gamma)$ w.r.t. the Poincaré duality. Assuming $\det \rho \neq 0$, with this choice of \mathcal{L}_z we produce an observable O_γ of degree $(2g - 2) \dim R$ which is well-defined and non-zero on an open subset of $\mathcal{F}_\Sigma|_\gamma$.

Expression (107) also is an observable for BF theory in dimension D . In this case n' should satisfy $-1 \leq n' \leq D - 1$. For $n' = 2k$ and γ a closed oriented submanifold of Σ of dimension $2k + 1$, we may take $m = 2k + 1$ and choose the gauge-fixing \mathcal{L}_z as

$$\mathcal{L}_z = (R[2k + 1] \oplus R^*[-1]) \otimes \mathcal{L}'$$

where $\mathcal{L}' \subset H^\bullet(\gamma)$ a Lagrangian subspace with the property

$$(110) \quad \dim \mathcal{L}'_j = \dim \mathcal{L}'_{2k-j}$$

where $\mathcal{L}'_j = \mathcal{L}' \cap H^j(\gamma)$, which is equivalent to

$$(111) \quad \dim \mathcal{L}'_j = B_j - B_{j-1} + \dots + (-1)^j B_0$$

where $B_j = \dim H^j(\gamma)$. For \mathcal{L}' to exist, we require that for γ the combinations of Betti numbers on the r.h.s. of (111) are non-negative. Assuming $\det \rho \neq 0$, property (110) ensures that the integral over zero-modes in (107) exists and is non-zero in an open subset of $\mathcal{F}_\Sigma|_\gamma$.

7.5. A codimension 2 (pre-)observable in BF theory. The following example is due to Cattaneo-Rossi [9]. Let $\mathcal{M} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[D - 2]$ with the structure of degree $D - 1$ Hamiltonian Q -manifold as in (23). One constructs a trivial degree $D - 3$ Hamiltonian Q -bundle over \mathcal{M} with fiber data

$$(112) \quad \mathcal{N} = \mathfrak{g} \oplus \mathfrak{g}^*[D - 3], \quad \mathcal{A} = \langle [\psi, q], \frac{\partial}{\partial q} \rangle + \langle \text{ad}^*_\psi p, \frac{\partial}{\partial p} \rangle + (-1)^D \langle \xi, \frac{\partial}{\partial p} \rangle,$$

$$\omega' = \langle \delta p, \delta q \rangle, \quad \alpha' = \langle p, \delta q \rangle, \quad \Theta' = \langle p, [\psi, q] \rangle + \langle \xi, q \rangle$$

Here q is the \mathfrak{g} -valued coordinate of degree 0 on \mathfrak{g} and p is the \mathfrak{g}^* -valued coordinate of degree $D - 3$ on $\mathfrak{g}^*[D - 3]$; ψ and ξ are coordinates on \mathcal{M} as in section 2.3.

Given a closed oriented D -manifold Σ and a closed oriented submanifold $i : \gamma \hookrightarrow \Sigma$ of codimension 2, by the construction of section 5 we get a pre-observable for the BF theory on Σ associated to the Hamiltonian Q -bundle (23,112):

$$(113) \quad \mathcal{F}_\gamma = \mathfrak{g} \otimes \Omega^\bullet(\gamma) \oplus \mathfrak{g}^*[D - 3] \otimes \Omega^\bullet(\gamma),$$

$$(114) \quad \Omega_\gamma = (-1)^D \int_\gamma \langle \delta \mathbf{p}, \delta \mathbf{q} \rangle,$$

$$(115) \quad S_\gamma = \int_\gamma \langle \mathbf{p}, d\mathbf{q} \rangle + \langle \mathbf{p}, [i^* A, \mathbf{q}] \rangle + \langle i^* B, \mathbf{q} \rangle$$

where $\mathbf{q} = \sum_{k=0}^{D-2} \mathbf{q}_{(k)}$ and $\mathbf{p} = \sum_{k=0}^{D-2} \mathbf{p}_{(k)}$ are the auxiliary superfields corresponding to q and p ; $\mathbf{q}_{(k)}$ and $\mathbf{p}_{(k)}$ are k -forms on γ with values in \mathfrak{g} and \mathfrak{g}^* respectively, having internal degrees $-k$ and $D - 3 - k$ respectively.

Push-forward of the pre-observable (113–115) to zero-modes (in the sense of proposition 2) is evaluated as in section 6.3:

$$(116) \quad \mathcal{F}_\gamma^z = \mathfrak{g} \otimes H^\bullet(\gamma) \oplus \mathfrak{g}^*[D - 3] \otimes H^\bullet(\gamma), \quad \Omega_\gamma^z = \int_\gamma \langle \delta \mathbf{p}_z, \delta \mathbf{q}_z \rangle,$$

$$S_\gamma^z = \int_\gamma \langle i^* B + (-1)^D \text{ad}^*_{i^* A} \mathbf{p}_z, (\text{id} + G \text{ad}_{i^* A})^{-1} \mathbf{q}_z \rangle - i \log \text{tor}(\gamma, i^* A, \text{ad})$$

where tor is as in (106), for $\rho = \text{ad}$ the adjoint representation of \mathfrak{g} .

In the case of abelian BF theory, $\mathfrak{g} = \mathbb{R}$, (116) yields

$$(117) \quad e^{iS_\gamma^z} = e^{i \int_\gamma \langle i^* B, \mathbf{q}_z \rangle}$$

which is an observable if we understand $\mathbf{q}_z \in H^\bullet(\gamma)$ as an external parameter (the crucial point here is that by construction (117) satisfies the semi-quantum master equation (42), but the last term in (42) vanishes since S_γ^z depends only on \mathbf{q}_z). Taking $\mathbf{q}_z = c \cdot 1 \in H^0(\gamma)$, we obtain the observable (90).

In [9] it is shown that in the case of “long knots” $\mathbb{R}^{D-2} \sim \gamma \subset \Sigma = \mathbb{R}^D$, embeddings of \mathbb{R}^{D-2} into \mathbb{R}^D with prescribed linear asymptotics, the push-forward of pre-observable (113–115) to zero-modes yields, upon fixing the values of zero-modes to $\mathbf{q}_z = c \otimes 1 \in \mathfrak{g}^* \otimes \Omega^0(\gamma)$ and $\mathbf{p}_z = 0$, an observable, generalizing (90) in non-abelian case.

8. FINAL REMARKS

In this paper we presented a two-step construction of classical observables in AKSZ sigma models, which consists of: (i) constructing an extension of the AKSZ theory to a BV theory on a larger space of fields using an extension of the target to a Hamiltonian Q -bundle, (ii) using the BV push-forward to the original space of fields to produce an observable. We also provided some examples, and in particular recovered the well-known Wilson loop observable in Chern-Simons theory together with its path integral representation [2] and the Cattaneo-Rossi “Wilson surface” observable in BF theory.

There are natural questions to this construction not answered here, to which we hope to return in a future publication:

- Extend the zoo of explicit examples. In particular,
 - give more explicit examples of “Wilson loops” in Poisson sigma model, section 7.3 (which requires a solution of (98) as the input for step (i) of the construction and its quantization satisfying (101) for step (ii)) other than those mentioned in remark 12;
 - give an example corresponding to a *non-trivial* Hamiltonian Q -bundle over the target of the AKSZ sigma model.
- Calculate the isotopy invariants of embeddings given by expectation values of our observables.

8.1. Extension to source manifolds with boundary. Our construction of observables possesses an extension to the case when the source manifold Σ has a boundary $\partial\Sigma$, and the submanifold $\gamma \subset \Sigma$ on which the observable is supported is also allowed to have boundary, $\partial\gamma \subset \partial\Sigma$. We will outline this extension here, treating gauge theories with boundary in “BV-BFV formalism” [8]. For a more detailed example, Chern-Simons theory on a manifold with boundary with Wilson lines ending on the boundary, the reader is referred to [4].

Recall [8] that a gauge theory on manifold Σ with boundary $\partial\Sigma$ in BV-BFV formalism is described by the following data:

- A degree 0 Hamiltonian Q -manifold $(\mathcal{F}_{\partial\Sigma}, Q_{\partial\Sigma}, \Omega_{\partial\Sigma} = \delta\alpha_{\partial\Sigma}, S_{\partial\Sigma})$ — “the BFV phase space” (or “the space of boundary fields”) associated to the boundary $\partial\Sigma$.
- A Q -manifold $(\mathcal{F}_\Sigma, Q_\Sigma)$ (“the space of bulk fields”) endowed with a Q -morphism $\pi : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_{\partial\Sigma}$ (restriction of fields to the boundary), a degree -1 symplectic form Ω_Σ and a degree 0 action $S_\Sigma \in C^\infty(\mathcal{F}_\Sigma)$ satisfying

$$(118) \quad \delta S_\Sigma = \iota_{Q_\Sigma} \Omega_\Sigma + \pi^* \alpha_{\partial\Sigma}$$

Equation (118) replaces the condition that S_Σ is the Hamiltonian function for Q_Σ in definition 4, and its consequence $Q_\Sigma S_\Sigma = \pi^*(\iota_{Q_\Sigma} \alpha_{\partial\Sigma} - 2S_{\partial\Sigma})$ or, equivalently,

$$\frac{1}{2} \underbrace{\{S_\Sigma, S_\Sigma\}_{\Omega_\Sigma}}_{:=\iota_{Q_\Sigma} \iota_{Q_\Sigma} \Omega_\Sigma} + \pi^* S_{\partial\Sigma} = 0$$

replaces the classical master equation.

AKSZ sigma models on manifolds with boundary fit naturally into BV-BFV formalism: one takes the standard construction (8,9,15,16) for the bulk and sets

$$\begin{aligned} \mathcal{F}_{\partial\Sigma} &= \text{Map}(T[1]\partial\Sigma, \mathcal{M}), \quad Q_{\partial\Sigma} = d_{\partial\Sigma}^{\text{lifted}} + Q^{\text{lifted}}, \quad \Omega_{\partial\Sigma} = \tau_{\partial\Sigma}(\omega), \\ \alpha_{\partial\Sigma} &= (-1)^{\dim \Sigma - 1} \tau_{\partial\Sigma}(\alpha), \quad S_{\partial\Sigma} = (-1)^{\dim \Sigma - 1} \left(\iota_{d_{\partial\Sigma}^{\text{lifted}}} \tau_{\partial\Sigma}(\alpha) + \tau_{\partial\Sigma}(\Theta) \right) \end{aligned}$$

for the boundary; Q -morphism $\pi : \text{Map}(T[1]\Sigma, \mathcal{M}) \rightarrow \text{Map}(T[1]\partial\Sigma, \mathcal{M})$ is the pull-back by the inclusion $\partial\Sigma \hookrightarrow \Sigma$.

A classical observable for a gauge theory in BV-BFV formalism (generalizing definition 5) can be defined as the following collection of data.

