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Generalized Reduction and Spinor Decomposition

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Introduction

0.1 Summary

This thesis records the results of an effort to understand the reduction process of Courant algebroid, Dirac structures and generalized complex structures first discussed in [6], the same process in the viewpoint of pure spinors first studied in [15] and the decomposition of spinors of generalized complex structure analyzed in [16, 10, 9]. Further work is put into combining these theories into a particularly interesting example, where the hypothesis that initiated this thesis is that the reduction process "commutes" with the decomposition of spinors, in the sense that doing either first will yield the same results, is tested. In the attempt at this example, although there was no concrete pattern or trend, there are some encouraging signs. Some minor changes suggested in the last Chapter of this thesis can be made which might yield concrete results. Furthermore, this idea of relating reduction to other properties in generalized complex geometry can be advanced, i.e. analyzing the relationship between reduction and the Hodge decomposition of generalized complex manifold [17].

Generalized complex geometry, as introduced by Hitchin [18] and further studied by Gualtieri [16], provides a unifying framework for both symplectic and complex geometry. The general idea of this unification is to treat classical geometric operations on the sum of the tangent and cotangent bundle $T \oplus T^*$, also known as *generalized tangent bundle*, instead of the usual tangent bundle. Notably, there exists an integrability condition for such structures that, on one hand, specializes to the closure of the symplectic form, and on the other hand, to the vanishing of the Nijenhuis tensor of a complex structure. Both cases can be encompassed into a single unified integrability condition with respect to a bracket called the *Courant bracket*, an extension of the Lie bracket of vector fields to smooth sections of $T \oplus T^*$, first introduced by Courant and Weinstein[13, 22], during which they delved deep into studying Dirac structures. Dirac structures bridge the gap between Poisson bivector fields and closed 2-forms; thus, generalized complex geometry can be viewed as analyzing complex Dirac structures.

Since generalized complex geometry encompasses the properties of symplectic and complex geometry, entities and properties of complex geometry have their counterpart in the generalized complex realm, for example properties such as the differential operators ∂ and $\bar{\partial}$ and the (p, q) decomposition of forms or subtopics such as Kähler manifolds and Calabi-Yau manifolds. On a complex manifold, a complex $k-$form can be decomposed uniquely into its sum of $(p, q)-$forms, these are wedge products of p differentials of holomorphic coordinates with q differentials of their complex conjugates. Similarly,a generalized complex structure has an Lie algebra action on differential form that decomposes differential forms into eigenspaces with respect to action, as shown in [16] . Subsequently, an explicit description of the decomposition of forms by constructing a formula for the subbundles is established in [10]. The decomposition of forms is one of the two main area of focus of this thesis.

Analogous to known geometries such as symplectic or Poisson geometry, there is the notion of obtaining a generalized complex structure via the process of reduction. The framework for reduction on generalized complex structures has been formulated by several authors, we will follow the procedure provided in [6] and [15]. As expected, one can spot many resemblance between reduction of a symplectic structure [20] and that of generalized complex structures [6]. In the former, one would need a compact Lie group G acting on a symplectic manifold such that it preserves the symplectic structure and a coisotropic submanifold such that it is compatible with the action. This reduction formula is the well-known Marsden-Weinstein reduction. Analogously, one can perform the reduction of Courant algebroids, Dirac structures and generalized complex structures formulated in [6] with ingredients resembling of the symplectic reduction such as a compact Lie group G by automorphisms on the Courant algebroid E over M (this is similar to symplectic diffeomorphisms), an invariant submanifold $N \subseteq M$, and an equivariant isotropic subbundle $K \subset E$. Subsequently, the reduced exact Courant algebroid E_{red} over N/G [6] is constructed using the reduction data. After that, a map can be defined by sending every invariant Dirac structure $L \subset E$, such that $L|_N \cap K$ has constant rank to a reduced Dirac structure $L_{red} \subset E_{red}$, i.e.

$$L \mapsto L_{red}. \tag{1}$$

This procedure is then explored in a different language in [15], namely in the realm of pure spinors. Suppose one has a Dirac structure $L \subset E$ and an isotropic splitting $\nabla : TM \to E$ satisfying the invariance and closedness conditions, then a pure spinor counterpart of 1, i.e. the following explicit map can be constructed

$$\Gamma(U_\nabla(L)) \to \Gamma(U_{\nabla_{red}}(L_{red})), \ \varphi \mapsto \varphi_{red} \tag{2}$$

where U_∇ and $U_{\nabla_{red}}$ are pure spinor line bundles of L and L_{red} respectively, establishing a correspondence between pure spinors of L and L_{red} with an isotropic splitting ∇_{red} associated to the reduced exact Courant algebroid E_{red}. The viewpoint of pure spinors of the generalized reduction is the one of the main object of study in this thesis.

0.2 Outline of Thesis

Chapter 1 is dedicated to the discussion of the necessary machinery to define generalized complex structures. In particular, we review the properties of $V \oplus V^*$, maximal isotropic subspaces and pure spinors.

Chapter 2 is a review of the necessary properties of Courant bracket which ultimately leads to a brief introduction to Dirac structures. In particular, we introduce Courant algebroids which serve as the bundle on which generalized complex structures are defined. We then discuss the symmetries of Courant algebroids which play role in the reduction process. Subsequently, we briefly present Dirac structures and some of their properties.

In Chapter 3, we begin by showing linear generalized complex structure and give known examples that has the structure. We then introduce generalized almost complex structure, similarly to almost complex structure, which, by Courant involutivity, it turns into generalized complex structure.

Chapter 4 is where the first focus of the thesis begins, where we review the decomposition of spinors for generalized complex structure of complex and symplectic type.

Chapter 5 is a review of the reduction of Courant algebroids, Dirac structures and generalized complex structures. We begin by introducing extended actions and subsequently provide the reduction procedure for all the structures mentioned.

Chapter 6 is the second focus of the thesis, where we review the reduction process in Chapter 5 in the viewpoint of pure spinors. The chapter emphasizes the formula for reducing a pure spinor under certain conditions.

Chapter 7 is where we study the example of reducing a generlized complex structure of complex type into a symplectic type in the viewpoint of pure spinors and record how the spinor decomposition changes after reduction.

Chapter 8 discusses the problem faced and probable tweaks and solutions to further the main ideas developed in this thesis.

Contents

Chapter 1

Linear algebra on $V \oplus V^*$

In this chapter, we recall several important algebraic and differential geometric machinery required to understand and tackle the subject.

We first analyze the space of $V \oplus V^*$ and the complexification of it as a direct sum of vector spaces where several of the properties are translated to $T \oplus T^*$. Next, we present the idea of maximal isotropic subspaces which will be an ingredient in the definition of a generalized complex structure. Another ingredient that is involved in the definition are pure spinors which are also used in our reduction process later in the thesis.

This chapter originates from and follows the structure of Chapter 2 of [16], while we also look to [12] and [21] on the theory of spinors.

1.1 Symmetries of $V \oplus V^*$

Let V be a real vector space of dimension m and denote its dual space V^*. As the title of the chapter suggests, our focus will be concentrated on $V \oplus V^*$. We shall furnish $V \oplus V^*$ with the following natural symmetric and skew-symmetric bilinear forms:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)) \tag{1.1}$$

where $X, Y \in V$ and $\xi, \eta \in V^*$. To show nondegeneracy; suppose it is degenerate, i.e. there exists $X + \xi$ non-zero such that $\langle X + \xi, Y + \eta \rangle = 0$ for all $Y + \eta$. Now take $X = e_i$ and $\xi = f_j$ (without the lost of generality) for $1 \leq i, j \leq n$ where $\{e_i\}$ are the basis of V while the $\{f_j\}$ are basis of V^* dual to $\{e_i\}$. However note that, $\langle e_i + f_j, e_i + f_j \rangle = 1$ contradicting the degeneracy assumption. Lastly, this symmetric inner product is of signature (m, m)

Next, we move our discussion to special symmetries of $V \oplus V^*$. We first note that the following:

$$\mathfrak{so}(V \oplus V^*) = \{T \in End(V \oplus V^*) | \langle Tx, y\rangle + \langle x, Ty\rangle = 0, \forall x, y \in \mathbb{V}\} \tag{1.2}$$

is the Lie algebra of $SO(V \oplus V^*)$. We may then decompose elements as follows via the splitting of $V \oplus V^*$ to the following transformation:

$$T = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} \tag{1.3}$$

where A being the endomorphism of V, B a linear map from V to V^*, and β a linear map from V^* to V such that $B^* = -B$ and $\beta^* = -\beta$, i.e. they are skew. Alternatively, we can take B as a 2-form while β a bivector. Due to this, we can derive that $\mathfrak{so}(V \oplus V^*) = End(V) \oplus \wedge^2 V^* \oplus \wedge^2 V$. subsequently, we can obtain orthogonal symmetries of $V \oplus V^*$ that might play a role in this thesis.

Example 1.1. Let B be given as above, and consider $X + \xi \in V \oplus V^*$, we have that

$$\begin{aligned} e^B(X + \xi) &= (1 + B + \frac{1}{2}B^2 + ...)(X + \xi) \\ &= (1 + B)(X + \xi) \\ &= X + \xi + i_\xi B. \end{aligned}$$

Note that the second line in the equation is due to B a 2-form, hence B^n is zero for all n bigger than 2. Hence, we can view

$$e^B = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \tag{1.4}$$

as an orthogonal transformation sending $X + \xi \mapsto X + \xi + i_X B$. This transformation is referred as the $B-$field transform.

Example 1.2. Similarly, we let β as above. Then,

$$e^\beta = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \tag{1.5}$$

an orthogonal transformation sending $X + \xi \mapsto X + \xi + i_\xi \beta$. It can again be regarded as a *shear transformation* but now shearing in the T direction instead of T^*. Similarly, we also refer this as $\beta-$ transform.

1.2 Maximal isotropic subspace

In the journey of exploring symplectic geometry, one will definitely encounter the idea of isotropic, coisotropic, and Lagrangian subspaces. Since generalized complex structures are innately and heavily influenced by symplectic geometry, we shall present these analogous subspaces for $V \oplus V^*$. We first recall the some definitions for notational purposes.

Definition 1.3 (Complements)**.** Let L be a subspace of $V \oplus V^*$ as a metric space. We define the complement of L as follows:

$$L^\perp = \{X + \xi \in V \oplus V^* \ \mid \ \langle X + \xi, Y + \eta \rangle = 0 \ \ \forall Y + \eta \in L\} \tag{1.6}$$

Definition 1.4 (Annihilator)**.** Let E a subspace of a vector space V. We define the annihilator of E as follows:

$$Ann(E) = \{\alpha \in V^* \ \mid \ \alpha|_E = 0\} \tag{1.7}$$

Here we present the first key definition of this section

Definition 1.5. A subspace $L \subset V \oplus V^*$ is

1. **isotropic** if $L \subset L^\perp$
2. **coisotropic** if $L^\perp \subset L$
3. **Lagrangian** if $L = L^\perp$

Definition 1.6. Let L be an isotropic subspace in $V \oplus V^*$ and the symmetric inner product on $V \oplus V^*$ has siganture (m, m). If L is of maximal dimension, i,e, $dim\ L = m$, then L is **maximal isotropic**.

Remark 1.7. A maximal isotropic subspace is also known as a linear Dirac structure.

It can be immediately deduced that V and V^* are maximal isotropic subspaces. We then have the following important example that provides us a characterization of maximal isotropic subspaces.

Example 1.8. Let E be a subspace of V and $\epsilon \in \wedge^2 E^*$. Treating ϵ as a map $E \to E^*$ via $X \mapsto i_X \epsilon$, we can define subspace from the graph of ϵ

$$L(E, \epsilon) = \{X + \xi \in E \oplus V^* \ :\ \xi|_E = \epsilon(X)\}.$$

where it is maximal isotropic. Indeed, let $X + \xi, Y + \eta \in L(E, \epsilon)$, then

$$\begin{aligned}\langle X + \xi, Y + \eta\rangle &= \frac{1}{2}(\xi(Y) + \eta(X)) \\ &= \frac{1}{2}(\epsilon(Y, X) + \epsilon(X, Y)) \\ &= 0\end{aligned} \tag{1.8}$$

Furthermore, if we let $\epsilon = 0$ we obtain another example i.e. $E \oplus Ann(E) < V \oplus V^*$ is a maximal isotropic subspace. Note that, via Example 1.8, we obtain a general formula in constructing maximal isotropic subspaces. First denote pr_V, pr_{V^*} to be the canonical projections from $V \oplus V^*$ to V and V^* respectively.

Proposition 1.9. [16] Every maximal isotropic subspace in $V \oplus V^*$ can be written as $L(E, \epsilon)$.

Proof. Let L be a maximal isotropic subspace. Define $E := pr_V(L) \leq V$ and let $W := L \cap V^*$ a subspace of V^*. It can be immediately deduced that W has dimension $m - dim\ E$ given that L is maximal isotropic and $dim\ L = m$. Take $X, Y \in E$ and $\xi, \eta \in W$, then we have that

$$\langle X + \xi, Y + \eta\rangle = 0$$

for all $X, Y \in E$ if and only if $\xi, \eta \in Ann(E)$ and $L = E \oplus Ann(E)$. Therefore, we have that $E \cong V^*/Ann(E)$ such that we can take the skew map $\epsilon : E \to E^*$ given by

$$e \mapsto pr_{V^*}(pr_V^{-1}(e) \cap L) \in V^*/Ann(E).$$

□

Now, we give the definition of a *type* which a keen observer will realize it comes from the proof above. This will provide another way of categorizing maximal isotropic subspaces:

Definition 1.10. The **type** of a maximal isotropic subspace is the codimension k of the projection $L \to V$, i.e.

$$k := dim\ Ann(E) = m - dim\ E$$

We can further determine more classifications of maximal isotropic subspaces via $End(V \oplus V^*)$, i.e. the transformations by B and β.

Proposition 1.11. [16] Let B and β defined as in the previous section.

1. $B-$transform does not change the type of a maximal isotropic subspace

2. $\beta-$transform changes the type of $L(E,\epsilon)$ by an even number

The natural follow up question would be "how does one determine whether two maximal isotropic subspaces have the same type?". The following propositions provides the answer.

Proposition 1.12. [12] Suppose that E is any maximally isotropic subspace of V and define $E^{'} = L(E,\epsilon)/E$. If $dim\ E = r$ and $dim\ L \cap V = h$, then the types of E and $E^{'}$ are equal if and only if

$$h \equiv r\ (mod 2)$$

We end this section with the following useful remark on Lagrangian subspaces.

Remark 1.13. Lagrangian subspaces are exactly maximal isotropic subspaces.

The point of the remark is to show how all the definitions given in this section has certain relation with each other and in this thesis, we might use the terms interchangebly.

1.3 Pure spinors and the Mukai pairing

In this section, we investigate the theory of spinors for our space $V \oplus V^*$. We first introduce the Clifford algebra for a general vector space V.

Definition 1.14. Let V be a vector space. $Cl(V, \langle\cdot,\cdot\rangle)$ is defined by

$$Cl(V) = T(V)/\langle v \otimes v - \langle v, v\rangle 1\rangle$$

where T(V) denotes the tensor algebra.

