Cohomological Ambiguities in General Relativity

by

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1 Introduction

The theory of General Relativity can be formulated in two different descriptions: the Einstein-Hilbert formulation based on the dynamics of metric field g and the Palatini-Cartan formulation which concerns the dynamics of two fields, the coframe, or so-called tetrad in 4 dimensional spacetime, and the connections defined on a $SO(3,1)$ bundle over a manifold M. This latter description is based on the works of Attilio Palatini in [8] and Élie Cartan in [9] and leads to a treatment of General Relativity as a gauge theory. These formalisms will be explained in Section 2. When a manifold with boundaries is considered, it is shown in [1] that the boundary variation of the Lagrangian considered in tetrad formalism differs from the metric one by an exact 3-form, $d\alpha$, where α is called dressing 2-form, defined on the corners of the manifold. In the context of covariant phase space this difference is read as a cohomological ambiguity of the symplectic potential defined up to an arbitrary 2-form, $\theta \to \theta + d\eta$. In the case of tetrad General Relativity this arbitrary 2-form is chosen to be the dressing one. Thus, choosing the right 2-form between the two formalisms affects the construction of Hamiltonian surface charges in covariant phase space method restoring the equivalence of the phase space (Hamiltonian charges, Noether charges, ect). Namely, the cohomological ambiguity of the symplectic potential can be an attempt to solve

the mismatch of Hamiltonian and Noether charges between different formulations of General Relativity.
On the other hand, the same ambiguity of the symplectic potential is treated in the case of BV-BFV formalism which is an approach to quantise classical field theories, in particular well-suitable for gauge field theories. After having defined two BV-BFV theories associated respectively with the Palatini-Cartan formalism (tetrad formalism) and Einstein-Hilbert metric formalution, differing by an d-exact local form, this difference is then studied. The BV-BFV structure of a general theory, as defined in [2] and [12], is invariant when the symplectic potential is changed by an d-exact local form, as it will be observed in Section 6.

Aim: This study interprets the cohomological ambiguity for pre-symplectic potentials in classical field theory, from the point of view of the BV-BFV construction.

2 Tetrad formalism and Metric formalism

In this section a brief review of tetrad and metric General Relativity is introduced.
The Einstein Hilbert classical formulation of General Relativity is based on the dynamics of the pseudo-Riemannian metric g defined over an m-dimensional spacetime manifold M (possibly with a non-trivial boundary) and the action functional is defined as

$$S_{EH} = \int_M d^m x \sqrt{-g}(R - \Lambda) \tag{1}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar of the metric g, $g := det(g)$ is the determinant of the metric g and Λ is the cosmological constant. The Einstein equations can be obtained by varying the Lagrangian $L_g = (g^{\mu\nu} R_{\mu\nu} - 2\Lambda)\epsilon$ (ϵ is the volume form) with respect to the metric:

$$\delta L_g = (G_{\mu\nu} + \Lambda)\delta g^{\mu\nu}\epsilon + d\theta_g(\delta) \tag{2}$$

where $d\theta_g(\delta)$ is the non-zero boundary term as a manifold with non-trivial boundary is considered. As well-known, General relativity in Einstein-Hilbert formulation is invariant under local diffeomorphism generated by vector fields ξ^μ.

An alternative description of General Relativity is the Palatini-Cartan formulation which manifests the gauge nature of the theory as it becomes a theory of principal connections on some spacetime. For any manifold M at each point $p \in M$ a "natural" basis of the tangent space T_pM is given by the partial derivative with respect to the coordinates at that point, i.e. ∂_μ (the partial derivative). However, at each point one can introduce any desired basis of the tangent space, not necessarily the coordinate basis . One considers a set of four independent **orthonormal** vector fields e_a, i.e. $g(e_a, e_b) = \eta_{ab}$, where g is the usual metric and η represents the Minkowski metric of the Lorentzian spacetime, also called *internal* metric. This was derived by the idea of Élie Cartan to describe the metric in terms of local frames in [9].

The other crucial idea of the formalism is to treat the connection as an independent variable, based on Palatini's observation [8] where the variation of Ricci tensor is described in terms of variation of the Christoffel symbols.

More precisely, consider a Lorentzian m-dimensional manifold M and a vector bundle $\mathcal{V}$ isomorphic to the tangent bundle TM endowed with a fibrewise Minkowski metric η. The vector bundle $\mathcal{V}$ is often called "fake" or "imitation" tangent bundle. A **coframe** e (called *tetrad* in 4-dimensional case) is defined as an orientation preserving bundle isomorphism covering the identity from the tangent bundle to the vector bundle $\mathcal{V}$, i.e. $e : TM \to \mathcal{V}$.

The **orthogonal connection** on $\mathcal{V}$ is denoted by ω and the space of such connections is denoted by $\mathcal{A}(M)$.

One of the main advantages of this formulation is that quantities can be seen as differential forms taking values in some exterior power of $\mathcal{V}$: $\Omega^{i,j} := \bigwedge^i T^*M \otimes \bigwedge^j \mathcal{V} = \Omega^i(M, \bigwedge^j \mathcal{V})$. In this case the coframe field $e \in \Omega^1(M, \mathcal{V})$ (as $\bigwedge^1 \mathcal{V} = \mathcal{V}$) and the connection $\omega \in \Omega^1(M, \bigwedge^2 \mathcal{V})$. Furthermore, if $P \longrightarrow M$ is a principal $SO(3,1)$ bundle associated to $\mathcal{V}$ as in the case of General Relativity, then ω is a local one-form taking values in group Lie Algebra $\mathfrak{so}(3,1) \simeq \bigwedge^2 \mathcal{V}$. To summarize the two dynamical fields of the theory are the coframe field e and the connection in an $SO(3,1)$ bundle. It is important to notice that this formulation of General Relativity will be still invariant under local diffeomorphisms, but also under internal Lorentz gauge symmetry.

The Palatini - Cartan action functional for a 4-dimensional manifold is defined as:

$$S_{PC} = \int_M \frac{1}{2} e \wedge e \wedge F_\omega + \frac{1}{4!} \Lambda e \wedge e \wedge e \wedge e = \int_M \frac{1}{2} eeF_\omega + \frac{1}{4!} \Lambda e^4 \tag{3}$$

where $F_\omega \in \Omega^2(M, \bigwedge^2 \mathcal{V})$ is the curvature and Λ is the cosmological constant. In the last equation we omit the wedge product symbol in order to have a more compact notation. The Euler-Lagrange equation obtained by a variation of ω is, by imposing non-degenerancy of e and as $\delta_\omega F_\omega = -d_\omega \delta\omega$:

$$e \wedge d_\omega e = 0 \rightarrow d_\omega e = 0 \tag{4}$$

where $d_\omega : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet+1,\bullet}$ is the covariant derivative associated to ω. This condition is equivalent to torsion-free condition of the affine connection Γ obtained by pulling back ω with the bundle isomorphism e, i.e. $\Gamma = e^*\omega$, which is also automatically a metric connection, i.e. preserving the metric g. Γ is then a Levi-Civita Connection $\nabla_\Gamma g = 0$. Hence, there is a unique connection ω for each e solving the equation (4). The connection $\omega \equiv \omega(e)$ is sometimes called *Levi-Civita Lorentz connection.*
The usual Einstein equations can be recovered from the variation with respect to e:

$$e \wedge F_\omega + \frac{1}{3!} e \wedge e \wedge e = 0 \tag{5}$$

It is important to notice that Palatini-Cartan theory is fully equivalent to Einstein-Hilbert formulation only when the equation (4) is determined. In this case, one can write the curvature F_ω in terms of the Levi-Civita Lorentz connection.