- A graded vector space (the auxiliary space of states) $\mathcal{H}_{\partial\gamma}$ and a degree 1 element

$$\delta_{\partial\gamma} \in C^\infty(\mathcal{F}_{\partial\Sigma}) \otimes \text{End}(\mathcal{H}_{\partial\gamma})$$

satisfying

$$Q_{\partial\Sigma} \delta_{\partial\gamma} + \delta_{\partial\gamma}^2 = 0$$

- A degree 0 $\mathcal{H}_{\partial\gamma}$ -valued function of bulk fields

$$O_\gamma \in C^\infty(\mathcal{F}_\Sigma) \otimes \mathcal{H}_{\partial\gamma}$$

satisfying

$$(Q_\Sigma + \pi^* \delta_{\partial\gamma}) O_\gamma = 0$$

The locality requirement is that O_γ only depends on the restrictions of bulk fields to γ ; likewise, $\delta_{\partial\gamma}$ should only depend on the restrictions of boundary fields to $\partial\gamma$.

We can also define a pre-observable in the BV-BFV context as the following data.

- A degree 0 Hamiltonian Q -bundle over $(\mathcal{F}_{\partial\Sigma}, Q_{\partial\Sigma})$ with fiber data $(\mathcal{F}_{\partial\gamma}, \mathcal{A}_{\partial\gamma}, \Omega_{\partial\gamma} = \delta\alpha_{\partial\gamma}, S_{\partial\gamma})$.
- A trivial Q -bundle over $(\mathcal{F}_\Sigma, Q_\Sigma)$ with fiber data $(\mathcal{F}_\gamma, \mathcal{A}_\gamma)$, with projection $\bar{\pi} : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_\gamma$ such that $d\bar{\pi}(\mathcal{A}_\gamma) = \mathcal{A}_{\partial\gamma}$, with fiber degree -1 symplectic structure Ω_γ and with degree 0 auxiliary action $S_\gamma \in C^\infty(\mathcal{F}_\Sigma \times \mathcal{F}_\gamma)$ satisfying

$$\delta^{\text{vert}} S_\gamma = \iota_{\mathcal{A}_\gamma} \Omega_\gamma + \bar{\pi}^* \alpha_{\partial\gamma}, \quad Q_\Sigma S_\gamma + \frac{1}{2} \underbrace{\{S_\gamma, S_\gamma\}_{\Omega_\gamma}}_{:=\iota_{\mathcal{A}_\gamma} \iota_{\mathcal{A}_\gamma} \Omega_\gamma} + \pi_{\text{tot}}^* S_{\partial\gamma} = 0$$

where δ^{vert} is the de Rham differential on \mathcal{F}_γ and $\pi_{\text{tot}} = \pi \times \bar{\pi} : \mathcal{F}_\Sigma \times \mathcal{F}_\gamma \rightarrow \mathcal{F}_{\partial\Sigma} \times \mathcal{F}_{\partial\gamma}$.

Passing from a pre-observable to observable is a quantization problem. We construct $\mathcal{H}_{\partial\gamma}$ as the geometric quantization of the symplectic manifold $(\mathcal{F}_{\partial\gamma}, \Omega_{\partial\gamma})$ and $\delta_{\partial\gamma}$ as the geometric quantization of $S_{\partial\gamma}$:

$$(119) \quad \mathcal{H}_{\partial\gamma} = \text{GeomQuant}(\mathcal{F}_{\partial\gamma}, \Omega_{\partial\gamma}), \quad \delta_{\partial\gamma} = i\hat{S}_{\partial\gamma}$$

For simplicity, assume that the Lagrangian polarization of $\mathcal{F}_{\partial\Sigma}$ used for the geometric quantization is the vertical polarization of a fibration $\mathfrak{p} : \mathcal{F}_{\partial\Sigma} \rightarrow \mathcal{B}$. Then O_γ is given by the following modification of the fiber BV integral:

$$(120) \quad O_\gamma(X, b) = \int_{\mathcal{L}_b \subset \bar{\pi}^{-1} \mathfrak{p}^{-1} b} \mathcal{D}Y e^{iS_\gamma(X, Y)}$$

for $b \in \mathcal{B}$, $X \in \mathcal{F}_\Sigma$; Y runs over \mathcal{F}_γ ; \mathcal{L}_b is a Lagrangian in $\bar{\pi}^{-1}\mathfrak{p}^{-1}b \subset \mathcal{F}_\gamma$.

Given an AKSZ theory on Σ with target \mathcal{M} and a Hamiltonian Q -bundle over \mathcal{M} , as a first step, we construct a pre-observable with the bulk data given by old formulae (53) and boundary data given by the same formulae with γ replaced by $\partial\gamma$:

$$\begin{aligned} \mathcal{F}_{\partial\gamma} &= \text{Map}(T[1]\partial\gamma, \mathcal{N}), \quad \mathcal{A}_{\partial\gamma} = d_{\partial\gamma}^{\text{lifted}} + p_\partial^* \mathcal{A}^{\text{lifted}}, \quad \Omega_{\partial\gamma} = \tau_{\partial\gamma}(\omega'), \\ \alpha_{\partial\gamma} &= (-1)^{\dim \gamma - 1} \tau_{\partial\gamma}(\alpha'), \quad S_{\partial\gamma} = (-1)^{\dim \gamma - 1} \left(\iota_{d_{\partial\gamma}^{\text{lifted}}} \alpha_{\partial\gamma} + p_\partial^* \tau_{\partial\gamma}^{\text{tot}}(\Theta') \right) \end{aligned}$$

where $p_\partial : \text{Map}(T[1]\partial\Sigma, \mathcal{M}) \rightarrow \text{Map}(T[1]\partial\gamma, \mathcal{M})$ is the restriction of ambient boundary fields to $\partial\gamma$.

As the second step, we pass from this pre-observable to an observable using construction (119,120).

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Chern-Simons Theory with Wilson Lines and Boundary in the BV-BFV Formalism

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Abstract

We consider the Chern-Simons theory with Wilson lines in 3D and in 1D in the BV-BFV formalism of Cattaneo-Mnev-Reshetikhin. In particular, we allow for Wilson lines to end on the boundary of the space-time manifold. In the toy model of 1D Chern-Simons theory, the quantized BFV boundary action coincides with the Kostant cubic Dirac operator which plays an important role in representation theory. In the case of 3D Chern-Simons theory, the boundary action turns out to be the odd (degree 1) version of the BF model with source terms for the B field at the points where the Wilson lines meet the boundary. The boundary space of states arising as the cohomology of the quantized BFV action coincides with the space of conformal blocks of the corresponding WZW model.

1 Introduction

In the case of complicated space-time topology, a promising approach to quantization of general field theories involves cutting the space-time manifold into simple pieces, where the problem is more easily solved, and then gluing back the individual elements to obtain the final answer. This method was proposed by Atiyah and successfully applied by Witten in [15] to study the quantization of the Chern-Simons theory with Wilson lines (cf. also [9]). Following this idea, a systematic program to understand quantization in the Batalin-Vilkovisky formalism for field theories with degeneracies on manifolds with boundaries has been initiated in [6]. As a part of the construction, a Batalin-Fradkin-Vilkovisky model is associated to the boundary of the space-time manifold. The canonical quantization of this boundary BFV model provides a space of boundary states, together with a cohomological invariance condition that defines the admissible quantum states of the theory among all boundary states. In the case of quantum field theories on manifolds without boundary, the partition function and

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other correlation functions are complex-valued. In the presence of a boundary, the correlators of the bulk theory take values in this boundary space of states.

The aim of this paper is to apply the BV-BFV formalism of [6] to the Chern-Simons theory on manifolds with boundary, with Wilson lines ending on the boundary. The BV formulation of this theory on closed manifolds is well understood (and served as the motivating example for the AKSZ construction). We will also consider the one-dimensional Chern-Simons model, obtained when the AKSZ construction is carried out in one dimension.

We include in our construction Wilson lines which may end on the boundary of the manifold. This requires some extra work in BV-BFV formalism. Our treatment is based on the path integral representation for Wilson loops suggested in [1], [7]. This is also an example of a more general construction of observables for AKSZ sigma models proposed in [12].

We compare our answers with those obtained using the geometric quantization for the boundary [15] and using canonical quantization [10], [8].

One of our main results is the boundary BFV action for Chern-Simons theory with Wilson lines, which has the form of an odd (degree 1) version of BF action modified by source terms for the B field at points where the Wilson lines meet the boundary. We also consider the toy model of one-dimensional Chern-Simons theory and derive the corresponding boundary action. Its quantization coincides with Kostant cubic Dirac operator. We compare the BV-BFV results for the one-dimensional model with the ones obtained in [3] on segments, and also see how Wilson lines can be added to the one-dimensional model. In the three-dimensional case, the boundary space of states, arising as the cohomology of the quantized BFV action, coincides with the space of conformal blocks of the WZW model on the boundary (in the picture of [10]).

We begin in section 2 with the treatment of Wilson lines in the BV formalism. We also provide a short introduction to the relevant aspects of the BV formalism. In section 3 we describe the BV formulation of the Chern-Simons theory with Wilson lines, applying the AKSZ construction to this special setting. Then we proceed to explain how the bulk model gets supplemented with a boundary BFV theory if the underlying manifold has a boundary. We repeat this procedure in section 4 for the one-dimensional Chern-Simons model. The \mathbb{Z}_2 -grading that replaces the usual \mathbb{Z} -grading in this case leads to certain subtleties with the master equation. In section 5 we present the quantization of the boundary BFV models and describe the arising spaces of quantum states, that we compare with known results for the quantization of the involved models.

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2 Wilson lines in the BV formalism

In this section, we start with a brief introduction to the BV formalism. The main point is to incorporate the Wilson line observables in this approach.

2.1 A short introduction to the BV formalism

We know that the path integral in quantum field theories is not well defined if the classical action S_{cl} defined over the space of classical fields \mathcal{F}_{cl} is degenerate, for instance due to gauge symmetries. The Batalin-Vilkovisky formalism provides a general method for the perturbative calculation of partition functions and correlators.

In the BV formalism, the space of fields is augmented to a BV space of fields \mathcal{F}_{BV} , a graded infinite-dimensional manifold equipped with a symplectic structure Ω_{BV} of degree -1 called the BV structure. The grading (usually \mathbb{Z} , sometimes \mathbb{Z}_2) is commonly referred to as “ghost number”, in relation with the Faddeev-Popov prescription. The BV bracket is defined as the Poisson bracket obtained by inverting the BV structure,

$$\{F, G\} = \Omega_{BV}^{-1}(\delta F, \delta G),$$

and obviously has a ghost number 1. Note that the variational operator δ can be interpreted as a de Rham differential in the space of fields. In many cases of interest, the BV space of fields is a cotangent bundle where the degree of the fibers is shifted to -1 , which ensures its canonical symplectic form has the proper degree. Coordinates along the cotangent fibers are then called antifields. In the case of gauge theories, where the degeneracy arises under the action of a gauge group, the BV space of fields is simply the shifted cotangent bundle of the BRST space of fields which contains all classical fields as well as the ghosts parametrizing the gauge symmetries (basically the infinitesimal gauge parameters with a ghost number shifted by one), $\mathcal{F}_{BV} = T^*[-1]\mathcal{F}_{BRST}$.

