General Relativity on Stratified Manifolds in the BV-BFV Formalism

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Abstract

The main goal of this thesis is to understand and solve some problems arising in the BV-BFV (Batalin–(Fradkin)–Vilkovisky) versions of Palatini–Cartan (PC) and Einstein–Hilbert (EH) theories.

After the introduction of the necessary background, the original work is divided in three main parts: one on the analysis of three dimensional PC theory, one on PC theory in dimension greater than or equal to four and one on EH theory in dimension greater than two.

In the first part we show that the BV version of three dimensional PC theory can be extended in the BV-BFV sense to boundaries and all higher-codimension strata. Furthermore, we show that such an extension is strongly equivalent to (non-degenerate) BF theory at all codimensions.

The second part is devoted to the construction of a BFV theory for PC theory in dimension greater than or equal to four. In particular we find an explicit description of reduced phase space of the theory on space/time-like and light-like boundaries. We describe it by means of a geometric construction developed by Kijowski and Tulczijew and obtain a presentation with first-class and second-class (in the light-like case) constraints, together with their Poisson brackets. From the BFV boundary theory we then develop the Aleksandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) construction, obtaining a BV theory for Palatini–Cartan theory on cylindrical manifolds. This theory is classically equivalent to the standard one but with a partial implementation of the torsion-free condition already on the space of fields.

The last part complements the known results about Einstein–Hilbert theory. From the BFV theory on the boundary, the same ASKZ construction allows us to obtain a BV theory on cylindrical manifolds. This theory represents a BV-extension of the first-order formulation of EH theory.

Finally, we attempt to induce corner data in the BV-BFV sense for both Einstein–Hilbert and Palatini–Cartan theories, showing that the standard procedure fails.

Original Content and Self-Plagiarism

The contents of the Chapters 2, 3, 4, 5 and 6 are original work and were previously published in [CS19a], [CCS20a; CCT20], [CCS20a], [CCT20] and [CCS20b] respectively. These papers were written in equal proportions by myself and my coauthors Alberto S. Cattaneo, Michele Schiavina and Manuel Tecchiolli. Parts of the Introduction and of Chapter 1 were originally published in the articles cited above and are inspired by previous article in the literature cited in the corresponding paragraphs. The remaining materials, including those of Chapter 7, are original and have not been published before.

Parts of this thesis are extracts from the preprints listed below, with minor changes:

[CS19a] G. Canepa and M. Schiavina. "Fully extended BV-BFV description of General Relativity in three dimensions" (2019). arXiv: `1905.09333 [math-ph]`.

[CCS20a] G. Canepa, A. S. Cattaneo, and M. Schiavina. "Boundary structure of General Relativity in tetrad variables" (2020). To appear in *Advances in Theoretical and Mathematical Physics*. arXiv: `2001.11004 [math-ph]`.

[CCS20b] G. Canepa, A. S. Cattaneo, and M. Schiavina. "General Relativity and the AKSZ construction" (2020). arXiv: `2006.13078 [math-ph]`.

[CCT20] G. Canepa, A. S. Cattaneo, and M. Tecchiolli. "Gravitational Constraints on a Light-like boundary" (2020). To appear in *Annales Henri Poincare*. arXiv: `2010.14871 [math-ph]`.

Contents

Introduction

This thesis is devoted to the analysis of various issues arising in the BV-BFV formulation of General Relativity (GR) and to explore the mathematical and physical structures involved in the construction. Indeed, casting GR in the BV-BFV formalism presents some difficulties and problems of nontrivial solution already at a classical (i.e. non-quantum) level. This work builds upon the previous results on the same topic [Sch15; CS16; CSS18; CS19b].

In order to introduce the BV-BFV formalism we first have to give an overview of the BV and BFV formalism separately.

The BV formalism is an approach to field theories with on-shell symmetries developed by Batalin and Vilkovisky (BV) [BV77; BV81] in order to deal with quantisation of theories with degenerate Lagrangians. It is a generalization of earlier constructions of Faddeev–Popov [FP67] and of Becchi–Rouet–Stora and Tyutin (BRST) [Tyu75; BRS76], meant to provide a general framework to gauge fix a field theory and make some progress towards its perturbative quantisation. The latter consists in a perturbative procedure based on the stationary phase formula for oscillatory integrals, with Feynman diagrams being the coefficients of asymptotic power series. However, such an expansion is only possible when the critical points of the action are isolated and this last hypothesis is not satisfied in theories with gauge symmetries (and, in particular, in General Relativity). The key idea of the BV approach to tackle this problem is to enlarge the space of fields to a graded supermanifold by adding non-physical fields (ghosts), keeping track in an appropriate way of the symmetries and to consider a new action, closely related to the classical one but with nicer properties. The gauge fixing is then achieved by the choice of a Lagrangian submanifold; the local independence of the results of this choice is guaranteed by theorems proved by Batalin and Vilkovisky. The data constituting a BV theory are a symplectic space of fields (a graded supermanifold equipped with a non-degenerate closed two form), an action functional satisfying the Classical Master Equation, and its Hamiltonian vector field. The relevant physical information is encoded within the cohomology of the latter vector field viewed as a derivation. For a systematic exposition of the BV formalism with examples we refer to [Mne17]. Other versions and intepretation of BV formalism can be found in [BBH95; FR12; CG16].

The BFV formalism — after Batalin, Fradkin and Vilkovisky [BV81; BF83] — is the counterpart of the BV formalism for constrained Hamiltonian systems. Historically, the Hamiltonian description of a field theory, which is a particular case of the study of a theory on the boundary of a space-time when the boundary is a Cauchy surface, has been done using the Dirac procedure through the analysis of the constraints of the theory [Dir58]. The reduced phase space of the theory, describing the boundary data as the symplectic reduction of some submanifold in a symplectic space of fields near the boundary, is obtained through a procedure involving the analysis of primary and secondary constraints which have to be then classified into first and second class constraints. In the Palatini–Cartan case this has been done (see for example [DI76; AS15]), but it turns out to be quite complicated and it involves second class constraints, which would make the construction of the BFV theory more complex[1].

In contrast, we use the Kijowski and Tulczijew (KT) construction to study the Hamiltonian formulation of Palatini–Cartan theory. The reduced phase space is here described as the symplectic reduction of the coisotropic submanifold defined by the constraints (assumed first class). The main advantage of this point of view is a clear geometric description of the phase space, and its compatibility with the BFV formalism. Furthermore, this construction avoids the introduction of the artificial classifications of constraints as primary, secondary, etc.

In order to prepare for the construction of a BFV theory, it is better to present the reduced phase space as the reduction of a submanifold determined by first-class contraints only (i.e. constraints that Poisson commute, on shell), when this is possible. One advantage of this is that the associated Hamiltonian vector fields can now be interpreted as generators of symmetries, so that the reduced phase space can be regarded as the quotient of a submanifold (fields satisfying the generalised Gauss laws) by gauge transformations. In some cases—e.g., Yang–Mills theory—this can be interpreted as a Marsden–Weinstein [MW74] reduction.

The reduction leading to the reduced phase space can be singular, and one possible description is by means of a cohomological resolution: one introduces a complex whose cohomology is the algebra of functions on the reduced phase space. In addition, one wants this resolution to feature also the symplectic/Poisson nature of the phase space, and a solution to this problem is provided by the Batalin–Fradkin–Vilkovisky (BFV) formalism [BF83] (see also [Sta97; Sch08; Sch09]). A BFV theory is a collection of data associated with the reduced phase space of a Lagrangian theory F (see Section 4.3.1). Namely, one considers some appropriate symplectic supermanifold and recasts the constraints into an odd functional (the BFV action) that Poisson commutes with itself. This yields a complex whose degree-zero cohomology is isomorphic, as a Poisson algebra, to the algebra of functions of the reduced phase space when the latter is smooth. The main advantage then is that one can take this procedure as a definition for the reduced phase space when it is not smooth. Moreover, one can attempt to quantise the reduced phase space in terms of an appropriate quantisation of the supersymplectic manifold (which is often geometrically simpler) and of the BFV action. The details of this construction are presented in Section 4.3.1.

Formally, the quantities involved in the classical version of the BFV formalism are the same as the ones constituting the BV formalism but with shifted degrees (where the degree is that of the space of fields viewed as a graded supermanifold). The original ideas have been further developed by Stasheff [Sta97], who gave a rigorous mathematical treatment of the formalism, providing formal proofs of existence and uniqueness based on homological perturbation theory, and treated a more general case based on Lie–Rinehart algebras. Later on, Schätz [Sch08] extended the result to general coisotropic submanifolds.

It is useful to fix here a piece of notation: given a classical Lagrangian field theory, denoted by F (i.e. the data of a space of fields and an action functional on it), we call F the BV data associated to it. Moreover, we will denote with F^∂ the BFV theory corresponding to the classical theory F.

The BV-BFV formalism is the combination of the BV and the BFV formalisms developed by Cattaneo, Mnev and Reshetikhin in [CMR11; CMR14]. It associates to a manifold with boundary a BV theory on the bulk and a BFV theory on the boundary, together with some compatibility and regularity assumptions. The construction is based on a constructive induction procedure, which builds the boundary BFV structure (denoted by $\mathit{BFV}(\mathsf{F})$) from the bulk BV data F in a way suitable for quantization [CMR18]. Using the notation introduced above, in regular cases it relates F and F^∂ in such a way that $\mathit{BFV}(\mathsf{F}) = \mathsf{F}^\partial$. The formalism is constructed so that the gauge invariance described by the corresponding BV-cohomology in the bulk is spoilt by the boundary data, but it is at the same time controlled by the induced cohomological

[1] Although they become necessary in the case of a light-like boundary

data associated to the boundary. Using this procedure it is then possible to describe a consistent quantisation scheme [CMR18] which is compatible with gluing along boundaries by construction. This last property is essential in the formulation of a quantum field theory in the axiomatization of Atiyah and Segal [Seg88; Ati89]. The output of this quantisation scheme is the cohomology of a quantum operator that directly encodes gauge invariance. Some very reliable and versatile examples have been developed in this formalism (see [CMR18; CMR20] for the quantisation of BF theory, [IM19a] for Yang–Mills theory in dimension 2, [CMW17] for split Chern–Simons theory, and [CMW19] for a general approach to a class of AKSZ models, including the Poisson sigma model).

It is possible to further extend the BV-BFV formalism to manifolds with boundaries and corners, or, more generally to stratified manifolds. In this case we can repeat the procedure above for each pair of consecutive strata and allow the $(k+1)$th stratum to spoil the gauge invariance on the kth stratum and keep track of the failure through the use of induced data. By construction the generalized BFV data on the kth stratum will satisfy similar axioms to the ones on the boundary but with further shifted degrees. Sometimes we will call such data BF^{k}V data, to emphasize the different degrees and the corresponding codimensions. We will denote by $\mathsf{F}^{(k)}$ the BF^{k}V data associated to the stratum of codimension k (e.g. $\mathsf{F}^{(0)} = \mathsf{F}$, $\mathsf{F}^{(1)} = \mathsf{F}^{\partial}$, etc.). The importance of such additional structures is related to the possibility of defining a gluing procedure in more general situations and on the physical information carried by the higher codimension data, such as for example the algebraic data connected to boundary insertions in codimension 2. The BV-BFV construction and its extension to higher codimensions are presented in Section 1.1 in an abridged but still sufficient form for the pourposes of this work. We are here interested only in the classical version of the BV-BFV formalism, the quantum counterpart being currently out of reach for the gravity theories.

Gravity as a BV-BFV theory

The first attempt to describe General Relativity within the BV-BFV framework was started by Schiavina in [Sch15]. His work shows that the BV-BFV procedure described above to induce a BFV theory on the boundary compatible with the data on the bulk does not always work. Indeed, obstructions are present in four dimensional General Relativity in the Palatini–Cartan (PC) coframe formulation[2][CS19b](see Section 1.2.4 for an introduction to this formulation) at least in a natural implementation of the BV formalism that works for the analogous three-dimensional case[3]. On the contrary, General Relativity in the Einstein–Hilbert formulation, i.e. as a field theory depending on the metric (see Section 1.2.1 for an introduction to this formulation), satisfies all the requirements (under the condition that the boundary metric be non-degenerate), thus providing a well defined BV-BFV theory in all spacetime dimensions $N \neq 2$ [CS16]. Obstructions similar to the PC case have been found in one-dimensional reparametrisation models [CS17] and are to be expected in certain supersymmetric models [GP19].

As a consequence, the choice of the formulation of General Relativity adopted plays an important role in the behaviour of the corresponding BV(-BFV) theory. Indeed, even though the aforementioned PC and EH formulations are classically equivalent, in the sense that they have the same space of solutions to the Euler–Lagrange equations modulo symmetries, they behave

[2] The physics literature for this version of GR seems to disagree on standard nomenclature with names arbitrary selected from the set Einstein{Sciama{Kibble{Palatini{Cartan{Holst. We will call this version simply Palatini{Cartan theory.

[3] There are examples of theories where one can modify the bulk BV formalism to make it compatible with the boundary BFV formalism [CS17]; as we will see later the AKSZ construction will do this job for cylindrical manifolds at the price of restricting the space of elds.

differently offshell when the symmetries are taken into account. On the one hand, Einstein–Hilbert theory behaves better in the BV-BFV formalism; on the other hand Palatini–Cartan formalism has several advantages. Indeed, the formulation of the theory through differential forms allows for a more natural restriction to boundaries, whose study is our main motivation. It is worth mentioning here, for future developments, other formulations classically equivalents to the PC and EH ones present in the literature. One such example is Plebanski theory, reconstructing GR from a BF theory, but this has also been shown to be obstructed. Ashtekar formalism provides yet another alternative[Ash86; dLV06]. Furthermore, the same problem can be analyzed in greater generality, [FP90] where no compatibility with either the coframe or the internal metric is required, and the ***parent formulation*** proposed in [BG11]. However, we will not explore these here but we postpone a more detailed analysis of these directions until future works.

Since, as we noted above, the behaviour of a theory under the BV-BFV construction can give important information in view of its quantisation (and in particular towards the BV quantisation with boundary), without the observations produced in this preliminary phase, the obstructions highlighted in this analysis might undermine an early attempt at directly quantising PC theory. Furthermore, the BV-BFV formulation gives new opportunities in the study of asymptotic symmetries, as initiated in [RS20].

In this sense, one of the driving motivation for this work is the crucial importance of a correct prepararation of a field theory at a classical level in view of its perturbative quantisation in order to idenitify and circumvent in advance problems that may arise when quantising gravity. Hence, besides the mathematical treatment, this work is of particular interest to the mathematical physics community involved in the study of quantum theories of gravity.

The first step in the direction of understanding the obstructions one may encounter is the analysis of three dimensional Palatini–Cartan theory. We can view such theory as a simpler scenario retaining the difficulties related to the presence of diffeomorphisms as a symmetry but still satisfying the BV-BFV axioms. Indeed, in this case it is possible to induce a boundary theory from the bulk by means of a (nontrivial) reduction procedure. Furthermore, the theory can be extended to higher codimension strata for which we show explicit expressions for the BV-BFV data at all steps. Besides the comparison with the higher dimension analogue, for which it constitutes a useful guideline, this theory is the first example of a fully extendable theory that features a nontrivial (symplectic) reduction at every step. Furthermore, three dimensional Palatini–Cartan theory is shown to be strongly equivalent in the BV-BFV sense (see Section 1.3 for a precise definition) to three dimensional BF theory. Roughly, this means that we are associating to the same classical theory two different but equivalent sets of symmetries. The details of this construction are presented in Chapter 2. Note that, as we shall show in Chapter 7, the corresponding Einstein–Hilbert formulation in dimension 3 satisfies only the first step, but not the extension to corners and hence to higher codimension strata.

This approach cannot be used to cast Palatini–Cartan theory in dimension N 4 in the BV-BFV formalism, as proved in [CS16], because it does not meet the regularity requirements needed for the induction of a BFV theory on the boundary (see Section 1.2.6 for more details). The idea to overcome the problem, proposed in [CS19c], is to study directly the BFV structure induced by the classical theory on the boundary, seen as a constrained Hamiltonian system.

This construction is described in full detail for Palatini–Cartan gravity in Chapter 4 for the time-like and space-like case and in Chapter 5 for the light-like case. While in the former case the construction allows the description of a BFV theory, in the latter the presence of second class constraints spoils the procedure. In order to simplify the analysis of these two cases, a number of technical lemmas and results are collected and proved in Chapter 3.

Although it is not possible to construct a BFV theory out of the reduced phase space in the light-like case, the importance of the description of this space is testified to by the number of previous works considering the structure of GR on null foliations, the first of which date back to Sachs and Penrose [Sac62; Pen80]. In particular the description of the Hamiltonian formulation of GR in the case of a null hypersurface has been studied for example in [Tor86; DS17] and in [Rei13; Rei18] in the Einstein–Hilbert formalism. This formulation would allow the construction of exact (but not unique) solutions starting from initial data on null hypersurfaces such as null horizons of black holes.

We can summarize the above discussion in the following table:

Theory	1-Extendable	2-Extendable
PC ($N = 3$)	yes	yes
PC ($N > 3$)	no	no
EH ($N > 2$)	yes	no

Closing the circle

As already mentioned, it is not always possible to construct directly a BV-BFV formulation out of a given theory. However, given a BFV theory associated to the boundary it is possible to reconstruct a BV-BFV theory in the special case of cylindrical manifolds. More precisely given a BFV theory F^{∂} associated to a manifold Σ, there is a standard way[4] to produce a BV theory on $\Sigma \times I$ by means of a construction due to Alexandrov, Kontsevich, Schwarz and Zaboronski (AKSZ [Ale+97]). The resulting BV theory, which we denote here by[5] $AKSZ(\mathsf{F}^{\partial})$, satisfies automatically the regularity assumptions of the BV-BFV formalism, and we also have $BFV(AKSZ(\mathsf{F}^{\partial})) = \mathsf{F}^{\partial}$ (see Section 6.1.1 for the details of this construction).

On the other hand, if from a BV theory F it is possible to induce a BFV theory $BFV(\mathsf{F})$, in general $AKSZ(BFV(\mathsf{F}))$ will not be the same as F. In fact, the AKSZ construction produces a theory that is invariant under reparametrization of I, which is certainly different from F if the latter does not enjoy this invariance. In this case $AKSZ(BFV(\mathsf{F}))$ is a version of F with "frozen time" and may be used to describe a change in the polarization chosen for the quantization of the reduced phase space (see [CMR18, Remark 2.38]). If F is reparametrization invariant — e.g. a topological field theory or GR — we may wonder whether $AKSZ(BFV(\mathsf{F}))$ and F are somehow related. In the case of AKSZ topological field theories, it turns out that $AKSZ(BFV(\mathsf{F}))$ and F are actually the same. For more general reparametrization invariant theories we might expect the two to be equivalent, in one of the possible ways presented in section 1.3.2.

In the case of General Relativity the two relevant notions considered here are those of BV inclusion and effective BV-equivalence. The first emerges when the space of fields of the two theories are one included in the other and the corresponding BV data are compatible (see Section 1.3.2 for the details). If F_2 is obtained from F_1 by a partial integration of the fields[6] (with some partial gauge fixing), we say that F_2 is an effective theory for F_1. We say that two BV theories are effectively BV-equivalent if one is (strongly BV-equivalent to) an effective theory for the other. Typical cases for this are Wilson renormalization or the passage to a second-order theory from its associated first-order formulation. Another important example is given by elimination of so-called auxiliary fields. In that case, one can argue that effective equivalence also preserves the BV cohomology [Hen90; BBH95].

[4] To the best of our knowledge, the first explicit application of the AKSZ construction to a BFV target to produce a BV structure in one dimension goes back to [GD00].

[5] This construction is clarified in Theorem 6.2, and $AKSZ(\mathsf{F}^{\partial})$ will be denoted $\mathsf{F}^{\mathrm{AKSZ}}(I; \mathsf{F}^{\partial})$.

[6] This is more appropriately called BV-pushforward or BV fiber integral, see [CMR18, Section 2.2.2].

This direction is explored in Section 6.4 for Palatini–Cartan theory. The resulting AKSZ theory turns out to be classically equivalent to the standard BV theory on a cylinder. Furthermore, one can build a BV inclusion between them.

Since a BFV formulation exists also in the Einstein–Hilbert formalism and this construction has not been attempted before, in Section 6.3 we carry out this construction, obtaining a theory which is an effective theory of original BV Einstein–Hilbert theory.

A first glance at codimension 2 strata

Having a BV-BFV theory, or a BFV theory on the boundary alone, one can consider its extension to a codimension 2 BV theory (or $\mathrm{BF}^2\mathrm{V}$ theory). The construction goes almost unchanged as in the first step from bulk to boundary theory, except for the degrees of the maps. As we have seen, off the codimension 1 data one can direclty read the algebra of constraints together with a cohomological presentation of the reduced phase space. In the same way, off the $\mathrm{BF}^2\mathrm{V}$ data one can read the representations carried by *boundary insertions*. Moreover, following [MSW19], a fully extended BV-BFV theory induces a solution of Witten descent equations [Wit88b] - a step towards understanding observables in General Relativity - and one can discuss the emergence of edge modes and holographic counterparts (see, e.g. [CHv95]), as well as asymptotic symmetries [RS20]. The knowledge of the $\mathrm{BF}^2\mathrm{V}$ is also useful for gluing along more complicated boundaries, presenting corners [Nar19].

A first exploration in this direction is carried out in Chapter 7 for Palatini–Cartan theory and for Einstein–Hilbert theory. As it turns out, both theories fail to generate an extended BV-BFV theory (in this case a 1-extended BFV and a 2-extended BV theory respectively, see Section 1.1 for the details).

Nonetheless, already the *pre-corner* structure, i.e. the data on the codimension 2 manifold before a symplectic reduction (which in this case is not possible due to the non-regularity of the pre-symplectic two form) contains interesting information. Furthermore, under certain assumptions on the diffeomorphisms symmetries, it is possible to perform a symplectic reduction and obtain a (partial) 2-extended BV theory in the Einstein–Hilbert case and a 1-extended BFV theory in the Palatini–Cartan case.

Diagrammatic overview

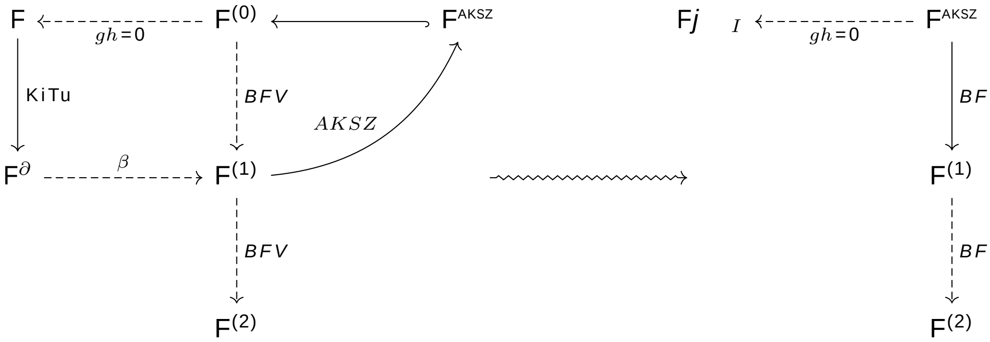
Let us summarize the content of this thesis by means of a diagrammatic overview. The work is divided into three parts, two concerning Palatini–Cartan theory (one for the three dimensional case and one for dimension greater than or equal to four) and one concerning Einstein–Hilbert theory. Note that these diagrams are just drawings for illustrative purpose and do not have the meaning of the standard commutative diagrams: here the arrows are not necessarily maps or functors but just *constructions* by which one can go from one node to the other.

We first consider the component of three dimensional Palatini–Cartan theory. As will be proved in Chapter 2, the corresponding BV theory can be fully extended to a BV-BFV theory. We represent this in a diagram by associating to each stratum a node representing the $\mathrm{BF}^k\mathrm{V}$ data assigned to it and by drawing an arrow (denoted by *BFV*) for each BFV step[7]:

$$\mathsf{F}^{(0)} \xrightarrow{BFV} \mathsf{F}^{(1)} \xrightarrow{BFV} \mathsf{F}^{(2)} \xrightarrow{BFV} \mathsf{F}^{(3)}.$$

[7] Here $\mathsf{F}^{(1)} = \mathsf{F}^{\partial}$.

This diagram also represent the best case scenario, when the reductions are possible at all strata[8]. This is not the case for Palatini–Cartan theory in dimension N 4, represented in the second diagram:

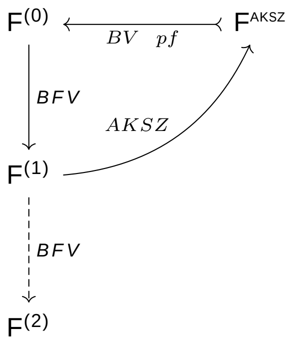


Indeed, it is not possible to induce a 1-extended BV theory on the boundary from the BV theory on the bulk ([CS19b], recap in Section 1.2.6). We denote this failure by a dashed arrow marked with symbols. As explained above, the workaround is to start over from the classical theory F (this corresponds to the ghost degree 0 part of the theory, first horizontal arrow from $\mathsf{F}^{(0)}$ to F) and the subsequently apply the Kijowski and Tulczijew (KiTu) algorithm to get the reduced phase space F^{∂}. In the time-like and space-like case one can then from this description construct a corresponding BFV theory $\mathsf{F}^{(1)}$ (β arrow in the diagram). See Chapter 4 for the details of this construction.

From the boundary BFV theory we can then try to induce a BF^{2}V structure on the corner. As established in Chapter 7, this is not possible, hence we represent the failure of the BV-BFV algorithm in codimension 2 with another dashed arrow marked with symbols. From the same BFV theory we can also construct an AKSZ theory in the bulk. We represent this construction as an arrow pointing to the AKSZ theory at the bulk level (see Section 6.4 for more details). Furthermore, in the same section we proved that there is a BV-inclusion between the so constructed AKSZ theory and the standard BV theory. We represent this BV-inclusion by an hooked arrow.

The AKSZ construction produces a theory, $\mathsf{F}^{\mathsf{AKSZ}}$, that satisfies automatically the requirements for the BV-BFV procedure. Hence on cylindrical manifolds we can draw a second diagram, on the left. This last goes closer to the best case scenario, at least for the first step, in which we have a fully extended theory (as in the three dimensional case above).

The last diagram is about Einstein–Hilbert theory:



[8] For typographical reasons we represent it horizontally, while in the next diagrams di erent strata will be in di erent vertical levels.

Here it is possible to induce the BFV theory directly from the BV theory on the bulk [CS16] (first vertical arrow). However, as in Palatini–Cartan theory, it is not possible to extend it to codimension 2 (Section 7.2). Finally, the arrows on the right of the diagram, depict the AKSZ construction. Note that the horizontal arrow connecting the AKSZ theory to the BV one is, this time, a BV pushforward (Section 6.3).

An alternative reading perspective

The reader more interested in the mathematical structures behind the BV-BFV formalism might prefer to read this thesis under a different light. Indeed, we can interpret the results presented here as a collection of examples and counterexamples in the BV-BFV formalism, the knowledge of which is of crucial importance in view of a possible axiomatisation in a categorical sense of the formalism.

In Section 1.3 we collected a list of possible notions of equivalences between BV(-BFV) theories: the gravity theories presented here, together with some lower dimensional analogues (from [Sim20]), form an almost complete set of example and counterexamples to these notions.

Furthermore, we can highlight the following features in the BV(-BFV) versions of the gravity theories:

1. Einstein–Hilbert theory provides an example of a BV theory which is 1-extendable [CS16] but not 2-extendable (Section 7.2). Moreover, the AKSZ theory built from the boundary BFV theory does not coincide with the original BV theory on the bulk but it is effectively equivalent to it (Section 6.3).

2. Palatini–Cartan theory in dimension three provides an example of a fully extendable theory that has nontrivial reductions at every step (Chapter 2).

3. In dimension greater than or equal to four, Palatini–Cartan theory provides an example of a theory that is not 1-extendable [CS19b]. A classically equivalent theory, the AKSZ one (Section 6.4) built from a BFV version (Section 4.3), can be 1-extended but not 2-extended (Section 7.1).

Yet another reading perspective

It is also possible to give another perspective for a more physics oriented reader. Indeed, we can extrapolate some relevant information from the BV(-BFV) formalism and express it in terms of more customary terminology from physics.

The main goal behind finding a BV-BFV version of a classical theory is to have a quantisation scheme for theories on manifolds with boundary compatible with cutting and gluing. Hence, in general the failure of a BV theory to induce a BFV theory gives information on the quantisability of a theory. However, a BV-BFV theory gives also information about the structure of the symmetries of the theory and of the structure of the reduced phase space on the boundary.

In particular, the following classical information can be extracted from the results given here. From Chapters 4 one can deduce a description of the reduced phase space of Palatini–Cartan theory for a space-like or time-like boundary. In the case of a light-like boundary a similar description can be found in Chapter 5. Namely we first give a description of the symplectic space of boundary fields within which we subsequently present the locus of constraints of the theory. The main difference between the two aforementioned cases is the presence of additional second class constraints in the light-like case, while in the space- and time-like case all constraints are found to be of first class.

It is also interesting to note that the AKSZ construction described in Chapter 6 produces a theory which is classically equivalent to the standard gravity theory (it has the same critical locus modulo symmetries) but has a slightly different set of symmetries.

In three dimensions the picture is different, since Palatini–Cartan theory is a topological field theory in the sense that its equations of motion are pure constraints, or—in other words—there are no propagating degrees of freedom. Besides the information that can be extracted from the BV-BFV structure mentioned above, the main physical meaning of the construction proposed in Chapter 2 is the interpretation of the set of symmetries of gravity as the different but yet equivalent generating set of the more straightforward and quantisable BF theory.

Finally, from the preliminary exploration of the corner $\mathrm{BF}^2\mathrm{V}$ structure in Chapter 7, we can deduce useful information about the corner fields and their structure.

Concluding remarks and future ideas

The results presented in this thesis offer new insight in the relation between General Relativity and the BV-BFV formalism. In particular, we can consider Palatini–Cartan theory under a new light: the construction made precise in Chapters 4, 5 and 6 outputs a BV-BFV theory of gravity in the Palatini–Cartan formalism which is (in some sense) better quantisable with respect to the original Palatini–Cartan theory, with the only additional hypothesis of working on a cylindrical manifold with a space-like or time-like boundary. Note that this hypothesis is always verified if we work in a neighbourhood of a Cauchy surface, which is the interesting physical case. We can hence proceed in the direction of quantisation in this framework.

Furthermore the BV-BFV formulation of Palatini–Cartan theory highlights even more clearly than the standard analysis the differences between the physical four dimensional case and the topological three dimensional case. Indeed this last case does not show any of the critical features of the physical one, as for example the distinction that has to be made between the construction on a space or time-like boundary and a light-like one. Nonetheless the relative similarity of the BV and BFV explicit structures make the three-dimensional case an interesting toy model to understand some of the problems that might arise in the physical case.

Concluding, we can also compare the Einstein–Hilbert case and the Palatini–Cartan one. While the first has a better behaviour in the BV-BFV formalism, the formulation of the second in terms of forms and connections (formulation that is preserved in the construction of the BFV theory and by the AKSZ construction) presents many more advantages than disadvantages. Furthermore the failure to induce an extended theory on codimension-two strata of both theories does not guarantee any further advantage to the Einstein–Hilbert formulation, which remains still too complicated for future application in the direction of quantisation. On the other hand it can be employed in future works to explore the possible definition of mass in General Relativity.

Chapter 1

Preliminary notions and known results

In this chapter we introduce the notions underlying the results of this thesis. These include an introduction to the classical (extended) BV-BFV formalism in Section 1.1 and an introduction to the Palatini–Cartan and Einstein–Hilbert classical theories together with the known results about their BV(-BFV) counterparts in Section 1.2.

Furthermore in Section 1.3 we analyse the various definitions proposed in the literature to compare different BV(-BFV) theories and provide examples and counterexample of couples of gravity theories (from the results of this thesis) satisfying (respectively not satisfying) such definitions.

1.1 The BV-BFV formalism in short

The purpose of this section is to give a working introduction to the BV-BFV formalism. We will give the fundamental definition in a compact way. For more insight on the structure and on the motivation behind this construction we refer to previous works [BV77; BV81; BF83; CMR11; CMR14; Mne17]. Furthermore we will only consider the classical version of this formalism and not consider its quantum counterpart.

We start with the definition of BV and BFV theories and give the notion of BV-BFV theory as a combination of the previous two. Afterwards we generalize these definitions to higher codimension obtaining BF^{k}V theories and k-extended BV theories.

Definition 1.1. A BV theory is a quadruple $\mathsf{F} = (\mathcal{F}, S, \varpi, Q)$ where $\mathcal{F}$ is a graded manifold (the *space of BV fields*) endowed with a degree -1 symplectic form ϖ, $S\colon \mathcal{F} \to \mathbb{R}$ is a degree 0 functional (the *BV action*) and Q is an odd Hamiltonian vector field of S with respect to ϖ of degree 1 satisfying $[Q, Q] = 0$.

Remark 1.2. Since Q is the Hamiltonian vector field of S, i.e. $\iota_Q\varpi = \delta S$ where δ is the de Rham differential on $\mathcal{F}$ and ι_Q is the contraction w.r.t. Q, we can rewrite the equation $[Q, Q] = 0$ as $(S, S) = 0$ where $(\,,\,)$ denotes the Poisson bracket defined by ϖ. This latter equation is called the Classical Master Equation (CME). Unless otherwise stated, all the actions satisfying the CME appearing in the various definitions concerning the BV formalism will be assumed to be proper solutions of the CME, i.e. solutions where all the gauge symmetries have been considered without any overcounting. The importance of considering proper solutions is related to the corresponding quantisation: indeed, if we do not consider all the symmetries we do not solve the problem to which BV aims i.e. avoid the degeneracies arising in the quantisation of gauge theories. However, since we are not interested in quantisation here, we will not deepen into details.

Remark 1.3. The fields have a parity and a $\mathbb{Z}$-grading, usually distinct. However, throughout this thesis the parity of the fields will always coincide with the parity of the degree.

Definition 1.4. An exact BFV theory is a quadruple $\mathsf{F}^\partial = \left(\mathcal{F}^\partial, S^\partial, \varpi^\partial, Q^\partial\right)$ where $\mathcal{F}^\partial$ is a graded manifold (the *space of boundary BFV fields*) endowed with a degree-0 exact symplectic form $\varpi^\partial = \delta\alpha^\partial$, $S^\partial : \mathcal{F}^\partial \to \mathbb{R}$ is a degree 1 functional and Q^∂ is the odd Hamiltonian vector field of S^∂ with respect to ϖ^∂ and such that $[Q^\partial, Q^\partial] = 0$.

Typical examples of BV and BFV theories are modeled on sections of bundles over differentiable manifolds, possibly with boundary, with $\varpi^\partial, S^\partial$ and Q^∂ respectively a local two-form, functional and vector field. Throughout the thesis, when specifying BV theories, we will assume that the equations $\iota_Q\varpi = \delta S$ and $(S, S) = 0$ are satisfied only up to boundary terms. The failure of said equations will be controlled by the data of a BV-BFV theory, as follows.

Definition 1.5 ([CMR14])**.** A BV theory $\mathsf{F} = (\mathcal{F}, S, \varpi, Q)$ is said to be 1-extended to an exact BFV theory $\mathsf{F}^\partial = \left(\mathcal{F}^\partial, S^\partial, \varpi^\partial, Q^\partial\right)$ if there exists a surjective submersion $\pi : \mathcal{F} \to \mathcal{F}^\partial$, such that the following compatibility relation is satisfied:

$$\iota_Q\varpi = \delta S + \pi^* \alpha^\partial \tag{1.1}$$

with $\varpi^\partial = \delta\alpha^\partial$ The data $\mathsf{F}^{\Uparrow 1} = \left(\mathsf{F}, \mathsf{F}^\partial, \pi\right)$ will be called 1-extended BV-BFV theory.

Remark 1.6. Notice that, from the data above, the following relation follows:

$$\iota_Q\iota_Q\varpi = 2\pi^* S^\partial. \tag{1.2}$$

Note also that, when α is a one-form, we can define a B(F)V theory using α instead of ϖ and recover this last quantity as $\varpi = \delta\alpha$. In the generalization below we will use this convention.

1.1.1 Generalization to higher codimension

We now generalize these notions to higher codimension, starting from the notion of BF^{k}V theory:

Definition 1.7. An exact BF^{k}V theory is a quadruple $\mathsf{F}^{(k)} = \left(\mathcal{F}^{(k)}, S^{(k)}, \varpi^{(k)}, Q^{(k)}\right)$ where $\mathcal{F}^{(k)}$ is a graded manifold endowed with a degree-$(k-1)$ exact symplectic form $\varpi^{(k)} = \delta\alpha^{(k)}$, $S^{(k)} : \mathcal{F}^{(k)} \to \mathbb{R}$ is a degree k functional and $Q^{(k)}$ is the odd Hamiltonian vector field of $S^{(k)}$ with respect to $\varpi^{(k)}$ and such that $[Q^{(k)}, Q^{(k)}] = 0$.

In particular for $k = 0$ and $k = 1$ we recover the standard definitions of BV and BFV theory respectively. For the generalization of the BV-BFV theory we need the notion of filtration. We define it as follows:

Definition 1.8. Let M be an m-dimensional smooth manifold. An n-stratification of M is a filtration of smooth manifolds (possibly with boundary) $\{M^{(k)}\}_{k=0\ldots n}$ of dimension $\dim(M^{(k)}) = m - k$, with $M^{(0)} = M$, such that there exists a smooth embedding $\iota^{(k+1)} \colon M^{(k+1)} \to M^{(k)}$ for every $0 \leq k < n$.

Remark 1.9. A particular example of a stratification is given by a manifold with boundaries and corners (and vertices, i.e. boundaries of corners), where the connected components of boundaries, corners and vertices compose the cells of a stratum $M^{(k)}$. For practical purposes the reader can consider this example as the main application, although our definitions allow for more general embedded submanifolds. Since in Chapters 4, 5, 6 and 7 we will be interested in this type of interpretation, we will use a different notation: given a manifold M, we will denote its boundary $M^{(1)} = \Sigma$ and its corner $M^{(2)} = \Gamma$.

The following definition generalizes the 1-extended BV theory to an n-extended one.