Making the indices explicit, the action functional is:

$$S_{PC} = \frac{1}{2} \epsilon_{IJKL} \int_M (e^I \wedge e^J \wedge F^{KL}(\omega) + \frac{\Lambda}{6} e^I \wedge e^J \wedge e^K \wedge e^L) \tag{6}$$

where $F^{IJ}(\omega) = d\omega^{IJ} + \omega^{IK} \wedge \omega_K^J$ is the curvature 2-form written in terms Levi-Civita Lorentz connection $\omega^{IJ} \equiv \omega^{IJ}(e)$ and Λ is the cosmological constant. The Lagrangian 4-form is then ($t = tetrad$):

$$L_t = \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge (F^{KL}(\omega) + \frac{\Lambda}{6} e^K \wedge e^L) \tag{7}$$

A manifold with boundary is now assumed, the variation of L_t becomes :

$$\delta L_t = \delta e^I \wedge \mathbb{E}_I^{(e)} + d\theta_t(\delta) \tag{8}$$

where $\mathbb{E}_I^{(e)}$ is the L.H.S of equations (5) and $d\theta_t$ is the the boundary term for the tetrad variation.

In the article [1] it is highlighted that the metric and the tetrad boundary forms, θ_g and θ_t respectively, differ by a particular exact 3-form, $d\alpha$, where α called *dressing 2-form* defined at the corners and d is the de Rham differential on the manifold M ($d^2 = 0$ and $d : \Omega^k \to \Omega^{k+1}$):

$$\boxed{\theta_g = \theta_t + d\alpha} \tag{9}$$

The boundary terms play an important role in covariant phase space method. Indeed, by taking the integral on a hypersurface $\Sigma = \partial M$ of the boundary terms obtained from the variation of the Lagrangian, one can define the symplectic potential (associated to the boundary) on the space of solutions to the fields equations:

$$\Theta := \int_{\Sigma=\partial M} \theta \tag{10}$$

Note: Sometimes θ is called the *symplectic potential current.* We will refer to it as symplectic potential.

Thus θ can be seen as a 1-form on the space of fields restricted to the boundary *on-shell*, as the integration over the boundary is concerned.
In the context of covariant phase space the relation (9) is translated as **cohomological ambiguity** of the symplectic potential which is defined only up to an exact form, i.e $\theta \to \theta + d\eta$. This freedom in choosing any arbitrary 2-form η strongly affects the symplectic structure of the theory and it is generally considered to play an important role in restoring the full equivalence within the context of Noether and Hamiltonian charges in the metric and tetrad formalisms of General Relativity. In fact, in [1] it is shown that if the arbitrary η is chosen to be the particular dressing 2-form α, this choice can solve the metric-tetrad mismacth. This suggests, as was argued in [1], that the cohomological ambiguity of the potential could have a relevant role as solution of a theoretical issue.

As the aim of this study is to understand how a theory, or more especially the structure of it, changes when such a cohomological ambiguity is applied in BV-BFV setting, we are not interested in explicitly calculations of these charges. Their definitions in terms of the symplectic structure can be found [13] and after calculated in case of General Relativity for both formalisms in section 6 of [1].

Remark: Explicitly, the symplectic potentials found in [1]:

$$\theta_t(\delta) = \frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \wedge \delta\omega^{KL} \tag{11}$$

$$\theta_g(\delta) = \frac{1}{3!}(2g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma})\epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{12}$$

where $g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma} = g^{\rho\sigma}\delta\Gamma^{\mu}_{\rho\sigma} - g^{\rho\mu}\delta\Gamma^{\sigma}_{\rho\sigma}$. It is important to observe that in the expression of θ_t the connection is a Levi-Civita Lorentz connection, so $\omega = \omega(e)$.

The dressing 2-form from [1] is:

$$\alpha(\delta) = \star_g(e_I \wedge \delta e^I) = -\frac{1}{2}\epsilon_{IJKL}e^I \wedge e^J(e^{\rho K}\delta e^L_\rho) \tag{13}$$

where the $\star_g$ is the Hodge star operator with respect the metric g of the space-time manifold M, i.e. $\star_g : \bigwedge^k \to \bigwedge^{n-k}$.

3 Geometry of the boundary structure

As introduced in the last section, in the context of covariant phase space method a one-form is extracted from the variation of the action, defined up to an arbitrary exact form. Thus, the definition of the one-form is arbitrary as well as one can make any arbitrary choice of the additional exact form. As argued in [1] this ambiguity could play a relevant role.

In the following section a geometrical construction of the space of boundary fields, following the Kijowski-Tulczyjew approach, is presented. This method leads to the construction of the space of boundary fields starting from the space of fields on the bulk and we will see that a similar method will be applied later in BV-BFV formalism. The first essential point is that in this geometrical approach the integrated one-form, obtained from the variation of the action (10), is identified as a one-form on the space given by the restriction of the fields and their derivatives (jets) from the bulk to the boundary of the theory. More precisely, when a manifold with boundary is considered, the non-zero contribution from the variation of the action is an integrated one-form $\tilde{\alpha}_{\partial M}$ defined on the space of **pre-boundary** fields $\tilde{F}_{\partial M}$ such that $\delta S_M = EL_M + (\tilde{\pi}^*_{\partial M}\tilde{\alpha}_{\partial M})$ where $\tilde{\pi}_{\partial M} := F_M \to \tilde{F}_{\partial M}$. As an example in the case of Palatini-Cartan, by taking the variation of the action (3), we get $\tilde{\alpha}_{\partial M} = \int_{\partial M} e \wedge e \wedge \delta\omega$ and the space of pre-boundary fields is given by $e|_{\partial M}$ and $\omega|_{\partial M}$[1]. Once a one-form is given, we derive the pre-symplectic 2-form as $\tilde{\omega}_{\partial M} = \delta\tilde{\alpha}_{\partial M}$ where δ is the de Rham differential in this space. If the kernel of this 2-form is regular[2], the space of **true boundary** fields is obtained by performing a (pre-)symplectic reduction of the space of pre-boundary fields by the kernel of $\tilde{\omega}_{\partial M}$, i.e. $F^{\partial}_{\partial M} = \tilde{F}_{\partial M}/ker(\tilde{\omega}_{\partial M})$. This space is also called *geometric phase space* of the theory. First of all, it is worth mentioning that in this construction the kernel of the pre-symplectic 2-form is not constituted by the gauge transformations, but is given by some general vector fields. In the case of Einstein-Hilbert these vectors are generated by the lapse and shift, see [6]. In the Palatini-Cartan formulation the kernel of the pre-symplectic form is given by the vector fields X acting as $\omega \to \omega + v$ such that $e \wedge v = 0$, see [3].

