From Graded Mathematics to Spin-Statistics and 3D Supergravity

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Master Thesis

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Preface

This master's degree thesis has a twofold nature. On the one hand, it is an opportunity to expose some of the knowledge that I enjoyed acquiring during this later part of my studies, and this both as a proof of my dedication and as a tribute to these domains of mathematics and physics. For this reason, much of the content in this document has no informative value for any active researcher in the field; it might, however, be of some use for any student or researcher new to the formalism presented here, or approaching for the first time the mathematical notions that this requires, notably graded mathematics. For those people, I choose to keep a slightly didactic tone probably unfit for the world of scientific journals, but that I deem valuable and aesthetically more pleasant. On the other hand, it is the result of a—dare I say—humble research, seeking to extend the existing knowledge on the application of the Batalin-Vilkovisky formalism to 3-dimensional gravity.

Additionally, this work includes a personal proof of the spin-statistics theorem, proof that itself has nothing to do with the BV formalism, but that however I decided to include for three reasons. Firstly, because it illustrates the *naturalness* of graded structures in physics by showing how in these terms the spin-statistics theorem follows readily from the postulates of any relativistic quantum theory, which is to say that it follows from our currently best and most successful tool to describe nature at its fundamental level. Secondly, because this theorem in its full generality has always seemed elusive to me as a student, so given the didactic aspect of this work it only made sense to include a simple yet general proof of it. Thirdly, because I wish to share this idea, that playfully came to mind while I was revisiting some books, almost as an epiphany.

Aknowledgments

Working on this topic and being introduced to the beautiful world of BV theories and graded mathematics is something that might never have happened to me if it was not for the generosity of Alberto Cattaneo, who both suggested the topic to me and proposed himself to supervise my work, as well as for Nima Moshayedi's, whose supervision was equally essential. For this reason, I choose to close this preface admitting my wholehearted gratitude for their time and help, and for the opportunity they offered me.

Notation

- We consider multiple types of numbers:
  - $\mathbb{N}$ denotes the set of natural numbers *including* 0,
  - $\mathbb{Z}$ denotes the integers,
  - $\mathbb{R}$ denotes the real numbers.
  - $\mathbb{C}$ denotes the complex numbers.
- If X is a topological space, $\tau(X)$ denotes its topology.
- Given a Lie group G, its associated Lie algebra is denoted by $\mathfrak{g}$.
- All forms of products are left implicit and deduced from the context, unless some ambiguity is present, e.g. $\lambda \cdot u \otimes v =: \lambda uv$.
- Unless parentheses are present, a derivation D acts only on the element directly adjacent to them: $aDbc =: aD(b)c$ and $ab\overleftarrow{D}c =: a(b)\overleftarrow{D}c$.
- In the context of group theory, $\langle a, b, c, ...|R_1, R_2, ...\rangle$ denotes the group generated by $a, b, c, ...$, subject to the relations $R_1, R_2, ...$
- If A is an element of a (Lie) group or a (Lie) algebra, its image under a representation is also written A, whenever there is no ambiguity.
- In the context of gravity or special relativity, Greek letters designate spacetime indices while Latin indices designate Lorentz bundle indices.
- Einstein's summation convention for pairs of upper and lower indices is generally assumed: $x^i y^i \neq x^i y_i := \sum_{i=j} x^i y_j$.
- Indices with a bar on them are not summed over: $x^{\bar{\imath}} y_{\bar{\imath}} \neq x^i y_i$.
- δ^i_j denotes Kronecker's delta: $\delta^i_j = 1$ for $i = j$, $\delta^i_j = 0$ otherwise.
- The equivalence sign "$\equiv$" is used to designate equality *on shell*.
- There are multiple fields associated to a field ψ:
  - ψ^* is it complex conjugate,
  - $\psi^\dagger$ is its Hermitian conjugate,
  - $\overline{\psi}$ is its Dirac conjugate,
  - $\psi^\daleth$ is its associated antifield.

Introduction

The content of this thesis is split among four chapters. The first of them will provide a basic introduction to graded mathematics, this through a presentation of graded algebra and graded derivations that should let a neophyte reader grasp what is the interest of, and the general procedure for, grading structures. This will be immediately followed by a chapter where I present my own proof of the spin-statistics theorem, which not only do I deem recomfortingly simple, but likewise provides an excellent opportunity to show how the most primitive concepts of graded mathematics are actually capable of addressing a quintessential problem of fundamental physics.

The third and fourth chapters will deviate from these general considerations and focus specifically on the Batalin-Vilkovisky[(1)] formalism and its application. To this end, the third chapter will, first, summarise those ideas in graded geometry that are essential to understand the classical BV formalism, introducing the relevant mathematical structures and their morphisms, always assuming that the reader is familiar with usual differential geometry. I recognize, however, that the treatment there will be partial and perhaps not rigorous enough, so I will not fail to refer to—what I believe it is—appropriate bibliography treating these topics more thoroughly. Furthermore, the third chapter will as well present the basics of the BV formalism, including an unfairly short description of the Alexandrov-Kontsevich-Schwarz-Zaboronsky[(2)] construction, that will be key to us in the final chapter.

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(1) Hereon I will always employ the initialism *BV*.

(2) Of course, systematically shortened to *AKSZ*.

In there, the fourth chapter, the formalism developed previously will be applied to an extension of 3-dimensional supergravity. The first section will present two strongly equivalent theories of 3-dimensional gravity, and the second, central section will exploit this equivalence and the AKSZ procedure to properly build a BV theory of 3-dimensional supergravity. With all this being said, it is time to delve into graded mathematics.

Chapter 1

Initiation to graded mathematics

The study of graded mathematical structures, as other topics in the history of mathematics, was initially motivated by the needs of theoretical physics and, despite being born in the context of supersymmetric theories, its reach goes beyond it. An example of this is given by the fact that that one employs this language in any rigorous description of the BRST[3] formalism and of its generalisation, the BV formalism. This chapter will not yet treat any of these formalisms, but rather lay the foundations on which they are built.

1.1 Graded algebra

In a certainly unconventional way, one could grade any category built upon sets by applying a general criterion: a structure is graded whenever it is split in minimally intersecting substructures. We might, for example, have a set S satisfying

$$S = \bigcup_i S_i, \qquad S_i \cap S_j = \emptyset \ \ \forall i \neq j \tag{1.1.1}$$

(3) Initialism for *Becchi-Rouet-Stora-Tyutin*.

and say that the degree of $s \in S$ is n if $s \in S_n$. However, the idea of grading has hardly any use in the absence of some sort of arithmetic, and grading a structure with less than two operations will either lead to contradictory definitions, or to definitions that are by no means more fruitful that the basic non-graded structure. For this reason, the usual starting point of graded algebra are unital rings or vector spaces, whose thorough treatment can respectively be found in the first chapters of Kessler's [Keß19] and of Rogers's [Rog07]. Both approaches lead to an equivalent graded geometry, yet I deem more practical to start from graded rings because, seen as structures that precede modules and hence vector spaces, they allow for a definition of graded linearity that is free of ambiguity

Definition 1.1.1. A unital ring $(R, +, \cdot)$ is **$\mathbb{Z}$-graded** whenever

$$R = \bigoplus_{i \in \mathbb{Z}} R_i, \qquad R_i \cdot R_j \subseteq R_{i+j} \ \forall i, j, \qquad R_i \cap R_j = \{0\} \ \forall i \neq j, \tag{1.1.2}$$

and $R_i \neq \{0\}$ only for finitely many i. If an element r belongs to a specific R_i, we say that it is **homogeneous** and posses a definite **degree**, in this context denoted by $\deg_{\mathbb{Z}}$ and given by

$$\deg_{\mathbb{Z}} r = n \iff r \in R_n. \tag{1.1.3}$$

Conventionally, $\deg_{\mathbb{Z}} 0 = \deg_{\mathbb{Z}} 1 = 0$.

Remark 1.1.2. This definition relies on the ring of integers as the indexing set, but it is often the case that we will consider the quotient set $\mathbb{Z}_2$ as the indexing set, case that is relevant enough as to deserve its own definition.

Definition 1.1.3. A ring R is **$\mathbb{Z}_2$-graded** or a **super**(4)**ring** whenever

$$R = R_0 \oplus R_1, \qquad R_i \cdot R_j \subseteq R_{i+j} \ \forall i, j, \tag{1.1.4}$$

where the sum of subscripts is understood to be modulo 2. Elements of R_0 are said to have **even parity** while elements of R_1 have **odd parity**.

(4) I prefer to avoid using the prefix *super* for $\mathbb{Z}_2$-graded structures because I consider it gives the false impression that such structures are only relevant to supersymmetric theories, and this without mentioning that I find uncomfortably ridiculous a text plagued with "super" things.

Remark 1.1.4. More generally we could imagine having other indexing abelian groups, and frequently a structure might be charged with a multiplicity of different gradings. For this reason, we might and we will often talk of graded structures without specifying the grading, either because it is understood by the context, either because the only degree that is relevant to us—for it will be the only one determining the graded commutativity—is the degree introduced in the following definition. It is worth mentioning, however, that we will be assuming that any indexing group is cyclic, hence identified with $\mathbb{Z}$ or some $\mathbb{Z}_n$ for $n \in \mathbb{N}$.

Definition 1.1.5. Given a graded structure X with n different gradings $\deg_1, \ldots, \deg_n$, we define the **total degree** $|x|$ of $x \in X$ as the sum of all its individual degrees:

$$|x| = \sum_{i=1}^{n} \deg_i x. \tag{1.1.5}$$

We mean this total degree whenever we speak of "degree" unspecificly.

Remark 1.1.6. Note that the total grading over V induces a $\mathbb{Z}_2$-grading:

$$\deg_{\mathbb{Z}_2} v = |v| \bmod 2 \quad \forall v \in V. \tag{1.1.6}$$

Now, analogously to conventional rings—and probably the main reason to consider gradings in the first place—graded rings can be commutative in a graded manner:

Definition 1.1.7. A graded ring is moreover **graded commutative** or **supercommutative** if for all $r, s \in R$ we have

$$rs = (-1)^{|r||s|} sr \tag{1.1.7}$$

This idea of commutativity can be generalised to a certain guiding principle that will establish the sign change whenever two adjacent mathematical objects are swapped.

Postulate 1. *Whenever two adjacent graded elements of any—and possibly different—kind exchange position, the sign should change according to their degrees, apart from any change of sign that the non-graded analogous structure includes.*

Graded commutativity is then nothing else than commutativity, in the graded context where this postulate holds, and the upcoming definitions are merely the application of this principle to create graded analogs of conventional structures. A graded module is a clear example of this.

Definition 1.1.8. A **graded module** V over R is a left and right module V over a graded commutative ring R such that for all $r \in R$ and $v \in V$

$$rv = (-1)^{|r||v|} vr, \qquad |rv| = |r| + |v| \iff r, v \neq 0. \tag{1.1.8}$$

Definition 1.1.9. Given a graded R-module V, a linearly independent generating set $\{e_{(i)}\}$ forms a **basis**; then we say that two subsets $\{l_v^i\}_i$ and $\{r_v^i\}_i$ of R are respectively the **left** and **right coordinates** of v whenever we have that

$$v = \sum_i l_v^i e_{(i)} = \sum_i e_{(i)} r_v^i \tag{1.1.9}$$

A graded module with a basis is **free**, and the commutativity of R implies that all possible bases of V have the same cardinality, which we call its **rank** or **dimension**—the latter case particularly if R is a field.

Remark 1.1.10. We could have impoverished the definition of graded module by separating left from right graded modules and by not assuming that the associated ring is commutative, but since the whole interest of grading structures is in considering the swap of adjacent symbols, a definition of graded modules that did not systematically lead to equation (1.1.8) would have felt incomplete, plus from the practical point of view we do not need that generality here. Actually, unless the contrary is specified, we will always assume that modules are free and that speaking of "coordinates" we mean left coordinates.

Remark 1.1.11. Note that the definition of the degree of a scalar product in (1.1.8) is the only reasonable one for graded structures. Indeed, commuting an element with a *pair* of odd elements is equal to commuting the former with each of the latter individually, so the total power of (-1) changing the sign should be proportional to the sum of two odd integers, that is an even integer—so pairs of odd elements behave as even elements. A similar reasoning can be made with any combination of even and odd elements, and the conclusion is the following postulate.

Postulate 2. *For each individual degree that two graded elements—possibly of different kind—have in common, each respective degree of their product should be equal to the sum of their individual degrees.*

Example 1.1.12. A notable example of the application of this principle are Cartesian products. Suppose that $\{O_i\}$ is a collection of graded objects in the same category. Then their Cartesian product is graded:

$$x = (x_1, x_2, ...) \in \prod_i O_i \quad \Longrightarrow \quad \deg_j x = \sum_i \deg_j x_i \tag{1.1.10}$$

for any of the degrees $\deg_j$ that each O_i is endowed with.

Remark 1.1.13. A reasonable question at this point, given that—for example in modules—we consider products of elements of different types, is how do we define the degree $\deg_G (x\xi)$ of a product where only ξ belongs to an G-graded structure. The answer is obvious: any structure X admits a trivial G-grading, such that $\deg_G x = 0$ for any $x \in X$.

Definition 1.1.14. Given any two graded structures X and Y, a **graded map** f is a map $f : X \to Y$ that has a definite degree $|f|$, given by the property that for all $x \in X$

$$|f(x)| = |f| + |x|. \tag{1.1.11}$$

Definition 1.1.15. Given the graded R-modules $V_1, V_2, \ldots, V_n$ and W, a map $f : V_1 \times \cdots \times V_n \to W$ is a **graded multilinear** or **graded n-linear map** if f is additive in each argument and such that for all $r \in R$ and $v_k \in V_k$

$$f(\ldots, rv_k, \ldots) = (-1)^{|r|(|v_1|+\cdots+|v_{k-1}|+|f|)}\, rf(\ldots, v_k, \ldots). \tag{1.1.12}$$

If $n = 1$, the map is simply **graded linear**.