Definition 1.10 ([MSW19]). A ***strict, n-extended, exact BV theory***, shorthanded with ***n-extended theory***, is the assignment, to the n-stratification $\{M^{(k)}\}_{k=0\dots n}$, of the data

$$\mathfrak{F}^{\Uparrow n} = (\mathcal{F}^{(k)}, S^{(k)}, \alpha^{(k)}, Q^{(k)}, \pi^{(k)})_{k=0\dots n},$$

such that, for every $0 \leq k \leq n$,

1. $\mathcal{F}^{(k)}$ is the space of sections of a graded vector bundle $E^{(k)} \to M^{(k)}$ and $\alpha^{(k)} \in \Omega^1_{\text{loc}}(\mathcal{F}^{(k)})$ is a degree-$(k-1)$ local form[1], such that $\varpi^{(k)} = \delta\alpha^{(k)}$ is symplectic on $\mathcal{F}^{(k)}$, and δ is the de Rham differential on $\mathcal{F}^{(k)}$,
2. $\pi^{(k)} \colon \mathcal{F}^{(k)} \to \mathcal{F}^{(k+1)}$ is a surjective submersion[2],
3. $Q^{(k)}$ is an evolutionary[3], cohomological, odd vector field of degree 1 on $\mathcal{F}^{(k)}$, i.e. $[L_{Q^{(k)}}, d] = [Q^{(k)}, Q^{(k)}] = 0$, that is also projectable: $Q^{(k+1)} = (\pi^{(k)})_* Q^{(k)}$,
4. $S^{(k)} \in \Omega^0_{\text{loc}}(\mathcal{F}^{(k)})$ is a (real-valued) degree-k local functional,

such that, for $0 \leq k \leq n-1$,

$$\iota_{Q^{(k)}}\varpi^{(k)} = \delta S^{(k)} + \pi^{(k)*}\alpha^{(k+1)} \tag{1.3}$$

whereas for $k = n$, we require

$$\iota_{Q^{(n)}}\varpi^{(n)} = \delta S^{(n)}. \tag{1.4}$$

When $n = \dim(M^{(0)})$ we say that the theory is ***fully extended***. When $n = 0$, the data defines a BV theory, when $n = 1$ a BV-BFV theory.

Remark 1.11. As for (1.2), notice that a direct consequence of (1.3) is

$$\frac{1}{2}\iota_{Q^{(k)}}\iota_{Q^{(k)}}\varpi^{(k)} = \pi^{(k)*}S^{(k+1)}, \tag{1.5}$$

as well as $\iota_{Q^{(n)}}\iota_{Q^{(n)}}\varpi^{(n)} = 0$. This is proven in [CMR14, Proposition 3.1].

Remark 1.12. A direct interpretation of Equation (1.3) is that, on every stratum, $Q^{(k)}$ is the Hamiltonian vector field of the action functional $S^{(k)}$ up to boundary terms (1.3), and that the classical master equation is satisfied up to boundary terms (1.5).

It is possible to further generalise this notion to include extension of BF^{k}V theory not arising from an extension of a BV theory.

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[1] In this thesis local forms are differential forms on $\mathcal{F}^{(k)}$ which depend only on a finite number of derivatives of fields, i.e. sections of E. See [MSW19].

[2] For $k = n$ the projection $\pi^{(n)}$ is the unique map from $\mathcal{F}^{(n)}$ to the empty set.

[3] A vector field on $\mathcal{F}$ is *evolutionary* if $L_X d = dL_X$ where d is the de Rahm differential on M and $L_X = [\iota_X, \delta]$.

Definition 1.13. A ***strict, n-extended, exact BF^{m}V theory*** is the assignment, to the n-stratification $\{M^{(k)}\}_{k=0\dots n}$, of the data

$$\mathsf{F}^{\partial^m\Uparrow n} = (\mathcal{F}^{(k)}, S^{(k)}, \alpha^{(k)}, Q^{(k)}, \pi^{(k)})_{k=m\dots n+m},$$

satisfying the same properties of definition 1.10.

We can recap the theories and the data associated to them in the following table:

Theory	Notation	Data	Strata
BV	F	$\mathsf{F}^{(0)}$	$M^{(0)} = M$
BFV	F^{∂}	$\mathsf{F}^{(1)}$	$M^{(1)} = \Sigma$
BF^{k}V	$\mathsf{F}^{(k)}$	$\mathsf{F}^{(k)}$	$M^{(k)}$
BV-BFV	$\mathsf{F}^{\Uparrow 1}$	$\mathsf{F}^{(0)}, \mathsf{F}^{(1)}$	$M^{(0)}, M^{(1)}$
n-extended BV	$\mathsf{F}^{\Uparrow n}$	$\mathsf{F}^{(0)}, \dots, \mathsf{F}^{(n)}$	$M^{(0)}, \dots M^{(n)}$
n-extended BF^{m}V	$\mathsf{F}^{\partial^m\Uparrow n}$	$\mathsf{F}^{(m)}, \dots, \mathsf{F}^{(n+m)}$	$M^{(0)}, \dots M^{(n)}$

where $\{M^{(k)}\}_{k=0\dots n}$ is an n-stratification. Note that a 1-extended BV theory is a BV-BFV theory.

We will be concerned here with classical field theories that enjoy symmetries given by Lie algebra actions, and our starting point to build an extended (exact) BV theory is a couple $(\mathcal{F}^{cl}, S^{cl})$. The space of classical fields $\mathcal{F}^{cl}$ is the space of sections[4] of some sheaf or bundle $E \to M$, while S^{cl} is a local functional on $\mathcal{F}^{cl}$, i.e. a function of the fields and a finite number of jets, called ***action functional***. The symmetry data[5] is sometimes encoded in an involutive distribution[6] $\mathcal{D}^{cl}$ on $\mathcal{F}^{cl}$.

The first step is to build a BV theory (or 0-extended BV) $\mathsf{F} = (\mathcal{F}, S, \varpi, Q)$, is to promote the space of classical fields to a (-1)-symplectic graded manifold $(\mathcal{F}, \varpi)$, extend the action S^{cl} to a BV action S satisfying the CME and to construct a cohomological vector field Q on $\mathcal{F}$, the Hamiltonian vector field of S w.r.t. ϖ [BV77; BV81]. It is then sometimes possible to build an m-extended BV theory using a constructive approach.

Let $\{M^{(k)}\}_{k=0\dots n}$ be an n-stratification of M and consider on it an n-extended BV theory. According to Definition 1.10 on the n-th stratum we have equation (1.4). If we allow an $(n+1)$-codimension stratum, Equation (1.4) will likely be spoiled, or we simply extend the theory by zero. In the former case, if we can find $\pi^{(n)}\colon \mathcal{F}^{(n)} \longrightarrow \mathcal{F}^{(n+1)}$, together with $\alpha^{(n+1)}$ and $S^{(n+1)}$ satisfying (1.3) (and (1.5)), and $\varpi^{(n+1)} = \delta\alpha^{(n+1)}$ non-degenerate, we will have extended the BV-BFV theory to codimension-$(n+1)$.

In a practical scenario, this goes through by integrating by parts the terms in $\delta S^{(n)}$, but the resulting data on the higher-codimension stratum does not automatically satisfy the axioms in Definition 1.10. In particular, the existence of $\mathcal{F}^{(n+1)}$ as a smooth symplectic manifold (and hence of $\pi^{(n)}$) is not always guaranteed (see [CS17; CS19b]). When this happens the theory is only ***n-extendable***. We summarise the previous discussion with the following definition:

Definition 1.14. Let M be an m-dimensional smooth manifold and let $\mathsf{F}^{\Uparrow 0}$ an exact BV theory on it. We say that the BV-theory $\mathsf{F}^{\Uparrow 0}$ is n-extendable if, for every n-stratification such that $M^{(0)} = M$, there exists an n-extended exact BV theory $\mathsf{F}^{\Uparrow n}$ associated to it. If $n = \dim(M)$ we will say that $\mathsf{F}^{\Uparrow 0}$ is fully extendable. Analogously, we say that a BF^{m}V theory on $M^{(m)}$ is n-extendable if there exists an n-extended exact BF^{m}V theory $\mathsf{F}^{\partial^m\Uparrow n}$ associated to it.

[4] In the present thesis we will consider principal connections as fields. However, we can reduce to this setting by expanding around an arbitrary reference connection.

[5] By symmetry data we mean a generating set of gauge symmetries.

[6] In full generality the BV formalism only requires that $\mathcal{D}^{cl}$ be involutive on the critical locus of S^{cl}, i.e. the space of solutions of the associated variational problem.

A more detailed description of the procedure to induce codimension $k+1$ BFV data from codimension k data is given in the next Section, while a full detailed example is in the proof of Proposition 2.10.

1.1.2 BV-BFV construction strategy

We outline here the strategy we employ to construct the maps $\pi^{(k)}$ of Definition 1.10, as was introduced in [CS16]. We analyse here the extension from a generic codimension-k stratum to a codimension-$(k+1)$ stratum.

The goal is to construct data corresponding to the codimension-$(k+1)$ stratum (the $(k+1)$-extended theory). We will assume here for simplicity that $M^{(k+1)} = \partial M^{(k)}$ is a boundary. The first step is to consider the variation of the action functional in the bulk and to construct appropriate data to satisfy (1.3) and the axioms in Definition 1.10. The variation of $\delta S^{(k)}$ will consist of two terms generated by integration by parts: the bulk term will be interpreted as $\iota_{Q^{(k)}}\varpi^{(k)}$ — defining the Euler–Lagrange equations for the variational problem — while the remainder, a boundary term, is interpreted as a one-form $\check{\alpha}^{(k+1)}$ on some appropriate space (see below). Namely, we have an equation formally equivalent to (1.3)

$$\delta S^{(k)} = \iota_{Q^{(k)}}\varpi^{(k)} \quad \check{\pi}^{(k)} \ \check{\alpha}^{(k+1)},$$

with the difference that the correction term $\check{\pi}^{(k)} \ \check{\alpha}^{(k+1)}$ lives on the intermediate space $\check{F}^{(k+1)}$, defined as the space of fields and transversal jets restricted to the boundary, with $\check{\pi}^{(k)} : F^{(k)} \to \check{F}^{(k+1)}$ being the restriction of fields to $M^{(k+1)}$. We will call $\check{F}^{(k+1)}$ space of ***pre-boundary*** fields.

We define $\check{\varpi}^{(k+1)} := \delta\check{\alpha}^{(k+1)}$, which is a closed form, but in general degenerate and hence not symplectic. However, one expects it to be pre-symplectic i.e. such that the kernel of the associated map

$$\begin{aligned} \check{\varpi}^{(k+1)\sharp} : T\check{F}^{(k+1)} &\to T \ \check{F}^{(k+1)} \\ X &\mapsto \check{\varpi}^{(k+1)\sharp}(X) = \check{\varpi}^{(k+1)}(X, \ ) \end{aligned}$$

is ***regular***. This condition, which has to be explicitly checked, shows that the kernel is a subbundle in $T\check{F}^{(k+1)}$, and one can define the space of boundary fields to be the symplectic reduction[7]:

$$F^{(k+1)} := \check{F}^{(k+1)}/\ker(\check{\varpi}^{(k+1)\sharp}) \tag{1.6}$$

Remark 1.15. This regularity condition of the kernel of $\check{\varpi}^{(k+1)\sharp}$ is the mathematical core of part of the content this thesis. Indeed, the non-regularity of this kernel is the only point where the BV-BFV procedure can fail.

The symplectic reduction map $\pi^{(k+1)}_{\mathsf{red}} : \check{F}^{(k+1)} \to F^{(k+1)}$ can be computed in a chart by explicitly flowing along the vertical vector fields (i.e. $X \in T\check{F}^{(k+1)}$ such that $\check{\varpi}^{(k+1)\sharp}(X) = 0$). The projection $\pi^{(k)} : F^{(k)} \to F^{(k+1)}$ to the true space of boundary fields, is then obtained by composing $\check{\pi}^{(k)}$ with the symplectic reduction map[8] $\pi^{(k+1)}_{\mathsf{red}}$, i.e. $\pi^{(k)} := \pi^{(k+1)}_{\mathsf{red}} \ \check{\pi}^{(k)}$. To

[7] Typically, the procedure we employ will provide a chart for the quotient, showing that the symplectic reduction is indeed smooth.

[8] The numbering convention we are using is such that the codimension index of the various maps coincides with that of their domain.

summarise, we have

$$\begin{array}{ccl} F^{(k)} & & \varpi^{(k)},\ (k-1)\text{-symplectic} \\ \downarrow \pi^{(k)} & & \\ \check{F}^{(k+1)} & & \check{\varpi}^{(k+1)},\ k\text{-presymplectic} \\ \downarrow \pi^{(k)}_{\mathrm{red}} & & \\ F^{(k+1)} & & \varpi^{(k+1)},\ k\text{-symplectic} \end{array} \qquad \text{(with } \pi^{(k)}: F^{(k)} \to F^{(k+1)} \text{ the composite)} \tag{1.7}$$

To recover the rest of the BV-BFV data on the higher codimension stratum, we first define a *pre-boundary* action functional $\check{S}^{(k+1)}$ on $\check{F}^{(k+1)}$ *via* an analogue of Equation (1.5):

$$\iota_{Q^{(k)}}\iota_{Q^{(k)}}\varpi^{(k)} = 2\check{\pi}^{(k)*}\check{S}^{(k+1)}.$$

It then follows from the BV-BFV theorems [CMR14, Section 3.1.3] that $\check{S}^{(k+1)}$ is basic with respect to $\pi^{(k+1)}_{\mathrm{red}} : \check{F}^{(k+1)} \to F^{(k+1)}$: namely, there exists $S^{(k+1)}$ on $F^{(k+1)}$ such that $\pi^{(k+1)*}_{\mathrm{red}} S^{(k+1)} = \check{S}^{(k+1)}$, and consequently

$$\iota_{Q^{(k)}}\iota_{Q^{(k)}}\varpi^{(k)} = 2\pi^{(k)*}S^{(k+1)}.$$

Moreover, [CMR14, Section 3.1.3] establishes that on $F^{(k+1)}$ one has the evolutionary, cohomological, projectable vector field $Q^{(k+1)} = \pi^{(k)}_{*}Q^{(k)}$, which will be Hamiltonian w.r.t. $S^{(k+1)}$ up to higher codimension terms.

1.2 Gravity theories in the BV-BFV formalism

In this section we introduce the gravity theories that we will consider in the thesis. We will be mainly interested in the Einstein–Hilbert formulation and in the Palatini–Cartan one and their BV(-BFV) counterparts. The state of the art for these theories is rather different, since for the first we have already a BV-BFV formulation, while for the second such a formulation on a generic manifold is missing.

The aim of this section is to introduce these theories in order to provide a solid starting point for the construction and results of the following chapters.

1.2.1 Einstein–Hilbert theory, Classical theory

The dynamical field of general relativity in the usual formulation by Einstein and Hilbert is a Lorentzian metric $g \in PR(M)$ where $PR(M)$ denotes the set of pseudo-Riemannian metrics on M and the action functional is

$$S^{cl} = \int_M (R - \Lambda)\sqrt{-g},$$

where M is a manifold, R the scalar curvature of g, $\Lambda \in \mathbb{R}$ the cosmological constant (a fixed parameter) and $\sqrt{-g}$ the density induced by g.

1.2.2 Einstein–Hilbert theory, BV theory

The BV formulation of Einstein–Hilbert theory has been first described in [Sch15; CS16].

The starting point is the identification of the symmetries of the classical theory. By construction the classical action is invariant under the action of diffeomorphisms. We can parametrize them with an (odd) vector field ξ *2* $\Gamma(T[1]M)$ in the following way:

$$\delta_\xi g \quad L_\xi g$$

where $\delta_\xi g$ denotes the infinitesimal variation of g under the action of the diffeomorphism ξ. We present the self-action of diffeomorphisms by $\delta_\xi \xi \quad \frac{1}{2}[\xi, \xi]$, the symmetry distribution generated by them is involutive. Hence the corresponding BV theory can be found by extending the space of fields as

$$F_{EH} = T\ [\quad 1]\,[PR(M) \quad \Gamma(T[1]M)]\,. \tag{1.8}$$

The correponding BV action is then

$$S_{EH} = \int_M (R \quad \Lambda)^{p}\overline{g} \quad L_\xi g g^{y} + \frac{1}{2}\iota_{[\xi,\xi]}\xi^{y} \tag{1.9}$$

where the fields in the fiber of F_{EH} are denoted by g^y *2* $\Gamma[\quad 1]\left(S^2TM\right) \quad \mathrm{Dens}(M)$ and ξ^y *2* $\Omega^1(M) \quad \mathrm{Dens}(M)$. Hence the canonical symplectic form reads

$$\varpi_{EH} = \int_M \delta g \delta g^y + \iota_{\delta\xi}\delta\xi^y.$$

It can be proved that the action (1.9) is a proper solution of the CME [CS16]. The last bit of information composing a BV theory, the cohomological vector field Q_{EH}, can be uniquely reconstructed as the Hamiltonian vector field of S_{EH} with respect to ϖ_{EH}.

1.2.3 Einstein–Hilbert theory, BV-BFV theory

From the BV version of Einstein–Hilbert theory, it is possible to construct a BV-BFV theory, by allowing the manifold M to have a boundary Σ. This fact has been proven in [CS16] under some mild assumption on the structure of the manifold, namely those admitting an ADM decomposition [ADM59]. We collect here only the most important results of the construction and we refer to the original article for all the details regarding the derivation of such a structure.

Theorem 1.16 ([CS16]). *For all* $\dim(M) = N > 2$, *the data* $(F_{EH}, S_{EH}, Q_{EH}, \varpi_{EH})$ *is 1-extendable.*

We will denote the induced data on the boundary by $(F^\partial_{EH}, S^\partial_{EH}, Q^\partial_{EH}, \varpi^\partial_{EH})$. In particular Theorem 1.16 can be translated as the possibility of finding a BV-BFV theory starting from a BV theory on the bulk of a manifold with boundary. We now describe the induced data on the boundary.

The BFV theory for GR in the Einstein–Hilbert formalism (as described in [CS16]) associates to any $(N \quad 1)$-dimensional (pseudo)-Riemannian manifold Σ the graded 0-symplectic manifold

$$F^\partial_{EH}(\Sigma) = T\ \underbrace{\left(S^2_{nd}(T\Sigma) \quad \mathsf{X}[1](\Sigma) \quad C^{1}[1](\Sigma)\right)}_{f\gamma,\xi^\partial,\xi^n g}, \tag{1.10}$$

where $S^2_{nd}(\Sigma)$ denotes the space of non-degenerate symmetric tensor fields of rank two (also known as co-metrics), with canonical exact symplectic structure:

$$\varpi^\partial_{EH}(\Sigma) = \delta\alpha^\partial_{EH}(\Sigma) = \delta \int h\Pi, \delta\gamma i + h\varphi_\partial, \delta\xi^\partial i + h\varphi_n, \delta\xi^n i, \tag{1.11}$$

where $f\Pi, \varphi_\partial, \varphi_n g$ denote variables in the cotangent fiber, respectively conjugate to $f\gamma, \xi^\partial, \xi^n g$.

Remark 1.17. The components $(\gamma)^{ab}$ of $\gamma \in S^2_{nd}(T\Sigma)$ can be thought of as the inverse of a (pseudo-)Riemannian metric on Σ (which we denote by γ^{-1}). With a slight abuse of notation[9] we will directly denote by $\sqrt{\gamma} = \sqrt{\det(\gamma_{ab})}$ the square root of the determinant of γ^{-1}. In other words, $\sqrt{\gamma}$ is the half density associated to the metric γ^{-1}. The conjugate field to γ is a section of the second symmetric tensor power of the cotangent bundle of Σ tensored with densities on Σ, $\Pi \in S^2(T^*\Sigma) \otimes \mathrm{Dens}(\Sigma)$. Observe that all fields in the fibres of the cotangent bundle (1.10) are sections of the respective dual bundles, tensored with densities.

In addition to $F^\partial_{EH}(\Sigma)$ and $\varpi^\partial_{EH}(\Sigma)$, the BV-BFV procedure outputs a functional of degree 1 on $F^\partial_{EH}(\Sigma)$, called BFV action. It is given by the local expression

$$S^\partial_{EH}(\Sigma) = \int_\Sigma \left\{ H_n\xi^n + \langle \Pi, L_{\xi^\partial}\gamma \rangle + \varphi_n L_{\xi^\partial}\xi^n - \gamma(\varphi_\partial, d\xi^n)\xi^n + \left\langle \varphi_\partial, \frac{1}{2}[\xi^\partial, \xi^\partial] \right\rangle \right\} \tag{1.12}$$

where we have denoted the ***Hamiltonian constraint density*** by

$$H_n(\gamma, \Pi) = \left(\frac{1}{\sqrt{\gamma}} \left(\mathrm{Tr}_\gamma[\Pi^2] - \frac{1}{d-1}\mathrm{Tr}_\gamma[\Pi]^2 \right) + \sqrt{\gamma}\left(R^\partial - 2\Lambda \right) \right) \tag{1.13}$$

with R^∂ is the trace of the Ricci tensor with respect to the metric γ^{-1}, $\Lambda \in \mathbb{R}$ is the cosmological constant, $\mathrm{Tr}_\gamma[\Pi^2] = \gamma^{ab}\gamma^{cd}\Pi_{bc}\Pi_{ad}$ and $\mathrm{Tr}_\gamma\Pi$ is the density $\gamma^{ab}\Pi_{ab}$. Observe that we can also denote the ***momentum constraint density*** as the density-valued one-form

$$H_\partial \colon \mathfrak{X}(\Sigma) \to \mathrm{Dens}(\Sigma) \qquad H_\partial(X) = \langle \Pi, L_X\gamma \rangle \tag{1.14}$$

for $X \in \mathfrak{X}(\Sigma)$.

Remark 1.18. One can integrate the density of equation (1.13) against a function $\lambda \in C^\infty(\Sigma)$, or integrate the density in (1.14) to get local functionals on fields. Then λ and X play the role of Lagrange multipliers, to enforce the so-called Hamiltonian and momentum constraints.

The Hamiltonian vector field $Q^\partial_{EH}(\Sigma)$ of $S^\partial_{EH}(\Sigma)$ with respect to $\varpi^\partial_{EH}(\Sigma)$ is a differential on $C^\infty(F^\partial_{EH}(\Sigma))$, the BFV differential, and its cohomology in degree zero is the reduced phase space defined by the constraints $\{H_n, H_\partial\}$.

Definition 1.19. We define BFV Einstein–Hilbert theory associated to be the assignment

$$\Sigma \mapsto \mathfrak{F}^\partial_{EH}(\Sigma) = (F^\partial_{EH}(\Sigma), S^\partial_{EH}(\Sigma), \varpi^\partial_{EH}(\Sigma), Q^\partial_{EH}(\Sigma)). \tag{1.15}$$

In order to complete the BV-BFV structure we have to provide also a surjective submersion $\pi : F_{EH} \to F^\partial_{EH}$. This can be found in [CS16], but we do not rewrite it here, as the expression is rather long and it does not have any particular importance in this work.

1.2.4 Palatini–Cartan theory, classical theory

In this section we focus on the formulation that goes under the name of Palatini–Cartan [Car22; Wit88a; Car98; Wis09](or also Palatini–Cartan–Holst in the four-dimensional case [Hol96]). It is based on Palatini's calculation of the variation of Riemann's tensor in terms of the Christoffel symbols (known as Palatini identity [Pal19]), later extended to the idea of treating the connection as an independent field, and on Cartan's observation [Car22] that a metric may be alternatively presented in terms of a local frame.

[9] This is not really problematic, since its variation reads $\delta\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\delta\gamma_{ab} = -\frac{1}{2}\sqrt{\gamma}\gamma_{ab}\delta\gamma^{ab}$ and we can use either formula according to our needs. If we wanted to use the correct notation we should simply replace $\sqrt{\gamma}$ with its reciprocal, in formula (1.13).

We consider an N-dimensional compact, oriented[10] smooth manifold M together with a reference Lorentzian structure so that we can reduce the frame bundle to an $SO(N-1,1)$-principal bundle $P \to M$. We denote by $\mathcal{V}$ the associated vector bundle by the standard representation. Each fibre of $\mathcal{V}$ is isomorphic to an N-dimensional vector space V with a Lorentzian inner product η on it. The inner product allows the identification $\mathfrak{so}(N-1,1) = \bigwedge^2 V$. Furthermore we use the shortened notation

$$\Omega^{i,j} := \Omega^i \left(M, \textstyle\bigwedge^j \mathcal{V}\right)$$

and omit the wedge product (both in $\bigwedge^\bullet T^*M$ and in $\bigwedge^\bullet \mathcal{V}$) in the formulas when no confusion can arise. The particular choice of reference Lorentzian metric is irrelevant; see also footnote 11.

Classical fields

The theory has two dynamical fields[11]. The first is a Cartan coframe, i.e., an orientation preserving bundle isomorphism covering the identity from the tangent bundle to $\mathcal{V}$

$$e \colon TM \xrightarrow{\sim} \mathcal{V}.$$

The coframe field is also known as the tetrad or vierbein in four dimensions. The metric of the Einstein–Hilbert approach is recovered as

$$g_{\mu\nu} = (e_\mu, e_\nu). \tag{1.16}$$

Note that there is more redundancy in e than in g, and this will be canceled by more gauge transformations in the BV formulation. We may regard the coframe e as an element of $\Omega^{1,1}$ (plus the non-degeneracy condition to be an orientation preserving isomorphism). Furthermore, throughout the article we use the shorthand notation e^k to denote kth wedge power of e.

The second dynamical field is an orthogonal connection ω on $\mathcal{V}$. The isomorphism e allows transforming the connection ω into an affine connection Γ that is automatically compatible with the metric g—a metric connection. The space of the P-connections, denoted with $\mathcal{A}(M)$, can be identified, via choosing a reference connection ω_0, with $\Omega^{1,2}$ thanks to $\mathfrak{so}(N-1,1) = \bigwedge^2 V$. We denote by d_ω and by $F_\omega \in \Omega^{2,2}$ respectively the covariant derivative $\Omega^{\bullet,\bullet} \to \Omega^{\bullet+1,\bullet}$ associated to a connection ω and its curvature.

Classical action

The classical action functional for Palatini–Cartan theory reads

$$S^{cl} = \int_M \left[\frac{1}{(N-2)!} e^{N-2} F_\omega - \frac{1}{N!} \Lambda e^N\right]. \tag{1.17}$$

Remark 1.20. Note that each term belongs to $\Omega^{N,N}$, which can be canonically identified, via $\sqrt{|\det \eta|}$, with the space of densities on M. Indeed, an element of $\Omega^{N,N}$ is a section of $\det T^*M \otimes \det \mathcal{V}$. On the other hand, $\sqrt{|\det \eta|}$ is a section of $|\det \mathcal{V}^*|$, so its product with an element of

[10] Extensions to noncompact manifolds are possible, but outside the main objective of this thesis. One possible way out is to define fields as compactly supported sections of the bundles on the noncompact components of a manifold. Otherwise it is possible to assume asymptotic conditions such as for example the ones in [RS20]. Orientability is not necessary, but we restrict to orientable manifolds for simplicity.

[11] The choice of V and η is immaterial. Different choices will produce equivalent field theories related by linear redefinitions of the fields.

$\Omega^{N,N}$ is a section of $\det T^*M \otimes \mathrm{or}(\mathcal{V})$, where $\mathrm{or}(\mathcal{V})$ is the orientation bundle of $\mathcal{V}$. Under our assumption that the isomorphism between TM and $\mathcal{V}$ is orientation preserving, we have $\mathrm{or}(TM) = \mathrm{or}(\mathcal{V})$, so the product of an element of $\Omega^{N,N}$ with $\sqrt{|\det \eta|}$ is a section of $|\det T^*M|$, i.e., a density. Furthermore it is actually possible to choose $\mathcal{V}$ in such a way that $\sqrt{|\det \eta|}$ is equal to one. Namely, pick a Lorentzian metric on M and reduce its frame bundle to the orthogonal frame bundle P. Then one can define $\mathcal{V}$ as the associate bundle $P \times_{O(N-1,1)} W$, where W is the fundamental representation, endowed with the Minkowski metric. With this choice η is the constant Minkowski metric, and the transitions functions of $\det \mathcal{V}$ are locally constant and equal to ± 1. Moreover, $\det T^*M \otimes \det \mathcal{V}$ is directly equal to $|\det T^*M|$, so that elements of $\Omega^{N,N}$ are canonically the same as densities. Hence we will omit writing down the factor $\sqrt{|\det \eta|}$ explicitly in the integrals.

Remark 1.21. Note that it is possible to consider other terms in the action, namely $e^{N-2k}F_\omega^k$ for every $k \leq N/2$. These other terms will however yield Euler–Lagrange equations involving higher derivatives of the fields, apart from the term $F_\omega^{N/2}$, which is topological (it is the Holst term in four dimensions). Since the preliminaries results in [CS19c] did not show any remarkable difference in the standard Palatini–Cartan and Holst theory, we do not consider in this thesis this generalization and postpone its analysis to future work.

The Euler–Lagrange equation obtained by a variation of ω is $\mathrm{d}_\omega(e^{N-2}) = 0$.[12] By the Leibniz rule this equation may be rewritten as $e^{N-3}\mathrm{d}_\omega e = 0$, which, by the non-degeneracy condition on e,[13] is equivalent to

$$\mathrm{d}_\omega e = 0. \tag{1.18}$$

It may be easily shown that this condition is equivalent to the condition that the affine connection Γ induced by ω be torsion free. Since Γ is also metric, it must then be the Levi-Civita connection, and this determines a unique ω_e solving (1.18) for a given e.

The Euler–Lagrange equation obtained by a variation of e is

$$\frac{1}{(N-3)!}e^{N-3}F_\omega - \frac{1}{(N-1)!}\Lambda e^{N-1} = 0. \tag{1.19}$$

Inserting ω_e, this equation turns out to be equivalent to Einstein's equation for the metric g defined in (1.16).

Remark 1.22. Truly, to obtain Equations (1.18) one needs injectivity of the map $e^{N-3}\wedge$. From the results of Lemma 3.3 will show that $e^{N-3} \wedge d_\omega e = 0$ is indeed equivalent to (1.18), while no further simplifications can be applied to (1.19). Moreover, this observation will turn out to be true only in the bulk, and will play a crucial role in the definition of boundary variables and constraints (see Sections 3.1 and 4.2.2).

We can collect all the information above in the following definition.

Definition 1.23. Classical N-dimensional General Relativity in the Palatini–Cartan formalism (PC) is the pair $(\mathcal{F}_{PC}^{cl}, S_{PC}^{cl})$ where

$$\mathcal{F}_{PC}^{cl} = \Omega_{nd}^1(M, \mathcal{V}) \oplus \mathcal{A}(M)$$

is the space of fields and the action functional reads

$$S_{PC}^{cl} = \int_M \left[\frac{1}{(N-2)!}e^{N-2}F_\omega - \frac{1}{N!}\Lambda e^N\right].$$

[12]One gets $\mathrm{d}_\omega(e^{N-2}) = 0$ directly with the choice of constant fibre metric as in remark 1.20. In general, the Euler–Lagrange equation is $\mathrm{d}_\omega(\sqrt{|\det \eta|}e^{N-2}) = 0$, but, since ω is an orthogonal connection, we have $\mathrm{d}_\omega \eta = 0$ and, therefore, $\mathrm{d}_\omega\sqrt{|\det \eta|} = 0$. By the Leibniz rule, we may omit the nonzero factor $\sqrt{|\det \eta|}$.

[13]See Remark 1.22. Note also that the non-degeneracy condition is obviously not necessary in case $N = 3$.

1.2.5 Palatini–Cartan theory, BV theory

In order to define a BV theory extending classical PC theory (Definition 1.23), we have to incorporate the symmetries by extending the space of fields to a graded manifold. The classical functional S_{PC}^{cl} is invariant under the action of *internal* gauge transformations $SO(N-1,1)$ and the action of spacetime diffeomorphisms. We parametrize their associated Lie-algebra actions with two *ghost* fields, $c \in \Omega^0[1](M, \wedge^2 \mathcal{V})$ and $\xi \in \Gamma[1](TM)$ respectively:

$$\delta_\xi e \mapsto L_\xi^\omega e \qquad \delta_\xi \omega \mapsto \iota_\xi F_\omega$$
$$\delta_c e \mapsto [c, e] \qquad \delta_c \omega \mapsto d_\omega c$$

where $L_\xi^\omega := [\iota_\xi, d_\omega]$ is the graded commutator between the contraction with respect to ξ (a degree-0 derivation), and d_ω (a degree-1 derivation). With these quantities we can also compute $\iota_{[\xi,\xi]} := [L_\xi^\omega, \iota_\xi]$. Note that by [CS19b, Lemma 18] $\iota_{[\xi,\xi]} = [L_\xi^\omega, \iota_\xi] = [L_\xi, \iota_\xi]$.

The BV structure associated to these symmetries has been studied in [CS19b, Section 3].

Definition 1.24. The BV theory for General Relativity in dimension N is given by the data $\mathfrak{F}_{PC}^{0} = (\mathcal{F}_{PC}, S_{PC}, \alpha_{PC}, Q_{PC})$ where the BV space of fields is

$$\mathcal{F}_{PC} = T^*[-1]\left(\Omega_{nd}^1(M, \mathcal{V}) \oplus \mathcal{A}(M) \oplus \Omega^0[1](M, \wedge^2 \mathcal{V}) \oplus \Gamma[1]TM\right)$$

denoting the corresponding fields in the cotangent fibre by $e^\dagger = \Omega^{N-1}[-1](M, \wedge^{N-1}\mathcal{V})$, $\omega^\dagger \in \Omega^{N-1}[-1](M, \wedge^{N-2}\mathcal{V})$, $c^\dagger = \Omega^{\mathrm{top}}[-2](M, \wedge^{N-2}\mathcal{V})$ and $\xi^\dagger \in \Omega^{\mathrm{top}}[-2](M, T^*M)$, and symmetry generators as $\xi \in \Gamma[1]TM$ and $c \in \Omega^0[1](M, \wedge^2\mathcal{V})$. The BV one-form and action functional[14] are[15]

$$\alpha_{PC} = \int_M e^\dagger \delta e + \omega^\dagger \delta\omega + c^\dagger \delta c + \iota_\xi \delta \xi^\dagger, \qquad \varpi_{PC} = \delta \alpha_{PC}$$

$$\begin{aligned} S_{PC} = \int_M & \frac{1}{(N-2)!} e^{N-2} F_\omega - \frac{1}{N!}\Lambda e^N + e^\dagger \left(L_\xi^\omega e - [c,e]\right) + \omega^\dagger \left(\iota_\xi F_\omega - d_\omega c\right) \\ & + \frac{1}{2} c^\dagger \left(\iota_\xi \iota_\xi F_\omega - [c,c]\right) + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger. \end{aligned} \tag{1.20}$$

The vector field Q_{PC}, satisfying $\iota_{Q_{PC}}\varpi_{PC} = \delta S_{PC}$ up to boundary terms, is given by[16]:

$$\begin{aligned} Qe &= L_\xi^\omega e - [c,e] & Q\omega &= \iota_\xi F_\omega - d_\omega c \\ Qc &= \frac{1}{2}\left(\iota_\xi \iota_\xi F_\omega - [c,c]\right) & Q\xi &= \frac{1}{2}[\xi,\xi] \\ Qe^\dagger &= -\frac{1}{(N-3)!} e^{N-3} F_\omega - \frac{1}{(N-1)!}\Lambda e^{N-1} - L_\xi^\omega e^\dagger - [c, e^\dagger] & Qc^\dagger &= -[e^\dagger, e] - d_\omega \omega^\dagger + [c^\dagger, c] \\ Q\omega^\dagger &= -\frac{1}{(N-3)!} e^{N-3} d_\omega e - \iota_\xi [e^\dagger, e] - d_\omega(\iota_\xi \omega^\dagger) - [c, \omega^\dagger] + \frac{1}{2} d_\omega(\iota_\xi \iota_\xi c^\dagger) & & \\ Q\xi^\dagger &= -e^\dagger d_\omega e - d_\omega e^\dagger e - \omega^\dagger F_\omega + \iota_\xi c^\dagger F_\omega + \partial_\bullet \xi^a \xi_a^\dagger + \partial_a \xi^a \xi_\bullet^\dagger, & & \end{aligned} \tag{1.21}$$

where we used the notation described in Remark 1.26 below, we dropped the PC-subscript and we denoted the 1-form coefficient of elements in $\Omega^{\mathrm{top}}[-2](M, T^*M)$ with a bullet.

[14] Note that S_{PC} is a proper solution of the CME.

[15] See Remarks 1.20 and 1.25 for the meaning of the integration and of the last term of the action respectively.

[16] We list here the *components* of the vector field Q_{PC}, see Remark 1.26

Remark 1.25. The *antighost* field ξ^{y} 2 $\Omega^{\text{top}}[\ 2](M, T\ M)$ is an element in the fiber of $T\ [\ 1](\Gamma[1]TM)$. In order to treat it homogeneously with respect to all other fields we can equivalently view it as $\xi^{y} = \chi\quad \mathsf{V}$ 2 $\Omega^{1}(M)\quad \Omega^{\text{top}}(M, \wedge^{N} V)$ multiplying it by a fixed normalised volume form V 2 $\Omega^{\text{top}}(M, \wedge^{N} V)$. Its integration will recover the original ξ^{y}, and it will be particularly useful to simplify the expressions appearing in Proposition 2.10.

Remark 1.26. We will describe the vector field Q here and in the following chapters by its component, using the following convention. Let φ_i 2 F the set of fields, then we can write Q acting on F as

$$Q = \int_M \sum_i Q\varphi_i \frac{\delta}{\delta \varphi_i}.$$

Hence in order to fully describe Q we can just give the expression for the components Q_{φ_i}.

1.2.6 Failure of the BV-BFV construction for N 4

We outline in this section the results of [CS19b], i.e. the failure of the BV-BFV construction in dimension N 4. This will be particularly useful in Chapter 4 to draw a comparison with the classical case. Furthermore it will constitute a reference point in Chapter 7, since in codimension 2 similar issues are found. We recap the results for $N = 4$, but they can be easily generalized for $N > 4$.