Going back to our construction, the pre-symplectic reduction allows us to find

[1] Writing $\Sigma = \partial M$, the restriction of the coframe to the boundary is a non-degenerate section of $T^*(\Sigma) \otimes \mathcal{V}|_\Sigma$, i.e. an injective bundle map $T(\Sigma) \to \mathcal{V}|_\Sigma$, and the restriction of the connection ω related to $\mathcal{V}|_\Sigma$. $(T^*(\Sigma) \otimes \bigwedge^2 \mathcal{V}|_\Sigma)$.

[2] A kernel is regular if it is subbundle of $T\tilde{F}_{\partial M}$. If the space has a finite dimension, this condition is equilavent to stating that the (pre-)symplectic 2-form ω has a constant rank.

the space of boundary fields endowed with a symplectic form. Defining the natural projection $\pi_M : F_M \to F^{\partial}_{\partial M}$ the restriction from the space of fields directly to the space of true boundary fields as the composition between $\tilde{\pi}_{\partial M}$ (bulk - preboundary) and $\tilde{\pi}^{\partial}_{\partial M} : \tilde{F}_{\partial M} \to F^{\partial}_{\partial M}$ (preboundary-boundary), the variation of the action functional is given by $\delta S_M = EL_M + (\pi^*_M \alpha_{\partial M})$ where now $\alpha_{\partial M}$ is the induced one-form on $F^{\partial}_{\partial M}$. Thus, we obtained a symplectic description of the space of true boundary, i.e. a symplectic manifold $(F^{\partial}_{\partial M}, \omega_{\partial M})$ where $\omega_{\partial M} = \delta \alpha_{\partial M}$.
At this point it is important to remark that this geometric phase space is not the true physical space of our theory. Indeed, inside the geometric phase space one wants to identify a coisotropic submanifold which is the space of solutions to some constraints generated by the equations of motion restricted to the boundary. Roughly speaking, the equations of motion defined in the bulk split into some equations describing the evolution and some constraints which are not evolution equations, i.e. they do not contain any time derivative. These constraints restrict to the boundary, but they have to be also satisfied by a field in the neighborhood of the boundary. On the other hand, a solution to Euler-Lagrange equations on the bulk has to satisfy necessarily the constraints on the boundary.
The space of Cauchy data is given by the space of boundary fields that can be extended to solutions of Euler-Lagrange equations in the bulk. In Electrodynamics, the space of Cauchy data are the solutions to the Gauss law[3] (divergence of electric field E equal to zero) restricted to the boundary.
In other words, defining the neighborhood of the boundary as $\partial M \times [0, \epsilon) \subset M$ (for a small ϵ), the space of Cauchy data C is the (sub)space of boundary fields of $F^{\partial}_{\partial M}$ that can be completed to a solution in $L_{\partial M \times [0,\epsilon]}$[4]. We conclude that inside this geometric phase space exists a coisotropic[5] submanifold $C \subset F^{\partial}_{\partial M}$ representing the space of Cauchy data. As the symplectic form restricted to C is degenerate, one could perform another symplectic reduction of the coisotropic submanifold by its kernel in order to obtain the real physical space, also called the *reduced phase space*. In this case the vector fields in the kernel of this symplectic form represent the gauge transformations[6]. As the quotient space obtained by this symplectic reduction is often a singular space in local field theory, the BFV formalism provides a good alternative to represent this space. Indeed, the main goal of BFV is to describe the quotient space cohomologically instead of performing the symplectic reduction of the coisotropic submanifold C. In very concise words, the BFV formalism produces a complex whose cohomology in degree zero is the space of functions on the quotient space (i.e. on the reduced phase space of the theory), endowed with a Poisson bracket coming

[3] In fact, the Gauss law obtained from the Maxwell's equations is not an evolution equation.
[4] $L_M := \pi_M(EL_M)$
[5] **Coisotropic submanifold:** Given a symplectic vector space (V,ω), a subspace $U \subset V$ is *coisotropic* if $U^{\perp} \subset U$ where $U^{\perp} = \{w \in V | \omega(w, v) = 0 \forall u \in U\}$
[6] One has to bear in mind that the kernel of the symplectic form restricted to C is spanned by the Hamiltonian vector fields of the constraints. The kernel is also called characteristic distribution. See Next Section: BFV formalism.

from the symplectic form on the reduction, as explained in the next section.

$C_{\partial M}$ **for Palatini-Cartan:** Without doing any calculation, let us sketch the construction of the coisotropic submanifold for Palatini-Cartan formulation of General Relativity. As already aforementioned, from the pre-boundary one-form obtained from the variation of the Palatini action (3), $\tilde{\alpha}_\partial = \frac{1}{2}\int_\Sigma ee\delta\omega$[7], one can define the pre-symplectic two-form $\tilde{\omega} = \delta\tilde{\alpha}_\partial$. Rememeber that the kernel of this latter form is given by the vector fields X acting as $\omega \to \omega + v$ such that $ev = 0$. When we perform the pre-symplectic reduction, the geometric phase space is the space of the coframe e and of the equivalence classes of ω, denoted $[\omega]_e$, under the e-dependent relation above. The next step will be the identification of the coisotropic submanifold, but one needs to be careful about the definitions of the constraints of the theory. The problem arises because one wants to identify a *unique* representative ω for each equivalence class $[\omega]_e$ under the e-depedent transformation and this can be achieved by imposing a so-called structural constraint. Namely, the constraints are the restriction to the boundary of the Euler-Lagrange equation (4) and (5), but they are not invariant under the v-translation generated by the vector in the kernel of the pre-symplectic form. So we might opt to look for a way to fix these translation completely. As argued in [4], the constraint (5), $eF_\omega = 0$, is invariant under the v-translation once we use the other constraint $d_\omega e = 0$[8]. It is important to observe that this latter constraint splits into two different parts: the *invariant* constraint $ed_\omega e = 0$, (invariant under a v-translation) and a remaining part (of $d_\omega e = 0$) called *structural* constraint. By making a choice of this latter constraint, we can fix the v-translations completely. Namely, one can choose a section e_n of $\mathcal{V}|_\Sigma$ for a completion of the basis (e_1, e_2, e_3) such that $e_n d_\omega e = e\sigma$ (i.e. the structural constraint) where σ is a one-form taking values in $\mathcal{V}|_\Sigma$, $\sigma \in \Omega^1(\Sigma, \mathcal{V}|_\Sigma)$. In other words, this choice of the structural constraint fixes a unique representative ω for each equivalence class $[\omega]_e$ [9]. Thus, this constraint is imposed on the space of pre-boundary fields, allowing us to take the restrictions of the Euler-Lagrange equations to the boundary. Namely, the set of constraints on the geometric phase space is[10]:

$$L_c = \int_\Sigma ced_\omega e$$

$$J_\mu = \int_\Sigma \mu(eF_\omega + \frac{1}{3!}\Lambda e^3)$$

[7] the wedge product is omitted.