Let $Cl(V \oplus V^*)$ be the Clifford algebra defined by the relation

$$v^2 = \langle v, v\rangle,\ \forall v \in V \oplus V^* \tag{1.9}$$

$V \oplus V^*$ has a natural representation on $S = \wedge^{\bullet} V^*$ given by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi \tag{1.10}$$

where $X + \xi \in V \oplus V^*$ and $\varphi \in \wedge^\bullet V^*$. With the following calculation,

$$\begin{aligned}(X+\xi)^2 \cdot \varphi &= (X+\xi)\cdot(i_X\varphi + \xi\wedge\varphi)\\ &= i_X(i_X\varphi) + \xi\wedge i_X\varphi + i_X(\xi\wedge\varphi)\\ &= i_X(i_X\varphi) + \xi\wedge i_X\varphi + i_X\xi\wedge\varphi - \xi\wedge i_X\varphi\\ &= i_X\xi\wedge\varphi\\ &= \xi(X)\varphi\\ &= \langle X+\xi, X+\xi\rangle\varphi.\end{aligned} \tag{1.11}$$

This is the standard spin representation. Therefore, we shall take elements of $\wedge^\bullet V^*$ as spinors. Observe that this is where one can tell how differential forms are perfect candidates to be taken as spinors.

Alternatively, we may consider the prospect of contravariant and covariant spinors.

Definition 1.15. Suppose we have $V \oplus V^*$ with the natural pairing $\langle\cdot,\cdot\rangle$, then the following

$$Cl(V\oplus V^*)/Cl(V\oplus V^*)V \cong \wedge^\bullet V^*,\;\; Cl(V\oplus V^*)/Cl(V\oplus V^*)V^* \cong \wedge^\bullet V \tag{1.12}$$

are the contravariant and covariant spinor modules respectively.

The spin representation decomposes according to helicity of the spinors due to signature (m,m), i.e. the ±1 eigenspaces

$$S = S^+ \oplus S^-.$$

which also coincides with the parity decomposition

$$\wedge^\bullet V^* = \wedge^{ev}V^* \oplus \wedge^{od}V^* \tag{1.13}$$

Note that the above decomposition is not preserved by the whole Clifford algebra, i.e. $Cl(V\oplus V^*)\cdot(S^+\oplus S^-) \neq Cl(V\oplus V^*)\cdot S^+ \oplus Cl(V\oplus V^*)\cdot S^-$, $S^\pm$ are irreducible representation of the spin group, which lives in the Clifford algebra as

$$Spin(V\oplus V^*) = \{v_1...v_k \mid v_i \in V\oplus V^*, \langle\cdot,\cdot\rangle = \pm1 \textit{ and } k \textit{ even}\}$$

and is a double cover of $SO(V\oplus V^*)$ via the homomorphism

$$\rho : Spin(V\oplus V^*) \to SO(V\oplus V^*)$$

$$\rho(x)(v) = xvx^{-1},\;\; x \in Spin(V\oplus V^*),\; v \in V\oplus V^*$$

In the previous section, we put a good amount of our focus on the symmetries of $V\oplus V^*$, i.e. the $B-$ and $\beta-$ transform. Note that both transforms are elements of $\mathfrak{so}(V\oplus V^)) = \wedge^2(V\oplus V^*)$ which in turn is a subset in $CL(V\oplus V^*)$. Hence, we give actions on spin representation.

Example 1.16. Denote e_i the basis of V and e^i the dual basis. Then the spinnorial action of a $B-$transform on a spinor $\varphi \in \wedge^\bullet V^*$ is given by:

$$B \cdot \varphi = \frac{1}{2} B_{ij} e^j \wedge (e^i \wedge \varphi) = -B \wedge \varphi$$

On the other hand, the $\beta-$transform is characterized by

$$e^\beta \varphi = (1 + i_\beta + \frac{1}{2} i_\beta^2 + \ldots)\varphi$$

These spinorial actions serve as an important role in determining the diffeomorphism group of a generalized complex manifold. We will then observe later that generalized complex geometry involves the specification of a maximal isotropic subbundle L and that these bundles correspond to certain spinors called pure spinors.

What follows is the description of a particular bilinear form on spinors such that they are well defined under spin representation too. For this mini section, we religiously follow the text by Chevalley [12]. Since our environment is in $V \oplus V^*$, our bilinear form corresponds to the Mukai pairing.

Definition 1.17. [12] The **Mukai Pairing** is defined as $(\cdot, \cdot) : S \otimes S \to detV^*$, $(s,t) \mapsto (\alpha(s) \wedge t)_{top}$, where α is the anti-automorphism of the Clifford algebra $Cl(V \oplus V^*)$ defined by the map of $v_1 \otimes \ldots \otimes v_k \mapsto v_k \otimes \ldots \otimes v_1$ and the *top* indicates the top degree of the form.

The following proposition provides some insights into the symmetries of $V \oplus V^*$ perform under this pairing.

Proposition 1.18. [12] The Mukai pairing is invariant under the identity component of $Spin(V \oplus V^*)$:

$$(g \cdot s, g \cdot t) = (s,t),\ \forall g \in Spin_0(V \oplus V^*)$$

Now to define pure spinors, let φ be any nonzero spinor. Define $L_\varphi < V \oplus V^*$ given by:

$$L_\varphi = \{v \in V \oplus V^* \ : \ v \cdot \varphi = 0\}$$

where

$$L_{g \cdot \varphi} = \rho(g) L_\varphi \ \forall g \in Spin(V \oplus V^*).$$

In other words, L_φ depends equivariantly on φ under the spin representation. We call L_φ the null space. Now let $e, f \in L_\varphi$, then

$$2\langle e, f \rangle \varphi = (ef + fe) \cdot \varphi = 0$$

implying that $\langle e, f \rangle = 0 \ \forall e, f \in L_\varphi$, i.e. null spaces are isotropic. Then,

Definition 1.19. A *spinor* φ is *pure* if L_φ is maximally isotropic, i.e. of dimension m.

Several interesting facts can be noted here: Let L be a maximal isotropic subspace of $V \oplus V^*$

1. There exists a pure spinor φ that L annihilates it
2. Conversely, if there are two pure spinors φ, ψ that L annihilates, then they are multiples of each other.

Combining the two statements shows that maximal isotropic subspaces are in $1-1$ correspondence with lines of pure spinors. The following proposition by Chevalley [12] willl be crucial in the later chapter.

Proposition 1.20. [12] Let φ_1, φ_2 be pure spinors. $L_{\varphi_1} \cap L_{\varphi_2} = \{0\} \Leftrightarrow (\varphi_1, \varphi_2) \neq 0$.

Here are some examples of pure spinors and the maximal isotropic subspaces.

Example 1.21. [16] The identity spinor $1 \in \wedge^\bullet V^*$ is a pure spinor. Indeed, note that

$$L(V, 0) = \{X + \xi \in V \oplus V^* \ : \ (X + \xi) \cdot 1 = 0\} < V \oplus V^*$$

is a maximal isotropic subspace. We can also extend this further and claim that any spin transformation applied to the identity spinor is also a pure spinor. To convince the reader, let B be a 2-form then the spinor $\varphi = e^B$ also has a maximal isotropic null space $L(V, -B) = \{X - i_X B : X \in V\}$.

Example 1.22. [16] A nonzero 1-form α of $\wedge^\bullet V^*$ is also a pure spinor. The maximal isotropic null space of α is the following

$$L(ker\ \alpha, 0) = ker\ \alpha \oplus \langle \alpha \rangle$$

Similar to the identity spinor, any spin transformation on applied to α is also a pure spinor.

Since any maximal isotropic subspace can be characterized by $L(E, \epsilon)$, it is natural to utilise this to build the pure spinor line bundle. To end this section, we provide a description of pure spinor lines associated to any maximal isotropics $L(E, \epsilon)$. We begin from the simplest example of $L(E, \epsilon)$. Suppose that E is a subspace of V and denote k its codimension.

Lemma 1.23. [12] $L(E, 0)$ defined by

$$E \oplus Ann(E)$$

is a maximal isotropic subspace of V. Its corresponding the pure spinor line is given by $det(Ann(E))$.

Proof. Let φ an element of $det(Ann(E))$ given by $\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_k$. Then a straightforward calculation yields that the Clifford action $(X+\xi)$ on φ is zero if and only if X is an element of E and ξ is an element of Ann(E). The result follows immediately. □

By Proposition 1.11, since $B-$field transform does not alter the type of a maximal isotropic subspace, we may be able to write any $L(E, \epsilon)$ in the form of a combination of a $B-$ transform and $L(E, 0)$ provided that $i^*B = \epsilon$. Bluntly, we may write

$$L(E, \epsilon) = e^{\epsilon}(L(E, 0)) \tag{1.14}$$

such that $i^*B = \epsilon$. From this, we can form the pure spinor line U_L of $L(E, \epsilon)$ and provide an expression for the spinors in U_L which will be important throughout this thesis.

Proposition 1.24. [12] Let $L(E, \epsilon)$ be any maximal isotropic. Then

$$U_L = e^{\epsilon}(det(Ann(E))) \tag{1.15}$$

is the pure spinor line of $L(E, \epsilon)$. Explicitly, suppose that $\alpha_1, ..., \alpha_k$ is a basis of $Ann(E)$ and B be any 2-form satisfying $i^*B = \epsilon$. Then any spinor φ_L (including pure) corresponding to the maximal isotropic $L(E, \epsilon)$ can be expressed as follows

$$\varphi_L = Ce^B \alpha_1 \wedge ... \wedge \alpha_k \tag{1.16}$$

where C is nonzero.

1.4 Complexification of $V \oplus V^*$

We end this chapter by providing the notion of *real index*. We first start with the complexfixied version of the ideas from previous sections.

The natural inner product $\langle \cdot, \cdot \rangle$ extends by complexification to $(V \oplus V^*) \otimes \mathbb{C}$. Moreover, all our results regarding properties of $V \oplus V^*$ are also adapted to the complexified version. The following theorem, took from [16], encapsulates this idea

Theorem 1.25. [16] Suppose $dim(V) = m$. A maximal isotropic subspace L of $(V \oplus V^*) \otimes \mathbb{C}$ such that it is of type k where $k \in \{0, ..., m\}$ is defined by any one of the following

1. A complex subspace $L < (V \oplus V^*) \otimes \mathbb{C}$, maximal isotropic with respect to $\langle \cdot, \cdot \rangle$, and such that $E = pr_{V \otimes \mathbb{C}} L$ has complex dimension $m - k$;

2. A complex subspace $E < V \otimes \mathbb{C}$ such that $dim_{\mathbb{C}}\ E = m - k$, together with a complex 2-form $\epsilon \in \wedge^2 E^*$;

3. A complex spinor line $U_L < \wedge^\bullet(V^* \otimes \mathbb{C})$ spanned by

$$\varphi_L = Ce^{B+i\omega}\alpha_1 \wedge ... \wedge \alpha_k, \tag{1.17}$$

   where $\alpha_1, ..., \alpha_k$ are complex 1-covectors in $V^* \otimes \mathbb{C}$ such that they are linearly independent, B and ω are real and imaginary parts of a complex 2-covectors in $\wedge^2(V^* \otimes \mathbb{C})$, and $C \in \mathbb{C}$ is a nonzero scalar.

Due to the complexification, an additional feature that arises is the complex conjugate, which acts on all relevant structures L, E and U_L. This is used to define *real index*, which plays a role in defining generalized complex structure.

Definition 1.26. [16] Let $L < (V \oplus V^*) \otimes \mathbb{C}$ be a maximal isotropic subspace. Then $L \cap \overline{L}$ is real, i.e. the complexification of a real space: $L \cap \overline{L} = K \otimes \mathbb{C}$, for $K < V \oplus V^*$. The real index r of the maximal isotropic L is defined by

$$r = dim_{\mathbb{C}}(L \cap \overline{L}) = dim_{\mathbb{R}} K.$$

Chapter 2

Courant Bracket

As mentioned in the introduction, to properly define generalized complex structures and also describe the reduction process, we require the notion of a bracket on $TM \oplus T^*M$ where M is a real smooth $m-$dimensional manifold.

Hence in this chapter, we review the notion of the Courant bracket. This bracket will be used to formulate the integrability in generalized complex structures. On top of that, we will also look into the symmetries of Courant algebroids which will be part of the setup of the reduction process. We then provide the definition of a Dirac structure and its properties. Subsequently, the chapter ends with the notion of contravariant and covariant pure spinors on exact Courant algebroids.

The material from this chapter originates from the sources, i.e. [16, 12, 13, 19].

2.1 Courant algebroid

We begin this section by introducing the canonical Courant bracket on $T \oplus T^*$. Let M be a smooth $n-$dimensional manifold. Consider vector fields X, Y and differential forms ξ, η on M, then $X + \xi, Y + \eta$ are smooth sections of the generalized tangent bundle $\Gamma(T \oplus T^*)$. The Courant bracket is then a skew-symmetric bracket defined on smooth sections of $T \oplus T^*$, given by

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) \tag{2.1}$$

An important fact to take into account is that the Courant bracket is not a Lie bracket as it fails to satisfy the Jacobi identity. The following operator provides a concrete quantity of how Courant brackets fail satisfy the Jacobi identity:

$$Jac(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B],$$

where $A, B, C \in \Gamma(T \oplus T^*)$. It is known as the Jacobiator operator and it can be expressed as the derivative of a quantity which we will call the *Nijenhuis* operator.

Hence we present several properties of the Courant bracket relating to the ideas of the Jacobiator and the Nijenhuis operator, which will lead us to the definition of a Courant algebroid.

Proposition 2.1. [19] Nij is the *Nijenhuis* operator given by:

$$Nij(A, B, C) = \frac{1}{3}(\langle [A, B], C\rangle + \langle [B, C], A\rangle + \langle [C, A], B\rangle).$$

We can then expressed the Jacobiator as follows:

$$Jac(A, B, C) = d(Nij(A, B, C)), \tag{2.2}$$

$\langle \cdot, \cdot \rangle$ is the same inner product defined in Chapter 1.

Due to the Newlander-Nirenberg theorem where it specifies that an almost complex structure I is integrable if and only if $Nij(I, A, B) = 0$, $\forall A, B \in \Gamma(T \oplus T^*)$, we are able to see a glimpse of how the Courant bracket plays a role in defining integrability. This notion will be explored in the next chapter when we define generalized complex manifolds. We can now give the definition of a Courant algebroid.

Definition 2.2. [19] A *Courant algebroid* over a manifold M is a vector bundle $E \xrightarrow{\pi} M$ equipped with a fiberwise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a bundle map $p : E \to TM$ called the *anchor*. This induces a natural differential operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ via $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\pi(A)f$, $\forall f \in C^\infty(M)$, $A \in \Gamma(E)$. The following compatibility conditions are required:
C1) $p([A, B]) = [p(A), p(B)]$, $\forall A, B \in \Gamma(E)$,
C2) $Jac(A, B, C) = \mathcal{D}(Nij(A, B, C))$, $\forall A, B, C \in \Gamma(E)$,
C3) $[A, fB] = f[A, B] + (p(A)f)B - \langle A, B\rangle \mathcal{D}f$, $\forall A, B \in \Gamma(E), f \in C^\infty(M)$,
C4) $p \circ \mathcal{D} = 0$,
C5) $p(A)\langle B, C\rangle = \langle [A, B] + \mathcal{D}\langle A, B\rangle, C\rangle + \langle B, [A, C] + \mathcal{D}\langle A, C\rangle\langle$, $\forall A, B, C \in \Gamma(E)$,
where $\mathcal{D} = \frac{1}{2}p^* d : C^\infty(M) \to \Gamma(E)$ (using $\langle \cdot, \cdot \rangle$ to identify E with E^*.)