The main result is the following theorem:

Theorem 1.27 ([CS19b]). *The BV data* $\mathsf{F}^{"0}_{PC} = (F_{PC}, S_{PC}, \alpha_{PC}, Q_{PC})$ *is not 1-extendable to a BV-BFV theory.*

We will not give here a full proof (for which we refer to the original article) but we highlight the main steps. The strategy and notation follow the ones described in Section 1.1.2 and we refer to it for the notation used here. The first step is to induce a closed two form $\breve{\varpi}$ on the boundary and to verify whether it is non-degenerate or if its associated map $(\breve{\varpi})^\sharp$ has a kernel and to check whether such kernel is regular. We have:

$$\begin{aligned}\breve{\varpi} = \int\quad & e\delta e \delta\omega + \delta e^{y}_{n}\xi^{n}\delta e + e^{y}_{n}\delta\xi^{n}\delta e + \delta e^{y}\delta(e_n\xi^n) + \delta e^{y}\iota_{\delta\xi}e + e^{y}\iota_{\delta\xi}\delta e \\ & + \iota_{\delta\xi}\omega^{y}\delta\omega + \delta(\omega^{y}_{n}\xi^{n})\delta\omega + \iota_{\xi}\delta\omega^{y}\delta\omega + \delta\omega^{y}\delta c \quad \delta\left(\iota_{\xi}c^{y}_{n}\xi^{n}\right)\delta\omega \\ & \delta\xi^{n}\iota_{\delta\xi}\chi\mathsf{V} \quad \xi^{n}\iota_{\delta\xi}\delta\chi\mathsf{V} \quad \delta\xi^{n}\delta\xi^{n}\chi_n\mathsf{V} + \xi^{n}\delta\xi^{n}\delta\chi_n\mathsf{V}.\end{aligned} \tag{1.22}$$

Among the equations defining the kernel of $(\breve{\varpi})^\sharp$ we have the following ones coming from the variation of e and ω:

$$\begin{aligned} e(X_e) &= \iota_{(X_\xi)}\omega^{y} \quad (X_{\xi^n})\omega^{y}_{n} + \iota_{\xi}(X_{\omega^{y}}) + (X_{\omega^{y}_{n}})\xi^{n} \quad (X_{\iota_\xi c^{y}_{n}\xi^{n}}) \\ e(X_\omega) &= \iota_{(X_\xi)}e^{y} + (X_{\xi^n})e^{y}_{n} + (X_{e^{y}_{n}})\xi^{n}. \end{aligned}$$

Since the map $e\ \wedge$ is injective but not surjective on (X_e) (see Lemma 3.4), the first equation is singular. Hence the BV theory does not induce a compatible BFV theory on the boundary.

Remark 1.28. As we will see in Section 2.2.2, the two form (1.22) differs from the one in the three dimensional case (2.9) just by the additional factor e in the first term. This factor is crucial in the singularity of the kernel for $N = 4$ since the corresponding equation in the three dimensional case are easily solvable. As we will see in Section 5.1 this is a characteristic that is retained also by the *classical* pre-boundary two form, i.e. that coming from the KT procedure there introduced.

1.3 Relations between BV-BFV theories

In this section we present some of the possible notions that can be considered in the BV-BFV formalism to compare two different theories. We will also provide examples of such notions by referring to some of the gravity theories presented in the following chapters.

Before introducing these notions, it is useful to define the BV complex and BV cohomology of a BV(-BFV) theory. This cohomology is introduced here in order to give a broader overview about all the possible definitions of equivalence of field theories in the BV BFV setting. However, it will not be used later. All the examples involving the BV-cohomology are taken from [Sim20] and we refer to it for a in-depth analysis. The same holds for the notion of lax BV-BFV theory (Section 1.3.4) and its examples. The application of the lax approach to a BV-BFV description of Gravity are postponed to future work.

Definition 1.29. Let $\mathsf{F}^{''0}$ be an BV theory and let $F^{(0)}$ and $Q^{(0)}$ be as in Definition 1.10. We define the BV-complex of $\mathsf{F}^{''0}$ to be the (co-)chain complex where the nth space is the space of local functionals $C^{1}_{\mathrm{loc}}(F)$ with ghost degree n and the (co-)chain map is the cohomological vector field $Q^{(0)}$. We denote the complex as B_0.

Definition 1.30. The BV cohomology $H\ (B_0)$ is the cohomology associated to the BV complex.

It has been proven that the BV complex is a resolution of the space of solution of the Euler-Lagrange equations modulo symmetries, i.e. its degree zero cohomology is isomorphic to the algebra of functions on the space of solution of the Euler-Lagrange equations modulo symmetries. Furthermore the cohomology in degree 1 is related to anomalies in quantization [CMR18].

1.3.1 BV equivalences

The rich structure of the BV(-BFV) theories allows the definition of various notions of equivalence between theories. The suitability of such notions depends on the level of depth of the investigation, e.g. if quantization of a theory is considered.

The most basic notion of equivalence is the *classical equivalence*. Classically two theories are equivalent when they have the same space of solutions modulo symmetries. This definition considers only the theory at the bulk level. In BV terms we can express such equivalence as in the following definition:

Definition 1.31. Two BV theories $\mathsf{F}_1^{(0)}$ and $\mathsf{F}_2^{(0)}$ with associated BV complexes $(B_0)_1$ and $(B_0)_2$ respectively are classically equivalent when

$$H^0((B_0)_1) = H^0((B_0)_2).$$

As we have seen the cohomology in degree zero describes the space of solutions of the Euler-Lagrange equation of a theory modulo symmetries. However this is not the only physical content of the cohomology if we consider the quantization of the theory. Indeed the following definition extends this equivalence to the whole cohomology.

Definition 1.32. Two BV theories $\mathsf{F}_1^{(0)}$ and $\mathsf{F}_2^{(0)}$ are (weakly) BV-equivalent if there exists a quasi-isomorphism between the corresponding BV complexes $(B_0)_1$ and $(B_0)_2$ and such that it preserves the class of the action and of the symplectic form.

Remark 1.33. We can rewrite explicitly this condition as follows: Two BV theories $\mathsf{F}_1^{(0)}$ and $\mathsf{F}_2^{(0)}$ are (weakly) BV-equivalent if there are two maps $f_1 : F_1^{(0)} \to F_2^{(0)}$ and $f_2 : F_2^{(0)} \to F_1^{(0)}$ such that their pullbacks are chain maps and satisfy the following conditions:

$f_1 \circ f_2$ and $f_2 \circ f_1$ are identities in $H^\bullet((B_0)_1)$ and $H^\bullet((B_0)_2)$ respectively.

The f_i^*s are symplectomorphisms up to terms in the image of $L_{Q_i^{(0)}}$ and they preserve the actions up to terms in $L_{Q_i^{(0)}}$.

We can go further and ask for this quasi-isomorphism to be a symplectomorphism *on the nose*. This goes under the name of strong equivalence and it is described by the following definition:

Definition 1.34. A strong equivalence between two BV theories $\mathsf{F}_1^{''0}$ and $\mathsf{F}_2^{''0}$ is a symplectomorphism

$$\Phi : (F_1^{(0)}, \varpi_1^{(0)}) \to (F_2^{(0)}, \varpi_2^{(0)})$$

preserving the BV action: $\Phi^* S_2^{(0)} = S_1^{(0)}$.

The following proposition clarifies the interdependence of these notions of equivalence.

Proposition 1.35. *If two BV theories are strongly BV equivalent then they are also weakly BV equivalent and if two BV theories are weakly BV equivalent, then they are also classically equivalent.*

We can express this graphically as:

$$\begin{array}{c}\text{Strong BV equivalence}\\ \Downarrow \\ \text{Weak BV equivalence}\\ \Downarrow \\ \text{Classical equivalence}\end{array}$$

Proof. Given a strong equivalence beteween two BV theories $\mathsf{F}_1^{''0}$ and $\mathsf{F}_2^{''0}$, consider the symplectomorphism Φ and its inverse Φ^{-1} and define $f_1 = \Phi$ and $f_2 = \Phi^{-1}$. Since Φ and Φ^{-1} are symplectomorphism and preserve the action, the pullback of the maps f_is are automatically chain maps because of the relation $\iota_{Q_i^{(0)}} \varpi_i^{(0)} = \delta S_i^{(0)}$. Furthermore the compositions $f_1 \circ f_2$ and $f_2 \circ f_1$ are identities and hence identities in cohomology.

A weak BV equivalence provides identities in cohomology at all level, so in particular also at level zero, hence it implies the classical equivalence. □

Example: PC theory and EH theory

An example of classical equivalence in gravity is the one between the Einstein-Hilbert formulation and the Palatini–Cartan one in dimension $N \geq 3$.

These formulations behave differently when considering a boundary. While the Einstein-Hilbert formalism admits a BV-BFV formulation on a manifold with boundary [CS16], the kernel of the preboundary one form is not regular and hence does not admit a BV-BFV formulation (Section 1.2.6, or [CS19b]).

Following the results [BBH95, Theorems 15.1 and 15.2] and using the fact that the connection in the PC formalism can be seen as an auxiliary field, we can also expect that Einstein–Hilbert and Palatini–Cartan theories are also weakly BV equivalent, although, to the best of our knowledge, an explicit construction of the maps constituting the equivalence is still missing.

Example: 1-dimensional gravity and Jacobi theory

A further example can be found by considering now the 1-dimensional gravity and Jacobi theory. We refer to [CS17] and to [Sim20] for an introduction to these theories. These two theories are classically equivalent as it can be seen by computing the Euler-Lagrange equations of the two theories. It has been proved that these two theories are also weakly BV equivalent [Sim20]. The corresponding Theorem reads:

Theorem 1.36 ([Sim20])**.** *The BV theories F_J and F_{GR1} of the Jacobi theory and 1D GR are BV-equivalent.*

The theorem is proved by showing an explicit quasi-isomorphism between the two corresponding complexes. The two theories are not strongly BV equivalent since the space of fields have different dimensions.

Example: Three-dimensional PC theory and BF theory

An example featuring all of these properties is given by the equivalence between PC theory and BF theory in three dimensions (this does not hold in dimensions greater than or equal to four). The following result shows that these two theories are strongly equivalent.

Theorem 1.37 ([CSS18])**.** *The BV theory for 3-dimensional Palatini–Cartan theory is strongly equivalent to 3-dimensional BF theory.*

We refer to [CSS18] for a proof. Alternatively a more direct proof of this theorem can be found in the proof of Lemma 2.22.

1.3.2 Other types of maps

We present in this section a few other notions to relate different BV theories that will be useful in Chapter 6.

Definition 1.38. Let F_1 and F_2 be two BV theories. A BV-inclusion $\mathsf{I} : \mathsf{F}_1 \to \mathsf{F}_2$ is a degree-0 inclusion of (super)manifolds $\iota : F_1 \to F_2$ such that $\iota(F_1)$ is closed in F_2, $\varpi_1 = \iota^* \varpi_2$ and $\iota^* Q_2 = Q_1 \iota^*$. In this case we say that F_1 is a BV-subspace of F_2.

Remark 1.39. Since Q_1 and Q_2 are the hamiltonian vector fields of S_1 and S_2 respectively, the condition $\iota^* Q_2 = Q_1 \iota^*$ is equivalent to the condition $\iota^* S_2 = S_1$.

Proposition 1.40. *The composition of a strong BV equivalence and a BV inclusion is in turn a BV inclusion.*

Proof. The map $\Phi \circ \iota$ satisfies trivially the properties of a BV inclusion. □

A notion that we will need to compare theories is that of BV-pushforward. This notion is usually phrased at the quantum level [Mne17; CMR18], where the additional data of a BV Laplacian needs to be provided. In order to introduce it at a classical level we first start with the notion of BV–Legendre transform.

Definition 1.41. Suppose that we have a splitting of a graded symplectic manifold (F, ϖ) so that $F = F' \times F''$, with $\varpi = \varpi' + \varpi''$, and let L be a Lagrangian submanifold of (F'', ϖ'') Denote coordinates (z', z'') respectively in F', F'', and let $z'' \in \{x, x^\dagger\}$ be Darboux adapted coordinates such that x parametrises $\mathcal{L}$ and x^y are transversal. We define the Batalin–Vilkovisky–Legendre transform of a functional $S \in C^\infty(\mathcal{F})$, with respect to the Lagrangian $\mathcal{L} \subset \mathcal{F}''$, as $S_{\mathsf{BVL}} \in C^\infty(\mathcal{F}')$:

$$S_{\mathsf{BVL}} = S(z, x_0, x^y = 0) \tag{1.23}$$

where x_0 is a critical point for S (assumed unique):

$$\left.\frac{\delta S}{\delta x}\right|_{x^y=0}(x_0) = 0.$$

This notion is the classical analogue of the BV-pushforward in the following sense. Starting from a BV theory on $(\mathcal{F}, \varpi)$ we build a theory on $(\mathcal{F}', \varpi')$ by means of a gauge-fixed fiber integral along $\mathcal{F}''$, (endowed with a half-density μ on $\mathcal{F}''$, which thus defines by restriction a measure $\mu_{\mathcal{L}}$ on $\mathcal{L}$) with gauge-fixing Lagrangian $\mathcal{L}$. In other words, if S denotes a BV action on $\mathcal{F}$ we consider the effective result of the BV pushforward (fiber integral) to be

$$\exp\left(\frac{i}{\hbar}S_{\mathsf{eff}}\right) := \int_{\mathcal{L}\subset\mathcal{F}''} \exp\left(\frac{i}{\hbar}S\right)\Big|_{\mathcal{L}} \mu_{\mathcal{L}} \tag{1.24}$$

where the integral is defined perturbatively as a power series in $\hbar$. Note that S_{BVL} is the dominant term of S_{eff}. When S depends only quadratically on the variables on $\mathcal{L}$, the only correction is $i\hbar/2$ times the logarithm of the determinant of the quadratic form. In these cases we say that one theory is the BV-pushforward of the other.

1.3.3 BV-BFV equivalences for extended theories

For the corresponding n-extended BV theories only the definition of strong BV equivalence has a counterpart.

Definition 1.42. A strong equivalence between two n-extended exact BV theories $\mathfrak{F}_1^{\uparrow n}$ and $\mathfrak{F}_2^{\uparrow n}$ is a collection of degree-0 symplectomorphisms

$$\Phi^{(k)} : (\mathcal{F}_1^{(k)}, \varpi_1^{(k)}) \to (\mathcal{F}_2^{(k)}, \varpi_2^{(k)})$$

preserving the k^{th} BFV action: $\Phi^* S_2^{(k)} = S_1^{(k)}$ and satisfying, for $0 \leq k \leq n-1$

$$\pi_2^{(k)} \circ \Phi^{(k)} = \Phi^{(k+1)} \circ \pi_1^{(k)}.$$

A similar definition holds for n-extended BF^{m}V theories.

The corresponding weak version has never been proposed before. A first idea might be to just an extend of the weak BV equivalence to the extended BV-BFV case, stratum by stratum. However Some compatibility issues between the projection of the different strata and the cohomology might arise. Hence such a definition will be object of future explorations.

Example: Three-dimensional PC theory and BF theory

It is here more difficult to find examples since not all the theories can be extended. Indeed while EH theory has a BV-BFV corresponding description ([CS16]), PC theory for dimensions greater than or equal to four does not allow such description ([CS19b]). The same holds for the couple Jacobi and 1 dimensional gravity coupled with gravity, where only the second admits a BV-BFV description ([CS17]).

The only example that we present here is the one between three-dimensional PC theory and BF theory. Indeed, Theorem 2.5 states that these theories are strongly BV-BFV equivalent.

1.3.4 Equivalences between lax BV-BFV theories

We complete this section by introducing the notion lax BV-BFV theory and the corresponding notion of equivalence. The aim is mainly to give a brief insight on possible future developments, as we will not use these notion elsewhere in the thesis. We introduce the notion of lax BV-BFV theory as described in [MSW19]. We first recall what we mean here by ***local***:

Definition 1.43 ([MSW19])**.** A ***local*** form on a (possibly graded) vector bundle $E \to M$ on an m-dimensional manifold M, is an element of

$$\left(\Omega^{\cdot}_{\mathsf{loc}}(\mathcal{F} \quad M), \delta, d\right) := (j^{\;})\ \ (\Omega^{\cdot}\ (J^{\;}(E)), d_V, d_H) \tag{1.25}$$

where $\mathcal{F} := \Gamma^{\;}(M, E)$, $j^{\;}$ is the limit of the maps $\{j^p : \mathcal{F} \quad M \to J^pE\}$ with J^pE the p-th jet bundle of E, $J^{\;}$ is the limit of the sequence

$$E \quad J^0E \qquad J^1E \qquad\qquad J^pE \qquad \ldots$$

and $\Omega^{\cdot}_{\mathsf{loc}}(\mathcal{F}, \quad M)$ is endowed with the differentials

$$\delta(j^{\;})\ \ \alpha := (j^{\;})\ \ d_V\alpha \tag{1.26}$$

$$d(j^{\;})\ \ \alpha := (j^{\;})\ \ d_H\alpha. \tag{1.27}$$

An element of $\Omega^{0,}_{\mathsf{loc}}(\mathcal{F} \quad M)$ is called ***Local Functional***.

Definition 1.44 ([MSW19])**.** A ***lax BV-BFV theory*** is the assignment, to a manifold M, of the data

$$\mathsf{F} = (\mathcal{F}, L\ \ , \theta\ \ , Q) \tag{1.28}$$

with

- $\mathcal{F}$ the space of $C^{\;}$ sections of a graded bundle (or sheaf) $E \to M$,
- $L \in \Omega^{0,}_{\mathsf{loc}}(\mathcal{F} \quad M)$, a local functional of total degree 0
- $\theta \in \Omega^{1,}_{\mathsf{loc}}(\mathcal{F} \quad M)$, a local one-form of total degree $\;1$,
- $Q \in \mathsf{X}_{\mathsf{evo}}(\mathcal{F} \quad M)[1]$, a degree-1, evolutionary, cohomological vector field on $\mathcal{F}$, i.e. $[\boldsymbol{L}_Q, d] = [Q, Q] = 0$,

such that

$$\iota_Q\varpi\ \ = \delta L\ \ + d\theta \tag{1.29a}$$

$$\frac{1}{2}\iota_Q\iota_Q\varpi\ \ = dL\ \ , \tag{1.29b}$$

where $\varpi\ \ := \delta\theta\ \ $ and $\boldsymbol{L}_Q = [\iota_Q, \delta]$.

As in the case of the strict BV-BFV theories we can define a corresponding complex and cohomology:

Definition 1.45 ([MSW19])**.** Let F be a lax BV-BFV theory. We define the lax BV-BFV complex to be the space of local forms with values in inhomogeneous forms on M, endowed with the combined differential $\boldsymbol{L}_Q \quad d$:

$$\Omega\ (\boldsymbol{L}_Q \quad d) \quad \Omega_{\mathsf{BV}\ \ \mathsf{BFV}}(\mathcal{F} \quad M, \boldsymbol{L}_Q \quad d) := \left(\left(\bigoplus_k \Omega^{\cdot,k}_{\mathsf{loc}}(\mathcal{F} \quad M)\right), \boldsymbol{L}_Q \quad d\right) \tag{1.30}$$

From a lax BV-BFV theory it is possible to get back an extended BV-BFV theory through the procedure outlined in the following definition:

Definition 1.46 ([MSW19]). An n-*strictification* of a lax BV-BFV theory F on a manifold M, is a pairing with an n-stratification $\{M^{(k)}\}_{k=0\dots n}$ to yield an n-extended, exact BV-BFV theory $\mathsf{F}^{''n} = (\mathcal{F}^{(k)}, S^{(k)}, \alpha^{(k)}, Q^{(k)}, \pi_{(k)})_{k=0\dots n}$ for which there are surjective submersions

$$p_{(k)} : \mathcal{F} \to \mathcal{F}^{(k)}$$

such that $p_{(k+1)} = \pi_{(k)} \circ p_{(k)}$, $Q^{(k)} = (p_{(k)})_* Q$, $\int_{M^{(k)}}[\theta]^{m-k} = p_{(k)}^*\alpha^{(k)}$ and $\int_{M^{(k)}}[L]^{m-k} = p_{(k)}^* L^{(k)}$.

We can now give some notion of equivalence for lax BV-BFV theories. The definition follows the (weak) BV equivalence.

Definition 1.47. Two lax BV-BFV theories F_1 and F_2 are lax-equivalent if there exists a quasi-isomorphism between the corresponding lax BV-BFV complexes $\Omega_1(L_Q - d)$ and $\Omega_2(L_Q - d)$ and such that the action functional and one form are preserved up to terms in the cohomology.

Example: 1-dimensional gravity and Jacobi theory

The only gravity-related lax BV-BFV theories that have been computed are the Jacobi theory and the one-dimensional gravity [CS17; Sim20]. We denote by F_J^{lax} and F_{GR1}^{lax} the lax theories corresponding to the Jacobi theory and to the 1d gravity respectively as described in [Sim20, Sections 2.2 and 2.3]. Despite these theories having a different behaviour with respect to the extension of the BV theory to a BV-BFV one, it turns out that these theories are lax BV-BFV equivalent as stated in the following theorem:

Theorem 1.48 ([Sim20]). *The lax theories F_J^{lax} and F_{GR1}^{lax} of the Jacobi theory and 1D gravity are lax-equivalent.*

In particular, since these two theories are equivalent but do not behave in the same way under the BV-BFV extension, we deduce that lax-equivalence does not give any information on the extendability of the corresponding BV theories. Hence we cannot also deduce anything about the BV(-BFV) equivalences.

As we have seen, there are many possible ways of considering two theories as *equivalent*. The use of one or the other depends on the type of analysis carried on. From a physical point of view this coincides with the possibility of detecting experimentally the differences between two theories. In particular one might focus on a classical (resp. quantum) setup and consider two theories as equivalent if it is impossible to separate them through a classical (resp. quantum) experiment. On the other hand, looking at the predictive power of the theory, different behaviours under different notions of equivalence of two distinct theories might give hints about where (and how) to find experimental differences between the two. One possible example are the (asymptotic) charges of two theories that can differ for classically equivalent theories (for example [OS20] for EH and PC). Hence if we are interested in such objects it is better to consider stronger notions of equivalence with respect to the classical one. The assessment of which notion is the most suitable (keeping in mind that it is always better to use the weakest possible equivalence able to catch these differences) is still a work in progress and it will be object of future work.

Chapter 2

A topological field theory: the three dimensional case

The goal of this chapter is to show that General Relativity in spacetime dimension $N = 3$, phrased in the triadic language or coframe formalism (i.e. in the Palatini–Cartan formalism, see Section 1.2.4 for a brief introduction), admits an extension to all higher codimension strata in the BV-BFV sense (Definition 1.10), in stark contrast with the N 4 analogue (where the procedure fails at codimension 1 [CS19b]) and with Einstein–Hilbert formalism (where the procedure fails at codimension 2 (Chapter 7).

This means not only that we can control the failure of gauge invariance of the theory upon introducing a boundary, but that the boundary gauge invariance is controlled by corner cohomology, and so on. More formally, using the definitions of Section 1.1 we can say that three dimensional Palatini–Cartan (3d PC) theory can be fully extended as a BV-BFV theory.

In order to prove this fact we obtain explicit expressions for the BV-BFV data at higher codimension strata. Besides the main claim, we also obtain that 3d PC is the first example of a fully extended theory that features a nontrivial symplectic reduction at every step. Furthermore, in Section 2.3 we show that the fully extended BV-BFV description of PC theory is strongly equivalent to that of *nondegenerate* BF theory. This means that, on every stratum, the data of PC theory and that of BF theory with an open condition on the fields differ by a change of coordinates at every codimension.

Although it is well known that 3d General Relativity is classically equivalent to nondegenerate BF theory [Wit88a], in this chapter we explicitly write down the symmetries in terms of diffeomorphisms, and show that this description is equivalent to the one coming from standard symmetries of BF theory, at all codimensions. This is the result of nontrivial calculations, interesting also as they can be taken as a guideline or bootcamp for the more involved case of 4d gravity, partially studied in the next chapters.

One interpretation of this result suggests thinking of BF theory as a (possibly degenerate) extension of PC theory (for $N = 3$), coinciding on an *open sector*, i.e. when the non-degeneracy condition is imposed. From this point of view, BV-BFV quantisation of BF theory (which has been carried out explicitly in [CMR18]) may be taken as a slightly more general quantisation of PC theory in three dimensions, compatible with cutting and gluing along submanifolds[1]. The non-degeneracy condition can then be imposed directly at the quantum level, without spoiling the quantisation procedure. Thus, the present result paves the way for a direct application of

[1] The adaptation of a quantisation of BF theory to include corner data has been proposed by [IM19b].

these extended quantisation techniques to the example of GR, bringing a model for gravity a step closer to functorial approaches to quantum field theory, by assigning compatible structure to higher-codimension strata, all the way down to points.

The content of this chapter is structured as follows. In section 2.1 we introduce BF theory and outline the main results. Then in Section 2.2 we describe in detail the constructive steps one needs in order to obtain the BV-BFV data at every codimension of a stratified manifold $\{M^{(k)}\}_{k=0\dots3}$. We divide the proof of the main Theorem, stating that PC theory in the BV formalism is fully extended, into three Propositions, each of which is aimed at recovering data one codimension further. Finally, in Section 2.3 we present explicit symplectomorphisms between the spaces of fields $\mathcal{F}^{(k)}_{PC/BF}$ at every codimension, and we show how they commute with the BV-BFV surjective submersion maps connecting the space of fields of each stratum. This proves that the BV-BFV construction commutes with equivalence at each codimension.

2.1 Setup

BF theory and three dimensional General Relativity in the PC formalism share the same framework, described in the first part of Section 1.2.4.

The BV formulation of General Relativity in the PC formalism has been fully developed in Section 1.2.5 and recollected in Definition 1.24. In this chapter we consider only the case $N = 3$ and $\Lambda = 0$ (made exception for Section 2.4).

We now introduce the BV version of three-dimensional BF theory, using the notation outlined in the first part of Section 1.2.4.

2.1.1 Three-dimensional BF theory

The fields of the theory are $B \in \Omega^1(M, \mathcal{V})$ and a connection $A \in \mathcal{A}(M)$. We will think of A as a connection form around the trivial connection, that is to say $A \in \Omega^1(M, \wedge^2\mathcal{V})$.

Definition 2.1. Classical BF theory is the pair $(\mathcal{F}^{cl}_{BF}, S^{cl}_{BF})$ where

$$\mathcal{F}^{cl}_{BF} = \Omega^1(M, \mathcal{V}) \times \Omega^1(M, \wedge^2\mathcal{V})$$

is the space of fields, and the action functional reads

$$S^{cl}_{BF} = \int_M BF_A,$$

with $F_A \in \Omega^2(M, \wedge^2\mathcal{V})$ being the curvature of the connection A. We can further require B to be non-degenerate as a map $B \colon TM \to \mathcal{V} \overset{\eta}{\cong} \wedge^2\mathcal{V}$. Denoting by $\Omega^1_{nd}(M, \wedge^2\mathcal{V})$ the space of non-degenerate B's, we will call the resulting theory *nondegenerate* BF theory, and denote it with the notation BF where relevant.

The symmetries of the theory comprise gauge trasformations, parametrized by[2] $\chi \in \Omega^0(M, \mathfrak{so}(2,1)) \simeq \Omega^0(M, \wedge^2\mathcal{V})$

$$\delta_\chi B = [\chi, B] \qquad \delta_\chi A = d_A\chi,$$

together with what is sometimes referred to as *shift symmetry*, a traslation of B parametrized by $\tau \in \Omega^0(M, \mathcal{V})$, $\delta_\tau B = d_A\tau$.

We recall the BV version of three-dimensional BF theory.

[2] We will denote here the action of a symmetry by the notation δ_χ with parameter χ. This is a notation historically used to denote a χ-dependent vector field acting on generators on the algebra of functions over F^{cl}. It will be replaced by a well-defined vector field when we pass to the BV formalism.

Definition 2.2. The BV-data for BF theory is given by

$$\mathsf{F}^{\partial 0}_{BF} = (\mathcal{F}_{BF}, \alpha_{BF}, S_{BF}, Q_{BF}),$$

where the BV space of fields can be written as

$$\mathcal{F}_{BF} = T^*[-1]\left(\Omega^1(M, \mathcal{V}) \oplus \mathcal{A}(M) \oplus \Omega^0[1](M, \wedge^2\mathcal{V}) \oplus \Omega^0[1](M, \wedge^1\mathcal{V})\right),$$

and, if we arrange the fields in the following convenient way

$$\mathcal{B} = \tau + B + A^\dagger + \chi^\dagger \in \Omega^\bullet(M, \mathcal{V})[1-\bullet], \qquad \mathcal{A} = \chi + A + B^\dagger + \tau^\dagger \in \Omega^\bullet(M, \wedge^2\mathcal{V})[1-\bullet],$$

the BV data reads[3]

$$\begin{aligned} \alpha_{BF} &= \int_M \mathcal{B}\delta\mathcal{A} \qquad \varpi_{BF} = \delta\alpha_{BF} \\ S_{BF} &= \int_M \mathcal{B}\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\right) \\ Q_{BF}\mathcal{B} &= d_{\mathcal{A}}\mathcal{B}; \qquad Q_{BF}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]. \end{aligned}$$

If $B \in \Omega^1_{nd}(M, \wedge^2\mathcal{V})$ we will denote the resulting BV theory by $\mathsf{F}^{\partial 0}_{BF}$.

As every AKSZ theory [Ale+97], BF theory can be fully extended:

Theorem 2.3. *[CMR14] The BV theory $\mathsf{F}^{\partial 0}_{BF} = (\mathcal{F}_{BF}, S_{BF}, \alpha_{BF}, Q_{BF})$ is fully extendable. The BV-BFV data of the fully extended theory $\mathsf{F}^{\partial 3}_{BF}$ is given by the following expressions ($i = 0 \dots 3$):*

$$\begin{aligned} \alpha^{(i)}_{BF} &= \int_{M^{(i)}} \mathcal{B}\delta\mathcal{A} \qquad \varpi^{(i)}_{BF} = \delta\alpha^{(i)}_{BF} \\ S^{(i)}_{BF} &= \int_{M^{(i)}} \mathcal{B}\left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\right) \\ Q^{(i)}_{BF}\mathcal{B} &= d_{\mathcal{A}}\mathcal{B}; \qquad Q^{(i)}_{BF}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \end{aligned}$$

where we used once again the convention that only the admissible terms appear in the integrands and $\pi^{(i)}_{BF}$ is the restriction of the fields to $M^{(i+1)}$.

Note that in this notation BF theory is *self-similar*, i.e. the action S, the symplectic two form ϖ and the cohomological vector field Q have the same expression on bulk (0-stratum), boundary (1-stratum) and every subsequent iteration.

2.1.2 Outline of the main results

We can now state the main results in this chapter. They refer to Definitions 1.14 and 1.42 respectively. Sections 2.2 and 2.3 will be devoted to their proof.

Theorem 2.4. *The BV theory $\mathsf{F}^{\partial 0}_{PC} = (\mathcal{F}_{PC}, S_{PC}, \alpha_{PC}, Q_{PC})$ is fully extendable.*

Theorem 2.5. *The fully extended BV theories $\mathsf{F}^{\partial 3}_{PC}$ and $\mathsf{F}^{\partial 3}_{BF}$ are strongly equivalent.*

[3] We use here the convention that only the admissible terms (i.e. the ones that are top forms) appear in the integrands. Note also the convention on the components of Q, see Remark 1.26.

Remark 2.6. The space of BV fields for classical PC theory and the action of symmetries presented in Definition 1.24 is essentially independent of spacetime dimensions. However, it was shown in [CS19b] that the 4-dimensional BV theory of General Relativity (in the tetrad formalism) cannot be extended without extra assumptions on the fields (see Section 1.2.6). Theorem 2.4, compared with the no-go result in [CS19b] marks a stark difference between 3 and 4 spacetime dimensions. GR in the Einstein–Hilbert formalism, instead, is independent of spacetime dimensions and is always at least 1-extendable (outside of $N = 2$) [CS16] but not 2-extendable (see Chapter 7).

Remark 2.7. These results are in good agreement with what one might expect from previous results and the structure of extended theory. Indeed BF theory and three dimensional gravity differ by the different set of generating symmetries, which are related by an invertible transformation.

2.2 Fully extended BV-BFV structure of three dimensional Palatini–Cartan theory

In this section we will prove Theorem 2.4 by extending the BV theory of definition 1.24 step by step, thus building the fully extended BV-BFV data for PC theory at every codimension.

First, let us introduce some useful notation and explain the common strategy employed at every step. Throughout the section, as we will only consider PC theory, we will drop the PC subscript everywhere, except in stating the results.

2.2.1 Notation and strategy

Consider $X \in \Omega^\bullet(M, \mathcal{V})$ and $Y \in \Omega^\bullet(M, \wedge^2\mathcal{V})$. Since the image of a non-degenerate triad e is a basis of $\mathcal{V}$ at every point, we can express X and Y as

$$X = \sum_{i=1}^{3} X^{\langle i \rangle} e_i \qquad Y = \sum_{i,j=1, i\neq j}^{3} Y^{\langle ij \rangle} e_i \wedge e_j$$

where $e_i = e(\partial_i)$, and we denote by $X^{\langle i \rangle}$ the i-th component of X and by $Y^{\langle ij \rangle}$ the ij-th component of Y. We can then define the following projections, $i = 1, 2, 3$:

$$\begin{aligned} p_i : \Omega^\bullet(M, \mathcal{V}) &\to \Omega^\bullet(M, \mathcal{V}) \\ X &\mapsto p_i(X) = X^{\langle i \rangle} e_i. \end{aligned}$$

It is also useful to define a ***dual*** map acting on elements of $\Omega^\bullet(M, \wedge^2\mathcal{V})$:

$$\begin{aligned} p_i^\dagger : \Omega^\bullet\left(M, \wedge^2\mathcal{V}\right) &\to \Omega^\bullet\left(M, \wedge^2\mathcal{V}\right) \\ Y &\mapsto p_i^\dagger(Y) = Y^{\langle hk \rangle} e_h \wedge e_k, \qquad h, k \neq i. \end{aligned} \tag{2.1}$$

Furthermore, given any element $Z \in \Omega^\bullet(M, \wedge^3\mathcal{V})$ we can expand it with respect to the basis spanned by the image of e: $Z = Z^{\langle abc \rangle} e_a \wedge e_b \wedge e_c$. We then define

$$Z^{\langle a \rangle} := Z^{\langle abc \rangle} e_b \wedge e_c \qquad Z^{\langle a \rangle} \in \Omega^k(M, \wedge^2\mathcal{V}). \tag{2.2}$$

Let M be a N-dimensional smooth manifold. Given a 1-stratification $\{M^{(k)}\}_{k=0,1}$ of M, consider a tubular neighbourhood $U \subset M^{(0)}$ of the embedding $\iota^{(1)} \colon M^{(1)} \to M^{(0)}$ and a local chart on U with coordinates $(x^1, \dots, x^{N-1}, x^n)$ where x^n is the coordinate along the normal direction of the tubular neighbourhood and $x^1, \dots, x^{N-1}$ are coordinates of $M^{(1)}$. Throughout the article we keep this notation, thus denoting with n the coordinate normal to the first stratum. We can expand forms $\mu \in \Omega^k(M)$ with respect to this coordinate system and get

$$\mu =: \mu^{//} + \mu_n dx^n, \tag{2.3}$$
$$\mu^{//} = \sum_{\substack{i_j=1 \\ j=1\dots k}}^{N-1} \mu_{i_1\dots i_k} dx^1 \wedge \dots \wedge dx^k$$
$$\mu_n = \sum_{\substack{i_j=1 \\ j=1\dots k-1}}^{N-1} \mu_{i_1\dots i_{k-1}n} dx^1 \wedge \dots \wedge dx^{k-1},$$

A similar prescription for the tangent bundle yields:

$$\sum_{\mu=1}^{3} \xi^\mu \frac{\partial}{\partial x^\mu} = \sum_{a=1}^{2} \xi^a \frac{\partial}{\partial x^a} + \xi^n \frac{\partial}{\partial x^n}, \tag{2.4}$$

and the contraction $\iota_\xi \mu = \iota_\xi \mu^{//} + \mu_n \xi^n + \iota_\xi \mu_n dx^n$ restricts[4] to $M^{(1)}$ as $\iota_\xi \mu|_{M^{(1)}} = \iota_\xi \mu^{//} + \mu_n \xi^n$. In particular, note that both $\mu^{//}$ and μ_n restrict to $M^{(1)}$. In case no confusion can arise we will simply denote $\mu^{//}$ with μ. Furthermore, we denote with μ_b the component of μ in direction of dx^b analogously to the notation μ_n in (2.3).

Remark 2.8. For higher codimension strata, (2.3) and (2.4) are modified accordingly. Though, for the sake of clarity, we use different letters: m (resp. a) for the direction transversal to the second (resp. third) stratum.

Remark 2.9. The decomposition defined above clearly depends on the choice of the coordinate system compatible with the topological structure of the tubular neighborhood. However, this choice becomes relevant only when one wants to explicitly write a map between spaces of different codimension in a coordinate chart. In such circumstance, as mentioned, this choice will depend on an embedding $M^{(k)} \to M^{(k+1)}$ (Corollaries 2.12, 2.16 and 2.19). The BFV data we will construct for each stratum in Propositions 2.10, 2.15 and 2.17, however, is coordinate independent at every codimension.

The strategy we employ to construct the maps $\pi^{(k)}$ has been described in detail in Section 1.1.2.

2.2.2 1-extended PC theory

If we allow M to bear a 1-stratification, and denote codimension-1 strata by $M^{(1)}$, denoting by $\Omega^1_{nd}(M^{(1)}, V)$ the space of maps $e\colon TM^{(1)} \to V$ such that the image of e is a linearly independent system, we have the following.

