Degenerate Boundary Structure of General Relativity Coupled to External Scalar and Yang-Mills Fields

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Abstract

The boundary structure of gravity coupled to external scalar and Yang–Mills fields is described by means of the Kijowski–Tulczijew construction. In particular, the 4-dimensional case of a degenerate boundary metric is covered and the algebra of the constraints is calculated.

Contents

Introduction

The aim of this thesis is to give a classical description of a theory which couples General Relativity to external fields and to study its boundary structure. In particular, we are interested in the reduced phase space, which can be found by means of a symplectic reduction. In [3] this has been done for the case of a space-like and time-like boundary. In this thesis the case of a light-like boundary will be covered.

More than one hundred years after its formulation, General Relativity is still the best theory of gravity which we have. Recent discoveries of gravitational waves strengthen it even further. Nevertheless, it cannot be the end of the story due to its incompatibility with modern physics. From a particle point of view, the idea of a supersymmetric theory which includes gravity is very promising since it would incorporate the successful standard model in an intuitive and elegant manner, using already known concepts such as gauge groups and symmetry breaking.

The approach taken in this work starts with the Palatini–Cartan formalism, which is a formulation of General Relativity different from the Einstein–Hilbert case, and is extended to the coupling of an external scalar and Yang–Mills field through the first order formalism. This results in a classical field theory describing the same space of solutions of the Einstein field equations and the respective equations of motion of scalar and Yang–Mills fields.

Furthermore, we assume to have a manifold with boundary. What follows is called the Kijowski–Tulczijew (KT) construction. It turns out that the space of fields restricted to this boundary has a presymplectic structure which arises naturally by the properties of a field theory. This presymplectic manifold can now be reduced further by quotienting out the zero locus of the equations of motion. This ends in a symplectic manifold, called the reduced phase space. Physically speaking, this space represents the set of all admissible initial conditions and its description is the main goal of this thesis.

It has been shown that under certain conditions there exists a cohomological description of the observables on the space of boundary fields, called Batalin–Fradkin–Vilkovisky (BFV) formalism. In particular, the algebra of constraints defining the reduced space should be of first-class. This means that each possible Poisson bracket is a linear combination of the constraints themselves. A more geometric interpretation would be that the submanifold they define is coisotropic.

It turns out that the properties of this boundary structure change significantly depending on whether the boundary is defined by a space-like, time-like or light-like direction. In the first two cases the induced metric on the boundary is still nondegenerate, while in the latter it becomes degenerate. Such a degenerate boundary metric has drastic implications also on the boundary tetrad, being its counterpart in the PC formalism. Indeed, by simply translating equations from the nondegenerate to the degenerate case, they might not define the same solution space anymore. This leads to a different set of constraints than in the nondegenerate case.

The nondegenerate cases of pure gravity and gravity coupled to external fields have been studied in [4] and [3], respectively. On the other hand, the degenerate case of pure gravity has been covered in [5]. This thesis aims to perform a similar analysis for the degenerate case of gravity coupled to external fields.

1 Bundle Structure

The theory of gravity with external fields which we will construct in chapter 3 will have as dynamical variables elements of fiber bundles. Furthermore, we want to formulate General Relativity as a gauge theory. In this chapter we will set up the geometric fundamentals to do so. We refer to [14], [19], [12] and [18].

The mathematical way to formulate physical theories with local symmetries, called gauge theories, is by means of fiber bundles. A fiber bundle E is a differentiable manifold which has several components and properties. Intuitively, E is made of a base space M such that at every point $x \in M$ there is the same fiber F attached to it. Moreover, there is a projection map π that maps every point of the fiber bundle to its respective point on the base space. The most simple case of a fiber bundle is a direct product $E = M \times F$ (such as the torus $T^2 = S^1 \times S^1$). Generally, however, it can only be written locally as a direct product (it can be trivialized). The easiest example of a non-trivial fiber bundle is the Möbius strip which has as a base space S^1 and as a fiber the interval $[0, 1]$. Locally we can always write it as the direct product of a subset of the circle with the interval but we cannot write the entire strip as $F = S^1 \times [0, 1]$, since it is twisted.

Going back to physics, fields are usually defined as sections of fiber bundles $\phi\colon M \to E$. The meaning of this is that at every point in spacetime $x \in M$ there is a particular field value. However, unless the fiber bundle is trivial, we can only describe the fields locally, yet with a well-defined prescription to relate overlapping local descriptions. In gauge theories there is a local symmetry described by a Lie group G. Its mathematical description leads to a special case of fiber bundle, called principal bundle, where the fiber is given by the Lie group itself (up to some technical deviations). A gauge transformation is just an automorphism of this principle bundle. The projection map can then be used to give rise to a notion of verticality inside the bundle. There is also a notion of horizontality, which however depends on an additional choice, called connection. The latter concept states how vectors defined at nearby points are connected to each other. By pulling back this connection onto the base manifold we receive our gauge field which is equipped with a natural local gauge transformation. Now we also need to describe external fields such that they transform under a change of gauge. This is achieved by means of associated bundles, which inherit the structure group of the principal bundle and transforms under a certain representation of it. Matter fields are therefore described by sections of associated bundles.

1.1 Principal Bundles and Connections

Definition 1.1 (Fiber Bundle)**.** Let E, M and F be differentiable manifolds and $\pi\colon E \to M$ a differentiable surjection. Then (E, M, π, F) is called a **fiber bundle** if $\forall x \in M$ there exists an open neighbourhood $U \subset M$ of x and a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \varphi\ } & U \times F \\ {\scriptstyle \pi}\big\downarrow & \swarrow{\scriptstyle \mathrm{proj}_1} & \\ U & & \end{array} \quad ,$$

where $\text{proj}_1 \colon U \times F \to U$ is the natural projection on the first element. The space M is called the **base space**, E the **total space**, F the **fiber** and π the **projection map**. The set of all $\{(U_i, \varphi_i)\}$ such that $\bigcup_i U_i = M$ is called a **local trivialization**.

Remark. Usually, the definition of a fiber bundle takes as a base space a general topological space and requires the projection and trivialization maps to be only continuous. However, for our purpose we will always consider differentiable structures and therefore it makes sense to incorporate differentiability in the definition instead of writing it always as an additional requirement.

Definition 1.2 (Bundle Morphism)**.** Let (E, M, π) and (E', M', π') be two fiber bundles and let $\Phi \colon E \to E'$ and $\phi \colon M \to M'$ be two smooth maps. Then Φ is called a **bundle morphism** covering ϕ if the following diagram commutes[1]

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\phi]{} & M' \end{array} .$$

If ϕ is a diffeomorphism and there exists a bundle morphism $\Phi^{-1} \colon E' \to E$ covering ϕ^{-1} such that $\Phi^{-1} \circ \Phi = \text{id}_E$, then Φ is called a **bundle isomorphism** and we write $E \sim E'$.

Definition 1.3 (Right Action)**.** Let G be a Lie group and M a smooth manifold. A **right action** of G on M is defined to be a smooth map

$$\begin{aligned} \mathcal{R} \colon M \times G &\to M \\ (x, g) &\mapsto \mathcal{R}_g(x) \end{aligned}$$

such that it satisfies

1. $\mathcal{R}_e(x) = x$, $\forall x \in M$, where e is the identity element in G,

2. $\mathcal{R}_{g_1} \circ \mathcal{R}_{g_2} = \mathcal{R}_{g_2 g_1}$, $\forall g_1, g_2 \in G$.

Remark. Similarly, we can define a **left action** of G by simply demanding that the second point translates to $\mathcal{R}_{g_1} \circ \mathcal{R}_{g_2} = \mathcal{R}_{g_1 g_2}$, $\forall g_1, g_2 \in G$. In the case of an abelian group any left action is also a right action and vice versa.

Definition 1.4 (Adjoint, Left Translation)**.** Let G be a Lie group and $g \in G$. The **adjoint map** and the **left translation map** are defined by

$$\begin{aligned} \text{Ad}_g \colon G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

and

$$\begin{aligned} l_g \colon G &\to G \\ h &\mapsto gh. \end{aligned}$$

[1] If the fibers have a particular mathematical structure we usually add the additional requirement that Φ preserves that structure. For example, in the case of a principal G-bundle we want Φ to be equivariant with respect to the two right actions.

Definition 1.5 (Orbit, Stabilizer)**.** Let G be a Lie group, M a smooth manifold and $\mathcal{R}$ a right action of G on M. For every $x \in M$ the **orbit** of x is defined to be

$$O_x = \{y \in M \mid \exists\, g \in G, \mathcal{R}_g(y) = x\}. \tag{1}$$

Furthermore, the **stabilizer** of x is defined by

$$S_x = \{g \in G \mid \mathcal{R}_g(x) = x\}. \tag{2}$$

Definition 1.6 (Free, Transitive)**.** Let G be a Lie group and M a smooth manifold. A right action $\mathcal{R}$ of G on M is said to be

- **free** if all stabilizers are trivial, that is $S_x = \{e\}$ for all $x \in M$,
- **transitive** if it has just one orbit, that is $O_x = M$ for all $x \in M$.

Definition 1.7 (Principal G-Bundle)**.** Let G be a Lie group. The fiber bundle (P, π, M) is called **principal G-bundle** if

1. there exists a free smooth right action $\mathcal{R}\colon P \times G \to P$, which acts transitively on the fibers;
2. (P, π, M) is isomorphic to $(P, \pi', P/G)$, where $\pi'\colon P \to P/G$ is the canonical projection on the quotient space. In other words the following diagram should commute (see Definition 1.2)

$$\begin{array}{ccc} P & & \\ {\scriptstyle \pi}\big\downarrow & \searrow^{\pi'} & \\ M & \xrightarrow[\phi]{\qquad} & P/G \end{array} \quad ,$$

   where $\phi\colon M \to P/G$ is a diffeomorphism.

Remark. A principal G-bundle is a fiber bundle which is isomorphic to a bundle whose fibers are the orbits under the right action of G, which are themselves isomorphic to G since the action is free. In particular, this implies that the local trivialization of a principal bundle satisfies the following condition. Let (U_i, φ_i) and (U_j, φ_j) be two overlapping local trivialization charts, meaning that $U_i \cap U_j \neq \emptyset$. Then the map

$$\varphi_i \circ \varphi_j^{-1}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$\varphi_i \circ \varphi_j^{-1}(x, f) = (x, t_{ij}(x) \cdot f),$$

where $t_{ij}\colon U_i \cap U_j \to G$ is called **transition function** and the dot indicates a smooth left action of the structure group G on F. When the local trivialization of a fiber bundle F satisfies this condition for some Lie group G, then G is called the **structure group** of F.

Definition 1.8 (Associated Bundle)**.** Let (P, π, M) be a principal G-bundle, F a smooth manifold and $\lambda\colon G \times F \to F$ a smooth left action. The **associated bundle** to P and λ is the fiber bundle $(P \times_\lambda F, \pi_\lambda, M)$, where

$$P \times_\lambda F := \frac{P \times F}{\sim} \tag{3}$$

with the equivalence relation

$$(p, a) \sim (q, b) \iff \exists\, g \in G\colon \begin{cases} q = \mathcal{R}_g(p) \\ b = \lambda_{g^{-1}}(a) \end{cases} \tag{4}$$

$\forall\, p, q \in P$ and $\forall a, b \in F$ and the projection

$$\begin{aligned} \pi_\lambda\colon P \times_\lambda F &\to M \\ [p, f] &\mapsto \pi(p). \end{aligned}$$

Remark. An important fact is that an associated bundle to a principal G-bundle has a different fiber F but same structure group G and same transition functions. In other words, for overlapping trivializations (U_i, φ_i), (U_j, φ_j) on the new fibers F we have

$$\varphi_i \circ \varphi_j^{-1}(x, f) = (x, \lambda_{t_{ij}(x)}(f)). \tag{5}$$

Definition 1.9 (Vertical Subbundle)**.** Let (E, π, M) be a fiber bundle and $x \in E$. The **vertical subspace at** x is defined to be the kernel of the pushforward of the projection map $d_x\pi\colon T_xE \to T_{\pi(x)}M$

$$V_xE := \ker(d_x\pi). \tag{6}$$

The **vertical subbundle** is then defined to be

$$VE := \bigsqcup_{x \in E} V_xE. \tag{7}$$

Definition 1.10 (Ehresmann Connection)**.** Let (E, π, M) be a fiber bundle. An **Ehresmann connection** is the choice of smooth subbundle $HE \subset TE$ such that

$$TE = HE \oplus VE, \tag{8}$$

where HE is called **horizontal subbundle** of the connection.

Definition 1.11 (Lift)**.** Let $\pi\colon E \to M$ be a fiber bundle, $x \in M$ and $e \in E$ such that $\pi(e) = x$. Given a smooth curve $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = x$, a **lift** of γ through e is defined to be a curve $\tilde{\gamma}\colon \mathbb{R} \to E$ such that

1. $\tilde{\gamma}(0) = e$,
2. $\pi(\tilde{\gamma}(t)) = \gamma(t) \quad \forall t \in \mathbb{R}$.

Furthermore, given an Ehresmann connection HE on E, the lift $\tilde{\gamma}$ is called **horizontal** if

$$\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}E \quad \forall t \in \mathbb{R}. \tag{9}$$

Remark. As in the case of an affine connection, which is uniquely determined by a parallel transport, an Ehresmann connection is uniquely determined by a horizontal lift at each point, and vice versa.

As the last definitions show, the concept of verticality and connection can be defined on any type of fiber bundle. Nevertheless, we are interested in the more specific case of a principal bundle. Restricting to the latter we can make use of some of its properties to find another way of characterizing the connection, namely by means of a connection 1-form. Doing so enables us to write the connection as a field variable which we can work with.

Definition 1.12 (Fundamental map)**.** Let (P, π, M) be a principal G-bundle. The **fundamental map** $\#\colon \mathfrak{g} \to \Gamma(TP)$ is defined such that

$$A_p^{\#} := \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{R}_{\exp(tA)}(p)]_{t=0}, \tag{10}$$

with $A \in \mathfrak{g}$ and $p \in P$. Furthermore, the set of all **fundamental vectors fields** is written as

$$\mathfrak{g}^{\#} := \{v \in \Gamma(TP) \mid \exists\, A \in \mathfrak{g}, v = A^{\#}\}.$$

Remark. We call the fundamental map **vertical** because

$$d\pi(A_p^{\#}) = \frac{\mathrm{d}}{\mathrm{d}t}[\pi(\mathcal{R}_{\exp(tA)}(p))]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\pi(p) = 0. \tag{11}$$

In other words the fundamental vector $A_p^{\#}$ is vertical, because it is parallel to the fibers.

Lemma 1.1. *Let G be a matrix Lie group with $g \in G$ and $A \in \mathfrak{g}$. Then the following identity holds*

$$d\mathcal{R}_g \circ A^{\#} = (\mathrm{Ad}_{g^{-1}}(A))^{\#}. \tag{12}$$

Proof. At $p \in P$ we have

$$d\mathcal{R}_g(A_p^{\#}) = \frac{\mathrm{d}}{\mathrm{d}t}[(\mathcal{R}_g \circ \mathcal{R}_{\exp(tA)})(p)]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[(\mathcal{R}_g \circ \mathcal{R}_{\exp(tA)} \circ \mathcal{R}_{g^{-1}} \circ \mathcal{R}_g)(p)]_{t=0}. \tag{13}$$

Using the fact that $\mathcal{R}_g \circ \mathcal{R}_{\exp(tA)} \circ \mathcal{R}_{g^{-1}} = \mathcal{R}_{\mathrm{Ad}_{g^{-1}}(\exp(tA))}$ and that for matrix Lie groups $\mathrm{Ad}_{g^{-1}}(\exp(tA)) = \exp(\mathrm{tAd_{g^{-1}}(A)})$ we find

$$d\mathcal{R}_g(A_p^{\#}) = \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{R}_{\exp(t\mathrm{Ad}_{g^{-1}}A))}(\mathcal{R}_g(p))]_{t=0} = (\mathrm{Ad}_{g^{-1}}A)^{\#}_{\mathcal{R}_g(p)}. \tag{14}$$

□

Definition 1.13 (Connection 1-Form)**.** Let P be a principal G-bundle and HP an Ehresmann connection. The associated **connection 1-form** (or **principal connection**) is the $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying for $v \in \Gamma(TP)$

$$\omega(v) = \begin{cases} A & \text{if } v = A^{\#} \text{ for some } A \in \mathfrak{g} \\ 0 & \text{if } v \text{ is horizontal with respect to } HP \end{cases}. \tag{15}$$

Remark. An alternative but equivalent definition of a connection 1-form would be to demand that it's kernel must be the horizontal subbundle defined by the Ehresmann connection: $\ker(\omega) = HP$. In that case we should change the definition of Ehresmann connection a little by demanding that it is G-equivariant. The G-equivariance condition is automatically satisfied in our definition of connection 1-form as the following proposition shows.

Proposition 1.2. *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form, then it is G-equivariant, which means*

$$\mathcal{R}_g^*(\omega) = \mathrm{Ad}_{g^{-1}} \circ \omega. \tag{16}$$

Proof. If $v \in \mathfrak{X}(P)$ is horizontal the equality clearly holds. So suppose $v = A^\#$ for some $A \in \mathfrak{g}$. Then by Lemma 1.1 we have

$$(\mathcal{R}_g^*(\omega))(A^\#) = \omega(d\mathcal{R}_g \circ A^\#) = \omega((\mathrm{Ad}_{g^{-1}}(A))^\#) = \mathrm{Ad}_{g^{-1}}(A). \tag{17}$$

Trivially, for the right-hand side we get the same result

$$\mathrm{Ad}_{g^{-1}}(\omega(A^\#)) = \mathrm{Ad}_{g^{-1}}(A). \tag{18}$$

□

Definition 1.14 (Curvature 2-Form)**.** Let ω be a connection 1-form on a principal G-bundle P, then the **curvature 2-form** of ω is defined to be

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega], \tag{19}$$

where the wedge product of two Lie algebra-valued differential forms $\omega, \eta \in \Omega^1(P, \mathfrak{g})$ is defined by

$$[\omega \wedge \eta](v_1, v_2) = \frac{1}{2}([\omega(v_2), \eta(v_2)] - [\omega(v_2), \eta(v_1)]), \tag{20}$$

with $[\cdot, \cdot]$ the Lie bracket on $\mathfrak{g}$ and the vector fields v_1 and v_2.

Remark. The connection 1-form measures in some sense how far a vector field is from being horizontal. The curvature 2-form does a similar thing for the commutator of two horizontal vector fields. Let $v_1, v_2 \in \Gamma(HP)$, then

$$\Omega(v_1, v_2) = -\omega([v_1, v_2]), \tag{21}$$

where we used $\mathrm{d}\omega(v_1, v_2) = v_1\omega(v_2) - v_2\omega(v_1) - \omega([v_1, v_2])$ and $\omega(v_1) = \omega(v_2) = 0$.

Definition 1.15 (Tensorial Form)**.** Let P be a principal G-bundle, V a vector space and $\rho\colon G \to \mathrm{Aut}(V)$ a representation. An element $\alpha \in \Omega^k(P, V)$ is called

- **horizontal**, if $\alpha(v_1, \ldots, v_k) = 0$ when at least one v_i is vertical;
- **equivariant**, if $\mathcal{R}_g^*(\alpha) = \rho(g^{-1}) \circ \alpha$.

If α is both horizontal and equivariant, it is called a **tensorial form**. The space of tensorial k-forms is denoted by $\Omega^k_{G,\rho}(P, V)$.

Remark. Let's consider the connection 1-form ω and the adjoint representation $\rho = \mathrm{Ad}$. Then Proposition 1.2 shows that it satisfies equivariance, but it does not satisfy horizontality and is therefore not a tensorial form. If, however, we take the difference of two connection 1-forms the resulting form is tensorial. This will be clear in the next section and is crucial for constructing the space of connections. The curvature 2-form on the other hand, satisfies both and is thus a tensorial form $\Omega \in \Omega^2_{G,\mathrm{Ad}}(P, \mathfrak{g})$.

Remark. Clearly, the usual exterior derivative $d\colon \Omega^k(P, V) \to \Omega^{k+1}(P, V)$ does not necessarily map tensorial forms to tensorial forms. In other words, there exists an $\alpha \in \Omega^k_{G,\rho}(P, V)$ such that $d\alpha \notin \Omega^{k+1}_{G,\rho}(P, V)$. Therefore we want to define a derivative which has exactly this property and we call it covariant derivative.

Definition 1.16 (Covariant Derivative)**.** The **covariant derivative**

$$d_\omega\colon \Omega^k_{G,\rho}(P, V) \to \Omega^{k+1}_{G,\rho}(P, V) \tag{22}$$

is defined for $\alpha \in \Omega^k_{G,\rho}(P, V)$ by

$$d_\omega \alpha := d\alpha + \omega \wedge_{d\rho} \alpha, \tag{23}$$

and

$$\omega \wedge_{d\rho} \alpha(v_1, \dots, v_{k+1}) = \frac{1}{(k+1)!} \sum_\sigma \mathrm{sign}(\sigma) d\rho(\omega(v_{\sigma(1)}))\alpha(v_{\sigma(2)}, \dots, v_{\sigma(k+1)}). \tag{24}$$

Theorem 1.3. *Let P be a principal G-bundle over M, V a vector space and $\rho\colon G \to \mathrm{Aut}(V)$ a representation. Then there exists an isomorphism*

$$\Omega^k_{G,\rho}(P, V) \simeq \Omega^k(M, P \times_\rho V). \tag{25}$$

Proof. See [14]. □

Remark. We can now extend the definition of covariant derivative of tensorial forms naturally to differential forms on M with values in the associated bundle $P \times_\rho V$ as

$$\begin{aligned} d_\omega\colon \Omega^k(M, P \times_\rho V) &\to \Omega^{k+1}(M, P \times_\rho V) \\ \alpha &\mapsto d\alpha + \omega \wedge_{d\rho} \alpha. \end{aligned}$$

Definition 1.17 (Soldering Form)**.** Let P be a principal G-bundle over an N-dimensional smooth manifold M, V an N-dimensional vector space and $\rho\colon G \to \mathrm{Aut}(V)$ a representation. A tensorial 1-form $\theta \in \Omega^1(M, P \times_\rho V)$ is called **soldering form** if the associated bundle morphism $\theta\colon TM \to P \times_\rho V$ is an isomorphism.

1.2 Gauge Field

Definition 1.18 (Gauge Field)**.** Let $\pi\colon P \to M$ be a principal G-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ a connection 1-form, $\{U_\alpha\}$ an open cover of M and $\sigma_\alpha\colon U_\alpha \to P$ a section. The **gauge field** (or **local connection 1-form**) is defined to be the pullback of ω along σ_α

$$A_\alpha := \sigma_\alpha^* \omega \in \Omega^1(U_\alpha, \mathfrak{g}). \tag{26}$$

Remark. In the case of a trivial bundle $P = M \times G$ it is possible to define a global gauge field $A \in \Omega^1(M, \mathfrak{g})$. However, generally it is only possible to build local gauge fields, which implies that we need to be able to change between patches U_α, U_β. An intuitive approach would be to use the one element in G whose right action brings us from $\sigma_\alpha(x)$ to $\sigma_\beta(x)$. It is always possible to find such an element since the fibers are the orbits under the right action and the map to find this element is called gauge map. It turns out that indeed the pullback of a tensorial form would transform under the action of the adjoint representation of this gauge map. However, ω is not tensorial. This adds another term in the transformation, which contains the Maurer–Cartan form.

Definition 1.19 (Maurer–Cartan Form)**.** Let G be a Lie group. The **Maurer–Cartan form** is the $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(G, \mathfrak{g})$ defined by

$$\theta_g \coloneqq d_g(l_{g^{-1}})\colon T_gG \to T_eG \tag{27}$$

for all $g \in G$.

Definition 1.20 (Gauge Map)**.** Let $\sigma_\alpha\colon U_\alpha \to P$ and $\sigma_\beta\colon U_\beta \to P$ be two sections such that $U_\alpha \cap U_\beta \neq \emptyset$. The **gauge map** is defined to be the map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ such that $\forall x \in U_\alpha \cap U_\beta$

$$\sigma_\beta(x) = R_{g_{\alpha\beta}(x)}\sigma_\alpha(x). \tag{28}$$

Theorem 1.4 (Gauge Transformation)**.** *Let ω be a connection 1-form and $\sigma_\alpha\colon U_\alpha \to P$ and $\sigma_\beta\colon U_\beta \to P$ be two sections such that $U_\alpha \cap U_\beta \neq \emptyset$. Then on $U_\alpha \cap U_\beta$ the following holds*

$$A_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ A_\alpha + g_{\alpha\beta}^*(\theta), \tag{29}$$

where $g_{\alpha\beta}^{-1}$ is the inverse of the group element $g_{\alpha\beta}$.

Proof. See [12]. □

Remark. In the special case when G is a matrix Lie group the Maurer–Cartan form is simply given by

$$\theta_g = g^{-1}\mathrm{d}g. \tag{30}$$

With the definition of adjoint action the gauge transformation rule reads

$$A_\beta = g_{\alpha\beta}A_\alpha g_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1}dg_{\alpha\beta}. \tag{31}$$

Definition 1.21 (Field Strength)**.** Let $\Omega \in \Omega^2(P, \mathfrak{g})$ be a curvature 2-form, $\{U_\alpha\}$ an open cover of M and $\sigma_\alpha\colon U_\alpha \to P$ a section. The **field strength** is defined to be the pullback of Ω along σ_α

$$F_\alpha \coloneqq \sigma_\alpha^*\Omega \in \Omega^2(U_\alpha, \mathfrak{g}). \tag{32}$$

Remark. By definition of Ω there is a connection 1-form ω such that equation (19) holds. It follows directly that the associated gauge field and field strength are connected by the equation

$$F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]. \tag{33}$$

To see how the field strength transforms under a change of trivialization chart we make use of the fact that the Maurer–Cartan form satisfies the Maurer–Cartan equation

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{34}$$

It follows that the transformation rule between two patches σ_α and σ_β reads

$$F_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ F_\alpha. \tag{35}$$

In general any pullback of a tensorial form transforms under the action of the adjoint representation. In other words the presence of the Maurer–Cartan form in the transformation rule of the gauge field reflects the non-horizontality of ω.

2 Field Theory and Reduction

In the previous chapter we have constructed the geometric structure necessary for formulating field theories. The next step is to study their dynamics and to define the concept of Lagrangian. One mathematical way for such a description is by using jet bundles and the variational bicomplex. For the theory of jet bundles we refer to [1] and [15]. For the part about Lagrangian field theory we refer to [20]. Lastly, the references used for the reduction are [6], [7], [17] and [13].

Physical theories, such as Newton's dynamics, Maxwell's laws and Schrödinger's equation, are often described using derivatives and differential equations. Thinking about these examples, the fundamental objects and their derivatives are all elements of the same space. For instance, take a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, its derivative $Df\colon \mathbb{R}^n \to \mathbb{R}^m$ is still a function with the same domain and codomain. This means that we can easily put them in the same equation and everything remains well-defined.

