Black Holes in (2+1)-dimensional Higher Spin Theory with Gauge Group $SO(3,2) \times SO(3,2)$

Mischa Knabenhans
Programme: Physics – Master Thesis
mischak@student.ethz.ch

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Abstract

The Bañados-Teitelboim-Zanelli (BTZ) black hole in three dimensional gravity is well-known. Classical General Relativity in 2+1 spacetime dimensions can be expressed as a Chern-Simons theory based on the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ while Chern-Simons theories based on higher rank groups containing $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ are equivalent to higher spin extensions of three dimensional gravity. Recently, new black hole solutions have been constructed in a Chern-Simons theory based on $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ generalizing the BTZ black hole to a black hole endowed with a spin 3 charge. In this thesis we generalize this result to a black hole solution charged under a spin 4 charge in a Chern-Simons theory based on $SO(3,2) \times SO(3,2)$.

The spectrum of a higher spin extension of three dimensional gravity theory is determined by the imposed boundary conditions. The latter are in turn associated to a choice of the $SL(2,\mathbb{R})$ subgroup of the gauge group of the equivalent Chern-Simons theory.

After a review of the basic theory of complex semi-simple Lie algebras, we classify the $SL(2)$-subgroups in the real simple Lie group $SO(3,2)$ according to the Dynkin-Kostant classification. Further, we investigate which of these subgroups will lead to a black hole solution charged under a higher spin field. Then, we explicitly construct a black hole solution in a Chern-Simons theory based on the gauge group $SO(3,2) \times SO(3,2)$. Finally we study the entropy of this black hole up to quadratic order in the higher spin field and compare it with its spin 3 counterpart.

Contents

1 Introduction **3**

2 Basic Lie Theory **5**
2.1 (Semi-)simple Lie Algebras and Cartan Subalgebras 5
2.2 Structure Theory 6
2.3 Nilpotent Orbits and Standard Triples 9
2.3.1 Properties of $\mathfrak{sl}(2)$ generators 9
2.3.2 Existence of Standard Triples 10
2.3.3 Conjugacy Classes and Nilpotent Orbits 11
2.3.4 Finiteness of Nilpotent Orbits 13
2.4 Dynkin Diagrams 13
2.4.1 Ordinary Dynkin Diagrams 13
2.4.2 Generalized Dynkin Diagrams 14

3 Construction and Classification of $\mathfrak{sl}(2)$ Embeddings **17**
3.1 Dynkin-Kostant Classification 17
3.2 The Construction of Standard Triples for Type B_n Algebras 19

4 Black Hole Solutions **24**
4.1 Three Dimensional Gravity 24
4.1.1 Action and Equation of Motion 24
4.1.2 The Asymptotic Symmetry Algebra 25
4.2 BTZ Black Hole 27
4.2.1 The Black Hole Solution 27
4.2.2 Black Hole Thermodynamics 31

5 $\mathfrak{so}(3,2)$ Black Hole Construction **34**
5.1 Construction of Standard Triples of $\mathfrak{so}(5,\mathbb{C})$ 34
5.2 Embeddings of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{so}(5,\mathbb{C})$ 37
5.3 Intersection of $\mathfrak{sl}(2,\mathbb{C})$ with $\mathfrak{so}(3,2)$ 39
5.4 The $\mathfrak{so}(3,2)$ Black Hole Solution 43
5.4.1 Generators of $\mathfrak{so}(3,2)$ 43
5.4.2 Connection Field and Asymptotic Symmetries 44
5.4.3 Black Hole 48
5.5 Thermodynamics of the $\mathfrak{so}(3,2)$ Black Hole 50
5.5.1 Regularity Condition and Holonomy 50

5.5.2 Entropy of the $\mathfrak{so}(3,2)$ Black Hole . . . 51
5.5.3 Entropies of the Spin 3 and Spin 4 Black Holes Compared and Contrasted . . . 53

6 Conclusion **57**

A Explicit Calculation of the Root System of $\mathfrak{sl}(3)$ (Example) **59**

B Dynkin Diagrams of the Simple Lie Algebras **65**
B.1 Classical Lie Algebras . . . 65
B.1.1 Type A_n: . . . 65
B.1.2 Type B_n: . . . 65
B.1.3 Type C_n: . . . 66
B.1.4 Type D_n: . . . 66
B.2 Exceptional Lie Algebras . . . 66
B.2.1 Type E_6: . . . 66
B.2.2 Type E_7: . . . 66
B.2.3 Type E_8: . . . 67
B.2.4 Type F_4: . . . 67
B.2.5 Type G_2: . . . 67

C Standard Triple Construction Algorithms for Lie Algebras of Type A_n, C_n and D_n **68**
C.1 Construction Algorithm for Type A_n Algebras . . . 68
C.2 Construction Algorithm for Type C_n Algebras . . . 71
C.3 Construction Algorithm for Type D_n Algebras . . . 73

D Pure Gravity Case **76**
D.1 Preparation . . . 76
D.2 Equation of Motion: The Flatness Condition . . . 76
D.3 Computation of $A_t^{\pm}$ . . . 78
D.3.1 Time Evolution of the Asymptotic Charges . . . 80
D.4 Functional Form of the Boundary Term . . . 80
D.5 Poisson Structure of the Asymptotic Symmetry Generators . . . 83

E Explicit Computations for the $\mathfrak{so}(3,2)$ Black Hole **87**
E.1 Normalization of the $\mathfrak{so}(3,2)$ algebra . . . 87
E.2 Boundary Conditions and Asymptotes of the Connection Field . . . 88
E.3 Evolution Equations of the Charges $\mathcal{L}^{\pm}$ and $\mathcal{W}^{\pm}$ . . . 91
E.4 Approximative Computation of the Black Hole Entropy . . . 92
E.4.1 Preliminary Derivations . . . 92
E.4.2 Linear and Quadratic Approximation . . . 93
E.4.3 Differential Equation Ansatz for an Exact Solution . . . 97

F Commutators in $\mathfrak{so}(3,2)$ **99**
F.1 L-L-Commutator Relations . . . 99
F.2 L-W-Commutator Relations . . . 99
F.3 W-W-Commutator Relations . . . 100

Chapter 1

Introduction

Three-dimensional classical General Relativity is known to be equivalent to a Chern-Simons gauge theory [1]. As it is described in more detail in the historical outline below, Chern-Simons theory is of great interest to study the thermodynamics of black holes that are charged under higher spin fields in 2+1 dimensional spacetime.

Black hole solutions with a spin 3 charge have been constructed recently in [2] in a Chern-Simons theory based on the gauge group $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$. It is our final aim in this thesis to generalize the black hole solutions provided in [2] by replacing the $SL(3,\mathbb{R})$ group by a real form of the group $SO(5,\mathbb{C})$ that also has rank 2. We will apply the Dynkin-Kostant classification in order to classify the $SL(2,\mathbb{C})$ subgroups in $SO(5,\mathbb{C})$ and identify the candidate that will lead to a black hole solution. Further, we use a technique based on Vogan diagrams in order to find the intersections of the $\mathfrak{sl}(2,\mathbb{C})$ algebra with the real simple Lie algebra $\mathfrak{so}(3,2)$.
We continue by constructing black hole solutions in the formalism of a Chern-Simons theory based on the gauge group $SO(3,2)$ in three dimensional Euclidean spacetime. Hereby we adhere entirely to the strategy presented in [2] and hence avoid any reference to a metric formulation. Starting from the Chern-Simons action, we compute step by step the solutions of the Chern-Simons field equations under a set of imposed boundary conditions and require these solutions to have thermodynamical properties. As we will find, the black hole solutions will be endowed with a spin 4 charge and the asymptotic symmetry algebra is the $\mathcal{W}(2,4)$-algebra. Finally, we derive the entropy of these black hole solutions directly from the Chern-Simons action and compare it to the spin 3 result provided in [2].

We start this thesis by providing a basic introduction to Lie theory and the notions of nilpotent orbits and standard triples. In chapter 3 the Dynkin-Kostant classification is explained as well as the construction of standard triples in the case of B_n algebras. We dedicate chapter 4 to a quick review of the analysis of pure gravity based on Chern-Simons theory as it is presented in [2] and follow analog trains of thought in chapter 5 where the black hole solutions in the $SO(3,2)$ Chern-Simons theory is explicitly constructed. We conclude this thesis with an investigation of the entropy of these black hole solutions.

Historical Outline

It is a prominent fact of modern physics that black holes must have an entropy in order to be consistent with classical thermodynamics [3, 4] (see [5] for an introduction). String theory provides a microscopic account for the Bekenstein-Hawking entropy of black holes [6]. One needs to go beyond supersymmetry to describe the microscopic origin of the entropy of Schwarzschild black holes in four spacetime dimensions [7]. Accordingly, it is natural to study a toy model such as gravity in (2+1) dimensional spacetime, which provides an exactly solvable theory on both classical and quantum level [8,9]. While it has been shown in [1] that classical General Relativity (GR) in three spacetime dimensions is equivalent to a Chern-Simons theory, Bañados, Teitelboim and Zanelli found that this theory allows for black hole solutions (nowadays known as BTZ black holes) that are simpler than their four dimensional counterparts but are non-trivial as they posses an event horizon and both, non-zero entropy and Hawking temperature [10].

In the late 70's, 80's and 90's of the last century, Fronsdal, Vasiliev, Fradkin and others started to study interaction theories of massive particles carrying higher spins [11–14]. These are of special interest in string theory as it contains massive states carrying higher spins. These could hint to a spontaneously broken phase of an underlying theory with a huge hidden gauge symmetry [15–22]. Moreover, some higher spin (HS) theories admit a new type of AdS-CFT duality and hence they should be classical limits of consistent theories of gravity together with HS theories [23–25]. In this setting it is further natural to generalize the BTZ black hole in classical, three-dimensional gravity theory to black hole solutions charged under higher spin fields. Results on this field of research have been provided e.g. in [2, 26] and a review is given in [7]. This kind of generalization in the setting of three-dimensional spacetime may provide further insight in the so far still not fully understood problems such as black hole thermodynamics.

Chapter 2

Basic Lie Theory

Before meeting the first goal of constructing and classifying embeddings of $\mathfrak{sl}(2)$ into (semi-)simple Lie algebras, the most important mathematical concepts shall be introduced. Hence, we start with an introduction to basic Lie theory. Most of the material covered here may either be considered as widely known or is taken from [27] and [28].

2.1 (Semi-)simple Lie Algebras and Cartan Subalgebras

Let us quickly recap their definitions:

Definition 2.1.1 (Lie algebra)
A vector space $\mathfrak{g}$ with an antisymmetric bilinear operation $[\ ,\ ]$ (called the commutator) mapping $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ is called a Lie algebra if it satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \qquad \text{for } X,Y,Z \in \mathfrak{g} \tag{2.1}$$

There are Lie algebras with additional properties, so called simple and semisimple Lie algebras. In order to define these we first need to introduce the concept of ideals:

Definition 2.1.2 (Ideal)
A subset $\mathfrak{e}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if $[\mathfrak{e},\mathfrak{g}] \subseteq \mathfrak{e}$.

And using this we can then define the concepts of simple and semisimple Lie algebras which play an important role this thesis.

Definition 2.1.3 (Simple and semisimple Lie algebras)
A Lie algebra that has no proper ideal is called a simple Lie algebra.
A Lie algebra that can be written as the direct sum of simple Lie algebras is called a semisimple Lie algebra.

Lie algebras are vector spaces and hence are spanned by a set of basis elements called the generators. These generators can now be represented as linear operators on a vector space V that are linearly independent. Since we can neglect the case of infinite-dimensional vector spaces for the purposes of this thesis, these generators can be represented as matrices. The number of the generators of an algebra is its dimension $\dim(\mathfrak{g})$. Each simple Lie algebra has a special subalgebra.

Definition 2.1.4 (Cartan subalgebra)
Let $\{H_1, \ldots, H_r\}$ be a maximal set of commuting Hermitian generators of a simple Lie algebra $\mathfrak{g}$. This set spans a "Cartan subalgebra $\mathfrak{h}$" of $\mathfrak{g}$.

The number r of generators of the Cartan subalgebra is called the "rank of the Lie algebra $\mathfrak{g}$", denoted by rank($\mathfrak{g}$). Notice also that the Cartan subalgebra is not unique. Furthermore, since Hermitian matrices are always diagonalizable and since all generators of a Cartan subalgebra commute with each other it follows directly from the simultaneous diagonalization theorem that the generators of the Cartan subalgebra can all be diagonalized simultaneously.

2.2 Structure Theory

Starting from the Cartan subalgebra of a simple Lie algebra its "root system" can be defined. This turns out to be a useful tool for our goal. Hence, there are some more notions we need to introduce. In this thesis, I always refer to *complex* Lie algebras if not stated explicitly otherwise.

Definition 2.2.1 (Roots & ladder operators)
Apart from the set of generators $\{H_1, \ldots, H_r\}$, the remaining generators of $\mathfrak{g}$ form the set $\{E_1, \ldots, E_{n-r}\}$ (where $n =$dim $\mathfrak{g}$) chosen to satisfy the following eigenvalue equation:

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.2}$$

The vector $\alpha = (\alpha_1, \ldots, \alpha_r)$ is called a root and E_α is the corresponding ladder operator. If $n =$dim $\mathfrak{g}$ and $r =$rank($\mathfrak{h}$), then there are $n - r$ roots each being an r-tuple.

Roots can be used to define a map from the Cartan subalgebra into the underlying field (i.e. here into $\mathbb{C}$) as follows:

$$\begin{aligned} \alpha : \ \mathfrak{h} &\to \mathbb{C} \\ H_i &\to \alpha(H_i) := \alpha_i \end{aligned} \tag{2.3}$$

Hence, roots live in the dual space $\mathfrak{h}^*$ of the Cartan subalgebra. Further, once we choose a root, we can associate a vector space to it which is a subspace of the Lie algebra $\mathfrak{g}$. These spaces are called "root spaces" and are defined as follows:

Definition 2.2.2 (Root space)
The set

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \quad \forall \ H \in \mathfrak{h}\} \tag{2.4}$$

forms a vector space, called the root space.

This allows to decompose the Lie algebra $\mathfrak{g}$ as follows:

Theorem 2.2.3 (Root space decomposition)
Let $\mathfrak{g}$ be a simple Lie algebra. For Φ being the set of all roots, $\mathfrak{g}_\alpha$ denoting the root space associated to the root α and $\mathfrak{h}$ denoting the Cartan subalgebra, we can write

$$\mathfrak{g} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{2.5}$$

Roots are linearly dependent. We thus can choose a basis $\{\beta_1, \ldots, \beta_k\}$ in the space $\mathfrak{h}^*$ so that the roots can be expanded as

$$\alpha = \sum_{\lambda=1}^{\kappa} n_\lambda \beta_\lambda \tag{2.6}$$

Remark Notice that latin indices denote a single component within a given root while greek indices refer to a root within a set of roots.

The root α is said to be "positive" , if the first nonzero number in the sequence $(n_1, \ldots, n_\kappa)$ is positive.

Definition 2.2.4 (Simple roots)
A root that cannot be written as the sum of two positive roots with non-negative coefficients only is called a "simple root".

There are r simple roots and their set forms the most convenient basis for the space of roots.

Definition 2.2.5 (Highest root)
There is a unique root for which, in the expansion of simple roots α_λ given by

$$\theta = \sum_{\lambda=1}^{r} m_\lambda \alpha_\lambda \tag{2.7}$$

the sum of the coefficients $\sum_{\lambda=1}^{r} m_\lambda$ is maximal. This unique root is called the "highest root".

All roots of a simple Lie algebra can be obtained by successive subtraction of simple roots α_λ from the highest root θ.

Definition 2.2.6 (Killing form and scalar product)
In a first step we define the "renormalized Killing form" by

$$K(X, Y) := \frac{1}{2g} \mathrm{tr}\left(ad(X) ad(Y)\right) \tag{2.8}$$

where g is a numerical prefactor called the dual coxeter number.

The Killing form defines a scalar product on the Cartan subalgebra. Recall now the linear forms defined in (2.3) mapping an element of the Cartan subalgebra to a number in the underlying field. The Killing form now serves as a tool in order to associate an element of the Cartan subalgebra to each root. To elaborate this fact, consider a certain root α_λ and an arbitrary element $H \in \mathfrak{h}$. The generators of $\mathfrak{h}$ being the elements H^i, we can expand the arbitrary element H in terms of these generators:

$$H = \sum_{i=1}^{r} n_i H^i \tag{2.9}$$

Acting with α_λ on this H yields:

$$\alpha_\lambda(H) = \alpha_\lambda\left(\sum_{i=1}^{r} n_i H^i\right) \overset{\text{linearity}}{=} \sum_{i=1}^{r} n_i \alpha_\lambda(H^i) \overset{(2.3)}{=} \sum_{i=1}^{r} n_i \alpha_{\lambda,i} := N_H \in \mathbb{C} \tag{2.10}$$

The element of the Cartan subalgebra that is associated with the root $\alpha_\lambda \in \mathfrak{h}^*$ is the element $H^{\alpha_\lambda} \in \mathfrak{h}$ such that $K(H^{\alpha_\lambda}, H) = N_H$. We thus can use the (renormalized) Killing form in order to define a scalar product in the space of roots:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \langle \alpha, \beta \rangle := K(H^\alpha, H^\beta) \end{aligned}$$

Thus we naturaly get $|\alpha|^2 := (\alpha, \alpha)$. The last important concept in the context of roots is the Weyl group.

Definition 2.2.7 (Weyl group)
Let us define an operation

$$\begin{aligned} s_\alpha : \mathfrak{h}^* &\to \mathfrak{h}^* \\ \beta &\mapsto s_\alpha \beta := \beta - \langle \alpha^\vee, \beta \rangle \alpha \end{aligned}$$

where both α and β are roots. Such an operation is called a "Weyl reflection" (since it is a reflection w.r.t. the hyperplane perpendicular to α). The set of all Weyl reflections forms the "Weyl group" W of the algebra.

The Weyl reflection with respect to a hyperplane perpendicular to a simple root are referred to as "simple Weyl reflections". These special Weyl reflections "span" the Weyl group in the sense that any Weyl group element can be considered as a chain of simple Weyl reflections. The quantities $\alpha^\vee$, called "coroots", are defined as follows:

Definition 2.2.8 (Coroots)
The quantity

$$\alpha_\lambda^\vee = \frac{2\alpha_\lambda}{|\alpha_\lambda|^2} \tag{2.11}$$

is called the "coroot" associated with the root α_λ.

Last but not least the Weyl group splits the space spanned by the simple roots naturaly into $\mid W \mid$ regions that are referred to as "Weyl chambers". A rigorous definition is given by

Definition 2.2.9 (Weyl chambers)
Let α_i for $i = 1, \ldots, r$ be the simple roots of a simple Lie algebra $\mathfrak{g}$ of rank r and $w \in W$ a Weyl reflection. Then the simplical cones defined by

$$C_w := \{\lambda \mid \langle w\lambda, \alpha_i \rangle \geq 0, i = 1, \ldots, r\} \tag{2.12}$$

are the "Weyl chamber". There is one distinguished Weyl chamber for w being the identity that is called the "fundamental Weyl chamber".

In appendix A the detailed computation of the root system of the Lie algebra $\mathfrak{sl}(3)$ is provided as an example.

2.3 Nilpotent Orbits and Standard Triples

When studying subalgebras of a semisimple Lie algebra, the semisimple subalgebras with minimal dimension are particularly important. Since the smallest possible dimension of a semisimple Lie algebra is 3 and since any 3-dimensional complex semisimple Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ (the same holds true for 3-dimensional real semisimple Lie algebras and $\mathfrak{sl}(2,\mathbb{R})$), it is enough to focus on the latter [29]. For $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, the simple Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ is well known. It is the simple Lie algebra of lowest dimension ($\dim[\mathfrak{sl}(2,\mathbb{F})] = 3$). In its matrix representation it is spanned by the following three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.13}$$

obeying the commutator relations given by

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H. \tag{2.14}$$

Throughout the rest of this thesis, we will write $\mathfrak{sl}(2)$ for $\mathfrak{sl}(2,\mathbb{F})$, i.e. if no field is mentioned explicitly, the respective statement holds for both fields $\mathbb{R}$ and $\mathbb{C}$.
The goal is to find all possible embeddings of $\mathfrak{sl}(2)$ into more complicated semisimple Lie algebras. Given such a Lie algebra $\mathfrak{g}$ of dimension n with a basis given by a set of generators $B = \{G_1,\dots,G_n\}$, this means that we want to find all possible subsets of B such that the elements of the subset span the algebra $\mathfrak{sl}(2)$. To accomplish this task we need to answer several questions in advance:

- What properties does a basis of $\mathfrak{sl}(2)$ have?
- How do we know whether or not there is such an embedding of $\mathfrak{sl}(2)$ into $\mathfrak{g}$?
- Are there several equivalent embeddings that can be collected into equivalence classes?
- Is there a finite or infinite number of embeddings or equivalence classes of embeddings, respectively?

This subsection is dedicated to finding the answers to these questions. On the way to this goal, several theorems are applied. These are all well-known and hence not proved here. For instance, their proofs can be found in [29].

2.3.1 Properties of $\mathfrak{sl}(2)$ generators

Definition 2.3.1 (Standard triple)
Any subset $\{H,X,Y\}$ of a semsimple Lie algebra $\mathfrak{g}$ obeying the commutator relations (2.14) is referred to as a "standard triple", where one referres to X as the "nilpositive", to Y as the "nilnegative" and to H as the "neutral" element.

Definition 2.3.2 (Adjoint representation)
There is a particular representation of a Lie algebra $\mathfrak{g}$ called its "adjoint representation" which is defined as

$$\begin{aligned} ad : \mathfrak{g} &\to End(\mathfrak{g}) \\ X &\mapsto ad_X \end{aligned} \tag{2.15}$$

where the "adjoint map" ad_X is defined as

$$\begin{aligned} ad_X : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto ad_X(Y) := [X, Y]. \end{aligned} \tag{2.16}$$

Using the adjoint map it is possible to define two properties of Lie algebra elements [29]:

Definition 2.3.3 (Nilpotent elements of Lie algebras)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called "nilpotent" if ad_X is a nilpotent endomorphism of the vector space $\mathfrak{g}$, i.e. if there is an integer $r > 0$ such that

$$X^r = \underbrace{X \circ \cdots \circ X}_{r \text{ times}} = 0 \quad \Leftrightarrow \quad (ad_X)^r = 0 \tag{2.17}$$

Definition 2.3.4 (Semisimple elements of Lie algebras)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called "semisimple" if ad_X is a semisimple operator on $\mathfrak{g}$, i.e. if ad_X is diagonalizable.

Notice that in the latter definition the fact that $\mathfrak{g}$ must be a vector space over the field of complex numbers is crucial. In the case of a real vector space the notions of semisimplicity and diagonalizability of an operator are no longer equivalent. We can summarize the following important properties of $\mathfrak{sl}(2, \mathbb{F})$ and its basis:

The simple Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ is 3-dimensional (over $\mathbb{F}$) and thus the smallest possible simple Lie algebra. Two basis elements (in (2.13) the ones called X and Y) are nilpotent and the third element (H) is semisimple.

2.3.2 Existence of Standard Triples

Next, we want to find out whether or not there is an embedding of $\mathfrak{sl}(2)$ if we are given a complex semisimple Lie algebra $\mathfrak{g}$. As a first step, recall the well-known Jordan decomposition theorem [30]:

Theorem 2.3.5 (Jordan Decomposition Theorem)
Let X be an endomorphism of a finite dimensional complex vector space V.

1. *There exist unique operators $S, N \in End(V)$ satisfying the conditions:*

   $$X = S + N$$

   where S is semisimple, N is nilpotent and S and N commute.

2. *Each of the operators S and N are polynomials in X with zero constant term. In particular, S and N commute with any operator commuting with X and stabilize any subspace of V that X stabilizes.*

Further, there is a theorem by N. Jacobson [31] and W. W. Morozov [32] making the following statement:

Theorem 2.3.6 (Jacobson-Morozov theorem)
Let $\mathfrak{g}$ be any complex semisimple Lie algebra. If X is a nonzero nilpotent element of $\mathfrak{g}$, then it is the nilpositive element of a standard triple. Equivalently, for any nilpotent element X, there exists a homomorphism

$$\phi : \mathfrak{sl}(2) \to \mathfrak{g} \tag{2.18}$$

such that

$$\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X. \tag{2.19}$$

The Jordan decomposition theorem tells us that any endomorphism of any finite dimensional vector space V can be decomposed into a semisimple and a nilpotent part. Hence, there is always a nilpotent element in any non-empty $End(V)$ for $\dim(V) < \infty$. The Jacobson-Morozov theorem then tells us that any nilpotent element in a complex semisimple Lie algebra is related to a standard triple of this Lie algebra. Consequently we are now able to answer the second question:

There exists always at least one standard triple in any complex semisimple Lie algebra.

2.3.3 Conjugacy Classes and Nilpotent Orbits

By now the basic properties of the $\mathfrak{sl}(2)$-algebra and semisimple Lie subalgebras of minimal dimension are known and an explanation for the existence of at least one standard triple in a complex semisimple Lie algebra is provided. Next, we need to address the question of how to find *all* standard triples, i.e. all $\mathfrak{sl}(2)$-embeddings into a given complex semisimple Lie algebra $\mathfrak{g}$. If the Jacobson-Morozov theorem would hold in the opposite direction, too, we could conclude that we would find all standard triples of a semisimple Lie algebra if we can find all nilpotent elements in it. An n dimensional, irreducible representation of $\mathfrak{sl}(2)$ can be defined by the following linear map:

$$\rho_n : \mathfrak{sl}(2) \to \mathfrak{sl}(n) \tag{2.20}$$

$$\rho_n(H) = \begin{pmatrix} n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n+3 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -n+1 \end{pmatrix}, \tag{2.21}$$

$$\rho_n(X) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.22}$$

$$\rho_n(Y) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \mu_1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \mu_{n-1} & 0 \end{pmatrix} \tag{2.23}$$

where $\mu_i = i(n-i)$ for $1 \leq i \leq n-1$. Hence, we now know that any nonzero nilpotent element of a complex semisimple Lie algebra $\mathfrak{g}$ is related to a standard triple and vice versa, i.e. any standard triple has one generator that is conjugate to a nonzero, nilpotent element. This leads to the following fact: We find all standard triples of a complex semisimple Lie algebra as soon as we have found all of its nonzero nilpotent elements. But we still do not know whether there is a one-to-one relation between the nilpotent elements and the standard triples or if there are either several standard triples related to the same nilpotent element or vice versa. To answer this question, let us consider the following two theorems:

Theorem 2.3.7 (Kostant theorem)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two standard triples $\{H, X, Y\}$ and $\{H', X', Y'\}$ with the same nilpositive element are conjugate by an element of G_{ad}.

Theorem 2.3.8 (Mal'cev theorem)
Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Any two standard triples $\{H, X, Y\}$ and $\{H', X', Y'\}$ with the same neutral element are conjugate by an element of G_{ad}.

Here, G_{ad}, called the "adjoint group", denotes the connected subgroup of $GL(\mathfrak{g})$, the group of linear, invertible maps on $\mathfrak{g}$, with Lie algebra $ad_{\mathfrak{g}}$. More precisely the action of G_{ad} on a Lie algebra element $Z \in \mathfrak{g}$ is given by

$$\mathfrak{g} \ni Z \mapsto G_{\mathrm{ad}} \cdot Z = ad_{\mathfrak{g}}(Z) = \{ad_X(Z) = [X, Z] \mid X \in \mathfrak{g}\} \in \mathfrak{g} \tag{2.24}$$

Kostant's theorem can thus be restated: Let $\{H, X, Y\}$ and $\{H', X', Y'\}$ be two standard triples with $X = X'$. Then holds: There is an element $P \in \mathfrak{g}$ such that $H' = [P, H]$ and $Y' = [P, Y]$. In order to obtain Mal'cev's theorem, just swap the roles of X and H and X' and H', respectively. Thus, several different nilpotent elements may be related to the same standard triple. It is important to notice that nilpotence is a property that is preserved under the action of an element of the adjoint group on an element of the Lie algebra. It follows hence that if Z is a nilpotent element of $\mathfrak{g}$, that all elements $G_{\mathrm{ad}} \cdot Z = \{[X, Z] \mid X \in \mathfrak{g}\}$ are also nilpotent. As a consequence, it is possible to consider all nilpotent elements that are related to the same standard triple at once and collect them into classes, so called nilpotent orbits that are defined as follows:

Definition 2.3.9 (Nilpotent orbit)
Let $\mathfrak{g}$ be a semisimple Lie algebra. Let Z be a nilpotent element of $\mathfrak{g}$ and let the adjoint map be defined by the commutator (as before):

$$ad_X : \mathfrak{g} \to \mathfrak{g} \tag{2.25}$$

$$Z \mapsto ad_X(Z) := [X, Z] \tag{2.26}$$

Then the set

$$N_Z := \{ad_X(Z) \mid X \in \mathfrak{g}\} \tag{2.27}$$

is called the "nilpotent orbit of Z".

There may be several nilpotent elements related to the same standard triple but these nilpotent elements can be collected into nilpotent orbits. Hence, it is possible to establish a one-to-one correspondence between the standard triples of a complex semisimple Lie algebra and its nilpotent orbits.

We thus find the following...

Conclusion: We find all standard triples of a complex semisimple Lie algebra $\mathfrak{g}$ if we find all of its nilpotent orbits.

The exact method to find the nilpotent orbits of a semisimple Lie algebra on to find a way to parametrize them depends on its type. These methods are provided in the subsection 3.2.

2.3.4 Finiteness of Nilpotent Orbits

So far we have seen that we can find all standard triples of a given semisimple, complex Lie algebra if we find all its nilpotent orbits. Thus, we are left with the task to answer the last of the four questions mentioned above: Is there only a finite number of nilpotent orbits or if there are infinitely many? At this point, we need to anticipate the result, since we the necessary tools to derive this result will be derived in subsection 3.1: As we will see, the Dynkin-Kostant classification proves that the nilpotent orbits are always finite in number and an upper bound for this number is given by $3^{\mathrm{rk}(\mathfrak{g})}$.

2.4 Dynkin Diagrams

Simple Lie algebras can be classified. A famous classification is the Killing-Cartan classification by W. Killing [33–36] and E. Cartan that lists four "classical" simple Lie algebra types and five "exceptional" simple Lie algebras that cannot be included in one of the first four types. A tool in order to classify simple Lie algebras are the Dynkin diagrams. In this section it is explained how Dynkin diagrams are set up, how they can be generalized and how the Dynkin diagrams of each of the nine Lie algebra types look like.

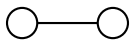
2.4.1 Ordinary Dynkin Diagrams

Drawing a Dynkin diagram is extremely easy. The recipe, e.g. given in [27], is:

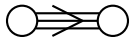
1. For each simple root α_λ draw a node. In a simply laced algebra, i.e. in an algebra that has only roots of equal length, all nodes are simply joined by a certain amount of edges. Otherwise one draws an additional arrow pointing from the longer towards the shorter root.

2. The nodes i and j are joined with $\langle\alpha_i, \alpha_j^\vee\rangle\langle\alpha_j, \alpha_i^\vee\rangle$ lines. Notice: Orthogonal roots are hence disconnected.

The example of $\mathfrak{sl}(3)$ mentioned in section 2.2 provides a very basic example here:



We can read off this diagram that $\mathfrak{sl}(3)$ has two simple roots (two nodes) which are equally long (no arrow) and sustain an angle of $2\pi/3$ (the nodes are joined by one line only).
A more advanced example would be the Dynkin diagram of the exceptional Lie algebra G_2:



From this diagram we can read off that the Lie algebra of the exceptional Lie group G_2 has two simple roots (two nodes) of different length (additional arrow pointing from longer to shorter root) that sustain an angle of $150° = 5\pi/6$ (the nodes are joined by three lines). Together with the information about the dimension of the Lie algebra, this is enough to draw the roots and to set up the Cartan matrix of the Lie algebra under study. In this example, $\dim(G_2)$=14 and thus we get a root system consisting of $\dim(G_2)-\text{rank}(G_2) = 14 - 2 = 12$ roots:

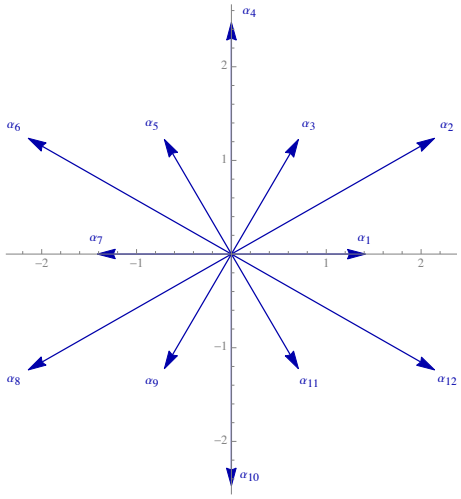


Figure 2.1: The twelve vectors represent the roots of the Lie algebra of G_2. There are two sets of roots, each containing all roots of equal length.

The two simple roots are α_1 and α_6, $\alpha_4 = 4\alpha_1 + 2\alpha_6$ is the highest root and the Cartan matrix is given by:

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \tag{2.28}$$

The classification of simple Lie algebra now boils down to the classification of Dynkin diagrams [27]. All classes of Dynkin diagrams are listed below in appendix B.

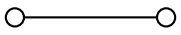
2.4.2 Generalized Dynkin Diagrams

Dynkin diagrams are usually generalized by adding more properties and thus more information to the nodes. There are several ways to do this. In the present context, only two kinds of generalizations are important and we will not consider the others.

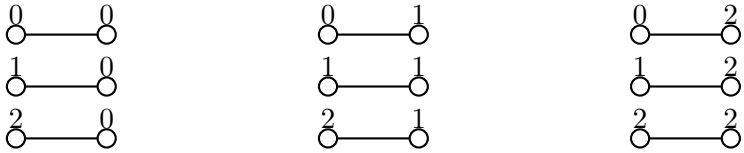
Weighted Dynkin Diagrams

These are Dynkin diagrams in which a number is associated to each node. This number represents the eigenvalue $\alpha(H)$ of an element H in the Cartan subalgebra $\mathfrak{h}$ such that H lies in the fundamental Weyl chamber with respect to a given basis of simple roots. It can be shown (cf. eg. [29]) that these numbers can only be 0, 1 or 2. Hence, if we consider the Dynkin diagram of a simple complex Lie algebra $\mathfrak{g}$ of rank $\mathrm{rk}(\mathfrak{g}) = r$, i.e. a diagram featuring r nodes, than we can label such a diagram in at most 3^r different ways. Consequently, there are at most 3^r different weighted Dynkin diagrams.

Example In this example we shall not go into the details but only illustrate what a weighted Dynkin diagram looks like. Further motivations and details can be found in subsection 3.2.
Let us again take a look at the Lie algebra $A_3 = \mathfrak{sl}(3)$ that was already considered in subsection 2.2. The corresponding Dynkin diagram is given by (omitting the numeration and the marks):



This diagram can only be labeled in maximally $3^2 = 9$ different ways. These are:



It is obvious that these are all possible weighted Dynkin diagrams of $\mathfrak{sl}(3)$. It can further be shown that in fact there are only 3 weighted Dynkin diagrams that are related to nilpotent orbits and one of them, the diagram labeled 00, corresponds to the zero orbit that by definition is not related to a standard triple. Hence, there are only two of these nine possible weighted Dynkin diagrams that are related to standard triples.

This kind of diagrams can be attached to nilpotent orbits in such a way that each of them is an invariant of the corresponding nilpotent orbit meaning that each pair of nilpotent orbits having the same weighted Dynkin diagram coincide [29].

Vogan Diagrams

Another possibility to generalize Dynkin Diagrams is to add information that allows to match them with real forms of a complex simple Lie algebra. We shall give here only a quick review of this concept and refer to [37] for more detail.
Before we are able to define Vogan diagrams, we need to introduce some more concepts:

Definition 2.4.1 (Cartan involution)
An involution θ such that the Killing form $B_\theta(X, Y) := -B(X, \theta Y)$ is a positive definite, bilinear form is called a Cartan involution.

Definition 2.4.2 (Real, imaginary and complex roots)
Let α be a root of a complex semi-simple Lie algebra $\mathfrak{g}$ with a real form $\mathfrak{g}_0$ and $\mathfrak{h}_0$ being a Cartan subalgebra of $\mathfrak{g}_0$. The root α is real if it takes real values on $\mathfrak{h}_0$, it is imaginary if it takes imaginary values on $\mathfrak{h}_0$ and it is complex otherwise [37].

Definition 2.4.3 (Compact and non-compact roots)
Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and θ a Cartan involution. Then the Cartan decomposition is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.29}$$

where $\mathfrak{k}$ is the eigenspace of θ corresponding to the eigenvalue $+1$ and $\mathfrak{p}$ the eigenspace to the eigenvalue -1. Let further α be a root of $\mathfrak{g}$. If the root space $\mathfrak{g}_\alpha \subset \mathfrak{k}$, α is said to be compact while if $\mathfrak{g}_\alpha \subset \mathfrak{h}$, it is called non-compact [37].

Let us now turn to the Vogan diagrams. Basically, a Vogan diagram is a Dynkin diagram with double arrows and colored nodes as additional information.

Definition 2.4.4 (Vogan diagram)
Let θ be a Cartan involution. Then, a Vogan diagram is associated to a triple $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ where $\mathfrak{g}_0$ denotes a real semi-simple Lie algebra, $\mathfrak{h}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ and Δ^+ is a set of positive roots. Then, the nodes corresponding to a 2-element orbit under θ are joined by a double arrow while the nodes of 1-element orbits are painted in black if the corresponding imaginary root is non-compact and not painted if it is compact [37].