Proposition 2.10. *The BV theory* $\mathfrak{F}^{\partial 0}_{PC} = (F_{PC}, S_{PC}, \varpi_{PC}, Q_{PC})$ *is 1-extendable to* $\mathfrak{F}^{\partial 1}_{PC}$. *The codimension-1 data are:*

The space of codimension-1 fields, given by the bundle

$$F^{(1)}_{PC} \to \Omega^1_{nd}(M^{(1)}, V), \tag{2.5}$$

[4] Here we obviously mean the pullback of forms along $\iota^{(1)}$.

with local trivialisation on an open $U^{(1)} \subset \Omega^1_{nd}(M^{(1)}, V)$

$$F^{(1)}_{PC} \simeq U^{(1)} \times \Omega^1(M^{(1)}, \wedge^2 V) \times T^*\Big(\Omega^0[1](M^{(1)}, \wedge^2 V) \times \mathfrak{X}[1](M^{(1)}) \times C^\infty[1](M^{(1)})\Big), \quad (2.6)$$

and fields denoted by $\widetilde{e} \in U^{(1)}$ *and* $\widetilde{\omega} \in \Omega^1(M^{(1)}, \wedge^2 V)$ *in degree zero,* $\widetilde{c} \in \Omega^0[1](M^{(1)}, \wedge^2 V)$, $\widetilde{\xi} \in \mathfrak{X}[1](M^{(1)})$ *and* $\widetilde{\xi}^n \in C^\infty[1](M^{(1)})$ *in degree one,* $\widetilde{\omega}^\dagger \in \Omega^3[-1](M^{(1)}, V)$ *and* $\widetilde{e}^\dagger \in \Omega^2[-1](M^{(1)}, \wedge^2 V)$ *in degree minus one, together with a fixed vector field* $\epsilon_n \in \Gamma(V)$, *completing the image of elements* $\widetilde{e} \in U^{(1)}$ *to a basis of* V*;*

The codimension-1 one-form, symplectic form and action functional

$$\alpha^{(1)}_{PC} = \int_{M^{(1)}} \widetilde{e}\delta\widetilde{\omega} + \widetilde{\omega}^\dagger\delta\widetilde{c} - \widetilde{e}^\dagger\epsilon_n\delta\widetilde{\xi}^n - (\iota_{\delta\widetilde{\xi}}\widetilde{e})\widetilde{e}^\dagger + \iota_{\widetilde{\xi}}\widetilde{\omega}^\dagger\delta\widetilde{\omega}, \quad (2.7a)$$

$$\varpi^{(1)}_{PC} = \int_{M^{(1)}} \delta\widetilde{e}\delta\widetilde{\omega} + \delta\widetilde{\omega}^\dagger\delta\widetilde{c} - \delta\widetilde{e}^\dagger\epsilon_n\delta\widetilde{\xi}^n + \iota_{\delta\widetilde{\xi}}\delta(\widetilde{e}\,\widetilde{e}^\dagger) + \delta(\iota_{\widetilde{\xi}}\widetilde{\omega}^\dagger)\delta\widetilde{\omega}, \quad (2.7b)$$

$$\begin{aligned} S^{(1)}_{PC} = \int_{M^{(1)}} & \iota_{\widetilde{\xi}}\widetilde{e}F_{\widetilde{\omega}} - \epsilon_n\widetilde{\xi}^n F_{\widetilde{\omega}} - \widetilde{c}d_{\widetilde{\omega}}\widetilde{e} + \frac{1}{2}[\widetilde{c},\widetilde{c}]\widetilde{\omega}^\dagger + \frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}F_{\widetilde{\omega}}\widetilde{\omega}^\dagger + \frac{1}{2}\iota_{[\widetilde{\xi},\widetilde{\xi}]}\widetilde{e}\widetilde{e}^\dagger \\ & + \widetilde{c}d_{\widetilde{\omega}}(\iota_{\widetilde{\xi}}\widetilde{\omega}^\dagger) + L^{\widetilde{\omega}}_{\widetilde{\xi}}(\epsilon_n\widetilde{\xi}^n)\widetilde{e}^\dagger - [\widetilde{c}, \epsilon_n\widetilde{\xi}^n]\widetilde{e}^\dagger, \end{aligned} \quad (2.7c)$$

where $\iota_{\delta\widetilde{\xi}}\delta(\widetilde{e}\,\widetilde{e}^\dagger)$ *indicates that the contraction is considered with respect to the one-form*[5] $\widetilde{e}$*;*

The cohomological vector field $Q^{(1)}_{PC}$

$$Q^{(1)}_{PC}\widetilde{e} = -L^{\widetilde{\omega}}_{\widetilde{\xi}}\widetilde{e} + d_{\widetilde{\omega}}(\epsilon_n\widetilde{\xi}^n) + [\widetilde{c},\widetilde{e}] - X_n^{\langle a\rangle}\widetilde{\omega}^\dagger_a + Y_n^{\langle a\rangle}\widetilde{\omega}^\dagger_a \quad (2.8a)$$

$$Q^{(1)}_{PC}\widetilde{\omega} = -\iota_{\widetilde{\xi}}F_{\widetilde{\omega}} + d_{\widetilde{\omega}}\widetilde{c} - X_n^{\langle a\rangle}\widetilde{e}^\dagger_a + Y_n^{\langle a\rangle}\widetilde{e}^\dagger_a \quad (2.8b)$$

$$Q^{(1)}_{PC}\widetilde{c} = \frac{1}{2}[\widetilde{c},\widetilde{c}] - \frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}F_{\widetilde{\omega}} - \iota_{\widetilde{\xi}}X_n^{\langle a\rangle}\widetilde{e}^\dagger_a + \iota_{\widetilde{\xi}}Y_n^{\langle a\rangle}\widetilde{e}^\dagger_a \quad (2.8c)$$

$$Q^{(1)}_{PC}\widetilde{\omega}^\dagger = -d_{\widetilde{\omega}}\widetilde{e} + [\widetilde{c},\widetilde{\omega}^\dagger] + d_{\widetilde{\omega}}\iota_{\widetilde{\xi}}\widetilde{\omega}^\dagger - [\epsilon_n\widetilde{\xi}^n, \widetilde{e}^\dagger] \quad (2.8d)$$

$$Q^{(1)}_{PC}\widetilde{e}^\dagger = -F_{\widetilde{\omega}} + [\widetilde{c},\widetilde{e}^\dagger] + d_{\widetilde{\omega}}\iota_{\widetilde{\xi}}\widetilde{e}^\dagger \quad (2.8e)$$

$$Q^{(1)}_{PC}\widetilde{\xi} = \frac{1}{2}[\widetilde{\xi},\widetilde{\xi}] - \left(X_n^{\langle a\rangle} + Y_n^{\langle a\rangle}\right)\frac{\partial}{\partial x^a} \quad (2.8f)$$

$$Q^{(1)}_{PC}\widetilde{\xi}^n = +X_n^{\langle n\rangle} - Y_n^{\langle n\rangle} \quad (2.8g)$$

where $X_n = L^{\widetilde{\omega}}_{\widetilde{\xi}}(\epsilon_n\widetilde{\xi}^n)$ *and* $Y_n = [\widetilde{c}, \epsilon_n\widetilde{\xi}^n]$*, and the superscript* $\langle n\rangle$ *denotes the component with respect to* ϵ_n*, following the notation of Section 2.2.1;*

The projection to codimension-1 fields $\pi^{(0)}_{PC} = \pi_{\text{red}}{}^{(1)} \circ \check{\pi}^{(0)}$ *is smooth, where* $\check{\pi}^{(0)} : F^{(0)} \to \check{F}^{(1)}$ *is the restriction of the codimension-0 fields to the 1-stratum* $M^{(1)}$ *and* $\pi_{\text{red}}{}^{(1)} : \check{F}^{(1)} \to F^{(1)}$ *is the symplectic reduction* (1.6)*.*

Remark 2.11. Before we produce the proof of Proposition 2.10, let us observe that all the expressions in (2.8) (and more manifestly in (2.7)) are coordinate independent. Indeed, expressions like $\iota_{\widetilde{\xi}}X_n^{\langle a\rangle}\widetilde{e}^\dagger_a$ or $X^{\langle a\rangle}\frac{\partial}{\partial x^a}$ are tensorial in a local chart of $M^{(1)}$, while the component $\langle n\rangle$ refers to the completion of a basis $\{\mathrm{Im}(\widetilde{e}), \epsilon_n\}$. Furthermore, the projection $\pi^{(0)}_{PC}$ as defined above is

[5] In a local chart we can write $\iota_{\delta\widetilde{\xi}}\delta(\widetilde{e}\,\widetilde{e}^\dagger) = \delta\xi^a\delta(e_a e^\dagger)$.

coordinate independent (both restrictions and symplectic reductions can be defined without the use of a coordinate system). However, the explicit expression of the BFV map $\pi_{PC}^{(0)}$ presented later, in Corollary 2.12, depends on the choice of an embedding $M^{(1)}$ in $M^{(0)}$ and a chart on a tubular neighbourhood (cf. Remark 2.9).

Proof. From the variation of the action (1.20), following the strategy outlined in Section 1.1.2, we get the pre-boundary one-form by isolating the boundary terms:

$$\breve{\alpha}^{(1)} = \int_{M^{(1)}} e\delta\omega + e_n^{\mathsf{y}}\xi^n\delta e + e^{\mathsf{y}}\delta(e_n\xi^n) + e^{\mathsf{y}}\iota_{\delta\xi}e + \iota_\xi\omega^{\mathsf{y}}\delta\omega + \omega_n^{\mathsf{y}}\xi^n\delta\omega + \omega^{\mathsf{y}}\delta c$$
$$\left(\iota_\xi c_n^{\mathsf{y}}\xi^n\right)\delta\omega \quad \xi^n\iota_{\delta\xi}\chi\mathsf{V} \quad \xi^n\delta\xi^n\chi_n\mathsf{V}$$

where $\xi^{\mathsf{y}} = \chi \quad \mathsf{V} \quad \chi\mathsf{V}$ (see Remark 1.25 for the definition of V). Then, we derive the two form $\breve{\varpi}^{(1)} = \delta\breve{\alpha}^{(1)}$:

$$\breve{\varpi}^{(1)} = \int_{M^{(1)}} \delta e\delta\omega + \delta e_n^{\mathsf{y}}\xi^n\delta e + e_n^{\mathsf{y}}\delta\xi^n\delta e + \delta e^{\mathsf{y}}\delta(e_n\xi^n) + \delta e^{\mathsf{y}}\iota_{\delta\xi}e + e^{\mathsf{y}}\iota_{\delta\xi}\delta e \tag{2.9}$$
$$+ \iota_{\delta\xi}\omega^{\mathsf{y}}\delta\omega + \delta(\omega_n^{\mathsf{y}}\xi^n)\delta\omega + \iota_\xi\delta\omega^{\mathsf{y}}\delta\omega + \delta\omega^{\mathsf{y}}\delta c \quad \delta\left(\iota_\xi c_n^{\mathsf{y}}\xi^n\right)\delta\omega$$
$$\delta\xi^n\iota_{\delta\xi}\chi\mathsf{V} \quad \xi^n\iota_{\delta\xi}\delta\chi\mathsf{V} \quad \delta\xi^n\delta\xi^n\chi_n\mathsf{V} + \xi^n\delta\xi^n\delta\chi_n\mathsf{V}.$$

We want to make sure that $\breve{\varpi}^{(1)}$ is pre-symplectic: the kernel of such a two-form is defined by the equations

$$(X_e) = \iota_{(X_\xi)}\omega^{\mathsf{y}} \quad (X_{\xi^n})\omega_n^{\mathsf{y}} + \iota_\xi(X_{\omega^{\mathsf{y}}}) + (X_{\omega_n^{\mathsf{y}}})\xi^n \quad (X_{\iota_\xi c_n^{\mathsf{y}}\xi^n}) \tag{2.10a}$$
$$(X_\omega) = \iota_{(X_\xi)}e^{\mathsf{y}} + (X_{\xi^n})e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n \tag{2.10b}$$
$$(X_{\xi^\mu})e_\mu = \quad (X_{e_n})\xi^n \tag{2.10c}$$
$$(X_c) = \iota_\xi(X_\omega) \tag{2.10d}$$
$$(X_{\omega^{\mathsf{y}}}) = 0 \tag{2.10e}$$
$$e_n(X_{e^{\mathsf{y}}}) = \quad (X_e)e_n^{\mathsf{y}} + (X_\omega)\omega_n^{\mathsf{y}} \quad (X_\omega)c_{nb}^{\mathsf{y}}\xi^b + (2(X_{\xi^n})\chi_n + (X_{\chi_n})\xi^n + (X_{\xi^a})\chi_a)\,\mathsf{V} \tag{2.10f}$$
$$e_a(X_{e^{\mathsf{y}}}) = \quad (X_e)e_a^{\mathsf{y}} + (X_\omega)\omega_a^{\mathsf{y}} \quad (X_\omega)c_{an}^{\mathsf{y}}\xi^n + ((X_{\xi^n})\chi_a + (X_{\chi_a})\xi^n)\,\mathsf{V}, \tag{2.10g}$$

together with

$$(X_{e^{\mathsf{y}}})\xi^n = 0 \tag{2.11a}$$
$$\iota_\xi(X_\omega)\xi^n = 0 \tag{2.11b}$$
$$(X_e)\xi^n = 0 \tag{2.11c}$$
$$(X_\omega)\xi^n = 0 \tag{2.11d}$$
$$(X_\xi)^\rho\xi^n = 0. \tag{2.11e}$$

We first solve (2.10c): expanding (X_{e_n}) in the basis $f e_\mu g_{\mu=1,2,n}$ we get a vector equation whose single components are

$$(X_{\xi^\mu}) = \quad (X_{e_n})^{h\mu i}\xi^n \text{ for } \mu = 1, 2, n.$$

It is then possible to solve equations (2.10) to yield

$$(X_{\omega^{\mathsf{y}}}) = 0 \tag{2.12a}$$
$$(X_{\xi^{\mu}}) = \quad (X_{e_n})^{\hbar\mu i}\xi^n \tag{2.12b}$$
$$(X_{\omega}) = \quad (X_{e_n})^{\hbar bi}\xi^n e_b^{\mathsf{y}} \quad (X_{e_n})^{\hbar ni}\xi^n e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n \tag{2.12c}$$
$$\begin{aligned}(X_e) &= (X_{e_n})^{\hbar bi}\xi^n \omega_b^{\mathsf{y}} + (X_{e_n})^{\hbar ni}\xi^n\omega_n^{\mathsf{y}} + (X_{\omega_n^{\mathsf{y}}})\xi^n \quad \iota_\xi (X_{c_n^{\mathsf{y}}})\xi^n \\ &\quad + (X_{e_n})^{\hbar ni}\iota_\xi c_n^{\mathsf{y}}\xi^n\end{aligned} \tag{2.12d}$$
$$(X_c) = \iota_\xi (X_\omega) \tag{2.12e}$$

$$\begin{aligned}p_a^{\mathsf{y}}(X_{e^{\mathsf{y}}}) = \Big[&\big((X_{e_n})^{\hbar bi}\xi^n\omega_b^{\mathsf{y}} + (X_{e_n})^{\hbar ni}\xi^n\omega_n^{\mathsf{y}} + (X_{\omega_n^{\mathsf{y}}})\xi^n \quad \iota_\xi(X_{c_n^{\mathsf{y}}})\xi^n\big)e_a^{\mathsf{y}} \\ &+ (X_{e_n})^{\hbar ni}\iota_\xi c_n^{\mathsf{y}}\xi^n e_a^{\mathsf{y}} + \big(\quad (X_{e_n})^{\hbar bi}\xi^n e_b^{\mathsf{y}} \quad (X_{e_n})^{\hbar ni}\xi^n e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n\big)\omega_a^{\mathsf{y}} \\ &\big(\quad (X_{e_n})^{\hbar bi}\xi^n e_b^{\mathsf{y}} \quad (X_{e_n})^{\hbar ni}\xi^n e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n\big)c_{an}^{\mathsf{y}}\xi^n\Big]^{\hbar ai} \\ &(X_{e_n})^{\hbar ni}\xi^n\chi_a \mathsf{V}^{\hbar ai} + (X_{\chi} \ )\xi^n\mathsf{V}^{\hbar ai}\end{aligned} \tag{2.12f}$$

$$\begin{aligned}p_n^{\mathsf{y}}(X_{e^{\mathsf{y}}}) = \Big[&\big((X_{e_n})^{\hbar bi}\xi^n\omega_b^{\mathsf{y}} + (X_{e_n})^{\hbar ni}\xi^n\omega_n^{\mathsf{y}} + (X_{\omega_n^{\mathsf{y}}})\xi^n \quad \iota_\xi(X_{c_n^{\mathsf{y}}})\xi^n\big)e_n^{\mathsf{y}} \\ &+ (X_{e_n})^{\hbar ni}\iota_\xi c_n^{\mathsf{y}}\xi^n e_n^{\mathsf{y}} + \big(\quad (X_{e_n})^{\hbar bi}\xi^n e_b^{\mathsf{y}} \quad (X_{e_n})^{\hbar ni}\xi^n e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n\big)\omega_n^{\mathsf{y}} \\ &\big(\quad (X_{e_n})^{\hbar bi}\xi^n e_b^{\mathsf{y}} \quad (X_{e_n})^{\hbar ni}\xi^n e_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n\big)c_{nb}^{\mathsf{y}}\xi^b \\ &2(X_{e_n})^{\hbar ni}\xi^n\chi_n\mathsf{V} + (X_{\chi_n})\xi^n\mathsf{V} \quad (X_{e_n})^{\hbar ai}\xi^n\chi_a\mathsf{V}\Big]^{\hbar ni}\end{aligned} \tag{2.12g}$$

and, since Equations (2.12) force the left hand sides to be proportional to the odd field ξ^n, Equations (2.11) are automatically satisfied. This shows that the kernel has *constant rank* — i.e. $\check{\varpi}^{(1)}$ is pre-symplectic — and we can perform symplectic reduction.

We now compute the BV-BFV data by presenting a chart for the symplectic reduction

$$F_{PC}^{(1)} := \check{F}_{PC}^{(1)}/\ker(\check{\varpi}^{(1)\sharp}).$$

We flow along vertical vector fields (i.e. vector fields in the kernel of $\check{\varpi}^{(1)\sharp}$) to obtain boundary coordinates. In other words, denoting by φ_Y the flow of a vector field Y at time $s = 1$, we define the change of coordinates on a field ψ to be given by

$$\widetilde{\psi} := (\varphi_{\mathbb{E}_n} \quad \varphi_{\mathbb{E}_n^{\mathsf{y}}} \quad \varphi_{\mathbb{C}_n^{\mathsf{y}}} \quad \varphi_{\Omega_n^{\mathsf{y}}} \quad \varphi_{\mathbb{X}_n} \quad \varphi_{\mathbb{X}_a})(\psi),$$

where the vertical vector fields $\mathbb{E}_n, \mathbb{E}_n^{\mathsf{y}}, \mathbb{C}_n, \Omega_n^{\mathsf{y}}, \mathbb{X}_n$ and $\mathbb{X}_a$ read

$$\begin{aligned}\mathbb{E}_n^{\mathsf{y}} &= (X_{e_n^{\mathsf{y}}})\frac{\delta}{\delta e_n^{\mathsf{y}}} + (X_{e_n^{\mathsf{y}}})\xi^n\frac{\delta}{\delta\omega} + \iota_\xi(X_{e_n^{\mathsf{y}}})\xi^n\frac{\delta}{\delta c} \\ &\quad + \Big[(X_{e_n^{\mathsf{y}}})\xi^n\omega_a^{\mathsf{y}}\Big]^{\hbar ai}\frac{\delta}{\delta p_a^{\mathsf{y}} e^{\mathsf{y}}} + \Big[(X_{e_n^{\mathsf{y}}})\xi^n\iota_\xi c_n^{\mathsf{y}} + (X_{e_n^{\mathsf{y}}})\xi^n\omega_n^{\mathsf{y}}\Big]^{\hbar ni}\frac{\delta}{\delta p_n^{\mathsf{y}} e^{\mathsf{y}}}\end{aligned} \tag{2.13a}$$

$$\mathbb{C}^{y}_{n} = (X_{c^{y}_{n}})\frac{\delta}{\delta c^{y}_{n}} \quad \iota_{\xi}(X_{c^{y}_{n}})\xi^{n}\frac{\delta}{\delta e} + \left(\iota_{\xi}(X_{c^{y}_{n}})\xi^{n}e^{y}_{n}\right)^{hni}\frac{\delta}{\delta p^{y}_{n}e^{y}}$$
$$+ \left(\iota_{\xi}(X_{c^{y}_{na}})\xi^{n}e^{y}\right)^{hai}\frac{\delta}{\delta p^{y}_{a}e^{y}} \tag{2.13b}$$

$$\mathbb{\Omega}^{y}_{n} = (X_{\omega^{y}_{n}})\frac{\delta}{\delta \omega^{y}_{n}} + (X_{\omega^{y}_{n}})\xi^{n}\frac{\delta}{\delta e} \quad \left((X_{\omega^{y}_{n}})\xi^{n}e^{y}_{n}\right)^{hni}\frac{\delta}{\delta p^{y}_{n}e^{y}} \quad \left((X_{\omega^{y}_{na}})\xi^{n}e^{y}\right)^{hai}\frac{\delta}{\delta p^{y}_{a}e^{y}} \tag{2.13c}$$

$$\mathbb{X}_{n} = (X_{\chi_{n}})\frac{\delta}{\delta \chi_{n}} + ((X_{\chi_{n}})\xi^{n}\mathsf{V})^{hni}\frac{\delta}{\delta p^{y}_{n}e^{y}} \tag{2.13d}$$

$$\mathbb{X}_{a} = (X_{\chi_{a}})\frac{\delta}{\delta \chi_{a}} + (X_{\chi_{a}})\xi^{n}\mathsf{V}^{hai}\frac{\delta}{\delta p^{y}_{a}e^{y}} \tag{2.13e}$$

$$\mathbb{E}_{n} = (X_{e_{n}})\frac{\delta}{\delta e_{n}} \quad \left((X_{e_{n}})^{hai}e^{y}_{a}\xi^{n} + (X_{e_{n}})^{hni}e^{y}_{n}\xi^{n}\right)\frac{\delta}{\delta \omega}$$
$$\iota_{\xi}\left((X_{e_{n}})^{hai}e^{y}_{a}\xi^{n} + (X_{e_{n}})e^{y}_{n}\xi^{n}\right)\frac{\delta}{\delta c}$$
$$+\left(\quad (X_{e_{n}})^{hai}\omega^{y}_{a}\xi^{n} \quad (X_{e_{n}})^{hni}\omega^{y}_{n}\xi^{n} + (X_{e_{n}})^{hni}\iota_{\xi}c^{y}_{n}\xi^{n}\right)\frac{\delta}{\delta e} \quad (X_{e_{n}})^{\rho}\xi^{n}\frac{\delta}{\delta \xi^{\rho}}$$
$$\Big[((X_{e_{n}})^{hbi}\xi^{n}\omega^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}\omega^{y}_{n} + (X_{e_{n}})^{hni}\iota_{\xi}c^{y}_{n}\xi^{n})e^{y}_{a}$$
$$+ ((X_{e_{n}})^{hbi}\xi^{n}e^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}e^{y}_{n})\omega^{y}_{a} + ((X_{e_{n}})^{hbi}\xi^{n}e^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}e^{y}_{n})c^{y}_{an}\xi^{n}$$
$$(X_{e_{n}})^{hni}\xi^{n}\chi_{a}\mathsf{V}\Big]^{hai}\frac{\delta}{\delta p^{y}_{a}e^{y}}$$
$$+\Big[((X_{e_{n}})^{hbi}\xi^{n}\omega^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}\omega^{y}_{n} + (X_{e_{n}})^{hni}\iota_{\xi}c^{y}_{n}\xi^{n})e^{y}_{n}$$
$$((X_{e_{n}})^{hbi}\xi^{n}e^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}e^{y}_{n})\omega^{y}_{n} + (\quad (X_{e_{n}})^{hbi}\xi^{n}e^{y}_{b} + (X_{e_{n}})^{hni}\xi^{n}e^{y}_{n})c^{y}_{nb}\xi^{b}$$
$$2(X_{e_{n}})^{hni}\xi^{n}\chi_{n}\mathsf{V} \quad (X_{e_{n}})^{hai}\xi^{n}\chi_{a}\mathsf{V}\Big]^{hni}\frac{\delta}{\delta p^{y}_{n}e^{y}} \tag{2.13f}$$

Using $\mathbb{X}_a$ and $\mathbb{X}_n$ we can eliminate χ_a and χ_n. From the differential equation $\dot{\chi}_{\rho}(s) = (X_{\chi_{\rho}})$ $(\rho = a, n)$, we conclude that

$$\chi_{\rho}(s=1) = 0 \quad (\;) \quad (X_{\chi_{\rho}}) = \quad \chi_{\rho}(0),$$

with s parametrising the flow of the vector field. The equation induced by the flow of $\mathbb{X}_n$ reads

$$\frac{d}{ds}(p^{y}_{n}e^{y}) = (\quad \chi_{\rho}(0)\xi^{n}\mathsf{V})^{hni}, \tag{2.14}$$

and similarly for $\mathbb{X}_a$. It follows that, if we define $p^{y}_{n}e^{y}[1] := (p^{y}_{n}e^{y})(s=1)$, we have

$$p^{y}_{n}e^{y}[1] = p^{y}_{n}e^{y} \quad \chi_{n}\xi^{n}\mathsf{V}^{hni}; \quad p^{y}_{a}e^{y}[1] = p^{y}_{a}e^{y} \quad \chi_{a}\xi^{n}\mathsf{V}^{hai}. \tag{2.15}$$

The numbered square bracket [1] denotes the step-by-step reconstruction of the change of variables: indeed, the variable $p^{y}_{n}e^{y}$ gets transformed first by (the flow of) $\mathbb{X}_n$, and then by $\mathbb{\Omega}^{y}_{n}$ and $\mathbb{E}^{y}_{n}$. Flowing along each of these vertical vector fields defines a temporary change of coordinates, which will be denoted by $[k]$.

Indeed, using Ω^{y}_{n} we can dynamically set $\omega^{y}_{n}(s=1)=0$ by picking $(X_{\omega^{y}_{n}}) = \quad \omega^{y}_{n}(0)$. This choice induces differential equations for the newly defined variables:

$$\frac{d}{ds}(p^{y}_{n}e^{y}[1]) = (\omega^{y}_{n}\xi^{n}e^{y}_{n})^{hai} \quad (\ ) \quad (p^{y}_{n}e^{y}[1])(s) = p^{y}_{n}e^{y}[1](0) + (\omega^{y}_{n}\xi^{n}e^{y}_{n})^{hni}s, \tag{2.16}$$

$$\frac{d}{ds}(p^{y}_{a}e^{y}[1]) = (\omega^{y}_{n}\xi^{n}e^{y}_{n})^{hai} \quad (\ ) \quad (p^{y}_{a}e^{y}[1])(s) = p^{y}_{a}e^{y}[1](0) + (\omega^{y}_{n}\xi^{n}e^{y}_{n})^{hai}s, \tag{2.17}$$

as well as

$$\frac{d}{ds}e = \quad \omega^{y}_{n}\xi^{n} \quad (\ ) \quad e(s) = e(0) \quad \omega^{y}_{n}\xi^{n}, \tag{2.18}$$

and defining the new temporary variables by $p^{y}_{n}e^{y}[2] := p^{y}_{n}e^{y}[1](s=1)$ (and similarly for $p^{y}_{a}e^{y}[2]$, we obtain:

$$\begin{gathered} e[1] := e(s=1) = e \quad \omega^{y}_{n}\xi^{n}; \\ (p^{y}_{n}e^{y})[2] = (p^{y}_{n}e^{y})[1] + \left(\omega^{y}_{n}\xi^{n}e^{y}_{n}\right)^{hni}; \quad (p^{y}_{a}e^{y})[2] = (p^{y}_{a}e^{y})[1] + \left(\omega^{y}_{na}\xi^{n}e^{y}\right)^{hai} \end{gathered} \tag{2.19}$$

Moving on to $\mathbb{E}^{y}_{n}$ with an analogous procedure, we solve the associated differential equations to yield

$$\begin{gathered} \omega[1] = \omega \quad e^{y}_{n}\xi^{n}; \qquad\qquad c[1] = c \quad \iota_{\xi}e^{y}_{n}\xi^{n}; \\ (p^{y}_{a}e^{y})[3] = (p^{y}_{a}e^{y})[2] \quad (e^{y}_{n}\xi^{n}\omega^{y}_{a})^{hai}; \quad (p^{y}_{n}e^{y})[3] = (p^{y}_{n}e^{y})[2] \quad (e^{y}_{n}\iota_{\xi}c^{y}_{n}\xi^{n})^{hni} \end{gathered} \tag{2.20}$$

while using $\mathbb{C}^{y}_{n}$ in the same fashion we can conclude:

$$e[2] = e[1] + \iota_{\xi}c^{y}_{n}\xi^{n}; \quad (p^{y}_{a}e^{y})[4] = (p^{y}_{a}e^{y})[3] \quad \left(\iota_{\xi}c^{y}_{na}\xi^{n}e^{y}\right)^{hai}. \tag{2.21}$$

Notice that we did not consider the coefficient of $\frac{\delta}{\delta e^{y}}$ in $\mathbb{E}^{y}_{n}$, for have fixed the values $c^{y}_{n} = \omega^{y}_{n} = 0$ at the internal parameter $s=1$ along the flow of the previously employed vector fields.
Now it is time to turn to $\mathbb{E}_{n}$. Its simplified expression after flowing along the other vector fields is

$$\begin{aligned} \mathbb{E}_{n} =& (X_{e_{n}})\frac{\delta}{\delta e_{n}} \quad (X_{e_{n}})^{hai}e^{y}_{a}\xi^{n}\frac{\delta}{\delta\omega} \quad \iota_{\xi}(X_{e_{n}})^{hai}e^{y}_{a}\xi^{n}\frac{\delta}{\delta c} \\ & (X_{e_{n}})^{hai}\omega^{y}_{a}\xi^{n}\frac{\delta}{\delta e} \quad (X_{e_{n}})^{\rho}\xi^{n}\frac{\delta}{\delta\xi^{\rho}} \end{aligned} \tag{2.22}$$

We want to flow along its integrating diffeomorphism and set the field e_{n} to a given value. In this case we cannot set $e_{n}(s=1)=0$ because this would violate the non-degeneracy requirement for the triad field. We will fix $e_{n}(1)$ to a vector ϵ_{n} 2 V proportional to the original e_{n}, pointwise, and thus linearly independent from (the vectors in the image of) e. Observe that for an open subset $U^{(1)} \quad \Omega^{1}_{nd}(M^{(1)}, V)$ the choice of ϵ_{n} is independent of e 2 $U^{(1)}$. The differential equation $\dot{e}_{n} = (X_{e_{n}})$ is solved as $e_{n}(s) = e_{n}(0) + (X_{e_{n}})s$, so that, fixing $e_{n}(s=1) \quad (1+\varepsilon)e_{n}(0)$ yields the flow:

$$(X_{e_{n}}) = \varepsilon e_{n}(0); \quad e_{n}(s) = e_{n}(0)(1+\varepsilon s), \tag{2.23}$$

with ε 2 $C^{1}(M^{(1)})$, $\varepsilon > 0$. In order to compute the other flows, we have to consider the components of $(X_{e_{n}})$ in the (varying) basis vectors $fe_{a}g$. With our choice we have $(X_{e_{n}})$ / $e_{n}(0)$ / $e_{n}(s)$. Hence we obtain

$$(X_{e_{n}})^{hai} = 0 \text{ and } (X_{e_{n}})^{hni}(s) = \frac{\varepsilon}{1+\varepsilon s}$$

Hence, looking at the expression for $\mathbb{E}_n$ the only equation that we have to consider is

$$\dot{\xi}^n = -(X_{e_n})^{\langle n \rangle}\xi^n \tag{2.24}$$

and we easily find

$$\xi^n(s) = \frac{1}{1+\varepsilon s}\xi^n(0)$$

so that

$$\xi^n[1] = \frac{1}{1+\varepsilon}\xi^n.$$

Gathering what we have done so far, defining $\epsilon_n := (1+\varepsilon)e_n$ (now a fixed vector field), we have constructed a chart on the symplectic reduction

$$F^{(1)} := \check{F}^{(1)}/\ker(\check{\varpi}^{(1)\sharp})$$

so that the map $\pi^{(1)}_{\mathsf{red}} \colon \check{F}^{(1)} \to F^{(1)}$ reads:

$$\pi^{(1)}_{\mathsf{red}} : \begin{cases} \widetilde{e} := e - \omega^{\partial}_n\xi^n + \iota_\xi c^{\partial}_n\xi^n \\ \widetilde{\omega} := \omega - e^{\partial}_n\xi^n \\ \widetilde{c} := c - \iota_\xi e^{\partial}_n\xi^n \\ \widetilde{\xi}^n := (1+\varepsilon)^{-1}\xi^n \\ \widetilde{\xi}^a := \xi^a \\ \tilde{f}_a^{\ \partial} := p^{\partial}_a e^{\partial} - \chi_a\xi^n V^{\langle a\rangle} + \left(\omega^{\partial}_{na}\xi^n e^{\partial}\right)^{\langle a\rangle} - (e^{\partial}_n\xi^n\omega^{\partial}_a)^{\langle a\rangle} - \left(\iota_\xi c^{\partial}_{na}\xi^n e^{\partial}\right)^{\langle a\rangle} \\ \tilde{f}_n^{\ \partial} := p^{\partial}_n e^{\partial} - \chi_n\xi^n V^{\langle n\rangle} + \left(\omega^{\partial}_n\xi^n e^{\partial}_n\right)^{\langle n\rangle} - (e^{\partial}_n\iota_\xi c^{\partial}_n\xi^n)^{\langle n\rangle} \\ \widetilde{\omega}^{\partial} := \omega^{\partial} \end{cases} \tag{2.25}$$

with $\widetilde{e} \in U \subset \Omega^1_{nd}(M^{(1)}, V)$, and the BV-BFV map[6] $\pi^{(0)} \colon F^{(0)} \to F^{(1)}$ is

$$\pi^{(0)} := \pi^{(1)}_{\mathsf{red}} \circ \check{\pi}^{(0)}.$$

This data defines a chart for the (locally trivialised) bundle

$$F^{(1)}_{PC} \simeq U^{(1)} \times \Omega^1(M^{(1)}, \wedge^2 V) \times T\left(\Omega^0[1](M^{(1)}, \wedge^2 V) \oplus \mathfrak{X}[1](M^{(1)}) \oplus C^{\infty}[1](M^{(1)})\right)$$

since, for all $\widetilde{e} \in U$ we can fix a completion ϵ_n that does not depend on $\widetilde{e}$.

An easy computation shows that $\check{\alpha}^{(1)}$ is not basic (in particular it is not horizontal, i.e. $\iota_{\mathbb{E}_n}\check{\alpha}^{(1)} \neq 0$), but it descends to the quotient upon adding the term $\delta(ee^{\partial}_n\xi^n)$. In the local chart defined by (2.25), we define:[7]

$$\alpha^{(1)} := \int_{M^{(1)}} \widetilde{e}\delta\widetilde{\omega} + \widetilde{\omega}^{\partial}\delta\widetilde{c} - \tilde{f}_n^{\ \partial}\epsilon_n\delta\widetilde{\xi}^n - \iota_{\delta\widetilde{\xi}}\widetilde{e}\tilde{f}_n^{\ \partial} - \iota_{\delta\widetilde{\xi}}\widetilde{e}\tilde{f}_a^{\ \partial} + \iota_{\widetilde{\xi}}\widetilde{\omega}^{\partial}\delta\widetilde{\omega},$$

so that $\check{\alpha}^{(1)} + \delta(ee^{\partial}_n\xi^n) = \pi^{(1)*}_{\mathsf{red}}\alpha^{(1)}$ and, consequently,

$$\begin{aligned}\varpi^{(1)} := \int_{M^{(1)}} & \delta\widetilde{e}\delta\widetilde{\omega} + \delta\widetilde{\omega}^{\partial}\delta\widetilde{c} - \delta\tilde{f}_n^{\ \partial}\epsilon_n\delta\widetilde{\xi}^n + \iota_{\delta\widetilde{\xi}}\delta(\widetilde{e}\tilde{f}_n^{\ \partial}) \\ & + \iota_{\delta\widetilde{\xi}}\delta(\widetilde{e}\tilde{f}_a^{\ \partial}) + \delta(\iota_{\widetilde{\xi}}\widetilde{\omega}^{\partial})\delta\widetilde{\omega}.\end{aligned}$$

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[6] Observe that, from now on, we will omit mentioning precomposition with π, when no confusion can arise.

[7] Since $\delta\epsilon_n = 0$ we obtain, in the space of preboundary fields, $\delta\varepsilon e_n = -(1+\varepsilon)\delta e_n$. Hence $\epsilon_n\delta\widetilde{\xi}^n = (1+\varepsilon)e_n\delta(1+\varepsilon)^{-1}\xi^n + e_n\delta\xi^n = -(1+\varepsilon)e_n(1+\varepsilon)^{-2}\delta\varepsilon\xi^n + e_n\delta\xi^n = \delta e_n\xi^n + e_n\delta\xi^n$.

Given any element $k \in \Omega^2(\partial M, \wedge^2 V)$ and a basis $\{v_i\}_{i=1,2,3}$ of the vector space V we can build a diffeomorphism

$$\phi : \Omega^2(\partial M, \wedge^2 V) \to \prod_{i=1}^{3} p_i^\dagger \Omega^2(\partial M, \wedge^2 V)$$
$$k \mapsto \left(p_1^\dagger k, p_2^\dagger k, p_3^\dagger k\right)$$

where $p_i^\dagger$ is the projection defined by (2.1) with inverse given by $\left(p_1^\dagger k, p_2^\dagger k, p_3^\dagger k\right) \mapsto p_1^\dagger k + p_2^\dagger k + p_3^\dagger k = k$. We now observe that the quantities $\tilde{f_a}^\dagger$ and $\tilde{f_n}^\dagger$ in (2.25) satisfy $p_a^\dagger \tilde{f_a}^\dagger = \tilde{f_a}^\dagger$ and $p_n^\dagger \tilde{f_n}^\dagger = \tilde{f_n}^\dagger$ where $p_a^\dagger$ and $p_n^\dagger$ are with respect to the basis (e_a, e_n). Hence defining $\widetilde{e}^\dagger := \tilde{f_a}^\dagger + \tilde{f_n}^\dagger$ we obtain that the image of $\widetilde{e}^\dagger$ under the diffeomorphism ϕ is the couple $\left(\tilde{f_a}^\dagger, \tilde{f_n}^\dagger\right)$. Since the codimension-1 quantities depend only on the $\widetilde{e}^\dagger$ as defined above, we conclude that we can use it as a basis independent field on $M^{(1)}$.

Hence the odimension-1 forms can be more conveniently rewritten as

$$\alpha^{(1)} := \int_{M^{(1)}} -\widetilde{e}\delta\widetilde{\omega} + \widetilde{\omega}^\dagger \delta\widetilde{c} - \widetilde{e}^\dagger \epsilon_n \delta\widetilde{\xi}^n - \iota_{\delta\widetilde{\xi}} \widetilde{e}\widetilde{e}^\dagger + \iota_{\widetilde{\xi}} \widetilde{\omega}^\dagger \delta\widetilde{\omega}$$
$$\varpi^{(1)} := \int_{M^{(1)}} -\delta\widetilde{e}\delta\widetilde{\omega} + \delta\widetilde{\omega}^\dagger \delta\widetilde{c} - \delta\widetilde{e}^\dagger \epsilon_n \delta\widetilde{\xi}^n + \iota_{\delta\widetilde{\xi}} \delta(\widetilde{e}\widetilde{e}^\dagger) + \delta(\iota_{\widetilde{\xi}} \widetilde{\omega}^\dagger)\delta\widetilde{\omega}.$$

where $\widetilde{e} \in U^{(1)}$, $\widetilde{\omega} \in \Omega^1(M^{(1)}, \wedge^2 V)$, $\widetilde{c} \in \Omega^0[1](M^{(1)}, \wedge^2 V)$, $\widetilde{\xi} \in \mathfrak{X}[1](M^{(1)})$, $\widetilde{\xi}^n \in C^\infty[1](M^{(1)})$, $\widetilde{\omega}^\dagger \in \Omega^3[-1](M^{(1)}, V)$ and $\widetilde{e}^\dagger \in \Omega^2[-1](M^{(1)}, \wedge^2 V)$.