The prime example of a Courant algebroid that will be utilised throughout this thesis is $T \oplus T^*$ with $\pi : T \oplus T^* \to TM$ as the anchor, $\langle \cdot, \cdot \rangle$ as the bilinear symmetric form given by the following

$$\langle X + \xi, Y + \eta \rangle = i_X \eta + i_Y \xi \tag{2.3}$$

and the bracket $[\![\cdot, \cdot]\!]$

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - i_Y d\eta. \tag{2.4}$$

for $X + \xi, Y + \eta \in \Gamma(T \oplus T^*)$

Remark 2.3. Note that bracket 2.4 coincides with the expression of the bracket first considered by and named after Dorfman [14]

$$[\![A, B]\!] = [A, B] + d\langle A, B \rangle, \tag{2.5}$$

in the context of complexes over Lie algebras, in order to characterize Dirac structures. It is often easier to calculate with the Dorfman bracket since it satisfies a Leibniz rule which resembles the Jacobi identity

$$[\![A, [\![B, C]\!]]\!] = [\![[\![A, B]\!], C]\!] + [\![B, [\![A, C]\!]]\!].$$

One can obtain the Courant bracket 2.1 by skew-symmetrization of the Dorfman bracket, i.e. they are related via 2.5. Hence we also obtain that $[A, B] = \frac{1}{2}([\![A, B]\!] - [\![B, A]\!])$.

Next. we introduce the notion of *exact* Courant algebroids which will be used throughout this thesis.

Definition 2.4. A Courant algebroid E is described as *exact* if the sequence

$$0 \to T^*M \xrightarrow{p^*} E \xrightarrow{p} TM \to 0 \tag{2.6}$$

is exact

Remark 2.5. Often on an exact Courant algebroid, we can choose an isotropic splitting $\nabla : TM \to E$ such that the splitting provides us a curvature 3-form $H \in \Omega^3_{cl}(M)$ defined as follows, for $X, Y \in \Gamma(TM)$:

$$H(X, Y, Z) = \langle [\![\nabla X, \nabla Y]\!], Z \rangle.$$

We then have the bundle isomorphism

$$\nabla + p^* : T \oplus T^* \to E \tag{2.7}$$

which can be used to endow the Courant algebroid structure onto $T \oplus T^*$. As before the pairing is nothing but the natural pairing defined in chapter 1, and for $X+\xi, Y+\eta \in \Gamma(T \oplus T^*)$, the bracket thus turns into the following

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X \eta - i_Y(d\xi - i_X H) \tag{2.8}$$

which is then called the $H-$twisted Courant bracket on $T \oplus T^*$. Note that the isotropic splitting derived from 2.6 differ by 2-forms $B \in \Omega^2(M)$:

$$(\nabla + B)(X) = \nabla X + p^*(i_X B). \tag{2.9}$$

Subsequently, this modifies the curvature H by the exact form dB, i.e. $H + dB$.

2.2 Symmetries of Courant bracket

In this section, we tackle the issue of determining the group of all transformation under which an exact Courant algebroid remains invariant, particularly, the group of bundle automorphisms such that the Courant algebroid structure is preserved. This plays a crucial role in the setup for the reduction of Courant algebroids.

Definition 2.6. [6] The group of symmetries, or also known as automorphism group, denoted $Aut(E)$ of a Courant algebroid E, with an underlying manifold M, consists of bundle automorphisms $F : E \to E$ covering a diffeomorphism $\psi : M \to M$ such that

1. **Orthogonality of F**: $\psi^* \langle F(\cdot), F(\cdot)\rangle = \langle \cdot, \cdot \rangle$,
2. **Bracket Preservation of F**: $[F(\cdot), F(\cdot)] = F[\cdot, \cdot]$,
3. **Anchor Compatibility of F**: $p \circ F = \psi_* \circ p$

Let $H \in \Omega^3_{cl}(M)$ and suppose we have $T \oplus T^*$ endowed with the structure of a $H-$twisted Courant algebroid. Let ψ be a diffeomorphism as defined in Definition 2.6 such that it preserves H, meaning $\psi^* H = H$. We are able to observe that ψ maintains the structure of $T \oplus T^*$ in the sense that the bundle automorphism $d\psi \oplus (\psi^{-1})^* : T \oplus T^* \to T \oplus T^*$ is an automorphism of the Courant algebroid $T \oplus T^*$.

Additionally, the Courant bracket possesses an extra symmetry induced by a $B-$field transformation, explicitly $e^B : X + \xi \mapsto X + \xi + i_X B$ from 1.1. Indeed an automorphism of an $H-$twisted Courant bracket must be the composition of a diffeomorphism ψ of M and a transformation defined by a 2-form B where $H - \psi^* H = dB$. This fact is summarized in the following.

Proposition 2.7. [6] The automorphism group of an exact Courant algebroid E is an extension of the group of diffeomorphisms preserving the cohomology class $[H]$ by the abelian group of closed 2-forms:

$$0 \to \Omega^2_{cl}(M) \to Aut(E) \to \text{Diff}_{[H]}(M) \to 0 \tag{2.10}$$

The infinitesimal version of an automorphism for an H-twisted exact Courant algebroid E is given by differentiating a 1-parameter family of automorphisms $F_t = \Psi_t e^{tB}$, $F_0 = id$. Then we get that the Lie algebra $Der(E)$ consists of pairs $(X, B) \in \Gamma(TM) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = dB$, which acts via

$$(X, B) \cdot (Y + \eta) = \mathcal{L}_X(Y + \eta) + i_Y B.$$

This leads to the following invariant presentation of derivations.

Proposition 2.8. [6] The Lie algebra of infinitesimal symmetries of an exact Courant algebroid E is an abelian extension of the Lie algebra of smooth vector fields by the closed 2-forms:

$$0 \to \Omega^2_{cl}(M) \to Der(E) \to \Gamma(TM) \to 0 \tag{2.11}$$

2.3 Dirac Structures

Now that we have establish our notion of bracket, a natural follow-up question to ask is "how does this bracket play a role in terms of involutivity". In realm of smooth manifolds, an example can be obtained of involutivity, coming from the Lie bracket where integrability is given by Frobenius' theorem. We can then raise the question whether is it possible to obtain a certain Courant involutive subbundles of $T \oplus T^*$ as an integrable structure. This is where *Dirac structure* comes into play.

Definition 2.9. [13] A real maximal isotropic subbundle $L < T \oplus T^*$ is called an **almost Dirac Structure**. If L is involutive with respect to the Courant bracket, then the almost Dirac structure is said to be integrable or a *Dirac* structure. Similarly, a maximal isotropic and involutive complex subbundle $L < T \oplus T^* \otimes \mathbb{C}$ is called a complex Dirac structure.

We combine Proposition 3.26 and 3.27 from [16] in the following proposition which justifies the hypothesis of real, maximal and isotropic.

Proposition 2.10. [16]

1. If $L < T \oplus T^*$ is involutive, then L can only be an isotropic subbundle or a bundle of type $U \oplus T^*$ for U a nontrivial involutive subbundle of T.

2. Given a maximal isotropic subbundle L of $T \oplus T^*$ then the following are equivalent:

   - L is involutive
   - $Nij|_L = 0$
   - $Jac|_L = 0$

Here are some examples of Dirac structures from well known areas of geometry.

Example 2.11. (Presymplectic Structure) A presymplectic structure on a manifold M is defined by a closed 2-form $\omega \in \Omega^2(M)$. The underlying Dirac structure is the graph of ω, denote by L_ω:

$$L_\omega = (X, i_X\omega) : X \in TM \tag{2.12}$$

This subbundle $L_\omega \subseteq TM \oplus T^*M$ is maximally isotropic and involutive with respect to the Courant Bracket

Example 2.12. (Poisson Structure) A Poisson structure on a manifold M is given by a bivector field $\pi \in \Gamma(\wedge^2 TM)$ under certain involutivity conditions. We then have the following Dirac structure:

$$L_\pi = (i_\pi\alpha, \alpha) : \alpha \in T^*M \tag{2.13}$$

Similarly, this is a subbundle of $TM \oplus T^*M$ where it is maximally isotropic and involutive with respect to the Courant bracket.

2.4 Pure Spinors on exact Courant algebroid

In this section, we provide the description of pure spinors in the setting of exact Courant algebroids E, where we obtain the notion of contravariant and covariant pure spinors. This will lead us to several important results that will aid us in later chapters particularly for the generalized reduction in the language of pure spinors.

Let E be an exact Courant algebroid over M and consider the Clifford bundle $Cl(E)$. Let $\nabla : TM \to E$ be an isotropic splitting as in the previous splitting then consider the isomorphism $F : E \to TM \oplus T^*M$. This isomorphism is the identification map from E to $TM \oplus T^*M$. Define the following

$$\xi_\nabla(e) = pr_{T^*M}(F_\nabla(e)). \tag{2.14}$$

for $e \in E$ and F_∇ is taken as the inverse of the map 2.7. We can obtain a Clifford bundle $Cl(E)$ representation corresponding to ∇, $\Upsilon_\nabla : Cl(E) \to End(\wedge^\bullet T^*M)$, defined on $e \in E$ by

$$\Upsilon_\nabla(e)\zeta = i_{p(e)}\zeta + \xi_\nabla(e) \wedge \zeta \tag{2.15}$$

for $\zeta \in \wedge^\bullet T^*M$, where p is the anchor map of E. Now let φ be a section of $\Gamma(\wedge^\bullet T^*M)$ and

$$\mathbb{L}_\nabla(\varphi) = \{e \in E \mid \Upsilon_\nabla(e)\varphi = 0\} \tag{2.16}$$

The Clifford relation

$$\langle e_1, e_2 \rangle = e_1e_2 + e_2e_1$$

where $e_1, e_2 \in E$ implies that $\mathbb{L}_\nabla(\varphi)$ is an isotropic subbundle of E provided that its rank is constant. So

Definition 2.13. $\varphi \in \Gamma(\wedge^\bullet T^*M)$ is a *contravariant pure spinor* if $\mathbb{L}_\nabla$ is a maximal isotropic subspace of E.

We then have that

Definition 2.14. Let L be a maximal isotropic subbundle of E. The following is the *pure spinor line bundle* of L.

$$U_\nabla(L) = \{\varphi \in \Gamma(\wedge^\bullet T^*M) \mid \Upsilon_\nabla(e)\varphi = 0, \quad \forall e \in L\}. \tag{2.17}$$

On the other hand, we can also define the Clifford bundle representation on multivector fields. The following

$$\Upsilon^{op}_\nabla : Cl(E) \to End(\wedge^\bullet TM)$$

is defined on $e \in E$ by

$$\Upsilon^{op}_\nabla(e)\mathfrak{X} = p(e) \wedge \mathfrak{X} + i_{\xi_\nabla(e)}\mathfrak{X}.$$

where $\mathfrak{X} \in \Gamma(\wedge^\bullet TM)$

Definition 2.15. A section $\mathfrak{X}$ of $\Gamma(\wedge^\bullet TM)$ is a *covariant pure spinor* if $\mathbb{L}_\nabla(\mathfrak{X}) := \{e \in E \mid \Upsilon^{op}_\nabla(e)\mathfrak{X} = 0 \; \forall e \in L\}$ is a maximal isotropic subbundle of E.

The covariant pure spinor line bundle corresponding to a maximal isotropic subbundle L of E is defined analogously $U^{op}_\nabla(L) = \{\mathfrak{X} \in \wedge^\bullet TM \mid \Upsilon^{op}_\nabla(e)\mathfrak{X} = 0, \quad \forall e \in L\}$.

There is a certain correspondence between the contravariant and covariant representations, namely they are isomorphic when the underlying manifold is orientable.

Remark 2.16. Let μ be a section of the corresponding determinant bundle $det(T^*M)$. Define

$$\Theta_\mu : \wedge^\bullet TM \to \wedge^\bullet T^*M \tag{2.18}$$

$$\mathfrak{X} \mapsto i_{\mathfrak{X}}\mu.$$

One has

$$\Theta_\mu \circ \Upsilon_\nabla^{op} = \Upsilon_\nabla \circ \Theta_\mu \tag{2.19}$$

such that Υ_∇^{op} and Υ_∇ are isomorphic locally.

On the other hand, The inverse of Θ_μ, denoted by $\Theta_{\mathfrak{v}}$

$$\Theta_{\mathfrak{v}} : \wedge^\bullet T^*M \to \wedge^\bullet TM \tag{2.20}$$

$$\beta \mapsto i_\beta \mathfrak{v}$$

where $\mathfrak{v} \in det(TM)$ such that $i_\mu \mathfrak{v} = 1$.

We end this section with a brief exposition into the symmetries of a Courant algebroid E, $Aut(E)$ as actions on the Clifford representation Υ_∇.

First, let ∇ be an isotropic splitting and $A = (F, \psi) \in Aut(E)$. Consider the associated splitting defined by $\nabla^A = F^{-1} \circ \nabla \circ \psi_*$ and we may also identify that $A = (\psi, B)$ by Proposition 2.7 where ψ is in the diffeomorphism group and B a closed 2-form such that $\nabla = \nabla^A + B$.

Given that the pairing $\langle \cdot, \cdot \rangle$ is preserved by F, one can construct a bundle map $Cl(F) : Cl(E) \to Cl(E)$ covering ψ, such that it is an isomorphism of algebras on the fibers. We can immediately identify that $Cl(F)$ provides a correspondence between the representations Υ_∇ and Υ_{∇^A}, In other words, we can construct the following diagram

$$\begin{array}{ccc} Cl(E) & \xrightarrow{\Upsilon_{\nabla^A}} & End(\wedge^\bullet T^*M) \\ \downarrow{\scriptstyle Cl(F)} & & \downarrow{\scriptstyle (\psi^{-1})\circ(\cdot)\circ\psi^*} \\ Cl(E) & \xrightarrow{\Upsilon_\nabla} & End(\wedge^\bullet T^*M) \end{array} \tag{2.21}$$

such that it commutes.

Let $\Xi_A : \wedge^\bullet T^*M \to \wedge^\bullet T^*M$ be a bundle isomorphism covering ψ by

$$\Xi_A = (\psi^{-1})^* \circ e^{-B}. \tag{2.22}$$

If A preserves the splitting ∇, then $\Xi_A = (\psi^{-1})^*$. Note that if we exchange the original splitting ∇ with $\nabla + B$ where B is a 2-form, the spin representation Υ_∇ modifies via

$$\Upsilon_{\nabla+B} \circ e^B = e^B \circ \Upsilon_\nabla \tag{2.23}$$

The corresponding maximal isotropic subbundle L of E then change accordingly

$$U_{\nabla+B}(L) = e^B(U_\nabla(L)) \tag{2.24}$$

Because of this, we are able to relate Υ_{∇^A} and Υ_∇, one may obtain

$$\Xi_A \circ \Upsilon_\nabla \circ \Xi_A^{-1} = \Upsilon_\nabla \circ Cl(F). \tag{2.25}$$

Hence, the map 2.22 induces an action of $Aut(E)$ on $\Gamma(\wedge^\bullet T^*M)$.

