Koszul Duality and Some of Its Physical Guises

Aleksandar Ivanov

February 17, 2023

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1 Introduction

Progress in physics in the past 60 years has put a spotlight on dualities; relations between two different mathematical descriptions of the same underlying physics. One duality that has appeared in seemingly disparate areas of mathematics has been Koszul duality, and more recently it has also made a crossover into physics. This is an algebraic type of duality (as opposed to a topological or geometric type) that relates two algebras of a certain type. On the physics side, these algebras are interpreted as the algebras of observables of some theory and thus, the dual algebras describe a different theory. One example is [2], which gives arguments that Koszul duality can be related to certain examples of the AdS/CFT correspondence.

The complications due to operator product expansions that arise in a lot of theories mean that we cannot just consider the algebras to be of the usual associative type. Rather they should be generalized to homotopy associative (and, if necessary, homotopy commutative) algebras. This simply means that associativity and commutativity don't hold 'on the nose', but rather they only hold up to homotopy, giving us the leeway necessary to describe these more complicated theories. Through this, the need arises to define this duality on a more general level, and this is done through operad theory. We will then specialize the examples to the more classical types of algebra for simplicity, but the formalism is of course general and continues to apply even in the case of homotopy associative algebras. In fact, operad theory is the main way used to formalize these concepts in the literature.

Koszul duality is also one part of the Poincaré–Koszul duality of Ayala and Francis [1] at the level of factorization homology. The implications to physics of this formalism are still under investigation.

A use of Koszul duality that we will explore is in coupling two algebras of observables, for example in the case of a theory with defects. Specifically, we elucidate the case of a one-dimensional, Euclidean, topological defect, an idea that is explored in [9] but in less detail. This formulation also provides an outlook to generalizations to other topologies and dimensions.

2 Operads

As we mentioned before, the way to state Koszul duality in the clarity we need is through the language of *operad theory*. In non-technical terms, this theory tries to mathematically encode what we mean by n-ary operations (i.e. operations with n inputs), like associative multiplication or the Lie bracket. In the same way that modules over an algebra (or representations over a group) are a way to make concrete the structure of the algebra, we should think of operads as one level above. Namely, they're the abstraction of the operations of an algebra, making the algebra a kind of representation of the operations. The large majority of constructions in this section can be found in a lot more detail in the paper [5]. Another, more recent source is the book "Algebraic operads" [7] as well as the good introductory presentation by one of its authors, which can be found in the talk [10].

2.1 Differential Graded Vector Spaces

For physical applications it will turn out that a lot of the time the setting for the whole following theory would be differential graded vector spaces. Roughly speaking this is because often we look at theories that have a BRST differential or possibly a supercharge; these are the possible differentials in question. That's why here we present a short introduction to differential graded vector spaces.

Definition 2.1. A graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V^i$ together with maps $\mathrm{d}_i : V^i \to V^{i+1}$ (often abbreviated as $\mathrm{d}\colon V \to V$), called the *differential*, such that $\mathrm{d}\circ\mathrm{d} = 0$ is called a *differential graded vector space.*

Remark. In the previous definition we've taken the grading to be a $\mathbb{Z}$ grading but, this condition is flexible. Common alternative choices include $\mathbb{N}$ and $\mathbb{Z}/2\mathbb{Z}$.

As usual, the tensor product of graded vector spaces is given by

$$(V\otimes W)^i := \bigoplus_{j+k=i} V^j \otimes W^k, \tag{1}$$

which can also inherit a differential graded structure from V and W by $\mathrm{d} := \mathrm{d}_V \otimes 1 + 1 \otimes \mathrm{d}_W$, which is a short form of

$$\forall v\in V^i\colon\ \forall w \in W^j\colon\ \mathrm{d}(v\otimes w) := \mathrm{d}_V(v)\otimes w + (-1)^i v\otimes \mathrm{d}_W(w). \tag{2}$$

The above is a general construction, which has two important special cases. Namely, if we specifically prescribe the tensor product to have the property[1]

$$v\otimes w \mapsto w\otimes v, \tag{3}$$

[1]The map here is the commutativity isomorphism of the symmetric monoidal structure that we are putting on the category of graded vector spaces under the tensor product, in categorical terms.

we get the case of commuting variables, as they're called in physics, while if we prescribe the property

$$v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v, \tag{4}$$

we get the case of anticommuting variables. The second case gives us super vector spaces (and further super structures).

For a differential graded vector space we will need the construction of the dual which we can define as

$$(V^*)^i := (V^{-i})^*, \qquad \mathrm{d}_{V^*} := (\mathrm{d}_V)^*. \tag{5}$$

As a simple case we can always degenerate to the case of usual vector spaces by taking $V^i = 0$ if $i \neq 0$ and $\mathrm{d} = 0$, namely we take the graded vector space to be concentrated in degree-0 and the differential to be trivial. Additionally, we will need the notation of a shift in grading, namely,

$$(V[i])^j := V^{i+j}, \qquad \mathrm{d}_{V[i]} := (-1)^i(\mathrm{d}_V). \tag{6}$$

All of the constructions we will do for operads can be considered to take place in any of the contexts we mentioned without much change. That is, we can take the underlying vector spaces to have the extra grading and differential structure if we want.

2.2 Definition of an Operad

There are very general definitions of operads, but for us, it will be enough to only look at linear operads here.[2]

Definition 2.2. Given a field k (of characteristic 0), a *(k-linear) operad* P is a collection $\{P(n) \mid n \geq 0\}$ of (possibly differential graded) k-vector spaces equipped with:

1. an action of the symmetric group Σ_n on each $P(n)$,
2. linear maps (which respect the possible differential graded structure) called *compositions*

$$\gamma_{\{m_1,\ldots,m_l\}} : P(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l) \to P(m_1 + \ldots m_l) \tag{7}$$

3. a *unit* $1 \in P(1)$ such that:

$$\forall l \geq 1\colon \forall \mu \in P(l)\colon \ \gamma_{\{1,\ldots,1\}}(\mu, 1, \ldots, 1) = \mu, \tag{8}$$

all such that they satisfy *associativity* and *equivariance*, which we will describe below.

Remark. We will abbreviate the compositions maps as $\gamma_{\{m_i\}}(\mu, v_1, \ldots, v_l) = \mu(v_1, \ldots v_l)$, because the idea that we're implementing is that the elements of $P(n)$ are n-ary operations, and the aptly named compositions maps are exactly the compositions of those operations.

Remark. There is a formal way to understand what associativity and equivariance imply, but the simplest explanation (which can be formalized, see [5]) is through trees and is what we will look into here.