At the classical level, infinitesimal gauge transformations and the classical action can be used to construct a differential acting on the functionals on the BV space of fields. Geometrically, this differential corresponds to a cohomological vector field Q on \mathcal{F}_{BV} . Moreover, Q is a Hamiltonian vector field,

$$\iota_Q \Omega_{BV} = \delta S_{BV},$$

with the Hamiltonian function being the BV action S_{BV} that reduces to the classical action when all antifields are set to zero. The condition $Q^2 = 0$ follows from the classical master equation $\{S_{BV}, S_{BV}\} = 0$, and determining the BV formulation of a given theory amounts to determining an extension of the classical action to the BV space of fields that satisfies this classical master equation.

2.2 The AKSZ construction

While it is usually difficult to find the BV formulation of a given field theory with a degenerate action, the study of the geometric interpretation of the clas-

sical master equation in [4] led to an insightful procedure to construct solutions thereof, called the AKSZ construction after its authors. A formalized more recent treatment can be found in [5].

In this construction, the target space of the theory is a graded manifold Y equipped with a symplectic structure ω_Y of degree $n - 1$ and a compatible cohomological vector field Q_Y of degree 1, in the sense that it preserves the symplectic structure, $\mathcal{L}_{Q_Y}\omega_Y = 0$. We also want ω_Y to be associated to a Liouville one-form α_Y (of degree $n - 1$ as well), namely $\omega_Y = \delta\alpha_Y$. Here δ denotes the de Rham exterior derivative on Y , while we keep the usual d for the one on the source manifold N of the model. For $n \neq 0$ (see for instance [13] for details), Q_Y can be shown to be Hamiltonian, i.e. there exists a function Θ_Y of degree n on Y such that $\iota_{Q_Y}\omega_Y = \delta\Theta_Y$. Like in the BV formalism, the nilpotency of Q_Y follows from the condition $\{\Theta_Y, \Theta_Y\}_Y = 0$, where the curly braces with a subscript Y denote the Poisson bracket on Y associated to its symplectic structure.

The BV space of fields is then given by maps between the odd tangent bundle of some n -dimensional manifold N and the graded manifold Y ,

$$\mathcal{F}_{AKSZ} = \text{Map}(T[1]N, Y).$$

The odd tangent bundle is naturally equipped with a cohomological vector field, the de Rham vector field, which can be expressed as

$$D = \theta^\mu \frac{\partial}{\partial x^\mu}$$

in coordinates x^μ of the base manifold N and θ^μ of the odd fibers. Notice also that real-valued functions on $T[1]N$ can be interpreted as differential forms on N (by expanding the function in powers of θ^μ),

$$C^\infty(T[1]N, \mathbb{R}) \simeq \Omega^\bullet(N),$$

which allows to define a canonical measure μ on $T[1]N$: the Berezinian integration along all odd fibers simply extracts the top-form out of this expansion and it remains to integrate it over the base N .

Roughly, the idea behind the AKSZ construction involves lifting the symplectic structure ω_Y from the target space to define the BV structure Ω_{AKSZ} on the space of fields. In effect, we replace functions and differential forms on Y by functionals on the space of fields with values in differential forms on N and their variations (which also explains the choice of δ to denote the exterior derivative on Y), and we integrate over the source space $T[1]N$ using its canonical measure μ ,

$$\Omega_{AKSZ} = \int_{T[1]N} \mu \tilde{\omega}_Y. \tag{1}$$

The tilde denotes the extension from function on Y to functional on the space of fields. The Berezinian integration along the fibers will lower the ghost number of ω_Y from $n - 1$ to -1 as required.

In a second stage we need to lift the cohomological vector field Q_Y on the target-space Y as well as the de Rham vector field D on the source-space $T[1]N$ to the space of fields, and combine them to form the BV cohomological vector field Q discussed above that will happen to be Hamiltonian. Its generating functional is nothing but the BV-AKSZ action

$$S_{\text{AKSZ}} = \int_{T[1]N} \mu \left(\iota_{Q_D} \tilde{\alpha}_Y + \tilde{\Theta}_Y \right), \quad (2)$$

where $Q_D = \Sigma_i D\phi^i \frac{\delta}{\delta\phi^i}$ is the lift of the de Rham vector field (ϕ^i denotes generic coordinates on the space of fields). This AKSZ action automatically solves the classical master equation as a consequence of the integrability condition on Θ_Y and the fact that the integral of exact forms vanishes provided $\partial N = \emptyset$.

As an example of the AKSZ construction, we derive here the BV formulation of the Chern-Simons theory, which corresponds to the special case $n = 3$ with $Y = \mathfrak{g}[1]$, where \mathfrak{g} is a Lie algebra equipped with an invariant scalar product. As required, the target space $\mathfrak{g}[1]$ supports a symplectic structure of degree 2 and a Hamiltonian cohomological vector field of degree 1 (sometimes called a Q-structure). If we denote with δ the exterior derivative on $\mathfrak{g}[1]$ and ψ a generic element, the symplectic form, its Liouville potential and the Hamiltonian of the cohomological vector field respectively can be written as

$$\omega_{\mathfrak{g}[1]} = -\frac{1}{2} (\delta\psi, \delta\psi), \quad (3)$$

$$\alpha_{\mathfrak{g}[1]} = -\frac{1}{2} (\psi, \delta\psi), \quad (4)$$

$$\Theta_{\mathfrak{g}[1]} = -\frac{1}{6} (\psi, [\psi, \psi]). \quad (5)$$

Note that Grassmanian variables ψ anticommute, but so do differential forms of odd degree, which explains why $\omega_{\mathfrak{g}[1]}$ may be built out of a symmetric product.

If we use coordinates $x^\mu, \mu = 1, 2, 3$ on N and corresponding Grassmanian coordinates θ^μ on the odd fibres of $T[1]N$, we can decompose the fields $\mathbf{A} \in \text{Map}(T[1]N, \mathfrak{g}[1])$ into \mathfrak{g} -valued differential forms of various degrees and grading,

$$\mathbf{A} = \gamma + A_\mu \theta^\mu + \frac{1}{2} A_{\mu\nu}^+ \theta^\mu \theta^\nu + \frac{1}{6} \gamma_{\mu\nu\sigma}^+ \theta^\mu \theta^\nu \theta^\sigma, \quad (6)$$

specifically

$$\begin{aligned} \gamma &\in \text{Map}(N, \mathfrak{g}[1]), \\ A &\in \Gamma(T^*N \otimes \mathfrak{g}), \\ A^+ &\in \Gamma(\bigwedge^2 T^*N \otimes \mathfrak{g}[-1]), \\ \gamma^+ &\in \Gamma(\bigwedge^3 T^*N \otimes \mathfrak{g}[-2]). \end{aligned}$$

These fields are endowed with two gradings, namely the ghost-grading (that stands in square brackets when non-zero) and the degree as a differential form. Their sum, the total degree, should amount to 1, since each θ^μ has a ghost number 1, and all terms in the decomposition (6) should have the same total

ghost number of 1. As usual, the fields of ghost number 0 are the classical fields, here a \mathfrak{g} -valued connection A , and the fields of ghost number 1 are simply called ghosts. The other two fields are their antifields (which in the BV formalism means canonically conjugated), as is clear when one computes the BV structure,

$$\begin{aligned}\Omega_{\text{BV}}^{\text{CS}} &= \int_{T[1]N} \mu \tilde{\omega}_{\mathfrak{g}[1]} = - \int_{T[1]N} \mu (\delta \mathbf{A}, \delta \mathbf{A}) \\ &= \int_N ((\delta \gamma^+, \delta \gamma) - (\delta A^+, \delta A)) = \int_N (-(\delta \gamma, \delta \gamma^+) + (\delta A, \delta A^+)).\end{aligned}\quad (7)$$

Note that the commutation rules for the fields (which are simultaneously functions on \mathcal{F} and differential forms on N) are determined by the total degree (the de Rham degree of the differential form plus the ghost number): two fields of odd total degree anti-commute, and commute if at least one field has even total degree.

It remains to compute the BV action, which is straightforward in the AKSZ scheme,

$$\begin{aligned}S_{\text{BV}}^{\text{CS}} &= \int_{T[1]N} \mu \left(\iota_{Q_D} \tilde{\alpha}_{\mathfrak{g}[1]} + \tilde{\Theta}_{\mathfrak{g}[1]} \right) \\ &= \int_{T[1]N} \mu \left((\mathbf{A}, D\mathbf{A}) + \frac{1}{6} (\mathbf{A}, [\mathbf{A}, \mathbf{A}]) \right) \\ &= \int_N \left(\frac{1}{2} (A, dA) + \frac{1}{6} (A, [A, A]) - (A^+, d\gamma + [A, \gamma]) + \left(\gamma^+, \frac{1}{2} [\gamma, \gamma] \right) \right).\end{aligned}\quad (8)$$

In the two terms involving only physical fields, we recognize the classical action of the Chern-Simons theory. The other terms complete the BV action, and by construction, it is clear that it satisfies the classical master equation. Nevertheless, we will show it explicitly, mainly to present an example of calculations in the space of fields, on which we will rely in the rest of this paper.

2.3 Calculations in the BV formalism

First of all, we need to find the BV bracket, simply by inverting the BV structure (7), without forgetting that the product between two differential forms in the space of fields really means the exterior product,

$$\delta \phi^+ \delta \phi = \delta \phi^+ \wedge \delta \phi = \delta \phi^+ \otimes \delta \phi \pm \delta \phi \otimes \delta \phi^+,$$

where the sign depends on the commutation rules between ϕ and ϕ^+ . We find the following expression for the BV bracket of two functionals F_1 and F_2 ,

$$\begin{aligned}\{F_1, F_2\} &= \int_N \left(\left(\frac{F_1 \overleftarrow{\delta}}{\delta \gamma}, \overrightarrow{\delta} F_2 \right) - \left(\frac{F_1 \overleftarrow{\delta}}{\delta \gamma^+}, \overrightarrow{\delta} F_2 \right) \right. \\ &\quad \left. - \left(\frac{F_1 \overleftarrow{\delta}}{\delta A}, \overrightarrow{\delta} F_2 \right) + \left(\frac{F_1 \overleftarrow{\delta}}{\delta A^+}, \overrightarrow{\delta} F_2 \right) \right)\end{aligned}\quad (9)$$

where the functional derivatives $\overrightarrow{\frac{\delta}{\delta\phi}}$ and $\overleftarrow{\frac{\delta}{\delta\phi}}$ for $\phi \in \{A, A^+, \gamma, \gamma^+, g^+\}$ are the duals of the differentials $\delta\phi$ in the space of fields (which can be interpreted as variations of fields in the framework of variational calculus). We need to make the difference between right- and left-derivatives due to the commutation rules that depend on ghost numbers and degrees of differential forms on N .