Let us look at two examples:

These are the Vogan diagram of the real simple Lie algebras $\mathfrak{so}(3,2)$ (left) and $\mathfrak{so}(4,1)$ (right). The difference comes from the fact that in the left diagram the long root is non-compact (painted) while it is compact (not painted) in the diagram on the right. We will come back to the diagram of $\mathfrak{so}(3,2)$ in section 5.3.

Chapter 3

Construction and Classification of $\mathfrak{sl}(2)$ Embeddings in Simple Lie Algebras

In this chapter we will review an algorithm for constructing standard triples in classical complex Lie algebras and we shall mainly follow [29].

3.1 Dynkin-Kostant Classification

In section 2.3 we have stated the important fact that all standard triples of a semisimple, complex Lie algebra can be found if one finds all nilpotent orbits and further that there is only a finite number of them. While the first part of this statement has been made reasonably plausible already there, we are lacking so far a reason for the finiteness statement. As rigorous proofs of all statements made in this context may again be found e.g. in [29], we shall only give a summary of the central statements of the classification here. According to [29] we will split the classification theorem into three steps:

Step 1:
Let $\mathfrak{g}$ again be a semisimple complex Lie algebra and $\mathrm{Hom}^{\times}(\mathfrak{sl}(2), \mathfrak{g})$ the set of nonzero homomorphisms from $\mathfrak{sl}(2)$ to $\mathfrak{g}$. Further recall the definition of the adjoint group given in expression (2.24). Then we can define the following two sets

$$\mathcal{A}_{\mathrm{Hom}} = \{G_{ad}\text{-conjugacy classes in } \mathrm{Hom}^{\times}(\mathfrak{sl}(2), \mathfrak{g})\} \tag{3.1}$$

$$\mathcal{A}_{\mathrm{Triple}} = \{G_{ad}\text{-conjugacy classes of standard triples in } \mathfrak{g}\} \tag{3.2}$$

and a map Ω:

$$\begin{aligned} \Omega : \quad & \mathcal{A}_{\mathrm{Triple}} \to \{\text{nonzero nilpotent orbits in } \mathfrak{g}\} \\ & \{H, X, Y\} \mapsto \Omega(\{H, X, Y\}) = \mathcal{O}_X \end{aligned} \tag{3.3}$$

This allows to formulate the first part of the theorem of the Dynkin-Kostant classification:

Theorem 3.1.1 (Dynkin-Kostant-Classification step 1)
The map Ω is a 1-to-1 correspondence between the set $\mathcal{A}_{Triple}$ and the set of nonzero nilpotent orbits in $\mathfrak{g}$.

Although the full proof is omitted here, we mention that the Jacobson-Morozov theorem 2.3.6 basically proves the surjectivity of this map while the theorem 2.3.7 of Kostant proves the injectivity.

Step 2:
Now define further the set $\mathcal{S}$ as the set of all semisimple elements of the Lie algebra $\mathfrak{g}$. There is a related set $\mathcal{S}_{\text{dist}}$ referred to as the set of "distinguished semisimple orbits in $\mathfrak{g}$" defined as the image of a map $\mathcal{Y}$ which in turn is defined as

$$\begin{aligned} \mathcal{Y}: \qquad & \mathcal{A}_{\text{Triple}} \to \{\text{nonzero semisimple orbits } \mathcal{O}_Z \mid Z \in \mathcal{S}\} \\ & \{H,X,Y\} \mapsto \mathcal{Y}(\{H,X,Y\}) = \mathcal{O}_H \end{aligned} \tag{3.4}$$

In other words, the elements in $\mathcal{S}_{\text{dist}}$ are those semisimple orbits in $\mathfrak{g}$ that are related to a standard triple. This allows then to formulate the second part of the classification theorem:

Theorem 3.1.2 (Dynkin-Kostant-Classification step 2)
The map $\mathcal{Y}$ yields a 1-to-1 correspondence between the set $\mathcal{A}_{Triple}$ and the set $\mathcal{S}_{dist}$ of distinguished semisimple orbits in $\mathfrak{g}$.

Notice that since $\mathcal{S}_{\text{dist}} := \text{Image}(\mathcal{Y})$ this map is automatically surjective such that it is enough to prove injectivity. This can be done using Mal'cev's theorem 2.3.8.

Notice that combining the theorems of these two classification steps again yields the already known fact that the set of nonzero nilpotent orbits in $\mathfrak{g}$ is in 1-to-1 correspondence with the set $\mathcal{S}_{\text{dist}}$, but we need the third step of this derivation to see that the set of nilpotent orbits is indeed finite.

Step 3:
Take $\mathfrak{h} \subset \mathfrak{g}$ as a Cartan subalgebra such that $H \in \mathfrak{h}$ and choose a basis Δ of simple roots such that H lies in the fundamental domain. Then draw the corresponding weighted Dynkin diagram (cf. subsection 2.4) and recall that there are maximally $3^{\text{rk}(\mathfrak{g})}$ different labelings for these diagrams. This yields the last step of the Dynkin-Kostant classification:

Theorem 3.1.3 (Dynkin-Kostant-Classification step 3)
There is a 1-to-1 correspondence between the set of nonzero nilpotent orbits in $\mathfrak{g}$ and weighted Dynkin diagrams of distinguished semisimple orbits. In particular, there are at most $3^{rk(\mathfrak{g})}$ weighted Dynkin diagrams and thus at most the same finite number of nilpotent orbits in $\mathfrak{g}$.

Notice that this is the statement about the finiteness of the set of nilpotent orbits we already anticipated at the very end of subsection 2.3.

3.2 The Construction of Standard Triples for Type B_n Algebras

In the following we follow closely [29] that is itself based on [38] and [39]. The reference to these texts should be understood as valid for the entire subsection as they will not be repeated everywhere.

We know now that it is possible to find all standard triples of a simple Lie algebra and that they can be classified due to their 1-to-1 relation with nilpotent orbits. Based on this fact it is our next goal to explicitly construct the standard triples. There are algorithms that tell us how to accomplish this task. In the main text we will treat only the case for type B_n algebras while the cases of the remaining three classical Lie algebras is addressed in the appendix C. We shall first introduce an important basic concept:

Definition 3.2.1 (Partition)
Let $n \in \mathbb{N}$ be a positive integer. A "partition of n" is a tuple $[d_1, d_2, \ldots, d_k]$ of positive integers with the following two properties:

- $d_1 \geq d_2 \geq \cdots \geq d_k > 0$
- $d_1 + d_2 + \cdots + d_k = n$

Notice that we will furthermore allow such a partition to have an arbitrary number of additional components $d_{k+1}, d_{k+2}, ...$ if all of them are equal to 0. We then identify two partitions if their non-zero parts are the same, e.g. for the positive integer 4 the two partitions $[2, 1, 1]$ and $[2, 1, 1, 0, 0]$ are identical.

At this point some notation needs to be introduced:

- We will use an exponential notation:

$$[t_1^{i_1}, \ldots, t_r^{i_r}] = [d_1, \ldots, d_n] \qquad \text{for} \qquad d_j = \begin{cases} t_1, & 1 \leq j \leq i_1 \\ t_2, & i_1 + 1 \leq j \leq i_1 + i_2 \\ t_3, & i_1 + i_2 + 1 \leq j \leq i_1 + i_2 + i_3 \\ \vdots & \vdots \end{cases} \tag{3.5}$$

  This allows to write chains of equal numbers in a partition in a shorter form, e.g. $[3, 2, 2, 1, 1, 1] = [3, 2^2, 1^3]$.

- A set of partitions of n shall be denoted by $\mathcal{P}(n)$, e.g. $\mathcal{P}(4) = \{[4], [3, 1], [2^2], [2, 1^2], [1^4]\}$.

In the following we will attach nilpotent orbits to partitions according to certain rules. There is one fact that is true for all cases: For all classical Lie algebras, i.e. for all types A_n, B_n, C_n and D_n, the partition $[1^n]$ corresponds to the zero-orbit which does not correspond to a standard triple.

Construction Algorithm

Setp 1: Parametrization of Nilpotent Orbits
In the case of B_n-type algebras the parametrization theorem states:

Theorem 3.2.2 *Nilpotent orbits in $\mathfrak{so}_{2n+1}$ are in 1:1 correspondence with the set of partitions of $2n+1$ in which even parts occur with even multiplicity [38].*

Example Let us consider the complex Lie algebra $\mathfrak{so}(5)$. Since $2 \cdot 2 + 1 = 5$ it follows that in this case $n = 2$. We have the following set of partitions

$$\mathcal{P}(5) = \{[5], [4,1], [3,2], [3,1^2], [2^2,1], [2,1^3], [1^5]\} \tag{3.6}$$

From this set we may not consider the partitions $[4,1]$, $[3,2]$ or $[2,1^3]$, since in this partitions the even parts occur with odd multiplicity. Since as always $[1^5]$ leads to the zero-orbit, we ignore that partition, too. The partitions relevant for the sequel hence are $[5], [3,1^2]$ and $[2^2,1]$.

Step 2: Cartan Subalgebra & Root Spaces
A type B_n-algebra $\mathfrak{g}$ may be realized as the following set of matrices:

$$\left\{ \begin{pmatrix} 0 & u & v \\ -v^t & Z_1 & Z_2 \\ -u^t & Z_3 & -Z^t \end{pmatrix} \mid u, v \in \mathbb{C}^n, Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \text{ skew-symmetric} \right\} \tag{3.7}$$

This is skew-**adjoint** matrices relative to the quadratic form

$$z_1^2 + 2(z_2 z_{n+2} + z_3 z_{n+3} + \cdots + z_{n+1} z_{2n+1})$$

on $\mathbb{C}^{2n+1}$. The set of diagonal matrices shall be our Cartan subalgebra $\mathfrak{h}_{B_n}$. The elements of $\mathfrak{h}_{B_n}$ take the form

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & -D \end{pmatrix} \quad \mid \quad D = \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_n \end{pmatrix} \tag{3.8}$$

Now we define linear functionals $e_i \in \mathfrak{h}_{B_n}^*$ by

$$e_i(H) = h_i \qquad 1 \leq i \leq n \tag{3.9}$$

Thus, e_i picks out the $(i+1)^{st}$ diagonal entry of a matrix in $H \in \mathfrak{h}_{B_n}$. The root system of $\mathfrak{g}$ is now

$$\{\pm e_i \pm e_j, \pm e_j \mid 1 \leq i, j \leq n, i \neq j\}. \tag{3.10}$$

We take

$$\Delta_{\mathfrak{g}} = \{e_i \pm e_j, e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\} \tag{3.11}$$

as our set of positive roots. The α-root space of $\mathfrak{g}$ is spanned by the vector E_α given by

$$\begin{aligned} E_{e_i-e_j} &= E_{i+1,j+1} - E_{j+n+1,i+n+1} \\ E_{e_i+e_j} &= E_{i+1,j+n+1} - E_{j+1,i+n+1} \quad (i<j) \\ E_{-e_i-e_j} &= E_{i+n+1,j+1} - E_{j+n+1,i+1} \quad (i<j) \\ E_{e_i} &= E_{1,i+n+1} - E_{i+1,1} \\ E_{-e_i} &= E_{1,i+1} - E_{i+n+1,1} \end{aligned} \tag{3.12}$$

Example Consider $\mathfrak{so}(5)$ as an example. Choosing the traceless diagonal matrices as the Cartan subalgebra, the following four diagonal matrices form a basis of it:

$$B(\mathfrak{h}_{\mathfrak{so}(5)}) = \{H_1 = \mathrm{diag}(0,1,0,-1,0), H_2 = \mathrm{diag}(0,0,1,0,-1)\} \tag{3.13}$$

The linear functionals are then e_1 and e_2 where $e_1(H_1) = 1, e_1(H_2) = 0, e_2(H_1) = 0$ and $e_2(H_2) = 1$. According to the prescription given, the positive roots are

$$\Delta_{\mathfrak{so}(5)} = \{e_1, e_2, e_1 - e_2, e_1 + e_2\} \tag{3.14}$$

Step 3: Attaching Explicit Standard Triples
Let again $\mathbf{d} \in \mathcal{P}(2n+1)$ according to the theorem 3.2.2 and break it up into chunks of the following three different types:

- pairs $\{r, r\}$ of equal parts
- pairs $\{2s+1, 2t+1\}$ of unequal odd parts and with $s > t$.
- a single odd part $\{2u+1\}$

There is always one unique chunk of the last type. As previously in the case of A_n-algebras we now attach blocks of consecutive indices to these chunks (again such that blocks attached to distinct chunks are disjoint) and subsequently we assign sets of positive roots to these blocks. But the rules according to which these blocks are matched with a given chunk depend on the type of the latter. These are the rules:

- For pairs $\{r, r\}$: Choose a block $\{j+1, \ldots, j+r\}$ of consecutive indices and let
$$\mathcal{C}^+(r,r) = \{e_{j+1} - e_{j+2}, e_{j+2} - e_{j+3}, \ldots, e_{j+r-1} - e_{j+r}\} \tag{3.15}$$
be the corresponding set of simple roots. If $r = 1$ there will be no simple roots.
- For pairs $\{2s+1, 2t+1\}$: Choose a block $B := \{m+1, \ldots, m+s+t+1\}$ of consecutive indices. Then define
$$\begin{aligned} H_{\mathcal{C}} = &\sum_{i=1}^{s-t} (2s+2-2i)(E_{m+i+1,m+i+1} - E_{n+m+i+1,n+m+i+1}) \\ &+ \sum_{j=1}^{t} (2t+2-2j) F_{m+s-t+2j} \end{aligned} \tag{3.16}$$
where
$$F_a := E_{a,a} - E_{a+n,a+n} + E_{a+1,a+1} - E_{a+n+1,a+n+1} \tag{3.17}$$
The corresponding set of simple roots $\mathcal{C}^+(2s+1, 2t+1)$ is then given by exactly those positive roots $\alpha = e_j \pm e_k \in \Delta_{\mathfrak{g}}$ such that $j, k \in B$ and $\alpha(H_{\mathcal{C}}) = 2$.

- For the single odd part $\{2u+1\}$: Choose a block $\{l+1,\ldots,l+u\}$ of consecutive indices and let

$$\mathcal{C}^+(2u+1) = \{e_{l+1} - e_{l+2}, e_{l+2} - e_{l+3}, \ldots, e_{l+u-1} - e_{l+u}, e_{l+u}\} \tag{3.18}$$

For each chunk of the form $\{r,r\}$ or $\{2u+1\}$ and for every simple root α in $\mathcal{C}^+(r,r)$ or $\mathcal{C}^+(2u+1)$ take X_α to be an arbitrary α-root vector. For each of the chunks of the second type and for every $\beta \in \mathcal{C}^+(2s+1,2t+1)$, choose a β-root vector X_β and a $-\beta$-root vector $X_{-\beta}$, sum over all β and compute the commutator:

$$H_{\mathcal{C}} = \left[\sum_\beta X_\beta, \sum_\beta X_{-\beta}\right] \tag{3.19}$$

The nilpositive element of the standard triple we are after is proportional to the sum of all these X_α and X_β. The sum of all $X_{-\alpha}$ and $X_{-\beta}$ in turn is proportional to the nilnegative element. The neutral element is given by $H = \sum_{\mathcal{C}} H_{\mathcal{C}}$ where the form of $H_{\mathcal{C}}$ again depends on the form of the chunk $\mathcal{C}$:

- For $\mathcal{C}$ of type $\{r,r\}$:

$$H_{\mathcal{C}} = \sum_{k=1}^{r}(r-2k+1)(E_{j+k+1,j+k+1} - E_{n+j+k+1,n+j+k+1}) \tag{3.20}$$

- For $\mathcal{C}$ of type $\{2s+1,2t+1\}$: As defined in equation (3.16) .

- For $\mathcal{C}$ of type $\{2u+1\}$:

$$H_{\mathcal{C}} = \sum_{k=1}^{u}(2u+2-2k)(E_{l+k+1,l+k+1} - E_{n+l+k+1,n+l+k+1}) \tag{3.21}$$

We are thus left with finding the proportionality factors for X and Y. We have

$$[H_s, X_s] = 2X_s \qquad \text{and} \qquad [H_s, Y_s] = -2Y_s \tag{3.22}$$

for $X_s := \sum_\alpha X_\alpha + \sum_\beta X_\beta$ and $Y_s := \sum_\alpha X_{-\alpha} + \sum_\beta X_{-\beta}$. These are two of the three defining commutator relations of $\mathfrak{sl}(2)$. These equations are invariant under a rescaling of X_s and Y_s. We thus can set up an equation

$$[aX_s, bY_s] = H \tag{3.23}$$

and solve for one of the scale parameter a or b in terms of the other. The nilpotent elements of the standard triple are finally given by $X := aX_s$ and $Y := bY_s$.

Example In the example of $\mathfrak{so}(5)$ the set of relevant partitions is given by

$$\tilde{\mathcal{P}}(5) = \{[5],[3,1^2],[2^2,1]\}. \tag{3.24}$$

Now we choose one partition from this set, say $\mathbf{d} = [3,1^2]$. There are now two chunks in this partition given by $\{3\}$ and $\{1,1\}$. We may split the set of indices $\{1,2,3,4,5\}$ into blocks according to the rule given above:

- For $\{1,1\}$: As $r=1$, this chunk does not lead to any simple roots.
- For $\{3\}$: In this case $u=1$ and thus the following block of indices consists actually of one index only, say 1 and thus $l=0$. The corresponding set of simple roots is given by $\Delta_{\mathfrak{so}(5)} = \{\alpha_1 = e_1\}$

Following the second but last identity in expression (3.12), this now leads to the only root vector $E_{e_1} = E_{1,1+2+1} - E_{2,1} = E_{14} - E_{21}$ such that the nilpositive element is finally proportional to

$$X_s = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.25}$$

and accordingly the nilnegative element is proportional to $E_{-e_1} = E_{12} - E_{41}$ such that

$$Y_s = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

The neutral element is given by:

$$\begin{aligned} H = H_{\mathcal{C}} &= (2+2-2)(E_{0+1+1,0+1+1} - E_{2+0+1+1,2+0+1+1}) \\ &\quad + (2+2-4)(E_{0+2+1,0+2+1} - E_{2+0+2+1,2+0+2+1}) \\ &= 2(E_{22} - E_{44}) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.27}$$

Solving the equation $[aX_s, bY_s] = H$ we find that $a = -2/b$. Setting $b=1$, the sought for standard triple of $\mathfrak{so}(5,\mathbb{C})$ related to the nilpotent orbit $\mathcal{O}_{[3,1^2]}$ is given by

$$\left\{ X = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \tag{3.28}$$

We will come back to the example of $\mathfrak{so}(5,\mathbb{C})$ in section 5.1.

Chapter 4

(2+1)-Dimensional Higher Spin Black Holes

The goal of the present thesis is to generalize the higher spin black hole described in [2]. In this chapter we review their train of thoughts considering the cases of pure gravity (see appendix D for computations) and the black hole solution of Bañados, Teitelboim and Zanelli (abbreviated as the *BTZ black hole*). We will not cover the generalizations provided in [2].

4.1 Three Dimensional Gravity

In a (2+1)-dimensional spacetime, general relativity (GR) is equivalent to a Chern-Simons (CS) formalism with the gauge group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2 = SO(2,2)$. We will simply use this formalism here and refer e.g. to [40–42] for an introduction to Chern-Simons theory and e.g. to [2, 22, 43] for its application in the present context.

4.1.1 Action and Equation of Motion

In standard GR a metric is used as the fundamental quantity to describe the geometry of the considered spacetime manifold. In the CS approach, in contrast, we consider two independent Lie-algebra valued one-forms

$$A^{\pm} = A^{\pm}_r dr \wedge A^{\pm}_\varphi d\varphi \wedge A^{\pm}_t dt \tag{4.1}$$

which are connections on a pair of principal $SL(2,\mathbb{R})$-bundles. For now, these two connections shall both be $\mathfrak{sl}(2,\mathbb{R})$-valued. A convenient basis of $\mathfrak{sl}(2,\mathbb{R})$ is given by the three matrices

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_{+1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

which satisfy the relation

$$[L_i, L_j] = (i-j)L_{i+j} \tag{4.3}$$

for $i, j = 0, \pm 1$ and where $[\cdot,\cdot]$ denotes the usual matrix commutator. The Chern-Simons action for pure gravity is given by [2]

$$I = I_{CS}[A^+] - I_{CS}[A^-] \tag{4.4}$$

where as usual

$$I_{CS}[A^{\pm}] = \frac{k}{4\pi}\int \mathrm{tr}\left(A^{\pm}\wedge dA^{\pm} + \frac{2}{3}A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right). \tag{4.5}$$

Later on, we will mostly use the Hamiltonian form of this action given by

$$I_{Ham} = I_{Ham}[A^{+}] - I_{Ham}[A^{-}] \tag{4.6}$$

with

$$I_{Ham}[A^{\pm}] = -\frac{k}{2\pi}\int dt\, dr\, d\varphi\ \mathrm{tr}\left(A^{\pm}_r \dot{A}^{\pm}_{\varphi} - A^{\pm}_t G^{\pm}\right) + B^{\pm}_{\infty} \tag{4.7}$$

where the boundary term $B^{\pm}_{\infty}$ is added such that the full asymptotic symmetry algebra is able to act (more information in [2]). The explicit form of this boundary term will later play a crucial role and hence we shall come back to it.
We define

$$G^{\pm} = F^{\pm}_{r\varphi} := \partial_r A^{\pm}_{\varphi} - \partial_{\varphi} A^{\pm}_r + [A^{\pm}_r, A^{\pm}_{\varphi}] \tag{4.8}$$

When the cosmological constant $\Lambda = -\frac{1}{l^2}$, then the prefactor $k = k_2 = \frac{l}{4G}$. From the variation of the action (4.5) it can be calculated that the equation of motion (EoM) reads

$$F^{\pm} = dA^{\pm} + A^{\pm}\wedge A^{\pm} = 0 \quad \Longrightarrow \quad \frac{\partial A^{\pm}_{\varphi}}{\partial t} = \frac{\partial A^{\pm}_t}{\partial \varphi} + [A^{\pm}_{\varphi}, A^{\pm}_t] \tag{4.9}$$

where F is the curvature. Hence, we will refer to this EoM sometimes as the "flatness condition". The computation proving that the flatness condition implies the second relation between the temporal and the angular component of the connection field is found in appendix D.2.
Even though this flatness condition basically states the fact that there are no propagating degrees of freedom (DoF) [2] such as gravitons, the theory is still non-trivial due to the presence of DoFs at infinity. The asymptotic symmetry algebra governs the dynamics of these asymptotic DoFs and hence it will be extensively studied below.

4.1.2 The Asymptotic Symmetry Algebra

Knowing the action and the equation of motion, we have two equations that define the connection fields up to boundary conditions. To fully determine the fields and thus the spacetime geometry we need to investigate these boundary conditions and thus their asymptotic symmetries. The latter form an algebra (the asymptotic symmetry algebra) that is spanned by generators that shall be called the "asymptotic charges". These charges are functions of state meaning that they are defined on each $t = const$ slice of the 3-spacetime-manifold.
The flatness condition (4.9) tells us that the temporal component of the connection field $A^{\pm}_t$ can be seen as a gauge transformation from one such slice at $x^0 = t$ to another slice at $x^0 = t + \delta t$. Hence, the charges are independent of $A^{\pm}_t$.

Choosing the boundary conditions such that in the metric formulation the spacetime is asymptotically Anti-de Sitter [44], we can write these boundary conditions for the radial and the angular component of the connection as [2, 45]:

$$A^{\pm}_{\varphi}(r,\varphi) \overset{r\to\infty}{\longrightarrow} A^{\pm}_{\varphi}(r,\varphi) = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1}. \tag{4.10}$$

where the asymptotic charges are denoted $\mathcal{L}^{\pm}$ and behave as

$$\mathcal{L}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{L}^{\pm}(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{4.11}$$

and the radial component behaves as

$$A_r^{\pm} \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right). \tag{4.12}$$

The most general form of $A_t^{\pm}$ is given by

$$A_t^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} A_t^{\pm}(r,\varphi) = \pm\big(\xi_{\mp 1}(r,\varphi)L_{\mp 1} + \xi_0(r,\varphi)L_0 + \xi_{\pm 1}(r,\varphi)L_{\pm 1}\big). \tag{4.13}$$

This gauge transformation alters the physical state at spatial infinity since it does not become the identity there. We shall refer to this class of gauge transformations as "improper gauge transformations" in contrast to "proper gauge transformations" that do not alter the physical state. We can now ask for the most general time translation $A_t^{\pm}$ that still preserves the boundary conditions of the spatial components given above. This is given by:

$$A_t^{\pm} \overset{r\to\infty}{\longrightarrow} \pm\left(\xi_{\pm}(r,\varphi)\left(L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1}\right) \mp \xi'_{\pm}(r,\varphi)L_0 + \frac{1}{2}\xi''_{\pm}(r,\varphi)L_{\mp 1}\right), \tag{4.14}$$

where the primes denote derivatives with respect to the angle φ. The explicit calculation is provided in appendix D. The time evolution equation of the asymptotic charge $\mathcal{L}$ is then easily derived:

$$\dot{\mathcal{L}}^{\pm} = \pm\left(\mathcal{L}'^{\pm}\xi_{\pm} + 2\mathcal{L}^{\pm}\xi'_{\pm} - \frac{k}{4\pi}\xi'''_{\pm}\right) \tag{4.15}$$

Recall that with any choice of coefficients in (4.14) the symmetries are preserved. We are thus free to choose the gauge corresponding to $\xi_{\pm} = 1$ at infinity which leads to the relation

$$\dot{\mathcal{L}}^{\pm} = \pm\mathcal{L}'^{\pm}. \tag{4.16}$$

Our next goal is to express the boundary term $B_{\infty}^{\pm}$ from the Hamiltonian action (4.28) in terms of the asymptotic charges. This can be done by varying the Hamiltonian action and performing an integration by parts due to which a boundary term arises. The B-term can now be chosen in a way that it compensates the boundary term from the integration by parts (as is also shown in appendix D.4). Performing this analysis one finds

$$B_{\infty}^{\pm} = -\int dt\, Q_{\infty}^{\pm}[\xi_{\pm}] \qquad | \quad Q_{\infty}^{\pm}[\xi_{\pm}] = \pm\int_{\partial\Sigma} d\varphi\, \xi_{\pm}(\varphi)\mathcal{L}^{\pm}(\varphi) \tag{4.17}$$

at spatial infinity on a $t = const$-slice called Σ.

Finally we want to find the Poisson structure of the asymptotic symmetry algebra. In appendix D.5 a detailed computation is provided. Thus, we expand the asymptotic charges and their coefficients in Fourier modes:

$$\mathcal{L}^{\pm} = \frac{1}{2\pi}\sum_n \mathcal{L}_n^{\pm} e^{in\varphi} \tag{4.18}$$

$$\xi_{\pm} = \frac{1}{2\pi}\sum_n \xi_n^{\pm} e^{in\varphi} \tag{4.19}$$

On the one hand, the variation of any phase space functional F under a gauge transformation parametrized by $\xi_\pm$ can be expressed in terms of the Poisson bracket of the boundary Q-term from above and itself, i.e.

$$\pm\, \delta_{\xi_\pm} \mathcal{L}^\pm = \{Q^\pm_\infty, F\}. \tag{4.20}$$

On the other hand, if we set the temporal variation of the charge equal to its derivative with respect to time

$$\pm\, \delta_{\xi_\pm} \mathcal{L}^\pm = \dot{\mathcal{L}}^\pm \tag{4.21}$$

the boundary conditions are preserved manifestly. Hence, we can set

$$\dot{\mathcal{L}}^\pm = \{Q^\pm_\infty, F\} \tag{4.22}$$

and plugging in the relation given (4.17) yields after some manipulation the following Poisson bracket relation for the Fourier transforms of the asymptotic charges:

$$i\left\{\mathcal{L}^\pm_n, \mathcal{L}^\pm_m\right\} = (n-m)\mathcal{L}^\pm_{n+m} + \frac{kn^3}{2}\delta_{n,-m} \tag{4.23}$$

We have obviously recovered the Virasoro algebra with the central charge $c = \frac{3l}{2G}$ that contains the $\mathfrak{sl}(2)$ as a subalgebra, the latter being spanned by $\mathcal{L}^\pm_{-1}, \mathcal{L}^\pm_0$ and $\mathcal{L}^\pm_{+1}$.

4.2 BTZ Black Hole

So far we considered only the vacuum solution. In the last section we managed to write down an equation of motion for the connection field which is given by the flatness condition. Since the action and the EoM do not uniquely determine the theory we considered a set of boundary conditions for the spatial components of the connection field. Further we found a parametrization of the gauge transformation per unit time that preserves those boundary conditions and we calculated the Poisson structure of the asymptotic symmetry algebra that is spanned by the asymptotic charges.
Next, we want to explore the case of a Bañados-Teitelboim-Zanelli (BTZ) black hole [10], i.e. the black hole that couples only to a spin 2 field such as gravity. A generalization to black holes endowed with higher spin charges and connections living in type A_n Lie algebras is presented in the paper of [2]. We shall not repeat their results here.

4.2.1 The Black Hole Solution

A Chern-Simons approach for the study of the thermodynamics of such a black hole is given in [46]. Since no gauge invariant metric formulation is available in the generalized case [2] we shall construct such a black hole from first principles through its thermodynamic properties. This requires us to work in Euclidean space from which one passes to the Lorentzian formulation by the inverse of a Wick rotation [2]. The exact prescription for how to switch between these two formulations is given below. For now it shall be enough to state that in the Euclidean formulation the two connection fields A^+ and A^- are merged into a single connection field A that lives in the complexified algebra $\mathfrak{sl}(2,\mathbb{C})$ according to the following rules [2]:

$$A^+ = A \tag{4.24}$$

$$A^- = -A^\dagger \tag{4.25}$$

Thus, the connection now is given by

$$A = A_r dr + A_\varphi d\varphi + A_\tau d\tau \tag{4.26}$$

where $\tau = it$ is the Euclidean time and the coefficients may be complex. As a consequence, the CS action as well as its Hamiltonian form read differently in the Euclidean setting. It holds that $I_E = iI_{\text{Lor}}$ and more explicitly:

$$I_E[A] = -2\text{Im}[I_{CS}(A)]. \tag{4.27}$$

The action $I_{CS}(A)$ is given in expression (4.5) and its Hamiltonian analog is

$$I_E^{Ham}[A] = \text{Im}\left[\frac{k_2}{\pi}\int d\tau dr d\varphi \text{tr}\left(A_r \dot{A}_\varphi - A_\tau G\right)\right] + B_E \tag{4.28}$$

$$G = \partial_r A_\varphi - \partial_\varphi A_r + [A_r, A_\varphi]. \tag{4.29}$$

Here, the term B_E comprises both the boundary term at infinity that was already discussed previously (cf. section 4.1) and the boundary term at the black hole horizon to which we will come back later.

The topology of such a (2+1)-dimensional black hole is given by a solid torus, i.e. it is equal to $\mathbb{R}^2 \times S^1$. [2, 46]: Considering r, φ and τ as Schwarzschild-like coordinates, the angle φ is 2π-periodic and the time coordinate τ is 1-periodic. Further, the τ-circles are contractible to a point while the φ-circles are not. This configuration is shown in figure 4.1. We shall thus define a (Euclidean) black hole is a solution to the CS field equations with

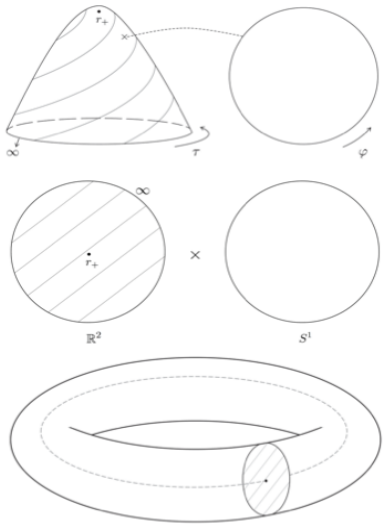


Figure 4.1: The topology of a 3-dimensional, Euclidean black hole is a torus. Two points given by (r, φ, τ) and $(r, \varphi + 2\pi, \tau + 1)$ are identified. [2]

the spacetime topology of a torus to which a non-vanishing entropy can be ascribed [2]. It is usual to investigate the thermodynamics of a black hole in its "rest frame", i.e. the frame where its only non-vanishing extensive parameters are the mass and the angular

momentum of the black hole. This is equivalent to stating that the only surviving Fourier modes of the asymptotic charges are the zero-modes $\mathcal{L}^{\pm}$ in the Lorentzian or $\mathcal{L}$ in the Euclidean language, respectively. It is always possible, to go to this frame. The gauge field configuration is then constant at spatial infinity, consequently, the boundary conditions analog to the equations (4.10), (4.11), (4.12) and (4.14) now read

$$A_{\varphi}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1} \tag{4.30}$$

$$\mathcal{L}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{L}_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{4.31}$$

$$A_r^{\pm} \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{4.32}$$

$$A_t^{\pm} \overset{r\to\infty}{\longrightarrow} \xi_{\pm}(r,\varphi)\left(L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1}\right) \tag{4.33}$$

$$\xi_{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\xi_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{4.34}$$

in Lorentzian spacetime and, in its Euclidean continuation, accordingly

$$A_{\varphi}(r,\varphi) \overset{r\to\infty}{\longrightarrow} L_{+1} - \frac{2\pi}{k}\mathcal{L}(r,\varphi)L_{-1} \tag{4.35}$$

$$\mathcal{L}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{L}_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{4.36}$$

$$A_r \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{4.37}$$

$$A_{\tau} \overset{r\to\infty}{\longrightarrow} -i\xi(r,\varphi)\left(L_{+1} - \frac{2\pi}{k}\mathcal{L}(r,\varphi)L_{-1}\right) \tag{4.38}$$

$$\xi(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\xi_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{4.39}$$

As already stated the Euclidean time is related to the Lorentzian time by $\tau = it$ and in accordance to continuation rules (4.24) and (4.25) holds $\mathcal{L}_0^{+} = \mathcal{L}_0$, $\mathcal{L}_0^{-} = \mathcal{L}_0^{*}$, $\xi_0^{+} = \xi_0$ and $\xi_0^{-} = \xi_0^{*}$, where * means complex conjugation. As is explained in [2] there is a simple relation between the zero-modes of the asymptotic charges and the two extensive parameters "mass" and "angular momentum" of the black hole. In the Lorentzian formalism holds

$$\mathcal{M}_{\mathrm{Lor}} = \frac{1}{l}(\mathcal{L}_0^{+} + \mathcal{L}_0^{-}) = \frac{M_{\mathrm{Lor}}}{8G} \tag{4.40}$$

$$\mathcal{J}_{\mathrm{Lor}} = \mathcal{L}_0^{+} - \mathcal{L}_0^{-} = \frac{J_{\mathrm{Lor}}}{8G}, \tag{4.41}$$

where M is the mass of the black hole and J the angular momentum and further for the lapse N and the shift N^{φ} at infinity

$$N(\infty) = \frac{1}{2\pi}N_0 = \frac{l}{4\pi}(\xi_0 + \xi^*) \tag{4.42}$$

$$N^{\varphi}(\infty) = \frac{1}{2\pi}N_0^{\varphi} = \frac{i}{4\pi}(\xi_0 - \xi^*) \tag{4.43}$$

Notice now that the asymptotic behavior of the connection field given by equations (4.35) to (4.39) still has a gauge DoF. One is thus free to choose a gauge where the radial field component vanishes, $A_r = 0$. In this gauge the solutions of the CS theory take their simplest form, since then the connection field behaves as prescribed to the boundary conditions not only at infinity but anywhere. Since the charges $\mathcal{L}$ and the parameters ξ are taken to be constant in r and φ and since therefore the partial derivatives in equation (4.8) vanish, it follows that the commutator $[A_\tau, A_\varphi] = 0$. This directly implies that the $\tau - \phi$ component of the curvature tensor F vanishes. In other words, we consider a solid torus with a flat connection.
We are now ready to write down a solution for a Euclidean black hole as it has been defined above:

$$A_r = 0 \tag{4.44}$$

$$A_\varphi = L_{+1} - \frac{1}{2k}(\mathcal{M}l + i\mathcal{J})L_{-1} \tag{4.45}$$

$$A_\tau = -i\xi\left(L_{+1} - \frac{1}{2k}(\mathcal{M}l + i\mathcal{J})L_{-1}\right) \tag{4.46}$$

The Lorentzian continuation is given by

$$A_r^\pm = 0 \tag{4.47}$$

$$A_\varphi^\pm = L_{\pm 1} - \frac{1}{2k}(\mathcal{M}_{\text{Lor}}l \pm \mathcal{J}_{\text{Lor}})L_{\mp 1} \tag{4.48}$$

$$A_\tau^\pm = \xi_\pm\left(L_{\pm 1} - \frac{1}{2k}(\mathcal{M}_{\text{Lor}}l \pm \mathcal{J}_{\text{Lor}})L_{\mp 1}\right) \tag{4.49}$$

But, this field configuration only describes a (2+1)-dimensional black hole if the condition

$$|\mathcal{J}_{\text{Lor}}| \leq \mathcal{M}_{\text{Lor}}l \tag{4.50}$$

is obeyed. This guarantees both, the existence of a horizon in the metric formulation as well as a real and positive entropy in the Euclidean formulation [2].