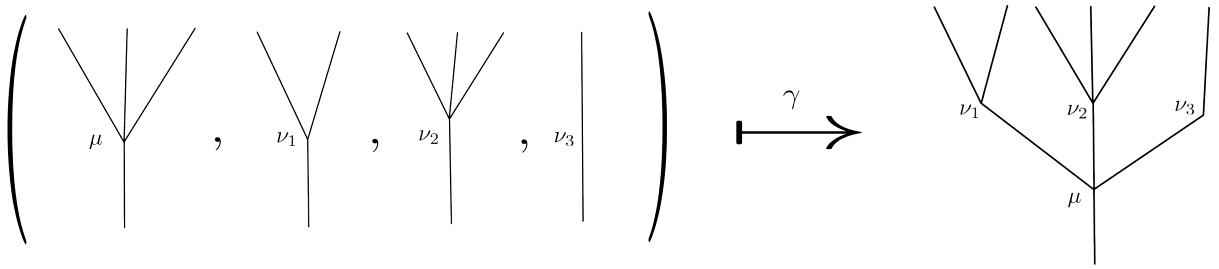

Figure 1: The composition maps provide a way to combine operations, which we visualize by tree concatenation. The image shows the example of $\mu \in P(3)$, $\gamma_{\{2,3,1\}}(\mu, \nu_1, \nu_2, \nu_3) \in P(6)$.

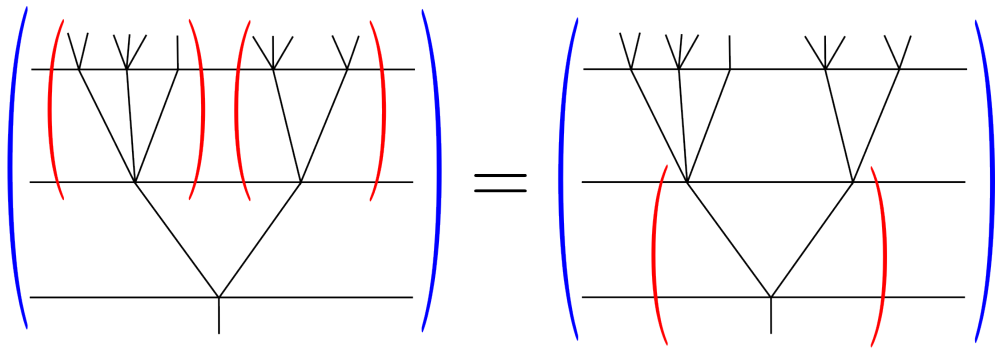

Figure 2: Associativity states that the order of concatenations does not matter. If we do the operations (red then blue / inside then outside) as on the left, this gives the same result as the order on the right

We visualize these operations as a tree that has a number of inputs and a single output. The composition maps then tell us how to concatenate these trees into a different tree (see fig. 1); plugging the outputs of l-many operations $v_1, \dots, v_l$ into the inputs of the l-ary operation μ. In this language, associativity simply states that whenever we have multiple compositions to do, the order in which we do the concatenations of trees does not matter (see fig. 2).

The symmetric group's involvement also becomes clear, because it says that we need to keep track of the order of the input legs of the tree. Namely, the equivariance requires that if the order or the inputs is permuted, then we need to change the operation that the tree represents by acting with this permutation through the specified action (see fig. 3).

Example 2.1. The prototypical operad is the operad of endomorphisms of a (k-)vector space V, denoted by $\mathcal{E}_V$. It is the operad whose collection of vector spaces are the maps

$$\mathcal{E}_V(n) := Hom(V^{\otimes n}, V). \tag{9}$$

The action of the symmetric group is by permutation of the arguments, and compositions are defined in the obvious way. The unit is given by $\mathrm{id}_V : V \to V$.

Now that we have the abstract notion of operad, we can connect it to the notion of an algebra, by the following definition.

Definition 2.3. Given an operad P, a P-algebra is a k-vector space A together with linear maps

$$f_n : P(n) \otimes A^{\otimes n} \to A, \tag{10}$$

[2]We mention that the more general theory, which uses categorical notions, acknowledges that the structure we use in the case of linear operads is only the symmetric monoidal structure of $(\mathrm{Vect}_k, \otimes)$. Therefore, we can export the definition to any other symmetric monoidal category (e.g. $(\mathrm{Top}, \times)$ for a topological operad). These are notions that are needed in the homotopy associative case, as suggested by the word homotopy.

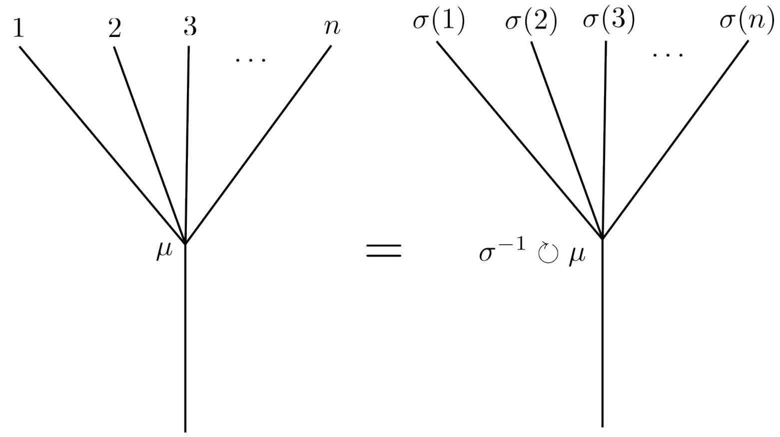

Figure 3: Equivariance states that if we permute the labels of the legs of a tree by $\sigma \in \Sigma_n$ we need to change the operation that the tree represents through the action of the symmetry group on the space of operations of the given arity.

abbreviated by $f_n(\mu, a_1, \ldots, a_n) = \mu(a_1, \ldots, a_n)$, such that:

1. they are equivariant with respect to the Σ_n action,
2. they commute with compositions, and
3. $f_1(1, -) = 1_A$.

Remark. If we were to develop the theory of operads further we would see that the previous data is given equivalently by the existence of a morphism of operads $f : P \to \mathcal{E}_A$.

Remark. If we're working in the context of super vector spaces then the algebras from the definition will be superalgebras, and if we're working with differential graded vector spaces, the algebras will be differential graded algebras. When we do not wish to differentiate between the cases, we will still use the terminology algebra, relying on the context to make things clear.

Example 2.2. The associative operad is defined as

$$Ass(n) := \text{span}_k\{x_{s(1)} \ldots x_{s(n)} \mid s \in \Sigma_n\} \qquad (n \geq 1), \tag{11}$$

and $Ass(0) = 0$, where the Σ_n action is given by permutations. Compositions are given by replacing some x_i by another specified string of multiple xs with some indices. These can then be relabeled if necessary. This operad models associativity, in that the algebras over this operad are exactly associative algebras. Roughly, associativity for algebras means that we don't need parentheses to write products, i.e. that at the level of n inputs there are as many distinct operations as there are permutations in Σ_n and no more (which would come about if different parenthesizations were different); this is exactly what the operad above encodes.

Example 2.3. The commutative operad is defined as

$$Com(n) := \text{span}_k\{\mu_n\} \qquad (n \geq 1), \tag{12}$$

and $Com(0) = 0$, where the Σ_n action is the trivial one. Compositions are defined on the generators as $\mu_l(\mu_{i_1}, \ldots, \mu_{i_l}) = \mu_{i_1+\cdots+i_l}$ and extended linearly. Algebras over this operad are exactly commutative (and associative) algebras; there is only one operation with n inputs and permutations don't change it.