Remark 2.17. By equation 2.25, suppose $\varphi \in \Gamma(\wedge^\bullet T^*M)$ is a pure spinor, then $\Xi_A(\varphi)$ is also a pure spinor. Additionally,

$$\mathbb{L}_\nabla(\Xi_A(\varphi)) = F(\mathbb{L}_\nabla(\varphi)).$$

In particular, F leaves $\mathbb{L}_\nabla(\varphi)$ invariant if and only if Ξ_A preserves the pure spinor line generated by φ.

Finally, for the covariant version, $A = (F, \psi) \in Aut(E)$ is an action on $\mathfrak{X} \in \wedge^\bullet TM$ by $\Xi_A^{op}(\mathfrak{X}) = \psi_*(i_{e^{-B}}\mathfrak{X})$. The isomorphism Θ_μ from 2.18 provides a correspondence between Ξ_A^{op} and Ξ_A as in 2.21 if ψ preserves a volume form $\mu \in det(T^*M)$. Further results follow similarly to the contravariant version.

Chapter 3

Generalized Complex Structures

In this chapter, we begin our exploration into the realm of generalized complex structures, the central theme of this thesis. The general idea involves building on and implementing notions from previous chapters such as Dirac structures and Courant algebroids to include both complex and symplectic geometry into a single geometrical definition.

We first define a generalized complex structure on $V \oplus V^*$ and present some properties of it. This will all translate to a generalized complex manifold when we endow a manifold with the structure. We then show that this structure induces a grading which will play a role in the decomposition of spinors in the next chapter.

The materials of this chapter originates from Chapter 4 of [16].

3.1 Linear generalized complex structures

We start by defining the idea of generalized complex structure on a real vector space. We first recall the definition of structures where generalized complex structures are build upon, namely the complex and symplectic structures. Let V be a real, finite-dimensional vector space.

Definition 3.1. A complex structure on V is an endomorphism $I : V \to V$ satisfying $I^2 = -1$.

On the other hand, one can utilize the following understanding of a symplectic structure

Definition 3.2. A symplectic structure consists of a linear isomorphism $\omega : V \to V^*$ via interior product

$$\omega : v \mapsto i_v\omega, \ v \in V$$

such that $\omega^* = -\omega$.

The goal is to include both definitions in a single higher algebra/geometrical structure. This is why we shift our focus on endomorphisms of $V \oplus V^*$. Therefore, we provide the definition as follows:

Definition 3.3. [16] A generalized complex structure on V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ satisfying the following condition:

1. it is complex, i.e. $\mathcal{J}^2 = -1$
2. it is symplectic , i.e. $\mathcal{J}^* = -\mathcal{J}$

Alternatively,

Proposition 3.4. [16] A generalized complex structure on V can also be defined as a complex structure on $V \oplus V^*$ such that it is orthogonal with respect to the natural inner product as defined in chapter 1.

In the following, one can show that the usual complex and symplectic structures are typical examples of generalized complex structures.

Example 3.5. [16] Consider the endomorphism

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$$

where I is a complex structure on V. We can immedatiely deduce that $\mathcal{J}_I^2 = -1$ and $\mathcal{J}_I^* = -\mathcal{J}_I$, i.e. satisying the definition of a generalized complex structure. Meanwhile, we also have the endomorphism,

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

where ω is a symplectic form and $\mathcal{J}_\omega$ also satisfies the condition of being a generalized complex structure.

Furthermore, the following proposition provides another characterization of generalized complex structures in the language of maximal isotropic subspaces.

Proposition 3.6. [16] A generalized complex structure on V can also defined by a complexified maximal isotropic subspace $L < (V \oplus V^*) \otimes \mathbb{C}$ such that $L \cap \bar{L} = \{0\}$ (L has real index zero).

Hence, instead of analyzing the endomorphisms of generalized complex structures, we can also study the corresponding complexified maximal isotropic subspaces with real index zero.

Remark 3.7. It can be immediately deduced that a generalized complex structure only exists on V if V is even dimensional. A detailed argument on this can be found in [16].

So far we have encountered two ways of characterizing generalized complex structures, we now introduce the third way, which involves the notion of spinors from Chapter 1. By propositions 3.6 and 1.25, generalized complex structures can equally be characterized by the complex spinor line $U_L < \wedge^\bullet V^* \otimes \mathbb{C}$, spanned by

$$\varphi_L = e^{B+i\omega}\alpha_1 \wedge ... \wedge \alpha_k \tag{3.1}$$

where $\alpha_1, ..., \alpha_k$ are linearly independent complex 1-covectors in $V^* \otimes \mathbb{C}$, and B, ω are the real and imaginary parts of a complex 2-covectors in $\wedge^2(V^* \otimes \mathbb{C})$. The complex line U_L is called the *canonical line* of the generalized complex structure. We also require that L to be of real index zero which adds an extra condition on the line U_L. The following theorem by Gualtieri [16] provides an accurate description of this.

Theorem 3.8. [16] Every complexified maximal isotropic subspace L of $(V \oplus V^*) \otimes \mathbb{C}$ can be determined by a pure spinor line spanned by $\varphi_L = e^{B+i\omega}\Omega$ such that $B, \omega \in \wedge^2 V^* \otimes \mathbb{C}$ and $\Omega = \alpha_1 \wedge ... \wedge \alpha_k$ for $(\alpha_1, ..., \alpha_k) \in \wedge^1 V^* \otimes \mathbb{C}$. k here denotes the type of the maximal isotropic defined in Chapter 1. The maximal isotropic is of real index zero if and only if

$$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$$

or in other words

1. The 1-covectors $(\alpha_1, ..., \alpha_k, \bar{\alpha_1}, ..., \bar{\alpha_k})$ are linearly independent

2. The 2-covector ω is nondegenerate when $(2n-2k)-$ dimensional subspace $U \leq V$ defined by $U = Ker(\Omega \wedge \bar{\Omega})$.

Example 3.9. [16] The following is a generalized complex structure governed by a symplectic structure ω

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{3.2}$$

has a maximal isotropic given by

$$L = \{X - i\omega(X) \ : \ X \in V \otimes \mathbb{C}\}$$

and a pure spinor line spanned by

$$\varphi_L = e^{i\omega}$$

showing how the information of a symplectic structure is incorporated in a generalized complex structure. Also, note that this example also presents the generalized complex structure of type $k = 0$.

Note that the maximum type of a generalized complex structure is n since $dim(V) = dim(V^*) = n$. Hence, we then have another example at the other extreme of the type:

Example 3.10. [16] The generalized complex structure consisting of a complex structure I.

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}$$

such that its maximal isotropic is given by

$$L = V_{0,1} \oplus V^*_{1,0}$$

where $V_{1,0} = \overline{V_{0,1}}$ is the $+i-$eigenspace of J. The pure spinor line is then spanned by

$$\varphi_L = \Omega^{n,0}$$

where $\Omega^{n,0}$ is the generator of $(n, 0)-$forms for the $n-$dimensional complex space (V, I). Therefore, this shows how the information of a complex structure is embedded in a generalized complex structure. This example gives a generalized complex structure of type $k = n$.

3.2 Generalized complex structure on a manifold

In this section, we want to furnish manifolds with generalized complex structures. Again, we look no further to building blocks of complex and symplectic manifolds for guidance where we can summarise the process into the following two steps:

1. the description of an algebraic or 'almost' structure on the generalized tangent bundle,

2. an integrability condition from the Courant bracket on this structure.

We begin tackling the above inquiries as follow with the following definition of an 'almost' structure. Note that, from the previous section, we have three different ways of defining generalized complex structures. We will utilize these ways accordingly in defining our algebraic structure building block.

Definition 3.11. [16] A generalized almost complex structure on a real $2n-$dimensional manifold M is characterized by the following:

1. a maximal isotropic subbundle $L < (T \oplus T^*) \otimes \mathbb{C}$ such that $L \cap \bar{L} = 0$, i.e. L is of real index zero.
2. A pure spinor line subbundle $U < \wedge^{\bullet} T^* \otimes \mathbb{C}$, also known as the canonical line bundle, generated by the pure spinor line as in Theorem 3.8, satisfying $(\varphi, \bar{\varphi}) \neq 0$ at each point $x \in M$ for any generator $\varphi \in U_x$.
3. An almost complex structure $\mathcal{J}$ on $T \oplus T^*$ such that it is orthogonal with respect to the natural inner product $\langle \cdot, \cdot \rangle$ from chapter 1.

Now, we tackle the second statement regarding the integrability condition on generalized almost complex structures. Yet again, we refer to the conditions from symplectic and complex geometry for inspiration such that our integrability condition has to include both the symplectic 'closedness' condition $d\omega = 0$ and the complex integrability condition that $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. Our answer to this is simply the Courant bracket.

Definition 3.12. The generalized almost complex structure $\mathcal{J}$ is integrable to a generalized complex structure if its $+i-$eigenbundle $L < (T \oplus T^*) \otimes \mathbb{C}$ is Courant involutive. Due to this, a generalized complex structure is also a complex Dirac structure of real index zero.

In the following examples, we show how our integrability condition on generalized almost complex structures encompass both the symplectic and complex condition.

Example 3.13. [16] The generalized almost complex structure given by a symplectic structure ω:

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{3.3}$$

gives us the $+i-$eigenbundle

$$L = \{X - i\omega(X) \ : \ X \in T \otimes \mathbb{C}\}. \tag{3.4}$$

We want that it is Courant involutive if and only if $d\omega = 0$. Indeed, letting $X, Y \in T \otimes \mathbb{C}$ where $X - i\omega(X), Y - i\omega(Y) \in L$, then we have

$$[X - i\omega(X), Y - i\omega(Y)] = [X, Y] - i\omega([X, Y]) - id\omega(X, Y)$$

where the last term reduces to zero as $d\omega = 0$ which gives us $[X, Y] \in L$ hence involutivity follows.

Example 3.14. [16] The generalized almost complex structure governed by a complex structure

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix} \tag{3.5}$$

provides the $+i$-eigenbundle

$$L = T_{0,1} \oplus T^*_{1,0}. \tag{3.6}$$

We can show that it is Courant involutive if and only if I is integrable as a complex structure. Indeed, suppose I is integrable. Let $X + \xi, Y + \eta \in \Gamma(T_{0,1} \oplus T^*_{1,0})$ then we can simplify the Courant bracket to the following:

$$[X + \xi, Y + \eta] = [X, Y] + i_X\overline{\partial}\eta - i_Y\overline{\partial}\xi,$$

where the right hand side is a section of $T_{0,1} \oplus T^*_{1,0}$. Therefore, L is involutive with respect to the Courant bracket. On the other hand, assume L is involutive under the Courant bracket, then notice that only taking into consideration of the vector part of the Courant bracket simplifies it to the Lie bracket, which in turn, makes $T_{0,1}$ involutive under the Lie bracket. Therefore I is integrable and hence $\mathcal{J}$ as well.

3.3 Integrability and Spinors

Due to the fact that a generalized complex structure can be characterized by maximal isotropic subbundles and the canonical line bundle, we obtain a alternative presentation of integrability and spinors. This can be seen as analogous to a complex structure inducing a $(p, q)-$decomposition of forms and a splitting $d = \partial + \overline{\partial}$.

Notice that, given the condition $L \cap \bar{L} = 0$ for $L < (T \oplus T^*) \otimes \mathbb{C}$, with L being maximal isotropic means

$$L \oplus L^* \cong ((T \oplus T^*) \otimes \mathbb{C}) \tag{3.7}$$

via the pairing $\langle \cdot, \cdot \rangle$ giving $\bar{L} = L^*$. This suggests an alternative grading on the spinors with $U < \wedge^\bullet T^* \otimes \mathbb{C}$ being the canonical line bundle of $\mathcal{J}$. i.e.

$$\wedge^\bullet T^* \otimes \mathbb{C} = U_n \oplus U_{n-1} \oplus ... \oplus U_0 \oplus ... \oplus U_{-n}$$

where $U_n = U$ and $U_k = \wedge^{n-k}\overline{L} \cdot U_n$ for $k = 1, ..., 2n$. Therefore, we obtain our first characterization of integrability in the language of canonical line bundles:

Proposition 3.15. [16] Let U_n be the canonical line bundle of the generalized almost complex structure $\mathcal{J}$.

1. $\mathcal{J}$ is integrable if and only if $d(\Gamma(U_{n-1})) \subseteq \Gamma(U_n)$
2. $\mathcal{J}$ is integrable if and only if every pure spinor φ of $\mathcal{J}$ satisfies the following:

$$d\varphi = i_X\varphi + \xi \wedge \varphi$$

   where $X + \xi \in T \oplus T^*$

An important result by Gualtieri [16] then show that the integrability of $\mathcal{J}$ can be characterized by the fact that the exterior derivative d splits into the sum $d = \partial + \overline{\partial}$ where for each $k = 0, ..., 2n - 1$,

$$\Gamma(U_k) \underset{\partial}{\overset{\bar{\partial}}{\rightleftarrows}} \Gamma(U_{k+1})$$

Theorem 3.16. [16] Let $\mathcal{J}$ be a generalized almost complex structure, and define

$$\bar{\partial} = pr_{k+1} \circ d : \Gamma(U_k) \to \Gamma(U_{k+1})$$

$$\partial = pr_{k-1} \circ d : \Gamma(U_k) \to \Gamma(U_{k-1})$$

where pr_k is the projection onto U_k, and $U_k = \{0\}$ for $k < 0, 2n$. Then $\mathcal{J}$ is integrable if and only if $d = \partial + \overline{\partial}$

Both proofs can be found in [16].

Chapter 4

Decomposition of Spinors

Here we begin to present the first main object of the thesis where concrete description of the spinorial decomposition of generalized complex manifold of *type* 0 and n is established. We begin the chapter with a description of how a generalized complex structure can be seen as an action and provide the neccessary examples.

Then the following section is divided into two parts where we present the decomposition for both extremal cases of generalized complex manifolds, i.e. the complex case (type 0) and the symplectic case (type n). We will first recall how a generalized complex structure on a manifold M induces a decomposition of spinors with respect to the subbundles.

This chapter is heavily influenced by materials from [10, 9].

4.1 Action of $\mathcal{J}$ on spinors

In chapter 1, we have determined how $\wedge^\bullet V^*$ can be canonically taken as the space of spinors, where the Lie algebra $\mathfrak{spin}(n,n)$ acts on $\wedge^\bullet V^*$. Interestingly, this Lie algebra action is established is also due to the fact that $\mathcal{J} \in \mathfrak{spin}(n,n)$ where $\mathcal{J}$ is a generalized complex structure. We can then utilize $\mathcal{J}$ to act on differential forms, or in our case, spinors. Therefore, we have the following complex and symplectic version of $\mathcal{J}$.

Definition 4.1. [9] Let $\mathcal{J}$ be of type 0 then $\mathcal{J}$ is the wedge product of 2-form ω and bivector $-\omega^{-1}$, acting on a spinor ψ yields

$$\mathcal{J}\psi = (-\omega \wedge -\omega^{-1})\psi$$

Definition 4.2. [9] On the other hand, suppose $\mathcal{J}$ is of type n it is a Lie algebra action represented by the traceless endomorphism $-I$. Hence, let γ be a (p, q) spinor, we have that

$$\mathcal{J}\gamma = I^*\gamma = i(p-q)\gamma$$

From Definition 4.1 and Example 3.14, we can deduce that suppose that a generalized complex structure is induced by a complex structure, the subspaces $U_k \subset \wedge^\bullet V^*$ are the $ik-$eigenspaces of the action $\mathcal{J}$. We can extend this to every generalized complex structure:

Proposition 4.3. [9] The spaces U_k are the ik-eigenspaces of the Lie algebra action of $\mathcal{J}$.