Calling the radius of the event horizon r_+, we know from [2, 46] that $A_\tau(r_+) \neq 0$ in (4.46). This is because the coefficient $\xi \neq 0$ due to the relations (4.42) and (4.43) and the fact that the lapse and the shift are both proportional to the inverse temperature β of the black hole. Let us repeat at this point that the temporal coordinate τ is an angular quantity in the Euclidean set up that measures the length of arcs around a center that we can refer to as r_+ (cf. figure 4.1, bottom). On the one hand, the center r_+ of a disk with φ =constant is obviously a singularity and the temporal component A_τ can only be defined in a coordinate patch excluding r_+ [2]. On the other hand, we need a gauge field that is defined on the entire disk. As a conclusion, we must remove the singularity r_+ which is possible by treating this point separately. We do so by defining a second connection field component A_τ^{reg} that is constant. This new field differs from the original gauge field by a regularizing gauge transformation. Then we need to glue these two gauge fields together at the horizon r_+. This leads to a regularity condition at r_+ that must be obeyed if A_τ shall be written as in (4.46). In order to find this regularity condition we diagonalize A_τ:

$$A_\tau = -2\pi i\nu(\xi, \mathcal{L})L_0 \qquad | \quad \nu(\xi, \mathcal{L}) = \xi\sqrt{\frac{2\mathcal{L}}{\pi k}} \tag{4.51}$$

The regularizing gauge transformation is given by an element of the group $SL(2,\mathbb{C})$ of the form

$$g(\tau) = e^{\tau A_\tau} = e^{-2\pi i \tau \nu(\xi,\mathcal{L}) L_0} \tag{4.52}$$

As the τ-cycles are 1-periodic, so must be this group element up to a global sign, i.e.

$$g(\tau+1) = \pm g(\tau), \tag{4.53}$$

which directly implies that $\nu(\xi,\mathcal{L})$ must be an integer (remember that there is a factor of $\frac{1}{2}$ included in the definition of L_0). The minus sign is allowed since the identity component the gauge group of (2+1)-dimensional Euclidean gravity is $SO^+(3,1)$ that is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$ such that $-\mathbb{1}$ is identified with $\mathbb{1}$ [2]. The very fact that $\nu(\xi,\mathcal{L}) \in \mathbb{Z}$ is the regularity condition, since only in this case $g(\tau) = \pm\mathbb{1}$. It has been shown in [2] that one must choose

$$\nu(\xi,\mathcal{L}) = \xi\sqrt{\frac{2\mathcal{L}}{\pi k}} = 1 \tag{4.54}$$

such that

$$\xi = \sqrt{\frac{\pi k}{2\mathcal{L}}}. \tag{4.55}$$

Whenever the right hand side of this latter equation is plugged in for ξ in the calculations below, this is where the regularity condition is implemented. The regularity condition implies the following holonomy:

$$H_\tau = e^{\int_{r_+} A_\tau d\tau}\Big|_{\text{on-shell}} = e^{A_\tau(r_+)}\Big|_{\text{on-shell}} = -\mathbb{1} \tag{4.56}$$

4.2.2 Black Hole Thermodynamics

Now we are ready to calculate the entropy of the (2+1)-dimensional black hole. It is obtained by evaluating the action on-shell in the microcanonical ensemble, i.e. having fixed the asymptotic charges $\mathcal{L}$ at infinity [2]. Hence, let us consider the Hamiltonian action given in equation (4.28). Notice that the black hole solution is time-independent since it represents a system in thermal equilibrium [2], such that the term proportional to $\dot{A}_\varphi$ vanishes. Simultaneously, the constraint $G = 0$ is obeyed, so the corresponding term drops out, too. Since further the asymptotic charges are fixed in the microcanonical ensemble, there is no boundary term at infinity to be included, $B_\infty = 0$. As mentioned in the sentence below (4.28), the boundary term B_E comprises both, the boundary term at infinity (B_∞) as well as the boundary term at the black hole horizon (B_{r_+}). As all other parts of the Hamiltonian action vanish, the black hole entropy is determined by the horizon boundary term that encodes the regularity condition established above which in turn assures that the EoM holds also at the origin r_+ (meaning "on-shell"). It thus became obvious that we now the black hole entropy as soon as we know the horizon boundary term B_{r_+}.
The variation of the hamiltonian action in the microcanonical ensemble (no boundary term at infinity) reads on-shell

$$\delta I_E^{Ham}[A] = \frac{k_2}{\pi}\text{Im}\left[\int_{r_+} d\tau d\varphi \text{tr}\left(A_\tau \delta A_\varphi\right)\right] + \delta B_{r_+} \tag{4.57}$$

This can be rewritten as

$$\delta I_E^{Ham}[A] = 2k_2 \mathrm{Im}\left[\mathrm{tr}\left(A_\tau(r_+)\delta A_\varphi(r_+)\right)\right] + \delta B_{r_+} \tag{4.58}$$

if we assume that A_φ and A_τ do neither depend on φ nor on τ near r_+ but they may depend on r, i.e. the previously mentioned gauge condition $A_r = 0$ is released at this point. The commutator $[A_\tau, A_\varphi]$ still vanishes such that these two components of the connection field can be simultaneously diagonalized. Accordingly, the trace of the product of these two components equals the sum of products of eigenvalues as we will see below. Introduce now a temporal connection field component A_τ^{reg} that obeys the regularity condition, and require the vanishing of the variation of the action $\delta I_E^{Ham}[A] = 0$ to imply that [2] (by $A_\tau^{\text{on-shell}}$ we will refer to the gauge field that was up to here called A_τ)

$$A_\tau^{\text{on-shell}}(r_+) = g A_\tau^{\mathrm{reg}}(r_+) g^{-1}. \tag{4.59}$$

Referring to the eigenvalues of $A_\tau(r_+)$ as μ_k this then yields $\mu_k^{\text{on-shell}} = \mu_k^{\mathrm{reg}}$ such that

$$\begin{aligned} &\delta I_E^{Ham}[A] = 2k_2 \mathrm{Im}\left[\mathrm{tr}\left(A_\tau(r_+)\delta A_\varphi(r_+)\right)\right] + \delta B_{r_+} = 0 \\ \iff \qquad & B_{r_+} = -\frac{k_2}{\pi}\mathrm{Im}\left[\int_{r_+} d\tau d\varphi \left(\sum_k \mu_k^{\mathrm{reg}} \lambda_k\right)\right] \end{aligned} \tag{4.60}$$

where the λ_k denote the eigenvalues of A_φ. In the latter equation consider the μ_k^{reg} as fixed such that only the λ_k can be varied and thus that

$$\mathrm{tr}\left(A_\tau(r_+)\delta A_\varphi(r_+)\right) = \sum_k \mu_k^{\mathrm{reg}} \delta\lambda_k. \tag{4.61}$$

Plugging this result for the horizon boundary term back into the action yields:

$$I_E^{Ham}[A] = \mathrm{Im}\left[\frac{k_2}{\pi}\int d\tau dr d\varphi \mathrm{tr}\left(A_r \dot{A}_\varphi - A_\tau G\right)\right] - \frac{k_2}{\pi}\mathrm{Im}\left[\int_{r_+} d\tau d\varphi \left(\sum_k \mu_k^{\mathrm{reg}} \lambda_k\right)\right] \tag{4.62}$$

Since, as mentioned previously, for the stationary case of a black hole in thermodynamical equilibrium the first term in this Hamiltonian action vanishes, we find the following expression for the entropy of a (2+1)-dimensional black hole in terms of the connection field components [2]:

$$S = -2k \mathrm{Im}\left[\mathrm{tr}\left(A_\tau A_\varphi\right)\right]_{\text{on-shell}} \tag{4.63}$$

In the Lorentzian language this reads as

$$S = k\left[\mathrm{tr}\left(A_t^+ A_\varphi^+\right) - \mathrm{tr}\left(A_t^- A_\varphi^-\right)\right]_{\text{on-shell}}. \tag{4.64}$$

As a last step we want now to recast this expression for the entropy in terms of the mass and the angular momentum of the black hole. To do so we first plug in the expressions (4.45) and (4.46) into (E.45) which leads to an expression of the entropy in terms of the asymptotic charges [2]

$$S = 2\pi n\sqrt{2\pi k_2}(\sqrt{\mathcal{L}} + \sqrt{\mathcal{L}^*}) \tag{4.65}$$

and thus

$$S = \pi l \sqrt{\frac{\mathcal{M}_{\text{Lor}}}{G}\left(1 + \sqrt{1 - \frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}} l^2}}\right)} = \frac{2\pi r_+}{4G}, \tag{4.66}$$

with the horizon radius given by

$$r^2_\pm = 4l^2 G \mathcal{M}_{\text{Lor}} \left(1 \pm \sqrt{1 - \frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}} l^2}}\right). \tag{4.67}$$

The last thing to do is to find the inverse temperature β and the chemical potential $\mu_{\mathcal{J}_{\text{Lor}}}$ that can be written as [2]:

$$\beta = \frac{l}{2}(\xi_+ + \xi_-) = \frac{2\pi r_+ l^2}{r^2_+ - r^2_-} \tag{4.68}$$

$$\beta \mu_{\mathcal{J}_{\text{Lor}}} = -\frac{1}{2}(\xi_+ - \xi_-) = \frac{r_-}{l r_+}, \tag{4.69}$$

where $\xi_\pm = \sqrt{\frac{\pi k}{2\mathcal{L}^\pm}}$. We thus obtained results that coincide with the ones from the metric formalism [2, 47].

Chapter 5

Construction of a $\mathfrak{so}(3,2)$ Black Hole Solution

In this chapter the construction of a $\mathfrak{so}(3,2)$ black hole shall be explained and its thermodynamics will be studied. We will follow the concepts introduced in the previous chapters. Starting from the computation of the root system and the standard triples of the complex simple Lie algebra $\mathfrak{so}(5,\mathbb{C})$ in section 5.1, we find the embeddings of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{so}(5,\mathbb{C})$ in section 5.2, followed in section 5.3 by the computation of the intersection of the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra with the real form $\mathfrak{so}(3,2)$. Then having collected all the preliminary mathematical results, we will build up a Chern-Simons theory with a connection field that lives in $\mathfrak{so}(3,2)$ and has boundary conditions that lead to a field of spin $s=4$. Then we construct a black hole solution that is charged under this spin 4 field. Ultimately, we will study the thermodynamics of this black hole in 5.5 focusing on its entropy.

5.1 Construction of Standard Triples of $\mathfrak{so}(5,\mathbb{C})$

The classical simple Lie algebra under consideration shall be $B_2 = \mathfrak{so}(5,\mathbb{C})$. In order to construct all its standard triples, we apply the algorithm given in subsection 3.2.

Step 1: Parametrization of Nilpotent Orbits of $\mathfrak{so}(5,\mathbb{C})$
We have already found in subsection 3.2 that the relevant partitions are $[5],[3,1^2]$ and $[2^2,1]$ which clearly parametrize the nilpotent orbits $\mathcal{O}_{[5]},\mathcal{O}_{[3,1^2]}$ and $\mathcal{O}_{[2^2,1]}$.

Step 2: Cartan Subalgebra & Root Spaces
We can realize $B_2 = \mathfrak{so}(5,\mathbb{C})$ explicitly as the set of matrices

$$\left\{ \begin{pmatrix} 0 & a & b & c & d \\ -c & e & f & 0 & i \\ -d & g & h & -i & 0 \\ -a & 0 & j & -e & -g \\ -b & -j & 0 & -f & -h \end{pmatrix} \mid a,b,c,d,e,f,g,h,i,j \in \mathbb{C} \right\}. \tag{5.1}$$

Notice that the Lie algebra under consideration has rank $\mathrm{rk}(B_2) = 2$. The Cartan subalgebra has thus dimension two and we choose its basis to be

$$B(\mathfrak{h}_{\mathfrak{so}(5,\mathbb{C})}) = \{H_1 = \mathrm{diag}(0,1,0,-1,0), H_2{=}diag(0,0,1,0,-1)\} \tag{5.2}$$

Define E_λ to be a matrix as parametrized by expression (5.1) with the additional requirement that all parameters $a,\ldots,j$ but λ vanish. Using elementary matrices we can write for these matrices:

$$\begin{array}{llll} E_a = E_{12} - E_{41}, & E_b = E_{13} - E_{51}, & E_c = E_{14} - E_{21}, & E_d = E_{15} - E_{31}, \\ E_f = E_{23} - E_{54}, & E_g = E_{32} - E_{45}, & E_i = E_{25} - E_{34}, & E_j = E_{43} - E_{52} \end{array} \tag{5.3}$$

and

$$E_e = E_{22} - E_{44} = H_1, \qquad E_h = E_{33} - E_{55} = H_2 \tag{5.4}$$

Using this notation, we state

$$\mathfrak{so}(5,\mathbb{C}) = \langle\{H_1, H_2, E_a, E_b, E_c, E_d, E_f, E_g, E_i, E_j\}\rangle_{\mathbb{C}} \tag{5.5}$$

Let us now compute the root system. Following the recipe for type B_n algebras, we shall define linear functionals $e_i \in \mathfrak{h}^*_{B_n}$ by

$$e_i(H) = h_i \qquad 1 \le i \le 2 \tag{5.6}$$

The root system of $\mathfrak{so}(5,\mathbb{C})$ is hence

$$\{e_1, e_1+e_2, e_2, e_2-e_1, -e_1, -e_1-e_2, -e_2, e_1-e_2\}. \tag{5.7}$$

Applying these functionals on the basis of the Cartan subalgebra leads to

$$\begin{array}{rr} e_1(H_1) = 1, & e_1(H_2) = 0 \\ (e_1+e_2)(H_1) = 1, & (e_1+e_2)(H_2) = 1 \\ e_2(H_1) = 0, & e_1(H_2) = 1 \\ (e_2-e_1)(H_1) = -1, & (e_2-e_1)(H_2) = 1 \\ -e_1(H_1) = -1, & -e_1(H_2) = 0 \\ (-e_1-e_2)(H_1) = -1, & (-e_1-e_2)(H_2) = -1 \\ -e_2(H_1) = 0, & -e_2(H_2) = -1 \\ (e_1-e_2)(H_1) = 1, & (e_1-e_2)(H_2) = -1 \end{array} \tag{5.8}$$

We thus can associate these root functionals with pairs of numbers according to:

$$\begin{array}{r} e_1 \to \alpha = (1,0), \\ e_1+e_2 \to \beta = (1,1), \\ e_2 \to \gamma = (0,1), \\ e_2-e_1 \to \delta = (-1,1), \\ -e_1 \to \varepsilon = (-1,0), \\ -e_1-e_2 \to \zeta = (-1,-1), \\ -e_2 \to \eta = (0,-1), \\ e_1-e_2 \to \vartheta = (1,-1) \end{array} \tag{5.9}$$

Using these identifications and considering the number pairs as components of a vector we can draw the roots as in the left panel of the following figure:

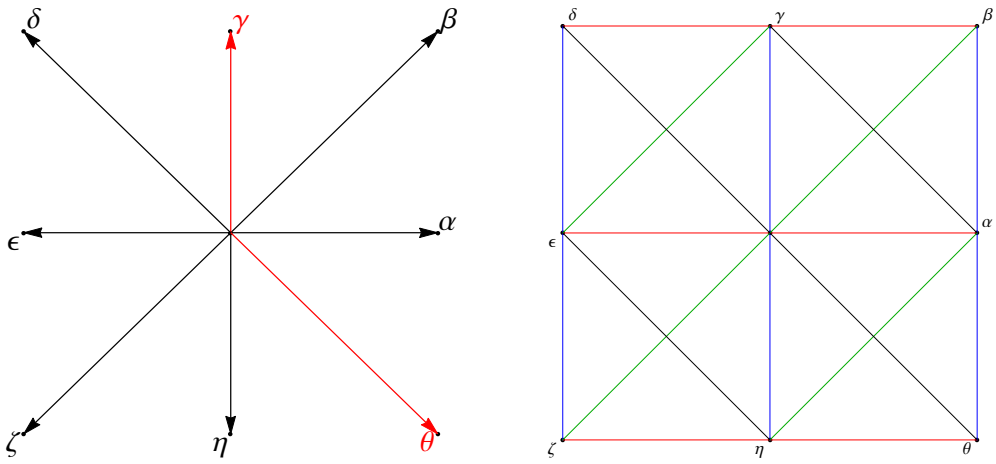


Figure 5.1: Left panel: The eight roots of $\mathfrak{so}(5,\mathbb{C})$ represented as vectors. The red vectors corresponding to the roots $\gamma = e_2$ and $\vartheta = e_1 - e_2$ can be chosen as simple roots. Right panel: The well-known root diagram of $\mathfrak{so}(5,\mathbb{C})$. The middle red line and the long black diagonal indicate already that $\mathfrak{sl}(2)$ is a subalgebra (see section 5.2).

The set

$$\Delta_{\mathfrak{so}}(5,\mathbb{C}) = \{e_1, e_2, e_1 + e_2, e_1 - e_2\} = \{\alpha, \beta, \gamma, \vartheta\} \tag{5.10}$$

is the set of positive roots suggested by the recipe. The simple roots are then γ and ϑ, as anticipated in fig. 5.1. It can be checked by direct calculation that the root vectors are given by the ladder operators defined above:

$$\begin{aligned} E_\alpha &= E_c, \quad E_\beta = E_i, \quad E_\gamma = E_d, \quad E_\delta = E_g, \\ E_\varepsilon &= E_a, \quad E_\zeta = E_j, \quad E_\eta = E_b, \quad E_\vartheta = E_f \end{aligned} \tag{5.11}$$

Step 3: Attaching Explicit Standard Triples
Now, we break up all three partitions into chunks and attach simple roots following in a first step the recipe:

- Partition $[5]$:
  There is one single chunk of a single odd part given by $\{5\}$. The corresponding set of simple roots assigned to this chunk is $\{e_1 - e_2, e_2\} = \{\vartheta, \gamma\}$.

- Partition $[3, 1^2]$:
  There is one chunk of a single odd part given by $\{3\}$ that we match with the set of one single root $\{e_1\} = \{\alpha\}$, and another chunk of two equal parts given by $\{1, 1\}$ to which no roots are matched, as stated in the recipe. Hence the unification of all attached simple roots corresponding to this partition is given by $\mathcal{C} = \{e_1\} = \{\alpha\}$.

- Partition $[2^2, 1]$:
  There is one chunk of two equal parts given by $\{2, 2\}$ and one single chunk given by $\{1\}$ to which again no roots are assigned. To the first chunk we assign the set of roots $\{e_1 - e_2\} = \{\vartheta\}$ that is also the unification of all simple roots related with this partition.

We thus find the following triples corresponding to the nilpotent orbits under study:

$$\begin{aligned}
\mathcal{O}_{[5]} &\Longrightarrow \{X_s^{[5]} = E_\gamma + E_\vartheta, Y_s^{[5]} = E_\zeta + E_\eta, H^{[5]} = 4E_{22} - 4E_{44} + 2E_{33} - 2E_{55}\} \\
\mathcal{O}_{[3,1^2]} &\Longrightarrow \{X_s^{[3,1^2]} = E_\alpha, Y_s^{[3,1^2]} = E_\varepsilon, H^{[3,1^2]} = 2E_{22} - 2E_{44}\} \\
\mathcal{O}_{[2^2,1]} &\Longrightarrow \{X_s^{[2^2,1]} = E_\vartheta, Y_s^{[2^2,1]} = E_\delta, H^{[2^2,1]} = E_{22} - E_{44} - E_{33} + E_{55}\}
\end{aligned} \tag{5.12}$$

For each orbit we now need to find the proportionality factors that scale the X_s and Y_s to the correct nilpotent elements. Solving the corresponding equations and scaling the matrices yields the following standard triples:

$$\{X,Y,H\}_{[5]} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 6 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \right\} \tag{5.13}$$

$$\{X,Y,H\}_{[3,1^2]} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \tag{5.14}$$

$$\{X,Y,H\}_{[2^2,1]} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \tag{5.15}$$

5.2 Embeddings of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{so}(5,\mathbb{C})$

We have thus found three embeddings of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{so}(5,\mathbb{C})$. Recall the root system of $\mathfrak{sl}(2)$:

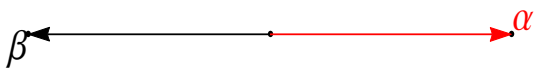


Figure 5.2: Root system of $\mathfrak{sl}(2)$. The root α (red arrow) is the simple root.

Hence, the corresponding root diagram is just a line. It is possible to determine how many different embeddings of a subalgebra $\mathfrak{sl}(2) \subset \mathfrak{so}(3,2)$ there are using root diagrams. There are exactly as many embeddings of the subalgebra $\mathfrak{sl}(2) \subset \mathfrak{so}(3,2)$ as there are possibilities to put the root diagram of $\mathfrak{sl}(2)$ into the root diagram of $\mathfrak{so}(3,2)$ in such a way that the orthogonal projections of the states of $\mathfrak{so}(3,2)$ onto the root diagram of $\mathfrak{sl}(2)$ are equidistant. This can be verified by direct calculation and geometrical considerations. In accordance with the result found in the previous section, there are three possibilities to

put a line (i.e. the root diagram of $\mathfrak{sl}(2)$) into the root diagram of $\mathfrak{so}(5,\mathbb{C})$ given in the right panel of figure 5.1. These possibilities are shown in the following figures 5.3, 5.4 and 5.5. From these projections we can read off the dimensions of irreducible representations of $\mathfrak{sl}(2)$ inside $\mathfrak{so}(5,\mathbb{C})$.

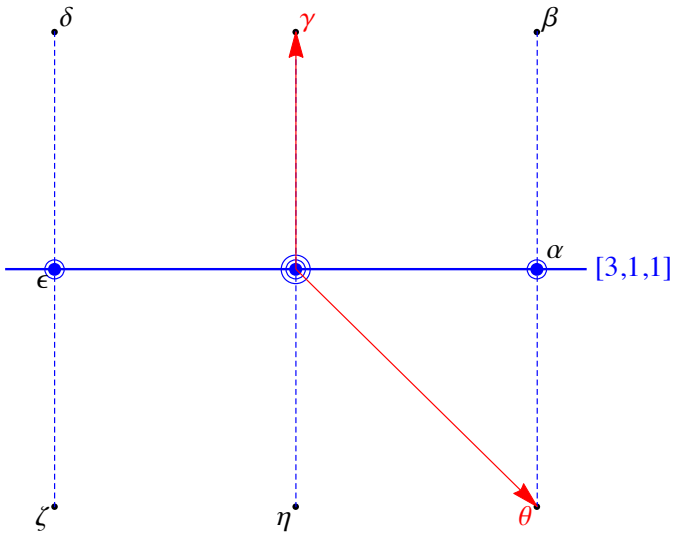


Figure 5.3: Projection of $\mathfrak{so}(5,\mathbb{C})$ onto $\mathfrak{sl}(2,\mathbb{C})$ corresponding to the partition $[3,1^2]$ leading to a decomposition of three representations of dimension 3 and one of dimension 1.

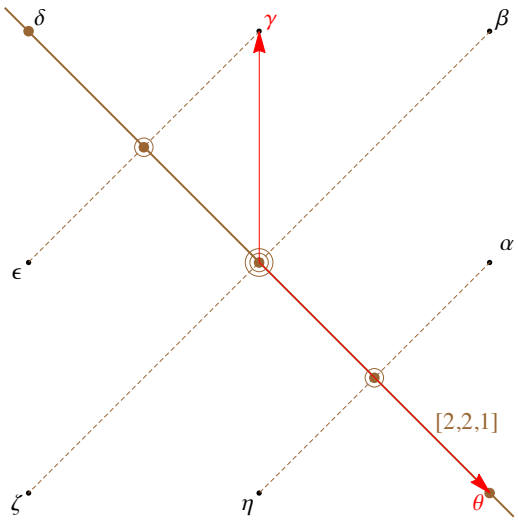


Figure 5.4: Projection of $\mathfrak{so}(5,\mathbb{C})$ onto $\mathfrak{sl}(2,\mathbb{C})$ corresponding to the partition $[2^2,1]$ leading to a decomposition of one representation of dimension 3, two of dimension 2 and three of dimension 1.

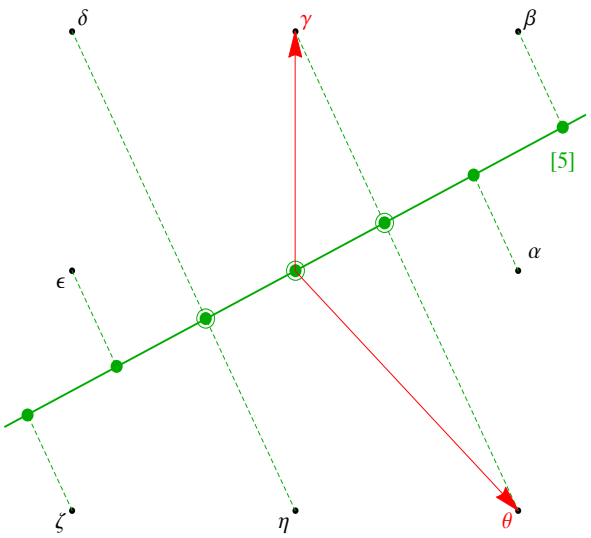


Figure 5.5: Projection of $\mathfrak{so}(5,\mathbb{C})$ onto $\mathfrak{sl}(2,\mathbb{C})$ corresponding to the partition [5] leading to a decomposition of one representations of dimension 7, one of dimension 2 and one of dimension 1.

We thus see immediately that there is only one irreducible representation with dimension larger than 3 that will lead to a higher spin field in the black hole analysis below: the 7-dimensional irreducible representation that appears in the case of the nilpotent orbit $\mathcal{O}_{[5]}$. Because of this, we will consider only this orbit in the sequel and abandon the other two.

5.3 Intersection of the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra of $\mathfrak{so}(5,\mathbb{C})$ with the Real Form $\mathfrak{so}(3,2)$

Let us recall what our final goal is: We want to find a black hole solution in terms of a CS-connection field that lives in real form of $SO(5,\mathbb{C})$. Since this connection field is a real quantity in the Lorentzian formulation, it thus lives in a real simple Lie algebra. Recall that classical GR is associated to the gauge group $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$. Since this is a non-compact group, the only real Lie algebras that lead to higher spin fields are non-compact real forms of a corresponding complex Lie algebra. We thus need to find the standard triples of these non-compact real forms of $\mathfrak{so}(5,\mathbb{C})$. There are two non-compact real forms of $\mathfrak{so}(5,\mathbb{C})$: $\mathfrak{so}(3,2)$ and $\mathfrak{so}(4,1)$. It is not straightforward to find a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra in $\mathfrak{so}(4,1)$. This fact will be explained under step 4 below, as soon as all required concepts have been introduced. We will hence treat the case of the former real form in this thesis. We thus need to look for the intersection of the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra in $\mathfrak{so}(5,\mathbb{C})$ with its real form $\mathfrak{so}(3,2)$. In this section we follow the construction presented in [37] without reference to any explicit basis.

Step 1: The Cartan Involution of $\mathfrak{so}(3,2)$
Let us refer to the Cartan involution of $\mathfrak{so}(3,2)$ by $\theta_{(3,2)}$. As any involution, this map has two eigenspaces: one called $\mathfrak{k}$ corresponding to the eigenvalue $+1$ and one called $\mathfrak{p}$

corresponding to the eigenvalue -1, such that

$$\theta_{(3,2)}(X) = \pm X \quad \forall\, X \in \mathfrak{so}(3,2). \tag{5.16}$$

The Cartan decomposition hence reads as

$$\mathfrak{so}(3,2) = \mathfrak{k} \oplus \mathfrak{p}. \tag{5.17}$$

We will call X "real" if and only if $\theta_{(3,2)}(X) = X$ and X is consequently "imaginary" if and only if $\theta_{(3,2)}(X) = -X$. From this we find

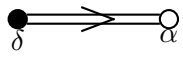
$$\theta_{(3,2)}(X) = X \qquad \Rightarrow \qquad \theta_{(3,2)}(iX) = -iX. \tag{5.18}$$

Further, a root α shall be called "compact" if the corresponding root space $\mathfrak{g}_\alpha$ is a subspace of $\mathfrak{k}$ and it shall be called "non-compact" if $\mathfrak{g}_\alpha \subset \mathfrak{p}$ [37].

Step 2: The Vogan Diagram of $\mathfrak{so}(3,2)$
In the sequel we will work again with the root system of $\mathfrak{so}(3,2)$. It is more convenient to change our choice of simple roots from $\{\gamma, \vartheta\}$ to $\{\alpha, \delta\}$.

Remark Notice that we can always change from one set of simple roots to another via a Weyl reflection without changing the results.

The Vogan diagram of $\mathfrak{so}(3,2)$ is known :



According to [37] the root that is assigned to the filled node is an imaginary non-compact root while the other one is an imaginary compact root. According to our new choice of simple roots δ is an imaginary non-compact root with respect to a Cartan involution $\theta_{(3,2)}$ of $\mathfrak{so}(3,2)$, i.e.:

$$\theta_{(3,2)}(E_\delta) = -E_\delta \qquad \text{and} \qquad \theta_{(3,2)}(E_\alpha) = E_\alpha \tag{5.19}$$

Step 3: Eigenspaces of $\theta_{(3,2)}$
We find the action of the Cartan involution on the remaining root vectors by acting successively with the adjoint representation of the simple roots on root vectors whose behavior under the action of $\theta_{(3,2)}$ is known. An example: If we want to find out how E_γ behaves under the action of $\theta_{(3,2)}$, we calculate

$$\begin{aligned} \theta_{(3,2)}(E_\gamma) &= \theta_{(3,2)}([E_\alpha, E_\delta]) \\ &= [\theta_{(3,2)}(E_\alpha), \theta_{(3,2)}(E_\delta)] \\ &= [E_\alpha, -E_\delta] \\ &= -[E_\alpha, E_\delta] \\ &= -E_\gamma \end{aligned} \tag{5.20}$$

The behavior of all root vectors under the action of $\theta_{(3,2)}$ is given by

$$\begin{aligned} \theta_{(3,2)}(E_\alpha) &= +E_\alpha, & \theta_{(3,2)}(E_\beta) &= -E_\beta, \\ \theta_{(3,2)}(E_\gamma) &= -E_\gamma, & \theta_{(3,2)}(E_\delta) &= -E_\delta, \\ \theta_{(3,2)}(E_\varepsilon) &= +E_\varepsilon, & \theta_{(3,2)}(E_\zeta) &= -E_\zeta, \\ \theta_{(3,2)}(E_\eta) &= -E_\eta, & \theta_{(3,2)}(E_\vartheta) &= -E_\vartheta \end{aligned} \tag{5.21}$$

and can be displayed as in the figure 5.6. Considering this figure, it becomes clear why the new choice of simple roots is more convenient: the nilpotent elements $X = E_\gamma + E_\vartheta$ and $Y = E_\delta + E_\eta$ in expression (5.13) are now well-behaved under the action of $\theta_{(3,2)}$ in the sense that $\theta_{(3,2)}(X) = -X$ and $\theta_{(3,2)}(Y) = -Y$. As a consequence, we have immediately that $\theta_{(3,2)}(iX) = iX$ and $\theta_{(3,2)}(iY) = iY$.

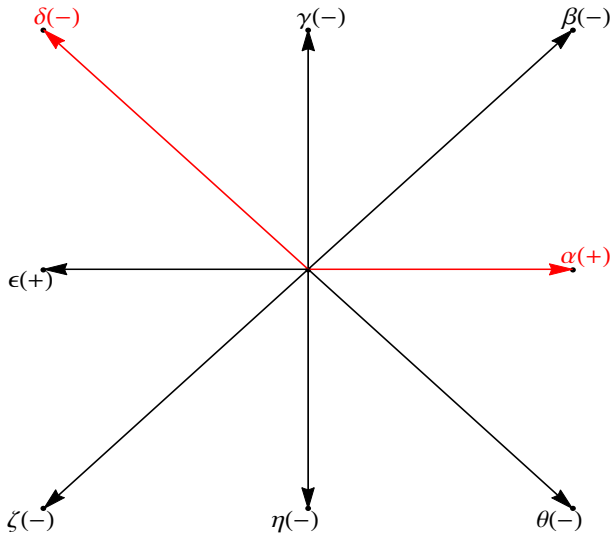


Figure 5.6: The signs in the brackets label the root vectors as "real" (+) or "imaginary" (−).

We are left with the statement that the Cartan subalgebra

$$\mathfrak{h}_{\mathfrak{so}(3,2)} = \langle\{H_1 = \mathrm{diag}(0,1,0,-1,0), H_2 = \mathrm{diag}(0,0,1,0,-1)\}\rangle_{\mathbb{R}} \tag{5.22}$$

is obviously invariant under $\theta_{(3,2)}$, i.e. $\theta_{(3,2)}(H) = H$ for all $H \in \mathfrak{h}$. It follows that $\theta_{(3,2)}(iH) = -iH$. Thus we have found the eigenspaces of $\theta_{(3,2)}$:

$$\mathfrak{k} := \langle\{E_\alpha, E_\varepsilon\}\rangle_{\mathbb{R}} \tag{5.23}$$
$$\mathfrak{p} := \langle\{E_\beta, E_\gamma, E_\delta, E_\zeta, E_\eta, E_\vartheta\}\rangle_{\mathbb{R}} \tag{5.24}$$

Step 4: Standard triples of $\mathfrak{so}(3,2)$
Further we have found that $\{iX, iY, H\} \in \mathfrak{k}$ and $\{X, Y, iH\} \in \mathfrak{p}$. Hence, $\{iX, iY, H\}$ is a $\theta_{(3,2)}$-stable standard triple of $\mathfrak{so}(5,\mathbb{C})$ and thus a standard triple of $\mathfrak{so}(3,2)$.
Now we are able to understand, why it is not straightforward to find a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra in $\mathfrak{so}(4,1)$: The Vogan diagram of $\mathfrak{so}(4,1)$ tells us that the short root α belongs to the eigenspace $\mathfrak{p}$ of the corresponding Cartan involution $\theta_{(4,1)}$ and the long root δ belongs to the other eigenspace $\mathfrak{k}$. Hence we can immediately conclude that

$$\mathfrak{k} := \langle\{E_\beta, E_\delta, E_\zeta, E_\vartheta\}\rangle_{\mathbb{R}} \tag{5.25}$$
$$\mathfrak{p} := \langle\{E_\alpha, E_\gamma, E_\varepsilon, E_\eta\}\rangle_{\mathbb{R}} \tag{5.26}$$

Now we see that the nilpotent elements X and Y are not well-behaved under the Cartan involution as they are neither real nor imaginary but complex:

$$\theta_{(4,1)}(X) = \theta_{(4,1)}(E_\gamma + E_\vartheta) = \theta_{(4,1)}(E_\gamma) + \theta_{(4,1)}(E_\vartheta) = -E_\gamma + E_\vartheta \tag{5.27}$$

and similar for Y, iX and iY . Hence, it is not straightforward to choose a $\theta_{(4,1)}$-stable standard triple.

Step 5: Consistency check
To verify this result we must check that

- the set $\{iX, iY, H\}$ indeed forms a standard triple,
- the set $\{iX, iY, H\}$ leads to an indefinite Killing metric.

The first point is easily proved by plugging in the elements of the triple into the commutator relations of $\mathfrak{sl}(2,\mathbb{R})$. We know that $\{X, Y, H\}$ obeys these relations from which we can deduce using the fact that the commutator is multiplicative in both arguments:

$$[H, X] = 2X \qquad \Rightarrow \qquad [H, iX] = 2iX \tag{5.28}$$
$$[H, Y] = -2Y \qquad \Rightarrow \qquad [H, iY] = -2iY \tag{5.29}$$
$$[X, Y] = H \qquad \Rightarrow \qquad [iY, iX] = H \tag{5.30}$$

In order to compute the Killing metric of this set we first need to compute their matrices corresponding to adjoint representation. These are given by

$$ad_{iX} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 0 & 0 \end{pmatrix}, \quad ad_{iY} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad ad_H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.31}$$

This can be checked by computing

$$ad_{iX} \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ -H \\ -2iX \end{pmatrix} \tag{5.32}$$
$$ad_{iY} \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} H \\ 0 \\ 2iY \end{pmatrix} \tag{5.33}$$
$$ad_H \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} iX \\ iY \\ H \end{pmatrix} = \begin{pmatrix} 2iX \\ -2iY \\ 0 \end{pmatrix} \tag{5.34}$$

which is exactly what we expect from the commutator relations (5.28) to (5.30).

Remark Notice that e.g. $ad_{iX}(H) = [iX, H] = -[H, iX]$. Hence there is sometimes an additional sign compared to the commutator relations.

Next, the Killing form can be written as $K(M, N) = \mathrm{tr}(ad_M ad_N)$ and the Killing metric of the above three elements iX, iY and H is given by

$$g_K := \begin{pmatrix} K(iX, iX) & K(iX, iY) & K(iX, H) \\ K(iY, iX) & K(iY, iY) & K(iY, H) \\ K(H, iX) & K(H, iY) & K(H, H) \end{pmatrix} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \tag{5.35}$$

The eigenvalues of g_K are 8,−4 and 4. The Killing metric is hence indefinite.