Similarly, we need a formalism for writing down differential equations for field theories. A field is defined to be a smooth local section of a fiber bundle $\sigma \in \Gamma(U, F)$. We can then define a bigger fiber bundle, called k-jet bundle, which in its fibers holds also information about the derivatives of the field. In particular, we are interested in the infinite jet bundle $J^\infty F$. An important fact is that smaller k-jet bundles and F itself are "contained" into the bigger ones. This allows a unified description with all possible field values and derivatives in terms of $J^\infty F$. Next, we define functions, vector fields and differential forms on it. Particularly, the differential forms will be of great interest since the Lagrangian should be a volume form on M. The set of these differential forms defines the variational bicomplex, which can essentially be seen as the de Rham complex of the jet bundle. However, it has the additional feature of being bigraded, meaning that its differential can be split into two smaller differentials $d = d_H + d_V$, each of which defining a cochain. Intuitively, the horizontal differential d_H differentiates along M and the vertical differential d_V differentiates along the fibers.

The differential forms on $J^\infty F$ have degrees in all possible spacetime, field and derivative directions and their coefficients are given by functions of, again, spacetime, field and derivative values $f(x, \phi, \partial\phi, \dots)$. From a physical point of view this is exactly what we seek for a Lagrangian, except for the fact that we only want differential forms along the spacetime manifold; their coefficients should nevertheless be functions of the field and their derivatives. Therefore we need to "pullback" these forms in a certain way. By doing so, the natural distinction between the differential d and the variation δ arises, where the first is the pullback of the horizontal differential and the second is the pullback of the vertical differential.

2.1 Variational Bicomplex

For the entire section we assume (F, M, π) to be a fiber bundle over an N-dimensional manifold M.

Definition 2.1 (k-Jet)**.** Let $x \in M$ and $k \in \mathbb{N}$. The **equivalence class of local**

sections at x of degree k, denoted by $j^k_x\sigma$, is defined by the equivalence relation

$$\sigma_1 \sim \sigma_2 \quad \Longleftrightarrow \quad \begin{cases} \partial_\mu \sigma_1|_x = \partial_\mu \sigma_2|_x & \forall \mu = 1, \dots, N \\ \vdots & \\ \partial_{\mu_1 \dots \mu_k} \sigma_1|_x = \partial_{\mu_1 \dots \mu_k} \sigma_2|_x & \forall \mu_1, \dots, \mu_k = 1, \dots, N \end{cases}$$

where $\sigma_1 \colon U_1 \to F$ and $\sigma_2 \colon U_2 \to F$ are two local sections and U_1, U_2 two neighbourhoods of $x \in M$. The k**-jet** at x is then defined to be the space of all such equivalence classes

$$J^k_x F = \frac{\Gamma(U, F)}{\sim} \tag{36}$$

and the **fiber bundle of** k**-jets** is

$$J^k F = \bigsqcup_{x \in M} J^k_x F. \tag{37}$$

Remark. The fiber bundle of k-jets can be seen as a fiber bundle over M with the projection map

$$\begin{aligned} \pi^k_M \colon J^k F &\to M \\ (x, j^k_x \sigma) &\mapsto x \end{aligned}$$

or as a fiber bundle over F with the projection

$$\begin{aligned} \pi^k_0 \colon J^k F &\to F \\ (x, j^k_x \sigma) &\mapsto \sigma(x). \end{aligned}$$

Furthermore, let $0 \leq l \leq k$, then $J^k F$ is also a fiber bundle over $J^l F$ with the projection map

$$\begin{aligned} \pi^k_l \colon J^k F &\to J^l F \\ (x, j^k_x \sigma) &\mapsto (x, j^l_x \sigma). \end{aligned}$$

Definition 2.2 (Infinite Jet Bundle)**.** The **infinite jet bundle** $J^\infty F$ is defined to be the inverse limit[2] of the sequence of projection maps $\pi^k_l \colon J^k F \to J^l F$, for $0 \leq l \leq k$. Its projection map onto the fiber bundle of k-jets is

$$\begin{aligned} \pi^\infty_k \colon J^\infty F &\to J^k F \\ (x, j^\infty_x \sigma) &\mapsto (x, j^k_x \sigma). \end{aligned}$$

[2] The formal definition of the inverse limit makes use of category theory. Even though it goes beyond the scope of this work, we can give a quick explanation of the concept. First of all, we need a family of groups $(A_i)_{i \in I}$ and a family of homomorphisms $f_{ij} \colon A_j \to A_i$, where $(I, \leq)$ is a partially ordered set. These should satisfy $f_{ii} = \mathrm{Id}$ and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k \in I$ (note how π^k_l satisfies these properties). The inverse limit of this system is then defined to be $A = \varprojlim A_i = \{\vec{a} \in \prod A_i \mid a_i = f_{ij}(a_j) \text{ for all } i \leq j \in I\}$. For further details see [10] and [16].

Remark. We want to find a coordinate description of the infinite jet bundle. To do so we can take a local trivialization chart (U, ϕ) on F, such that for all $x \in U$ and $p \in \pi^{-1}(x)$ we have $\phi(p) = (x^\mu, y^i)$. We can now use this to write local sections of the bundle in coordinates, meaning that $\phi(\sigma(x)) = (x^\mu, \sigma^i(x))$, and then lift the local trivialization on F to a local trivialization on $J^\infty F$ by defining (U_J, ϕ_J) with

$$U_J := (\pi_0^\infty)^{-1}(\pi^{-1}(U)) \tag{38}$$

and with ϕ_J such that given[3]

$$\begin{aligned} u^i &= \sigma^i(x) \\ u^i_\mu &= \partial_\mu \sigma^i(x) \\ &\vdots \\ u^i_{\mu_1 \ldots \mu_k} &= \partial_{\mu_1} \ldots \partial_{\mu_k} \sigma^i(x) \\ &\vdots \end{aligned}$$

we have

$$\phi_J(x, j^\infty_x \sigma) := (x^\mu, u^i, u^i_\mu, \ldots, u^i_{\mu_1 \ldots \mu_k}, \ldots). \tag{39}$$

In summary, an element of the infinite jet bundle is given by a point x on the spacetime manifold M and by an equivalence class containing information about the field value of a section and all its derivatives at that point, which in coordinates are given by u^i and $u^i_{\mu_1 \ldots \mu_k}$, respectively.

Definition 2.3 (k-Jet Prolongation)**.** Let $\sigma \in \Gamma(U, F)$ be a local section for $U \subset M$. The **k-jet prolongation** of σ is defined to be the section $j^k\sigma \in \Gamma(U, J^k F)$ such that

$$\begin{aligned} j^k\sigma \colon U &\to J^k F \\ x &\mapsto (x, j^k_x \sigma). \end{aligned}$$

Definition 2.4 (Infinite Prolongation)**.** Let (F, π, M) and $/F', \pi', M')$ be two fiber bundles, $\phi \colon F \to F'$ a bundle morphism and $\phi_0 \colon M \to M'$ its restriction on the base manifolds. The **infinite prolongation** $j^\infty \phi$ of ϕ is defined to be

$$\begin{aligned} j^\infty \phi \colon J^\infty F &\to J^\infty F' \\ (x, j^\infty_x \sigma) &\mapsto (x, j^\infty_{\phi_0(x)}(\phi \circ \sigma \phi_0^{-1})). \end{aligned}$$

Remark. We have defined the fundamental object for our construction: the infinite jet bundle $J^\infty F$. In order to define vectors and differential forms on it we use the projection map. We start with the vector fields on the k-jet bundle which are defined as usual. Then we define the vector fields on the infinite jet bundle by saying that $X \in \mathfrak{X}(J^k F)$ if and only if there exists a $X_k \in \mathfrak{X}(J^k F)$ such that

$$X_k = (\pi_0^\infty)_*(X).$$

[3] Note that when writing $u^i = \sigma^i(x)$ we implicitly assume a trivialization ϕ.

Using the coordinates in (39) a general vector field can be represented as

$$X = a^{\mu}\frac{\partial}{\partial x^{\mu}} + b^{i}\frac{\partial}{\partial u^{i}} + \sum_{k=1}^{\infty} b^{i}_{\mu_1\ldots\mu_k}\frac{\partial}{\partial u^{i}_{\mu_1\ldots\mu_k}} \tag{40}$$

Similarly, the we define the p-forms on the infinite jet bundle by saying that $\omega \in \Omega^p(J^\infty F)$ if and only if there exists an $\omega_k \in \Omega^p(J^k F)$ such that

$$\omega = (\pi_k^\infty)^*(\omega_k).$$

Definition 2.5 (Contact Form)**.** A p-form $\omega \in \Omega^p(J^k F)$ is called **contact form** if for all k-jet prolongations $j^k\sigma$

$$(j^k\sigma)^*(\omega) = 0. \tag{41}$$

Remark. We denote the set of all contact forms on $J^\infty F$ by $\mathcal{C}(J^\infty F)$. Note that the latter is an ideal in the exterior algebra $\bigwedge(J^\infty F)$. In particular, it is also a differential ideal, meaning that

$$d\mathcal{C}(J^\infty F) \subset \mathcal{C}(J^\infty F). \tag{42}$$

Definition 2.6 (Vertical Form)**.** Let $p \in \mathbb{N}$ and $s = 0, 1, \ldots, p+1$. The set of **vertical** (s,p)**-forms** is defined to be

$$\Omega_V^{(s,p)} = \mathcal{C}(J^\infty F)^{\wedge s} \cap \Omega^p(J^\infty F), \tag{43}$$

where $\mathcal{C}(J^\infty F)^{\wedge s}$ is the s-th wedge product of $\mathcal{C}(J^\infty F)$.

Definition 2.7 (Horizontal Form)**.** Let $p \in \mathbb{N}$ and $s = 0, 1, \ldots, p+1$. A p-form $\omega \in \Omega^p(J^\infty F)$ is called a **horizontal** (r,p)**-form** if for all sets of tangent vectors $\{X_i\}_{i=1,\ldots,p}$ where at least $p - r + 1$ vectors are vertical it holds that

$$\omega(X_1, \ldots, X_p) = 0. \tag{44}$$

The set of all such horizontal forms is defined by $\Omega_H^{(r,p)}$.

Proposition 2.1. *The exterior differential* $d\colon \Omega^p(J^\infty F) \to \Omega^{p+1}(J^\infty F)$ *can be split into*

$$d = d_H + d_V \tag{45}$$

such that $d_H\colon \Omega_H^{(r,p)} \to \Omega_H^{(r+1,p+1)}$ *increases the horizontal degree of a form and* $d_V\colon \Omega_V^{(s,p)} \to \Omega_H^{(s+1,p+1)}$ *increases the vertical degree of a form. Furthermore,* $d_H^2 = d_V^2 = 0$ *and, consequently,* $d_H d_V = -d_V d_H$*. Then* d_H *is called the* ***horizontal differential*** *and* d_V *the* ***vertical differential****.*

Definition 2.8 (Variational Bicomplex)**.** We define the **variational bicomplex** to be the following chain complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
\cdots & \xrightarrow{d_H} & \Omega^{(r-1,s-1)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r,s-1)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r+1,s-1)}(J^\infty F) & \xrightarrow{d_H} & \cdots \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
\cdots & \xrightarrow{d_H} & \Omega^{(r-1,s)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r,s)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r+1,s)}(J^\infty F) & \xrightarrow{d_H} & \cdots \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
\cdots & \xrightarrow{d_H} & \Omega^{(r-1,s+1)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r,s+1)}(J^\infty F) & \xrightarrow{d_H} & \Omega^{(r+1,s+1)}(J^\infty F) & \xrightarrow{d_H} & \cdots \\
 & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_V} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where

$$\Omega^{(r,s)}(J^\infty F) = \Omega_H^{(r,p)}(J^\infty F) \cap \Omega_V^{(s,p)}(J^\infty F), \tag{46}$$

where r is the **horizontal degree**, s is the **vertical degree** and $p = r + s$ is the degree of the differential form.

2.2 Lagrangian Field Theory

After this formal construction of the variational bicomplex we will introduce the physically relevant concept of field theory. The space of fields, which is the fundamental object of a field theory, is defined to be the set of local sections of a fiber bundle F. In general, this is an infinite-dimensional manifold which inherits the structure of a Fréchet space.

Definition 2.9 (Space of Fields)**.** Let M be an N-dimensional manifold and (F, π, M) a fiber bundle. The **space of fields** is defined to be the set of local sections of F[4]

$$F_M := \Gamma(U, F). \tag{47}$$

Definition 2.10 (Local Variational Bicomplex)**.** Define the **evaluation map**

$$\begin{aligned} \mathrm{ev}\colon M \times F_M &\to F \\ (x, \phi) &\mapsto \phi(x) \end{aligned}$$

and the map

$$\begin{aligned} e_\infty\colon M \times F_M &\to J^\infty F \\ (x, \phi) &\mapsto \mathrm{ev}(x, j^\infty(\phi)). \end{aligned}$$

Then the **local variational bicomplex** is defined to be

[4] $\Gamma(U, F) = \{\phi\colon U \to F \mid U \subset M \text{ open}\}$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\cdots & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r-1,s-1)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r,s-1)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r+1,s-1)} & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\cdots & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r-1,s)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r,s)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r+1,s)} & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\cdots & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r-1,s+1)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r,s+1)} & \xrightarrow{d} & \Omega_{\mathrm{loc}}^{(r+1,s+1)} & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where the set of **local** (r,s)**-forms** is defined to be

$$\Omega_{\mathrm{loc}}^{(r,s)}(M \times F_M) \coloneqq e_\infty^*(\Omega^{(r,s)}(J^\infty F)) \tag{48}$$

and with the **differential**

$$
\begin{aligned}
d\colon \Omega_{\mathrm{loc}}^{(r,s)}(M \times F_M) &\to \Omega_{\mathrm{loc}}^{(r+1,s)}(M \times F_M) \\
e_\infty^*(\alpha) &\mapsto e_\infty^*(d_H\alpha)
\end{aligned}
$$

and the **variation**

$$
\begin{aligned}
\delta\colon \Omega_{\mathrm{loc}}^{(r,s)}(M \times F_M) &\to \Omega_{\mathrm{loc}}^{(r,s+1)}(M \times F_M) \\
e_\infty^*(\alpha) &\mapsto e_\infty^*(d_V\alpha).
\end{aligned}
$$

Definition 2.11 (Field Theory)**.** Let M be an N-dimensional manifold with boundary ∂M. A **Field Theory** is the assignment of a space of fields F_M and an **action**

$$S_M[L] = \int_M L, \tag{49}$$

where the **Lagrangian** $L \in \Omega_{\mathrm{loc}}^{(N,0)}(M \times F_M)$ is a local $(N,0)$-form, to the manifold M.

Definition 2.12 (Source Form)**.** A local $(N,1)$-form $\alpha \in \Omega_{\mathrm{loc}}^{(N,1)}(M \times F_M)$ is called a **source form**, if for all (x,ϕ) the element $\alpha(x,\phi)$ only depends on the 0-jet $J_x^0\phi$. The space of all such forms is labelled by $\Omega_{\mathrm{source}}^{(N,1)}(M \times F_M)$.

Lemma 2.2. *The space of local* $(N,1)$*-forms is given by the following direct sum*

$$\Omega_{\mathrm{loc}}^{(N,1)}(M \times F_M) = \Omega_s^{(N,1)}(M \times F_M) \oplus d\,\Omega_{\mathrm{loc}}^{(N-1,1)}(M \times F_M). \tag{50}$$

Theorem 2.3. *Let* $L \in \Omega_{\mathrm{loc}}^{(N,0)}(M \times F_M)$ *be a Lagrangian. Then there exists a unique source form* $E(L) \in \Omega_{\mathrm{source}}^{(N,1)}(M \times F_M)$ *and a local* $(N-1,1)$*-form* $\alpha \in \Omega_{\mathrm{loc}}^{(N-1,1)}(M \times F_M)$ *such that*

$$\delta L = E(L) - d\alpha, \tag{51}$$

*where $E(L)$ is called the **Euler–Lagrange operator** and is uniquely defined from L. Moreover, α is unique up to d-exact terms and $E(L+dK)=E(L)$ for all $K\in\Omega_{\text{loc}}^{(N-1,0)}$.*

Definition 2.13 (Euler–Lagrange Equation)**.** The **Euler–Lagrange equation** is given by

$$E(L)(x,\phi)=0. \tag{52}$$

Furthermore, we define the space EL_M to be the subspace of fields which satisfy this equation

$$EL_M \coloneqq \{(x,\phi) \mid E(L)(x,\phi)=0\} \subset F_M. \tag{53}$$

2.3 Symplectic Reduction

We will give a quick introduction to symplectic geometry, Hamiltonian mechanics and how to reduce this structure to submanifolds, given by level sets of momentum maps.

Definition 2.14 (Symplectic Manifold)**.** A **symplectic manifold** (M,ω) is a smooth manifold M together with a **symplectic form** ω, that is a nondegenerate, closed 2-form on M.

Remark. Non-degeneracy of the symplectic form implies that there exists an isomorphism

$$\begin{aligned}\flat\colon TM &\to T^*M\\ X &\mapsto \iota_X\omega\end{aligned}$$

associated to ω with inverse $\sharp\colon T^*M\to TM$. This means that for every 1-form there is an associated vector field, which is necessary to define the Hamiltonian vector field.[5]

Definition 2.15 (Hamiltonian Vector Field)**.** Let $H\in C^\infty(M)$ be a smooth function on a symplectic manifold (M,ω). Then $dH\in T^*M$ and so there exists a vector field $X_H\in TM$, called **Hamiltonian vector field** of H, such that

$$dH=\iota_{X_H}\omega. \tag{54}$$

Definition 2.16 (Poisson Manifold)**.** A **Poisson manifold** is a smooth manifold M together with a Poisson bracket, that is a bilinear map $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ satisfying

1. $\{f,g\}=-\{g,f\}$ (Skew symmetry),
2. $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$ (Jacobi identity),
3. $\{fg,h\}=f\{g,h\}+g\{f,h\}$ (Leibniz rule),

$\forall f,g,h\in C^\infty(M)$.

[5] In the infinite dimensional case only injectivity of $\flat$ is required.

Remark. In the definition of Poisson structure the first two conditions imply that $\{\cdot,\cdot\}$ is a Lie bracket on $C^\infty(M)$ while the third condition guarantees that for each $f \in C^\infty(M)$ the linear map $X_f \coloneqq \{f,\cdot\}\colon C^\infty(M) \to C^\infty(M)$ is a derivation, i.e. it defines a vector field $X_f \in \mathfrak{X}(M)$. It turns out that if the Poisson manifold derives from a symplectic structure this vector field is the Hamiltonian vector field of f.[6] Indeed, one can easily show that on every symplectic manifold (M,ω) there is a Poisson bracket $\{\cdot,\cdot\}_\omega$ defined by

$$\{f,g\}_\omega \coloneqq \omega(X_f,X_g), \tag{55}$$

where $f,g \in C^\infty(M)$ and $X_f, X_g \in TM$ are their Hamiltonian vector fields. The skew symmetry and Leibniz rule follow directly from the antisymmetry of 2-forms and the fact that $d(fg) = fdg + gdf$ implies $X_{fg} = fX_g + gX_f$. The steps to show the Jacobi identity make use of the fact that $d\omega = 0$ and are somewhat more laborious.

Definition 2.17 (Symplectic Action)**.** Let (M,ω) be a symplectic manifold and G a Lie group. A **symplectic action** Ψ is a left action of G on M which acts by symplectomorphisms[7]

$$\begin{aligned} \Psi\colon G &\to \mathrm{Sympl}(M) \subset \mathrm{Diff}(M) \\ g &\mapsto \Psi_g. \end{aligned}$$

Definition 2.18 (Momentum Map)**.** Let Ψ be a symplectic action of G on M. The map $\mu\colon M \to \mathfrak{g}^*$ is called **momentum map** if

1. For every Lie algebra element $X \in \mathfrak{g}$

$$d\mu^X = \iota_{X^\#}\omega, \tag{56}$$

   where $\mu^X\colon M \to \mathbb{R}, p \mapsto \mu(p)(X)$ and $X^\#$ is the fundamental vector field associated to X (see Definition 1.12)[8];

2. μ is equivariant with respect to the action Ψ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, that is $\forall g \in G$[9]

$$\mu \circ \Psi_g = \mathrm{Ad}^*_g \circ \mu. \tag{57}$$

Then Ψ is said to be a **Hamiltonian action**[10].

[6] In the infinite dimensional case only some functions, called Hamiltonian functions, possess a Hamiltonian vector field, which is however uniquely defined. Then the Poisson bracket of two functions is defined whenever at least one function is Hamiltonian.

[7] A symplectomorphism is a diffeomorphism $\Psi_g\colon M \to M$ such that it leaves the symplectic form invariant, i.e. $(\Psi_g)^*\omega = \omega$.

[8] In the previous definition we were treating principal bundles which have an incorporated action $\mathcal{R}$. Here instead we have written the action in terms of diffeomorphisms, which leads to the equivalent definition $X^\#(p) = \frac{d}{dt}[\Psi_{\exp(tX)}(p)]_{t=0}$.

[9] This is the usual notation symplectic geometry literature. However, we can rephrase this condition using a notation more similar to the previous chapter, that is $\Psi^*_g\mu = \mathrm{Ad}^*_g\mu$ where $\mu \in \Omega^0(M,\mathfrak{g})$.

[10] Some authors only require the first point and call equivariant momentum maps those which satisfy also 2.

Remark. The fundamental vector field $X^\#$ is a symplectic vector field for every $X \in \mathfrak{g}$, which means that $(X^\#)^\flat$ is closed ($d\iota_{X^\#}\omega = 0$). If $(X^\#)^\flat$ is also exact, meaning that $\iota_{X^\#}\omega = df$ for some $f \in C^\infty(M)$, the vector field is Hamiltonian. If $X^\#$ is Hamiltonian then it is also symplectic since $d^2 = 0$.

Now we are interested in the zero level set of a momentum map

$$C = \mu^{-1}(0).$$

By definition this is a subset of M which is mapped to the zero vector $0 \in \mathfrak{g}^*$. The equivariance condition ensures that, if 0 is a regular value of the momentum map, then C is preserved by the group action[11]

$$\Psi_g(C) \subset C, \qquad \forall g \in G.$$

We can now study the orbit space C/G given by the set of all orbits[12]

$$O_p = \{q \in M \mid \exists\, g \in G, \Psi_g(q) = p\}.$$

Theorem 2.4. *Let G be a compact Lie group and $\Psi \colon G \to \mathrm{Diff}(M)$ a free and transitive action on a smooth manifold M. Then the orbit space M/G is a smooth manifold and $(M, \pi, M/G)$ is a principal G-bundle with the canonical projection $\pi \colon M \to M/G$.*

Theorem 2.5 (Marsden–Weinstein)**.** *Let (M, ω) be a symplectic manifold, G a compact Lie group, μ a momentum map with 0 as regular value, whose action is free and transitive on $C = \mu^{-1}(0)$ and with the inclusion map $i \colon C \hookrightarrow M$. Define the orbit space $\underline{C} := C/G$. Then*

1. *$\underline{C}$ is a smooth manifold,*
2. *$(C, \pi, \underline{C})$ is a principal G-bundle, with the canonical projection π,*
3. *there is a unique symplectic form ω_{red} on $\underline{C}$ such that $i^*\omega = \pi^*\omega_{\mathrm{red}}$.*

Proof. Both proofs can be found in [17]. □

2.4 KT Construction

Now we will go back to what we have learned about field theories and combine it with the last section about symplectic reduction. It turns out that the space of fields on the boundary ∂M of a field theory is naturally endowed with a presymplectic structure. The latter is given by the variation of the boundary term which

[11] More precisely, the action of G does not preserve the level sets of μ in general. Equivariance of μ implies that the elements of G which do preserve C are exactly those that fix $0 \in \mathfrak{g}$ under the coadjoint action.

[12] Physically speaking, quotienting out G of C is equivalent to fixing the symmetry of the system. In particular, when P is a principal bundle, and is therefore equipped with a natural action, it is equivalent to the process of gauge fixing.

arises together with the Euler–Lagrange equations. Let's take some action $S_M[L]$ and apply Theorem 2.3. We then find

$$\delta S_M = E(L)_M - \int_{\partial M} \alpha, \tag{58}$$

where we used Stokes' theorem $\int_M d\alpha = \int_{\partial M} \alpha$ and defined $E(L)_M := \int_M E(L)$.

Definition 2.19 (Presymplectic Manifold)**.** A **presymplectic manifold** (M, ω) is a smooth manifold M together with a **presymplectic form** ω, that is a closed 2-form on M whose kernel has constant rank[13].

Definition 2.20 (Germ)**.** Let M be a smooth manifold, $U, V \subset M$ two neighbourhoods of $x \in M$ and $f \in C^\infty(U), g \in C^\infty(V)$. Define the equivalence relation on the set of smooth functions on a neighbourhood of x such that $f \sim g$ if there exists a neighbourhood $W \subset U \cap V$ of x such that

$$f|_W = g|_W. \tag{59}$$

A **germ** at x is then an equivalence class under this relation.

Definition 2.21 (Preboundary Fields)**.** The space of **preboundary fields** $\tilde{F}_{\partial M}$ on ∂M is defined to be the space of germs of fields at $\partial M \times \{0\}$ on $\partial M \times [0, \varepsilon]$ for some $\varepsilon > 0$.

Remark. In other words, the space of preboundary fields is just the space of possible field values near the boundary. We still need to include the N-th dimension in the form of a short interval $[0, \varepsilon]$ to keep all derivatives well-defined. Furthermore, there is a canonical submersion $\tilde{\pi}_{\partial M} \colon F_M \to \tilde{F}_{\partial M}$ which maps the fields to their restriction on the boundary, allowing us to rewrite equation (58) as stated in the following theorem. This is important since it allows us to study this boundary term where also the fields are restricted on the boundary.

Theorem 2.6. *The space of preboundary fields $\tilde{F}_{\partial M}$ is a smooth manifold. Furthermore, there exists a $(0,1)$-form $\tilde{\alpha}_{\partial M}$ on $\tilde{F}_{\partial M}$ such that*

$$\delta S_M = E(L)_M - \tilde{\pi}^*_{\partial M}(\tilde{\alpha}_{\partial M}). \tag{60}$$

Moreover, define the following $(0,2)$-form on $\tilde{F}_{\partial M}$

$$\tilde{\varpi}_{\partial M} := \delta \tilde{\alpha}_{\partial M}, \tag{61}$$

where δ is the variation defined in 2.10. Then $(\tilde{F}_{\partial M}, \tilde{\varpi}_{\partial M})$ is a presymplectic manifold.