Observe that, due to the arbitrariness of ϵ_n, and since all expressions involving $\widetilde{e}_a$ are coordinate independent on $M^{(1)}$, the above discussion ensures the independence of $\alpha^{(1)}$ from the choice of a specific set of coordinates.

Finally, we can compute $\check{S}^{(1)}$ such that $\iota_{Q^{(0)}}\iota_{Q^{(0)}}\varpi^{(0)} = 2\check{\pi}^{(0)*}\check{S}^{(1)}$, using Equation (1.21) for the bulk Q. A straightforward calculation yields:

$$\begin{aligned}\check{S}^{(1)} = \int_{M^{(1)}} & -\iota_\xi e F_\omega - e_n \xi^n F_\omega - c d_\omega e + \frac{1}{2}[c,c]\omega^\dagger + c[e_n^\dagger, e]\xi^n + c[e^\dagger, e_n]\xi^n \\ & + c d_\omega(\iota_\xi \omega^\dagger + \omega_n^\dagger \xi^n) - c d_\omega(\iota_\xi c_n^\dagger \xi^n) + \iota_\xi F_\omega \iota_\xi c_n^\dagger \xi^n - \frac{1}{2}\iota_\xi \iota_\xi F_\omega \omega^\dagger \\ & - \iota_\xi F_\omega \omega_n^\dagger \xi^n - \iota_\xi F_\omega \iota_\xi \omega^\dagger + \frac{1}{2}\iota_{[\xi,\xi]} e e^\dagger + e_n \xi^n d_\omega \iota_\xi e^\dagger + \iota_\xi e d_\omega(e_n^\dagger \xi^n) \\ & + e_n^\dagger \xi^n \iota_\xi d_\omega e + e_n \xi^n d_\omega(e_n^\dagger \xi^n) - \frac{1}{2}\iota_{[\xi,\xi]}\chi V \xi^n - \iota_\xi d\xi^n \chi_n V \xi^n.\end{aligned}$$

Using the projection (2.26) we can find the codimension-1 action functional to be

$$\begin{aligned}S^{(1)} = \int_{M^{(1)}} & -\iota_{\widetilde{\xi}} \widetilde{e} F_{\widetilde{\omega}} - \epsilon_n \widetilde{\xi}^n F_{\widetilde{\omega}} - \widetilde{c} d_{\widetilde{\omega}} \widetilde{e} + \frac{1}{2}[\widetilde{c}, \widetilde{c}]\widetilde{\omega}^\dagger + \frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}} F_{\widetilde{\omega}} \widetilde{\omega}^\dagger + \frac{1}{2}\iota_{[\widetilde{\xi},\widetilde{\xi}]}\widetilde{e}\widetilde{e}^\dagger \\ & + \widetilde{c} d_{\widetilde{\omega}}(\iota_{\widetilde{\xi}} \widetilde{\omega}^\dagger) + L_{\widetilde{\xi}}^{\widetilde{\omega}}(\epsilon_n \widetilde{\xi}^n)\widetilde{e}^\dagger - [\widetilde{c}, \epsilon_n \widetilde{\xi}^n]\widetilde{e}^\dagger\end{aligned}$$

satisfying $\check{S}^{(1)} = \pi_{\mathsf{red}}^{(1)*} S^{(1)}$. As a direct consequence, then, we have

$$\iota_{Q^{(0)}}\iota_{Q^{(0)}}\varpi^{(0)} = 2\pi^{(0)*} S^{(1)}.$$

Having found the codimension-1 action functional we can compute the cohomological vector field $Q^{(1)}$ on the 1-stratum. Since the coordinates we are using are not a Darboux chart, some complications in the computation arise. Nonetheless, the non-degeneracy of $\varpi^{(1)}$ guarantees that starting from the variation of the action and using the equation $\iota_{Q^{(1)}}\varpi^{(1)} = \delta S^{(1)}$, we can compute the cohomological codimension-1 vector field $Q^{(1)}$. □

Corollary 2.12 (to the proof of Proposition 2.10)**.** *An explicit expression of the projection to codimension-*1 *fields, given a coordinate system adapted to the embedding* $M^{(1)} \to M^{(0)}$ *is given by*

$$\pi^{(0)}_{PC} : \begin{cases} \widetilde{e} = e - \omega_n^\dagger \xi^n + \iota_\xi c_n^\dagger \xi^n \\ \widetilde{\omega} = \omega - e_n^\dagger \xi^n \\ \widetilde{c} = c - \iota_\xi e_n^\dagger \xi^n \\ \widetilde{\xi}^n = (1+\varepsilon)^{-1}\xi^n \\ \widetilde{\xi}^a = \xi^a \\ \widetilde{e}^\dagger = e^\dagger - \chi_a \xi^n \mathsf{V}^{\langle a \rangle} + \left(\omega_{na}^\dagger \xi^n e^\dagger\right)^{\langle a \rangle} - (e_n^\dagger \xi^n \omega_a^\dagger)^{\langle a \rangle} - \left(\iota_\xi c_{na}^\dagger \xi^n e^\dagger\right)^{\langle a \rangle} \\ \qquad - \chi_n \xi^n \mathsf{V}^{\langle n \rangle} + \left(\omega_n^\dagger \xi^n e_n^\dagger\right)^{\langle n \rangle} - (e_n^\dagger \iota_\xi c_n^\dagger \xi^n)^{\langle n \rangle} \\ \widetilde{\omega}^\dagger = \omega^\dagger \\ \epsilon_n = (1+\varepsilon) e_n \end{cases} ; \tag{2.26}$$

where $\mathsf{V} \in \Omega^3(\partial M, \wedge^3 V)$ *is a fixed volume form,* $\varepsilon \in C^\infty(M^{(1)})$ *is a smooth function,* $\chi \in \Omega(M)$ *is the one form component of* $\xi^\dagger = \chi \otimes \mathsf{V}$*, and we used the notation explained in* (2.2)*,* (2.3) *and* (2.4) *for the restriction of field and their normal components where the superscript* $^{\langle n \rangle}$ *here denotes the component with respect to* e_n*.*

Remark 2.13. Observe that, strictly speaking, Equation (1.3) is satisfied by the above data only if we modify S_{PC} by the boundary term $\int_{M^{(0)}} d(e e_n^\dagger \xi^n)$, so that the associated *pre-boundary one-form* $\check{\alpha}^{(1)}$ is automatically basic. Indeed, this is necessary only if we insist on the BV-BFV data to be *exact*, i.e. such that the symplectic forms are exact at every codimension $\varpi^{(k)} = \delta\alpha^{(k)}$ *and* that their symplectic potential $\alpha^{(k)}$ is pulled back to the respective term in (1.3). A picture suitable to situations like the present one, where symplectic reduction is possible, but non-trivial, at every codimension, is to consider the symplectic forms instead of their potentials, and the equation

$$L_Q \varpi^{(k)} = \pi^{(k)}_* \varpi^{(k+1)},$$

which follows from (1.3) by differentiating w.r.t. δ. However, modifying $S^{(k)}$ by a term concentrated in codimension-$(k+1)$ does not change the BV-BFV structure (cf. [MSW19]).

Remark 2.14. The expression (2.7b) is not in Darboux form. The change of coordinates to a Darboux chart will turn out to coincide with the boundary symplectomorphism between PC theory and BF theory (see Section 2.3.2). The same symplectomorphism will also turn the local trivialisation (2.6) into a global one, showing that the bundle $F^{(1)}_{PC}$ is indeed trivial (since $F^{(1)}_{BF}$ is).

2.2.3 2-extended PC theory

We are now ready to compute the structure induced on codimension-2 strata when M carries a 2-stratification, for example in the presence of corners. Building up from the codimension-1 BV-BFV structure found in Proposition 2.10, denoting again by $\Omega^1_{nd}(M^{(2)}, V)$ the space of maps $e\colon TM^{(2)} \to V$ whose image defines a linearly independent system, we have the following result.

Proposition 2.15. *The BV theory $\mathsf{F}_{PC}^{''0}$ is 2-extendable to $\mathsf{F}_{PC}^{''2}$. The codimension-2 data are:*

The space of codimension-2 fields, given by the bundle

$$F_{PC}^{(2)} \to \Omega_{nd}^{1}(M^{(2)}, V), \tag{2.27}$$

with local trivialisation on a open subset $U^{(2)} \subset \Omega_{nd}^{1}(M^{(2)}, V)$

$$F_{PC}^{(2)} = U^{(2)} \times \Omega^1(M^{(2)}, \wedge^2 V) \times \Omega^0[1](M^{(2)}, \wedge^2 V) \times \mathfrak{X}[1](M^{(2)}) \times C^{\infty}[1](M^{(2)})^{\times 2},$$

and fields denoted by $\tilde{\tilde{e}} \in U^{(2)}$, $\tilde{\tilde{\omega}} \in \Omega^1(M^{(2)}, \wedge^2 V)$ in degree zero, $\tilde{\tilde{c}} \in \Omega^0[1](M^{(2)}, \wedge^2 V)$, $\tilde{\tilde{\xi}} \in \mathfrak{X}[1](M^{(2)})$ and $\widetilde{\widetilde{\xi^m}}, \widetilde{\widetilde{\xi^n}} \in C^{\infty}[1](M^{(2)})$ in degree one, together with two linearly independent, fixed vector fields $\epsilon_m, \epsilon_n \in \Gamma(V)$, completing the image of elements $\tilde{\tilde{e}} \in U^{(2)} \subset \Omega_{nd}^{1}(M^{(2)}, V)$ to a basis of V;

The codimension-2 one-form, symplectic form and action functional

$$\alpha_{PC}^{(2)} = \int_{M^{(2)}} \tilde{\tilde{c}}\delta\tilde{\tilde{e}} + \iota_{\tilde{\tilde{\xi}}}\tilde{\tilde{e}}\delta\tilde{\tilde{\omega}} + \epsilon_m\widetilde{\widetilde{\xi^m}}\delta\tilde{\tilde{\omega}} + \epsilon_n\widetilde{\widetilde{\xi^n}}\delta\tilde{\tilde{\omega}}, \tag{2.28a}$$

$$\varpi_{PC}^{(2)} = \int_{M^{(2)}} \delta\tilde{\tilde{e}}\delta\tilde{\tilde{c}} + \iota_{\delta\tilde{\tilde{\xi}}}\tilde{\tilde{e}}\delta\tilde{\tilde{\omega}} + \iota_{\tilde{\tilde{\xi}}}\delta\tilde{\tilde{e}}\delta\tilde{\tilde{\omega}} + \epsilon_m\delta\widetilde{\widetilde{\xi^m}}\delta\tilde{\tilde{\omega}} + \epsilon_n\delta\widetilde{\widetilde{\xi^n}}\delta\tilde{\tilde{\omega}}, \tag{2.28b}$$

$$S_{PC}^{(2)} = \int_{M^{(2)}} \frac{1}{2}[\tilde{\tilde{c}}, \tilde{\tilde{c}}]\tilde{\tilde{e}} - \iota_{\tilde{\tilde{\xi}}}\tilde{\tilde{e}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}} - \epsilon_m\widetilde{\widetilde{\xi^m}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}} - \epsilon_n\widetilde{\widetilde{\xi^n}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}}, \tag{2.28c}$$

The cohomological vector field $Q_{PC}^{(2)}$

$$Q_{PC}^{(2)}\tilde{\tilde{e}} = -d_{\tilde{\tilde{\omega}}}(e_a\tilde{\tilde{\xi}}^a) - d_{\tilde{\tilde{\omega}}}(\epsilon_m\widetilde{\widetilde{\xi^m}}) - d_{\tilde{\tilde{\omega}}}(\epsilon_n\widetilde{\widetilde{\xi^n}}) - [\tilde{\tilde{c}}, \tilde{\tilde{e}}] \tag{2.29a}$$

$$Q_{PC}^{(2)}\tilde{\tilde{\omega}} = -d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}} \tag{2.29b}$$

$$Q_{PC}^{(2)}\tilde{\tilde{c}} = -\frac{1}{2}[\tilde{\tilde{c}}, \tilde{\tilde{c}}] \tag{2.29c}$$

$$Q_{PC}^{(2)}\tilde{\tilde{\xi}} = \frac{1}{2}[\tilde{\tilde{\xi}}, \tilde{\tilde{\xi}}] + X_n^{\langle a \rangle} - Y_n^{\langle a \rangle} + X_m^{\langle a \rangle} - Y_m^{\langle a \rangle} \tag{2.29d}$$

$$Q_{PC}^{(2)}\widetilde{\widetilde{\xi^n}} = X_n^{\langle n \rangle} - Y_n^{\langle n \rangle} + X_m^{\langle n \rangle} - Y_m^{\langle n \rangle} \tag{2.29e}$$

$$Q_{PC}^{(2)}\widetilde{\widetilde{\xi^m}} = X_n^{\langle m \rangle} - Y_n^{\langle m \rangle} + X_m^{\langle m \rangle} - Y_m^{\langle m \rangle} \tag{2.29f}$$

where

$$X_n = L_{\tilde{\tilde{\xi}}}^{\tilde{\tilde{\omega}}}(\epsilon_n\widetilde{\widetilde{\xi^n}}), \qquad X_m = L_{\tilde{\tilde{\xi}}}^{\tilde{\tilde{\omega}}}(\epsilon_m\widetilde{\widetilde{\xi^m}}), \qquad Y_n = [\tilde{\tilde{c}}, \epsilon_n\widetilde{\widetilde{\xi^n}}], \qquad Y_m = [\tilde{\tilde{c}}, \epsilon_m\widetilde{\widetilde{\xi^m}}],$$

and the superscripts $^{\langle n \rangle}, ^{\langle m \rangle}$ denote components with respect to ϵ_n, ϵ_m as defined in Section 2.2.1;

The projection to codimension-2 fields $\pi_{PC}^{(1)} = \pi_{\mathrm{red}}^{(2)} \circ \tilde{\pi}^{(1)}$ is smooth, where $\tilde{\pi}^{(1)} : F^{(1)} \to \check{F}^{(2)}$ is the restriction of the codimension-1 fields to the 2-stratum $M^{(2)}$ and $\pi_{\mathrm{red}}^{(2)} : \check{F}^{(2)} \to F^{(2)}$ is the symplectic reduction (1.6).

Proof. We proceed as before, following the strategy outlined in Section 1.1.2. Let $\check{\alpha}^{(2)}$ be such that $\iota_{Q^{(1)}}\varpi^{(1)} = \delta S^{(1)} + \check{\pi}^{(1)}\check{\alpha}^{(2)}$:

$$\check{\alpha}^{(2)} = \int_{M^{(2)}} \iota_{\widetilde{\xi}}\widetilde{e}\delta\widetilde{\omega} + \widetilde{e}_m\widetilde{\xi}^m\delta\widetilde{\omega} + \epsilon_n\widetilde{\xi}^n\delta\widetilde{\omega} + \widetilde{c}\delta\widetilde{e} \quad \delta\widetilde{\omega}\iota_{\widetilde{\xi}}\widetilde{\omega}^{y}_m\widetilde{\xi}^m + \iota_{\delta\widetilde{\xi}}\widetilde{e}\widetilde{e}^{y}_m\widetilde{\xi}^m \\ \widetilde{e}_m\delta\widetilde{\xi}^m\widetilde{e}^{y}_m\widetilde{\xi}^m \quad \widetilde{c}\delta(\widetilde{\omega}^{y}_m\widetilde{\xi}^m) \quad \epsilon_n\delta\widetilde{\xi}^n\widetilde{e}^{y}_m\widetilde{\xi}^m \tag{2.30}$$

$$\check{\varpi}^{(2)} = \delta\check{\alpha}^{(2)} = \int_{M^{(2)}} \iota_{\delta\widetilde{\xi}}\widetilde{e}\delta\widetilde{\omega} + \iota_{\widetilde{\xi}}\delta\widetilde{e}\delta\widetilde{\omega} + \delta\widetilde{e}_m\widetilde{\xi}^m\delta\widetilde{\omega} \quad \widetilde{e}_m\delta\widetilde{\xi}^m\delta\widetilde{\omega} \quad \epsilon_n\delta\widetilde{\xi}^n\delta\widetilde{\omega} + \delta\widetilde{c}\delta\widetilde{e} \\ \delta\widetilde{\omega}\iota_{\delta\widetilde{\xi}}\widetilde{\omega}^{y}_m\widetilde{\xi}^m \quad \delta\widetilde{\omega}\iota_{\widetilde{\xi}}\delta\widetilde{\omega}^{y}_m\widetilde{\xi}^m + \delta\widetilde{\omega}\iota_{\widetilde{\xi}}\widetilde{\omega}^{y}_m\delta\widetilde{\xi}^m \quad \delta\widetilde{e}_m\delta\widetilde{\xi}^m\widetilde{e}^{y}_m\widetilde{\xi}^m \\ + \widetilde{e}_m\delta\widetilde{\xi}^m\delta\widetilde{e}^{y}_m\widetilde{\xi}^m + \widetilde{e}_m\delta\widetilde{\xi}^m\widetilde{e}^{y}_m\delta\widetilde{\xi}^m \quad \iota_{\delta\widetilde{\xi}}\delta\widetilde{e}\widetilde{e}^{y}_m\widetilde{\xi}^m \quad \iota_{\delta\widetilde{\xi}}\widetilde{e}\delta\widetilde{e}^{y}_m\widetilde{\xi}^m \\ \iota_{\delta\widetilde{\xi}}\widetilde{e}\widetilde{e}^{y}_m\delta\widetilde{\xi}^m \quad \delta\widetilde{c}\delta(\widetilde{\omega}^{y}_m\widetilde{\xi}^m) + \epsilon_n\delta\widetilde{\xi}^n\delta\widetilde{e}^{y}_m\widetilde{\xi}^m + \epsilon_n\delta\widetilde{\xi}^n\widetilde{e}^{y}_m\delta\widetilde{\xi}^m$$

The forms $\check{\alpha}^{(2)}, \check{\varpi}^{(2)}$ are defined on $\check{F}^{(2)}$, the space of restrictions of codimension-1 fields (and their normal jets) to the codimension-2 stratum, with $\check{\pi}^{(1)}\colon F^{(1)} \; ! \; \check{F}^{(2)}$.

We have to show that the symplectic reduction of $\check{\varpi}^{(2)}$ is possible. The equations that define the kernel of $(\check{\varpi}^{(2)})^\sharp$ are:

$$\delta\widetilde{c}: \quad X_{\widetilde{e}} + (X_{\widetilde{\omega}^{y}_m})\widetilde{\xi}^m \quad \widetilde{\omega}^{y}_m(X_{\widetilde{\xi}^m}) = 0 \tag{2.31a}$$

$$\delta\widetilde{\omega}: \quad \iota_{(X_{\widetilde{\xi}})}\widetilde{e} + \iota_{\widetilde{\xi}}X_{\widetilde{e}} + X_{\widetilde{e}_m}\widetilde{\xi}^m \quad \widetilde{e}_m(X_{\widetilde{\xi}^m}) \quad \epsilon_n(X_{\widetilde{\xi}^n}) \\ + \iota_{(X_{\widetilde{\xi}})}\widetilde{\omega}^{y}_m\widetilde{\xi}^m + \iota_{\widetilde{\xi}}(X_{\widetilde{\omega}^{y}_m})\widetilde{\xi}^m \quad \iota_{\widetilde{\xi}}(\widetilde{\omega}^{y}_m)X_{\widetilde{\xi}^m} = 0 \tag{2.31b}$$

$$\delta\widetilde{\xi}: \quad \widetilde{e} \; p^{y}_p(X_{\widetilde{\omega}}) \quad (X_{\widetilde{\omega}})\widetilde{\omega}^{y}_m \; \widetilde{\xi}^m + \widetilde{e} \; (X_{\widetilde{e}^{y}_m})\widetilde{\xi}^m + \widetilde{e} \; \widetilde{e}^{y}_m(X_{\widetilde{\xi}^m}) = 0 \tag{2.31c}$$

$$\delta\widetilde{\xi}^n: \quad \epsilon_n p^{y}_n(X_{\widetilde{\omega}}) + \epsilon_n\widetilde{e}^{y}_m(X_{\widetilde{\xi}^m}) + \epsilon_n(X_{\widetilde{e}^{y}_m})\widetilde{\xi}^m = 0 \tag{2.31d}$$

$$\delta\widetilde{\xi}^m: \quad \widetilde{e}_m p^{y}_m(X_{\widetilde{\omega}}) + (X_{\widetilde{\omega}})\iota_{\widetilde{\xi}}\widetilde{\omega}^{y}_m \quad \iota_{(X_{\widetilde{\xi}})}\widetilde{e}\widetilde{e}^{y}_m \quad 2e_m\widetilde{e}^{y}_m(X_{\widetilde{\xi}^m}) \\ (X_{\widetilde{e}_m})\widetilde{e}^{y}_m\widetilde{\xi}^m + \widetilde{e}_m(X_{\widetilde{e}^{y}_m})\widetilde{\xi}^m + (X_{\widetilde{c}})\widetilde{\omega}^{y}_m + \epsilon_n(X_{\widetilde{\xi}^n})\widetilde{e}^{y}_m = 0 \tag{2.31e}$$

$$\delta\widetilde{e}_m: \quad \widetilde{\xi}^m(X_{\widetilde{\omega}}) + (X_{\widetilde{\xi}^m})\widetilde{e}^{y}_m\widetilde{\xi}^m = 0 \tag{2.31f}$$

$$\delta\widetilde{e}: \quad \iota_{\widetilde{\xi}}(X_{\widetilde{\omega}}) + (X_{\widetilde{c}}) = 0 \tag{2.31g}$$

$$\delta\widetilde{\omega}^{y}_m: \quad \iota_{\widetilde{\xi}}(X_{\widetilde{\omega}})\widetilde{\xi}^m \quad (X_{\widetilde{c}})\widetilde{\xi}^m = 0 \tag{2.31h}$$

$$\delta\widetilde{e}^{y}_m: \quad \iota_{(X_{\widetilde{\xi}})}\widetilde{e}\widetilde{\xi}^m \quad \widetilde{e}_m(X_{\widetilde{\xi}^m})\widetilde{\xi}^m \quad \epsilon_n(X_{\widetilde{\xi}^n})\widetilde{\xi}^m = 0 \tag{2.31i}$$

From (2.31a) we get

$$X_{\widetilde{e}} = \quad (X_{\widetilde{\omega}^{y}_m})\widetilde{\xi}^m + \widetilde{\omega}^{y}_m(X_{\widetilde{\xi}^m}).$$

Inserting this result into (2.31b) we get

$$\iota_{(X_{\widetilde{\xi}})}\widetilde{e} + X_{\widetilde{e}_m}\widetilde{\xi}^m \quad \widetilde{e}_m(X_{\widetilde{\xi}^m}) \quad \epsilon_n(X_{\widetilde{\xi}^n}) + \iota_{(X_{\widetilde{\xi}})}\widetilde{\omega}^{y}_m\widetilde{\xi}^m = 0$$

which is a vector equation. Using the basis $f\widetilde{e}, \widetilde{e}_m, \epsilon_n g$ we can write it equivalently as

$$\widetilde{e}^a X_{\widetilde{\xi}^a} + (X^{hai}_{e_m})\widetilde{e}_a\widetilde{\xi}^m + (X^{hmi}_{e_m})\widetilde{e}_m\widetilde{\xi}^m + (X^{hni}_{e_m})\epsilon_n\widetilde{\xi}^m \quad \widetilde{e}_m(X_{\widetilde{\xi}^m}) \quad \epsilon_n(X_{\widetilde{\xi}^n}) \\ + \widetilde{\omega}^{yhai}_{mb}\widetilde{e}_a X_{\widetilde{\xi}^b}\widetilde{\xi}^m + \widetilde{\omega}^{yhmi}_{mb}\widetilde{e}_m X_{\widetilde{\xi}^b}\widetilde{\xi}^m + \widetilde{\omega}^{yhni}_{mb}\epsilon_n X_{\widetilde{\xi}^b}\widetilde{\xi}^m = 0.$$

In matrix notation we get

$$\begin{pmatrix} 1 \quad \widetilde{\omega}^{y\hbar a i}_{mb}\widetilde{\xi}^m & 0 & 0 \\ \quad \widetilde{\omega}^{y\hbar m i}_{mb}\widetilde{\xi}^m & 1 & 0 \\ \quad \widetilde{\omega}^{y\hbar n i}_{mb}\widetilde{\xi}^m & 0 & 1 \end{pmatrix} \begin{pmatrix} (X_{\widetilde{\xi}^b}) \\ (X_{\widetilde{\xi}^m}) \\ (X_{\widetilde{\xi}^n}) \end{pmatrix} = \begin{pmatrix} (X^{\hbar a i}_{e_m}) \\ (X^{\hbar m i}_{e_m}) \\ (X^{\hbar n i}_{e_m}) \end{pmatrix} \widetilde{\xi}^m. \tag{2.32}$$

We can write equation (2.32) as $(1+Y)x = b$ where Y is a nilpotent matrix of index 2. Hence the equation can be inverted by $x = (1 \quad Y + Y^2)b$ and we get:

$$\begin{pmatrix} (X_{\widetilde{\xi}^a}) \\ (X_{\widetilde{\xi}^m}) \\ (X_{\widetilde{\xi}^n}) \end{pmatrix} = \begin{pmatrix} 1 + \widetilde{\omega}^{y\hbar a i}_{ma}\widetilde{\xi}^m & 0 & 0 \\ +\widetilde{\omega}^{y\hbar m i}_{ma}\widetilde{\xi}^m & 1 & 0 \\ +\widetilde{\omega}^{y\hbar n i}_{ma}\widetilde{\xi}^m & 0 & 1 \end{pmatrix} \begin{pmatrix} (X^{\hbar a i}_{e_m}) \\ (X^{\hbar m i}_{e_m}) \\ (X^{\hbar n i}_{e_m}) \end{pmatrix} \widetilde{\xi}^m. \tag{2.33}$$

Since $(\widetilde{\xi}^m)^2 = 0$ we get

$$(X_{\widetilde{\xi}^\mu}) = (X^{\hbar \mu i}_{e_m})\widetilde{\xi}^m \qquad \mu = a, m, n,$$

and, for the same reason, (2.31i) is satisfied. From (2.31d) we get

$$p^y_n(X_{\widetilde{\omega}}) = p^y_n \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m + p^y_n (X_{\widetilde{e}^y_m})\widetilde{\xi}^m. \tag{2.34}$$

Equation (2.31g) is easily solved as

$$(X_{\widetilde{c}}) = \quad \iota_{\widetilde{\xi}}(X_{\widetilde{\omega}})$$

which in turn solves also (2.31h). Using equation (2.31g) we can solve (2.31e):

$$p^y_m(X_{\widetilde{\omega}}) = p^y_m \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m + p^y_m (X_{\widetilde{e}^y_m})\widetilde{\xi}^m. \tag{2.35}$$

Turning to (2.31c) we get

$$p^y_a(X_{\widetilde{\omega}})(1 + \widetilde{\omega}^{y\hbar a i}_m \widetilde{\xi}^m) = p^y_a(X_{\widetilde{e}^y_m})\widetilde{\xi}^m + p^y_a \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m,$$

hence

$$p^y_a(X_{\widetilde{\omega}}) = p^y_a(X_{\widetilde{e}^y_m})\widetilde{\xi}^m + p^y_a \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m. \tag{2.36}$$

Collecting (2.34), (2.35) and (2.36), we get that $(X_{\widetilde{\omega}})$ is proportional to $\widetilde{\xi}^m$, hence also (2.31f) is solved. This shows that it is possible to perform symplectic reduction.

Collecting all remaining nontrivial equations together we obtain

$$\begin{aligned} (X_{\widetilde{\xi}^\mu}) &= (X^{\hbar \mu i}_{e_m})\widetilde{\xi}^m \qquad \mu = a, m, n \\ (X_{\widetilde{\omega}}) &= (X_{\widetilde{e}^y_m})\widetilde{\xi}^m + \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m \\ (X_{\widetilde{c}}) &= \iota_{\widetilde{\xi}}(X_{\widetilde{e}^y_m})\widetilde{\xi}^m + \iota_{\widetilde{\xi}}\widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m \\ (X_{\widetilde{e}}) &= \quad (X_{\widetilde{\omega}^y_m})\widetilde{\xi}^m + \widetilde{\omega}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m. \end{aligned}$$

The vertical vector fields are

$$\begin{aligned} \mathbb{E}_m =& (X_{e_m})\frac{\delta}{\delta \widetilde{e}_m} + \widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m \frac{\delta}{\delta \widetilde{\omega}} + \iota_{\widetilde{\xi}}\widetilde{e}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m \frac{\delta}{\delta \widetilde{c}} \\ &+ \widetilde{\omega}^y_m (X^{\hbar m i}_{e_m})\widetilde{\xi}^m \frac{\delta}{\delta e} \quad (X_{e_m})^\mu \widetilde{\xi}^m \frac{\delta}{\delta \widetilde{\xi}^\mu} \\ \mathbb{E}^y_m =& (X_{\widetilde{e}^y_m})\frac{\delta}{\delta \widetilde{e}^y_m} + (X_{\widetilde{e}^y_m})\widetilde{\xi}^m \frac{\delta}{\delta \widetilde{\omega}} + \iota_{\widetilde{\xi}}(X_{\widetilde{e}^y_m})\widetilde{\xi}^m \frac{\delta}{\delta \widetilde{c}} \\ \Omega^y_n =& (X_{\widetilde{\omega}^y_m})\frac{\delta}{\delta \widetilde{\omega}^y_m} \quad (X_{\widetilde{\omega}^y_m})\widetilde{\xi}^m \frac{\delta}{\delta \widetilde{e}}. \end{aligned}$$

With an analogous procedure to Proposition 2.10 we flow along these vertical vector fields to obtain the new corner variables. Using Ω^{ψ}_m we have that $\widetilde{\omega}^{\psi}_m(s=1)=0$ ( ) $(X_{\widetilde{\omega}^{\psi}_m}) = -\widetilde{\omega}^{\psi}_m$ which leads to

$$\widetilde{e}[1] = \widetilde{e} - \widetilde{\omega}^{\psi}_m \widetilde{\xi}^m.$$

Analogously, with $\mathbb{E}^{\psi}_m$ we get

$$\widetilde{\omega}[1] = \widetilde{\omega} - \widetilde{e}^{\psi}_m \widetilde{\xi}^m; \quad \widetilde{c}[1] = \widetilde{c} - \iota_{\widetilde{\xi}} \widetilde{e}^{\psi}_m \widetilde{\xi}^m.$$

Now we have to consider

$$\mathbb{E}_m = (X_{e_m})\frac{\delta}{\delta \widetilde{e}_m} - (X_{e_m})^{\mu} \widetilde{\xi}^m \frac{\delta}{\delta \widetilde{\xi}^{\mu}}.$$

As before, this sets the vector $\widetilde{e}_m$ to a constant $\epsilon_m = (1+\varepsilon^0)\widetilde{e}_m$ for $\varepsilon^0 \in C^{\infty}(M^{(1)})$, $\varepsilon^0 > 0$, and transforms $\widetilde{\xi}^m[1] = (1+\varepsilon^0)^{-1}\widetilde{\xi}^m$, while leaving $\widetilde{\xi}^a$ and $\widetilde{\xi}^n$ unchanged. Once again, fixing a linearly independent vector ϵ_m can be done in an open subset $U^{(2)} \subset \Omega^1_{nd}(M^{(2)}, V)$ independently of $\widetilde{e}$. Summarizing, if we denote the space of codimension-2 fields by

$$F^{(2)} := \check{F}^{(2)}/\ker(\check{\varpi}^{(2)\sharp})$$

we then obtain the following projection $\pi^{(2)}_{\text{red}} : \check{F}^{(2)} \to F^{(2)}$:

$$\pi^{(2)}_{\text{red}} : \begin{cases} \widetilde{\widetilde{e}} := \widetilde{e} - \widetilde{\omega}^{\psi}_m \widetilde{\xi}^m \\ \widetilde{\widetilde{\omega}} := \widetilde{\omega} - \widetilde{e}^{\psi}_m \widetilde{\xi}^m \\ \widetilde{\widetilde{c}} := \widetilde{c} - \iota_{\widetilde{\xi}} \widetilde{e}^{\psi}_m \widetilde{\xi}^m \\ \widetilde{\widetilde{\xi}}^a := \widetilde{\xi}^a \\ \widetilde{\widetilde{\xi}^n} := \widetilde{\xi}^n \\ \widetilde{\widetilde{\xi}^m} := (1+\varepsilon^0)^{-1} \widetilde{\xi}^m \end{cases} \tag{2.37}$$

together with $\epsilon_m = (1+\varepsilon^0)\widetilde{e}_m$, and once more we define the BV-BFV map to be $\pi^{(1)} := \pi^{(2)}_{\text{red}} \circ \check{\pi}^{(1)}$.

The one-form $\check{\alpha}^{(2)}$ is not basic w.r.t. $\pi^{(2)}_{\text{red}}$, and it descends to the quotient only upon adding the exact term $\delta(\iota_{\widetilde{\xi}} \widetilde{e} \widetilde{e}^{\psi}_m \widetilde{\xi}^m + \epsilon_n \widetilde{\xi}^n \widetilde{e}^{\psi}_m \widetilde{\xi}^m)$. The codimension-2 one-form is then given by

$$\alpha^{(2)} = \int_{M^{(2)}} \widetilde{\widetilde{c}} \delta \widetilde{\widetilde{e}} + \iota_{\widetilde{\widetilde{\xi}}} \widetilde{\widetilde{e}} \delta \widetilde{\widetilde{\omega}} + \epsilon_m \widetilde{\widetilde{\xi}^m} \delta \widetilde{\widetilde{\omega}} + \epsilon_n \widetilde{\widetilde{\xi}^n} \delta \widetilde{\widetilde{\omega}}.$$

From the defining formula $\iota_{Q^{(1)}} \iota_{Q^{(1)}} \varpi^{(1)} = 2\check{\pi}^{(1)*} \check{S}^{(2)}$ and the expression for $Q^{(1)}$ (2.8), we compute the ***pre-codimension***-2 action functional $\check{S}^{(2)}$.

$$\begin{aligned} \check{S}^{(2)} = \int_{M^{(2)}} & \widetilde{e}_m \widetilde{\xi}^m d_{\widetilde{\omega}} c + \epsilon_n \widetilde{\xi}^n d_{\widetilde{\omega}} c + \iota_{\widetilde{\xi}} \widetilde{e} d_{\widetilde{\omega}} \widetilde{c} - \frac{1}{2}[\widetilde{c}, \widetilde{c}]\widetilde{e} + \iota_{\widetilde{\xi}} \widetilde{\omega}^{\psi}_m \widetilde{\xi}^m d_{\widetilde{\omega}} \widetilde{c} \\ & + \frac{1}{2}[\widetilde{c}, \widetilde{c}] \widetilde{\omega}^{\psi}_m \widetilde{\xi}^m + [\widetilde{c}, \epsilon_n \widetilde{\xi}^n] \widetilde{e}^{\psi}_m \widetilde{\xi}^m - \iota_{\widetilde{\xi}} d_{\widetilde{\omega}}(\epsilon_n \widetilde{\xi}^n) \widetilde{e}^{\psi}_m \widetilde{\xi}^m \\ & - \iota_{\widetilde{\xi}} d \widetilde{\xi}^m \widetilde{e}_m \widetilde{e}^{\psi}_m \widetilde{\xi}^m - \iota_{\widetilde{\xi}} d \widetilde{\xi}^a \widetilde{e}_a \widetilde{e}^{\psi}_m \widetilde{\xi}^m. \end{aligned}$$

It is easy to show that $\check{S}^{(2)}$ is basic, and, defining

$$S^{(2)} = \int_{M^{(2)}} -\frac{1}{2}[\widetilde{\widetilde{c}}, \widetilde{\widetilde{c}}]\widetilde{\widetilde{e}} + \iota_{\widetilde{\widetilde{\xi}}} \widetilde{\widetilde{e}} d_{\widetilde{\widetilde{\omega}}} \widetilde{\widetilde{c}} + \epsilon_m \widetilde{\widetilde{\xi}^m} d_{\widetilde{\widetilde{\omega}}} \widetilde{\widetilde{c}} + \epsilon_n \widetilde{\widetilde{\xi}^n} d_{\widetilde{\widetilde{\omega}}} \widetilde{\widetilde{c}}$$

we obtain that $\pi^{(2)}_{\mathrm{red}}\; S^{(2)} = \check{S}^{(2)}$ and

$$\iota_{Q^{(1)}}\iota_{Q^{(1)}}\varpi^{(1)} = 2\pi^{(1)}\; S^{(2)}.$$

□

Corollary 2.16 (to the proof of Proposition 2.15)**.** *The projection to codimension-2 fields, given a coordinate system adapted to the embedding $M^{(2)} \hookrightarrow M^{(1)}$ (see Remark 2.9):*

$$\pi^{(1)}_{PC} : \begin{cases} \widetilde{\widetilde{e}} := \widetilde{e} - \widetilde{\omega}^{y}_{m}\widetilde{\xi}^{m} \\ \widetilde{\widetilde{\omega}} := \widetilde{\omega} - \widetilde{e}^{y}_{m}\widetilde{\xi}^{m} \\ \widetilde{\widetilde{c}} := \widetilde{c} - \iota_{\widetilde{\xi}}\widetilde{e}^{y}_{m}\widetilde{\xi}^{m} \\ \widetilde{\widetilde{\xi}}^{a} := \widetilde{\xi}^{a} \\ \widetilde{\widetilde{\xi^{n}}} := \widetilde{\xi}^{n} \\ \widetilde{\widetilde{\xi^{m}}} := (1+\varepsilon^{0})^{-1}\widetilde{\xi}^{m} \\ \epsilon_{m} := (1+\varepsilon^{0})\widetilde{e}_{m} \end{cases} ; \tag{2.38}$$

where $\varepsilon^0 \in C^\infty(M^{(2)})$ and we used the notation explained in (2.3) *and* (2.4) *for the restriction of field and their tranversal components, adapted as described in Remark 2.8.*

2.2.4 3-extended PC theory

Finally, we allow M to bear a 3-stratification, i.e. we consider $\{M^{(k)}\}_{k=0\ldots3}$. We iterate once again the BV-BFV procedure from the previously obtained, codimension-2 data.