Example 2.4. The Lie operad is defined as

$$Lie(n) := \mathrm{span}_k\{[x_{s(1)}, [\ldots, x_{s(n)}] \ldots] \mid s \in \Sigma_n\} \qquad (n \geq 1), \tag{13}$$

and $Lie(0) = 0$, where the Σ_n action is given by permutations. The square brackets in the definition are the commutator $[x_1, x_2] := x_1x_2 - x_2x_1$, where the multiplication is formal. This means that unlike until now, the elements in the argument of the span are not linearly independent, because of the antisymmetry and Jacobi properties. It can be shown by proper counting that $\dim_k Lie(n) = (n-1)!$. Compositions, as in the associative case, are given by replacing some x_i by another specified string of multiple bracketed xs. The algebras over this operad are exactly Lie algebras.

Remark. If we want to make sure that the algebras over the previous operads are unital then we simply have to modify the operad by making $uP(0) = k$ instead of $P(0) = 0$. The map $f_0 : uP(0) = k \to A$ then specifies the unit.

2.3 Quadratic Operads and the Koszul Dual

The type of operad for which we will define Koszul duality is called a quadratic operad. The notion can be generalized, but we will not look into that here. Roughly speaking, the idea is to have an operad with (mainly) binary operations and some ternary relations that these operations satisfy. This is the case for the most common operads; for example *Ass*, *Com* and *Lie* from before.

The general idea for the construction is to look at the 'free operad' generated by the binary operations and then mod out the 'operadic ideal' generated by the relations. These terms require a some explanation, which we will partly give, however for the full details [5, 7] are recommended.

Definition 2.4 (Ideal)**.** Given an operad P, I is a (double-sided) ideal of P if:

1. $I(n) \subseteq P(n)$ is preserved by the Σ_n action, and
2. it holds that *any* argument of a composition γ being in the ideal guarantees that the result of the composition is also in the ideal.

The full construction of the free operad requires more work, and we will not give all the details, but the following informal definition captures the idea in the quadratic case.

Definition 2.5 (Quadratic Free Operad)**.** Given a Σ_2 representation space E (representing our binary operations) we construct the free operad as

$$F(E)(n) := \bigoplus_{\text{binary } n\text{-trees } T} \; \bigotimes_{v \in T} E, \tag{14}$$

where v runs through all the vertices in T. (Thus we construct the space of all possible n-ary operations arising from a set of given binary operations)[3]

Given these ingredients we can construct the notion of a quadratic operad.

Definition 2.6 (Quadratic Operad)**.** Given E as before, let $R \subseteq F(E)(3)$ be a subspace, and (R) the operadic ideal generated by it. These data define a *quadratic operad* by

$$P(E, R) := F(E)/(R). \tag{15}$$

[3]The categorical definition of a free operad given a collection $\{M(n) \mid M(n) \text{ is a } \Sigma_n\text{-repr.}\}$ follows the usual procedure by universal property. (see [7, Chap. 5])

We again underline the important point that the constructing data of a quadratic operad is some number of binary operations (parametrized by E) and some number of relations of these operations (parametrized by R). This allows us the important construction of dualization. Namely, we can consider the dual E^* of the vector space of binary operations and, after noting that $F(E^*)(3) = F(E)(3)^*$, we can consider the orthogonal complement $R^\perp \subseteq F(E^*)(3)$.

Remark. Here the orthogonal complement is a subspace of the dual, namely if $W \subseteq V$ then

$$W^\perp := \{f \in V^* \mid \forall w \in W\colon\ f(w) = 0\}. \tag{16}$$

The dual spaces also need an action of Σ_2 to make them usable to us, thus we fix it to be

$$\forall \sigma \in \Sigma_2\colon\ \forall f \in E^*\colon\ \forall e \in E\colon\ \sigma(f)(e) := \operatorname{sgn}(\sigma) f(\sigma^{-1}(e)). \tag{17}$$

Remark. At first, it might seem unnatural to choose the action with the extra sign factor from the permutation, but we will later see that this is a necessary choice to get a duality.

Definition 2.7 (Koszul Dual)**.** The Koszul dual of the quadratic operad $P(E, R)$ is the quadratic operad

$$P(E, R)^! := P(E^*, R^\perp). \tag{18}$$

Remark. The symbol ! is usually pronounced 'shriek'.

The idea of the Koszul dual is that it represents "the same" binary operations, but satisfying the "orthogonal" relations.

We want to bring this construction down to the algebra level, which requires defining what a quadratic algebra is.

Definition 2.8 (Free Algebra)**.** Given a vector space V, the free algebra over an operad P generated by V is given by

$$F_P(V) := \bigoplus_{n \geq 0} \left(P(n) \otimes V^{\otimes n}\right)_{\Sigma_n}, \tag{19}$$

where the Σ_n acts diagonally over the tensor product and the subscript notation denotes the elements that are not invariant under the action. The compositions in P induce maps $P(n) \otimes F_P(V)^{\otimes n} \to F_P(V)$, which are the maps that make $F_P(V)$ into an algebra in the sense of definition 2.3.

The idea of the free algebra over an operad is that we 'fill in' the slots of the operations of the operad with some undetermined inputs, and we're then allowed to consider the compositions in the operad as operations in the algebra.

Remark. From the definition of the free algebra it is obvious that it has a natural *weight* grading, inherited from P and unrelated to any possible grading that V might have. This will play an important role later on.

Until now, the fact that the underlying vector spaces could have the extra structure of a grading and/or differential has not played any particular role. However, we will now get the first instance where this makes a difference. Namely, it will turn out that the dual of an algebra (even grading) is a superalgebra (odd grading).

Definition 2.9 (Quadratic Algebra)**.** Let P be a quadratic operad and V a vector space of generators. Further, let $S \subseteq \left(P(2) \otimes V^{\otimes 2}\right)_{\Sigma_2}$ be a subspace of relations. Then we define the quadratic algebra

$$A(V, S) := F_P(V)/(S), \tag{20}$$

where (S) again denotes the ideal generated by S.

Remark. If we're working in the context of differential graded vector spaces, then the algebras are differential graded algebras (dgas), which will play a big role for us. Looking back to the two special cases of even and odd variables eqs. (3) and (4), and denoting them with $\pm$ respectively, we have a different action of the symmetric group

$$\forall \sigma \in \Sigma_2 \colon \forall \mu \in P(2) \colon \forall v_1, v_2 \in V \colon \sigma(\mu \otimes (v_1 \otimes v_2)) = (\pm 1)^{\deg(v_1)\deg(v_2)} \sigma(\mu) \otimes (v_2 \otimes v_1). \quad (21)$$

This makes a difference, for example, in the definition of the free algebra.

Remark. Since S is homogeneous in the weight grading of $F_P(V)$, $A(V, S)$ inherits this.

Definition 2.10. Given a quadratic operad P and a quadratic algebra $A = A(V, S)$ we define the quadratic algebra that is Koszul dual to A as

$$A^! := A(V^*[1], S^\perp[2]). \quad (22)$$

Remark. Recall that the square brackets denote a shifting of the degree of the differential graded vector space. This is exactly why we needed to choose an action that had an extra factor of $\mathrm{sgn}(\sigma)$ on the dual space (see eq. (17)). It is necessary to make the following key computation work

$$\left(\left(E \otimes V^{\otimes 2} \right)_{\Sigma_2} \right)^* = \left(E^* \otimes (V^*)^{\otimes 2} \right)_{\Sigma_2}, \quad (23)$$

where it's important to note that the actions are different. This choice is what allows for the well-definition of the dual.