Conclusion
We thus find the following standard triple of $\mathfrak{so}(3,2)$:

$$\{iX, iY, H\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4i \\ 6i & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \right\} \tag{5.36}$$

5.4 The $\mathfrak{so}(3,2)$ Black Hole Solution

In [2] a generalization of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ theory to a $SL(N,\mathbb{R})\times\mathfrak{sl}(N,\mathbb{R})$ theory was suggested. Here, we shall construct a different generalization considering the $SO(3,2)\times SO(3,2)$ theory. As we will find later, the generators of the asymptotic symmetry algebra have spin 4. Detailed computations related to this topic can be found in appendix E.

5.4.1 Generators of $\mathfrak{so}(3,2)$

We proceed as in [2]: The Lorentzian action of the $SO(3,2)\times SO(3,2)$ theory is given by

$$I_{CS}[A^{\pm}] = \frac{k_3}{4\pi}\int \mathrm{tr}\left(A^{\pm}\wedge dA^{\pm} + \frac{2}{3}A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right) \tag{5.37}$$

where the $A^{\pm}$ lives in the $\mathfrak{so}(3,2)$ algebra and $k_3 = \frac{1}{20}k = \frac{l}{80G}$ (cf. appendix E.1). The Hamiltonian version is still given by expression (4.28). The real simple Lie algebra $\mathfrak{so}(3,2)$ is 10-dimensional and it shall be spanned by the basis given by $\{L_i, W_m\}$ where $i = 0, \pm 1$ and $m = 0, \pm 1, \pm 2, \pm 3$. The basis matrices L_0 and $L_{\pm 1}$ must simultaneously be a standard triple of $\mathfrak{so}(3,2)$ and satisfy the commutator relation

$$[L_i, L_j] = (i-j)L_{i+j} \qquad \text{for } i,j = -1, 0, +1. \tag{5.38}$$

such that the asymptotic symmetry algebra contains the Virasoro algebra. We thus take the standard triple found in the last section and rescale its elements such that the latter relation is obeyed. This is the case for

$$L_{-1} = iY, \qquad L_0 = -\frac{1}{2}H, \qquad L_{+1} = iX \tag{5.39}$$

or explicitly

$$\{L_{+1}, L_0, L_{-1}\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4i \\ 6i & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

These three matrices form a 3-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(3,2)$. In order to span the entire Lie algebra, we thus need to find seven more generators. We look for them by imposing the following commutator condition:

$$[L_j, W_m] = (3j - m)W_{j+m} \quad \text{for } j = 0, \pm 1, m = 0, \pm 1, \pm 2, \pm 3 \tag{5.40}$$

the following seven W-matrices form a 7-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(3,2)$ that obeys this latter commutator relation:

$$W_{-3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 720i & 0 & 0 \\ 0 & -720i & 0 & 0 & 0 \end{pmatrix}, \qquad W_{-2} = \begin{pmatrix} 0 & 120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -120 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{-1} = \begin{pmatrix} 0 & 0 & -24i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -24i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24i \\ 24i & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad W_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix},$$

$$W_{+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 4i \\ 0 & 0 & -6i & 0 & 0 \\ -4i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6i & 0 \end{pmatrix}, \qquad W_{+2} = \begin{pmatrix} 0 & 0 & 0 & -5 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_{+3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5i \\ 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, we consider the real simple Lie algebra

$$\mathfrak{so}(3,2) = \langle \{L_i, W_m \mid i = 0, \pm 1; m = 0, \pm 1, \pm 2, \pm 3\} \rangle_{\mathbb{R}} \tag{5.41}$$

We will need also a second copy of this algebra which shall be spanned by the generators $\{\tilde{L}_i, \tilde{W}_m\}$ where $i = 0, \pm 1$ and $m = 0, \pm 1, \pm 2, \pm 3$ where $L_i^t = \tilde{L}_{-i}$ and $W_m^t = \tilde{W}_{-m}$. Then, the above generators obey the satisfy the following identities

$$L_i^\dagger = (-1)^i \tilde{L}_{-i} \tag{5.42}$$

$$W_m^\dagger = (-1)^m \tilde{W}_{-m} \tag{5.43}$$

All commutator relations can be found in the appendix F.

5.4.2 Connection Field and Asymptotic Symmetries

Analogue to asymptotic behavior stated in the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$-generalization treated in [2], we require the following limits for the connection field in the present case under

study:

$$A_{\varphi}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} A_{\varphi}^{\pm}(r,\varphi) = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1} - \frac{\pi}{2k}\mathcal{W}^{\pm}(r,\varphi)W_{\mp 3} \tag{5.44}$$

$$A_{r}^{\pm} \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right). \tag{5.45}$$

with

$$\mathcal{L}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{L}^{\pm}(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.46}$$

$$\mathcal{W}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{W}^{\pm}(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.47}$$

Remark It shall be understood always that in the case of the "-"-components of the connection field A^{-}, the Lie algebra generators L_i and W_m are always replaced by $\tilde{L}_i$ and $\tilde{W}_m$.

Similar to the ansatz in expression (4.13), the most general form of the asymptotic behavior of $A_t^{\pm}$ is given now by

$$A_{t}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} A_{t}^{\pm}(r,\varphi) = \pm\sum_{i=-1}^{1}\xi_i(r,\varphi)L_i \pm \sum_{m=-3}^{3}\eta_m(r,\varphi)W_m. \tag{5.48}$$

Requiring that $A_t^{\pm}$ preserves the boundary conditions, i.e. imposing the equation of motion (D.12), its most general form reads as:

$$
\begin{aligned}
&A_t^{\pm} \\
&= \pm \Bigg\{ \left(\frac{1}{2}\xi_{\pm}'' - \frac{2\pi}{k}\xi_{\pm}\mathcal{L}^{\pm} - \frac{540\pi}{k}\eta_{\pm}\mathcal{W}^{\pm} \right) L_{\mp 1} \mp \xi_{\pm}' L_0 + \xi_{\pm} L_{\pm 1} \\
&\quad + \Bigg[\frac{1}{720}\eta_{\pm}^{VI} - \frac{41\pi}{180k}\eta_{\pm}^{IV}\mathcal{L}^{\pm} - \frac{5\pi}{12k}\eta_{\pm}'''(\mathcal{L}^{\pm})' - \frac{17\pi}{60k}\eta_{\pm}''(\mathcal{L}^{\pm})'' - \frac{\pi}{12k}\eta_{\pm}'(\mathcal{L}^{\pm})''' - \frac{\pi}{60k}\eta_{\pm}(\mathcal{L}^{\pm})^{IV} \\
&\quad + \frac{24\pi^2}{5k^2}\eta_{\pm}''(\mathcal{L}^{\pm})^2 + \frac{77\pi^2}{15k^2}\eta_{\pm}'\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' + \frac{6\pi^2}{5k^2}\eta_{\pm}(\mathcal{L}^{\pm})'^2 + \frac{23\pi^2}{15k^2}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' - \frac{3\pi}{2k}\eta_{\pm}''\mathcal{W}^{\pm} \\
&\quad - \frac{2\pi}{k}\eta_{\pm}'(\mathcal{W}^{\pm})' - \frac{\pi}{2k}\eta_{\pm}(\mathcal{W}^{\pm})'' - \frac{8\pi^3}{k^3}\eta_{\pm}(\mathcal{L}^{\pm})^3 + \frac{22\pi^2}{k^2}\eta_{\pm}\mathcal{L}^{\pm}\mathcal{W}^{\pm} - \frac{\pi}{2k}\xi_{\pm}\mathcal{W}^{\pm} - \frac{\pi}{k}\eta_{\pm}''\mathcal{W}^{\pm} \Bigg] W_{\mp 3} \\
&\quad \pm \Bigg[-\frac{1}{120}\eta_{\pm}^{V} + \frac{6\pi}{5k}\eta_{\pm}'''\mathcal{L}^{\pm} + \frac{13\pi}{10k}\eta_{\pm}''(\mathcal{L}^{\pm})' + \frac{2\pi}{5k}\eta_{\pm}'(\mathcal{L}^{\pm})'' + \frac{\pi}{10k}\eta_{\pm}(\mathcal{L}^{\pm})''' - \frac{44\pi^2}{5k^2}\eta_{\pm}'(\mathcal{L}^{\pm})^2 \\
&\quad - \frac{36\pi^2}{5k^2}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' + \frac{9\pi}{k}\eta_{\pm}'\mathcal{W}^{\pm} + \frac{3\pi}{k}\eta_{\pm}(\mathcal{W}^{\pm})' \Bigg] W_{\mp 2} \\
&\quad + \Bigg[\frac{1}{24}\eta_{\pm}^{IV} - \frac{5\pi}{k}\eta_{\pm}''\mathcal{L}^{\pm} - \frac{3\pi}{2k}\eta_{\pm}'(\mathcal{L}^{\pm})' - \frac{\pi}{2k}\eta_{\pm}(\mathcal{L}^{\pm})'' + \frac{12\pi^2}{k^2}\eta_{\pm}(\mathcal{L}^{\pm})^2 - \frac{15\pi}{k}\eta_{\pm}\mathcal{W}^{\pm} \Bigg] W_{\mp 1} \\
&\quad \pm \Bigg[-\frac{1}{6}\eta_{\pm}''' + \frac{16\pi}{3k}\eta_{\pm}'\mathcal{L}^{\pm} + \frac{2\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm})' \Bigg] W_0 \\
&\quad + \Bigg[\frac{1}{2}\eta_{\pm}'' - \frac{6\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm}) \Bigg] W_{\pm 1} \\
&\quad \mp \eta_{\pm}' W_{\pm 2} \\
&\quad + \eta_{\pm} W_{\pm 3} \Bigg\}
\end{aligned}
\tag{5.49}
$$

Also from the equation of motion we can deduce the time evolution equations of the asymptotic charges:

$$
\dot{\mathcal{L}}^{\pm} = \pm \left(2\xi_{\pm}'\mathcal{L}^{\pm} + \xi_{\pm}(\mathcal{L}^{\pm})' - \frac{k}{4\pi}\xi_{\pm}''' + 90[3\eta_{\pm}(\mathcal{W}^{\pm})' + 4\eta_{\pm}'\mathcal{W}^{\pm}] \right) \tag{5.50}
$$

$$
\begin{aligned}
\dot{\mathcal{W}}^{\pm} = \pm \Bigg[& -\frac{k}{360\pi}\eta_{\pm}^{VII} + \frac{22}{45}\eta_{\pm}^{V}\mathcal{L}^{\pm} + \frac{58}{45}\eta_{\pm}^{IV}(\mathcal{L}^{\pm})' + \frac{7}{5}\eta_{\pm}'''(\mathcal{L}^{\pm})'' + \frac{11}{15}\eta_{\pm}''(\mathcal{L}^{\pm})''' \\
& + \frac{1}{5}\eta_{\pm}'(\mathcal{L}^{\pm})^{IV} + \frac{1}{30}\eta_{\pm}(\mathcal{L}^{\pm})^{V} - \frac{72\pi}{5k}\eta_{\pm}'''(\mathcal{L}^{\pm})^2 - \frac{442\pi}{15k}\eta_{\pm}''\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' \\
& - \frac{38\pi}{3k}\eta_{\pm}'(\mathcal{L}^{\pm})'^2 - \frac{154\pi}{15k}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' - \frac{118\pi}{15k}\eta_{\pm}(\mathcal{L}^{\pm})'(\mathcal{L}^{\pm})'' - \frac{14\pi}{3k}\eta_{\pm}'\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' \\
& - \frac{52\pi}{15k}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})''' + \frac{256\pi^2}{5k^2}\eta_{\pm}'(\mathcal{L}^{\pm})^3 + \frac{384\pi^2}{k^2}\eta_{\pm}(\mathcal{L}^{\pm})^2(\mathcal{L}^{\pm})' \\
& - \frac{112\pi}{k}\eta_{\pm}'\mathcal{L}^{\pm}\mathcal{W}^{\pm} - \frac{56\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm}\mathcal{W}^{\pm})' + 4\eta_{\pm}'''\mathcal{W}^{\pm} + 7\eta_{\pm}''(\mathcal{W}^{\pm})' \\
& + \xi_{\pm}(\mathcal{W}^{\pm})' + 4\xi_{\pm}'(\mathcal{W}^{\pm}) + 5\eta_{\pm}'(\mathcal{W}^{\pm})'' + \eta_{\pm}(\mathcal{W}^{\pm})''' \Bigg]
\end{aligned}
\tag{5.51}
$$

where

$$\xi_\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} \xi_\pm(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.52}$$

$$\eta_\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} \eta_\pm(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.53}$$

Choosing the gauge parameters $\xi_\pm = 1$ and $\eta_\pm = 0$, we obtain:

$$\dot{\mathcal{L}}^\pm = \mathcal{L}^{\pm\prime} \tag{5.54}$$

$$\dot{\mathcal{W}}^\pm = \mathcal{W}^{\pm\prime} \tag{5.55}$$

which corresponds to the chiral equations we also found in the pure gravity case.

Now we are ready to find the functional form of the boundary term. The corresponding calculation in the present case is identical to the pure gravity scenario given in section D.4 up to equation (D.62). Hence, this is where we set in. We want to evaluate the expression

$$\frac{k_3}{2\pi}\int_{\partial\Sigma} d\varphi \,\mathrm{tr}\left(A_t^\pm \delta A_\varphi^\pm\right). \tag{5.56}$$

Notice that $A_t^\pm \delta A_\varphi^\pm$ is a sum of terms each of which is proportional to a product of two generators. Accordingly, $\mathrm{tr}(A_t^\pm \delta A_\varphi^\pm)$ is a sum of terms each of which is proportional to the trace of the product of two generators. Hence, we need only to consider the terms proportional to a non-vanishing trace. These are the ones that are proportional to

$$\mathrm{tr}\left(L_{\pm1}L_{\mp1}\right) = -20 \quad \text{and} \quad \mathrm{tr}\left(W_{\pm3}W_{\mp3}\right) = -7200 \tag{5.57}$$

Writing equation (5.56) in terms of the generators and plugging in these traces we find the equation analogue to (D.78)

$$\mathrm{tr}\left(A_t^\pm \delta A_\varphi^\pm\right) = \pm\frac{40\pi}{k}\left(\xi_\pm \delta\mathcal{L}^\pm + 90\eta_\pm\delta\mathcal{W}^\pm\right) = \pm\frac{2\pi}{k_3}\left(\xi_\pm \delta\mathcal{L}^\pm + 90\eta_\pm\delta\mathcal{W}^\pm\right) \tag{5.58}$$

one obtains finally the functional form of the boundary term:

$$B_\infty^\pm = -\int dt\, Q_\infty^\pm \qquad | \quad Q_\infty^\pm = \pm\int_{\partial\Sigma} d\varphi \left(\xi^\pm(\varphi)\mathcal{L}^\pm(\varphi) + 90\eta^\pm(\varphi)\mathcal{W}^\pm(\varphi)\right) \tag{5.59}$$

Notice that this result is in accordance with the pure gravity result in the sense that we recover the latter given by expressions (D.82) and (D.83) up to a purely numerical prefactor that comes from the fact that the traces are different for different Lie algebras.

As a last step in analyzing the $\mathfrak{so}(3,2)$-connection field before we start constructing the black hole solution we want to find Poisson structure of the Fourier modes of the asymptotic symmetry generators. Perform analog calculations as the one given in the appendix

D.5 we find for the structure of the asymptotic symmetry algebra:

$$i\{\mathcal{L}_n^{\pm},\mathcal{L}_m^{\pm}\} = \frac{1}{20}\left[(n-m)\mathcal{L}_{n+m}^{\pm} + \frac{kn^2}{2}\delta_{n,-m}\right] \tag{5.60}$$

$$i\{\mathcal{L}_n^{\pm},\mathcal{W}_m^{\pm}\} = \frac{1}{20}\left[(3n-m)\mathcal{W}_{n+m}^{\pm}\right] \tag{5.61}$$

$$\begin{aligned} i\{\mathcal{W}_n^{\pm},\mathcal{W}_m^{\pm}\} = \frac{i}{1800}\Bigg[&\frac{1}{3}(3n^5-5mn^4+6m^2n^2-24m^3n^2+3m^4n-5m^5)\mathcal{L}_{m+n}^{\pm} \\ &-(n^3+2m^2n-m^3)\mathcal{W}_{m+n}^{\pm} \\ &+\frac{\pi}{15k}\sum_q(72n^3-442nq+190mnq+190n^2q \\ &\qquad -118mq^2-84nq^2+66q^3)\mathcal{L}_{m+n-q}^{\pm}\mathcal{L}_q^{\pm} \\ &+\frac{56\pi}{k}(m-n)\mathcal{L}_{m+n-q}^{\pm}\mathcal{W}_q^{\pm} \\ &+\frac{128\pi^2}{5k^2}\sum_q\sum_r(2n-3r)\mathcal{L}_{m+n-q-r}^{\pm}\mathcal{L}_q^{\pm}\mathcal{L}_r^{\pm}+n^7\delta_{n,-m}\Bigg] \end{aligned} \tag{5.62}$$

The asymptotic symmetry algebra is the $\mathcal{W}(2,4)$-algebra, as is explained in [48] and discussed in [49–53]. Recall the following definition:

Definition 5.4.1 (Conformal weight)
A field $\phi(z)$ has conformal weight or conformal dimension J if under a coordinate transformation $z \to z'(z)$ the field ϕ transforms as [2]:

$$\phi'(z') = \left(\frac{z}{z'}\right)^J \phi(z) \tag{5.63}$$

In terms of Fourier modes and Poisson brackets we thus obtain for infinitesimal transformations $z \to z' = z + \epsilon$:

$$i\{L_n,\phi_m\} = [n(J-1)-m]\phi_{n+m} \tag{5.64}$$

Notice that equation (5.61) is of the form (5.64) if we set $J = 4$ (up to a numerical prefactor) leading to the fact, that the $\mathcal{W}$-field is a field of conformal weight 4 and thus indeed a higher spin field. Further, one can again perform a consistency check: Recall the corresponding Poisson structure of the asymptotic symmetry algebra in the pure gravity case that is given by equation (4.23) which is equal in equation (5.60) for the $\mathfrak{so}(3,2)$ case up to a purely numerical prefactor.

5.4.3 Black Hole

In order to construct the higher spin black hole we still want the system to have the topology of a solid torus with flat connection. Hence, in the Euclidean formulation, we must write down a Euclidean connection field, i.e. a complexified connection living in $\mathfrak{so}(5,\mathbb{C})$, that obeys the flatness condition. The rules for continuing the Lorentzian to the Euclidean formulation are the same as in the case of the BTZ-black hole (cf. equations (4.24) and (4.25)). The flat Euclidean connection in the rest frame (the only extensive parameters

characterizing the black hole are thus again its mass and its angular momentum) is given by (the arguments are supressed on the right hand side and again $t = -i\tau$):

$$A_\varphi(r,\varphi) \overset{r\to\infty}{\longrightarrow} L_{+1} - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-3} \tag{5.65}$$

$$A_\tau(r,\varphi) \overset{r\to\infty}{\longrightarrow} -i\xi\left(L_{+1} - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-3}\right) \tag{5.66}$$

$$-i\eta\left(W_{+3} - \frac{6\pi}{k}\mathcal{L}W_{+1} + \frac{12\pi^2}{k^2}\mathcal{L}^2W_{-1} - \frac{15\pi}{k}\mathcal{W}W_{-1}\right. \tag{5.67}$$

$$\left. -\frac{8\pi^3}{k^3}\mathcal{L}^3W_{-3} + \frac{22\pi^2}{k^2}\mathcal{L}\mathcal{W}W_{-3} - \frac{540\pi}{k}\mathcal{W}L_{-1}\right) \tag{5.68}$$

$$A_r \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{5.69}$$

$$\mathcal{L}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{L}_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.70}$$

$$\mathcal{W}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{W}_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.71}$$

$$\xi(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\xi_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.72}$$

$$\eta(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\eta_0 + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.73}$$

where all quantities $\mathcal{L}^\pm, \mathcal{W}^\pm, \xi^\pm$ and $\eta^\pm$ are constant at radial infinity. Notice that we still have the topology of a torus and the connection satisfies the equation of motion (4.9). If we are able to ascribe a reasonable entropy to this solution, it indeed describes a Euclidean black hole. Its Lorentzian formulation reads as

$$A^\pm_\varphi(r,\varphi) \overset{r\to\infty}{\longrightarrow} L_{\pm1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp1} - \frac{\pi}{2k}\mathcal{W}^\pm W_{\mp3} \tag{5.74}$$

and the asymptotic temporal component, i.e. the gauge transformation, on the other hand, simplifies significantly from equation (5.49) to

$$\begin{aligned} A^\pm_t(r,\varphi) \overset{r\to\infty}{\longrightarrow} \pm\Bigg[&\xi_\pm\left(L_{\pm1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp1} - \frac{\pi}{2k}\mathcal{W}^\pm W_{\mp3}\right) \\ &+ \eta_\pm\left(W_{\pm3} - \frac{6\pi}{k}\mathcal{L}^\pm W_{\pm1} + \frac{12\pi^2}{k^2}\mathcal{L}^{\pm 2}W_{\mp1} - \frac{15\pi}{k}\mathcal{W}^\pm W_{\mp1}\right. \\ &\left. -\frac{8\pi^3}{k^3}\mathcal{L}^{\pm 3}W_{\mp3} + \frac{22\pi^2}{k^2}\mathcal{L}^\pm\mathcal{W}^\pm W_{\mp3} - \frac{540\pi}{k}\mathcal{W}^\pm L_{\mp1}\right)\Bigg]. \end{aligned} \tag{5.75}$$

Further we have

$$A_r^{\pm} \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{5.76}$$

$$\mathcal{L}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{L}_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.77}$$

$$\mathcal{W}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\mathcal{W}_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.78}$$

$$\xi_{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\xi_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.79}$$

$$\eta_{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \frac{1}{2\pi}\eta_0^{\pm} + \mathcal{O}\left(\frac{1}{r}\right) \tag{5.80}$$

5.5 Thermodynamics of the $\mathfrak{so}(3,2)$ Black Hole

5.5.1 Regularity Condition and Holonomy

We are once more faced with the fact that the temporal component A_τ is not well-defined at the horizon, i.e. along the r_+-circle in the solid torus representing the topology of the system under study. We thus need to regularize it as in subsection 4.2.1. We will construct the regularity condition as follows:

Remember that in the $SL(2,\mathrm{R}) \times SL(2,\mathrm{R})$ theory we started the regularization by diagonalizing the temporal connection component:

$$A_\tau = -2\pi i\nu(\xi,\mathcal{L})L_0 \tag{5.81}$$

In the present $\mathfrak{so}(3,2)$-setting, one recovers the pure gravity result by "turning off" the higher spin field, i.e. by setting $\eta = 0$ and $\mathcal{W} = 0$ such that

$$A_\tau = -i\xi\left(L_{+1} - \frac{2\pi}{k}\mathcal{L}L_{-1}\right) \tag{5.82}$$

which in turn can be diagonalized as before (where $L_0 \in \mathfrak{so}(3,2)$):

$$A_\tau = 2\pi i\nu(\xi,\mathcal{L})L_0 \qquad | \quad \nu(\xi,\mathcal{L}) = \sqrt{\frac{2\mathcal{L}}{k\pi}}\xi \tag{5.83}$$

While in the former case a factor of $1/2$ in the definition of $L_0 \in \mathfrak{sl}(2,\mathbb{R})$ allowed a regularizing gauge transformation given by a group element

$$g = e^{2\pi i\nu L_0} \tag{5.84}$$

to assume both values, $\mathbb{1}$ and $-\mathbb{1}$, in the present case the corresponding group element must be unity, since $L_0 = \mathrm{diag}(0,-2,-1,2,1) \in \mathfrak{so}(3,2)$ has only integer and no half-integer entries. Consequently, the holonomy is given by $g = \mathbb{1}$. As before, the regularity condition is given by

$$\nu(\xi,\mathcal{L}) \in \mathbb{Z} \tag{5.85}$$

which is the same condition as in the $\mathfrak{sl}(2,\mathbb{R})$ set-up.

5.5.2 Entropy of the $\mathfrak{so}(3,2)$ Black Hole

Now we are heading towards the final result: the entropy of the $\mathfrak{so}(3,2)$ black hole in the microcanonical ensemble. As the derivation leading to equation (E.45) holds in general, we may start calculating the entropy in the present case starting from this very equation:

$$S_E = -2k_3 \mathrm{Im}\left[\mathrm{tr}\left(A_\tau A_\varphi\right)\right]_{\text{on-shell}} \tag{5.86}$$

or again in the Lorentzian case:

$$S_{\text{Lor}} = k_3\left[\mathrm{tr}\left(A_t^+ A_\varphi^+\right) - \mathrm{tr}\left(A_t^- A_\varphi^-\right)\right]_{\text{on-shell}}. \tag{5.87}$$

Plugging in the expressions for A_φ given in expression (5.65) and A_τ in expression (5.66) into the Euclidean entropy formula leads to

$$\begin{aligned} S_E &= 2\,\mathrm{Re}[4\pi\xi\mathcal{L} + 720\pi\eta\mathcal{W}] && (5.88)\\ &= 4\pi(\xi\mathcal{L} + \xi^*\mathcal{L}^*) + 720\pi(\eta\mathcal{W} + \eta^*\mathcal{W}^*) && (5.89)\end{aligned}$$

In order to express this entropy in terms of the asymptotic charges only, we need to relate the coeffictions ξ and η to these and further we need to implement the regularity condition. Adapting the procedure presented in the spin 3-generalization of BTZ black hole studies provided in [2] we can accomplish this task as follows: We introduce a second connection field A_τ^{reg} that is required to fulfill the regularity condition $\nu = 1$ (so far, this is even equivalent to the procedure given for the BTZ black hole in section 4.2.2 below equation (4.58)). Then require that the invariant polynomials of A_τ^{reg} and $A_\tau^{\text{on-shell}}$ evaluated at the horizon r_+ coincide. The invariant polynomials of lowest order of both versions of A_τ that do not vanish upon evaluation at the horizon are given by $\mathrm{tr}\left(A_\tau^2\right)$ and $\mathrm{tr}\left(A_\tau^4\right)$. As given in the previous section, the eigenvalues of A_τ^{reg} are $0, \pm 2\pi i, \pm 4\pi i$ (cf. equation (5.84)) such that

$$\mathrm{tr}\left(A_\tau^{\text{reg}\,2}\right) = 2(2i\pi)^2 + 2(4i\pi)^2 = -40\pi^2 \tag{5.90}$$

$$\text{and} \qquad \mathrm{tr}\left(A_\tau^{\text{reg}\,4}\right) = 2(2i\pi)^4 + 2(4i\pi)^4 = 544\pi^4. \tag{5.91}$$

Hence we need to solve the following system of equations

$$\mathrm{tr}\left(A_\tau^{\text{on-shell}\,2}\right) + 40\pi^2 = 0 \tag{5.92}$$

$$\mathrm{tr}\left(A_\tau^{\text{on-shell}\,4}\right) - 544\pi^4 = 0. \tag{5.93}$$

Explicitly, these system of equation reads:

$$\begin{aligned} 0 &= 2\frac{\mathcal{L}}{k\pi}\xi^2 + 2^4 3^2 5\frac{\mathcal{W}}{k\pi}\xi\eta + 2^5 3^2\frac{\mathcal{L}}{k^3}\left(2^4\mathcal{L}^2\pi - 5\cdot 7k\mathcal{W}\right)\eta^2 - 1\\ 0 &= \quad 2\left(2\cdot 17\pi k^4\mathcal{L}^2 - 3^2 5k^5\mathcal{W}\right)\xi^4 - 2^5 3^2\pi\left(2^4\pi k^3\mathcal{L}^3 - 5\cdot 41k^4\mathcal{L}\mathcal{W}\right)\xi^3\eta\\ &\quad + 2^6 3^3\pi k^2\left(2^8\pi^2\mathcal{L}^4 - 2^8 5\pi k\mathcal{L}^2\mathcal{W} + 3^2 5^4 k^2\mathcal{W}^2\right)\xi^2\eta^2\\ &\quad + 2^9 3^4\pi^2 k\left(2^8\pi^2\mathcal{L}^5 + 2^5 5\cdot 17\pi k\mathcal{L}^3\mathcal{W} - 2^9 5^4 23k^2\mathcal{L}\mathcal{W}^2\right)\xi\eta^3\\ &\quad + 2^9 3^4\pi^2\left(2^9 17\pi^3\mathcal{L}^6 - 2^6 5\cdot 11\pi^2 k\mathcal{L}^4\mathcal{W} - 2\cdot 5^4 7\pi k^2\mathcal{L}^2\mathcal{W}^2 - 3^2 5^7 k^3\mathcal{W}^3\right)\eta^4\\ &\quad - 17k^6\pi^3 \end{aligned} \tag{5.94}$$

Since the exact solution of this equation is extremely complicated, we will consider perturbative solutions close to the BTZ black hole where the field $\mathcal{L}$ is fixed and $\mathcal{W}$ is very small. For the latter we write from now on $\delta\mathcal{W}$. We want to find the functional expressions of η and ξ in terms of the fields $\mathcal{L}$ and $\delta\mathcal{W}$. We make the ansatz

$$\xi(\mathcal{L},\delta\mathcal{W}) = \xi_{\text{BTZ}} + \sum_{k=1}^{\infty} \delta\mathcal{W}^k \xi_k \tag{5.95}$$

$$\eta(\mathcal{L},\delta\mathcal{W}) = \eta_{\text{BTZ}} + \sum_{k=1}^{\infty} \delta\mathcal{W}^k \eta_k, \tag{5.96}$$

where $\xi_{\text{BTZ}} = \sqrt{k\pi/2\mathcal{L}}$ and $\eta_{\text{BTZ}} = 0$.

Linear Approximation:
As a first step, we want to find an expression of the entropy linear in $\delta\mathcal{W}$. Recalling expression (5.88), it becomes clear that we need to expand $\xi(\mathcal{L},\delta\mathcal{W})$ up to linear order and η up to zeroth order, because η is multiplied with $\delta\mathcal{W}$. Since $\eta_{\text{BTZ}} = 0$, the latter drops out. We thus consider:

$$\xi(\mathcal{L},\delta\mathcal{W}) = \xi_{\text{BTZ}} + \delta\mathcal{W}\xi_1 \tag{5.97}$$

Plugging this into the first line of (5.94) and solving for ξ_1 yields that $\xi_1 = 0$. Accordingly, the entropy of a black hole with a weak $\mathcal{W}$-field is the BTZ-entropy up to first order:

$$S_E^{\text{lin}} = 4\pi(\xi_{\text{BTZ}}\mathcal{L} + \xi^*_{\text{BTZ}}\mathcal{L}^*) = 2\pi\sqrt{2\pi k}(\sqrt{\mathcal{L}} + \sqrt{\mathcal{L}^*}) \tag{5.98}$$

This result coincides with expression (4.65).

Quadratic Approximation:
As the linear approximation does not provide any information about how the entropy depends on the higher spin field, we shall proceed by calculating an approximation up to second order (see appendix E.4.2 for detailed computations). Now, we need to expand ξ up to second and η up to linear order in $\delta\mathcal{W}$:

$$\xi(\mathcal{L},\delta\mathcal{W}) = \xi_{\text{BTZ}} + \delta\mathcal{W}\xi_1 + \delta\mathcal{W}^2\xi_2 \tag{5.99}$$

$$\eta(\mathcal{L},\delta\mathcal{W}) = \eta_{\text{BTZ}} + \delta\mathcal{W}\eta_1 \tag{5.100}$$

We already know that $\eta_{\text{BTZ}} = 0$, $\xi_{\text{BTZ}} = \sqrt{k_3\pi/2\mathcal{L}}$ and $\xi_1 = 0$. This simplifies the latter expressions to

$$\xi(\mathcal{L},\delta\mathcal{W})\sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}^2\xi_2 \tag{5.101}$$

$$\eta(\mathcal{L},\delta\mathcal{W}) = \delta\mathcal{W}\eta_1 \tag{5.102}$$

Plugging these expressions again into (5.94) and ignoring all terms of third or higher order in $\delta\mathcal{W}$, we obtain:

$$\xi_2 = \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^9\pi^3}} \tag{5.103}$$

$$\eta_1 = -\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}} \tag{5.104}$$

Plugging now the found expressions

$$\xi(\mathcal{L},\delta\mathcal{W})=\sqrt{\frac{k\pi}{2\mathcal{L}}}+\frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^9\pi^3}}\delta\mathcal{W}^2 \quad \text{and} \quad \eta(\mathcal{L},\delta\mathcal{W})=-\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W} \tag{5.105}$$

into equation (5.88) yields for the Euclidean black hole endowed with a spin 4 field:

$$S_E^{quad}=2\,\mathrm{Re}\left[2\pi\sqrt{2\pi k\mathcal{L}}\left(1-\frac{225}{512}\frac{k^2}{\pi^2\mathcal{L}^4}\delta\mathcal{W}^2\right)\right] \tag{5.106}$$

In the Lorentzian formalism the entropy reads up to quadratic order:

$$S_{\mathrm{Lor}}^{quad}= \quad 2\pi\sqrt{2\pi k}\left(\sqrt{\mathcal{L}^+}+\sqrt{\mathcal{L}^-}\right)-\frac{225}{128}\sqrt{\frac{k^5}{2\pi}}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{\mathcal{L}^{+7}}}+\frac{\delta\mathcal{W}^{-2}}{\sqrt{\mathcal{L}^{-7}}}\right) \tag{5.107}$$

We can alternatively, analog to the case of the spin 2-black hole discussed in section 4.2, express the entropy in terms of the Lorentzian mass and the Lorentzian angular momentum:

$$\begin{aligned}S^{quad}=&\,\pi l\sqrt{\frac{\mathcal{M}_{\mathrm{Lor}}}{G}\left(1+\sqrt{1-\frac{\mathcal{J}_{\mathrm{Lor}}^2}{\mathcal{M}_{\mathrm{Lor}}^2l^2}}\right)}\\&-\frac{225}{32}\pi^3\sqrt{\frac{l^5}{2G^5}}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{(\mathcal{M}_{\mathrm{Lor}}l+\mathcal{J}_{\mathrm{Lor}})^7}}+\frac{\delta\mathcal{W}^{-2}}{\sqrt{(\mathcal{M}_{\mathrm{Lor}}l-\mathcal{J}_{\mathrm{Lor}})^7}}\right)\end{aligned} \tag{5.108}$$

Using this equation, we can calculate the inverse temperature β and the chemical potentials associated to the angular momentum and the spin 4 field $\mathcal{W}$ in the microcanonical ensemble as it is done in [2] for the spin 3 black hole:

$$\beta=\left(\frac{\partial S}{\partial\mathcal{M}_{\mathrm{Lor}}}\right)_{\mathcal{J}_{\mathrm{Lor}},\mathcal{W}_0^{\pm}} \tag{5.109}$$

$$\beta\mu_{\mathcal{J}_{\mathrm{Lor}}}=-\left(\frac{\partial S}{\partial\mathcal{J}_{\mathrm{Lor}}}\right)_{\mathcal{M}_{\mathrm{Lor}},\mathcal{W}_0^{\pm}} \tag{5.110}$$

$$\beta\mu_{\mathcal{W}^{\pm}}=-\left(\frac{\partial S}{\partial\mathcal{W}_0^{\pm}}\right)_{\mathcal{M}_{\mathrm{Lor}},\mathcal{J}_{\mathrm{Lor}}} \tag{5.111}$$

where $\mathcal{W}_0^{\pm}=2\pi\mathcal{W}^{\pm}$.

Remark As outlined in appendix E.4.3, it has been tried to find an exact expression for the entropy of the spin 4 black hole by transforming the system of equations (5.94) into a differential equation analog to the strategy presented in [26]. The differential equation obtained is both non-linear and non-separable and hence a solution could not be found yet.

5.5.3 Entropies of the Spin 3 and Spin 4 Black Holes Compared and Contrasted

Let us now investigate the similarities and the differences of the black hole entropies in the case of the spin 3 black hole treated in [2] and the spin 4 black hole just derived. This

is most easily done in the Euclidean language. The Euclidean entropy of the spin 3 black hole is given by [2]:

$$S_{\text{Spin3}} = 4\pi\sqrt{2\pi k}\ \text{Re}\left\{\sqrt{\mathcal{L}}\cos\left[\frac{1}{3}\arcsin\left(\frac{3}{8}\sqrt{\frac{3k_2}{2\pi\mathcal{L}^3}}\mathcal{W}\right)\right]\right\} \tag{5.112}$$

Expanding this expression up to quadratic order in $\mathcal{W}$ leads to:

$$S_{\text{Spin3}} = 2\ \text{Re}\left(2\pi\sqrt{2\pi k\mathcal{L}} - \frac{3}{64}\sqrt{\frac{\pi k^3}{2\mathcal{L}^5}}\mathcal{W}^2 + \mathcal{O}(\mathcal{W}^3)\right) \tag{5.113}$$

Hence we get for small $\delta\mathcal{W}$

$$S_{\text{Spin3}} = 2\ \text{Re}\left(2\pi\sqrt{2\pi k\mathcal{L}} - \frac{3}{64}\sqrt{\frac{\pi k^3}{2\mathcal{L}^5}}\delta\mathcal{W}^2\right) \tag{5.114}$$

$$= 4\pi\sqrt{2\pi k}\ \text{Re}\left[\sqrt{\mathcal{L}}\left(1 - \frac{3}{256}\frac{k}{\pi}\frac{\delta\mathcal{W}^2}{\mathcal{L}^3}\right)\right] \tag{5.115}$$

On the other hand, we can rewrite expression (5.106) as

$$S_{\text{Spin4}} = 4\pi\sqrt{2\pi k}\ \text{Re}\left[\sqrt{\mathcal{L}}\left(1 - \frac{225}{512}\frac{k^2}{\pi^2}\frac{\delta\mathcal{W}^2}{\mathcal{L}^4}\right)\right] \tag{5.116}$$

Dependence of the Entropy on the Mass and Angular Momentum:
We have already seen the dependence off the spin 4 black hole entropy in terms of Lorentzian mass and angular momentum (cf. (5.108)). The analog expression for the spin 3 black hole entropy reads as:

$$\begin{aligned} S^{quad}_{\text{Spin3}} = {} & \pi l\sqrt{\frac{\mathcal{M}_{\text{Lor}}}{G}\left(1 + \sqrt{1 - \frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}}l^2}}\right)} \\ & - \frac{3}{32}\pi^3\sqrt{\frac{2l^3}{G^3}}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l + \mathcal{J}_{\text{Lor}}\right)^5}} + \frac{\delta\mathcal{W}^{-2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l - \mathcal{J}_{\text{Lor}}\right)^5}}\right) \end{aligned} \tag{5.117}$$

Let us consider for a moment only the contributions coming from the higher spin fields:

$$S^{\pm}_{\text{spin 3, HSC}} = -\frac{3}{32}\pi^3\sqrt{\frac{2l^3}{G^3}}\left(\frac{\delta\mathcal{W}^{\pm 2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l \pm \mathcal{J}_{\text{Lor}}\right)^5}}\right) \tag{5.118}$$

$$S^{\pm}_{\text{spin 4, HSC}} = -\frac{225}{32}\pi^3\sqrt{\frac{l^5}{2G^5}}\left(\frac{\delta\mathcal{W}^{\pm 2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l \pm \mathcal{J}_{\text{Lor}}\right)^7}}\right), \tag{5.119}$$

the subscript "HSC" indicating that we consider the higher spin contribution only.