[8] Remember that $ev = 0$ and $d_\omega e = 0$ by the second constraint, the variation w.r.t v is: $\delta_v(eF_\omega) = ed_\omega v = d_\omega(ev) - d_\omega ev = 0$.

[9] Let consider the following linear map: $W_k^{\partial,(i,j)}(X) = X \wedge e \wedge ... \wedge e$, where wedge product repeated k times ($W_k^{\partial,(i,j)} : \Omega_\partial^{i,j} \to \Omega_\partial^{i+k,j+k}$). After choosing a section e_n that is a completion of (e_1, e_2, e_3), it can be proved that the constraint $d_\omega e = 0 \iff ed_\omega e = 0$ and $e_n d_\omega e \in \mathrm{Im} W_1^{\partial,(1,1)}$. The second requirement is equivalent to the existence of $\sigma \in \Omega_\partial^{1,1}$ such that $e_n d_\omega e = e\sigma$. Moreover, it can be shown that it fixes a *unique* representative $\omega \in [\omega]_e$ without imposing any other condition on $[\omega]$. See [4] and [5] for a formal derivation.

[10] $\omega \in [\omega]_e$ is the representative one for each class, i.e. satisfying the structural constraint.

where c and μ are Lagrange multipliers. As it turns out that these constraints are first-class, a coisotropic submanifold is then defined, considering the symplectic form defined in the geometric phase space.

4 BFV formalism

Given a symplectic manifold (M, ω), we aim to achieve a cohomological description of the symplectic reduction of coisotropic submanifold, called quotient space. As shown in the last section, the symplectic manifold in our case is the $(\mathcal{F}^{\partial}_{\partial M}, \omega_{\partial})$ and the coisotropic submanifold is $C \subset \mathcal{F}^{\partial}_{\partial M}$, the space of Cauchy data. Let be the coisotropic submanifold of (M, ω) constituted by zero locus of a set of independent constraints ϕ_i. Hence, $C = \{x \in M \text{ such that } \phi_i = 0 \ \forall i\}$. First of all, it is important to observe that the coisotropy of the submanifold implies that the constraints are first-class, i.e. $\{\phi_i, \phi_j\} = f^k_{ij}\phi_k$, where the brackets are the Poisson brackets induced by the symplectic form [11] and the $f^k_{ij} \in C^\infty(M)$ (smooth function on M). Afterwards, the restriction of the symplectic form to C is degenerate and its kernel is spanned by the Hamiltonian vector field X_i of the constraint ϕ_i. Instead of performing the symplectic reduction of C by this kernel defining the symplectic quotient $\underline{C}$, we try to describe $\underline{C}$ cohomologically. In order to achieve this description, one states that the original symplectic manifold is the *body* of a symplectic supermanifold $\mathcal{F}$ with an additional ($\mathbb{Z}$-)grading endowed with an appropriate symplectic form $\omega_{\mathcal{F}}$. The additional internal grading is called *ghost number*. Namely, the symplectic manifold (M, ω) is extended to a graded (super)manifold $\mathcal{F} = M \times T^*W$, where W is an odd vector space. So, new odd variables are added, one for each constraint ϕ_i: c_i, called *ghost* with ghost number $+1$, and their conjugate momenta b_i, called *anti-ghost* as their ghost number is -1. Some *anti-fields* (ghost number -1) corresponding to conjugate momenta of the classical fields are also added. One often introduces other extra variables: some extra ghosts with ghost number greater than 1 and consequently some extra anti-ghosts with ghost number less than -1. As one aims to obtain the structure of a symplectic supermanifold, the classical symplectic form ω is also extended to a more general one involving the extra fields: $\omega_{\mathcal{F}} = \omega + \delta b_i \delta c^i$ (degree of $\omega_{\mathcal{F}}$ is zero). Moreover, a definition of a local functional [12] is introduced: $S = c^i\phi_i + \frac{1}{2}f^k_{ij}b_k c^i c^j +$ higher terms. S is a function of degree $+1$ and it has to satisfy the so-called BFV master equation $(S, S) = 0$, where $(,)$ are the odd-Poisson bracket defined by the symplectic form $\omega_{\mathcal{F}}$. The last step is the definition of a cohomological vector fields Q on $\mathcal{F}$ which is the Hamiltonian vector field of S with degree 1 with respect to $\omega_{\mathcal{F}}$

[11] Let a symplectic manifold (M, ω), the Hamiltonian vector field of a function H on M is defined as: $\iota_{X_H} = dH$ where ι denotes the *contraction operator*: $\iota : \Omega^p(M) \to \Omega^{p-1}(M)$ and $\Omega^p(M)$ is the set of all p-differential form on M. H is called *Hamiltonian*. So, $\{f, g\} = \omega(X_f, X_G)$ and $X_{f,g}$ are the Hamiltonian vector fields associated respectively to the functions f, g.

[12] A formal definition of local functional, or more generally of a local form, will be given in section 6. Conceptually, a local functional associated to a supermanifold plays the same role of an action functional associated to a ordinary manifold.

and it satisfies $[Q,Q]=0$[13].
This method defines an alternative way to look at the symplectic quotient. Indeed, it manages to replace the symplectic reduction of the coisotropic submanifold, which can be singular, with the BFV data ($\mathcal{F}=M\times T^*W$, $\omega_{\mathcal{F}}$, S) and it produces a complex ($C^\infty(M\times T^*W)$, Q) whose cohomology in degree zero is the space of function on the symplectic quotient[14].

In conclusion the BFV formalism consists in the assignment of the BFV collection of data $(\mathcal{F}_{\partial M}, S_{\partial M}, \omega_{\partial M}, Q_{\partial M})$ to boundary theory.[15]

5 BV and BFV formalism

Similar to the BFV formalism, a BV theory on the bulk manifold associated to a local field theory is the assignment of the data $(\mathcal{F}_M, S_M, \omega_M, Q_M)$ where $(\mathcal{F}_M, \omega_M)$ is a ($\mathbb{Z}$-)graded symplectic manifold (degree of ω_M equal -1), S_Q a function of degree 0 and Q_M is cohomological vector field of degree 1, i.e $[Q_M, Q_M]=0$ and also the corresponding Hamiltonian vector field of the function S_M, $\iota_{QM}\omega_M=\delta S_M$. Moreover, the symplectic structure induces a odd-Poisson bracket $(,)$ and S_M satisfies the Classical Master Equation $(S_M, S_M)=0$.