Remark. The construction follows the template from the operad construction, but we do learn the interesting fact that algebras are dual to superalgebras if we specialize to those special cases. This again comes about as a consequence of the shifting and choice of action on E^*.

The construction of the Koszul dual makes the following theorem clear:

Theorem 2.1. *At both the operad and algebra level, for finite dimensional vector spaces, we have*

$$(P^!)^! \cong P \qquad \text{and} \qquad (A^!)^! \cong A. \quad (24)$$

For the common examples of operads that we looked at in the examples above we have the following theorem, which we state here without proof. The proof can be found in both [5] and [7].

Theorem 2.2. *The operads Ass, Com and Lie are quadratic, and it holds that*

$$Ass^! \cong Ass \qquad \qquad Com^! \cong Lie. \quad (25)$$

Remark. The theorem remains valid even if we modify all operads to their unital versions.

2.4 Some Calculations

Example 2.5. The most famous (and motivating) example of Koszul duality for algebras is the duality between the symmetric and exterior algebras. We first choose the operad to be $uAss$, which is the unital ($uAss(0) = k$) version of the associative operad. Furthermore, we work in the context of non-differential vector spaces. In this case $uAss$-algebras and $uAss$-superalgebras are the same thing, which is related to eq. (25); they both give the usual notion of a unital graded associative algebra. Since we're in the unital case the free algebra $F_{uAss}(V)$

is nothing but the tensor algebra of V (in the non-unital case it would have been the reduced tensor algebra), where we choose V to be finite dimensional.

We can now look at the specific case of taking the relations

$$S = \text{span}\{x \otimes y - y \otimes x \mid x, y \in V\} \subset V^{\otimes 2}. \tag{26}$$

The algebra generated in this way is the symmetric algebra of a vector space $Sym(V) = A(V, S)$. To compute the dual we need to find the orthogonal complement $S^{\perp}$. It's easy to see that

$$S^{\perp} = \{\hat{x} \otimes \hat{y} + \hat{y} \otimes \hat{x} \mid \hat{x}, \hat{y} \in V^*\} \subset (V^*)^{\otimes 2}, \tag{27}$$

which can be seen as the equivalent of the linear algebra fact that $A_{ij}B^{[ij]} = 0 \Rightarrow A_{ij} = A_{(ij)}$. Thus, the Koszul dual algebra is

$$A(V^*[1], S^{\perp}[2]) = \Lambda(V^*) \qquad \Rightarrow \qquad Sym(V)^! = \Lambda(V^*). \tag{28}$$

In this example the shifting plays no role since at this level, of non-differential vector spaces that don't have a notion of commutativity, it doesn't make any difference.

The example can also be viewed in the context of the $uCom$ operad. The free algebra in this operad is exactly the symmetric algebra. By the definition of the duality we know that the dual should be a $uLie$-superalgebra in V^*, which the exterior algebra is. The shifting is now relevant, and leads to the anti-commutativity of the exterior algebra.

Physically, this has the flavor of a boson-fermion duality, since if we choose a basis $\{x_i\}$ of V, where all the x_i are even (commuting) variables, then $Sym(V) \cong k[\{x_i\}]$ the polynomials in x_i over the field k. On the other hand, if we choose a basis $\{\eta_i\}$ of V^*, where all the η_i are now odd (anti-commuting) variables, then $\Lambda(V^*) \cong k[\{\eta_i\}]$. Namely, the polynomials in even variables are dual to the polynomials in odd variables.

Example 2.6. Another physically important example is the example of the universal algebra $U(\mathfrak{g})$ of a given Lie algebra $\mathfrak{g}$. We should remember here that an enveloping algebra is always an associative algebra, which means that this example again takes place in the context of $uAss$. It is actually related to the previous example, but there is a difficulty. The problem is that the relations that define the universal enveloping algebra of $\mathfrak{g}$ are not of the kind that we considered in the theory above, rather they are

$$S = \text{span}\{[x, y] - x \otimes y + y \otimes x \mid x, y \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2}. \tag{29}$$

One consequence is that $U(\mathfrak{g})$ does not inherit a grading from the free algebra since we don't quotient by a homogeneous ideal to get it. This inhomogeneity can be dealt with though (for the details see [7, Chap. 3.6]) and the result ends up being

$$U(\mathfrak{g})^! = C(\mathfrak{g}) = \Lambda(\mathfrak{g}^*), \tag{30}$$

where $C(\mathfrak{g})$ is the Chevalley–Eilenberg algebra of $\mathfrak{g}$ [8]. The final equals sign doesn't specify anything about the differential, which is why the new notation is needed.

Through these examples we have seen that a lot of the time there is a natural grading on the algebras we're considering and furthermore they can be endowed with a differential, upgrading them to differential graded algebras. Even though the two examples above are (up to the inhomogeneity) the same at the algebra level, their differential structures are different. This is why we chose to set the theory in the differential graded context from the beginning.

There are also other ways to calculate the Koszul dual of a given algebra that we haven't mentioned here. One way involves considering the base field as a module of the algebra, taking a projective resolution (with other modules) and calculating the cohomology of maps from this resolution to the base field. A description of this can be found in [6], and calculations using this method can be found in [9].

3 Differential Graded Algebras and Coupling

One problem that we will look into that has the Koszul dual appearing is the problem of coupling two differential graded algebras (dgas) like for example in the case of defects in a theory. Differential graded algebras are just algebras over an operad in the context of differential graded vector spaces. The physical interpretation of dgas is that they're the algebras of observables of a theory, where the differential can play the role of a BRST differential or of a supercharge. We will look at associative dgas here, but more properly the logic should be extended to homotopy associative dgas. Mathematical accounts of this are given in [7, Chap. 10] and [3], while a physical perspective is provided in [4], especially in its appendices.

Given two dgas (A, Q_A) and (B, Q_B), where the Qs are the differentials, we can form the tensor product dga $(A \otimes B, Q)$ as described in eqs. (1) and (2). This is our model of considering the two systems that the algebras describe at the same time, but without letting them interact. The next step is to couple the algebras, and one way to do this is by modifying the differential

$$Q_\alpha := Q + [\alpha, -] \tag{31}$$

into a differential *twisted* by a suitable α. We see that $\alpha \in (A \otimes B)^1$, i.e. it has to be a degree-1 element in the combined algebra, to keep the differential of homogeneous degree. We denote this new dga as $A \otimes_\alpha B := (A \otimes B, Q_\alpha)$. The discussion of how this relates to more traditional physical ways to couple systems is postponed to section 4.

The important point is that not all $\alpha \in (A \otimes B)^1$ will give a well-defined dga since the new differential has to square to zero again. A straightforward calculation shows that this condition on Q_α is equivalent to

$$0 \overset{!}{=} Q\alpha + \alpha^2 = Q\alpha + \frac{1}{2}[\alpha, \alpha]. \tag{32}$$

This is the Maurer–Cartan equation for α in the untwisted algebra. We call such an $\alpha \in (A \otimes B)^1$ that satisfies the equation a *Maurer–Cartan element.*

At this point we also need to introduce the concept of an augmentation. In the graded context we have the following.