All the fields of the Chern-Simons model are \mathfrak{g} -valued, so taking the functional derivative of a real-valued functional F on \mathcal{F} by one of these fields should produce a \mathfrak{g}^* -valued result, but we can use the non-degenerate scalar product (\cdot, \cdot) to identify \mathfrak{g} with its dual. If F is constructed as an integral, like an action, the left- and right-derivatives by a field $\phi \in \{A, A^+, \gamma, \gamma^+, g^+\}$ can be defined as the components of the exterior derivative with respect to the local frame induced by these coordinate-fields of \mathcal{F} ,

$$\delta F(\phi_1, \dots, \phi_n) = \int_{\partial N} \sum_{j=1}^n \left(\delta\phi_j, \overrightarrow{\frac{\delta F}{\delta\phi_j}} \right) = \int_{\partial N} \sum_{j=1}^n \left(\overleftarrow{\frac{F}{\delta\phi_j}}, \delta\phi_j \right).$$

If on the other hand F is a local functional, we may still express it as an integral provided we filter its position with a Dirac distribution, a distribution that will stick to the functional derivative.

At a later stage, we will need to consider Lie group-valued fields of the form $g \in \text{Map}(N, G)$. The problem with such a field is that its variation does not take value in \mathfrak{g} , but rather in the tangent space at g of the Lie group G . Natural coupling with the other \mathfrak{g} -valued fields via the invariant scalar product involves the right multiplication by g^{-1} to bring it back to the Lie algebra, explicitly $\delta g g^{-1}$. The dual derivative $\frac{\delta}{\delta g}$ assumes its value in $T_{g^{-1}}^*G$, which is isomorphic to $T_{g^{-1}}G$ thanks to the invariant non-degenerate scalar product, and we need to apply this time left-multiplication by g to get back to the Lie algebra. If the functional F depends also on g , we find for the derivative

$$\delta F(\phi_1, \dots, \phi_n, g) = \int_{\partial N} \sum_{j=1}^n \left(\delta\phi_j, \overrightarrow{\frac{\delta F}{\delta\phi_j}} \right) + \left(\delta g g^{-1}, g \overleftarrow{\frac{\delta F}{\delta g}} \right).$$

To compute $\{S_{\text{BV}}^{\text{CS}}, S_{\text{BV}}^{\text{CS}}\}$, we need the derivatives of the Chern-Simons BV action. We find

$$\begin{aligned} \delta S_{\text{BV}}^{\text{CS}} = \int_N & \left(\left(\delta A, dA + \frac{1}{2} [A, A] + [A^+, \gamma] \right) \right. \\ & + (\delta\gamma, -d_A A^+ - [\gamma^+, \gamma]) \\ & \left. + (\delta A^+, -d_A \gamma) + \left(\delta\gamma^+, \frac{1}{2} [\gamma, \gamma] \right) \right), \end{aligned} \quad (10)$$

where we introduced the covariant derivative $d_A = d + [A, \cdot]$. Note that we need to integrate by parts to find the contribution of the exterior derivatives, such as

$$\int_N (A, \delta dA) = \int_N (A, d\delta A) = - \int_{\partial N} (A, \delta A) + \int_N (dA, \delta A),$$

where the boundary term vanishes for a closed manifold N . When we consider source spaces with boundaries, these terms will no longer vanish, and they will contribute to a one-form in the boundary space of fields. For now we can check the classical master equation,

$$\begin{aligned} \frac{1}{2} \{S_{\text{BV}}^{\text{CS}}, S_{\text{BV}}^{\text{CS}}\} &= \int_N \left(\left(\frac{S_{\text{BV}}^{\text{CS}} \overleftarrow{\delta}}{\delta\gamma}, \overrightarrow{\delta} S_{\text{BV}}^{\text{CS}} \right) - \left(\frac{S_{\text{BV}}^{\text{CS}} \overleftarrow{\delta}}{\delta A}, \overrightarrow{\delta} S_{\text{BV}}^{\text{CS}} \right) \right) \\ &= \int_N \left(\left(d_A A^+ + [\gamma^+, \gamma], \frac{1}{2} [\gamma, \gamma] \right) \right. \\ &\quad \left. - \left(dA + \frac{1}{2} [A, A] + [A^+, \gamma], -d_A \gamma \right) \right) \\ &= 0. \end{aligned}$$

In the last step we make repeated use of the invariance of the scalar product, the Jacobi identity for \mathfrak{g} , and the Stokes theorem.

We are now fully prepared to describe Wilson lines in the BV formalism.

2.4 Wilson Lines

In gauge theories, a degeneracy arises under the local action of a Lie group on the space of fields, the gauge group. In what follows, we will denote by G the gauge group, and \mathfrak{g} its associated Lie algebra. Their so-called gauge field is a connection A in a principal G -bundle over some manifold N . The gauge symmetry is parametrized in the BV (and BRST) formalism by a ghost field $\gamma \in \text{Map}(N, \mathfrak{g}[1])$. The BV variation of these two fields depends only on their behavior under gauge transformations and not on the specific type of the underlying ambient theory. We assume that the dynamics and the gauge structure of this ambient theory is encoded in the BV action S^{amb} and the corresponding BV structure Ω^{amb} (both defined as integrals over N), and of course that S^{amb} solves the ambient classical master equation $\{S^{\text{amb}}, S^{\text{amb}}\}_{\text{amb}} = 0$. The part of this ambient BV bracket involving the gauge connection and the ghost field relevant for our further investigation is defined by the BV variation of these fields, namely

$$\begin{cases} \{S^{\text{amb}}, A\} &= Q A &= d_A \gamma, \\ \{S^{\text{amb}}, \gamma\} &= Q \gamma &= \frac{1}{2} [\gamma, \gamma]. \end{cases} \quad (11)$$

Natural non-local observables to consider in gauge theories are given by Wilson-loops, traces of the holonomy of the connection A along a curve Γ embedded in N in given representations of the Lie algebra,

$$W_{\Gamma, R}[A] = \text{Tr}_R \text{Pexp} \left(\int_{\Gamma} A \right),$$

where P stands for the path-ordering and R labels the representation of \mathfrak{g} .

This cumbersome path-ordering can be removed at the price of integrating over all inequivalent gauge transformations along the loop [1],

$$W_{\Gamma,R}[A] = \int \mathcal{D}g \exp \left(\int_{\Gamma} \langle T_0, g^{-1} A g + g^{-1} dg \rangle \right). \quad (12)$$

The dual algebra element $T_0 \in \mathfrak{g}^*$ encodes the representation R , along the lines of the orbit method [11] that links unitary irreducible representations of Lie groups and their coadjoint orbits. This expression for the Wilson loop can be absorbed into an extended action by adding the auxiliary term

$$S_{\text{Wilson}} = \int_{\Gamma} \langle \text{Ad}_g^*(T_0), A + dg g^{-1} \rangle \quad (13)$$

to the ambient action S^{amb} of the model under consideration. In this last step we replaced the adjoint action on the second factor of the product by the coadjoint action on the first factor, to emphasize the role of the coadjoint orbit \mathcal{O} of T_0 .

Now we would like to find a BV formulation of this contribution, so as to obtain a BV action of the full model with Wilson loops. The partition function of such a model with an action extended to take into account a Wilson line as an auxiliary term actually corresponds to the expectation value of this Wilson line in the pure theory,

$$Z_{S^{\text{amb}}+S^{\text{aux}}} = \langle W_{\Gamma,R} \rangle_{S^{\text{amb}}}.$$

We note that the coadjoint orbit \mathcal{O} supports the Kirillov symplectic structure $\omega_{\mathcal{O}}$, of ghost number 0, and that the curve Γ carrying the Wilson line has dimension 1, the first two main ingredients for the AKSZ construction for $n = 1$. It is thus tempting to try to apply the prescription proposed in [12] to construct observables within the AKSZ formalism. Nonetheless, [12] treats exclusively the case of an ambient theory of the AKSZ type, whereas we want to consider gauge theories, with the sole requirement that their space of fields contains a gauge connection and an associated ghost field obeying the relations (11). The obvious solution is to study a gauge theory of the AKSZ type, a condition fulfilled by the Chern-Simons model, the main subject of this paper. The BV formulation of the Wilson line contribution will happen to remain valid for other gauge theories.

So following [12], the auxiliary fields are the maps between the odd tangent bundle of the curve Γ and the coadjoint orbit,

$$\mathcal{F}^{\text{aux}} = \text{Map}(T[1]\Gamma, \mathcal{O}).$$

This auxiliary space of fields needs to be equipped with its own BV structure, Ω^{aux} , that once added to the BV structure Ω^{amb} of the ambient theory will provide the BV structure $\Omega = \Omega^{\text{amb}} + \Omega^{\text{aux}}$ of the full model with space of fields $\mathcal{F} = \mathcal{F}^{\text{amb}} \oplus \mathcal{F}^{\text{aux}}$. Then it will be possible to add to the ambient action S^{amb} an auxiliary term S^{aux} that obeys certain constraints to obtain a solution of the master equation of the full model.

The definition of Ω^{aux} is similar to the one of the AKSZ-BV structure (1), we just need to change the source space and the symplectic structure of the target,

$$\Omega^{\text{aux}} = \int_{T[1]\Gamma} \mu_\Gamma \tilde{\omega}_\mathcal{O}.$$

Here μ_Γ is obviously the canonical measure on $T[1]\Gamma$.

Unfortunately, the Kirillov symplectic form on the coadjoint orbit is in general not exact, so it is in general not possible to find a Liouville one-form, which we would normally use to construct the kinetic term of the auxiliary action. However, in the case of integrable orbits, we may pick a line bundle (the pre-quantum line bundle in the language of geometric quantization) and a connection $\alpha_\mathcal{O}$ thereon with curvature $\omega_\mathcal{O}$,

$$\delta\alpha_\mathcal{O} = \omega_\mathcal{O}$$

(we recall that in the target spaces of AKSZ theories, we denote by δ the exterior derivative), and we can simply use this connection to construct the kinetic term of the auxiliary action.

This formulation is not very practical to carry out calculations. To find expressions easier to deal with, we apply the defining property of the Kirillov symplectic form, that the pullback by the projection map

$$\pi : G \rightarrow \mathcal{O} \simeq G/\text{Stab}(T_0) \tag{14}$$

brings it to an explicit presymplectic form ω_G on G ,

$$\pi^*(\omega_\mathcal{O}) = \omega_G = -\langle \text{Ad}_g^*(T_0), \frac{1}{2} [\delta g g^{-1}, \delta g g^{-1}] \rangle. \tag{15}$$

This two-form is the contraction of T_0 with the exterior derivative of the Maurer-Cartan one-form on G . It thus admits a potential

$$\alpha_G = -\langle \text{Ad}_g^*(T_0), \delta g g^{-1} \rangle. \tag{16}$$

As it happens, the pullback by the projection map π brings the connection $\alpha_\mathcal{O}$ over to the one-form α_G ,

$$\pi^*(\alpha_\mathcal{O}) = \alpha_G,$$

that we will use in the place of the more cumbersome connection to compute certain quantities. Since ω_G is degenerate, it is not possible to construct a Poisson bracket out of it, unless we restrict it to invariant functions on G , such as the ones obtained by pullback of functions on the coadjoint orbit \mathcal{O} by the projection map π .