Remark. This $(0,2)$-form[14] is the starting point for the symplectic description which we will build. It is presymplectic in the sense that it is closed by construction, since $\delta^2 = 0$, but not necessarily nondegenerate.

[13] In the infinite dimensional case the kernel is a subbundle $\ker(\omega) \subset TM$.

[14] All the forms which we will consider from now on will all have the form $(0, s)$, meaning that they are differential form along the fibers. To simplify notation we will simply call them s-forms.

Definition 2.22 (Geometric Phase Space)**.** The **geometric phase space** of a field theory is defined to be the quotient of the space of preboundary fields by the kernel of its presymplectic form

$$F_{\partial M} \coloneqq \frac{\tilde{F}_{\partial M}}{\ker(\tilde{\varpi}_{\partial M})}. \tag{62}$$

Remark. Again, there is a canonical submersion $\pi_{\partial M}\colon F_M \to F_{\partial M}$, which can simply be extended from $\tilde{\pi}_{\partial M}$ by additionally mapping an element to the equivalence class it belongs to. Then we can find a 1-form $\alpha_{\partial M}$ on $F_{\partial M}$ such that

$$\delta S_M = E(L)_M - \pi^*_{\partial M}(\alpha_{\partial M}). \tag{63}$$

Next, we make an assumption which has to be verified for each field theory separately, which is that $(F_{\partial M}, \varpi_{\partial M})$ is a symplectic manifold, where we defined the following 2-form on $F_{\partial M}$

$$\varpi_{\partial M} \coloneqq \delta\alpha_{\partial M}. \tag{64}$$

We have obtained a symplectic manifold of fields restricted to the boundary. In order to get to the lowest possible state of reduction we need to fix the constraints, given by the Euler–Lagrange equations, and perform a symplectic reduction. The result is the so called reduced phase space.

Definition 2.23 (Reduced Phase Space)**.** Define the zero locus of the Euler–Lagrange equations on the boundary $L_{\partial M} \coloneqq \pi_{\partial M}(EL_M)$. The **Cauchy Data** $C_{\partial M}$ is then defined to be the subspace of $F_{\partial M}$ that can be completed to a pair belonging to $L_{\partial M\times[0,\varepsilon]}$ for all $\varepsilon > 0$. If $C_{\partial M}$ is a submanifold, then the **reduced phase space** $\underline{C}_{\partial M}$ is defined to be the symplectic reduction of $C_{\partial M}$.

The reduced phase space is the final object of this construction. One can assume that it is a submanifold, but usually it is not. Physically, it indicates the space of possible initial conditions of the theory modulo symmetries. A trivial example is the case of classical mechanics defined on a manifold M, where the reduced phase space for a free particle is just the cotangent bundle T^*M with the canonical symplectic structure. In cases where equations of motion present also derivatives which are not transverse to the boundary (or the space of initial conditions), namely they are not evolution equations, some of these equations turn into constraints of the theory and the reduced phase space is then built in the aforementioned way.[15] Historically, the description of this space has been performed in terms of Dirac's analysis, where one first analyses the primary and secondary constraints which are then regrouped into constraints of first- and second-class. The KT construction instead allows for a more geometrical description of the reduced phase space and it turns out to be much more natural in the case of the Palatini–Cartan formalism. A summary of this construction can be seen in figure 1.

[15] This is the case for every gauge theory, where not all equations of motion are transverse. More precisely, all systems with local symmetries according to the second Noether's theorem possess constraints on their momenta, since these are not all independent. Systems of this kind are also called *singular* in the literature.

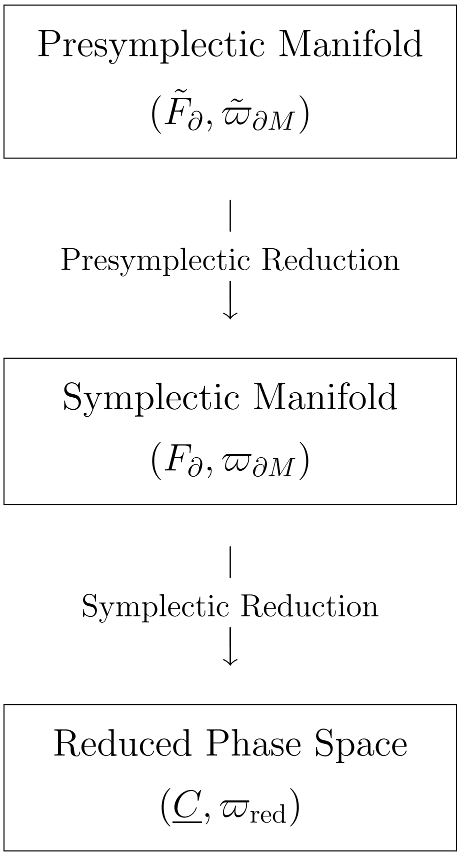

Figure 1: Summary of the KT construction. We start with the space of pre-boundary fields which is naturally endowed with a presymplectic structure. By quotienting out the kernel of the presymplectic form, we find the geometric phase space which is consequently symplectic. Finally, we perform a symplectic reduction by the symmetries of the theory. What one may also do in the last step, in order to reach the reduced phase space, is to perform a coisotropic reduction (Appendix A) by the constraints of the theory coming from the field equations.

3 Palatini–Cartan Formalism

In this chapter we will introduce the Palatini–Cartan formalism, which is an alternative formulation of General Relativity almost equivalent to the Einstein formulation. The main benefit is that it is formulated in terms of differential forms, allowing a more natural restriction to the boundary. Furthermore, it is a gauge theory. The references for this chapter are [4], [8] and [5].

Usually, General Relativity is written in terms of the Einstein–Hilbert action

$$S_{EH} = \int_M (R - \Lambda)\sqrt{g},$$

which leads to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

with the metric tensor g as a dynamical field variable.

The Palatini–Cartan formalism is an alternative approach to gravity equivalent to the latter, in the sense that both give the same space of solutions of the Euler–Lagrange equations modulo symmetries. There is some more freedom in the PC formulation, since it has an additional gauge symmetry. The difference is that, instead of considering the metric tensor as variable field, one introduces two new field variables. The first is a coframe field, or vielbein, that is an orientation preserving isomorphism $e\colon TM \to \mathcal{V}$ such that one can recover the Lorentzian metric with $g = e^*\eta$. The second field is a principal connection ω on P which turns out to have values in the so-called Minkowski bundle $\mathcal{V}$.

3.1 Coframe Field

The goal of this section is to formally introduce the tetrad, vielbein or coframe field, which is one of the two variables of the Palatini–Cartan action. As we will conclude, the tetrad is a bundle isomorphism $TM \to \mathcal{V}$. We could define it right away from the soldering form on M with respect to the group $\mathrm{SO}(N-1,1)$ (see Definition 1.17). However, we will take the more physical approach by defining it from the coframe bundle. This will show how one can recover the spacetime metric from it.

For the entire section we will assume that M is an oriented N-dimensional pseudo-Riemannian manifold with a metric g that has signature $(N-1,1)$.

Definition 3.1 (Coframe Bundle)**.** For every $x \in M$ the **coframe space** at x is defined to be

$$L_x^*M := \{e = (e^1, \ldots, e^N) \mid \{e^a\} \text{ is a basis of } T_x^*M\}. \tag{65}$$

The **coframe bundle** is then defined to be

$$L^*M := \bigsqcup_{x\in M} L_x^*M. \tag{66}$$

Proposition 3.1. *The coframe bundle* (L^*M, π, M) *is a principal* $\mathrm{GL}(N, \mathbb{R})$*-bundle with the canonical projection* $\pi\colon L^*M \to M$*.*

Proof. See [11]. □

Definition 3.2 (Orthonormal Coframe Bundle)**.** For every $x \in M$ the **orthonormal coframe space**[16] at x with respect to the metric g is defined to be

$$\mathrm{SO}^*_x(M, g) \coloneqq \{e = (e^1, \dots, e^N) \mid \{e^a\} \text{ is an orthonormal basis of } T^*_x M\}. \tag{67}$$

The **orthonormal coframe bundle** is then defined to be

$$\mathrm{SO}^*(M, g) \coloneqq \bigsqcup_{x \in M} \mathrm{SO}^*_x(M, g). \tag{68}$$

Proposition 3.2. *The orthonormal coframe bundle* $(\mathrm{SO}^*(M, g), \pi, M)$ *is a principal* $\mathrm{SO}(N-1, 1)$*-bundle with the canonical projection* $\pi\colon \mathrm{SO}^*(M, g) \to M$*.*

Proof. See [11]. □

Remark. We have defined the coframe bundle and orthornormal coframe bundle over M to be the fiber bundles where the fibers at some $x \in M$ are the spaces of possible bases, respectively orthonormal bases, of the cotangent space at that x. Clearly, the orthonormal coframe bundle is a subbundle of the coframe bundle. Furthermore, propositions 3.1 and 3.2 state that these also define principal bundles with respect to the structure group of their fibers. The meaning of this is that for every two bases $e, \tilde{e} \in L^*_x M$ there exists some $A \in \mathrm{GL}(N, \mathbb{R})$ such that $e = A\tilde{e}$ (similarly for the orthonormal case). As for every principal bundle, the fibers can therefore be identified by the structure group itself.

Remark. One can define in an equivalent way the frame bundle LM and the orthonormal frame bundle $\mathrm{SO}(M, g)$. It is straightforward to show that they are also principal bundles.

Definition 3.3 (Vielbein Map)**.** Let (P, p, M) be a principal $\mathrm{SO}(N-1, 1)$-bundle. A **vielbein map** on M is defined to be a principal bundle morphism $e\colon P \to LM$ together with the canonical embedding $i\colon \mathrm{SO}(N-1, 1) \to \mathrm{GL}(N, \mathbb{R})$.

Remark. In order to be a principal bundle morphism the vielbein map has to satisfy two conditions:

1. Verticality: $\pi \circ e = p$
2. Equivariance: $\mathcal{R}_{i(g)} \circ e = e \circ \mathcal{R}_g$ for all $g \in G$.

This is equivalent to asking that the following two diagrams should commute

$$\begin{array}{ccc} P & \xrightarrow{e} & LM \\ & \searrow^{p} & \downarrow \pi \\ & & M \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{e} & LM \\ \mathcal{R}_g \downarrow & & \downarrow \mathcal{R}_{i(g)} \\ P & \xrightarrow[e]{} & LM \end{array} .$$

[16]Orthonormality with respect to the metric g means that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$.

Definition 3.4 (Minkowski Bundle)**.** Consider the principal $\mathrm{SO}(N-1,1)$-bundle (P, p, M) and let (V, η) be a real N-dimensional vector space with signature $(N-1, 1)$. The **Minkowski bundle** on M is the associated bundle

$$\mathcal{V} := P \times_\rho V, \tag{69}$$

where $\rho\colon \mathrm{SO}(N-1,1) \to \mathrm{Aut}(V)$ is the fundamental representation.

Remark. Let us briefly analyse what we have just constructed. First of all we have shown that the fiber bundle of orthonormal frames is a principal bundle with structure group $\mathrm{SO}(N-1,1)$. By construction, one can recover the spacetime metric with

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab},$$

where a, b indicate the basis vector and μ, ν its spacetime component. Then we have a vielbein map, that is a bundle morphism, which "changes fibers" from a general $\mathrm{SO}(N-1,1)$-bundle to the specific frame bundle while keeping the structure group. What this map does is setting a rule for the choice of an orthonormal basis. Since it is kind of cumbersome to work with bases as fundamental object we want to describe them in terms of a general vector field. The vielbein map can now induce such a "selection rule" for the associated vector bundles. In particular, we are interested in a map going from the tangent bundle TM to the Minkowski bundle $\mathcal{V}$. It is important to notice that the tangent bundle is associated to the frame bundle. The existence of such an induced map can be shown using the universal property of the quotient.[17] Due to the equivariance condition of the vielbein map the universal property of the quotient for the bundle morphism $\pi' \circ e\colon P \to TM$, where $\pi'\colon LM \to TM$ is the projection on the tangent bundle, guarantees the existence and uniqueness of a map $\hat{e}\colon \mathcal{V} \to TM$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\;e\;} & LM \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ \mathcal{V} & \xrightarrow[\;\hat{e}\;]{} & TM \end{array} \;.$$

Locally, we can give an explicit construction of this map and show that it is an isomorphism. Let's assume we have a vielbein map, then there is a specific choice of basis $\{e_a\}$ at any point in M. Now we can define the isomorphism (and call it e by abuse of notation)

$$\begin{aligned} e(x)\colon T_xM &\to \mathcal{V}_x \\ v &\mapsto v^a, \end{aligned} \tag{70}$$

where $v = v^a e_a \in T_xM$. This map forms a bundle isomorphism[18] covering the identity on M. Now we can also rewrite the metric in terms of the Minkowski

[17] Let F be a fiber bundle, $\sim$ an equivalence relation on F and $\pi\colon F \to F/\sim$ the canonical projection on the quotient. For every fiber bundle E and bundle morphism $f\colon F \to E$ such that $p \sim q$ implies $f(p) = f(q)$ for all $p, q \in F$, there exists a unique bundle morphism $g\colon F/\sim \to E$ such that $f = g \circ \pi$.

[18] By construction, all fibers are isomorphic and this implies that there exists a vector bundle isomorphism e.

metric in V, that is

$$g = e^*\eta. \tag{71}$$

We will take the map in (70) as a definition for the vielbein.

Definition 3.5 (Vielbein)**.** A **vielbein**[19] (or **tetrad** for $N = 4$) is a smooth section $e \in \Gamma(T^*M \otimes \mathcal{V})$ such that $e(x)$ defines an isomorphism $T_xM \to \mathcal{V}_x$ for all $x \in M$.

3.2 Connection Field

Now we will introduce the second dynamical variable of the theory: the connection field ω. Contrary to the vielbein, we do not have to artificially construct this object since it already exists for any principal bundle P.

One important consequence of Chapter 1 was that the principal connection is not a tensorial form. This is reflected by the fact that the gauge field does not exactly transform in the adjoint representation since there arises another term containing the Maurer–Cartan form. However, the difference of two connection forms is a tensorial form. Clearly, also the difference of two gauge fields transforms in the adjoint, since the two Maurer–Cartan forms cancel. This means that with the help of some reference connection ω_0 we can define the space of connections to be the space of tensorial 1-forms with respect to the adjoint representation.

Definition 3.6 (Space of Connections)**.** Let P be a principal G-bundle and $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation. The **space of connections** on P is defined to be the space of tensorial 1-forms on P with respect to the adjoint representation

$$\mathcal{A}(P) \coloneqq \Omega^1_{G,\mathrm{Ad}}(P, \mathfrak{g}). \tag{72}$$

Remark. As mentioned above, the space of connections $\mathcal{A}(P)$ only describes the differences to some reference connection 1-form ω_0. In other words, for every connection 1-form $\omega_1 \in \Omega^1(P, \mathfrak{g})$ there exists an element $\omega \in \mathcal{A}(P)$ such that $\omega_1 - \omega_0 = \omega$.[20]

Corollary 3.3. *Let P be a principal G-bundle over M. The space of connections on P is isomorphic to the space of 1-forms on M with values in the associated bundle with respect to the adjoint representation*

$$\mathcal{A}(P) \simeq \Omega^1(M, P \times_{\mathrm{Ad}} \mathfrak{g}). \tag{73}$$

Proof. This follows directly from Theorem 1.3. □

Let us now come back to the case where M is an oriented N-dimensional pseudo-Riemannian manifold and P is a principal $\mathrm{SO}(N-1, 1)$-bundle over M.

Proposition 3.4. *Let V be a real N-dimensional vector space, then*

$$\wedge^2 V \simeq \mathfrak{so}(N-1, 1). \tag{74}$$

[19] The German term vielbein translates to "many legs" and is often used for a general N-dimensional case. In the 4-dimensional case it is often replaced by vierbein, meaning "four legs", or tetrad.

[20] Therefore the space of connections is an affine space modelled over $\mathcal{A}(P)$.

Proof. Let $\eta = \text{diag}(1, \dots, 1, -1)$ be the Minkowski metric on V and $\{v_i\}_{i=1}^N$ a basis of V. Then a basis of $\wedge^2 V$ is given by $v_i \wedge v_j$ and contraction with η gives a basis of $\mathfrak{so}(N-1, 1)$[21]

$$\omega_i^j = v_i \wedge v_k \eta^{jk} \tag{75}$$

since $v_i \wedge v_j \eta^{jk}\eta^{il} = -v_i \wedge v_j \eta^{jl}\eta^{ik}$. □

Corollary 3.5. *The space of connections on P is isomorphic to the space of 1-forms on M with values in the second order exterior product of the Minkowski bundle*

$$\mathcal{A}(P) \simeq \Omega^1(M, \wedge^2 \mathcal{V}). \tag{76}$$

Remark. In equation (76) the second order exterior product of $\mathcal{V}$ indicates the associated bundle $P \times_\rho \wedge^2 V$.

3.3 Palatini–Cartan Action

We are now ready to state the Palatini–Cartan formalism by using its main ingredients: the coframe and the connection field. The idea is to write both as differential forms on M with values in the Minkowski bundle. It is important to note that, although we will now make use of the Lagrangian theory constructed in Chapter 2, the (i, j)-forms which are our dynamical objects are fundamentally different from the local (r, s)-forms $\Omega^{(r,s)}_{\text{loc}}(M \times F_M)$.

Definition 3.7 ((i, j)-forms)**.** An (i, j)**-form** on M is a differential i-form with values in the j-th exterior power of $\mathcal{V}$

$$\Omega^{(i,j)}(M) := \Omega^i(M, \wedge^j \mathcal{V}). \tag{77}$$

Furthermore, we define some maps on the set of (i, j)-forms. First of all we define the k-th wedge product of the tetrad $e \in \Omega^{(1,1)}(M)$ as[22]

$$\begin{aligned} W_k^{(i,j)} \colon \Omega^{(i,j)}(M) &\to \Omega^{(i+k,j+k)}(M) \\ \alpha &\mapsto e^k \wedge \alpha. \end{aligned}$$

Next, recalling equation (74) we define

$$\begin{aligned} [\cdot, \cdot] \colon \Omega^{(0,2)} \times \Omega^{(0,2)} &\to \Omega^{(0,2)} \\ (\alpha, \beta) &\mapsto [\alpha, \beta], \end{aligned}$$

where $[\alpha, \beta]$ is defined by the Lie bracket of the corresponding elements in $\mathfrak{so}(N-1, 1)$. We can also extend this bracket to the following map

$$\begin{aligned} [\cdot, \cdot] \colon \Omega^{(i,2)} \times \Omega^{(k,2)} &\to \Omega^{(i+k,2)} \\ (\alpha, \beta) &\mapsto [\alpha, \beta], \end{aligned}$$

which in coordinates is given by

$$[\alpha, \beta]^{a_1 a_2}_{\mu_1 \dots \mu_{i+k}} = \sum_{\sigma_{i+k}} \text{sign}(\sigma_{i+k}) \alpha^{a_1 a_3}_{\mu_{\sigma(1)} \cdots \mu_{\sigma(i)}} \beta^{a_2 a_4}_{\mu_{\sigma(i+1)} \cdots \mu_{\sigma(i+k)}} \eta_{a_3 a_4}. \tag{78}$$

[21] Recall that the Lie algebra $\mathfrak{so}(N-1, 1)$ is the set of all matrices ω_a^b satisfying $\omega_c^a \eta^{cb} + \eta^{ac}\omega_c^b = 0$. Furthermore, with $\omega^{ab} = \omega_c^a \eta^{cb}$ this condition becomes $\omega^{ab} = -\omega^{ba}$.

[22] Note that here the powers of e are defined by wedge products $e^k := \underbrace{e \wedge \cdots \wedge e}_{k-\text{times}}$.

Remark. Note the equivalence $\Omega^{(i,j)}(M) = \Gamma(\wedge^i T^*M \otimes \wedge^j \mathcal{V})$. Therefore one can rewrite the vielbein as an element $e \in \Omega^{(1,1)}_{\text{nd}}(M)$ and the principal connection as an element $\omega \in \Omega^{(1,2)}(M)$. For the vielbein we require it to be a nondegenerate form since it should define an isomorphism according to Definition 3.5. Furthermore, by Theorem 1.3, every (i,j)-form is a tensorial form. Since the covariant derivative d_ω from Definition 1.16 maps tensorial forms of degree k to tensorial forms of degree $k+1$ it is also a well-defined map from (i,j)-forms to $(i+1,j)$-forms.

Definition 3.8 (Classical Palatini–Cartan)**.** Let M be an N-dimensional pseudo-Riemannian manifold. The classical **Palatini–Cartan** theory is the assignment of the pair $(\mathcal{F}_{PC}, S_{PC})_M$ to M, with the space of fields

$$\mathcal{F}_{PC} = \Omega^{(1,1)}_{\text{nd}}(M) \times \Omega^{(1,2)}(M) \tag{79}$$

and the action[23]

$$S_{PC} = \int_M \frac{1}{(N-2)!} e^{N-2} \wedge F_\omega + \frac{\Lambda}{N!} e^N, \tag{80}$$

where $(e, \omega) \in \mathcal{F}_{PC}$.

The variation of the action δS_{PC} leads to the Euler–Lagrange equations of motion

$$e^{N-3} d_\omega e = 0, \tag{81}$$

$$\frac{1}{(N-3)!} e^{N-3} F_\omega + \frac{\Lambda}{(N-1)!} e^{N-1} = 0 \tag{82}$$

and to the boundary term

$$\tilde{\alpha} = \int_\Sigma \frac{1}{(N-2)!} e^{N-2} \delta\omega. \tag{83}$$

3.4 Restriction on the Boundary

Now we will consider the boundary of the spacetime manifold $\Sigma = \partial M$ with the inclusion map $i \colon \Sigma \to M$ and the natural metric

$$g^\partial := i^* g. \tag{84}$$

The reduction of the fields in this bundle setting is quite intuitive. The fiber bundles on the whole manifold M are simply restricted to its boundary Σ, while keeping the fibers unchanged. Mathematically this is achieved by the pullback bundle.

Definition 3.9 (Pullback Bundle)**.** Let (F, π, M) be a fiber bundle, M' a smooth manifold and $\phi \colon M' \to M$ a smooth map. The **pullback bundle** of F by ϕ is defined to be

$$\phi^* F := \{(x', p) \in M' \times F \mid \phi(x') = \pi(p)\}. \tag{85}$$

[23] Here F_ω is the field strength stated in equation (33).

Definition 3.10 (Boundary (i,j)-forms)**.** First define the pullback principal bundle

$$P|_\Sigma \coloneqq i^*P \tag{86}$$

on M by the inclusion map $i\colon \Sigma \to M$. The latter induces the Minkowski bundle on the boundary

$$\mathcal{V}|_\Sigma \coloneqq P|_\Sigma \times_\rho V. \tag{87}$$

Then we define the space of **boundary** (i,j)**-form** on Σ to be

$$\Omega_\partial^{(i,j)} \coloneqq \Omega^i(\Sigma, \wedge^j \mathcal{V}|_\Sigma). \tag{88}$$

Again, we define similar maps as in Definition 3.7 on the set of boundary (i,j)-forms:

$$\begin{aligned} W_k^{\partial(i,j)}\colon \Omega_\partial^{(i,j)} &\to \Omega_\partial^{(i+k,j+k)} \\ \alpha &\mapsto e^k \wedge \alpha, \end{aligned}$$

$$\begin{aligned} [\cdot,\cdot]\colon \Omega_\partial^{(i,2)} \times \Omega_\partial^{(k,2)} &\to \Omega_\partial^{(i+k,2)} \\ (\alpha,\beta) &\mapsto [\alpha,\beta], \end{aligned}$$

where the tetrad is a nondegenerate boundary form $e \in \Omega_\partial^{(1,1)}$.[24] Furthermore, we define a generalization of the $\mathfrak{so}(N-1,1)$ action on $\mathcal{V}|_\Sigma$ and can therefore write the bracket with the boundary tetrad as

$$\begin{aligned} \varrho^{(i,j)}\colon \Omega_\partial^{(i,j)} &\to \Omega_\partial^{(i+1,j-1)} \\ \alpha &\mapsto [\alpha, e], \end{aligned}$$

which in coordinates is defined by

$$[\alpha, e]^{a_1\dots a_{j-1}}_{\mu_1\dots\mu_{i+1}} = \sum_{\sigma_{i+1}} \operatorname{sign}(\sigma_{i+1}) \alpha^{a_1\dots a_j}_{\mu_{\sigma(1)}\dots\mu_{\sigma(i)}} e^b_{\mu_{\sigma(i+1)}} \eta_{a_j b}. \tag{89}$$

Definition 3.11 (Lie Derivative)**.** Let $\alpha \in \Omega_\partial^{(i,j)}$, $\omega \in \Omega_\partial^{(1,2)}$ a boundary connection and $\xi \in \mathfrak{X}(\Sigma)$ a vector field. The **Lie derivative** of α along ξ with respect to ω is defined to be

$$\mathcal{L}_\xi^\omega \alpha = \iota_\xi d_\omega \alpha - d_\omega \iota_\xi \alpha. \tag{90}$$

Remark. Again, we can see the boundary forms as the tensor product $\Omega_\partial^{(i,j)} = \Gamma(\wedge^i T^*\Sigma \otimes \wedge^j \mathcal{V}|_\Sigma)$. The vielbein $e \in \Omega_{\text{nd}}^{(1,1)}(M) = T^*M \otimes \mathcal{V}$ can also be seen as an isomorphism $e\colon TM \xrightarrow{\sim} \mathcal{V}$. When restricted to the boundary one gets an object in $\Omega_\partial^{(1,1)} = T^*\Sigma \otimes \mathcal{V}|_\Sigma$. By the previous definition $\mathcal{V}|_\Sigma$ is locally isomorphic to $\Sigma \times V$. Thus when taking the old field as a map and restricting it to the boundary $e\colon T\Sigma \to \mathcal{V}|_\Sigma$, it is not an isomorphism anymore. Indeed for all $u \in \Sigma$ it spans a 3-dimensional subspace $e(T_u\Sigma)$ of $\mathcal{V}_u$ which can be given the basis (e_1, e_2, e_3). Therefore it makes sense to define another field

$$e_n \in \mathcal{V}|_\Sigma \tag{91}$$

which completes the above basis to (e_1, e_2, e_3, e_n) such that it spans the whole space. The latter is called the transversal component to the boundary.

[24] In this case nondegenerate means that it induces an injective map $T\Sigma \to \mathcal{V}|_\Sigma$.

As explained in section 2.4, we want to quotient out the kernel of the presymplectic form which arises from the boundary term (83) (see Definition 2.22). Thus, for the fields which lie not trivially in the kernel of this form, equivalence classes arise in the geometric phase space. As we will see later, a major problem will be to find a set of equations (if possible arising from the equations of motion) which fix a representative of the equivalence class of $[\omega]$ uniquely. This procedure is the same for the case of pure gravity, scalar coupling and Yang–Mills coupling. However, it strongly depends on whether the boundary metric g^∂ is nondegenerate or degenerate. In the following we will present the results of [4] for the nondegenerate case which we will recall to in Chapters 4 and 5.