Definition 1.1.16. The **dual** graded R-module V^* of a graded R-module V is the module of graded linear maps $V \to R$. Given a basis $\{e_i\}$ of V, the associated **dual basis** $\{e^i\}$ is defined by the relation

$$e^i (e_j) = \delta^i_j \quad \forall i, j \in [\![1, \dim V]\!]. \tag{1.1.13}$$

Remark 1.1.17. The definition (1.1.13) implies that for each degree $\deg_j$ on V, dual basis pairs have opposite degrees:

$$\deg_j e^i = - \deg_j e_i. \tag{1.1.14}$$

Definition 1.1.18. Given a graded R-module V, we define its **tensor algebra** $T(V)$ as in the non-graded case:

$$T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i} \tag{1.1.15}$$

where $V^{\otimes k}$ is the k^{th} tensor power of V

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \tag{1.1.16}$$

and $V^{\otimes 0} := R$. Besides any grading that V might have, $T(V)$ is endowed with an additional $\mathbb{Z}$-grading, whose associated **tensor degree** $\deg_{\mathrm{T}}$ is defined for any $t \in T(V)$ as

$$\deg_{\mathrm{T}} t = k \qquad \Longleftrightarrow \qquad t \in V^{\otimes k}. \tag{1.1.17}$$

Definition 1.1.19. We denote by $I_\pm$ the two-sided ideal of $T(V)$ generated by graded (anti-)symmetric elements of $T(V)$, that is

$$I_\pm := \langle \{ u \otimes v \pm (-1)^{|u||v|} v \otimes u \; : \; u, v \text{ are homogeneous} \} \rangle. \tag{1.1.18}$$

Then the **symmetric algebra** $\mathrm{Sym}(V)$ and the **exterior algebra** $\bigwedge V$ of V are given by the quotients

$$\mathrm{Sym}(V) := T(V) \big/ I_- \,, \qquad\qquad \bigwedge V := T(V) \big/ I_+ \,. \tag{1.1.19}$$

In general an analogous ideal $I^n_\pm$ exists in any n^{th} tensor power of V, and in particular we call the $\boldsymbol{n}^{\textbf{th}}$ **exterior power** of V the space

$$V^{\wedge n} := V^{\otimes n} \big/ I^n_+ \,. \tag{1.1.20}$$

Remark 1.1.20. In the case of a $\mathbb{Z}_2$-graded module $V = V_0 \oplus V_1$—and hence of any graded module where we have induced a $\mathbb{Z}_2$-grading as in remark 1.1.6—we have one important property:

$$\mathrm{Sym}(V) = \mathrm{Sym}(V_0) \otimes \bigwedge V_1. \tag{1.1.21}$$

Exterior algebras are as essential for graded geometry as they are for differential geometry, through objects that will be constructed in the next section. That being said, we could keep providing definitions of graded analogs to common mathematical structures, but these definitions follow straightforwardly from the application of the two previous postulates to non-graded structures. Nevertheless, before we pass to the next section I explicitly give one last definition—that, hopefully, will already seem natural—due to its relevance for us.

Definition 1.1.21. A **graded Lie algebra** $\boldsymbol{A}$ is a graded module $\boldsymbol{A}$ with a bilinear product $[\bullet, \bullet] : \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ that is **graded antisymmetric**, meaning that

$$[a, b] = -(-1)^{|a||b|}[b, a], \tag{1.1.22}$$

and satisfies the **graded Jacobi identity**, so

$$[a, [b, c]] + (-1)^{|a|(|b|+|c|)}[b, [c, a]] + (-1)^{|c|(|b|+|a|)}[c, [a, b]] = 0 \tag{1.1.23}$$

for all $a, b, c \in \boldsymbol{A}$. The degree of such product satisfies Postulate 2:

$$\deg\, [a, b] = \deg a + \deg b. \tag{1.1.24}$$

1.2 Graded derivations

The topic of graded derivations is covered extensively in any book on graded geometry, myself employing Kessler's [Keß19] and DeWitt's [DeW92], and as with the previous section, I give here the principal aspects that we are interested in.

Definition 1.2.1. A **derivation** over a graded R-algebra(5) A is, for some A-module V, a graded linear map $D : A \to V$ that for all $a, b \in A$ satisfies the **graded Leibniz rule**:

$$D(ab) = Dab + (-1)^{|D||a|}aDb. \;^{(6)} \tag{1.2.1}$$

The collection of such graded derivations forms a graded A-module $\mathrm{Der}_R(A, E)$, which whenever $E = A$ will be denoted $\mathrm{Der}_R(A)$ or even $\mathrm{Der}(A)$, because rarely will there be any ambiguity.

(5) Unless explicitly stated, by *algebra* I will always mean unital associative algebras, whose grading satisfies postulates 1 and 2 in the obvious manner.

(6) The reader should be aware of the notation guide in page iv.

Definition 1.2.2. A **graded differential form of degree n**—or simply **n-form**—over A is an element of

$$\Omega^n(A) := \big(\mathrm{Der}(A)^*\big)^{\wedge n} \tag{1.2.2}$$

for any $n \in \mathbb{N}$. The exterior A-algebra of these spaces is

$$\Omega(A) := \bigoplus_{i=0}^{\infty} \Omega^i(A) \tag{1.2.3}$$

and to it we associate the **cohomological degree**

$$\deg_{\Omega(A)} \omega = n \qquad \Longleftrightarrow \qquad \omega \in \Omega^n(A), \tag{1.2.4}$$

written $\deg_\Omega$ when no ambiguity is possible.

Remark 1.2.3. A first observation is that the cohomological degree is nothing other than the tensorial degree, but we choose a specific term in the context of differential forms because of their role in cohomology, and also because it is never bad to have a term which immediately signals that we are dealing with differential forms, which are central.

Remark 1.2.4. A second thing to observe, more substantial, is that one recovers the usual construction of differential forms by setting to zero every degree *except for the cohomological one.* This is due to the fact that conventional exterior algebras *are* graded algebras, whose grading is given by the tensorial degree, fact that should be a guiding principle and is encapsulated in the following postulate.

Postulate 3. *The grading of algebraic and differential-geometric structures should be made such that the non-graded structure is recovered by setting to zero every degree except for the tensorial one.*

Definition 1.2.5. A **graded symplectic form** ω is a graded 2-form that is **closed** and **non-degenerate**, i.e. respectively such that $d\omega = 0$ and

$$\omega(X, Y) = 0 \; \forall \; Y \qquad \Longleftrightarrow \qquad X = 0. \tag{1.2.5}$$

Definition 1.2.6. A **de Rham differential** or **cohomological derivation** over A is a derivation $d_\Omega \in \mathrm{Der}(\Omega(A))$ with three defining properties:

$$|d_\Omega| = 1, \qquad d_\Omega \Omega^n(A) \subseteq \Omega^{n+1}(A), \qquad d_\Omega^2 = 0. \tag{1.2.6}$$

Definition 1.2.7. The **canonical derivation** or **differential** d over A is the unique de Rham differential such that for all $f \in \Omega^0(A) = A$ and $X \in \mathrm{Der}(A)$

$$df(X) = X(f). \tag{1.2.7}$$

Definition 1.2.8. A **de Rham complex** over A is a tuple $(\Omega(A), d_\Omega)$ for some de Rham differential d_Ω over A. Whenever we talk about *the* de Rham complex, we mean the one with $d_\Omega = d$.

Definition 1.2.9. The **interior derivative** or **contraction** with respect to a derivation $X \in \mathrm{Der}(A)$ is the unique derivation $\iota_X \in \mathrm{Der}(\Omega(A))$ such that for any $f \in A$ and any degree $\deg_j$ on A it satisfies

$$\iota_X f = 0, \qquad \iota_X df = X(f), \tag{1.2.8a}$$

$$\deg_j \iota_X = \deg_j X, \qquad \deg_\Omega \iota_X = -1. \tag{1.2.8b}$$

The associated map

$$\iota : A \times \Omega(A) \to \Omega(A) : (X, \omega) \mapsto \iota_X \omega \tag{1.2.9}$$

is called the **interior product** because it is 2-linear, which translates into a sort of distributivity law.

Remark 1.2.10. In terms of the interior product, (1.2.8b) reduces to

$$|\iota| = -1. \tag{1.2.10}$$

Proposition 1.2.11. *If* $X \in \mathrm{Der}(A)$ *is even, then* $\iota_X^2 = 0$.

Proof. Given $f, g \in A$ and $X \in \mathrm{Der}(A)$ we have

$$\begin{aligned}
\iota_X^2(df\,dg) &= \iota_X\Big(X(f)dg + (-1)^{(|X|-1)(|f|+1)} df\,X(g)\Big) \\
&= \Big((-1)^{(|X|-1)(|f|+|X|)} + (-1)^{(|X|-1)(|f|+1)}\Big) X(f)X(g) \\
&= (-1)^{|f|(|X|-1)}\Big(1 + (-1)^{|X|-1}\Big) X(f)X(g)
\end{aligned} \tag{1.2.11}$$

which is zero whenever $|X| \in 2\mathbb{Z}$. This applies to any n-form with $n \geq 2$ because they are generated by 1-forms such as df and dg. ♡

Remark 1.2.12. Many other properties concerning contractions and differentials will be analogous to the ones in the non-graded context, and we will introduce only some of them in chapter 3.

Definition 1.2.13. A **left derivation** $\overleftarrow{D}$ is associated to each derivation $D \in \mathrm{Der}(A)$ and for all $f \in A$ it acts as follows:

$$f\overleftarrow{D} = (-1)^{|D|(|f|+1)} Df. \tag{1.2.12}$$

By contraposition, the usual derivation D is also called **right derivation** and sometimes noted $\overrightarrow{D}$.

Remark 1.2.14. Evidently, left and right derivations agree for even derivations, and the sign is chosen to ensure that derivatives with respect to a function f act the same way on f from both sides, i.e.

$$D_f f = 1 \quad \Longleftrightarrow \quad f\overleftarrow{D}_f = 1, \tag{1.2.13}$$

and also to guarantee that left derivations follow the same Leibniz rule as right derivations—and hence respect postulate 1—namely

$$(fg)\overleftarrow{D} = fg\overleftarrow{D} + (-1)^{|D||g|} f\overleftarrow{D}g. \tag{1.2.14}$$

With this, we are ready to conclude this chapter, because—as we shall see—the notions of graded mathematics presented so far find already an interesting application, which we treat next.

Chapter 2

Spin and statistics

This chapter will be different from the preceding one, in that it will consist precisely in the application of those definitions and postulates to proofs, and this in order to illustrate both the utility and the naturalness of graded mathematics in quantum theories. Indeed, and as we shall see in what follows, the Hilbert spaces describing physical states of a relativistic quantum theory—such as quantum field theory—admit a canonical grading which, in combination with well known results, make the spin-statistics theorem appear as little more than a tautology.

We will follow here standard conventions in physics. Consequently, if v is a 4-vector, its first component will be written v^0 and its remaining three components will be collected in a 3-vector $\vec{v}$; in other words,

$$v = (v^0, v^1, v^2, v^3) \quad \Longrightarrow \quad \vec{v} := (v^1, v^2, v^3). \tag{2.1}$$

Moreover, we will use the same symbol for an operator $M : H \to H$ over a Hilbert space H and for any of the tensor powers of M, keeping the product symbol implicit, so in the end we write

$$M(\Psi_1 \cdots \Psi_n) := M^{\otimes n}(\Psi_1 \otimes \cdots \otimes \Psi_n) \quad \forall n \in \mathbb{N},\ \{\Psi_i\}_{i=1}^n \subset H. \tag{2.2}$$

Finally, J_3 will denote the component of the angular momentum along the quantisation axis, as well as its image under a Lie algebra representation, and P^μ will denote the μ^{th} component of the linear momentum operator. Being aware of this, we are ready to begin.

Definition 2.1. A **single-particle state** is an eigenstate of J_3 that lives in an irreducible representation of $\mathrm{ISO}(3,1)$, specified by a number $s \in \frac{1}{2}\mathbb{N}$ called its **spin**. Its J_3-eigenvalue σ will be called its **spinoid**.

Remark 2.2. Single particles in nature will be described by linear combinations of what I call "single-particle states", and most authors such as Weinberg in [Wei95] use this term to describe these combinations. However, restricting the definition to those irreducible eigenstates will lighten the writing *without* sacrificing generality.

Remark 2.3. Most often authors call also *spin* what I call *spinoid*, but to me there is a critical reason to not confuse them: the spinoid of a single-particle state depends on the representation theory of its associated little group, which differs for massive and massless particles(7).

Definition 2.4. A spin or a spinoid is **partial** if it is half an odd integer. The **partiality** is 1 if the spin or spinoid is partial, and it is 0 otherwise.

Definition 2.5. The **Fock space** $\mathcal{H}$ of a relativistic quantum theory is the tensor algebra of the Hilbert space H spanned by the single-particle states of the theory, that is

$$\mathcal{H} = \bigoplus_{i=0}^{\infty} H^{\otimes i}. \tag{2.3}$$

Definition 2.6. Given a single-particle state Ψ in a Fock space $\mathcal{H}$ with vacuum state Υ, the operator $a^\dagger : \mathcal{H} \to \mathcal{H}$ is the **creator** of Ψ if

$$\Psi = a^\dagger \Upsilon. \tag{2.4}$$

Definition 2.7. The **natural grading** of a space of physical fields is the $\mathbb{Z}_2$-grading where bosonic and fermionic states are assigned even and odd parity respectively.

Proposition 2.8. *The Fock space of a relativistic quantum theory admits the natural grading, thus becoming a graded commutative algebra.*

(7) The **little group** of a state living in an irreducible unitary representation of $\mathrm{ISO}(3,1)$ is the subgroup of the latter that leaves invariant the 4-momentum of the given state. This group is $\mathrm{SO}(3)$ for massive particles and $\mathrm{ISO}(2)$ for massless ones.

Proof. Let the Fock space be denoted by $\mathcal{H}$, the vacuum state be Υ, and let Ψ be the single-particle state associated to the creator $a^\dagger$. We define the degree of both Ψ and $a^\dagger$ as follows:

$$|\Psi| := |a^\dagger| := \begin{cases} 0 & \text{for bosons,} \\ 1 & \text{for fermions.} \end{cases} \tag{2.5}$$

Note that, under this definition, the (anti-)commutation properties of creation and annihilation operators can be expressed under the common umbrella of a graded Lie bracket, so that for any two—possibly different—creation operators $a^\dagger$, $a'^\dagger$ we will in particular have the following *graded* commutation relation:

$$[a^\dagger, a'^\dagger] = 0. \tag{2.6}$$

This degree will extend into a consistent $\mathbb{Z}_2$-grading of $\mathcal{H}$ if

1. single particle states generate $\mathcal{H}$,
2. we have $\mathcal{H}_i \otimes \mathcal{H}_j \subseteq \mathcal{H}_{i+j}$ for all $i, j \in \mathbb{Z}_2$.