It remains to define the interaction term. The idea of [12] is to construct a function $\Theta_\mathcal{O}$ on $\mathfrak{g}[1] \times \mathcal{O}$ that will generate together with $\Theta_{\mathfrak{g}[1]}$ a Hamiltonian cohomological vector field on $\mathfrak{g}[1] \times \mathcal{O}$. While $\Theta_{\mathfrak{g}[1]}$ already satisfies an integrability condition on its own and generates a cohomological vector field $\mathcal{Q}_{\mathfrak{g}[1]}$, the

integrability condition for $\Theta_{\mathcal{O}}$ needs to be slightly adapted to account for the mixed term, namely

$$\mathcal{Q}_{\mathfrak{g}[1]}\Theta_{\mathcal{O}} + \frac{1}{2}\{\Theta_{\mathcal{O}}, \Theta_{\mathcal{O}}\}_{\mathcal{O}} = 0.$$

As it happens, the function

$$\Theta_{\mathcal{O}} = \langle \text{Ad}_g^*(T_0), \psi \rangle \quad (17)$$

satisfies this requirement and naturally extends the term $\langle \text{Ad}_g^*(T_0), A \rangle$ that already appeared in the classical part (13). This integrability condition is most easily checked by pulling it back by π to a function on $\mathfrak{g}[1] \times G$, where the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{O}}$ becomes $\{\cdot, \cdot\}_G$, which can be explicitly determined by inverting ω_G .

We now have all the ingredients to construct the auxiliary BV action, that we just need to combine into a formula similar as the usual AKSZ action (2),

$$S^{\text{aux}} = \int_{T[1]\Gamma} \mu_{\Gamma} \left(\iota_{Q_D} \tilde{\alpha}_{\mathcal{O}} + \tilde{\Theta}_{\mathcal{O}} \right).$$

By construction, if S^{amb} is the AKSZ action of the Chern-Simons model, the total action

$$S = S^{\text{amb}} + S^{\text{aux}}$$

automatically satisfies the classical master equation generated by the total BV structure

$$\Omega = \Omega^{\text{amb}} + \Omega^{\text{aux}}.$$

We claimed that it remains true when S^{amb} is the BV action of a generic gauge theory with gauge group G . To verify this assertion, we need to compute

$$\frac{1}{2}\{S, S\} = \frac{1}{2}\{S^{\text{amb}}, S^{\text{amb}}\} + \{S^{\text{amb}}, S^{\text{aux}}\} + \frac{1}{2}\{S^{\text{aux}}, S^{\text{aux}}\}. \quad (18)$$

The first two terms involve only the ambient BV structure, since S^{amb} does not depend on the auxiliary fields. The first one vanishes due to the master equation of the BV ambient model. To compute the second term, we should know the exact dependence of the auxiliary term S^{aux} on the ambient fields A and γ , and to compute the last one, we need an expression of the auxiliary BV structure Ω^{aux} that we know how to invert.

These two issues can be addressed by using the projection map (14) to define an extended space of fields,

$$\hat{\mathcal{F}}_G^{\text{aux}} = \pi^*(\mathcal{F}^{\text{aux}}) = \{(g, g^+) | g \in \text{Map}(\Gamma, G), g^+ \in \Omega^1(\Gamma) \otimes g^*(T\mathcal{O})[-1]\}.$$

The subscript G emphasizes the fact that the coadjoint orbit is replaced by the whole group.

This projection map, now seen as a map between spaces of fields,

$$\pi : \hat{\mathcal{F}}_G^{\text{aux}} \rightarrow \mathcal{F}^{\text{aux}},$$

acts on the group-valued component g by sending it to its image $\text{Ad}_g^*(T_0)$ in the coadjoint orbit of T_0 . It can be used to pull back differential forms on the auxiliary space of fields (such as the auxiliary BV structure, a two-form, or the auxiliary BV action, a zero-form) to this extended space of fields, where it is easier to compute BV brackets of G -invariant functionals given the explicit formulas for the pullbacks of the auxiliary BV structure and of the auxiliary action.

In $\hat{\mathcal{F}}_G^{\text{aux}}$, both fields g and g^+ can be combined into a superfield of total degree 0 that we can use to express the pullback of the auxiliary BV structure and action,

$$\mathbf{H}(x, \theta) = \text{Ad}_{g(x)}^*(T_0) - \theta g^+(x).$$

Here x is a coordinate of Γ and θ a Grassmanian coordinate on the odd fibers of $T[1]\Gamma$, and $g^+(x)$ is the component of the one-form g^+ expressed in this coordinate system, $g^+ = g^+(x)dx$.

We now have all the tools to compute the pullback of the auxiliary BV structure,

$$\begin{aligned} \hat{\Omega}_G^{\text{aux}} &= \pi^*(\Omega^{\text{aux}}) = \int_{T[1]\Gamma} \mu_\Gamma \pi^*(\tilde{\omega}_\mathcal{O}) = \int_{T[1]\Gamma} \mu_\Gamma \tilde{\omega}_G \\ &= - \int_{T[1]\Gamma} \mu \delta \langle \mathbf{H}, \delta g g^{-1} \rangle = \int_\Gamma \delta \langle g^+, \delta g g^{-1} \rangle \\ &= \int_\Gamma \langle \delta g^+, \delta g g^{-1} \rangle + \langle g^+, \frac{1}{2} [\delta g g^{-1}, \delta g g^{-1}] \rangle, \end{aligned} \quad (19)$$

and of the auxiliary BV action,

$$\begin{aligned} \hat{S}_G^{\text{aux}} &= \pi^*(S^{\text{aux}}) = \int_{T[1]\Gamma} \mu_\Gamma \left(\iota_{Q_D} \tilde{\alpha}_G + \pi^*(\tilde{\Theta}_\mathcal{O}) \right) \\ &= \int_{T[1]\Gamma} \mu_\Gamma \left(\langle \mathbf{H}, Dg g^{-1} \rangle + \langle \mathbf{H}, \mathbf{A} \rangle \right) \\ &= \int_\Gamma \left(\langle \text{Ad}_g^*(T_0), A + dg g^{-1} \rangle - \langle g^+, \gamma \rangle \right). \end{aligned} \quad (20)$$

The additional term $\langle g^+, \gamma \rangle$ encodes the action of gauge transformations of the ambient model on the auxiliary classical field $\text{Ad}_g^*(T_0)$.

We insist on the fact that $\hat{\Omega}_G^{\text{aux}}$ is only a pre-BV structure in $\hat{\mathcal{F}}_G^{\text{aux}}$, it is closed but degenerate. Nevertheless, like its finite dimensional counterpart ω_G , it can be used to define a BV bracket on invariant functionals, such as the ones obtained by pullback from \mathcal{F}^{aux} , for instance \hat{S}_G^{aux} .

We can turn our attention to the pullback of the last two terms in (18). To compute the second one, we notice that $\{S^{\text{amb}}, \cdot\}$ acts as a differential (namely Q) on the fields of the ambient model, of which only A and γ appear in the

auxiliary action (20), so that we may simply use the relations (11) to find

$$\begin{aligned} \{S^{\text{amb}}, \hat{S}_G^{\text{aux}}\} &= \int_{\Gamma} (\langle \text{Ad}_g^*(T_0), Q(A) \rangle - \langle g^+, Q(\gamma) \rangle) \\ &= \int_{\Gamma} \left(\langle \text{Ad}_g^*(T_0), d_A \gamma \rangle - \langle g^+, \frac{1}{2} [\gamma, \gamma] \rangle \right). \end{aligned} \quad (21)$$

Next, the bracket of the third term contains only contributions from the auxiliary structure, and we may compute

$$\begin{aligned} \frac{1}{2} \{ \hat{S}_G^{\text{aux}}, \hat{S}_G^{\text{aux}} \} &= \int_{\Gamma} \left(\langle g \frac{\hat{S}_G^{\text{aux}} \overleftarrow{\delta}}{\delta g}, \frac{\overrightarrow{\delta} \hat{S}_G^{\text{aux}}}{\delta g^+} \rangle + \langle g^+, \frac{1}{2} \left[\frac{\hat{S}_G^{\text{aux}} \overleftarrow{\delta}}{\delta g^+}, \frac{\overrightarrow{\delta} \hat{S}_G^{\text{aux}}}{\delta g^+} \right] \rangle \right) \\ &= \int_{\Gamma} \left(\langle (d + \text{ad}_A^*) \text{Ad}_{g^{-1}}^* T_0, \gamma \rangle + \langle g^+, \frac{1}{2} [\gamma, \gamma] \rangle \right). \end{aligned} \quad (22)$$

The first line displays the BV bracket of invariant functionals on $\hat{\mathcal{F}}_G^{\text{aux}}$ constructed out of the pre-BV structure $\hat{\Omega}_G^{\text{aux}}$.

The sum of these two terms yields the integral of an exact term that vanishes since Γ is closed (for now), and the pullback of the classical master equation is satisfied,

$$\pi^* \left(\frac{1}{2} \{ S^{\text{amb}} + S^{\text{aux}}, S^{\text{amb}} + S^{\text{aux}} \} \right) = 0.$$

Furthermore, since the left-hand side of this last equality is G -invariant, it behaves nicely enough under the projection π so that $S^{\text{amb}} + S^{\text{aux}}$ still solves the classical master equation at the level of $\mathcal{F} = \mathcal{F}^{\text{amb}} \oplus \text{Map}(T[1]\Gamma, \mathcal{O})$, also for generic gauge theories.

2.5 Quadratic Lie algebras

In many cases of interest, the Lie algebra \mathfrak{g} is equipped with a non-degenerate scalar product (\cdot, \cdot) , which we can use to define an isomorphism $\beta : \mathfrak{g}^* \rightarrow \mathfrak{g}$, that we can apply to T_0 , H and g^+ . The relation $\beta(\text{Ad}_g^*(T_0)) = \text{Ad}_g(\beta(T_0))$ will be very useful, in particular we can replace the canonical pairing between \mathfrak{g} and \mathfrak{g}^* with the scalar product,

$$\langle \text{Ad}_g^*(T_0), \cdot \rangle = (\text{Ad}_g(\beta(T_0)), \cdot),$$

and thus identify coadjoint orbits with adjoint orbits.

