Fractional Quantum Hall Effect and BV-BFV Formalism
DRAFT

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Master of Science in Applied Mathematics (ETH)

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An attempt is made to derive the boundary action governing the fractional quantum Hall effect in the spirit of the BV-BFV formalism. The necessary prerequisites for this approach are presented. A graphical calculus for dealing with signs in graded algebra, based on string diagrams, is sketched out. The standard formulas of QFT revolving around Wick's lemma are proven in the context of graded vector spaces. Chern-Simons theory is constructed as an AKSZ model, and its canonical quantization on the boundary is related to a boundary action. The fundamental expressions for the currents, bulk and boundary actions that govern the fractional quantum Hall effect in the effective field theory approach are derived.

Contents

List of Symbols

$\mathcal{F}^\partial_N$ Space of boundary fields on N.
$\mathcal{F}_M$ Space of bulk fields on M.
$\mathcal{Q}$ Cohomological vector field.
$⨍$ Formal integration using Wick's theorem.
$*$ Hodge star operator.
$\langle \cdot, \cdot \rangle$ Hodge inner product, $\langle \alpha, \beta \rangle = \int \alpha \wedge *\beta$.
$\{\cdot, \cdot\}$ Poisson bracket.
Σ Hall sample, domain in $\mathbb{R}^2$.
ω Symplectic form, typically the BV symplectic form.
δ Field variation, i.e. de Rham differential on $\mathcal{F}$ or $\mathcal{F}^\partial$.
d^* de Rham codifferential, $\mathrm{d}^* = *^{-1}\,\mathrm{d}\,*\,(-1)^p$.

Introduction

> This situation is not an uncommon one in mathematical physics: The mathematically precise quantities are not equal to experimental quantities but are related to them by some nonrigorous argument. There is thus an important element of extramathematical taste in certain problems of mathematical physics.
>
> Reed and Simon, Volume IV

> He that bloweth not his own trumpet, his trumpet shall not be blown.
>
> J. E. Littlewood

0.1. This thesis records the results of an effort to understand some of the literature on the theoretical physics of the fractional quantum Hall effect, in particular the work of Jürg Fröhlich and collaborators. An attempt was made to explore whether the BV-BFV formalism of Cattaneo, Mnev and Reshetikhin provides the right mathematical framework for the description of the FQHE via effective field theory and Chern-Simons theory. The central question was whether the various different edge actions which are used in the physics literature could be derived as effective edge actions directly from the bulk (Chern-Simons) action.

The physicist reading this may reasonably object that such a proposal is not plausible. The bulk Chern-Simons action is topological, whereas to write down the edge action one needs to introduce new structures, for instance a metric or a complex structure on the boundary. These new structures are not determined by the bulk action, but must be input by hand. For instance, the metric on the boundary encodes the propagation velocity of the edge modes, which depends on microscopic physics not accessible to the effective field theory approach. However, even in the BV-BFV formalism, where one principle is that the bulk data should directly generate as much of the boundary data as possible, there are some choices that must be made in computing the effective edge action.

One of these choices is the polarization on the space of boundary fields. The natural point of attack is therefore to see if by a choice of polarization (using for instance a boundary metric) one can generate the correct boundary action from the bulk data.

In this attempt, although there I cannot present any complete results, there are some encouraging signs, for instance the computation in Section 4.3 of Chapter 4. This almost gives the correct boundary action; the term $\int_M a \wedge *a = \int_M a_+ \wedge a_-$ is still missing. However, it can harmlessly be added to the action at any point since it does not depend on any fields that are integrated over. Another problem is that imposing the crucial constraint $\mathrm{d}\pi_M(B) = 0$ by inserting a delta functional is somewhat awkward. Perhaps a more elegant description can be found by more systematically treating the "fat edge" (Section 3 of Chapter 4) differently from the bulk.

0.2. Apart from this general theme, the material in this thesis is somewhat haphazard, and perhaps suffers from a degree of superficiality. Much of the material is, in a sense, well-trodden, especially the recapitulation of classical and quantum field theory. However, I hope that the presentation is in some ways idiosyncratic and despite its fundamental familiarity "presents some features of interest". I might point out here that although my proofs of Wick's lemma and related results are completely in the style of the standard physics computations, they are as far as possible expressed in a way which places graded and ungraded vector spaces on equal footing, which leads to some departures from the standard presentation.

One tool that I employ in this aim is the use of *string diagrams*. Here, too, there is nothing fundamentally new. However, I believe that string diagrams have never quite been employed as in this thesis, to work out signs in graded algebra.

After searching for a long time for the best way to typeset string diagrams in LaTeX, I eventually settled on Aleks Kissinger's Ti*k*ZiT [**46**].

Acknowledgements. It is a pleasure to thank Alberto Cattaneo and Konstantin Wernli for suggesting the general theme of this thesis and for guidance in completing this thesis. I wish to especially thank Konstantin Wernli for infinite patience with my many slow-witted and dull questions. I am also grateful to Jürg Fröhlich for useful and critical comments. Finally, I wish to thank my family for supporting me in my studies and in all my life.

CHAPTER 1

Preliminaries on Graded Vector Spaces.

The central constructions of the BV-BFV formalism involve the notion of a *graded vector space*, a refinement of the notion of *super vector space*. The algebra of functions on a graded vector space contains not only functions of commuting variables, as in the ungraded case, but also *anticommuting variables* satisfying $\theta_i\theta_j = -\theta_j\theta_i$. Such variables are familiar to physicists for their use in describing fermionic fields in the path integral formalism.

We shall need to formulate Wick's lemma in this context, which requires us to define the notions of multilinear algebra on graded vector spaces, which we do in some detail. An excellent account of $\mathbb{Z}_2$-graded vector spaces (super vector spaces) and the signs involved can be found in [**29**]. A lucid introduction can also be found in [**49**] and in [**28**]. A treatment of super algebra and super geometry from a categorical perspective is [**65**]. The reader may also find the text [**74**] useful.

1. Basic definitions.

1.1. In the following, all vector spaces are finite-dimensional, and $\mathbb{K}$ is a field of characteristic 0. Unless mentioned otherwise, all vector spaces will be over $\mathbb{K}$, and accordingly we will generally suppress the field.

Definition 1.1. Let A be an abelian group and C a category. We turn A into a category (the *discrete category* on A) A by taking as objects the elements of A and morphisms

$$\mathrm{Hom}(m,n) = \begin{cases} \emptyset, & m \neq n \\ \mathrm{id}_m, & m = n \end{cases} . \tag{1.1.1}$$

Then the category C^{gr} of *graded objects in* C is the functor category C^{A}, whose objects are functors $F : \mathsf{A} \to \mathsf{C}$ and whose morphisms are natural transformations between functors.

Concretely, a graded object $F : \mathsf{A} \to \mathsf{C}$ is a family $\{F_k = F(k)\}_{k\in A}$ of objects $F_k \in \mathrm{obj}(\mathsf{C})$ indexed by A. Let us write such families using a notation reminiscent of that for chain complexes: when the group A is understood, we write $F_\bullet$ for $\{F_k\}_{k\in A}$.

A morphism between graded objects $F_\bullet, G_\bullet$ is a natural transformation between the two functors, hence a family of morphisms $\eta_\bullet$, with $\eta_m \in \mathrm{Hom}_{\mathsf{C}}(F(m), G(m))$. Of course, for $\eta_\bullet$ to be a natural transformation, these morphisms must be such that the diagram

$$\begin{array}{ccc} F(m) & \xrightarrow{F(\phi)} & F(n) \\ \big\downarrow{\scriptstyle \eta_m} & & \big\downarrow{\scriptstyle \eta_n} \\ G(m) & \xrightarrow{G(\phi)} & G(n) \end{array} \tag{1.1.2}$$

commutes for every morphism $\phi \in \mathrm{Hom}_{\mathsf{A}}(m,n)$. But this condition is vacuous for functors $F : \mathsf{A} \to \mathsf{C}$.

DEFINITION 1.2. Let Vect be the category of vector spaces with linear maps, A an abelian group. An A-*graded vector space* is an object of the category Vect^A. The *total space* of a graded vector space $V_\bullet$ is the vector space

$$V_\oplus = \bigoplus_{k\in\mathbb{Z}} V_k. \tag{1.1.3}$$

DEFINITION 1.3. An A-graded vector space $V_\bullet$ is *locally finite* if V_k is finite for all k. A graded vector space is *finite-dimensional* if its total space is finite-dimensional.

REMARKS 1.4.

i) If no group is specified, we take the phrase *graded vector space* to mean $\mathbb{Z}$-*graded vector space.*
ii) Typically, $\mathbb{Z}_2$-graded vector spaces are referred to as *super vector spaces.*
iii) A graded vector space $V_\bullet$ is said to be *concentrated* in degree k if $V_m = 0$ for $m \neq k$. Then an ungraded vector space is simply a graded vector space concentrated in degree 0.

DEFINITION 1.5. An element v of an A-graded vector space $V_\bullet$ is called *homogeneous* if $v \in V_i$ for some $i \in A$. In that case, the *degree* of v is $|v| = i$.

The following definition is a very useful generalization of morphisms between graded vector spaces.

DEFINITION 1.6. Let $V_\bullet, W_\bullet$ be A-graded vector spaces. For $k \in A$, a k-morphism from $V_\bullet$ to $W_\bullet$ is a family $f_\bullet$ of linear maps, $f_m : V_m \to V_{m+k}$. That is, f shifts the grading by k.

DEFINITION 1.7. The graded vector space $\hom(V, W)$, with $\hom(V, W)_k$ the set of k-morphisms from V to W is called the *space of maps* or *inner hom* between V and W. The space of morphisms is the 0-part of the maps and is denoted $\mathrm{Hom}(V, W) = \hom(V, W)_0$.

DEFINITION 1.8. The graded vector space $V^\vee := \hom(V, \mathbb{K})$ is the *dual space* of V.

REMARKS 1.9.

i) Let V be a graded vector space, and let $(V_k)^* := \mathrm{Hom}_{\mathsf{finVect}}(V_k, \mathbb{K})$ be the dual of V_k as an ungraded vector space. Then $(V^\vee)_k = (V_{-k})^*$.
ii) For graded vector spaces V, W we have an isomorphism

$$\begin{aligned} W \otimes V^\vee &\cong \hom(V, W) \\ w \otimes l &\mapsto (v \mapsto wl(v)) \end{aligned} \tag{1.1.4}$$

EXAMPLES 1.10.

i) Any chain complex is a $\mathbb{Z}$-graded vector space.
ii) Similarly, the homology of a chain complex is a $\mathbb{Z}$-graded vector space.
iii) For any vector space V, the Grassmann algebra $\Lambda^\bullet(V)$ is a $\mathbb{Z}$-graded vector space.

REMARK 1.11. There is an obvious functor from $\mathbb{Z}$-graded vector spaces to super vector spaces. Denoting by $\overline{0}, \overline{1}$ the equivalence classes of 0 and 1 in $\mathbb{Z}_2$, we simply put

$$V_{\overline{0}} = \bigoplus_{i \in \mathbb{Z} \text{ even}} V_i \tag{1.1.5}$$

$$V_{\overline{1}} = \bigoplus_{i \in \mathbb{Z} \text{ odd}} V_i. \tag{1.1.6}$$

DEFINITION 1.12. Let V be a graded vector space. The *dual space* is defined to be $V^\vee := \hom(V, \mathbb{K})$.

DEFINITION 1.13. For V, W graded vector spaces, their tensor product and direct sum are defined as

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j \tag{1.1.7}$$

$$(V \oplus W)_k = V_k \oplus W_k. \tag{1.1.8}$$

DEFINITION 1.14. Let $V_\bullet$ be a $\mathbb{Z}$-graded vector space. Then there is a "shifted" graded vector space $s^k(V)_\bullet$ with $s^k(V)_m = V_{m-k}$. We $[p] := s^p(\mathbb{K})$, with $\mathbb{K}$ considered as a graded vector space concentrated in degree 0. The 1-morphism s is called the *suspension map*, $s(V)$ is the *suspension of* V, s^{-1} is the *desuspension map* and $s^{-1}(V)$ is the *desuspension of* V.

The construction in the previous definition is perhaps somewhat awkward, since it involves manipulating "internals" of an object $V_\bullet$. In a more categorical fashion, one can construct vector spaces isomorphic to the shifted vector space by tensor products. Indeed, consider the tensor product $[p] \otimes V$. In components, it is

$$([p] \otimes V)_m = \mathbb{K} \otimes V_{m-p} \cong V_{m-p} \tag{1.1.9}$$

so that it is isomorphic to $s^p(V)$ via the isomorphism $\mathbb{K} \otimes U \cong U$ for ungraded $\mathbb{K}$-vector spaces U. We therefore make the following definition.

DEFINITION 1.15. Let V be a graded vector space. Then we define

$$[p]V := [p] \otimes V \tag{1.1.10}$$

$$V[p] := V \otimes [p]. \tag{1.1.11}$$

We also identify the suspension map s with the map $V \mapsto [p]V$.

REMARK 1.16. In the literature, only the notation $V[p]$ for the shifted vector space is common, though whether the tensor product is applied from the left or the right varies. One exception is [28], where the shift is written from the left. Here we keep the distinction and use both notations, though we favour the left-sided shift by its identification with the suspension map.

So far, what we have seen of graded vector spaces is quite unremarkable. The differences to plain vector spaces only appear once we multiply elements of graded vector spaces, that is, once we consider *algebras*. In this case, we will always enforce the *Koszul sign rule*: for homogeneous elements a, b, we want to impose

$$ab = (-1)^{|a| \cdot |b|} ba. \tag{1.1.12}$$

The first example of this rule is the *braiding* of the tensor product of graded vector spaces.

DEFINITION 1.17. The isomorphism $c^+_{V,W} : V \otimes W \to W \otimes V$ defined on homogeneous elements by

$$c^+_{V,W}(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v \tag{1.1.13}$$

is called the *braiding* on graded vector spaces. The map $c^-_{V,W} = -c^+_{V,W}$ we will call the *anti-braiding*. Note that

$$c^+_{V,W} \circ c^+_{W,V} = c^-_{V,W} \circ c^-_{W,V} = \mathrm{id}\,. \tag{1.1.14}$$

We say that the braiding is *symmetric*.

2. The category of graded vector spaces.

2.1. Let us denote

i) by $\mathsf{finVect}_{\mathbb{K}}$ the category of (ungraded) finite-dimensional $\mathbb{K}$-vector spaces with linear maps as morphisms.
ii) by $\mathsf{grFinVect}_{\mathbb{K}}$ the category of finite-dimensional graded $\mathbb{K}$-vector spaces with degree-preserving linear maps as morphisms.

From basic linear algebra it follows that both of these categories are *symmetric monoidal categories*. We will not give the precise definition (which is somewhat lengthy) here, but describe only the rough idea. Monoidal categories are equipped with a *tensor product* which is associative and a unit object I such that $I \otimes V \cong V$ for any object V. In the present case, the unit object is the field $\mathbb{K}$. A symmetric monoidal category is additionally equipped with an isomorphism (the braiding) $V \otimes W \cong W \otimes V$ which is its own inverse.

2.2. In Section 1 we defined the *inner hom* of graded vector spaces. This is one example of a general notion of *inner hom* in a symmetric monoidal category. Note that the set of morphisms $\mathrm{Hom}(X, Y)$ is a functor

$$\mathrm{Hom}(\cdot,\cdot) : \mathsf{C}^{\mathrm{op}} \otimes \mathsf{C} \to \mathsf{Set}. \tag{1.2.1}$$

In some cases, we may want to treat the sets $\mathrm{Hom}(X, Y)$ as being objects of the category C. In the variational calculus, one often wants to think of spaces of functions between manifolds as forming an infinite-dimensional manifold. When this is not possible, it may still be possible to construct an auxiliary object $\mathrm{hom}(X, Y)$ which in some ways behaves like a set of morphisms, but is also an object of the category C.

In $\mathsf{finVect}$, the morphisms between objects are again a vector space. In $\mathsf{grFinVect}$, the morphisms also form a vector space, and therefore trivially a graded vector space. But we will never obtain a non-trivially graded vector space, and therefore it is necessary to consider an inner hom construction.

DEFINITION 2.1. A *left inner hom* in a symmetric monoidal category $(\mathsf{C}, \otimes)$ is a functor

$$\mathrm{hom}(\cdot,\cdot) : \mathsf{C}^{\mathrm{op}} \otimes \mathsf{C} \to \mathsf{C} \tag{1.2.2}$$

together with a natural isomorphism c, called *currying*

$$c_{X,Y,Z} : \mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X, \mathrm{hom}(Y, Z)). \tag{1.2.3}$$

A *right inner hom* is a functor as above, with a *currying* isomorphism

$$c_{X,Y,Z} : \mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(Y, \mathrm{hom}(X, Z)). \tag{1.2.4}$$

If a left (right) inner hom exists, we call the category *left (right) closed*.

In a symmetric monoidal category, existence of a left inner hom is equivalent to the existence of a right inner hom – we just precompose with the braiding. However, if the braiding is non-trivial, it is still useful to keep the two notions distinct.

PROPOSITION 2.2. The inner hom in Definition 1.7 satisfies Equation 1.2.3.

PROOF. We abbreviate $\mathsf{finVect}$ and $\mathsf{grFinVect}$ by Vect and grVect only for this computation.

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{grVect}}(X \otimes Y, Z) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Vect}}((X \otimes Y)_k, Z_k) \\
&= \bigoplus_{\substack{i,j,k \in \mathbb{Z} \\ i+j=k}} \mathrm{Hom}_{\mathsf{Vect}}(X_i \otimes Y_j, Z_k) = \bigoplus_{\substack{i,j,k \in \mathbb{Z} \\ i=k-j}} \mathrm{Hom}_{\mathsf{Vect}}(X_i, \mathrm{Hom}_{\mathsf{Vect}}(Y_j, Z_k)) \\
&= \bigoplus_{\substack{i,j,k \in \mathbb{Z} \\ i=k-j}} \mathrm{Hom}_{\mathsf{Vect}}\left(X_i, \mathrm{hom}_{\mathsf{grVect}}(Y, Z)_{k-j}\right) \\
&= \mathrm{Hom}_{\mathsf{grVect}}\left(X, \mathrm{hom}_{\mathsf{grVect}}(Y, Z)\right)
\end{aligned}$$

■

REMARK 2.3. On homogeneous elements

$$x_i \in X_i, y_j \in Y_j, f_k \in \mathrm{Hom}_{\mathsf{Vect}}((X \otimes Y)_k, Z_k),\ i + j = k,$$

we can write the currying isomorphism as

$$c(f_k)(x_i)(y_j) = f_k(x_i \otimes y_j) \tag{1.2.5}$$

and $c(f_k)(x_i)$ is indeed in $\mathrm{hom}(Y, Z)_i$ since Y_j is mapped into Z_k and $k - j = i$.

We can phrase the definition of inner hom in terms of adjoint functors. Denote by $R_Y = Y \otimes (-)$ the functor $R_Y(X) = Y \otimes X$ and similarly $L_Y(X) = X \otimes Y$. Then a functor $\mathrm{hom}(-, -)$ is a left inner hom if for any object Y, the functor $\mathrm{hom}(Y, -)$ is a right adjoint to L_Y. Similarly, $\mathrm{hom}(-, -)$ is a right inner hom if $\mathrm{hom}(Y, -)$ is a right adjoint to R_Y for any object Y.

This definition of the inner hom means that in a closed symmetric category, we have two units and two counits corresponding to the left/right tensor product–inner-hom adjunctions. But the two units are just related by precomposition with the braiding, as are the two counits, so it suffices to give just one version of each. The counit of this adjoint pair is the evaluation map

$$\mathrm{eval}_{X,Y} : \mathrm{hom}(X, Y) \otimes X \to Y. \tag{1.2.6}$$

The unit is a map

$$\eta_{X,Y} : Y \to \mathrm{hom}(X, Y \otimes X) \tag{1.2.7}$$

PROPOSITION 2.4. In $\mathsf{grFinVect}$, the evaluation map is given on homogeneous elements by

$$\mathrm{eval}(f_i, x_j) = f_i(x_j), \qquad f_i \in \mathrm{hom}(X, Y)_i, v_j \in X_j \tag{1.2.8}$$

and so is "really" an evaluation of morphisms on elements. The unit on homogeneous elements is

$$\eta(y_i)(x_j) = y_i \otimes x_j. \tag{1.2.9}$$

PROOF. Using Remark 2.3, we can compute on homogeneous elements

$$c(\mathrm{eval})(f_i)(x_j) = \mathrm{eval}(f_i \otimes x_j) = f_i(x_j) \tag{1.2.10}$$

which shows that

$$c(\mathrm{eval}) = \mathrm{id}_{\hom(X,Y)}\,. \tag{1.2.11}$$

∎

2.3. The notion of inner hom is closely related to that of a *dual object*.

DEFINITION 2.5. Let $(\mathsf{C}, \otimes, I)$ be a monoidal category. An *exact pairing* of a pair (A, B) of objects is given by morphisms

$$\eta : I \to B \otimes A \tag{1.2.12}$$

$$\epsilon : A \otimes B \to I. \tag{1.2.13}$$

such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X \otimes \eta"] \arrow[dr, "\mathrm{id}_X"'] & X \otimes Y \otimes X \arrow[d, "\epsilon \otimes \mathrm{id}_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta \otimes \mathrm{id}_Y"] \arrow[dr, "\mathrm{id}_Y"'] & Y \otimes X \otimes Y \arrow[d, "\mathrm{id}_Y \otimes \epsilon"] \\ & Y \end{tikzcd} \,. \tag{1.2.14}$$

If there is an exact pairing for (A, B) then A is called a *left dual* of B, and B is called a *right dual* of A. If every object in C has both a left and a right dual, C is called *compact*.

REMARKS 2.6.

i) If a left (right) dual object exists, it is unique up to unique isomorphism, and hence one can speak of *the* left (right) dual object.
ii) A compact category is automatically closed. The inner hom can be defined as

$$\hom(X, Y) = X^\vee \otimes Y \tag{1.2.15}$$

$$\hom(X, Y) = Y \otimes X^\vee \tag{1.2.16}$$

iii) In a symmetric monoidal category, we can pass between left and right duals via the braiding (just as for left and right inner hom).

PROPOSITION 2.7. Let V be a finite-dimensional graded vector space. Then the dual space $V^\vee = \hom(V, \mathbb{K})$ is a left dual of V in the sense of Definition 2.5.

2.4. With the previous definitions, we can summarize the most important properties of (graded) vector spaces in the following proposition.

PROPOSITION 2.8. The categories $\mathsf{finVect}$ and $\mathsf{grFinVect}$ enjoy the following properties:

i) The space of maps or inner hom which we previously introduced is indeed right adjoint to the tensor product functor, i.e. there are natural isomorphisms

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \hom(V, W)) \tag{1.2.17}$$

for U, V, W (graded) vector spaces.
ii) The dual space which we defined is indeed a dual object. Hence both categories are compact.

3. The language of string diagrams.

To facilitate reasoning about graded vector spaces, we now introduce a graphical notation which exists in any monoidal category, namely *string diagrams*. These diagrams have their origins in Penrose's graphical notation for tensors [**62**].

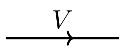
Subsequently, Penrose's diagrams were generalized and largely subsumed as part of category theory. As a result, the string diagrams used in category theory usually represent not any particular tensor, but rather vector spaces of all tensors of a particular type. The categorical framework allows one to prove *coherence theorems* which guarantee (vaguely put) that valid string diagrams are in bijection with valid categorical constructions. The rules of the string diagram calculus can be extended if the category in question possesses further structure, leading to a variety of coherence theorems, an overview of which is given in [**68**].

In this thesis, our use of string diagrams will be somwhat informal; for us, the main advantage of the graphical notation is to hide signs arising from the nontrivial brading of graded vector spaces. This use of graphical notation in this thesis was inspired in part by the notes [**43**]. The reader interested in details should consult the references in [**68**], especially the paper of Joyal and Street [**44**].

Applications of string diagrams are manifold; a good starting point are the overview papers [**4, 6**] (which also serve as introductions to string diagrams) and the many references therein. The reader who is not completely comfortable with categorical language may find the "Birdtracks" used in [**27**] somewhat more accessible; while they are essentially string diagrams, they are introduced in that monograph by analogy with Feynman diagrams, and not via category theory. Applications in Physics include Topological Quantum Field Theory [**7**] and Quantum Mechanics [**20, 21, 19**].

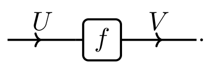
3.1. We now introduce the basic elements of the string diagram calculus. Objects are represented by labelled and directed lines called *strings* or *wires*. Thus a graded vector space V is drawn like this:

(1.3.1)



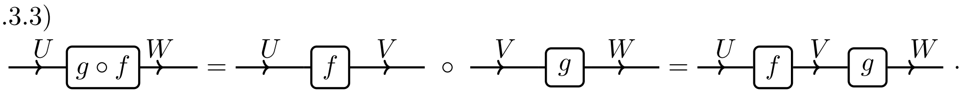
For now all arrows are drawn from left to right. Morphisms between objects are represented by boxes with one arrow going in and one coming out. For instance, $f \in \mathrm{Hom}(V, W)$ is represented by

(1.3.2)



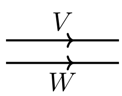
Composition is given by left-to-right juxtaposition. Let $f : U \to V$, $g : V \to W$ be morphisms. Then $g \circ f$ is written

(1.3.3)

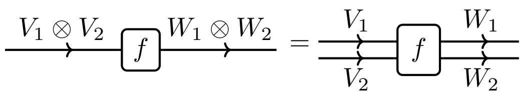


The tensor product $V \otimes W$ is represented by top-to-bottom juxtaposition: the wire for V is drawn above the wire for W (left-to-right translates into top-to-bottom).

(1.3.4)



In the same vein, morphisms from a tensor product $V_1 \otimes V_2$ to $W_1 \otimes W_2$ are drawn as boxes

(1.3.5)


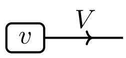
The field $\mathbb{K}$ is the identity object for the tensor product and is suppressed in the string diagram notation.

3.2. Not every category consists of "sets with extra structure", and therefore it does not always make sense to speak of an object containing "elements". The following generalization, however, exists in any category and can be quite useful.

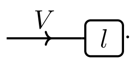
Definition 3.1. Let C be a category and V an object of C. A *generalized element* of of V is a morphism $x : X \to V$, where X is another object of C. The generalized elements from an object X into V are called *the X-points of V*. In a monoidal category with unit object I, an I-point of V is called a *global point* of V.

The Yoneda lemma says that an object is determined by its generalized elements. But frequently, one needs to consider only points from a small number of objects (rather than all of them). For instance, in $\mathsf{finVect}$, any morphism $x : \mathbb{K} \to V$ determines a vector $v = x(1)$, so the global points are enough to determine the object.

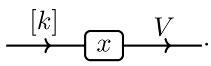
This is quite important for the applications to quantum mechanics in the work of Coecke et. al. [**19**, **20**, **21**], where Dirac's notation for quantum mechanics is expressed through string diagrams. Since we suppress $\mathbb{K}$, a vector v in an ungraded vector space can be represented as

(1.3.6)


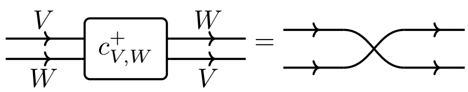
and similarly, a covector $l \in \mathrm{Hom}(V, \mathbb{K})$ as

(1.3.7)


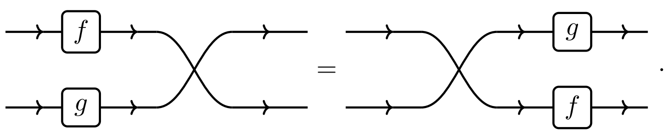
In $\mathsf{grFinVect}$, global points only give the degree-0 homogeneous elements of a graded vector space. The homogeneous elements of degree k are exactly the $[k]$-points of V:

(1.3.8)


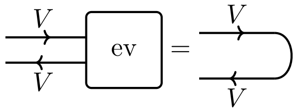
3.3. Let us go through the categorical definitions of Section 2 in the language of string diagrams. The braiding morphism is represented by twisting strings across one another. Note that because the braiding is symmetric, it does not matter which string overlaps the other, and no attempt is made to distinguish these possibilities graphically.

(1.3.9)


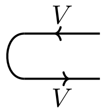
Naturality of the braiding implies that it commutes with morphisms:

(1.3.10)


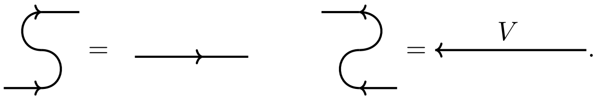
In a category with duals, the dual object is represented by an arrow *going to the left*. The evaluation map $\mathrm{ev} : V^\vee \otimes V \to \mathbb{K}$ can then be drawn as a turn:

(1.3.11)


Similarly, we have an opposite turn which comes from the *unit* (rather than counit) of the inner hom/tensor product adjunction.

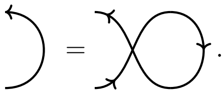
(1.3.12)


The unit-counit equations can then be drawn as the following *zig-zag equations*:

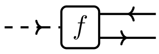
(1.3.13)


This shows one advantage of the string diagram calculus: key equations reduce to "topologically true" (and in that sense obvious) relations of strings.