Proposition 2.17. *The BV theory $\mathsf{F}^{\uparrow 0}_{PC}$ is 3-extendable to $\mathsf{F}^{\uparrow 3}_{PC}$. The codimension-3 data are:*

• *The space of fields*

$$F^{(3)}_{PC} = \Omega^0[1](M^{(3)}, \wedge^2 V) \oplus C^\infty[1](M^{(3)})^{\oplus 3}$$

together with a basis of V denoted by $\{\epsilon_m, \epsilon_n, \epsilon_a\}$;

• *The codimension-3 one-form, symplectic form and action functional*

$$\alpha^{(3)}_{PC} = \int_{M^{(3)}} -\epsilon_n\widetilde{\widetilde{\widetilde{\xi^n}}}\delta\widetilde{\widetilde{c}} - \epsilon_m\widetilde{\widetilde{\widetilde{\xi^m}}}\delta\widetilde{\widetilde{c}} - \epsilon_a\widetilde{\widetilde{\xi}}^a\delta\widetilde{\widetilde{c}}, \tag{2.39}$$

$$\varpi^{(3)}_{PC} = \int_{M^{(3)}} -\epsilon_n\delta\widetilde{\widetilde{\widetilde{\xi^n}}}\delta\widetilde{\widetilde{c}} - \epsilon_m\delta\widetilde{\widetilde{\widetilde{\xi^m}}}\delta\widetilde{\widetilde{c}} - \epsilon_a\delta\widetilde{\widetilde{\xi}}^a\delta\widetilde{\widetilde{c}}, \tag{2.40}$$

$$S^{(3)}_{PC} = \int_{M^{(3)}} \frac{1}{2}[\widetilde{\widetilde{c}},\widetilde{\widetilde{c}}]\epsilon_a\widetilde{\widetilde{\xi}}^a + \frac{1}{2}[\widetilde{\widetilde{c}},\widetilde{\widetilde{c}}]\epsilon_m\widetilde{\widetilde{\widetilde{\xi^m}}} + \frac{1}{2}[\widetilde{\widetilde{c}},\widetilde{\widetilde{c}}]\epsilon_n\widetilde{\widetilde{\widetilde{\xi^n}}}; \tag{2.41}$$

• *The cohomological vector field $Q^{(3)}_{PC}$*

$$Q^{(3)}_{PC}\,\widetilde{\widetilde{c}} = \frac{1}{2}[\widetilde{\widetilde{c}},\widetilde{\widetilde{c}}] \tag{2.42a}$$

$$Q^{(3)}_{PC}\,\widetilde{\widetilde{\widetilde{\xi^\mu}}} = [\widetilde{\widetilde{c}},\epsilon_a\widetilde{\widetilde{\xi}}^a]^{\langle\mu\rangle} + [\widetilde{\widetilde{c}},\epsilon_m\widetilde{\widetilde{\widetilde{\xi^m}}}]^{\langle\mu\rangle} + [\widetilde{\widetilde{c}},\epsilon_n\widetilde{\widetilde{\widetilde{\xi^n}}}]^{\langle\mu\rangle}; \tag{2.42b}$$

The projection to codimension-3 elds $\pi_{PC}^{(2)} = \pi_{\mathrm{red}}^{(3)} \quad \check{\pi}^{(2)}$ is smooth, where $\check{\pi}^{(2)} : F^{(2)} \; ! \; \check{F}^{(3)}$ is the restriction of the codimension-2 elds to the 3-stratum $M^{(3)}$ and $\pi_{\mathrm{red}}^{(3)} : \check{F}^{(3)} \; ! \; F^{(3)}$ is the symplectic reduction (1.6).

Remark 2.18. Since $M^{(3)}$ is a set of points, integration is here intended as algebraic sum over such points with induced orientation, with the space of fields on a single point being given by

$$\wedge^2 V[1] \quad \mathbb{R}^3[1].$$

Proof. In order to keep the notation light and readable we drop the tildes. Once again, from the defining equation $\iota_{Q^{(2)}}\varpi^{(2)} = \delta S^{(2)} + \check{\pi}^{(2)}\check{\alpha}^{(3)}$ we get

$$\check{\alpha}^{(3)} = \int_{M^{(3)}} \epsilon_n \xi^n \delta c \quad \epsilon_m \xi^m \delta c \quad e_a \xi^a \delta c$$

and

$$\check{\varpi}^{(3)} = \delta\check{\alpha}^{(3)} = \int_{M^{(3)}} \epsilon_n \delta\xi^n \delta c + \epsilon_m \delta\xi^m \delta c \quad \delta e_a \xi^a \delta c + e_a \delta\xi^a \delta c,$$

with $\check{\alpha}^{(3)} \; 2 \; \Omega^1(\check{F}^{(3)})$, the space of restrictions of fields in $F^{(2)}$ (and their normal jets) to the stratum $M^{(3)}$, with $\check{\pi}^{(2)} \colon F^{(2)} \quad ! \; \check{F}^{(3)}$. There is only one nontrivial equation defining the kernel of $(\check{\varpi}^{(3)})^\sharp$:

$$\epsilon_n X_{\xi^n} + \epsilon_m X_{\xi^m} + e_a X_{\xi^a} \quad X_{e_a} \xi^a = 0.$$

This is solved as in the previous cases by

$$X_{\xi}^{\mu} = (X_{e_a})^{\langle\mu\rangle} \xi^a$$

and, since this defines a subbundle of $T\check{F}^{(3)}$, it is possible to perform the symplectic reduction. The kernel of $\check{\varpi}^{(3)}$ is spanned by

$$\mathbb{E}_a = (X_{e_a})\frac{\delta}{\delta e_a} + (X_{e_a})^{\langle\mu\rangle}\xi^a \frac{\delta}{\delta\tilde{\xi}^\mu},$$

and ance again, flowing along $\mathbb{E}_a$, we are able to fix the vector e_a to $\epsilon_a = (1 + \varepsilon^{00})e_a$ for some $\varepsilon^{00} > 0$ (a function on a finite set of points) and consequently, $\xi^a[1] = (1 + \varepsilon^{00})\xi^a$, while the rest is left unchanged. This defines the symplectic reduction $\pi_{\mathrm{red}}^{(3)} \colon \check{F}^{(3)} \quad ! \; F^{(3)}$, and the BV-BFV map $\pi^{(2)} := \pi_{\mathrm{red}}^{(3)} \quad \check{\pi}^{(2)}$. Then, the expression

$$\alpha^{(3)} = \int_{M^{(3)}} \epsilon_n \xi^n \delta c \quad \epsilon_m \xi^m \delta c \quad \epsilon_a \xi^a \delta c$$

is such that $\pi_{\mathrm{red}}^{(3)} \; \alpha^{(3)} = \check{\alpha}^{(3)}$. Lastly, we compute the vertex action. With a calculation completely analogous to what was done in Propositions 2.10 and 2.15 we compute $\check{S}^{(3)}$ such that $\iota_{Q^{(2)}}\iota_{Q^{(2)}}\varpi^{(2)} = 2\check{\pi}^{(2)} \; \check{S}^{(3)}$:

$$\check{S}^{(3)} = \int_{M^{(3)}} \frac{1}{2}[c,c]e_a\xi^a + \frac{1}{2}[c,c]\epsilon_m\xi^m + \frac{1}{2}[c,c]\epsilon_n\xi^n$$

Then, we get that the vertex action

$$S^{(3)} = \int_{M^{(3)}} \frac{1}{2}[c,c]\epsilon_a\xi^a + \frac{1}{2}[c,c]\epsilon_m\xi^m + \frac{1}{2}[c,c]\epsilon_n\xi^n$$

satisfies $\check{S}^{(3)} = \pi_{\mathsf{red}}^{(3)*} S^{(3)}$, and consequently

$$\iota_{Q^{(2)}}\iota_{Q^{(2)}}\varpi^{(2)} = 2\pi^{(2)*} S^{(3)}.$$

□

Corollary 2.19 (to the proof of Proposition 2.17)**.** *The projection to codimension-3 fields, given a coordinate system adapted to the embedding $M^{(3)} \to M^{(2)}$ (see Remark 2.9):*

$$\pi_{PC}^{(2)} : \begin{cases} \tilde{\tilde{\tilde{c}}} := \tilde{\tilde{c}} \\ \tilde{\tilde{\tilde{\xi}}}^a := (1+\varepsilon^{00})^{-1}\tilde{\tilde{\xi}}^a \\ \widetilde{\widetilde{\widetilde{\xi^n}}} := \widetilde{\widetilde{\xi^n}} \\ \widetilde{\widetilde{\widetilde{\xi^m}}} := \widetilde{\widetilde{\xi^m}} \\ \epsilon_a := (1+\varepsilon^{00})\tilde{\tilde{e}}_a \end{cases} \tag{2.43}$$

where $\varepsilon^{00} \in \Gamma(M^{(3)})$ and we used the notation explained in (2.3) and (2.4) for the restriction of fields and their normal components adapted as described in Remark 2.8.

2.3 BV-BFV equivalence

The goal of this section is to prove Theorem 2.5, based on Definition 1.42. We recall it here:

Theorem 2.20. *The fully extended BV theories $\mathsf{F}_{PC}^{\prime\prime 3}$ and $\mathsf{F}_{BF}^{\prime\prime 3}$ are strongly equivalent.*

Remark 2.21. Explicitly, we have to prove the existence of invertible symplectomorphisms $\psi^{(k)}$ that make the following diagram commute.

$$\begin{array}{ccccccc} F_{PC}^{(0)} & \xrightarrow{\pi_{PC}^{(0)}} & F_{PC}^{(1)} & \xrightarrow{\pi_{PC}^{(1)}} & F_{PC}^{(2)} & \xrightarrow{\pi_{PC}^{(2)}} & F_{PC}^{(3)} \\ \downarrow{\scriptstyle \psi^{(0)}} & & \downarrow{\scriptstyle \psi^{(1)}} & & \downarrow{\scriptstyle \psi^{(2)}} & & \downarrow{\scriptstyle \psi^{(3)}} \\ F_{BF}^{(0)} & \xrightarrow{\pi_{BF}^{(0)}} & F_{BF}^{(1)} & \xrightarrow{\pi_{BF}^{(1)}} & F_{BF}^{(2)} & \xrightarrow{\pi_{BF}^{(2)}} & F_{BF}^{(3)} \end{array} \tag{2.44}$$

Note that the vertical symplectomorphisms preserve the action functionals, i.e. they satisfy $(\psi^{(k)})^* S_{BF}^{(k)} = S_{PC}^{(k)}$, and that the horizontal arrows on both lines have been already described in Section 2.1.1 and 2.2 respectively. The symplectomorphisms $\psi^{(k)}$ are non-canonical, as they depend on the the choice of a basis in V.

2.3.1 Equivalence on the bulk

A strong equivalence (see Definition 1.34) between the BV data associated to BF theory and PC theory was proven in [CSS18, Theorem 10], provided that on B is imposed a non-degeneracy condition. We denote by BF non-degenerate BF theory (see Definition 2.1). An explicit generating function for the canonical transformation between the two (-1)-symplectic spaces of fields has been given as well. We recall here the most important steps of the construction.

Using the notation introduced in Subsections 2.1.1 and 1.2.5 the generating function[8] reads

$$H = -B^{\dagger}\left(e - \iota_\xi \omega^{\dagger} + \frac{1}{2}\iota_\xi^2 c^{\dagger}\right) - \tau^{\dagger}\left(-\iota_\xi e + \frac{1}{2}\iota_\xi^2\omega^{\dagger} - \frac{1}{3}\iota_\xi^3 c^{\dagger}\right) - A\omega^{\dagger} + \chi c^{\dagger}. \tag{2.45}$$

Starting from this generating function, we recover an explicit expression of the transformation $\psi^{(0)}: F^{(0)}_{PC} \to F^{(0)}_{BF}$. It can be found using the standard rules

$$p = -(-1)^{|q|}\frac{\delta H}{\delta q}; \qquad Q = (-1)^{|P|}\frac{\delta H}{\delta P} \tag{2.46}$$

where $P = (\tau^{\dagger}, B^{\dagger}, A, \chi)$, $Q = (\tau, B, A^{\dagger}, \chi^{\dagger})$, $p = (\xi^{\dagger}, e^{\dagger}, \omega, c)$ and $q = (\xi, e, \omega^{\dagger}, c^{\dagger})$.

Lemma 2.22. *The symplectomorphism $\psi^{(0)}: F^{(0)}_{PC} \to F^{(0)}_{BF}$ is given by*

$$\psi^{(0)}: \begin{cases} B = e - \iota_\xi\omega^{\dagger} + \frac{1}{2}\iota_\xi^2 c^{\dagger} \\ B^{\dagger} = e^{\dagger} - \iota_\xi\tau^{\dagger} \\ A = \omega - \iota_\xi e^{\dagger} + \frac{1}{2}\iota_\xi^2\tau^{\dagger} \\ A^{\dagger} = \omega^{\dagger} \\ \chi = -c + \frac{1}{2}\iota_\xi^2 e^{\dagger} - \frac{1}{6}\iota_\xi^3\tau^{\dagger} \\ \chi^{\dagger} = -c^{\dagger} \\ \tau = -\iota_\xi e + \frac{1}{2}\iota_\xi^2\omega^{\dagger} - \frac{1}{3}\iota_\xi^3 c^{\dagger} \\ p_a^{\dagger}\tau^{\dagger} = \xi_a^{\dagger\langle a\rangle} - [e^{\dagger}\omega_a^{\dagger}]^{\langle a\rangle} + [e^{\dagger}\iota_\xi c_a^{\dagger}]^{\langle a\rangle} \end{cases}, \tag{2.47}$$

whereas its inverse is given by

$$(\psi^{(0)})^{-1}: \begin{cases} e = B + \iota_\xi A^{\dagger} + \frac{1}{2}\iota_\xi^2\chi^{\dagger} \\ \xi^a = -\tau^{\langle a\rangle} - \tau^{\langle b\rangle}\tau^{\langle c\rangle}A^{\dagger(a)}_{bc} + \tau^{\langle b\rangle}\tau^{\langle c\rangle}\tau^{\langle d\rangle}A^{\dagger\langle e\rangle}_{cd}A^{\dagger\langle a\rangle}_{eb} + \frac{1}{2}\tau^{\langle b\rangle}\tau^{\langle c\rangle}\tau^{\langle d\rangle}\chi^{\dagger\langle a\rangle}_{bcd} \\ \omega^{\dagger} = A^{\dagger} \\ c^{\dagger} = -\chi^{\dagger} \\ \omega = A + \iota_\xi B^{\dagger} + \frac{1}{2}\iota_\xi^2\tau^{\dagger} \\ c = -\chi + \frac{1}{2}\iota_\xi^2 B^{\dagger} + \frac{1}{3}\iota_\xi^3\tau^{\dagger} \\ \xi_a^{\dagger} = B^{\dagger}(A_a^{\dagger} + \iota_\xi\chi_a^{\dagger}) + \tau^{\dagger}(B_a - \frac{1}{2}\iota_\xi^2\chi_a^{\dagger}) \\ e^{\dagger} = B^{\dagger} + \iota_\xi\tau^{\dagger} \end{cases}, \tag{2.48}$$

where the indices $\langle a\rangle$ denote components with respect to the basis $\{B_a\}$ as defined in (2.2).

Proof. We first apply (2.46) to the generating function (2.45) and get

$$B = e - \iota_\xi\omega^{\dagger} + \frac{1}{2}\iota_\xi^2 c^{\dagger} \tag{2.49a}$$

$$A^{\dagger} = \omega^{\dagger} \tag{2.49b}$$

$$\tau = -\iota_\xi e + \frac{1}{2}\iota_\xi^2\omega^{\dagger} - \frac{1}{3}\iota_\xi^3 c^{\dagger} \tag{2.49c}$$

[8] Some signs differ from the formula given in [CSS18, Theorem 10] because we are using a different convention for the signs in (1.20).

$$\chi^{y} = \quad c^{y} \tag{2.49d}$$
$$\xi^{y}_{a} = B^{y}\omega^{y}_{a} \quad B^{y}\iota_{\xi}c^{y}_{a} + \tau^{y}e_{a} \quad \tau^{y}\iota_{\xi}\omega^{y}_{a} + \tau^{y}\iota^{2}_{\xi}c^{y}_{a} \tag{2.49e}$$
$$e^{y} = B^{y} + \iota_{\xi}\tau^{y} \tag{2.49f}$$
$$\omega = \iota_{\xi}B^{y} + \frac{1}{2}\iota^{2}_{\xi}\tau^{y} + A \tag{2.49g}$$
$$c = \frac{1}{2}\iota^{2}_{\xi}B^{y} + \frac{1}{3}\iota^{3}_{\xi}\tau^{y} \quad \chi. \tag{2.49h}$$

Equation (2.49f) yields $B^{y} = e^{y} \quad \iota_{\xi}\tau^{y}$ that in turn, inserted into (2.49e), gives

$$\tau^{y}e_{a} = \xi^{y}_{a} \quad e^{y}\omega^{y}_{a} + e^{y}\iota_{\xi}c^{y}_{a}.$$

Since this is an equation between objects valued in $\wedge^{3}V$, we can extract an e_a factor and get

$$p^{y}_{a}\tau^{y} = \xi^{y\,h a i}_{a} \quad [e^{y}\omega^{y}_{a}]^{h a i} + [e^{y}\iota_{\xi}c^{y}_{a}]^{h a i}.$$

Having τ^{y}, we can now invert all other equations. An easy but lengthy computation shows that this symplectomorphism correctly satisfies $\varpi^{(0)}_{PC} = \psi^{(0)} \ \varpi^{(0)}_{BF}$ and it preserves the action functionals, as was shown in [CSS18]. □

Remark 2.23. In order to build the inverse symplectomorphism $(\psi^{(0)}) \ ^{1}$ we must require that the image of B be a basis of V at every point. Thus the two theories are strongly equivalent only if also B satisfies the non-degeneracy condition in Definition 2.1. An extension of this equivalence to the degenerate case is presented in [Ćir+20], within the context of L_{1} algebras.

2.3.2 Equivalence on boundaries, corners, vertices

Since the form $\varpi^{(1)}_{PC}$ of Equation (2.7b) is not in Darboux form, it is not possible to find a generating function. We can nonetheless produce an explicit symplectomorphism and its inverse:

$$\psi^{(1)}: \begin{cases} B = \widetilde{e} \quad \iota_{\widetilde{\xi}}\widetilde{\omega}^{y} \\ B^{y} = \widetilde{e}^{y} \\ A = \omega \quad \iota_{\widetilde{\xi}}\widetilde{e}^{y} \\ A^{y} = \widetilde{\omega}^{y} \\ \chi = \quad \widetilde{c} + \frac{1}{2}\iota^{2}_{\widetilde{\xi}}\widetilde{e}^{y} \\ \tau = \quad \iota_{\widetilde{\xi}}\widetilde{e} \quad \epsilon_{n}\widetilde{\xi}^{n} + \frac{1}{2}\iota^{2}_{\widetilde{\xi}}\widetilde{\omega}^{y} \end{cases} \qquad (\psi^{(1)}) \ ^{1}: \begin{cases} \widetilde{e} = B + \iota_{\widetilde{\xi}}A^{y} \\ \widetilde{\xi}^{a} = \quad \tau^{h a i} \quad \tau^{h a i}\tau^{h b i}A^{y h a i}_{ab} \\ \widetilde{\xi}^{n} = \quad \tau^{h n i} \quad \tau^{h a i}\tau^{h b i}A^{y h n i}_{ab} \\ \widetilde{\omega} = A + \iota_{\widetilde{\xi}}B^{y} \\ \widetilde{c} = \quad \chi + \frac{1}{2}\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}B^{y} \\ \widetilde{\omega}^{y} = A^{y} \\ \widetilde{e}^{y} = B^{y} \end{cases} \tag{2.50}$$

where superscripts $h n i$ and $h a i$ denote the components with respect to $f\epsilon_{n}, B_{a}g$. It is straightforward to check that $\psi^{(1)} \ (\psi^{(1)}) \ ^{1} = id$, $(\psi^{(1)}) \ ^{1} \ \psi^{(1)} = id$ and $\varpi^{(1)}_{PC} = \psi^{(1)} \ \varpi^{(1)}_{BF}$. Analogously, on the corner (the codimension-2 stratum) we have the explicit transformation

$$\psi^{(2)}: \begin{cases} B = \widetilde{\widetilde{e}} \\ A = \widetilde{\widetilde{\omega}} \\ \chi = \quad \widetilde{\widetilde{c}} \\ \tau = \quad \iota_{\widetilde{\widetilde{\xi}}}\widetilde{\widetilde{e}} \quad \epsilon_{m}\widetilde{\widetilde{\xi}}^{m} \quad \epsilon_{n}\widetilde{\widetilde{\xi}}^{n}. \end{cases} \qquad (\psi^{(2)}) \ ^{1}: \begin{cases} \widetilde{\widetilde{e}} = B \\ \widetilde{\widetilde{\omega}} = A \\ \widetilde{\widetilde{c}} = \quad \chi \\ \widetilde{\widetilde{\xi}}^{m} = \tau^{h m i} \\ \widetilde{\widetilde{\xi}}^{n} = \tau^{h n i} \\ \widetilde{\widetilde{\xi}}^{a} = \tau^{h a i}, \end{cases} \tag{2.51}$$

while, on the vertex, we have

$$\psi^{(3)} : \begin{cases} \chi = \quad \widetilde{\widetilde{\widetilde{c}}} \\ \tau = \quad \epsilon_a \widetilde{\widetilde{\widetilde{\xi}}}^a \quad \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^m \quad \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^n \end{cases} \tag{2.52}$$

with inverses given by $\widetilde{\widetilde{\widetilde{\xi}}}^m = \tau^{hmi}$, $\widetilde{\widetilde{\widetilde{\xi}}}^n = \tau^{hni}$ and $\widetilde{\widetilde{\widetilde{\xi}}}^a = \tau^{hai}$, i.e. the components of τ with respect to ϵ_m, ϵ_n and ϵ_a respectively. Finally, it is straightforward to check that $(\psi^{(k)})\ S^{(k)}_{BF}\ = S^{(k)}_{PC}$ for $k = 1, 2, 3$.

2.3.3 Commutativity

In this section we prove the commutativity of the three square subdiagrams of the diagram (2.44). This is sufficent to prove commutativity as a whole. For the sake of clarity, we denote the BF variables on the 1-stratum (and subsequent 2- and 3-strata) with a tilde, analogously to the PC notation. To avoid confusion, we explicitly denote the restriction to the 1-stratum (resp. 2- and 3-stratum) with an apex ', e.g. $e^{\prime} \quad ej_{M^{(1)}}$ and $\widetilde{e}^{\prime} \quad \widetilde{e}j_{M^{(2)}}$.

Proposition 2.24. *Diagram* (2.44) *is commutative.*

Proof. The first square is

$$\begin{array}{ccc} F^{(0)}_{PC} & \xrightarrow{\pi^{(0)}_{PC}} & F^{(1)}_{PC} \\ \downarrow{\scriptstyle \psi^{(0)}} & & \downarrow{\scriptstyle \psi^{(1)}} \\ F^{(0)}_{BF} & \xrightarrow{\pi^{(0)}_{BF}} & F^{(1)}_{BF} \end{array}$$

The left-bottom composition $\pi^{(0)}_{BF} \quad \psi^{(0)}$ reads

$$\begin{aligned} \widetilde{B} &= (e \quad \iota_\xi \omega^y + \frac{1}{2}\iota_\xi^2 c^y)^{\prime} = e^{\prime} \quad \iota_{\xi^{\prime}} \omega^{y^{\prime}} \quad \omega_n^{y^{\prime}} \xi^{n^{\prime}} + \iota_{\xi^{\prime}} c_n^{y^{\prime}} \xi^{n^{\prime}} \\ \widetilde{B}^y &= (e^y \quad \iota_\xi \tau^y)^{\prime} = e^{y^{\prime}} \quad \tau_n^{y^{\prime}} \xi^{n^{\prime}} \\ \widetilde{A} &= (\omega \quad \iota_\xi e^y + \frac{1}{2}\iota_\xi^2 \tau^y)^{\prime} = \omega^{\prime} \quad \iota_{\xi^{\prime}} e^{y^{\prime}} \quad e_n^{y^{\prime}} \xi^{n^{\prime}} + \iota_{\xi^{\prime}} \tau_n^{y^{\prime}} \xi^{n^{\prime}} = \omega^{\prime} \quad \iota_{\xi^{\prime}} \widetilde{B}^y \quad e_n^{y^{\prime}} \xi^{n^{\prime}} \\ \widetilde{A}^y &= \omega^{y^{\prime}} \end{aligned}$$

$$\begin{aligned} \widetilde{\chi} &= (\quad c + \frac{1}{2}\iota_\xi^2 e^y \quad \frac{1}{6}\iota_\xi^3 \tau^y)^{\prime} = \quad c^{\prime} + \frac{1}{2}\iota_{\xi^{\prime}}^2 e^{y^{\prime}} + \iota_{\xi^{\prime}} e_n^{y^{\prime}} \xi^{n^{\prime}} \quad \frac{1}{2}\iota_{\xi^{\prime}}^2 \tau_n^{y^{\prime}} \xi^{n^{\prime}} \\ &= \quad c^{\prime} + \frac{1}{2}\iota_{\xi^{\prime}}^2 \widetilde{B}^y + \iota_{\xi^{\prime}} e_n^{y^{\prime}} \xi^{n^{\prime}} \\ \widetilde{\tau} &= (\quad \iota_\xi e + \frac{1}{2}\iota_\xi^2 \omega^y \quad \frac{1}{3}\iota_\xi^3 c^y)^{\prime} = \quad \iota_{\xi^{\prime}} e^{\prime} \quad e_n^{\prime} \xi^{n^{\prime}} + \frac{1}{2}\iota_{\xi^{\prime}}^2 \omega^{y^{\prime}} + \iota_{\xi^{\prime}} \omega_n^{y^{\prime}} \xi^{n^{\prime}} \quad \frac{1}{2}\iota_{\xi^{\prime}}^2 c_n^{y^{\prime}} \xi^{n^{\prime}} \end{aligned}$$

where

$$\widetilde{\tau}_n^y = (\chi_a V_n^{hai} \quad [e_n^y \omega_a^y \quad e^y \omega_{an}^y]^{hai} + [e^y \iota_\xi c_{an}^y]^{hai} + \chi_n V_n^{hni} \quad [e_n^y \omega_n^y]^{hni} + [e_n^y \iota_\xi c_n^y]^{hni})^{\prime}.$$

The top-right composition $\psi^{(1)} \quad \pi_{PC}^{(0)}$ reads:

$$
\begin{aligned}
\widetilde{B} &= \widetilde{e} \quad \iota_{\widetilde{\xi}}\widetilde{\omega}^{y} = e^{0} \quad \omega_{n}^{y^{0}}\xi^{n^{0}} + \iota_{\xi^{0}}c_{n}^{y^{0}}\xi^{n^{0}} \quad \iota_{\xi^{0}}\omega^{y^{0}} \\
\widetilde{B}^{y} &= \widetilde{e}^{y} = e^{y} \quad \chi_{a}\xi^{n}\mathsf{V}^{hai} + \left(\omega_{na}^{y}\xi^{n}e^{y}\right)^{hai} \quad (e_{n}^{y}\xi^{n}\omega_{a}^{y})^{hai} \quad \left(\iota_{\xi}c_{na}^{y}\xi^{n}e^{y}\right)^{hai} \\
&\quad \chi_{n}\xi^{n}\mathsf{V}^{hni} + \left(\omega_{n}^{y}\xi^{n}e_{n}^{y}\right)^{hni} \quad (e_{n}^{y}\iota_{\xi}c_{n}^{y}\xi^{n})^{hni} \\
\widetilde{A} &= \widetilde{\omega} \quad \iota_{\widetilde{\xi}}\widetilde{e}^{y} = \omega \quad e_{n}^{y}\xi^{n} \quad \iota_{\xi}\widetilde{e}^{y} \\
\widetilde{A}^{y} &= \widetilde{\omega}^{y} = \omega^{y^{0}} \\
\widetilde{\chi} &= \quad \widetilde{c} + \frac{1}{2}\iota_{\widetilde{\xi}}^{2}\widetilde{e}^{y} = c \quad \iota_{\xi}e_{n}^{y}\xi^{n} + \frac{1}{2}\iota_{\xi}^{2}\widetilde{e}^{y} \\
\widetilde{\tau} &= \quad \iota_{\widetilde{\xi}}\widetilde{e} \quad \epsilon_{n}\widetilde{\xi}^{n} + \frac{1}{2}\iota_{\widetilde{\xi}}^{2}\widetilde{\omega}^{y} = \quad \iota_{\xi^{0}}e^{0} + \iota_{\xi^{0}}\omega_{n}^{y^{0}}\xi^{n^{0}} \quad \iota_{\xi^{0}}^{2}c_{n}^{y^{0}}\xi^{n^{0}} \quad e_{n}^{0}\xi^{n^{0}} + \frac{1}{2}\iota_{\xi^{0}}^{2}\omega^{y^{0}},
\end{aligned}
$$

where we used that $e_{n}^{0}\xi^{n^{0}} = \epsilon_{n}\widetilde{\xi}^{n}$. The rows of $\widetilde{B}$, $\widetilde{A}^{y}$ and $\widetilde{\tau}$ coincide in both cases. The expressions of $\widetilde{B}^{y}$ coincide as well, hence also the rows of $\widetilde{A}$ and $\widetilde{\chi}$ give the same result. The second square is:

$$
\begin{array}{ccc}
F_{PC}^{(1)} & \xrightarrow{\pi_{PC}^{(1)}} & F_{PC}^{(2)} \\
\downarrow{\scriptstyle \psi^{(1)}} & & \downarrow{\scriptstyle \psi^{(2)}} \\
F_{BF}^{(1)} & \xrightarrow{\pi_{BF}^{(1)}} & F_{BF}^{(2)}
\end{array}
$$

The left-bottom composition $\pi_{BF}^{(1)} \quad \psi^{(1)}$ is

$$
\begin{aligned}
\widetilde{\widetilde{B}} &= (\widetilde{e} \quad \iota_{\widetilde{\xi}}\widetilde{\omega}^{y})^{0} = \widetilde{e}^{0} \quad \widetilde{\omega}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{A}} &= (\widetilde{\omega} \quad \iota_{\widetilde{\xi}}\widetilde{e}^{y})^{0} = \widetilde{\omega}^{0} \quad \widetilde{e}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{\chi}} &= (\quad \widetilde{c} + \frac{1}{2}\iota_{\widetilde{\xi}}^{2}\widetilde{e}^{y})^{0} = \quad \widetilde{c}^{0} + \iota_{\widetilde{\xi}^{0}}\widetilde{e}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{\tau}} &= (\quad \iota_{\widetilde{\xi}}\widetilde{e} \quad \epsilon_{n}\widetilde{\xi}^{n} + \frac{1}{2}\iota_{\widetilde{\xi}}^{2}\widetilde{\omega}^{y})^{0} = \quad \iota_{\widetilde{\xi}^{0}}\widetilde{e}^{0} \quad \epsilon_{n}\widetilde{\xi}^{n^{0}} \quad \widetilde{e}_{m}^{0}\widetilde{\xi}^{m^{0}} + \iota_{\widetilde{\xi}^{0}}\widetilde{\omega}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}}
\end{aligned}
$$

while for the top-right composition $\psi^{(2)} \quad \pi_{PC}^{(1)}$ we have

$$
\begin{aligned}
\widetilde{\widetilde{B}} &= \widetilde{\widetilde{e}} = \widetilde{e}^{0} \quad \widetilde{\omega}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{A}} &= \widetilde{\widetilde{\omega}} = \widetilde{\omega}^{0} \quad \widetilde{e}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{\chi}} &= \quad \widetilde{\widetilde{c}} = \quad \widetilde{c}^{0} + \iota_{\widetilde{\xi}^{0}}\widetilde{e}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}} \\
\widetilde{\widetilde{\tau}} &= \quad \iota_{\widetilde{\widetilde{\xi}}}\widetilde{\widetilde{e}} \quad \epsilon_{m}\widetilde{\widetilde{\xi^{m}}} \quad \epsilon_{n}\widetilde{\widetilde{\xi^{n}}} = \quad \iota_{\widetilde{\xi}^{0}}(\widetilde{e}^{0} \quad \widetilde{\omega}_{m}^{y^{0}}\widetilde{\xi}^{m^{0}}) \quad \widetilde{e}_{m}^{0}\widetilde{\xi}^{m^{0}} \quad \epsilon_{n}\widetilde{\xi}^{n^{0}},
\end{aligned}
$$

(again we used $\widetilde{e}_{m}^{0}\widetilde{\xi}^{m^{0}} = \epsilon_{m}\widetilde{\widetilde{\xi^{m}}}$) and the expressions are identical. The last square subdiagram is

$$
\begin{array}{ccc}
F^{(2)}_{PC} & \xrightarrow{\pi^{(2)}_{PC}} & F^{(3)}_{PC} \\
\downarrow \psi^{(2)} & & \downarrow \psi^{(3)} \\
F^{(2)}_{BF} & \xrightarrow{\pi^{(2)}_{BF}} & F^{(3)}_{BF}
\end{array}
$$

The two compositions are $\pi^{(2)}_{BF} \quad \psi^{(2)}$:

$$
\begin{aligned}
\widetilde{\widetilde{\widetilde{\chi}}} &= \quad \widetilde{\widetilde{c}}^{0} \\
\widetilde{\widetilde{\widetilde{\tau}}} &= (\quad \iota_{\widetilde{\widetilde{\xi}}}\widetilde{\widetilde{e}} \quad \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^m \quad \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^n)^0 = \quad \widetilde{\widetilde{e}}^{0}_a \widetilde{\widetilde{\xi}}^{a^0} \quad \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^{m^0} \quad \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^{n^0}
\end{aligned}
$$

and $\psi^{(3)} \quad \pi^{(2)}_{PC}$:

$$
\begin{aligned}
\widetilde{\widetilde{\widetilde{\chi}}} &= \quad \widetilde{\widetilde{\widetilde{c}}} = \quad \widetilde{\widetilde{c}}^{0} \\
\widetilde{\widetilde{\widetilde{\tau}}} &= \quad \epsilon_a \widetilde{\widetilde{\widetilde{\xi}}}^a \quad \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^m \quad \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^n = \quad \widetilde{\widetilde{e}}^{0}_a \widetilde{\widetilde{\xi}}^{a^0} \quad \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^{m^0} \quad \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^{n^0}
\end{aligned}
$$

using once again $\widetilde{\widetilde{e}}^{0}_a \widetilde{\xi}^{a^0} = \epsilon_a \widetilde{\widetilde{\xi}}^a$. □

2.4 Cosmological constant

In this section we consider BF theory and PC theory with the addition of what is generally known as the cosmological term: a cubic term in B (respectively e). Classically this amounts to considering the functionals

$$
S^{cl}{}_{BF} = \int_M B \wedge F_A + \frac{1}{6}\Lambda B \wedge B \wedge B, \qquad S^{cl}{}_{PC} = \int_M e \wedge F_\omega + \frac{1}{6}\Lambda e \wedge e \wedge e
$$

where $\Lambda \; 2 \; \mathbb{R}$ is a constant. The corresponding BV expression are ([CSS18])

$$
S \;\; {}_{BF} = S_{BF} + \frac{1}{6}\int_M \Lambda \mathsf{B} \wedge \mathsf{B} \wedge \mathsf{B}, \qquad S_{PC} = S \;\; {}_{PC} + \frac{1}{6}\int_M \Lambda e \wedge e \wedge e
$$

BF theory with this additional term is still fully extendable and self-similar, so, using the notation of Theorem 2.3 we get

$$
S^{(k)}{}_{BF} = S_{BF} + \frac{1}{6}\int_{M^{(k)}} \Lambda \mathsf{B} \wedge \mathsf{B} \wedge \mathsf{B}.
$$

Since the additional cosmological term does not contain any derivative, also PC theory is fully extendable and the reductions are not modified. The actions in higher codimensions are

$$
\begin{aligned}
S^{(1)}{}_{PC} &= S^{(1)}_{PC} \quad \frac{1}{2}\int_{M^{(1)}} \Lambda \epsilon_n \widetilde{\xi}^n \widetilde{e}\widetilde{e} \\
S^{(2)}{}_{PC} &= S^{(2)}_{PC} + \int_{M^{(2)}} \Lambda \epsilon_n \widetilde{\widetilde{\xi}}^n \epsilon_m \widetilde{\widetilde{\xi}}^m \widetilde{\widetilde{e}} \\
S^{(3)}{}_{PC} &= S^{(3)}_{PC} \quad \int_{M^{(3)}} \Lambda \epsilon_n \widetilde{\widetilde{\widetilde{\xi}}}^n \epsilon_m \widetilde{\widetilde{\widetilde{\xi}}}^m \epsilon_a \widetilde{\widetilde{\widetilde{\xi}}}^a.
\end{aligned}
$$

The two fully extended theories are still strongly equivalent and the map realizing the equivalence (namely the ones appearing in the diagram (2.44)) remain unchanged. We have just to check that the actions are still preserved by the corresponding symplectomorphisms. In the bulk this has been proved in [CSS18, section 2.3]. The same argument can be adapted to higher codimension actions. The equations $(\psi^{(k)})\ S^{(k)}_{BF} = S^{(k)}_{PC}$ for $k = 0, \ldots 3$ can also be verified by direct computation.

Chapter 3

Technical Results for the Palatini–Cartan boundary structure

The description of the reduced phase space of Palatini–Cartan gravity that we will propose in Chapters 4 and 5 respectively for a time-like or space-like and a light-like boundary is based on a number of lemmas concerning the properties of some maps that can be built out of the coframe e. Since the proofs of these lemmas are quite long, we collect all the relevant technical results in this chapter in order to have a cleaner exposition later. Furthermore the analysis of the extension to the $\mathrm{BF}^2\mathrm{V}$ structure carried out in Chapter 7 for Palatini–Cartan theory requires similar results.

It is worth noting here that all the results of this chapter are based on the requirement of the coframe e being an isomorphism between the tangent bundle and the vector bundle V, described in Section 1.2.4.

The chapter is organized as follows: in Section 3.1 we introduce the maps needed for the construction and state some of their properties. In Sections 3.2 and 3.3 we present the results that we will use in Chapters 4 and 5 respectively. Finally, the long proofs of the Lemmas of this chapter are collected in Appendix 3.A.

3.1 Relevant maps and their properties

The goal of this section is to define the relevant quantities and maps, to establish the conventions and to summarize the technical results needed in the following chapters. Some of the proofs will be postponed to Appendix 3.A.