Now, the first requirement is trivial, since it follows from the definition of a Fock space, while the second requirement follows from the commutativity relation of the creation operators. To see this let Ψ and Ψ', with associated creators $a^\dagger$ and $a'^\dagger$, be two—possibly different—single-particle states. Firstly, (2.6) implies that their tensor product is graded commutative:

$$\Psi\Psi' = a^\dagger a'^\dagger \Upsilon = (-1)^{|a^\dagger||a'^\dagger|} a'^\dagger a^\dagger \Upsilon = (-1)^{|\Psi||\Psi'|} \Psi'\Psi. \tag{2.7}$$

Secondly, this in turn implies that the degree of their product is the sum of their degrees. Indeed, taking two identical copies $\Psi^{(1)} = \Psi^{(2)} = \Psi$ and $\Psi'^{(1)} = \Psi'^{(2)} = \Psi'$ of each state, numbered only to keep track of their position as we permute them, we realise that

$$\begin{aligned} (-1)^{|\Psi\Psi'|^2} \Psi^{(1)}\Psi'^{(1)}\Psi^{(2)}\Psi'^{(2)} &\overset{(2.7)}{=} \Psi^{(2)}\Psi'^{(2)}\Psi^{(1)}\Psi'^{(1)} \\ &= (-1)^{|\Psi|(|\Psi|+|\Psi'|)}(-1)^{|\Psi'|(|\Psi|+|\Psi'|)} \Psi^{(1)}\Psi'^{(1)}\Psi^{(2)}\Psi'^{(2)} \\ &= (-1)^{(|\Psi|+|\Psi'|)^2} \Psi^{(1)}\Psi'^{(1)}\Psi^{(2)}\Psi'^{(2)}. \end{aligned} \tag{2.8}$$

Modulo 2, nonetheless, there is the stringent equivalence

$$x^2 = y^2 \bmod 2 \iff x = y \bmod 2 \tag{2.9}$$

that we can apply to (2.8) to finally deduce that

$$|\Psi\Psi'| = |\Psi| + |\Psi'| \bmod 2. \tag{2.10}$$

This, combined with the graded commutativity (2.7), allows to further derive that for

$$H_j := \operatorname{span}\{\Psi \in H : |\Psi| = j\} \quad \forall j \in \mathbb{Z}_2 \tag{2.11a}$$

the Fock space $\mathcal{H}$ satisfies

$$\begin{aligned} \mathcal{H} &= \bigoplus_{i\in\mathbb{N}} (H_0 \oplus H_1)^{\otimes i} = \bigoplus_{i,j\in\mathbb{N}} H_0^{\otimes i} \otimes H_1^{\otimes j} \\ &= \underbrace{\bigoplus_{\substack{i\in\mathbb{N}\\ j\in 2\mathbb{N}}} H_0^{\otimes i} \otimes H_1^{\otimes j}}_{\mathcal{H}_0} \oplus \underbrace{\bigoplus_{\substack{i\in\mathbb{N}\\ j\in 2\mathbb{N}+1}} H_0^{\otimes i} \otimes H_1^{\otimes j}}_{\mathcal{H}_1}, \end{aligned} \tag{2.11b}$$

from which we finally conclude that

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \qquad \mathcal{H}_i \otimes \mathcal{H}_j \subseteq \mathcal{H}_{i+j}, \tag{2.12}$$

meaning that $\mathcal{H}$ is a naturally graded commutative algebra. ♡

Theorem 2.9. *In a relativistic quantum theory, the spin of bosons is an integer and that of fermions is partial.*

Proof. Assume that the Fock space of the theory has been naturally graded by virtue of proposition (2.8). Meanwhile, Wigner's results on the unitary representations of $\mathrm{ISO}(3,1)$ [Wig89] implies that a *free* single-particle state is fully characterised by its spinoid σ and its 4-momentum p [BW48], so as a result their degree must be a function of at most their spinoid and their 4-momentum. We will focus on these states because the spin of a particle is an intrinsic feature, independent of whether the particle is interacting or not. Moreover, given that the natural grading must be compatible with the action of the Lorentz group—for otherwise we could boost or rotate fermions into bosons and vice versa, which violates multiple conservation laws—it must be the case too that the natural degree depends only on Lorentz invariant quantities, namely its spin, the sign $\text{sgn}(p^0)$ of the zeroth component of its 4-momentum, and its invariant mass

$$m^2(p) := -p^\mu p_\mu = p^0 \cdot p^0 - \vec{p} \cdot \vec{p}. \tag{2.13}$$

We will make the physically trivial assumption that all single-particle states have non-negative energy, which in turn establishes that the grading can only depend on the spin and on the invariant mass.

With this, let us consider two—potentially different—free single-particle states $\Psi_\sigma(p)$ and $\Psi_{\sigma'}(p')$, denoted respectively Ψ and Ψ' to lighten the equations. The first thing to realise is that the grading doesn't have any effect on the representation of the angular momentum operator J_3 or of the 4-momentum operator P^μ, since the algebra $\mathfrak{iso}(3,1)$ is not graded, reason why—by definition of a Lie algebra representation, and of spinoid and momentum—we still have

$$\begin{aligned} J_3(\Psi\Psi') &= (J_3\Psi)\Psi' + \Psi(J_3\Psi') = (\sigma + \sigma')\Psi\Psi', \\ P^\mu(\Psi\Psi') &= (P^\mu\Psi)\Psi' + \Psi(P^\mu\Psi') = (p^\mu + p'^\mu)\Psi\Psi'. \end{aligned} \tag{2.14}$$

Now, because the problem only concerns the partiality of the spin, and given the interdependence between the spin and the possible values of the spinoid, we will consider the grading function γ as depending on the spinoid rather than on the spin:

$$\gamma(\sigma, m^2(p)) := |\Psi_\sigma(p)|. \tag{2.15}$$

Since the grading satisfies postulate 2, this function satisfies

$$\begin{aligned} \gamma(\sigma + \sigma', m^2(p + p')) &= |\Psi\Psi'| = |\Psi| + |\Psi'| \\ &= \gamma(\sigma, m^2(p)) + \gamma(\sigma', m^2(p')). \end{aligned} \tag{2.16}$$

The immediate consequence of this is that γ cannot be a function of the invariant mass, because the first line in equation (2.16) depends on the angular configuration of the 3-momenta $\vec{p}$ and $\vec{p}'$, while the last does not—since $m^2(p^0, \vec{p})$ depends on $\vec{p}$ only through its squared modulus. Consequently, γ must be a map

$$\gamma : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}_2 \tag{2.17}$$

with the property that

$$\gamma(\sigma + \sigma') = \gamma(\sigma) + \gamma(\sigma'). \tag{2.18}$$

This exactly means that γ is a group homomorphism between additive groups. Moreover, both groups happen to be cyclic,

$$\left(\frac{1}{2}\mathbb{Z}, +\right) = \left\langle \frac{1}{2} \right\rangle, \qquad \left(\mathbb{Z}_2, +\right) = \langle 1 \rangle, \tag{2.19}$$

so γ is entirely determined by the image of $1/2$ under it. Of course, γ is not trivial because fermions and bosons are by definition assigned different degrees, thus the only option is to have $\gamma(1/2) = 1$, that is

$$\gamma(\sigma) = 2\sigma \bmod 2. \tag{2.20}$$

Due to the relation between the partiality of the spin s and the one of the spinoid σ, this allows us to conclude that

$$|\Psi| = \begin{cases} 0 \iff s \in \mathbb{Z}, \\ 1 \iff s \in \frac{1}{2}(2\mathbb{Z} + 1), \end{cases} \tag{2.21}$$

which, by definition of the natural grading, means that a state corresponds to a boson or a fermion if and only if its spin is an integer or a half-integer respectively. ♡

> Under the light of this result, I give the following interpretation to the spin-statistics theorem: the existence in Nature of two types of particles with radically different quantum-statistical properties can canonically be accounted for in a quantum theory by grading the space of physical states, and ultimately this grading can only be determined by the spin of a state. In this sense, the spin-statistics theorem might be the direct mathematical consequence of recognising and formalising the existence of a *natural parity*.

Chapter 3

The BV formalism

3.1 Elements of graded geometry

Except for the last definition, none of the notions presented in chapter 1 were really *new*, but merely the application of the grading postulates to familiar notions of algebra and differential calculus. Graded geometry is, in the same spirit, a generalisation of differential geometry, but the extension of non-graded structures and operations becomes at some point far from trivial, and particularly the integration theory on graded manifolds requires some care. Nevertheless, since the present work only goes as far as to find the BV extension of a classical theory, the integration theory on graded manifolds—that would have been crucial if we wanted to quantise such BV extension—will not be treated here. Anything concerning it, as well as a more thorough treatment of the contents of this section, can be found in Kessler's [Keß19], Rogers's [Rog07], DeWitt's [DeW92] or Mnev's [Mne19]; I will mostly follow the latter; besides, the reader is assumed to be familiarised with differential geometry and have basic understanding of sheaf theory; otherwise, a very succinct—but helpful—introduction to the former is found in Nakahara's [Nak03] while a treatment of the second is found in Warner's [War10]. As usual, we start with some definitions.

Definition 3.1.1. Given a topological space (X, τ), a graded **presheaf** over X is a collection $\mathcal{O}_X$ of graded commutative algebras $\{A(U)\}_{U\in\tau}$ that, for any open subsets U, V and W of X such that $W \subseteq V \subseteq U$, it satisfies three properties:

1. There is a linear map $\rho_{UV} : A(U) \to A(V)$, called the **restriction map** from U to V.

2. The restriction of U to U is the identity on $A(U)$: $\rho_{UU} = \mathrm{id}_{A(U)}$.

3. The restriction maps are transitive: $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

Example 3.1.2. It is easy to show that the collection of smooth functions $\{C^\infty(U)\}_{U\in\tau(M)}$ on the open subsets of a conventional manifold M forms a (non-graded) presheaf, taking as restriction maps the restriction of the functions: $\rho_{UV} : C^\infty(U) \to C^\infty(V \subseteq U) : f \mapsto f|_V$.

Remark 3.1.3. Presheaves are probably the result of trying to generalise the idea in the previous example to more complicated manifold-like spaces; in fact, we will generally denote the image under a restriction as a "functional" restriction: $\rho_{UV} f =: f|_V$. Additionally, presheaves of smooth functions satisfy one further property, namely that a function is fully characterised by its images under the restriction maps, and this is formalised by the next definition.

Definition 3.1.4. A graded **sheaf** is a graded presheaf over X whose restrictions maps, for every open U of X and open cover $\{U_i\}$ of U, satisfy two additional properties:

1. For all $f, g \in A(U)$
$$f|_{U_i} = g|_{U_i}\ \forall i \quad \iff \quad f = g. \tag{3.1.1}$$

2. Every collection $\{f_i\}$ such that $f_i \in A(U_i)$ satisfies
$$f_i|_{U_i U_{ij}} = f_j|_{U_j U_{ij}}\ \forall i, j \quad \iff \quad \exists f \in A(U) : f_i = f|_{U_i} \tag{3.1.2}$$
using the notation $U_{ij} := U_i \cap U_j$.

Now, one of our greatest incentives to study graded geometry is to formalise what is an anticommuting field, such as those that appear in any quantum field theory with fermions or ghosts. If we consider the classical case, fields in a lagrangian theory can be understood as differential forms, which are spanned by the canonical differential of the corresponding coordinates on a conventional manifold. This suggests an approach: to produce a theory with graded fields we need to be able to generalise manifolds enough as to incorporate graded coordinates. Analogously to how an n-dimensional manifold is locally homeomorphic to an euclidean space $\mathbb{R}^n$, a $\mathbb{Z}_2$-graded manifold should be locally homeomorphic to some $\mathbb{Z}_2$-graded generalisation of euclidean space.

Definition 3.1.5. The **supereuclidean space** $\mathbb{R}^{m|n}$ of dimension $(m|n)$ is the supermanifold $(\mathbb{R}^m, \mathcal{O}_{m|n})$ for

$$\mathcal{O}_{m|n} := \mathbb{R}^m \otimes \bigwedge \mathbb{R}^n. \tag{3.1.3}$$

Then a (real) $\mathbb{Z}_2$-graded manifold is analogous to a manifold, but instead of being locally euclidean, it is locally supereuclidean, and this is formalised in the next definition.

Definition 3.1.6. A (real) **$\mathbb{Z}_2$-graded manifold** or **supermanifold** of **dimension $(m|n)$** is a pair $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$, where $\mathcal{O}_\mathcal{M}$ is a graded sheaf over an m-dimensional manifold M, known as the **body** of $\mathcal{M}$, whose algebras are isomorphic to the smooth functions over an open of $\mathbb{R}^{m|n}$. That is to say that for every open U of M and $A(U) \in \mathcal{O}_\mathcal{M}$

$$A(U) \cong C^\infty(U) \otimes \bigwedge V^* \tag{3.1.4}$$

is a graded algebra isomorphism for some n-dimensional real vector space V. Coordinates $\{x^i\}$ on U are called **even coordinates**, while a basis $\{\theta^j\}$ of V^* provides the **odd coordinates**. The real vector space V is called the **odd fibre**, while the **even fibre** corresponds to the real vector space W that is linearly isomorphic to the span of the images of U under the even coordinates, for any open U; that is

$$W \simeq \mathrm{span}\{x^i(U)\} \quad \forall\, U \in \tau(M). \tag{3.1.5}$$

The isomorphism (3.1.4) is realised through **graded charts**, namely through an open cover $\{U_i\}$ of M together with an identically indexed family $\{\varphi_i\}$ of graded algebra isomorphisms known as **chart maps**.

Remark 3.1.7. It will often be the case that we refer to the even and odd coordinates pair (x^i, θ^j) of an $(m|n)$-manifold with a single coordinate (z^k) such that for all $i \in [\![1, m]\!]$, $j \in [\![1, n]\!]$ and $k \in [\![1, m+n]\!]$

$$z^i = x^i, \qquad z^{m+j} = \theta^j, \qquad \partial_k := \frac{\partial}{\partial z^k}. \tag{3.1.6}$$

Remark 3.1.8. The core of such a definition, which might seem more convoluted than that of a non-graded manifold, is actually quite intuitive: locally the elements of the algebras in the sheaf will have the structure of a non-graded differential form, but where the role of the generators "dx^i" is taken by a basis $\{\theta^i\}$ of V^*. If $\mathcal{M}$ is $(m|n)$-dimensional this translates formally to the fact that, given any algebra $A(U) \in \mathcal{O}_\mathcal{M}$, a function $f \in A(U)$ will decompose as

$$f(p) = \sum_{\substack{0 \le k \le n \\ 1 \le i_1 \le \cdots \le i_k \le n}} f_{i_1 \dots i_k} \circ x(p) \otimes \theta^{i_1} \cdots \theta^{i_k} \quad \forall p \in U \tag{3.1.7}$$

for some collection of smooth real functions $\{f_{i_1 \dots i_k}\} \subseteq C^\infty(\mathbb{R}^m)$ and a non-graded chart map $x : U \to \mathbb{R}^m$ over the body.

Remark 3.1.9. A chart map φ over a chart (U, φ) of $\mathcal{M}$ also takes the form (3.1.7), but for φ each function $f_{i_1 \dots i_k}$ it must be a constant $c_{i_1 \dots i_k}$ specifying an element in $\bigwedge V^*$. As a result, for every $p \in U$

$$\varphi(p) = x(p) \sum_{\substack{0 \le k \le n \\ 1 \le i_1 \le \cdots \le i_k \le n}} c_{i_1 \dots i_k} \otimes \theta^{i_1} \cdots \theta^{i_k} \in W \otimes \bigwedge V^*, \tag{3.1.8}$$

which makes manifest the local $\mathbb{R}^{m|n}$ homeomorphy. Moreover, this also shows that conventional manifolds are supermanifolds for an odd fibre equal to the trivial vector space $\{0\}$.

Example 3.1.10. Given any non-graded vector bundle $E \to M$ over a conventional manifold M, we can construct a supermanifold ΠE by taking as body M and letting the algebras in its sheaf $\mathcal{O}_{\Pi E}$ be isomorphic to the sections of the exterior algebra of its dual E^*. That is, for all $U \in \tau(M)$ and $A(U) \in \mathcal{O}_{\Pi E}$ we set

$$A(U) \cong \Gamma\big(M, \bigwedge E^*\big). \tag{3.1.9}$$

Definition 3.1.11. Given a vector bundle $E \to M$, the supermanifold ΠE is called its **odd shifted bundle**.

Definition 3.1.12. A $(m|n)$-dimensional supermanifold $\mathcal{M}$ is called **$\mathbb{Z}$-graded**—or simply **graded**—if

1. Its even fibre W and its odd fibre V are both graded.
2. This grading is global: for any two intersecting charts (U_i, φ_i) and (U_j, φ_j) of $\mathcal{M}$, the map $\varphi_i \circ \varphi_j^{-1}$ preserves this grading.

This grading effectively induces a $\mathbb{Z}$-grading of the functions in each local algebra, which are already $\mathbb{Z}_2$-graded, and we say that these gradings are **compatible** whenever the $\mathbb{Z}_2$-grading issued from the superstructure coincides with the $\mathbb{Z}_2$-grading issued from the reduction modulo 2 of the $\mathbb{Z}$-grading over V and W as in (1.1.6). In such case, the $\mathbb{Z}$-degree is called **ghost number** and is denoted by gh .