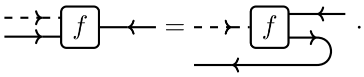
In the ungraded case, it does not matter whether the pairing between V and $V^\vee$ is in the dual of $V^\vee \otimes V$ or of $V \otimes V^\vee$. In the graded case, there is a difference of sign coming from the nontrivial braiding. In terms of string diagrams, we can construct a turn going the opposite way by composing the braiding with a regular turn to get a loop:

(1.3.14)


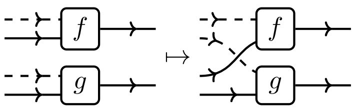
3.4. In the diagrammatic language presented thus far, we have treated of objects and morphisms, but not k-morphisms for $k \neq 0$. To integrate these into string diagrams, we must work with shifted vector spaces and indeed both the left shift functor $[k] \otimes (-)$ and right shift functor $(-) \otimes [k]$ are relevant. In the sequel, we will represent the shifted vector spaces $[k]$ by *dashed lines*. A k-morphism f from U to V is a $[k]$-point of $\hom(U, V) \cong V \otimes U^\vee$ is therefore

(1.3.15)


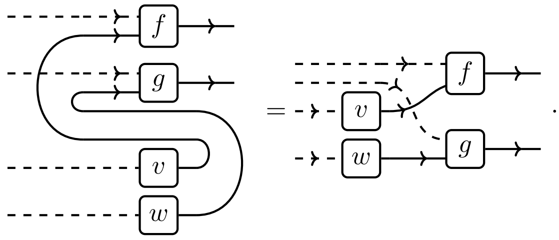
To make it look more like a morphism, we bend back the wire corresponding to $U^\vee$:

(1.3.16)


This equation just says that we can equivalently view a k-morphism as a morphism from a k-shifted vector space. Let us say that a k-morphism is in *standard form* if all dashed lines are at the top, i.e. we only use a single shift functor from the left. Then to get the tensor product of two standard-form k-morphisms into standard form, we need to braid some lines:

(1.3.17)


The "standard" form suggests itself if one wants to evaluate the tensor product of k-morphisms on generalized elements; essentially, one is forced to use it if one wants evaluation on generalized elements to the same as composition:

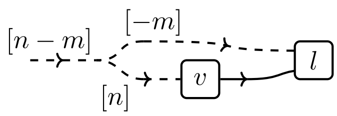
(1.3.18)


In traditional notation:

$$(T \otimes T')(v \otimes w) = (-1)^{|T'|\cdot|v|}(Tv) \otimes (Tw). \tag{1.3.19}$$

In general, rearranging tensor products of shifted vector spaces into what we have called *standard form* is known as the *décalage isomorphism* (see e.g. [28]). The necessity of all these braidings and the décalage isomorphism is a reflection of the fact that the shift functors $V \mapsto [p]V$ are not monoidal functors.

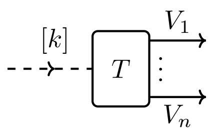
Note also that if we have homogeneous vector $v \in V_m$ and a homogeneous covector $l \in (V^\vee)_{-n}$, $l(v)$ has degree $m - n$. But $\mathbb{K}$ is concentrated in degree 0, so $l(v) = 0$ unless $m = n$.

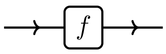
(1.3.20)


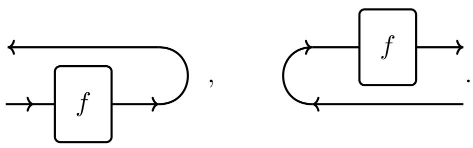
More generally, if we have a string diagram where all solid lines are contracted and only dashed lines remain, the diagram represents a morphism $\mathrm{Hom}([k], \mathbb{K})$, where k is the sum of all the degrees of the dashed lines. This can be nonzero only if $k = 0$.

We have used dashed lines to represent elements of the inner hom of graded vector spaces. A slightly different use of dashed lines is to represent odd vector spaces in terms of even ones. In this case, one can interpret the use of explicit dashed lines as a diagrammatic application of the "even rules" principle (see [29], §1.7), but we do not discuss this here as it has no relevance to this thesis and would take us too far afield.

3.5. The graphical calculus affords easy transitions between tensors and multilinear maps, by using the turns (unit and counit) to twist strings around. Indeed, a tensor of degree k is an element of $(V_1 \otimes \ldots \otimes V_n)_k$, and is therefore considered as a morphism $[k] \to V_1 \otimes \ldots \otimes V_n$

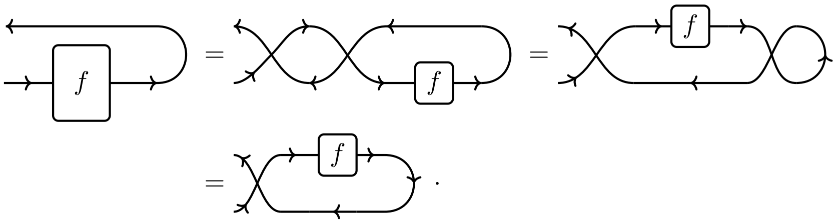
(1.3.21)


Now given a morphism , we can bend wires forward or back:

(1.3.22)


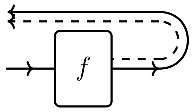
Of course, we can bend the string "around the top" or "around the bottom" of the morphism; the difference is only an additional braiding:

(1.3.23)



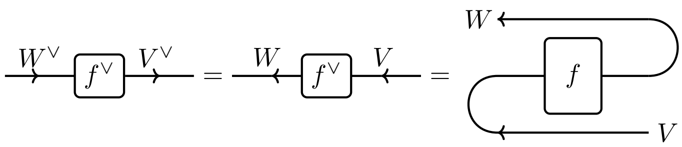
Notice that if one has dashed lines in the diagram, flipping switches between left shift functor and right shift functor, e.g.

(1.3.24)



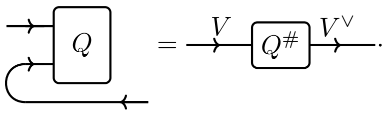
Bending both wires around gives rise to the *dual morphism*:

(1.3.25)



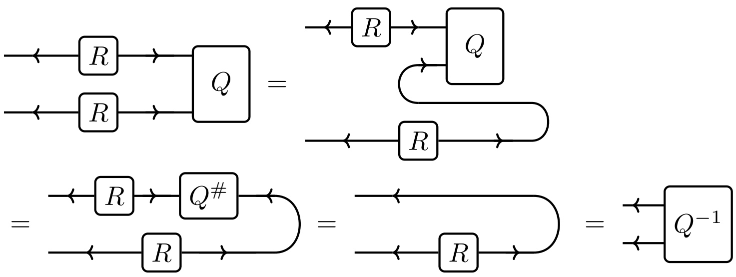
Bending wires allows us to view a quadratic form $Q \in \mathrm{S}^2(V^\vee)$ of degree 0 as a map $Q^\# : V \to V^\vee$:

(1.3.26)



If Q is nondegenerate, then $Q^\#$ is an isomorphism and there is an inverse (which cannot be obtained by operations on diagrams) $R := (Q^\#)^{-1} : V^\vee \to V$. Associated to R there is a tensor $Q^{-1} \in \mathrm{S}^2(V)$ by bending strings. Now let $j \in V^\vee$ and $j^\# = Rj$. What is $Q(j^\#)$? In the graphical notation, this is simple:

(1.3.27)



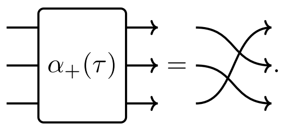
Hence $Q(j^\#) = Q^{-1}(j)$.

4. Multilinear algebra.

4.1. We will now discuss algebra structures for tensors, symmetric tensors and antisymmetric tensors. We denote by S_k the permutation group on k letters. In general it is important to distinguish between (anti-) symmetric tensor defined as *subspaces* or as *quotients* of tensors (see [22] for a discussion of this point in the ungraded case). While the quotients are more natural to deal with from an algebraic point of view, it is the subspaces which are more easily represented using string diagrams.

Let V be a graded vector space. Let us write $T^k(V) := V \otimes \ldots \otimes V$, and consider $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$. This graded vector space has the structure of an algebra; there is an associative multiplication map $m : T(V) \otimes T(V) \to T(V)$ coming from the tensor product $T^k(V) \otimes T^l(V) \to T^{k+l}(V)$. However, $T(V)$ is almost always infinite-dimensional, and so takes us outside our categorical setting. This issue may be sidestepped by considering $T^\bullet(V)$ as a graded object in the category of graded vector spaces. Then properties (such as associativity) of morphisms $T^\bullet(V) \otimes T^\bullet(V) \to T^\bullet(V)$ reduce to those of morphisms $T^k(V) \otimes T^l(V) \to T^{k+l}(V)$, i.e. we can treat everything "degreewise" in the new "tensor degree" m of $T^m(V)$. We will not go into more detail on this point, as it is not essential to the following developments.

Since the braiding is symmetric, it induces an action of S_k on $T^k(V)$; when using the braiding c^+ of Definition 1.17, we call this action α_+. When using the braiding c^-, we call the action α_-. Using string diagrams, we can represent for instance the action of the permutation $\tau = (123)$ as



Remark 4.1. The actions $\alpha_\pm$ intertwine permutations of tensor products of morphisms, that is

$$\alpha_\pm(\sigma)(f_1(v_1) \otimes \ldots \otimes f_k(v_k)) = (f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(k)})(\alpha_\pm(\sigma)(v_1 \otimes \ldots \otimes v_k)). \tag{1.4.1}$$

This is simply a consequence of the naturality of the braiding (cf. Equation (1.3.10)), but one can also verify it directly on homogeneous elements $v, w \in V_1, W_1$ and morphisms $f : V_1 \to V_2, g : W_1 \to W_2$:

$$\begin{aligned} c^+(f(v) \otimes g(w)) &= (-1)^{|f(v)|\cdot|f(w)|} g(w) \otimes f(v) = (g \otimes f)((-1)^{|v|\cdot|w|} w \otimes v) \\ &= (g \otimes f)(c^+(v \otimes w)). \end{aligned}$$

Let I_+^k, $I_-^k =$ be the subspaces in $T^k(V)$ spanned by elements $x - \alpha_+(x)$ and $x - \alpha_-(x)$, respectively. Then we define

$$\mathrm{S}^\bullet(V) = \{\mathrm{S}^k(V)\}_{k \in \mathbb{N}}, \quad \mathrm{S}^k(V) = T^k(V)/I_+^k \tag{1.4.2}$$

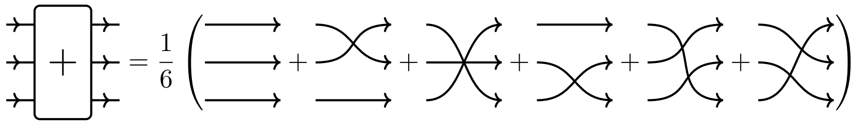
$$\Lambda^\bullet(V) = \{\Lambda^k(V)\}_{k \in \mathbb{N}}, \quad \Lambda^k(V) = T^k(V)/I_-^k. \tag{1.4.3}$$

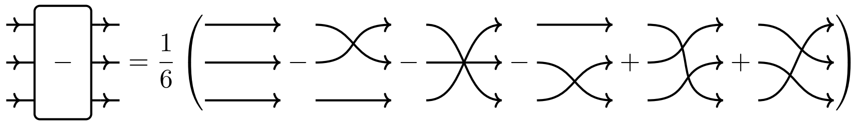
We shall denote the quotient maps by $q_+^k : T^k(V) \to \mathrm{S}^k(V)$ and $q_-^k : T^k(V) \to \Lambda^k(V)$. Now consider the maps

$$\begin{aligned} &\pi_+ : T^k(V) \to \mathrm{Sym}^k(V) \subset T^k(V) \\ &\pi_+(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha_+(\sigma)(v_1 \otimes \ldots \otimes v_k) \end{aligned} \tag{1.4.4}$$

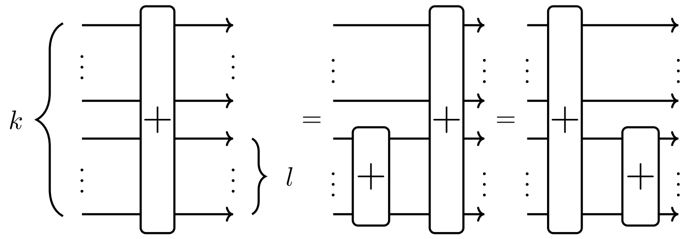
$$\begin{aligned} &\pi_- : T^k(V) \to \mathrm{Alt}^k(V) \subset T^k(V) \\ &\pi_-(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha_-(\sigma)(v_1 \otimes \ldots \otimes v_k) \end{aligned} \tag{1.4.5}$$

The subspaces $\mathrm{Sym}^k(V) := \operatorname{Ran} \pi_+$, $\mathrm{Alt}^k(V) := \operatorname{Ran} \pi_-$ are the subspaces invariant under the action of α_+, α_-. In terms of string diagrams, we represent π_+, π_- by drawing boxes spanning k wires and labelled by $+$ or $-$:

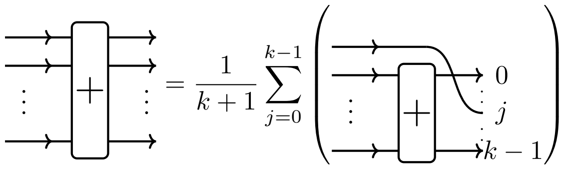
(1.4.6) 

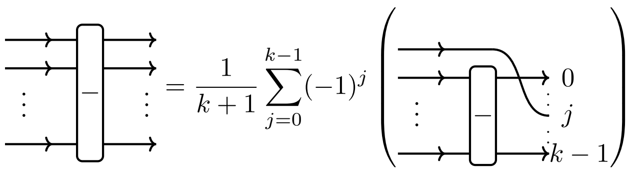
(1.4.7) 

These maps are projections, and the composition of these operators on k and $l < k$ wires is simply the operator on k wires:

(1.4.8) 

Furthermore, a projector on k wires can be expressed as an average of projectors on $k-1$ wires:

(1.4.9) 

(1.4.10) 

Then we get maps

$$p_+ : \mathrm{S}^k(V) \overset{T}{\hookrightarrow}{}^{k} (V) \xrightarrow{\pi_+} \mathrm{Sym}(V) \tag{1.4.11}$$

$$p_- : \Lambda^k(V) \overset{T}{\hookrightarrow}{}^{k} (V) \xrightarrow{\pi_-} \mathrm{Alt}(V). \tag{1.4.12}$$

Proposition 4.2. The maps p_+, p_- are isomorphisms.

Proof. It is clear that

$$p_+(v_1 \dots v_k) = \pi_+(v_1 \otimes \dots \otimes v_k) \tag{1.4.13}$$

is well-defined. Then the following simple computation shows $q_+ \circ p_+ = \mathrm{id}$.

$$q_+ \circ \pi_+(v_1 \otimes \dots \otimes v_k) = q_+ \left(\frac{1}{k!} \sum v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)} \right) \tag{1.4.14}$$

$$= \frac{1}{k!} \sum v_{\sigma(1)} \dots v_{\sigma(k)} = v_1 \dots v_k. \tag{1.4.15}$$

The computation to show $p_+ \circ q_+ = \mathrm{id}$, and the analogous computations with p_- are similar. ■

4.2. We seek to construct nondegenerate pairings

$$T^k(V^\vee)\otimes T^k(V)\to\mathbb{K},\quad \mathrm{S}^k(V^\vee)\otimes \mathrm{S}^k(V)\to\mathbb{K},\quad \Lambda^k(V^\vee)\otimes\Lambda^k(V)\to\mathbb{K} \tag{1.4.16}$$

which extend the pairing $\mathrm{ev}: V^\vee\otimes V\to\mathbb{K}$. This is equivalent to constructing isomorphisms

$$T^k(V^\vee)\cong T^k(V)^\vee,\quad \mathrm{S}^k(V^\vee)\cong \mathrm{S}^k(V)^\vee,\quad \Lambda^k(V^\vee)\cong\Lambda^k(V)^\vee. \tag{1.4.17}$$

There is no unique choice of such pairings, though some more or less obvious choices present themselves. Indeed, consider the following pairings on $T^k(V^\vee)\otimes T^k(V)$,

$$B_1(l_1\otimes\ldots\otimes l_k, v_1\otimes\ldots\otimes v_k) = (-1)^\epsilon\prod_{i=0}^{k} l_i(v_i)$$
$$B_2(l_1\otimes\ldots\otimes l_k, v_1\otimes\ldots\otimes v_k) = \prod_{i=0}^{k-1} l_{k-i}(v_{i+1})$$

where the sign ϵ is induced by the braiding isomorphism which moves l_i next to v_i. The second pairing B_2 may appear strange at first sight, but is just the pairing "induced by juxtaposition"; one contracts only vectors and covectors that are next to each other, i.e. first l_k and v_1, then l_{k-1} and v_2 and so on, i.e.

$$B_1 : (V^\vee)^{\otimes k}\otimes V^{\otimes k}\xrightarrow{\text{braiding}}(V^\vee\otimes V)^{\otimes k}\xrightarrow{\mathrm{ev}^{\otimes k}}\mathbb{K} \tag{1.4.18}$$
$$B_2 : (V^\vee)^{\otimes k}\otimes V^{\otimes k}\xrightarrow{\mathrm{ev}}(V^\vee)^{\otimes (k-1)}\otimes V^{\otimes(k-1)}\xrightarrow{\mathrm{ev}}\ldots\xrightarrow{\mathrm{ev}}\mathbb{K}. \tag{1.4.19}$$

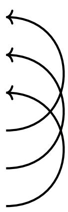
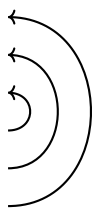
It is easy to see that these pairings are nondegenerate by writing down dual bases. If $\{e_i^m\}$ is a basis for V_m, then $T^k(V)_m$ has a basis given by elements of the form

$$e_{i_1}^{m_1}\otimes\ldots\otimes e_{i_k}^{m_k} \tag{1.4.20}$$

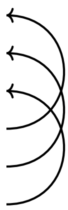
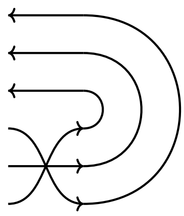
with $\sum m_i = m$. The dual element to this one is then

$$\pm(e_{i_1}^{m_1})^\vee\otimes\ldots\otimes(e_{i_k}^{m_k})^\vee\quad\text{with respect to } B_1, \tag{1.4.21}$$
$$(e_{i_k}^{m_k})^\vee\otimes\ldots\otimes(e_{i_1}^{m_1})^\vee\quad\text{with respect to } B_2. \tag{1.4.22}$$

Nondegeneracy then implies $T^k(V^\vee)\cong T^k(V)^\vee$. In terms of string diagrams, the difference between the pairings is how we stack the wire turns:

(1.4.23) $B_1 =$  $B_2 =$ 

The difference amounts simply to a permutation of the wires:

(1.4.24)  $=$ 

4.3. The symmetric algebra affords a description of polynomial functions and formal power series on a vector space which is independent of any choice of coordinates. Indeed, homogeneous degree-k polynomial functions on a vector space V are given by $\mathrm{S}^k(V^\vee)$. In any basis $\{v_1, \ldots, v_n\}$, the coordinate functions

$$x_i : v = \sum_i \lambda_i v_i \mapsto x_i(v) = \lambda_i \tag{1.4.25}$$

are linear functionals, so that a basis provides a non-canonical isomorphism between $\mathrm{S}^k(V^\vee)$ and degree-k homogeneous elements of $\mathbb{R}[x_1, \ldots, x_n]$. The formal power series on V are the completion of the polynomials; we denote them abstractly by $\mathrm{S}(V^\vee)$ and in coordinates by $\mathbb{R}[[x_1, \ldots, x_n]]$. We will also use the notation $\mathcal{O}(V) := \mathrm{S}(V^\vee)$.

We will generally consider functions to be formal power series or polynomials.

4.4. The pairings on $\mathrm{S}^\bullet(V)$ and $\Lambda^\bullet(V)$ encode the combinatorics of the product rule for graded derivations. Typically, the pairings are simply written down and then various facts proven about them. Here we derive the pairings as extensions of the simple *directional derivative*. The definition of the directional derivative for graded vector spaces is given by analogy with the ungraded case. Because of this, there is also a choice of signs in the composition of directional derivatives.

Let V be an ungraded vector space. Note that for a linear functional $l \in V^\vee$ and a vector $v \in V$, taking the directional derivative (at any point) is the same as contraction: $\partial_v(l) = l(v)$. Now let V be a graded vector space. Then to enforce the Koszul sign rule we need the derivative to be $\partial_v(l) = (-1)^{|l|\cdot|v|} l(v)$.

Hence the operator

$$\partial : V \otimes V^\vee \to \mathbb{K}$$
$$\partial(v \otimes l) = \partial_v(l)$$

is simply the evaluation pairing on $V \otimes V^\vee$. More precisely, we have $\partial = \mathrm{ev} \circ c_{V,V^\vee}$.

In defining the directional derivative, it seems natural to take for the direction a vector, acting on a covector (a linear *function*). The appearence of the sign, however, indicates that the roles should really be reversed. Indeed, let us use the isomorphism $V \cong V^{\vee\vee}$ and then defining a derivative *by a covector on a vector*, we have no extraneous sign:

$$\partial_l(v) := \partial_l(v^{\vee\vee}) = (-1)^{|v|\cdot|l|} v^{\vee\vee}(l) = l(v). \tag{1.4.26}$$

This simply reflects the fact that the directional derivative is the evaluation functional, which is more naturally defined as $V^\vee \otimes V \to \mathbb{K}$, rather than $V \otimes V^\vee \to \mathbb{K}$.

Now we extend the directional derivative to $\mathrm{S}^\bullet(V)$ and $\Lambda^\bullet(V)$ by making it a derivation.

Definition 4.3. Let V be a graded vector space. For any $v \in V$ there are derivations ∂_v^+, ∂_v^- on $\mathrm{S}(V^\vee)$, $\Lambda(V^\vee)$ defined on homogeneous elements by

$$\partial_l^+(v_1 \ldots v_k) = \sum_{j=1}^k (-1)^{\epsilon_j^+} l(v_j) v_1 \ldots \hat{v}_j \ldots v_k \tag{1.4.27}$$

$$\partial_l^-(v_1 \wedge \ldots \wedge v_k) = \sum_{j=1}^k (-1)^{\epsilon_j^-} l(v_j) v_1 \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_k \tag{1.4.28}$$

where the hat on $\hat{v}_j$ indicates the *omission* of the vector v_j, and $\epsilon_j^+, \epsilon_j^-$ are signs induced by the braidings c^+ and c^-, respectively. Explicitly, they are

$$\epsilon_j^+ = \sum_{i<j} |l| \cdot |v_i| \tag{1.4.29}$$

$$\epsilon_j^- = \sum_{i<j} (-|l| \cdot |v_i|). \tag{1.4.30}$$

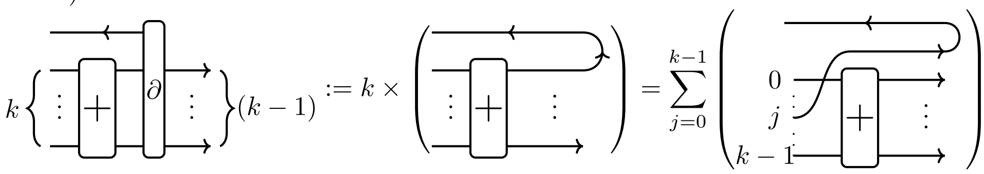
Remark 4.4. The signs $\epsilon_j^\pm$ are such that

$$\partial_l^+(f_1 f_2) = \partial_l^+(f_1) f_2 + (-1)^{|l|\cdot|f_1|} f_1 \partial_l^+(f_2)$$
$$\partial_l^-(g_1 \wedge g_2) = \partial_l^-(g_1) \wedge g_2 - (-1)^{|l|\cdot|g_1|} g_1 \wedge \partial_l^-(g_2)$$

where $l \in V^\vee$, $f_{1,2}$ homogeneous elements of $\mathrm{S}^\bullet(V)$ and $g_{1,2}$ homogeneous elements of $\Lambda^\bullet(V)$.

Definition 4.5. We define the map ∂^+ on $V^\vee \otimes \mathrm{S}^k(V)$:

(1.4.31)



and ∂^- is defined analogously (with π_- in place of π_+).

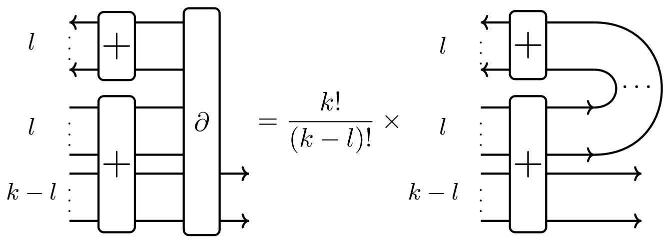
Remark 4.6. One can also define a map $\partial : V \otimes \mathrm{S}^k(V^\vee) \to \mathbb{K}$ by simply turning around all the edges in the above diagram. Note that in this case the extra signs involved are hidden in the definition of the right turn (left turn composed with braiding).

Now we extend the operator ∂ to an action of $\mathrm{S}^l(V)$ on $\mathrm{S}^k(V^\vee)$, where $l < k$. By this we simply mean that we compose derivatives, i.e.

$$\partial_{v_1 \ldots v_l}(f) = (\partial_{v_1} \ldots \partial_{v_l})(f), \quad f \in \mathrm{S}^k(V^\vee). \tag{1.4.32}$$

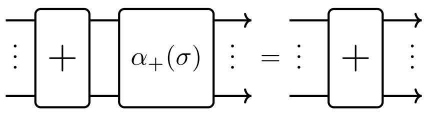
Or, in string diagrams:

(1.4.33)



Composing derivatives as in Equation 1.4.32 translates to using the pairing B_2 on $V^{\otimes l} \otimes (V^\vee)^{\otimes l}$. However, in the case of $\mathrm{S}^k(V^\vee)$, it makes no difference whether we use B_2 or B_1 because the difference between the two is only a permutation of the wires (see Equation 1.4.24), and for any permutation σ, we have the identity

(1.4.34)



However, in the case of the exterior algebra $\Lambda(V^\vee)$, one obtains a difference in sign, and should therefore take care that it is the pairing B_2 which is used in Definition 4.3.

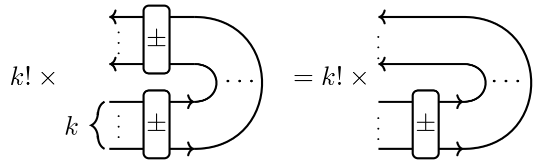
Finally, we can define the pairings.

DEFINITION 4.7. The pairings $\mathrm{S}^\bullet(V^\vee) \otimes \mathrm{S}^\bullet(V) \to \mathbb{K}$ and $\Lambda^\bullet(V^\vee) \otimes \Lambda^\bullet(V) \to \mathbb{K}$ are defined as

$$(A|B)_\pm = \partial_A^\pm B. \tag{1.4.35}$$

In string diagrams:

(1.4.36)



REMARKS 4.8.

i) It follows immediately from Definition (4.7) that the pairings satisfy

$$(A \,|\, \partial_l^+ B)_+ = (Al \,|\, B)_+ \tag{1.4.37}$$

$$(A \,|\, \partial_l^- B)_- = (A \wedge l \,|\, B)_-. \tag{1.4.38}$$

ii) There are analogous pairings with vectors on the left, covectors on the right, which pick up additional signs via the braiding. We also denote these by $(\cdot \,|\, \cdot)_\pm$.

4.5. From the string diagrams of Equation (1.4.36) we can read off explicit expressions for the pairings on $\mathrm{S}^\bullet(V^\vee) \otimes \mathrm{S}^\bullet(V)$ and $\Lambda^\bullet(V^\vee) \otimes \Lambda^\bullet(V)$. Note that for ungraded vector spaces V, W, we have

$$\mathrm{S}^\bullet(V \oplus W) \cong \mathrm{S}^\bullet(V) \otimes \mathrm{S}^\bullet(W), \qquad \Lambda^\bullet(V \oplus W) \cong \Lambda^\bullet(V) \otimes \Lambda^\bullet(W). \tag{1.4.39}$$

Similarly, let V be a graded vector space. Then

$$\mathrm{S}^\bullet(V) \cong \mathrm{S}^\bullet(V_{\bar{0}}) \otimes \Lambda^\bullet(V_{\bar{1}}), \qquad \Lambda^\bullet(V) \cong \Lambda^\bullet(V_{\bar{0}}) \otimes \mathrm{S}^\bullet(V_{\bar{1}}) \tag{1.4.40}$$

Thus it suffices to compute these pairings for V an ungraded vector space.