As it turns out, the equivalence class of $[\omega]$ is defined by $\omega \sim \omega' \Leftrightarrow \omega' = \omega + v$. The problem is that (81) is not invariant under a change of representative, meaning that if $d_\omega e = 0$ then $d_{\omega'} e = [v, e]$ is not necessarily 0. Therefore we use the following lemma.

Lemma 3.6. *Let $\alpha \in \Omega_\partial^{(2,1)}$ and g^∂ be nondegenerate, then*

$$\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e\alpha = 0 \\ e_n\alpha \in \mathrm{Im} W_1^{\partial(1,1)} \end{cases} . \tag{92}$$

We can apply this to $d_\omega e \in \Omega_\partial^{(2,1)}$ and find two new constraints $ed_\omega e = 0$ and $e_n d_\omega e = e\sigma$ for some $\sigma \in \Omega_\partial^{(1,1)}$. The second constraint will be used to fix uniquely the representative. The question is whether $e_n d_{\omega'} e = 0$ implies $e_n d_\omega e = 0$ and the answer is yes since there is no $\sigma \in \Omega_\partial^{(1,1)}$ such that $e_n[v, e] = e\sigma$, as the following lemma shows.

Lemma 3.7. *Let g^∂ be nondegenerate, then the map*

$$\begin{aligned} \chi \colon \ker W_1^{\partial(1,1)} &\to \Omega_\partial^{(2,2)} \\ v &\mapsto e_n[v, e] \end{aligned}$$

is injective and in particular

$$\mathrm{Im}\chi \cap \mathrm{Im} W_1^{\partial(1,1)} = \{0\}. \tag{93}$$

Proof. Both proofs can be found in [4]. □

From here it follows that there exists a unique $v \in \ker W_1^{\partial(1,2)}$ with $e_n d_{\omega'} e = e\sigma + e_n[v, e]$. Since the corresponding ω satisfies $e_n d_\omega e = e\sigma$ the fixing is complete.

3.5 Light-like Boundary

In this section we clarify the implications of choosing a boundary with a light-like direction.

Let's for a while consider the so called Minkowski space time with the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$. We start by picking a set of coordinates $(x^0, x^1, x^2, \dots, x^{N-1})$

such that $x^2 = \eta_{\mu\nu}x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + \cdots + (x^{N-1})^2$, and we build another set of coordinates $(x^+, x^-, x^2, \ldots, x^{N-1})$, called light-cone coordinates, which are defined by

$$x^+ := \frac{1}{\sqrt{2}}(x^1 + x^0),$$
$$x^- := \frac{1}{\sqrt{2}}(x^1 - x^0).$$

We set now the invariance of $x^2 = x^\mu x^\nu \eta_{\mu\nu}$ by allowing the metric η to change[25], which implies that

$$\begin{aligned} x^2 &= -(x^0)^2 + (x^1)^2 + (x^2)^2 + \cdots + (x^{N-1})^2 \\ &= -\frac{1}{2}(x^+ - x^-)^2 + \frac{1}{2}(x^+ + x^-)^2 + (x^2)^2 + \cdots + (x^{N-1})^2 \\ &= 2x^+x^- + (x^2)^2 + \cdots + (x^{N-1})^2. \end{aligned}$$

Then one recovers the Minkowski metric in light-cone coordinates

$$\eta = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{94}$$

It is easy to see that the directions x^+ and x^- are light-like directions. In other words, a photon starting at $(0, 0, 0, \ldots, 0)$ and moving forward in the direction of x^1 will be parameterized by the coordinate x^+.
Now consider a manifold whose boundary $\Sigma = \partial M$ moves along the light-like component x^- and all the spatial components $x^2, \ldots, x^{N-1}$, namely it is characterized by the equation $x^+ = 0$. The latter description corresponds to the case we will be dealing with, where the dimension of the degeneracy of the boundary metric is 1.
What we actually do is to follow a similar reasoning pointwise with coframes. Namely, we can consider the Minkowski metric on the fibers of $\mathcal{V}$ and choose non-degenerate vielbein[26] $(e_0, e_1, e_2, ..., e_{N-1})$. Then, as in the previous example with Minkowski coordinates, we build the set of vielbein $(e_+, e_-, e_2, ..., e_{N-1})$ and we pull them back to the null boundary and choose the boundary vielbein $(e_+, e_2, ..., e_{N-1})$. Then, in this set of coordinates, the metric on the fibers will be given by (94), since we want to maintain the condition $g = e^*\eta$, and the boundary metric is degenerate with a 1-dimensional degeneracy. Notice that one can always "diagonalize" vielbein in this way point wise on a light like boundary.

Now let us approach the problem of choosing a unique representative of $[\omega]$ in the case of a degenerate boundary metric. We will present the main results of [5] which again we will refer to in Chapters 4 and 5.

[25]This is equivalent to taking vector fields instead of coordinates and the full space time metric instead of η and working point wise, or equivalently to working with coframes as we will see later.
[26]We shall refer to vielbein such that $\eta_{ab}e^a_\mu e^b_\nu = 0$ as degenerate vielbein or degenerate tetrads.

The crucial difference to the nondegenerate case is that we cannot find a unique representative of $[\omega]$ satisfying $e_n d_\omega e = e\sigma$ for some $\sigma \in \Omega_\partial^{(1,1)}$. The impossibility of this task comes from the fact that there is no such lemma as 3.7 in the degenerate case. This is because in the nondegenerate case the map $\varrho|_{\ker W_{N-3}^{\partial(1,2)}}$ is injective, which is a key condition for this lemma. In the degenerate case, however, $\dim(\ker\varrho^{(1,2)}|_{\ker W_{N-2}^{\partial(1,2)}}) = \frac{N(N-3)}{2}$, meaning that $\varrho|_{\ker W_{N-3}^{\partial(1,2)}}$ is not injective and, hence, neither is the map $v \in \ker(W_1^{\partial(1,2)}) \mapsto e_n[v,e] \in \Omega_\partial^{(2,2)}$. In other words, v has components which might be associated to more than one element $e_n[v,e]$ and we need to deal with these components separately. To do so we define the following spaces:

$$\mathcal{T} := \ker W_1^{\partial(2,1)} \cap J \subset \Omega_\partial^{(2,1)}, \tag{95}$$

$$\mathcal{K} := \ker W_1^{\partial(1,2)} \cap \ker\varrho^{(1,2)} \subset \Omega_\partial^{(1,2)}, \tag{96}$$

$$\mathcal{S} := \ker W_1^{\partial(1,3)} \cap \ker\tilde{\varrho}^{(1,3)} \subset \Omega_\partial^{(1,3)}, \tag{97}$$

where $J \subset \Omega_\partial^{(2,1)}$ is the orthogonal complement of $\mathrm{Im}\varrho^{(1,2)}|_{\ker W_1^{\partial(1,2)}}$. All these three spaces are zero in the nondegenerate case. By definition of $\mathcal{T}$ it is clear that there exists an element $\theta \in \mathcal{T}$ such that $e_n d_{\omega'} e = e\sigma + e_n[v,e] + e_n\theta$ (considering again $\omega' = \omega + v$ as above). This does still not fix the representative uniquely because there are elements of ω which do not appear either in $ed_\omega e$ or in $e_n d_\omega e$ but do appear in $d_\omega e$. These lie exactly in $\mathcal{K}$. Hence, we also require $p_\mathcal{K}\omega = 0$. Furthermore, the space $\mathcal{S}$ plays the role of dual of $\mathcal{T}$, as the following lemma shows.

Lemma 3.8. *Let $\alpha \in \Omega_\partial^{(2,1)}$. Then*

$$\int_\Sigma \tau\alpha = 0 \quad \forall \tau \in \mathcal{S} \qquad \Longrightarrow \qquad p_\mathcal{T}(\alpha) = 0. \tag{98}$$

Now we can write a trivial generalization of Lemma 3.6.

Lemma 3.9. *Let $\alpha \in \Omega_\partial^{(2,1)}$ and g^∂ be degenerate, then*

$$\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e\alpha = 0 \\ e_n\alpha - e_n p_\mathcal{T}\alpha \in \mathrm{Im}W_1^{\partial(1,1)} \\ p_\mathcal{T}\alpha = 0 \end{cases}. \tag{99}$$

Proof. Both proofs can be found in [5]. □

With the considerations from above, by applying this lemma to $\alpha = d_\omega e$ a representative of $[\omega]$ is uniquely fixed. However, there is a residual equation of the form $p_\mathcal{T} d_\omega e = 0$ which must be taken as an additional constraint.

4 Scalar Coupling

In this chapter we move away from the case of pure gravity and add coupling to a real massless scalar field to our theory. This coupling is described in the first order formalism.

A scalar field is just a smooth function on spacetime $\phi \in C^\infty(M)$. Normally, in field theories a Lagrangian density for a free scalar field on an arbitrary background is described by the Klein–Gordon term

$$\mathcal{L}_{KG} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi,$$

where we assumed the scalar field to be massless. This then leads to the Klein–Gordon equation

$$g^{\mu\nu} \partial_\mu \partial_\nu \phi = 0.$$

We want to incorporate this description into our PC theory. We could use the KG term but dealing with the metric in terms of the vielbein makes the computations laborious. Therefore we use the first order formalism, by introducing a new field Π, which physically speaking takes the role of momentum field. The difference from the usual momentum is that at first glance the fields Π and ϕ are independent of each other. However, by construction of the theory they are connected by the equations of motion, leading back to the usual momentum field.

4.1 Scalar Palatini–Cartan Action

In this first section we give the general description of the theory in the general case where M is an N-dimensional pseudo-Riemannian manifold.

Definition 4.1 (Pairing)**.** Let $\{v_i\}_{\mu=1}^N$ be an orthonormal[27] basis of V and let $A = A^\mu v_\mu \in V$ and $B = B^\nu v_\nu \in V$. The **pairing** of A and B is then defined as

$$(A, B) := A^\mu B^\nu \eta_{\mu\nu}. \tag{100}$$

Definition 4.2 (Scalar Palatini–Cartan)**.** Let M be an N-dimensional pseudo-Riemannian manifold with boundary $\Sigma := \partial M$. The **scalar Palatini–Cartan** theory (SPC) is the assignment of the pair $(\mathcal{F}_{SPC}, S_{SPC})_M$ to M, with the space of fields

$$\mathcal{F}_{SPC} = \Omega^{(1,1)}_{\text{nd}}(M) \times \mathcal{A}(M) \times C^\infty(M) \times \Omega^{(0,1)}(M) \tag{101}$$

and the action

$$S_{SPC} = S_{PC} + S_{scal}, \tag{102}$$

with

$$S_{PC} = \int_M \frac{1}{(N-2)!} e^{N-2} \wedge F_\omega + \frac{\Lambda}{N!} e^N, \tag{103}$$

$$S_{scal} = \int_M \frac{1}{(N-1)!} e^{N-1} \wedge \Pi \wedge d\phi + \frac{1}{2N!} e^N (\Pi, \Pi) \tag{104}$$

where $(e, \omega, \phi, \Pi) \in \mathcal{F}_{SPC}$.

[27] Here orthonormal means with respect to the Minkowski metric in V. So $\eta_{\mu\nu} v_a^\mu v_b^\nu = \eta_{ab}$ for all basis vectors v_a and v_b.

Following what we have studied in Chapter 2, the variation of the action S_{SPC} leads to the Euler–Lagrange equations of motion

$$d_\omega e = 0, \tag{105}$$

$$\tfrac{1}{(N-3)!}e^{N-3}F_\omega + \tfrac{\Lambda}{(N-1)!}e^{N-1} + \tfrac{1}{(N-2)!}e^{N-2}\Pi d\phi + \tfrac{1}{2(N-1)!}e^{N-1}(\Pi,\Pi) = 0, \tag{106}$$

$$d(e^{N-1}\Pi) = 0, \tag{107}$$

$$e^{N-1}(d\phi - (e,\Pi)) = 0. \tag{108}$$

Furthermore, we get the boundary term

$$\tilde{\alpha} := \int_\Sigma \frac{1}{(N-2)!}e^{N-2}\delta\omega + \frac{1}{(N-1)!}e^{N-1}\Pi\delta\phi. \tag{109}$$

Remark. We can simplify equation (108) by using that the map $W_{N-1}^{(1,0)}$ is injective (see Lemma B.1), meaning that

$$e^{N-1}(d\phi - (e,\Pi)) = 0 \quad \Leftrightarrow \quad d\phi - (e,\Pi) = 0.$$

By inversion of the metric we find the usual momentum term for scalar fields

$$\Pi = \pi^\mu v_\mu = -g^{\mu\nu}\partial_\nu\phi v_\mu.$$

Furthermore, by using that $\frac{e^N}{N!} = \mathrm{Vol}_g$ we recover the Klein–Gordon action

$$S_{scal} = -\int_M \frac{1}{2}\mathrm{Vol}_g g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

4.2 Scalar Boundary Structure in $N = 4$

In this section we study the boundary structure of the theory, following section 2.4. Our goal is to find the reduced phase space of the theory, which is the space of admissible initial field configurations. To do so we first define the space of pre-boundary fields by restricting the space of fields onto the boundary of the manifold. The bundle construction makes it easy to do so. Furthermore, it turns out that by considering $\tilde{\varpi} = \delta\tilde{\alpha}$ we already have something which resembles a symplectic form, since $\delta^2 = 0$ implies that it is closed. However, it might still be degenerate. This means that we have to quotient out the kernel that is, perform a presymplectic reduction, to get a symplectic manifold which is called the geometric phase space of the theory.

In order to get the reduced phase space we need the fields to obey the equations of motion restricted on the boundary and to fix the representative of the equivalence classes. In general, however, these equations are not invariant under a change of representative. In other words, the restriction onto the boundary has as a consequence that the equations do not lead to unique solutions. In these cases we need to find an equivalent version of the constraint which at the same time fixes the representative. We write these constraints as Lagrange multipliers. The reduced phase space is the quotient of the geometric phase space with respect to these constraints.

Definition 4.3 (Preboundary Fields)**.** Define the space of **preboundary fields** of the SPC theory and its **presymplectic form** to be

$$\tilde{F}_\partial = \Omega_\partial^{(1,1)} \times \mathcal{A}(\Sigma) \times C^\infty(\Sigma) \times \Omega_\partial^{(0,1)}, \tag{110}$$

$$\tilde{\varpi} := \delta\tilde{\alpha} = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\delta(e^3\Pi)\delta\phi. \tag{111}$$

Remark. Clearly $\tilde{\varpi}$ is a 2-form on the space of preboundary fields $\tilde{F}_\partial$. It is also closed $\delta\tilde{\varpi} = 0$ since $\delta^2 = 0$. However, it is not nondegenerate because $\ker(\tilde{\varpi}) \neq \{0\}$. Thus after quotienting out the kernel the latter defines a symplectic form.

Proposition 4.1. *From Definition 2.22 we know that the geometric phase space is the space of preboundary fields modulo its kernel*

$$F_\partial = \frac{\tilde{F}_\partial}{\ker(\tilde{\varpi})}.$$

For the SPC theory we therefore have

$$F_\partial = \Omega_\partial^{(1,1)} \times \frac{\mathcal{A}(\Sigma)}{\sim} \times C^\infty(\Sigma) \times \frac{\Omega_\partial^{(0,1)}}{\sim} \tag{112}$$

where the equivalence classes $[\omega]$ *and* $[\Pi]$ *satisfy*

$$\omega \sim \omega' \iff \omega' = \omega + v, \quad \text{with} \quad ev = 0, \tag{113}$$

$$\Pi \sim \Pi' \iff \Pi' = \Pi + \gamma, \quad \text{with} \quad e^3\gamma = 0. \tag{114}$$

Proof. The kernel of a 2-form is defined through the inner product

$$\ker(\tilde{\varpi}) = \{X \in \mathfrak{X}(\tilde{F}_\partial) | \iota_X\tilde{\varpi} = 0\}. \tag{115}$$

Consider now a generic vector field $X = \mathbb{X}_e\frac{\delta}{\delta e} + \mathbb{X}_\omega\frac{\delta}{\delta\omega} + \mathbb{X}_\phi\frac{\delta}{\delta\phi} + \mathbb{X}_\Pi\frac{\delta}{\delta\Pi}$. This is a vector field with respect to the variation δ (see Definition 2.10), meaning that the components are along the field values $\mathbb{X}_e \in \Omega_\partial^{(1,1)}, \mathbb{X}_\omega \in \mathcal{A}(\Sigma), \mathbb{X}_\phi \in C^\infty(\Sigma), \mathbb{X}_\Pi \in \Omega_\partial^{(0,1)}$. Contracting it with equation (111) gives

$$\begin{aligned}
\iota_X\tilde{\varpi} = \int_\Sigma & e\mathbb{X}_e\delta\omega + \frac{1}{2}e^2\Pi\mathbb{X}_e\delta\phi + e\mathbb{X}_\omega\delta e \\
&+ \frac{1}{3!}e^3\mathbb{X}_\Pi\delta\phi + \frac{1}{2}e^2\Pi\mathbb{X}_\phi\delta e + \frac{1}{3!}e^3\mathbb{X}_\phi\delta\Pi \\
= \int_\Sigma & \Big(e\mathbb{X}_\omega + \frac{1}{2}e^2\Pi\mathbb{X}_\phi\Big)\delta e + e\mathbb{X}_e\delta\omega \\
&+ \frac{1}{3!}e^3\mathbb{X}_\phi\delta\Pi + \Big(\frac{1}{2}e^2\Pi\mathbb{X}_e + \frac{1}{3!}e^3\mathbb{X}_\Pi\Big)\delta\phi.
\end{aligned}$$

Therefore we find that the kernel is defined by the set of equations

$$e\mathbb{X}_\omega + \frac{1}{2}e^2\Pi\mathbb{X}_\phi = 0, \tag{116}$$

$$e\mathbb{X}_e = 0, \tag{117}$$

$$\frac{1}{3!}e^3\mathbb{X}_\phi = 0, \tag{118}$$

$$\frac{1}{2}e^2\Pi\mathbb{X}_e + \frac{1}{3!}e^3\mathbb{X}_\Pi = 0. \tag{119}$$

Using that $W_1^{\partial(1,1)}$ and $W_3^{\partial(0,0)}$ are both injective (see Lemma B.3) equations (117) and (118) are equal to $\mathbb{X}_e = 0$ and $\mathbb{X}_\phi = 0$[28], which leads to trivial equivalence relations for e and ϕ. Furthermore equations (116) and (119) are equivalent to $e\mathbb{X}_\omega = 0$ and $e^3\mathbb{X}_\Pi = 0$, proving the statement. □

Theorem 4.2. *(F_∂, ϖ) is a symplectic manifold with symplectic form*

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{3!}\delta(e^3[\Pi])\delta\phi. \tag{120}$$

Remark. The physical space of the theory will be the geometric phase space with fixed representative modulo the constraints. The constraints are given by the equations of motion (105)–(108) restricted on the boundary. Note that (107) will not be fixed since it is an evolution equation containing transversal derivatives. Furthermore (108) simplifies because $W_{N-1}^{(1,0)}$ is injective (see Lemma B.1). In 4 dimensions the constraints become

$$\mathrm{d}_\omega e = 0, \tag{121}$$

$$eF_\omega + \frac{\Lambda}{6}e^3 + \frac{1}{2}e^2\Pi\mathrm{d}\phi + \frac{1}{12}e^3(\Pi,\Pi) = 0, \tag{122}$$

$$\mathrm{d}\phi - (e,\Pi) = 0. \tag{123}$$

4.2.1 Nondegenerate Case

The fixing of the representative leads to some problems. The constraints are not invariant under a change of representative. For example take equation (121) and two elements of the same equivalence class $\omega, \omega' \in [\omega]$ with $\omega' = \omega + v$ and $ev = 0$. Then if $\mathrm{d}_\omega e = 0$ it follows that

$$\mathrm{d}_{\omega'} e = \mathrm{d}_\omega e + [v, e] = [v, e] \tag{124}$$

which is not necessarily equal to 0. Thus it is useful to rephrase the constraint into another weaker constraint plus a condition which fixes the representative uniquely, which is done in the following two theorems.

Theorem 4.3. *Let $\omega \in \mathcal{A}(\Sigma)$ and $e \in \Omega_\partial^{(1,1)}$ and g^∂ be nondegenerate, then*

$$d_\omega e = 0 \quad \Longleftrightarrow \quad \begin{cases} ed_\omega e = 0 \\ e_n d_\omega e \in \mathrm{Im} W_1^{\partial(1,1)} \end{cases}, \tag{125}$$

where the first equation is called ***invariant constraint*** *and the second one is called* ***structural constraint****.*

Proof. Follows trivially from Lemma 3.6 by substituting $\alpha = d_\omega e$. □

Theorem 4.4. *Let g^∂ be nondegenerate, then $\forall\, \omega' \in A(\Sigma)$ there is a unique decomposition*

$$\omega' = \omega + v, \tag{126}$$

such that $ev = 0$ and $e_n d_\omega e \in \mathrm{Im} W_1^{\partial(1,1)}$.

[28] Recall that $\Omega_\partial^{(0,0)} = C^\infty(\Sigma)$.

Theorem 4.5. *Let g^∂ be nondegenerate, then $\forall\, \Pi' \in \Omega_\partial^{(0,1)}$ there is a unique decomposition*

$$\Pi' = \Pi + \gamma, \tag{127}$$

such that $e^3\gamma = 0$ and $\mathrm{d}\phi - (e, \Pi) = 0$.

Proof. See [11]. □

Remark. We see that imposing the structural constraint is equivalent to choosing a representative for $[\omega]$. Similarly, choosing a representative for $[\Pi]$ already fixes the constraint (123). This means that the only remaining part of the equations (121) and (123) is the invariant constraint. We will use Lagrange multipliers to fix the constraints. This leads us to the following functions

$$L_c := \int_\Sigma c e d_\omega e, \tag{128}$$

$$J_{\tilde{\mu}} := \int_\Sigma \tilde{\mu}\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{12}e^3(\Pi,\Pi)\Big), \tag{129}$$

with $c \in \Omega_\partial^{(0,2)}[1]$[29] and $\tilde{\mu} \in \Omega_\partial^{(0,1)}[1]$ in order to make the integral well-defined. It is convenient to split the element $\tilde{\mu}$ into a normal component and a tangential component. For the latter we simply use e and contract it with a vector field in Σ. Therefore we have the decomposition $\tilde{\mu} = \iota_\xi e + \lambda e_n$ with $\xi \in \mathfrak{X}(\Sigma)[1]$ and $\lambda \in C^\infty(\Sigma)[1]$. This replaces $J_{\tilde{\mu}}$ with two different constraints

$$P_\xi := \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi(e^3\Pi)d\phi + \iota_\xi(\omega - \omega_0)ed_\omega e, \tag{130}$$

$$H_\lambda := \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{12}e^3(\Pi,\Pi)\Big). \tag{131}$$

The cosmological constant only appears in the normal constraint since $\iota_\xi e^4 = 0$. Furthermore we added a term proportional to $ed_\omega e$ with the help of a reference connection ω_0. This is not necessary but it largely simplifies computations.

Theorem 4.6. *Let g^∂ be nondegenerate. Then the Poisson brackets of the constraints in the SPC theory read*

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]}, \tag{132}$$

$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}, \tag{133}$$

$$\{H_\lambda, H_\lambda\} = 0, \tag{134}$$

$$\{L_c, P_\xi\} = L_{\mathcal{L}_\xi^{\omega_0}c}, \tag{135}$$

$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}, \tag{136}$$

$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(a)}}, \tag{137}$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to the frame (e_a, e_n).

Proof. See [11]. □

[29]Here the number 1 in the square brackets indicates that there is an additional odd (ghost) degree of freedom. This is not necessary for the computations of the Poisson brackets but it simplifies them slightly. It would be necessary, however, for the incorporation of the BFV formalism.

4.2.2 Degenerate Case

The main difference to the nondegenerate case is that we cannot find a unique representative of the equivalence class $[\omega]$ satisfying the structural constraint. This means that we need to find another set of weaker equations which are able to fix the representative. The idea is to subtract the problematic part leading to a new structural constraint. However, by modifying the latter it does not define the same zero locus as $d_\omega e = 0$ anymore. Thus another constraint has to be added which accounts for the missing part of the weakened structural constraint.

Theorem 4.7. *Let $\omega \in \mathcal{A}(\Sigma)$ and $e \in \Omega_\partial^{(1,1)}$ and g^∂ degenerate, then*

$$d_\omega e = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e d_\omega e = 0 \\ e_n d_\omega e - e_n p_\mathcal{T} d_\omega e \in \mathrm{Im} W_1^{\partial(1,1)} \\ p_\mathcal{T} d_\omega e = 0 \end{cases}, \tag{138}$$

*where the last equation is called **degeneracy constraint**.*

Proof. Follows trivially from Lemma 3.9 by substituting $\alpha = d_\omega e$. □

Theorem 4.8. *Let g^∂ be degenerate, then $\forall\, \omega' \in \mathcal{A}(\Sigma)$ there is a unique decomposition*

$$\omega' = \omega + v, \tag{139}$$

such that $ev = 0$, $e_n d_\omega e - e_n p_\mathcal{T} d_\omega e \in \mathrm{Im} W_1^{\partial(1,1)}$ and $p_\mathcal{K} v = 0$.

Remark. Let us compare this to Theorem 4.4. The point is that in the nondegenerate case the map

$$\begin{aligned} \ker\left(W_1^{\partial(1,2)}\right) &\to \Omega_\partial^{(2,2)} \\ v &\mapsto e_n[v, e] \end{aligned}$$

is injective, meaning that the decomposition is unique. This is no longer the case in the degenerate case. Thus we need to consider the components of $d_\omega e$ in $\mathcal{T}$ and the components of ω in $\mathcal{K}$ separately.

Theorem 4.9. *Let g^∂ be degenerate with $\dim(\ker(g^\partial)) = 1$, $e_n \in \mathcal{V}|_\Sigma$ (as defined in equation (91)) and $[\Pi] \in \frac{\Omega_\partial^{(0,1)}}{\sim}$ an equivalence class. Then there is a unique $\pi \in \ker W_3^{\partial(0,1)}$ and π^n with*

$$\Pi = \pi^n e_n + \pi, \tag{140}$$

such that

1. *$d\phi - (e, \Pi) = 0$,*
2. *$p_\mathcal{W} \Pi = 0$,*
3. *$\xi(\phi) = 0$,*

where $\mathcal{W} = e(\ker g^\partial)$, $\xi \in \ker g^\partial$ and $\ker g^\partial = \{\xi \in \mathfrak{X}(\Sigma) \mid \iota_\xi g^\partial = 0\} \subset \mathfrak{X}(\Sigma)$.