Remark The reader should keep in mind, that the spin 3 and the spin 4 field are in fact considered in two different theories. Hence, we may not a priori assume that a black hole is charged under both fields simultaneously. The analysis we are performing at this point shall only give a rough qualitative comparison between these two fields.

In order to investigate in which mass and angular momentum regimes which field dominates the entropy contribution, we define the relative entropy contribution as follows:

$$S^{\pm}_{\text{rel}} := \frac{S^{\pm}_{\text{spin 4, HSC}} - S^{\pm}_{\text{spin 3, HSC}}}{S^{\pm}_{\text{spin 3, HSC}}} \tag{5.120}$$

Further, we want to analyze two cases separately: on the one hand the case of a non-rotating black hole ($\mathcal{J}_{\text{Lor}} = 0$) that has only a non-vanishing mass and one of the higher spin fields and on the other hand the case of a rotating ($\mathcal{J}_{\text{Lor}} \neq 0$) black hole of some unit mass ($\mathcal{M}_{\text{Lor}} = 1$). We will refer to the former type as "Schwarzschild-like" while the latter type shall be called "Kerr-like". Figures 5.7 and 5.8 show the relative energy contributions in these two cases where

for the Schwarzschild-like black hole: $$S^{\pm}_{\text{rel}}(\mathcal{M}_{\text{Lor}}) = \frac{75}{2\mathcal{M}_{\text{Lor}}} - 1 \tag{5.121}$$

for the Kerr-like black hole: $$S^{\pm}_{\text{rel}}(\mathcal{J}_{\text{Lor}}) = \frac{75}{2\mathcal{J}_{\text{Lor}} + 2} - 1 \tag{5.122}$$

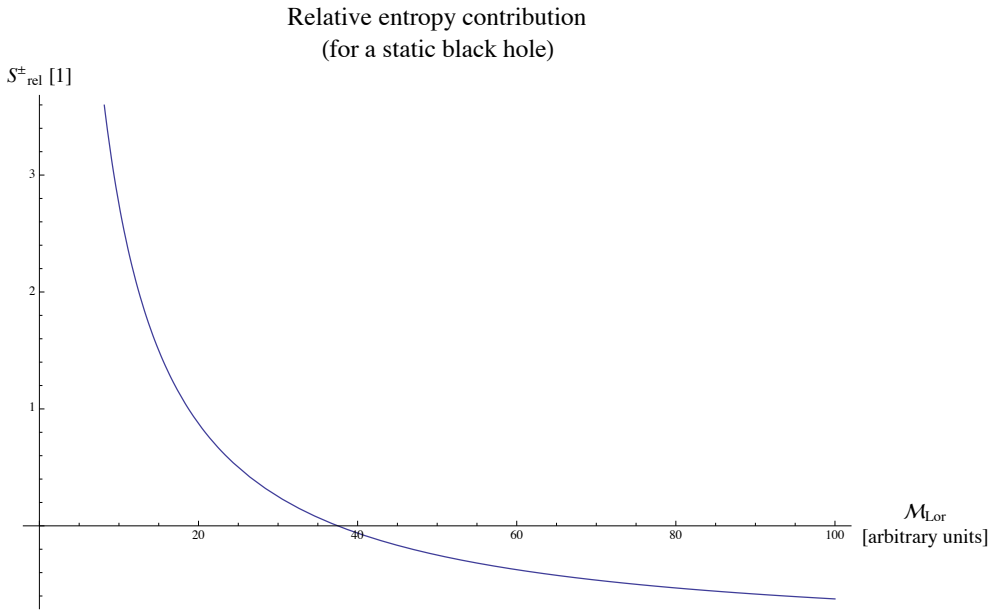


Figure 5.7: Relative energy contribution of the spin 4 field with respect to the spin 3 field for a static, Schwarzschild-like black hole. This plot nicely shows that the entropy contribution due to the spin 4 field dominates over the contribution over the spin 3 field for lower mass black holes.

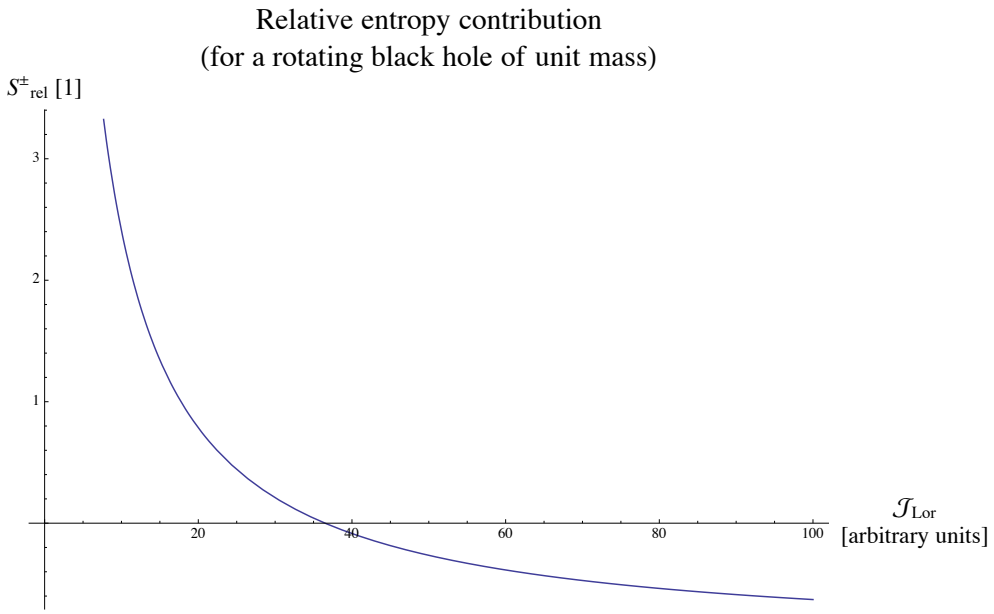


Figure 5.8: Relative energy contribution of the spin 4 field with respect to the spin 3 field for a rotating, Kerr-like black hole of unit mass. In this plot we see that the entropy contribution due to the spin 4 field dominates over the contribution over the spin 3 field for black holes with lower angular momentum.

We can thus state that a lower mass, quasi-static black hole has a higher entropy contribution due to the spin 4 field in the $SO(3,2) \times SO(3,2)$ theory than due to the spin 3 field in the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ theory.

Chapter 6

Conclusion

We shall conclude by giving a short summary of the ideas and results provided in this thesis followed by listing some interesting open questions.

Based on theorems of Jacobson-Morozov, Kostant and Mal'cev, the Dynkin-Kostant classification allows to classify the $\mathfrak{sl}(2,\mathbb{C})$ subalgebras in a classical semi-simple Lie algebra $\mathfrak{g}$. In this thesis, we have chosen $\mathfrak{g} = B_2 = \mathfrak{so}(5,\mathbb{C})$. This classification yields three non-equivalent embeddings of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{so}(5,\mathbb{C})$. We find that only one of these embeddings decomposes into irreducible representations such that one of the latter has dimension higher than 3. We therefore identify that specific embedding as a starting point for the construction of a higher spin black hole. We focus on the intersection of this embedding with the real form $\mathfrak{so}(3,2)$. This is constructed using Vogan diagrams.
Having found a standard triple of $\mathfrak{so}(3,2)$ in this way, we complete the set of generators of this Lie algebra and derive a black hole solution in the Chern-Simons theory based on the gauge group $SO(3,2)\times SO(3,2)$ under a set of imposed boundary conditions. We find that this black hole solution is endowed with a spin 4 charge. Finally, the thermodynamics of this black hole is studied focusing on the entropy contribution due to the higher spin charge in a weak field approximation. We draw the conclusion that the entropy contribution due to the spin 4 charge in the $SO(3,2) \times SO(3,2)$ Chern-Simons theory dominates over the spin 3 charge contribution in the $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ for quasi-static lower mass black holes.

This latter conclusion suggests immediately to compare the entropy contribution due to the spin 4 charges in both Chern-Simons theories, the one based on $SO(3,2) \times SO(3,2)$ and the one based on $SL(4,\mathbb{R}) \times SL(4,\mathbb{R})$, respectively.
Furthermore, it is natural to ask for an exact solution of the entropy as given e.g. in [2,26]. So far it turned out not to be straightforward to find an exact and simultaneously compact solution.
There are moreover two canonical generalizations of the investigation performed in this thesis: On the one hand one can study the extension to higher spin N theories in the context of type B_n Lie algebras analog to the generalization sketched in [2] for arbitrary $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons theories. The large N limit in this kind of generalization is particularly interesting due to the fact that is involved in the Gaberdiel-Gopakumar duality [24]. Further, as stated in [43], such a large N limit could probably simplify the comparison to Vasiliev theory [54,55] describing an infinite tower of interacting higher spin

gauge fields in AdS backgrounds of dimension four and higher. On the other hand, one could investigate this kind of Chern-Simons theories based on different finite dimensional gauge groups such as the classical Lie groups of Type C_n and D_n as well as the exceptional Lie groups such as e.g. G_2. Work done by D. Djokovic [56–58] serves as a starting point for analyses similar to the one presented in this thesis since a full list of standard triples for all exceptional Lie groups is provided there.

Acknowledgement

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Appendix A

Explicit Calculation of the Root System of $\mathfrak{sl}(3)$ (Example)

Let us look at an example now. Consider the simple Lie algebra $\mathfrak{sl}(3)$. It may be represented as the set of all traceless 3×3 matrices. The following set of matrices would hence be a basis:

$$\left\{ E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \ E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \right.$$
$$\left. E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \ H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

Accordingly, we get directly $\dim[\mathfrak{sl}(3)] = 8$. A choice of a maximal set of commuting Hermitian generators and hence a choice for the set of generators of the Cartan subalgebra of $\mathfrak{sl}(3)$ would be the subset containing the two diagonal matrices H_1 and H_2. Hence:

$$\mathfrak{h}_{\mathfrak{sl}(3)} = \langle H_1, H_2 \rangle_{\mathbb{C}} \tag{A.1}$$

As the Cartan subalgebra is obviously spanned by two elements, we have

$$\dim(\mathfrak{h}_{\mathfrak{sl}(3)}) = \operatorname{rank}[\mathfrak{sl}(3)] = 2.$$

Now we calculate the $8 - 2 = 6$ roots:

$$[H_1, E_{12}] = 2E_{12} \qquad [H_2, E_{12}] = -E_{12} \tag{A.2}$$
$$[H_1, E_{13}] = E_{13} \qquad [H_2, E_{13}] = E_{13} \tag{A.3}$$
$$[H_1, E_{21}] = -2E_{21} \qquad [H_2, E_{21}] = E_{21} \tag{A.4}$$
$$[H_1, E_{23}] = -E_{23} \qquad [H_2, E_{23}] = 2E_{23} \tag{A.5}$$
$$[H_1, E_{31}] = -E_{31} \qquad [H_2, E_{31}] = -E_{31} \tag{A.6}$$
$$[H_1, E_{32}] = E_{32} \qquad [H_2, E_{32}] = -2E_{32} \tag{A.7}$$

Hence, we find the following roots for $\mathfrak{sl}(3)$:

$$\begin{aligned} \alpha_1 &= (+2,-1), \quad \alpha_2 = (+1,+1), \quad \alpha_3 = (-2,+1), \\ \alpha_4 &= (-1,+2), \quad \alpha_5 = (-1,-1), \quad \alpha_6 = (+1,-2) \end{aligned} \tag{A.8}$$

Notice at this point that we can use these results in order to define linear maps according to

$$\begin{aligned} \alpha_\lambda : \ \mathfrak{h} &\to \mathbb{C} \qquad , \qquad \lambda = 1, \ldots, 6 \\ H_i &\mapsto \alpha_{\lambda,i}. \end{aligned} \tag{A.9}$$

For example we get $\alpha_1(H_1) = 2, \alpha_2(H_2) = 1$ or $\alpha_4(H_1) = -1$. We will come back to this. Let us first consider ways of visualizations of these roots. They can be pictured in a two-dimensional plane with standard Euclidean scalar product as follows:

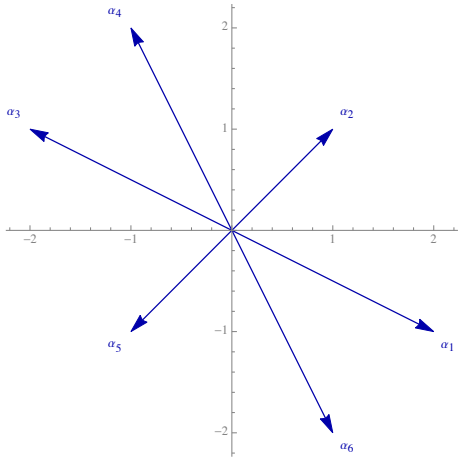


Figure A.1: The six vectors represent the roots of the Lie algebra $\mathfrak{sl}(3)$ in a two-dimensional plane with standard Euclidean scalar product.

The root spaces are 1-dimensional. They are spanned by the ladder operators, i.e.

$$\mathfrak{g}_{\alpha_\lambda} = \langle E_{\mu\nu} \rangle_{\mathbb{C}} \qquad \lambda, \mu, \nu = 1, \ldots, 6; \mu \neq \nu \tag{A.10}$$

That the root space decomposition holds true is now obvious:

$$\mathfrak{sl}(3) = \mathfrak{h} \oplus \bigoplus_{\lambda=1}^{6} \mathfrak{g}_{\alpha_\lambda} \tag{A.11}$$

$$= \langle H_1, H_2 \rangle_{\mathbb{C}} \oplus \bigoplus_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{6} \langle E_{\mu\nu} \rangle_{\mathbb{C}} \tag{A.12}$$

$$= \langle H_1, H_2, E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32} \rangle_{\mathbb{C}} \tag{A.13}$$

Next, let us choose a set of positive roots, e.g. $\{\alpha_2, \alpha_3, \alpha_4\}$. These are separated from the negative roots by a line cutting the plane in two half-spaces. In the left panel of figure A.1 this line is given by the horizontal axis. From this subset of positive roots we can further

extract a set of simple roots. Recall that all simple roots cannot be decomposed into a sum of simple roots with positive coefficients only. Hence, in our case, the simple roots are $\{\alpha_2, \alpha_3\}$, since $\alpha_4 = \alpha_2 + \alpha_3$. So, there are 2 simple roots in $\mathfrak{sl}(3)$, as expected. It is also clear, that this fact is independent of how we choose the set of positive roots. Further we have α_4 as the highest root in this choice of simple roots.

Let us now look for elements in the Cartan subalgebra $\mathfrak{h}$ that correspond to our chosen set of positive roots in the sense that the following equivalence holds true:

$$\alpha_\lambda \stackrel{\wedge}{=} H^{\alpha_\lambda} \quad \Leftrightarrow \quad K[H^{\alpha_\lambda}, H_i] = \alpha_\lambda(H_i) \tag{A.14}$$

where $H^\alpha \in \mathfrak{h}$ denotes the element of the Cartan subalgebra associated to the root $\alpha \in \mathfrak{h}^*$ and $K(\cdot,\cdot)$ is the Killing form. We claim:

$$H^{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H^{\alpha_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^{\alpha_4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.15}$$

These equivalences can be checked by direct calculation:

$$K[H^{\alpha_2}, H_1] = \mathrm{tr}\,(H^{\alpha_2}, H_1) = +1 = \alpha_2(H_1) \tag{A.16}$$
$$K[H^{\alpha_2}, H_2] = \mathrm{tr}\,(H^{\alpha_2}, H_2) = +1 = \alpha_2(H_2) \tag{A.17}$$
$$K[H^{\alpha_3}, H_1] = \mathrm{tr}\,(H^{\alpha_3}, H_1) = -2 = \alpha_3(H_1) \tag{A.18}$$
$$K[H^{\alpha_3}, H_2] = \mathrm{tr}\,(H^{\alpha_3}, H_2) = +1 = \alpha_3(H_2) \tag{A.19}$$
$$K[H^{\alpha_4}, H_1] = \mathrm{tr}\,(H^{\alpha_4}, H_1) = -1 = \alpha_4(H_1) \tag{A.20}$$
$$K[H^{\alpha_4}, H_2] = \mathrm{tr}\,(H^{\alpha_4}, H_2) = +2 = \alpha_4(H_2) \tag{A.21}$$

The Killing form thus provides a scalar product in the root space that shall be denotes by $\langle\cdot,\cdot\rangle$. Hence, we can write $K[H^{\alpha_i}, H^{\alpha_j}] = \langle\alpha_i, \alpha_j\rangle$. W. r. t. this scalar product, we can now compute the length of each root and the angles between them. For the length we obtain:

$$\langle\alpha_2, \alpha_2\rangle = \langle\alpha_3, \alpha_3\rangle = \langle\alpha_4, \alpha_4\rangle = 2 \tag{A.22}$$

For the angles we get:

$$\angle(\alpha_2, \alpha_3) = \frac{2\pi}{3}, \quad \angle(\alpha_3, \alpha_4) = \frac{\pi}{3}, \quad \angle(\alpha_2, \alpha_4) = \frac{\pi}{3} \tag{A.23}$$

This is enough information to draw a new picture where the roots are pictured again in a two-dimensional plane but this time w. r. t. the new scalar product $\langle\cdot,\cdot\rangle$ induced by the Killing form K:

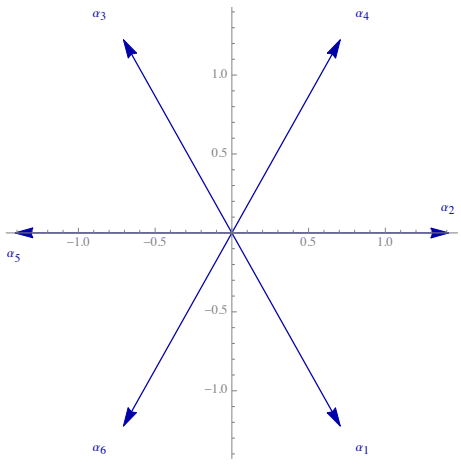


Figure A.2: The six vectors represent the roots of the Lie algebra $\mathfrak{sl}(3)$ in a two-dimensional plane with the scalar product induced by the Killing form. This regular hexagon is the characteristic root vector pattern for $\mathfrak{sl}(3)$.

Notice that the the our choice of positive roots is still valid and that both, the set of simple roots and the highest root are the same in the new space with the new scalar product as in the old space with the standard Euclidean scalar product. Expanded in the basis of simple roots, the roots have the following coordinates:

$$\begin{aligned} &\alpha_1 = (0,-1), \quad \alpha_2 = (1,0), \quad \alpha_3 = (0,1), \\ &\alpha_4 = (1,1), \quad \alpha_5 = (-1,0), \quad \alpha_6 = (-1,-1) \end{aligned} \tag{A.24}$$

By direct calculation it is easily verified that all six roots are found by successive subtraction of the simple roots $\{\alpha_2, \alpha_3\}$ from the highest root α_4:

$$\alpha_4 = \alpha_4 \tag{A.25}$$
$$\alpha_3 = \alpha_4 - \alpha_2 \tag{A.26}$$
$$\alpha_2 = \alpha_4 - \alpha_3 \tag{A.27}$$
$$\alpha_5 = \alpha_4 - 2\alpha_2 - \alpha_3 \tag{A.28}$$
$$\alpha_1 = \alpha_4 - 2\alpha_3 - \alpha_2 \tag{A.29}$$
$$\alpha_6 = \alpha_4 - 2\alpha_2 - 2\alpha_3 \tag{A.30}$$

The Weyl group is given by all reflections that send one root to another. This corresponds to the symmetry group of this hexagon which is given by the reflections w. r. t. the red lines s_1, s_2 and s_3 in the next figure. The triangular regions between two red rays are the Weyl chambers. The gray shaded region is the fundamental Weyl chamber.

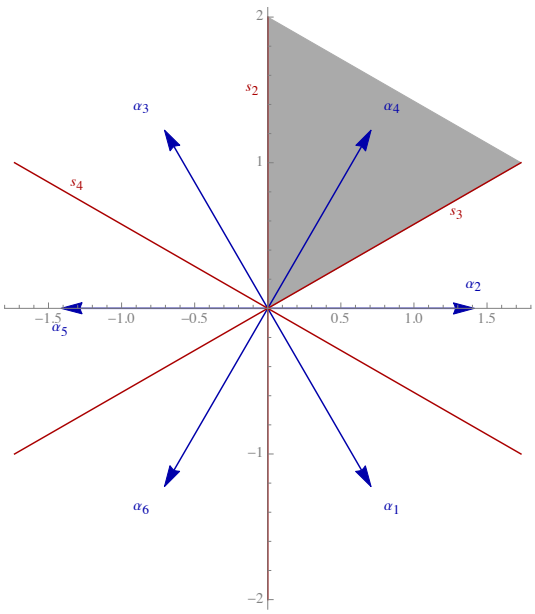


Figure A.3: The Weyl group of $\mathfrak{sl}(3)$ consists of all reflections with the red lines as the reflection hyperplanes and the gray shaded region is the fundamental Weyl chamber.

Let us compute the Weyl group explicitly: Recall the definition (2.2.7) of a Weyl reflection. First we obviously need to compute all the coroots. To do so, recall that all roots $\alpha_1, \dots, \alpha_6$ have length $\sqrt{2}$. Hence we obtain:

$$\alpha_\lambda^\vee = \frac{2\alpha_\lambda}{|\alpha_\lambda|^2} = \alpha_\lambda \qquad \forall\ \lambda = 1, \dots, 6 \tag{A.31}$$

Further, this leads to:

$$s_{\alpha_1}\alpha_1 = \alpha_1 - \langle \alpha_1^\vee, \alpha_1\rangle \alpha_1 = \alpha_1 - \langle \alpha_1, \alpha_1\rangle \alpha_1 \tag{A.32}$$
$$s_{\alpha_1}\alpha_2 = \alpha_2 - \langle \alpha_1^\vee, \alpha_2\rangle \alpha_1 = \alpha_2 - \langle \alpha_1, \alpha_2\rangle \alpha_1 \tag{A.33}$$
$$s_{\alpha_1}\alpha_3 = \alpha_3 - \langle \alpha_1^\vee, \alpha_3\rangle \alpha_1 = \alpha_3 - \langle \alpha_1, \alpha_3\rangle \alpha_1 \tag{A.34}$$
$$s_{\alpha_1}\alpha_4 = \alpha_4 - \langle \alpha_1^\vee, \alpha_4\rangle \alpha_1 = \alpha_4 - \langle \alpha_1, \alpha_4\rangle \alpha_1 \tag{A.35}$$
$$s_{\alpha_1}\alpha_5 = \alpha_5 - \langle \alpha_1^\vee, \alpha_5\rangle \alpha_1 = \alpha_5 - \langle \alpha_1, \alpha_5\rangle \alpha_1 \tag{A.36}$$
$$s_{\alpha_1}\alpha_6 = \alpha_6 - \langle \alpha_1^\vee, \alpha_6\rangle \alpha_1 = \alpha_6 - \langle \alpha_1, \alpha_6\rangle \alpha_1 \tag{A.37}$$

Now we need to compute the scalar products:

$$\begin{aligned} H^{\alpha_1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H^{\alpha_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ H^{\alpha_4} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H^{\alpha_5} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H^{\alpha_6} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{A.38}$$

Remark Notice that we now have found the relations $H^{\alpha_1} = -H^{\alpha_3}$, $H^{\alpha_2} = -H^{\alpha_5}$ and $H^{\alpha_4} = -H^{\alpha_6}$. These can be found in several ways: We can compute them directly or read them off from one of the figures shown above or from the relations (A.8) or (A.24).

The scalar products are given by:

$$\begin{aligned}
\langle\alpha_1,\alpha_1\rangle &= K(H^{\alpha_1},H^{\alpha_1}) = 2, & \langle\alpha_1,\alpha_2\rangle &= K(H^{\alpha_1},H^{\alpha_2}) = 1,\\
\langle\alpha_1,\alpha_3\rangle &= K(H^{\alpha_1},H^{\alpha_3}) = -2, & \langle\alpha_1,\alpha_4\rangle &= K(H^{\alpha_1},H^{\alpha_4}) = -1,\\
\langle\alpha_1,\alpha_5\rangle &= K(H^{\alpha_1},H^{\alpha_5}) = -1, & \langle\alpha_1,\alpha_6\rangle &= K(H^{\alpha_1},H^{\alpha_6}) = 1
\end{aligned} \tag{A.39}$$

Thus we have:

$$\begin{aligned}
s_{\alpha_1}\alpha_1 &= \alpha_1 - 2\alpha_1 = \alpha_3, & s_{\alpha_1}\alpha_2 &= \alpha_2 - \alpha_1 = \alpha_4,\\
s_{\alpha_1}\alpha_3 &= \alpha_3 + 2\alpha_1 = \alpha_1, & s_{\alpha_1}\alpha_4 &= \alpha_4 + \alpha_1 = \alpha_2,\\
s_{\alpha_1}\alpha_5 &= \alpha_5 + \alpha_1 \ = \alpha_6, & s_{\alpha_1}\alpha_6 &= \alpha_6 - \alpha_1 = \alpha_5
\end{aligned} \tag{A.40}$$

These results correspond to the reflection w. r. t. the hyperplane s_3 in figure (A.3) as expected since s_3 is the hyperplane perpendicular to α_1. Without computing the remaining results explicitly, we anticipate the relations:

$$s_{\alpha_1} \leftrightarrow s_{\alpha_3} \leftrightarrow R(s_3) \tag{A.41}$$

$$s_{\alpha_2} \leftrightarrow s_{\alpha_5} \leftrightarrow R(s_2) \tag{A.42}$$

$$s_{\alpha_4} \leftrightarrow s_{\alpha_6} \leftrightarrow R(s_4) \tag{A.43}$$

where $R(s_i)$ denotes the reflection w. r. t. s_i. The simple Weyl reflections are accordingly the reflections w. r. t. the simple roots, i.e. $\{R(s_2), R(s_3)\}$. Since the Weyl group W is generated by the simple Weyl reflections, we have now found the entire group:

$$W = \langle\{R(s_2), R(s_3)\}\rangle \tag{A.44}$$

$$= \{\mathrm{Id}, R(s_2), R(s_3), R(s_2)R(s_3), R(s_3)R(s_2), R(s_3)R(s_2)R(s_3)\} \tag{A.45}$$

Appendix B

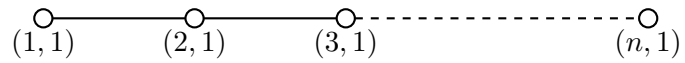
Dynkin Diagrams of the Simple Lie Algebras

In this subsection a list of all nine classes of Dynkin Diagrams is provided. This list and more information about the structure of simple Lie algebras can be found in [27].

B.1 Classical Lie Algebras

B.1.1 Type A_n:

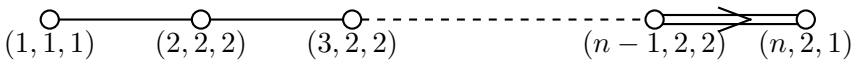
This class of complex simple Lie algebras comprises the special unitary algebras of rank $n \geq 2$, i.e. $\mathfrak{sl}(n)$. The dimension is $d = n(n+2)$. The Dynkin Diagram looks as follows:



The numbers next to the nodes are the number of the node, its mark (and in the cases below the third number will be the comark). This is the simplest case of a simple Lie algebra: There are n nodes, each pair joined by exactly one single edge. Accordingly, these algebras feature n simple roots, they are simply laced and each pair of roots sustain an angle of $2\pi/3 = 120°$. All roots that are not directly joined by an edge are orthogonal to each other.

B.1.2 Type B_n:

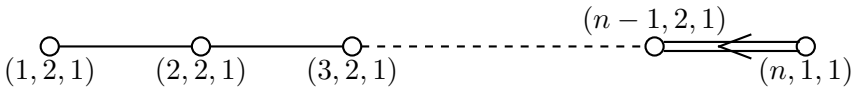
This is the class of special orthogonal, complex algebras of rank $n \geq 3$ and dimension $d = n(2n+1)$, i.e. $\mathfrak{so}(2n+1)$.



Of course, there are again n nodes, but these algebras are not simply laced because there are $n-1$ long simple roots and a single short one (notice the arrow pointing towards a single node). The $n-1$ long roots sustain again pairwise an angle of $2\pi/3$, while the short and a long root sustain an angle of $3\pi/4 = 135°$.

B.1.3 Type C_n:

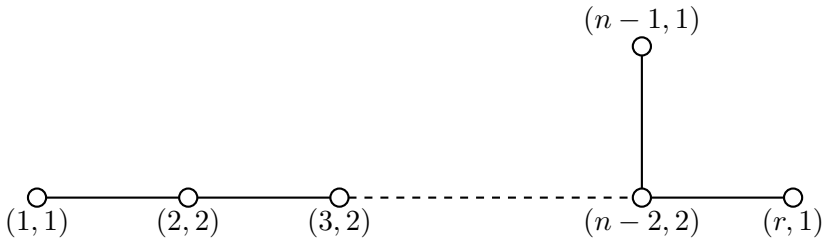
The type C_n is the class of symplectic algebras of rank $n \geq 2$ and again of dimension $d = n(2n+1)$, i.e. $\mathfrak{sp}(2n)$:



These algebras have a very similar structure compared with the type B_n algebras. The difference is that in C_n-type algebras there are $n-1$ short and only one single long root.

B.1.4 Type D_n:

The special orthogonal, complex algebras of rank $n \geq 4$, $\mathfrak{so}(2n)$ having dimension $d = n(2n-1)$ form the algebras of type D_n:

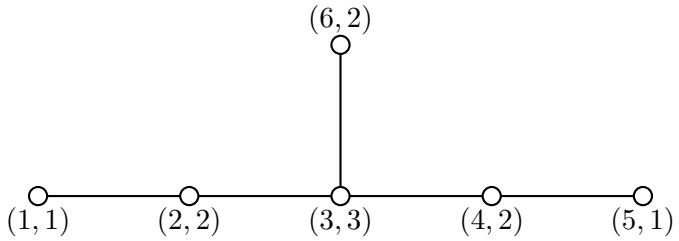


These algebras are simply laced. There is an obvious new feature: These algebras have a distinct simple root that sustains an angle of $2\pi/3$ not only with one other root (as it was the case in all other types so far) but with two.

B.2 Exceptional Lie Algebras

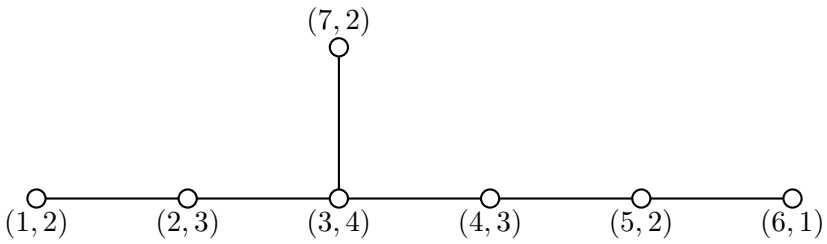
B.2.1 Type E_6:

As the index suggests, this algebra has rank 6. It has dimension $d = 78$ and is simply laced. Its Dynkin Diagram is given by



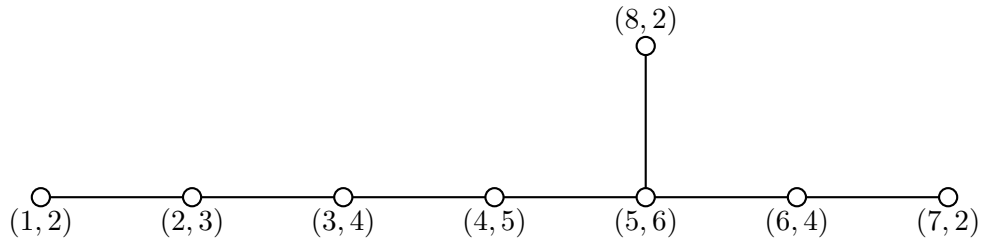
B.2.2 Type E_7:

The next algebra, E_7, has rank 7, dimension d=133 and is also simply laced.



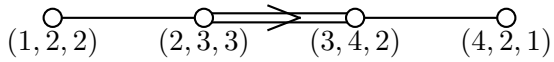
B.2.3 Type E_8:

The Lie algebra E_8 is the largest execptional Lie algebra. It is simply laced, has rank 8 and is of dimension $d = 248$.



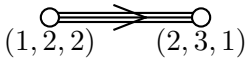
B.2.4 Type F_4:

This Lie algebra is one of the two non-simply laced exceptional Lie algebras. It has rank 4 and dimensiion $d = 52$.



B.2.5 Type G_2:

G_2 is the smallest of all exceptional Lie groups. It is not simply laced, has rank 2 and dimension $d = 14$.



Appendix C

Standard Triple Construction Algorithms for Lie Algebras of Type A_n, C_n and D_n

C.1 Construction Algorithm for Type A_n Algebras

Step 1: Parametrization of Nilpotent Orbits
We start by stating a theorem due to M. Gerstenhaber:

Theorem C.1.1 *Nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$ are in 1-to-1 correspondence with the set $\mathcal{P}(n)$ of partitions of n [38].*

This yields a natural parametrization of the nilpotent orbits if we write $\mathcal{O}_{[d_1,\ldots,d_r]}$ for an orbit that is attached to the partition $[d_1,\ldots,d_r]$.

Example According to the previous theorem, all nilpotent orbits in $\mathfrak{sl}(3)$ are in 1-to-1 correspondence with the set $\mathcal{P}(3) = \{[3],[2,1],[1^3]\}$. Since this set has only three elements, consequently, there are only three nilpotent orbits. Since moreover the zero-orbit corresponds to the partition $[1^3]$ and since the zero-orbit is not attached to a standard triple, we are left with the orbits $\mathcal{O}_{[3]}$ and $\mathcal{O}_{[2,1]}$ that are in relation with such triples. This explains why at the end of subsection 2.4 where the weighted Dynkin diagrams were introduced, it was stated in the example that although there are nine different labelings for the diagrams, there are in fact only three diagrams that correspond to a nilpotent orbit and even only two of them are in relation with a standard triple.

Step 2: Cartan Subalgebra & Root Spaces
We choose the Cartan subalgebra $\mathfrak{h}_{A_n}$ to be the set of traceless diagonal matrices of side length n. An obvious basis of this set is given by

$$B(\mathfrak{h}_{A_n}) = \{\mathrm{diag}(d_1,\cdots,d_n) \mid d_i = 1 \wedge 1 \leq i < n-1 \;\&\; d_j = -1 \wedge i < j \leq n\} \tag{C.1}$$

Then one defines linear functionals $e_i \in \mathfrak{h}^*_{A_n}, 1 \leq i \leq n$, by the rule

$$\begin{aligned} e_i : \mathfrak{h}_{A_n} &\to \mathbb{C} \\ H &\mapsto d_i \end{aligned} \tag{C.2}$$

where d_i stands for the i-th diagonal entry of H, i.e. $e_i(H) = d_i$. Then, the root system of $\mathfrak{g} = \mathfrak{sl}(n)$ is given by the set

$$\{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\}. \tag{C.3}$$

We can now choose $\Delta_\mathfrak{g} = \{e_i - e_j \mid i < j\}$ as the set of positive roots. The $(e_i - e_j)$-root space is spanned by the elementary matrix $E_{i,j}$ having a 1 as its ij-entry and zeros elsewhere.