Summarizing, we recovered two independent theories: one for the bulk (BV theory) and one for the boundary (BFV theory). A question which arises naturally is to know how a BFV theory on the boundary is related to a BV theory on the bulk. In many situations, when a theory with a manifold with boundaries is concerned, it is shown that compatible BFV theory on the boundary can be derived from a BV description on the bulk by a surjective submersion $\pi_M : \mathcal{F}_M \to \mathcal{F}_{\partial M}$. This relation is achieved by using a method equivalent to the approach implemented in the previous section to construct the space of true boundary fields (the Kijowski-Tulczyjew approach). Conceptually, this method provides a way to derive the (non-classical) BFV space on the boundary from (non-classical) BV space on the bulk.
Starting from the bulk, the local functional S_M satisfies $\iota_{QM}=\delta S_M$. In presence of the boundary this latter equation is no more true due to a non-zero contribution on the boundary generated by the integration by parts. Considering the natural surjective submersion $\tilde{\pi}:\mathcal{F}_M\to\tilde{\mathcal{F}}_{\partial M}$ which restricts the fields and their jets (the derivatives) to the boundary, i.e. the pre-boundary space,

[13] This is the reason why is called cohomological vector fields. It can be shown that every cohomological vector field on a supermanifold $\mathcal{F}$ is equivalent to a differential on the graded algebra of the function $C^\infty(\mathcal{F})$. See [10]

[14] To be more precisely: the cohomology in degree 0 is isomorphic as a Poisson algebra to the algebra of functions on the coisotropic submanifold C which are invariant under the vector fields spanning the kernel of the symplectic form (invariant under the characteristic distribution), i.e. the algebra of the functions on the reduced phase space

[15] The boundary is considered as a closed manifold

the equation $\iota_{Q} M = \delta S_M$ becomes:

$$\iota_{Q_M}\omega_M = \delta S_M + \tilde{\pi}^*\tilde{\alpha}$$

where the boundary terms is the pullback of a one-form $\tilde{\alpha}$ on $\tilde{\mathcal{F}}_{\partial M}$, called the pre-boundary one-form. Consider the pre-boundary two-form $\tilde{\omega} = \delta\tilde{\alpha}$. If this form is pre-symplectic, i.e. its kernel form has a constant rank, the true space of boundary fields is achieved by the following symplectic reduction:

$$\mathcal{F}^{\partial}_{\partial M} = \tilde{\mathcal{F}}_{\partial M}/ker(\tilde{\omega})$$

with the corresponding surjective submersion $\pi : \tilde{\mathcal{F}}_{\partial M} \to \mathcal{F}^{\partial}_{\partial M}$, a symplectic reduction map. Taking the compisition of $\tilde{\pi}$ with π, we define the surjective submersion $\pi_M = \pi \circ \tilde{\pi}$ and the reduced two-form $\omega^{\partial}_{\partial M} := \underline{\tilde{\omega}}$ is a symplectic form of degree 0. Furthermore, the cohomological vector field Q_M projects to a cohomological vector field $Q_{\partial M}$ on the space of boundary fields $\mathcal{F}^{\partial}_{\partial M}$ which will be the Hamiltonian vector field of the function $S_{\partial M}$, i.e. the boundary action. Thus, when a d-dimensional manifold with boundary is considered $(M, \partial M)$, a BV-BFV theory is the assignment of a BV-BFV pair $(\mathcal{F}_M, \mathcal{F}^{\partial}_{\partial M})_{\pi_M}$ where π_M is the fibration, $\mathcal{F}^{\partial}_{\partial M}$ is smooth and it can be obtained by the above construction. Thus we see that a BV-theory of local field theory can be extended to a BV-BFV theory on the manifold with boundaries.
By iterating this machinery in order to relate a BFV theory to a theory describing the corners and so on, a BF-BFV structure can be delineated when a manifold carries several strata. Our goal is to study the consequences of the ambiguity of the symplectic potential θ, considered in this context as a local form, when a change by an exact (local) form is performed and try to figure out how the entire BV-BFV structure "reacts" under this variation. This question will be treated in the next section.
In conclusion, the goal of BV-BFV formalism is *to provide a cohomological description of gauge theories* over a manifold with boundaries theories as we are able to describe at the same time what happens in the bulk and in the boundary of a local field theory. However, is important to notice that gauge symmetries are considered because of most interest in modern physics, but such construction is valid for more general symmetry.
Moreover, it can be possible to show that BV-BFV formalism proceeds to an approach to quantization of gauge theories.

6 Cohomological ambiguites in BV-BFV setting

The same cohomological ambiguity is treated in the context of BV-BFV formalism. In this section this latter ambiguity is implemented for any general theory, not only for General Relativity. Before that, some more technical definitions are briefly presented (following [2]) in order to formally describe the entire BV-BFV structure associated to a local field theory when a manifold carrying a stractification is considered.

The space of the *fields*, $\mathcal{F}$, will be identified as the space of smooth section of a graded vector bundle over the manifold M. Moreover, combining together the manifold M and $\mathcal{F}$, one obtains the structure of *variational bicomplex* where two exterior operators are defined: the horizontal derivative d_H which is identified with the d de Rham differential on (on forms on) the ordinary manifold M and the vertical exterior derivative d_V associated to δ de Rham differential on (on forms on) the space of fields $\mathcal{F}$ (see [7]). We will be interested in forms on the variational bicomplex which are local. Schematically this construction can be seen in the Figure 1.

$$\begin{array}{ccccccccccccc}
 & & & & \uparrow{\scriptstyle d_V} & & & & & & & & \uparrow{\scriptstyle d_V} \\
 & & & 0 \longrightarrow & \Omega^{0,3} & & & & & \cdots & & & \Omega^{n,3} \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle d_V} \\
 & & & 0 \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \Omega^{2,2} & \xrightarrow{d_H} \cdots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & & 0 \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \Omega^{2,1} & \xrightarrow{d_H} \cdots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \Omega^{2,0} & \xrightarrow{d_H} \cdots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0}
\end{array}$$

Figure 1: The variational bicomplex for the fibered manifold $\pi : E \longrightarrow M$: $(\Omega^{\bullet,\bullet}_{loc}(\mathcal{F} \times M), \delta, d) := (j^\infty)^*(\Omega^{\bullet,\bullet}(J^\infty(E), d_V, d_H)$

More precisely, a *local* form on a (possibly) graded vector bundle $E \longrightarrow M$ on a m-dimensional manifold M, is an element of

$$(\Omega^{\bullet,\bullet}_{loc}(\mathcal{F} \times M), \delta, d) := (j^\infty)^*(\Omega^{\bullet,\bullet}(J^\infty(E), d_V, d_H)$$

where $\mathcal{F} := \Gamma^\infty(M, E)$, j^∞ is the limit of the maps $\{j^p : \mathcal{F} \times M \longrightarrow J^pE\}$, J^pE the p-the jet bundle of E, $J^\infty E$ is the limit of the sequence $\{E \equiv J^0E \leftarrow J^1E \leftarrow .... \leftarrow J^pE \leftarrow ...\}$ and $\Omega^{\bullet,\bullet}_{loc}(\mathcal{F} \times M)$ is endowed with the differentials

$$\delta(j^\infty)^*\alpha := (j^\infty)^* d_V\alpha$$

$$d(j^\infty)^*\alpha := (j^\infty)^* d_H\alpha$$

A *local* form is called *Local Functional* if it is an element of $\Omega^{0,\bullet}_{loc}(\mathcal{F} \times M)$, the first column in the Figure 1.