In the rest of this article, we will assume that \mathfrak{g} admits such a non-degenerate scalar product. We will make use of it to write down all actions and BV structures, and we will simply consider T_0 , H and g^+ to be elements of \mathfrak{g} instead of its dual, \mathfrak{g}^* (we drop the β for simplicity). Coadjoint orbits will therefore be identified with adjoint orbits. To summarize the main results of this section with this new convention, we can re-write the auxiliary BV structure (19) as

$$\hat{\Omega}_G^{\text{aux}} = \int_{\Gamma} (\delta g^+, \delta g g^{-1}) + (g^+, \frac{1}{2} [\delta g g^{-1}, \delta g g^{-1}]) \quad (23)$$

and the auxiliary BV action (20) as

$$\hat{S}_G^{\text{aux}} = \int_{\Gamma} ((\text{Ad}_g(T_0), A + dg g^{-1}) - (g^+, \gamma)). \quad (24)$$

3 3D Chern-Simons Theory with a Wilson line

Our main goal in this paper is to study the behaviour of Chern-Simons models on a manifold N with boundaries and Wilson lines ending on these boundaries, both in three and one dimension, in the BV formalism. The presence of a boundary requires either a careful choice in the boundary conditions for the fields, so as to keep the classical master equation under control, or the application of the recently developed BV-BFV formalism for gauge theories with boundaries [6], which presents the big advantage that it allows to glue pieces together along their boundary.

In order to obtain the BV-BFV formulation of the three-dimensional Chern-Simons model with a boundary supporting some Wilson lines, we first need to determine the BV theory of the bulk. Actually we already know all its ingredients. Obviously, we will treat the Wilson lines as an auxiliary part of the action, as described in section 2, added to the ambient Chern-Simons action (8),

$$S^{\text{amb}} = S_{BV}^{CS}.$$

To include Wilson lines to our model, say n of them, we need to extend the ambient space of fields

$$\mathcal{F}^{\text{amb}} = \text{Map}(T[1]N, \mathfrak{g}[1])$$

carrying the ambient BV structure $\Omega^{\text{amb}} = \Omega_{BV}^{CS}$ with an auxiliary part

$$\mathcal{F}^{\text{aux}} = \bigoplus_{k=1}^n \text{Map}(T[1]\Gamma_k, \mathcal{O}_k)$$

made of n components, one for each Wilson line labeled by k . We recall that we denote by \mathcal{O}_k the (co)adjoint orbit of a Lie algebra element $T_{0,k}$ encoding the representation in which the k -th Wilson line is computed and by Γ_k the curve embedded in N supporting this Wilson line. The BV structure of this auxiliary space of fields is the sum of n copies of the auxiliary BV structure of a single Wilson line,

$$\Omega^{\text{aux}} = \sum_{k=1}^n \int_{T[1]\Gamma_k} \mu_{\Gamma_k} \tilde{\omega}_{\mathcal{O}_k}.$$

The auxiliary BV action is similarly constructed as a sum,

$$S^{\text{aux}} = \sum_{k=1}^n \int_{T[1]\Gamma_k} \mu_{\Gamma_k} \left(\iota_{Q_D} \tilde{\alpha}_{\mathcal{O}_k} + \tilde{\Theta}_{\mathcal{O}_k} \right).$$

From now on, unless specified otherwise, a superscript “amb” will always describe a quantity associated to the BV formulation of the bare Chern-Simons model, be it a BV structure, a BV action or a BV space of fields, “aux” will always describe a quantity associated to the BV formulation of the auxiliary contribution of n Wilson lines, and no superscript will mean a BV quantity of the full model, namely $\mathcal{F} = \mathcal{F}^{\text{amb}} \oplus \mathcal{F}^{\text{aux}}$, $\Omega = \Omega^{\text{amb}} + \Omega^{\text{aux}}$ and $S = S^{\text{amb}} + S^{\text{aux}}$.

Before we consider the case of a source manifold with boundary, we need to compute the Hamiltonian vector field Q generated by S , i.e. satisfying

$$\iota_Q \Omega = \delta S. \quad (25)$$

Once the BV structure Ω is inverted to form the BV bracket $\{\cdot, \cdot\}$, this is equivalent to

$$Q = \{S, \cdot\}.$$

This relation is linear in Q and S , namely $\iota_{Q^{\text{amb}}} \Omega = \delta S^{\text{amb}}$ and $\iota_{Q^{\text{aux}}} \Omega = \delta S^{\text{aux}}$. We will again use the projection maps $\pi_k : G \rightarrow \mathcal{O}_k$ to pull back the auxiliary action and the auxiliary BV structure to the space of fields $\bigoplus_{k=1}^n \hat{\mathcal{F}}_G^{\text{aux}}$ where the calculations are easier. Note that we need one copy of $\hat{\mathcal{F}}_G^{\text{aux}}$ for each Wilson line. The results can then be easily brought over to the actual auxiliary space of fields by the n projections π_k .

For the ambient Hamiltonian vector field we obtain

$$\begin{aligned} Q^{\text{amb}} = \{S^{\text{amb}}, \cdot\} = & \left((d_A A^+ + [\gamma^+, \gamma]), \frac{\overrightarrow{\delta}}{\delta \gamma^+} \right) - \frac{1}{2} \left([\gamma, \gamma], \frac{\overrightarrow{\delta}}{\delta \gamma} \right) \\ & - \left(\left(dA + \frac{1}{2} [A, A] + [A^+, \gamma] \right), \frac{\overrightarrow{\delta}}{\delta A^+} \right) + \left(d_A \gamma, \frac{\overrightarrow{\delta}}{\delta A} \right), \end{aligned} \quad (26)$$

and for its auxiliary counterpart

$$\begin{aligned} \hat{Q}_G^{\text{aux}} = \{\hat{S}_G^{\text{aux}}, \cdot\} = & - \sum_k \left(\text{Ad}_g (T_{0,k}) \delta(\Gamma_k), \frac{\overrightarrow{\delta}}{\delta A^+} \right) - \left(d_A (\text{Ad}_g (T_{0,k})), \frac{\overrightarrow{\delta}}{\delta g^+} \right) \\ & - \left(\gamma, g \frac{\overrightarrow{\delta}}{\delta g} \right) - \sum_k \left(g^+ \delta(\Gamma_k), \frac{\overrightarrow{\delta}}{\delta \gamma^+} \right). \end{aligned} \quad (27)$$

Here $\delta(\Gamma_k)$ denotes a Dirac distribution two-form centred on Γ_k to filter the curve out of the whole manifold N . The fields g and g^+ that appear in front of these Dirac two-forms are defined only on the Wilson lines. The functional derivatives appearing right after them act on functionals in the bulk, but their results are zero- and one-forms that make sense on the curves Γ_k . Since the ambient and the total actions solve classical master equations, we know that Q^{amb} and $Q = Q^{\text{amb}} + Q^{\text{aux}}$ are cohomological, but not Q^{aux} .

If the source space has a boundary, on which might end an open Wilson line along one or more curves Γ_k (for simplicity, we will assume all of them), these

cohomological vector fields cease to be Hamiltonian due to boundary effects affecting the variation of the differentiated terms in the action. The integration by part required to compute the contribution of these terms to the variation of the action now contains a surface integral. The BV-BFV formalism is based on the observation that this correction can be seen as a one-form in the boundary space of fields \mathcal{F}_∂ . This boundary space of fields contains the restriction of the fields of the bulk BV theory to their value on the boundary of the source manifold, ∂N in the case of our ambient theory, $\bigcup_{k=1}^n \partial\Gamma_k$ for the auxiliary model describing the Wilson lines. We denote by

$$\pi_\partial : \mathcal{F} \rightarrow \mathcal{F}_\partial$$

the projection corresponding to this restriction. The correction to the Hamiltonian condition (25) can be expressed as

$$\delta S = \iota_Q \Omega + \pi_\partial^*(\alpha_\partial). \quad (28)$$

The exterior derivative of this one-form,

$$\Omega_\partial = \delta\alpha_\partial,$$

happens to be symplectic and is called the BFV structure. It is a two-form of ghost number 0 in the boundary space of fields. The corrected Hamiltonian condition (28) is linear in α_∂ , too, so we may decompose the boundary BFV structure

$$\Omega_\partial = \Omega_\partial^{\text{amb}} + \Omega_\partial^{\text{aux}}$$

and compute it in two parts. The ambient one corresponds to the Chern-Simons model,

$$\Omega_\partial^{\text{amb}} = \int_{\partial N} \left(\frac{1}{2} (\delta A, \delta A) + (\delta\gamma, \delta A^+) \right), \quad (29)$$

and could actually be derived in a two-dimensional adaptation of the AKSZ construction. To compute the auxiliary part, we make use of the usual trick to do calculations in the augmented space of fields. The result

$$\hat{\Omega}_{\partial, G}^{\text{aux}} = \sum_k \int_{\partial\Gamma_k} \left(\text{Ad}_g(T_{0,k}), \frac{1}{2} [\delta g g^{-1}, \delta g g^{-1}] \right) \quad (30)$$

is the sum of $2k$ copies of the symplectic form ω_G of the target space of the augmented space of fields $\hat{\mathcal{F}}_{\partial, G}^{\text{aux}}$, one carried by each extremity of every Wilson line. Using the relation (15), we immediately find the BFV structure on the actual auxiliary boundary space of fields $\mathcal{F}_\partial^{\text{aux}} = \bigoplus_{k=1}^n \text{Map}(\partial\Gamma_k, \mathcal{O}_K)$,

$$\Omega_\partial^{\text{aux}} = \sum_k \int_{\partial\Gamma_k} \tilde{\omega}_{\mathcal{O}_k}, \quad (31)$$

where $\tilde{\omega}_{\mathcal{O}_k}$ is evidently the Kirillov symplectic form on the k -th (co)adjoint orbit \mathcal{O}_k lifted to the space of fields.

The curve Γ_k being one dimensional, we have $\partial\Gamma_k = \{z_k, z'_k\} \subset \partial N$, and what the last integral really means is $\int_{\partial\Gamma_k} \tilde{\omega}_{\mathcal{O}_k} = \tilde{\omega}_{\mathcal{O}_k}(z_k) - \tilde{\omega}_{\mathcal{O}_k}(z'_k)$.

In the last step of the construction of the boundary BFV model, we know that the restriction of Q to the boundary surface ∂N is Hamiltonian with respect to the BFV structure, and the boundary BFV action is defined as its generating functional,

$$\iota_{Q_\partial} \Omega_\partial = \delta S_\partial.$$

Ghost number counting shows that the BFV action has ghost number 1.