The proof of the proposition can be found in [9].

4.2 Spinor Decomposition

From Chapter 3, we will use the fact that for each $k = 1, ..., 2n$, the U_k can be computed using Proposition 3.15 and $U_k = \wedge^{n-k}\bar{L} \cdot U_n$ with U_n being the canonical line bundle of $\mathcal{J}$.

4.2.1 Complex Decomposition

For a complex manifold (M, I) as a generalized complex structure, we have that, from previous chapters, $L = T_{1,0} \oplus T^*_{0,1}$. Hence using Proposition 3.15 and the spinor $\varphi_L = \Omega^{n,0}$ where $\Omega^{p,q}$ is the $(p, q)-$decomposition of complex forms, we have that

$$U_k = \bigoplus_{p-q=k} \wedge^{p,q} T^*M. \tag{4.1}$$

Indeed, as a quick sanity check, if we take $X + \xi \in \overline{L}$, then we obtain

$$(X + \xi) \cdot \Omega^{n,0} = i_X\Omega^{n,0} + \xi \wedge \Omega^{n,0} \tag{4.2}$$

which yields a sum of $(n-1, 0)$ and $(n, 1)$ form respectively. Therefore, we have 4.1 by using $\overline{L}$ to act on $\Omega^{n,0}$ k times inductively.

4.2.2 Symplectic Decomposition

The case for generalized complex structure induced by a symplectic form is not as obvious as the complex case. In this section, we work on the symplectic vector space

(V, ω) and subsequently, we can transfer this to a manifold as describe in Chapter 4 via generalized complex structure. We require several notions before explicitly presenting how the U_k spaces look like. Note that define Λ to be the interior product with the bivector ω^{-1} where ω is the symplectic 2-form, then we have the following operator due to Brylinski [4]:

$$d^{\mathcal{J}} = [d, \mathcal{J}] = d(\omega - \Lambda) - (\omega - \Lambda)d = \Lambda d - d\Lambda \tag{4.3}$$

We then have the following theorem for the symplectic version of the decomposition. We show how this lemma plays a role in giving the explicit expression for U_k.

Theorem 4.4. [9] The decomposition of $\wedge^\bullet V \otimes \mathbb{C}$ for a symplectic vector space (V, ω) is

$$U_{n-k} = \{e^{i\omega}(e^{\frac{\Lambda}{2i}}\alpha) \mid \alpha \in \wedge^k V \otimes \mathbb{C}\} \tag{4.4}$$

We also have the natural isomorphism

$$\varphi : \wedge^\bullet V \otimes \mathbb{C} \to \wedge^\bullet V \otimes \mathbb{C} \;\; \varphi(\alpha) = e^{i\omega}e^{\frac{\Lambda}{2i}}\alpha \tag{4.5}$$

such that $\varphi : \wedge^k V \otimes \mathbb{C} \cong U_{n-k}$

We will utilize the above theorem in the example used to test the main hypothesis of this thesis. Finally, we present the operators ∂ and $\bar{\partial}$ to complete the description of the symplectic decomposition.

Theorem 4.5. [9] For any form α,

$$d(e^{i\omega}e^{\frac{\Lambda}{2i}}\alpha) = e^{i\omega}e^{\frac{\Lambda}{2i}}(d\alpha - \frac{1}{2i}d^{\mathcal{J}}\alpha) \tag{4.6}$$

Hence,

$$\partial(e^{i\omega}e^{\frac{\Lambda}{2i}}\alpha) = -e^{i\omega}e^{\frac{\Lambda}{2i}}\frac{1}{2i}d^{\mathcal{J}}\alpha \tag{4.7}$$

$$\bar{\partial}(e^{i\omega}e^{\frac{\Lambda}{2i}}\alpha) = e^{i\omega}e^{\frac{\Lambda}{2i}}d\alpha. \tag{4.8}$$

Therefore, the natural isomorphism φ of Theorem 4.4 is such that

$$\varphi(d\alpha) = \bar{\partial}\varphi(\alpha) \;\; \varphi(d^{\mathcal{J}}\alpha) = -2i\partial\varphi(\alpha). \tag{4.9}$$

Chapter 5

Generalized Reduction

The theory presented in this chapter have major similarity to the reduction of symplectic manifolds via symplectomorphisms of which demonstrates the possibility of needing to consider submanifolds in addition to the quotient by the G action in order to obtain a manifold with the desired structure. This concept serves as a key aspect in the reduction of structures presented in this chapter, prompting the need to define an action notion that encompasses the inclusion of submanifolds.

The ultimate objective in generalized reduction is to establish a reduction process on generalized complex structures. However a fundamental question that must be first addressed is the reduction of the Courant algebroid E, a matter crucial to this theory.

In simplifying the concept, it is important to note that an action of a Lie group G on a manifold M can be fully described by the infinitesimal action of its Lie algebra, say, $\phi : \mathfrak{g} \to \Gamma(TM)$, functioning as a Lie algebra morphism. Extending this notion to a Courant algebroid E over M, the aim is to represent a G action on E that aligns with the G action on M through a map $\Phi : \mathfrak{a} \to \Gamma(E)$. This endeavor introduces two challenges:

1. What is $\mathfrak{a}$? i.e. a Lie algebra?

2. How do we guarentee that E and M can be acted by the same group G

Question one prompts the consideration of *Courant algebras* in order for $\mathfrak{a}$ to possess the characteristics of a Courant algebroid. The second question directs our attention towards *extended actions*.

Upon establishing an extended action on a Courant algebroid, a foliation is determined on M with leaves that remain invariant under the G-action. The 'quotient' algebroid is an algebroid defined over the quotient of a leaf of this distribution by G which results in the *reduced manifold*. The algebroid itself can be derived as the quotient of a subspace of E, leading to the *reduced Courant algebroid*.

Once reduction on Courant algebroids is successfully completed, with E_{red} explicitly defined, the majority of the work is finished as Dirac structures and subsequently generalized complex structures can be reduced from E.

The materials of this chapter originate from [6], focusing on ideas related to the reduction of Courant algebroids, Dirac structures, and generalized complex structures as these concepts will be explored further in studying the reduction process under the viewpoint of pure spinors. Some contents including definitions and examples are also from [8] and [15]

5.1 Courant Algebras and Extended Actions

In this section, we first address the question of "what is $\mathfrak{a}$ by introducing the notion of a Courant algebra, then the second question above by giving the ideas of an extended action. We assume the setup regarding the symmetries of Courant algebroid from Chapter 2.

When a Lie group G acts on a manifold M, we have the following Lie algebra homomorphism:

$$\psi : \mathfrak{g} \to \Gamma(TM)$$

The objective here is to adapt this notion of an action to a Courant algebroid E. This section demonstrates the methodology for achieving this by selecting an extension $\mathfrak{g}$ as a *Courant algebra*, and identifying a homomorphism from this extension to the Courant algebroid E.

Definition 5.1. [6] A *Courant algebra* over a Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ endowed with a bilinear bracket $[\![\cdot,\cdot]\!] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ and a map $\pi : \mathfrak{a} \to \mathfrak{g}$, such that for all $a_1, a_2, a_3 \in \mathfrak{a}$:
c1) $[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!]$,
c2) $\pi([\![a_1, a_2]\!]) = [\![\pi(a_1), \pi(a_2)]\!]$.

A quick verification, we can see that letting $\mathfrak{a} = \Gamma(E)$ provides us that a Courant algebroid is an example of a Courant algebra over $\mathfrak{g} = \Gamma(TM)$.

Definition 5.2. [6] An Courant algebra is *exact* if π is surjective and $\mathfrak{h} = ker\ \pi$ is abelian. Alternatively, the following is satisfied: $[\![h_1, h_2]\!] = 0$ for all $h_1, h_2 \in \mathfrak{h}$

Remark 5.3. We can also immediately see that E is an exact Courant algebroid if and only if $\Gamma(E)$ is an exact Courant algebra.

Example 5.4. [6] Let $\mathfrak{g}$ be a Lie algebra action on $\mathfrak{h}$ as a vector space. Then by letting $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ we obtain our first non-trivial example of a Courant algebra over $\mathfrak{g}$ via the bracket

$$[\![(g_1, h_1), (g_2, h_2)]\!] = ([\![g_1, g_2]\!], g_1 \cdot h_2), \tag{5.1}$$

where $g \cdot h$ is the $\mathfrak{g}-$action. This example is also known as the *hemisemidirect* product of $\mathfrak{g}$ with $\mathfrak{h}$.

Now that we have briefly developed our structure, as always, the next issue to tackle is a notion of mapping or relation in this structure. We then introduce the idea of morphism between Courant algebra.

Definition 5.5. [8] A Courant algebra morphism sending $(\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}, [\![\cdot,\cdot]\!], \theta)$ to $(\mathfrak{a}' \xrightarrow{\pi'} \mathfrak{g}', [\![\cdot,\cdot]\!]', \theta')$ is a diagram of maps below:

$$\begin{CD} \mathfrak{a} @>{\pi}>> \mathfrak{g} \\ @V{\Psi}VV @VV{\phi}V \\ \mathfrak{a}' @>{\pi'}>> \mathfrak{g}' \end{CD} \tag{5.2}$$

such that it is commutative, where ψ is a Lie algebra homomorphism, $\Psi([\![a_1, a_2]\!]) = [\![\Psi(a_2), \Psi(a_2)]\!]'$ and $\Psi(\theta(a_1, a_2)) = \theta'(\Psi(a_1), \Psi(a_2))$ for all $a_i \in \mathfrak{a}$.

With the notion of morphism defined, we can proceed to introduce the idea of an action on Courant algebroids, namely an extended action.

Definition 5.6. [6] Let G be a connected Lie group acting on a manifold M with an infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TM)$. An *extended action* on a Courant algebroid E over M is an exact Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ with a Courant algebra morphism $\rho : \mathfrak{a} \to \Gamma(E)$:

$$\begin{CD} 0 @>>> \mathfrak{h} @>>> \mathfrak{a} @>>> \mathfrak{g} @>>> 0 \\ @. @. @V{\rho}VV @VV{\psi}V \\ @. @. \Gamma(E) @>>> \Gamma(TM) \end{CD} \tag{5.3}$$

such that the following is satisfied:

- $(ad \circ \rho)(\mathfrak{h}) = 0$

- the induced action of $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ on $\Gamma(E)$ integrates to a $G-$action on the total space of E.

A crucial example of an extended action that we will use in this thesis is the trivialy extended action , also known as the $G-$lifted action. This will leads us to the notion of an equivariant form which will play a crucial role in deriving an important component of the formula for the pure spinor reduction of Dirac and generalized complex structures. Assume we have a Lie group G being compact and connected, together with an exact Courant algebroid E.

Example 5.7. A $G-$lifted action on E is an extended action as in Definition 5.6 with the Courant algebra $\mathfrak{a}$ is the Lie algebra $\mathfrak{g}$. i.e. the following diagram commutes

$$\begin{CD} \mathfrak{g} @>>> \mathfrak{g} \\ @V{\rho}VV @VV{\psi_g}V \\ \Gamma(E) @>>> \Gamma(TM) \end{CD} \tag{5.4}$$

Alternatively, supposed we have a homomorphism $\mathcal{H} : G \to Aut(E)$ and a bracket-preserving map $\lambda : \mathfrak{g} \to \Gamma(E)$. Note that if for $g \in G$, $\mathcal{H}_g$ covers a certain $\psi_g \in \mathrm{Diff}(M)$ and the infinitesimal action $\rho : \mathfrak{g} \to Der(E)$ corresponding to $\mathcal{H}$ factorizes into the following

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho} & Der(E) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle ad} \\ & & \Gamma(E) \end{array} \tag{5.5}$$

then the pair $(\mathcal{H}, \lambda)$ also defines a $G-$lifted action. Furthermore, for a $G-$lifted action to be isotropic, it has to satisfy $\langle \lambda(v), \lambda(v) \rangle = 0$ for $v \in \mathfrak{g}$.

Definition 5.8. We say that an isotropic splitting is $G-$invariant if $\mathcal{H}_g$ preserves ∇ for all $g \in G$. We can always obtain a $G-$invariant splittings since G is compact.

Example 5.9. [15] Suppose that $H \in \Omega^3(M)$, i.e. a closed 3-form. The following map

$$\mathcal{H} : g \in G \mapsto \mathcal{H}_g = \begin{pmatrix} (\psi_g)_* & 0 \\ 0 & \psi^*_{g^{-1}} \end{pmatrix} \tag{5.6}$$

takes values in $Aut(T \oplus T^*, [\![\cdot, \cdot]\!]_H)$ if and only H is invariant. Therefore, $\mathcal{H}$ defines an homomorphism. The infinitesimal action $\rho : \mathfrak{g} \to Der(T \oplus T^*)$ corresponding to $\mathcal{H}$ is given, for $Y + \eta \in \Gamma(T \oplus T^*)$, by

$$\rho(a)(Y + \eta) = [\![a_M, Y]\!] + \mathcal{L}_{a_M}\eta,$$

where a_M is the infinitesimal generator of the $G-$action on M corresponding to $a \in \mathfrak{g}$. The existence of bracket preserving map $\lambda : \mathfrak{g} \to \Gamma(T \oplus T^*)$ such that $\rho = ad \circ \lambda$ is equivalent to the existence of a linear map $\xi : \mathfrak{g} \to \Omega^1(M)$ such that

1. $\rho(a) = [\![a_M + \xi(a), \cdot]\!]$,
2. $\xi([a, b]) = \mathcal{L}_{a_M}\xi(b)$,

for every $a, b \in \mathfrak{g}$. In this case, $\lambda(a) = a_M + \xi_a$. Furthermore, $\langle\lambda(a), \lambda(a)\rangle = 0$ if and only if

$$i_{a_M}\xi(a) = 0.$$

We call $\xi : \mathfrak{g} \to \Omega^1(M)$ the moment one-form for $(\mathcal{H}, \lambda)$.

Remark 5.10. The condition $\rho(a) = [\![a_M + \xi(a), \cdot]\!]$ can also be rephrase to $i_{a_M}H - d\xi(a) = 0$ where it represents the obstruction to lift the action.

For a concrete way to compute the equivariant form ξ , we would need to turn our attention to the theory of equivariant cohomology. We consult [3] for some details on it. In a simplified manner, the Cartan model can be summarised with the following data:

$$\Omega_G(M) = (S\mathfrak{g}^* \otimes \Omega(M))^G \tag{5.7}$$

where $S\mathfrak{g}^*$ consists of $G-$equivariant polynomials on $\mathfrak{g}$ as a vector space taking values in $\Omega(M)$, i.e.

$$f : \mathfrak{g} \to \Omega(M).$$

together with the equivariant derivative $d_G : \Omega_G(M) \to \Omega_G(M)$ is defined to be:

$$(d_G f)(a) = d(f(a)) - i_{a_M} f(a), \tag{5.8}$$

for all $a \in \mathfrak{g}$. For further constructions, one can turn to the amazing source [3].