Remark 3.1.13. We will always assume that both gradings are compatible, which in practice can be achieved by demanding that

$$V = \bigoplus_{i \in 2\mathbb{N}} V_i, \qquad W = \bigoplus_{i \in 2\mathbb{N}+1} W_i. \tag{3.1.10}$$

When compatibility takes place, local functions still take the form (3.1.7), with the added possibility of having a definite ghost number.

Definition 3.1.14. A **morphism of graded manifolds** $\mathcal{M} \to \mathcal{N}$ is a morphism of smooth manifolds $\Phi : M \to N$ and an associated morphism of graded sheaves $\Phi_\flat : \mathcal{O}_\mathcal{N} \to \mathcal{O}_\mathcal{M}$ such that for every $U \in \tau(N)$ the following diagram commutes:

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{\Phi^*} & C^\infty\left(\Phi^{-1}(U)\right) \\ \downarrow & & \downarrow \\ A_N(U) & \xmapsto{\Phi_\flat} & A_M(\Phi^{-1}(U)) \end{array} \tag{3.1.11}$$

Remark 3.1.15. We generally use the same symbol for both the morphism between graded manifolds and the morphism between their bodies, so in the definition above one would write $\Phi : \mathcal{M} \to \mathcal{N}$.

Once these ideas are set in mind and we are able to picture intuitively what a graded manifold locally looks like—which is little more than a graded euclidean space—we are ready to grade usual differential-geometric objects, in a manner that it should seem trivial now, given our previous treatment of graded algebra and derivations.

Definition 3.1.16. Given a graded manifold $\mathcal{M}$, the

- **tangent space** $T_p\mathcal{M}$ and **cotangent space** $T_p^*\mathcal{M}$ at $p \in \mathcal{M}$,
- **tangent bundle** $T\mathcal{M}$ and **cotangent bundle** $T^*\mathcal{M}$,
- space of **vector fields** $\mathfrak{X}(\mathcal{M})$ over $\mathcal{M}$,
- space of **multi-vector fields** $\bigwedge \mathfrak{X}(\mathcal{M})$ over $\mathcal{M}$,
- space of **differential forms** $\Omega(\mathcal{M})$ over $\mathcal{M}$,
- and in general the notion of **tensor field** over $\mathcal{M}$

are defined identically as they are in differential geometry, but employing the graded variant of the corresponding algebraic structure. In particular, given even-odd coordinates (z^i), the **partial derivative** with respect to, and the differential of, each coordinate are defined by the following equalities:

$$\partial_i(z^j) := \delta_i^j, \tag{3.1.12a}$$
$$dz^j(\partial_i) := \iota_{\partial_i} dz^j := \delta_i^j. \tag{3.1.12b}$$

Remark 3.1.17. The use of the contraction in the definition (3.1.12b) is essential to be consistent with postulates 2 and 3, because one needs to explain why, given that $|d| = 1$ and that $|\partial_i| = -|z^i|$, we still have

$$0 = |dz^{\bar{\imath}}(\partial_{\bar{\imath}})| \neq |d| + |z^i| + |\partial_i| = 1. \tag{3.1.13}$$

The consistency is imposed recurring to the contraction, that satisfies $|\iota| = -1$, hence implying that

$$0 = |dz^{\bar{\imath}}(\partial_{\bar{\imath}})| = |\iota_{\partial_{\bar{\imath}}} dz^{\bar{\imath}}| = |\iota| + |\partial_i| + |dz^i| = -1 - |z^i| + |z^i| + 1. \tag{3.1.14}$$

Example 3.1.18. Consequently $\mathfrak{X}(\mathcal{M})$ is a graded Lie algebra of derivations, so that for all $X, Y \in \mathfrak{X}(\mathcal{M})$ and $f, g \in C^\infty(U)$, $U \in \tau(\mathcal{M})$

$$[X, Y] = XY - (-1)^{|X||Y|} YX \tag{3.1.15a}$$

$$X(fg) = Xfg + (-1)^{|X||f|} fXg \tag{3.1.15b}$$

Remark 3.1.19. Notice that tangent and cotangent bundles over a manifold M are non-graded vector bundles, so that the local algebras $A(U) \in \mathcal{O}_{\Pi TM}$ and $A^*(U) \in \mathcal{O}_{\Pi T^*M}$ of their associated odd shifted bundles ΠTM and ΠT^*M satisfy

$$A(U) \cong \Gamma\big(M, \textstyle\bigwedge T^*M\big) = \Omega(M), \tag{3.1.16a}$$

$$A^*(U) \cong \Gamma\big(M, \textstyle\bigwedge TM\big) = \textstyle\bigwedge \mathfrak{X}(M). \tag{3.1.16b}$$

Definition 3.1.20. The **grading k-shift** of a graded object O is an object $O[k]$ in the same category, whose elements are identical to those of O modulo the grading, but with their degree increased by k:

$$x \in O, \ \deg x = j \qquad \Longleftrightarrow \qquad x \in O[k], \ \deg x = j + k \tag{3.1.17}$$

except for $x = 0$ and, if a product is defined, $x = 1$.

Definition 3.1.21. The **k-shifted (co-)tangent bundle** $T^{(*)}[k]\mathcal{M}$ is given as the graded (co-)tangent bundle with *only* the fibre being k-shifted and *after* omitting the cohomological degree.

Example 3.1.22. Let z^i be a coordinate on $\mathcal{M}$, with degree $|z^i| = j$. Associated to it there will be a coordinate dz^i on the fibre of $T\mathcal{M}$ and a coordinate ∂_i on the fibre of $T^*\mathcal{M}$, with respective degrees

$$|dz^i| = j + 1, \qquad |\partial_i| = -j. \tag{3.1.18}$$

To avoid confusion in what concerns these degrees, in $T[k]\mathcal{M}$ and $T^*[-k]\mathcal{M}$—take good note of the sign—the corresponding coordinates are renamed as $dz^i \mapsto \zeta^i$ and $\partial_i \mapsto z_i^*$, and their degrees are

$$|\zeta^i| = j + k, \qquad |z_i^*| = -j - k. \tag{3.1.19}$$

Hence, in particular, $\Pi TM \cong T[1]M$ and $\Pi T^*M \cong T^*[-1]M$ for any non-graded smooth manifold M.

Remark 3.1.23. The fact that the shift is taken *after omission of the cohomological degree* is again a matter of consistency with postulates 2 and 3, and the fact that ζ^i is not seen as a differential form acting on z_i^*. See remark 3.1.17

Definition 3.1.24. The **graded Lie derivative** $\mathcal{L}_X$ with respect to a graded vector field $X \in \mathfrak{X}(\mathcal{M})$ is a graded derivation that, for any $f \in C^\infty(\mathcal{M})$, $Y \in \mathfrak{X}(\mathcal{M})$, $\omega \in \Omega(\mathcal{M})$ and tensor field T, satisfies the following properties:

$$\mathcal{L}_X f := Xf \tag{3.1.20a}$$

$$\mathcal{L}_X Y := [X, Y] \tag{3.1.20b}$$

$$\begin{aligned}(\mathcal{L}_X T)(\omega, Y, ...) := &\mathcal{L}_X\big(T(\omega, Y, ...)\big)\\ &- (-1)^{|X||T|}T(\mathcal{L}_X\omega, Y, ...)\\ &- (-1)^{|X|(|T|+|\omega|)}T(\omega, \mathcal{L}_X Y, ...) - \cdots\end{aligned} \tag{3.1.20c}$$

Remark 3.1.25. Notice how the definition of the Lie derivative here reduces to the algebraic definition of the non-graded one once we set to zero every degree except for the cohomological one.

Corollary 3.1.26. A direct consequence of its definition is that

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \quad \forall\ X, Y \in \mathfrak{X}(\mathcal{M}). \tag{3.1.21}$$

Proposition 3.1.27. ***Cartan's identity****—or Cartan's* ***magic formula****—holds in the graded context:*

$$[\iota_X, d] = \mathcal{L}_X. \tag{3.1.22}$$

Remark 3.1.28. The proof is verbatim the one leading to the formula in conventional differential geometry, but keeping track of the signs implied by a possible grading. The reader might object that the conventional formula involves the anticommutator, but note that this follows from (3.1.22) given that

$$[\iota_X, d] = \iota_X d - (-1)^{|\iota_X||d|} d\iota_X \quad \xmapsto{|X|=0} \quad \iota_X d + d\iota_X, \tag{3.1.23}$$

recalling that $|\iota_X| = |X| - 1$ and that $|d| = 1$.

Proposition 3.1.29. *For any graded vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ and form $\omega \in \Omega(\mathcal{M})$ over a graded manifold $\mathcal{M}$ we have*

$$[\mathcal{L}_X, \iota_Y]\omega = \iota_{[X,Y]}\omega \tag{3.1.24a}$$

Proof. The behaviour of the interior and Lie derivatives, as for any derivation, will be fully determined by their action on an arbitrary function $f \in C^\infty(\mathcal{M})$ and on its differential df. Now, on the one hand it is obvious that

$$[\mathcal{L}_X, \iota_Y]f = \iota_{[X,Y]}f = 0 \tag{3.1.25}$$

because $\mathcal{L}_X$ doesn't affect the cohomological degree and the interior derivative maps functions to zero. On the other hand, recalling that $|\iota_X| = |X| - 1$, Cartan's identity and $d^2 = 0$ together imply that

$$\begin{aligned} \iota_Y(\mathcal{L}_X df) &= \iota_Y([\iota_X, d]df) = -(-1)^{|\iota_X|}\iota_Y d(Xf) \\ &= (-1)^{|X|}Y(Xf). \end{aligned} \tag{3.1.26a}$$

This, jointly with $|\mathcal{L}_X| = |X|$, lets us conclude that

$$\begin{aligned} [\mathcal{L}_X, \iota_Y]df &= \mathcal{L}_X(\iota_Y df) - (-1)^{|\mathcal{L}_X||\iota_Y|}\iota_Y(\mathcal{L}_X df) \\ &= X(Yf) - (-1)^{|X|(|Y|-1)}(-1)^{|X|}Y(Xf) = [X,Y]f \\ &= \iota_{[X,Y]}df, \end{aligned} \tag{3.1.26b}$$

thus proving the proposition. ♡

Definition 3.1.30. A **graded symplectic manifold** or **P-manifold** is a graded manifold $\mathcal{M}$ of dimension $(n|n)$ endowed with a graded symplectic form $\omega \in \Omega^2(\mathcal{M})$.

Remark 3.1.31. If the number of even and odd coordinates did not agree, then one could find a non-zero vector $X \in \mathfrak{X}(\mathcal{M})$ such that $\omega(X, \bullet) = 0$ everywhere on $\mathcal{M}$, for the same reason that all the invertible matrices are square. Consequently, every graded symplectic manifold must have dimension $(n|n)$.

Definition 3.1.32. Associated to any symplectic form $\omega = dz^i \omega_{ij} dz^j$, for coordinates (z^i) over $\mathcal{M}$, there is a **Poisson bracket** $\{\bullet, \bullet\}$ defined for all $f, g \in C^\infty(\mathcal{M})$ as

$$\{f, g\} = f\overleftarrow{\partial}_i(\omega^{-1})^{ij}\overrightarrow{\partial}_j g. \tag{3.1.27}$$

This induced poissonian structure explains the term *P-manifold.*

Remark 3.1.33. Observe that, for the cohomological degree $\deg_\Omega$ and any other degree $\deg_j \neq \deg_\Omega$, we have

$$\deg_\Omega \{\bullet, \bullet\} = -\deg_\Omega \omega + 2, \qquad \deg_j \{\bullet, \bullet\} = -\deg_j \omega. \tag{3.1.28}$$

Definition 3.1.34. A **canonical symplectic form** ω is a symplectic form that in coordinates (x^i, θ^i) is written

$$\omega = \sum_i dx^i d\theta^i. \tag{3.1.29}$$

Coordinates in which ω takes this form are called **Darboux coordinates**.

Remark 3.1.35. Due to its relevance, particularly given the proposition 3.1.38 below, one should be aware of the form of the Poisson bracket associated to a canonical symplectic form, in coordinates (x^i, θ^i):

$$\{f, g\} = \sum_i \left(f \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\partial}{\partial \theta^i} g - (-1)^{|x^i||\theta^i|} f \frac{\overleftarrow{\partial}}{\partial \theta^i} \frac{\partial}{\partial x^i} g \right). \tag{3.1.30}$$

Example 3.1.36. The archetypical example of a supermanifold admitting a global, canonical symplectic form is the odd shifted cotangent bundle $\Pi T^* M$ of a conventional manifold M. See proposition 3.1.38.

Definition 3.1.37. A **symplectomorphism** $\Phi : (\mathcal{M}, \omega_\mathcal{M}) \to (\mathcal{N}, \omega_\mathcal{N})$ is an isomorphism of graded symplectic manifolds, hence an isomorphism $\Phi : \mathcal{M} \to \mathcal{N}$ that preserves the symplectic form:

$$\Phi^* \omega_\mathcal{N} = \omega_\mathcal{M}. \tag{3.1.31}$$

Proposition 3.1.38 (Schwarz's theorem)**.** *For any symplectic supermanifold $(\mathcal{M}, \omega)$ with body M*

1. *There are local Darboux coordinates in the vicinity of any $p \in \mathcal{M}$.*
2. *There is a symplectomorphism $(\mathcal{M}, \omega) \to (\Pi T^* M, \omega')$ where ω' is canonical.*

This result is key to us, for it allows us to see every supermanifold—or the supermanifold structure of a graded manifold—locally as the odd shifted cotangent bundle of a conventional manifold. In fact, this result is fundamental to justify the validity of the AKSZ construction.

3.2 Classical BV formalism

The path integral formalism for quantum field theory relies on the possibility of integrating out the quadratic terms in the lagrangian density defining the action, what is achieved through a generalisation to field theory of the saddle point method—called *Feynman-Laplace method* in Mnev's [Mne19], that I followed for this section together with Cattaneo's and Moshayedi's [CS19a]—which requires the critical points of the action to form a finite subset of its support. However, precisely because of the "continuity"—as opposed to "discreteness"—of topological groups, in theories described by a lagrangian with gauge freedom one can smoothly deform a critical point into another critical point, resulting in critical loci that are themselves submanifolds. This spoils the applicability of the aforementioned method, making manifest the need for a machinery selecting discrete subsets of the critical locus of an action; and this is precisely the issue that both the BRST and the BV formalisms address.

We choose to employ the latter because it has a greater range of applicability than the former, and also because of its relation to the Batalin-Fradkin-Vilkovisky (BFV) formalism, which allows for a perturbative quantisation of field theories on the possible boundary of a manifold. Despite all this, though, we will not be concerned with any quantisation in this document, so our treatment of the theory will be minimal; an exhaustive approach is found in Mnev's [Mne19], and a review of the BFV formalism is in [CS19a]. Assuming the reader has a basic understanding of quantum field theory, we begin.

Definition 3.2.1. Given a graded manifolds $\mathcal{M}$, a **cohomological vector field** Q is a an element $Q \in \mathfrak{X}(\mathcal{M})$ such that

$$Q^2 = 0, \qquad |Q| = 1 \bmod 2, \qquad \mathrm{gh}\, Q = 1, \tag{3.2.1}$$

where one understands Q^2 as $Q \circ Q$, being a map $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$. A manifold endowed with such a vector field is a **Q-manifold**.