In the case of the Chern-Simons model with Wilson lines, we calculate the restriction of $\hat{Q}_G = Q^{\text{amb}} + \hat{Q}_G^{\text{aux}}$ in the extended space of fields,

$$\begin{aligned} \hat{Q}_{\partial,G} = & - \left(\frac{1}{2} [\gamma, \gamma], \frac{\vec{\delta}}{\delta\gamma} \right) - \left(\left(dA + \frac{1}{2} [A, A] + [A^+, \gamma] \right), \frac{\vec{\delta}}{\delta A^+} \right) \\ & + \left(d_A \gamma, \frac{\vec{\delta}}{\delta A} \right) - \sum_k \left(\text{Ad}_g(T_{0,k}) \delta(\Gamma_k), \frac{\vec{\delta}}{\delta A^+} \right) - \left(\gamma, g \frac{\vec{\delta}}{\delta g} \right), \end{aligned} \quad (32)$$

which leads to the two contributions

$$S_\partial^{\text{amb}} = - \int_{\partial N} \left(\left(dA + \frac{1}{2} [A, A], \gamma \right) + \left(A^+, \frac{1}{2} [\gamma, \gamma] \right) \right) \quad (33)$$

and

$$\hat{S}_{\partial,G}^{\text{aux}} = - \sum_k \int_{\partial\Gamma_k} (\text{Ad}_g(T_{0,k}), \gamma) \quad (34)$$

to the boundary BFV action. As expected, the G -valued field g appears only in a (co)adjoint action, so the projection to the auxiliary space of fields is straightforward, and we obtain the BFV action

$$\begin{aligned} S_\partial = & - \int_{\partial N} \left(\left(dA + \frac{1}{2} [A, A], \gamma \right) + \left(A^+, \frac{1}{2} [\gamma, \gamma] \right) \right. \\ & \left. + \sum_k (\text{Ad}_g(T_{0,k}), \gamma) (\delta^{(2)}(z_k) - \delta^{(2)}(z'_k)) \right). \end{aligned} \quad (35)$$

In the last line, we have cast everything into the integral over ∂N by making use of Dirac distributions centered on the extremities of the Wilson lines.

We recognize in the boundary BFV action of the Chern-Simons model with Wilson lines an odd version of the two-dimensional BF model with sources, where the role of the B field is taken over by the restriction to the boundary of the ghost field γ of the bulk theory.

We conclude this section with a short remark regarding the insertions (labeled by z_k and z'_k) of the boundary model. In our setting, with Wilson lines ending on the boundary, these insertions always come in pairs of points carrying the same representation, one insertion at each end of a Wilson line. If we consider Wilson graphs in the bulk model, which are a natural generalizations of

the Wilson lines, we can obtain any configuration of points and representations as insertions. Wilson graphs are observables modeled after Wilson lines, but based on oriented graphs instead of curves. Each edge carries a representation of \mathfrak{g} and contributes with a similar term as a Wilson line to the total action, while each vertex carries an intertwining operator between the representations of the attached edges. If the formulation of these intertwining operators is straightforward in the operator formalism, their description is more involved in the path-integral formalism and goes beyond the scope of this paper, where we will for simplicity consider only Wilson loops and open Wilson lines.

4 1D Chern-Simons Theory with a Wilson line

The AKSZ construction for the Chern-Simons model can also be carried out in one dimension [3]. In this section, we will see how to add a Wilson line to this model, by following the same procedure as in the previous section. The main difference comes from the fact that the Wilson line is now a space-filling observable, and that the BV bracket of the auxiliary term with itself will pick up terms from the ambient part of the BV structure. Moreover, as stated before, we will now use a \mathbb{Z}_2 -grading, since a \mathbb{Z} -grading is not possible in one dimension, so instead of denoting the ghost number in square brackets, we will use the parity-reversing operator Π .

Given a one-dimensional manifold Γ (in general a disjoint union of circles and open segments), the space of fields is

$$\mathcal{F}^{\text{amb}} = \text{Map}(\Pi T\Gamma, \Pi\mathfrak{g}), \quad (36)$$

where \mathfrak{g} is again assumed to be equipped with an invariant scalar product (\cdot, \cdot) .

The target space $\Pi\mathfrak{g}$ supports the same geometric structures as before, that we may again transpose to the space of fields.

If x is a coordinate on Γ and θ a Grassmanian coordinate on the odd fibers of $\Pi T\Gamma$, we can decompose the fields $\Psi \in \text{Map}(\Pi T\Gamma, \Pi\mathfrak{g})$ into a \mathfrak{g} -valued fermion ψ and a \mathfrak{g} -valued one-form $A = A(x)dx$,

$$\Psi = \psi + \theta A(x). \quad (37)$$

We repeat the same procedure to find the BV structure

$$\Omega^{\text{amb}} = - \int_{\Pi\Gamma} \mu(\delta\Psi, \delta\Psi) = \int_{\Gamma} (\delta\psi, \delta A) \quad (38)$$

and the BV action

$$S^{\text{amb}} = \int_{\Pi\Gamma} \mu \left(\frac{1}{2} (\Psi, D\Psi) + \frac{1}{6} (\Psi, [\Psi, \Psi]) \right) = \int_{\Gamma} \frac{1}{2} (\psi, d_A\psi). \quad (39)$$

The \mathfrak{g} -valued one-form A can be interpreted as a connection for some principal G -bundle over Γ , where G is a Lie group integrating \mathfrak{g} . The odd \mathfrak{g} -valued scalar

ψ serves simultaneously as a ghost for the gauge symmetry and an antifield for A .

In order to add Wilson lines to this model, we need to extend the space of fields, the BV structure and the action with precisely the same auxiliary structure Ω^{aux} and action S^{aux} as in the three-dimensional case, except that they are now supported directly by the base manifold of the ambient source space Γ . We consider a single Wilson line for simplicity, it is easy to add similar terms for additional lines. Furthermore we assume it covers the whole source space Γ . Actually it could involve only some of the connected components of Γ , and the other ones would support a bare (in the sense that there are no Wilson lines) one-dimensional Chern-Simons model. Notice also that instead of γ we write ψ to emphasize the fact that it plays simultaneously the role of γ and A^+ of the previous model.

We insist once more that since the Wilson line is a space-filling observable, we need to check that the classical master equation is solved, a result which is not guaranteed by the AKSZ construction due to the term $\{S^{\text{aux}}, S^{\text{aux}}\}_{\text{ambient}}$ coming from the auxiliary part of the action and the ambient part of the BV structure. If we use again the projection map $\pi : G \rightarrow \mathcal{O}$ to pull differential forms from the auxiliary space of fields \mathcal{F}^{aux} to the extended one $\hat{\mathcal{F}}_G^{\text{aux}}$, we can calculate

$$\begin{aligned}
\frac{1}{2} \{ \hat{S}_G, \hat{S}_G \} &= \int_{\Gamma} \left(\left(\left(\frac{\hat{S}_G \overleftarrow{\delta}}{\delta A}, \frac{\overrightarrow{\delta} \hat{S}_G}{\delta \psi} \right) + \left(g \frac{\hat{S}_G \overleftarrow{\delta}}{\delta g}, \frac{\overrightarrow{\delta} \hat{S}_G}{\delta g^+} \right) \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \left(\frac{\hat{S}_G \overleftarrow{\delta}}{\delta g^+}, \left[g^+, \frac{\overrightarrow{\delta} \hat{S}_G}{\delta g^+} \right] \right) \right) \right) \\
&= \int_{\Gamma} \left(-\frac{1}{2} [\psi, \psi] + \text{Ad}_g T_0, d_A \psi + g^+ \right) + (-d_A(\text{Ad}_g T_0), -\psi) \\
&\quad - \frac{1}{2} (\psi, [g^+, -\psi]) \tag{40} \\
&= \int_{\Gamma} (\text{Ad}_g T_0, g^+) \\
&= -\frac{1}{2} \int_{\Pi T \Gamma} \mu(\mathbf{H}, \mathbf{H}) \\
&= -\frac{1}{2} \int_{\Pi T \Gamma} \mu \left(\frac{S^{\text{aux}} \overleftarrow{\delta}}{\delta \Psi}, \frac{\overrightarrow{\delta} S^{\text{aux}}}{\delta \Psi} \right),
\end{aligned}$$

and the last line shows it explicitly. Nevertheless, this term vanishes, since g^+ takes value in the tangent space $T_H \mathcal{O}$ at $H = \text{Ad}_g T_0$ to the adjoint orbit, which is easily seen to be orthogonal to H with respect to the invariant scalar product on \mathfrak{g} .

Again, before we turn to the case of a source space with a boundary, we need to compute the Hamiltonian cohomological vector field Q generated by S ,

or more accurately its counterpart in the extended space of fields, namely

$$\begin{aligned} \hat{Q}_G = & \left((d_A \psi + g^+), \frac{\delta}{\delta A} \right) + \left(\left(-\frac{1}{2} [\psi, \psi] + \text{Ad}_g(T_0) \right), \frac{\delta}{\delta \psi} \right) \\ & - \left(\psi, g \frac{\delta}{\delta g} \right) - \left(([\psi, g^+] + d_A(\text{Ad}_g(T_0))), \frac{\delta}{\delta g^+} \right). \end{aligned} \quad (41)$$

If the source space has a boundary, in other words if some of its components are segments, we can repeat the procedure to construct the BFV boundary model. We first calculate the image of the symplectic potential of the boundary BFV structure in the augmented space of fields from the variation of the BV action,

$$\hat{\alpha}_{\partial, G} = \int_{\partial\Gamma} \left(\frac{1}{2} (\psi, \delta\psi) + (T_0, g^{-1} \delta g) \right), \quad (42)$$

and the corresponding pre-BFV structure,

$$\hat{\Omega}_{\partial, G} = \int_{\partial\Gamma} \left(\frac{1}{2} (\delta\psi, \delta\psi) - \frac{1}{2} (\text{Ad}_g(T_0), [\delta g g^{-1}, \delta g g^{-1}]) \right). \quad (43)$$

We see that the second term is connected to the pullback by the projection map $\pi : G \rightarrow \mathcal{O}$ of the Kirillov-Kostant-Souriau symplectic structure, and we obtain as a BFV structure in the proper boundary space of fields

$$\Omega_{\partial} = \int_{\partial\Gamma} \left(\frac{1}{2} (\delta\psi, \delta\psi) + \tilde{\omega}_{\mathcal{O}} \right). \quad (44)$$

In all these expressions, the integral over $\partial\Gamma$ is nothing but a sum over the boundary points, with each term carrying a sign given by the orientation of its segment.

The restriction of the cohomological vector field \hat{Q}_G to the boundary,

$$\hat{Q}_{\partial, G} = \left(\left(-\frac{1}{2} [\psi, \psi] + \text{Ad}_g(T_0) \right), \frac{\delta}{\delta \psi} \right) - \left(\psi, g \frac{\delta}{\delta g} \right), \quad (45)$$

is Hamiltonian with respect to the BFV structure, and it is generated by the BFV action of the boundary model,

$$S_{\partial} = \int_{\partial I} \left(-\frac{1}{6} (\psi, [\psi, \psi]) + (\text{Ad}_g(T_0), \psi) \right). \quad (46)$$

Finally we show that S_{∂} solves the master equation of the BFV model,

$$\{S_{\partial}, S_{\partial}\} = Q_{\partial} S_{\partial} = \int_{\partial\Gamma} (T_0, T_0) = 0. \quad (47)$$

As stated before, the last integral is really a sum over the boundary elements of $\partial\Gamma$ with a sign assigned to their orientation, and since they come in pairs, at each end of every segment, the overall sum vanishes.