Analogous to the case of symplectic geometry, there is a notion of a moment map of an extended action. We end this section with the following definition

Definition 5.11. A moment map for an extended action is a $\mathfrak{g}-$equivariant map μ such that it factors ν through $C^\infty_{\mathbb{R}}(M)$, i.e.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\ \nu\ } & \Omega_{cl}(M) \\ & \searrow^{\mu} & \uparrow{\scriptstyle d} \\ & & C^\infty_{\mathbb{R}}(M) \end{array} \tag{5.9}$$

5.2 Reduction of Courant Algebroid

In this section, we begin our development of the reduction of Courant algebroids. An initial realization that has to be taken account is that an extended action on an exact Courant algebroid E over a manifold M may not result in an exact Courant algebroid structure on M/G. One can take the crucial step of passing it over to a submanifold $P \subset M$ of a certain condition, then we have the reduced space P/G furnished with the structure of an exact Courant algebroid.

Due to an extended action $\rho : \mathfrak{a} \to E$, we can immediately obtain the following:

Definition 5.12. [6] Let $K = \rho(\mathfrak{a})$ and denotes the orthogonal space. Then *big distribution* is defined as $\triangle_b = \pi(K + K^{\perp}) \subset TM$ and the *small distribution* $\triangle_s = \pi(K^{\perp}) \subset TM$. $K^{\perp}$

Observe that

1. $\triangle_s = Ann(\rho(\mathfrak{h}))$ is locally integrable where $\rho(\mathfrak{h})$ is constant rank since $\rho(\mathfrak{h})$ is spanned by closed 1-forms.

2. With a moment map $\mu : M \to \mathfrak{h}^*$, we can view

   - $\triangle_s$ is the tangential distribution of the level sets of μ
   - $\triangle_b$ is tangent to the orbit (with respect to G) of the $\mu-$level sets.

Now suppose we have $P \subset M$ be a leaf of $\triangle_b$, i.e. a big leaf where it is acted by G freely and properly, and $\rho(\mathfrak{h})$ has constant rank along P. To carry our the procedure, we require the following lemma and theorem from [6].

Lemma 5.13. [6] K and $K^{\perp}$ has constant rank along P.

Proof. [6] Lemma 3.2 □

Theorem 5.14. [6] An exact Courant algebroid E reduces to E_{red} and it is a Courant algebroid over $M_{red} = P/G$ with a surjective anchor.

Proof. [6] Theorem 3.3 □

Furthermore, we obtain an exact Courant algebroid on the reduced structure if K is isotropic.

Example 5.15. [6] Let a Lie group G act on a manifold M freely and properly, with infinitesimal action $\psi : \mathfrak{g} \to \Gamma(TM)$ and consider the Courant algebroid $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!]_H)$. Utilizing the inclusion map from tangent bundle to the generalized tangent bundle, we can obtain an action of the Courant algebra $\mathfrak{a} = \mathfrak{g}$. (which gives us a trivially extended action and the splitting is preserved by 5.10). We then have that $K^\perp = TM \oplus Ann(\psi(\mathfrak{g}))$ and hence M is the only leaf of $\triangle_b = \pi(K^\perp + K) = TM$. Therefore, M/G is the reduced manifold structure and the following

$$E_{red} = \frac{K^\perp/G}{K/G} = TM/\psi(\mathfrak{g}) \oplus Ann(\psi(\mathfrak{g})) \cong TM_{red} \oplus T^*M_{red}.$$

defines the reduced Courant algebroid.

5.3 Reduction of Dirac Structures

We now move on to the reduction process of the Dirac structure. Analogous to our previous goal, we would like to carry over Dirac structures under an extended action from E to E_{red} such that the structure is preserved under the action.

Given an extended action of a connected Lie group G on the Courant algebroid E, we utilised the following definition to open our discussion to the reduction of Dirac structures:

Definition 5.16. [6] We say that a Dirac structure $D \subset E$ is preserved by an extended action ρ if and only if $[\![\rho(\mathfrak{a}), \Gamma(D)]\!] \subset \Gamma(D)$

With the above, we proceed to present how to transport the structure from E to E_{red}. We begin the process on a linear algebra level. With an isotropic subspace $K \subset V \oplus V^*$, it provides a way to carry over linear Dirac structures from $V \oplus V^*$ to the quotient $K^\perp/K$. Let $D \subset V \oplus V^*$ be a Dirac structure, then we can define the following

$$D_{red} = \frac{D \cap K^\perp + K}{K}. \tag{5.10}$$

One can claim that D_{red} is a Dirac structure on $K^\perp/K$ which can be shown by:

$$D_{red}^\perp = \frac{(D \cap K^\perp + K)^\perp}{K} = \frac{(D + K) \cap K^\perp}{K} = \frac{D \cap K^\perp + K}{K} = D.$$

Finally, here is the reduction procedure. Let $\rho : \mathfrak{a} \to \Gamma(E)$ be an extended action such that the reduced Courant algebroid over a reduced manifold M_{red} is exact. If a

Dirac structure D is preserved by ρ as in 5.16, then we can furnish the Dirac structure onto the reduced Courant algebroid as follows:

$$D_{red} = \frac{(D \cap K^{\perp} + K)^G}{K^G}|_{M_{red}} \subset E_{red} \tag{5.11}$$

Now the remaining condition to verify is integrability of D_{red}. Given sections $v_1, v_2 \in \Gamma(D_{red})$, let $\tilde{v_1}, \tilde{v_2} \in \Gamma((D \cap K^{\perp} + K)|_P)$ be their respective $G-$invariant representatives in $K^{\perp G}$. Then, We can express

$$\tilde{v_i} = \tilde{v_i}^D + \tilde{v_i}^K,$$

such that $\tilde{v_i}^D \in \Gamma(D \cap K^{\perp}|_P)$ and $\tilde{v_i}^K \in \Gamma(K|_P)$ are smooth sections.
Now extend $\tilde{v_i}^D$ to invariant sections of $D \cap K^{\perp}$ and $\tilde{v_i}^K$ to invariant sections in K, so that

$$[\![\tilde{v_1}, \tilde{v_2}]\!] = [\![v_1^D, v_2^D]\!] + [\![v_1^D, v_2^K]\!] + [\![v_1^K, v_2^D]\!] + [\![v_1^K, v_2^K]\!].$$

Observe that $[\![v_1^D, v_2^D]\!] \subseteq D \cap K^{\perp}$, by the fact that D and $\Gamma(K^{\perp})^G$ are closed under the bracket and v_i^D are elements of $\Gamma(D \cap K^{\perp})^G$. With the fact that this is an ideal of $\Gamma(K^{\perp})^G$ The remaining terms are elements of $\Gamma(K)^G$. Therefore,

$$[\![v_1, v_2]\!] = [\![\tilde{v_1}, \tilde{v_2}]\!] + K \subset \Gamma(D \cap K^{\perp} + K)^G$$

proving D_{red} is closed under the bracket in E_{red}, which completes our description of a reduced Dirac struture.

Remark 5.17. The reduction of the complexified version of Dirac structures follows the same procedure with the difference being K is replaced by its complexification. This is important for the central theme of this thesis since generalized complex structures are defined to be complex Dirac structures.

5.4 Reduction of Generalized Complex Structures

Analogous to the issue encountered in the previous sections, the reduced Dirac structure L_{red} may not have a generalized complex structure as $L_{red} \cap \overline{L_{red}} = \{0\}$ may not be fulfilled. Thus, we have the following lemma that rephrases this particular condition:

Lemma 5.18. [6] The reduced Dirac distribution L_{red} is of real index zero, i.e. $L_{red} \cap \overline{L}_{red} = \{0\}$ if and only if

$$\mathcal{J}K \cap K^{\perp} \subset K \quad over \quad P. \tag{5.12}$$

The following theorem subsequently completes the procedure:

Theorem 5.19. [6] Consider ρ an extended $G-$action on an exact Courant algebroid E. Suppose that P is a big leaf (i.e. over $\triangle_b$), over which K is isotropic and G acts freely and properly on it. If the action preserves a generalized complex structure $\mathcal{J}$ on E and $\mathcal{J}K = K$ over P then $\mathcal{J}$ reduces to E_{red}.

Proof. [6] Theorem 5.2. □

Theorem 5.19 uses the compatibility condition $\mathcal{J}K = K$ for the reduction of $\mathcal{J}$. The following theorem shows that reduction also works in the extreme opposite case.

Theorem 5.20. [6] With the same setup as Theorem 5.19. If K is isotropic over P and $\langle\cdot,\cdot\rangle : K \times \mathcal{J}K \to \mathbb{R}$ is nondegenerate then $\mathcal{J}$ reduces.

Example 5.21. [6] Consider $\mathbb{C}^2$ equipped with its standard holomorphic coordinates $z_j = x_j + iy_j$ for $j = 1, 2$, and let Ψ be the extended $\mathbb{R}^2$ action on $\mathbb{C}^2$ defined by:

$$\Psi(a_1) = \partial_{x_1} + d_{x_2}, \qquad \Psi(a_2) = \partial_{y_2} + dy_1$$

where $\{a_1, a_2\}$ is the standard basis for $\mathbb{R}^2$. Note that $K = \Psi(\mathbb{R}^2)$ is isotropic which also implies that the reduced Courant algebroid over $\mathbb{C}^2/\mathbb{R}^2$ is exact. Since the natural pairing between K and $\mathcal{J}_I K$ is nondegenerate, Theorem 5.20 implies that $\mathcal{J}_I$ can be redued by this extended action. Hence, one can compute that

$$K^{\perp} = Span\{\partial_{x_1}, dy_1, \partial_{x_2} - dx_1,\ \partial_{y_1} - dy_2, \partial_{y_2}, dx_2\}.$$

Thus

$$K_{\mathbb{C}}^{\perp} \cap L = Span\{\partial_{x_1} + i\partial_{y_1} - dx_2 - idy_2,\ \partial_{y_2} - i\partial_{x_2} - dy_1 + idx_1\},$$

and $K_{\mathbb{C}}^{\perp} \cap L \cap K_{\mathbb{C}} = \{0\}$. As a result, we can write $L_{red} \cong K_{\mathbb{C}}^{\perp} \cap L$. Then we have an injection $\pi : L_{red} \to \mathbb{C}^2/\mathbb{R}^2$ sending (x_1, y_1, x_2, y_2) to $\mathbb{R}^2_{[y_1,x_2]}$, in other words we express

$$L_{red} = Span\{\partial_{y_1} + idx_2, \quad \partial_{x_2} - idy_1\} = Span\{e^{i\omega}(\partial_{y_1}), e^{i\omega}(\partial_{x_2})\}$$

Therefore, we show that $L_{red} = \Gamma_{e^{i\omega}} \subset T\mathbb{R}_{[y_1,x_2]} \oplus T^*\mathbb{R}_{[y_1,x_2]}$, which gives us the spinor $\rho_{red} = e^{i\omega}$ where we let $\omega = dx_2 \wedge dy_1$ hence the reduced generalized complex structure $\mathcal{J}_{red}$ is of type zero, i.e. symplectic type.

The main obstacle in the above example lies in the computations of $K^{\perp}$ and $K_{\mathbb{C}}^{\perp} \cap L$. The process is summarized as follows:

1. The steps to computing $K^{\perp}$ involves trial and error. Since we know that the elements $\partial_{x_1} + dx_2$ and $\partial_{y_2} + dy_1$ are in K, we test linear combinations of

$\partial_{x_j}, dx_j, \partial_{y_j}, dy_j$ for $j = 1, 2$ with these two elements of K by using the skew-symmetric bilinear form

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$

from Chapter 1. Explicitly, as examples, take ∂_{x_2} and $\partial_{x_2} - dx_1$, we then have that

$$\langle \partial_{x_1} + dx_2, \partial_{x_2} \rangle = \frac{1}{2}, \quad \langle \partial_{x_1} + dx_2, \partial_{x_2} - dx_1 \rangle = 0$$

By definition of $K^\perp$ as an orthogonal complement of K, $\partial_{x_2} - dx_1$ is in $K^\perp$ and ∂_{x_2} is not.

2. For the search of $K_{\mathbb{C}}^\perp \cap L$, we have to find $u \in K^\perp$ such that $u - i\mathcal{J}_I u \in K_{\mathbb{C}}^\perp \cap L$. To do this, we first note that the linear complex structure I has the following properties:

   (a) $-I(\partial_{x_j}) = -\partial_{y_j}, \quad -I(\partial_{y_j}) = \partial_{x_j}$,

   (b) $I^*(dx_j) = -dy_j, \quad I^*(dy_j) = dx_j$.

   With this, again by trial and error, we obtain the span of $K_{\mathbb{C}}^\perp \cap L$. Explicitly, say $u = \partial_{x_1} - dx_2$ then

$$\begin{aligned}(\partial_{x_1} - dx_2) - i\mathcal{J}_I(\partial_{x_1} - dx_2) &= (\partial_{x_1} - dx_2) - i(-\partial_{y_1} + dy_2) \\ &= \partial_{x_1} + i\partial_{y_1} - dx_2 - idy_2\end{aligned}$$

   which is the first element in of the span of $K_{\mathbb{C}}^\perp \cap L$. The other element can be computed similarly.

Chapter 6

Generalized Reduction via Pure Spinors

In this chapter, we describe the reduction procedure under the viewpoint of pure spinors. The general idea of this chapter is to establish an alternative link between the pure spinor line bundle of L and L_{red} as a Dirac structure from the pure spinors point of view.

Recalling our set up towards a reduction procedure, let G be a compact Lie group. Similarly to coisotropic reduction, where we pick an invariant submanifold $N \subset M$ and an equivariant isotropic subbundle $K \subset E|_N$, we can obtain a map $L \to L_{red}$ involving every invariant Dirac structure $L \subset E$ such that $L \cap K$ satisfying the condition of a constant rank gets mapped to a reduced Dirac structure $L_{red} \subset E_{red}$, where E_{red} is the reduced exact Courant algebroid over N/G. Here, with an invariant Dirac structure $L \subset E$, together with an isotropic splitting $\nabla : TM \to E$ that is invariant and closed, we want to derive a similar map that associates the pure spinor line bundle of L with that of L_{red}. Ultimately, we will arrive at the following result, letting φ be the pure spinor of L,

$$\varphi_{red} = q_*(e^B \wedge j^*\varphi),$$

where q_* is the pushforward map on the pure spinors, j the inclusion map of the submanifold, and B here is a 2-form obtained as a consequence of choosing a connection θ from the submanifold. In this chapter, we focus more on introducing what each symbol of the formula does. An example is provided in the next chapter on how computation works. For detailed construction of the formula, see [15].

This content of this chapter are the materials from [15] and the structure of a paper also follows [15].

6.1 Final Ingredients

In this section, we derive the final ingredients to obtain the formula for the pure spinor reduction. We will first compute the 2-form B due to the connection θ chosen as mentioned before. The formula will require an equivariant form ξ (some authors call it a moment 1-form). Following the setup from section 5.1, we will use the equivariant form defined in Example 5.9 in defining the 2-form B. We look to [15] for the lemma that provides the environment to compute B. This lemma essentially offers a helpful tool in streamlining the explanation of an isotropic $G-$lifted action.