Remark 3.2.2. Given that Q is odd, saying that $Q^2 = 0$ is equivalent to saying that $[Q, Q] = 0$.

Remark 3.2.3. Let me point out that even if the cohomological vector field was named something else than "Q" the structure would still be called Q-structure. The name follows from the fact that the idea is closely related to the BRST formalism, where conventionally the cohomological vector field has been denoted by Q.

Example 3.2.4. A paradigmatic example of Q-manifold is given by the odd tangent bundle ΠTM of any non-graded manifold M. If (x^i) are coordinates on M and (θ^i) the (odd) coordinates on its fibre, then a Q-structure is given by

$$Q = \theta^i \partial_i. \tag{3.2.2}$$

Definition 3.2.5. Given a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a form $\omega \in \Omega(\mathcal{M})$, we say that ω is **X-invariant** if

$$\mathcal{L}_X \omega = 0. \tag{3.2.3}$$

Definition 3.2.6. A **PQ-manifold** is a graded symplectic manifold with a cohomological vector field Q under which the symplectic form ω is invariant. That is,

$$\mathcal{L}_Q \omega = 0. \tag{3.2.4}$$

Definition 3.2.7. Given a P-manifold $\mathcal{M}$ and a function $f \in C^\infty(U)$ over some open $U \in \tau(\mathcal{M})$, the **hamiltonian vector field** associated to f is the field $\vec{f} \in \mathfrak{X}(\mathcal{M})$ such that

$$\vec{f} = (-1)^{|f|+1}\{f, \bullet\}. \tag{3.2.5}$$

Definition 3.2.8. A **hamiltonian manifold of degree k** $(\mathcal{M}, H, Q, \omega)$ is a QP-manifold $\mathcal{M}$ where the symplectic form ω has ghost number k and the cohomological vector field Q is given as the hamiltonian vector field of some degree $k+1$ function H, called its **hamiltonian function**:

$$Q = (-1)^k\{H, \bullet\}. \tag{3.2.6}$$

Definition 3.2.9. A **classical BV theory** is a hamiltonian manifold of degree -1. That is, a tuple $(\mathcal{F}, S, Q, \omega)$ such that

1. $\mathcal{F}$ is a graded symplectic $(n|n)$-manifold, called the **field space**.

2. S is a function over $\mathcal{F}$ of degree 0, called its **BV action**.

3. Q is the cohomological, hamiltonian vector field of S.

4. ω is the symplectic form over $\mathcal{F}$, of degree -1 and Q-invariant.

Given that $\mathcal{F}$ has dimension $(n|n)$, we can and we do associate each field $\phi \in \mathcal{F}$ to another field $\phi^{\daleth} \in \mathcal{F}$, known as its **antifield**, that by construction satisfies

$$\deg_\Omega \phi^{\daleth} = n - \deg_\Omega \phi, \qquad \operatorname{gh} \phi^{\daleth} = -\operatorname{gh} \phi - 1. \tag{3.2.7}$$

Remark 3.2.10. This definition, hardly obvious to anyone approaching the topic for the first time, gains intuition by considering some facts. First, the graded manifold $\mathcal{F}$ will consist in the space of fields that will need to be graded in order to account for the non-zero ghost numbers that characterise the ghost fields introduced to quantise the theory. In turn, following the Feynman integral formalism, after quantisation the dynamics of these fields will be determined by the perturbative expansion of some integral

$$\int_L \exp(S) \tag{3.2.8}$$

over an appropriate submanifold $L \subseteq M$ of the spacetime M, and where S is to be understood as the action of the theory, in the sense of lagrangian field theory. Now, a recurring—if not essential—feature of (quantum) field theories is gauge invariance, which is formalised by the fact that, given any element $\boldsymbol{g}$ in the Lie algebra $\mathfrak{g}$ of the given gauge group G, the fundamental vector field $\vec{\boldsymbol{g}}$ of $\boldsymbol{g}$ over $\mathcal{F}$ will satisfy

$$\vec{\boldsymbol{g}}(S) = \mathcal{L}_{\vec{\boldsymbol{g}}} S = 0. \tag{3.2.9}$$

The vector field Q encodes this invariance, and its cohomological properties mimic those of the canonical differential; in particular, the fact that Q^2 must be zero follows from the demand that the map $\boldsymbol{g} \mapsto \vec{\boldsymbol{g}}$ be a Lie algebra morphism. Meanwhile, that S have degree zero is imposed by necessity of having terms in S that correspond to a classical action—which is not graded—and, consequently, ω must have ghost number -1 simply to ensure that Q has ghost number and parity equal to 1—see remark 3.1.33. This becomes clearer after considering the fact that BV theories are most often BV extensions of a classical theory.

Definition 3.2.11. A BV theory $(\mathcal{F}, S, Q, \omega)$ is a **BV extension** of a classical theory described by a space of fields $\mathcal{F}_{\mathrm{cl}}$ and an action S_{cl}, if the ghost number zero part of $\mathcal{F}$ and of S correspond to $\mathcal{F}_{\mathrm{cl}}$ and to S_{cl}.

Remark 3.2.12. In general, constructing BV extensions is far from trivial but, fortunately, in certain cases straightforward methods such as the AKSZ construction exist, which will be treated later.

Definition 3.2.13. Two BV theories are **weakly equivalent** if both of them are BV extensions of a same classical theory.

These theories will be **strongly equivalent** if there is a graded symplectomorphism $\Phi : \mathcal{F} \to \mathcal{F}'$ between their respective field spaces that pulls the action of one theory back to the action of the other:

$$\phi^* S' = S. \tag{3.2.10}$$

Such a symplectomorphism is known as a **canonical transformation**.

Definition 3.2.14. Let (q, p) and (q', p') be the respective even-odd coordinates of two different BV field spaces $\mathcal{F}$ and $\mathcal{F}'$. A **graded generating function of type j**, for $j \in [\![1, 4]\!]$, is a graded function G_j that we use to define two coordinates among q, p, q', p' as a function of the remaining two, in one of the four following ways:

$$p = (-1)^{|q|+1} \frac{\partial G_1(q, q')}{\partial q}, \qquad p' = (-1)^{|q'|} \frac{\partial G_1(q, q')}{\partial q'}, \tag{3.2.11a}$$

$$p = (-1)^{|q|+1} \frac{\partial G_2(q, p')}{\partial q}, \qquad q' = (-1)^{|p'|} \frac{\partial G_2(q, p')}{\partial p'}, \tag{3.2.11b}$$

$$q = (-1)^{|p|+1} \frac{\partial G_3(p, q')}{\partial p}, \qquad p' = (-1)^{|q'|} \frac{\partial G_3(p, q')}{\partial q'}, \tag{3.2.11c}$$

$$q = (-1)^{|p|+1} \frac{\partial G_4(p, p')}{\partial p}, \qquad q' = (-1)^{|p'|} \frac{\partial G_4(p, p')}{\partial p'}. \tag{3.2.11d}$$

Remark 3.2.15. Generating functions owe their name to the fact that the map $(q, p) \mapsto (q', p')$ is guaranteed to be a canonical transformation, whenever q, p, q' and p' are related by the the differential equations in (3.2.11) corresponding to a specific type of generating function.

Remark 3.2.16. By design, generating functions have top cohomological degree and ghost number -1.

Definition 3.2.17. Given two graded manifolds $\mathcal{M}$, $\mathcal{N}$, and letting $\mathrm{Mor}(\mathcal{M}, \mathcal{N})$ be the manifold of grade-preserving morphisms $\mathcal{M} \to \mathcal{N}$ in the category of graded manifolds, the **mapping space** $\mathrm{Map}(\mathcal{M}, \mathcal{N})$ is the extension of $\mathrm{Mor}(\mathcal{M}, \mathcal{N})$ that includes grade-shifting maps.

Remark 3.2.18. If $\mathcal{N}$ is a graded vector space then

$$\mathrm{Map}(\mathcal{M}, \mathcal{N}) = C^{\infty}(\mathcal{M}) \otimes \mathcal{N}, \tag{3.2.12}$$

so locally the mapping space will have the form of such a tensor product of graded spaces. Details of this definition can be found in [CMR14].

Definition 3.2.19. An **AKSZ theory in n dimensions** $(M, \mathcal{N}, H, Q, \alpha)$ is the combination of two things:

1. A **source** consisting in a closed and oriented n-manifold M.
2. A **target** consisting in a $(n-1)$-hamiltonian manifold $(\mathcal{N}, H, Q, \omega)$ whose symplectic form ω is exact:

$$\omega = d_{\mathcal{N}}\alpha \tag{3.2.13}$$

for $d_{\mathcal{N}}$ the canonical differential on $\mathcal{N}$.

Definition 3.2.20. Given an AKSZ theory $(M, \mathcal{N}, H_{\mathcal{N}}, Q_{\mathcal{N}}, \alpha_{\mathcal{N}})$ in n dimensions, we define the **AKSZ fields space**

$$\mathcal{F} = \mathrm{Map}(\Pi TM, \mathcal{N}). \tag{3.2.14}$$

Employing the notation in remark 3.2.21, we take next the evaluation map $\mathrm{ev} : \Pi TM \times \mathcal{F} \to \mathcal{N}$ and define, for all $k \in \mathbb{N}$ and for coordinates ξ on $\mathcal{N}$ and X on $\mathcal{F}$, its pullback as

$$\mathrm{ev}^* : \Omega^k(\mathcal{N}) \to \Omega(M) \otimes \Omega^k(\mathcal{F}) : \beta(\xi) \mapsto \beta(X), \tag{3.2.15}$$

and we define the pushed forward projection

$$\begin{aligned} \pi_* : \Omega(M) \otimes \Omega^k(\mathcal{F}) \;&\to\; \Omega^k(\mathcal{F}), \\ \varphi \otimes \Phi \;&\mapsto \int_M \varphi^{\mathrm{top}} \otimes \Phi \quad \forall\, \varphi \in \Omega(M),\; \Phi \in \Omega^k(\mathcal{F}). \end{aligned} \tag{3.2.16}$$

These two are then combined to produce the **transgression map**

$$\mathcal{T} = \pi_* \mathrm{ev}^*, \tag{3.2.17}$$

with which we construct coordinates on $\mathcal{F}$—the so called **AKSZ superfields**—defined for coordinates (x^i) on M as

$$X^i := \mathcal{T} x^i. \tag{3.2.18}$$

Letting d_M be the canonical differential on M, and further letting $\widetilde{d}_M, \widetilde{Q}_\mathcal{N} \in \mathfrak{X}(\mathcal{F})$ be the respective lifts to $\mathcal{F}$ of d_M and of $Q_\mathcal{N}$, we finally define the **AKSZ construction** associated to this theory as the tuple $(\mathcal{F}, S, Q, \omega)$ for

1. the **AKSZ action**

$$S = \iota_{\widetilde{d}_M} \mathcal{T}\alpha_\mathcal{N} + \mathcal{T}H_\mathcal{N}, \tag{3.2.19}$$

2. the **AKSZ vector field**

$$Q = \widetilde{d}_M + \widetilde{Q}_\mathcal{N}, \tag{3.2.20}$$

3. and the **AKSZ symplectic form**

$$\omega = (-1)^n \mathcal{T}\omega_\mathcal{N}. \tag{3.2.21}$$

Remark 3.2.21. Given a form $\beta \in \Omega(\mathcal{N})$, and coordinates ξ over $\mathcal{N}$ and X over $\mathcal{F}$, by $\beta(X)$ one understands the coordinates expression for $\beta(\xi)$, but symbolically replacing ξ by X.

Given any manifold M and form $\varphi \in \Omega(M)$, φ^{top} denotes the components of φ with top cohomological degree, that is, those components such that $\deg_{\Omega(M)} \varphi = \dim M$.

Remark 3.2.22. The maps π_* and $\mathcal{T}$ are graded:

$$|\pi_*| = |\mathcal{T}| = -\dim M. \tag{3.2.22}$$

Remark 3.2.23. By the definition 3.2.17, locally

$$\mathcal{F} \cong \Omega(M) \otimes \mathcal{N}, \tag{3.2.23}$$

so in practice we can write S, Q and ω explicitly:

$$S = \int_M \Big((\alpha_\mathcal{N})_i(X)\, d_M X^i + H_\mathcal{N}(X) \Big), \tag{3.2.24}$$

$$Q = \int_M \Big(d_M X^i + Q^i_\mathcal{N}(X) \Big) \frac{\partial}{\partial X^i}, \tag{3.2.25}$$

$$\omega = (-1)^n \int_M (\omega_\mathcal{N})_{ij}(X)\, d_\mathcal{N} X^i \wedge d_\mathcal{N} X^j. \tag{3.2.26}$$

Given this, together with the fact that each superfield X^i can be decomposed over summands $\{X^i_{(j)}\}_j$ of definite cohomological degree $j \in [\![0, n]\!]$, at the end of the day the AKSZ action is the action that we would obtain by

1. symbolically replacing the original fields with their associated superfields[8],

2. expanding those in components of definite cohomological degree,

3. keeping only the terms which have top cohomological degree.

An example of such construction is given in next chapter, so let us proceed without further ado.

(8) Associated in the sense that the original fields can be assigned to coordinates on the space of fields, and those coordinates define the superfields.

Chapter 4

3D supergravity

The Palatini-Cartan-Holst (PCH) formalism was conceived in order to bring Einstein's general relativity closer to the language of gauge theories, and based on the work of Cattaneo and Schiavina in [CS19b] it was proven in [CSS18] that in three dimensions PCH gravity is strongly equivalent to a BF theory. The goal of this chapter is to show that this strong equivalence persists in three dimensions even when we incorporate the Rarita-Schwinger term to the action, and thus allows us to define a BV extension of 3D supergravity. Before we begin, I point out that a review of the PCH formalism can be found in [Rom93; CS19b], and that an careful treatment of supergravity is given in [WB92].

4.1 3D gravity and BF theory in vacuum

Definition 4.1.1. An n-dimensional **spacetime** is a closed n-manifold with a **mostly positive** metric, that is, of signature $(p = n-1, n = 1)$.

Definition 4.1.2. Given a principal $\mathrm{SO}(n-1,1)$-bundle P over an n-dimensional manifold, its **Minkowski bundle** is the associated vector bundle $(\mathcal{V}, \eta)$ with typical fibre $\mathbb{R}^n$, endowed with the **Minkowski metric**

$$\eta := (-\mathbb{I}_1) \oplus \mathbb{I}_{(n-1)}, \tag{4.1.1}$$

for $\mathbb{I}_k$ the identity matrix in k dimensions.

Remark 4.1.3. Here we will focus on the case where $n = 3$, so

$$\eta = \mathrm{diag}(-1, 1, 1). \tag{4.1.2}$$

Definition 4.1.4. Given a Minkowski bundle $(\mathcal{V}, \eta)$, a **cotriad** or **coframe field** over a 3-spacetime M is a non-degenerate 1-form e over M valued in $\mathcal{V}^{\wedge 1}$. Associated to it there is a **triad** or **frame field** e^{-1}, which is its inverse in the following sense:

$$e \in \Omega^1(M, \mathcal{V}^{\wedge 1}), \qquad e^{-1} \in \Omega^1(M, \mathcal{V}^{\wedge 1})^*, \qquad e^{-1}(e) = 1. \tag{4.1.3}$$

Remark 4.1.5. We talk of $\mathcal{V}^{\wedge 1}$ and not simply of $\mathcal{V}$ because we associate to the (co-)triads a multivector degree

$$\deg_{\mathcal{V}} e = -\deg_{\mathcal{V}} e^{-1} = 1. \tag{4.1.4}$$

From now on, we will always conceive $\mathcal{V}$ as $\mathcal{V}^{\wedge 1}$.