Example Consider $\mathfrak{sl}(5)$ as an example. Choosing the traceless diagonal matrices as the Cartan subalgebra, the following four diagonal matrices form a basis of it:

$$\begin{aligned} B(\mathfrak{h}_{\mathfrak{sl}(5)}) = \{&H_1 = \operatorname{diag}(1,-1,0,0,0), H_2 = \operatorname{diag}(0,1,-1,0,0), \\ &H_3 = \operatorname{diag}(0,0,1,-1,0), H_4 = \operatorname{diag}(0,0,0,1,-1)\} \end{aligned} \tag{C.4}$$

The linear functionals are the $e_1, \ldots, e_5$ where e.g. $e_1(H_1) = 1$ and $e_4(H_3) = -1$ and so on. The positive roots are

$$\Delta_{\mathfrak{sl}(5)} = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5\} \tag{C.5}$$

such that for instance

$$(e_2 - e_3)(H_3) = e_2(H_3) - e_3(H_3) = 0 - 1 = -1 \tag{C.6}$$

Further, we can identitfy the root $e_2 - e_3$ with the elementary matrix E_{23}. Let us convince ourselves that this identification is valid. According to definition 2.2.1 the equation $[H_3, E_{23}] = (e_2 - e_3)(H_3)E_{23}$ must hold.

$$[H_3, E_{23}] = \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0 \end{pmatrix} \tag{C.7}$$

$$= \begin{pmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \tag{C.8}$$

$$= \begin{pmatrix} 0&0&0&0&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix} \tag{C.9}$$

$$= -E_{23} \tag{C.10}$$

$$= (e_2 - e_3)(H_3)E_{23} \tag{C.11}$$

Step 3: Attaching Explicit Standard Triples
Take $\mathbf{d} \in \mathcal{P}(n)$, i.e. $\mathbf{d}$ is a certain partition of n. We shall now take this partition and break it up into "chunks" $\{d_i\}$, where each chunk contains the part d_i of the partition $\mathbf{d}$ as often as its multiplicity dictates. For each chunk $\{d_i\}$, choose a block of consecutive indices $\{N_i + 1, \ldots, N_i + d_i\} \subset \{1, \ldots, n\}$ in such a way that disjoint blocks are attached to distinct chunks. To every chunk $\{d_i\}$ attach the set of simple roots given by

$$\mathcal{C}^+ = \mathcal{C}^+(d_i) = \{e_{N_i+1} - e_{N_i+2}, e_{N_i+2} - e_{N_i+3}, \ldots, e_{N_i+d_i-1} - e_{N_i+d_i}\}. \tag{C.12}$$

Notice that this set $\mathcal{C}^+$ is empty whenever $d_i = 1$. For every simple root α in $\mathcal{C} := \bigcup_i \mathcal{C}^+(d_i)$, let E^α be an α-root vector. Let X be the sum of the E^α. The nilnegative element of the standard triple Y is obtained by the sum of $E^{-\alpha}$, one for every $\alpha \in \mathcal{C}$. The last element of the standard triple $H \in \mathfrak{h}$ is then given by

$$H = \sum_i H_{\mathcal{C}(d_i)} \qquad | \quad H_{\mathcal{C}(d_i)} = \sum_{i \le l \le d_i} (d_i - 2l + 1) E_{N_i+l, N_i+l}. \tag{C.13}$$

Example In the example of $\mathfrak{sl}(5)$ the set of partitions is given by

$$\mathcal{P}(5) = \{[5], [4,1], [3,2], [3,1^2], [2^2,1], [2,1^3], [1^5]\}. \tag{C.14}$$

Now we choose one partition from this set, say $\mathbf{d} = [2^2, 1]$. There are now two chunks in this partition given by $\{1\}$, $\{2,2\}$ where the second chunk contains the part 2 twice. We may split the set of indeces $\{1,2,3,4,5\}$ into blocks corresponding to these chunks in the following way:

$$\{1\}, \{\{2,3\}, \{4,5\}\} \quad \Rightarrow \quad \begin{cases} N_1 = 0 \\ \{N_2^1 = 1, N_2^2 = 3\} \end{cases} \tag{C.15}$$

Here, we have chosen the first block of indices to correspond to the chunk $\{1\}$ (notice that the chunk and the block of indices are not the same, although they appear the same way here) while the second block corresponds to the chunk $\{2,2\}$ with multiplicity 2. The corresponding set of positive roots is thus given by:

$$\mathcal{C}^+(1) = \{\ \} \tag{C.16}$$

$$\mathcal{C}_1^+(2) = \{\underbrace{e_2 - e_3}_{\alpha_2}\} \tag{C.17}$$

$$\mathcal{C}_2^+(2) = \{\underbrace{e_4 - e_5}_{\alpha_4}\} \tag{C.18}$$

We know take the unification of these sets

$$\mathcal{C} = \{\underbrace{e_2 - e_3}_{\alpha_2}, \underbrace{e_4 - e_5}_{\alpha_4}\} = \{\alpha_2, \alpha_4\} \tag{C.19}$$

This leads further to two root-vector $E^{\alpha_2} = E_{23}$ and $E^{\alpha_4} = E_{45}$. The nilpositive element of our standard triple is given by the sum of these root vectors:

$$X = E_{23} + E_{45} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.20}$$

The nilnegative element is given by the sum of $E^{-\alpha_2} = E_{32}$ and $E^{-\alpha_4} = E_{54}$:

$$Y = E_{32} + E_{54} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{C.21}$$

Finally, the neutral element is given according to the rule mentioned above by

$$\begin{aligned} H &= H_{\mathcal{C}_1^+(2)} + H_{\mathcal{C}_2^+(2)} && \text{(C.22)} \\ &= (2-2+1)E_{22} + (2-4+1)E_{33} + (2-2+1)E_{44} + (2-4+1)E_{55} && \text{(C.23)} \\ &= E_{22} - E_{33} + E_{44} - E_{55} && \text{(C.24)} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} && \text{(C.25)} \end{aligned}$$

It can be easily veryfied by direct calculation that these three matrices X, Y and H obey the defining commutation relations (2.14) of $\mathfrak{sl}(2)$. Thus we conclude that

$$\left\{ X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \right\} \tag{C.26}$$

span a $\mathfrak{sl}(2)$-subalgebra in $\mathfrak{sl}(5)$ and thus form the standard triple associated to the nilpotent orbit $\mathcal{O}_{[2^2,1]}$.

C.2 Construction Algorithm for Type C_n Algebras

Step 1: Parametrization of Nilpotent Orbits
The last parametrization theorem due to Gerstenhaber consideres the symplectitc algebras:

Theorem C.2.1 *Nilpotent orbits in $\mathfrak{sp}_{2n}$ are in 1:1 correspondence with the set of partitions of $2n$ in which odd parts occur with even multiplicity [38].*

Example Consider the symplectic algebra $\mathfrak{sp}(6)$. All possible partitions of 6 form the set

$$\mathcal{P}(6) = \{[6], [5,1], [4,2], [4,1^2], [3^2], [3,2,1], [3,1^3], [2^3], [2^2,1^2], [2,1^4], [1^6]\} \tag{C.27}$$

Removing the partitions from this set in which odd parts occur with odd multiplicity yields:

$$\tilde{\mathcal{P}}(6) = \{[6], [4,2], [4,1^2], [3^2], [2^3], [2^2,1^2], [2,1^4], [1^6]\} \tag{C.28}$$

Step 2: Cartan Subalgebra & Root Spaces
Any element of such a symplectic complex Lie algebra $\mathfrak{sp}(2n)$ can be represented as a matrix of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \quad \text{with} \quad Z_i \in M_n(\mathbb{C}) \text{ and } Z_2, Z_3 \text{ symmetric} \tag{C.29}$$

We choose again the diagonal matrices to form the Cartan subalgebra $\mathfrak{h}_{C_n}$ whose elements take a similar form as previously in the case of B_n algebras:

$$H = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \quad | \quad D = \begin{pmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_n \end{pmatrix} \tag{C.30}$$

Now we define linear functionals $e_i \in \mathfrak{h}_{C_n}^*$ by

$$e_i(H) = h_i \qquad 1 \leq i \leq n \tag{C.31}$$

Thus, e_i picks out the 1^{st} diagonal entry of a matrix in $H \in \mathfrak{h}_{C_n}$. The root system of $\mathfrak{g}$ is now

$$\{\pm e_i \pm e_j, \pm 2e_i \mid 1 \leq i, j \leq n, i \neq j\}. \tag{C.32}$$

The standard choice for the set of positive roots is

$$\Delta_{\mathfrak{g}} = \{e_i \pm e_j, 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}. \tag{C.33}$$

The α-root space of $\mathfrak{g}$ is spanned by the vector E_α given by

$$\begin{aligned} E_{e_i - e_j} &= E_{i,j} - E_{j+n,i+n} \\ E_{e_i + e_j} &= E_{i,j+n} - E_{j,i+n} \quad (i < j) \\ E_{-e_i - e_j} &= E_{i+n,j} - E_{j+n,i} \quad (i < j) \\ E_{2e_i} &= E_{i,i+n} \\ E_{-2e_i} &= E_{i+n,i} \end{aligned} \tag{C.34}$$

Example Let us consider the algebra $\mathfrak{sp}(6)$ such that $n = 3$. The Cartan algebra $\mathfrak{h}_{\mathfrak{sp}(6)}$ has then the basis

$$\begin{aligned} B(\mathfrak{h}_{\mathfrak{sp}(6)}) = \{&H_1 = \mathrm{diag}(1,0,0,-1,0,0), H_2 = \mathrm{diag}(0,1,0,0,-1,0), \\ &H_3 = \mathrm{diag}(0,0,1,0,0,-1)\} \end{aligned} \tag{C.35}$$

The linear functionals are

$$e_1(H_1) = 1, \qquad e_1(H_2) = 0, \qquad e_1(H_3) = 0, \tag{C.36}$$
$$e_2(H_1) = 0, \qquad e_2(H_2) = 1, \qquad e_2(H_3) = 0, \tag{C.37}$$
$$e_3(H_1) = 0, \qquad e_3(H_2) = 0, \qquad e_3(H_3) = 1 \tag{C.38}$$

which yields the following set of positive roots:

$$\Delta_{\mathfrak{sp}(6)} = \{e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3, 2e_1, 2e_2, 2e_3\} \tag{C.39}$$

Step 3: Attaching Explicit Standard Triples
Given a partition **d** according to the parametrization theorem C.2.1, break it up into chunks of the following two types, similar to as we did for B_n-type algebras:

- pairs $\{2r+1, 2r+1\}$ of equal odd parts
- single even parts $\{2q\}$

We again attach sets of roots to these chunks, but in contrast to the previous cases, the assigned roots are only positive and need not to be simple.

- For pairs $\{2r+1, 2r+1\}$: Choose a block $\{l+1, \ldots, l+2r+1\}$ of consecutive indices and let

$$\mathcal{C}^+(2r+1, 2r+1) = \{e_{l+1} - e_{l+2}, e_{l+2} - e_{l+3}, \ldots, e_{l+2r} - e_{l+2r+1}\} \tag{C.40}$$

  be the corresponding set of roots. Notice that this set is empty if $r = 0$ such that the corresponding chunk is $\{1, 1\}$.

- For single even parts $\{2q\}$: Choose a block $\{j+1, \ldots, j+q\}$ of consecutive indices and let

$$\mathcal{C}^+(2q) = \{e_{j+1} - e_{j+2}, e_{j+2} - e_{j+3}, \ldots, e_{j+q-1} - e_{j+q}, 2e_{j+q}\} \tag{C.41}$$

As always the assigned blocks of consecutive indices most be disjoint for distinct chunks. The computation of the standard triple elements proceeds as in the previous two cases: We take take the unification $\mathcal{C} := \bigcup_r \mathcal{C}^+(2r+1, 2r+1) \cup \bigcup_q \mathcal{C}^+(2q)$ and let E_α be an α-root vector for each $\alpha \in \mathcal{C}$. The nilpositive standard triple element X is given by the sum of all these α-root vectors and the nilnegative element Y by the sum of the corresponding $-\alpha$-root vectors. The neutral element is given by $H = \sum_{\mathcal{C}} H_{\mathcal{C}}$ where

$$H_{\mathcal{C}} := \sum_{l=1}^{q} (2q - 2l + 1)(E_{j+l,j+l} - E_{n+j+l,n+j+l}) \tag{C.42}$$

if $\mathcal{C}$ is of the type $\mathcal{C}^+(2q)$ and

$$H_{\mathcal{C}} := \sum_{m=0}^{2r} (2r - 2m)(E_{l+1+m,l+1+m} - E_{n+l+1+m,n+l+1+m}) \tag{C.43}$$

if $\mathcal{C}$ is of the type $\mathcal{C}^+(2r+1, 2r+1)$.

C.3 Construction Algorithm for Type D_n Algebras

Step1: Parametrization of Nilpotent Orbits
The parametrization theorem considering the D_n-type algebras is due tue Springer and Steinberg:

Theorem C.3.1 *Nilpotent orbits in $\mathfrak{so}_{2n}$ are parametrized by partitions of $2n$ in which even parts occur with even multiplicity, excepts that "very even" partitions $[d_1, \ldots, d_{2n}]$ (those with only even parts, each having even multiplicity) correspond to two orbits, denoted $\mathcal{O}^I_{[d_1,\ldots,d_{2n}]}$ and $\mathcal{O}^{II}_{[d_1,\ldots,d_{2n}]}$ [39].*

Example Here we shall consider the complex Lie algebra $\mathfrak{so}(8)$. All 22 partitions of 8 are in the set

$$\begin{aligned}\mathcal{P}(8) =&\{[8],[7,1],[6,2],[6,1^2],[5,3],[5,2,1],[5,1^3],\\ &[4^2],[4,3,1],[4,2^2],[4,2,1^2],[4,1^4],[3^2,2],[3^2,1^2],\\ &[3,2^2,1],[3,2,1^3],[3,1^5],[2^4],[2^3,1^2],[2^2,1^4],[2,1^6],[1^8]\}\end{aligned} \tag{C.44}$$

Removing all partitions whose even parts occur with odd multiplicity leads to the following set of 10 relevant partitions:

$$\tilde{\mathcal{P}}(8) = \{[7,1],[5,3],[5,1^3],[4^2],[3^2,1^2],[3,2^2,1],[3,1^5],[2^4],[2^2,1^4],[1^8]\} \tag{C.45}$$

Matching two orbits with the "very even" partitions $[4^2]$ and $[2^4]$ leads to the 12 nilpotent orbits:

$$\mathcal{O}_{[7,1]},\mathcal{O}_{[5,3]},\mathcal{O}_{[5,1^3]},\mathcal{O}^I_{[4^2]},\mathcal{O}^{II}_{[4^2]},\mathcal{O}_{[3^2,1^2]},\mathcal{O}_{[3,2^2,1]},\mathcal{O}_{[3,1^5]},\mathcal{O}^I_{[2^4]},\mathcal{O}^{II}_{[2^4]},\mathcal{O}_{[2^2,1^4]},\mathcal{O}_{[1^8]} \tag{C.46}$$

Step 2: Cartan Subalgebra & Root Spaces
Algebras $\mathfrak{g}$ of type D_n can be realized in the form

$$\left\{\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \in M_n(\mathbb{C}), Z_2, Z_3 \text{ skew-symmetric}\right\}, \tag{C.47}$$

i.e. as the set of skew-adjoint matrices relative to the form

$$z_1 z_{n+1} + \cdots + z_n z_{2n} \text{ on } \mathbb{C}^{2n} \tag{C.48}$$

As in all cases before, we the Cartan subalgebra $\mathfrak{h}_\mathfrak{g}$ shall be the set of diagonal matrices which take exactly the same form as in the case of the type C_n algebras and the linear functionals are as well defined as in that case. The set of positive roots is the taken to be

$$\Delta_\mathfrak{g} = \{e_i \pm e_j \mid 1 \leq i,j \leq n, i \neq j\} \tag{C.49}$$

The α-root vectors E_α are then given by

$$E_{e_i - e_j} = E_{i,j} - E_{nj,n+i} \tag{C.50}$$
$$E_{e_i + e_i} = E_{i,n+j} - E_{j,n+i} \tag{C.51}$$
$$E_{-e_i - e_j} = E_{n+i,j} - E_{n+j,i} \tag{C.52}$$

Step 3: Attaching Explicit Standard Triples
We start again with a partition $\mathbf{d} \in \mathcal{P}(2n)$ according to the parametrization theorem C.3.1, break it up into chunks as in the case of B_n-type algebras with one difference: In that case we required that there must be one unique chunk of a single odd part $\{2u+1\}$. This condition is relaxed here. Now let us consider two cases separately:

- If $\mathbf{d}$ is not very even: We construct the standard triple $\{X,Y,H\}$ exactly as for the $\mathfrak{so}(2n+1) = B_n$ algebras under the adaption that all indices in the formulae for $H_\mathcal{C}$ need to be reduces by one.

- If $\mathbf{d}$ is very even: In the theorem C.3.1 it was mentioned that for very even partitions a Roman numeral I or II must be attached in order to establish a correspondence to a unique nilpotent orbit. Here we shall state the rules how to handle these numerals:

  – If the numeral is I: Again proceed as in the case of B_n-algebras for constructing the standard triples, in particular, the neutral element of the standard triple is defined according to expression (3.20) (where r is replaced by $2r$ just emphasizing that r is an even number and the indices are reduced by 1 as already required)

$$H_{\mathcal{C}} = \sum_{k=1}^{2r} (2r-2k+1)(E_{j+k,j+k} - E_{n+j+k,n+j+k}) \tag{C.53}$$

$$\begin{aligned} &= \sum_{k=1}^{2r-1} (2r-2k+1)(E_{j+k,j+k} - E_{n+j+k,n+j+k}) \\ &\quad + (-2r+1)(E_{j+2r,j+2r} - E_{n+j+2r,n+j+2r}) \end{aligned} \tag{C.54}$$

  – If the numeral is II: As all chunks in this case are of the form $\{2r, 2r\}$, choose an arbitrary chunk $\mathcal{C}$ and attach a block $\{j+1, \ldots, j+2r\}$ of indices. Now replace the positive root $e_{j+2r-1} - e_{2j+r}$ by ${}_{j+2r-1} + e_{2j+r}$ while the rest of the roots remains unchanged. Finally, the construction of the nilpositive and nilnegative elements X and Y of the standard triple remains unaltered. There is, however, a slight difference in the computation of the neutral element compared to the equation (3.20):

$$\begin{aligned} H_{\mathcal{C}} &= \sum_{k=1}^{2r-1} (2r-2k+1)(E_{j+k,j+k} - E_{n+j+k,n+j+k}) \\ &\quad + (2r-1)(E_{j+2r,j+2r} - E_{n+j+2r,n+j+2r}) \end{aligned} \tag{C.55}$$

  Obviously, the coefficient of the last term changed from $-2r+1$ to $2r-1$.

Appendix D

Explicit Calculation in the Case of Pure Gravity

D.1 Preparation

Consider the $\mathfrak{sl}(2,\mathbb{R})$ algebra with the basis $\{L_{-1}, L_0, L_{+1}\}$ (that obeys the Virasoro relation) and two connections living there: $A^{\pm} = A^{\pm}_r dr + A^{\pm}_\varphi d\varphi + A^{\pm}_t dt$, where

$$A^{\pm}_\varphi(r,\varphi) \overset{r\to\infty}{\longrightarrow} A^{\pm}_\varphi(r,\varphi) = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1}. \tag{D.1}$$

In turn

$$\mathcal{L}^{\pm}(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{L}^{\pm}(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{D.2}$$

and the most general from of A_t is given by

$$A^{\pm}_t(r,\varphi) \overset{r\to\infty}{\longrightarrow} A^{\pm}_t(r,\varphi) = \pm\big(\xi_{\mp 1}(r,\varphi)L_{\mp 1} + \xi_0(r,\varphi)L_0 + \xi_{\pm 1}(r,\varphi)L_{\pm 1}\big) \tag{D.3}$$

Last but not least, the radial component behaves as

$$A^{\pm}_r \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{D.4}$$

D.2 Equation of Motion: The Flatness Condition

Let us start with the Chern-Simons action which is given by:

$$I_{CS}[A^{\pm}] = \frac{k}{4\pi}\int \mathrm{tr}\left(A^{\pm}\wedge dA^{\pm} + \frac{2}{3}A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right) \tag{D.5}$$

In order to find the equations of motion for the fields $A^{\pm}$ we perform the usual trick: extremize the action.

$$\delta I_{CS}[A^{\pm}] = \frac{k}{4\pi}\int \mathrm{tr}\left(\delta A^{\pm}\wedge dA^{\pm} + A^{\pm}\wedge d\delta A^{\pm} + 2\delta A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right) \tag{D.6}$$

In this step we used for the second term the linearity of δ and d and the consequence that they hence commute with each other and in the third term the fact that the trace is

invariant under cyclic permutation of its arguments. Next we integrate by parts (which is possible since the integral is a linear operator, too, and hence commutes with the trace) and get:

$$\delta I_{CS}[A^{\pm}] = \frac{k}{4\pi}\int \mathrm{tr}\left(\delta A^{\pm}\wedge dA^{\pm} + \delta A^{\pm}\wedge dA^{\pm} + 2\delta A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right) \tag{D.7}$$

$$= \frac{k}{2\pi}\int \mathrm{tr}\left(\delta A^{\pm}\wedge dA^{\pm} + \delta A^{\pm}\wedge A^{\pm}\wedge A^{\pm}\right) \tag{D.8}$$

$$= \frac{k}{2\pi}\int \mathrm{tr}\left(\delta A^{\pm}\wedge (dA^{\pm} + A^{\pm}\wedge A^{\pm})\right) \tag{D.9}$$

The curvature in the Chern-Simons theory is defined as

$$F^{\pm} = dA^{\pm} + A^{\pm}\wedge A^{\pm}. \tag{D.10}$$

So we recognize that

$$\delta I_{CS}[A^{\pm}] = \frac{k}{2\pi}\int \mathrm{tr}\left(\delta A^{\pm}\wedge F^{\pm}\right) \stackrel{!}{=} 0. \tag{D.11}$$

Due to the non-degeneracy of the trace the equation of motion for the connection fields $A^{\pm}$ reads as

$$F^{\pm} = dA^{\pm} + A^{\pm}\wedge A^{\pm} \stackrel{!}{=} 0 \tag{D.12}$$

For the rest of this section we will omit the superscript "$\pm$", it shall be understood anyway.

$$F = dA + A\wedge A \tag{D.13}$$

$$= d(A_r dr + A_\varphi d\varphi + A_t dt) + (A_r dr + A_\varphi d\varphi + A_t dt)\wedge(A_r dr + A_\varphi d\varphi + A_t dt) \tag{D.14}$$

$$\begin{aligned} &= dA_r\wedge dr + dA_\varphi\wedge d\varphi + dA_t\wedge dt + A_r^2 dr\wedge dr + A_\varphi^2 d\varphi\wedge d\varphi + A_t^2 dt\wedge dt \\ &\quad + A_r A_\varphi dr\wedge d\varphi + A_r A_t dr\wedge dt + A_\varphi A_r d\varphi\wedge dr + A_\varphi A_t d\varphi\wedge dt \\ &\quad + A_t A_r dt\wedge dr + A_t A_\varphi dt\wedge d\varphi \end{aligned} \tag{D.15}$$

$$\begin{aligned} &= dA_r\wedge dr + dA_\varphi\wedge d\varphi + dA_t\wedge dt + A_r^2 dr\wedge dr + A_\varphi^2 d\varphi\wedge d\varphi + A_t^2 dt\wedge dt \\ &\quad + [A_r, A_\varphi] dr\wedge d\varphi + [A_r, A_t] dr\wedge dt + [A_\varphi, A_t] d\varphi\wedge dt \end{aligned} \tag{D.16}$$

$$\begin{aligned} &= \left(\frac{\partial A_r}{\partial r}dr + \frac{\partial A_r}{\partial \varphi}d\varphi + \frac{\partial A_r}{\partial t}dt\right)\wedge dr + \left(\frac{\partial A_\varphi}{\partial r}dr + \frac{\partial A_\varphi}{\partial \varphi}d\varphi + \frac{\partial A_\varphi}{\partial t}dt\right)\wedge d\varphi \\ &\quad + \left(\frac{\partial A_t}{\partial r}dr + \frac{\partial A_t}{\partial \varphi}d\varphi + \frac{\partial A_t}{\partial t}dt\right)\wedge dt + A_r^2 dr\wedge dr + A_\varphi^2 d\varphi\wedge d\varphi + A_t^2 dt\wedge dt \\ &\quad + [A_r, A_\varphi] dr\wedge d\varphi + [A_r, A_t] dr\wedge dt + [A_\varphi, A_t] d\varphi\wedge dt \end{aligned} \tag{D.17}$$

$$\begin{aligned} &= \left(A_r^2 + \frac{\partial A_r}{\partial r}\right) dr\wedge dr + \left(A_\varphi^2 + \frac{\partial A_\varphi}{\partial \varphi}\right) d\varphi\wedge d\varphi + \left(A_t^2 + \frac{\partial A_t}{\partial t}\right) dt\wedge dt \\ &\quad + \left(\frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi} + [A_r, A_\varphi]\right) dr\wedge d\varphi + \left(\frac{\partial A_t}{\partial r} - \frac{\partial A_r}{\partial t} + [A_r, A_t]\right) dr\wedge dt \\ &\quad + \left(\frac{\partial A_t}{\partial \varphi} - \frac{\partial A_\varphi}{\partial t} + [A_\varphi, A_t]\right) d\varphi\wedge dt \end{aligned} \tag{D.18}$$

All coefficients must vanish in order to make the curvature flat. In particular the $\varphi - t$-component must vanish, too:

$$\frac{\partial A_t}{\partial \varphi} - \frac{\partial A_\varphi}{\partial t} + [A_\varphi, A_t] = 0 \tag{D.19}$$

So, finally we find that the flatness condition $F = 0$ implies that the time derivative of A_φ is a gauge transformation with gauge parameter A_t:

$$\frac{\partial A_\varphi}{\partial t} = \frac{\partial A_t}{\partial \varphi} + [A_\varphi, A_t] \tag{D.20}$$

D.3 Computation of $A_t^{\pm}$

The latter equation yields a constraint for the coefficients ξ_i in the general form of $A_t^{\pm}$. Let's derive it. To do so, we start by calculating the derivatives:

$$\frac{\partial A_\varphi^{\pm}}{\partial t} = -\frac{2\pi}{k}\dot{\mathcal{L}}^{\pm}(r,\varphi)L_{\mp 1} \tag{D.21}$$

$$\frac{\partial A_t^{\pm}}{\partial \varphi} = \pm \sum_{i=-1}^{1} \xi_i'(r,\varphi)L_i \tag{D.22}$$

Notice that dots ($\dot{}$) denote derivatives w.r.t. time and primes ($'$) denote derivatives w.r.t. the angle φ. The next step is to calculate the commutator:

$$[A_\varphi^{\pm}, A_t^{\pm}] = \left[L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)L_{\mp 1}, \pm \sum_{i=-1}^{1} \xi_i(r,\varphi)L_i\right] \tag{D.23}$$

$$= \pm\left[L_{\pm 1}, \sum_{i=-1}^{1} \xi_i(r,\varphi)L_i\right] \mp \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi)\left[L_{\mp 1}, \sum_{i=-1}^{1} \xi_i(r,\varphi)L_i\right] \tag{D.24}$$

$$= \pm \sum_{i=-1}^{1} \xi_i(r,\varphi)[L_{\pm 1}, L_i] \mp \frac{2\pi}{k}\mathcal{L}^{\pm}(r,\varphi) \sum_{i=-1}^{1} \xi_i(r,\varphi)[L_{\mp 1}, L_i] \tag{D.25}$$

Using now the Virasoro condition

$$[L_i, L_j] = (i-j)L_{i+j} \tag{D.26}$$

we get:

$$[L_{\pm 1}, L_{\pm 1}] = 0 \tag{D.27}$$

$$[L_{+1}, L_{-1}] = 2L_0 \tag{D.28}$$

$$[L_{-1}, L_{+1}] = -2L_0 \tag{D.29}$$

$$[L_{+1}, L_0] = L_{+1} \tag{D.30}$$

$$[L_{-1}, L_0] = -L_{-1} \tag{D.31}$$

and hence (suppressing the arguments r and φ):

(a) "+"-case:

$$\begin{aligned}[A_\varphi^{+}, A_t^{+}] = \quad & \xi_{-1}[L_{+1}, L_{-1}] + \xi_0[L_{+1}, L_0] + \xi_{+1}[L_{+1}, L_{+1}] \\ & - \frac{2\pi}{k}\mathcal{L}^{+}\Big(\xi_{-1}[L_{-1}, L_{-1}] + \xi_0[L_{-1}, L_0] + \xi_{+1}[L_{-1}, L_{+1}]\Big)\end{aligned} \tag{D.32}$$

$$= \quad 2\xi_{-1}L_0 + \xi_0 L_{+1} - \frac{2\pi}{k}\mathcal{L}^{+}\Big(-\xi_0 L_{-1} - 2\xi_{+1}L_0\Big) \tag{D.33}$$

$$= \quad \frac{2\pi}{k}\mathcal{L}^{+}\xi_0 L_{-1} + 2\Big(\xi_{-1} + \frac{2\pi}{k}\mathcal{L}^{+}\xi_{+1}\Big)L_0 + \xi_0 L_{+1} \tag{D.34}$$

(b) "−"-case:

$$\begin{aligned}[A_\varphi^-, A_t^-] &= -\big(\xi_{-1}[L_{-1}, L_{-1}] + \xi_0[L_{-1}, L_0] + \xi_{+1}[L_{-1}, L_{+1}]\big) \\ &\quad + \frac{2\pi}{k}\mathcal{L}^-\Big(\xi_{-1}[L_{+1}, L_{-1}] + \xi_0[L_{+1}, L_0] + \xi_{+1}[L_{+1}, L_{+1}]\Big) \qquad \text{(D.35)} \\ &= 2\xi_{+1}L_0 + \xi_0 L_{-1} + \frac{2\pi}{k}\mathcal{L}^-\Big(\xi_0 L_{+1} + 2\xi_{-1}L_0\Big) \qquad \text{(D.36)} \\ &= \frac{2\pi}{k}\mathcal{L}^-\xi_0 L_{+1} + 2\Big(\xi_{+1} + \frac{2\pi}{k}\mathcal{L}^-\xi_{-1}\Big)L_0 + \xi_0 L_{-1} \qquad \text{(D.37)}\end{aligned}$$

We can combine these two cases to:

$$[A_\varphi^\pm, A_t^\pm] = \frac{2\pi}{k}\mathcal{L}^\pm\xi_0 L_{\mp1} + 2\Big(\xi_{\mp1} + \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1}\Big)L_0 + \xi_0 L_{\pm1} \qquad \text{(D.38)}$$

Plugging now the equations (D.21), (D.22) and (D.38) into the equation of motion (4.9) we obtain:

$$-\frac{2\pi}{k}\dot{\mathcal{L}}^\pm L_{\mp1} = \pm\sum_{i=-1}^{1}\xi_i' L_i + \frac{2\pi}{k}\mathcal{L}^\pm\xi_0 L_{\mp1} + 2\Big(\xi_{\mp1} + \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1}\Big)L_0 + \xi_0 L_{\pm1} \qquad \text{(D.39)}$$

Once more we need to have component-wise equality such that we get the following system of equations:

$$\begin{aligned}&(I) \quad -\frac{2\pi}{k}\dot{\mathcal{L}}^\pm = \pm\xi_{\mp1}' + \frac{2\pi}{k}\xi_0\mathcal{L}^\pm \\ &(II) \quad 0 = \pm\xi_0' + 2\left(\xi_{\mp1} + \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1}\right) \qquad \text{(D.40)} \\ &(III) \quad 0 = \pm\xi_{\pm1}' + \xi_0\end{aligned}$$

From equation (III) we get:

$$-\xi_0 = \pm\xi_{\pm1}' \qquad \Rightarrow \qquad \underline{\xi_0 = \mp\xi_{\pm1}'} \qquad \text{(D.41)}$$

On the other hand, from (II) we obtain:

$$\begin{aligned}0 &= \pm\frac{1}{2}\xi_0' + \xi_{\mp1} + \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1} \qquad \text{(D.42)} \\ \Rightarrow \quad \xi_{\mp1} &= \mp\frac{1}{2}\xi_0' - \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1} \qquad \text{(D.43)} \\ \overset{(D.41)}{\Rightarrow} \quad \xi_{\mp1} &= \frac{1}{2}\xi_\pm'' - \frac{2\pi}{k}\mathcal{L}^\pm\xi_{\pm1} \qquad \text{(D.44)}\end{aligned}$$

Now we can go back to the beginning and plug these expressions for ξ_0 and $\xi_{\mp1}$ into the general form of $A_t^\pm$ and get:

$$\begin{aligned}A_t^\pm(r,\varphi) &= \pm\big(\xi_{\mp1}(r,\varphi)L_{\mp1} + \xi_0(r,\varphi)L_0 + \xi_{\pm1}(r,\varphi)L_{\pm1}\big) \qquad \text{(D.45)} \\ &= \pm\left[\frac{1}{2}\xi_{\pm1}''(r,\varphi) - \frac{2\pi}{k}\mathcal{L}^\pm(r,\varphi)\xi_{\pm1}(r,\varphi)\right]L_{\mp1} - \xi_{\pm1}'(r,\varphi)L_0 \pm \xi_{\pm1}(r,\varphi)L_{\pm1} \qquad \text{(D.46)}\end{aligned}$$

And thus we finally have found expression (2.13) in the paper by Bunster et al. (notice that we redefine $\xi_{\pm1} := \xi_\pm$ and write $A_t^\pm$ for $A_t^\pm(r,\varphi)$):

$$\underline{\underline{A_t^\pm \overset{r\to\infty}{\longrightarrow} \pm\left(\xi_\pm(r,\varphi)\left(L_{\pm1} - \frac{2\pi}{k}\mathcal{L}^\pm(r,\varphi)L_{\mp1}\right) \mp \xi_\pm'(r,\varphi)L_0 + \frac{1}{2}\xi_\pm''(r,\varphi)L_{\mp1}\right)}} \qquad \text{(D.47)}$$

D.3.1 Time Evolution of the Asymptotic Charges

Further we can also use equation (I) from the system of equations above in order to derive equation (2.15) in the paper:

$$-\frac{2\pi}{k}\dot{\mathcal{L}}^{\pm} = \pm\xi'_{\mp 1} + \frac{2\pi}{k}\xi_0\mathcal{L}^{\pm} \tag{D.48}$$

$$\overset{(D.41)}{=} \pm\xi'_{\mp 1} \mp \frac{2\pi}{k}\xi'_{\pm 1}\mathcal{L}^{\pm} \tag{D.49}$$

$$\overset{(D.44)}{=} \pm\left(\frac{1}{2}\xi''_{\pm} - \frac{2\pi}{k}\mathcal{L}^{\pm}\xi_{\pm 1}\right)' \mp \frac{2\pi}{k}\xi'_{\pm 1}\mathcal{L}^{\pm} \tag{D.50}$$

$$= \pm\left(\frac{1}{2}\xi'''_{\pm 1} - \frac{2\pi}{k}\left(\mathcal{L}'^{\pm}\xi_{\pm 1} + \mathcal{L}^{\pm}\xi'_{\pm 1}\right)\right) \mp \frac{2\pi}{k}\xi'_{\pm 1}\mathcal{L}^{\pm} \tag{D.51}$$

Which leads directly to equation (2.15):

$$\underline{\underline{\dot{\mathcal{L}}^{\pm} = \pm\left(\mathcal{L}'^{\pm}\xi_{\pm} + 2\mathcal{L}^{\pm}\xi'_{\pm} - \frac{k}{4\pi}\xi'''_{\pm}\right)}} \tag{D.52}$$

Notice that choosing $\xi_{\pm} = 1$ at infinity leads to the chiral equations

$$\dot{\mathcal{L}}^{\pm} = \pm\mathcal{L}'^{\pm} \tag{D.53}$$

D.4 Functional Form of the Boundary Term

This time, let us start from the Hamiltonian formulation of the Chern-Simons action:

$$I_{Ham}[A^{\pm}] = -\frac{k}{2\pi}\int dt\, dr\, d\varphi\ \mathrm{tr}\left(A^{\pm}_r\dot{A}^{\pm}_{\varphi} - A^{\pm}_t G^{\pm}\right) + B^{\pm}_{\infty} \tag{D.54}$$

where the $B^{\pm}_{\infty}$ is needed in order to compensate for a boundary term that appears due to presence of a spatial boundary of the considered spacetime. Further, we define

$$G^{\pm} = F^{\pm}_{r\varphi} := \partial_r A^{\pm}_{\varphi} - \partial_{\varphi}A^{\pm}_r + [A^{\pm}_r, A^{\pm}_{\varphi}] \tag{D.55}$$