Lemma 6.1. [15] Supposed that $N \subset M$ is an invariant submanifold such that G acts freely on it and $j : N \to M$ be the inclusion map. Let $\theta \in \Omega^1(N, \mathfrak{g})$ be a connection. Let $H \in \Omega^3(M)$ be a 3-form and $\xi \in S^1\mathfrak{g}^* \otimes \Omega^1(M)$ an equivariant form such that $H + \xi \in \Omega^3_G(M)$. If $d_G(H + \xi) = 0$, then

$$B(X, Y) = \langle \xi(\theta(Y)), \theta(X)_N - X\rangle + \langle \xi(\theta(X)), Y\rangle, \tag{6.1}$$

for $X, Y \in T_xN,\ \ x \in N$, defines an element of $\Omega^2(N)^G$ such that

$$i_{a_N} B = j^*\xi(a), \ \ \forall a \in \mathfrak{g}$$

where a_N is the infinitesimal generator. In this case, $j^*H + dB = j^*(H + \xi) + d_G B$ is a basic 3-form on N.

Remark 6.2. A differential form α basic if and only if it satisfies $i_Y\alpha$ and $\mathcal{L}_Y\alpha$ for every vector field Y. A differential form that satisfies the first condition is called *horizontal* and *invariant* for the second condition.

Definition 6.3. Let $(\mathcal{H}, \lambda)$ be an isotropic $G-$lifted action and $\nabla : TM \to E$ an invariant isotropic splitting. $\lambda_\nabla : \mathfrak{g} \to \Gamma(TM \oplus T^*M)$ is *purely tangent* on N if

$$j^*\xi_\nabla = 0. \tag{6.2}$$

Lemma 6.4. For any invariant splitting ∇ where λ_∇ is purely tangent on N. j^*H is basic on N.

Proof.

$$d_G(j^*H) = d_G(j^*(H + \xi_\nabla)) = 0$$

Result follows from remark 6.3 □

6.2 Relation between L and L_{red}

In this section, we provide an alternative way of presenting L_{red} from L. Assume we have the same set up as before, and fix an isotropic $G-$lifted action $(\mathcal{H}, \lambda)$ on E. Suppose we are given an invariant submanifold of M, $j : N \to M$, on which G acts freely. Let $x \in N$ then we can take, on the fibers, $K_x = \{\lambda(a)(x) \mid a \in \mathfrak{g}\} + p^*(Ann(T_xN))$ as a subbundle of E_x. Furthermore, if we are provided with a splitting ∇ where it is invariant and λ_∇ is purely tangent on N then we can take

$$K = \{\nabla a_M \mid a \in \mathfrak{g}\}|_N \oplus p^*(Ann(TN)). \tag{6.3}$$

as an equivariant isotropic subbundle (see [5]). Using this, following the procedure in the previous chapter, we are also able to obtain an exact Courant algebroid over the quotient manifold $M_{red} = N/G$.

Before presenting the alternate way of describing L_{red}, we start from the Courant algebroid. Note that via reduction theorem from section 5.2, we have determined that the reduced space thus indeed have a Courant algebroid structure. Hence, our definition of a Courant algebroid, denote p_{red}, $\langle\cdot,\cdot\rangle_{red}$, and $[\![\cdot,\cdot]\!]_{red}$ the anchor, bilinear form, and bracket corresponding to the reduced exact Courant algebroid E_{red}. With the aid of these, we can now describe L_{red} in the language of spinor bundles.

We begin with $q : N \to M_{red}$ as the quotient map and $x \in N$. Suppose that $k^\perp \in K_x^\perp$, then denote $[k^\perp + K]$ the orbit (with respect to G) in $E_{red}|_{q(x)}$. Let $k_1^\perp, k_2^\perp \in K_x^\perp$ and $\kappa \in T^*_{q(x)}M_{red}$ where M_{red} is the corresponding reduced manifold on E_{red}, then we can deduce that

$$\langle [k_1^\perp + K], [k_2^\perp + K]\rangle_{red} = \langle k_1^\perp, k_2^\perp\rangle \tag{6.4}$$

and

$$p^*_{red}\kappa = [p^* dq_x^* \kappa + K]. \tag{6.5}$$

The equation 6.5 is well defined (see [15] Remark 2.12)From here on, we can ubiquitously assume this splitting exists and chosen.

Now, we choose ∇ an invariant splitting such that λ_∇ is purely tangent on N. We claim that $\nabla(TN) \subset K^\perp$. Indeed let $X \in TN$ and $\xi_\nabla : \mathfrak{g} \to \Omega^1(M)$ be the equivariant form of $(\mathcal{H}_\nabla, \lambda_\nabla)$. An element of K can be described as

$$k = \nabla a_M + p^*(\xi_\nabla(a) + \kappa)$$

for $a \in \mathfrak{g}$ and $\kappa \in Ann(TN)$. Hence,

$$\langle \nabla X, k\rangle = \langle \nabla X, \nabla a_M + p^*(\xi_\nabla(a) + \kappa)\rangle = i_X(\xi_\nabla(a) + \kappa) = 0,$$

as $j^*(\xi_\nabla(a) + \kappa) = 0$. This proves that $\nabla X \in K^\perp$.

We can now provide an explicit description of ∇_{red}.

Lemma 6.5. [15] Let $\theta \in \Omega^1(N, \mathfrak{g})$ be a connection on $N \subset M$. Suppose that ∇ is an invariant splitting, then we can define the reduced splitting $\nabla_{red} : TM_{red} \to E_{red}$ via:

$$\nabla_{red} dq(X) = [\nabla X + p^* i_X B + K], \quad X \in TN, \tag{6.6}$$

where $B \in \Omega^2(N)$ as described in 6.1 for $\xi = \xi_\nabla$. On top of that, ∇_{red} is also an isotropic splitting for E_{red}. Furthermore, its curvature $H_{red} \in \Omega^3(M_{red})$ is given by

$$q^* H_{red} = j^* H + dB, \tag{6.7}$$

where H is the curvature of ∇.

We defer the reader to [15] for the complete proof of the lemma.

We can now provide a different interpretation of L_{red} for $L \subset E$ an invariant Dirac structure. Let ∇ be an invariant splitting, $\theta \in \Omega^1(N, \mathfrak{g})$ a connection, and the 2-form $B \in \Omega^2(N)$ as in 6.1 for $\xi = \xi_\nabla$. Fix $x \in N$ and define

$$\mathcal{B}j(L_x) = \{X + dj_x^*\beta - i_X B \in T_x N \oplus T_x^* N \mid \nabla X + p^*\beta \in L_x\} \tag{6.8}$$

Proposition 6.6. [15] Let $x \in N$ and $Y + \eta \in T_{q(x)} M_{red} \oplus T^*_{q(x)} M_{red}$. One has that $\nabla_{red} Y + p^*_{red}\eta \in L_{red}|_{q(x)}$ if and only if there exists $X \in T_x N$ such that

1. $Y = dq_x(X)$;
2. $X + dq_x^*\eta \in \mathcal{B}j(L_x)$.

Proof. The proof is given in [15] and paraphrased here. Suppose we have that $X' \in T_x N$ such that $dq_x(X') = Y$. From the definition of p_{red} 6.5, and ∇_{red} 6.6, then

$$\nabla_{red} Y + p^*_{red}\eta = [\nabla X' + p^*(i_{X'} B + dq_x^*\eta) + K].$$

where $\nabla X' + p^*(i_{X'} B + d_x^*\eta) = k^\perp$. From definition of L_{red}, the following statement holds

$$\nabla_{red} Y + p^*_{red}\eta \in L_{red}|_{q(x)} \Leftrightarrow \exists k \in K_x \textit{ such that } k^\perp + k \in L_x$$

By Remark 6.3, $k = \nabla u_M|_x + p^*(i_X B + \alpha)$, for some $u \in \mathfrak{g}$ and $\alpha \in Ann(T_x N)$ since $\lambda_{\nabla+B}$ is purely tangent on N. Define $X' = X + u_N(x) \in T_x N$ and $\beta = i_X B + dq_x^*\eta + \alpha \in T_x^* M$. Then,

$$\nabla X + p^*\beta = k^\perp + k \in L_x$$

Observe that $dq_x(X) = dq_x(X') = Y$ and as α belong to $Ann(T_x N)$, one has $dj_x^*\beta - i_X B = dq_x^*\eta$ as required. ∎

Remark 6.7. [15] $\mathcal{B}j(L_x)$ can be characterized as a Dirac structure with respect to the j^*H-twisted Courant bracket provided it is smooth as a vector bundle (see [13]). In this way, Proposition 6.6 states that the map q can be defined as a forward Dirac map (see [7] Definition 2.8) as follows

$$q : (N, j^*H, \mathcal{B}j(L_x)) \to (M_{red}, H_{red}, F_{\nabla_{red}}(L_{red}))$$

where H_{red} represents the curvature of ∇_{red} and $F_{\nabla_{red}}$ is the inverse map of 2.7 on the level of the reduced structure.

We now proceed to determine a relation between the pure spinor line bundle of L and L_{red}. We begin by providing a description of $U_\nabla(L_x)$. As always, we start on a linear algebra level of the problem. We recall several facts of from [12] regarding the spinors. Let $S_x \subset T_xM$ be the image of L_x under the anchor map p. Define $\omega_S \in \wedge^2 S^*$ by

$$\omega_S(X, Y) = \xi(Y), \tag{6.9}$$

where $\xi \in T_x^*M$ is such that $\nabla X + p^*\xi \in L_x$. The fact that L is isotropic implies that ω is antisymmetric.

Remark 6.8. 6.9 does not depend on the choice of ξ. Since, for every $p^*\eta \in L_x \cap p^*(T_x^*M)$,

$$\eta(Y) = 0, \forall Y \in S_x$$

Recall that we can express any spinor of $U_\nabla(L_x)$ in the form of $\varphi_x = e^{-\omega} \wedge \Omega$ where Ω is the wedge product of 1-forms such that the number of 1-forms is the codimension of L_x. This can be summarised in the following proposition.

Proposition 6.9. [12] Let $0 \neq \Omega \in det(Ann(S_x)) \subset \wedge^\bullet T_x^*M$. One has that

$$\varphi_x = e^{-\omega} \wedge \Omega \tag{6.10}$$

is a nonzero generator of $U_\nabla(L_x)$, where $\omega \in \wedge^2 T_x^*M$ is any extension of ω_S.

Since L is isotropic, we have the following

$$Ann(S_x) = L_x \cap p^*(T_x^*M). \tag{6.11}$$

In the case of covariant spinor, interchanging T_x^*M with T_xM will lead to the representation of $U_\nabla^{op}(L_x)$. The covariant pure spinor line of L_x is then spanned by $e^{-\pi} \wedge \mathfrak{X}$, where $\pi \in \wedge^2 T_xM$ is a bivector and $\mathfrak{X} \in det(L_x \cap \nabla T_xM)$.

In the following theorem by Drummond [15], we present the result of obtaining a link between the canonical line bundles of L and L_{red}.

Theorem 6.10. [15] For $\varphi_x \in U_\nabla(L_x)$,

$$\overline{\omega}_x^{op} := dq_x \circ \Theta_{\mathfrak{v}}(e^B \wedge dj_x^* \varphi_x) \neq 0 \Leftrightarrow L_x \cap K_x = 0, \tag{6.12}$$

where $\Theta_{\mathfrak{v}} : \wedge^\bullet T_x^* N \to \wedge^\bullet T_x N$ is the map 2.20 corresponding to $\mathfrak{v} \in det(T_x N)$. In this case, $\overline{\omega}_x^{op}$ is a generator of the covariant canonical line bundle $U_\nabla^{op}(L_{red}|_{q(x)})$.

Proof. [15] Theorem 3.3 □

On the other hand, we also have the contravariant version of Theorem 6.10, i.e. the contravariant pure spinor line bundle of L_{red} is as follows

$$\Theta_\mu \circ dq_x \circ \Theta_{\mathfrak{v}}(e^B \wedge dj_x^* \varphi_x)$$

which lies in $\wedge^\bullet T_{q(x)} M_{red}$. Note that it is the map from 2.19. Hence, since the spaces of contravariant and covariant pure spinor line bundles are isomorphic, we can utilize any one of them. Therefore, we will drop the U^{op} notation and just use U.

Lastly, we provide the steps in computing the push-forward of the quotient map $q : N \to M_{red}$. The digression of this small but crucial part is summarised from [2]. We first provide a short comment on a particular description of differential forms. Suppose G is compact and connected. Denote $\mathcal{W}$ an open subset of M_{red} such that $N|_{q^{-1}(\mathcal{W})}$ is trivial and let $pr_G : N|_{q^{-1}(\mathcal{W})} \to G$ be the projection onto the fiber. Any forms on N from a local perspective is given by:

$$f q^* \gamma \wedge pr_G^* \zeta$$

where γ a form on M_{red} and f is a continuously differentiable function on $q^{-1}(\mathcal{W})$. Furthermore, ζ belongs to two different types depending on whether or not ζ is top form, i.e.

1. $\zeta \in \Omega^r(G)$, type (I)
2. $\zeta \in \Omega^k(G)$ where $k < r$, type (II)

where r is the dimension of the group G.

Proposition 6.11. [2] The push-forward is then

$$q_* : \Omega(N) \to \Omega(M_{red})$$

given by:

$$\tau = f q^* \gamma \wedge pr_G^* \zeta \mapsto \begin{cases} (\int_G f(\cdot, g)\zeta)\gamma, & \text{if } \tau \text{ is type(I);} \\ 0, & \text{if } \tau \text{ is type (II).} \end{cases} \tag{6.13}$$

The integral symbol above denotes the fiber integration in [2].

6.3 Reduction Process

With the neccessary background and setup, we can now present theorem for the reduction of pure spinors. Most of the important derivation and work are shown in the previous sections with details such as proofs can be consulted from [15]. Essentially, the theorem provides a comprehensive formula to compute the pure spinor of a Dirac structure after the process of "quotient-ing out" the symmetries via the ∇ an invariant isotropic splitting and the associated lifted action. This process also require one to pick a connection, due to the combination of pull-back of the inclusion j and push-forward of the principal bundle q to achieve L_{red} from L. This connection then comes at a cost of introducting a 2-form B which we can be obtained via Lemma 6.1. Then the formula is as follows:

Theorem 6.12. [15] Suppose we have a connection $\theta \in \Omega^1(N, \mathfrak{g})$ hence a 2-form B given by 6.1 with $\xi = \xi_\nabla$ an equivariant form. Let φ be a nowhere-zero invariant section of $U_\nabla(L)$. Then

$$\varphi_{red} = q_*(e^B \wedge j^*\varphi) \tag{6.14}$$

is a section of $U_\nabla(L_{red})$.

Some important examples and applications can be found in [15], including the reduction of generalized Calabi-Yau structures and a reinvention of a result of T-duality by Cavalcanti and Gualtieri [11]. In our case, we will immediately apply the result in the following chapter. We end the chapter with a remark that potentially have a key role to play in exploring further computations outside of the scope of this thesis.

Remark 6.13. Suppose we have connections $\theta_1, \theta_2 \in \Omega^1(N, \mathfrak{g})$, then we can also derive that $B_1 - B_2$ is a basic 2-form. Now letting $\tilde{B} \in \Omega^2(M_{red})$ where $q^*\tilde{B} = B_1 - B_2$, then we can show that,

$$\varphi_1 = q_* \circ (e_1^B \wedge j^*\varphi) = q_*(e^{(B_1 - B_2)} \wedge e^{B_2} \wedge j^*\varphi) = e^B \wedge \varphi_2.$$

We end this chapter with a short comment regarding the condition $L \cap K = 0$. It is known that there exists some cases where this condition may not be satisfied. Hence, there is a version of Theorem 6.12, that engages with these cases. See [15] for the complete details.