If the manifold M carries a stratification $\{M^{(k)}\}_{k=0...m}$ [16](boundary, corners, ect., called strata), *local* forms and *local* functional can be integrate and

[16] A stractification of manifold M is a filtration of M by CW-complexes $\{M^{(k)}\}_{k=0...m}$ such that for each codimension k, $M^{(k)} \backslash M^{(k+1)}$ is a smooth $(m-k)$-dimensional manifold.

this will allow us to obtain the usual BV-BFV data at each strata.
Thus an *integrated local form* on $E \longrightarrow M$ is the integral along an $(m-k)$-dimensional (sub)manifold $M^{(k)} \longrightarrow M$ of an element of $\Omega^{\bullet, m-k}_{loc}(\mathcal{F} \times M)$. The complex of integrated local forms is $(\Omega^{\bullet}_{loc}(\mathcal{F}, M), \delta)$.

After the definitions of the space of the fields, the variational bicomplex ("ordinary manifold + $\mathcal{F}$ ") and the corresponding local forms on it, we are able to define the BV-BFV data for a local field theory, taking into account classical fields, ghosts, anti-ghosts and anti-fields in the space of fields $\mathcal{F}$.

Definition: A (*lax*) *BV-BFV theory* is the assignment, to a (m-dimensional) manifold M, of the data:

$$\mathfrak{F} = (\mathcal{F}, L^{\bullet}, \theta^{\bullet}, Q)$$

with

- $\mathcal{F}$ the space of C^{∞} sections of a graded bundle (or sheaf) $E \longrightarrow M$, the space of *fields*
- $L^{\bullet} \in \Omega^{0,\bullet}_{loc}(\mathcal{F} \times M)$, a local functional of total degree 0,
- $\theta^{\bullet} \in \Omega^{1,\bullet}_{loc}(\mathcal{F} \times M)$, a one-form of total degree -1,
- $Q \in \mathfrak{X}_{evo}(\mathcal{F} \times M)[1]$ a degree-1, evolutionary, cohomological vector field on $\mathcal{F}$, i.e. $[L_Q, d] = 0$ (evolutionary) and $[Q, Q] = 0$ (cohomology condition for Q),

such that

$$\iota_Q \omega^{\bullet} = \delta L^{\bullet} + d\theta^{\bullet} \tag{14}$$

$$\frac{1}{2} \iota_Q \iota_Q \omega^{\bullet} = dL^{\bullet} \tag{15}$$

where $\omega^{\bullet} := \delta \theta^{\bullet}$.

$\omega^{\bullet}, \theta^{\bullet}, L^{\bullet}$ are local forms defined on a graded vector bundle $E \longrightarrow M$ on an m-dimensional ordinary manifold M with values in inhomogeneous differential forms on M.
A BV-BFV theories in the sense of the above definition in the case of Palatini-Cartan formalism can be defined as described in [3] and [4] and for Einstein-Hilbert General Relativity in [6].

Cohomological ambiguities For the purpose of this study the invariance of the equations (14) e (15) for a general BV-BFV theory under the following transformations has to be verified:

- f-transformation : $L^{k+1} \to L^{k+1} + d(\eta^k)$ and $\theta^k \to \theta^k + \delta\eta^k$.
- Cohomological ambiguity: $\theta^k \to \theta^k + d(\eta^{k-1})$

where k is the differential form index, not the codimension.

Remark: The De Rham differential δ defined on the space of field $\mathcal{F}$ (the space of C^∞ sections of a graded bundle) and the contraction ι_Q (as Q is defined on $\mathcal{F}$, not on M)leave the differential form index k invariant.

This procedure will apply also in the case of BV-BFV theory of Palatini-Cartan formulation of General Relativity defined in [3], [4].
Additionally, it is important to notice that in the case of 4-dimensional manifold ($m = 4$) if $k = m - 1 = 3$ and $k - 1 = m - 2 = 2$ it refers respectively to the boundary and to the corners and one recovers exactly the case presented in the first part of this study. Then $\theta^k = \theta^{(3)} = \theta_i$ (i= g or t) will be the symplectic potential defined on the boundary and $\eta^{k-1} = \eta^{(2)} = \alpha$ dressing 2-form (corner) considered in [1]. In conclusion, the same cohomological issue is considered in BV-BFV context.

6.1 Cohomological ambiguity $\theta^k \to \theta^k + d(\eta^{k-1})$

In the following the cohomological ambiguity of the symplectic potential pointed out in [1] is explicitly calculated in BV-BFV formalism.
Let $\theta^k \to \theta^k + d(\eta^{k-1})$ (the round brackets will be omitted). By definition $\omega^k = \delta\theta^k$ becomes :

$$\tilde{\omega}^k = \delta\tilde{\theta}^k = \delta\theta^k + \delta d\eta^{k-1} = \omega^k - d\delta\eta^{k-1}$$

Considering the first equation (14) of the BV-BFV structure:

$$\iota_Q\tilde{\omega}^k = \iota_Q(\omega^k - d\delta\eta^{k-1}) = \iota_Q\omega^k - \iota_Q d\delta\eta^{k-1} = \delta L^k + d\theta^{k-1} - d\iota_Q\delta\eta^{k-1}$$

It can be rewritten as:

$$\iota_Q\tilde{\omega}^k = \delta L^k + d(\theta^{k-1} - \iota_Q\delta\eta^{k-1})$$

We conclude the following:

$$\boxed{\theta^{k-1} \to \theta^{k-1} - \iota_Q\delta\eta^{k-1}} \tag{16}$$

From this last equation, one can infer that cohomological ambiguity corresponding to a certain strata reflects in a cohomological ambiguity of the potential at a strata of codimension +1. Then a question which arises naturally is how the BV-BFV data of this latter strata changes. Given the transformation (16), how do ω^{k-1} and L^{k-1} transform? ($\omega^{k-1} \to ?$ and $L^{k-1} \to ?$) Which assumptions on η are needed in order to have an invariant theory?