Proof. We can write a general element $\Pi \in \Omega_\partial^{(0,1)}$ along any basis as $\Pi = \pi^n e_n + \pi^a v_a$ with $a = 1, 2, 3$. It is clear that $\pi = \pi^a v_a$ is in $\ker W_3^{\partial(0,1)}$ because $e^3 \wedge v_a = 0$ for every a due to antisymmetry of the wedge product. This directly implies that the component π^n in (140) is already uniquely defined by the equivalence class $[\Pi]$.
It remains to show that the conditions define a unique π. Since $\dim(\ker(g^\partial)) = 1$ and e is injective we have that $\mathcal{W} \subset \mathcal{V}|_\Sigma$ is a 1-dimensional subspace. Now we can choose a basis $\{v_1, v_2, v_3\}$ of $\mathcal{V}|_\Sigma$ such that $v_3 \in \mathcal{W}$. Then if we write $\pi = \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3$ the condition $p_\mathcal{W}\Pi = 0$ implies that $\pi^3 = 0$. Next, consider the condition $\xi(\phi) = 0$, which implies that the scalar field ϕ is constant along the vector fields which lie in the kernel of the metric. Then we can use the vielbein to write components of the tangent space in terms of components of V which results in

$$d\phi = \frac{\partial\phi}{\partial x^i} dx^i = e_i^1 \partial_1 \phi dx^i + e_i^2 \partial_2 \phi dx^i,$$

since e maps the component along ξ to the component along v_3. Furthermore, by definition of $\mathcal{W}$, g is invertible on the complement of $\mathcal{W}$ in $\mathcal{V}|_\Sigma$. So it exists $(g^{-1})^{ab} =: g^{ab}$ such that $g_{ab}g^{ab} = 1$ for $a, b = 1, 2$. Then

$$\begin{aligned} d\phi - (e, \Pi) &= \frac{\partial\phi}{\partial x^i} dx^i - (e_i^a dx^i v_a, \pi^b v_b + \pi^n e_n) \\ &= e_i^a \partial_a \phi dx^i - e_i^a \pi^b g_{ab} dx^i - e_i^a \pi^n g_{an} dx^i \\ &= e_i^a (\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an}) dx^i = 0. \end{aligned}$$

And since $a, b \neq 3$

$$\partial_a \phi - \pi^b g_{ab} - \pi^n g_{an} = 0 \quad \Longrightarrow \quad \pi^b = g^{ab}(\partial_a \phi - \pi^n g_{an}).$$

So with $\pi^3 = 0$ and π^n already fixed, we have that Π is uniquely determined in terms of its components. □

Remark. From theorems 4.7 and 4.8 we see that the new structural constraint, together with some additional equations, fixes the representative for $[\omega]$. Simultaneously the constraints (121) splits into the invariant constraint and the degeneracy constraint. On the other hand, constraint (123) together with conditions 2. and 3. from Theorem 4.9 fix the representative for $[\Pi]$. This leads to the following Lagrange multipliers

$$\begin{aligned} L_c &= \int_\Sigma c e d_\omega e, \\ P_\xi &= \int_\Sigma \frac{1}{2}\iota_\xi(e^2) F_\omega + \frac{1}{3!}\iota_\xi(e^3 \Pi) d\phi + \iota_\xi(\omega - \omega_0) e d_\omega e, \\ H_\lambda &= \int_\Sigma \lambda e_n \Big(e F_\omega + \frac{\Lambda}{3!} e^3 + \frac{1}{2} e^2 \pi d\phi + \frac{1}{2 \cdot 3!} e^3 (\Pi, \Pi) \Big) - p_\mathcal{S}(\lambda e_n(\omega - \omega_0)) d_\omega e, \\ R_\tau &= \int_\Sigma \tau d_\omega e, \end{aligned}$$

with $c \in \Omega_\partial^{(0,2)}[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$. Notice that in H_λ we added a term which is proportional to $R_{\tau'}$, with $\tau' = p_\mathcal{S}(\lambda e_n(\omega - \omega_0))$. This is of course optional but it simplifies calculations. There is one crucial difference in the newly added constraint, which is that τ is not an element of the entire space $\Omega_\partial^{(1,3)}$ but only of the subspace $\mathcal{S}$ and the latter depends on e. As a consequence a variation of this Lagrange multiplier is not independent of a variation of e. The following lemma describes this property.

Lemma 4.10. *The variation of an element $\tau \in \mathcal{S}$ is constrained by the following equations*

$$p'_{\tilde{\rho}}\delta\tau = \tilde{\rho}^{-1}\Big(\frac{\delta\tilde{\rho}}{\delta e}(\tau)\delta e\Big),$$
$$p'_W \delta\tau = W_1^{-1}(\tau\delta e),$$

where $\tilde{\rho}^{-1}$ and W_1^{-1} are defined on their images and $p'_{\tilde{\rho}}$ and p'_W are respectively the projections to a complement of the kernel of $\tilde{\rho}$ and $W_1^{\partial,(1,3)}$.

Proof. See [5]. □

Theorem 4.11. *Let g^∂ be degenerate with* $\dim(\ker g^\partial) = 1$. *Then the Poisson brackets of the constraints in the SPC theory read*

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]}, \tag{141}$$
$$\{R_\tau, R_\tau\} \approx F_{\tau\tau}, \tag{142}$$
$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}}, \tag{143}$$
$$\{H_\lambda, H_\lambda\} \approx F_{\tau'\tau'}, \tag{144}$$
$$\{L_c, R_\tau\} = -R_{p_{\mathcal{S}}[c,\tau]}, \tag{145}$$
$$\{L_c, P_\xi\} = L_{\mathcal{L}_\xi^{\omega_0}c}, \tag{146}$$
$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} + R_{p_{\mathcal{S}}(X^{(a)}e_a(\omega-\omega_0)-\lambda e_n d_{\omega_0}c)}, \tag{147}$$
$$\{R_\tau, P_\xi\} = R_{p_{\mathcal{S}}\mathcal{L}_\xi^{\omega_0}\tau}, \tag{148}$$
$$\{R_\tau, H_\lambda\} \approx F_{\tau\tau'} + G_{\lambda\tau} + K_{\lambda\tau}^S \tag{149}$$
$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(a)}} - R_{p_{\mathcal{S}}(Y^{(a)}e_a(\omega-\omega_0)-\lambda e_n\iota_\xi F_{\omega_0})}, \tag{150}$$

with $\tau' = p_{\mathcal{S}}(\lambda e_n(\omega-\omega_0))$, $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to the frame (e_a, e_n). Furthermore, $F_{\tau\tau}$, $F_{\tau\tau'}$, $F_{\tau'\tau'}$, $G_{\lambda\tau}$ and $K_{\lambda\tau}^S$ are functions of $e, \omega, \Pi, \phi, \tau, \tau'$ and λ defined in the proof which are not proportional to any other constraint.

Remark. The symbol "≈" indicates the identity on the zero locus of the constraints. In particular, this means that those brackets written with this symbol are not a linear combination of the constraints themselves. On the other hand, all the brackets written with a "=" vanish on the zero locus, for example $\{L_c, L_c\} \approx 0$.

Proof. We notice that L_c and R_τ remain the same as in the free theory case. Thus we write the other two as a sum of a free and an interacting part

$$P_\xi = P_\xi^0 + P_\xi^I,$$
$$H_\lambda = H_\lambda^0 + H_\lambda^I,$$

with

$$\begin{aligned} P_\xi^0 &= \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega-\omega_0)ed_\omega e, \\ P_\xi^I &= \int_\Sigma \iota_\xi(p)d\phi, \\ H_\lambda^0 &= \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3\Big) - p_{\mathcal{S}}(\lambda e_n(\omega-\omega_0))d_\omega e, \\ H_\lambda^I &= \int_\Sigma \lambda e_n\Big(\frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi)\Big). \end{aligned}$$

Here we have introduced the new field $p := \frac{1}{3!}e^3\Pi$. Making use of previous results we can easily calculate their variations and the Hamiltonian vector fields. For L_c we get

$$\delta L_c = \int_\Sigma -\frac{1}{2}c[\delta\omega, e^2] + \frac{1}{2}cd_\omega\delta(e^2) = \int_\Sigma [c,e]e\,\delta\omega + d_\omega ce\,\delta e,$$

and therefore the Hamiltonian vector field reads

$$\begin{aligned} \mathbb{L}_e &= [c,e], & \mathbb{L}_\omega &= d_\omega c + \mathbb{V}_L, \\ \mathbb{L}_p &= 0, & \mathbb{L}_\phi &= 0, \end{aligned}$$

where $\mathbb{V}_L \in \ker(W_1^{\partial,(1,2)})$. Next, the variation of R_τ reads

$$\begin{aligned} \delta R_\tau &= \int_\Sigma \delta_e\tau d_\omega e - \tau[\delta\omega, e] + \tau d_\omega\delta e \\ &= \int_\Sigma (g(\tau,\omega,e) + d_\omega\tau)\delta e + [\tau,e]\delta\omega, \end{aligned}$$

where we have introduced the formal expression $g(\tau,\omega,e)$ which encodes the dependence of τ on e, that is (see Lemma 4.10)

$$\delta e g(\tau,\omega,e) = \left(p'_{\tilde\rho}\tilde\rho^{-1}\Big(\frac{\delta\tilde\rho}{\delta e}(\tau)\delta e\Big) + p'_W W_1^{-1}(\tau\delta e) - p'_X\tilde\rho^{-1}\Big(\frac{\delta\tilde\rho}{\delta e}(\tau)\delta e\Big)\right)d_\omega e. \quad (151)$$

and the vector fields

$$\begin{aligned} e\mathbb{R}_e &= [\tau,e], & e\mathbb{R}_\omega &= g(\tau,\omega,e) + d_\omega\tau, \\ \mathbb{R}_p &= 0, & \mathbb{R}_\phi &= 0. \end{aligned}$$

For P_ξ we get

$$\begin{aligned} \delta P_\xi^0 &= \int_\Sigma \iota_\xi(e\delta e)F_\omega + \frac{1}{2}\iota_\xi(e^2)\delta F_\omega + \iota_\xi(\delta\omega)ed_\omega e - \iota_\xi(\omega-\omega_0)\delta(ed_\omega e) \\ &= \int_\Sigma -e\delta e\iota_\xi F_\omega + \frac{1}{2}d_\omega\iota_\xi(e^2)\delta\omega - \frac{1}{2}\delta\omega\iota_\xi d_\omega(e^2) + \frac{1}{2}\delta\omega[\iota_\xi(\omega-\omega_0),e^2] \\ &\quad + \frac{1}{2}d_\omega\iota_\xi(\omega-\omega_0)\delta(e^2) \\ &= \int_\Sigma -e\delta e\iota_\xi F_\omega - (\mathcal{L}_\xi^\omega e)e\delta\omega + e\delta\omega[\iota_\xi(\omega-\omega_0),e] + d_\omega\iota_\xi(\omega-\omega_0)e\delta e \\ &= \int_\Sigma -e\,\delta e(\mathcal{L}_\xi^{\omega_0}(\omega-\omega_0) + \iota_\xi F_{\omega_0}) - (\mathcal{L}_\xi^{\omega_0}e)e\,\delta\omega, \end{aligned}$$

and

$$\begin{aligned}
\delta P_\xi^I &= \int_\Sigma \iota_\xi(\delta p) d\phi + \iota_\xi p d(\delta\phi) \\
&= \int_\Sigma \delta p \iota_\xi(d\phi) + d(\iota_\xi p)\delta\phi \\
&= \int_\Sigma -\xi(\phi)\,\delta p - \mathcal{L}_\xi^{\omega_0} p\,\delta\phi,
\end{aligned}$$

and the vector field

$$\begin{aligned}
&\mathbb{P}_e^0 = -\mathcal{L}_\xi^{\omega_0} e, && \mathbb{P}_\omega^0 = -\mathcal{L}_\xi^{\omega_0}(\omega - \omega_0) - \iota_\xi F_{\omega_0}, \\
&\mathbb{P}_p^0 = 0, && \mathbb{P}_\phi^0 = 0, \\
&\mathbb{P}_e^I = 0, && \mathbb{P}_\omega^I = 0, \\
&\mathbb{P}_p^I = -\mathcal{L}_\xi^{\omega_0} p, && \mathbb{P}_\phi^I = -\xi(\phi).
\end{aligned}$$

Finally, for the variation of H_λ we find

$$\begin{aligned}
\delta H_\lambda^0 = &\int_\Sigma \lambda e_n\Big(\delta e F_\omega + e\delta F_\omega + \frac{\Lambda}{2}e^2\delta e\Big) - p_{\mathcal{S}}(\lambda e_n \delta\omega) d_\omega e - \delta(p_{\mathcal{S}}(\lambda e_n(\omega - \omega_0)) d_\omega e \\
&- p_{\mathcal{S}}(\lambda e_n(\omega-\omega_0))\delta(d_\omega e) \\
= &\int_\Sigma \Big(\lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2}e^2\Big)\delta e - \lambda e_n e d_\omega \delta\omega - p_{\mathcal{S}}(\lambda e_n\delta\omega) d_\omega e \\
&- \delta_e \tau' d_\omega e + \tau'[\delta\omega, e] - \tau' d_\omega \delta e \\
= &\int_\Sigma \Big(\lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2}e^2\Big)\delta e + d_\omega(\lambda e_n) e\delta\omega + \lambda e_n d_\omega e \delta\omega - \lambda e_n \delta\omega p_{\mathcal{T}}(d_\omega e) \\
&- g(\tau', \omega, e)\delta e - [\tau', e]\delta\omega - d_\omega \tau' \delta e \\
= &\int_\Sigma \Big(\lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2}e^2 - g(\tau',\omega,e) - d_\omega\tau'\Big)\delta e \\
&+ \big(e d_\omega(\lambda e_n) + \lambda e\sigma - [\tau', e]\big)\delta\omega,
\end{aligned}$$

where we have used $\tau' = p_{\mathcal{S}}(\lambda e_n(\omega - \omega_0))$ and $e_n d_\omega e - e_n p_{\mathcal{T}} d_\omega e = e\sigma$ with $\sigma \in$

$\mathrm{Im} W_1^{\partial(1,1)}$ due to Theorem 4.7. Furthermore, for the interacting part we find

$$\begin{aligned}
\delta H^I_\lambda &= \int_\Sigma \lambda e_n \Big(\frac{e^2}{4}(\Pi,\Pi)\delta e + e\Pi d\phi \delta e + \frac{e^3}{3!}(\Pi,\delta\Pi) + \frac{e^2}{2} d\phi\delta\Pi + \frac{e^2}{2}\Pi d\delta\phi \Big) \\
&= \int_\Sigma \lambda e_n \Big(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi \Big)\delta e + \frac{1}{2} d_\omega(\lambda e_n e^2 \Pi)\delta\phi \\
&\quad + \frac{1}{2}\lambda e_n e^2 (d\phi - (e,\Pi))\delta\Pi - \lambda\frac{e^3}{3!}(e_n,\Pi)\delta\Pi - \frac{\lambda}{2} e^2 (e_n,\Pi)\Pi\delta e \\
&= \int_\Sigma \Big(\lambda e_n \frac{e^2}{4}(\Pi,\Pi) + \lambda e_n e \Pi d\phi - \frac{\lambda}{2}e^2(e_n,\Pi)\Pi \Big)\delta e \\
&\quad - \lambda(e_n,\Pi)\delta p + \frac{1}{2} d_\omega(\lambda e_n e^2\Pi)\delta\phi,
\end{aligned}$$

and consequently the vector fields read

$$\begin{aligned}
e\mathbb{H}^0_e &= e d_\omega(\lambda e_n) + \lambda e\sigma - [\tau', e], & e\mathbb{H}^0_\omega &= \lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2} e^2 - g(\tau',\omega,e) - d_\omega \tau', \\
\mathbb{H}^0_p &= 0, & \mathbb{H}^0_\phi &= 0, \\
\mathbb{H}^I_e &= 0, & \mathbb{H}^I_\omega &= \lambda e_n \Pi d\phi + \frac{e}{4}\lambda e_n(\Pi,\Pi) - \frac{\lambda}{2} e\Pi(\Pi, e_n), \\
\mathbb{H}^I_p &= \frac{1}{2} d_\omega(\lambda e_n e^2 \Pi), & \mathbb{H}^I_\phi &= -\lambda(e_n,\Pi).
\end{aligned}$$

Now we are ready to calculate the Poisson structure of the constraints. Since the constraints L_c and P_ξ are identical to the nondegenerate case we can simply take their brackets from [11]

$$\begin{aligned}
\{L_c, L_c\} &= \int_\Sigma [c,e] e d_\omega c = \int_\Sigma \frac{1}{2}[c,e^2] d_\omega c \\
&= \int_\Sigma \frac{1}{4} d_\omega [c,c] e^2 = \int_\Sigma -\frac{1}{2}[c,c] e d_\omega e = -\frac{1}{2} L_{[c,c]},
\end{aligned}$$

$$\begin{aligned}
\{L_c, P_\xi\} &= \int_\Sigma -[c,e]e(\mathcal{L}^{\omega_0}_\xi(\omega-\omega_0) + \iota_\xi F_{\omega_0}) - d_\omega c e \mathcal{L}^{\omega_0}_\xi e \\
&= \int_\Sigma \frac{1}{2}\Big(\mathcal{L}_\xi \omega_0 c[\omega-\omega_0, e^2] + c[\omega-\omega_0, \mathcal{L}_\xi\omega_0(e^2)] \\
&\qquad - c[e^2, \iota_\xi F_{\omega_0}] - d_\omega \mathcal{L}^{\omega_0}_\xi (e^2) c \Big) \\
&= \int_\Sigma \frac{1}{2}\mathcal{L}^{\omega_0}_\xi c[\omega, e^2] - \frac{1}{2} d c \iota_\xi d(e^2) + \frac{1}{2}[\iota_\xi \omega_0, d(e^2)]c \\
&= \int_\Sigma \frac{1}{2}\mathcal{L}^{\omega_0}_\xi c d_\omega(e^2) = \int_\Sigma \mathcal{L}^{\omega_0}_\xi c e d_\omega e = L_{\mathcal{L}^{\omega_0}_\xi c},
\end{aligned}$$

$$\begin{aligned}
\{P_\xi, P_\xi\} &= \{P^0_\xi, P^0_\xi\} + 2\{P^0_\xi, P^I_\xi\} + \{P^I_\xi, P^I_\xi\} \\
&= \frac{1}{2} P_{[\xi,\xi]} - \frac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}},
\end{aligned}$$

where we have used

$$\begin{aligned}
\{P_\xi^0, P_\xi^0\} &= \int_\Sigma \mathcal{L}_\xi^{\omega_0}(\omega-\omega_0)\mathcal{L}_\xi^{\omega_0}(e)e + \iota_\xi F_{\omega_0}\mathcal{L}_\xi^{\omega_0}(e)e \\
&= \int_\Sigma \frac{1}{2}\mathcal{L}_\xi^{\omega_0}(\omega-\omega_0)\mathcal{L}_\xi^{\omega_0}(e^2) + \frac{1}{2}\iota_\xi F_{\omega_0}\mathcal{L}_\xi^{\omega_0}(e^2) \\
&= \int_\Sigma \frac{1}{4}\mathcal{L}_{[\xi,\xi]}^{\omega_0}(e^2)(\omega-\omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, e^2](\omega-\omega_0) + \frac{1}{2}\mathcal{L}_\xi^{\omega_0}(e^2)\iota_\xi F_{\omega_0} \\
&= \int_\Sigma \frac{1}{4}\iota_{[\xi,\xi]}d_{\omega_0}(e^2)(\omega-\omega_0) + \frac{1}{4}d_{\omega_0}\iota_{[\xi,\xi]}(e^2)(\omega-\omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, e^2](\omega-\omega_0) \\
&\quad + \frac{1}{2}\mathcal{L}_\xi^{\omega_0}(e^2)\iota_\xi F_{\omega_0} \\
&= \int_\Sigma \frac{1}{4}\iota_{[\xi,\xi]}d_\omega(e^2)(\omega-\omega_0) - \frac{1}{4}\iota_{[\xi,\xi]}[\omega-\omega_0, e^2](\omega-\omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(e^2)d_{\omega_0}(\omega-\omega_0) \\
&\quad + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, e^2](\omega-\omega_0) + \frac{1}{2}\mathcal{L}_\xi^{\omega_0}(e^2)\iota_\xi F_{\omega_0} \\
&= \int_\Sigma \frac{1}{4}d_\omega(e^2)\iota_{[\xi,\xi]}(\omega-\omega_0) - \frac{1}{4}[\omega-\omega_0, e^2]\iota_{[\xi,\xi]}(\omega-\omega_0) - \frac{1}{4}\iota_{[\xi,\xi]}(e^2)F_{\omega_0} \\
&\quad + \frac{1}{4}\iota_{[\xi,\xi]}(e^2)F_\omega - \frac{1}{8}\iota_{[\xi,\xi]}(e^2)[\omega_0-\omega, \omega_0-\omega] + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, e^2](\omega-\omega_0) \\
&\quad + \frac{1}{2}\mathcal{L}_\xi^{\omega_0}(e^2)\iota_\xi F_{\omega_0} \\
&= \int_\Sigma \frac{1}{2}\iota_{[\xi,\xi]}(\omega-\omega_0)ed_\omega e + \frac{1}{4}\iota_{[\xi,\xi]}(e^2)F_\omega - \frac{1}{4}d_\omega(e^2)\iota_\xi\iota_\xi F_{\omega_0} \\
&= \frac{1}{2}P_{[\xi,\xi]}^0 - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}},
\end{aligned}$$

$$\{P_\xi^0, P_\xi^I\} = 0$$

and

$$\begin{aligned}
\{P_\xi^I, P_\xi^I\} &= \int_\Sigma \xi(\phi)\mathcal{L}_\xi^{\omega_0}p = \int_\Sigma d_{\omega_0}(\iota_\xi p)\iota_\xi(d\phi) \\
&= \int_\Sigma \iota_\xi(d_{\omega_0}\iota_\xi p)d\phi = \int_\Sigma \frac{1}{2}\iota_{[\xi,\xi]}(p)d\phi \\
&= \frac{1}{2}P_{[\xi,\xi]}^I.
\end{aligned}$$

Similarly L_c and R_τ are identical to the case of pure gravity with a degenerate metric. The results of [5] give us directly the following two brackets

$$\{R_\tau, R_\tau\} \approx F_{\tau\tau}, \tag{152}$$

$$\begin{aligned}
\{L_c, R_\tau\} &= \int_\Sigma [c,e]g(\tau,\omega,e) + [c,e]d_\omega\tau + d_\omega c[\tau,e] \\
&= \int_\Sigma [c,e]g(\tau,\omega,e) - [c,\tau]d_\omega e = \int_\Sigma p'_{\mathcal{S}}[c,\tau]d_\omega e - [c,\tau]d_\omega e \\
&= \int_\Sigma -p_{\mathcal{S}}[c,\tau]d_\omega e = -R_{p_{\mathcal{S}}[c,\tau]}.
\end{aligned}$$

The bracket $\{P_\xi, R_\tau\}$ divides into the following

$$\{P_\xi, R_\tau\} = \{P_\xi^0, R_\tau\} + \{P_\xi^I, R_\tau\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau},$$

where we have used

$$\begin{aligned}\{P_\xi^0, R_\tau\} &= \int_\Sigma -[\tau, e]\mathcal{L}_\xi^{\omega_0}(\omega - \omega_0) - [\tau, e]\iota_\xi F_{\omega_0} - \mathcal{L}_\xi^{\omega_0} e g(\tau, \omega, e) - \mathcal{L}_\xi^{\omega_0} e d_\omega \tau \\ &= \int_\Sigma -\mathcal{L}_\xi^{\omega_0} e g(\tau, \omega, e) + \mathcal{L}_\xi^{\omega_0} \tau d_\omega e = \int_\Sigma -p'_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau d_\omega e + \mathcal{L}_\xi^{\omega_0}\tau d_\omega e \\ &= \int_\Sigma p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau d_\omega e = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}\end{aligned}$$

and

$$\{P_\xi^I, R_\tau\} = 0. \tag{153}$$

Let us now look at all the brackets containing H_λ. Starting with $\{L_c, H_\lambda\}$ which divides into two parts, where the first is given by

$$\begin{aligned}\{L_c, H_\lambda^0\} = &\int_\Sigma [c, e]\lambda e_n F_\omega + \frac{\Lambda}{2}[c, e]\lambda e_n e^2 - [c, e]g(\tau', \omega, e) - [c, e]d_\omega \tau' \\ &+ e d_\omega c d_\omega(\lambda e_n) + e d_\omega c \lambda \sigma - d_\omega c[\tau', e] \\ = &\int_\Sigma -[c, \lambda e_n]e F_\omega - \frac{\Lambda}{3!}[c, \lambda e_n]e^3 + p_\mathcal{S}([c, \tau'] - \lambda e_n d_\omega c)d_\omega e \\ = &\int_\Sigma -\Big([c, \lambda e_n]^{(a)} e_a e F_\omega - [c, \lambda e_n]^{(n)} e_n e F_\omega\Big) - \frac{\Lambda}{3!}[c, \lambda e_n]^{(n)} e_n e^3 \\ &p_\mathcal{S}(\lambda e_n d_{\omega_0} c)d_\omega e + p_\mathcal{S}([c, \lambda e_n]^{(a)} e_a(\omega - \omega_0) + [c, \lambda e_n]^{(n)} e_n(\omega - \omega_0))d_\omega e \\ = &-P^0_{X^{(a)}} + L_{X^{(a)}(\omega - \omega_0)_a} - H^0_{X^{(n)}} + R_{p_\mathcal{S}(X^{(a)} e_a(\omega - \omega_0) - \lambda e_n d_{\omega_0} c)}.\end{aligned} \tag{154}$$

Note that we have separated the element $X = [c, \lambda e_n] \in \Omega_\partial^{(0,1)}$ into a tangential component $[c, \lambda e_n]^{(a)}$ and a normal component $[c, \lambda e_n]^{(n)}$, where the first is a vector field in $\mathfrak{X}(\Sigma)$ since it contracts with the vielbein and the second is a smooth function in $C^\infty(\Sigma)$ since it contracts with the fourth basis vector in V. Similarly, we get for the second bracket

$$\begin{aligned}
\{L_c, H^I_\lambda\} &= \int_\Sigma \frac{\lambda e_n}{4} e^2(\Pi,\Pi)[c,e] + \lambda e_n e \Pi d\phi [c,e] - \frac{\lambda}{2} e^2 \Pi (\Pi, e_n)[c,e] \\
&= \int_\Sigma \lambda e_n \Big(\frac{1}{2\cdot 3!}[c,e^3](\Pi,\Pi) + \frac{1}{2}[c,e^2]\Pi d\phi\Big) - \frac{\lambda}{3!}[c,e^3]\Pi(\Pi,e_n) \\
&= \int_\Sigma -[c,\lambda e_n]\Big(\frac{1}{2\cdot 3!} e^3(\Pi,\Pi) + \frac{1}{2} e^2 \Pi d\phi\Big) \\
&\qquad - \frac{\lambda e_n}{2} e^2 [c,\Pi] d\phi - \frac{\lambda}{3!} e^3 [c,\Pi](\Pi, e_n) \\
&= \int_\Sigma -[c,\lambda e_n]^{(a)} e_a \Big(\frac{1}{2\cdot 3!} e^3(\Pi,\Pi) + \frac{1}{2} e^2 \Pi d\phi\Big) \\
&\qquad - [c,\lambda e_n]^{(n)} e_n \Big(\frac{1}{2\cdot 3!} e^3(\Pi,\Pi) + \frac{1}{2} e^2 \Pi d\phi\Big) \\
&= -P^I_{X^{(a)}} - H^I_{X^{(n)}}.
\end{aligned} \tag{155}$$