We first recall and introduce some useful shorthand notation. Let M be a stratified manifold of dimension N as in Definition 1.8. As already said in Remark 1.9, we will denote by $\Sigma = M^{(1)}$ its N 1-dimensional stratum, and consider it as *the boundary* of M. We will also denote by $\Gamma = M^{(2)}$ its N 2-dimensional stratum, when present. Furthermore we will use the notation V for the restriction of V to Σ and V for the restriction of V to Γ. We also define

$$\Omega^{i,j} := \Omega^i \left(M, \textstyle\bigwedge^j V\right) \qquad \Omega_\partial^{i,j} := \Omega^i \left(\Sigma, \textstyle\bigwedge^j V\right) \qquad \Omega_{\partial\partial}^{i,j} := \Omega^i \left(\Gamma, \textstyle\bigwedge^j V\right).$$

Remark 3.1. Throughout the thesis we will refer to the dimensions of the spaces $\Omega^{i,j}$ as the number of degrees of freedom of the space. Note that this dimension is also the same as their rank as (as C^∞ modules) and of the dimension of their typical fiber. Hence for example $\dim(\Omega^{i,j}) := \dim \bigwedge^i (T_x M) \otimes \bigwedge^j \mathcal{V}_x = \binom{N}{i}\binom{N}{j}$.

On $\Omega^{i,j}$, $\Omega_\partial^{i,j}$ and $\Omega_{\partial\partial}^{i,j}$ we define the following maps:

$$
\begin{aligned}
W_k^{(i,j)} : \Omega^{i,j} &\longrightarrow \Omega^{i+k,j+k} \\
X &\longmapsto X \wedge \underbrace{e \wedge \cdots \wedge e}_{k \ times}, \\
W_k^{\partial,(i,j)} : \Omega_\partial^{i,j} &\longrightarrow \Omega_\partial^{i+k,j+k} \\
X &\longmapsto X \wedge \underbrace{e|_\Sigma \wedge \cdots \wedge e|_\Sigma}_{k \ times}. \\
W_k^{\partial\partial,(i,j)} : \Omega_{\partial\partial}^{i,j} &\longrightarrow \Omega_{\partial\partial}^{i+k,j+k} \\
X &\longmapsto X \wedge \underbrace{e|_\Gamma \wedge \cdots \wedge e|_\Gamma}_{k \ times}.
\end{aligned}
$$

Remark 3.2. Usually we will omit writing the restriction of e to the corresponding manifold Σ or Γ.

Furthermore, generalizing the action of the Lie algebra $\mathfrak{so}(N-1,1)$ on $\mathcal{V}$ (or $\mathcal{V}_\Sigma$) we can also introduce the following maps:

$$
\begin{aligned}
\varrho^{(i,j)} : \Omega_\partial^{i,j} &\longrightarrow \Omega_\partial^{i+1,j-1} \\
X &\longmapsto [X, e|_\Sigma].
\end{aligned}
$$

In coordinates they are defined as

$$
X \mapsto \sum_{\sigma \in S_{i+1}} X^{a_1 \ldots a_j}_{\mu_{\sigma(1)} \ldots \mu_{\sigma(i)}} \eta_{a_j b} e^b_{\mu_{\sigma(i+1)}}.
$$

The properties of these maps will be clarified by the following results. They will turn out to be crucial in shaping the boundary and corner structure of Palatini–Cartan theory. We will consider elements in $\Omega^{i,j}$ to have total degree $i+j$ and the wedge product will define a graded commutative associative algebra $\Omega^{\bullet,\bullet}$ with respect to the total degree.[1]

Lemma 3.3. *Let* $N = \dim(M) \geq 4$. *Then*

1. $W_{N-3}^{(2,1)}$ *is bijective;*
2. $\dim \mathrm{Ker} W_{N-3}^{(2,2)} \neq 0$.

Lemma 3.4. *The maps* $W_k^{\partial,(i,j)}$ *have the following properties for* $N \geq 4$*:*

1. $W_{N-3}^{\partial,(2,1)}$ *and* $W_{N-3}^{\partial,(1,2)}$ *are surjective but not injective;*
2. $W_{N-3}^{\partial,(1,1)}$ *is injective;*
3. $\dim \mathrm{Ker} W_{N-3}^{\partial,(1,2)} = \dim \mathrm{Ker} W_{N-3}^{\partial,(2,1)}$;

[1] For $\alpha \in \Omega^{i,j}$ and $\beta \in \Omega^{k,l}$ we have $\alpha \wedge \beta = (-1)^{(i+j)(k+l)} \beta \wedge \alpha$. In particular note that e is an even element of $\Omega^{\bullet,\bullet}$.

4. $W_{N\ 4}^{\partial,(2,1)}$ *is injective. ($N \quad 5$)*

Lemma 3.5. *The maps $W_k^{\partial\partial,(i,j)}$ have the following properties for $N \quad 4$:*

1. $W_{N\ 3}^{\partial\partial,(1,1)}$ *is neither injective nor surjective;*

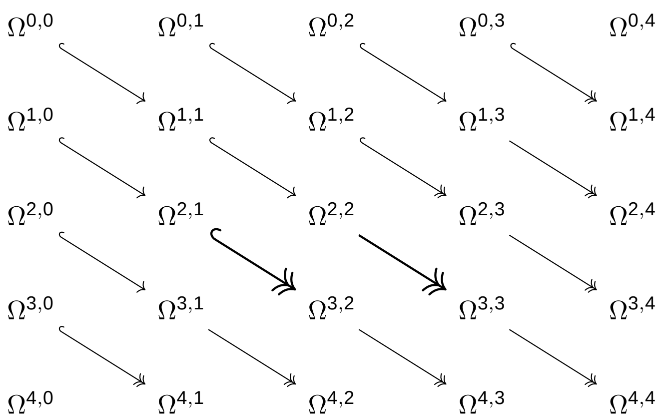
2. $W_{N\ 3}^{\partial\partial,(0,2)}$ *is neither injective nor surjective;*

Lemma 3.6. *If $\dim \mathrm{Ker}(g^\partial) = k$ for $k = 0, 1$, then $\varrho^{(1,2)} j_{\mathrm{Ker} W_{N\ 3}^{\partial,(1,2)}}$ has a kernel of dimension $\frac{N(N\ 3)}{2} k$. In particular if g^∂ is non-degenerate, then $\varrho j_{\mathrm{Ker} W_{N\ 3}^{\partial,(1,2)}}$ is injective.*

The proofs of these three lemmas are quite long and technical and are hence postponed to Appendix 3.A.

Remark 3.7. Some of the properties in Lemmas 3.3, 3.4 and 3.6 have already been proven in [CS19c] for $N = 4$. In Appendix 3.A we will follow a similar strategy for their proofs, adapting them to the different dimensions. In [CS19c, Lemma 4.12], a map similar to ϱ was used, denoted by ϕ_e, which is the restriction of ϱ to the kernel $\mathrm{Ker} W_1^{\partial,(1,2)}$, composed with the projection $p_{2,1}$ to $\mathrm{Ker} W_1^{\partial,(2,1)}$, that is to say $\phi_e \quad p_{(2,1)} \quad \varrho j_{\mathrm{ker} W_1^{\partial,(1,2)}}$.

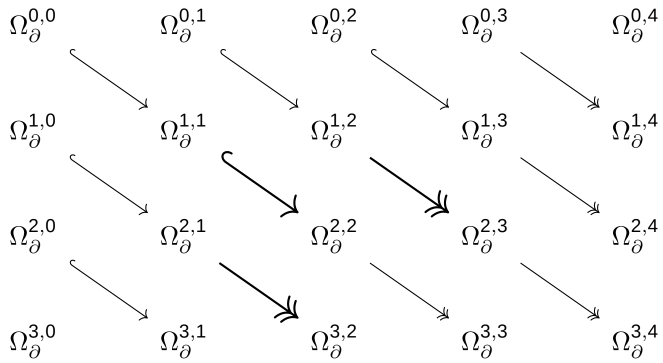
For $N = 4$, the results of Lemmas 3.3, 3.4 and 3.5 can be visualized pictorially in the following set of tables[2], one for each codimension. We organize the Ω^{ij} spaces in an array and connect them with arrows corresponding to the maps $W_1{}^{(i,j)}$: a hooked arrow denotes an injective map while a two headed arrow denotes a surjective map. The tables contain more information than that proved in the lemmas above, depicting the injectivity and/or surjectivity of the maps $W_1{}^{(i,j)}$ for each possible i and j. The proofs of these properties are similar to those proved above and are left to the reader. We highlight the maps mentioned in Lemmas 3.3, 3.4 and 3.5 with thicker arrows. We present the diagrams just for $N = 4$, but it is straightforward to generalize the corresponding properties to higher dimensions. We start with codimension 0, i.e. with the table of the maps in the bulk:



(3.1)

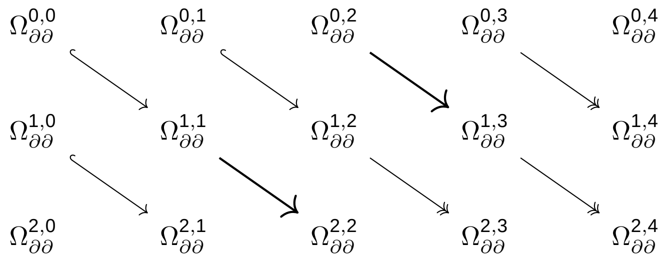
On the boundary, i.e. in codimension 1, the index i runs only between 1 and 3. Note that the properties of the maps heavily depend on the codimension in which we are working. We have the following table:

[2] Note that these are not commutative diagrams, hence no concatenation of arrows are taken into account.



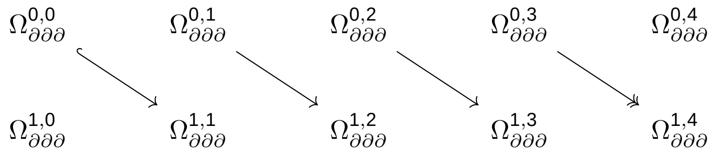
(3.2)

In codimension 2 we have the following table:



(3.3)

We conclude with the table for codimension 3, denoting the spaces by $\Omega_{\partial\partial\partial}^{i,j}$ defined in a similar way to the ones in lower codimension. These properties are not used in this thesis but are presented here for future convenience.



(3.4)

Remark 3.8. In the tables it is also possible to draw the arrows corresponding to the maps $\varrho^{(i,j)}$: these would be arrows pointing upwards and right, instead of downwards and right. However, since we will only need the property listed in Lemma 3.6 we do not show them.

The coframe e viewed as an isomorphism $e\colon TM$! V defines, given a set of coordinates on M, a preferred basis on V. If we denote by ∂_i the vector field in TU, where U is a coordinate neighborhood in M, corresponding to the coordinate x_i, we get a basis on Vj_U by $e_i := e(\partial_i)$. On the boundary, since $T\Sigma$ has one dimension less than V , we can complement the linear independent set e_i with another independent vector that we will call e_n. On the corner Γ the tangent space loses one further dimension, hence we will have to introduce one more additional independent vector that will be denoted by e_m. Fixed a coordinate system on M (or Σ), we call this basis the *standard basis* and, unless otherwise stated, the components of the fields will always be taken with respect to this basis.

3.1.1 Results for the *degeneracy constraint*

In order to define the space to which the Lagrange multiplier of the *degeneracy constraint* belongs (see Section 5.2 for more details), it is useful to consider the following construction.

If a metric g^∂ is degenerate, we can find a vector field X on Σ such that $\iota_X g^\partial = 0$. Using a reference metric g_0, we can locally complete the vector field X_0 (with $\iota_{X_0} g_0^\partial = 0$) to a basis X_0, Y_0^i of TM. If we then choose a coframe e *near* the original one, the same Y_0^is would also be a completion of X to a basis of TM.

Let now $\beta \in \Omega_\partial^{1,0}$ a one form such that $\iota_X \beta = 1$. We then define $\widehat{e} := \beta \iota_X e$ and fix β by requiring that $\widetilde{e} := e - \widehat{e}$ satisfies[3]

$$\iota_{Y_0^1} \dots \iota_{Y_0^{N-2}} (\widetilde{e} \wedge e^{N-4} \wedge v) = 0$$

for all $v \in \Omega_\partial^{1,2}$ such that $e^{N-3} \wedge v = 0$. Using this notation we can define another set of maps

$$\begin{aligned} \widetilde{\varrho}^{(i,j)} : \Omega_\partial^{i,j} &\to \Omega_\partial^{i+1,j-1} \\ X &\mapsto [X, \widetilde{e}] \end{aligned}$$

which in coordinate reads

$$X \mapsto \sum_{\sigma_{i+1}} X^{a_1 \dots a_j}_{\mu_{\sigma(1)} \dots \mu_{\sigma(i)}} \eta_{a_j b} \widetilde{e}^b_{\mu_{\sigma(i+1)}}.$$

Let J be a complement of the space $\mathrm{Im}\, \varrho^{(1,2)}|_{\mathrm{Ker} W_{N-3}^{\partial,(1,2)}}$ in $\Omega_\partial^{2,1}$. We now consider the following spaces:

$$\mathcal{T} = \mathrm{Ker} W_{N-3}^{\partial(2,1)} \cap J \subset \Omega_\partial^{2,1}, \tag{3.5a}$$

$$\mathcal{K} = \mathrm{Ker} W_{N-3}^{\partial(1,2)} \cap \mathrm{Ker} \varrho^{(1,2)} \subset \Omega_\partial^{1,2}, \tag{3.5b}$$

$$\mathcal{S} = \mathrm{Ker} W_1^{\partial(N-3,N-1)} \cap \mathrm{Ker} \widetilde{\varrho}^{(N-3,N-1)} \subset \Omega_\partial^{N-3,N-1}. \tag{3.5c}$$

We also denote by $p_{\mathcal{T}} : \Omega_\partial^{2,1} \to \mathcal{T}$, by $p_{\mathcal{K}} : \Omega_\partial^{1,2} \to \mathcal{K}$ and by $p_{\mathcal{S}} : \Omega_\partial^{N-3,N-1} \to \mathcal{S}$ some corresponding projections to them. The spaces $\mathcal{T}$ and $\mathcal{K}$ are not empty because of the results of Lemmas 3.4.(1) and 3.6, while $\mathcal{S}$ is characterized by the following Proposition, in which we also give a local-coordinate expression, since it will be useful in the computation of the Poisson brackets of the constraints.

Proposition 3.9. *The dimension of* $\mathcal{S}$ *is*

$$\dim \mathcal{S} = \frac{N(N-3)}{2}.$$

Let $p \in \Sigma$ *and* U *a neighbourhood of* p*, then in normal coordinates centered in* p *and in the standard basis of* V*, an element* $\tau \in \mathcal{S}$ *is parametrized by functions*

$$\begin{aligned} Y_\mu &:= \tau^{NN-1\mu_1 \dots \mu_{N-3}}_{\mu_1 \dots \mu_{N-3}} \text{ where } \mu \neq \mu_1 \dots \mu_{N-3}, \\ X^{\mu_2}_{\mu_1} &:= \tau^{NN-1\mu_3 \dots \mu_{N-2}\mu_2}_{\mu_3 \dots \mu_{N-2}\mu_1}, \end{aligned}$$

such that

$$\sum_{\mu=1}^{N-2} Y_\mu = 0 \text{ and } X^{\mu_2}_{\mu_1} = f(\widetilde{g}^\partial, X^{\mu_1}_{\mu_2}, Y_\mu)$$

for $\mu_1 < \mu_2$ *and some linear function* f *with* $\widetilde{g}^\partial := \eta(\widetilde{e}, \widetilde{e})$.

[3] The fact that the required condition is sufficient and well defined will be analyzed later in Lemma 3.21.

The proof of this Proposition is postponed to Appendix 3.A.

Remark 3.10. In order to compute the structure of the Poisson brackets between the constraints we will need to know the equations defining S not only point-wise but also in a small neighbourhood, since we will need to take derivatives. Despite being in principle computable for every dimension, we do not need the explicit expression of f. It is also worth noting that at the base point p of the normal coordinates the last set of equations reduces to

$$X_{\mu_1}^{\mu_2} = \quad X_{\mu_2}^{\mu_1}.$$

While the space K and T arise naturally while considering the symplectic reduction of the boundary two form, the importance of the space S resides in the following Proposition that shows that S plays the role of a *dual space* of T.

Lemma 3.11. *Let* α *2* $\Omega_\partial^{2,1}$. *Then*

$$\int \tau\alpha = 0 \ \ 8\tau \ 2 \ S =) \ \ p_T(\alpha) = 0.$$

We conclude this section with a result that will be necessary in the computation of the Hamiltonian vector fields of the constraints and in their Poisson brackets.

Lemma 3.12.

$$\mathrm{Im}\,\varrho^{(N \quad 1, N \quad 3)}j_S \qquad \mathrm{Im}\,W_{N \quad 3}^{\partial,(1,1)}.$$

Corollary 3.13. *The free components of* $W_N{}^1{}_3([\tau, e])$ *are*

$$[W_N{}^1{}_3([\tau, e])]_{\mu_1}^{\mu_2} \ / \ X_{\mu_1}^{\mu_2}$$
$$[W_N{}^1{}_3([\tau, e])]_{\mu}^{\mu} \ / \ Y_{\mu}$$

such that $\sum_{\mu=1}^{N \quad 2}[W_N{}^1{}_3([\tau, e])]_{\mu}^{\mu} = 0$ *and* $[W_N{}^1{}_3([\tau, e])]_{\mu_1}^{\mu_2} = \quad [W_N{}^1{}_3([\tau, e])]_{\mu_2}^{\mu_1}$.

As before, the proofs of these three lemmas are quite long and technical and are hence postponed to Appendix 3.A.

3.2 Characterization of v 2 $\mathrm{Ker} W_{N \quad 3}^{\partial,(1,2)}$, non-degenerate case

Recalling the definition of e_n in Section 4.1.2 as a section of V that is a completion of the basis e_1, e_2, e_3, we can state the following:

Lemma 3.14. *Let* α *2* $\Omega_\partial^{2,1}$. *Then*

$$\alpha = 0 \qquad (\ ) \qquad \begin{cases} e^{N \quad 3}\alpha = 0 \\ e_n e^{N \quad 4}\alpha \ 2 \ \mathrm{Im}\,W_{N \quad 3}^{\partial,(1,1)} \end{cases} . \tag{3.6}$$

Proof. We first note that the second requirement corresponds to the existence of a σ 2 $\Omega_\partial^{1,1}$ such that $e_n e^{N \quad 4}\alpha = e^{N \quad 3}\sigma$. Let now $I \quad \mathbb{R}$ be an interval and let x^n be the coordinate along it. We define $\widetilde{M} = \Sigma \quad I$ and rewrite (3.6) as conditions on the pullbacks of e, e_n, σ and α to $\widetilde{M}$, which we will keep denoting with the same letters. We now define the following forms on $\widetilde{M}$:

$$E = e^{N \quad 3} + e_n e^{N \quad 4}dx^n, \qquad A = \alpha + \sigma dx^n.$$

Hence the system (3.6) corresponds to the single equation $E \wedge A = 0$. Since e_n has been chosen to be linearly independent from e as vectors in $\mathcal{V}$, E is an isomorphism $T\widetilde{M} \to \mathcal{V}$. Hence we can use Lemma 3.3.(1) and deduce that

$$E \wedge \cdot : \Omega^2(\widetilde{M}, \mathcal{V}) \to \Omega^{N-1}(\widetilde{M}, \textstyle\bigwedge^{N-2} \mathcal{V})$$

is injective. Hence $A = 0$, which in turn implies $\alpha = 0$. □

Corollary 3.15.

$$d_\omega e = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e^{N-3} d_\omega e = 0 \\ e_n e^{N-4} d_\omega e \in \mathrm{Im}\, W_{N-3}^{\partial,(1,1)} \end{cases}.$$

Proof. Trivial application of Lemma 3.14 to $\alpha = d_\omega e$. □

Corollary 3.16. *If g^∂ is non-degenerate, the map*

$$\begin{aligned} \chi \colon \mathrm{Ker}W_{N-3}^{\partial,(1,2)} &\to \Omega_\partial^{(N-2,N-2)} \\ v &\mapsto e_n e^{N-4}[v, e] \end{aligned}$$

is injective and in particular

$$\mathrm{Im}\,\chi \cap \mathrm{Im}\, W_{N-3}^{\partial,(1,1)} = \{0\}. \tag{3.7}$$

Proof. Consider $0 \neq v \in \mathrm{Ker}W_{N-3}^{\partial,(1,2)}$, i.e. such that $e^{N-3} v = 0$. We get

$$e^{N-3}[v, e] = [e^{N-3} v, e] - v[e^{N-3}, e] = (N-3) v e^{N-4}[e, e] = 0.$$

Suppose now by contradiction that $e_n e^{N-4}[v, e] \in \mathrm{Im}\, W_{N-3}^{\partial,(1,1)}$; then, applying lemma 3.14 to $\alpha = [v, e]$, we get $[v, e] = 0$. From Lemma 3.6 we know that if g^∂ is non-degenerate, $[v, e] \neq 0$ which contradicts the previous assertion. □

Lemma 3.17. *Let $\beta \in \Omega_\partial^{N-2,N-2}$. If g^∂ is non-degenerate, there exist a unique $v \in \mathrm{Ker}W_{N-3}^{\partial,(1,2)}$ and a unique $\gamma \in \Omega_\partial^{1,1}$ such that*

$$\beta = e^{N-3}\gamma + e_n e^{N-4}[v, e].$$

Proof. From Lemma 3.4.(2) and Lemma 3.4.(3) we know that $W_{N-3}^{\partial,(1,1)}$ is injective and that the sum of the dimensions of $\mathrm{Ker}W_{N-3}^{\partial,(1,2)}$ and of $\mathrm{Im}\, W_{N-3}^{\partial,(1,1)}$ agrees with dimension of $\Omega_\partial^{N-2,N-2}$. Using Corollary 3.16, we deduce that $\Omega_\partial^{N-2,N-2}$ is the direct sum of $\mathrm{Im}\,\chi$ and $\mathrm{Im}\, W_{N-3}^{\partial,(1,1)}$. Hence every $\beta \in \Omega_\partial^{N-2,N-2}$ can be written as $\beta = e^{N-3}\gamma + \theta$ with $\gamma \in \Omega_\partial^{1,1}$ and $\theta = e_n e^{N-4}[v, e]$. Uniqueness of v and γ follows from the injectivity of χ and $W_{N-3}^{\partial,(1,1)}$. □

3.3 Characterization of $v \in \mathrm{Ker}W_{N-3}^{\partial,(1,2)}$, degenerate case

In this section we present the generalizations of the lemmas of the previous section to encompass the differences of the degenerate case.

Lemma 3.18. *Let $\alpha \in \Omega_\partial^{2,1}$. Then $\alpha = 0$ if and only if*

$$\begin{cases} e^{N-3}\alpha = 0 \\ e_n e^{N-4}\alpha - e_n e^{N-4} p_{\mathcal{T}}\,\alpha \in \operatorname{Im} W_{N-3}^{\partial,(1,1)} \\ p_{\mathcal{T}}\,\alpha = 0 \end{cases}. \tag{3.8}$$

Proof. Trivial generalization of Lemma 3.14. □

Lemma 3.19. *Let $\beta \in \Omega_\partial^{N-2,N-2}$. If g^∂ is degenerate, there exist a unique $v \in \mathrm{Ker} W_{N-3}^{\partial,(1,2)}$, a unique $\gamma \in \Omega_\partial^{1,1}$ and a unique $\theta \in \mathcal{T}$ such that*

$$\beta = e^{N-3}\gamma + e_n e^{N-4}[v,e] + e_n e^{N-4}\theta.$$

Proof. By definition of $\mathcal{T}$ it is clear that for each element $\alpha \in \mathrm{Ker} W_{N-3}^{\partial(2,1)}$ it is possible to find $\theta \in \mathcal{T}$ and $v \in \mathrm{Ker} W_{N-3}^{\partial,(1,2)}$ such that $\alpha = [v,e] + \theta$. From the proof of Lemma 3.17 we also know that each element $\beta \in \Omega_\partial^{N-2,N-2}$ can be written as $\beta = e^{N-3}\gamma + e_n e^{N-4}\alpha$ for some $\alpha \in \mathrm{Ker} W_{N-3}^{\partial(2,1)}$. Combining these two results we get the claim. □

Appendix

3.A Lengthy proofs of Section 3.1

In this appendix we collect the proofs of Lemmas 3.3, 3.4, 3.5, 3.6, 3.11, 3.12, of Proposition 3.9 and of Corollary 3.13.

Proof of Lemma 3.3. For each of the properties we use the following demonstrative scheme. We first compute the number of independent equations that a quantity must satisfy in order to lie in the kernel of the map under consideration. We then compare it to the dimension of the domain. If they agree, then the function is injective, otherwise comparing it with the dimension of the codomain we can deduce whether the map is surjective.

1. Consider $W_{N-3}^{(2,1)} : \Omega_N^{2,1} \to \Omega_N^{N-1,N-2}$: the dimension of the spaces are $\dim \Omega_N^{2,1} = \binom{N}{2}\binom{N}{1}$ and $\dim \Omega_N^{N-1,N-2} = \binom{N}{N-2}\binom{N}{N-1}$. Notice that the two dimensions agree. The kernel of $W_{N-3}^{(2,1)}$ is defined by the following set of equations:

$$X^a_{\mu_1\mu_2} e_a \wedge e_{\mu_3} \wedge \cdots \wedge e_{\mu_{N-1}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-1}} = 0$$

where we used the vectors $e_a = e(\partial_a)$ as a basis for V. Let now $1 \leq k \leq N$. Since $\{dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-1}}\}$ is a basis for $\Omega^{N-1}(M)$ we obtain N equations of the form

$$\sum_\sigma X^a_{\mu_{\sigma(1)}\mu_{\sigma(2)}} e_a \wedge e_{\mu_{\sigma(3)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-1)}} = 0$$

where σ runs on all permutations of $N-1$ elements and $1 \leq \mu_i \leq N$, $\mu_i \neq k$ for all $1 \leq i \leq N-1$. Recall now that $e_a \wedge e_{\mu_{\sigma(3)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-1)}}$ is a basis of $\wedge^{N-2}V$. Hence we obtain the following equations:

$$\begin{aligned} X^k_{ij} &= 0 \qquad 1 \leq i,j \leq N\ i \neq j,\ i,j \neq k \\ \sum_{i \neq k, i \neq j} X^i_{ij} &= 0 \qquad \forall 1 \leq j \leq N\ j \neq k \end{aligned}$$

Letting now k vary in $\{1,\dots,N\}$ we obtain the following equations:

$$X^k_{ij}=0 \qquad 1\le i,j,k\le N \quad i\neq j\neq k\neq i$$
$$\sum_{i\neq k,i\neq j} X^i_{ij}=0 \qquad \forall j\neq k,\ 1\le k,j\le N.$$

It is easy to check that these equations are independent. The total number of equations defining the kernel is then $\frac{N(N-1)(N-2)}{2}+(N-1)N=\frac{(N-1)N^2}{2}$ which coincides with both the dimensions of the domain and codomain. Hence $W^{(2,1)}_{N-3}$ is bijective.

2. $W^{(2,2)}_{N-3}:\Omega^{2,2}_N\to\Omega^{N-1,N-1}_N$ cannot be injective since $\Omega^{2,2}_N$ has $\frac{N^2(N-1)^2}{4}$ degrees of freedom, while $\Omega^{N-1,N-1}_N$ has just N^2 degrees of freedom and $N\ge 4$.

□

Proof of Lemma 3.4. For each of the properties we use the same scheme of the proof of Lemma 3.3.

1. The proof of $W^{\partial,(2,1)}_{N-3}$ is analogous to that of $W^{(2,1)}_{N-3}$ with the difference that now k is fixed to be the transversal direction (conventionally $k=N$). Hence we get the following set of equations:

$$X^N_{ij}=0 \qquad 1\le i,j\le N-1 \quad i\neq j$$
$$\sum_{i\neq j} X^i_{ij}=0 \qquad \forall 1\le j\le N-1$$

which are $\frac{(N-1)(N-2)}{2}+(N-1)=\frac{N(N-1)}{2}$ which is exactly the number of degrees of freedom of $\Omega^{N-1,N-2}_\partial$. Hence $W^{\partial,(2,1)}_{N-3}$ is surjective but not injective. In particular $\dim \mathrm{Ker} W^{\partial,(2,1)}_{N-3}=\frac{N(N-1)(N-2)}{2}-\frac{N(N-1)}{2}=\frac{N(N-1)}{2}(N-3)$.

Consider now $W^{\partial,(1,2)}_{N-3}:\Omega^{1,2}_\partial\to\Omega^{N-2,N-1}_\partial$: the dimensions of domain and codomain are $\dim\Omega^{1,2}_\partial=(N-1)\frac{N(N-1)}{2}$ and $\dim\Omega^{N-2,N-1}_\partial=(N-1)N$. The kernel of $W^{\partial,(1,2)}_{N-3}$ is defined by the following set of equations:

$$X^{ab}_{\mu_1}e_ae_b\wedge e_{\mu_2}\wedge\cdots\wedge e_{\mu_{N-2}}dx^{\mu_1}dx^{\mu_2}\dots dx^{\mu_{N-2}}=0$$

where we used e_a as a basis for V. Let now $k=N$ be the transversal direction and let $k'\in\{1,\dots N-1\}$. Since $\{dx^{\mu_1}dx^{\mu_2}\dots dx^{\mu_{N-2}}\}$ is a basis for $\Omega^{N-2}(\Sigma)$ we obtain $N-1$ equations of the form

$$\sum_\sigma X^{ab}_{\mu_{\sigma(1)}}e_ae_b\wedge e_{\mu_{\sigma(2)}}\wedge\cdots\wedge e_{\mu_{\sigma(N-2)}}=0$$

where σ runs on all permutations of $N-2$ elements and $1\le\mu_i\le N-1$, $\mu_i\neq k'$ for all $1\le i\le N-2$. Recall now that $e_ae_b\wedge e_{\mu_{\sigma(2)}}\wedge\cdots\wedge e_{\mu_{\sigma(N-2)}}$ is a basis of $\wedge^{N-1}V$. Hence we obtain the following equations:

$$X^{Nk'}_i=0 \qquad 1\le i\le N-1 \quad i\neq k'$$
$$\sum_{i\neq N,i\neq k'} X^{iN}_i=0 \qquad \sum_{i\neq N,i\neq k'} X^{ik'}_i=0$$

Letting now k' vary in $\{1, \dots, N-1\}$ we obtain the following equations:

$$X_i^{Nj} = 0 \qquad 1 \leq i,j \leq N-1 \; i \neq j \tag{3.9a}$$

$$\sum_{i \neq N, i \neq j} X_i^{iN} = 0 \qquad 1 \leq j \leq N-1 \tag{3.9b}$$

$$\sum_{i \neq N, i \neq j} X_i^{ij} = 0 \qquad 1 \leq j \leq N-1 \tag{3.9c}$$

It is easy to check that these equations are independent. The total number of equations defining the kernel is then $(N-1)+(N-1)(N-2)+(N-1) = (N-1)N$ which coincides with number of degrees of freedom of the codomain. Hence $W_{N-3}^{\partial,(1,2)}$ is surjective but not injective. In particular $\dim \mathrm{Ker} W_{N-3}^{\partial,(1,2)} = (N-1)\frac{N(N-1)}{2} - N(N-1) = \frac{N(N-1)}{2}(N-3)$.

2. Consider $W_{N-3}^{\partial,(1,1)} : \Omega_\partial^{1,1} \to \Omega_\partial^{N-2,N-2}$: the dimension of the spaces are $\dim \Omega_\partial^{1,1} = (N-1)N$ and $\dim \Omega_\partial^{N-2,N-2} = (N-1)\frac{N(N-1)}{2}$. The kernel of $W_{N-3}^{\partial,(1,1)}$ is defined by the following set of equations:

$$X_{\mu_1}^a e_a \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_{N-2}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}} = 0$$

where we used e_a as a basis for V. Let now $k = N$ be the transversal direction and let $k' \in \{1, \dots N-1\}$. Since $\{dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}\}$ is a basis for $\Omega^{N-2}(\Sigma)$ we obtain $N-1$ equations of the form

$$\sum_\sigma X_{\mu_{\sigma(1)}}^a e_a \wedge e_{\mu_{\sigma(2)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}} = 0$$

where σ runs on all permutations of $N-2$ elements and $1 \leq \mu_i \leq N-1$, $\mu_i \neq k'$ for all $1 \leq i \leq N-2$. Recall now that $e_a \wedge e_{\mu_{\sigma(2)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}}$ is a basis of $\wedge^{N-2} V$. Hence we obtain the following equations:

$$X_i^k = 0 \qquad 1 \leq i \leq N-1 \; i \neq k'$$

$$X_i^{k'} = 0 \qquad 1 \leq i \leq N-1 \; i \neq k'$$

$$\sum_{i \neq k, i \neq k'} X_i^i = 0$$

Letting now k' vary in $\{1, \dots, N-1\}$ we obtain the following equations:

$$X_i^k = 0 \qquad 1 \leq i \leq N-1$$

$$X_i^j = 0 \qquad 1 \leq i,j \leq N-1 \; i \neq j$$

$$\sum_{i \neq k, i \neq j} X_i^i = 0 \qquad 1 \leq j \leq N-1$$

It is easy to check that these equations are independent. The total number of equations defining the kernel is then $(N-1)+(N-1)(N-2)+(N-1) = (N-1)N$ which coincides with number of degrees of freedom of the domain. Hence $W_{N-3}^{\partial,(1,1)}$ is injective but not surjective.

3. Is a direct consequence of the previous parts.

4. Consider $W_{N-4}^{\partial,(2,1)} : \Omega_{\partial}^{2,1} \to \Omega_{\partial}^{N-2,N-3}$:
the dimension of domain and codomain are $\dim \Omega_{\partial}^{2,1} = \frac{(N-2)(N-1)}{2} N$ and $\dim \Omega_{\partial}^{N-2,N-3} = (N-1)\frac{N(N-1)(N-2)}{6}$. The kernel of $W_{N-4}^{\partial,(2,1)}$ is defined by the following set of equations:

$$X^a_{\mu_1\mu_2} e_a \wedge e_{\mu_3} \wedge \cdots \wedge e_{\mu_{N-2}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}} = 0$$

where we used e_a as a basis for V. Let now $k = N$ be the transversal direction and let $k' \in \{1, \dots N-1\}$. Since $\{dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}\}$ is a basis for $\Omega^{N-2}(\Sigma)$ we obtain $N-1$ equations of the form

$$\sum_\sigma X^a_{\mu_{\sigma(1)}\mu_{\sigma(2)}} e_a \wedge e_{\mu_{\sigma(3)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}} = 0$$

where σ runs on all permutations of $N-2$ elements and $1 \leq \mu_i \leq N-1$, $\mu_i \neq k'$ for all $1 \leq i \leq N-2$. Recall now that $e_a \wedge e_{\mu_{\sigma(3)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}}$ is a basis of $\wedge^{N-3} V$. Hence we obtain the following equations:

$$\begin{aligned} X^N_{ij} &= 0 \qquad 1 \leq i,j \leq N-1 \; i,j \neq k' \\ X^{k'}_{ij} &= 0 \qquad 1 \leq i,j \leq N-1 \; i,j \neq k' \\ \sum_{i \neq N, i \neq k'} X^i_{ij} &= 0 \qquad 1 \leq j \leq N-1 \; j \neq k' \end{aligned}$$

Letting now k' vary in $\{1, \dots, N-1\}$ we obtain the following equations:

$$\begin{aligned} X^N_{ij} &= 0 \qquad 1 \leq i,j \leq N-1 \\ X^{j'}_{ij} &= 0 \qquad 1 \leq i,j,j' \leq N-1 \; i,j \neq j' \; i \neq j \\ \sum_{i \neq k, i \neq j'} X^i_{ij} &= 0 \qquad 1 \leq j,j' \leq N-1 \; j \neq j' \end{aligned}$$

It is easy to check that these equations are independent. The total number of equations defining the kernel is then $\frac{(N-2)(N-1)}{2} + \frac{(N-3)(N-2)(N-1)}{2} (N-2)(N-1) = \frac{(N-2)(N-1)N}{2}$ which coincides with number of degrees of freedom of the domain. Hence $W_{N-4}^{\partial,(2,1)}$ is injective but not surjective.

□

Proof of Lemma 3.5. For each of the properties we use the same scheme of the proof of Lemma 3.3.

1. Consider $W_{N-3}^{\partial\partial(1,1)} : \Omega_{\partial\partial}^{1,1} \to \Omega_{\partial\partial}^{N-2,N-2}$: the dimension of the spaces are $\dim \Omega_{\partial\partial}^{1,1} = \binom{N-2}{1}\binom{N}{1}$ and $\dim \Omega_{\partial\partial}^{N-2,N-2} = \binom{N-2}{N-2}\binom{N}{N-2}$. The kernel of $W_{N-3}^{\partial\partial(1,1)}$ is defined by the following set of equations:

$$X^a_{\mu_1} e_a \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_{N-2}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}} = 0$$

where we used the vectors $e_a = e(\partial_a)$ together with two linearly independent vectors e_n, e_m as a basis for V. Since $\{dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-1}}\}$ is a basis for $\Omega^{N-2}(\Gamma)$ we obtain an equations of the form

$$\sum_\sigma X^a_{\mu_{\sigma(1)}} e_a \wedge e_{\mu_{\sigma(2)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}} = 0$$

where σ runs on all permutations of $N-2$ elements and $1 \leq \mu_i \leq N-2$ for all $1 \leq i \leq N-2$. Recall now that $e_a \wedge e_{\mu_{\sigma(2)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}}$ is a basis of $\wedge^{N-2}V$. Hence we obtain the following equations:

$$X_i^m = 0 \qquad X_i^m = 0 \qquad 1 \leq i \leq N-2$$
$$\sum_{i=1}^{N-2} X_i^i = 0$$

These equations are clearly independent. The total number of equations defining the kernel is then $2(N-2)+1$ which is smaller than the dimension of the domain. Furthermore the dimension of the image is smaller than the dimension of the codomain. Hence $W_{N-3}^{\partial\partial(1,1)}$ is neither injective nor surjective.