5 Boundary Quantum States

Upon quantization, the partition function and correlators of a field theory defined on a manifold N without boundary are complex numbers. In the presence of a boundary, one should rather expect quantum states, elements of a Hilbert space associated to each component of the boundary ∂N , according to the Atiyah-Segal picture of quantum field theory. The disjoint union of boundary components corresponds to the tensor product of the associated Hilbert spaces. Then gluing together a pair of components of ∂N corresponds to taking the scalar product of the two corresponding factors of the tensor product.

For instance, if $N = [0, 1]$ is an interval, the partition function of the BV-BFV model should take value in some Hilbert space of the form $\mathcal{H} \otimes \mathcal{H}$, with one factor for each component of the boundary $\partial N = \{0, 1\}$, and upon gluing the two ends, contracting this tensor product using the scalar product on \mathcal{H} should yield the BV partition function of the same model constructed on the circle S^1 .

If the bulk theory is studied in the BV formalism, the boundary information is encoded in the associated BFV model, at least at the classical level, as we saw in the particular cases of the Chern-Simons theory in one and three dimensions, possibly with Wilson lines.

To pass to the quantum level, we first observe that a BFV boundary model can be canonically quantized. The BFV structure, a symplectic structure of ghost number 0 in the space of fields, is used to define the (anti)commutation rules for the quantized fields. These act on the Hilbert space $\mathcal{H}_\partial^{\text{BFV}}$ associated to the boundary where the partition function of the bulk takes value. This Hilbert space inherits a grading from the ghost number of the classical fields. In this picture, the BFV action S_∂ , which was the generator of the cohomological vector field on the boundary Q_∂ , can be quantized by replacing the classical fields with their quantized counterparts, and we obtain the quantized BFV charge \hat{S}_∂ . Its action on the boundary space of states squares to zero and it roughly encodes the gauge transformations. At the classical level, a physical observable is a functional annihilated by the cohomological vector field Q generated by the BV action in the bulk and the BFV action S_∂ on the boundary, and two observables are gauge equivalent if they differ by a Q -exact term. At the quantum level, the role of Q is taken over by the BFV charge \hat{S}_∂ : a gauge invariant boundary state should be annihilated by the BFV charge, and two states are gauge-equivalent if they differ by a BFV-exact term. Moreover, we require physical states to depend only on physical quantum fields, and not on the ghosts or the antifields. In other words, the space of boundary quantum states should correspond to the BFV-cohomology at ghost number zero $H_{\hat{S}_\partial}^0(\mathcal{H}_\partial^{\text{BFV}})$.

The relation with the quantized bulk theory is that the partition function (and all the other correlators) should obviously be gauge invariant and therefore belong to this cohomology $H_{\hat{S}_\partial}^0(\mathcal{H}_\partial^{\text{BFV}})$. Its determination thus becomes a subject of interest.

We will start with the one-dimensional Chern-Simons theory, a simpler model where all calculations can be done until the end, before we study the

more interesting three-dimensional model.

5.1 1D Chern-Simons Theory

The zero-dimensional boundary model of the one-dimensional Chern-Simons theory contains \mathfrak{g} -valued fermions and bosonic fields $H = \text{Ad}_g(T_0)$ which take value in the (co)adjoint orbit \mathcal{O} . Once quantized, the fermions form a Clifford algebra $Cl(\mathfrak{g})$. If $(t^a)_{a=1}^{\dim \mathfrak{g}}$ is an orthonormal basis of \mathfrak{g} with structure constants f_{abc} , we obtain the anticommutation rules

$$[\hat{\psi}_a, \hat{\psi}_b] = \hbar \delta_{ab} \quad (48)$$

for the quantized fermions.

For the bosonic content of the model, the Kirillov symplectic form on the (co)adjoint orbits is the inverse of the restriction from $\mathfrak{g}^* \simeq \mathfrak{g}$ to \mathcal{O} of the Kirillov-Kostant-Souriau Poisson structure, so that the commutator of two \mathcal{O} -valued quantized fields is simply given by their Lie bracket. If we use the basis (t_a) of the Lie algebra to write

$$H = \text{Ad}_g(T_0) = X_a t_a,$$

we can express the commutation rules with the structure constants of the Lie algebra,

$$[\hat{X}_a, \hat{X}_b] = \hbar f_{abc} \hat{X}_c. \quad (49)$$

The corresponding sector of the algebra of quantum operators is a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra \mathfrak{g} , namely $\rho_R(\mathcal{U}(\mathfrak{g})) \subset \text{End}(V_R)$. This representation is simply the representation R in which we computed the Wilson loops in the previous sections.

We can use these operators $\hat{\psi}$ and \hat{X} to construct the expectation value of the Wilson line $\langle W_{\Gamma, R} \rangle$ in the operator formalism, such as in [3], where the partition function for the one-dimensional Chern-Simons model is derived in both the path-integral and the operator formalism. If the curve Γ is open, this expectation value maps the space of fields to the boundary space of quantum states which is the cohomology at level 0 of the quantum BFV charge,

$$\langle W_{I, R} \rangle \in H_{\hat{S}_\partial}^0(\mathcal{H}_\partial).$$

We need to find this cohomology.

The BFV charge

$$\hat{S}_\partial = \int_{\partial \Gamma} \hat{X}_a \hat{\psi}_a - \frac{1}{6} f_{abc} \hat{\psi}_a \hat{\psi}_b \hat{\psi}_c \quad (50)$$

carries one copy of the cubic Dirac operator [2]

$$\mathfrak{D} = \hat{X}_a \hat{\psi}_a - \frac{1}{6} f_{abc} \hat{\psi}_a \hat{\psi}_b \hat{\psi}_c$$

at each boundary point of Γ . This operator squares to

$$\mathfrak{D}^2 = \frac{1}{2} [\mathfrak{D}, \mathfrak{D}] = \frac{1}{2} \hat{X}_a \hat{X}_a - \frac{1}{48} f_{abc} f_{abc},$$

a central element in the quantum Weil algebra $\mathcal{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g})$, which guarantees that the action of the BFV charge squares to zero.

It is known that this cohomology is trivial ([2],[3]). The resulting quantized BV-BFV is therefore not very interesting, so we should turn to the more involved problem of the three-dimensional Chern-Simons theory.

5.2 3D Chern-Simons Theory

We may now repeat the same procedure for the three-dimensional model. The first observation is that the treatment of the part coming from the extremities of the Wilson lines, namely the terms in the insertion points labeled by z_k and z'_k , is essentially the same as in the one-dimensional model. Each insertion contributes to the overall BFV structure with a term in (31), that when canonically quantized gives the algebra of operators (49) we encountered in the quantization of the one-dimensional model. We can formally express the quantization map

$$\text{Ad}_{g(z_k)}(T_{k,0}) = X_{k,a}(z_k)t_a \mapsto \rho_k(\hat{X}_a(z_k))t^a.$$

Even though the orbits might be different for different insertions, the commutation rules (49) are identical for all of them, only the representation ρ_k differs, as we emphasized on the right-hand side.

In the next step, if we choose a complex structure on the boundary surface $\Sigma = \partial N$, we get a polarization of the connection

$$A = A_z dz + A_{\bar{z}} d\bar{z}, \tag{51}$$

which allows us to rewrite the ambient part of the BFV structure (29) in Darboux coordinates of the corresponding sector $\mathcal{F}_{\partial}^{\text{amb}}$ of the BFV space of boundary fields,

$$\Omega_{\partial}^{\text{amb}} = \int_{\partial N} dz d\bar{z} ((\delta A_z \delta A_{\bar{z}}) + (\delta \gamma, \delta A^+)). \tag{52}$$

Consequently, we may perform the canonical quantization by choosing among each pair of conjugated fields one quantum field and replace the other one by the corresponding functional differential, for instance

$$\begin{aligned} A_{\bar{z}} &\rightarrow a, \\ A_z &\rightarrow -\frac{\delta}{\delta a}, \\ \gamma &\rightarrow \gamma, \\ A^+ &\rightarrow \frac{\delta}{\delta \gamma}, \end{aligned} \tag{53}$$

so as to obtain canonical (anti)commutation rules. Note that a is a boson and γ a fermion.

The Hilbert space $\mathcal{H}_\partial^{\text{BFV}}$ of boundary states on which act all these operators is therefore the space of functionals in a and γ with value in a tensor product of all the representation space associated to each insertion,

$$\mathcal{H}_\partial^{\text{BFV}} = \text{Fun} \left(a, \gamma; \bigotimes_k V_{\rho_k} \otimes V_{\rho_k} \right).$$

We recall it is graded by the ghost number.

Among these states, we want to determine the cohomology $H_{\hat{S}_\partial}^0(\mathcal{H}_\partial^{\text{BFV}})$ of the BFV charge at ghost number zero, made up of the quantum states of the BV-BFV model. At degree zero, we are considering functionals ψ of the \mathfrak{g} -valued $(0, 1)$ -form a , independent of the ghosts γ , which take value in the tensor product of all representation spaces associated to the extremities of the Wilson lines of the models.

The BFV charge

$$\begin{aligned} \hat{S}_\partial = & - \int_{\partial N} dz d\bar{z} \left(\left(\partial a + \bar{\partial} \frac{\delta}{\delta a} + \left[a, \frac{\delta}{\delta a} \right], \gamma \right) \right. \\ & - \sum_k \left(\rho_k(\hat{X}_a(z_k)) \delta(z - z_k) - \rho_k(\hat{X}_a(z_{k'})) \delta(z - z_{k'}) \right) (t^a, \gamma) \\ & \left. + \left(\frac{1}{2} [\gamma, \gamma], \frac{\delta}{\delta \gamma} \right) \right) \end{aligned} \quad (54)$$

acts on the Hilbert space $\mathcal{H}_\partial^{\text{BFV}}$ via multiplication and differentiation by the quantum fields a and γ and via the obvious action of the representation ρ_k on its representation space V_{ρ_k} .

At ghost number zero, BFV quantum states ψ are therefore subject to the condition

$$\left(\partial a + \bar{\partial} \frac{\delta}{\delta a} + \left[a, \frac{\delta}{\delta a} \right] - \sum_k \left(\rho_k(\hat{X}_a(z_k)) \delta_{z_k}(z) - \rho_k(\hat{X}_a(z_{k'})) \delta_{z_k}(z) \right) t^a \right) \psi = 0. \quad (55)$$

This actually coincides with the constraint (1) in [10] imposed to the Schrödinger picture states in the canonical quantization of the Chern-Simons model on $\Sigma \times \mathbb{R}$ at genus 0, or the constraint (2.2) in [8] in the same situation at genus 1, where it is found that the cohomology $H_{\hat{S}_\partial}^0(\mathcal{H}_\partial^{\text{BFV}})$ coincides with the space of conformal blocks in the WZW model for a correlator of fields inserted at the extremities of the Wilson lines.

The condition (55) for quantum states also appears in the geometric quantization framework, see for instance constraint (3.4) in [15], therefore the space of states in geometric quantization coincides with the space of quantum boundary states in BV-BFV quantization.

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