PROPOSITION 4.9. Let V be an ungraded vector space.

i) The pairings on the symmetric and exterior algebra are related to the pairing on the tensor algebra by

$$\begin{aligned}(l_0 \dots l_k \,|\, v_0 \dots v_k)_+ &= k! \cdot B_2(p_+(l_0 \dots l_k) \otimes p_+(v_0 \dots v_k)) \\ &= k! \cdot B_2(l_0 \otimes \dots \otimes l_k \otimes \pi_+(v_0 \otimes \dots \otimes v_k)) = \sum_{\sigma \in S_k} \prod_{i=1}^k l_i(v_{\sigma(i)})\end{aligned} \tag{1.4.41}$$

$$\begin{aligned}(l_0 \wedge \dots \wedge l_k \,|\, v_0 \wedge \dots \wedge v_k)_- &= k! \cdot B_2(p_-(l_0 \wedge \dots \wedge l_k) \otimes p_-(v_0 \wedge \dots \wedge v_k)) \\ &= k! \cdot B_2(l_0 \otimes \dots \otimes l_k \otimes \pi_-(v_0 \otimes \dots \otimes v_k)) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \prod_{i=1}^k l_{k-i}(v_{\sigma(i)}).\end{aligned} \tag{1.4.42}$$

ii) The symmetric and antisymmetric pairings $(\cdot \,|\, \cdot)_\pm$ are nondegenerate.

PROOF. As mentioned above, i) can simply be read off the string diagrams in Equation (1.4.36). For ii), we follow [22]. It is easy to see that the pairing on the exterior algebra is nondegenerate; a basis of $\Lambda^k(V)$ is given by wedge products $e_{i_1} \wedge \dots \wedge e_{i_k}$ with $i_1 < \dots < i_k$, with dual basis

$$(e_{i_1} \wedge \dots \wedge e_{i_k})^\vee = e_{i_k}^\vee \wedge \dots \wedge e_{i_1}^\vee. \tag{1.4.43}$$

The symmetric pairing is somewhat more difficult, because pairing $e_{i_1} \dots e_{i_k}$ with $e_{i_1}^\vee \dots e_{i_k}^\vee$ need not give 1 since there are possibly multiple permutations contributing to the sum. Indeed, let $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$ be multiindices and consider elements

$$e_I^J = e_{i_1}^{j_1} \dots e_{i_k}^{j_k}. \tag{1.4.44}$$

Define $J! = \Pi_{m=1}^k j_m!$. Then it is shown in [22] that

$$(e_I^J \,|\, e_{I'}^{J'}) = \begin{cases} 0 & e_I^J \neq e_{I'}^{J'} \\ J! & e_I^J = e_{I'}^{J'} \end{cases} . \tag{1.4.45}$$

Thus the pairing is nondegenerate if $J! \neq 0 \in \mathbb{K}$ for any multiindex J. Since we assumed char $\mathbb{K} = 0$, this is the case. ■

5. Integration on graded vector spaces.

5.1. We need to define integration on a graded vector space. Of course, in an invariant setting it is clear that one does not integrate *functions*. Rather, one integrates *densities*. Vaguely put, these are tensor products of functions and *volume elements*. Let $\mathbb{K} = \mathbb{R}$. The idea is that there is a function assigning to each automorphism $A \in GL(V)$ a number for the "amount of stretching" that A does. Plausible reasoning suggests that the function should determine a group homomorphism

$$\mathrm{vol} : GL(V) \to \mathbb{R}^{>0} \leq GL(\mathrm{vol}(V)) \tag{1.5.1}$$

where $\mathrm{vol}(V)$ is a line, i.e. a 1-dimensional vector space. Since vector spaces are orientable, working with *signed volume elements* is more convenient. The same line of thought then leads us to look for a representation

$$\det : GL(V) \to GL(\det(V)) \tag{1.5.2}$$

where again $\det(V)$ is a line, but the representation takes values in both connected components of $\mathbb{R}^x$. Of course, in the ungraded case, such a representation is given by the top forms of a vector space, which can then be shown to give the usual definition of the determinant. Hence we think of integration as a linear map

$$\int_V : C^\infty(V) \otimes \det(V^\vee) \to \mathbb{K} \tag{1.5.3}$$

where $\det(V)$ is the one-dimensional vector space of top-degree exterior forms, $\det(V) = \Lambda^{\dim V} V$. Unfortunately, this construction does not immediately carry over to the graded case, because the exterior algebra on an odd vector space has no top degree.

Instead, we will sketch a construction below (which the author first found in Kazhdan [45], but which according to [69] is originally due to Manin [52]) which gives a *graded line* carrying an action of $\mathrm{Aut}(V)$ for V a graded vector space. This line, the *Berizinian*, is given in terms of even and odd parts as

$$\mathrm{Ber}(V) \cong \Lambda^{\dim V_{\bar{0}}}(V_{\bar{0}}) \otimes \Lambda^{\dim V_{\bar{1}}}(V_{\bar{1}}^\vee). \tag{1.5.4}$$

Thus integration on graded vector spaces becomes a map

$$\int_V \in (\mathcal{O}(V) \otimes \mathrm{Ber}(V^\vee))^\vee \cong \left(\mathcal{O}(V) \otimes \Lambda^{\dim V_{\bar{0}}}(V_{\bar{0}}^\vee) \otimes \Lambda^{\dim V_{\bar{1}}}(V_{\bar{1}})^\vee\right)^\vee . \tag{1.5.5}$$

To construct the Berizinian line, let V be a graded vector space and consider $W := V \oplus V^\vee[1]$. There is a canonical odd symplectic form ω on W, which is the odd version of the canonical pairing of V and $V^\vee$. Then one has that $\omega \wedge \omega = 0$, so $\alpha \mapsto \omega \wedge \alpha$ defines a differential on $\Lambda^\bullet(W)$, turning it into a chain complex.

PROPOSITION 5.1. The cohomology $B_W := H^\bullet(W)$ of $(W, \omega\wedge)$ is 1-dimensional and generated by $y_1 \wedge \ldots \wedge y_k$ with $\{y_i\}$ a basis of the even part of W.

Since the construction $V \mapsto (W, \omega)$ is canonical, there is an inclusion $\mathrm{Aut}(V) \hookrightarrow \mathrm{Symp}(W, \omega)$ which in turn acts on the cohomology and hence on the line B_W.

DEFINITION 5.2. Let V be an n-dimensional vector space, so that $W = V[1]$ is purely odd. The *Berezin integral* on W is the linear map

$$\int_W : \Lambda(V^\vee) \otimes \Lambda^n(V) \to \mathbb{K} \tag{1.5.6}$$

$$\int_W f\mu = (f_n, \mu), \qquad f = \sum_k f_k, f_k \in \Lambda^k(V^\vee), \mu \in \Lambda^n(V). \tag{1.5.7}$$

One notes that the Berezin *integral* is precisely equal to the *derivative* of odd variables, and that the Berezin integral for any function is therefore finite. Questions of convergence are completely absent. Finally, the integral on a graded vector space is defined by combining the integrals over the even and odd parts in the obvious way.

DEFINITION 5.3. Let V be a graded vector space. Under the isomorphism

$$C^\infty(V) \otimes \mathrm{Ber}(V) \cong C^\infty(V_{\bar{0}}) \otimes \det(V_{\bar{0}}) \otimes C^\infty(V_{\bar{1}}) \otimes \mathrm{Ber}(V_{\bar{1}}) \tag{1.5.8}$$

the integral on V corresponds to the map

$$\int_V = \int_{V_{\bar{0}}} \otimes \int_{V_{\bar{1}}}. \tag{1.5.9}$$

6. Basic graded geometry.

6.1. We will introduce some notions of graded geometry, but we will only give definitions for geometry on graded vector spaces. That is, we provide only the *local picture* of graded geometry.

DEFINITION 6.1. A *graded vector bundle* $\pi : E \to M$ over a manifold M is a collection of vector bundles $E_i \to M$. The fibre at p of such a bundle is the graded vector space $E_p = \bigoplus_i (E_i)_p[i]$.

DEFINITION 6.2. Let V be a graded vector space. The *differential forms* on V are given by

$$\Omega^\bullet(V) = C^\infty(V) \otimes \Lambda^\bullet(V). \tag{1.6.1}$$

DEFINITION 6.3. Let M be a manifold. The *shifted tangent bundle* $T[1]M$ of M is the graded vector bundle $E \to M$ with $E_1 = TM$ and $E_i = 0$ for $i \neq 1$. The fibres of the shifted tangent bundle are just $(T_pM)[1]$.

If V is a vector space, the shifted tangent bundle is just

$$T[1]V = V \oplus V[1]. \tag{1.6.2}$$

REMARK 6.4. If we take $C^\infty(V)$ to mean $\mathcal{O}(V) = \mathrm{S}(V^\vee)$, then the differential forms on V are simply $\mathcal{O}(T[1]V)$. One can similarly relate differential forms on a manifold M to functions on $T[1]M$, but must take care to distinguish between smooth functions on M and formal power series.

Definition 6.5. Let V be a graded vector space. A *symplectic form* on V is a 2-form $\omega \in C^\infty(V) \otimes \Lambda^2(V^\vee)$ such that $\omega^\# : V \to V^\vee$ is an isomorphism.

7. Infinite-dimensional vector spaces.

7.1. As we have seen, the categories of ungraded and graded vector spaces are very easy to handle. The rich structure of these categories gives rise to a string diagram calculus with a great number of permissible operations, which, as we shall see, lead to Feynman diagrams. However, the vector spaces encountered in quantum theory (especially field theory) are generally infinite-dimensional. In this case it is clear that much of the structure has to be abandoned.

For instance, compactness seems basically untenable in an infinite-dimensional setting. Compact categories are, in particular, traced. But on infinite-dimensional spaces there are plenty of operators (such as the identity) whose trace diverges.

This does not mean that no graphical calculus is possible in these categories, just that it cannot be naively based on compactness. Indeed, if string diagrams are immediately well-defined due to general properties of our category of vector spaces, then it follows that all Feynman diagrams are finite on the nose – this means that if we are describing a theory with divergencies, the purported categorical structure already incorporates the regularization and renormalization of all Feynman diagrams. It cannot be ruled out that such a category exists, but its construction would clearly not be trivial.

Conversely, even for theories where renormalization is well-understood and leads to finite values for Feynman diagrams, it is not clear (to me at least) how to turn the renormalization algorithm into the definition of a useful category. I am not aware of any particular reason that Feynman diagrams should naturally fit into a categorical framework, except that it works quite well in the finite-dimensional case. In the end, the structure of Feynman diagrams may fundamentally be quite different from string diagrams, and the resemblance in finite dimensions only spurious.

7.2. Working with "bare" vector spaces without further structure is unsatisfactory because the bare tensor product does not suit the study of vector spaces of functions, such as $L^2(X)$ or $C^\infty(X)$. One desirable property of the tensor product should be

$$L^2(X \times Y) \cong L^2(X) \otimes L^2(Y) \tag{1.7.1}$$

but this fails if we use the bare tensor product; instead, one must use a completed tensor product.

Apart from this, the category of vector spaces is still closed symmetric monoidal. But of course compactness fails.

Let U, V be vector spaces. The algebraic tensor product $U \otimes V$ is spanned by *finite sums* $\sum_i u_i \otimes v_i$.

$$U^\vee \otimes V \to \mathrm{Hom}(U, V) \tag{1.7.2}$$

$$l \otimes v \mapsto (w \mapsto l(w)v) \tag{1.7.3}$$

cannot be an isomorphism since its image consists only of finite-rank maps.

7.3. To meet the needs of functional analysis, one uses topological vector spaces, most commonly Hilbert, Banach or Fréchet spaces. Of course, the choice of topology is not unique; indeed, it is quite common to equip the same vector space with different topologies to suit the needs of analysis.

For Hilbert spaces, there is a straightforward way of constructing a topological tensor product. If U, V are Hilbert spaces, then there is a natural sequilinear form on the algebraic tensor product $U \otimes V$. Taking the completion of $U \otimes V$ in the topology generated by this form then gives a Hilbert space $U \hat{\otimes} V$.

Note, however, that the completed tensor product does not satisfy the universal property of the tensor product. Equivalently, closedness fails. Indeed, there is no way to turn the category of Hilbert spaces with bounded linear maps into a closed symmetric monoidal category [37].

For Banach or Fréchet spaces, things are more subtle. Given normed spaces U, V, norms on the algebraic tensor product $U \otimes V$ abound, and there is no single distinguished choice of product norm. There are at least two choices that recommend themselves: the *projective cross norm* and the *injective cross norm*, leading to the projective and injective tensor products, respectively.

One can show in any case that these topological tensor products allow the construction of symmetric monoidal categories of infinite-dimensional vector spaces.

7.4. There is a reasonably distinguished choice of topological vector space, if one is willing to sacrifice the powerful geometry of Banach or Hilbert spaces. That is the category of *nuclear Fréchet spaces*, which is used for instance in [23] (see in particular Appendix 2 therein).

Essentially, a locally convex Hausdorff space is *nuclear* if the projective and injective tensor product agree. Hence there is "only one reasonable tensor product" on nuclear spaces. Many common spaces of functions are nuclear, such as $C^\infty(U)$ and $C_c^\infty(U)$, for $U \subset \mathbb{R}^n$ open, Schwartz functions on $\mathbb{R}^n$, distributions and tempered distributions on $\mathbb{R}^n$. Some details and examples can be found in the already cited Appendix of [23].

7.5. Going beyond topological vector spaces, there is a large body of research dealing with what are known as *convenient vector spaces*. The designation "convenient" is due to Steenrod [70] and refers in that context to cartesian closed categories of topological spaces which satisfy some other useful conditions. By analogy, "convenient" categories of vector spaces are at least closed monoidal categories with further good properties depending on context.

A monograph of central importance in this line of thought is [48]. As far as categories of vector spaces for use in Quantum Field Theory is concerned, we refer the reader to Appendix B of [24] and references therein. The reader may also find the textbook [41] to be of use.

It may be possible to extend the string diagram methods of this thesis to the case of categories that are merely closed and not compact. In [4], it is shown that by decorating string diagrams with "bubbles" and "clasps", the difference between compact and closed categories can be ameliorated (as far as string diagrams are concerned).

CHAPTER 2

Perturbative Quantum Field Theory.

The aim of Quantum Field Theory is to resolve incompatibilities between two successful physical theories: quantum mechanics and classical field theory. Since the first efforts in this direction in the 1920s, this has proved to be an immensely difficult project which is nearing the centenary of its inception without giving any indication of attaining completion by this date.

The purpose of this section is therefore not to give a self-contained exposition of Quantum Field Theory (which would be impossible in such a limited space), but rather to recapitulate the line of thought leading to the BV-BFV formalism, which is employed in this thesis. We do so by summarizing briefly the ingredients that go into the pie (without trying to reconstruct the recipe).

1. Classical Field Theory.

1.1. There are two complementary approaches to classical mechanics: the *Lagrangian* and the *Hamiltonian* formalisms. The Lagrangian approach is perhaps the more natural (cf. [**26**]), but when the Lagrangian is *regular* (in the sense of its *Legendre transform* being an isomorphism), the formalisms are quivalent. Roughly, in the Lagrangian formalism the possible trajectories (forming the *covariant phase space*) of a system are picked out as points of a space of *system histories* (system configurations over time). This is particularly convenient for relativistic theories because it is fully covariant. In the Hamiltonian formalism, one fixes a certain parameterization (called the *phase space*) of the trajectories (related to their initial conditions). One then obtains a *Hamiltonian flow* on this phase space, describing the time evolution of the system.

In classical field theory, it becomes more difficult to obtain regular Lagrangians, and the differences between the Lagrangian and Hamiltonian formalisms become greater. And because fundamental field theories must be relativistic, in field theory the Lagrangian approach tends to be favored.

On the other hand, a major advantage of the Hamiltonian approach is that it lends itself more readily to quantization, because one axiom of quantum theory is that there is a time-evolution operator on the Hilbert space of states which is *unitary.* While it is possible to quantize Lagrangian field theories directly via *path-integral quantization*, proving unitarity of the time-evolution usually requires passing to a Hamiltonian picture.

It thus seems promising to investigate the nature of time-evolution in the Lagrangian framework and generalize the Hamiltonian flow to some operation which is both relativistically covariant and readily implemented in the case of non-regular Lagrangians.

1.2. After these general considerations, let us consider a concrete problem. One important application of Lagrangian field theory is the description of Lagrangian mechanics. Let us consider a free particle moving on a Riemannian manifold $(X, \langle\cdot,\cdot\rangle)$. To start with, the basic data consist of the following.

i) A space of fields, $\mathcal{F} = C^\infty(I, X), I = [t_0, t_1]$
ii) An action functional, $S \in C^\infty(\mathcal{F}, \mathbb{R})$, $S(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t$ with the Lagrangian

$$\mathcal{L} : TX \to \mathbb{R}, \quad \mathcal{L}(q, v_q) = \frac{1}{2m}\langle v_q, v_q\rangle - V(q)$$

and $V : X \to \mathbb{R}$ a potential.
iii) The de Rham differential on $\mathcal{F}$ as δ, which allows us to pose the variational problem. That is to find a subset $\mathcal{N} \subset \mathcal{F}$ such that

$$\delta S|_p = 0. \tag{2.1.1}$$

for all $p \in \mathcal{N}$.

Remarks 1.1.

i) The description of tangent spaces, de Rham differentials and similar objects on the infinite-dimensional space $\mathcal{F}$ presents some problems. Although in the present case, $\mathcal{F}$ may be endowed with the structure of a Frechet manifold, this is not true for all mapping spaces between manifolds. For the present, we take a naive view concerned only with computation. We view the tangent space $T_\gamma\mathcal{F}$ at γ as defined by equivalence classes of germs of curves at γ, but we do not care to be precise about the equivalence relation defining these classes. For now, it suffices that a curve ϕ through γ *represents* a tangent vector $[\phi]$ at γ, and we will simply assume that all operations defined on representatives are actually well-defined on the equivalence class.
ii) There is a "currying" isomorphism

$$\mathrm{curry} : C^\infty((-1, 1), \mathcal{F}) \cong C^\infty((-1, 1) \times I, X)$$
$$\phi(s)(t) = \mathrm{curry}(\phi)(s, t).$$

In the following, we suppress this and similar isomorphisms and switch freely between $\phi_s(t)$, $\phi(s, t)$ and $\phi(s)(t)$.
iii) Consequent to the last two remarks, δS_γ is defined on curves $\phi(s, t)$ with $\phi(0, t) = \gamma(t)$ as

$$\delta S_\gamma([\phi]) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} S(\phi_s). \tag{2.1.2}$$

A standard computation then shows that

$$\delta S_\gamma([\phi]) = \int_{t_0}^{t_1} \langle -m\ddot{\gamma}(t) - \nabla V, \partial_s\phi(t, 0)\rangle\, \mathrm{d}t + \langle \dot{\gamma}(t), \partial_s\phi(t, 0)\rangle|_{t_0}^{t_1}. \tag{2.1.3}$$

It is significant that we obtain on the right-hand side of 2.1.3 two distinct contributions, a "bulk" term and a "boundary" term. To some extent, these terms, treated individually, are more important than δS itself.

Definition 1.2. Let $\phi(s, t)$ be a representative for the tangent vector $[\phi] \in T_\gamma\mathcal{F}$. The *Euler-Lagrange* 1-form $EL \in \Omega^1(\mathcal{F})$ is defined by

$$EL([\phi]) = \int_0^1 \langle \mathcal{EL}(\gamma)(t), (\pi_t^*\delta q)([\phi])\rangle\, \mathrm{d}t \tag{2.1.4}$$

where $\mathcal{EL}$ is the operator

$$\mathcal{EL}(\gamma)(t) = -m\ddot{\gamma}(t) - \nabla V(\gamma(t)). \tag{2.1.5}$$

1.3. Suppose we impose boundary conditions by restricting to the submanifold L consisting of curves γ such that $\gamma(t_0) = x_0, \gamma(t_1) = x_1$. Then the solutions of $\delta S_\gamma = 0, \gamma \in L$ are given by solutions of the usual Newton equation

$$m\ddot{\gamma}(t) = -\nabla V(\gamma(t)). \tag{2.1.6}$$

However, instead of imposing some set of boundary conditions, we wish to find a geometric interpretation for the boundary term. The space of Cauchy data for Equation 2.1.6 is simply $C = TX$ (it suffices to specify an initial position and velocity to select a unique solution). But uniqueness can also be obtained by specifying initial and terminal data together, so let us consider boundary data at *both ends* of the time interval by defining $\mathcal{F}_\partial := C \times C$. We have a surjective map

$$\pi = \pi_0 \times \pi_1 : \mathcal{F} \to \mathcal{F}_\partial$$
$$\pi_s(\gamma) = (\gamma(s), \dot{\gamma}(s))$$

and we can interpret the boundary term in 2.1.3 as a pulled back 1-form.

Definition 1.3. Let $c : \mathbb{R} \to TX$ be a curve representing the tangent vector $[c] \in T_{c(0)}C$, and write $c = (c_1(t), c_2(t))$ with $c_1(t) \in X$, $c_2(t) \in T_{c_1(t)}X$. Then the 1-form $\alpha = \langle \dot{q}, \delta q \rangle$ is defined by its action on tangent vectors $[c]$:

$$\alpha([c]) = \langle c_2(0), \dot{c}_1(0) \rangle. \tag{2.1.7}$$

Let $p_1, p_2 : C \times C \to C$ be the projections onto the first and second factor, respectively, and $\alpha_1 := p_1^*\alpha, \alpha_2 := p_2^*\alpha$, as well as

$$\alpha_{C\times C} = p_2^*\alpha - p_1^*\alpha. \tag{2.1.8}$$

We can now write the boundary term as an evaluation of the pulled-back form on the tangent vector represented by the curve ϕ.

$$\langle \dot{\gamma}(t), \partial_s \phi(t, 0) \rangle |_0^1 = \pi_1^*(\alpha_2)([\phi]) - \pi_0^*(\alpha_1)([\phi]) = \pi^*(\alpha_{C\times C})([\phi]). \tag{2.1.9}$$

Let us denote the zero-set of $\mathcal{EL}$ by L_{EL}. A priori, solving the variational problem is something that must be done “in the bulk” (that is, the interior) of the space-time manifold. If we have a submanifold $L \subset \mathcal{F}$ such that $\alpha|_L = 0$ and intersecting L_{EL} transversally, then $\delta S|_p = 0$ for all $p \in \mathcal{N}$.

However, we can also attempt to transfer the problem to the boundary of space-time. The crucial element of this transfer is the 2-form $\omega = \mathrm{d}\alpha$. In the happiest of cases, this form will be *symplectic*, and we can find solutions to the variational problem in terms of the symplectic geometry of the boundary.

1.4. Let us recall some relevant definitions.

Definition 1.4. Let V be a vector space.

i) A bilinear form $\omega : V \otimes V \to \mathbb{R}$ on a vector space V is *symplectic* if it is alternating ($\omega(v, v) = 0$ for all $v \in V$) and the induced homomorphism

$$\omega^\# : V \to V^\vee, \quad \omega^\#(w)(v) = \omega(w, v) \tag{2.1.10}$$

is an isomorphism. Equivalently, $\omega(u, v) = 0$ for all $v \in V$ implies $u = 0$.

ii) A *presymplectic form* is a closed 2-form of constant rank.

iii) A *symplectic form* ω on a manifold M is a closed 2-form such that (T_pM, ω_p) is a symplectic vector space for every $p \in M$.

Let (V, ω) be a symplectic vector space and W a subspace. Then W^ω is the ω-orthogonal subspace, i.e.

$$W^\omega = \{v \in V |\ \omega(v, w) = 0\ \forall\ w \in W\} \tag{2.1.11}$$

DEFINITION 1.5. A subspace W of a symplectic vector space (V, ω) is called

i) *isotropic* if $W \subset W^\omega$,
ii) *coisotropic* if $W^\omega \subset W$,
iii) *Lagrangian* if $W = W^\omega$ (i.e.isotropic and coisotropic)
iv) *symplectic* if $W \cap W^\omega = \{0\}$.

REMARK 1.6. A symplectic vector space always has even dimension, and a Lagrangian subspace W of a symplectic vector space V always has dimension $\dim W = \frac{1}{2} \dim V$.

DEFINITION 1.7. A submanifold N of a symplectic manifold M is called isotropic (coisotropic, Lagrangian, symplectic) if $T_pN \subset T_pM$ is an isotropic (coisotropic, Lagrangian, symplectic) subspace.

PROPOSITION 1.8. Let $(M_1, \omega_1), (M_2, \omega_2)$ be symplectic manifolds. Then $(M_1 \times M_2, p_1^*\omega_1 + p_2^*\omega_2)$ is also a symplectic manifold (denoted just by $M_1 \times M_2$ where no confusion is possible).

DEFINITION 1.9. Let (M, ω) be a symplectic manifold. Then $\overline{M}$ denotes the symplectic manifold $(M, -\omega)$.

1.5. We have seen that the boundary term in the variation of S is the pull-back of a 1-form α on $C = TX$. The 2-form $\omega = \mathrm{d}\alpha$ is certainly closed. If ω is presymplectic, it becomes possible to speak of isotropic and coisotropic submanifolds. Let us assume that ω is actually symplectic.

PROPOSITION 1.10. Suppose Σ is an $(n-1)$-manifold and space-time is a cylinder, $M = \Sigma \times [0, \epsilon]$. If ϵ is small enough, the submanifold $\pi(L_{EL}) \subset \overline{C} \times C$ is Lagrangian.

PROPOSITION 1.11. Let $L \subset \overline{C} \times C$ be a submanifold such that α vanishes on L. Then L is isotropic.

PROOF. Let $\iota : L \hookrightarrow \overline{C} \times C$ be the inclusion map. Then

$$\omega|_L = \iota^*(\mathrm{d}\alpha) = \mathrm{d}(\iota^*\alpha) = \mathrm{d}(\alpha|_L) = 0. \tag{2.1.12}$$

Hence $\omega_p(v_p, w_p) = 0$ trivially for all $v_p, w_p \in T_pL$, and hence L is isotropic. ■

As mentioned above, in order to conclude that δS vanishes *in all directions* at a point p of the intersection $L \cap \pi(L_{EL})$, the intersection needs to be transverse. In particular, when $\mathcal{F}$ is finite-dimensional, we need $\dim L + \dim \pi(L_{EL}) \geq \dim \mathcal{F}$. Since $\pi(L_{EL})$ is Lagrangian, this implies that L (which is isotropic) too must be Lagrangian.

We might therefore conclude that *a choice of boundary conditions corresponds to a Lagrangian submanifold in $\overline{C} \times C$ transverse to $\pi(L_{EL})$*. This is feasible if we fix a space-time manifold (in our case just the interval I). However, we also want to consider time-evolution as gluing of space-time manifolds. In this case, it is natural to choose boundary conditions on each component of the boundary separately, i.e.we have Lagrangian submanifolds $L_1 \subset \overline{C}$, $L_2 \subset C$ and form $L = L_1 \times L_2$. Furthermore, we need to be more flexible in choosing boundary conditions. Roughly, rather than, say, fixing $x(t_0) = x_0, x(t_1) = x_1$, we choose families of boundary conditions, along the lines of "fix the initial position $x(t_0)$ and the terminal position $x(t_1)$."

More precisely, we fix a *Lagrangian fibration of the boundary fields.* For our purposes, this is simply a fibre bundle $p_\partial : \mathcal{F}_\partial \to B_\partial$ for some submanifold $B_\partial \subset \mathcal{F}_\partial$, such that each fibre of this bundle is a Lagrangian submanifold of $\mathcal{F}_\partial$.

2. Quantum Field Theory and the Path Integral.

2.1. The path integral approach to quantum field theory revolves around manipulating and interpreting the fundamental expression

$$\langle \Psi_1 \,|\, U \,|\, \Psi_2 \rangle = \int_{\phi_{\Sigma_1}=\Psi_1}^{\phi_{\Sigma_2}=\Psi_2} [\mathrm{D}\phi] \exp(\frac{i}{\hbar} S[\phi]). \tag{2.2.1}$$

Here, U is the time-evolution operator and Ψ_1, Ψ_2 are states in the Hilbert space at times t_1, t_2 (corresponding to time-slices Σ_1, Σ_2 of the space-time manifold). The integral is over a space of fields with given boundary conditions and the "Lebesgue measure" $[\mathrm{D}\phi]$.

In the infinite-dimensional case, such a measure does not exist. Instead, one can sometimes construct a Gaussian measure corresponding to $[\mathrm{D}\phi] \exp(S_0[\phi])$, where $S_0[\phi]$ is the free (quadratic) part of the action. The research programme associated with such constructions is known as *constructive field theory*; an introduction can be found in [**38**].

In this thesis, we work in the purely *perturbative* setting. This means that we associate to expressions such as (2.2.1) a formal power series whose individual terms are constructed by analogy with an asymptotic series approximating the integral in the finite-dimensional case. This sounds quite simple, but there are several difficulties. In the first place, it is not even always clear what power series to associate to a finite-dimensional integral; this is the case with degenerate actions, such as in gauge theories. Secondly, the tensor contractions defining the terms of the power series are already divergent integrals in the infinite-dimensional case. One must therefore *regularize* and *renormalize.*

2.2. An important step in the mathematization of the path integral is the discovery of a suitable set of axioms. The axiomatic method allows one to seperate the question of existence from the rules for symbolic manipulation. Axioms have been proposed by Atiyah [**3**] and Segal [**67**] which express mainly two important principles: covariance under space-time transformations and locality. In [**15**], the following axioms are used (for a slightly different approach, see for instance [**60, 61**]).