In the last computation we have used that

$$\begin{aligned}
-\frac{\lambda e_n}{2} e^2 [c,\Pi] d\phi - \frac{\lambda}{3!} e^3 [c,\Pi](\Pi,e_n) &= \frac{\lambda e_n}{3!} e^3 \Big([c,\Pi]^{(a)}(\Pi,e_a) + [c,\Pi]^{(n)}(\Pi,e_n)\Big) \\
&= \frac{\lambda e_n}{3!} e^3 ([c,\Pi],\Pi) = \frac{\lambda e_n}{2\cdot 3!} e^3 [c,(\Pi,\Pi)] = 0.
\end{aligned}$$

This means that the complete bracket reads

$$\begin{aligned}
\{L_c, H_\lambda\} &= \{L_c, H^0_\lambda\} + \{L_c, H^I_\lambda\} \\
&= -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} + R_{p_\mathcal{S}(Y^{(a)} e_a(\omega-\omega_0) - \lambda e_n d_{\omega_0} c)}.
\end{aligned} \tag{156}$$

Next, for $\{P_\xi, H_\lambda\}$ we first calculate

$$\begin{aligned}
\{P^0_\xi, H^0_\lambda\} = &\int_\Sigma -\mathcal{L}^{\omega_0}_\xi e \Big(\lambda e_n F_\omega + \frac{\Lambda}{2}\lambda e_n e^2 - g(\tau',\omega,e) - d_\omega \tau'\Big) \\
&\quad - (\mathcal{L}^{\omega_0}_\xi(\omega-\omega_0) + \iota_\xi F_{\omega_0})\Big(e d_\omega(\lambda e_n) + \lambda e \sigma - [\tau', e]\Big) \\
= &\int_\Sigma \mathcal{L}^{\omega_0}_\xi(\lambda e_n) e F_\omega + \frac{\Lambda}{3!} e^3 \mathcal{L}^{\omega_0}_\xi(\lambda e_n) \\
&\quad + p_\mathcal{S}\Big(- \mathcal{L}^{\omega_0}_\xi \tau' + \lambda e_n (\mathcal{L}^{\omega_0}_\xi(\omega-\omega_0) + \iota_\xi F_{\omega_0})\Big) d_\omega e \\
= &\int_\Sigma \mathcal{L}^{\omega_0}_\xi(\lambda e_n)^{(a)} e_a e F_\omega + \mathcal{L}^{\omega_0}_\xi(\lambda e_n)^{(n)} e_n e F_\omega + \frac{\Lambda}{3!} e^3 \mathcal{L}^{\omega_0}_\xi(\lambda e_n)^{(n)} e_n \\
&\quad + p_\mathcal{S}(\lambda e_n \iota_\xi F_{\omega_0}) d_\omega e - p_\mathcal{S}(\mathcal{L}^{\omega_0}_\xi(\lambda e_n)^{(a)} e_a(\omega-\omega_0)) d_\omega e \\
&\quad - p_\mathcal{S}(\mathcal{L}^{\omega_0}_\xi(\lambda e_n)^{(n)} e_n(\omega-\omega_0))) d_\omega e \\
= &\; P^0_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H^0_{Y^{(n)}} - R_{p_\mathcal{S}(Y^{(a)} e_a(\omega-\omega_0) - \lambda e_n \iota_\xi F_{\omega_0})},
\end{aligned}$$

with $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$. The remaining part can be calculated together as

$$\begin{aligned}
&\{P_\xi^0, H_\lambda^I\} + \{P_\xi^I, H_\lambda^0\} + \{P_\xi^I, H_\lambda^I\} \\
&\quad = \int_\Sigma -\mathcal{L}_\xi^{\omega_0} e\Big(\lambda e_n \frac{e^2}{4}(\Pi,\Pi) + \lambda e_n e \Pi d\phi - \frac{\lambda}{2} e^2 (e_n,\Pi)\Pi\Big) \\
&\qquad + \mathcal{L}_\xi^{\omega_0} p \lambda (e_n, \Pi) - \xi(\phi)\frac{1}{2} d_\omega(\lambda e_n e^2 \Pi) \\
&\quad = \int_\Sigma -\frac{\lambda e_n}{2\cdot 3!}(\Pi,\Pi)\mathcal{L}_\xi^{\omega_0}(e^3) - \frac{\lambda e_n}{2}\Pi d\phi \mathcal{L}_\xi^{\omega_0}(e^2) + \frac{\lambda}{3!}\Pi(\Pi, e_n)\mathcal{L}_\xi^{\omega_0}(e^3) \\
&\qquad + \lambda \mathcal{L}_\xi^{\omega_0}(p)(\Pi, e_n) - \frac{1}{2} d_\omega(\lambda e_n e^2 \Pi)\iota_\xi d\phi \\
&\quad = \int_\Sigma \mathcal{L}_\xi^{\omega_0}\Big(\frac{\lambda e_n}{2\cdot 3!}\Big)(\Pi,\Pi)e^3 + \frac{\lambda e_n}{2\cdot 3!}\mathcal{L}_\xi^{\omega_0}(\Pi,\Pi)e^3 - \frac{\lambda e_n}{2}\Pi d\phi \mathcal{L}_\xi^{\omega_0}(e^2) \\
&\qquad + \frac{\lambda}{3!}\Pi(\Pi, e_n)\mathcal{L}_\xi^{\omega_0}(e^3) + \frac{\lambda}{3!}\mathcal{L}_\xi^{\omega_0}(\Pi^n e_n)e^3(\Pi, e_n) - \frac{\lambda e_n}{3!}\mathcal{L}_\xi^{\omega_0}(e^3)\Pi^n \\
&\qquad \mathcal{L}_\xi^{\omega_0}\Big(\frac{\lambda e_n}{2}\Big)e^2 \Pi d\phi + \frac{\lambda e_n}{2}\mathcal{L}_\xi^{\omega_0}(e^2)\Pi d\phi + \frac{\lambda e_n}{2} e^2 \mathcal{L}_\xi^{\omega_0}(\Pi) d\phi \\
&\quad = \int_\Sigma \mathcal{L}_\xi^{\omega_0}(\lambda e_n)\Big(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\Big) + \lambda e_n \Big(\frac{e^3}{3!}(\Pi, \mathcal{L}_\xi^{\omega_0}(\Pi)) + \frac{e^3}{2}(e,\Pi)\mathcal{L}_\xi^{\omega_0}(\Pi)\Big) \\
&\qquad + \frac{\lambda}{3!} e^3 (\Pi, e_n)\mathcal{L}_\xi^{\omega_0}(\Pi) \\
&\quad = \int_\Sigma \mathcal{L}_\xi^{\omega_0}(\lambda e_n)\Big(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\Big) \\
&\qquad - \frac{\lambda e_n}{2} e^3 (e,\Pi)\mathcal{L}_\xi^{\omega_0}(\Pi) - \frac{\lambda}{3!} e^3 (\Pi, e_n)\mathcal{L}_\xi^{\omega_0}(\Pi) \\
&\qquad + \frac{\lambda e_n}{2} e^3 (e,\Pi)\mathcal{L}_\xi^{\omega_0}(\Pi) + \frac{\lambda}{3!} e^3 (\Pi, e_n)\mathcal{L}_\xi^{\omega_0}(\Pi) \\
&\quad = \int_\Sigma \mathcal{L}_\xi^{\omega_0}(\lambda e_n)\Big(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\Big) \\
&\quad = P^I_{Y^{(a)}} + H^I_{Y^{(n)}},
\end{aligned}$$

where we used that $(e, \Pi) = d\phi$. Once more the complete bracket is given by

$$\begin{aligned}
\{P_\xi, H_\lambda\} &= \{P_\xi^0, H_\lambda^0\} + \{P_\xi^0, H_\lambda^I\} + \{P_\xi^I, H_\lambda^0\} + \{P_\xi^I, H_\lambda^I\} \\
&= P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}} - R_{p_\mathcal{S}(Y^{(a)}e_a(\omega-\omega_0) - \lambda e_n \iota_\xi F_{\omega_0})}.
\end{aligned}$$

Also $\{H_\lambda, H_\lambda\}$ splits into three parts, which are given by

$$\{H_\lambda^0, H_\lambda^0\} \approx F_{\tau'\tau'},$$

$$\begin{aligned}
\{H_\lambda^I, H_\lambda^I\} &= \int_\Sigma -\frac{\lambda}{2}(e_n,\Pi) d_\omega(\lambda e_n e^2 \Pi) \\
&= \int_\Sigma -\frac{\lambda}{2}(e_n,\Pi) d\lambda e_n e^2 \Pi - \frac{\lambda}{2}(e_n,\Pi)\lambda d_\omega(e_n e^2 \Pi) \\
&= \int_\Sigma -\frac{\lambda}{2}(e_n,\Pi) d\lambda e_n e^2 \Pi,
\end{aligned}$$

$$\begin{aligned}
\{H^0_\lambda, H^I_\lambda\} &= \int_\Sigma \Big(\lambda e_n \frac{e}{4}(\Pi,\Pi) + \lambda e_n \Pi d\phi - \frac{\lambda}{2} e(e_n,\Pi)\Pi\Big)(ed_\omega(\lambda e_n) + \lambda e\sigma - [\tau', e]) \\
&= \int_\Sigma \lambda d\lambda e_n^2 \frac{e^2}{4}(\Pi,\Pi) + \lambda d\lambda e_n^2 e\Pi d\phi + \frac{\lambda}{2}(e_n,\Pi) d\lambda e_n e^2 \Pi \\
&\quad - \lambda^2 \Big(e_n \frac{e}{4}(\Pi,\Pi) + e_n \Pi d\phi - \frac{1}{2} e(e_n,\Pi)\Pi\Big) e\sigma \\
&\quad - \frac{\lambda^2}{4} p_{\mathcal{S}}(e_n(\omega-\omega_0))[e_n e, e](\Pi,\Pi) - \lambda^2 p_{\mathcal{S}}(e_n(\omega-\omega_0))[e_n\Pi, e] d\phi \\
&\quad + \frac{\lambda^2}{2} p_{\mathcal{S}}(e_n(\omega-\omega_0))[e\Pi, e](e_n,\Pi) \\
&= \int_\Sigma \frac{\lambda}{2}(e_n,\Pi) d\lambda e_n e^2 \Pi,
\end{aligned}$$

where we have used that $\lambda^2 = 0$ and $e_n^2 = 0$. Consequently, the complete bracket reads

$$\{H_\lambda, H_\lambda\} = \{H^0_\lambda, H^0_\lambda\} + 2\{H^0_\lambda, H^I_\lambda\} + \{H^I_\lambda, H^I_\lambda\} \approx F_{\tau'\tau'}. \tag{157}$$

The last remaining bracket to be calculated is $\{H_\lambda, R_\tau\}$, which splits into two parts given by

$$\{H^0_\lambda, R_\tau\} \approx F_{\tau\tau'} + G_{\lambda\tau}$$

and

$$\{H^I_\lambda, R_\tau\} = \int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] - \frac{e}{4}\lambda e_n(\Pi,\Pi)[\tau,e] + \frac{\lambda}{2} e\Pi(\Pi, e_n)[\tau, e].$$

The last two terms are zero because $e[\tau, e] = 0$ (see Lemma B.5). The first term instead does not vanish in general:

$$\begin{aligned}
\int_\Sigma -\lambda e_n \Pi d\phi[\tau, e] &= \int_\Sigma -\lambda \Pi d\phi \tau [e_n, e] = \int_\Sigma -\lambda \Pi d\phi \tau (e_n, e) = \int_\Sigma -\lambda \Pi d\phi \tau e^n_k dx^k \\
&= \int_\Sigma \Big(- \lambda \Pi^a \tau^{bcd}_i (d\phi)_j e^n_k \Big) v_a v_b v_c v_d dx^i dx^j dx^k \\
&=: K^S_{\lambda\tau},
\end{aligned}$$

where we have used $e_n[\tau, e] = \tau[e_n, e]$ (again Lemma B.5) and $[e_n, e] = (e_n, e) = e^n_k dx^k$ for $k = 1, 2, 3$. The complete bracket reads

$$\{H_\lambda, R_\tau\} \approx F_{\tau\tau'} + G_{\lambda\tau} + K^S_{\lambda\tau}.$$

□

4.3 Conclusion SPC Theory

We have analysed the boundary structure of the SPC theory in $N = 4$ for the case of a degenerate boundary metric by calculating its Poisson algebra in Theorem 4.11. We have also stated the results following from a nondegenerate boundary metric in Theorem 4.6. We will now state the consequences of the constrained algebra for the reduced phase space. In particular, following Definition A.1 we can detect the number of first- and second-class constraints.

Corollary 4.12. *Let g^∂ be non-degnerate. Then the constraints $\{L_c, P_\xi, H_\lambda\}$ in the SPC theory define a first-class system and the submanifold $C \subset F_\partial$ given by the zero locus of the constraints is coisotropic.*

Proof. From Theorem 4.6 we see that the Poisson brackets of the constraints are all given by a linear combination of the constraints themselves, hence vanishing on the zero locus. It therefore defines a system of first-class and by Proposition A.3 C is coisotropic. □

Remark. In the nondegenerate case the constraints on the geometric phase space are all of first-class. Hence, they define a coisotropic submanifold C of F_∂ and the reduced phase space can be found by means of a coisotropic reduction $\underline{C} = C/\sim$ (see Theorem A.4). Furthermore, all constraints are associated to gauge transformations, meaning that their Hamiltonian flows leave the subspace invariant. In particular, L_c generates the internal gauge symmetry (recall that $c \in \Omega_\partial^{(0,2)}$ and $\wedge^2\mathcal{V} \simeq \mathfrak{so}(3,1)$), P_ξ is associated to the local diffeomorphisms tangential to the boundary and H_λ is the generator of the local diffeomorphisms normal to the boundary.

Corollary 4.13. *Let g^∂ be degnerate with* $\dim(\ker(g^\partial)) = 1$. *Then, in the SPC theory, we have first- and second-class constraints. In particular, L_c, P_ξ, H_λ by themselves are first-class constraints.*

Proof. From Theorem 4.11 we see that not all the Poisson brackets vanish on the zero locus of the constraints, meaning that they do not define a first-class system. In particular, the brackets $\{R_\tau, R_\tau\}$, $\{R_\tau, H_\lambda\}$ and $\{H_\lambda, H_\lambda\}$, which we denote with the matrices D, B and C, respectively, are different from zero. We extend the proof done in [5] for the pure gravity degenerate case. We want to proof that $B^T D^{-1} B = -C$. Note, that the matrices D and C are identical to the pure gravity case and the only additional term in B is $K^S_{\lambda\tau}$ which can be found in the proof of Theorem 4.11. In particular, $K^S_{\lambda\tau}$ is proportional to the odd quantity λ and does not contain derivatives. Then by Lemma A.2 the contribution to $B^T D^{-1} B$ coming from $K^S_{\lambda\tau}$ vanishes and hence leaving the same terms as in the case of pure gravity. Therefore by Proposition A.1 the number of second-class constraints is the rank of the matrix D which is 1 and the second-class constraint is R_τ. Furthermore, by Definition A.1 there are 3 constraints of first-class, which must be L_c, P_ξ and H_λ. □

In the degenerate case, due to the different outcome of Theorem 4.7, another constraint R_τ appears, which turns out to be of second-class. The constrained system does therefore not define a coisotropic submanifold of F_∂. The differences between the nondegenerate and degenerate cases in the SPC theory have been outlined in figure 2.

Scalar Palatini–Cartan

Geometric Phase Space

$$F_\partial = \Omega_\partial^{(1,1)} \times \frac{\mathcal{A}(\Sigma)}{\sim} \times C^\infty(\Sigma) \times \frac{\Omega_\partial^{(0,1)}}{\sim}$$

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{3!}\delta(e^3[\Pi])\delta\phi$$

Equations of motion

$$(1) \quad d_\omega e = 0$$

$$(2) \quad eF_\omega + \frac{\Lambda}{6}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{12}e^3(\Pi,\Pi) = 0$$

$$(3) \quad d\phi - (e,\Pi) = 0$$

Nondegenerate case

$(1) \rightsquigarrow$ Fixes $[\omega] + L_c$

$(2) \rightsquigarrow P_\xi, H_\lambda$

$(3) \rightsquigarrow$ Fixes $[\Pi]$

L_c, P_ξ, H_λ are of first-class and $\{L_c, P_\xi, H_\lambda\}$ defines a coisotropic submanifold

Degenerate case

$(1) + p_{\mathcal{K}}v = 0 \rightsquigarrow$ Fix $[\omega] + L_c, R_\tau$

$(2) \rightsquigarrow P_\xi, H_\lambda$

$(3) + p_{\mathcal{W}}\Pi = 0 + \xi(\phi) = 0 \rightsquigarrow$ Fix $[\Pi]$

L_c, P_ξ, H_λ are of first-class, R_τ is of second-class and $\{L_c, R_\tau, P_\xi, H_\lambda\}$ does not define a coisotropic submanifold

Figure 2: Summary of the boundary structure of the SPC theory in $N = 4$ dimensions and the differences between the nondegenerate and degenerate case.

5 Yang–Mills Coupling

We start this chapter by reminding how the usual Yang–Mills theory is stated in the gauge field framework which we have established in chapter 1. A Yang–Mills theory refers to any gauge theory based on a Lie group whose Lie algebra is compact and reductive.[30] So let G be such a Lie group, $\mathfrak{g}$ its Lie algebra, M the usual N-dimensional pseudo-Riemannian manifold, (P, π, M) a principal G-bundle and $\omega \in \Omega^1(P, \mathfrak{g})$ a principal connection on it. If we consider a section $\sigma\colon U \to P$ we can recover the local version of the connection, that is the gauge field $A = \sigma^*(\omega)$. In coordinates it reads

$$A(x) = A^I_\mu(x) T_I dx^\mu, \tag{158}$$

where $\{T_I\}$ is a basis of the Lie algebra $\mathfrak{g}$. The field strength is then $F_A = dA + \frac{1}{2}[A \wedge A]$ which leads to[31]

$$\begin{aligned} F_A(x) &= \frac{1}{2}(\partial_\mu A^I_\nu - \partial_\nu A^I_\mu + f^I_{JK} A^J_\mu A^K_\nu) T_I dx^\mu \wedge dx^\nu && (159) \\ &= \frac{1}{2} F^I_{\mu\nu}(x) T_I dx^\mu \wedge dx^\nu, && (160) \end{aligned}$$

where f^I_{JK} are the structure constants of $\mathfrak{g}$ defined by $[T_J, T_K] = f^I_{JK} T_I$. On every Lie algebra (over some field $\mathbb{K}$) there is a unique invariant symmetric bilinear form $K\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, called Killing form, given by the trace in some representation $K(a, b) = \mathrm{Tr}(\rho(a)\rho(b))$. In the case of a compact reductive Lie algebra the Killing form of the basis elements defines a metric[32]

$$k_{IJ} \coloneqq \mathrm{Tr}(T_I T_J). \tag{161}$$

The final goal is to construct a gauge invariant Lagrangian, that is some volume form which does not change under gauge transformations. We can make use of the fact that the field strength transforms under the adjoint action. In order to receive a volume form out of it we can simply take the wedge product of the field strength with its Hodge star dual $\star F_A$[33], which also transforms under the adjoint action. Gauge invariance is achieved by taking the trace of the resulting object

$$\mathrm{Tr}(F_A \star F_A) = \mathrm{Vol}_g F^I_{\mu\nu} F^J_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} k_{IJ}, \tag{162}$$

where $\mathrm{Vol}_g = \sqrt{|\det(g)|} dx^1 \dots dx^N$.

5.1 Yang–Mills Palatini–Cartan Action

Again, we start by giving a general description of the theory for an N-dimensional pseudo-Riemannian manifold M.

[30] For the considerations of this chapter it is actually enough to assume that the Lie algebra possesses an invariant nondegenerate symmetric bilinear form.

[31] $[A \wedge A] = [A^I_\mu T_I dx^\mu \wedge A^J_\nu T_J dx^\nu] = A^I_\mu A^J_\nu [T_I, T_J] dx^\mu \wedge dx^\nu$.

[32] Where we used the fundamental representation $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \rho(a) = a$.

[33] On every pseudo-Riemannian manifold M with metric g the Hodge star operator is defined as $\star\colon \Omega^k(M) \to \Omega^{N-k}(M)$ such that for $\alpha = \frac{1}{k!}\alpha_{\mu_1 \dots \mu_k} dx^{\mu_1} \dots dx^{\mu_k}$ the Hodge star dual of α is $\star\alpha = \frac{1}{k!(N-k)!}\sqrt{|\det(g)|}\alpha_{\nu_1 \dots \nu_k} g^{\nu_1\mu_1} \dots g^{\nu_k\mu_k} \varepsilon_{\mu_1 \dots \mu_k \mu_{k+1} \dots \mu_N} dx^{\mu_{k+1}} \dots dx^{\mu_N}$.

Definition 5.1 (Pairing)**.** Let $\{v_i\}_{\mu=1}^N$ be an orthonormal basis of V and let $A = A^{\mu\nu} v_\mu \wedge v_\nu \in \wedge^2 V$ and $B = B^{\rho\sigma} v_\rho \wedge v_\sigma \in \wedge^2 V$. The **pairing** of A and B is then defined as

$$(A, B) \coloneqq A^{\mu\nu} B^{\rho\sigma} \eta_{\mu\rho} \eta_{\nu\sigma}. \tag{163}$$

Definition 5.2 (Yang–Mills Palatini–Cartan)**.** Let M be an N-dimensional pseudo-Riemannian manifold with boundary $\Sigma \coloneqq \partial M$. The **Yang-Mills Palatini–Cartan** theory (YMPC) is the assignment of the pair $(\mathcal{F}_{YMPC}, S_{YMPC})_M$ to M, with the space of fields

$$\mathcal{F}_{YMPC} = \Omega^{(1,1)}_{\text{nd}}(M) \times \mathcal{A}(M) \times \Omega^{(0,2)}(M) \otimes \mathfrak{g} \times \Omega^{(1,0)}(M) \otimes \mathfrak{g} \tag{164}$$

and the action

$$S_{YMPC} = S_{PC} + S_{YM}, \tag{165}$$

with

$$S_{PC} = \int_M \frac{1}{(N-2)!} e^{N-2} \wedge F_\omega + \frac{\Lambda}{N!} e^N, \tag{166}$$

$$S_{YM} \coloneqq \int_M \frac{1}{(N-2)!} e^{N-2} \mathrm{Tr}(B F_A) + \frac{1}{2N!} e^N \mathrm{Tr}(B, B) \tag{167}$$

where $(e, \omega, B, A) \in \mathcal{F}_{YMPC}$.

The variation of the action S_{YMPC} leads to the equations of motion

$$d_\omega e = 0, \tag{168}$$

$$\frac{e^{N-3}}{(N-3)!}(F_\omega + \mathrm{Tr}(B F_A)) + \frac{e^{N-1}}{(N-1)!}(\Lambda + \frac{1}{2}\mathrm{Tr}(B, B)) = 0, \tag{169}$$

$$e^{N-2}\big(F_A + \frac{1}{2}(e^2, B)\big) = 0, \tag{170}$$

$$d_A(e^{N-2} B) = 0 \tag{171}$$

and the boundary term

$$\tilde{\alpha}_{YM} = \int_{\partial M} \frac{e^{N-2}}{(N-2)!} \delta\omega + \frac{e^{N-2}}{(N-2)!} \mathrm{Tr}(B \delta A). \tag{172}$$

Remark. By Lemma B.1 we know that the map $W_{N-2}^{(2,0)} \colon \Omega^{(2,0)} \to \Omega^{(N,N-2)}$ is injective. Thus equation (170) can be simplified to

$$F_A + \frac{1}{2}(e^2, B) = 0.$$

In coordinates the B-field reads

$$B = B^{\mu\nu} v_\mu v_\nu = (-1)^N g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} v_\mu v_\nu$$

which leads to the usual Yang–Mills action

$$S_{YM} = -\int_M \frac{1}{2} \mathrm{Vol}_g \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}).$$

5.2 Yang–Mills Boundary Structure in $N = 4$

The procedure is identical as in the scalar coupling. First we restrict the fields to the boundary, then we quotient out the kernel of the presymplectic form and finally we impose the constraints using a suitable choice of representative.