Chapter 7

Decomposition and Reduction

As the title of the thesis suggest and mentioned in the introduction, we want to look at the consequences of generalized reduction on the decomposition of spinors of generalized complex manifolds. The Example 5.21 is used as our experiment as it is an interesting example in the sense where the reduction performed does not preserve the same type on the generalized complex structure, i.e. from complex to symplectic type.

We first perform the spinor decomposition on both $\mathbb{C}^2$ as a generalized complex structure of type n and $\mathbb{R}^2$ as a generalized complex structure of type 0.

Then we show the reduction in the sense of Theorem 6.12. There are several steps to consider, i.e. we first have to choose a connection $\theta \in \Omega^1(N, \mathfrak{g})$ and use this together with $\xi \in S\mathfrak{g} \otimes \Omega^1(M)$ to compute B as described in Lemma 6.1. Note that we refer to Theorem 6.12 since the condition of $L|_N \cap K = 0$ is satisfied as described in the example.

Finally, we combine both ideas of our example.

7.1 Decomposition

In this section we want to apply the the Decomposition Theorems in Chapter 4 to our example of generalized complex structure of type 2 and type 0. We first determine the spinors of complex type

Example 7.1. Recall we have the following grading

$$\wedge^\bullet T^*\mathbb{C}^2 \otimes \mathbb{C} = U_2 \oplus U_1 \oplus U_0 \oplus U_{-1} \oplus U_{-2}$$

and from 4.1, we have that $U_k = \bigoplus_{p-q=k} \wedge^{p,q} T^*$. Let $\Omega^{(p,q)}$ denote the (p,q)-forms. Then we have that

1. $U_2 = \Omega^{(2,0)}$
2. $U_1 = \Omega^{(2,1)} \oplus \Omega^{(1,0)}$
3. $U_0 = \Omega^{(2,2)} \oplus \Omega^{(1,1)} \oplus \Omega^{(0,0)}$
4. $U_{-1} = \Omega^{(1,2)} \oplus \Omega^{(0,1)}$
5. $U_{-2} = \Omega^{(0,2)}$

Locally, let $\{z_1, z_2\}$ be a basis of $\mathbb{C}^2$ then a form in $\Omega^{(2,0)}$ is a span of $dz_1 \wedge dz_2$, which is also the span of the canonical line bundle, hence the pure spinor of $\mathbb{C}^2$ as a generalized complex structure of type 2.

Alternatively, this decomposition can be summarized in the following diamond diagram:

$$\begin{array}{ccccc} & & \Omega^{2,2} & & \\ & \Omega^{2,1} & & \Omega^{1,2} & \\ \Omega^{2,0} & & \Omega^{1,1} & & \Omega^{0,2} \\ & \Omega^{1,0} & & \Omega^{0,1} & \\ & & \Omega^{0,0} & & \end{array}$$

It can be observed that going from left to right coresponds to the change in eigenvalue while going from bottom to top corresponds to the $n-$forms where $n = p + q$.

Remark 7.2. It can be observed as well that the diagram resembles the Hodge diamond.

In the case of the symplectic decomposition, the explicit expression of the spinors in each U_k only requires simplifying the expression $e^{i\omega}(e^{\frac{\Lambda}{2i}}\alpha)$ with different α based on which U_k we are dealing with.

Example 7.3. Using Theorem 4.4, we can immediately determine the U_k decomposition. Let ω be the canonical symplectic form on $\mathbb{R}^2$,

1. For $k = 0$, we take α to be smooth functions on $\mathbb{R}^2$, which gives us
$$U_1 = \{e^{i\omega} f \mid f \in C^\infty(\mathbb{R}^2)\}$$
2. For $k = 1$, we take α to be 1-forms on $\mathbb{R}^2$, which gives us,
$$U_0 = \{\alpha \mid \alpha \in \Omega(\mathbb{R}^2)\}$$

3. For $k = 2$, we take α to be 2-forms on $\mathbb{R}^2$, which gives us

$$U_{-1} = \{-fe^{-i\omega} \mid f \in C^\infty(\mathbb{R}^2)\}$$

7.2 Reduction

Recall that from Example 5.21, we managed to reduced the generalized complex structure of complex type on $\mathbb{C}^2$ to a generalized comples structure of symplectic type on $\mathbb{R}^2$. In short, we obtain the complex Dirac structure L_{red} by identifying it with $K^\perp \cap L$ and this complex Dirac structure provided us the spinor of symplectic type $\rho = e^{i\omega}$.

Now we begin the process of reduction using Theorem 6.12. The first step to this is to compute the 2-form B of Lemma 6.1. To do this, let $\theta = dx_1 \otimes (1,0) + dy_2 \otimes (0,1)$, $\xi(t_1,t_2) = t_1 dx_2 + t_2 dy_1$ and $H = 0$ there is no 3-forms in $\mathbb{R}^2$ by dimensionality.
Let $\{e_1, e_2\}$ be basis of $\mathbb{R}^2$ then we have $\{e^1, e^2\}$ the dual basis. A quick computation confirms that

$$\begin{aligned} d_G(H+\xi) = d_G\xi &= d\xi - i_{X_{e_i}}\xi \otimes e^i \\ &= 0 - i_{\partial_{x_1}}\xi \otimes e^1 - i_{\partial_{y_2}}\xi \otimes e^2 \\ &= 0 \end{aligned} \tag{7.1}$$

which shows the ξ chosen is valid. Then the 2-form B can be computed using the following formula from 6.1:

$$B(X,Y) = \langle \xi(\theta(Y)), \theta(X)_N - X\rangle + \langle \xi(\theta(X)), Y\rangle,$$

We perform the computations case by case involving x_1, x_2, y_1, y_2:
For $\partial_{x_1}, \partial_{x_2}$, We have that

$$\begin{aligned} B(\partial_{x_1}, \partial_{x_2}) &= \langle \xi(\theta(\partial_{x_2}), X(\theta(\partial_{x_1})) - \partial_{x_1}\rangle + \langle \xi(\theta(\partial_{x_1}))\partial_{x_2}\rangle \\ &= \langle 0, 0\rangle + \langle dx_2, \partial_{x_2}\rangle \\ &= 1 \end{aligned} \tag{7.2}$$

For $\partial_{y_1}, \partial_{y_2}$, We have that

$$\begin{aligned} B(\partial_{y_1}, \partial_{y_2}) &= \langle \xi(\theta(\partial_{y_2}), X(\theta(\partial_{y_1})) - \partial_{y_1}\rangle + \langle \xi(\theta(\partial_{y_1}))\partial_{y_2}\rangle \\ &= \langle dy_1, \partial_{y_1}\rangle + \langle 0, \partial_{y_2}\rangle \\ &= -1 \end{aligned} \tag{7.3}$$

For $\partial_{x_1}, \partial_{y_1}$, We have that

$$\begin{aligned} B(\partial_{x_1}, \partial_{y_1}) &= \langle \xi(\theta(\partial_{y_1}), X(\theta(\partial_{x_1})) - \partial_{x_1} \rangle + \langle \xi(\theta(\partial_{x_1})), \partial_{y_1} \rangle \\ &= \langle 0, \partial_{x_1} - \partial_{x_1} \rangle + \langle dx_2, \partial_{y_1} \rangle \\ &= 0 \end{aligned} \tag{7.4}$$

For $\partial_{x_1}, \partial_{y_2}$, We have that

$$\begin{aligned} B(\partial_{x_1}, \partial_{y_2}) &= \langle \xi(\theta(\partial_{y_2}), X(\theta(\partial_{x_1})) - \partial_{x_1} \rangle + \langle \xi(\theta(\partial_{x_1})), \partial_{y_2} \rangle \\ &= \langle dy_2, \partial_{x_1} - \partial_{x_1} \rangle + \langle dx_2, \partial_{y_2} \rangle \\ &= 0 \end{aligned} \tag{7.5}$$

For $\partial_{x_2}, \partial_{y_1}$, We have that

$$\begin{aligned} B(\partial_{x_2}, \partial_{y_1}) &= \langle \xi(\theta(\partial_{y_1}), X(\theta(\partial_{x_2})) - \partial_{x_2} \rangle + \langle \xi(\theta(\partial_{x_2})), \partial_{y_1} \rangle \\ &= \langle 0, 0 - \partial_{x_2} \rangle + \langle 0, \partial_{y_1} \rangle \\ &= 0 \end{aligned} \tag{7.6}$$

For $\partial_{x_2}, \partial_{y_2}$, We have that

$$\begin{aligned} B(\partial_{x_2}, \partial_{y_2}) &= \langle \xi(\theta(\partial_{y_2}), X(\theta(\partial_{x_2})) - \partial_{x_2} \rangle + \langle \xi(\theta(\partial_{x_2})), \partial_{y_2} \rangle \\ &= \langle dy_1, 0 - \partial_{x_2} \rangle + \langle 0, \partial_{y_2} \rangle \\ &= 0 \end{aligned} \tag{7.7}$$

Each of the above are coefficient of the 2-forms with respect to the local coordinates. Hence we obtain B by summing over these coefficients together with the 2-forms. Therefore we get

$$B = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$$

Thus, we have all the ingredients to explicitly calculate the reduce pure spinor φ_{red} of the reduction process stated in Theorem 6.12. First, recall from previous section that, we have U_2 as the canonical line bundle where spanned by the $(2,0)$ forms of the decomposition,i.e. $dz_1 \wedge dz_2$ which can be expressed in terms x_1, y_1, x_2, y_2

$$\begin{aligned} dz_1 \wedge dz_2 &= (dx_1 + idy_1) \wedge (dx_2 + idy_2) \\ &= (dx_1 \wedge dx_2 - dy_1 \wedge dy_2) + i(dx_1 \wedge dy_2 + dy_1 \wedge dx_2) \end{aligned} \tag{7.8}$$

which we denote φ as in the expression 6.14. Now in our case, we choose $\mathbb{C}^2$ itself to be the invariant submanifold, hence j^* is the identity map. Thus, we have to compute the following

$$\begin{aligned} e^B \wedge \varphi &= (1 + B) \wedge \varphi \\ &= \varphi + B \wedge \varphi \\ &= \varphi + (-2dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2) \\ &= \varphi + (-2dx_2 \wedge dy_1 \wedge dx_1 \wedge dy_2). \end{aligned} \tag{7.9}$$

where the last line is just interchanging the terms so that it is convenient when we compute the pushforward q_*. Performing the pushforward on the above then yields

$$\begin{aligned}\varphi_{red} &= q_*(\varphi + (-2dx_2 \wedge dy_1 \wedge dx_1 \wedge dy_2)) \\ &= i - 2dx_2 \wedge dy_1 \\ &= i(1 + 2idx_2 \wedge dy_1) \\ &= ie^{i\omega}.\end{aligned} \tag{7.10}$$

where $\omega = 2dx_2 \wedge dy_1$ and $ie^{i\omega}$ lies in U_1, the canonical line bundle of the generalized complex structure of type 0 on $\mathbb{R}^2$, which is what we expected from Example 5.21.

7.3 Decomposition+Reduction

To proceed to see how this reduction process affects the spinor decomposition, we perform the same procedure on all U_k of generalized complex structure of type 2 on $\mathbb{C}^2$ and see in which U_k it is in on the generalized complex structure of type 0 on $\mathbb{R}^2$. To summarize it, We have the following to summarize the result:

1. A spinor in U_2 of $\mathbb{C}^2$ reduces to $ie^{i\omega}$ which lies in U_1 of $\mathbb{R}^2$;
2. A spinor in U_1 of $\mathbb{C}^2$ reduces to dy_1 which lies in U_0 of $\mathbb{R}^2$;
3. A spinor in U_0 of $\mathbb{C}^2$ reduces to $-2ie^{i\omega}$ whcih lies in U_1 of $\mathbb{R}^2$;
4. A spinor in U_{-1} of $\mathbb{C}^2$ reduces to $-dy_1$ which lies in U_0 of $\mathbb{R}^2$;
5. A spinor in U_{-2} of $\mathbb{C}^2$ reduces to $-ie^{i\omega}$ which lies in U_{-1} of $\mathbb{R}^2$.

Indeed, for $U_1 = \Omega^{2,1} \oplus \Omega^{1,0}$ we choose $dz_1 \wedge dz_2 \wedge d\bar{z}_1 + dz_1$ as our spinor which locally we can simplify

$$\begin{aligned}dz_1 \wedge dz_2 \wedge d\bar{z}_1 + dz_1 &= (dx_1 + idy_1) \wedge (dx_2 + idy_2) \wedge (dx_1 - idy_1) + (dx_1 + idy_1) \\ &= (2idx_1 \wedge dy_1 \wedge dx_2 + 2dy_1 \wedge dx_1 \wedge dy_2) + (dx_1 + idy_1)\end{aligned} \tag{7.11}$$

Denote this by φ, and we have that

$$\begin{aligned}e^B \wedge \varphi &= \varphi + B \wedge \varphi \\ &= \varphi + i(dx_1 \wedge dx_2 \wedge dy_1) - dy_1 \wedge dy_2 \wedge dx_1\end{aligned} \tag{7.12}$$

Subsequently the push-forward q_* sends terms with $dx_2 \wedge dy_1$ to zero, we have that

$$q_*(\varphi) = dy_1 \tag{7.13}$$

which is in U_0 of the symplectic decomposition. Repeat the same procedure to the spinors of the remaining U_k yields the result above.

Remark 7.4. It can be seen that the reduction process could be seen collapsing the diamond diagram 7.1 of the complex decomposition into a single line of U_k symplectic decomposition. However this speculation is not concrete since U_0 of the complex decomposition did not reduce to the expected eigenbundle in the symplectic decomposition. Suggestions on changes in computations is provided in the next chapter in hope of noticing a trend in relating the reduction process to the decomposition of spinors.

Chapter 8

Suggestions, Outlook and Potential Future Work

As shown in the previous chapter, due to where the pure spinor of U_0 of $\mathbb{C}^2$ reduces to, it is not possible to conclude how the reduction process affects the decomposition of spinors. The result is very close to a certain pattern however not fruitful at the end. However, further attempts can be made, provided certain changes are made in the computations.

First, note that, Remark 6.13 shows that the if a different 2-form B is used, the overall reduced spinor φ_{red} could be different, subsequently the decomposition could be different as well. Besides that, more examples could be explored using Theorem 6.12, which could provide a trend or pattern in terms of how the decomposition of spinors will become after reduction. Furthermore, an attempt on developing a equivariant model for spinors might also be helpful.

Moreover, there are also several potential future work to be explored in terms of how reduction process affects generalized complex structures. After spinor decomposition, the next interesting topic to explore would be the Hodge decomposition of generalized Kahler manifold [17]. It would be interesting to see how the reduction process of generalized Kahler structure [6] affects the Hodge decomposition. Potentially, a result similar to Proposition 6.14 of [1] is desired as it shows the commutativity of performing reduction and decomposition in simple terms.

All in all, there are definitely more to explore in terms of both reduction and decomposition.

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