Definition 3.1. Let A be a dga over the field k. An augmentation of A is a map of dgas $\epsilon : A \to k$, where k is considered as a dga concentrated in degree 0. Equivalently, it's a linear map that respects grading and satisfies

$$\forall a, b \in A\colon\ \epsilon(ab) = \epsilon(a)\epsilon(b) \quad \text{and} \quad \epsilon(\mathrm{d}a) = 0. \tag{33}$$

Remark. If we're working over an operad with $P(0) = k$ then a quadratic algebra $A(V, S)$ is naturally augmented by the projection onto the weight-0 subspace. This is because $F_P(V)(0) = P(0) = k$ and the quotient by the relations S does not disturb this fact.

Mathematically, the augmentation plays a role in the following theorem:

Theorem 3.1. *Let (A, ϵ_A) and (B, ϵ_B) be augmented, quadratic, finite-dimensional dgas over k, where ϵ_A and ϵ_B are the augmentations. There is a bijective correspondence between:*

1. *Maurer–Cartan elements $\alpha \in (A \otimes B)^1$ satisfying*

$$(\epsilon_A \otimes 1)(\alpha) = 0 \qquad \text{and} \qquad (1 \otimes \epsilon_B)(\alpha) = 0. \tag{34}$$

2. *Homomorphisms of augmented dgas*

$$\phi_\alpha : A^! \to B. \tag{35}$$

i.e. dga homomorphisms satisfying $\epsilon_{A^!} = \epsilon_B \circ \phi_\alpha$

Remark. The previous theorem can be modified, while remaining true, by removing the augmentations altogether. Namely, if A and B are simply quadratic dgas without any augmentations then there is direct correspondence between Maurer–Cartan elements and dga homomorphisms, without any further conditions.

Proof. First we prove the statement in the remark, namely that the presence of the augmentation is optional. Given an augmented algebra (A, ϵ_A) over a field k, we have a canonical split $A \cong k \oplus \overline{A}$, where $\overline{A} = \ker(\epsilon_A)$ is the reduced algebra. The conditions of eq. (34) in the first part then say that $\alpha \in \overline{A} \otimes \overline{B}$.

On the other hand, the condition that homomorphisms respect the augmentation in the second part can be understood as the fact that every such homomorphism $\phi : A \to B$ uniquely factorizes through a homomorphism of non-augmented algebras $\overline{\phi} : \overline{A} \to \overline{B}$ by

$$A \xrightarrow{\cong} k \oplus \overline{A} \xrightarrow{(0, \overline{\phi})} k \oplus \overline{B} \xrightarrow{\cong} B. \tag{36}$$

Thus, we can state the theorem by using the reduced versions and no mention of augmentations. The other direction also has to be checked, but given an algebra A we can form $k \oplus A$, with the augmentation being the projection to k, and extend the maps by 0; by the same argument as above the remark holds.

For the theorem itself we will only give a rough sketch of a proof. We do this in the non-augmented case since we can choose to do so. Furthermore, like in the whole section, we work with associative algebras. It turns out that we can form a model for the Koszul dual by the following construction. The multiplication $\mu_A : A \otimes A \to A$ in our algebra gives rise to a map of the shifted algebras

$$\mu'_A : A[1] \otimes A[1] \to A[1], \tag{37}$$

which we dualize into a map

$$\mu'^*_A : A[1]^* \to A[1]^* \otimes A[1]^*. \tag{38}$$

This map extends uniquely to a map of the free tensor algebra generated by $A[1]^*$, $Q_2 : F(A[1]^*) \to F(A[1]^*)$, by virtue of it being free, and it can be shown that the associativity of μ_A implies exactly that $Q_2 \circ Q_2 = 0$. Moreover, the differential Q_A, again extends uniquely to a differential Q_1 on $F(A[1]^*)$ that anti-commutes with Q_2. Collecting them together we get the dga

$$(F(A[1]^*), Q = Q_1 + Q_2), \tag{39}$$

which can be shown to be a model for the Koszul dual $A^!$ if A is quadratic. That is, there is an isomorphism of dgas between them. The full details can be found in [7, Chap. 2].

Given the construction above we move on to the bijective correspondence. Since our model for the dual is a free algebra, any map $\Phi : F(A[1]^*) \to B$ is fully determined by its restriction to the generating space $A[1]^*$. After shifting, that means that the maps $\Phi : F(A[1]^*) \to B$ are bijectively related to maps $\phi : A^* \to B$ up to the restriction that Φ is a dga homomorphism. Exploring this restriction more, we see that the condition is:

$$Q_B \circ \Phi = \Phi \circ (Q_1 + Q_2), \tag{40}$$

but after restricting it to the level of the generating space A^* we have

$$Q_B \circ \phi = -\phi \circ Q_A - \mu_B \circ (\phi \otimes \phi) \circ \mu^*_A, \tag{41}$$

where each term comes from the corresponding one in eq. (40). Here we have also used the multiplication $\mu_B : B \otimes B \to B$. The minus signs appear because of the shifting of the generating space.

Finally, we use the fact that the maps $\phi : A^* \to B$ are bijectively related to elements $\alpha \in A \otimes B$ and translate the restriction eq. (41) into an equation for α. This gives exactly

$$(Q_A \otimes 1 + 1 \otimes Q_B)\alpha = -\alpha^2, \tag{42}$$

i.e. exactly the Maurer–Cartan equation. □

This key theorem classifies algebraic couplings (i.e. ones gotten through a twisting of the differential) through maps from the Koszul dual of one of the algebras to the other, which makes them more easily understood.

One might wonder which version, with or without the augmentations, is more amenable to physical interpretation. One way to think about the question, suggested by Paquette and Williams in [9], is that the augmentation provides a choice of vacuum of the theory. Their citing of theorem 3.1 needs a slight correction to match with the one presented above and the mathematical literature, however the interpretation is still plausible. Under some assumptions discussed in their note, being a choice of vacuum can be understood as the augmentation's role. The idea is that a choice of vacuum allows us to evaluate the vacuum expectation value of every operator in the algebra and get a complex number, where we assume that the field of interest is $k = \mathbb{C}$.

Example 3.1. The simplest, physically-interesting example to illustrate the above formalism is the case of Chern–Simons theory with a Lie algebra $\mathfrak{g}$ on say $\mathbb{R}^3$ that has a line defect carrying a trivial gauge theory. The defect being trivial means that its dga B is concentrated in degree 0 and has trivial differential. This assumption makes the example easy to work with, but the discussion can easily be extended to arbitrary B. The local algebra of observables of the Chern–Simons theory is $A = C(\mathfrak{g})$, the Chevalley–Eilenberg algebra. This is because if we choose a basis $\{\mathfrak{c}_a\}$ of $\mathfrak{g}$, and the corresponding dual basis $\{\mathfrak{c}^a\}$ of $\mathfrak{g}^*$, then the BRST differential is known to be given by

$$Q\mathfrak{c}^a = -\frac{1}{2} f^a{}_{bc} \mathfrak{c}^b \wedge \mathfrak{c}^c, \tag{43}$$

where $f^a{}_{bc}$ are the structure constants of $\mathfrak{g}$ with respect to $\{\mathfrak{c}_a\}$. This is exactly the Chevalley–Eilenberg algebra of the Lie algebra $\mathfrak{g}$, which we met already in the examples of example 2.6. The one dimensional defect (which we can see as a quantum mechanical system, as opposed to a quantum field theoretic one) has 0 differential, so by slight abuse of notation, the Maurer–Cartan elements are solutions of