Definition 5.3 (Preboundary Fields)**.** The space of **preboundary fields** of the YMPC theory and its **presymplectic form** are defined to be[34]

$$\tilde{F}_\partial = \Omega_\partial^{(1,1)} \times \mathcal{A}(\Sigma) \times \Omega_{\partial,\mathfrak{g}}^{(0,2)} \times \Omega_{\partial,\mathfrak{g}}^{(1,0)}, \tag{173}$$

$$\tilde{\varpi} := \delta\tilde{\alpha} = \int_\Sigma e\delta e\delta\omega + \mathrm{Tr}(eB\delta e\delta A) + \frac{1}{2}\mathrm{Tr}(e^2\delta B\delta A). \tag{174}$$

Proposition 5.1. *The geometric phase space for the YMPC theory reads*

$$F_\partial = \Omega_\partial^{(1,1)} \times \frac{\mathcal{A}(\Sigma)}{\sim} \times \frac{\Omega_{\partial,\mathfrak{g}}^{(0,2)}}{\sim} \times \Omega_{\partial,\mathfrak{g}}^{(1,0)} \tag{175}$$

where the equivalence classes $[\omega]$ *and* $[B]$ *satisfy*

$$\omega \sim \omega' \iff \omega' = \omega + v, \quad \text{with} \quad ev = 0, \tag{176}$$

$$B \sim B' \iff B' = B + C, \quad \text{with} \quad e^2C = 0. \tag{177}$$

Proof. As in the scalar case, we consider now a generic vector field $X = \mathbb{X}_e\frac{\delta}{\delta e} + \mathbb{X}_\omega\frac{\delta}{\delta\omega} + \mathbb{X}_B\frac{\delta}{\delta B} + \mathbb{X}_A\frac{\delta}{\delta A}$. This is a vector field with respect to the variation δ (see Definition 2.10), meaning that the components are along the field values $\mathbb{X}_e \in \Omega_\partial^{(1,1)}, \mathbb{X}_\omega \in \mathcal{A}(\Sigma), \mathbb{X}_B \in \Omega_{\partial,\mathfrak{g}}^{(0,2)}, \mathbb{X}_A \in \Omega_{\partial,\mathfrak{g}}^{(1,0)}$. Contracting it with equation (174) gives

$$\begin{aligned}\iota_X\tilde{\varpi} = \mathrm{Tr}\int_\Sigma & e\mathbb{X}_e\delta\omega + eB\mathbb{X}_e\delta A + e\mathbb{X}_\omega\delta e \\ & + \frac{1}{2}e^2\mathbb{X}_B\delta A + eB\mathbb{X}_A\delta e + \frac{1}{2}e^2\mathbb{X}_A\delta B \\ = \mathrm{Tr}\int_\Sigma & \Big(e\mathbb{X}_\omega + eB\mathbb{X}_A\Big)\delta e + e\mathbb{X}_e\delta\omega \\ & + \frac{1}{2}e^2\mathbb{X}_A\delta B + \Big(eB\mathbb{X}_e + \frac{1}{2}e^2\mathbb{X}_B\Big)\delta A.\end{aligned}$$

Therefore we find that the kernel is defined by the set of equations

$$e\mathbb{X}_\omega + e\Pi\mathbb{X}_A = 0, \tag{178}$$

$$e\mathbb{X}_e = 0, \tag{179}$$

$$\frac{1}{2}e^2\mathbb{X}_A = 0, \tag{180}$$

$$eB\mathbb{X}_e + \frac{1}{2}e^2\mathbb{X}_B = 0. \tag{181}$$

[34]We defined the $\mathfrak{g}$-valued boundary fields $\Omega_{\partial,\mathfrak{g}}^{(i,j)} := \Omega_\partial^{(i,j)} \otimes \mathfrak{g}$. The fact that they are $\mathfrak{g}$-valued will not play a crucial role for computations.

As in the scalar case, using that $W_1^{\partial(1,1)}$ and $W_2^{\partial(1,0)}$ are both injective (see Lemma B.3) equations (179) and (180) are equivalent to $\mathbb{X}_e = 0$ and $\mathbb{X}_A = 0$, which leads to trivial equivalence relations for e and ϕ. Furthermore equations (116) and (119) are equivalent to $e\mathbb{X}_\omega = 0$ and $e^2\mathbb{X}_\Pi = 0$, proving the statement. □

Theorem 5.2. *(F_∂, ϖ) is a symplectic manifold with symplectic form*

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{2}\mathrm{Tr}(\delta(e^2[B])\delta A). \tag{182}$$

Remark. The next step is to simultaneously fix the representatives and impose the constraints, which are given by the equations of motion for $N = 4$:

$$d_\omega e = 0, \tag{183}$$

$$eF_\omega + \frac{\Lambda}{6}e^3 + \mathrm{Tr}\Big(eBF_A + \frac{1}{2\cdot 3!}e^3(B,B)\Big) = 0, \tag{184}$$

$$d_A(e^2B) = 0, \tag{185}$$

$$F_A + \frac{1}{2}(e^2, B) = 0. \tag{186}$$

Notice how the equivalence relation of $[\omega]$ and the equation of motion (183) remain the same as in the scalar case (and pure gravity). This implies that for both the nondegenerate and degenerate case the same discussion as in the previous chapter apply.

5.2.1 Nondegenerate Case

Theorem 5.3. *Let g^∂ be nondegenerate, then $\forall B' \in \Omega^{(0,2)}_{\partial,\mathfrak{g}}$ there is a unique decomposition*

$$B' = B + C, \tag{187}$$

such that $e^2C = 0$ and $F_A + \frac{1}{2}(e^2, B) = 0$.

Proof. See [3]. □

Remark. The previous theorem shows that imposing the constraint coming from equation (186) is equivalent to choosing a representative for $[B] \in \Omega^{(0,2)}_{\partial,\mathfrak{g}}/\sim$.
In order to fix the representative of the equivalence classes $[\omega] \in \mathcal{A}(\Sigma)/\sim$ we use again Theorems 4.3 and 4.4 as in the case of the scalar coupling. Hence, equation (183) again splits into the invariant and structural constraint, where the first becomes a Lagrange multiplier and the second chooses a representative. Equation (184) takes a similar role as (129) in the scalar case. However, in the case of Yang–Mills coupling we get an additional constraint coming from equation (185). Using

Darboux coordinates $\rho = \frac{1}{2}e^2 B$ the Lagrange multipliers read

$$L_c := \int_\Sigma c e d_\omega e, \tag{188}$$

$$M_\mu := \int_\Sigma \frac{1}{2}\mathrm{Tr}(\mu d_A \rho), \tag{189}$$

$$P_\xi := \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{2}\mathrm{Tr}(\iota_\xi \rho F_A) + \iota_\xi(\omega - \omega_0) e d_\omega e + \mathrm{Tr}\big(\iota_\xi(A - A_0) d_A \rho\big), \tag{190}$$

$$H_\lambda := \int_\Sigma \lambda e_n \Big(e F_\omega + \frac{\Lambda}{3!}e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!}e^3 \mathrm{Tr}(B,B)\Big), \tag{191}$$

with $c \in \Omega_\partial^{(0,2)}[1]$, $\mu \in C^\infty(\Sigma, \mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$ and $\lambda \in C^\infty(\Sigma)[1]$. These constraints define smooth functions on the symplectic manifold F_∂ and with the symplectic form

$$\varpi = \int_\Sigma e\delta e \delta\omega + \mathrm{Tr}(\delta\rho\delta A) \tag{192}$$

we can calculate the Poisson structure of these constraints, which is stated in the following theorem.

Theorem 5.4. *Let g^∂ be nondegenerate. Then the Poisson brackets of the constraints in the YMPC theory read*

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]}, \tag{193}$$

$$\{M_\mu, M_\mu\} = -\frac{1}{2}M_{[\mu,\mu]}, \tag{194}$$

$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{A_0}}, \tag{195}$$

$$\{H_\lambda, H_\lambda\} = 0, \tag{196}$$

$$\{L_c, M_\mu\} = 0, \tag{197}$$

$$\{L_c, P_\xi\} = L_{L_\xi^{\omega_0} c}, \tag{198}$$

$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}}, \tag{199}$$

$$\{M_\mu, P_\xi\} = M_{\mathcal{L}_\xi^{A_0}\mu}, \tag{200}$$

$$\{M_\mu, H_\lambda\} = 0, \tag{201}$$

$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(a)}}, \tag{202}$$

with $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ and where the superscripts (a) and (n) describe their components with respect to the frame (e_a, e_n).

Proof. See [3]. □

5.2.2 Degenerate Case

Theorem 5.5. *Let g^∂ be degenerate with* $\dim(\ker(g^\partial)) = 1$*, $e_n \in \mathcal{V}|_\Sigma$ (as defined in equation* (91)*) and $[B] \in \frac{\Omega_{\partial,\mathfrak{g}}^{(0,2)}}{\sim}$ an equivalence class. Then there exist unique $b \in \ker(W_2^{\partial,(0,2)})$ and b^{an} with*

$$B = 2b^{an} v_a e_n + b, \tag{203}$$

such that

1. $F_A + \frac{1}{2}(e^2, B) = 0$,
2. $\iota_\xi F_A = 0$,
3. $p_{\mathcal{W}'} B = 0$,

where $\xi \in \ker(g^\partial)$ *and* $\mathcal{W}'$ *is the space spanned by elements which can be written as a wedge product* $A = B \wedge C$ *such that* $B \in \Omega_\partial^{(0,1)}$ *and* $C \in \mathcal{W} = e(\ker g^\partial) \subset \Omega_\partial^{(0,1)}$.

Proof. We can write a general element $B \in \Omega_{\partial,\mathfrak{g}}^{(0,2)}$ along any basis as $B = 2b^{an} v_a e_n + b^{ab} v_a v_b$ with $a, b = 1, 2, 3$. It is clear that $b = b^{ab} v_a v_b \in \ker(W_2^{\partial,(0,2)})$ because $e^2 \wedge v_a \wedge v_b = 0$ for all a, b due to antisymmetry of the wedge product. This directly implies that the components b^{an} in (203) are already uniquely determined by the equivalence class $[B]$.
It remains to show that the conditions define a unique b. Since $\dim(\ker(g^\partial)) = 1$ and e is injective we have that $\mathcal{W} \subset \mathcal{V}|_\Sigma$ is a 1-dimensional subspace. Now we can choose a basis $\{v_1, v_2, v_3\}$ of $\mathcal{V}|_\Sigma$ such that $v_3 \in \mathcal{W}$. Then a basis of $\mathcal{W}'$ is given by $\{v_1 v_3, v_2 v_3\}$ and with $b = b^{12} v_1 v_2 + b^{13} v_1 v_3 + b^{23} v_2 v_3$ the condition $p_{\mathcal{W}'} B = 0$ implies that $b^{13} = b^{23} = 0$.
We can use the vielbein to rewrite the components of the curvature in terms of components of elements in $\wedge^2 V$

$$F_A = F_{ij} dx^i dx^j = F_{ab} e_i^a e_j^b dx^i dx^j. \tag{204}$$

Next, consider the condition $\iota_\xi F_A = 0$, which implies that the Yang–Mills curvature F_A contracted with any vector field which lies in the kernel of the metric vanishes. This means that

$$F_A = F_{12} e_i^1 e_j^2 dx^i dx^j, \tag{205}$$

since e maps the components along ξ to the component along v_3. Furthermore, by definition of $\mathcal{W}$, g is invertible on the complement of $\mathcal{W}$ in $\mathcal{V}|_\Sigma$. So it exists $(g^{-1})^{ab} =: g^{ab}$ such that $g_{ab} g^{ab} = 1$ for $a, b = 1, 2$. Then

$$\begin{aligned} F_A + \frac{1}{2}(e^2, B) &= F_{ij} dx^i dx^j + \frac{1}{2}(e_i^a e_j^b dx^i dx^j v_a v_b, b^{cd} v_c v_d + 2b^{cn} v_c e_n) && (206)\\ &= F_{ab} e_i^a e_j^b dx^i dx^j + b^{cd} e_i^a e_j^b g_{ac} g_{bd} dx^i dx^j + b^{cn} e_i^a e_j^b g_{ac} g_{bd} dx^i dx^j && (207)\\ &= e_i^a e_j^b (F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn}) dx^i dx^j = 0. && (208) \end{aligned}$$

And since $a, b, c, d \neq 3$

$$F_{ab} + b^{cd} g_{ac} g_{bd} + b^{cn} g_{ac} g_{bn} = 0 \quad \Longrightarrow \quad b^{cd} = -g^{ac} g^{bd} F_{ab} - g^{bd} g_{bn} b^{cn}. \tag{209}$$

So with $b^{13} = b^{23} = 0$ and b^{an} fixed for all a, we have that B is uniquely determined in terms of its components. □

Remark. Theorem 5.5 shows that constraint (186) together with conditions 2. and 3. is equivalent to choosing a unique representative for $[B] \in \Omega_{\partial,\mathfrak{g}}^{(0,2)} / \sim$.
In order to fix the representative of the equivalence class $[\omega] \in \mathcal{A}(\Sigma) / \sim$ we use the

same two theorems (4.7 and 4.8) as in the scalar coupling case. Hence, equation (183) again splits into structural, invariant and degeneracy constraints, where the first fixes the representative of $[\omega]$. Altogether, using Darboux coordinates the constraint functions in the degenerate case read

$$L_c := \int_\Sigma c e d_\omega e, \tag{210}$$

$$M_\mu := \int_\Sigma \mathrm{Tr}(\mu d_A \rho), \tag{211}$$

$$P_\xi := \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{2}\mathrm{Tr}(\iota_\xi \rho F_A) + \iota_\xi(\omega-\omega_0) e d_\omega e + \mathrm{Tr}\big(\iota_\xi(A-A_0)d_A\rho\big), \tag{212}$$

$$H_\lambda := \int_\Sigma \lambda e_n \Big(eF_\omega + \frac{\Lambda}{3!}e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!}e^3\mathrm{Tr}(B,B) \tag{213}$$
$$- (\omega-\omega_0)p_\mathcal{T}(d_\omega e)\Big),$$

$$R_\tau := \int_\Sigma \tau d_\omega e, \tag{214}$$

with $c \in \Omega_\partial^{(0,2)}[1]$, $\mu \in C^\infty(\Sigma,\mathfrak{g})[1]$, $\xi \in \mathfrak{X}(\Sigma)[1]$, $\lambda \in C^\infty(\Sigma)[1]$ and $\tau \in \mathcal{S}[1]$.

Theorem 5.6. *Let g^∂ be degenerate with* $\dim(\ker g^\partial) = 1$. *Then the Poisson brackets of the constraints in the YMPC theory read*

$$\{L_c, L_c\} = -\frac{1}{2}L_{[c,c]}, \tag{215}$$
$$\{R_\tau, R_\tau\} \approx F_{\tau\tau}, \tag{216}$$
$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{\omega_0}}, \tag{217}$$
$$\{H_\lambda, H_\lambda\} \approx F_{\tau'\tau'}, \tag{218}$$
$$\{M_\mu, M_\mu\} = -\frac{1}{2}M_{[\mu,\mu]}, \tag{219}$$
$$\{L_c, R_\tau\} = -R_{p_\mathcal{S}[c,\tau]}, \tag{220}$$
$$\{L_c, P_\xi\} = L_{\mathcal{L}_\xi^{\omega_0}c}, \tag{221}$$
$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} \tag{222}$$
$$+ R_{p_\mathcal{S}(X^{(a)}e_a(\omega-\omega_0)-\lambda e_n d_{\omega_0}c)} + M_{X^{(a)}(A-A_0)_{(a)}},$$
$$\{L_c, M_\mu\} = 0, \tag{223}$$
$$\{R_\tau, P_\xi\} = R_{p_\mathcal{S}\mathcal{L}_\xi^{\omega_0}\tau}, \tag{224}$$
$$\{R_\tau, H_\lambda\} \approx F_{\tau\tau'} + G_{\lambda\tau} + K_{\lambda\tau}^{YM} \tag{225}$$
$$\{R_\tau, M_\mu\} = 0, \tag{226}$$
$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(a)}} - R_{p_\mathcal{S}(Y^{(a)}e_a(\omega-\omega_0)-\lambda e_n\iota_\xi F_{\omega_0})}, \tag{227}$$
$$\{P_\xi, M_\mu\} = M_{\mathcal{L}_\xi^{A_0}\mu}, \tag{228}$$
$$\{H_\lambda, M_\mu\} = 0, \tag{229}$$

with $\tau' = p_\mathcal{S}(\lambda e_n(\omega-\omega_0))$, $X = [c, \lambda e_n]$, $Y = \mathcal{L}_\xi^{\omega_0}(\lambda e_n)$ *and where the superscripts* (a) *and* (n) *describe their components with respect to the frame* (e_a, e_n). *Further-*

more, $F_{\tau\tau}$, $F_{\tau\tau'}$, $F_{\tau'\tau'}$, $G_{\lambda\tau}$ and $K^{YM}_{\lambda\tau}$ are functions of $e, \omega, B, \tau, \tau'$ and λ defined in the proof which are not proportional to any other constraint.

Proof. As in the scalar case, we split the functions into a free and an interacting component. L_c and R_τ only have a free part, whereas M_μ only has an interacting part since it does not exist in the free theory. The remaining two constraints split into

$$P_\xi = P^0_\xi + P^I_\xi,$$
$$H_\lambda = H^0_\lambda + H^I_\lambda,$$

with

$$\begin{aligned} P^0_\xi &= \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e, \\ P^I_\xi &= \int_\Sigma \mathrm{Tr}\Big(\iota_\xi \rho F_A + \iota_\xi(A - A_0)d_A\rho\Big), \\ H^0_\lambda &= \int_\Sigma \lambda e_n\Big(eF_\omega + \frac{\Lambda}{3!}e^3 - (\omega - \omega_0)p_{\mathcal{T}}(d_\omega e)\Big), \\ H^I_\lambda &= \int_\Sigma \lambda e_n \mathrm{Tr}\Big(eBF_A + \frac{1}{2\cdot 3!}e^3(B, B)\Big). \end{aligned}$$

Most of the variations and correspondent Hamiltonian vector fields are identical to the scalar case. The only different ones are those with interacting part, which are δM_μ, δP^I_ξ and δH^I_λ.
For L_c we get

$$\delta L_c = \int_\Sigma [c, e]e\,\delta\omega + d_\omega c e\,\delta e, \tag{230}$$

and thus

$$\mathbb{L}_e = [c, e], \qquad \mathbb{L}_\omega = d_\omega c + \mathbb{V}_L,$$
$$\mathbb{L}_\rho = 0, \qquad \mathbb{L}_A = 0.$$

For M_μ we get

$$\begin{aligned} \delta M_\mu &= \int_\Sigma \mathrm{Tr}(\mu\delta(d_A\rho)) = \int_\Sigma \mathrm{Tr}(-\mu([\delta A, \rho] + d_A(\delta\rho)) \\ &= \int_\Sigma \mathrm{Tr}\big([\mu, \rho]\delta A + d_A\mu\,\delta\rho\big), \end{aligned}$$

and therefore

$$\mathbb{M}_e = 0, \qquad \mathbb{M}_\omega = 0,$$
$$\mathbb{M}_\rho = [\mu, \rho], \qquad \mathbb{M}_A = d_A\mu.$$

For P_ξ we get

$$\delta P^0_\xi = \int_\Sigma -e\,\delta e(\mathcal{L}^{\omega_0}_\xi(\omega - \omega_0) + \iota_\xi F_{\omega_0}) - (\mathcal{L}^{\omega_0}_\xi e)e\,\delta\omega,$$

and

$$\begin{aligned}
\delta P^I_\xi &= \int_\Sigma \mathrm{Tr}\Big(\delta(\iota_\xi \rho F_A) + \delta(\iota_\xi (A - A_0) d_A \rho)\Big)\\
&= \int_\Sigma -\mathrm{Tr}\Big(\iota_\xi \delta\rho F_A - \iota_\xi \rho d_A \delta A - \iota_\xi(\delta A) d_A \rho - \iota_\xi (A - A_0)[\delta A, \rho] + \iota_\xi (A - A_0) d_A \delta\rho\Big)\\
&= \int_\Sigma -\mathrm{Tr}\Big(\big(\iota_\xi F_A - d_A \iota_\xi (A - A_0)\big)\delta\rho + \big(\iota_\xi d_A \rho + d_A \iota_\xi \rho + [\iota_\xi (A - A_0), \rho]\big)\delta A\Big)\\
&= \int_\Sigma -\mathrm{Tr}\Big(\big(\mathcal{L}^{A_0}_\xi (A - A_0) + \iota_\xi F_{A_0}\big)\delta\rho + \mathcal{L}^{A_0}_\xi \rho \delta A\Big),
\end{aligned}$$

and the vector fields

$$\begin{aligned}
&\mathbb{P}^0_e = -\mathcal{L}^{\omega_0}_\xi e, && \mathbb{P}^0_\omega = -\mathcal{L}^{\omega_0}_\xi (\omega - \omega_0) - \iota_\xi F_{\omega_0},\\
&\mathbb{P}^0_\rho = 0, && \mathbb{P}^0_A = 0,\\
&\mathbb{P}^I_e = 0, && \mathbb{P}^I_\omega = 0,\\
&\mathbb{P}^I_\rho = -\mathcal{L}^{A_0}_\xi \rho, && \mathbb{P}^I_A = -\mathcal{L}^{A_0}_\xi (A - A_0) - \iota_\xi F_{A_0}.
\end{aligned}$$

For H_λ we get

$$\begin{aligned}
\delta H^0_\lambda = \int_\Sigma &\Big(\lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2} e^2 - g(\tau', \omega, e) - d_\omega \tau'\Big)\delta e\\
&+ \big(d_\omega(\lambda e_n e) + \lambda\sigma e - [\tau', e]\big)\delta\omega,
\end{aligned}$$

and

$$\begin{aligned}
\delta H^I_\lambda &= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n \delta(eBF_A) + \frac{\lambda e_n}{2\cdot 3!}\delta\big(e^3(B, B)\big)\Big)\\
&= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n \big(BF_A + \frac{e^2}{4}(B, B)\big)\delta e + \lambda e_n e\delta B F_A - \lambda e_n e B d_A(\delta A) + \frac{\lambda e_n}{3!} e^3 (B, \delta B)\Big)\\
&= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n \big(BF_A + \frac{e^2}{4}(B, B)\big)\delta e + \lambda e_n e\delta B F_A + d_A(\lambda e_n e B)\delta A\\
&\qquad + \frac{\lambda e_n}{2} e(e^2, B)\delta B + \frac{\lambda}{2}(B, e_n e) e^2 \delta B\Big)\\
&= \int_\Sigma \mathrm{Tr}\Big(\big(\lambda e_n (BF_A + \frac{e^2}{4}(B, B)) - \lambda e B(B, e_n e)\big)\delta e\\
&\qquad + d_A(\lambda e_n e B)\delta A + \lambda (B, e_n e)\delta\rho\Big),
\end{aligned}$$

where in the last step we used $F_A + \frac{1}{2}(e^2, B) = 0$ and $\rho = \frac{1}{2}e^2 B$ and in the step before the identity

$$\frac{\lambda e_n}{3!} e^3 (C, D) = \frac{\lambda}{2}(C, e_n e) e^2 D + \frac{\lambda e_n}{2} e(e^2, C) D, \tag{231}$$

which can be recovered from Lemma B.4 for $N = 4$. Then the vector fields read

$$
\begin{aligned}
e\mathbb{H}^0_e &= ed_\omega(\lambda e_n) + \lambda e\sigma - [\tau', e], & e\mathbb{H}^0_\omega &= \lambda e_n F_\omega + \lambda e_n \frac{\Lambda}{2} e^2 - g(\tau', \omega, e) - d_\omega \tau', \\
\mathbb{H}^0_\rho &= 0, & \mathbb{H}^0_A &= 0, \\
\mathbb{H}^I_e &= 0, & e\mathbb{H}^I_\omega &= \mathrm{Tr}\big(\lambda e_n B F_A + \frac{e^2}{4}\lambda e_n (B, B) - \lambda e B(B, e_n e)\big), \\
\mathbb{H}^I_\rho &= d_A(\lambda e_n e B), & \mathbb{H}^I_A &= \lambda(B, e_n e).
\end{aligned}
$$

Finally, for R_τ we get

$$
\delta R_\tau = \int_\Sigma (g(\tau, \omega, e) + d_\omega \tau)\delta e + [\tau, e]\delta\omega, \tag{232}
$$

and the vector fields

$$
\begin{aligned}
e\mathbb{R}_e &= [\tau, e], & e\mathbb{R}_\omega &= g(\tau, \omega, e) + d_\omega \tau, \\
\mathbb{R}_p &= 0, & \mathbb{R}_\phi &= 0.
\end{aligned}
$$

Now we can calculate the Poisson brackets. All brackets expect the ones containing M_μ, P^I_ξ and H^I_λ remain the same as in the scalar case. The following are all identical to the scalar case

$$
\begin{aligned}
\{L_c, L_c\} &= -\frac{1}{2} L_{[c,c]}, \\
\{R_\tau, R_\tau\} &\approx F_{\tau\tau}, \\
\{L_c, R_\tau\} &= -R_{p_{\mathcal{S}}[c,\tau]}
\end{aligned}
$$

and

$$
\{L_c, P_\xi\} = L_{\mathcal{L}^{\omega_0}_\xi c},
$$

where the last one does not change because $\{L_c, P^I_\xi\} = 0$. The constraints L_c, P_ξ and M_μ are identical to the nondegenerate case and their brackets have therefore already been calculated in [11]. We will first consider all brackets which do not contain H_λ. So

$$
\{L_c, M_\mu\} = 0;
$$

$$
\begin{aligned}
\{M_\mu, M_\mu\} &= \int_\Sigma \mathrm{Tr}(d_A \mu [\mu, \rho]) = \int_\Sigma -\mathrm{Tr}([\mu, d_A \mu]\rho) \\
&= \frac{1}{2}\int_\Sigma \mathrm{Tr}(d_A [\mu, \mu]\rho) = -\frac{1}{2}\int_\Sigma ([\mu, \mu] d_A \rho) \\
&= -\frac{1}{2} M_{[\mu,\mu]};
\end{aligned}
$$

$$
\begin{aligned}
\{M_\mu, P_\xi\} &= \int_\Sigma -\mathrm{Tr}\Big([\mu, \rho]\big(\mathcal{L}^{A_0}_\xi (A - A_0) + \iota_\xi F_{A_0}\big) + \mathcal{L}^{A_0}_\xi \rho\, d_A \mu\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}^{A_0}_\xi \mu [A - A_0, \rho] + \mu [A - A_0, \mathcal{L}^{A_0}_\xi \rho] - \mu[\rho, \iota_\xi F_{A_0}] - d_A \mathcal{L}^{A_0}_\xi \rho\, \mu\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}^{A_0}_\xi \mu [A, \rho] - d\mu \iota_\xi d\rho + [\iota_\xi A_0, d\rho]\mu\Big) = \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}^{A_0}_\xi \mu\, d_A \rho\Big) \\
&= M_{\mathcal{L}^{A_0}_\xi \mu};
\end{aligned}
$$

$$\begin{aligned}\{P_\xi, P_\xi\} &= \{P^0_\xi, P^0_\xi\} + 2\{P^0_\xi, P^I_\xi\} + \{P^I_\xi, P^I_\xi\} \\ &= \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi \iota_\xi F_{A_0}},\end{aligned}$$

where we used that $\{P^0_\xi, P^0_\xi\} = \frac{1}{2}P^0_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}}$ is the same computation as in the scalar case and $\{P^0_\xi, P^I_\xi\}$. Furthermore, the calculation of the interacting part simplifies if one notes that with the substitution $\frac{1}{2}e^2 \mapsto \rho$ and $\omega \mapsto A$ it is equivalent to the free part $\{P^0_\xi, P^0_\xi\}$

$$\begin{aligned}\{P^I_\xi, P^I_\xi\} &= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}^{A_0}_\xi \rho\big(\mathcal{L}^{A_0}_\xi (A - A_0) + \iota_\xi F_{A_0}\big)\Big) \\ &= \int_\Sigma \mathrm{Tr}\Big(\frac{1}{2}d_A(\rho)\iota_{[\xi,\xi]}(A - A_0) + \frac{1}{2}\iota_{[\xi,\xi]}(\rho)F_A - \frac{1}{2}d_A(\rho)\iota_\xi \iota_\xi F_{A_0}\Big) \\ &= \frac{1}{2}P^I_{[\xi,\xi]} - \frac{1}{2}M_{\iota_\xi \iota_\xi F_{A_0}}.\end{aligned}$$