Definition 4.1.6. The **Palatini-Cartan formalism** in three dimensions, or simply **3D gravity**, consists of

1. an orientable spacetime M, assumed to have no boundary,
2. a principal $\mathrm{SO}(2,1)$-bundle P over M, with Minkowski bundle $\mathcal{V}$,
3. a cotriad $e \in \Omega^1(M, \mathcal{V})$ over M,
4. a connection 1-form $\Gamma \in \mathrm{Conn}(P) \cong \Omega(M, \mathcal{V}^{\wedge 2})$ over P,
5. a possibly non-zero **cosmological constant** $\Lambda \in \mathbb{R}$.

With these we define the classical field space

$$\mathcal{F}^0_{\mathrm{GR}} = \Omega^1(M, \mathcal{V}) \times \mathrm{Conn}(P) \tag{4.1.5}$$

and the classical action

$$S^0_{\mathrm{GR}}(\Lambda) = \int_M \left\langle e \wedge F_\Gamma + \frac{\Lambda}{6} e^{\wedge 3} \right\rangle, \tag{4.1.6}$$

where the angle brackets $\langle \bullet \rangle$ designate the appropriate contraction of any indices other than those over $\Omega(M)$.

Remark 4.1.7. The reason why we can say that connections take values in $\mathcal{V}^{\wedge 2}$ follows from the fact that, since we work in three dimensions, $V^{\wedge 2} \cong \mathfrak{so}(2,1)$ as a Lie algebra.

Remark 4.1.8. For convenience and when it is possible, we often express a particular action S_i as a the integral of a *density* $\mathcal{L}_i$:

$$S_i = \int_M \left\langle \mathcal{L}_i \right\rangle. \tag{4.1.7}$$

Remark 4.1.9. By *appropriate contractions* one understands that ultimately S^0_{GR} must be a scalar, so any internal index must be contracted. This is to say that we must correctly "trace over" the multivector indices that both e and F_Γ have over $\mathcal{V}$, leading specifically to

$$\left\langle e \wedge F_\Gamma \right\rangle = \epsilon_{abc}\, e^a \wedge F_\Gamma^{bc}, \qquad \left\langle e^{\wedge 3} \right\rangle = \epsilon_{abc}\, e^a \wedge e^b \wedge e^c, \tag{4.1.8}$$

for the 3-dimensional Levi-Civita symbol ϵ_{abc} and leaving implicit the decomposition over the generators of $\Omega(M)$.

Definition 4.1.10. Given a connection 1-form Γ and its associated covariant derivative D_Γ, the **Lie covariant derivative** $\mathcal{L}^\Gamma_\xi$ with respect to a vector field ξ is defined as

$$\mathcal{L}^\Gamma_\xi = [\iota_\xi, D_\Gamma]. \tag{4.1.9}$$

Construction 4.1.11 (BV extension of 3D gravity)**.** Based on the data from 3D gravity, let $\mathcal{F}_{\text{GR}}$ be the field space

$$\mathcal{F}_{\text{GR}} = T^*[-1]\big(\mathcal{F}^0_{\text{GR}} \times \mathfrak{X}(M)[1] \times \Omega^0(M, \text{ad}(P)[1])\big), \tag{4.1.10}$$

where $\mathfrak{X}(M)[1]$ is the space of vector fields over M with shifted degree and $\text{ad}(P)[1]$ stands for the degree-shifted adjoint bundle of P. On the body of this graded manifold we define the fields

$$(e, \Gamma, \xi, \chi) \in \mathcal{F}^0_{\text{GR}} \times \mathfrak{X}(M)[1] \times \Omega^0(M, \text{ad}(P)[1]), \tag{4.1.11}$$

where ξ encodes the diffeomorphism invariance of general relativity and χ is the ghost field required for a successful quantisation. In turn, the odd fibre includes their associated antifields, and the fact that $\mathcal{F}_{\text{GR}}$ is an odd shifted bundle allows us to define the canonical symplectic form

$$\omega_{\text{GR}} = d\phi^i d\phi^\daleth_i, \qquad \phi^i \in \{e, \Gamma, \xi, \chi\}. \tag{4.1.12}$$

Finally, we can define an action $S_{\mathrm{GR}}(\Lambda) := S^0_{\mathrm{GR}}(\Lambda) + S^1_{\mathrm{GR}}$ for

$$S^1_{\mathrm{GR}} = \int_M \Big\langle e^\daleth\big(\mathcal{L}^\Gamma_\xi e + [\chi, e]\big) + \Gamma^\daleth\big(D_\Gamma \chi + \iota_\xi F_\Gamma\big) \\ + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\daleth + \frac{1}{2}\chi^\daleth\big([\chi,\chi] - \iota^2_\xi F_\Gamma\big)\Big\rangle, \tag{4.1.13}$$

and an associated cohomological vector field Q_{GR} that acts as

$$Q_{\mathrm{GR}}(e) = \mathcal{L}^\Gamma_\xi e + [\chi, e], \tag{4.1.14a}$$
$$Q_{\mathrm{GR}}(\Gamma) = D_\Gamma \chi + \iota_\xi F_\Gamma, \tag{4.1.14b}$$
$$Q_{\mathrm{GR}}(\xi) = \frac{1}{2}[\xi, \xi] \tag{4.1.14c}$$
$$Q_{\mathrm{GR}}(\chi) = \frac{1}{2}\big([\chi,\chi] - \iota^2_\xi F_\Gamma\big), \tag{4.1.14d}$$

and englobe everything in a tuple $\mathscr{T}_{\mathrm{GR}} := (\mathcal{F}_{\mathrm{GR}}, S_{\mathrm{GR}}, Q_{\mathrm{GR}}, \omega_{\mathrm{GR}})$.

Proposition 4.1.12. *The tuple $\mathscr{T}_{\mathrm{GR}}$ is a BV extension of 3D gravity, that we shall call* 3*-dimensional* ***BV gravity****.*

Definition 4.1.13. A **BF theory** in $n \geq 2$ dimensions is defined as a tuple $(\mathcal{F}, S)$ where, given

1. an n-dimensional spacetime M,
2. a finite dimensional Lie group G and a G-bundle P over M,
3. a form $B \in \Omega^{n-2}(M, \mathrm{ad}^*(P))$ valued in the coadjoint bundle,
4. a connection form $A \in \mathrm{Conn}(P)$,

one sets the field space to be

$$\mathcal{F} = \mathrm{ad}^*(P) \times \mathrm{Conn}(P) \tag{4.1.15}$$

and the action to be

$$S = \int_M \langle B, F_A \rangle, \tag{4.1.16}$$

where F_A is the curvature of A and $\langle \bullet, \bullet \rangle$ is the pairing of dual maps.

Definition 4.1.14. We define **BF gravity** as the BF theory $(\mathcal{F}^0_{\text{BF}}, S^0_{\text{BF}})$ where $n = 3$, and where P is the SO(2, 1)-bundle over M with associated Minkowsky bundle $\mathcal{V}$.

Moreover, we interpret B and A as 1-forms valued in $\mathcal{V}^*$ and $\mathcal{V}^{\wedge 2}$ respectively, so that the field space is given by

$$\mathcal{F}^0_{\text{BF}} = \Omega^1(M, \mathcal{V}^*) \times \Omega^1(M, \mathcal{V}^{\wedge 2}), \tag{4.1.17}$$

and we incorporate a cosmological term in the action, resulting in

$$S^0_{\text{BF}}(\Lambda) = \int_M \langle BF_A + \frac{\Lambda}{6} B^{\wedge 3} \rangle. \tag{4.1.18}$$

Proposition 4.1.15. *BF gravity is an AKSZ theory and its AKSZ extension* $\mathfrak{T}_{\text{BF}} := (\mathcal{F}_{\text{BF}}, S_{\text{BF}}, Q_{\text{BF}}, \omega_{\text{BF}})$ *is strongly equivalent to* 3*-dimensional BV gravity. A canonical transformation* $\Phi_{\text{BF}} : S_{\text{GR}} \to S_{\text{BF}}$ *is provided by the following type* 2 *generating function:*

$$\begin{aligned} G(q, p')_{BF} = -B^{\daleth}(e - \iota_\xi \Gamma^{\daleth} - \frac{1}{2}\iota_\xi^2 \chi^{\daleth}) - A\Gamma^{\daleth} \\ - \tau^{\daleth}(-\iota_\xi e + \frac{1}{2}\iota_\xi^2 \Gamma^{\daleth} - \frac{1}{3}\iota_\xi^3 \chi^{\daleth}) - c\chi^{\daleth} \end{aligned} \tag{4.1.19}$$

for $q := (e, \Gamma^{\daleth}, \xi, \chi^{\daleth})$ *and* $p' := (B^{\daleth}, A, \tau^{\daleth}, c)$.

Remark 4.1.16. The AKSZ construction is provided in the next section, while the strong equivalence is proven in [CSS18]. Here we content ourselves with giving the resulting decomposition of the fields, keeping $\tau^{\daleth}$ implicit to avoid cramping the equations:

$$B = e - \iota_\xi \Gamma^{\daleth} - \frac{1}{2}\iota_\xi^2 \chi^{\daleth}, \qquad B^{\daleth} = e^{\daleth} - \iota_\xi \tau^{\daleth}, \tag{4.1.20a}$$

$$A = \Gamma - \iota_\xi e^{\daleth} + \frac{1}{2}\iota_\xi^2 \tau^{\daleth}, \qquad A^{\daleth} = \Gamma^{\daleth}, \tag{4.1.20b}$$

$$\tau = -\iota_\xi e + \frac{1}{2}\iota_\xi^2 \Gamma^{\daleth} + \frac{1}{3}\iota_\xi^3 \chi^{\daleth}, \qquad \tau^{\daleth} = e^{-1}(\xi^{\daleth} - e^{\daleth}\Gamma^{\daleth} + \iota_\xi e^{\daleth} \chi^{\daleth}), \tag{4.1.20c}$$

$$c = \chi + \frac{1}{2}\iota_\xi^2 e^{\daleth} - \frac{1}{6}\iota_\xi^3 \tau^{\daleth}, \qquad c^{\daleth} = \chi^{\daleth}. \tag{4.1.20d}$$

4.2 BV supergravity

Definition 4.2.1. We define **3D supergravity** and **BF supergravity** by extending respectively $\mathcal{F}^0_{\mathrm{GR}}$ and $\mathcal{F}^0_{\mathrm{BF}}$ to

$$\mathcal{F}^0_{\mathrm{GR}\Phi} = \mathcal{F}^0_{\mathrm{GR}} \times \Pi S(P) \qquad \text{and} \qquad \mathcal{F}^0_{\mathrm{BF}\Phi} = \mathcal{F}^0_{\mathrm{BF}} \times \Pi S(P), \tag{4.2.1}$$

for $S(P)$ the spinor bundle associated to the principal bundle P, and then extending their actions by a **Rarita-Schwinger action** term

$$\begin{aligned} S^0_{\mathrm{GR}\Phi}(\Lambda) &= S^0_{\mathrm{GR}}(\Lambda) + \int_M \frac{1}{2}\overline{\psi}\, D_\Gamma \psi, \\ S^0_{\mathrm{BF}\Phi}(\Lambda) &= S^0_{\mathrm{BF}}(\Lambda) + \int_M \frac{1}{2}\overline{\varphi}\, D_A \varphi. \end{aligned} \tag{4.2.2}$$

Here the fields $\psi, \varphi \in \Pi S(P)$ are spin $\frac{3}{2}$ Majorana spinors.

Remark 4.2.2. The Majorana fields in the Rarita-Schwinger action account for the *gravitino*, which is the supersymmetric partner of the *graviton* encoded by the coframe field.

Remark 4.2.3. The connection forms will act on the spinor fields through the spin $\frac{3}{2}$ real (Majorana) representation of the algebra $\mathfrak{spin}(2,1)$, which is a real form of $\mathfrak{sl}(2,\mathbb{C})$. As a result, this representation is generated by a set of three 4×4 matrices $\{\rho^a\}$ that generalise the 2×2 Pauli matrices. Assuming that $\{v_a\}$ is a basis of the sections of the Minkowski bundle $\mathcal{V}$, we write

$$\rho := \rho^a v_a, \tag{4.2.3}$$

with which the equations of motion that follow from $S^0_{\mathrm{BF}\Phi}$ take the following form:

$$F_A + \frac{\Lambda}{2} B \wedge B \equiv 0, \tag{4.2.4a}$$

$$D_A \varphi \equiv 0, \tag{4.2.4b}$$

$$D_A B + \frac{1}{2}\overline{\varphi}\, \rho \varphi \equiv 0. \tag{4.2.4c}$$

Here I chose to use the sign "≡" instead of the equality sign to insist on the fact that those equalities only hold *on-shell*. Meanwhile, the equations of motion issued from $S^0_{\mathrm{GR}\Phi}$ are analogous, after replacing

$$(B, A, \varphi) \;\leftrightarrow\; (e, \Gamma, \psi). \tag{4.2.5}$$

Proposition 4.2.4. *BF supergravity is an AKSZ theory.*

Proof. We take the spacetime M as the source manifold, and we set (b, a, f) as coordinates on the target

$$\mathcal{N} \cong \mathcal{V}^*[1] \oplus \mathcal{V}^{\wedge 2}[1] \oplus \Pi S(P)[1] \tag{4.2.6}$$

We then see that $\mathcal{N}$ is endowed with the symplectic form

$$\omega_{\mathcal{N}} = d_{\mathcal{N}} \alpha_{\mathcal{N}}, \tag{4.2.7}$$

$$\Omega^1(\mathcal{N}) \ni \alpha_{\mathcal{N}} = b\, d_{\mathcal{N}} a + \frac{1}{2} \overline{f}\, d_{\mathcal{N}} f, \tag{4.2.8}$$

the hamiltonian function

$$H_{\mathcal{N}} = \left\langle \frac{1}{2} b[a, a] + \frac{\Lambda}{6} b^3 \right\rangle + \frac{1}{2} \overline{f}\, a f, \tag{4.2.9}$$

and the cohomological vector field $Q_{\mathcal{N}} = -\{H_{\mathcal{N}}, \bullet\}$ associated to $H_{\mathcal{N}}$ through the Poisson bracket induced by $\omega_{\mathcal{N}}$. ♡

Remark 4.2.5. One might decide to build the BV extension of 3D supergravity applying the AKSZ construction to this theory; the problem, however, is that the resulting theory would only account implicitly for diffeomorphism invariance, since this procedure—as we will see next—doesn't introduce a vector field ξ as the one found in the previous BV extension 4.1.11 of 3D gravity. Our proposed solution to the problem consists in complementing the AKSZ extension of BF supergravity with an addition to the generating function (4.1.19), that we will employ to incorporate to S_{GR} a BV extensions of the Rarita-Schwinger action.