•$\boldsymbol{\omega^{k-1}}$: By definition: $\omega^{k-1} := \delta\theta^{k-1}$. If $\tilde{\theta}^{k-1} = \theta^{k-1} - \iota_Q \delta\eta^{k-1}$, we obtain:

$$\tilde{\omega}^{k-1} = \delta(\theta^{k-1} - \iota_Q \delta\eta^{k-1}) = \omega^{k-1} - \delta\iota_Q\delta\eta^{k-1}$$

Given the definition of the Lie derivative L_Q, $\delta\iota_Q = \iota_Q\delta - L_Q$ and thus $\delta\iota_Q\delta = (\iota_Q\delta - L_Q)\delta = -L_Q\delta$, the above equation can be written as:

$$\boxed{\tilde{\omega}^{k-1} = \omega^{k-1} + L_Q\delta\eta^{k-1}} \tag{17}$$

•$\boldsymbol{L^{k-1}}$: By using the second equation (15) of the BV-BFV theory and the commutation $\iota_Q d = d\iota_Q$: $\frac{1}{2}\iota_Q\iota_Q\tilde{\omega}^k = dL^{k-1} - \frac{1}{2}\iota_Q\iota_Q d\delta\eta^{k-1} = dL^{k-1} - \frac{1}{2}d(\iota_Q\iota_Q\delta\eta^{k-1})$. So,

$$\boxed{\tilde{L}^{k-1} = L^{k-1} - \frac{1}{2}\iota_Q\iota_Q\delta\eta^{k-1}} \tag{18}$$

6.1.1 Compatibility of the BV-BFV structure

The transformation $\theta^k \to \theta^k + d\eta^{k-1}$ (so $\omega^k \to \omega^k - d\delta\eta^{k-1}$) implies the following transformations at the strata of codimension +1 (corresponding to a term of form degree $k-1$):

- $\theta^{k-1} \to \theta^{k-1} - \iota_Q\delta\eta^{k-1}$
- $\omega^{k-1} \to \omega^{k-1} + L_Q\delta\eta^{k-1}$
- $L^{k-1} \to L^{k-1} - \frac{1}{2}\iota_Q\iota_Q\delta\eta^{k-1}$

In order to prove the compatibility of these transformations with the BV-BFV structure, the invariance of the equations (14) e (15) and the invariance of the condition $L_Q\omega^k = d\omega^{k-1}$, called *failure of Q-invariance*, have to be verified.

Structure's equations: Applying the above transformations $\tilde{\omega}^{k-1}$ e $\tilde{L}^{k-1}$ to the **first** equation $\iota_Q\omega^{k-1} = \delta L^{k-1} + d\theta^{k-2}$:

$$\iota_Q\tilde{\omega}^{k-1} = \delta\tilde{L}^{k-1} + d\theta^{k-2} \tag{19}$$

Combining L.H.S and R.H.S together given in Appendix 8.2:

$$\iota_Q\omega^{k-1} - \cancel{\delta\iota_Q L_Q\eta^{k-1}} = \delta L^{k-1} - \cancel{\delta\iota_Q L_Q\eta^{k-1}} + d\theta^{k-2}$$

We conclude that the equation (14) is invariant: **LHS =RSH** $\Longrightarrow \iota_Q\omega^{k-1} = \delta L^{k-1} + d\theta^{k-2}$.

Concerning the **second** equation (15):

$$\frac{1}{2}\iota_Q\iota_Q\tilde{\omega}^{k-1} = \frac{1}{2}\iota_Q\iota_Q(\omega^{k-1} + L_Q\delta\eta^{k-1}) = dL^{k-2} + \frac{1}{2}\iota_Q\iota_Q L_Q\delta\eta^{k-1}$$

Thus the second term of the R.H.S can be written as, using the identity given in Appendix 8.3:

$$\frac{1}{2}\iota_Q\iota_Q L_Q\delta\eta^{k-1} = -\frac{1}{2}\iota_Q\delta\iota_Q\delta\iota_Q\eta^{k-1} = 0$$

In fact it can be noticed that the first contraction of the operator $\iota_Q\delta\iota_Q\delta\iota_Q$ applied to η^{k-1} is a function, i.e. $\iota_Q\eta^{k-1} = f$. Moreover, $\iota_Q\delta\iota_Q\delta\iota_Q\eta^{k-1} = \iota_Q\delta\iota_Q\delta f = \iota_Q\delta L_Q f = \iota_Q\delta g = L_Q g = L_Q(L_Q f) = L_Q^2 f = 0$.

Failure of Q-invariance $L_Q\omega^k = d\omega^{k-1}$**:** the transformed equation is

$$L_Q\tilde{\omega}^k = d\tilde{\omega}^{k-1}$$

L.H.S: $L_Q\tilde{\omega}^k = L_Q(\omega^k - d\delta\eta^{k-1}) = L_Q\omega^k + dL_Q\delta\eta^{k-1}$, $(L_Q d = -dL_Q)$.

R.H.S: $d\tilde{\omega}^{k-1} = d(\omega^{k-1} + L_Q\delta\eta^{k-1}) = d\omega^{k-1} + dL_Q\delta\eta^{k-1}$.

Thus **L.H.S** = **R.H.S**:

$$L_Q\omega^k + \cancel{dL_Q\delta\eta^{k-1}} = d\omega^{k-1} + \cancel{dL_Q\delta\eta^{k-1}} \tag{20}$$

The equation is invariant.

6.1.2 Other relations of the BV-BFV structure

For completion, the other relations between different strata are taken into account and their invariance is verified.

$\boldsymbol{L_Q\omega^{k-1} = d\omega^{k-2}}$: The R.H.S of this equation remains invariant as θ^{k-2} does not change.
Using $L_Q^2 = 0$, the L.H.S is: $L_Q\tilde{\omega}^{k-1} = L_Q(\omega^{k-1} + L_Q\delta\eta^{k-1}) = L_Q\omega^{k-1}$.

$\boldsymbol{L_Q\omega^{k+1} = d\omega^{k}}$: The L.H.S is invariant as ω^{k+1} does not change.
The R.H.S is also invariant as it transforms as: $d\tilde{\omega}^k = d(\omega^k - d\delta\eta^{k-1}) = d\omega^k$.

The variation of the (symplectic) local form $\theta^\bullet$ by an d-exact local form in the space of fields is then absorbed by the BV-BFV structure. After a strictification of the theory is defined, the entire BV-BFV structure remains invariant under this cohomological ambiguity for any arbitrary η. Moreover, it is important to notice that no special assumptions are required for η. This cohomological ambiguity is generated by any arbitrary η, independently on this additional form. These calculations can be applied to Palatini-Cartan(-Holst) General Relativity defined in [3],[4].

6.1.3 f-transformation

In order to give a more complete description of cohomological ambiguities, this part deals with the f-transformation, i.e $L^{k+1} \to L^{k+1}+d(\eta^k)$ and $\theta^k \to \theta^k+\delta\eta^k$. In this case, the first equation (14) of the BV-BFV structure is the only equation considered :

$$\iota_Q\omega^{k+1} = \delta(L^{k+1}+d\eta^k)+d(\theta^k+\delta\eta^k) = \delta L^{k+1}+\delta d\eta^k+d\theta^k+d\delta\eta^k = \delta L^{k+1}+d\theta^k$$

7 Conclusion

In [1], the cohomological ambiguity of the symplectic potential is studied in order to solve the mismatch in metric-tetrad formalisms between the Noether and Hamiltonian charges. The symplectic potential θ_t (t = tetrad) is dressed by the so-called dressing 2-form α. When this is added, the Noether charge for diffeomorphism calculated in tetrad coincide with the respectively one in metric. The same is true for the Hamiltonian charges for diffeomorphism in tetrad-metric formulations. The charges associated to internal Lorentz symmetry of tetrads are set to zero by the dressing 2-form.