Since R_τ does not have an interacting part its bracket with M_μ and P^I_ξ vanishes. Therefore we have

$$\{M_\mu, R_\tau\} = 0$$

and

$$\{P_\xi, R_\tau\} = R_{p_\mathcal{S} \mathcal{L}^{\omega_0}_\xi \tau}.$$

Now we consider the brackets containing H_λ and we will start with $\{L_c, H_\lambda\}$, where the part $\{L_c, H^0_\lambda\}$ is identical to the scalar case and the second term is given by

$$\begin{aligned}\{L_c, H^I_\lambda\} &= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n\Big(\frac{1}{4}e^2(B,B)[c,e] + BF_A[c,e]\Big) + \lambda e B(B, e_n e)[c,e]\Big) \\ &= \int_\Sigma \mathrm{Tr}\Big(- [c, \lambda e_n]\Big(\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\Big) - \lambda e_n e[c,B]F_A \\ &\qquad + \frac{\lambda}{2}e^2(B, e_n e)[c,B]\Big) \\ &= \int_\Sigma \mathrm{Tr}\Big(- [c, \lambda e_n]\Big(\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\Big) - \lambda e_n e F_A[c,B] \\ &\qquad + \frac{\lambda e_n}{2\cdot 3!}e^3[c,(B,B)] - \frac{\lambda e_n}{2}e(e^2,B)[c,B]\Big) \\ &= \int_\Sigma -[c,\lambda e_n]\mathrm{Tr}\Big(\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\Big) \\ &= -P^I_{X^{(a)}} - H^I_{X^{(n)}} + M_{X^{(a)}(A-A_0)_{(a)}},\end{aligned}$$

with $X = [c, \lambda e_n]$. Then the total bracket reads

$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_{(a)}} - H_{X^{(n)}} + R_{p_\mathcal{S}(X^{(a)}e_a(\omega-\omega_0) - \lambda e_n d_{\omega_0}c)} + M_{X^{(a)}(A-A_0)_{(a)}}.$$

Next, we consider the bracket $\{P_\xi, H_\lambda\}$. The free part $\{P^0_\xi, H^0_\lambda\}$ is again identical to the scalar case. The remaining three terms can be calculated together as follows

$$
\begin{aligned}
&\{P_\xi^0, H_\lambda^I\} + \{P_\xi^I, H_\lambda^0\} + \{P_\xi^I, H_\lambda^I\} \\
&= \int_\Sigma -\mathcal{L}_\xi^{\omega_0} e \mathrm{Tr}\Big(\lambda e_n\Big(BF_A + \frac{e^2}{4}(B,B)\Big) - \lambda e B(B, e_n e)\Big) \\
&\qquad - \mathcal{L}_\xi^{A_0}\rho \mathrm{Tr}(\lambda(B, e_n e)) - (\mathcal{L}_\xi^{A_0}(A - A_0) + \iota_\xi F_{A_0})\mathrm{Tr}(d_A(\lambda e_n e B)) \\
&= \int_\Sigma \mathrm{Tr}\Big(- \mathcal{L}_\xi^{\omega_0 + A_0}(e^3)\frac{\lambda e_n}{2\cdot 3!}(B,B) - \mathcal{L}_\xi^{\omega_0+A_0} e\lambda e_n BF_A + \mathcal{L}_\xi^{\omega_0}(e^2)\frac{\lambda}{2}B(B, e_n e) \\
&\qquad - \mathcal{L}_\xi^{A_0}(e^2)\frac{\lambda}{2}B(B, e_n e) - \mathcal{L}_\xi^{A_0} B\frac{\lambda}{2}e^2(B, e_n e) \\
&\qquad - \mathcal{L}_\xi^{\omega_0}(e^2)\frac{\lambda}{2}B(B, e_n e) - \mathcal{L}_\xi^{\omega_0} B\frac{\lambda}{2}e^2(B, e_n e) \\
&\qquad + \lambda e_n e B d_A\big(- \iota_\xi F_A + d_A \iota_\xi(A - A_0)\big)\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}_\xi^{\omega_0}(\lambda e_n)\frac{e^3}{2\cdot 3!}(B,B) + \frac{\lambda e_n}{3!}e^3(B, \mathcal{L}_\xi^{\omega_0+A_0}B) + \mathcal{L}_\xi^{\omega_0}(\lambda e_n)eBF_A \\
&\qquad + \lambda e_n e \mathcal{L}_\xi^{\omega_0+A_0}(BF_A) - \frac{\lambda e_n}{2}(B, e_n e)e^2\mathcal{L}_\xi^{\omega_0+A_0}B \\
&\qquad + \lambda e_n e B\big(- d_A\iota_\xi F_A + [F_A, \iota_\xi(A - A_0)]\big)\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}_\xi^{\omega_0}(\lambda e_n)\frac{e^3}{2\cdot 3!}(B,B) + \frac{\lambda e_n}{3!}e^3(B, \mathcal{L}_\xi^{\omega_0+A_0}B) + \mathcal{L}_\xi^{\omega_0}(\lambda e_n)eBF_A \\
&\qquad + \lambda e_n e F_A \mathcal{L}_\xi^{\omega_0+A_0}B + \lambda e_n e B \mathcal{L}_\xi^{A_0}(F_A) \\
&\qquad - \frac{\lambda e_n}{3!}e^3(B, \mathcal{L}_\xi^{\omega_0+A_0}B) + \lambda e_n e\frac{(e^2, B)}{2}\mathcal{L}_\xi^{\omega_0+A_0}B \\
&\qquad - \lambda e_n e B\big(- d_{A_0}\iota_\xi F_A + \iota_\xi[A - A_0, F_A]\big)\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\mathcal{L}_\xi^{\omega_0}(\lambda e_n)\Big(\frac{e^3}{2\cdot 3!}(B,B) + eBF_A\Big) \\
&\qquad + \lambda e_n e B \mathcal{L}_\xi^{A_0}F_A - \lambda e_n e B(\iota_\xi d_{A_0}F_A - d_{A_0}\iota_\xi F_A)\Big) \\
&= \int_\Sigma \mathcal{L}_\xi^{\omega_0}(\lambda e_n)\mathrm{Tr}\Big(\frac{e^3}{2\cdot 3!}(B,B) + eBF_A\Big) \\
&= P_{\mathcal{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{\mathcal{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}} - M_{\mathcal{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega-\omega_0)_{(a)}}.
\end{aligned}
$$

In the last computation we used several times that the fields e and F_A transform in the trivial representation of $\mathfrak{g}$ and $\mathfrak{so}(N-1,1)$, respectively. This means that $[A_0, e] = 0$ and $[\omega_0, F_A] = 0$ and consequently $\mathcal{L}_\xi^{A_0}e = 0$ and $\mathcal{L}_\xi^{\omega_0}F_A = 0$. Now we calculate the following bracket

$$
\begin{aligned}
\{M_\mu, H_\lambda\} &= \int_\Sigma \mathrm{Tr}\Big([\mu,\rho]\lambda(B, e_n e) + d_A \mu d_A(\lambda e_n e B)\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\frac{\lambda}{2} e^2[\mu, B](B, e_n e) + d(\lambda e_n e B)[A,\mu] \\
&\qquad + [A, \lambda e_n e B] d\mu + [A, \lambda e_n e B][A,\mu]\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(- \lambda e_n e B[dA, \mu] + \lambda e_n e B[A, d\mu] - \lambda e_n e B[A, d\mu] \\
&\qquad + \frac{\lambda e_n}{2} e B[\mu, [A,A]] + \frac{\lambda e_n}{3!} e^3 (B, [\mu, B]) - \frac{\lambda e_n}{2} e(B, e^2)[\mu, B]\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n e B\big([\mu, F_A] + \frac{1}{2}[\mu, (e^2, B)]\big) + \frac{\lambda e_n}{2\cdot 3!} e^3 [\mu, (B,B)]\Big) = 0,
\end{aligned}
$$

where in the last step we used $F_A + \frac{1}{2}(e^2, B) = 0$ and $\mathrm{Tr}[\mu, (B,B)] = 0$. Next, consider $\{H_\lambda, H_\lambda\}$ which splits into three parts which are given by

$$
\{H^0_\lambda, H^0_\lambda\} \approx F_{\tau'\tau'},
$$

$$
\{H^I_\lambda, H^I_\lambda\} = \int_\Sigma \mathrm{Tr}(\lambda(B, e_n e) d_A(\lambda e_n e B)) = \int_\Sigma \mathrm{Tr}(\lambda e B(B, e_n e) d\lambda e_n),
$$

and

$$
\begin{aligned}
\{H^0_\lambda, H^I_\lambda\} &= \int_\Sigma \mathrm{Tr}\bigg(\Big(\lambda e_n B F_A + \lambda e_n \frac{e^2}{4}(B,B) - \lambda e B(B, e_n e)\Big) \\
&\qquad \cdot \Big(d_\omega(\lambda e_n) + \lambda\sigma + W_1^{-1}[\tau', e]\Big)\bigg) \\
&= \int_\Sigma \mathrm{Tr}(-\lambda e B(B, e_n e) d\lambda e_n).
\end{aligned}
$$

Therefore the complete bracket reads

$$
\{H_\lambda, H_\lambda\} = \{H^0_\lambda, H^0_\lambda\} + 2\{H^0_\lambda, H^I_\lambda\} + \{H^I_\lambda, H^I_\lambda\} \approx F_{\tau'\tau'}.
$$

The last remaining bracket is once again $\{H_\lambda, R_\tau\}$ which splits into two parts given by

$$
\{H^0_\lambda, R_\tau\} \approx F_{\tau\tau'} + G_{\lambda\tau}
$$

and

$$
\{H^I_\lambda, R_\tau\} = \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] + \lambda e_n \frac{e}{4}(B,B)[\tau, e] - \lambda B(B, e_n e)[\tau, e]\Big).
$$

The second term is zero because $e[\tau, e] = 0$. The first and third terms instead do not vanish in general:

$$
\begin{aligned}
\{H^I_\lambda, R_\tau\} &= \int_\Sigma \mathrm{Tr}\Big(\lambda e_n B F_A W_1^{-1}[\tau, e] - \lambda B(B, e_n e)[\tau, e]\Big) \\
&= \int_\Sigma \mathrm{Tr}\Big(\frac{\lambda e_n}{2} B(B, e^2) - \lambda B(B, e_n e)e\Big) W_1^{-1}[\tau, e] \\
&= \int_\Sigma \mathrm{Tr}\Big(\lambda B\Big(\frac{e_n}{2}(B, e^2) - (B, e_n e)e\Big)\Big) W_1^{-1}[\tau, e] \\
&= \int_\Sigma \mathrm{Tr}\Big(\lambda(b^{ab} e_a e_b + 2b^{an} e_a e_n)\Big(\frac{e_n}{2}(B, e^2) - (B, e_n e)e\Big)\Big) W_1^{-1}[\tau, e] \\
&=: K^{YM}_{\lambda\tau}.
\end{aligned}
$$

Then the complete bracket reads

$$\{H_\lambda, R_\tau\} \approx F_{\tau\tau'} + G_{\lambda\tau} + K^{YM}_{\lambda\tau}.$$

□

5.3 Conclusion YMPC Theory

As for the SPC case, we have analysed the boundary structure of the YMPC theory in $N = 4$ for the case of a degenerate boundary metric by calculating its Poisson algebra in Theorem 5.6 and we have also stated the results following from a nondegenerate boundary metric in Theorem 5.4. Once again, we will state the consequences of the constrained algebra for the reduced phase space.

Corollary 5.7. *Let g^∂ be nondegenerate. Then the constraints $\{L_c, P_\xi, H_\lambda, M_\mu\}$ in the SPC theory define a first-class system and the submanifold $C \subset F_\partial$ given by the zero locus of the constraints is coisotropic.*

Proof. From Theorem 5.4 we see that the Poisson brackets of the constraints are all given by a linear combination of the constraints themselves, hence vanishing on the zero locus. It therefore defines a system of first-class and by Proposition A.3 C is coisotropic. □

Remark. As in the scalar case, L_c, P_ξ, H_λ are the generators of the internal gauge transformations of $\mathfrak{so}(3,1)$, of the local diffeomorphisms tangential to the boundary and of the local diffeomorphisms normal to the boundary, respectively. The new constraint M_μ is associated to the gauge transformation of the Yang–Mills field with respect to the Lie algebra $\mathfrak{g}$. Indeed, notice the analogy between M_μ and L_c which both generate the respective gauge transformations of the compact Lie group G and $\mathrm{SO}(3,1)$.

Corollary 5.8. *Let g^∂ be degenerate with $\dim(\ker(g^\partial)) = 1$. Then, in the YMPC theory, we have first- and second-class constraints. In particular, $L_c, P_\xi, H_\lambda, M_\mu$ by themselves are first-class constraints.*

Proof. The proof can be copied *mutatis mutandis* from Corollary 4.13, by noting that also $K^{YM}_{\lambda\tau}$ is proportional to λ and does not contain derivatives. □

Once again, the presence of the second-class constraint R_τ implies that the constrained system does not define a coisotropic submanifold of F_∂. The differences between the nondegenerate and degenerate cases in the YMPC theory have been outlined in figure 3.

Yang–Mills Palatini–Cartan

Geometric Phase Space

$$F_\partial = \Omega_\partial^{(1,1)} \times \frac{\mathcal{A}(\Sigma)}{\sim} \times \frac{\Omega_{\partial,\mathfrak{g}}^{(0,2)}}{\sim} \times \Omega_{\partial,\mathfrak{g}}^{(1,0)}$$

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{2}\mathrm{Tr}(\delta(e^2[B])\delta A)$$

Equations of motion

$$(1)\quad d_\omega e = 0$$

$$(2)\quad eF_\omega + \frac{\Lambda}{6}e^3 + \mathrm{Tr}\Big(eBF_A + \frac{1}{2\cdot 3!}e^3(B,B)\Big) = 0$$

$$(3)\quad d_A(e^2B) = 0$$

$$(4)\quad F_A + \frac{1}{2}(e^2, B) = 0$$

Nondegenerate case	Degenerate case
(1) $\rightsquigarrow$ Fixes $[\omega] + L_c$	(1) $+ p_\mathcal{K} v = 0 \rightsquigarrow$ Fix $[\omega] + L_c, R_\tau$
(2) $\rightsquigarrow P_\xi, H_\lambda$	(2) $\rightsquigarrow P_\xi, H_\lambda$
(3) $\rightsquigarrow M_\mu$	(3) $\rightsquigarrow M_\mu$
(4) $\rightsquigarrow$ Fixes $[B]$	(4) $+ p_{\mathcal{W}'} B = 0 + \iota_\xi F_A = 0 \rightsquigarrow$ Fix $[B]$
$L_c, P_\xi, H_\lambda, M_\mu$ are of first-class and $\{L_c, P_\xi, H_\lambda, M_\mu\}$ defines a coisotropic submanifold	$L_c, P_\xi, H_\lambda, M_\mu$ are of first-class, R_τ is of second-class and $\{L_c, R_\tau, P_\xi, H_\lambda, M_\mu\}$ does not define a coisotropic submanifold

Figure 3: Summary of the boundary structure of the YMPC theory in $N = 4$ dimensions and the differences between the nondegenerate and degenerate case.

A First- and Second-Class Constraints

We will give a quick overview of the concept of first- and second-class constraints and their connection to symplectic reduction. In simple (and not totally correct) words, a constraint is of first-class if its Poisson bracket with all other constraints vanishes on the constrained surface, and of second-class if it does not vanish. They have also different physical interpretations. The first-class constraints are in one to one correspondence with the generators of gauge transformations. Take for example a first-class constraint ϕ, then it generates the following gauge transformation

$$\delta_\varepsilon = \varepsilon\{\phi, \cdot\}. \tag{233}$$

Intuitively, this means that the constraint flows all commute with each other on the constrained submanifold or in other words, if we start at one point on the constrained subspace, all the constrained flows bring us to another point on the submanifold. If all constraints of a system are of first-class, one can canonically quantize the theory by promoting the Poisson bracket to a commutator. On the other hand, the second-class constraints are just identities through which we can express some canonical variables in terms of the others and they are not related to gauge transformations. Indeed, a similar transformation as above, would move a point on the constrained submanifold out of it. One can still try to canonically quantize the system by replacing the Poisson bracket by the so called Dirac bracket $\{\cdot,\cdot\}_D$ [9]. Furthermore, from a geometric point of view, a constrained system of first-class defines a coisotropic submanifold C and its leaf space naturally inherits a symplectic structure. Indeed, even though the restriction of the symplectic form ω onto C is degenerate, its kernel, called the characteristic distribution, is spanned by the Hamiltonian vector fields X_i of the constraints ϕ_i. The fact that the system is of first-class then implies that the characteristic distribution is involutive and hence, the symplectic reduction $\underline{C}$, which is the quotient of C by the characteristic distribution, is endowed with a unique symplectic form $\underline{\omega}$. If only second-class constraints are present, the constrained submanifold is symplectic and no reduction is needed. If both first- and second-class constraints are present, then the submanifold is presymplectic.

Definition A.1 (First- and Second-Class)**.** Let M be a symplectic manifold and let $\{\phi_i \in C^\infty(M)\}_{i=1,\dots,n}$ be a set of smooth functions on it. Denote with $C_{ij} = \{\phi_i, \phi_j\}$ the matrix of the Poisson brackets of the functions. Then the number s of **second-class** functions of the set is the rank (assumed to be constant on the zero locus) of the matrix C_{ij} on the zero locus of the functions. The number of **first-class** functions is then $f = n - s$. In particular if $C_{ij} \approx 0\,\forall i, j = 1, \dots, n$ then we say that the system is of first-class.

Remark. This definition stands in agreement with the usual definition of first- and second-class constraints but it allows for a more general treatment, since it is invariant under rearranging the constraints by linear combinations. In particular, if we have a set of constraints $\phi_i = 0$ for $i = 1, \dots, n$ and we can write all the Poisson brackets as linear combinations

$$\{\phi_i, \phi_j\} = \sum_k f_{ij}^k \phi_k, \tag{234}$$

then all Poisson brackets vanish on the constrained subspace and the system is of first-class.

Proposition A.1. *Let M be a symplectic manifold and let $\{\psi_i \in C^\infty(M)\}_{i=1,\dots,n}$ and $\{\phi_j \in C^\infty(M)\}_{j=1,\dots,m}$ be two sets of smooth functions. Denote with $C_{jj'}, B_{ij}, D_{ii'}$ respectively the matrices representing the Poisson brackets $\{\phi_j, \phi_{j'}\}, \{\psi_i, \phi_j\}$ and $\{\psi_i, \psi_{i'}\}$ with $i, i' = 1, \dots, n$ and $j, j' = 1, \dots, m$. Then, if D is invertible and $C = -B^T D^{-1} B$, the number of second-class constraints is n, i.e. the rank of the matrix D.*

Lemma A.2. *Let D be an invertible matrix such that D^{-1} does not contain any derivatives and let B be some matrix proportional to the odd parameter λ such that it does not contain derivatives. Then $B^T D^{-1} B = 0$.*

Proof. Both proofs can be found in [5]. □

Definition A.2 (Coisotropic Submanifold)**.** Let (M, ω) be a symplectic manifold and let $N \subset M$ be a submanifold. Then N is said to be **coisotropic** if $T_pN^\perp \subset T_pN$ for all $p \in N$, where $T_pN^\perp := \{v \in T_pM \mid \omega(v, w) = 0 \,\forall w \in T_pN\}$. Similarly, N is said to be **isotropic** if $T_pN \subset T_pN^\perp$ and **Lagrangian** if $T_pN^\perp = T_pN$.

Proposition A.3. *Let M be a symplectic manifold and let $C \subset M$ be a submanifold defined by the zero locus of a set of smooth functions $\{\phi_i \in C^\infty(M)\}_{i=1,\dots,n}$*

$$C := \{x \in M \mid \phi_i(x) = 0 \,\forall i\}. \tag{235}$$

Then C is a coisotropic submanifold of M if and only if the constraints define a system of first-class.

Proof. See [2]. □

Theorem A.4 (Coisotropic reduction)**.** *Let (M, ω) be a symplectic manifold and $i \colon C \hookrightarrow M$ a coisotropic submanifold. Then the space of leaves $\underline{C}$ of the characteristic foliation inherits a unique symplectic form $\underline{\omega}$ such that $\pi^*(\underline{\omega}) = i^*(\omega)$.*

Proof. See [11]. □

This quotient space $\underline{C}$ from Theorem A.4 might not be smooth. A better alternative is therefore the BFV formalism which describes the quotient space by a cohomological resolution. In particular, one extends the original symplectic manifold to a graded supersymplectic manifold and introduces the odd BFV action S on it, such that the classical master equation (CMS) holds[35]

$$\{S, S\} = 0. \tag{236}$$

The Hamiltonian vector field Q of S is an odd operator defined by $Q = \{S, \cdot\}$. One can then show, that the space of smooth functions on the reduced phase space is isomorphic its the degree zero cohomology

$$H^0(\mathcal{C}) \simeq C^\infty(\underline{C}), \tag{237}$$

where $\mathcal{C}$ is the BFV complex associated to Q.

[35] The odd degree comes from the new variables c^i and $c_i^\dagger$, called ghosts and antighost. Both of them are odd with a ghost degree of +1 and -1, respectively. The BFV action is then $S = \int_\Sigma c^i \phi_i + \frac{1}{2} f_{ij}^k c_k^\dagger c^i c^j + R$, where R assures that S satisfies the CMS.

B Important Lemmas

Recall definitions 3.7 and 3.10 where we defined the (i,j)-forms

$$W_k^{(i,j)} \colon \Omega^{(i,j)}(M) \to \Omega^{(i+k,j+k)}(M)$$
$$\alpha \mapsto e^k \wedge \alpha$$

and the boundary forms

$$W_k^{\partial(i,j)} \colon \Omega_\partial^{(i,j)} \to \Omega_\partial^{(i+k,j+k)}$$
$$\alpha \mapsto e^k \wedge \alpha.$$

Lemma B.1. *The following propositions are true*

1. $W_{N-2}^{(2,0)}$ *is injective,*
2. $W_{N-1}^{(1,0)}$ *is injective.*

Proof. See [5]. □

Lemma B.2. *Let $\omega \in \Omega^{(2,0)}$ and $B \in \Omega^{(0,2)}$, then*

$$e^{N-2}\omega B = -\frac{2(N-2)!}{N!} e^N \omega_{\mu\nu} B^{\mu\nu}. \tag{238}$$

Proof. See [5]. □

Lemma B.3. *The following propositions are true for $N \geq 4$*

1. $W_{N-3}^{\partial(2,1)}$ *is surjective,*
2. $W_{N-3}^{\partial(1,1)}$ *is injective,*
3. $W_{N-3}^{\partial(1,2)}$ *is injective,*
4. $W_{N-2}^{\partial(1,0)}$ *is injective,*
5. $W_k^{\partial(0,0)}$ *is injective for all k.*

Proof. All proofs can be found in [4], [8] and [3]. □

The bracket $[\cdot,\cdot] \colon \Omega^{(1,1)} \times \Omega^{(1,1)} \to \Omega^{(2,0)}$ of two $(1,1)$-forms $X, Y \in \Omega^{(1,1)}$ is defined by

$$[X,Y]_{\mu\nu} = X^a_{[\mu} Y^b_{\nu]} g_{ab}. \tag{239}$$

Then the bracket of the vielbein with itself vanishes

$$[e,e]_{\mu\nu} = \frac{1}{2}(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu) g_{ab} = 0, \tag{240}$$

since g is symmetric.

The bracket $[\cdot,\cdot] \colon \Omega^{(1,2)} \times \Omega^{(1,2)} \to \Omega^{(2,2)}$ of two $(1,2)$-forms $X, Y \in \Omega^{(1,2)}$ is defined by

$$[X,Y]^{ab}_{\mu\nu} = X^{[ac}_{[\mu} Y^{b]d}_{\nu]} g_{cd}. \tag{241}$$

Then the following holds

- The bracket of the connection field with itself does not vanish in general

$$[\omega,\omega]^{ab}_{\mu\nu} = \frac{1}{4}\Big(\omega^{ac}_{\mu}\omega^{bd}_{\nu} - \omega^{bc}_{\mu}\omega^{ad}_{\nu} - \omega^{ac}_{\nu}\omega^{bd}_{\mu} + \omega^{bc}_{\nu}\omega^{ad}_{\mu}\Big) g_{cd} \tag{242}$$

- The bracket of two connection fields is symmetric under exchange

$$\begin{aligned} [\omega,\omega_0]^{ab}_{\mu\nu} &= \frac{1}{4}\Big(\omega^{ac}_{\mu}\omega^{bd}_{0,\nu} - \omega^{bc}_{\mu}\omega^{ad}_{0,\nu} - \omega^{ac}_{\nu}\omega^{bd}_{0,\mu} + \omega^{bc}_{\nu}\omega^{ad}_{0,\mu}\Big) g_{cd} \\ &= \frac{1}{4}\Big(\omega^{ad}_{0,\mu}\omega^{bc}_{\nu} - \omega^{bd}_{0,\mu}\omega^{ac}_{\nu} - \omega^{ad}_{0,\nu}\omega^{bc}_{\mu} + \omega^{bd}_{0,\nu}\omega^{ac}_{\mu}\Big) g_{cd} = [\omega_0,\omega]^{ab}_{\mu\nu}. \end{aligned} \tag{243}$$

Lemma B.4. *Let $A, B \in \Omega^{(0,1)}_{\partial}$ and $C, D \in \Omega^{(0,2)}_{\partial}$. Then the following identities hold*

1. $e_n \frac{e^{N-1}}{(N-1)!}(A,B) = (-1)^{|A|+|B|}\Big[e_n \frac{e^{N-2}}{(N-2)!}(e,A)B + \frac{e^{N-1}}{(N-1)!}(e_n,A)B\Big]$,
2. $e_n \frac{e^{N-1}}{(N-1)!}(C,D) = \Big[e_n \frac{e^{N-3}}{2(N-3)!}(e^2,C)D + \frac{e^{N-2}}{(N-2)!}(e_n e,C)D\Big]$.

Proof. See [5]. □

Lemma B.5. *Let $\tau \in \mathcal{S}$. Then the following identities hold*

1. $e_n \wedge \tau = 0$,
2. $e_n[\tau, e] = \tau[e_n, e]$,
3. $e[\tau, e] = 0$.

Proof. The first identity has been shown in [5]. The second follows trivially from $[\tau e_n, e] = \tau[e_n, e] - e_n[\tau, e] = 0$. For the last identity note that by definition of $\mathcal{S}$ $\tau \in \ker(W_1^{\partial(1,3)})$, so $e\tau = 0$. Hence also $[e\tau, e] = e[\tau, e] + \tau[e, e] = 0$ and using $[e, e] = 0$ the statement follows. □

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