Construction 4.2.6 (AKSZ extension of BF supergravity)**.** Given the target established in the proof of proposition 4.2.4, the field space is

$$\mathcal{F}_{\text{BF}\Phi} \cong \Omega\big(M, \mathcal{V}^*\big)[1] \oplus \Omega\big(M, \mathcal{V}^{\wedge 2}\big)[1] \oplus \Omega\big(M, \Pi S(P)\big)[1]. \tag{4.2.10}$$

On it, the coordinates are given by the superfields

$$\widetilde{b} = \tau + B + A^{\urcorner} + c^{\urcorner}, \tag{4.2.11a}$$

$$\widetilde{a} = c + A + B^{\urcorner} + \tau^{\urcorner}, \tag{4.2.11b}$$

$$\widetilde{f} = \gamma + \varphi + \varphi^{\urcorner} + \gamma^{\urcorner}, \tag{4.2.11c}$$

where summands are ordered in increasing cohomological degree, from 0 to 3, and decreasing ghost number, from 1 to -2; moreover, since superfields have the total degree of their associated field—that is, of the classical field included among their components—every field in $\widetilde{a}$ and in $\widetilde{b}$ have respectively two and one internal indices over $\mathcal{V}^{(*)}$, while those in $\widetilde{f}$ have a spinor internal index over $\Pi S(P)$. Replacing the classical fields in $S^0_{\mathrm{BF}\Phi}$ by their associated superfield, keeping only those terms of cohomological degree 3 and rearranging them, we find the BV action for BF supergravity:

$$S_{\mathrm{BF}\Phi}(\Lambda) = \int_M \Big\langle \mathscr{L}^0_{\mathrm{BF}\Phi}(\Lambda) + \mathscr{L}^1_{\mathrm{BF}}(\Lambda) + \mathscr{L}^2_{\mathrm{BF}\Phi}, \Big\rangle, \tag{4.2.12a}$$

where

$$\mathscr{L}^0_{\mathrm{BF}\Phi}(\Lambda) = BF_A + \frac{\Lambda}{6}B^{\wedge 3} + \frac{1}{2}\overline{\varphi}\, D_A\varphi, \tag{4.2.12b}$$

$$\mathscr{L}^1_{\mathrm{BF}}(\Lambda) = B^\daleth\big([c,B] + D_A\tau\big) + A^\daleth\big(D_Ac + \Lambda B\tau\big) + \frac{1}{2}c^\daleth\big([c,c] + \Lambda\tau\tau\big) + \tau^\daleth[c,\tau], \tag{4.2.12c}$$

$$\mathscr{L}^2_{\mathrm{BF}\Phi} = \overline{\gamma}\, B^\daleth\varphi + \frac{1}{2}\overline{\gamma}\,\tau^\daleth\gamma + \overline{\varphi^\daleth}\,\big(D_A\gamma + c\varphi\big) + \overline{\gamma^\daleth}\, c\gamma. \tag{4.2.12d}$$

The symplectic form is given by (4.2.7) when we replace the coordinates by their corresponding superfields and keep only the terms of ghost number -1, which then leads to the Poisson bracket

$$\{\bullet,\bullet\} = \frac{\overleftarrow{\partial}}{\partial x^i}\cdot\frac{\overrightarrow{\partial}}{\partial x^\daleth_i} - \frac{\overleftarrow{\partial}}{\partial x^\daleth_i}\cdot\frac{\overrightarrow{\partial}}{\partial x^i} + 2\frac{\overleftarrow{\partial}}{\partial \overline{y^i}}\cdot\frac{\overrightarrow{\partial}}{\partial y^\daleth_i} - 2\frac{\overleftarrow{\partial}}{\partial \overline{y^\daleth_i}}\cdot\frac{\overrightarrow{\partial}}{\partial y^i} \tag{4.2.13}$$

for $x = (B, A, \tau, c)$ and $y = (\varphi, \gamma)$. With it we can read off the action the components of the cohomological field:

$$Q_{\mathrm{BF}\Phi}(B) = [c,B] + D_A\tau + \overline{\gamma}\,\rho\varphi, \qquad Q_{\mathrm{BF}\Phi}(\tau) = [c,\tau] + \frac{1}{2}\overline{\gamma}\,\rho\gamma, \tag{4.2.14a}$$

$$Q_{\mathrm{BF}\Phi}(A) = D_Ac + \Lambda B\tau, \qquad Q_{\mathrm{BF}\Phi}(c) = \frac{1}{2}[c,c] + \frac{1}{2}\Lambda\tau\tau, \tag{4.2.14b}$$

$$Q_{\mathrm{BF}\Phi}(\varphi) = 2D_A\gamma + 2c\varphi, \qquad Q_{\mathrm{BF}\Phi}(\gamma) = 2c\gamma. \tag{4.2.14c}$$

Thus we conclude the construction collecting everything in a tuple

$$\mathscr{T}_{\mathrm{BF}\Phi} := (\mathcal{F}_{\mathrm{BF}\Phi}, S_{\mathrm{BF}\Phi}, Q_{\mathrm{BF}\Phi}, \omega_{\mathrm{BF}\Phi}). \tag{4.2.15}$$

Remark 4.2.7. To enforce that the total degree of a spinor and its corresponding antifield defer by 1, we incorporate to the components of the superfield $\widetilde{f}$ a spinorial degree $\deg_S$, such that

$$\deg_S s^\daleth = \deg_S s + 1 \tag{4.2.16}$$

for any spinor field s. This assumption is necessary for one goal: to make the type-2 generating function associated to the identity be given by a sum of terms $-\phi_i^\daleth \phi^i$, where ϕ^i is *any* field, spinorial or not. Without this grading, the spinors $\{s_i\}$ in the AKSZ construction above would satisfy $|s_i^\daleth| = |s^i|$, which implies that a term $-s_i^\daleth s^i$ in a generating function of type 2 would correspond to a map $(s, s^\daleth) \mapsto (\pm s, \mp s^\daleth)$, and hence the identity for fermions could not be implemented.

Remark 4.2.8. I point your attention to the fact that S^1_{BF} doesn't contain any fermionic term and that, in fact, the AKSZ extension of S^0_{BF} is given by

$$S_{\text{BF}}(\Lambda) = S^0_{\text{BF}}(\Lambda) + S^1_{\text{BF}}(\Lambda), \tag{4.2.17}$$

so the last construction 4.2.6 generalizes the AKSZ extension of BF gravity; to obtain the latter one ignores the fermionic contributions, thus proving the first claim in proposition 4.1.15.

Lemma 4.2.9. For any degree 1 vector field $\xi \in \mathfrak{X}(M)[1]$, principal connection 1-form $\Gamma \in \text{Conn}(P)$ and field $\varphi \in \Omega(M, \mathcal{V})$ valued in an associated vector bundle $\mathcal{V}$, the following identity holds:

$$\left[\mathcal{L}^\Gamma_\xi, \iota_\xi\right]\varphi = \iota_{[\xi,\xi]}\varphi. \tag{4.2.18}$$

This is proven in [CS19b].

Proposition 4.2.10. *There is a ghost fermion ε such that on shell*

$$Q_{\text{GR}\Phi}(\xi) \equiv Q_{\text{GR}}(\xi) + \frac{1}{2}\overline{\varepsilon}\, e^{-1}(\rho)\varepsilon. \tag{4.2.19}$$

Proof. Let us call $Q' := Q_{\text{GR}\Phi} - Q_{\text{GR}}$ the on-shell extension of Q_{GR}. Now, on shell all antifields are set to zero, so (4.1.20) reduces to

$$(B, A, \tau, c) = (e, \Gamma, -\iota_\xi e, \chi), \tag{4.2.20}$$

which we extend additionally with $(\varphi, \gamma) = (\psi, \kappa)$, thus being able to translate on shell the first part of (4.2.14a) to

$$\begin{aligned} Q_{\mathrm{GR}\Phi}(e) &\equiv -D_\Gamma(\iota_\xi e) + [\chi, e] + \overline{\kappa}\,\rho\psi \\ &= -\iota_\xi D_\Gamma e + \mathcal{L}^\Gamma_\xi e + [\chi, e] + \overline{\kappa}\,\rho\psi. \end{aligned} \tag{4.2.21}$$

Meanwhile, the fact that $|e| = 2$, together with (3.1.24) and the definition of $\mathcal{L}^\Gamma_\xi$, imply that

$$\iota_\xi\big(\mathcal{L}^\Gamma_\xi e\big) = \frac{1}{2}\big([\iota^2_\xi, D_\Gamma] - \iota_{[\xi,\xi]}\big)e. \tag{4.2.22}$$

Recalling that $Q_{\mathrm{GR}}(\xi) = \frac{1}{2}[\xi, \xi]$, we use all this to further find that

$$\begin{aligned} Q_{\mathrm{GR}\Phi}(\iota_\xi e) &= [Q_{\mathrm{GR}\Phi}, \iota_\xi]e + \iota_\xi(Q_{\mathrm{GR}\Phi} e) = \iota_{Q_{\mathrm{GR}\Phi}(\xi)} e + \iota_\xi(Q_{\mathrm{GR}\Phi} e) \\ &\equiv \iota_{Q_{\mathrm{GR}\Phi}(\xi)} e - \iota^2_\xi D_\Gamma e + \iota_\xi\big(\mathcal{L}^\Gamma_\xi e\big) + \iota_\xi[\chi, e] + \overline{\kappa}\,\rho\iota_\xi\psi \\ &= \iota_{Q'(\xi)} e + \frac{1}{2}\iota_{[\xi,\xi]} e - \iota^2_\xi D_\Gamma e + \frac{1}{2}\iota^2_\xi D_\Gamma e - \frac{1}{2} D_\Gamma(\iota^2_\xi e) \\ &\qquad\qquad - \frac{1}{2}\iota_{[\xi,\xi]} e + \iota_\xi[\chi, e] + \overline{\kappa}\,\rho\iota_\xi\psi \\ &= \iota_{Q'(\xi)} e - \frac{1}{2}\iota^2_\xi D_\Gamma e + \iota_\xi[\chi, e] + \overline{\kappa}\,\rho\iota_\xi\psi. \end{aligned} \tag{4.2.23}$$

Moreover, after adapting to on-shell 3D supergravity both the equation of motion (4.2.4c) and the second part of (4.2.14a), from what precedes we deduce that

$$\begin{aligned} Q_{\mathrm{GR}\Phi}(\iota_\xi e) &\equiv \iota_{Q'(\xi)} e + \frac{1}{2}\iota_\xi\overline{\psi}\,\rho\iota_\xi\psi + \iota_\xi[\chi, e] + \overline{\kappa}\,\rho\iota_\xi\psi \\ &\equiv -Q_{\mathrm{GR}\Phi}(\tau) \equiv \iota_\xi[\chi, e] - \frac{1}{2}\overline{\kappa}\,\rho\kappa, \end{aligned} \tag{4.2.24}$$

which holds if and only if

$$\iota_{Q'(\xi)} e \equiv -\frac{1}{2}(\overline{\kappa} + \iota_\xi\overline{\psi})\,\rho(\kappa + \iota_\xi\psi). \tag{4.2.25}$$

With this we finally conclude that

$$\begin{aligned} Q_{\mathrm{GR}\Phi}(\xi) &= Q_{\mathrm{GR}}(\xi) + e^{-1}\big(e(Q'\xi)\big) = Q_{\mathrm{GR}}(\xi) - e^{-1}\big(\iota_{Q'(\xi)} e\big) \\ &\equiv Q_{\mathrm{GR}}(\xi) + \frac{1}{2}\overline{\varepsilon}\, e^{-1}(\rho)\varepsilon \end{aligned} \tag{4.2.26}$$

for the ghost Majorana fermion $\varepsilon := \kappa + \iota_\xi\psi$. ♡

Remark 4.2.11. This property is expected, since the generators of supersymmetry square to translation generators, and the former are to be encoded by ghost fermions while the latter are realised through ξ.

Remark 4.2.12. Our current goal being to extend (4.1.19) as to find a BV theory of 3D supergravity that encodes explicitly both supersymmetry and diffeomorphism invariance, this last proposition will serve us as guiding principle. Indeed, we will be searching for a type 2 generating function $G_{\mathrm{BF}\Phi}$ that decomposes as

$$G_{\mathrm{BF}\Phi} = G_{\mathrm{BF}} + G_{\mathrm{BF}}^{\mathrm{ext}}, \tag{4.2.27}$$

and evidently we would like $G_{\mathrm{BF}}^{\mathrm{ext}}$ to be a minimal extension, that is, as simple as possible without being ineffective. This *without being ineffective* is precisely what the proposition 4.2.10 addresses: we must ensure that the extended symplectomorphism $\Phi_{\mathrm{BF}\Phi} : \mathcal{F}_{\mathrm{GR}\Phi} \to \mathcal{F}_{\mathrm{BF}\Phi}$ leads to a cohomological vector field that on shell is equal to (4.2.19). Fortunately, finding such extension is eased by the next proposition.

Proposition 4.2.13. *A minimal extension* $G_{\mathrm{BF}}^{\mathrm{ext}}$ *of* G_{BF} *ensuring that equation* (4.2.19) *holds on shell can only depend on spinorial fields or on contractions of those with respect to* ξ.

Proof. As before, we denote by (φ, γ) the spinorial field and ghost in $\mathcal{F}_{\mathrm{BF}\Phi}$ and by (ψ, ε) the corresponding pair on $\mathcal{F}_{\mathrm{GR}\Phi}$. Since the extension we are looking for aims at being minimal, all its terms must be spinorial scalars, because if any term in $G_{\mathrm{BF}}^{\mathrm{ext}}$ didn't include spinors, it would effectively amount to a modification of G_{BF} spoiling the known canonical transformation between 3D and BF gravities in the absence of fermions. Consequently, all terms in $G_{\mathrm{BF}}^{\mathrm{ext}}$ should take the form

$$\overline{y}xy' \tag{4.2.28}$$

where y and y' are spinors in $\mathcal{F}_{\mathrm{BF}\Phi}$ and in $\mathcal{F}_{\mathrm{GR}\Phi}$ respectively, and x is any product of non-spinorial fields—possibly including contractions—in either theory. Of course, not any such combination is valid, given that every such product must have cohomological degree 3 and ghost number -1, and every internal index must be contracted. In fact, under these constraints there will be at most two kinds of valid products $\overline{y}xy'$. The first kind will have $x = 1$ and an appropriate distribution of contractions ι_ξ, consisting of only—up to redistribution of the ι_ξ—the following possible pairs (y, y'):

$$\begin{aligned} &(\gamma^\daleth, \varepsilon), && (\varphi^\daleth, \psi), && (\gamma^\daleth, \iota_\xi\psi), \\ &(\iota_\xi\varphi^\daleth, \psi^\daleth), && (\iota_\xi^2\varphi^\daleth, \varepsilon^\daleth), && (\iota_\xi^3\gamma^\daleth, \varepsilon^\daleth), \end{aligned} \tag{4.2.29}$$

or the analogous pairings exchanging the roles of the fields in $\mathcal{F}_{\mathrm{GR}\Phi}$ by those in $\mathcal{F}_{\mathrm{BF}\Phi}$ and vice versa. Meanwhile, the second kind of product will have $x \neq 1$, yet it is evident that any product of this type, to be valid, should be obtained from a product of the first kind by replacing any number of contractions with a product x of non-spinorial fields that have the same cohomological degree and ghost number as the power of ι_ξ that they are replacing. In other words,

$$\begin{pmatrix} \deg_\Omega x \\ \mathrm{gh}\ x \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix} \tag{4.2.30}$$

for some $k \in \mathbb{N}$. Now, every non-spinorial field—including the contraction ι_ξ—has its pair $(\deg_\Omega \bullet, \mathrm{gh}\, \bullet)$ among the following:

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix},\ v_4 = \begin{pmatrix} 3 \\ -2 \end{pmatrix},\ v_5 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{4.2.31}$$

that respectively correspond to the degree pairs of c, A, $B^\daleth$, $c^\daleth$ and ι_ξ, in that order. The question, then, reduces to solving the simple equation

$$k^i v_i = 0 \quad \text{for} \quad \{k^i\}_{i=1}^4 \subset \mathbb{N},\ k^5 \in \mathbb{Z}, \tag{4.2.32}$$

which holds if and only if $k^i = 0$ for all i. This is equivalent to saying that any valid product $\overline{y}xy$ is of the first kind, that is, a Dirac product of spinors or of contractions of those with respect to ξ. ♡

Remark 4.2.14. The proposition 4.2.13 facilitates the labour notably by making $G^{\mathrm{ext}}_{\mathrm{BF}}$ include at most four terms that, moreover, they will show a convenient property: they will only fix the spinorial fields and, at most, modify the expression for $\xi^\daleth$ as a function of the fields in $\mathcal{F}_{\mathrm{BF}\Phi}$. Finding an appropriate extension, then, is letting

$$G^{\mathrm{ext}}_{\mathrm{BF}} = \sum_i k_i \overline{y_i} y'_i \tag{4.2.33}$$

and determining the—at most—four parameters k_i that will lead to a $Q_{\mathrm{GR}\Phi}$ that on shell satisfies (4.2.19) and to an action $S_{\mathrm{GR}\Phi}$ whose classical spinorial part is $\frac{1}{2}\psi\, D_\Gamma \psi$.