- Locality of boundary states:

$$\mathcal{H}(\emptyset) = \mathbb{C}, \quad \mathcal{H}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2). \tag{2.2.2}$$

- Locality of the partition function:

$$Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2} \in \mathcal{H}(\partial M_1) \otimes \mathcal{H}(\partial M_2). \tag{2.2.3}$$

- Functoriality: an orientation preserving isomorphism (in our space-time category, i.e. preserving geometric structure) $f : \Sigma_1 \to \Sigma_2$ induces an isomorphism

$$T_f : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2} \tag{2.2.4}$$

  with $T_{f \sqcup g} = T_f \otimes T_g$ and $T_{f \circ g} = T_f T_g$. For any isomorphism $f : M_1 \to M_2$ inducing an isomorphism $f_\partial : \partial M_1 \to \partial M_2$, we should have

$$T_{f_\partial} Z_{M_1} = Z_{M_2}. \tag{2.2.5}$$

- Inner product: There is a sesquilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma \to \mathbb{C}$ which turns $\mathcal{H}_\Sigma$ into a Hilbert space.
- Involution: There is an antilinear map $i_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$ satisfying $i_\Sigma \circ i_{\overline{\Sigma}} = \mathrm{id}$.
- There is a nondegenerate bilinear pairing

$$(\cdot,\cdot)_\Sigma : \mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_\Sigma \to \mathbb{C} \tag{2.2.6}$$

such that

$$(\cdot,\cdot)_{\Sigma_1 \sqcup \Sigma_2} = (\cdot,\cdot)_{\Sigma_1} \otimes (\cdot,\cdot)_{\Sigma_2} \tag{2.2.7}$$

and that is compatible with the inner product, i.e.

$$\langle\phi_1,\phi_2\rangle_\Sigma = (i_{\overline{\Sigma}}(\phi_1),\phi_2)_\Sigma. \tag{2.2.8}$$

- Cutting/gluing: Suppose that $\partial M = \overline{\Sigma}_i \sqcup \Sigma_o$ and let Σ be some section of M, so that we can write $M = M_1 \times_\Sigma M_2$, i.e. M is the gluing along Σ of M_1 and M_2 with $\partial M_1 = \overline{\Sigma}_i \sqcup \Sigma$, $\partial M_2 = \overline{\Sigma} \sqcup \Sigma_o$. Then $Z_{M_1} \in \mathcal{L}(\mathcal{H}_{\Sigma_i},\mathcal{H}_\Sigma)$, $Z_{M_2} \in \mathcal{L}(\mathcal{H}_\Sigma,\mathcal{H}_{\Sigma_o})$ and $Z_M \in \mathcal{L}(\mathcal{H}_{\Sigma_i},\mathcal{H}_{\Sigma_o})$. The gluing axiom requires

$$Z_M = Z_{M_2} Z_{M_1}. \tag{2.2.9}$$

Remarks 2.1.

i) Instead of requiring existence of a nondegenerate bilinear pairing $\mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_\Sigma \to \mathbb{C}$ and an antilinear involution, we might equally well require the existence of an antilinear isomorphism $\mathcal{H}_{\overline{\Sigma}} \cong \mathcal{H}_\Sigma^\vee$.

ii) Especially with regard to the point on functoriality above, it is necessary to choose a category of space-time manifolds to work in, e.g. smooth manifolds, Riemannian manifolds, semi-Riemannian manifolds etc. If the category is that of smooth manifolds (without additional structure), one speaks of topological quantum field theory.

iii) With enough care, these axioms allow one to construct a functor $Z : \mathsf{Cob} \to \mathsf{Vect}$, where Cob is a category of cobordisms between manifolds in our chosen space-time category. For this point of view on topological quantum field theories, see [7, 12, 14].

2.3. Note that these axioms say nothing about the structure of observables, and therefore capture only a part of the properties of the functional integral. Axioms which include observables must be considerably more sophisticated, for instance there are the *factorization algebras* of [39, 24, 25].

However, one simple approach to working with observables is to suppose that one can absorb the observable into the action, so that any observable $\mathcal{O}$ yields a new action $S_\mathcal{O}$:

$$e^{\frac{i}{\hbar}S(\phi)}\mathcal{O}(\phi) = e^{\frac{i}{\hbar}S_\mathcal{O}(\phi)}. \tag{2.2.10}$$

Then expectation values of observables define new partition functions $Z_\mathcal{O} := \langle\mathcal{O}\rangle$ which satisfy all the axioms above. It is reasonable to assume furthermore that the spaces of fields and the Hilbert spaces for the observable expecation "partition functions" are the same as for the original partition function Z_M. Then for a fixed manifold M with boundary $\partial M = \Sigma$, any observable defines a state $Z_M^\mathcal{O}$ in $\mathcal{H}(\Sigma)$.

Assumptions along these lines seem to be tacitly present when states are in some way identified with certain observables. For instance, in Chern-Simons theory, states are sometimes identified with Wilson line observables ending in a fixed time-slice of the space-time manifold [9].

2.4. Let us attempt to make sense of (2.2.1) in the finite-dimensional case before moving on to the infinite-dimensional setting. We take as given the data of a classical field theory; in particular, associated to any n-manifold M and any $(n-1)$-manifold Σ we have spaces of bulk fields $\mathcal{F}_M$ and boundary fields $\mathcal{B}_\Sigma$, as well as boundary maps $\pi_\partial : \mathcal{F}_M \to \mathcal{B}_{\partial M}$, so that $\mathcal{F}_M$ is a fibre bundle over $\mathcal{B}_{\partial M}$. Let us denote the fibre over $b \in \mathcal{B}_{\partial M}$ by $F(b)$. We have Lebesgue measure $\mathrm{d}\phi$ and $\mathrm{d}b$ on the spaces of fields. There is a natural measure $\mathrm{d}f$ on the fibre, $\mathrm{d}f = \frac{\mathrm{d}\phi}{\mathrm{d}b}$ so that $\mathrm{d}\phi = \mathrm{d}f\,\mathrm{d}b$.

Then in finite dimensions, the fibre integral $Z_M(b) = \int_{F(b)} \exp(iS(f))\,\mathrm{d}f$ is a well-defined function on $\mathcal{B}_{\partial M}$. Using Fubini's theorem to change the order of integration, we have

$$\int_{\mathcal{B}_{\partial M}} Z_{M_1}(b) Z_{M_2}(b)\,\mathrm{d}b = Z_{M_1 \times_\Sigma M_2}. \tag{2.2.11}$$

Let us suppose that under disjoint union of underlying manifolds the spaces of fields are direct sums

$$\mathcal{F}_{M_1 \sqcup M_2} \cong \mathcal{F}_{M_1} \oplus \mathcal{F}_{M_2} \tag{2.2.12}$$

$$\mathcal{B}_{\Sigma_1 \sqcup \Sigma_2} \cong \mathcal{B}_{\Sigma_1} \oplus \mathcal{B}_{\Sigma_2}, \tag{2.2.13}$$

as is the case when the fields are sections of vector bundles over the given manifolds. Then if we define $\mathcal{H}_\Sigma := L^2(\mathcal{B}_\Sigma)$, we have

$$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} \cong \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}. \tag{2.2.14}$$

2.5. One important class of quantities derived from the path integral are *effective actions*. Suppose for now that M has no boundary, and that the space of fields splits, $\mathcal{F}_M = \mathcal{F}_1 \times \mathcal{F}_2$. Then one can integrate the Gaussian measure over $\mathcal{F}_2$, leaving a function on $\mathcal{F}_1$.

$$\exp(S_{\text{eff}}(f)) = \int_{\Phi \in \mathcal{F}_2} \exp(S(f, \Phi))[\mathrm{D}\Phi]. \tag{2.2.15}$$

Effective actions come in many different varieties. For instance, one may split fields by considering an energy cut-off and consequently a low-energy effective action. The dependence of this effective action on the energy cut-off is what underlies the Wilsonian approach to renormalization. In the BV-BFV formalism, one considers effective actions for boundary fields as well as for topological "zero-modes."

3. Perturbative Path Integrals and the Wick Lemma.

In this section, we compute moments of Gaussian integrals. We then extend these results to define an operation on formal power series, which we call the *formal integral*.

The Wick lemma is the basis for the technique of *Feynman diagrams*. There are quite a few expository publications concerning Feynman diagrams, and directed at a mathematical audience [63, 58, 55, 57, 56, 40]. It has also frequently been remarked that string diagrams for symmetric monoidal categories form a mathematical basis for Feynman diagrams [4, 6, 5].

Detailed expositions of this fact, however, are much scarcer, one example being [32]. The authors of that publication employ more sophisticated mathematical machinery (operads and PROPs) than is to be found in this thesis; but the relation between graphical

TABLE 1. Comparison of coordinate and abstract notation.

NOTION	IN COORDINATES	ABSTRACT
Space of fields	$\mathbb{R}^n$	V
Polynomials	$\mathbb{R}[x_1, \dots x_n]$	$\mathrm{S}^n(V^\vee)$
Formal power series	$\mathbb{R}[[x_1, \dots, x_n]]$	$\mathcal{O}(V) := \mathrm{S}(V^\vee)$
Free field action	$S_0(x) = \langle x, Ax \rangle, A \in GL_n(\mathbb{R})$	quadratic form $Q \in \mathrm{S}^2(V^*)$
Propagator	$\eta = A^{-1} \in GL_n(\mathbb{R})$	$\eta = Q^{-1} \in \mathrm{S}^2(V)$

calculus and Gaussian integrals appears to be in the same spirit as what is done here. There is also the paper [59], which treats quantum field theory in more general braided (not necessarily symmetric) categories. I have not attempted to compare my results with those in that publication.

3.1. We refrain from choosing a basis, preferring instead to use a coordinate-free notation for all kinds of functions on vector spaces. An overview is given in Table 1. The most common notational schemes in physics make free use of coordinates; in such a system, the kinds of integrals we want to compute are the *moments*

$$\langle x_{i_1} \dots x_{i_k} \rangle = \int_{\mathbb{R}^n} x_{i_1} \dots x_{i_k} e^{\frac{i}{\hbar} S(x)} \, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{2.3.1}$$

with $S(x) = \langle x, Ax \rangle + I(x)$. Expressed without coordinates, we would write this as

$$\langle l_1 \dots l_k \rangle = \int_V l_1 \dots l_k e^{\frac{1}{2} Q + I} \mu \tag{2.3.2}$$

with V a vector space, $l_i \in V^\vee$ are linear functionals, $Q \in \mathrm{S}^2(V)$ a quadratic form, $I \in \mathrm{S}^\bullet(V^\vee)$ a polynomial, and μ a volume form. Notice that we do not choose an inner product on V. Throughout the following discussion we will keep the volume form μ fixed.

Though we find the coordinate-free notation appealing for its own sake, it is also hides the differences between graded and ungraded vector spaces, allowing us to treat them almost on equal footing. It also avoids a common source of confusion for students: in introductory lectures on QFT, one often hears the question how the left-hand side of (2.3.1) can depend on the x_i, when they are integrated over on the right-hand side.

3.2. Let V be a graded vector space, $Q \in \mathrm{S}^2(V^\vee)$ a non-degenerate positive-definite quadratic form, $j \in V^\vee$, $I \in \mathrm{S}^k(V^\vee)$ a monomial of degree k.

DEFINITION 3.1. The function $Z_0 \in \mathcal{O}(V^\vee)$ given by

$$Z_0(j) = \int_V e^{-\frac{1}{2} Q + j} \mu \tag{2.3.3}$$

is called the *free generating function*. The *full* or *interacting generating function* is

$$Z(j) = \int_V e^{-\frac{1}{2} Q + I + j} \mu. \tag{2.3.4}$$

LEMMA 3.2. The free generating functional is simply

$$Z(j) = Z(0) \exp\left(\frac{1}{2} \eta(j)\right). \tag{2.3.5}$$

PROOF. In the ungraded case, this is a simple matter of “completing the square”. For V a graded vector space, the computation is similar, but slightly more subtle, and demonstrates some of the pitfalls involved in evaluating at a point functions on graded spaces.

For instance, suppose that V is concentrated in degree 1, and let $Q \in \mathrm{S}^2(V^\vee)$ be a quadratic form. Then $Q(v, w) = (-1)^{|v|\cdot|w|}Q(w, v)$, and therefore $Q(v, v) = 0$ for any $v \in V$. The point is that a function on a graded space is not determined by its evaluation at points. This does not contradict the Yoneda lemma; as a morphism, a quadratic form Q has domain $V \otimes V$ and can therefore be evaluated at generalized points $x \otimes y$. As a function, it can only be evaluated at diagonal points $v \otimes v$.

First we add a zero:

$$-\frac{1}{2}Q + j = -\frac{1}{2}Q + j - \frac{1}{2}Q^{-1}(j) + \frac{1}{2}Q^{-1}(j). \tag{2.3.6}$$

Defining $R := (Q^{\#})^{-1}$, Equation (1.3.26) gives $Q^{-1}(j) = Q(Rj, Rj)$. Similarly,

$$j(v) = Q(Rj, v) = Q(v, Rj). \tag{2.3.7}$$

This is somewhat surprising; one might expect a sign of $(-1)^{|Rj|\cdot|v|}$ to appear here. This is due to a measure of ambiguity in the notation involved in evaluation of functions on points. Note that if f, g are maps then

$$Q(f, g) = Q \circ (f \otimes g) = (-1)^{|f|\cdot|g|}Q \circ (g \otimes f). \tag{2.3.8}$$

hence

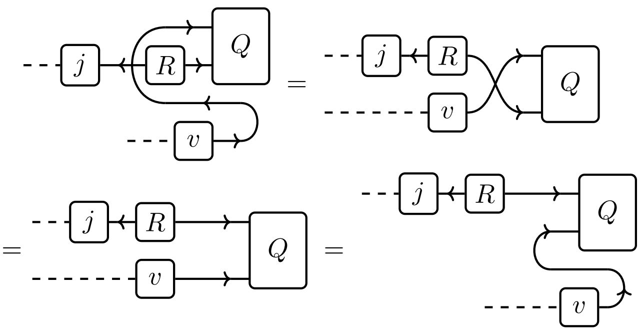
$$Q(-, Rj) = Q \circ (\mathrm{id} \otimes Rj) = Q \circ (Rj \otimes \mathrm{id}) = Q(Rj, -). \tag{2.3.9}$$

Then we find

$$\mathrm{ev}(Q(-, Rj), v) = \mathrm{ev}(Q(Rj, -), v) = j(v). \tag{2.3.10}$$

This is perhaps more easily seen in terms of string diagrams.

(2.3.11)



The first term is the string diagram representation of $\mathrm{ev}(Q(-, Rj), v)$. In the subsequent steps we use naturality of the braiding and symmetry of Q, and the final term is $\mathrm{ev}(Q(Rj, -), v)$. The point is that in *evaluating* a function, one has to decide whether the point at which one is evaluating sits to the right or the left of the function. Hence

$$-\frac{1}{2}Q + j = -\frac{1}{2}Q \circ ((\mathrm{id} + Rj) \otimes (\mathrm{id} + Rj)) + \frac{1}{2}Q^{-1}(j). \tag{2.3.12}$$

Now a simple coordinate transformation of shifting by a constant does the trick. ■

Now for a polynomial $p \in \mathrm{S}^\bullet(V^\vee)$, we define the *expectation value* as the normalized moment:

$$\langle p \rangle := \frac{1}{Z(0)} \int_V p e^{-\frac{1}{2}Q + I} \mu. \tag{2.3.13}$$

We denote the expectation value with respect to the free action by a subscript 0, i.e.

$$\langle p\rangle_0 := \frac{1}{Z_0(0)}\int_V pe^{-\frac{1}{2}Q}\mu. \tag{2.3.14}$$

Observe that expectations with respect to the full action can be expressed in terms of free expectation values:

$$\langle p\rangle = \langle e^I p\rangle_0. \tag{2.3.15}$$

The reason for $Z(j)$ being called the generating function is the following fact.

Proposition 3.3. All moments can be computed as derivatives of $Z(j)$; explicitly

$$\langle l_1 \dots l_k\rangle = \frac{1}{Z(0)}\partial_{l_1}\dots\partial_{l_k}Z(j)|_{j=0}. \tag{2.3.16}$$

Proof. Because the integral is absolutely convergent we can exchange derivative and integral. Then in order to differentiate with respect to $l \in V^\vee$, we must consider the functions under the integral as joint functions of $v \in V$ and $j \in V^\vee$. Thus

$$e^j = e^{(-\,|\,-)}\circ(j,\mathrm{id}). \tag{2.3.17}$$

Now $\partial_l(-\,|\,-) = (l\,|\,-) = l$. Thus

$$\partial_l e^j = \left(\partial_l e^{(-\,|\,-)}\right)\circ(j,\mathrm{id}) = \left(l\cdot e^{(-\,|\,-)}\right)\circ(j,\mathrm{id}) = l\cdot e^j. \tag{2.3.18}$$

and in particular

$$\left(\partial_{l_1}\cdots\partial_{l_k}e^{(-\,|\,-)}\right)\circ(0,\mathrm{id}) = l_1\cdots l_k. \tag{2.3.19}$$

Thus

$$\frac{1}{Z(0)}\partial_{l_1}\dots\partial_{l_k}Z(j)|_{j=0} = \frac{1}{Z(0)}\int_V l_1\cdots l_k e^{-\frac{1}{2}Q+I}\mu = \langle l_1\cdots l_k\rangle. \tag{2.3.20}$$

■

Lemma 3.4. Let $l_i \in V^\vee, i = 1,\dots,n$. Then

$$\langle l_1\dots l_n\rangle_0 = \begin{cases}\frac{1}{2^m m!}(l_1\dots l_{2k}\,|\,\eta^k) & n = 2k\\ 0 & n \text{ odd}\end{cases} \tag{2.3.21}$$

Proof.

$$\begin{aligned}\langle l_1\dots l_{2k}\rangle_0 &= \frac{1}{Z(0)}\int_{v\in V} e^{-\frac{1}{2}Q(v)}l_1(v)\dots l_{2k}(v)\\ &= \frac{1}{Z(0)}\int_{v\in V} e^{-\frac{1}{2}Q(v)}\left(\partial_{l_1}\dots\partial_{l_{2k}}|_{j=0}e^{j(v)}\right)\\ &= (\partial_{l_1}\dots\partial_{l_{2k}})\,|_{j=0}\frac{1}{Z(0)}\int_{v\in V} e^{-\frac{1}{2}Q(v)+j(v)} = (\partial_{l_1}\dots\partial_{l_{2k}})\,|_{j=0}e^{\frac{1}{2}\eta}\\ &= (\partial_{l_1}\dots\partial_{l_{2k}})\,\frac{1}{k!2^k}\eta^k = \frac{1}{k!2^k}(l_1\dots l_{2k}\,|\,\eta^k)\end{aligned}$$

■

Lemma 3.5. Let $f \in \mathrm{S}(V^\vee)$. Then

$$\langle f\rangle_0 = \partial_f e^{\frac{1}{2}\eta(j)}|_{j=0} = e^{\frac{1}{2}\partial_\eta}f(0). \tag{2.3.22}$$

The first equality in (2.3.22) is referred to by Zee as the *fundamental identity of Quantum Field Theory* [76].

PROOF. For $f = l_1 \dots l_{2k}$ we compute:

$$\begin{aligned} e^{\frac{1}{2}\partial_\eta} l_1 \dots l_{2k} &= \frac{1}{k!}\left(\frac{1}{2}\partial_\eta\right)^k l_1 \dots l_{2k} = \frac{1}{2^k k!}\partial_{\eta^k} l_1 \dots l_{2k} \\ &= \frac{1}{2^k k!}(\eta^k \,|\, l_1 \dots l_{2k}) = \langle l_1 \dots l_{2k} \rangle. \end{aligned}$$

With the previous lemma, we have proven the claim whenever $f = l_1 \dots l_n$. But by linearity, this is sufficient. ■

3.3. Now we pass from proper integrals to formal ones, and in the process drop our restrictions on Q.

DEFINITION 3.6. Let V be a graded vector space, $Q \in \mathrm{Sym}^2(V^\vee)$ a nondegenerate (but not necessarily positive-definite) quadratic form on V, $\eta = Q^{-1}$ and $f \in \mathcal{O}(V)$. Since what we are generalizing is the expectation value (where we divide by $Z(0)$), we can also drop the measure μ from the notation. Then the *formal integral of* f is

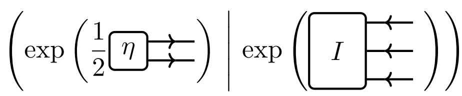
$$\fint_V e^{-\frac{1}{2}Q} : \; f \mapsto \fint_V e^{-\frac{1}{2}Q} f := \partial_f \exp\left(\frac{1}{2}\eta\right)\bigg|_{j=0} = \sum_k \frac{1}{k!2^k}(f_{2k} \,|\, \eta^k). \tag{2.3.23}$$

REMARK 3.7. The exponential factor $e^{-\frac{1}{2}Q}$ is part of the definition of the formal integral, and should stay attached to the integral sign.

4. Feynman diagrams

4.1. Suppose that we have an interaction term $I \in \mathrm{S}^3(V^\vee)$. Then we represent the expectation of e^I, using the fundamental identity as

$$\langle e^I \rangle = \left(\exp\left(\frac{1}{2}\eta\right) \,\middle|\, \exp(I)\right) = \tag{2.4.1}$$

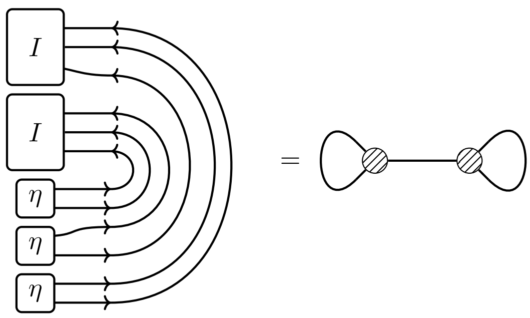


Thus this expectation is given by a sum over diagrams, where each diagram is a complete contraction of tensors (no loose strings) and therefore represents an element of $\mathrm{Hom}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$, i.e. a number.

The string diagram calculus of Chapter 1, while useful for *general considerations*, is somewhat clumsy in representing these contractions. For this reason we will present a condensed notation, such that any diagram in the new notation represents a unique string diagram. The tensor I is represented by a hashed blob, and the propagator η is simply written as a line. The directionality of string diagrams is redundant information, and is therefore dropped.

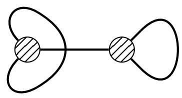
Then the first term of the expansion above looks like

(2.4.2)



If the tensors involved are indeed symmetric, then line crossings have no effect, so that the diagram above gives the same contribution as the following diagram:

(2.4.3)

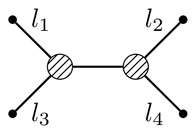
 .

Other types of diagrams appear in the computation of expectation values like the 4-point function

$$\langle l_1 l_2 l_3 l_4 \rangle = \langle l_1 l_2 l_3 l_4 e^I \rangle_0 \tag{2.4.4}$$

where one has *external* legs labelled by the covectors l_i, such as

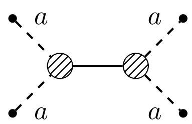
(2.4.5)



.

In the calculation of effective actions, one also sometimes has diagrams where the interaction tensor I is contracted directly with a vector a, without being coupled via a propagator. One needs a different type of line to express this, for instance a dashed line (not representing shifted vector spaces here):

(2.4.6)



.

The story of Feynman diagrams is told in many books. We refer the reader to [**23**] for a good account of the facts.

4.2. Let us sketch how the above story should play out for an infinite-dimensional vector space $F = \Gamma(M, E)$, where E is a vector bundle equipped with a smooth inner product $(\cdot, \cdot)$ on the fibres. The dual of this space is the set of distributional sections of E, but we adhere to the notational convention of treating every distribution as a function. That is, every distribution l has a formal kernel $l(x)$ and for any $f \in \Gamma(M, E)$

$$l(f) = \int_M (l(x), f(x))\,\mathrm{dVol}(x).$$

Similarly, every operator $T : F \to F$ has a kernel $T(x, y)$ and

$$(Tf)(y) = \int_M (T(x, y), f(x))\,\mathrm{dVol}(x).$$

On a formal level, we are identifying $F \cong F^\vee$ via the inner product

$$(f, g) = \int_M (f(x), g(x))\,\mathrm{dVol}(x)$$

but since F is not a Hilbert space, this is purely a notational device. However, the consequence is that our care in distinguishing between vectors and covectors is again somewhat obscured by the notation. We must take care to remember that the "functions" representing covectors can be very singular distributions.

The upshot is that tensor contractions are written as integrals over M. For instance, the contraction in (2.4.2) is given by the integral

$$\int_{M^6} \eta(x_1, x_2) I(x_1, x_2, x_3) \eta(x_3, x_4) I(x_4, x_5, x_6) \eta(x_5, x_6)\,\mathrm{d}^6 x.$$

A typical interaction with 3 legs is

$$I(\phi) = \lambda \int_M \phi^3(x)\,\mathrm{d}x$$

that is, I has "integral kernel"

$$I(x_1, x_2, x_3) = \lambda\delta(x_1 - x_2)\delta(x_2 - x_3)\delta(x_3 - x_1).$$

One then obtains the integral

$$\int_{M^2} \eta(x, x)\lambda\eta(x, y)\lambda\eta(y, y)\,\mathrm{d}x\,\mathrm{d}y$$

but since η tends to be singular on the diagonal, the integral is not defined.

4.3. There is a conventional notation for derivatives in infinite dimensions which goes under the name of *functional derivative*. The idea is the following. Let $\Phi \in \mathrm{S}^\bullet(F^\vee)$ and $f \in F$. Then taking the derivative of Φ at f defines a covector

$$\delta_f\Phi : g \mapsto (\partial_g\Phi)(f) \tag{2.4.7}$$

Then the functional derivative $\frac{\delta\Phi}{\delta f}(x)$ is the "integral kernel" of this covector, i.e.

$$(\partial_g\Phi)(f) = \int_M \frac{\delta\Phi}{\delta f}(x)g(x)\,\mathrm{d}x. \tag{2.4.8}$$

For instance, if we have a covector $\Phi_f(g) = \int f(x)g(x)\,\mathrm{d}x$, then we simply recover the (genuine) integral kernel

$$\frac{\delta\Phi}{\delta f}(x) = f(x).$$

5. The BV-BFV Formalism.

5.1. So far, our discussion has been restricted to the case where the quadratic form Q defining the kinetic term of the action S is non-degenerate. This hypothesis has been necessary as the Wick expansion proceeds in terms of contractions of the *propagator*, which is the tensor Q^{-1}.

A method for dealing with degenerate action functionals in the path integral was published by Faddeev and Popov in their two page note [30]. Their method was later generalized to the so-called *BRST formalism*, which in turn was extended to the *BV formalism*. The BV formalism, then, is the most powerful general method of perturbatively quantizing action functionals with degeneracies. As such, it has attracted a great deal of interest, resulting in an enormous literature on the subject. Let us mention as introductory works only [23, 54, 31]. For the relation of the BV formalism to homological algebra (in particular, homological perturbation theory) see [39, 40, 42].

The BV formalism as such does not deal with theories defined on manifolds with boundary. The natural extension to this setting goes by the name of BV-BFV formalism and was developed by Cattaneo, Mnev and Reshetikhin in a series of papers [12, 13, 14, 15].

The formalism has been applied to study (split) Chern-Simons theory and associated topological invariants [75, 16, 17], General Relativity [66], Poisson Sigma Models [18], and abelian BF theory [14].

5.2. We proceed to state the basic definitions of the BV formalism.

DEFINITION 5.1 ([17], 3.1). A *BV vector space* is a quadruple $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S})$, where $\mathcal{F}$ is a $\mathbb{Z}$-graded vector space, ω is a symplectic form on $\mathcal{F}$ of degree -1, $\mathcal{Q}$ is a vector field of degree +1 and $\mathcal{S}$ is a function of degree 0, such that

i) $\mathcal{Q}^2 = 0$
ii) $\iota_{\mathcal{Q}}\omega = \delta\mathcal{S}$.

REMARKS 5.2.

i) δ denotes the de Rham differential on $\mathcal{F}$.
ii) Condition ii) just says that $\mathcal{Q}$ is the Hamiltonian vector field of $\mathcal{S}$. Equivalently, ω induces a Poisson bracket $\{\cdot, \cdot\}$ and for any function F on $\mathcal{F}_M$, $\mathcal{Q}F = \{\mathcal{S}, F\}$. The condition i) is then just the *classical master equation* $\{\mathcal{S}, \mathcal{S}\} = 0$.
iii) The symplectic form is generally a constant form and can therefore be considered as an element $\omega \in \Lambda^2(V^\vee)$.