$$Q\alpha + \alpha^2 = 0, \tag{44}$$

which necessary lies in the degree 1 subspace $(A \otimes B)^1$. This means that $\alpha \in \mathfrak{g}^* \otimes B$, by noticing the degrees and using the defining eq. (1). Let's try to solve the equation in the chosen basis. Letting $\alpha = \mathfrak{c}^a \otimes \alpha_a$, where $\alpha_a \in B$, so that we can calculate

$$\begin{aligned} Q\alpha &= (Q\mathfrak{c}^a) \otimes \alpha_a = -\frac{1}{2} f^a{}_{bc} (\mathfrak{c}^b \wedge \mathfrak{c}^c) \otimes \alpha_a \\ \alpha^2 &= (\mathfrak{c}^b \otimes \alpha_b)(\mathfrak{c}^c \otimes \alpha_c) = (\mathfrak{c}^b \wedge \mathfrak{c}^c) \otimes (\alpha_b \alpha_c), \end{aligned} \tag{45}$$

where we denote the multiplication in $C(\mathfrak{g})$ by $\wedge$ (considering its definition), and the multiplication in B by juxtaposition. The Maurer–Cartan equation then gives

$$\begin{aligned} Q\alpha + \alpha^2 = (\mathfrak{c}^b \wedge \mathfrak{c}^c) \otimes (-\frac{1}{2} f^a{}_{bc} \alpha_a + \alpha_b \alpha_c) &= -\frac{1}{2} (\mathfrak{c}^b \wedge \mathfrak{c}^c) \otimes (f^a{}_{bc} \alpha_a - \alpha_b \alpha_c + \alpha_b \alpha_c) = 0 \\ \Rightarrow \alpha_b \alpha_c - \alpha_b \alpha_c &= f^a{}_{bc} \alpha_a. \end{aligned} \tag{46}$$

By simple linear algebra the αs are bijectively related to maps $\tilde{\phi}_\alpha : \mathfrak{g} \to B$, since we're working with finite vector spaces. In the basis, $\tilde{\phi}_\alpha(x) = x^a \alpha_a$, where $x = x^a \mathfrak{c}_a$. Thus, if we contract eq. (46) with arbitrary x^b and y^c we get the basis independent equation

$$\forall x, y \in \mathfrak{g} \colon\ \tilde{\phi}_\alpha([x,y]) = \tilde{\phi}_\alpha(x)\tilde{\phi}_\alpha(y) - \tilde{\phi}_\alpha(y)\tilde{\phi}_\alpha(x), \tag{47}$$

which says that $\tilde{\phi}_\alpha$ is a Lie algebra homomorphism, where the Lie algebra structure on B is the induced commutator.

On the other hand, from example 2.6 we know that

$$A^! = U(\mathfrak{g}), \tag{48}$$

so that the Maurer–Cartan elements are classified by dga homomorphisms $\phi_\alpha : U(\mathfrak{g}) \to B$. Because both are concentrated in degree 0 the differential graded requirements are automatic, so that it only needs to be an algebra homomorphism. But by the universal property of the enveloping algebra, the maps ϕ_α exactly determine maps $\tilde{\phi}_\alpha : \mathfrak{g} \to B$, which are Lie algebra homomorphisms as above, and bijectively at that. This is through $\tilde{\phi}_\alpha = \phi_\alpha \circ e$, where $e : \mathfrak{g} \to U(\mathfrak{g})$ is the canonical embedding of the Lie algebra in its universal enveloping algebra.

If we look at this example in the augmented context, then for $C(\mathfrak{g})$ we have the canonical augmentation $\mathrm{prj} : C(\mathfrak{g}) \to C^0(\mathfrak{g}) = k$, which projects to its weight-0 component. The condition that the augmentation annihilates α is automatic since $\alpha \in \mathfrak{g} \otimes B$. On the other side, we augment B with $\epsilon : B \to k$. The requirement we have to satisfy is that $\epsilon \circ \phi_\alpha = \mathrm{prj}$. It suffices if we evaluate this on an arbitrary product of basis elements

$$\begin{aligned} \epsilon(\phi_\alpha(\mathfrak{c}_{a_1} \ldots \mathfrak{c}_{a_n})) &= \epsilon(\tilde{\phi}_\alpha(\mathfrak{c}_{a_1}) \ldots \tilde{\phi}_\alpha(\mathfrak{c}_{a_n})) = \epsilon(\alpha_{a_1} \ldots \alpha_{a_n}) = \epsilon(\alpha_{a_1}) \ldots \epsilon(\alpha_{a_n}) \\ \mathrm{prj}(\mathfrak{c}_{a_1} \ldots \mathfrak{c}_{a_n}) &= \delta_{n,0}, \end{aligned} \tag{49}$$

so that we get the compatibility to be $\forall a :\ \epsilon(\alpha_a) = 0$, but this is exactly the annihilation condition of the first part of the theorem for B.

4 Algebraic to Physical Couplings

The deformation viewpoint presented above is mathematically clean, but it is not yet clear if it has a physical interpretation. The physics perspective couples systems though finding interaction Lagrangians. Thus, to make it more physically meaningful we try to find a Lagrangian coupling $\mathcal{L}$ that is somehow constructed from a Maurer–Cartan element. In this section we make this precise for the case where B is the algebra of observables of a one-dimensional, Euclidean, topological defect. The outline of the calculations in this section was given in [9, Sect. 4].

4.1 Setup of the Problem

Let (B, Q) be a dga, which we interpret, for concreteness only, as a BRST complex, which means that we call the grading of this dga the ghost number. Furthermore, we are working on $\mathbb{R}$ with operators valued in B. The interaction Lagrangian density should be a top form of ghost number 0. Thus, we want

$$\mathcal{L} \in \Omega^1(\mathbb{R}, B). \tag{50}$$

For a quantum theory our evolution operator related to the coupling would be given by a path-ordered exponential of the Lagrangian

$$\mathrm{Pexp} \int_{\mathbb{R}} \mathcal{L}. \tag{51}$$

A path-ordered exponential (POE), from its very definition, is just a sum of integrals over simplices of increasing dimension, the orientation of a simplex providing the ordering. This means that

$$\operatorname{Pexp}\int_{\mathbb{R}}\mathcal{L} := \sum_{n=0}^{\infty}\int_{\tilde{\Delta}^n}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_n^n, \tag{52}$$

where the sub- and superscript notation on any form means pullback by the projections, namely

$$\mathcal{L}_k^n := \operatorname{proj}_k^{n*}\mathcal{L}, \tag{53}$$

for $\operatorname{proj}_k^n : \mathbb{R}^n \to \mathbb{R}$, $(t_1,\ldots,t_n)\mapsto t_k$. The superscript determining the dimension in which the simplex lives might seem like notational overkill, but we will later see that it is a necessary part to keep track of terms. To be precise about $\tilde{\Delta}^n$, it is the subspace

$$\tilde{\Delta}^n := \{(t_1,\ldots,t_n)\,|\,t_i\in(-\infty,\infty), t_1>t_2>\cdots>t_n\}\subset\mathbb{R}^n. \tag{54}$$

The path-ordered exponential needs to be a quantity that passes to cohomology in Q, so it needs to be closed; we require

$$Q\operatorname{Pexp}\int_{\mathbb{R}}\mathcal{L}\overset{!}{=}0. \tag{55}$$

Writing this out gives

$$Q\operatorname{Pexp}\int_{\mathbb{R}}\mathcal{L} = \sum_{n=0}^{\infty}Q\int_{\tilde{\Delta}^n}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_n^n \tag{56}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\tilde{\Delta}^n}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_{k-1}^n\wedge(Q\mathcal{L})_k^n\wedge\mathcal{L}_{k+1}^n\wedge\cdots\wedge\mathcal{L}_n^n, \tag{57}$$

because Q is a derivation acting on a degree 0 element. The next step is to find a way to compute $Q\mathcal{L}$.