Theorem 4.2.15. *A BV extension of 3D supergravity is provided by the tuple* $\mathcal{T}_{\mathrm{GR}\Phi} := (\mathcal{F}_{\mathrm{GR}\Phi}, S_{\mathrm{GR}\Phi}, Q_{\mathrm{GR}\Phi}, \omega_{\mathrm{GR}\Phi})$ *for*

$$\mathcal{F}_{\mathrm{GR}\Phi} = \mathcal{F}_{\mathrm{GR}} \times T^*[-1]\Omega(M, \Pi S(P)), \tag{4.2.34a}$$
$$S_{\mathrm{GR}\Phi} = \Phi_{\mathrm{BF}\Phi}{}^* S_{\mathrm{BF}\Phi}, \tag{4.2.34b}$$
$$\omega_{\mathrm{GR}\Phi} = \Phi_{\mathrm{BF}\Phi}{}^* \omega_{\mathrm{BF}\Phi}, \tag{4.2.34c}$$
$$Q_{\mathrm{GR}\Phi} = \{\bullet, S_{\mathrm{GR}\Phi}\}, \tag{4.2.34d}$$

where the Poisson bracket is defined by $\omega_{\mathrm{GR}\Phi}$ *and the canonical transformation* $\Phi_{\mathrm{BF}\Phi}$ *is generated by*

$$G_{\mathrm{BF}\Phi} = G_{\mathrm{BF}} + G^{\mathrm{ext}}_{\mathrm{BF}}, \tag{4.2.35}$$

where G_{BF} *is the generating function* (4.1.19) *and* $G^{\mathrm{ext}}_{\mathrm{BF}}$ *is given as*

$$G^{\mathrm{ext}}_{\mathrm{BF}}(q, p') = -\overline{\varphi^\daleth}\,\psi - \overline{\gamma^\daleth}\,(\varepsilon - \iota_\xi \psi) \tag{4.2.36}$$

for $q := (e, \Gamma^\daleth, \xi, \chi^\daleth, \psi, \varepsilon)$ *and* $p' := (B^\daleth, A, \tau^\daleth, c, \varphi^\daleth, \gamma^\daleth)$.

Proof. This generating function leads to

$$\varphi = \psi, \qquad \varphi^\daleth = \psi^\daleth - \iota_\xi \varepsilon^\daleth, \tag{4.2.37a}$$
$$\gamma = \varepsilon - \iota_\xi \psi, \qquad \gamma^\daleth = \varepsilon^\daleth, \tag{4.2.37b}$$

so following the previous remark 4.2.14, we only have to attend some of the terms in $S_{\mathrm{BF}\Phi}$ (4.2.12) to check whether it produces an extension of classical 3D supergravity. Firstly, since the definition (4.1.20b) of A in terms of fields in $\mathcal{F}_{\mathrm{GR}\Phi}$ remains unchanged, the expansion of the classical spinorial field gives

$$\frac{1}{2}\overline{\varphi}\, D_A \varphi = \frac{1}{2}\overline{\psi}\, D_\Gamma \psi - \frac{1}{2}\overline{\psi}\,(\iota_\xi e^\daleth - \frac{1}{2}\iota_\xi^2 \tau^\daleth)\psi, \tag{4.2.38}$$

so indeed the classical spinorial term is recovered on shell—where, remember, antifields are set to zero. Secondly, since the only terms modifying $Q_{\mathrm{GR}}(\xi)$ are those spinorial terms in $S_{\mathrm{BF}\Phi}$ that include a factor of $\tau^\daleth$—because only these depend on $\xi^\daleth$, specifically through $e^{-1}(\xi^\daleth)$—we only verify the following terms in $\mathcal{L}_{\mathrm{BF}\Phi}$:

$$\frac{1}{2}\overline{\psi}\, A\psi + \overline{\gamma}\, B^\daleth \varphi + \frac{1}{2}\overline{\gamma}\, \tau^\daleth \gamma + \overline{\varphi^\daleth}\, A\gamma + \overline{\varphi^\daleth}\, c\varphi + \overline{\gamma^\daleth}\, c\gamma. \tag{4.2.39}$$

After expansion—that is rendered explicit below, in definition 4.2.17—one verifies that

$$Q_{\mathrm{GR}\Phi}(\xi) = \frac{1}{2}[\xi, \xi] + \frac{1}{2}\overline{\varepsilon}\, e^{-1}(\rho)\varepsilon + \cdots \tag{4.2.40}$$

omitting all terms that contain antifields, so indeed $Q_{\mathrm{GR}\Phi}$ satisfies (4.2.19). Finally, (4.2.36) holds necessarily, since (4.2.34a) merely accounts for the incorporation of fermions, while equations (4.2.34b) to (4.2.34d) follow from the definition of a canonical transformation and the fact that $G_{\mathrm{BF}\Phi}$ is a generating function. Therefore, we have constructed a theory $\mathscr{T}_{\mathrm{GR}\Phi}$ that is symplectomorphic to BF supergravity, and moreover $\mathscr{T}_{\mathrm{GR}\Phi}$ produces classical 3D supergravity on shell; in other words, $\mathscr{T}_{\mathrm{GR}\Phi}$ is a BV extension of 3D supergravity. ♡

Remark 4.2.16. Due to the fact that it only involves spinors and their contraction, the only equation in (4.1.20) that the extension $G_{\mathrm{BF}}^{\mathrm{ext}}$ modifies is the one corresponding to $\tau^{\daleth}$, giving

$$\tau^{\daleth} = e^{-1}\big(\xi^{\daleth} - e^{\daleth}\Gamma^{\daleth} - \overline{\gamma^{\daleth}}\psi + \iota_{\xi} e^{\daleth}\chi^{\daleth}\big). \tag{4.2.41}$$

Conclusion

Perhaps unusually, I will conclude both this chapter and my thesis with a definition. My reasons to do so are, on the one hand, because defining this BV extension was precisely the object of the chapter, which up until that point was nothing more than a justification of this definition; on the other hand, because I certainly admit that such construction is the beginning of further possible work. Indeed, I can think of three immediate lines of further research:

1. What does this model teach us about 4-dimensional BV supergravity?

2. We have made the assumption that spacetime has no boundary, but what if we didn't and we decided to study the field theory on its boundary?

3. The theory we have obtained is a classical BV theory, but what about its quantisation?

Nevertheless, time did not allow me to treat any of these questions, so they will remain, for now, little more than invitations. That being said, let us conclude with the anticipated definition.

Definition 4.2.17. We will call 3-dimensional **BV supergravity** the theory built in theorem 4.2.15. Its action is given by

$$S_{\mathrm{GR}\Phi} = S_{\mathrm{GR}} + \int_M \left\langle \mathscr{L}^2_{\mathrm{GR}\Phi} \right\rangle, \tag{4.2.42a}$$

for the density

$$\mathscr{L}^2_{\mathrm{GR}\Phi} = \frac{1}{2}\Big(\overline{\psi}\, D_\Gamma \psi + \overline{\psi}\, e^{\urcorner} \varepsilon + \overline{\varepsilon}\, \tau^{\urcorner} \varepsilon + \overline{\psi^{\urcorner}}\, Q_{\mathrm{GR}\Phi}(\psi) + \overline{\varepsilon^{\urcorner}}\, Q_{\mathrm{GR}\Phi}(\varepsilon) \Big), \tag{4.2.42b}$$

where $\tau^{\urcorner}$—as given in (4.2.41)—is kept implicit for the sake of readability. In turn, the cohomological vector field decomposes as

$$Q_{\mathrm{GR}\Phi} = Q_{\mathrm{GR}} + Q^{\mathrm{ext}}_{\mathrm{GR}}, \tag{4.2.43a}$$

for an extension that acts in the following manner:

$$Q^{\mathrm{ext}}_{\mathrm{GR}}(\Gamma) = Q^{\mathrm{ext}}_{\mathrm{GR}}(\chi) = 0, \tag{4.2.43b}$$

$$\begin{aligned} Q^{\mathrm{ext}}_{\mathrm{GR}}(e) = \frac{1}{2}\overline{\psi}\, \rho\varepsilon - \iota_\xi \overline{\psi^{\urcorner}}\, \rho\kappa - \frac{1}{2}\iota^2_\xi \overline{\psi^{\urcorner}}\, \rho\psi \\ - \iota_\xi \overline{\psi^{\urcorner}}\, \rho \iota_\xi \psi + \frac{1}{2}\iota^2_\xi \overline{\varepsilon^{\urcorner}}\, \rho(\varepsilon - 2\iota_\xi\psi), \end{aligned} \tag{4.2.43c}$$

$$Q^{\mathrm{ext}}_{\mathrm{GR}}(\xi) = \frac{1}{2}\overline{\varepsilon}\, \underline{\rho}\varepsilon + \frac{1}{2}\iota^2_\xi \overline{\psi^{\urcorner}}\, \underline{\rho}\varepsilon - \frac{1}{6}\iota^3_\xi \overline{\varepsilon^{\urcorner}}\, \underline{\rho}(2\varepsilon - 3\iota_\xi\psi), \tag{4.2.43d}$$

$$Q_{\mathrm{GR}\Phi}(\psi) = 2\chi\psi + 2D_\Gamma\kappa - 2\iota_\xi e^{\urcorner}\kappa + \iota^2_\xi e^{\urcorner}\psi + \iota^2_\xi \tau^{\urcorner}\kappa - \frac{1}{3}\iota^3_\xi \tau^{\urcorner}\psi, \tag{4.2.43e}$$

$$Q_{\mathrm{GR}\Phi}(\varepsilon) = 2\chi\varepsilon + 2\iota_\xi D_\Gamma\kappa - \iota^2_\xi e^{\urcorner}(\varepsilon - 2\iota_\xi\psi) + \frac{1}{3}\iota^3_\xi \tau^{\urcorner}(2\varepsilon - 3\iota_\xi\psi), \tag{4.2.43f}$$

writing $\kappa := (\varepsilon - \iota_\xi\psi)$ and $\underline{\rho} := e^{-1}(\rho)$.

Bibliography

[BW48] V. Bargmann and E. P. Wigner. "Group Theoretical Discussion of Relativistic Wave Equations". en. In: *Proc Natl Acad Sci U S A* 34.5 (May 1948), pp. 211–223.

[CMR14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin. "Classical BV Theories on Manifolds with Boundary". In: *Communications in Mathematical Physics* 332.2 (Aug. 2014), pp. 535–603. ISSN: 1432-0916. DOI: 10.1007/s00220-014-2145-3. URL: http://dx.doi.org/10.1007/s00220-014-2145-3.

[CS19a] A. S. Cattaneo and M. Schiavina. "BV-BFV Approach to General Relativity: Palatini–Cartan–Holst Action". In: *Advances in Theoretical and Mathematical Physics* 23.8 (2019), pp. 2025–2059. ISSN: 1095-0753. DOI: 10.4310/atmp.2019.v23.n8.a3. URL: http://dx.doi.org/10.4310/ATMP.2019.v23.n8.a3.

[CS19b] A. S. Cattaneo and M. Schiavina. "BV-BFV Approach to General Relativity: Palatini–Cartan–Holst Action". In: *Advances in Theoretical and Mathematical Physics* 23.8 (2019), pp. 2025–2059. ISSN: 1095-0753. DOI: 10.4310/atmp.2019.v23.n8.a3. URL: http://dx.doi.org/10.4310/ATMP.2019.v23.n8.a3.

[CSS18] A. S. Cattaneo, M. Schiavina, and I. Selliah. "BV Equivalence between Triadic Gravity and BF Theory in Three Dimensions". In: *Letters in Mathematical Physics* 108.8 (Feb. 2018), pp. 1873–1884. ISSN: 1573-0530. DOI: 10.1007/

s11005-018-1060-5. URL: http://dx.doi.org/10.1007/s11005-018-1060-5.

[DeW92] B. DeWitt. *Supermanifolds*. 2nd ed. Cambridge monographs on mathematical physics. Cambridge University Press, 1992. ISBN: 9780521413206.

[Keß19] E. Keßler. *Supergeometry, Super Riemann Surfaces and the Superconformal Action Functional*. Springer, 2019. ISBN: 9783030137571. DOI: 10.1007/978-3-030-13758-8.

[Mne19] P. Mnev. *Quantum Field Theory: Batalin-Vilkovisky Formalism and its Applications*. University Lecture Series (Book 72). American Mathematical Society, 2019. ISBN: 9781470452711.

[Nak03] M. Nakahara. *Geometry, Topology, and Physics*. 2nd ed. Graduate student series in physics. Institute of Physics Publishing, 2003. ISBN: 9780750306065.

[Rog07] A. Rogers. *Supermanifolds: Theory and Applications*. World Scientific Publishing Company, 2007. ISBN: 9789810212285.

[Rom93] J. D. Romano. "Geometrodynamics versus Connection Dynamics". In: *General Relativity and Gravitation* 25.8 (Aug. 1993), pp. 759–854. ISSN: 1572-9532. DOI: 10.1007/bf00758384. URL: http://dx.doi.org/10.1007/BF00758384.

[War10] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. 1st ed. 1971. 2nd printing. Graduate Texts in Mathematics. Springer, 2010. ISBN: 9781441928207.

[WB92] J. Wess and J. Bagger. *Supersymmetry and Supergravity*. 2nd ed., rev. and expanded. Princeton series in physics. Princeton University Press, 1992. ISBN: 9780691085562.

[Wei95] S. Weinberg. *Quantum Theory of Fields I: Foundations*. 1st ed. Cambridge University Press, 1995.

[Wig89] E. Wigner. "On Unitary Representations of the Inhomogeneous Lorentz Group". In: *Nuclear Physics B - Proceedings Supplements* 6 (1989), pp. 9–64. ISSN: 0920-5632. DOI: https://doi.org/10.1016/0920-5632(89)90402-7. URL: https://www.sciencedirect.com/science/article/pii/0920563289904027.

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