The basic mechanism of gauge-fixing in the BV formalism consists of two steps:

i) Extend the space of fields F_M to a BV vector space $\mathcal{F}_M$, with $(\mathcal{F}_M)_0 = F_M$ (in physics terms: one adds ghosts and antifields). Find an extended action functional $\mathcal{S}$ on $\mathcal{F}_M$ such that $\mathcal{S}|_{F_M} = S$.
ii) Choose a gauge-fixing subspace $\mathcal{L} \subset \mathcal{F}_M$ which is a *Lagrangian* subspace (for the symplectic form ω) and such that $\mathcal{S}$ restricted to $\mathcal{L}$ is nondegenerate. Gauge-fixed integrals are integrals over $\mathcal{L}$, e.g.

$$Z_M = \int_{\mathcal{L}} e^{\frac{i}{\hbar}\mathcal{S}[\phi]}[\mathrm{D}\phi]. \tag{2.5.1}$$

5.3. On the face of it, the partition function Z_M now depends on the choice of gauge-fixing Lagrangian. By imposing a further requirement, the *Quantum Master Equation*, on $\mathcal{S}$, this dependency can be removed.

From the symplectic form $\omega \in \Lambda^2(\mathcal{F}_M^\vee)$ we can construct a form $\omega^\vee$ in the following way. By nondegeneracy, ω induces an isomorphism

$$\omega^\# : \mathcal{F}_M[-1] \to \mathcal{F}_M^\vee \tag{2.5.2}$$

with inverse

$$(\omega^\#)^{-1} : \mathcal{F}_M^\vee \to \mathcal{F}_M[-1]. \tag{2.5.3}$$

Applying the functor Λ^2, we obtain an isomorphism

$$\Lambda^2((\omega^\#)^{-1}) : \Lambda^2(\mathcal{F}_M^\vee) \to \Lambda^2(\mathcal{F}_M[-1]) \cong \mathrm{S}^2(\mathcal{F}_M)[-2]. \tag{2.5.4}$$

The image of ω under this map is an element of degree 1 in $\mathrm{S}^2(\mathcal{F}_M)$ and is denoted $\omega^\vee$.

DEFINITION 5.3. The *BV Laplacian* associated to a symplectic form ω is the operator

$$\Delta_{\mathrm{BV}} = \partial_{\omega^\vee}. \tag{2.5.5}$$

DEFINITION 5.4. The *Quantum Master Equation* (QME) is the condition

$$\Delta_{\mathrm{BV}} \exp\left(\frac{1}{\hbar}S\right) = 0 \tag{2.5.6}$$

or, equivalently,

$$\frac{1}{2}\{S, S\} + i\hbar\Delta_{\mathrm{BV}}S = 0. \tag{2.5.7}$$

If $\mathcal{S}$ satisfies the Quantum Master Equation, then Z_M does not depend on the choice of gauge-fixing Lagrangian.

5.4. We now move on to the BV-BFV formalism.

DEFINITION 5.5 ([17], 3.2). A *BFV vector space* is a triple $(\mathcal{F}^\partial, \omega^\partial, \mathcal{Q}^\partial)$ where $\mathcal{F}^\partial$ is a $\mathbb{Z}$-graded vector space, ω is a symplectic form on $\mathcal{F}^\partial$ of degree 0, $\mathcal{Q}^\partial$ is a vector field of degree $+1$ which is symplectic and satisfies $(\mathcal{Q}^\partial)^2 = 0$.

LEMMA 5.6. Let $\mathcal{Q}$ be a symplectic degree-1 vector field with $\mathcal{Q}^2 = 0$. Then $\mathcal{Q}$ is Hamiltonian.

DEFINITION 5.7 ([17], 3.3). Let $(\mathcal{F}^\partial, \omega^\partial, \mathcal{Q}^\partial)$ be a BFV vector space with exact BFV form $\omega^\partial = \delta\alpha^\partial$. A *BV-BFV vector space over* $\mathcal{F}^\partial$ is a quintuple $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S}, \pi)$, where $(\mathcal{F}, \omega, \mathcal{Q}, \mathcal{S})$ is a BV vector space and $\pi : \mathcal{F} \to \mathcal{F}^\partial$ is a subjective submersion such that

i) $\delta\pi\mathcal{Q} = \mathcal{Q}^\partial$
ii) $\iota_{\mathcal{Q}}\omega = \delta\mathcal{S} + \pi^*\alpha^\partial$.

Because of the previous lemma, any BV-BFV vector space automatically has a function $\mathcal{S}^\partial$, the Hamiltonian function for $\mathcal{Q}^\partial$.

The quantization procedure is somewhat more involved than in the purely BV case. It consists of the following steps:

i) Choose a polarization $\mathcal{P}$ on $\mathcal{F}^\partial_{\partial M}$, that is a fibration $p_\partial : \mathcal{F}^\partial_{\partial M} \to \mathcal{B}^\partial$ such that the fibres are Lagrangian. For our purposes we may assume that the polarization is actually $\mathcal{F}^\partial = \mathcal{B}^\partial_1 \oplus \mathcal{B}^\partial_2$ with both subspaces being Lagrangian. Either subspace may then be chosen as base or as fibre of the polarization.
ii) Modify the 1-form $\alpha^\partial \mapsto \tilde{\alpha}^\partial$ so that $\tilde{\alpha}^\partial$ vanishes on the fibres of $\mathcal{P}$. The action $\mathcal{S}$ needs to be shifted correspondingly by a boundary term.
iii) Choose an extension of boundary fields, that is a section $e : \mathcal{B}^\partial \to \mathcal{F}_M$ of $\mathcal{F}_M \to \mathcal{F}^\partial_{\partial M} \to \mathcal{B}^\partial$. Generally, this is the discontinuous extension $e(\phi) = \phi\chi_{\partial M}$ where $\chi_{\partial M}$ is the characteristic function of ∂M. Formally, this splits $\mathcal{F}_M$ into boundary fields and the rest, i.e. "bulk fields", $\mathcal{F}_M = e(\mathcal{B}^\partial) \times \mathcal{Y}$.
iv) The partition function is formally

$$Z_M(b) = \int_{\phi\in\mathcal{L}\subset\mathcal{Y}} \exp\left(\frac{1}{\hbar}\mathcal{S}[b, \phi]\right) \tag{2.5.8}$$

for $\mathcal{L}$ a gauge-fixing Lagrangian subspace of $\mathcal{Y}$.

Not only is the construction somewhat lengthier than in the case without boundary, but the question of independence of the choice of gauge-fixing Lagrangian is also subtler. We come now to a step which, in the scope of the BV-BFV formalism, remains unclarifie, perhaps even mysterious. The boundary action S^∂ must be quantized to yield an operator $\widehat{\Omega}$. How this quantization is to proceed is not specified by the formalism, though the idea is to employ methods of geometric quantization.

One condition on the quantization is that it yields a coboundary operator, $\widehat{\Omega}^2 = 0$. The partition function is then a cocycle in the $\widehat{\Omega}$-complex

$$\widehat{\Omega}\psi_M = 0. \tag{2.5.9}$$

Finally, one can show that a change of gauge-fixing subspace changes the partition function by an $\widehat{\Omega}$-exact term. Thus, the partition function is well-defined as an element of the $\widehat{\Omega}$-cohomology.

5.5. We want to discuss a class of examples of the BV-BFV formalisms which is broad enough to exhibit the general features of the formalism, without being hampered too much by the need for full generality. Let us call the data we describe in this section the *generic first-order model* of the BV-BFV formalism. The term *first-order* refers to the fact that the differential operator in the quadratic form will be assumed to be first-order.

The classical space of fields on the manifold M is $\Gamma(M, E)$ and is the body of $\mathcal{F}_M = \Gamma(M, \mathcal{E})$ where $\mathcal{E}$ is a super vector bundle with body $E = \mathcal{E}_0$. Suppose that the boundary fields are similarly $\mathcal{F}_{\partial M} = \Gamma(\partial M, \mathcal{E}')$ for some super vector bundle $\mathcal{E}'$. We assume that fibres $\mathcal{E}_p$ are equipped with a nondegenerate bilinear form $(\cdot,\cdot)$ inducing a nondegenerate bilinear form on sections,

$$(\phi_1, \phi_2) = \int_M (\phi_1(x), \phi_2(x))\, \mathrm{dVol}(x). \tag{2.5.10}$$

Suppose further that $\mathcal{E}'$ splits as $\mathcal{E}' = \mathcal{E}_b \oplus \mathcal{E}_f$ and that there is a corresponding polarization of boundary fields, $\mathcal{F}_M = \Gamma(\partial M, \mathcal{E}_b) \oplus \Gamma(\partial M, \mathcal{E}_f)$. We consider the first summand to be the base B_∂ of the polarization and the second summand to represent the fibres. Hence any particular fibre may be written as

$$p_\partial^{-1}(b) = b + \Gamma(\partial M, \mathcal{E}_f), \quad b \in \Gamma(\partial M, \mathcal{E}_b).$$

In order to represent the fibres $\pi^{-1}p_\partial^{-1}(b)$, we suppose that we have a section of π, i.e.an extension map $e : \mathcal{F}_{\partial M} \to \mathcal{F}_M$ such that $\pi \circ e = \mathrm{id}_{\mathcal{F}_{\partial M}}$. We define a space of bulk fields with boundary values in a subspace,

$$\mathcal{F}_M(V) := \{\phi \in \mathcal{F}_M | \pi(\phi) \in V\}. \tag{2.5.11}$$

Then the fibres in the bulk can be represented as

$$\pi^{-1}p_\partial^{-1}(b) = e(b) + \mathcal{F}_M(\Gamma(\partial M, \mathcal{E}_f)). \tag{2.5.12}$$

Thus in order to define the formal path integral, we need to fix a gauge by picking a Lagrangian subspace $\mathcal{L} \subset \mathcal{F}_M(\Gamma(\partial M, \mathcal{E}_f))$. For instance, one may have some subspace $\mathcal{L}_{\mathrm{gf}}$ which lends itself to constructing $\mathcal{L} = \mathcal{L}_{\mathrm{gf}} \cap \mathcal{F}_M(\Gamma(\partial M, \mathcal{E}_f))$. We then have

$$\begin{aligned}\psi_M(b) &= \int_{\pi^{-1}p_\partial^{-1}(b)} \exp\left(\frac{1}{2}q(\phi) + I(\phi)\right) = \int_{\phi\in\mathcal{L}} \exp\left(\frac{1}{2}q(\phi + e(b)) + I(\phi + e(b))\right)\\ &= e^{\frac{1}{2}q(e(b))} \int_{\mathcal{L}} \exp\left(\frac{1}{2}q(\phi) + I(\phi + e(b)) + \frac{1}{2}q(\phi, e(b)) + \frac{1}{2}q(e(b), \phi)\right).\end{aligned}$$

We see that in principle, the boundary fields act as sources for the bulk fields (recall that $q(\cdot, \phi)$ is a covector, i.e. a source). However, these terms, as well as $q(e(b), e(b))$, tend to be singular and ill-defined if handled naively. This is because, on general principles, the extension $e(b)$ must be the discontinuous extension by zero in the bulk (see [14] for a discussion of this point).

CHAPTER 3

Chern-Simons Theory.

1. Generalities on Chern-Simons theory.

1.1. Chern-Simons theory proper is a topological field theory whose fields are connection 1-forms of some G-principal bundle on a 3-manifold M. In this thesis, we will consider only the rather simpler case in which the pricipal bundle is trivial. We may then identify the connection 1-forms with $\Omega^1(M, \mathfrak{g})$, where $\mathfrak{g}$ the Lie algebra of G.

The *Chern-Simons* action is the function $S \in C^\infty(\Omega^1(M, \mathfrak{g}))$ defined by

$$S(A) = \int_M \operatorname{Tr}\left(\frac{1}{2} A \wedge \mathrm{d}A + \frac{1}{6}[A, A] \wedge A\right). \tag{3.1.1}$$

We will consider only abelian Chern-Simons theory, that is Chern-Simons with abelian Lie algebra $\mathfrak{g}$. The cubic term then vanishes. One elegant way of constructing the correct BV description of Chern-Simons theory is to exhibit it as a special case of the *AKSZ construction* (originally due to Alexandrov, Kontsevich, Schwarz and Zaboronsky [2]), which automatically outputs a BV theory. While in some sense the power of the AKSZ construction is overkill for something as simple as abelian Chern-Simons theory, it is useful to unpack the definitions of AKSZ in a simple case in order to gain understanding of more complex situations where the power of the construction really is needed.

1.2. There are two important pairings in Chern-Simons theory, the *quadratic form* appearing as the free part of the action and the *sympletic form* which is the starting point for the BV formalism. They are given by

$$q(A, B) = \int_M A \wedge \mathrm{d}B \tag{3.1.2}$$

$$\omega(A, B) = \int_M A \wedge B. \tag{3.1.3}$$

These pairings satisfy the following symmetry relations (denoting by $\deg A$ the form degree of A):

$$\omega(A, B) = (-1)^{\deg A \cdot \deg B} \omega(B, A) = \omega(B, A) \tag{3.1.4}$$

$$q(A, B) = (-1)^{\deg A \cdot \deg B + 1} q(B, A) = (-1)^{(\deg A - 1) \cdot (\deg B - 1)} q(B, A). \tag{3.1.5}$$

To verify these relations, note that $\omega(A, B) = 0$ unless $\deg A + \deg B = \dim M = 3$. Similarly, $q(A, B) = 0$ unless $\deg A + \deg B = 2$.

Now it is necessary to interpret these pairings as a symmetric bilinear $q \in \mathrm{S}^2(\mathcal{F}_{CS}^\vee)$ and a 2-form $\omega \in \Lambda^2(\mathcal{F}_{CS}^\vee)$. To this end, we define the graded vector space $\mathcal{F}_{CS}$ degree-wise in the following way.

$\Omega^0(M)$	degree -1; "ghosts"
$\Omega^1(M)$	degree 0; "fields"
$\Omega^2(M)$	degree 1; "anti-fields"
$\Omega^3(M)$	degree 2; "anti-ghosts"

If we consider $\Omega^\bullet(M)$ as a graded vector space with grading given by the form degree, this is simply $\Omega^\bullet(M)[-1]$. Considering q as a linear map on $\mathcal{F}_{CS}^{\otimes 2}$, we see that it is supported in degree 0 of $\mathcal{F}_{CS}^{\otimes 2}$. Similarly, ω is supported in degree 1 of $\mathcal{F}_{CS}^{\otimes 2}$. If we denote the new grading of a form A by $|A| = \deg A - 1$, we then have

$$q(A \otimes B) = (-1)^{|A|\cdot|B|} q(B \otimes A) \tag{3.1.6}$$

$$\omega(A \otimes B) = -(-1)^{|A|\cdot|B|} \omega(B \otimes A). \tag{3.1.7}$$

2. AKSZ construction of Chern-Simons theory.

2.1. What we will give as the AKSZ construction is a simplified version of lesser generality; in particular, the target is a graded vector space (rather than a graded manifold) and the source is fixed to be $T[1]M$ for some manifold M. These assumptions, which are not necessary for the full AKSZ construction, save us from having to discuss graded manifolds, which would go beyond the scope of this thesis. Chern-Simons theory as an AKSZ model is discussed in [**12**, **54**, **1**].

DEFINITION 2.1. By *AKSZ data*, we mean a collection consisting of the following:

i) A source: an n-manifold M modeling spacetime.
ii) A target: a graded vector space Y,
iii) equipped with a symplectic form $\omega_Y = \mathrm{d}\alpha_Y$ of degree $n-1$ and a symplectic vector field $\mathcal{Q}_Y$ of degree 1. Symplectic vector field here means $L_{\mathcal{Q}_Y}\omega_Y = 0$.
iii) A graded vector space of fields:

$$\mathcal{F}_{\text{AKSZ}} = \Omega^\bullet(M) \otimes Y \cong C^\infty(T[1]M, Y). \tag{3.2.1}$$

In the following, we will need to make computations which fully bring out the difficulties in handling the signs of graded algebra. Frequently, the precise way in which some familiar object (such as $\Omega^\bullet(M)$) is viewed as a graded vector space differs within the literature or is not explicitly specified. Furthermore, there is the common abuse of notation which consists in writing the linear functional $f \mapsto f(x)$ as "$f(x)$" itself. Both of these sources of confusion are related to the *superfield formalism*, which makes use of the observation that the suspension (or desuspension) map can be written as an element of a tensor product, sometimes called the *generic element*. We will come back to this point later.

DEFINITION 2.2. Let $\{T_\mu\}$ be a basis for $\mathfrak{g}$, θ_μ the dual basis of $\mathfrak{g}^\vee$ and let s be the suspension map $s : V \mapsto V[1]$. Then $\psi := T_\mu s^{-1}(\theta^\mu)$ is the *shifted identity map* $\mathrm{id}_{\mathfrak{g}}[-1] : \mathfrak{g}[1] \to \mathfrak{g}$. Sometimes the expression *generic element* of $\mathfrak{g}[1]$ is also used.

Recall that if $l \in V^\vee$, then $\mathrm{d}l \in \Omega^1(V) \cong \Gamma(V, V \oplus V^\vee)$ is the constant 1-form $\mathrm{d}l_p(v) = l(v)$.

Definition 2.3. The Chern-Simons-AKSZ data are given by taking $Y = \mathfrak{g}[1]$, where $\mathfrak{g}$ is a Lie algebra equipped with an Ad-invariant inner product $(\cdot, \cdot)$ and

$$\omega = \frac{1}{2}(\mathrm{d}\psi, \mathrm{d}\psi) \tag{3.2.2}$$

$$\alpha = \frac{1}{2}(\psi, \mathrm{d}\psi). \tag{3.2.3}$$

2.2. There is a method, called *transgression*, for transporting structure from the target to the space of fields.

Let $p : E = F \times B \to B$ be a product bundle, (F stands for *fibre*, B for *base* and E, the total space, has no mnemonic). Then we can pull back forms from the base to the bundle:

$$p^* : \Omega^\bullet(B) \to \Omega^\bullet(E) \tag{3.2.4}$$

We now look for a pushforward of forms (rather than vectors), $p_* : \Omega^\bullet(E) \to \Omega^\bullet(B)$. Such constructions of *integration along the fibre* are common in algebraic topology (c.f.§4 of Bott and Tu's textbook [11]). Let $l \in \hom(C^\infty(F), \mathbb{R})$ be a linear functional. This is thought of as an integration map, but we have no need for any integration theory at this point.

Definition 2.4. The pushforward of forms is defined as

$$p_* : \Omega^\bullet(F) \otimes \Omega^\bullet(B) \cong C^\infty(F) \otimes \Omega^\bullet(B) \oplus \Omega^{p>0} \otimes \Omega^\bullet(B) \tag{3.2.5}$$

$$p_* = (l \otimes 1, 0). \tag{3.2.6}$$

Let us present this map somewhat more concretely. We call the vector spaces in the above direct sum decomposition the forms of *type I* and *type II*, $\Omega^\bullet(E) = \Omega_I(E) \oplus \Omega_{II}(E)$. The types look like

$$\omega_I = p^*(\phi)g, \qquad g \in C^\infty(E) \tag{3.2.7}$$

$$\omega_{II} = \sum_\alpha p^*(\phi_\alpha) f_\alpha dx^\alpha \tag{3.2.8}$$

where the sum runs over multiindices α with $|\alpha| > 0$ and $\phi \in \Omega^\bullet(B)$. The pushforward p_* acts on the forms ω_I, ω_{II} by

$$p_* \omega_I = \phi \cdot l(g) \tag{3.2.9}$$

$$p_* \omega_{II} = 0. \tag{3.2.10}$$

Proposition 2.5. The pushforward has the following properties:

i) p_* is linear
ii) $p_* p^* \phi = \phi \int_F 1$
iii) $\mathrm{d}_Y p_* \omega = p_* \, \mathrm{d}_E \omega$

where for clarity we denote the space the exterior derivative acts on by a subscript.

PROOF. Property iii) is the only one which is not immediately clear, but it can still be obtained by a simple calculation:

(3.2.11)
$$\mathrm{d}_E\omega_I = (\mathrm{d}_E p^*\phi)g + (-1)^{|p^*\phi|}p^*(\phi)\,\mathrm{d}_E g = p^*(\mathrm{d}_B\phi)g + (-1)^{|\phi|}p^*(\phi)(\mathrm{d}_F g + \mathrm{d}_B g)$$

(3.2.12)
$$p_*\,\mathrm{d}_E\omega_I = \mathrm{d}_Y(\phi)\int_F g + (-1)^{|\phi|}\phi\int_F \mathrm{d}_B g = \mathrm{d}_B\left(\phi\int_F g\right) = \mathrm{d}_B p_*\omega_I$$

■

Consider now the diagrams

$$\begin{array}{ccc} T[1]M\times\mathcal{F} & \xrightarrow{\mathrm{ev}} & Y \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{F} & & \end{array} \qquad \begin{array}{ccc} C^\infty(\mathcal{F})\otimes\Omega^\bullet(M) & \xleftarrow{\mathrm{ev}^*} & Y \\ {\scriptstyle\pi^*}\uparrow & & \\ C^\infty(\mathcal{F}) & & \end{array}$$

where ev: $(x,\phi)\mapsto\phi(x)$ is the evaluation mapping and $\pi((x,\phi))=\phi$ is the projection onto the second factor. We construct the pushforward π_* as above, using the following integration map (canonical Berizinian) for functions on $T[1]M$. Recall that $C^\infty(T[1]M)\cong\Omega^\bullet(M)$.

(3.2.13)
$$\int_{T[1]M}\mu\beta = \int_M \beta$$

for β a top-degree form and $\int\mu\beta = 0$ otherwise.

DEFINITION 2.6. The *transgression map* is defined as

(3.2.14)
$$\mathbb{T} = \pi_*\mathrm{ev}^* : \Omega^p(Y)_j \to \Omega^p(\mathcal{F})_{j-n}.$$

LEMMA 2.7. Transgression is a chain map, i.e. $\mathbb{T}\delta_Y = \delta_{\mathcal{F}}\mathbb{T}$, where we now denote the exterior derivative on $Y,\mathcal{F}$ by $\delta_Y,\delta_{\mathcal{F}}$ respectively.

PROOF. Both ev* (by virtue of being a pullback) and π_* (as we proved above) are chain maps. ■

2.3. Lifting the target data to the space of fields via the transgression map turns the AKSZ model into a BV theory. Before we can give a theorem to this effect, we need to make one more auxiliary definition.

DEFINITION 2.8. The lifted de Rham vector field is the derivation D on

(3.2.15)
$$C^\infty(T[1]M\times\mathcal{F})\cong C^\infty(\mathcal{F})\otimes\Omega^\bullet(M,\mathfrak{g})$$

given by $D = 1\otimes\mathrm{d}$.

Equivalently, viewing $C^\infty(T[1]M\times\mathcal{F})$ as $C^\infty(\mathcal{F},\Omega^\bullet(M))$, it is $(DF)(\omega)=\mathrm{d}F(\omega)$ for ω some (possibly inhomogeneous) form.

Suppose that $F\in C^\infty(T[1]M\times\mathcal{F})$ and δF is a 1-form on $T[1]M\times\mathcal{F}$. Then $(\iota_D\delta F)(\omega) = (DF)(\omega) = \mathrm{d}F(\omega)$.

THEOREM 2.9 ([54], 4.108). Define

(3.2.16)
$$\omega_{\mathrm{AKSZ}} = \mathbb{T}\omega_Y$$

(3.2.17)
$$S_{\mathrm{AKSZ}} = \iota_D\mathbb{T}\alpha_Y.$$

Then

i) ω_{AKSZ} is odd-symplectic,
ii) S_{AKSZ} satisfies the CME $\{S, S\} = 0$,
iii) S_{AKSZ} is Hamiltonian with Hamiltonian vector field $\mathcal{Q}$ and
iv) $\mathcal{Q}^2 = 0$.

2.4. The theorem of the previous section assures us that Chern-Simons is a BV theory. However, we should look more closely at the transgression map, in particular the intermediate stage on $T[1]M \times \mathcal{F}$ and find concrete expressions for the data $S, \omega, \mathcal{Q}$.

In particular, we need to understand the role of ψ, which gives rise to the so-called *superfield formalism.* Since $\psi \in \hom(\mathfrak{g}[1], \mathfrak{g})$, its pullback is a degree-1 element

$$\mathbf{A} := \mathrm{ev}^*\psi \in C^\infty(T[1]M \times \mathcal{F}) \otimes \mathfrak{g} \cong C^\infty(\mathcal{F}) \otimes \Omega^\bullet(M, \mathfrak{g}) \tag{3.2.18}$$

which we call the *superfield.* Let us denote the space of superfields as $\mathscr{F}$. Then

$$\mathscr{F} = C^\infty(\mathcal{F}) \otimes \Omega^\bullet(M, \mathfrak{g}) \cong C^\infty(\mathcal{F}, \mathcal{F}[1]) \tag{3.2.19}$$

since $\mathcal{F}$ is the (-1)-shift of $\Omega^\bullet(M, \mathfrak{g})$. This space naturally has a distinguished element, the shifted identity $\mathrm{id}[1]$.

Other elements of $\mathscr{F}$ which one might naturally consider are shifts of the projections $\Omega^\bullet(M) \to \Omega^i(M)$. Fix i and let $\{T^\mu\}$ be a basis for $\Omega^i(M)$ with dual basis $\{\theta_\mu\}$. If $\Omega^k(M)$ is considered as a graded vector space concentrated in degree k, its dual (and therefore θ_μ) is concentrated in degree $-k$. Then using the suspension map we place θ_μ in degree $1-k$ to get shifted projections, which are degree-1 elements

$$A^{(i)} := \sum_\mu s(\theta_\mu) \otimes T^\mu \in C^\infty(\mathcal{F}, \mathcal{F}[1]). \tag{3.2.20}$$

Because the direct sum of the Ω^i is $\Omega^\bullet$, the sum of the projections is the identity. Hence the superfield decomposes according to the degree of forms

$$\mathbf{A} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} \tag{3.2.21}$$

Typically we write $\gamma = A^{(0)}, A = A^{(1)}, A^+ = A^{(2)}, \gamma^+ = A^{(3)}$. For emphasis, we repeat that these are not themselves forms; they are shifted projections onto homogeneous components of differential forms, or in other words, form-valued *functions* on the space of fields $\mathcal{F}$.

Hence when we form expressions consisting of *products* (in some sense) of the $A^{(i)}$, the signs involved in commuting these components are not the signs of the natural graded algebra structure of $\Omega^\bullet(M)$, but rather of the tensor product of graded algebras, $\mathrm{S}^\bullet(\mathcal{F}) \otimes \Omega^\bullet(M)$. Thus $A^{(k)}A^{(l)} = -A^{(l)}A^{(k)}$, for any k, l.

The de Rham differential d on M lifts to a derivation $1 \otimes \mathrm{d}$ on $\mathscr{F}$, and the de Rham differential on $\mathcal{F}$ lifts to a derivation $\delta \otimes 1$ on $\mathscr{F}$, i.e. both of these lifted derivations pick up a sign according to the grading of $\mathscr{F}$, not of $\Omega^\bullet(M)$ or of $\mathcal{F}$ individually.

2.5. The action in abelian Chern-Simons is given by $S = \iota_D \mathbb{T}\alpha$. Now since $\mathbf{A} = \mathrm{ev}^*\psi$ is just a shifted identity map, the action of D on ψ is very simple:

$$(\iota_D \delta \mathbf{A})(s^{-1}\omega) = (D\mathbf{A})(s^{-1}\omega) = \mathrm{d}\omega. \tag{3.2.22}$$

By the definition of the pullback,

$$S = \frac{1}{2}\iota_D \pi_*(\mathrm{ev}^*\psi, \delta \mathrm{ev}^*\psi) = \frac{1}{2}\int_{T[1]M} (\mathbf{A}, D\mathbf{A}). \tag{3.2.23}$$

The Ad-invariant scalar product $(\cdot,\cdot)$ on $\mathfrak{g}$ lifts to a pairing

$$(\cdot,\cdot):\Omega^k(M,\mathfrak{g})\times\Omega^l(M,\mathfrak{g})\to\Omega^{k+l}(M)$$
$$(\alpha\otimes v,\beta\otimes w)=(\alpha\wedge\beta)\cdot(v,w)\qquad \alpha\in\Omega^k(M),\beta\in\Omega^l(M),\ v,w\in\mathfrak{g}$$

The pairing is further extended to forms on $\mathcal{F}$:

$$(\cdot,\cdot):\Omega^k(M,\mathfrak{g})\otimes\Omega^\bullet(\mathcal{F})\otimes\Omega^l(M,\mathfrak{g})\otimes\Omega^\bullet(\mathcal{F})\to\Omega^{k+l}(M)\otimes\Omega^\bullet(\mathcal{F}) \tag{3.2.24}$$
$$(\omega\otimes\phi,\eta\otimes\psi)=(-1)^{|\eta|\cdot|\phi|}(\omega,\eta)\phi\psi,\qquad \phi,\psi\in\Omega^\bullet(\mathcal{F}). \tag{3.2.25}$$

The pushforward π_* annihilates all terms which do not have form degree 3 (the dimension of M). Hence we find

$$S=\frac{1}{2}\int_M\left[(\gamma,\mathrm{d}A^+)+(A,\mathrm{d}A)+(A^+,\mathrm{d}\gamma)\right]. \tag{3.2.26}$$

We can simplify further by noting that since A^+ and $\mathrm{d}\gamma$ have different total degrees, they commute. Since we are simply integrating plain forms over a plain (ungraded) manifold, we employ Stokes' theorem as usual. But note that since d is actually $\mathrm{d}\otimes\mathrm{id}$, the sign that appears is the *total degree* of the first field:

$$\int_{\partial M}(\phi,\psi)=\int_M\mathrm{d}(\phi,\psi)=\int_M\Big((\phi,\psi)+(-1)^{|\phi|}(\phi,\mathrm{d}\psi)\Big). \tag{3.2.27}$$

Thus assuming for now that $\partial M=\emptyset$, we can compute

$$\int_M(A^+,\mathrm{d}\gamma)=(-1)^{|A^+|+1}\int_M(\mathrm{d}A^+,\gamma)=\int_M(\gamma,\mathrm{d}A^+), \tag{3.2.28}$$

and hence

$$S=\int_M\left[\frac{1}{2}(A,\mathrm{d}A)+(\gamma,\mathrm{d}A^+)\right]. \tag{3.2.29}$$

The BV symplectic form $\omega_{CS}=\mathbb{T}\omega$ is given by

(3.2.30)
$$\begin{aligned}\omega_{CS}&=\frac{1}{2}\int_M(\delta\mathbf{A},\delta\mathbf{A})=\frac{1}{2}\int_M\left[(\delta\gamma,\delta\gamma^+)+(\delta A,\delta A^+)+(\delta A^+,\delta A)+(\delta\gamma^+,\delta\gamma)\right]\\&=\int_M\left[(\delta\gamma,\delta\gamma^+)+(\delta A,\delta A^+)\right].\end{aligned}$$

So we recover the form ω from Section 1 of this chapter.