4.2 The Topological Assumption

To continue making progress it's clear that we need some assumption on how Q acts on the Lagrangian. One choice is to assume that we're working with a *topological theory*. This term can mean different things in the literature, but in almost all cases the minimal requirement is that we're guaranteed the existence of a K such that

$$QK + KQ = \frac{\partial}{\partial t}. \tag{58}$$

The equation above should be interpreted at the level of $\Omega^0(\mathbb{R}, B)$, as a chain complex with differential Q. There it says that there exists a chain homotopy K, which makes the chain map $\partial/\partial t$ homotopic to 0; the physical interpretation being that there is no 'evolution' along $\mathbb{R}$ at the cohomology level.

We will later see that we also need to make the assumption that K is a differential in the ghost grading of $\Omega^0(\mathbb{R}, B)$.

K allows us the construction of the Lagrangian as $\mathcal{L} = (K\mathcal{O})\,\mathrm{d}t$, for some *scalar* function $\mathcal{O}$. This allows for the key computation

$$Q\mathcal{L} = Q(KO)\,\mathrm{d}t = \left(\frac{\partial}{\partial t}\mathcal{O}\right)\mathrm{d}t - (KQ\mathcal{O})\,\mathrm{d}t = \mathrm{d}\mathcal{O} - (KQ\mathcal{O})\,\mathrm{d}t, \tag{59}$$

which expresses $Q\mathcal{L}$ as a part that is a total differential and a part that is in the image of K. Intuitively, we expect the total differential to give some boundary terms after integration by parts; we make this precise in the following subsection.

4.3 Integration by Parts

Focusing on the $\mathrm{d}\mathcal{O}$ term and bringing it into the expression for the POE we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\tilde{\Delta}^n}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_{k-1}^n\wedge\mathrm{d}\mathcal{O}_k^n\wedge\mathcal{L}_{k+1}^n\wedge\cdots\wedge\mathcal{L}_n^n \tag{60}$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\int_{\tilde{\Delta}^n}\mathrm{d}\big(\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_{k-1}^n\wedge\mathcal{O}_k^n\wedge\mathcal{L}_{k+1}^n\wedge\cdots\wedge\mathcal{L}_n^n\big) \tag{61}$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\int_{\partial\tilde{\Delta}^n}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_{k-1}^n\wedge\mathcal{O}_k^n\wedge\mathcal{L}_{k+1}^n\wedge\cdots\wedge\mathcal{L}_n^n, \tag{62}$$

The second line above follows because d commutes with pullbacks and because $\mathcal{L}\in\Omega^1(\mathbb{R},B)$ is a top form. The third line is just an application of Stokes' theorem. Thus, we need to analyze the boundary of the simplex $\tilde{\Delta}^n$. Since it is a simplex its boundary is a union of simplices of one dimension lower. Looking at its defining eq. (54) we see that the boundary is gotten when adjacent t_is become equal, i.e. the boundary is $\partial\tilde{\Delta}^n=\cup_l(-1)^{l+1}s_l^{n-1}$ $(l=1,\ldots,n-1)$ (the alternating sign is there to get the right orientation), where

$$s_l^{n-1}:=\{(t_1,\ldots,t_n)\,|\,t_i\in\mathbb{R},t_1<\cdots<t_{l-1}<t_l=t_{l+1}<\cdots<t_n\}. \tag{63}$$

This makes the expression that we need to calculate equal to

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{l=1}^{n-1}(-1)^{l+1}\int_{s_l^{n-1}}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{L}_{k-1}^n\wedge\mathcal{O}_k^n\wedge\mathcal{L}_{k+1}^n\wedge\cdots\wedge\mathcal{L}_n^n, \tag{64}$$

where the n-sum now starts at 2 because $\partial\tilde{\Delta}^1=\emptyset$. Since the s_l^{n-1} are just $n-1$-simplices in $\mathbb{R}^n$, we introduce the maps $\phi_l:\mathbb{R}^{n-1}\to\mathbb{R}^n$ to show how they are embedded. Namely,

$$\phi_l:(t_1,\ldots,t_{n-1})\mapsto(t_1,\ldots,t_{l-1},t_l,t_l,t_{l+1},\ldots,t_{n-1}), \tag{65}$$

and consequently $\phi_l(\tilde{\Delta}^{n-1})=s_l^{n-1}$. This means that we can rewrite the integral from before as

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{n-1}(-1)^{k+l}\int_{s_l^{n-1}}\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{O}_k^n\wedge\cdots\wedge\mathcal{L}_n^n$$

$$=\sum_{n=2}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{n-1}(-1)^{k+l}\int_{\tilde{\Delta}^{n-1}}\phi_l^*\left(\mathcal{L}_1^n\wedge\cdots\wedge\mathcal{O}_k^n\wedge\cdots\wedge\mathcal{L}_n^n\right). \tag{66}$$

Since the pullback distributes over the wedge and since

$$(\phi\circ\psi)^*=\psi^*\phi^*, \tag{67}$$

we're prompted to consider the composite map $\mathrm{proj}_m^n\circ\phi_l$. It can easily be seen that

$$\mathrm{proj}_m^n\circ\phi_l=\begin{cases}\mathrm{proj}_m^{n-1} & m\le l\\ \mathrm{proj}_{m-1}^{n-1} & m>l\end{cases}. \tag{68}$$

Returning to the integral in eq. (66) we see that we need to split the sum over k into three cases: (1) $k<l$, (2) $k>l+1$ and (3) $k=l$ or $k=l+1$.

Case (1): $k<l$

By applying eq. (68) the integrand in eq. (66) in this case becomes

$$\phi_l^*\left(\bigwedge_{m=1}^{k-1}\mathcal{L}_m^n\wedge\mathcal{O}_k^n\wedge\bigwedge_{m=k+1}^{n}\mathcal{L}_m^n\right)=\bigwedge_{m=1}^{k-1}\mathcal{L}_m^{n-1}\wedge\mathcal{O}_k^{n-1}\wedge\bigwedge_{m=k+1}^{l-1}\mathcal{L}_m^{n-1}\wedge\mathcal{L}_l^{n-1}\wedge\mathcal{L}_l^{n-1}\wedge\bigwedge_{m=l+2}^{n}\mathcal{L}_{m-1}^{n-1}. \tag{69}$$

The important part of this is that we have the combination $\mathcal{L}_l^{n-1}\wedge\mathcal{L}_l^{n-1}=(\mathcal{L}\wedge\mathcal{L})_l^{n-1}$, since, again, the wedge commutes with pullbacks. Because $\mathcal{L}$ is a top form this product has to be 0 and consequently this doesn't contribute to out integral.