We can find the functional form of the boundary term $B^{\pm}_{\infty}$ by imposing the flatness condition and thus by extremizing the action:

$$\begin{aligned}\delta I_{Ham}[A^{\pm}] &= -\frac{k}{2\pi}\delta\int_{\mathcal{M}} dt\, dr\, d\varphi\ \mathrm{tr}\left(A^{\pm}_r\dot{A}^{\pm}_{\varphi} - A^{\pm}_t\left[\partial_r A^{\pm}_{\varphi} - \partial_{\varphi}A^{\pm}_r + [A^{\pm}_r, A^{\pm}_{\varphi}]\right]\right) \\ &\quad + \delta B^{\pm}_{\infty}\end{aligned} \tag{D.56}$$

$$\begin{aligned}&= -\frac{k}{2\pi}\int_{\mathcal{M}} dt\, dr\, d\varphi\ \mathrm{tr}\Big(\delta A^{\pm}_r\dot{A}^{\pm}_{\varphi} + A^{\pm}_r\delta\dot{A}^{\pm}_{\varphi} \\ &\qquad - \delta A^{\pm}_t\Big\{\partial_r A^{\pm}_{\varphi} - \partial_{\varphi}A^{\pm}_r + [A^{\pm}_r, A^{\pm}_{\varphi}]\Big\} \\ &\qquad - A^{\pm}_t\delta\Big\{\partial_r A^{\pm}_{\varphi} - \partial_{\varphi}A^{\pm}_r + [A^{\pm}_r, A^{\pm}_{\varphi}]\Big\}\Big) + \delta B^{\pm}_{\infty}\end{aligned} \tag{D.57}$$

$$
\begin{aligned}
&= -\frac{k}{2\pi}\int_{\mathcal{M}} dt\; dr\; d\varphi\; \mathrm{tr}\Big(\delta A_r^\pm \dot{A}_\varphi^\pm + A_r^\pm \delta\dot{A}_\varphi^\pm - \delta A_t^\pm \partial_r A_\varphi^\pm + \delta A_t^\pm \partial_\varphi A_r^\pm \\
&\qquad\qquad - \delta A_t^\pm \big[A_r^\pm, A_\varphi^\pm\big] - A_t^\pm \partial_r \delta A_\varphi^\pm + A_t^\pm \partial_\varphi \delta A_r^\pm \\
&\qquad\qquad - A_t^\pm \delta\big[A_r^\pm, A_\varphi^\pm\big]\Big) - \int dt\; \delta Q_\infty^\pm
\end{aligned} \tag{D.58}
$$

$$
\begin{aligned}
&= \int dt \Bigg\{ -\frac{k}{2\pi}\int_\Sigma dr\; d\varphi\; \mathrm{tr}\Big(\delta A_r^\pm \dot{A}_\varphi^\pm + A_r^\pm \delta\dot{A}_\varphi^\pm - \delta A_t^\pm \partial_r A_\varphi^\pm + \delta A_t^\pm \partial_\varphi A_r^\pm \\
&\qquad\qquad - \delta A_t^\pm \big[A_r^\pm, A_\varphi^\pm\big] - A_t^\pm \partial_r \delta A_\varphi^\pm + A_t^\pm \partial_\varphi \delta A_r^\pm \\
&\qquad\qquad - A_t^\pm \delta\big[A_r^\pm, A_\varphi^\pm\big]\Big) - \delta Q_\infty^\pm \Bigg\}
\end{aligned} \tag{D.59}
$$

where $\mathcal{M}$ denotes the full 2+1 dimensional spacetime and Σ a $t = const$-slice with boundary $\partial\Sigma$. Restricting our further calculations onto one such slice we can ignore the time integral and consider the purely spatial part only, which we now integrate by parts in the radial direction:

$$
\begin{aligned}
&= -\frac{k}{2\pi}\int_\Sigma dr\; d\varphi\; \mathrm{tr}\Big(\delta A_r^\pm \dot{A}_\varphi^\pm + A_r^\pm \delta\dot{A}_\varphi^\pm - \delta A_t^\pm \partial_r A_\varphi^\pm + \delta A_t^\pm \partial_\varphi A_r^\pm \\
&\qquad\qquad -\delta A_t^\pm [A_r^\pm, A_\varphi^\pm] - A_t^\pm \partial_r \delta A_\varphi^\pm + A_t^\pm \partial_\varphi \delta A_r^\pm - A_t^\pm \delta[A_r^\pm, A_\varphi^\pm]\Big) \\
&\qquad\qquad - \delta Q_\infty^\pm
\end{aligned} \tag{D.60}
$$

$$
\begin{aligned}
&= -\frac{k}{2\pi}\int_\Sigma dr\; d\varphi\; \mathrm{tr}\Big(\delta A_r^\pm \dot{A}_\varphi^\pm + A_r^\pm \delta\dot{A}_\varphi^\pm - \delta A_t^\pm \partial_r A_\varphi^\pm + \delta A_t^\pm \partial_\varphi A_r^\pm \\
&\qquad\qquad -\delta A_t^\pm [A_r^\pm, A_\varphi^\pm] + \partial_r A_t^\pm \delta A_\varphi^\pm + A_t^\pm \partial_\varphi \delta A_r^\pm - A_t^\pm \delta[A_r^\pm, A_\varphi^\pm]\Big) \\
&\quad + \frac{k}{2\pi}\int_{\partial\Sigma} d\varphi\; \mathrm{tr}\left(A_t^\pm \delta A_\varphi^\pm\right) \\
&\quad - \delta Q_\infty^\pm
\end{aligned} \tag{D.61}
$$

Now let us focus on the boundary integral only

$$
\frac{k}{2\pi}\int_{\partial\Sigma} d\varphi\; \mathrm{tr}\left(A_t^\pm \delta A_\varphi^\pm\right) \tag{D.62}
$$

and recall the expressions for $A_\varphi^\pm$ and $A_t^\pm$:

$$
A_\varphi^\pm = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp 1} \tag{D.63}
$$

$$
A_t^\pm = \pm\left(\xi_\pm\left(L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp 1}\right) \mp \xi'_\pm L_0 + \frac{1}{2}\xi''_\pm L_{\mp 1}\right) \tag{D.64}
$$

The most general variation of the field component A_φ that respects the boundary condition is

$$
\delta A_\varphi^\pm = -\frac{2\pi}{k}\delta\mathcal{L}^\pm L_{\mp 1} \tag{D.65}
$$

Now, plugging this into the integrand of expression (D.62) we get:

$$\begin{aligned}
&\operatorname{tr}\left(A_t^{\pm}\delta A_\varphi^{\pm}\right) && \text{(D.66)}\\
&= \operatorname{tr}\left(\pm\left[\xi_\pm\left(L_{\pm1}-\frac{2\pi}{k}\mathcal{L}^{\pm}L_{\mp1}\right)\mp\xi'_\pm L_0+\frac{1}{2}\xi''_\pm L_{\mp1}\right]\left[-\frac{2\pi}{k}\delta\mathcal{L}^{\pm}L_{\mp1}\right]\right) && \text{(D.67)}\\
&= \pm\operatorname{tr}\left(\left[\xi_\pm L_{\pm1}-\frac{2\pi}{k}\xi_\pm\mathcal{L}^{\pm}L_{\mp1}\mp\xi'_\pm L_0+\frac{1}{2}\xi''_\pm L_{\mp1}\right]\left[-\frac{2\pi}{k}\delta\mathcal{L}^{\pm}L_{\mp1}\right]\right) && \text{(D.68)}\\
&= \pm\operatorname{tr}\left(-\frac{2\pi}{k}\xi_\pm\delta\mathcal{L}^{\pm}L_{\pm1}L_{\mp1}+\frac{4\pi^2}{k^2}\xi_\pm\mathcal{L}^{\pm}\delta\mathcal{L}^{\pm}L_{\mp1}L_{\mp1}\pm\frac{2\pi}{k}\xi'_\pm\delta\mathcal{L}^{\pm}L_0L_{\mp1}\right.\\
&\quad\left.-\frac{\pi}{k}\xi''_\pm\delta\mathcal{L}^{\pm}L_{\mp1}L_{\mp1}\right) && \text{(D.69)}
\end{aligned}$$

Now we need to calculate the matrix products:

$$L_{+1}L_{+1}=\begin{pmatrix}0&-1\\0&0\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix}=0 \tag{D.70}$$

$$L_{-1}L_{-1}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=0 \tag{D.71}$$

$$L_{+1}L_{-1}=\begin{pmatrix}0&-1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&0\end{pmatrix} \tag{D.72}$$

$$L_{-1}L_{+1}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&-1\end{pmatrix} \tag{D.73}$$

$$L_0L_{-1}=\begin{pmatrix}-\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}0&-1\\0&0\end{pmatrix}=\begin{pmatrix}0&\frac{1}{2}\\0&0\end{pmatrix} \tag{D.74}$$

$$L_0L_{+1}=\begin{pmatrix}-\frac{1}{2}&0\\0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\\frac{1}{2}&0\end{pmatrix} \tag{D.75}$$

Hence the trace simplifies to

$$\begin{aligned}
&\pm\operatorname{tr}\left(-\frac{2\pi}{k}\xi_\pm\delta\mathcal{L}^{\pm}L_{\pm1}L_{\mp1}\pm\frac{2\pi}{k}\xi'_\pm\delta\mathcal{L}^{\pm}L_0L_{\mp1}\right) && \text{(D.76)}\\
&=\pm\left[-\frac{2\pi}{k}\xi_\pm\delta\mathcal{L}^{\pm}\underbrace{\operatorname{tr}\left(L_{\pm1}L_{\mp1}\right)}_{=-1}\pm\frac{2\pi}{k}\xi'_\pm\delta\mathcal{L}^{\pm}\underbrace{\operatorname{tr}\left(L_0L_{\mp1}\right)}_{=0}\right] && \text{(D.77)}
\end{aligned}$$

Thus we get:

$$\operatorname{tr}\left(A_t^{\pm}\delta A_\varphi^{\pm}\right)=\pm\frac{2\pi}{k}\xi_\pm\delta\mathcal{L}^{\pm} \tag{D.78}$$

Plugging this back into expression (D.61) and remembering that in the pure gravity case $k=k_2$ we get:

$$\begin{aligned}
\delta I_{Ham}[A^{\pm}]=-\frac{k_2}{2\pi}\int_\Sigma dr\,d\varphi\,\operatorname{tr}\Big(&\delta A_r^{\pm}\dot{A}_\varphi^{\pm}+A_r^{\pm}\delta\dot{A}_\varphi^{\pm}-\delta A_t^{\pm}\partial_rA_\varphi^{\pm}+\delta A_t^{\pm}\partial_\varphi A_r^{\pm}\\
&-\delta A_t^{\pm}[A_r^{\pm},A_\varphi^{\pm}]+\partial_rA_t^{\pm}\delta A_\varphi^{\pm}+A_t^{\pm}\partial_\varphi\delta A_r^{\pm}\\
&-A_t^{\pm}\delta[A_r^{\pm},A_\varphi^{\pm}]\Big)\\
\pm\frac{k_2}{2\pi}\int_{\partial\Sigma}d\varphi\,\frac{2\pi}{k_2}\xi_\pm\delta\mathcal{L}^{\pm}&-\delta Q_\infty^{\pm}
\end{aligned} \tag{D.79}$$

Since we extremize the action, we expect this expression to vanish which in turn means that the explicitly imposed boundary term and the boundary integral must cancel each other:

$$-\delta Q_\infty^\pm \pm \int_{\partial\Sigma} d\varphi\ \xi_\pm \delta\mathcal{L}^\pm \stackrel{!}{=} 0 \tag{D.80}$$

$$\delta Q_\infty^\pm = \pm \int_{\partial\Sigma} d\varphi\ \xi_\pm \delta\mathcal{L}^\pm \tag{D.81}$$

The latter equation is the variation of

$$Q_\infty^\pm[\xi_\pm] = \pm \int_{\partial\Sigma} d\varphi\ \xi_\pm \mathcal{L}^\pm. \tag{D.82}$$

The full boundary term is then given by

$$B_\infty^\pm = -\int dt\ Q_\infty^\pm[\xi_\pm] = -\int dt\ \left(\pm \int_{\partial\Sigma} d\varphi\ \xi_\pm \mathcal{L}^\pm\right) = \mp \int dt d\varphi\ \xi_\pm \mathcal{L}^\pm \tag{D.83}$$

D.5 Poisson Structure of the Asymptotic Symmetry Generators

It is well known that for any phase space functional F we can write the following equation for its variation under a gauge transformation parametrized by ξ:

$$\pm\, \delta_{\xi_\pm} F = \{Q_\infty^\pm, F\} \tag{D.84}$$

Notice that the fact that we consider the gauge transformation as being parametrized by $\xi_\pm$ (instead of $A_t^\pm$) directly implies that the boundary conditions discussed in section D.2 are preserved. Furthermore, we define the Poisson bracket as:

$$\{F, G\} := \int_\Sigma dx^i dx^j \mathrm{tr}\left(\frac{\delta F}{\delta A_i(x)} \frac{\delta G}{\delta A_j(x)}\right) \tag{D.85}$$

Now take the equation (2.17) for the boundary Q-term and plug it into the Poisson bracket together with the asymptotic symmetry generator $\mathcal{L}^\pm$ (which in this case is considered as a coordinate function in the phase space and hence as a special case of F):

$$\{Q_\infty^\pm, \mathcal{L}^\pm\} = \pm\delta_{\xi_\pm}\mathcal{L}^\pm \tag{D.86}$$

$$\left\{\int_{\partial\Sigma} d\varphi\ \xi_\pm(\varphi)\mathcal{L}^\pm(\varphi), \mathcal{L}^\pm(\tilde{\varphi})\right\} = \pm\delta_{\xi_\pm}\mathcal{L}^\pm(\tilde{\varphi}) \tag{D.87}$$

$$\int_{\partial\Sigma} d\varphi\ \xi_\pm(\varphi)\left\{\mathcal{L}^\pm(\varphi), \mathcal{L}^\pm(\tilde{\varphi})\right\} = \pm\delta_{\xi_\pm}\mathcal{L}^\pm(\tilde{\varphi}) \tag{D.88}$$

On the other hand we can now take equation (D.52)

$$\dot{\mathcal{L}}^\pm = \pm\left(\mathcal{L}'^\pm\xi_\pm + 2\mathcal{L}^\pm\xi'_\pm - \frac{k}{4\pi}\xi'''_\pm\right) \tag{D.89}$$

and recognize that setting $\dot{\mathcal{L}}^{\pm} = \pm\delta_{\xi_{\pm}}\mathcal{L}^{\pm}$ is a manifestation of preserving the boundary conditions, i.e. the latter relation holds true as long as equations (D.41) and (D.44) are respected. Accordingly, we can set

$$\int_{\partial\Sigma} d\varphi\, \xi_{\pm}(\varphi)\left\{\mathcal{L}^{\pm}(\varphi), \mathcal{L}^{\pm}(\tilde{\varphi})\right\} = \mathcal{L}'^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi}) + 2\mathcal{L}^{\pm}(\tilde{\varphi})\xi'_{\pm}(\tilde{\varphi}) - \frac{k}{4\pi}\xi'''_{\pm}(\tilde{\varphi}) \tag{D.90}$$

which is solved by

$$\left\{\mathcal{L}^{\pm}(\varphi), \mathcal{L}^{\pm}(\tilde{\varphi})\right\} = -\left[\delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi) + 2\mathcal{L}^{\pm}(\varphi)\delta'(\varphi-\tilde{\varphi}) + \frac{k}{4\pi}\delta'''(\varphi-\tilde{\varphi})\right]. \tag{D.91}$$

Let us prove this:

$$\int_{\partial\Sigma} d\varphi\, \xi_{\pm}(\varphi)\left\{\mathcal{L}^{\pm}(\varphi), \mathcal{L}^{\pm}(\tilde{\varphi})\right\} \tag{D.92}$$

$$= -\int_{\partial\Sigma} d\varphi\, \xi_{\pm}(\varphi)\left(\delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi) + 2\mathcal{L}^{\pm}(\varphi)\delta'(\varphi-\tilde{\varphi}) + \frac{k}{4\pi}\delta'''(\varphi-\tilde{\varphi})\right) \tag{D.93}$$

$$\begin{aligned} = &-\int_{\partial\Sigma} d\varphi\, \delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi)\xi_{\pm}(\varphi) - 2\int_{\partial\Sigma} d\varphi\, \delta'(\varphi-\tilde{\varphi})\mathcal{L}^{\pm}(\varphi)\xi_{\pm}(\varphi) \\ &- \frac{k}{4\pi}\int_{\partial\Sigma} d\varphi\, \xi_{\pm}(\varphi)\delta'''(\varphi-\tilde{\varphi}) \end{aligned} \tag{D.94}$$

$$\begin{aligned} \overset{ibp}{=} &-\int_{\partial\Sigma} d\varphi\, \delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi)\xi_{\pm}(\varphi) + 2\int_{\partial\Sigma} d\varphi\, \delta(\varphi-\tilde{\varphi})\left[\mathcal{L}^{\pm}(\varphi)\xi_{\pm}(\varphi)\right]' \\ &+ \frac{k}{4\pi}\int_{\partial\Sigma} d\varphi\, \xi'''_{\pm}(\varphi)\delta(\varphi-\tilde{\varphi}) \end{aligned} \tag{D.95}$$

$$= -\mathcal{L}'^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi}) + 2\left[\mathcal{L}^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi})\right]' + \frac{k}{4\pi}\xi'''_{\pm}(\tilde{\varphi}) \tag{D.96}$$

$$= -\mathcal{L}'^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi}) + 2\mathcal{L}'^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi}) + 2\mathcal{L}^{\pm}(\tilde{\varphi})\xi'_{\pm}(\tilde{\varphi}) + \frac{k}{4\pi}\xi'''_{\pm}(\tilde{\varphi}) \tag{D.97}$$

$$= \mathcal{L}'^{\pm}(\tilde{\varphi})\xi_{\pm}(\tilde{\varphi}) + 2\mathcal{L}^{\pm}(\tilde{\varphi})\xi'_{\pm}(\tilde{\varphi}) + \frac{k}{4\pi}\xi'''_{\pm}(\tilde{\varphi}) \tag{D.98}$$

where "*ibp*" means "integration by parts". This proves equation (D.91). We now take equation (D.91) and integrate over φ in a first step in order to get a handle on the δ-

functions:

$$\begin{aligned}\left\{\mathcal{L}^{\pm}(\varphi),\mathcal{L}^{\pm}(\tilde{\varphi})\right\}e^{in\varphi} = &-\Big[\delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi)+2\mathcal{L}^{\pm}(\varphi)\delta'(\varphi-\tilde{\varphi})\\ &+\frac{k}{4\pi}\delta'''(\varphi-\tilde{\varphi})\Big]e^{in\varphi}\end{aligned} \tag{D.99}$$

$$\begin{aligned}\int_{-\pi}^{\pi}d\varphi\left\{\mathcal{L}^{\pm}(\varphi),\mathcal{L}^{\pm}(\tilde{\varphi})\right\}e^{in\varphi} = &-\int_{-\pi}^{\pi}d\varphi\Big[\delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi)+2\mathcal{L}^{\pm}(\varphi)\delta'(\varphi-\tilde{\varphi})\\ &+\frac{k}{4\pi}\delta'''(\varphi-\tilde{\varphi})\Big]e^{in\varphi}\end{aligned} \tag{D.100}$$

$$\begin{aligned}\left\{\int_{-\pi}^{\pi}d\varphi\mathcal{L}^{\pm}(\varphi)e^{in\varphi},\mathcal{L}^{\pm}(\tilde{\varphi})\right\} = &-\int_{-\pi}^{\pi}d\varphi\Big[\delta(\varphi-\tilde{\varphi})\mathcal{L}'^{\pm}(\varphi)e^{in\varphi}\Big]\\ &-2\int_{-\pi}^{\pi}d\varphi\Big[\mathcal{L}^{\pm}(\varphi)\delta'(\varphi-\tilde{\varphi})e^{in\varphi}\Big]\\ &-\frac{k}{4\pi}\int_{-\pi}^{\pi}d\varphi\Big[\delta'''(\varphi-\tilde{\varphi})e^{in\varphi}\Big]\end{aligned} \tag{D.101}$$

where we have used the linearity of the integral and the Poisson bracket on the LHS. Via an integration by parts in the second and third term on the RHS (remember that the boundary terms vanish due to the periodicity of the functions) and the definition of the Fourier transform in the LHS we obtain:

$$-\left\{\mathcal{L}_n^{\pm},\mathcal{L}^{\pm}(\tilde{\varphi})\right\} = -\Big[\ \mathcal{L}'^{\pm}(\varphi)\big|_{\varphi=\tilde{\varphi}}\,e^{in\tilde{\varphi}}\Big]+2\Big[\mathcal{L}^{\pm}(\varphi)e^{in\varphi}\Big]'_{\varphi=\tilde{\varphi}}+\frac{k}{4\pi}\Big[(in)^3e^{in\tilde{\varphi}}\Big] \tag{D.102}$$

$$= (2-1)\Big[\ \mathcal{L}'^{\pm}(\varphi)\big|_{\varphi=\tilde{\varphi}}\,e^{in\tilde{\varphi}}\Big]+2in\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{in\tilde{\varphi}}\Big]+\frac{k}{4\pi}\Big[(in)^3e^{in\tilde{\varphi}}\Big] \tag{D.103}$$

$$= \Big[\ \mathcal{L}'^{\pm}(\varphi)\big|_{\varphi=\tilde{\varphi}}\,e^{in\tilde{\varphi}}\Big]+2in\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{in\tilde{\varphi}}\Big]-i\frac{kn^3}{4\pi}e^{in\tilde{\varphi}} \tag{D.104}$$

We continue now by first multiplying another exponential factor and a second integration over φ:

$$\begin{aligned}-\left\{\mathcal{L}_n^{\pm},\mathcal{L}^{\pm}(\tilde{\varphi})\right\}e^{im\tilde{\varphi}} = &\Big[\ \mathcal{L}'^{\pm}(\varphi)\big|_{\varphi=\tilde{\varphi}}\,e^{in\tilde{\varphi}}\Big]e^{im\tilde{\varphi}}+2in\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{in\tilde{\varphi}}\Big]e^{im\tilde{\varphi}}\\ &-i\frac{kn^3}{4\pi}e^{in\tilde{\varphi}}e^{im\tilde{\varphi}}\end{aligned} \tag{D.105}$$

$$\begin{aligned}= &\Big[\ \mathcal{L}'^{\pm}(\varphi)\big|_{\varphi=\tilde{\varphi}}\,e^{i(n+m)\tilde{\varphi}}\Big]+2in\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{i(n+m)\tilde{\varphi}}\Big]\\ &-i\frac{kn^3}{4\pi}e^{i(n+m)\tilde{\varphi}}\end{aligned} \tag{D.106}$$

$$-\int_{-\pi}^{\pi} d\tilde{\varphi}\left\{\mathcal{L}_n^{\pm},\mathcal{L}^{\pm}(\tilde{\varphi})\right\}e^{im\tilde{\varphi}} = \int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\left.\mathcal{L}'^{\pm}(\varphi)\right|_{\varphi=\tilde{\varphi}} e^{i(n+m)\tilde{\varphi}}\Big] + 2in\int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{i(n+m)\tilde{\varphi}}\Big] - i\frac{kn^3}{4\pi}\int_{-\pi}^{\pi} d\tilde{\varphi}e^{i(n+m)\tilde{\varphi}} \tag{D.107}$$

$$-\left\{\mathcal{L}_n^{\pm},\int_{-\pi}^{\pi} d\tilde{\varphi}\mathcal{L}^{\pm}(\tilde{\varphi})e^{im\tilde{\varphi}}\right\} = \int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\left.\mathcal{L}'^{\pm}(\varphi)\right|_{\varphi=\tilde{\varphi}} e^{i(n+m)\tilde{\varphi}}\Big] + 2in\int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{i(n+m)\tilde{\varphi}}\Big] - i\frac{kn^3}{4\pi}2\pi\delta_{n,-m} \tag{D.108}$$

Again according to the definition of the Fourier transform we get:

$$\left\{\mathcal{L}_n^{\pm},\mathcal{L}_m^{\pm}\right\} = \int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\left.\mathcal{L}'^{\pm}(\varphi)\right|_{\varphi=\tilde{\varphi}} e^{i(n+m)\tilde{\varphi}}\Big] - 2in\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.109}$$

Another integration by parts in the first term of the RHS yields:

$$\left\{\mathcal{L}_n^{\pm},\mathcal{L}_m^{\pm}\right\} = \int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\left.\mathcal{L}'^{\pm}(\varphi)\right|_{\varphi=\tilde{\varphi}} e^{i(n+m)\tilde{\varphi}}\Big] - 2in\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.110}$$

$$= -i(n+m)\int_{-\pi}^{\pi} d\tilde{\varphi}\Big[\mathcal{L}^{\pm}(\tilde{\varphi})e^{i(n+m)\tilde{\varphi}}\Big] - 2in\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.111}$$

$$= i(n+m)\mathcal{L}_{n+m}^{\pm} - 2in\pi\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.112}$$

$$= in\mathcal{L}_{n+m}^{\pm} + im\mathcal{L}_{n+m}^{\pm} - 2in\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.113}$$

$$= -in\mathcal{L}_{n+m}^{\pm} + im\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.114}$$

$$= -i(n-m)\mathcal{L}_{n+m}^{\pm} - i\frac{kn^3}{2}\delta_{n,-m} \tag{D.115}$$

Thus we get:

$$i\left\{\mathcal{L}_n^{\pm},\mathcal{L}_m^{\pm}\right\} = (n-m)\mathcal{L}_{n+m}^{\pm} + \frac{kn^3}{2}\delta_{n,-m} \tag{D.116}$$

Appendix E

Explicit Computations for the $\mathfrak{so}(3,2)$ Black Hole

E.1 Normalization of the $\mathfrak{so}(3,2)$ algebra

Let us compare the trace scalar products of the two simple Lie algebras $\mathfrak{so}(3,2)$ and $\mathfrak{sl}(2,\mathbb{R})$. To do so, we recall the basis of the latter algebra as defined in [2]:

$$\mathfrak{sl}(2,\mathbb{R}) = \left\langle \left\{ L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, L_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right\} \right\rangle_{\mathbb{R}} \tag{E.1}$$

while $\mathfrak{so}(3,2)$ is spanned by

$$\{L_{+1}, L_0, L_{-1}\} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -6i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4i \\ 6i & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

Let us compare the normalizations of these two algebras. The following condition must be obeyed:

$$k_3 \mathrm{Tr}(X_{32}^2) = k_2 \mathrm{tr}\left(X^2\right) \tag{E.2}$$

where k_3 and k_2 are normalization factors, Tr denotes the trace in the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of $\mathfrak{so}(3,2)$ and tr the trace in the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$. The matrices X represent the same Lie algebra element on both sides of the equation but in different representations. In what follows we will drop the numerals in the subscript since it becomes clear from the choice of trace which representation is the relevant one. We can choose an arbitrary element for X whose square has a non-vanishing trace. We find for $X = L_0$:

$$\mathrm{Tr}(L_0^2) = 10 \qquad \text{and} \qquad \mathrm{tr}(L_0^2) = \frac{1}{2} \tag{E.3}$$

such that

$$10k_3 = \frac{1}{2}k_2 \qquad \Rightarrow \qquad k_3 = \frac{1}{20}k_2 \tag{E.4}$$

Hence we finally find

$$k_3 = \frac{1}{20}\frac{l}{4G} \tag{E.5}$$

where $\Lambda = -\frac{1}{l^2}$, Λ the cosmological constant.

E.2 Boundary Conditions and Asymptotes of the Connection Field

We follow the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ theory computations given in [2]. Our starting point hence shall be:

$$A_\varphi^\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^\pm(r,\varphi)L_{\mp 1} - \frac{\pi}{2k}\mathcal{W}^\pm(r,\varphi)W_{\mp 3} \tag{E.6}$$

$$A_r^\pm \overset{r\to\infty}{\longrightarrow} \mathcal{O}\left(\frac{1}{r}\right) \tag{E.7}$$

$$\mathcal{L}^\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{L}^\pm(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{E.8}$$

$$\mathcal{W}^\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} \mathcal{W}^\pm(\varphi) + \mathcal{O}\left(\frac{1}{r}\right) \tag{E.9}$$

and the most general from of $A_t^\pm$ shall now be

$$A_t^\pm(r,\varphi) \overset{r\to\infty}{\longrightarrow} \pm\left(\sum_{k=-1}^{1}\xi_k L_k + \sum_{m=-3}^{3}\eta_m W_m\right) \tag{E.10}$$

where $\xi_k = \xi_k(r,\varphi)$ and $\eta_m = \eta_m(r,\varphi)$. We can compute directly the following derivatives:

$$\frac{\partial A_\varphi^\pm}{\partial t} \overset{r\to\infty}{\longrightarrow} -\frac{2\pi}{k}\dot{\mathcal{L}}^\pm L_{\mp 1} - \frac{\pi}{2k}\dot{\mathcal{W}}^\pm W_{\mp 3} \tag{E.11}$$

$$\frac{\partial A_t^\pm}{\partial \varphi} \overset{r\to\infty}{\longrightarrow} \pm\left(\sum_{k=-1}^{1}\xi_k' L_k + \sum_{m=-3}^{3}\eta_m' W_m\right) \tag{E.12}$$

where $\dot{}$ denotes derivative w.r.t. the time coordinate t and $'$ stands for the derivative w. r. t. φ. It is our first goal in this section to find the most general form of the gauge transformation $A_t^\pm$ which preserves the asymptotes (E.6) to (E.9). To do so, we recall the relation (4.9)

$$\frac{\partial A_\varphi^\pm}{\partial t} = \frac{\partial A_t^\pm}{\partial \varphi} + [A_\varphi^\pm, A_t^\pm] \tag{E.13}$$

that will yield a constraint on $A_t^{\pm}$ as in the case of pure gravity treated in appendix D.3. The commutator of the expressions (E.6) and (E.10) is then given as follows:

$$\begin{aligned}
[A_\varphi^{\pm}, A_t^{\pm}] = \quad & \left(\frac{2\pi}{k}\xi_0\mathcal{L}^{\pm} + \frac{180\pi}{k}\mathcal{W}^{\pm}\right) L_{\mp 1} \\
& + \left(2\xi_{\mp 1} + \frac{4\pi}{k}\xi_{\pm}1\mathcal{L}^{\pm} + \frac{1080\pi}{k}\eta{\pm}3\mathcal{W}^{\pm}\right) L_0 \\
& + \xi_0 L_{\pm 1} \\
& + \left(\frac{2\pi}{k}\eta_{\mp 2}\mathcal{L}^{\pm} + \frac{3\pi}{2k}\xi_0\mathcal{W}^{\pm} + \frac{3\pi}{k}\eta_0\mathcal{W}^{\pm}\right) W_{\mp 3} \\
& + \left(6\eta_{\mp 3} + \frac{4\pi}{k}\eta_{\mp 1}\mathcal{L}^{\pm} + \frac{3\pi}{k}\xi_{\pm 1}\mathcal{W}^{\pm} + \frac{12\pi}{k}\eta_{\pm 1}\mathcal{W}^{\pm}\right) W_{\mp 2} \\
& + \left(5\eta_{\mp 2} + \frac{6\pi}{k}\eta_0\mathcal{L}^{\pm} + \frac{30\pi}{k}\eta_{\pm 2}\mathcal{W}^{\pm}\right) W_{\mp 1} \\
& + \left(4\eta_{\mp 1} + \frac{8\pi}{k}\eta_{\pm 1}\mathcal{L}^{\pm} + \frac{6\pi}{k}\eta_{\pm 3}\mathcal{W}^{\pm}\right) W_0 \\
& + \left(3\eta_0 + \frac{10\pi}{k}\eta_{\pm 2}\mathcal{L}^{\pm}\right) W_{\pm 1} \\
& + \left(2\eta_{\pm 1} + \frac{12\pi}{k}\eta_{\pm 3}\mathcal{L}^{\pm}\right) W_{\pm 2} \\
& + \eta_{\pm 2} W_{\pm 3}
\end{aligned} \tag{E.14}$$

We have now all ingredients we need: Plugging the expression for the two derivatives and the commutator derived above into equation (4.9) we can solve it componentwise. In other words, we obtain a system of equations:

$$-\frac{2\pi}{k}\dot{\mathcal{L}}^{\pm} = \pm\xi'_{\mp 1} + \left(\frac{2\pi}{k}\xi_0\mathcal{L}^{\pm} + \frac{180\pi}{k}\mathcal{W}^{\pm}\right) \tag{E.15}$$

$$0 = \pm\xi'_0 + \left(2\xi_{\mp 1} + \frac{4\pi}{k}\xi_{\pm}1\mathcal{L}^{\pm} + \frac{1080\pi}{k}\eta{\pm}3\mathcal{W}^{\pm}\right) \tag{E.16}$$

$$0 = \pm\xi'_{\pm 1} + \xi_0 \tag{E.17}$$

$$-\frac{\pi}{2k}\dot{\mathcal{W}}^{\pm} = \pm\eta'_{\mp 3} + \left(\frac{2\pi}{k}\eta_{\mp 2}\mathcal{L}^{\pm} + \frac{3\pi}{2k}\xi_0\mathcal{W}^{\pm} + \frac{3\pi}{k}\eta_0\mathcal{W}^{\pm}\right) \tag{E.18}$$

$$0 = \pm\eta'_{\mp 2} + \left(6\eta_{\mp 3} + \frac{4\pi}{k}\eta_{\mp 1}\mathcal{L}^{\pm} + \frac{3\pi}{k}\xi_{\pm 1}\mathcal{W}^{\pm} + \frac{12\pi}{k}\eta_{\pm 1}\mathcal{W}^{\pm}\right) \tag{E.19}$$

$$0 = \pm\eta'_{\mp 1} + \left(5\eta_{\mp 2} + \frac{6\pi}{k}\eta_0\mathcal{L}^{\pm} + \frac{30\pi}{k}\eta_{\pm 2}\mathcal{W}^{\pm}\right) \tag{E.20}$$

$$0 = \pm\eta'_0 + \left(4\eta_{\mp 1} + \frac{8\pi}{k}\eta_{\pm 1}\mathcal{L}^{\pm} + \frac{6\pi}{k}\eta_{\pm 3}\mathcal{W}^{\pm}\right) \tag{E.21}$$

$$0 = \pm\eta'_{\pm 1} + \left(3\eta_0 + \frac{10\pi}{k}\eta_{\pm 2}\mathcal{L}^{\pm}\right) \tag{E.22}$$

$$0 = \pm\eta'_{\pm 2} + \left(2\eta_{\pm 1} + \frac{12\pi}{k}\eta_{\pm 3}\mathcal{L}^{\pm}\right) \tag{E.23}$$

$$0 = \pm\eta'_{\pm 3} + \eta_{\pm 2} \tag{E.24}$$

We start solving this equation considering the two equations (E.17) and (E.24) leading to

$$\xi_0 = \mp\xi'_{\pm 1} \tag{E.25}$$

$$\eta_{\pm 2} = \mp\eta'_{\pm 3}. \tag{E.26}$$

Plugging these relations back in will eliminate ξ_0 and $\eta_{\pm 2}$. We can continue afterwards in a similar way and eliminate successively the coefficients ξ_k and η_m until we have expressed all the relations in terms of the coefficients $\xi_\pm := \xi_{\pm 1}$ and $\eta_\pm := \eta_{\pm 3}$. This procedure will yield the following relations:

$$\xi_{\mp 1} = \pm\left(\frac{1}{2}\xi''_\pm - \frac{2\pi}{k}\xi_\pm\mathcal{L}^{\pm} - \frac{540\pi}{k}\eta_\pm\mathcal{W}^{\pm}\right) \tag{E.27}$$

$$\xi_0 = -\xi'_\pm \tag{E.28}$$

$$\eta_{\mp 3} = \pm\left[\frac{1}{720}\eta_\pm^{VI} - \frac{41\pi}{180k}\eta_\pm^{IV}\mathcal{L}^{\pm} - \frac{5\pi}{12k}\eta'''_\pm(\mathcal{L}^{\pm})' - \frac{17\pi}{60k}\eta''_\pm(\mathcal{L}^{\pm})'' - \frac{\pi}{12k}\eta'_\pm(\mathcal{L}^{\pm})'''\right. \tag{E.29}$$

$$- \frac{\pi}{60k}\eta_\pm(\mathcal{L}^{\pm})^{IV} + \frac{24\pi^2}{5k^2}\eta''_\pm(\mathcal{L}^{\pm})^2 + \frac{77\pi^2}{15k^2}\eta'_\pm\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' + \frac{6\pi^2}{5k^2}\eta_\pm(\mathcal{L}^{\pm})'^2 \tag{E.30}$$

$$+ \frac{23\pi^2}{15k^2}\eta_\pm\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' - \frac{3\pi}{2k}\eta''_\pm\mathcal{W}^{\pm} - \frac{2\pi}{k}\eta'_\pm(\mathcal{W}^{\pm})' - \frac{\pi}{2k}\eta_\pm(\mathcal{W}^{\pm})'' \tag{E.31}$$

$$\left. - \frac{8\pi^3}{k^3}\eta_\pm(\mathcal{L}^{\pm})^3 + \frac{22\pi^2}{k^2}\eta_\pm\mathcal{L}^{\pm}\mathcal{W}^{\pm} - \frac{\pi}{2k}\xi_\pm\mathcal{W}^{\pm} - \frac{\pi}{k}\eta''_\pm\mathcal{W}^{\pm}\right] \tag{E.32}$$

$$\eta_{\mp 2} = \left[-\frac{1}{120}\eta_\pm^{V} + \frac{6\pi}{5k}\eta'''_\pm\mathcal{L}^{\pm} + \frac{13\pi}{10k}\eta''_\pm(\mathcal{L}^{\pm})' + \frac{2\pi}{5k}\eta'_\pm(\mathcal{L}^{\pm})'' + \frac{\pi}{10k}\eta_\pm(\mathcal{L}^{\pm})'''\right. \tag{E.33}$$

$$\left. - \frac{44\pi^2}{5k^2}\eta'_\pm(\mathcal{L}^{\pm})^2 - \frac{36\pi^2}{5k^2}\eta_\pm\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' + \frac{9\pi}{k}\eta'_\pm\mathcal{W}^{\pm} + \frac{3\pi}{k}\eta_\pm(\mathcal{W}^{\pm})'\right] \tag{E.34}$$

$$\eta_{\mp 1} = \pm \left[\frac{1}{24}\eta_{\pm}^{IV} - \frac{5\pi}{k}\eta_{\pm}''\mathcal{L}^{\pm} - \frac{3\pi}{2k}\eta_{\pm}'(\mathcal{L}^{\pm})' - \frac{\pi}{2k}\eta_{\pm}(\mathcal{L}^{\pm})'' + \frac{12\pi^2}{k^2}\eta_{\pm}(\mathcal{L}^{\pm})^2 \right. \tag{E.35}$$

$$\left. - \frac{15\pi}{k}\eta_{\pm}\mathcal{W}^{\pm} \right] \tag{E.36}$$

$$\eta_0 = \left[-\frac{1}{6}\eta_{\pm}''' + \frac{16\pi}{3k}\eta_{\pm}'\mathcal{L}^{\pm} + \frac{2\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm})' \right] \tag{E.37}$$

$$\eta_{\pm 1} = \pm \left[\frac{1}{2}\eta_{\pm}'' - \frac{6\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm}) \right] \tag{E.38}$$

$$\eta_{\pm 2} = -\eta_{\pm}' \tag{E.39}$$

Plugging these relations into the ansatz (E.10) yields the most general form of $A_t^{\pm}$ such that the boundary conditions are conserved. This form is given in expression (5.49).