The cohomological vector field in the bulk, $\mathcal{Q}$, can be found from the Poisson bracket induced by ω and S:

$$\mathcal{Q}F=\{\mathcal{S},F\}=\int_M\left[\left(\mathrm{d}A^+,\frac{\delta F}{\delta\gamma^+}\right)+\left(\mathrm{d}A,\frac{\delta F}{\delta A^+}\right)-\left(\mathrm{d}\gamma,\frac{\delta F}{\delta A}\right)\right]. \tag{3.2.31}$$

2.6. We now let M be a 3-manifold with boundary and aim to construct the BV-BFV theory for the AKSZ-Chern-Simons model discussed above. We start with the variation of the action. To compute the variation, we take the de Rham differential δ on $\mathcal{F}$ and implicitly extend it as $\mathrm{id}\otimes\delta$ to $C^\infty(T[1]M\times\mathcal{F})$. Next, we drop the assumption that the boundary is empty, and compute

$$\begin{aligned}&(-1)^{|A|}\int_M(A,\delta\,\mathrm{d}A)=(-1)^{|A|+1}\int_M(A,\mathrm{d}\delta A)\\=&(-1)^{|A|+1}(-1)^{|A|}\left(\int_{\partial M}(A,\delta A)-\int_M(\mathrm{d}A,\delta A)\right)=\quad\int_M(\mathrm{d}A,\delta A)-\int_{\partial M}(A,\delta A).\end{aligned}$$

Using an analagous computation for $(\gamma, \delta\, \mathrm{d}A^+)$, we obtain the variation

$$\begin{aligned}(3.2.32)\quad \delta\mathcal{S} &= \int_M \left[\frac{1}{2}(\delta A, \mathrm{d}A) + \frac{1}{2}(-1)^{|A|}(A, \delta\, \mathrm{d}A) + (\delta\gamma, \mathrm{d}A^+) + (-1)^{|\gamma|}(\gamma, \delta\, \mathrm{d}A^+)\right] \\ &= \int_M \left[(\delta A, \mathrm{d}A) + (\delta\gamma, \mathrm{d}A^+) + (\delta A^+, \mathrm{d}\gamma)\right] - \int_{\partial M}\left[\frac{1}{2}(A, \delta A) + (\gamma, \delta A^+)\right]\end{aligned}$$

from which we can read off the functional derivatives of S. Note that since the total degree of $\delta\phi$ is 1 for $\phi \in \{\gamma, A, A^+, \gamma^+\}$, the left- and right-sided functional derivatives are equal.

$$(3.2.33)\qquad \frac{\delta\mathcal{S}}{\delta A} = \mathrm{d}A, \quad \frac{\delta\mathcal{S}}{\delta\gamma} = \mathrm{d}A^+, \quad \frac{\delta\mathcal{S}}{\delta A^+} = \mathrm{d}\gamma.$$

The boundary term is

$$(3.2.34)\qquad \pi_\partial^*(\alpha_\partial) = -\int_{\partial M}\left[(\gamma, \delta A^+) + \frac{1}{2}(A, \delta A)\right].$$

From the boundary one-form we find the BFV form

$$(3.2.35)\qquad \Omega_\partial = \delta\alpha_\partial = -\int_{\partial M}\left[(\delta\gamma, \delta A^+) + \frac{1}{2}(\delta A, \delta A)\right].$$

The cohomological vector field on the boundary, $\mathcal{Q}_\partial$, is simply the restriction of $\mathcal{Q}$, i.e. the pullback by the inclusion map $\iota_\partial : \mathcal{F}_\partial \to \mathcal{F}$. Since restricting to the boundary kills top-forms, we drop the term with $\mathrm{d}A^+$ to get

$$(3.2.36)\qquad \mathcal{Q}_\partial = \int_{\partial M}\left[\left(\mathrm{d}A, \frac{\delta}{\delta A^+}\right) - \left(\mathrm{d}\gamma, \frac{\delta}{\delta A}\right)\right]$$

which gives

$$\begin{aligned}(3.2.37)\qquad \delta S_\partial = \iota_{\mathcal{Q}_\partial}\Omega_\partial &= -\int_{\partial M}\left[\frac{1}{2}(\mathcal{Q}_\partial A, \delta A) + \frac{1}{2}(\delta A, \mathcal{Q}_\partial A) + (\delta\gamma, \mathcal{Q}_\partial A^+)\right] \\ &= \int_{\partial M}\left[(\delta A, \mathrm{d}\gamma) - (\mathrm{d}\gamma, \delta A)\right] = -\delta\int(\gamma, \mathrm{d}A).\end{aligned}$$

And we deduce

$$(3.2.38)\qquad S_\partial = -\int_{\partial M}(\gamma, \mathrm{d}A).$$

2.7. An important class of observables in any gauge theory are the *Wilson lines*. Here we just give the basic definitions; a more systematic treatment can be found in [53]/

Definition 2.10. Let $\gamma : [0,1] \to M$ be a smooth curve. The *line observable* associated to γ is the linear functional

$$(3.2.39)\qquad u_\gamma \in \mathcal{F}_M^\vee,\ u_\gamma(A) = \operatorname{tr}\int_\gamma A.$$

Since γ is 1-dimensional, u_γ is non-trivial only on 1-forms.

Definition 2.11. Let $\gamma : [0,1] \to M$ be a smooth curve. The *Wilson observable* associated to γ is the observable

$$(3.2.40)\qquad \mathcal{W}_\gamma = \exp(u_\gamma) \in \mathcal{O}(\mathcal{F}_M).$$

As long as the space of fields consists only of smooth forms, Wilson observables are well-defined. However, in the present form they are not amenable to extending our considerations to distributional forms. This may be remedied by associating to a curve γ a 2-form B_γ such that

$$\int_\gamma A = \int_M A \wedge B_\gamma, \tag{3.2.41}$$

i.e. B_γ should be Poincaré dual to γ. We also want the support of B_γ to be localized near γ. It is a standard result of algebraic topology that such a form can be found for any smooth curve.

Proposition 2.12. *Let $N \subset M$ be a submanifold and choose a tubular neighbourhood U of N in M. Let τ be the Thom form of the normal bundle of N in M and Φ the pullback of τ to U. Then Φ is Poincaré dual to N and has support in U.*

In the following, we will always assume that any Wilson operator $\mathcal{W}_\gamma$ is expressed in the form (3.2.41) in a way such that the support of B_γ is "small enough". In particular, when treating of multiple Wilson observables for different (disjoint) curves, we assume that the supports of the Poincaré duals do not intersect.

Classically (i.e. evaluated on closed 1-forms, $\mathrm{d}A = 0$), if the curve γ does not intersect ∂M, the Wilson observable $\mathcal{W}_\gamma$ only depends on the cohomology class of B_γ (and hence on the homology class of γ). In particular, if γ is a contractible loop, $\mathcal{W}_\gamma(A) = 1$ for any closed 1-form A. But quantum mechanically, the expectation values $\langle \mathcal{W}_\gamma \rangle$ contain more information, because we integrate also over forms which may not be closed.

3. Canonical quantization on the boundary.

3.1. The next step is the canonical quantization of boundary theory and the derivation of the constraint equation. Let $\Sigma = \partial M$. A choice of complex structure on Σ induces a polarization of the connection 1-form

$$A = A_z \,\mathrm{d}z + A_{\bar z}\,\mathrm{d}\bar z. \tag{3.3.1}$$

In these coordinates, the boundary BFV form is

$$\Omega_\partial = \int_{\partial M} \mathrm{d}z\,\mathrm{d}\bar z \left[(\delta A_z, \delta A_{\bar z}) + (\delta\gamma, \delta A^+)\right] \tag{3.3.2}$$

Canonical quantization then proceeds by the operator assignments

$$A_{\bar z} \mapsto \mathcal{M}_a, \qquad A_z \mapsto -\frac{\delta}{\delta a},$$
$$\gamma \mapsto \mathcal{M}_\gamma, \qquad A^+ \mapsto \frac{\delta}{\delta\gamma}$$

in order to obtain canonical commutation relations. $\mathcal{M}_a$ stands for the operator of multiplication by a. The vector space of states on which these operators act is the space of functions on the boundary space of fields. In other words, it is the space of functionals in a, γ, $\mathcal{H} = \mathrm{Fun}(a,\gamma) = C^\infty(\mathcal{F}_\partial)$. The physical states are those functionals independent of γ and annihilated by the "BFV charge" $\hat S_\partial$, that is the quantization of the boundary action S_∂. Since $\mathrm{d}A = (\partial_{\bar z} A_z)\,\mathrm{d}\bar z\,\mathrm{d}z + (\partial_z A_{\bar z})\,\mathrm{d}z\,\mathrm{d}\bar z$, we have

$$\hat S_\partial = \int_{\partial M} \mathrm{d}z\,\mathrm{d}\bar z (\partial a - \bar\partial \frac{\delta}{\delta a}, \gamma). \tag{3.3.3}$$

The constraint equation for a state $\psi \in \mathcal{H}$ is therefore

$$\left[\partial a - \bar{\partial}\frac{\delta}{\delta a}\right]\psi(a) = 0. \tag{3.3.4}$$

Proposition 3.1. Let $T : C^\infty(\mathcal{F}_\partial) \to C^\infty(\mathcal{F}_\partial)$ be an operator such that

$$\partial_{\bar{z}}(Ta) = \partial_z a. \tag{3.3.5}$$

Such an operator is commonly written suggestively as

$$T = \frac{\partial_z}{\partial_{\bar{z}}} = 2\frac{\partial_z^2}{\Box}. \tag{3.3.6}$$

A solution to 3.3.4 is then given by $\psi(a) = \exp(\Gamma(a))$, where

$$\Gamma(a) = \frac{1}{2}\int_{\partial M}(a, Ta). \tag{3.3.7}$$

Proof. The functional derivative at a point x_0 satisfies the following rule:

$$\frac{\delta}{\delta a(x_0)}\int a(x)K(x,y)a(y)\,\mathrm{d}x\,\mathrm{d}y = 2\int K(x_0,y)a(y)\,\mathrm{d}y. \tag{3.3.8}$$

Hence for an operator T with an integral kernel $(Ta)(x) = \int K(x,y)a(y)\,\mathrm{d}y$, we have

$$\frac{\delta}{\delta a(x)}\int a(y)(Ta)(y)\,\mathrm{d}y = (Ta)(x). \tag{3.3.9}$$

The functional derivative also acts on exponentials just as the normal derivative does; for a functional $F(a)$, we have

$$\frac{\delta}{\delta a(x)}e^{F(a)} = \left(\frac{\delta}{\delta a(x)}F(a)\right)e^{F(a)}. \tag{3.3.10}$$

Hence

$$\partial_{\bar{z}}\frac{\delta}{\delta a(x)}\psi(a) = \partial_{\bar{z}}(Ta)(x)\psi(a) = (\partial_z a)(x)\psi(a). \tag{3.3.11}$$

■

This is an effective action; indeed

$$\psi(a) = \int_{\mathcal{F}_\partial}\exp(iS(\phi,a))[\mathrm{D}\phi] \tag{3.3.12}$$

with

$$S(\phi,a) = \int_{\partial M}(\phi,\Box\phi) + (\phi,\mathrm{d}a). \tag{3.3.13}$$

4. Chern-Simons propagator.

4.1. In order to actually compute observables of the CS theory, we need to fix a gauge. We follow [34] in choosing a so-called *axial gauge* on the cylinder $M = \mathbb{R}\times\Sigma$. Let us consider first the case $\Sigma = \mathbb{R}^2$. We choose a complex structure on $\mathbb{R}^2$ and hence consider the base of the cylinder to be $\Sigma = \mathbb{C}$. The 1-forms on $\mathbb{C}$ decompose as

$$\Omega^1(\mathbb{C}) \cong \Omega_z \oplus \Omega_{\bar{z}} \tag{3.4.1}$$

where Ω_z is spanned by 1-forms $f\,\mathrm{d}z$, $f \in C^\infty(\mathbb{C})$ and $\Omega_{\bar{z}}$ is spanned by elements of the form $f\,\mathrm{d}\bar{z}$. The 2-forms on $M = \mathbb{R}\times\mathbb{C}$ similarly decompose into forms which are products of $\mathrm{d}t, \mathrm{d}z$ and $\mathrm{d}\overline{z}$. We single out the subspace Ω^2_z spanned by elements of the form $f\,\mathrm{d}t \wedge \mathrm{d}z$. Recall that the space of fields in Chern-Simons theory is

$$\mathcal{F}_{CS} = \Omega^0(M,\mathfrak{g})[-1] \oplus \Omega^1(M,\mathfrak{g}) \oplus \Omega^2(M,\mathfrak{g})[1] \oplus \Omega^3(M,\mathfrak{g})[2]. \tag{3.4.2}$$

The idea is to pick a subspace which enforces the condition $\mathrm{d}\overline{z} = 0$.

Proposition 4.1. The subspace

$$\begin{aligned} \mathcal{L} &\subset \Omega^0(M,\mathfrak{g})[-1] \oplus C^\infty(M) \otimes \big(\Omega_z(\mathbb{C},\mathfrak{g}) \oplus \Omega^1(\mathbb{R},\mathfrak{g})\big) \oplus \Omega^2_z(\mathbb{C},\mathfrak{g})[1] && (3.4.3)\\ &= \ker(\mathrm{d}^*_{\overline{z}}), && (3.4.4)\end{aligned}$$

consisting of forms of compact support, is a Lagrangian subspace of $\mathcal{F}_{CS}$.

Proof. A simple calculation shows that $\mathcal{L}$ is isotropic and $\mathcal{L} \cap \star\mathcal{L} = \emptyset$. ■

4.2. Now consider the abelian Chern-Simons action, i.e. the quadratic form

$$Q(\mathbf{A}) = \int_M (\mathbf{A}, \mathrm{d}\mathbf{A}) \tag{3.4.5}$$

where $\mathbf{A}$ is the superfield. Pick a superfield in $\mathcal{L}$, i.e. of the form

$$\mathbf{A} = f + g_1\,\mathrm{d}z + g_2\,\mathrm{d}t + h\,\mathrm{d}t\,\mathrm{d}z \tag{3.4.6}$$

with $f \in \Omega^0(M,\mathfrak{g})[-1]$, $g_1, g_2 \in \Omega^0(M,\mathfrak{g})$, $h \in \Omega^0(M,\mathfrak{g})[1]$. Then

$$\begin{aligned} Q(\mathbf{A}) &= \int_M [(f, \partial_{\overline{z}} h) + (h, \partial_{\overline{z}} f) + (g_1, \partial_{\overline{z}} g_2) - (g_2, \partial_{\overline{z}} g_1)]\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\overline{z} \\ &= \int_M (\mathbf{A}, \mathrm{d}_{\overline{z}}\mathbf{A}). \end{aligned}$$

Thus in order to find the propagator for Q in this gauge, we need to find the Green's function for the operator $\mathrm{d}_{\overline{z}}$. In fact, this is provided by the Cauchy-Pompeiu formula (for details we refer the reader to [8] and references therein).

Definition 4.2. The *Pompeiu operator* $\mathcal{T}$ is defined by

$$(\mathcal{T}f)(w) = \frac{1}{\pi}\int_D \frac{f(z)}{w-z}\,\mathrm{d}x \wedge \mathrm{d}y. \tag{3.4.7}$$

The *Cauchy operator* $\mathcal{C}$ is defined by

$$(\mathcal{C}f)(w) = \frac{1}{2\pi i}\int_{\partial D} f(z)\frac{\mathrm{d}z}{z-w}. \tag{3.4.8}$$

The image of the Cauchy operator is an analytic function in D.

Proposition 4.3 (Cauchy-Pompeiu formula). Let $D \subset \mathbb{C}$ be an open disk and let f be differentiable on $\overline{D}$. Then

$$f(w) = (\mathcal{C}\, f|_{\partial D})(w) + (\mathcal{T}\partial_{\overline{z}} f)(w). \tag{3.4.9}$$

Proposition 4.4. Let $f \in C^\infty(D)$ and suppose that f vanishes on the boundary. Then $\mathcal{T}$ is both a left and a right inverse to $\partial_{\overline{z}}$, i.e.

$$f(z) = \partial_{\overline{z}}\mathcal{T}(f)(z) = \mathcal{T}(\partial_{\overline{z}} f)(z) \tag{3.4.10}$$

for $z \in D$.

Thus we obtain $\mathrm{d}_{\overline{z}}^{-1}$ by defining

$$\mathrm{d}_{\overline{z}}^{-1}: \begin{cases} f\,\mathrm{d}\overline{z} \mapsto \mathcal{T}(f) \\ f\,\mathrm{d}\overline{z}\wedge\mathrm{d}z \mapsto \mathcal{T}(f)\,\mathrm{d}z \\ f\,\mathrm{d}\overline{z}\wedge\mathrm{d}t \mapsto \mathcal{T}(f)\,\mathrm{d}t \\ f\,\mathrm{d}\overline{z}\wedge\mathrm{d}z\wedge\mathrm{d}t \mapsto \mathcal{T}(f)\,\mathrm{d}z\wedge\mathrm{d}t. \end{cases}$$

4.3. Let us examine the *propagator*, that is the integral kernel of the inverse to $\mathrm{d}_{\overline{z}}$. This is a differential form $\eta \in \Omega^\bullet(M\times M)$ such that

$$\int_{M\times M} \alpha\wedge\eta\wedge\beta = \int_M \alpha\wedge \mathrm{d}_{\overline{z}}^{-1}(\beta) \tag{3.4.11}$$

for any $\alpha,\beta\in\mathcal{L}_{\mathrm{gf}}^\vee$. Let us write $\mathcal{G}(z,t,w,s) := \frac{1}{\pi(w-z)}\delta(t-s)$.

PROPOSITION 4.5. The propagator

$$\eta(z,t,w,s) = -\mathcal{G}(z,t,w,s)(\mathrm{d}z+\mathrm{d}w)\wedge(\mathrm{d}t+\mathrm{d}s) \tag{3.4.12}$$

satisfies (3.4.11).

PROOF. Let $\alpha,\beta\in\mathcal{L}_{\mathrm{gf}}^\vee$ and write

$$\begin{aligned}\alpha &= f_{\overline{z}}\,\mathrm{d}\overline{z} + f_{t\overline{z}}\,\mathrm{d}t\wedge\mathrm{d}\overline{z} + f_{z\overline{z}}\,\mathrm{d}z\wedge\mathrm{d}\overline{z} + f_3\,\mathrm{d}t\wedge\mathrm{d}z\wedge\mathrm{d}\overline{z}\\ \beta &= g_{\overline{z}}\,\mathrm{d}\overline{z} + g_{t\overline{z}}\,\mathrm{d}t\wedge\mathrm{d}\overline{z} + g_{z\overline{z}}\,\mathrm{d}z\wedge\mathrm{d}\overline{z} + g_3\,\mathrm{d}t\wedge\mathrm{d}z\wedge\mathrm{d}\overline{z}.\end{aligned}$$

Then we compute

$$\int_M \alpha\wedge\mathrm{d}_{\overline{z}}^{-1}\beta = \int_M \left[f_{\overline{z}}\mathcal{T}(g_3) + f_{t\overline{z}}\mathcal{T}(g_{z\overline{z}}) - f_{z\overline{z}}\mathcal{T}(g_{t\overline{z}}) + f_3\mathcal{T}(g_{\overline{z}})\right]\mathrm{d}t\wedge\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

Next, we expand $\alpha(\mathrm{d}z+\mathrm{d}w)(\mathrm{d}t+\mathrm{d}s)\beta$. We can discard any terms which have form degree less than 6, since they do not contribute to the integral. By also discarding any terms where any form appears twice, we get

$$\begin{aligned}&\alpha_3\,\mathrm{d}z\,\mathrm{d}t\beta_{\overline{z}} + \alpha_{s\overline{w}}\,\mathrm{d}w\,\mathrm{d}t\beta_{z\overline{z}} + \alpha_{w\overline{w}}\,\mathrm{d}z\,\mathrm{d}s\beta_{t\overline{z}} + \alpha_{\overline{w}}\,\mathrm{d}w\,\mathrm{d}s\beta_3\\ =&\left[-f_3 g_{\overline{z}} - f_{s\overline{w}}g_{z\overline{z}} + f_{w\overline{w}}g_{t\overline{z}} - f_{\overline{w}}g_3\right]\mathrm{d}s\,\mathrm{d}w\,\mathrm{d}\overline{w}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\overline{z}\end{aligned}$$

Then, with

$$\int_{M\times M} f(w,s)\mathcal{G}(w,s,z,t)g(z,t)\,\mathrm{dVol}_{M\times M} = \int_M f(z,s)(\mathcal{T}g)(z,s)\,\mathrm{dVol}_M \tag{3.4.13}$$

the claim is shown. ∎

5. The Knizhnik-Zamolodchikov equation.

We now derive a differential equation for the vev's of Wilson lines, the *Knizhnik-Zamolodchikov equation* or KZ equation for short. This equation has been used to apply Chern-Simons theory to knot theory in [36].

Recall that we defined Wilson line operators in Section 2.7. Here vary the notion a little bit. Let spacetime be a product, $M=\Sigma\times\mathbb{R}$, and consider a path $\gamma(t)=(\sigma(t),t)$ with $\sigma:\ \mathbb{R}\to\Sigma$. Let ρ be an irreducible representation of G and $\mathrm{d}\rho$ the corresponding representation of $\mathfrak{g}$. Then we define the *line observable*

$$u_{\gamma,\rho}(A) = \int_\gamma \mathrm{d}\rho(A) \tag{3.5.1}$$

and the *Wilson observable*

$$\mathcal{W}_{\gamma,\rho}(A) = \exp\left(u_{\gamma,\rho}(A)\right). \tag{3.5.2}$$

Next, we take time slices. Let $\gamma(t)$ be the path γ truncated at t. Then we define $\mathcal{W}_i(t) := \mathcal{W}_{\gamma_i(t),\rho_i}$ and $\phi(t) = \langle \mathcal{W}_1(t) \otimes \ldots \otimes \mathcal{W}_n(t)\rangle$. The KZ equation is a simple first-order ODE for $\phi(t)$ (and thus allows us to write down an explicit solution quite easily).

We rely on the following lemma, which says that the propagator is regular enough that joint expectations of line observables whose lines only touch at endpoints do not contribute.

LEMMA 5.1. Let $\gamma(t) = (\sigma(t), t)$ be a curve. We define a family of curves which are segments of γ:

$$\gamma_{t_1,t_2}(s) = \gamma(t_1 + (t_2 - t_1)s). \tag{3.5.3}$$

Denote $u_{t_0,t_1} = u_{\gamma_{t_0,t_1}}$. Then

$$\langle u_{t_0,t_1} u_{t_1,t_2}\rangle = 0. \tag{3.5.4}$$

PROOF. Denote the two sections of the path γ as γ_1 and γ_2 and choose Poincaré dual 2-forms B_1 and B_2, so that

$$u_{\gamma_i}(A) = \int_M B_i \wedge A. \tag{3.5.5}$$

Now we write

$$B_i = f_i \, \mathrm{d}t \wedge \mathrm{d}\overline{z} + g_i \, \mathrm{d}z \wedge \mathrm{d}\overline{z} \tag{3.5.6}$$

where f_1, g_1 are supported on $[t_0, t_1]$ and f_2, g_2 are suppored on $[t_1, t_2]$. Pairing the axial gauge propagator η with u_{t_0,t_1}, u_{t_1,t_2} gives an integral

$$\int_{M\times M} B_1 \wedge \eta \wedge B_2 = \int_M \left[f_1 \mathcal{T}(g_2) - g_1 \mathcal{T}(f_2)\right] \mathrm{d}t \, \mathrm{d}z \, \mathrm{d}\overline{z}. \tag{3.5.7}$$

Thus the integrand is smooth and its support is contained in $\{t_1\} \times \Sigma$, a set of measure 0. Hence the integral vanishes. ■

We shall also need the two-point functions of Chern-Simons theory in axial gauge fixing.

LEMMA 5.2. Let us write $A = A_0 \, \mathrm{d}t + A_1 \, \mathrm{d}z + A_2 \, \mathrm{d}\overline{z}$ and $A_i = \sum_\alpha A_i^\alpha T^\alpha$, $i = t, z, \overline{z}$ with $\{T^\alpha\}$ a basis for $\mathfrak{g}$.

$$\langle A_a^\alpha(z,s) A_b^\beta(w,s')\rangle = \delta^{\alpha\beta}\delta_{a0}\delta_{b1}\delta(s-s')\frac{1}{z-w}. \tag{3.5.8}$$

PROOF. This is just a rewriting of the propagator in axial gauge. ■

Note that time derivatives do not commute with taking values of observables, so we must be somewhat pedestrian and expand $\phi(t+\epsilon)$ in ϵ. We go up to order ϵ^2 because the propagator gives a term of order $\frac{1}{\epsilon}$. First we expand $\mathcal{W}_i$:

$$\begin{aligned} \mathcal{W}_i(t+\epsilon) &= \exp\left(\int_0^{t+\epsilon} A(\dot\sigma_i(s))\, \mathrm{d}s\right) = \mathcal{W}_i(t)\exp\left(\int_t^{t+\epsilon} A(\dot\sigma_i(s))\, \mathrm{d}s\right) \\ &= \mathcal{W}_i(t)\left(1 + \int_t^{t+\epsilon} A(\dot\sigma_i(s))\, \mathrm{d}s + \frac{1}{2}\int_t^{t+\epsilon}\int_t^{t+\epsilon} A(\dot\sigma_i(s))A(\dot\sigma_i(s'))\, \mathrm{d}s\, \mathrm{d}s' + O(\epsilon^3)\right) \end{aligned} \tag{3.5.9}$$

whence
(3.5.10)

$$\phi(t+\epsilon) = \phi(t) + \sum_i \left\langle \mathcal{W}_1(t) \otimes \ldots \otimes \mathcal{W}_i(t) \int_t^{t+\epsilon} A(\dot\sigma_i(s))\,\mathrm{d}s \otimes \ldots \otimes \mathcal{W}_n(t) \right\rangle$$

$$+ \sum_{i<j} \left\langle \mathcal{W}_1(t) \otimes \ldots \otimes \mathcal{W}_i(t) \int_t^{t+\epsilon} A(\dot\sigma_i(s))\,\mathrm{d}s \otimes \ldots \otimes \mathcal{W}_j(t) \int_t^{t+\epsilon} A(\dot\sigma_j(s))\,\mathrm{d}s \otimes \ldots \otimes \mathcal{W}_n(t) \right\rangle$$

$$+ \frac{1}{2} \sum_i \left\langle \mathcal{W}_1(t) \otimes \ldots \otimes \mathcal{W}_i(t) \int_t^{t+\epsilon} \int_t^{t+\epsilon} A(\dot\sigma_i(s)) A(\dot\sigma_i(s'))\,\mathrm{d}s\,\mathrm{d}s' \otimes \ldots \otimes \mathcal{W}_n(t) \right\rangle + O(\epsilon^3).$$

Using Wick's lemma and Lemma 5.1 we can factorize this expression. The term with a single integral over $[t, t+\epsilon]$ vanishes and we get

$$\phi(t+\epsilon) - \phi(t) = \Bigg[\sum_{i<j} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \langle A(\dot\sigma_i(t)) \otimes A(\dot\sigma_j(s')) \rangle \,\mathrm{d}s\,\mathrm{d}s'$$

$$+ \frac{1}{2} \sum_i \int_t^{t+\epsilon} \int_t^{t+\epsilon} \langle A(\dot\sigma_i(s)) A(\dot\sigma_i(s')) \rangle \,\mathrm{d}s\,\mathrm{d}s' \Bigg] \phi(t) \tag{3.5.11}$$

where we suppressed a number of factors of the identity in the tensor product.