As shown in [1] , after a non-trivial calculation of the charges for diffeomorphism in both formalisms (which coincide once the dressign form is chosen) and the charges associated to the additional internal Lorentz symmetry in tetrad formalism only (which are set to zero by choosing the dressing form), the equivalence appears to be restored.

On the other hand, using the Kijowski-Tulczyjew geometrical approach to construct the space of boundary fields starting from the bulk, the symplectic potential can be considered as a one-form on the space of true boundary fields. Inside this geometric phase space, a coisotropic submanifold - the space of Cauchy data - is identified. At this point the reduced phase space, or the real physical space, could be achieved by performing a symplectic reduction of the coistotropic submanifold, but one prefers to use BFV formalism to represent this space *cohomologically*, in a non-classical way. As BV theory yields to a similar cohomological description on the bulk manifold, a relation between BV data and BFV data can be established by using a method equivalent to the geometrical approach as constructed in section 5. Hence, a BV-BFV theory for a manifold with boundary is achieved. Moreover, a BV-BFV structure can be obtained for a manifold which carries a stractification by iterating this process. Consequently, the same cohomological ambiguity of the symplectic potential is investigated in BV-BFV formalism. In section 6, it is proved that this cohomological ambiguity is irrelevant, even without imposing any condition on the arbitrary additional exact form: once a BV-BFV theory and a strictification of it are defined the entire structure is invariant under this ambiguity. We conclude that if a BV-BFV for Palatini-Cartan General Relativity is given, as in [3] and [4], and the associated symplectic local form is changed by any arbitrary d-exact local form, this difference leave the theory invariant.

In conclusion, in the context of BV-BFV setting the cohomological ambiguity of the symplectic potential associated to the boundary, considered in [1], can be interpreted as a cohomological ambiguity of the symplectic potential defined on the corners, whose BV-BFV structure's equations remain invariant without imposing any further conditions. Furthermore, the cohomological ambiguity can be generated by any arbitrary exact form as no special assumptions are required and it is absorbed by the BV-BFV structure.

8 Appendix

8.1 Some useful relations

- $L_Q = \iota_Q \delta - \delta \iota_Q$ implies $\iota_Q \delta = L_Q + \delta \iota_Q$ and $\delta \iota_Q = \iota_Q \delta - L_Q$
- $\delta d = -d\delta$
- $\delta L_Q = -L_Q \delta$
- $L_Q d = -d L_Q$
- $\iota_Q d = d \iota_Q$
- $\iota_Q L_Q = L_Q \iota_Q$
- $L_Q^2 = d^2 = \delta^2 = 0$

8.2 Invariance of $\iota_Q \omega^{k-1} = \delta L^{k-1} + d\theta^{k-2}$

Calculating separately L.H.S and R.H.S:

L.H.S:

$$\begin{aligned} \iota_Q \tilde{\omega}^{k-1} &= \iota_Q \omega^{k-1} + \iota_Q L_Q \delta \eta^{k-1} \\ &= \iota_Q \omega^{k-1} - \iota_Q \delta L_Q \eta^{k-1} \\ &= \boldsymbol{\iota_Q \omega^{k-1} - \delta \iota_Q L_Q \eta^{k-1}} \\ &= \delta(L^{k-1} - \iota_Q L_Q \eta^{k-1}) + d\theta^{k-2} \end{aligned} \tag{A.1}$$

where the following relation is used:

- $\boxed{\iota_Q \delta L_Q = \delta \iota_Q L_Q = \delta L_Q \iota_Q}$.

proof: $\iota_Q \delta L_Q = (L_Q + \delta \iota_Q) L_Q = \cancel{L_Q^2} + \delta \iota_Q L_Q = \delta \iota_Q L_Q$ and $\delta \iota_Q L_Q = \delta L_Q \iota_Q$

R.H.S: Considering the first term $\delta\tilde{L}^{k-1}$

$$
\begin{aligned}
\delta\tilde{L}^{k-1} &= \delta(L^{k-1} - \frac{1}{2}\iota_Q\iota_Q\delta\eta^{k-1}) \\
&= \delta L^{k-1} - \frac{1}{2}\delta\iota_Q\iota_Q\delta\eta^{k-1} \\
&= \delta L^{k-1} - \frac{1}{2}(\iota_Q\delta - L_Q)(L_Q + \delta\iota_Q)\eta^{k-1} \\
&= \delta L^{k-1} - \frac{1}{2}(\iota_Q\delta L_Q + \cancel{\iota_Q\delta^2\iota_Q} - \cancel{L_Q^2} - L_Q\delta\iota_Q)\eta^{k-1} \\
&= \delta L^{k-1} - \frac{1}{2}(\iota_Q\delta L_Q - L_Q\delta\iota_Q)\eta^{k-1} \\
&= \delta L^{k-1} - \frac{1}{2}(\iota_Q\delta L_Q + \delta L_Q\iota_Q)\eta^{k-1} \\
&= \delta L^{k-1} - \frac{1}{2}(2\iota_Q\delta L_Q)\eta^{k-1} \\
&= \delta L^{k-1} - \iota_Q\delta L_Q\eta^{k-1} \\
&= \boldsymbol{\delta L^{k-1} - \delta\iota_Q L_Q\eta^{k-1}} \\
&= \delta(L^{k-1} - \iota_Q L_Q\eta^{k-1})
\end{aligned} \tag{A.2}
$$

$$\textbf{RHS} = \delta L^{k-1} - \delta\iota_Q L_Q\eta^{k-1} + d\theta^{k-2} \tag{A.3}$$

8.3 Operator $\iota_Q\iota_Q L_Q\delta$

Using the definition of the Lie derivative ($L_Q = \iota_Q\delta - \delta\iota_Q$), it can be noticed that:

$$
\begin{aligned}
\iota_Q\iota_Q L_Q\delta &= \iota_Q\iota_Q(\iota_Q\delta - \delta\iota_Q)\delta \\
&= -\iota_Q\iota_Q\delta\iota_Q\delta \\
&= -\iota_Q(L_Q + \delta\iota_Q)(L_Q + \delta\iota_Q) \\
&= -\iota_Q(\cancel{L_Q^2} + L_Q\delta\iota_Q + \delta\iota_Q L_Q + \delta\iota_Q\delta\iota_Q) \\
&= -\iota_Q(L_Q\delta\iota_Q - L_Q\delta\iota_Q + \delta\iota_Q\delta\iota_Q) \\
&= \boldsymbol{-\iota_Q\delta\iota_Q\delta\iota_Q}
\end{aligned} \tag{A.4}
$$

where $\delta\iota_Q L_Q = \delta L_Q\iota_Q = -L_Q\delta\iota_Q$.

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