E.3 Evolution Equations of the Charges $\mathcal{L}^{\pm}$ and $\mathcal{W}^{\pm}$

Recall equation (E.15) and replace all the coefficients by $\xi_{\pm}$ and $\eta_{\pm}$ according to the previously found relations:

$$-\frac{2\pi}{k}\dot{\mathcal{L}}^{\pm} = \pm\frac{1}{2}\xi_{\pm}''' \mp \frac{4\pi}{k}\xi_{\pm}'\mathcal{L}^{\pm} \mp \frac{2\pi}{k}\xi_{\pm}(\mathcal{L}^{\pm})' \mp \frac{540\pi}{k}\eta_{\pm}(\mathcal{W}^{\pm})' \mp \frac{720\pi}{k}\eta_{\pm}'\mathcal{W}^{\pm} \tag{E.40}$$

Solving this for $\dot{\mathcal{L}}^{\pm}$ yields:

$$\dot{\mathcal{L}}^{\pm} = \pm\left(2\xi_{\pm}'\mathcal{L}^{\pm} + \xi_{\pm}(\mathcal{L}^{\pm})' - \frac{k}{4\pi}\xi_{\pm}''' + 90[3\eta_{\pm}(\mathcal{W}^{\pm})' + 4\eta_{\pm}'\mathcal{W}^{\pm}] \right) \tag{E.41}$$

Choosing $\eta_{\pm} = 0$ and $\xi_{\pm} = 1$ we obtain relation:

$$\dot{\mathcal{L}}^{\pm} = \pm(\mathcal{L}^{\pm})' \tag{E.42}$$

We can proceed similarly for the asymptotic charge $\mathcal{W}$. We start by again substituting all the previously found relations amongst the coefficients ξ_k and η_k into equation (E.18). After a straight forward but rather tedious and lengthy manipulation of the equation we end up with

$$\begin{aligned}
\dot{\mathcal{W}}^{\pm} = \pm \Big[& -\frac{k}{360\pi}\eta_{\pm}^{VII} + \frac{22}{45}\eta_{\pm}^{V}\mathcal{L}^{\pm} + \frac{58}{45}\eta_{\pm}^{IV}(\mathcal{L}^{\pm})' + \frac{7}{5}\eta_{\pm}'''(\mathcal{L}^{\pm})'' + \frac{11}{15}\eta_{\pm}''(\mathcal{L}^{\pm})''' \\
& + \frac{1}{5}\eta_{\pm}'(\mathcal{L}^{\pm})^{IV} + \frac{1}{30}\eta_{\pm}(\mathcal{L}^{\pm})^{V} - \frac{72\pi}{5k}\eta_{\pm}'''(\mathcal{L}^{\pm})^2 - \frac{442\pi}{15k}\eta_{\pm}''\mathcal{L}^{\pm}(\mathcal{L}^{\pm})' \\
& - \frac{38\pi}{3k}\eta_{\pm}'(\mathcal{L}^{\pm})'^2 - \frac{154\pi}{15k}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' - \frac{118\pi}{15k}\eta_{\pm}(\mathcal{L}^{\pm})'(\mathcal{L}^{\pm})'' - \frac{14\pi}{3k}\eta_{\pm}'\mathcal{L}^{\pm}(\mathcal{L}^{\pm})'' \\
& - \frac{52\pi}{15k}\eta_{\pm}\mathcal{L}^{\pm}(\mathcal{L}^{\pm})''' + \frac{256\pi^2}{5k^2}\eta_{\pm}'(\mathcal{L}^{\pm})^3 + \frac{384\pi^2}{k^2}\eta_{\pm}(\mathcal{L}^{\pm})^2(\mathcal{L}^{\pm})' \\
& - \frac{112\pi}{k}\eta_{\pm}'\mathcal{L}^{\pm}\mathcal{W}^{\pm} - \frac{56\pi}{k}\eta_{\pm}(\mathcal{L}^{\pm}\mathcal{W}^{\pm})' + 4\eta_{\pm}'''\mathcal{W}^{\pm} + 7\eta_{\pm}''(\mathcal{W}^{\pm})' \\
& + \xi_{\pm}(\mathcal{W}^{\pm})' + 4\xi_{\pm}'(\mathcal{W}^{\pm}) + 5\eta_{\pm}'(\mathcal{W}^{\pm})'' + \eta_{\pm}(\mathcal{W}^{\pm})'''\Big]
\end{aligned} \tag{E.43}$$

With the same choice of parameters, $\eta_{\pm} = 0$ and $\xi_{\pm} = 1$, we get:

$$\dot{\mathcal{W}}^{\pm} = \pm(\mathcal{W}^{\pm})' \tag{E.44}$$

E.4 Approximative Computation of the Black Hole Entropy

E.4.1 Preliminary Derivations

Let us repeat a result encountered in subsection 4.2.2:

$$S = -2k_3 \mathrm{Im}\left[\mathrm{tr}\left(A_\tau A_\varphi\right)\right]_{\text{on-shell}} \tag{E.45}$$

This formula is independent of the algebra the connection field is defined on. Let us first evaluate the trace in the case of $A_{\tau,\varphi} \in \mathfrak{so}(3,2)$:

$$\begin{aligned}\mathrm{tr}\left(A_\tau A_\varphi\right) =& \mathrm{tr}\left\{\left(-i\xi\left[L_{+1} - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-3}\right]\right.\right.\\ &- i\eta\left[W_{+3} - \frac{6\pi}{k}\mathcal{L}W_{+1} + \frac{12\pi^2}{k^2}\mathcal{L}^2W_{-1} - \frac{15\pi}{k}\mathcal{W}W_{-1}\right.\\ &\left.\left.- \frac{8\pi^3}{k^3}\mathcal{L}^3W_{-3} + \frac{22\pi^2}{k^2}\mathcal{L}\mathcal{W}W_{-3} - \frac{540\pi}{k}\mathcal{W}L_{-1}\right]\right)\\ &\left.\times\left(L_{+1} - \frac{2\pi}{k}\mathcal{L}L_{-1} - \frac{\pi}{2k}\mathcal{W}W_{-3}\right)\right\}\end{aligned} \tag{E.46}$$

To continue we need only to consider the products of generators with non-vanishing trace. We have

$$\mathrm{tr}\left(L_{\pm1}L_{\mp1}\right) = -20 \tag{E.47}$$

$$\mathrm{tr}\left(W_{\pm3}W_{\mp3}\right) = -7200 \tag{E.48}$$

and all other traces are equal to zero. Consequently, we obtain

$$\mathrm{tr}\left(A_\tau A_\varphi\right) = 2\frac{2i\pi}{k}\xi\mathcal{L}(-20) + \frac{540i\pi}{k}\eta\mathcal{W}(-20) + \frac{i\pi}{2k}\eta\mathcal{W}(-7200) \tag{E.49}$$

$$= -\frac{80i\pi}{k}\xi\mathcal{L} - \frac{14400i\pi}{k}\eta\mathcal{W} \tag{E.50}$$

Plugging this into the Euclidean version of the entropy yields with $k = 20k_3$:

$$S = -2k_3\mathrm{Im}\left(-\frac{80i\pi}{20k_3}\xi\mathcal{L} - \frac{14400i\pi}{20k_3}\eta\mathcal{W}\right) \tag{E.51}$$

$$= 2\,\mathrm{Re}\left[4\pi\xi\mathcal{L} + 720\pi\eta\mathcal{W}\right] \tag{E.52}$$

In the next step, let us derive the corresponding result in the Lorentzian formalism. Starting from

$$S = k_3\left[\mathrm{tr}\left(A_t^+A_\varphi^+\right) - \mathrm{tr}\left(A_t^-A_\varphi^-\right)\right]_{\text{on-shell}} \tag{E.53}$$

we again need to consider only the non vanishing traces:

$$\mathrm{tr}\left(A_t^\pm A_\varphi^\pm\right) = \pm\left[-2\frac{2\pi}{20k_3}\xi_\pm\mathcal{L}^\pm(-20) - \frac{\pi}{2\cdot 20k_3}\eta_\pm\mathcal{W}^\pm(-7200) - \frac{540\pi}{20k_3}\eta_\pm\mathcal{W}^\pm(-20)\right] \tag{E.54}$$

$$= \pm\frac{80\pi}{20k_3}\xi_\pm\mathcal{L}^\pm \pm \frac{14400\pi}{20k_3}\eta_\pm\mathcal{W}^\pm \tag{E.55}$$

This leads to the following intermediate expression for the Lorentzian entropy:

$$S = 4\pi\xi_+\mathcal{L}^+ + 720\pi\eta_+\mathcal{W}^+ + 4\pi\xi_-\mathcal{L}^- + 720\pi\eta_-\mathcal{W}^- \tag{E.56}$$

E.4.2 Linear and Quadratic Approximation

Euclidean Entropy
Recall the system of equation (5.94)

$$\begin{aligned}
0 &= 2\frac{\mathcal{L}}{k\pi}\xi^2 + 2^4 3^2 5\frac{\mathcal{W}}{k\pi}\xi\eta + 2^5 3^2\frac{\mathcal{L}}{k^3}\left(2^4\mathcal{L}^2\pi - 5\cdot 7k\mathcal{W}\right)\eta^2 - 1 \\
0 &= \ 2\left(2\cdot 17\pi k^4\mathcal{L}^2 - 3^2 5k^5\mathcal{W}\right)\xi^4 - 2^5 3^2\pi\left(2^4\pi k^3\mathcal{L}^3 - 5\cdot 41k^4\mathcal{L}\mathcal{W}\right)\xi^3\eta \\
&\quad + 2^6 3^3\pi k^2\left(2^8\pi^2\mathcal{L}^4 - 2^8 5\pi k\mathcal{L}^2\mathcal{W} + 3^2 5^4 k^2\mathcal{W}^2\right)\xi^2\eta^2 \\
&\quad + 2^9 3^4\pi^2 k\left(2^8\pi^2\mathcal{L}^5 + 2^5 5\cdot 17\pi k\mathcal{L}^3\mathcal{W} - 2^9 5^4 23k^2\mathcal{L}\mathcal{W}^2\right)\xi\eta^3 \\
&\quad + 2^9 3^4\pi^2\left(2^9 17\pi^3\mathcal{L}^6 - 2^6 5\cdot 11\pi^2 k\mathcal{L}^4\mathcal{W} - 2\cdot 5^4 7\pi k^2\mathcal{L}^2\mathcal{W}^2 - 3^2 5^7 k^3\mathcal{W}^3\right)\eta^4 \\
&\quad - 17k^6\pi^3
\end{aligned} \tag{E.57}$$

and make the ansatz

$$\begin{aligned}
\mathcal{W} &\to \delta\mathcal{W} \\
\xi &\to \xi_{\mathrm{BTZ}} + \delta\mathcal{W}\xi_1 + \delta\mathcal{W}^2\xi_2 = \sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}\xi_1 + \delta\mathcal{W}^2\xi_2 \\
\eta &\to \eta_{\mathrm{BTZ}} + \delta\mathcal{W}\eta_1 \qquad\quad = \qquad\quad \delta\mathcal{W}\eta_1
\end{aligned} \tag{E.58}$$

where we used that $\eta_{\mathrm{BTZ}} = 0$. Now plug these expansions into the first line of the system of equations above and, in a first step, retain only terms up to linear order in $\delta\mathcal{W}$:

$$\begin{aligned}
0 = \ & 2\frac{\mathcal{L}}{k\pi}\left(\sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}\xi_1\right)^2 + 2^4 3^2 5\frac{\delta\mathcal{W}}{k\pi}\left(\sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}\xi_1\right)\delta\mathcal{W}\eta_1 \\
& + 2^5 3^2\frac{\mathcal{L}}{k^3}\left(2^4\mathcal{L}^2\pi - 5\cdot 7k\delta\mathcal{W}\right)\delta\mathcal{W}^2\eta_1^2 - 1
\end{aligned} \tag{E.59}$$

$$= \ \frac{4\mathcal{L}}{k\pi}\sqrt{\frac{k\pi}{2\mathcal{L}}}\delta\mathcal{W}\xi_1 \tag{E.60}$$

Hence we obtain $\xi_1 = 0$ such that

$$\xi(\mathcal{L}, \delta\mathcal{W}) = \xi_{\mathrm{BTZ}} + \mathcal{O}(\delta\mathcal{W}^2) \tag{E.61}$$

and we already know that $\eta(\mathcal{L}, \delta\mathcal{W}) = \mathcal{O}(\delta\mathcal{W})$. We can now plug in the linearized expressions for ξ and η into the general relation for the $\mathfrak{so}(3,2)$ black hole:

$$S^{\mathrm{lin}} = 4\pi(\xi\mathcal{L} + \xi^*\mathcal{L}^*) \tag{E.62}$$

$$= 4\pi(\xi_{\mathrm{BTZ}}\mathcal{L} + \xi^*_{\mathrm{BTZ}}\mathcal{L}^*) \tag{E.63}$$

$$= 4\pi\left(\sqrt{\frac{k\pi}{2\mathcal{L}}}\mathcal{L} + \sqrt{\frac{k\pi}{2\mathcal{L}}}^*\mathcal{L}^*\right) \tag{E.64}$$

$$= 2\pi\sqrt{2k\pi}\left(\sqrt{\mathcal{L}} + \sqrt{\mathcal{L}^*}\right) \tag{E.65}$$

Based on this linear result we can go one step further and compute an expression for the entropy of to quadratic order in $\delta\mathcal{W}$. To do so, we plug the ansatz (E.58) into the second

equation of the system given at the beginning of this section and again retain terms up to linear order in $\delta\mathcal{W}$:

$$\begin{aligned}0 = \quad & 2\left(2\cdot 17\pi k^4\mathcal{L}^2 - 3^2 5 k^5\delta\mathcal{W}\right)\xi^4 - 2^5 3^2\pi\left(2^4\pi k^3\mathcal{L}^3 - 5\cdot 41 k^4\mathcal{L}\delta\mathcal{W}\right)\xi^3\eta \\ & + 2^6 3^3\pi k^2\left(2^8\pi^2\mathcal{L}^4 - 2^8 5\pi k\mathcal{L}^2\delta\mathcal{W} + 3^2 5^4 k^2\delta\mathcal{W}^2\right)\xi^2\eta^2 \\ & + 2^9 3^4\pi^2 k\left(2^8\pi^2\mathcal{L}^5 + 2^5 5\cdot 17\pi k\mathcal{L}^3\delta\mathcal{W} - 2^9 5^4 23 k^2\mathcal{L}\delta\mathcal{W}^2\right)\xi\eta^3 \\ & + 2^9 3^4\pi^2\left(2^9 17\pi^3\mathcal{L}^6 - 2^6 5\cdot 11\pi^2 k\mathcal{L}^4\delta\mathcal{W} - 2\cdot 5^4 7\pi k^2\mathcal{L}^2\delta\mathcal{W}^2\right. \\ & \left. -3^2 5^7 k^3\delta\mathcal{W}^3\right)\eta^4 - 17k^6\pi^3 \end{aligned} \tag{E.66}$$

$$\begin{aligned} = \quad & 2\left(2\cdot 17\pi k^4\mathcal{L}^2 - 3^2 5 k^5\delta\mathcal{W}\right)\xi^4 - 2^5 3^2\pi\left(2^4\pi k^3\mathcal{L}^3 - 5\cdot 41 k^4\mathcal{L}\delta\mathcal{W}\right)\xi^3\eta \\ & - 17k^6\pi^3 \end{aligned} \tag{E.67}$$

where we used the fact that all terms of quadratic or higher order in η will as well be of higher than linear order in $\delta\mathcal{W}$. Using that $\xi_1 = 0$ we can write:

$$\begin{aligned}0 = \quad & 2\left(2\cdot 17\pi k^4\mathcal{L}^2 - 3^2 5 k^5\delta\mathcal{W}\right)\left(\sqrt{\frac{k\pi}{2\mathcal{L}}}\right)^4 \\ & - 2^5 3^2\pi\left(2^4\pi k^3\mathcal{L}^3 - 5\cdot 41 k^4\mathcal{L}\delta\mathcal{W}\right)\left(\sqrt{\frac{k\pi}{2\mathcal{L}}}\right)^3\delta\mathcal{W}\eta_1 \\ & - 17k^6\pi^3 \end{aligned} \tag{E.68}$$

$$\approx -2\cdot 3^2 5 k^5\delta\mathcal{W}\frac{k^2\pi^2}{4\mathcal{L}^2} - 2^9 3^2\pi^2 k^3\mathcal{L}^3\sqrt{\frac{k^3\pi^3}{8\mathcal{L}^3}}\delta\mathcal{W}\eta_1 \tag{E.69}$$

$$= -2\cdot 3^2 5\frac{k^7\pi^2}{4\mathcal{L}^2}\delta\mathcal{W} - 2^8 3^2\pi^3 k^4\mathcal{L}^2\sqrt{\frac{k\pi}{2\mathcal{L}}}\delta\mathcal{W}\eta_1 \tag{E.70}$$

where we used the result that $\xi_1 = 0$ in the last equation. Hence we obtain

$$0 = -2\cdot 3^2\pi^2 k^4\left(5\frac{k^3}{4\mathcal{L}^2} + 2^7\pi\mathcal{L}^2\sqrt{\frac{k\pi}{2\mathcal{L}}}\eta_1\right)\delta\mathcal{W} \tag{E.71}$$

which is solved by

$$\eta_1 = -\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}. \tag{E.72}$$

Consider once more the first equation but this time up to quadratic order in $\delta\mathcal{W}$. Using the expressions for η_1 just found and the fact that $\xi_1 = 0$ we can compute an expression for ξ_2:

$$\begin{aligned}0 = \quad & 2\frac{\mathcal{L}}{k\pi}\left(\sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}^2\xi_2\right)^2 + 2^4 3^2 5\frac{\delta\mathcal{W}}{k\pi}\left(\sqrt{\frac{k\pi}{2\mathcal{L}}} + \delta\mathcal{W}^2\xi_2\right)\delta\mathcal{W}\eta_1 \\ & + 2^5 3^2\frac{\mathcal{L}}{k^3}\left(2^4\mathcal{L}^2\pi - 5\cdot 7k\delta\mathcal{W}\right)\delta\mathcal{W}^2\eta_1^2 - 1 \end{aligned} \tag{E.73}$$

$$\approx \quad \frac{4\mathcal{L}}{k\pi}\sqrt{\frac{k\pi}{2\mathcal{L}}}\delta\mathcal{W}^2\xi_2 + \frac{2^4 3^2 5}{\sqrt{2k\pi\mathcal{L}}}\delta\mathcal{W}^2\eta_1 + 2^9 3^2\frac{\mathcal{L}^3}{k^3}\pi\delta\mathcal{W}^2\eta_1^2 \tag{E.74}$$

$$= \quad \left(\sqrt{\frac{8\mathcal{L}}{k\pi}}\xi_2 - \frac{3^2 5^2 k^2}{2^5\mathcal{L}^4\pi^2} + \frac{3^2 5^2 k^2}{2^8\mathcal{L}^4\pi^2}\right)\delta\mathcal{W}^2 \tag{E.75}$$

This equation is solved by

$$\xi_2 = \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^9\pi^3}} \tag{E.76}$$

Thus we obtain the following functional expressions for ξ and η in terms of $\mathcal{L}$ and $\delta\mathcal{W}$ up to quadratic order in $\delta\mathcal{W}$:

$$\xi(\mathcal{L},\delta\mathcal{W}) = \sqrt{\frac{k\pi}{2\mathcal{L}}} + \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^9\pi^3}}\delta\mathcal{W}^2 + \mathcal{O}(\delta\mathcal{W}^3) \tag{E.77}$$

$$\eta(\mathcal{L},\delta\mathcal{W}) = -\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W} + \mathcal{O}(\delta\mathcal{W}^2) \tag{E.78}$$

Before we plug this into equation (5.88), we compute the products:

$$\xi\mathcal{L} = \sqrt{\frac{k\pi\mathcal{L}}{2}} + \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W}^2 \tag{E.79}$$

$$\eta\delta\mathcal{W} = -\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W}^2 \tag{E.80}$$

This finally leads to the Euclidean entropy up to quadratic order for the $\mathfrak{so}(3,2)$ black hole:

$$S^{quad} = 2\,\mathrm{Re}\left[4\pi\left(\sqrt{\frac{k\pi\mathcal{L}}{2}} + \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W}^2\right) - 720\pi\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^7\pi^3}}\delta\mathcal{W}^2\right] \tag{E.81}$$

$$= 2\,\mathrm{Re}\left[2\pi\sqrt{2\pi k\mathcal{L}} - \frac{225}{128}\sqrt{\frac{k^5}{2\pi\mathcal{L}^7}}\delta\mathcal{W}^2\right] \tag{E.82}$$

$$= 2\,\mathrm{Re}\left[2\pi\sqrt{2\pi k\mathcal{L}}\left(1 - \frac{225}{512}\frac{k^2}{\pi^2\mathcal{L}^4}\delta\mathcal{W}^2\right)\right] \tag{E.83}$$

Lorentzian Entropy

To find the Lorentzian expression we recall equation (E.56) which can be reorganized to

$$S^{quad} = 4\pi(\xi_+\mathcal{L}^+ + \xi_-\mathcal{L}^-) + 720\pi(\eta_+\delta\mathcal{W}^+ + \eta_-\delta\mathcal{W}^-) \tag{E.84}$$

Using the relations of the Euclidean-Lorentzian continuation given in [2]

$$\mathcal{L}^+ = \mathcal{L}, \qquad \delta\mathcal{W}^+ = \delta\mathcal{W}, \qquad \xi^+ = \xi, \qquad \eta^+ = \eta \tag{E.85}$$

$$\mathcal{L}^- = \mathcal{L}^*, \qquad \delta\mathcal{W}^- = -\delta\mathcal{W}^*, \qquad \xi^- = \xi^*, \qquad \eta^- = -\eta^* \tag{E.86}$$

we find by plugging in the expressions (E.79) and (E.80) yields:

$$\begin{aligned} S^{quad} = \quad & 4\pi\left(\sqrt{\frac{k\pi\mathcal{L}^+}{2}} + \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^{+7}\pi^3}}\delta\mathcal{W}^{+2} + \sqrt{\frac{k\pi\mathcal{L}^-}{2}} + \frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^{-7}\pi^3}}\delta\mathcal{W}^{-2}\right) \\ & + 720\pi\left(-\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^{+7}\pi^3}}\delta\mathcal{W}^{+2} - \frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^{-7}\pi^3}}\delta\mathcal{W}^{-2}\right) \end{aligned} \tag{E.87}$$

$$
\begin{aligned}
= \quad & 4\pi\left(\sqrt{\frac{k\pi\mathcal{L}^{+}}{2}}+\frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^{+7}\pi^3}}\delta\mathcal{W}^{+2}\right)-720\pi\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^{+7}\pi^3}}\delta\mathcal{W}^{+2} \\
& +4\pi\left(\sqrt{\frac{k\pi\mathcal{L}^{-}}{2}}+\frac{1575}{512}\sqrt{\frac{k^5}{2\mathcal{L}^{-7}\pi^3}}\delta\mathcal{W}^{-2}\right)-720\pi\frac{5}{256}\sqrt{\frac{k^5}{2\mathcal{L}^{-7}\pi^3}}\delta\mathcal{W}^{-2}
\end{aligned}
\tag{E.88}
$$

This ends up in the following expression:

$$
S^{quad} = \quad 2\pi\sqrt{2\pi k}\left(\sqrt{\mathcal{L}^{+}}+\sqrt{\mathcal{L}^{-}}\right)-\frac{225}{128}\sqrt{\frac{k^5}{2\pi}}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{\mathcal{L}^{+7}}}+\frac{\delta\mathcal{W}^{-2}}{\sqrt{\mathcal{L}^{-7}}}\right) \tag{E.89}
$$

Entropy in Terms of Lorentzian Charges

We finally want to express the entropy in terms of the Lorentzian mass $\mathcal{M}_{\text{Lor}}$ and the Lorentzian angular momentum $\mathcal{J}_{\text{Lor}}$. Recall that

$$
\mathcal{L}=\frac{1}{4\pi}(\mathcal{M}l+i\mathcal{J}) \qquad \text{and} \qquad \mathcal{L}^{*}=\frac{1}{4\pi}(\mathcal{M}l-i\mathcal{J}). \tag{E.90}
$$

Let us consider first only the BTZ-entropy:

$$
S^{BTZ} = 2\text{Re}\left[2\pi\sqrt{2\pi k\mathcal{L}}\right] \tag{E.91}
$$

$$
= 2\pi\sqrt{2\pi k}\left(\sqrt{\mathcal{L}}+\sqrt{\mathcal{L}^{*}}\right) \tag{E.92}
$$

$$
= 2\pi\sqrt{2\pi k}\left(\sqrt{\frac{1}{4\pi}(\mathcal{M}l+i\mathcal{J})}+\sqrt{\frac{1}{4\pi}(\mathcal{M}l-i\mathcal{J})}\right) \tag{E.93}
$$

$$
\begin{aligned}
\Rightarrow \quad \left(S^{BTZ}\right)^2 = {} & 4\pi^2(2\pi k)\left(\frac{1}{4\pi}(\mathcal{M}l+i\mathcal{J})+\frac{2}{4\pi}\sqrt{(\mathcal{M}l+i\mathcal{J})(\mathcal{M}l-i\mathcal{J})}\right. \\
& \left.+\frac{1}{4\pi}(\mathcal{M}l-i\mathcal{J})\right)
\end{aligned}
\tag{E.94}
$$

$$
= \pi(2\pi k)\left(2\mathcal{M}l+2\sqrt{\mathcal{M}^2l^2+\mathcal{J}^2}\right) \tag{E.95}
$$

$$
= \pi(2\pi k)2\mathcal{M}l\left(1+\sqrt{1+\frac{\mathcal{J}^2}{\mathcal{M}^2l^2}}\right) \tag{E.96}
$$

Recall now that $k=\frac{l}{4G}$, $\mathcal{M}=\mathcal{M}_{\text{Lor}}$ and $i\mathcal{J}=\mathcal{J}_{\text{Lor}}$. Plugging these relations in the latter expression for the Eulerian BTZ-entropy squared yields:

$$
\left(S^{BTZ}\right)^2 = 4\pi^2k\mathcal{M}l\left(1+\sqrt{1+\frac{\mathcal{J}^2}{\mathcal{M}^2l^2}}\right) \tag{E.97}
$$

$$
= 4\pi^2\frac{l}{4G}\mathcal{M}_{\text{Lor}}l\left(1+\sqrt{1-\frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}}l^2}}\right) \tag{E.98}
$$

$$
= \pi^2l^2\frac{\mathcal{M}_{\text{Lor}}}{G}\left(1+\sqrt{1-\frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}}l^2}}\right) \tag{E.99}
$$

Hence we find in analogy to the spin 2 case:

$$S^{BTZ} = \pi l \sqrt{\frac{\mathcal{M}_{\text{Lor}}}{G}\left(1+\sqrt{1-\frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}} l^2}}\right)} \tag{E.100}$$

To obtain the quadratic approximation in terms of the Lorentzian variables we manipulate the entropy expression as follows:

$$S^{quad} = 2 \text{ Re}\left[2\pi\sqrt{2\pi k\mathcal{L}} - \frac{225}{128}\sqrt{\frac{k^5}{2\pi\mathcal{L}^7}}\delta\mathcal{W}^2\right] \tag{E.101}$$

$$= S^{BTZ} - \frac{225}{128}\sqrt{\frac{1}{2\pi}\left(\frac{l}{4G}\right)^5}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{\left(\frac{1}{4\pi}(\mathcal{M}l+i\mathcal{J})\right)^7}} + \frac{\delta\mathcal{W}^{-2}}{\sqrt{\left(\frac{1}{4\pi}(\mathcal{M}l-i\mathcal{J})\right)^7}}\right) \tag{E.102}$$

Hence we obtain for the entropy of the spin-4 black hole up to quadratic order in $\delta\mathcal{W}$ the following final expression:

$$\begin{aligned} S^{quad} = {} & \pi l \sqrt{\frac{\mathcal{M}_{\text{Lor}}}{G}\left(1+\sqrt{1-\frac{\mathcal{J}^2_{\text{Lor}}}{\mathcal{M}^2_{\text{Lor}} l^2}}\right)} \\ & - \frac{225}{32}\pi^3\sqrt{\frac{l^5}{2G^5}}\left(\frac{\delta\mathcal{W}^{+2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l+\mathcal{J}_{\text{Lor}}\right)^7}} + \frac{\delta\mathcal{W}^{-2}}{\sqrt{\left(\mathcal{M}_{\text{Lor}}l-\mathcal{J}_{\text{Lor}}\right)^7}}\right) \end{aligned} \tag{E.103}$$

E.4.3 Differential Equation Ansatz for an Exact Solution

We have found a linearized solution for the black hole entropy. This allows us to proceed as in [26]. Make the following ansatz:

$$S = 2\pi\sqrt{2\pi k}\left[\sqrt{\mathcal{L}}f(y) + \sqrt{\mathcal{L}^*}f(y^*)\right] \tag{E.104}$$

with

$$y := Ck^n\pi^m\frac{\mathcal{W}}{\mathcal{L}^2} \qquad \text{and} \qquad y^* := Ck^n\pi^m\frac{\mathcal{W}^*}{(\mathcal{L}^*)^2} \tag{E.105}$$

and

$$f(0) = 1. \tag{E.106}$$

Here, C denotes a numerical prefactor and n and m are suitable exponents, all of which are to be determined. This ansatz is motivated by the fact that it continuously deforms into the entropy of the BTZ black hole for $y \propto \mathcal{W} \to 0$ and that the ratio $\mathcal{W}/\mathcal{L}^2$ is dimensionless as can be seen from the scaling relations given by the Poisson bracket (5.61). Accordingly, any function of y will be dimensionless, too, such that multiplying it to any quantity will preserve the dimension of the latter. In particular, the quantity S in (E.104) is guaranteed to be an entropy. As it consists of two terms with equal structure (one depending on complex-conjugated quantities), we are free to consider just the first term. In order to obtain a final result, we then can simply add the second term in the end. Recall the relations given in [26] (notice that we use a slightly different notation):

$$\xi = -\frac{i}{4\pi^2}\frac{\partial S}{\partial\mathcal{L}} \qquad \text{and} \qquad \eta = \frac{i}{4\pi^2}\frac{\partial S}{\partial\mathcal{W}} \tag{E.107}$$

Plugging the ansatz into these relations and the results into the first equation of the system (5.94) first makes clear that a suitable choice for the parameters to be determined is given by $n = 1$, $m = -1$ and $C = 1/96$ such that

$$y := \frac{k\mathcal{W}}{96\pi\mathcal{L}^2}. \tag{E.108}$$

Second, it yields the following differential equation:

$$f^2(y) + 728yf(y)f'(y) - (2896y^2 - 210y + 1)[f'(y)]^2 + 4\pi^2 = 0 \tag{E.109}$$

This is a non-separable, non-linear differential equation. Unfortunately, so far no exact and simultaneously compact solution could be found.

Appendix F

Explicit Commutator Relations for $\mathfrak{so}(3,2)$ Generators

F.1 L-L-Commutator Relations

$$
\begin{array}{lll}
[L_{+1}, L_{+1}] = 0, & [L_{+1}, L_0] = L_{+1}, & [L_{+1}, L_{-1}] = 2L_0 \\
[L_0, L_{+1}] = -L_{+1}, & [L_0, L_0] = 0, & [L_0, L_{-1}] = L_{-1} \\
[L_{-1}, L_{+1}] = -2L_0, & [L_{-1}, L_0] = -L_{-1}, & [L_{-1}, L_{-1}] = 0
\end{array}
$$

F.2 L-W-Commutator Relations

$$
\begin{array}{lll}
[L_{+1}, W_{+3}] = 0, & [L_0, W_{+3}] = -3W_{+3}, & [L_{-1}, W_{+3}] = -6W_{+2} \\
[L_{+1}, W_{+2}] = W_{+3}, & [L_0, W_{+2}] = -2W_{+2}, & [L_{-1}, W_{+2}] = -5W_{+1} \\
[L_{+1}, W_{+1}] = 2W_{+2}, & [L_0, W_{+1}] = -W_{+1}, & [L_{-1}, W_{+1}] = -4W_0 \\
[L_{+1}, W_0] = 3W_{+1}, & [L_0, W_0] = 0, & [L_{-1}, W_0] = -3W_{-1} \\
[L_{+1}, W_{-1}] = 4W_0, & [L_0, W_{-1}] = W_{-1}, & [L_{-1}, W_{-1}] = -2W_{-2} \\
[L_{+1}, W_{-2}] = 5W_{-1}, & [L_0, W_{-2}] = 2W_{-2}, & [L_{-1}, W_{-2}] = -W_{-3} \\
[L_{+1}, W_{-3}] = 6W_{-2}, & [L_0, W_{-3}] = 3W_{-3}, & [L_{-1}, W_{-3}] = 0
\end{array}
$$

F.3 W-W-Commutator Relations

$$
\begin{aligned}
[W_{+3}, W_{+3}] &= 0, & [W_{+2}, W_{+3}] &= 0,\\
[W_{+3}, W_{+2}] &= 0, & [W_{+2}, W_{+2}] &= 0,\\
[W_{+3}, W_{+1}] &= 0, & [W_{+2}, W_{+1}] &= -4W_{+3},\\
[W_{+3}, W_{0}] &= 6W_{+3}, & [W_{+2}, W_{0}] &= -6W_{+2},\\
[W_{+3}, W_{-1}] &= 24W_{+2}, & [W_{+2}, W_{-1}] &= -120L_{+1},\\
[W_{+3}, W_{-2}] &= 60W_{+1} + 360L_{+1}, & [W_{+2}, W_{-2}] &= 20W_0 - 240L_0,\\
[W_{+3}, W_{-3}] &= 120W_0 + 2160L_0, & [W_{+2}, W_{-3}] &= 60W_{-1} + 360L_{-1},
\end{aligned}
$$

$$
\begin{aligned}
[W_{+1}, W_{+3}] &= 0, & [W_{0}, W_{+3}] &= -6W_{+3},\\
[W_{+1}, W_{+2}] &= 4W_{+3}, & [W_{0}, W_{+2}] &= 6W_{+2},\\
[W_{+1}, W_{+1}] &= 0, & [W_{0}, W_{+1}] &= 6W_{+1} - 72L_{+1},\\
[W_{+1}, W_{0}] &= -6W_{+1} + 72L_{+1}, & [W_{0}, W_{0}] &= 0,\\
[W_{+1}, W_{-1}] &= -8W_0 + 48L_0, & [W_{0}, W_{-1}] &= -6W_{-1} + 72L_{-1},\\
[W_{+1}, W_{-2}] &= -120L_{-1}, & [W_{0}, W_{-2}] &= -6W_{-2},\\
[W_{+1}, W_{-3}] &= 24W_{-2}, & [W_{0}, W_{-3}] &= 6W_{-3},
\end{aligned}
$$

$$
\begin{aligned}
[W_{-1}, W_{+3}] &= -24W_{+2}, & [W_{-2}, W_{+3}] &= -60W_{+1} - 360L_{+1},\\
[W_{-1}, W_{+2}] &= 120L_{+1}, & [W_{-2}, W_{+2}] &= -20W_0 + 240L_0,\\
[W_{-1}, W_{+1}] &= 8W_0 - 48L_0, & [W_{-2}, W_{+1}] &= 120L_{-1},\\
[W_{-1}, W_{0}] &= 6W_{-1} - 72L_{-1}, & [W_{-2}, W_{0}] &= 6W_{-2},\\
[W_{-1}, W_{-1}] &= 0, & [W_{-2}, W_{-1}] &= 4W_{-3},\\
[W_{-1}, W_{-2}] &= -4W_{-3}, & [W_{-2}, W_{-2}] &= 0,\\
[W_{-1}, W_{-3}] &= 0, & [W_{-2}, W_{-3}] &= 0,
\end{aligned}
$$

$$
\begin{aligned}
[W_{-3}, W_{+3}] &= -120W_0 - 2160L_0,\\
[W_{-3}, W_{+2}] &= -60W_{-1} - 360L_{-1},\\
[W_{-3}, W_{+1}] &= -24W_{-2},\\
[W_{-3}, W_{0}] &= -6W_{-3},\\
[W_{-3}, W_{-1}] &= 0,\\
[W_{-3}, W_{-2}] &= 0,\\
[W_{-3}, W_{-3}] &= 0,
\end{aligned}
$$

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