Since $\dot\sigma_i(t) = (1, \dot z_i(t))$, we have $A(\dot\sigma_i(t)) = A_0(t) + A_1(t)\dot z_i(t) + A_2(t)\dot{\bar z}_i(t)$. Therefore

$$\langle A(\dot\sigma_i(s)) \otimes A(\dot\sigma_j(s')) \rangle = \sum_{a,\alpha,b,\beta} \langle A_a^\alpha(t) A_b^\beta(t) \rangle \dot\sigma_i(t)^a \dot\sigma_j(t)^b T^\alpha \otimes T^\beta$$

$$= \delta(s-s') \sum_\alpha T^\alpha \otimes T^\alpha \left(\frac{\dot z_i(t) - \dot z_j(t)}{z_i(t) - z_j(t)} \right) \tag{3.5.12}$$

Thus we find $\phi(t+\epsilon) - \phi(t) = S\phi(t)$ with

$$S = \sum_{i<j} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \langle \mathrm{id} \otimes \ldots \otimes A_i(s) \otimes \ldots \otimes A_j(s') \otimes \ldots \otimes \mathrm{id} \rangle \,\mathrm{d}s\,\mathrm{d}s'$$

$$= \sum_{i<j} \Omega_{ij} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \delta(s-s') \frac{\dot z_i(s) - \dot z_j(s')}{z_i(s) - z_j(s')} \,\mathrm{d}s\,\mathrm{d}s'$$

$$= \epsilon \sum_{i<j} \Omega_{ij} \frac{\dot z_i(t) - \dot z_j(t)}{z_i(t) - z_j(t)} + O(\epsilon^2) \tag{3.5.13}$$

with

$$\Omega_{ij} = \sum_\alpha \mathrm{id} \otimes \ldots \otimes T^\alpha \otimes \ldots T^\alpha \otimes \ldots \otimes \mathrm{id}\,. \tag{3.5.14}$$

Hence we find that

$$\lim_{\epsilon\to 0} \frac{\phi(t+\epsilon) - \phi(t)}{\epsilon} = \lambda \sum_{i<j} \Omega_{ij} \frac{\dot z_i(t) - \dot z_j(t)}{z_i(t) - z_j(t)} \phi =: \Omega(t)\phi(t). \tag{3.5.15}$$

If we define the *Knizhnik-Zamolodchikov* connection

$$\omega = \lambda \sum_{i<j} d\log(z_i - z_j)\Omega_{ij} \tag{3.5.16}$$

we can express the differential equation for ϕ as

$$\mathrm{d}\phi = \omega\phi. \tag{3.5.17}$$

CHAPTER 4

The Fractional Quantum Hall Effect.

We turn now from mathematics to physics. Unfortunately, we cannot claim this chapter as a strict application of the BV-BFV quantization of Chern-Simons theory, as the calculations in this chapter take place entirely in degree 0 (or rather: on ungraded vector spaces). There are, however, some points where a deeper relationship of the physical formulae to the BV-BFV formalism may yet be obtained in future work. The reader may also consider that here we only deal with abelian Chern-Simons theory, where gauge-fixing is in any case extremely simple. Dealing with nonabelian Chern-Simons theory may therefore justify the use of the BV-BFV formalism. Furthermore, the Knizhnik-Zamolodchikov Equation makes an appearence due to its relation to the Laughlin wavefunction.

The classical Hall effect was discovered by Edwin Hall in 1879. The quantum Hall effect was discovered by Klaus von Klitzing et. al. in 1980 [47], and the fractional quantum Hall effect in 1982 by Tsui, Störmer and Gossard [73]. To a large degree, the FQHE is described in terms of the classical Hall effect, and therefore it seems expedient to begin there, moving on to the integer QHE and then to the fractional QHE. The essential text on the physics of the QHE is [64]. The lecture notes of Tong [72] are also a valuable source for the student.

1. Basic aspects of the Hall effect.

1.1. The experimental setup is sketched in Figure 1. We consider an approximately 2-dimensional conducting solid which extends over a compact region $\Lambda \subset \mathbb{R}^2$ of the plane.

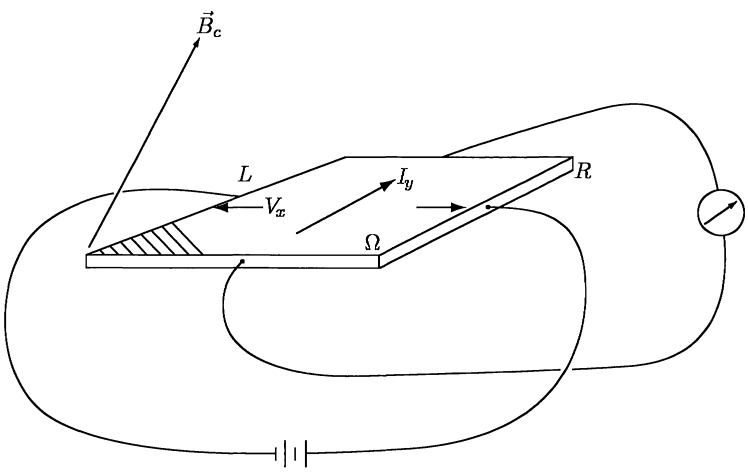


FIGURE 1. Sketch of the experimental setup, taken from [34].

Typically, Λ has the topology of a disc or an annulus. Space-time is for our purposes simply a "cylinder" $M = \mathbb{R} \times \Lambda$. Of course, M has the topology of a genuine (infinite) full cylinder only when Λ is a disk.

The electromagnetic field is described by a closed 2-form F, called the *electromagnetic field strength tensor*. Because space-time is a product, we can define electric and magnetic fields separately; the electric field component of F is the part which contains the form $\mathrm{d}t$; the magnetic field is the rest. In other words, we have forms $E \in \Omega^1(M) \otimes C^\infty(\Lambda), B \in \Omega^2(M) \otimes C^\infty(\Lambda)$ such that $F = E \wedge \mathrm{d}t + B$. Note that since M is 2-dimensional, B is a top-degree form. If we embed the plane into $\mathbb{R}^3$ via $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}$, then B is equivalently described by a vector field normal to M.

We will mostly be interested in constant electric and magnetic fields, i.e.the forms

$$B = |B| \,\mathrm{d}x \wedge \mathrm{d}y \tag{4.1.1}$$

$$E = E_x \,\mathrm{d}t \wedge \mathrm{d}y + E_y \,\mathrm{d}t \wedge \mathrm{d}x. \tag{4.1.2}$$

It is often the case (for instance if the cohomology of M is trivial) that F is not just closed, but exact. In this case we introduce a *potential* (or *gauge form*), i.e.a 1-form A such that $F = \mathrm{d}A$. For instance, potentials describing the constant magnetic and electric field above are, respectively

$$A_0 = \frac{|B|}{2}\,(x\,\mathrm{d}y - y\,\mathrm{d}x) \tag{4.1.3}$$

$$A_1 = -(E_x y + E_y x)\,\mathrm{d}t \tag{4.1.4}$$

Let us suppose that the system is prepared in the presence of the gauge field A_0 as a "background field". Then we apply the gauge field A_1 as a perturbation and measure the response of the system. In general terms, the electric field exerts a force on the electrons in the sample. Because the sample is a conductor, the electrons are fairly free to move in the material, and thus the collective motion of the electrons gives rise to a *transport of charge* and thus a measurable *current* in the material.

1.2. Describing this charge transport and current is, on the microscopic level of the individual particles, an immensely complex problem of many-body quantum theory. One has to consider the potential generated by the particles forming the solid, as well as the interactions between the conduction electrons. It is crucial both that the background potential of the solid is *roughly* periodic as well as disturbed by *impurities* so that the potential is not *precisely* periodic. At non-zero temperatures, the vibration of the solid itself (*phonons*) must in general be taken into account as well.

From a microscopic view there is thus a host of complexities to be accounted for. On the macroscopic level, however, one frequently (though not always) observes very simple relations. One of these is *Ohm's law.* Let j denote the current density in response to the electric field E. Then Ohm's law states that there is a linear relationship between the two:

$$\rho \cdot j = E \tag{4.1.5}$$

with ρ the 2×2 *resistivity matrix*

$$\rho = \begin{pmatrix} R_L & -R_H \\ R_H & R_L \end{pmatrix}. \tag{4.1.6}$$

Notwithstanding the fact that Ohm's "law" is less of a law and more of a crude simplification, it can be a surprisingly accurate approximation. Sometimes it may be justified by techniques known collectively as *linear response* and in particular by the *Kubo formula*. A simple classical model for electron transport leading to Ohm's law is the *Drude model*.

The classical Hall effect is a special form of Ohm's law. In the presence of the background magnetic gauge field A_0 (as defined above), the response current to an electric field E satisfies Ohm's law with the resistivity matrix

$$\rho = \begin{pmatrix} 0 & -R_H \\ R_H & 0 \end{pmatrix} \tag{4.1.7}$$

where R_H is called the *Hall resistivity* and its inverse $\sigma_H = R_H^{-1}$ is known as the *Hall conductivity*. The vanishing of the longitudinal resistivity R_L means that the system is *incompressible*, which is crucial to any explanation of the QHE.

1.3. Let us look at a simple model for the classical Hall effect. A number N of charge carrier of charge q moves with a velocity $\mathbf{v}$ in the sample. In a stationary state, we have $\dot{\mathbf{v}} = 0$. In the presence of a magnetic field, there is a Lorentz force on the particle, and to model particle scattering, we introduce a friction term $\frac{q\mathbf{v}}{\mu}$ (this comes from the *Drude model*), with μ the *mobility*. Then the stationary state equation is

$$0 = \dot{\mathbf{v}} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) - \frac{q}{\mu}\mathbf{v}. \tag{4.1.8}$$

The solution is $\mathbf{E} = \rho\mathbf{j}$ with $\mathbf{j} = qn\mathbf{v}$, n the charge carrier density N/V and

$$\rho = \frac{1}{qn}\begin{pmatrix} \mu^{-1} & -\frac{B}{c} \\ \frac{B}{c} & \mu^{-1} \end{pmatrix}. \tag{4.1.9}$$

Comparing with (4.1.7) in the limit $\mu \to \infty$, we find

$$R_H = \frac{|B|}{qnc}. \tag{4.1.10}$$

Thus R_H classically depends *linearly* on the magnetic field strength $|B|$. In a metal, the current may be carried either by electrons, or indeed by holes in the electron gas, so q can be $\pm e$, with e the charge of the electron.

Typically, the Hall conductivity σ_H is expressed not in terms of the magnetic field strength, but the *filling factor* ν, defined as

$$\nu = \frac{nch}{|q| \cdot |B|} \tag{4.1.11}$$

where h is Planck's constant (not divided by 2π). In terms of the filling factor σ_H is

$$\sigma_H = \frac{q^2}{h}\nu. \tag{4.1.12}$$

1.4. The filling factor obtains its name from the spectral theory of the Landau Hamiltonian. The single-particle Landau Hamiltonian is the operator

$$H_{\mathrm{L}} = \frac{1}{2m}\left(p - \frac{q}{c}A_0\right)^2 \tag{4.1.13}$$

on $L^2(\Sigma)$ with Σ a 2-dimensional domain. It turns out that the eigenvalues of H_{L} are equally spaced and all have the same degeneracy d_L. On a flat torus obtained from the rectangle $[0, L_x] \times [0, L_y]$ one finds [72]

$$d_L = \frac{|q|L_xL_y|B|}{2\pi\hbar c}. \tag{4.1.14}$$

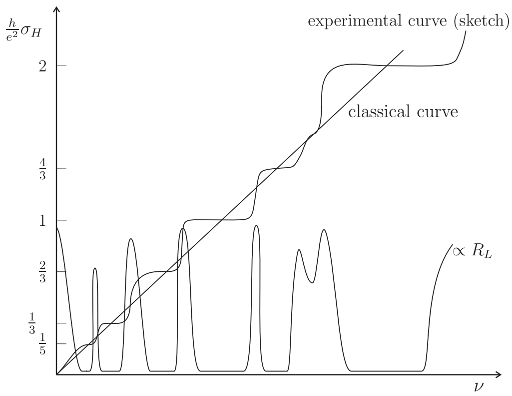


FIGURE 2. From [10].

Hence if one has N non-interacting particles, then in the ground state the first N/d_L levels are filled, giving

$$\nu = \frac{N}{d_L} = \frac{nch}{|q| \cdot |B|} \tag{4.1.15}$$

where $n = N/(L_x L_y)$ is the particle density.

In a proper treatment of the free electron gas in a magnetic field, one has to take the thermodynamic limit, $N \to \infty$, $\Sigma \to \mathbb{R}^2$ while the particle density n is kept fixed. Since N and Σ only enter into the filling factor through the particle density, it is clear that ν is unaffected by the thermodynamic limit.

1.5. In the quantum Hall effect, one observes a departure from the classical behaviour. In the quantum Hall regime, i.e. at low temperatures ($\sim 1^\circ$ K), in strong magnetic fields (~ 1 T), in very clean samples and with weak currents, the following observations can be made.

i) The Hall conductance σ_H, rather than depending linearly on ν, shows a step-like behaviour, forming plateaus over certain intervals of ν and jumping between discrete values (sketched in Figure 2).
ii) The values σ_H can take on these plateaus are strongly restricted; they can be integers (integer quantum Hall effect) or certain rational numbers (fractional quantum Hall effect).
iii) The longitudinal resistance R_L vanishes on plateaus.
iv) The cleaner the sample, the more plateaus are observed and the narrower they are.

1.6. The electromagnetic fields applied to any physical system must satisfy the Maxwell equations. In addition, the *response current* (whether classical or quantum-mechanical) to those electromagnetic fields must satisfy the continuity equation.

The response current is not the same thing as the current which may appear as a *source* in the inhomogeneous Maxwell equations. This source current is irrelevant to what follows; whenever we speak of *current* in the following, we mean the response current.

We will write the response current density as a 2-form j. Equivalently, we can consider it as the 1-form J with $*_M J = j$. The Hodge star is induced by the Minkowskian metric (signature (-, +, +)) with volume form $\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y$, so that $*_M\, \mathrm{d}t = -\,\mathrm{d}x \wedge \mathrm{d}y$ and $*_M^2 = -1$ (on forms of any degree). The current forms j, J can be decomposed into spatial current densities and charge densities. As in Appendix A, the product structure of M allows us to decompose

$$J = J_\Lambda + \rho\,\mathrm{d}t \tag{4.1.16}$$

$$j = j_\Lambda - \rho\,\mathrm{d}x \wedge \mathrm{d}y \tag{4.1.17}$$

where $J_\Lambda = \iota_\Lambda^* J$ with $\iota_\Lambda : \Lambda \hookrightarrow M$ the inclusion and $j_\Lambda = *_M J_\Lambda$.

The laws obeyed by the external electromagnetic field and the response current can be summarized in the following way.

i) The Maxwell-Faraday induction law, $\mathrm{d}F = 0$. In particular, $\dot{B} = -\,\mathrm{d}_\Lambda E$.
ii) The continuity equation, $\mathrm{d}j = \mathrm{d} *_M J = 0$.

In any system exhibiting the Hall effect, we also have

iii) Hall's law. When $R_L = 0$, the electric current is perpendicular to the electric field:

$$J_\Lambda = -\sigma_H *_\Lambda E. \tag{4.1.18}$$

Proposition 1.1. Suppose the relations i) – iii) are satisfied and $j_0(t=0) = B(t=0)$. Then we also have

iv) Chern-Simons-Gauss law $\rho = \sigma_H B$.
v) The Hall response current

$$J = \sigma_H *_M \mathrm{d}A. \tag{4.1.19}$$

Proof. From Equation (A.0.1) we have

$$*_M (E \wedge \mathrm{d}t) = -*_\Lambda E. \tag{4.1.20}$$

The Hall law now states

$$j_\Lambda = *_M J_\Lambda = -\sigma_H *_M *_\Lambda E = -\sigma_H *_M^2 (-E \wedge \mathrm{d}t) = -\sigma_H E \wedge \mathrm{d}t. \tag{4.1.21}$$

Therefore we may conclude

$$j = -\sigma_H E \wedge \mathrm{d}t - \rho\,\mathrm{d}x \wedge \mathrm{d}y. \tag{4.1.22}$$

Now we use the continuity equation:

$$0 = \mathrm{d}j = -\sigma_H\,\mathrm{d}_\Lambda E \wedge \mathrm{d}t - \dot{\rho}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \tag{4.1.23}$$

which gives, using the homogeneous Maxwell equations

$$-\,\sigma_H \dot{B} = \sigma_H\,\mathrm{d}_\Lambda E = -\dot{\rho}\,\mathrm{d}x \wedge \mathrm{d}y. \tag{4.1.24}$$

Then iv) follows. To obtain v), note that from the above we have

$$j = -\sigma_H (E \wedge \mathrm{d}t + B) = -\sigma_H\,\mathrm{d}A, \tag{4.1.25}$$

and therefore $J = -*\,j = \sigma_H * \mathrm{d}A$. ■

2. Effective bulk theory.

Some aspects of the FQHE can be understood in terms of an effective Chern-Simons theory coupled to the external magnetic field. This is basically a phenomenological fact, though there are some results connecting the phenomenological Chern-Simons theory to the microscopic theory of interacting electrons in a solid. There is a vast literature even on just this effective field theory approach to the FQHE, but in this thesis I have concentrated on research conducted by Jürg Fröhlich and collaborators since the late 1980s.

If the reader follows me in this self-restriction, she will still find some dozens of relevant papers. The topic of this thesis was especially sparked by the articles [**34**, **33**]. Subsequently, I have also found the papers [**10**, **9**, **36**] to be particularly useful.

2.1. Suppose that S_m (m for *microscopic*) is an action describing a system of nonrelativistic charged particles in a magnetic potential A. Then the particle current density has to be coupled to the gauge field A, i.e. S_m contains a term

$$S_J = \int_M *\mathcal{J} \wedge A = \int_M \mathcal{J}_\mu A^\mu \,\mathrm{dVol} \tag{4.2.1}$$

where $\mathcal{J}$ is the current density 1-form, which is coclosed, $\mathrm{d} * \mathcal{J} = 0$. We use the calligraphic letter $\mathcal{J}$ here because we reserve the plain J for the current expectation $J = \langle \mathcal{J} \rangle$. The relation of $\mathcal{J}$ to the particle fields depends precisely on what type of fields we consider. For instance, in [**35**] the following action is given for a system of non-relativistic, non-interacting electrons (spin 1/2 fermions) for a chemical potential μ in d space dimensions:

$$S_\mu(\psi, \psi^*, A) = \int \mathrm{d}t \int \mathrm{d}x \Big[i\hbar c \psi^*(x) D_0(A)\psi(x) - \mu\psi^*(x)\psi(x) - \frac{\hbar^2}{2m}\sum_{k=1}^{d} (D_k(A)\psi(x))^* D_k(A)\psi(x) \Big] \tag{4.2.2}$$

$$= \int \mathrm{d}t \int \mathrm{d}x \Big[i\hbar c\psi^*(x)\partial_0\psi(x) - \frac{\hbar^2}{2m}\sum_{k=1}^{d}(\partial_k\psi(x))^*\partial_k\psi(x) - \mu\psi^*(x)\psi(x) - \sum_{\rho=0}^{d} A_\rho(x)\mathcal{J}^\rho[\psi,\psi^*,A](x) \Big]. \tag{4.2.3}$$

where $D_\mu(A) := \partial_\rho - i\frac{e}{\hbar c}A_\mu$ are covariant derivatives, and the current density is

$$\begin{aligned} &\mathcal{J}^0[\psi^*,\psi] = -e\psi^*(x) \\ &\mathcal{J}^k[\psi^*,\psi,A] = -\frac{e\hbar}{i2mc}\left[\psi^*(x)D_k(A)\psi(x) - (D_k(A)\psi(x))^*\psi(x)\right],\ k = 1,\dots,d. \end{aligned} \tag{4.2.4}$$

In this action, things are complicated somewhat because the current depends on the gauge field A. If we split $A = A_0 + A_1$ where A_0 is the background magnetic field and A_1 is the electric field which induces the Hall current, we assume that A_1 can be treated as a small perturbation, and that the dependence of the current on A_1 is negligible. For details, the reader should consult [**35**].

From now on, the background field A_0 is absorbed into the action and we write A for the perturbing field, and suppose that the the current in $S_m[\phi, A]$ does not depend on A. Then

$$\frac{\delta}{\delta A}S_m[\phi,A]=\mathcal{J}[\phi]S_m[\phi,A]. \tag{4.2.5}$$

Taking the functional derivative of $\Gamma[A]$ gives the expecation value of the current:

$$\begin{aligned}\frac{\delta}{\delta A}\Gamma[A]=\frac{\hbar}{i}e^{-\frac{i}{\hbar}\Gamma[A]}\frac{\delta}{\delta A}e^{\frac{i}{\hbar}\Gamma[A]}=\frac{\hbar}{i}\frac{Z[0]}{Z[A]}\frac{\delta}{\delta A}\frac{Z[A]}{Z[0]}=\frac{\hbar}{i}\frac{1}{Z[A]}\frac{\delta}{\delta A}Z[A]\\ =\frac{\hbar}{i}\frac{1}{Z[A]}\frac{\delta}{\delta A}\int e^{\frac{i}{\hbar}S_m[\phi,A]}[\mathrm{D}\phi]=⨍ e^{\frac{i}{\hbar}S_m[\phi,A]}\mathcal{J}[\phi][\mathrm{D}\phi]=\langle\mathcal{J}\rangle.\end{aligned} \tag{4.2.6}$$

More generally,

$$\frac{\delta}{\delta A(x_1)}\cdots\frac{\delta}{\delta A(x_k)}\Gamma[A]=\langle\mathcal{J}(x_1)\dots\mathcal{J}(x_k)\rangle. \tag{4.2.7}$$

2.2. The phenomenological laws of Section 1.6 already determine the effective action.

$$\sigma_H * \mathrm{d}A=J=\langle\mathcal{J}\rangle_A=\frac{\delta}{\delta A}\Gamma[A]. \tag{4.2.8}$$

The solution, up to a constant, is

$$\Gamma[A]=\frac{\sigma_H}{2}\int_M A\wedge\mathrm{d}A. \tag{4.2.9}$$

In order to quantize, we need to revive at least some of the degrees of freedom that have been integrated out in the effective action.

Lemma 2.1. Let $\Lambda\in\mathbb{R}^2$ be a disk with a finite number of punctures. Then $H^2(\Lambda\times\mathbb{R})=0$.

Proof. By the Künneth formula,

$$H^2(\Lambda\times\mathbb{R})\cong H^1(\Lambda)\otimes H^1(\mathbb{R})\oplus H^2(\Lambda)\otimes H^0(\mathbb{R}). \tag{4.2.10}$$

Now $H^1(\mathbb{R})$ is trivial by the fundamental theorem of calculus and $H^2(\Lambda)=0$ by Mayer-Vietoris. ■

Since the de Rham cohomology is trivial, $\mathrm{d}*\mathcal{J}=0$ implies $\mathcal{J}=*\mathrm{d}B$ for some 1-form $B\in\Omega^1(M)$. The 1-form B will represent the revived degrees of freedom.

Proposition 2.2. Consider the action

$$S[B,A]=-\frac{1}{2}\int_M B\wedge\mathrm{d}B+\sqrt{\sigma_H}\int_M A\wedge\mathrm{d}B. \tag{4.2.11}$$

Then $\Gamma[A]$ is the effective action of $S[B,A]$ after formally integrating out B, i.e.

$$\exp(\Gamma[A])=⨍[\mathrm{D}B]e^{-\frac{1}{2}\int B\wedge\mathrm{d}B}\exp\left(\sqrt{\sigma_H}\int A\wedge\mathrm{d}B\right). \tag{4.2.12}$$

Proof. The inverse of d is not defined without gauge fixing, but the special form of the integral makes this unnecessary. Indeed,

$$S[B,A]=-\frac{1}{2}q(B)+(q^{\#}\sigma_H A)(B) \tag{4.2.13}$$

where q is the Chern-Simons quadratic form

$$q(A)=\int_M A\wedge\mathrm{d}A. \tag{4.2.14}$$

Then the formal integral over B gives

$$\Gamma[A] = \frac{1}{2}q^{-1}(q^{\#}\sqrt{\sigma_H}A) = \frac{\sigma_H}{2}q(A) = \frac{\sigma_H}{2}\int_M A \wedge \mathrm{d}A. \tag{4.2.15}$$

■

We will allow a small generalization of the above. We will let the Hall current be a sum of independently preserved currents. This is the same thing as allowing the current to be a form with values in an abelian Lie algebra $\mathfrak{g}$. We will assume that $\mathfrak{g}$ is equipped with an Ad-invariant bilinear pairing $(\cdot,\cdot)$. Note that the background gauge field is still only an element of $\Omega^1(M)$ and does not take values in $\mathfrak{g}$. We therefore need to pick some vector $q \in \mathfrak{g}$ so that we can write

$$S[B, A] = \frac{1}{2}\int_M [(B, \mathrm{d}B) + (A \otimes q, \mathrm{d}B)]\,. \tag{4.2.16}$$

Integrating out B now gives the effective action

$$\Gamma[A] = \frac{1}{2}(q, q)\int A \wedge \mathrm{d}A \tag{4.2.17}$$

Comparing this with the previous form of the effective action implies the condition $(q, q) = \sigma_H$.

3. Bulk and edge currents.

3.1. Let us make the idealization that the electrons in the sample are perfectly confined; the potential barrier at the edge is infinite, and hence there is no current through the edge, i.e. $\tilde{\pi}_M(J) = 0$.

This requirement conflicts with the Hall law iii) in Subsection 1.6, since

$$\tilde{\pi}_M(\sigma_H * \mathrm{d}A) = \sigma_H \pi_M(\mathrm{d}A) = \sigma_H\, \mathrm{d}a, \tag{4.3.1}$$

with $a = \pi_M(A)$, is generally nonzero (this is the component of the electric field which is parallel to the boundary of the sample). The Hall law iii) therefore only has restricted validity, away from the edge. In the vicinity of the edge, the confining potential of the sample boundary governs the behaviour of electrons and eclipses the effect of the bulk current.

We will account for this behaviour with a heuristic model, where we partition the sample Σ into two subsets $\Sigma = \Sigma_B \cup \Sigma_E$, representing the bulk and a thickened edge (or, as we will say, the *fat edge*), respectively. We partition M in a corresponding way, $M_B = \Sigma_B \times \mathbb{R}$, $M_E = \Sigma_E \times \mathbb{R}$. We can then enforce the Hall law for the bulk current on M_B and the correct boundary behaviour on M_E. In the limit where $\Sigma_B \to \Sigma$ and the thickness of the edge vanishes, we obtain a modified description of the current which should give the correct behaviour in the bulk and near the boundary.

3.2. Fix a collar $\Sigma_E \cong \partial\Sigma \times [0, 1]$ of $\partial\Sigma$ (we denote the standard coordinate on $[0, 1]$ by r) such that $\partial\Sigma \times 1 \cong \partial\Sigma$. This collar is the fat edge and the complement $\Sigma_B = (\Sigma\backslash\Sigma_E)^{\mathrm{cl}}$ represents the bulk. We will generically write ω_E, ω_B for the restrictions of a form ω to M_E and M_B, respectively.

For every connected component of $\partial\Sigma$, Σ_E has two boundary components, corresponding to $r = 0$ and $r = 1$. We therefore write $\partial\Sigma_E = (\partial\Sigma_E)_0 \cup (\partial\Sigma_E)_1$ with $(\partial\Sigma_E)_i \cong \partial\Sigma \times \{i\}$.

As in Appendix A, on M_E we can decompose forms

$$\omega = \omega_{\|}(r) + \omega_{\perp}(r) \wedge \mathrm{d}r \tag{4.3.2}$$

with $\omega_{\|}(r), \omega_{\perp}(r)$ in ∂M. The inclusion $\partial\Sigma \times \{r\} \hookrightarrow \Sigma$ induces parallel and perpendicular boundary projections of forms

$$\pi_R(\omega) = \omega_{\|}(R) \tag{4.3.3}$$

$$\tilde{\pi}_R(\omega) = -(-1)^{|\omega|}\omega_{\perp}(R). \tag{4.3.4}$$

In particular, we can decompose the current J and the gauge field A:

$$J_E = J_{\|}(r) + J_{\perp}\,\mathrm{d}r \tag{4.3.5}$$

$$A_E = A_{\|}(r) + A_{\perp}(r)\,\mathrm{d}r. \tag{4.3.6}$$

Note that since J and A are 1-forms, $J_{\perp}$ and $A_{\perp}$ are just functions (0-forms).

3.3. Now we impose the Hall laws i) - iii) of Subsection 1.6 on M_B. On M_E we require that the $\mathrm{d}^*J = 0$ and that $\tilde{\pi}_M(J) = 0$.