Case (2): $k > l+1$

This case proceeds similarly to case (1), but now on the other side of the $\mathcal{O}$. Thus, it also doesn't contribute to the overall integral.

Case (3): $k = l$ or $k = l+1$

In the case of $k=l$, in the same manner as before, we calculate

$$\begin{aligned}\phi_l^*\left(\bigwedge_{m=1}^{k-1}\mathcal{L}_m^n\wedge\mathcal{O}_k^n\wedge\bigwedge_{m=k+1}^{n}\mathcal{L}_m^n\right)&=\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n-1}\wedge\mathcal{O}_l^{n-1}\wedge\mathcal{L}_l^{n-1}\wedge\bigwedge_{m=l+2}^{n}\mathcal{L}_{m-1}^{n-1}\\&=\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n-1}\wedge(\mathcal{O}\mathcal{L})_l^{n-1}\wedge\bigwedge_{m=l+1}^{n-1}\mathcal{L}_m^{n-1}.\end{aligned} \tag{70}$$

And similarly in the case of $k=l+1$ we get

$$\begin{aligned}\phi_l^*\left(\bigwedge_{m=1}^{l}\mathcal{L}_m^n\wedge\mathcal{O}_{l+1}^n\wedge\bigwedge_{m=l+2}^{n}\mathcal{L}_m^n\right)&=\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n-1}\wedge\mathcal{L}_l^{n-1}\wedge\mathcal{O}_l^{n-1}\wedge\bigwedge_{m=l+2}^{n}\mathcal{L}_{m-1}^{n-1}\\&=\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n-1}\wedge(\mathcal{L}\mathcal{O})_l^{n-1}\wedge\bigwedge_{m=l+1}^{n-1}\mathcal{L}_m^{n-1}.\end{aligned} \tag{71}$$

With this we have moved the whole integral eq. (66) one dimension down and in doing so we have picked up two 'boundary terms' in the form of eqs. (70) and (71), with the relative minus coming from the alternating $(-1)^{k+l}$. With further shifting of the index n by 1 we get

$$\sum_{n=2}^{\infty}\sum_{l=1}^{n-1}\int_{\tilde{\Delta}^{n-1}}\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n-1}\wedge(\mathcal{O}\mathcal{L}-\mathcal{L}\mathcal{O})_l^{n-1}\wedge\bigwedge_{m=l+1}^{n-1}\mathcal{L}_m^{n-1} \tag{72}$$

$$=\sum_{n=1}^{\infty}\sum_{l=1}^{n}\int_{\tilde{\Delta}^{n}}\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n}\wedge(\mathcal{O}\mathcal{L}-\mathcal{L}\mathcal{O})_l^{n}\wedge\bigwedge_{m=l+1}^{n}\mathcal{L}_m^{n}. \tag{73}$$

This is the boundary term that we get from the integration by parts. It is a contribution that appears explicitly because we have the POE, and thus it has a quantum flavor; it would not appear in the classical case as argued in [9].

4.4 Recovering the Algebraic Coupling

Finally, we can combine the boundary term of the previous subsection with the term in the image of K, which we set aside in eq. (59) so that together we get

$$Q\,\mathrm{Pexp}\int_{\mathbb{R}}\mathcal{L}=-\sum_{n=1}^{\infty}\sum_{l=1}^{n}\int_{\tilde{\Delta}^{n}}\bigwedge_{m=1}^{l-1}\mathcal{L}_m^{n}\wedge\big((KQ\mathcal{O})\,\mathrm{d}t-\mathcal{O}\mathcal{L}+\mathcal{L}\mathcal{O}\big)_l^{n}\wedge\bigwedge_{m=l+1}^{n}\mathcal{L}_m^{n} \tag{74}$$

Thus, we get the condition that we need to satisfy for the POE to be closed,

$$(KQ\mathcal{O})\,\mathrm{d}t - \mathcal{O}\mathcal{L} + \mathcal{L}\mathcal{O} = 0. \tag{75}$$

As we commented on before, eq. (58) further allows for the assumption that K is a derivation, and we make this assumption. With its use we can rewrite the above as

$$(KQ\mathcal{O})\,\mathrm{d}t - \mathcal{O}(K\mathcal{O})\,\mathrm{d}t + (K\mathcal{O})\mathcal{O}\,\mathrm{d}t = K(Q\mathcal{O} + \mathcal{O}\cdot\mathcal{O})\,\mathrm{d}t = 0, \tag{76}$$

since the ghost number of $\mathcal{O}$ is 1. This obviously gets us the Maurer–Cartan equation

$$Q\mathcal{O} + \mathcal{O}\cdot\mathcal{O} = 0, \tag{77}$$

establishing the connection to the algebraic couplings of the previous section.

Specifically, the above construction shows that in the case of a one-dimensional, Euclidean, topological theory (or defect) the algebraic deformation coupling is related to the physical coupling with a Lagrangian. Namely, if α is a Maurer–Cartan element then we can construct a Lagrangian $\mathcal{L} = K\alpha\,\mathrm{d}t$, with K the chain homotopy that trivializes the evolution.

Lastly, we mention that a similar construction to the above, namely introducing an operator K and acting with it to obtain $\mathcal{L}$ was first introduced in [11], and now goes by the name topological descent. Our situation, though, is not exactly the same since we only ever want to do one 'step'.

5 Remarks and Further Questions

We have introduced Koszul duality in the setting of operads — a necessary level of generality for theories in which the algebras of observables are only homotopy associative for example — and we have looked at some more down to Earth examples where we can calculate it. Koszul duality seems to appear in different ways in physical theories. One interesting thing we observed is that algebras are dual to superalgebras, so that one of its guises is a type of fermion–boson duality, which we saw in the example of $Sym(V)^! = \Lambda(V^*)$

Another area that we explore in great detail was the coupling of defects to a bulk theory, specifically in the one-dimensional, Euclidean, topological defect case. Here we've established a connection between physical couplings in the sense of a Lagrangian and deformation couplings of the algebras of observables $\alpha \in A \otimes B$, which are in turn related to maps out of the Koszul dual $\phi : A^! \to B$. This way of phrasing things also has the advantage that it suggests possible generalizations to higher dimensions or non-trivial topologies of the manifold à la topological descent, which would have to be examined subsequently.

A related question that needs to be explored further is the connection to Poincaré–Koszul duality [1], which manages to combine the discussed algebraic notion of Koszul duality with the geometric notions of Poincaré duality into one formalism. From the physics perspective, this is obviously desirable, since Koszul duality tells us about the algebraic side of things, namely the observables of the theory, but it doesn't say anything about the manifold of the theory, on which the observables live and how it might be dualized.

In this note we have mostly stayed away from categorical constructions to keep the prerequisite level of understanding relatively low. But to make all the things we have mentioned more precise it's helpful to have them be expressed in categorical language. It also has the advantage of making the constructions — for example in operad theory — less lengthy and more clear. In generalizing to Poincaré–Koszul duality, it becomes basically the only way forward.

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