Since we require $\tilde{\pi}_M(J) = 0$, we must have $J_{\perp}(r = 1) \equiv 0$. On the inner boundary ∂M_B, the currents J_E and J_B must agree, hence

$$\pi_0(*\,\mathrm{d}A) = J_{\|}(0) \tag{4.3.7}$$

$$\tilde{\pi}_0(*\,\mathrm{d}A) = \pm\,\mathrm{d}_{\partial}A_{\|}(0) = J_{\perp}(0). \tag{4.3.8}$$

Closedness of J_E means

$$0 = \mathrm{d}^*J_E = \mathrm{d}^*_{\partial}J_{\|} + \partial_r J_{\perp}. \tag{4.3.9}$$

The function $J_{\perp}$ smoothly interpolates between the component of the bulk current parallel to the edge and 0. More explicitly,

$$J_{\perp}(p,r) = \Xi(p,r)\left(*_{\partial}\,\mathrm{d}A_{\|}(0)\right)(p), \quad p \in \partial M,\ r \in [0,1]. \tag{4.3.10}$$

where Ξ is a smooth function on $\partial M \times [0,1]$ such that $\Xi(p,0) = 1$ and $\Xi(p,R) = 0$ for any $p \in \partial M$, i.e. Ξ is a smoothed step function. It is clear that the conditions we have imposed on J_E are not sufficient to determine Ξ. The exact form of this function depends on microscopic physics of the sample near the edge. Possibly the form of Ξ can be related to the hydrodynamic approach to edge physics taken for instance in [**51**].

As we shrink the collar Σ_E, the function Ξ approaches a sharp step function. Heuristically, $\Xi \to \chi_{\Sigma}$, and thus $\partial_r\Xi \to -\delta_{\partial\Sigma}$ and

$$\partial_r J_{\perp} \to -\delta_{\partial\Sigma} *_{\partial} \mathrm{d}A_{\|}(0). \tag{4.3.11}$$

Now let us look at the other component of the current, $J_{\|}$. In the thin edge limit, $J_{\|}$ is singular, with

$$\mathrm{d}^*_{\partial}J_{\|} = -\partial_r J_{\perp} = \delta_{\partial\Sigma} *_{\partial} \mathrm{d}A_{\|}(0). \tag{4.3.12}$$

Since the δ-distribution cannot be a result of d^*_{∂}, which does not act on the r-variable, $J_{\|} = \delta_{\partial\Sigma}J^{\partial}$, with $\mathrm{d}^*_{\partial}J^{\partial} = *_{\partial}\,\mathrm{d}A_{\|}(0)$.

We therefore arrive at the description of the currents given in [**10**]. We have a total current J, which is conserved, $\mathrm{d}*J = 0$. The total current is the sum of bulk and edge currents, $J = J_{\mathrm{bulk}} + J_{\mathrm{edge}}$, with $J_{\mathrm{bulk}} = \sigma_H\,\mathrm{d}*A$ and

$$\mathrm{d}*J_{\mathrm{edge}} = \sigma_H\delta_{\partial M} * \mathrm{d}a \tag{4.3.13}$$

where $a = \pi_M(A)$.

Remark 3.1. In [10], a similar, but slightly different argument is presented. There, it is argued that σ_H is not a constant coefficient but rather a multiple of a step function, $\sigma_H \cdot \chi_\Sigma$, and that therefore $\mathrm{d} * (\sigma_H \chi_\Sigma * \mathrm{d}A)$ contains a term $\sigma_H \,\mathrm{d}\chi_\Sigma \wedge \mathrm{d}A$, which is in contradiction to conservation of J.

4. Edge action.

4.1. In the previous section we saw that in addition to the bulk current, for samples with boundary there is an additional boundary current. For a quantum mechanical treatment of this current, we need to find a boundary action functional which allows us to generate the boundary current by functional derivatives. In order to find this boundary action, we exploit gauge invariance.

On manifolds with boundary, the effective action (4.2.9) is not gauge invariant. Indeed,

$$\Gamma[A + \mathrm{d}\chi] - \Gamma[A] = \int_{\partial M} A \wedge \mathrm{d}\chi. \tag{4.4.1}$$

The right-hand side of this equation is the *gauge anomaly*. One can correct this by adding to $\Gamma[A]$ a boundary term which is itself an effective action of a local action functional on ∂M.

In the following, the pairing $(\cdot,\cdot)$ will always be the Hodge pairing $(\alpha,\beta) = \alpha \wedge *\beta$. Note that this pairing is symmetric. To express the boundary term, and to prove that it generates the correct term canceling the anomaly, we shall require some facts on the Hodge star operator, which are summarized in the Appendix. Fix some arbitrary $1+1$ metric on the boundary ∂M. Crucially, in this case $*^2 = 1$ on 1-forms. We write $\Box = \Delta$ since the Laplace-Beltrami operator is the d'Alembert operator. On functions, $\Box = \mathrm{d}^*\mathrm{d}$. On top-degree forms, $\Box = \mathrm{d}\,\mathrm{d}^*$.

The fact that the Hodge operator squares to the identity on 1-forms allows us to decompose them in the following way. Let $a \in \Omega^1(\partial M)$. Define $a_+ = a + *a$, $a_- = a - *a$. Then $*a_+ = a_+, *a_- = -a_-$ and $a = \frac{1}{2}(a_+ + a_-)$.

Proposition 4.1. Consider the functional

$$\Gamma_\partial[a] = \frac{1}{2}\int_{\partial M} \left[(\mathrm{d}a_-, \Box^{-1}\,\mathrm{d}a_-) + (a,a)\right]. \tag{4.4.2}$$

Then $\Gamma(a + \mathrm{d}\chi) - \Gamma(a) = -\int a \wedge \mathrm{d}\chi$.

Proof. Let $\Gamma_1 = \frac{1}{2}\int (\mathrm{d}a_-, \Box^{-1}\,\mathrm{d}a_-)$ and $\Gamma_2 = \frac{1}{2}\int (a,a)$. In $\Gamma[a + \mathrm{d}\chi]$, the terms with $\mathrm{d}\chi$ are paired with coexact terms and hence vanish. We are left with

$$2\left(\Gamma_1(a+\mathrm{d}\chi) - \Gamma_1(a)\right) = (*\,\mathrm{d}\chi, \mathrm{d}^*\Box^{-1}\,\mathrm{d}a_-) + (\mathrm{d}a_-, \Box^{-1}\,\mathrm{d} * \mathrm{d}\chi) + (*\,\mathrm{d}\chi, \mathrm{d}^*\Box^{-1}\,\mathrm{d} * \mathrm{d}\chi). \tag{4.4.3}$$

We demonstrate how to simplify the first term and leave the rest as an excercise the interested reader. We have

$$(*\,\mathrm{d}\chi, \mathrm{d}^*\Box^{-1}\,\mathrm{d}a_-) = (\Box^{1}\,\mathrm{d}^*\,\mathrm{d} * \mathrm{d}\chi, a_-) = -(\mathrm{d}^*\Box^{-1} * *^{-1}\,\mathrm{d} * (-1)\,\mathrm{d}\chi, a_-) \tag{4.4.4}$$

$$= -\,(\mathrm{d}^* * \Box^{-1}\,\mathrm{d}^*\,\mathrm{d}\chi, a_-) = -(*\chi, \mathrm{d}a_-). \tag{4.4.5}$$

Similarly, we find

$$(\mathrm{d}a_-, \Box^{-1}\,\mathrm{d} * \mathrm{d}\chi) = (\mathrm{d}a_-, *\chi)$$
$$(*\,\mathrm{d}\chi, \mathrm{d}^*\Box^{-1}\,\mathrm{d} * \mathrm{d}\chi) = -(\mathrm{d}\chi, \mathrm{d}\chi)$$

and clearly

$$2(\Gamma_2[a+\mathrm{d}\chi]-\Gamma_2[a]) = 2(\mathrm{d}\chi, a) + (\mathrm{d}\chi, \mathrm{d}\chi). \tag{4.4.6}$$

Putting it all together, we have

$$\Gamma[a+\mathrm{d}\chi]-\Gamma[a] = (a, \mathrm{d}\chi) - (a_-, \mathrm{d}\chi) = -(a, *\,\mathrm{d}\chi) = -\int_{\partial M} a\wedge \mathrm{d}\chi. \tag{4.4.7}$$

■

4.2. Because of the inverse of the d'Alembertian, the expression (4.4.2) is non-local. By introducing a scalar field ϕ, we can write down a local action functional whose effective action is (4.4.2).

PROPOSITION 4.2. Consider the action

$$S[\phi, a] = \int_{\partial M}\left[-\frac{1}{2}(\mathrm{d}\phi, \mathrm{d}\phi) + (\phi, \mathrm{d}a_-) + \frac{1}{2}(a,a)\right]. \tag{4.4.8}$$

Then the effective action of $S[\phi, a]$ is the boundary term canceling the anomaly,

$$\exp(\Gamma_\partial[a]) = ⨍[\mathrm{D}\phi]\exp(S[\phi, a]) \tag{4.4.9}$$

$$= \exp\left(\frac{1}{2}\int_{\partial M}(a,a)\right)⨍[\mathrm{D}\phi]\exp\left(-\frac{1}{2}\int_{\partial M}(\mathrm{d}\phi,\mathrm{d}\phi)\right)\exp\left(\int_{\partial M}(\phi, \mathrm{d}a_-)\right). \tag{4.4.10}$$

PROOF. The computation is similar to the one in Proposition 2.2. We have a quadratic form

$$q(\phi) = \int_{\partial M}(\phi, \Box\phi) \tag{4.4.11}$$

and the action can be written as

$$S[\phi, a] = -\frac{1}{2}q(\phi) + q^{\#}(\mathrm{d}\Box^{-1}a_-)(\phi) + \frac{1}{2}\int_{\partial M}(a,a). \tag{4.4.12}$$

Then the formal integral gives

$$\frac{1}{2}q(\mathrm{d}\Box^{-1}a_-) = \frac{1}{2}\int_{\partial M}(\mathrm{d}\Box^{-1}a_-, \Box\,\mathrm{d}\Box^{-1}a_-) = \frac{1}{2}\int_{\partial M}(\mathrm{d}a_-, \Box^{-1}\,\mathrm{d}a_-). \tag{4.4.13}$$

■

4.3. The boundary action which corrects the anomaly can be obtained as a boundary effective action by integrating out bulk fields, as in the BV-BFV formalism. But we have already seen that the bulk action on its own does not "know" about the requirement that there be no current across the boundary of the sample (this was the reason for the lengthy discussion of Section 3). We will implement this condition by inserting a delta functional in the formal path integral.

Recall that on boundary 1-forms, $*^2 = 1$, and 1-forms therefore split as $a = a_+ + a_-$ with $*a_\pm = \pm a_\pm$. We choose a polarization by splitting 1-forms into the $*$-eigenspaces by writing

Let us choose $\mathcal{B}_+$ as the base of the fibration. Splitting the space of fields allows us to write $B = \mathbb{B} + \beta$ where $B \in \mathcal{F}_M$ and $\mathbb{B} \in \mathcal{B}_-$ and $\beta \in \mathcal{F}_M$ is a fluctuation.

The next step should be to choose some Lagrangian gauge-fixing subspace, but this proves to be unnecessary. Any choice is suitable to the following computation, and would give the same result. First we expand the action.

$$
\begin{aligned}
&\int_{\pi^{-1}(p_\partial^{-1}(\mathbb{B}))}[\mathrm{D}B]\exp(S[B,A])\delta(\mathrm{d}\pi_M(B))\\
&=\int[\mathrm{D}\beta]\exp\left(\frac{1}{2}\int_M(\beta+\mathbb{B})\wedge\mathrm{d}(\beta+\mathbb{B})-\int\mathrm{d}(\beta+\mathbb{B})\wedge A\right)\delta(\mathrm{d}\pi_M(B))\\
&=\int[\mathrm{D}\beta]\exp\left(\frac{1}{2}\int_M\beta\wedge\mathrm{d}\beta+\frac{1}{2}\int_{\partial M}\beta\wedge\mathbb{B}-\int_M\mathrm{d}\beta\wedge A-\int_{\partial M}\mathbb{B}\wedge A\right)\delta(\mathrm{d}\pi_M(B)).
\end{aligned}
$$

We can solve the constraint $\mathrm{d}\pi(B)=0$ explicitly, and in the process show that $\pi_M(\beta)$ is completely determined by $\mathbb{B}$. As a result, we can pull the boundary term out of the integral.

To see this, note that $\mathrm{d}\pi(B)=0$ implies $\mathrm{d}\beta^\partial=-\mathrm{d}\mathbb{B}$, where $\beta^\partial=\pi_M(\beta)$. Thus $\mathbb{B}=-\beta^\partial+\gamma$, where γ is some closed form. But since β^∂ is in the fibre and $\mathbb{B}$ in the base of the boundary fibration, $*\mathbb{B}=\mathbb{B}$ and $*\beta^\partial=-\beta^\partial$ and therefore

$$\mathbb{B}=*\mathbb{B}=*(-\beta^\partial+\gamma)=\beta^\partial+*\gamma. \tag{4.4.14}$$

By adding and subtracting this relation from $\mathbb{B}=-\beta^\partial+\gamma$, we obtain

$$\mathbb{B}=\frac{1}{2}(\gamma+*\gamma)=\gamma_+ \tag{4.4.15}$$

$$\beta^\partial=\frac{1}{2}(\gamma-*\gamma)=\gamma_-. \tag{4.4.16}$$

Thus we can pull boundary terms out of the integral to get

$$
\begin{aligned}
&\exp\left(-\int_{\partial M}\mathbb{B}\wedge A+\frac{1}{2}\int_{\partial M}\gamma_-\wedge\gamma_+\right)\int[\mathrm{D}\beta]\exp\left(\frac{1}{2}\int_M\beta\wedge\mathrm{d}\beta-\int_M\mathrm{d}\beta\wedge A\right)\\
&=\exp\left(-\int_{\partial M}\gamma_+\wedge a_-+\frac{1}{2}\int_{\partial M}\gamma_-\wedge\gamma_+\right)\exp\left(-\frac{1}{2}\int_M A\wedge\mathrm{d}A\right).
\end{aligned}
$$

If we suppose that γ is not only closed but exact with $\gamma=\mathrm{d}\phi$, we obtain the effective action

$$S_{\text{eff}}=-\int_{\partial M}(\mathrm{d}\phi)_+\wedge a_-+\frac{1}{2}\int_{\partial M}(\mathrm{d}\phi)_-\wedge(\mathrm{d}\phi)_+-\frac{1}{2}\int_M A\wedge\mathrm{d}A. \tag{4.4.17}$$

5. Laughlin wavefunction.

5.1. The Laughlin wavefunction is a famous guess for the wavefunction describing the ground state of a quantum Hall fluid (see for instance [64], Chapter 7). It is also related to the solutions of the Knizhnik-Zamolodchikov equation.

PROPOSITION 5.1. The function

$$\phi(z_1,q_1,\ldots,z_n,q_n)=\lambda\prod_{i<j}(z_i-z_j)^{(q_i,q_j)} \tag{4.5.1}$$

solves the Knizhnik-Zamolodchikov equation.

PROOF.

$$
\begin{aligned}
\dot{\phi} &= \sum_{k<l} \frac{\mathrm{d}}{\mathrm{d}t}\left((z_k - z_l)^{(q_k,q_l)}\right) \prod_{k\neq i<j\neq l} (z_i - z_j)^{(q_i,q_j)} \\
&= \sum_{k<l} \langle q_k, q_l\rangle (\dot{z}_k - \dot{z}_l)(z_k - z_l)^{(q_k,q_l)-1} \prod_{k\neq i<j\neq l} (z_i - z_j)^{(q_i,q_j)} \\
&= \sum_{k<l} \langle q_i, q_j\rangle \frac{\dot{z}_k - \dot{z}_l}{z_k - z_l} \prod_{i<j} (z_i - z_j)^{(q_i,q_j)} = \sum_{k<l} (q_i, q_j) \frac{\dot{z}_k - \dot{z}_l}{z_k - z_l} \phi.
\end{aligned}
$$

■

We can also include static sources by adding a constant path $z_{n+1}(t) = z_s$ and a vector $q_{n+1} := q_s$ of source charges. Then we obtain the form of the KZ equation given as Equation 3.10 in [34]:

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left\{ \sum_{1\leq i<j\leq n} (q_i, q_j) \frac{\dot{z}_i - \dot{z}_j}{z_i - z_j} + \sum_{i=1}^{n} (q_i, q_s) \frac{\dot{z}_i}{z_i - z_s} \right\} \phi. \tag{4.5.2}
$$

To compare with [34], simply take $h(z) = \ln(z - z_s)$ and $q_\partial = q_s$.

It is easy to see that the solution of this equation is

$$
\phi(z_1, q_1, \ldots, z_n, q_n) = \lambda \prod_{i<j} (z_i - z_j)^{(q_i,q_j)} \prod_{i=1}^{n} (z_i - z_s)^{(q_i,q_s)}. \tag{4.5.3}
$$

DEFINITION 5.2. A *Laughlin wavefunction* is any $\psi \in L^2(\mathbb{R}^N)$ with

$$
\psi(z_1, \ldots, z_N) = f(z_1, \ldots, z_n) \exp\left(- \sum_{i=1}^{N} |z_i|^2 \right) \tag{4.5.4}
$$

where f solves the KZ equation.

Conclusions, Future Research, Open Problems.

One of the difficulties in writing this thesis has been the diversity of topics that the main physical subject matter touches on. Indeed, as far as the FQHE is concerned, there are vast areas of research that we have entirely neglected: the connection to conformal field theory, the effects of disorder, microscopic analysis in general, nonabelian states, the list goes on.

A mathematical treatment of any of these areas in particular would provide more than enough material for a Master's thesis. I focused on the effective field theory approach, but even this restriction is not enough to narrow the field of investigation to the scope of a student paper.

As a result, the present work is, to a certain extent, somewhat superficial. In order to make some amends for this defect, I will indicate some problems which I wish to investigate in more depth in the future.

Chapter 1 was written with the purpose of providing tools to handle the sign issues that arise in graded algebra and graded geometry. At present, there appears to be no "canonical" way of handling signs. Of course, in handling signs there will always remain a certain ambiguity which cannot be eliminated due to the different conventions (unless everyone were to agree to one convention). It should be possible, however, to hide most or all signs by categorical methods, and while all the tools for doing so exist in the literature (or on MathOverflow), there exists at present no complete guide to this approach, and much of the literature makes no attempt to handle signs systematically.

Chapter 1 represents a first attempt in this direction, and it appears to me that string diagrams are an excellent way not only of mitigating the effects of different conventions, but also of representing the different conventions. However, there is still much work to be done. For instance, the question of "graded graded objects" versus bigraded objects has not been discussed at all. Graded geometry has only been discussed superficially; here there are still many pitfalls related to signs which could be cleared up. One point is that in exterior algebra on a graded vector space V, one works in $\Lambda^\bullet(V^\vee)$, but if one considers exterior forms as constant differential forms, they live more naturally in $\mathrm{S}^\bullet([1]V^\vee)$. Passing between these descriptions involves copious use of signs.

More generally, I believe there is still much scope for extending the use of diagrammatic tools in quantum theory. One problem which has only been hinted at in this thesis is extending the string diagram calculus to vector spaces of infinite dimension. The discussion of Feynman diagrams as simpler representations of string diagrams could also be pursued further than was done in this thesis.

Chapter 4 hides a number of attempts which did not quite work out. For instance, I spent some time working on "axial" gauge fixings adapted to the boundary of $M = \Sigma \times \mathbb{R}$ (rather than adapted to Σ), without any results which would be worthwhile presenting in this thesis. Another line of thought which did not quite "deliver the goods" is to parameterize the gauge degrees of freedom in Chern-Simons theory by use of the Hodge decomposition. One problem here is that one aims to describe the boundary degrees of freedom using a $1 + 1$ metric, and therefore one naturally works with a $1 + 2$ metric in the bulk. But Hodge theory breaks down completely for pseudo-Riemannian metrics.

Despite these setbacks, there are some encouraging results (as I mentioned in the introduction), such as the computation in Section 3 of Chapter 4 leading to the boundary effective action. It still seems to me that with patience, care and some cleverness these indications can be fleshed out into satisfying results.

APPENDIX A

Computations with the Hodge $*$ Operator.

Differential forms are defined on any smooth manifold; they are therefore *topological*, rather than geometric objects. To speak of the *geometry* of forms therefore requires introducing additional structure, concretely: a metric. Once a metric g has been specified, there is an associated Hodge star operator which captures a plethora of geometric operations. In particular, the Hodge star can be used to relate the calculus of differential forms to traditional vector calculus.

If g is Riemannian, there is a powerful geometric result, namely the *Hodge decomposition* of forms. However, we will not discuss this here, and show instead merely some computations with the Hodge star which do not rely on g being Riemannian.

PROPOSITION 0.1. Let $M = N \times \mathbb{R}$, and let t be a coordinate on $\mathbb{R}$. Suppose $\dim M = m, \dim N = n = m-1$. Fix a metric on N, and suppose that $g_M^2 = g_N^2 + (\mathrm{d}t)^2$. Let $\alpha \in \Omega(N)$. Then

$$*_M\,(\alpha \wedge \mathrm{d}t) = (-1)^{|\alpha|} *_N \alpha \tag{A.0.1}$$

$$*_M\,\alpha = (*_N\alpha) \wedge \mathrm{d}t \tag{A.0.2}$$

PROOF. Orient M so that the volume form is $\mathrm{dVol}_M = \mathrm{d}t \wedge \mathrm{dVol}_N$. Then

$$(\alpha \wedge \mathrm{d}t) \wedge *_M(\alpha \wedge \mathrm{d}t) = \mathrm{d}t \wedge \mathrm{dVol}_N = \mathrm{d}t \wedge \alpha \wedge *_N\alpha = (-1)^{|\alpha|}\alpha \wedge \mathrm{d}t \wedge *_N\alpha \tag{A.0.3}$$

so $*_N\alpha = (-1)^{|\alpha|} *_M (\alpha \wedge \mathrm{d}t)$.

Using this result, let $\alpha = *_N^{-1}\beta$ and take the $*_M$ of both sides of A.0.1 to get

$$*_M^2(*_N^{-1}\beta \wedge \mathrm{d}t) = (-1)^{n-|\beta|} *_M \beta. \tag{A.0.4}$$

The $*_N^{-1}$ contributes a sign $(-1)^{|\beta|(n-|\beta|)+s}$, and $*_M^2$ contributes $(-1)^{(n-|\beta|)(1+|\beta|))+s}$. All the signs cancel and we end up with just

$$*_M\beta = (*_N \wedge \beta) \wedge \mathrm{d}t. \tag{A.0.5}$$

■

DEFINITION 0.2. The *Hodge codifferential* d^* acts on p-forms as

$$\mathrm{d}^* = *^{-1}\,\mathrm{d} * (-1)^p. \tag{A.0.6}$$

The *Laplace-Beltrami operator* is defined as

$$\Delta = \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}. \tag{A.0.7}$$

REMARK 0.3. Of course, if $s = 1$, then Δ is the *d'Alembert operator* $\Box$ in $1 + (n-1)$ dimensions.

LEMMA 0.4. Let g be a metric on an n-manifold with signature $(s, n-s)$ (s negative eigenvalues). Then we have the following facts:

i) $\alpha \wedge *\beta = \beta \wedge *\alpha$.
ii) On p-forms, $*^2 = (-1)^{p(n-p)+s}$.
iii) $(*\alpha, *\beta) = (-1)^s(\alpha, \beta)$.
iv) Δ, Δ^{-1} commute with $*, \mathrm{d}, \mathrm{d}^*$.
v) On manifolds without boundary, d^* is the formal adjoint of d, i.e. $(\mathrm{d}^*\alpha, \beta) = (\alpha, \mathrm{d}\beta)$.

Proof. The proofs of i), ii) and iv) can be found in [71]. Item iii) is a corollary of i) and ii). For v), if there is no boundary, we can integrate by parts without boundary terms to compute

$$(\mathrm{d}^*\alpha, \beta) = \int \mathrm{d}^*\alpha \wedge *\beta = \int \beta \wedge * \mathrm{d}^*\alpha = (-1)^{|\alpha|} \int \beta \wedge \mathrm{d} * \alpha \tag{A.0.8}$$

$$= (-1)^{|\beta|+|\alpha|+1} \int \mathrm{d}\beta \wedge *\alpha = \int \alpha \wedge * \mathrm{d}\beta = (\alpha, \mathrm{d}\beta). \tag{A.0.9}$$

Here we used that $|\mathrm{d}^*\alpha| = |\beta|$, i.e. $|\alpha| = |\beta| + 1$. Similarly, one shows $(\alpha, \mathrm{d}^*\beta) = (\mathrm{d}\alpha, \beta)$. ∎

Definition 0.5. The *Hodge pairing* is the bilinear map $\Omega^k(M) \otimes \Omega^k(M) \to \mathbb{R}$ defined by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta. \tag{A.0.10}$$

Proposition 0.6. If g is Riemannian, then the Hodge pairing is nondegenerate, and therefore an inner product on $\Omega^\bullet(M)$.

Let M be a manifold with boundary. A neighbourhood of the boundary which is a product of the boundary and a "radial" coordinate is called a collar. More precisely:

Definition 0.7. Let M be a smooth manifold with boundary. A *collar* of M is a neighbourhood U (called *collar neighbourhood*) of ∂M together with a diffeomorphism $\phi : U \to \partial M \times [0, 1)$.

Proposition 0.8 ([50], 9.25). If M is a smooth manifold with nonempty boundary, then M has a collar.

Hence, "near the boundary", i.e. on a collar neighbourhood, one can decompose any form ω as

$$\omega = \omega_{\|}(r) + \omega_{\perp}(r) \wedge \mathrm{d}r \tag{A.0.11}$$

where r is the pullback of the standard coordinate on $[0, 1))$ and $\omega_{\|}(r), \omega_{\perp}(r)$ are forms on ∂M.

Proposition 0.9. Consider a form ω as in A.0.11. Then

$$\mathrm{d}\omega = \mathrm{d}_\partial\omega_{\|} + (-1)^{|\omega|}\partial_r\omega_{\|} \wedge \mathrm{d}r + \mathrm{d}_\partial\omega_{\perp} \tag{A.0.12}$$

$$\mathrm{d}^*\omega = \mathrm{d}^*_\partial\omega_{\|} + (\mathrm{d}^*_\partial\omega_{\perp}) \wedge \mathrm{d}r - (-1)^{m+p}\partial_r\omega_{\perp}. \tag{A.0.13}$$

In particular, if ω is a 1-form, then

$$\mathrm{d}^*\omega = \mathrm{d}^*_\partial\omega_{\|} + (-1)^m\partial_r\omega_{\perp}. \tag{A.0.14}$$

PROOF. The first formula is trivial. The second is obtained from Proposition 0.1, and is, in a sense, trivial up to signs. I will take this opportunity to remind the reader that any correct result can be explained by an even number of sign errors in the computation[1].

Now let $|\omega| = p$, $\dim M = m$ and hence $\dim \partial M = m-1$. Then $|\omega_\perp| = p-1$, $|*_\partial \omega_\perp| = m-p$ and $|*_\partial^{-1} \mathrm{d}_\partial *_\partial \omega_\perp| \equiv p \bmod 2$. Hence

$$\begin{aligned}
\mathrm{d}^*\omega &= (-1)^p *^{-1} \mathrm{d} * (\omega_\parallel + \omega_\perp \wedge \mathrm{d}r) = (-1)^p *^{-1} \mathrm{d} \left[(*_\partial \omega_\parallel) \wedge \mathrm{d}r + (-1)^{p-1} *_\partial \omega_\perp \right] \\
&= (-1)^p *^{-1} \left[(\mathrm{d}_\partial *_\partial \omega_\parallel) \wedge \mathrm{d}r + (-1)^{m-1} \partial_r (*_\partial \omega_\perp) \wedge \mathrm{d}r + (-1)^{p-1} \mathrm{d}_\partial *_\partial \omega_\perp \right] \\
&= *^{-1} \left[*(*_\partial^{-1} \mathrm{d}_\partial *_\partial (-1)^p \omega_\parallel) + (-1)^{m+p-1} * (\partial_r \omega_\perp) + *(*_\partial^{-1} \mathrm{d}_\partial *_\partial (-1)^{p-1} \omega_\perp \wedge \mathrm{d}r) \right] \\
&= \mathrm{d}_\partial^* \omega_\parallel - (-1)^{m+p} \partial_r \omega_\perp + (\mathrm{d}_\partial^* \omega_\perp) \wedge \mathrm{d}r.
\end{aligned}$$

■

DEFINITION 0.10. Let $\iota : \partial M \hookrightarrow M$ be the inclusion. Define

$$\pi = \iota^* : \Omega^\bullet(M) \to \Omega^\bullet(\partial M) \tag{A.0.15}$$

$$\tilde{\pi}(\alpha) = *_\partial^{-1} \pi(*\alpha) \tag{A.0.16}$$

PROPOSITION 0.11. The projections $\pi, \tilde{\pi}$ satisfy

$$\mathrm{d}\pi = \pi\, \mathrm{d} \tag{A.0.17}$$

$$\mathrm{d}^* \tilde{\pi} = \tilde{\pi}\, \mathrm{d}^*. \tag{A.0.18}$$

PROOF. The first relation is by virtue of π being a pullback. For the second,

$$\tilde{\pi}(\mathrm{d}^*\alpha) = *_\partial^{-1} \pi(* *^{-1} \mathrm{d} * \alpha) = *_\partial^{-1} \mathrm{d}_\partial \pi(*\alpha) = \mathrm{d}_\partial^* \tilde{\pi}(*\alpha), \tag{A.0.19}$$

where we used that π commutes with d. ■

[1] An observation for which I am indebted to K. H. Rehren

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