2. Consider $W_{N-3}^{\partial\partial(0,2)} : \Omega_{\partial\partial}^{0,2} \to \Omega_{\partial\partial}^{N-3,N-1}$: the dimension of the spaces are $\dim \Omega_{\partial\partial}^{0,2} = \binom{N}{2}$ and $\dim \Omega_{\partial\partial}^{N-3,N-1} = \binom{N-2}{N-3}\binom{N}{N-1}$. The kernel of $W_{N-3}^{\partial\partial(0,2)}$ is defined by the following set of equations:

$$X^{ab} e_a e_b \wedge e_{\mu_1} \wedge \cdots \wedge e_{\mu_{N-3}} dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_{N-3}} = 0$$

where we used the vectors $e_a = e(\partial_a)$ together with two linearly independent vectors e_n, e_m as a basis for V. Since $\{dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_{N-3}}\}$ is a basis for $\Omega^{N-3}(\Gamma)$ we obtain $N-2$ equations of the form

$$\sum_\sigma X^{ab} e_a e_b \wedge e_{\mu_{\sigma(1)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-3)}} = 0$$

where σ runs on all permutations of $N-3$ elements and $1 \leq \mu_i \leq N-2$ for all $1 \leq i \leq N-3$. Recall now that $e_a e_b \wedge e_{\mu_{\sigma(1)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-3)}}$ is a basis of $\wedge^{N-1}V$. Hence we obtain the following equations:

$$X^{km} = 0 \qquad X^{kn} = 0 \qquad 1 \leq k \leq N-2$$
$$X^{mn} = 0$$

These equations are clearly independent. The total number of equations defining the kernel is then $2(N-2)+1$ which is smaller than the dimension of the domain. Furthermore the dimension of the image is smaller than the dimension of the codomain. Hence $W_{N-3}^{\partial\partial(0,2)}$ is neither injective nor surjective.

□

Proof of Lemma 3.6. Consider $\varrho|_{\mathrm{Ker}W_{N-3}^{\partial,(1,2)}} : \mathrm{Ker}W_{N-3}^{\partial,(1,2)} \to \Omega_\partial^{2,1}$. From 3.4.(1) we know that $\dim \mathrm{Ker}W_{N-3}^{\partial,(2,1)} = \frac{N(N-1)}{2}(N-3)$. An element $v \in \mathrm{Ker}W_{N-3}^{\partial,(1,2)}$ must satisfy equations (3.9). The kernel of ϱ is defined by the following set of equations:[4]

$$[v,e]_{\mu_1\mu_2}^a = v_{\mu_1}^{ab} g_{b\mu_2}^\partial - v_{\mu_2}^{ab} g_{b\mu_1}^\partial = 0.$$

Using now normal geodesic coordinates, we can diagonalise g^∂ with eigenvalues on the diagonal $\alpha_\mu \in \{1, -1, 0\}$:

$$[v,e]_{\mu_1\mu_2}^a = v_{\mu_1}^{a\mu_2} \alpha_{\mu_2} - v_{\mu_2}^{a\mu_1} \alpha_{\mu_1} = 0 \tag{3.10}$$

[4] Here we use that in every point we can find a basis in V such that $e_\mu^i = \delta_\mu^i$: $[v,e]_{\mu_1\mu_2}^a = v_{\mu_1}^{ab}\eta_{bc}e_{\mu_2}^c - v_{\mu_2}^{ab}\eta_{bc}e_{\mu_1}^c = v_{\mu_1}^{ab}e_b^d\eta_{dc}e_{\mu_2}^c - v_{\mu_2}^{ab}e_b^d\eta_{dc}e_{\mu_1}^c$

If g^∂ is non-degenerate these equations become $v^{a\mu_2}_{\mu_1} = -v^{a\mu_1}_{\mu_2}$. Namely, using $v \in \mathrm{Ker}W^{\partial,(2,1)}_{N-3}$ we get

$$v_i^{ij} = 0 \qquad 1 \le i,j \le N-1,\ i \ne j$$
$$v^{i_2 i_3}_{i_1} = -v^{i_1 i_3}_{i_2} \qquad 1 \le i_1, i_2, i_3 \le N-1\ i_1, i_2 \ne i_3,\ i_2 \ne i_1$$

It is easy to check that these equations are independent. The total number of equations defining the kernel is then $(N-1)(N-3) + \frac{(N-1)(N-2)(N-3)}{2} = \frac{N(N-1)}{2}(N-3)$ which coincides with number of degrees of freedom of the domain. Hence, in the non-degenerate case $\varrho|_{\mathrm{Ker}W^{\partial,(1,2)}_{N-3}}$ is injective.

Let now $\alpha_\mu = 0$ for $\mu = N-1$ and $\alpha_\mu = -1$ for $1 \le \mu \le N-2$. We adopt now the following convention on indices for $m, p \in \mathbb{N}$: $1 \le i_m \le N-2$, $i_m \ne i_p$ iff $m \ne p$. Using $v \in \mathrm{Ker}W^{\partial,(2,1)}_{N-3}$, from (3.10) we get

$$[v,e]^{i_3}_{i_1 i_2} = v^{i_3 i_2}_{i_1} - v^{i_3 i_1}_{i_2} \qquad [v,e]^{i_1}_{i_1 i_2} = v^{i_1 i_2}_{i_1} \tag{3.11a}$$
$$[v,e]^{N-1}_{i_1 i_2} = v^{N-1 i_2}_{i_1} - v^{N-1 i_1}_{i_2} \qquad [v,e]^{N}_{i_1 i_2} = 0 \tag{3.11b}$$
$$[v,e]^{i_2}_{i_1 N-1} = -v^{i_2 i_1}_{N-1} \qquad [v,e]^{N-1}_{i N-1} = -v^{N-1 i}_{N-1} \tag{3.11c}$$
$$[v,e]^{i}_{i N-1} = 0 \qquad [v,e]^{N}_{i N-1} = 0. \tag{3.11d}$$

By imposing that every component vanishes, we get the corresponding equations for the kernel. It is easy to check that these equations are independent but the second of (3.11a) and the first of (3.11c) which are connected by the third of (3.9). The total number of equations defining the kernel is then

$$\frac{(N-2)(N-3)(N-4)}{2} + (N-3)(N-4) + (N-2)(N-3).$$

Since $\frac{N(N-1)}{2}(N-3)$ is the number of degrees of freedom of the domain, the kernel has dimension

$$\dim(\mathrm{Ker}\varrho|_{\mathrm{Ker}W^{\partial,(1,2)}_{N-3}}) = \frac{N(N-3)}{2}.$$

□

Proof of Proposition 3.9. We split the proof into simpler lemmas. First we compute the dimension of S and the equations defining it at the point p in Lemmas 3.20 and 3.21.

Collecting these results we get that S is defined by the following equations:

$$\sum_{\mu_1 \dots \mu_{N-3}=1}^{N-2} X^{NN-1\mu_1\dots\mu_{N-3}}_{\mu_1\dots\mu_{N-3}} = 0$$
$$X^{\mu_1\dots\mu_{N-1}}_{N-1 i_1 \dots i_{n-4}} = 0 \quad 1 \le \mu_a \le N,\ 1 \le i_a \le N-2$$
$$X^{N i_1\dots i_{N-2}}_{i_1\dots i_{N-3}} = 0 \quad 1 \le i_a \le N-2$$
$$X^{N-1 i_1\dots i_{N-2}}_{i_1\dots i_{N-3}} = 0 \quad 1 \le i_a \le N-2$$
$$X^{NN-1 i_1\dots i_{N-4} i_{N-2}}_{i_1\dots i_{N-4} i_{N-3}} + X^{NN-1 i_1\dots i_{N-4} i_{N-3}}_{i_1\dots i_{N-4} i_{N-2}} = 0 \quad 1 \le i_a \le N-2.$$

Counting them and subtracting the result to the total dimension of $\Omega_\partial^{N-3,N-1}$ we get the claimed result. Then in Lemma 3.22 we express the equations defining the kernel of $\widetilde{\varrho}^{(N-3,N-1)}$ in the geodesic neighbourhood U in terms of the components of $\tau \in S$ and those of the modified metric $\widetilde{g}^\partial := \eta(\widetilde{e}, \widetilde{e})$ and find the corresponding free components. Note also that the equations in Lemma 3.20 hold in a neighbourhood since we are not using normal geodesic coordinates in the proof. □

Lemma 3.20. *The space* $\mathrm{Ker} W_1^{\partial(N-3,N-1)} \subset \Omega_\partial^{N-3,N-1}$ *in the standard basis is defined by the following* $N-1$ *equations*

$$\sum_{\substack{\mu_1\dots\mu_{N-3}=1\\ \mu_i\neq k,\mu_i\neq\mu_j}}^{N-1} X^{Nk\mu_1\dots\mu_{N-3}}_{\mu_1\dots\mu_{N-3}} = 0 \qquad 1 \leq k \leq N-1.$$

Proof. Consider $W_1^{\partial,(N-3,N-1)} : \Omega_\partial^{N-3,N-1} \to \Omega_\partial^{N-2,N}$: the kernel of it is defined by the following set of equations:

$$X^{a\dots a_{N-1}}_{\mu_1\dots\mu_{N-3}} e_a \wedge \cdots \wedge e_{a_{N-1}} \wedge e_{\mu_{N-2}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}} = 0$$

where we used e_a as a basis for V. Since $\{dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}\}$ is a basis for $\Omega^{N-2}(\Sigma)$ we obtain $N-1$ equations of the form

$$\sum_\sigma X^{a\dots a_{N-1}}_{\mu_{\sigma(1)}\dots\mu_{\sigma(N-3)}} e_a \wedge \cdots \wedge e_{a_{N-1}} \wedge e_{\mu_{\sigma(N-2)}} = 0$$

where σ runs on all permutations of $N-2$ elements and $1 \leq \mu_i \leq N-1$ and denote by k the missing index: $\mu_i \neq k$ for all $1 \leq i \leq N-2$. Recall now that $e_a \wedge e_{\mu_{\sigma(2)}} \wedge \cdots \wedge e_{\mu_{\sigma(N-2)}}$ is a basis of $\wedge^N V$. Hence we obtain the following $N-1$ equations:

$$\sum_{\substack{\mu_1\dots\mu_{N-3}=1\\ \mu_i\neq k,\mu_i\neq\mu_j}}^{N-1} X^{Nk\mu_1\dots\mu_{N-3}}_{\mu_1\dots\mu_{N-3}} = 0 \qquad 1 \leq k \leq N-1.$$

□

Lemma 3.21. *The space* $\mathrm{Ker}\,\tilde\rho|_{\partial}^{N-3,N-1} \subset \Omega_\partial^{N-3,N-1}$ *is defined by the following four sets of equations in the standard basis:*

$$\begin{aligned}
X^{\mu_1\dots\mu_{N-1}}_{N-1 i_1\dots i_{n-4}} &= 0 \quad 1 \leq \mu_a \leq N, 1 \leq i_a \leq N-2\\
X^{Ni_1\dots i_{N-2}}_{i_1\dots i_{N-3}} &= 0 \quad 1 \leq i_a \leq N-2\\
X^{i_1\dots i_{N-2}N-1}_{i_1\dots i_{N-3}} &= 0 \quad 1 \leq i_a \leq N-2\\
X^{NN-1i_1\dots i_{N-4}i_{N-2}}_{i_1\dots i_{N-4}i_{N-3}} + X^{NN-1i_1\dots i_{N-4}i_{N-3}}_{i_1\dots i_{N-4}i_{N-2}} &= 0 \quad 1 \leq i_a \leq N-2.
\end{aligned}$$

Proof. In normal geodesic coordinates the boundary metric g^∂ at the base point is diagonal, and we can assume that its eigenvalues α_i are such that $\alpha_a = 1$ for $1 \leq a \leq N-2$ and $\alpha_{N-1} = 0$. Since X is such that $\iota_X g^\partial = 0$ we get $X = \partial_{N-1}$. Let now be $\beta = \sum_{i=1}^{N-1} \beta_i dx^i$ a generic one form. From the equation $\iota_X \beta = 1$ we get $\beta_{N-1} = 1$. Hence $\widehat{e} = \sum_{i=1}^{N-1} \beta_i dx^i e_{N-1}$ and $\widetilde{e} = \sum_{i=1}^{N-2} (e_i + \beta_i e_{N-1}) dx^i$. We now impose the last condition to find an explicit expression for $\widetilde{e}$ in the standard basis.

Using these coordinates, since $X = \partial_{N-1}$ we can take Y_0^i as $Y_0^i = \partial_i$. Let now $v \in \Omega_\partial^{1,2}$ such that $e^{N-3}v = 0$ i.e. its components must satisfy (3.9). Using the same techniques as in the proof of Lemma 3.4.(3), we get

$$\begin{aligned}
ve^{N-4}\widetilde{e} &= X^{ab}_{\mu_1} e_a e_b \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_{N-3}} \widetilde{e}^c_{\mu_{N-2}} e_c dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}\\
&= X^{ab}_{\mu_1} e_a e_b \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_{N-3}} e_{\mu_{N-2}} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}\\
&\quad X^{ab}_{\mu_1} e_a e_b \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_{N-3}} \beta_{\mu_{N-2}} e_{N-1} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_{N-2}}
\end{aligned}$$

where in the second and third line μ_{N-2} cannot take the value $N-1$. Equating this quantity to zero, we get the following equations for the components:

$$\sum_{\sigma} X^{ab}_{\mu_{\sigma(1)}} e_a e_b e_{\mu_{\sigma(2)}} \dots e_{\mu_{\sigma(N-2)}} - X^{ab}_{\mu_{\sigma(1)}} e_a e_b e_{\mu_{\sigma(2)}} \dots e_{\mu_{\sigma(N-3)}} e_{N-1} \beta_{\mu_{\sigma(N-2)}} = 0$$

where $\mu_{\sigma(N-2)} \neq N-1$. Now, letting $\{\mu_1 \dots \mu_{N-2}\} = \{1 \dots N-2\}$ we get

$$\sum_{i \neq j, N-1, N} \left(X_j^{N-1 N} - X_j^{iN}\beta_i + X_i^{iN}\beta_j \right) = 0 \qquad j = 1 \dots N-2$$

$$\sum_{i \neq N-1, N} X_i^{iN} = 0$$

$$\sum_{i \neq N-1, N} X_i^{iN-1} - \sum_{i,j \neq N-1, N} X_i^{ij}\beta_j = 0.$$

Using the properties (3.9), we can deduce from the very first equation that $\beta_i = 0$ for $i = 1 \dots N-2$. Plugging this result into the others we do not get any further condition, as all the quantities vanish automatically. We deduce that, with this choice of the coordinates x_i, $\widetilde{e} = \sum_{i=1}^{N-2} e_i dx^i$.

Now, using the same procedure as in Lemma 3.6 we obtain the following equations defining the kernel of $\tilde{\rho}$:

$$[\tau, e]^{\nu_1 \dots \nu_{N-2}}_{\mu_1 \dots \mu_{N-2}} = \sum_{\sigma_{N-2}} \tau^{\nu_1 \dots \nu_{N-2} \mu_{\sigma(N-2)}}_{\mu_{\sigma(1)} \dots \mu_{\sigma(N-3)}} \alpha_{\mu_{\sigma(N-2)}} = 0$$

where $1 \le \mu_a \le N-1$, $1 \le \nu_a \le N$, $\alpha_a = 1$ for $1 \le a \le N-2$, $\alpha_{N-1} = 0$ and σ_{N-2} represents the permutation of $N-2$ elements. Using the properties of the αs we get

$$\sum_{\sigma_{N-3}} \tau^{\nu_1 \dots \nu_{N-2} i_{\sigma(N-3)}}_{N-1\, i_{\sigma(1)} \dots i_{\sigma(N-4)}} = 0 \quad 1 \le i_a \le N-2 \tag{3.12a}$$

$$\sum_{\sigma_{N-2}} \tau^{\nu_1 \dots \nu_{N-2} i_{\sigma(N-2)}}_{i_{\sigma(1)} \dots i_{\sigma(N-3)}} = 0 \quad 1 \le i_a \le N-2 \tag{3.12b}$$

for $1 \le \nu_a \le N$. Let us consider the first set of equations. If $\{\nu_1, \dots, \nu_{N-2}\} \supset \{i_1, \dots, i_{N-3}\}$ no term survives and we do not get equations. Let now n be an index in $\{i_1, \dots, i_{N-3}\}$ but not in $\{\nu_1, \dots, \nu_{N-2}\}$: then only one term survives and we have the following equations:

$$\tau^{\nu_1 \dots \nu_{N-2} n}_{N-1\, i_1 \dots i_{N-4}} = 0$$

where $1 \le i_a, n \le N-2$ and $\{\nu_1, \dots, \nu_{N-2}\} \supset \{i_1, \dots, i_{N-4}\}$. The only other case that is left is when there are two indices n_1, n_2 in $\{i_1, \dots, i_{N-3}\}$ but not in $\{\nu_1, \dots, \nu_{N-2}\}$: here two terms of the sum are surviving and we get:

$$\tau^{N N-1 i_1 \dots i_{N-5} n_3 n_2}_{N-1\, i_1 \dots i_{N-5} n_1} + \tau^{N N-1 i_1 \dots i_{N-5} n_3 n_1}_{N-1\, i_1 \dots i_{N-5} n_2} = 0$$

where $1 \le i_a, n \le N-2$ and n_3 is the only index left different from all the others. Because of the arbitrariness of n_1, n_2, n_3, this set of equations will contain also the ones corresponding to permutations of them:

$$\tau^{N N-1 i_1 \dots i_{N-5} n_2 n_3}_{N-1\, i_1 \dots i_{N-5} n_1} + \tau^{N N-1 i_1 \dots i_{N-5} n_2 n_1}_{N-1\, i_1 \dots i_{N-5} n_3} = 0$$

$$\tau^{N N-1 i_1 \dots i_{N-5} n_1 n_3}_{N-1\, i_1 \dots i_{N-5} n_2} + \tau^{N N-1 i_1 \dots i_{N-5} n_1 n_2}_{N-1\, i_1 \dots i_{N-5} n_3} = 0.$$

Composing these three equations we get that

$$\tau_{N\ \ 1 i_1 \dots i_{N\ \ 5} n_1}^{NN\ \ 1 i_1 \dots i_{N\ \ 5} n_3 n_2} = 0.$$

Together with the first case this proves the first set of equations in the statement. We proceed in the same way for the second set in (3.12). If $\{\nu_1, \dots, \nu_{N\ \ 2}\}\ \ \ \{i_1, \dots, i_{N\ \ 3}\}$ no term survives and we do not get equations. Let now n be an index in $\{i_1, \dots, i_{N\ \ 3}\}$ but not in $\{\nu_1, \dots, \nu_{N\ \ 2}\}$. We get

$$X_{i_1 \dots i_{N\ \ 3}}^{N i_1 \dots i_{N\ \ 3} n} = 0 \quad 1 \quad i_a \quad N \quad 2$$
$$X_{i_1 \dots i_{N\ \ 3}}^{N\ \ 1 i_1 \dots i_{N\ \ 3} n} = 0 \quad 1 \quad i_a \quad N \quad 2$$

which are respectively the second and the third set of equations in the statement. When there are two indices n_1, n_2 in $\{i_1, \dots, i_{N\ \ 3}\}$ but not in $\{\nu_1, \dots, \nu_{N\ \ 2}\}$ we get the fourth set of equations:

$$X_{i_1 \dots i_{N\ \ 4} n_1}^{NN\ \ 1 i_1 \dots i_{N\ \ 4} n_2} + X_{i_1 \dots i_{N\ \ 4} n_2}^{NN\ \ 1 i_1 \dots i_{N\ \ 4} n_1} = 0 \qquad 1 \quad i_a \quad N \quad 2.$$

□

Lemma 3.22. *Let $p \in \Sigma$ and U an open neighbourhood of p. Then, in the standard basis of V , the equations defining the space $\mathrm{Ker}\tilde{\rho}|_{\Omega_\partial^{N\ \ 3, N\ \ 1}}$ $\Omega_\partial^{N\ \ 3, N\ \ 1}$ are*

$$X_{N\ \ 1 i_1 \dots i_{n\ \ 4}}^{\mu_1 \dots \mu_{N\ \ 1}} = 0 \quad 1 \quad \mu_a \quad N,\ 1 \quad i_a \quad N \quad 2$$
$$X_{i_1 \dots i_{N\ \ 3}}^{N i_1 \dots i_{N\ \ 2}} = 0 \quad 1 \quad i_a \quad N \quad 2$$
$$X_{i_1 \dots i_{N\ \ 3}}^{N\ \ 1 i_1 \dots i_{N\ \ 2}} = 0 \quad 1 \quad i_a \quad N \quad 2$$
$$X_{i_1 \dots i_{N\ \ 4} i_{N\ \ 3}}^{NN\ \ 1 i_1 \dots i_{N\ \ 4} i_{N\ \ 2}} = f(\tilde{g}^\partial, X_{i_1 \dots i_{N\ \ 4} i_{N\ \ 2}}^{N i_1 \dots i_{N\ \ 4} N\ \ 1 i_{N\ \ 3}}, X_{i_1 \dots i_{N\ \ 3}}^{NN\ \ 1 i_1 \dots i_{N\ \ 3}})$$

for some function f.

Proof. Using the standard basis of V , we obtain the following equations for the kernel of $\tilde{\rho}$:

$$[\tau, \tilde{e}]_{j_1 \dots j_{N\ \ 2}}^{i_1 \dots i_{N\ \ 2}} = \sum_{\sigma, \mu} \tau_{j_{\sigma(1)} \dots j_{\sigma(N\ \ 3)}}^{i_1 \dots i_{N\ \ 2} \mu} \tilde{g}^\partial_{\mu j_{\sigma(N\ \ 2)}} = 0 \tag{3.13}$$

where σ runs over the permutations of order $N \ \ 2$ and $\mu = 1 \dots N \ \ 2$, $i_k \in \{1 \dots N \ \ 1\}$, $j_k \in \{1 \dots N\}$. Using normal geodesic coordinates, $\tilde{g}^\partial$ is diagonal in the point p, with diagonal entries different from zero. Hence using continuity, in the whole neighbourhood U (eventually shrinking it if necessary) the diagonal component will be non-zero. Furthermore, $\det \tilde{g}^\partial \neq 0$, since $\tilde{g}^\partial$ is non-degenerate by construction.
We first analyse the case when $N \ \ 1 \in \{i_1, \dots, i_{N\ \ 2}\}$ and prove the first set of equations in the statement. Expanding the equations (3.13) in all possible choices of indexes, one finds a overdetermined system of equations, and expressing it in its matricial form, it is always possible to find a square submatrix whose determinant is equal to $\det \tilde{g}^\partial \neq 0$. This implies that all the variables must be zero.
Let now $N \ \ 1 \notin \{i_1, \dots, i_{N\ \ 2}\}$. If $N, N \ \ 1 \notin \{j_1, \dots, j_{N\ \ 2}\}$ no equations are generated. Let then $N \in \{j_1, \dots, j_{N\ \ 2}\}$ or $N \ \ 1 \in \{j_1, \dots, j_{N\ \ 2}\}$ but not $N, N \ \ 1 \in \{j_1, \dots, j_{N\ \ 2}\}$. We proceed as in the previous case and obtain a system of equations whose only solution is the zero

one. Hence we deduce the second and the third set of equations in the statement. Let now $N, N-1 \in \{j_1,\dots,j_{N-2}\}$. Expanding equations (3.13) we get

$$[\tau,\tilde e]^{NN-1\mu_3\dots\mu_{N-2}}_{\mu_1\mu_2\mu_3\dots\mu_{N-2}} = \sum_\sigma \tau^{NN-1\mu_3\dots\mu_{N-2}\mu_1}_{\mu_{\sigma(1)}\dots\mu_{\sigma(N-3)}}\widetilde{g}^\partial_{\mu_1\mu_{\sigma(N-2)}} + \tau^{NN-1\mu_3\dots\mu_{N-2}\mu_2}_{\mu_{\sigma(1)}\dots\mu_{\sigma(N-3)}}\widetilde{g}^\partial_{\mu_2\mu_{\sigma(N-2)}} = 0.$$

Inverting some of the equation exploiting the properties of $\widetilde{g}^\partial$ we can express the components $\tau^{NN-1\mu_3\dots\mu_{N-2}\mu_1}_{\mu_3\dots\mu_{(N-3)}\mu_2}$ with $\mu_1 < \mu_2$ in function of the components of $\widetilde{g}^\partial$, $\tau^{NN-1\mu_3\dots\mu_{N-2}\mu_2}_{\mu_3\dots\mu_{(N-3)}\mu_1}$ (with $\mu_1 < \mu_2$) and $\tau^{NN-1\mu_2\dots\mu_{N-3}}_{\mu_2\dots\mu_{(N-3)}}$.

□

Proof of Lemma 3.11. From the proof of Lemma 3.4.(3), the free components of an element in $\mathcal{T}$ are:

$$X^{i_1}_{N-1i_2} \quad 1 \le i_1, i_2 \le N-2,\ i_1 \neq i_2$$
$$X^{i}_{iN-1} \quad 1 \le i \le N-2$$

such that

$$X^{i_1}_{ji_2} = -X^{i_2}_{ji_1}; \quad \sum_{\mu=1}^{N-1} X^{\mu}_{\mu j} = 0.$$

From Proposition 3.9 the free components of an element $\tau \in \mathcal{S}$ are Y_μ and $X^{\mu_2}_{\mu_1}$ satisfying

$$\sum_{\mu=1}^{N-2} Y_\mu = 0 \text{ and } X^{\mu_2}_{\mu_1} = -X^{\mu_1}_{\mu_2}$$

for $\mu_1, \mu_2 = 1\dots N-2$. Let us now consider some particular choices of τ. First we consider τ such that the only non-zero components are $\tau^{\mu_2}_{\mu_1} = -\tau^{\mu_1}_{\mu_2}$ for some particular μ_1 and μ_2. Then

$$\int \tau\alpha = \int (X^{\mu_2}_{\mu_1}\alpha^{\mu_1}_{N-1\mu_2} + X^{\mu_1}_{\mu_2}\alpha^{\mu_2}_{N-1\mu_1})\mathbf{V} = \int X^{\mu_2}_{\mu_1}(\alpha^{\mu_1}_{N-1\mu_2} - \alpha^{\mu_2}_{N-1\mu_1})\mathbf{V} = 0.$$

Hence we deduce that $\alpha^{\mu_1}_{N-1\mu_2} - \alpha^{\mu_2}_{N-1\mu_1} = 0$. Furthermore the components of $p_{\mathcal{T}}(\alpha)$ satisfy $p_{\mathcal{T}}\alpha^{\mu_1}_{N-1\mu_2} + p_{\mathcal{T}}\alpha^{\mu_2}_{N-1\mu_1} = 0$. Hence we conclude $p_{\mathcal{T}}\alpha^{\mu_1}_{N-1\mu_2} = 0$ for all μ_1 and μ_2. Now consider τ such that the only non-zero components are Y_μ. Hence now

$$\int \tau\alpha = \int \sum_{\mu=1}^{N-2} Y_\mu \alpha^\mu_{N-1\mu}\mathbf{V} = \int \sum_{\mu=1}^{N-3} Y_\mu(\alpha^\mu_{N-1\mu} - \alpha^{N-2}_{N-1N-2})\mathbf{V} = 0.$$

By the arbitrariness of τ we deduce that $\alpha^\mu_{N-1\mu} - \alpha^{N-2}_{N-1N-2} = 0$ for each $\mu = 1,\dots N-3$. Furthermore the components of $p_{\mathcal{T}}(\alpha)$ satisfy $\sum_{\mu=1}^{N-2} p_{\mathcal{T}}(\alpha)^\mu_{N-1\mu} = 0$. Hence we deduce that $p_{\mathcal{T}}(\alpha)^\mu_{N-1\mu} = 0$ for all μ. This proves the claim. □

Proof of Lemma 3.12. Let $\tau \in \mathcal{S}$. Then we want to prove that $[\tau, e] \in \operatorname{Im} W^{\partial,(1,1)}_{N-3}$. Using the results of Lemma 3.4, we know that the free components of $\operatorname{Im} W^{\partial,(1,1)}_{N-3}$ are

$$X^{\mu_1\dots\mu_{N-2}}_{\mu_1\dots\mu_{N-2}},\ X^{\mu_1\dots\mu_{N-3}\mu_{N-1}}_{\mu_1\dots\mu_{N-3}\mu_{N-2}} \text{ and } X^{\mu_1\dots\mu_{N-3}\mu_{N-2}}_{\mu_1\dots\mu_{N-3}N}$$

such that $X^{\mu_1\dots\mu_{N-3}\mu_{N-2}}_{\mu_1\dots\mu_{N-3}N} = X^{\mu'_1\dots\mu'_{N-3}\mu_{N-2}}_{\mu'_1\dots\mu'_{N-3}N}$.

From Proposition 3.9 we deduce that the free components of $\tau \in S$ are

$$\tau^{NN-1\mu_1\dots\mu_{N-3}}_{\mu_1\dots\mu_{N-3}} \text{ and } \tau^{NN-1\mu_1\dots\mu_{N-4}\mu_{N-2}}_{\mu_1\dots\mu_{N-4}\mu_{N-3}}$$

such that

$$\sum_{\mu_i=1}^{N-2} \tau^{NN-1\mu_1\dots\mu_{N-3}}_{\mu_1\dots\mu_{N-3}} = 0,$$
$$\tau^{NN-1\mu_1\dots\mu_{N-4}\mu_{N-2}}_{\mu_1\dots\mu_{N-4}\mu_{N-3}} + \tau^{NN-1\mu_1\dots\mu_{N-4}\mu_{N-3}}_{\mu_1\dots\mu_{N-4}\mu_{N-2}} = 0.$$

Recalling that $[\tau, \tilde{e}] = 0$, we deduce that $[\tau, e]$ has components[5]

$$[\tau, e]^{\nu_1\dots\nu_{N-2}}_{\mu_1\dots\mu_{N-3}N-1} = \tau^{\nu_1\dots\nu_{N-2}N}_{\mu_1\dots\mu_{N-3}}.$$

Plugging into this expression the free components of τ we get the free components of $[\tau, e]$:

$$[\tau, e]^{N-1\mu_1\dots\mu_{N-3}}_{N-1\mu_1\dots\mu_{N-3}} \text{ and } [\tau, e]^{N-1\mu_1\dots\mu_{N-4}\mu_{N-2}}_{N-1\mu_1\dots\mu_{N-4}\mu_{N-3}}$$

such that

$$\sum_{\mu_i=1}^{N-2} [\tau, e]^{N-1\mu_1\dots\mu_{N-3}}_{N-1\mu_1\dots\mu_{N-3}} = 0,$$
$$[\tau, e]^{N-1\mu_1\dots\mu_{N-4}\mu_{N-2}}_{N-1\mu_1\dots\mu_{N-4}\mu_{N-3}} + [\tau, e]^{N-1\mu_1\dots\mu_{N-4}\mu_{N-3}}_{N-1\mu_1\dots\mu_{N-4}\mu_{N-2}} = 0.$$

It is straightforward to check that these components are in the image of $W_{N-3}^{\partial,(1,1)}$. □

Proof of Corollary 3.13. Using the standard basis we have that

$$(X \wedge e^{N-3})^{j\mu_1\dots\mu_{N-3}}_{i\mu_1\dots\mu_{N-3}} = X_i^j \text{ for } i \neq j$$
$$(X \wedge e^{N-3})^{\mu_1\dots\mu_{N-2}}_{\mu_1\dots\mu_{N-2}} = \sum_{\mu} X^{\mu}_{\mu} \text{ with } \mu \in \{\mu_1 \dots \mu_{N-2}\}.$$

Comparing these expressions with the ones in the proof Lemma 3.12 we deduce that

$$[W_{N-3}^{-1}([\tau, e])]^{\mu_2}_{\mu_1} \propto X^{\mu_2}_{\mu_1}$$

and that

$$\sum_{\mu=1,\mu\neq\nu}^{N-1} [W_{N-3}^{-1}([\tau, e])]^{\mu}_{\mu} = Y_{\nu}.$$

Summing for $\nu = 1 \dots N-1$ and remembering that $Y_{N-1} = 0$ and that $\sum_{\nu} Y_{\nu} = 0$ we deduce the claim. □

[5] We use here the same trick of footnote 4 but since τ can have components in the direction $N-1$ in the standard basis, the metric is the one of the bulk and not the one of the boundary, In particular, since we diagonalized the metric on the boundary we can choose coordinates on the bulk such that g has the form

$$g = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Chapter 4

Palatini–Cartan Boundary structure, non-degenerate case

In this Chapter we investigate the geometry of the boundary structure of general relativity—in particular, the reduced phase space—(in any space–time dimension greater than three) in the Palatini–Cartan formalism and develop its Batalin–Fradkin–Vilkovisky (BFV) formulation.

This is a first step in a plan to overcome the problem encountered in [CS19b]: namely, reversing the BV-BFV procedure by first studying the BFV formalism for Palatini–Cartan gravity on the boundary and then inducing a compatible BV formalism in the bulk (See Chapter 6 for this last step). Another motivation for our study of the BFV structure is to extend the analysis of General Relativity to strata of higher codimension as was successfully done for other BV-BFV theories [CMR14; MSW19]. In Chapter 2, and more precisely in Proposition 2.10, we explored this analysis in the three-dimensional case, while in Chapter 7 we will give some insight of the general case in the Einstein–Hilbert and Palatini–Cartan formalism.

Our solution to the above problem is based on a geometric construction of the reduced phase space, as introduced by Kijowski and Tulczijew [KT79]. This procedure has several advantages, giving a geometric approach to the reduced phase space. Moreover, it usually produces the reduced phase space as a coisotropic reduction — i.e., only first class constraints appear — and, finally, it is compatible to the BV-BFV construction in the sense that the BFV structure associated to the boundary is the resolution of the reduced phase space [Sta97; Sch08].

This construction was firstly applied to the current problem in [CS19c] in the non-null boundary case, showing in particular that only first-class constraints appear. However, some of the expressions presented in [CS19c] were not quite as explicit as one might have liked. Although this does not hinder the theorems on the classical (Hamiltonian) structure, a more explicit description would be desirable when writing down the BFV action for the theory, or for further computations. In this chapter we provide such a description, improving the understanding of the reduced phase space of Palatini–Cartan theory and extending all the results to higher dimensions. One main advantage of this explicit presentation is that it will allow us to construct the BFV action for PC theory in dimension $N \quad 4$.

In this Chapter we discuss only the case when the induced boundary metric is non-degenerate, corresponding to time-like or space-like boundaries. The remaining light-like case, i.e. when the induced boundary metric is degenerate, is slightly different and requires a somewhat more complicated description. This last analysis is postponed to Chapter 5 in which an adaptation of the construction presented here is shown.

The chapter is organized as follows: in Section 4.1 we review the construction of the reduced phase space of PC theory (following [CS19c]), and present the new idea that will be used throughout to simplify the boundary structure.

In Section 4.2 we construct the reduced phase space of PC theory using the choice presented in Section 4.1.2: we show that the constraints are first class, and compute their Poisson brackets explicitly.

Section 4.3 is devoted to the construction of the BFV data for Palatini–Cartan theory in dimension 4, while Section 4.4 generalises all the previous results to $N \geq 5$.

Section 4.4.1 depends on Section 4.2 but is completely independent of Section 4.3, which is required only by 4.4.2 and can be ignored by a reader who is interested in purely classical (non-BFV) considerations.

4.1 A (short) overview

Our geometric construction is based on the paper [CS19c] where Cattaneo and Schiavina treated the four-dimensional case, applying the construction of Kijowski and Tulczijew [KT79]. In this context, the reduced phase space is obtained as the reduction by first class constraints of an appropriate space of boundary fields.

The aim of this chapter is to supplement the construction of [CS19c] with a more explicit choice of boundary data. In this section we will summarise the main results of [CS19c], and present the new idea from which this result stems.

Although the solution is proposed for a generic dimension $N \geq 4$, the remainder of this overview will focus on the four-dimensional case: $N = 4$. In this case the Palatini–Cartan action functional is simply

$$S = \int_M \frac{1}{2} eeF_\omega + \frac{1}{4!}\Lambda e^4 \tag{4.1}$$

and its Euler–Lagrange equations are (equivalent to)

$$d_\omega e = 0, \qquad eF_\omega + \frac{1}{3!}\Lambda e^3 = 0. \tag{4.2}$$

4.1.1 Reduced phase space for Palatini–Cartan theory

We apply to this theory the construction due to Kijowski and Tulczijew [KT79], which allows to investigate its phase space and the Hamiltonian formulation. Let now M have a boundary Σ.

We begin by observing that, when varying the action (4.1), one gets a boundary term[1]

$$\widetilde{\alpha}^\partial = \frac{1}{2}\int ee\delta\omega.$$

We view the restrictions of e and ω to the boundary as, respectively, a non-degenerate section of $T^*\Sigma \otimes V$ — i.e., as an injective bundle map $T\Sigma \to V$ — and an orthogonal connection associated to V. Again, we may view the space of these connections as modeled on $T^*\Sigma \otimes \bigwedge^2 V$.

We can then regard $\widetilde{\alpha}^\partial$ as a one-form on the space $\widetilde{F}_{PC}$ of the ***pre-boundary*** fields $e|_\Sigma$ and $\omega|_\Sigma$.[2] Thus, we might then think of $\widetilde{\varpi} = \delta\widetilde{\alpha}^\partial$ as a closed degenerate two form on the space of

[1] This is precisely so with the choice of constant fibre metric as in Remark 1.20. In general, in this formula, as well as in the formulas for the constraints that will appear later, there is a hidden factor $\sqrt{|\det \eta|}$ — which has no effects on the computations and results — but is required to produce densities, which can be canonically integrated. From now on we will no longer mention this factor.

[2] As already said in Chapter 3, we will omit the restrictions to the boundary from now on.

pre-boundary fields. In fact, the two-form $\tilde{\tilde{\varpi}}$ is degenerate: a vector field X in the kernel of $\tilde{\tilde{\varpi}}$ acts as $\omega \to \omega + v$ with

$$ev = 0. \tag{4.3}$$

Remark 4.1. We stress that this transformation implicitly depends on e and, under the non-degeneracy assumption on e, the v's satisfying (4.3) and hence the X's in the kernel of $\tilde{\tilde{\varpi}}$ have exactly 6 local components. If we mod-out the space of pre-boundary fields by the kernel of $\tilde{\tilde{\varpi}}$ we get a space parametrized by e and equivalence classes of ω under the e-dependent transformation above.[3] This defines the map

$$\pi_{PC} \colon \tilde{F}_{PC} \to F_{PC}. \tag{4.4}$$

On this quotient space, the two-form $\tilde{\tilde{\varpi}}$ determines a non-degenerate, closed two-form: the manifold (F_{PC}, ϖ_{PC}) will be given the name ***geometric phase space*** of the theory.

The symplectic manifold defined by (4.4) is not yet the "physical" phase space of the theory, usually called ***reduced phase space***. Indeed, the Euler–Lagrange equations (4.2) split into evolution equations, which contain derivatives of the pre-boundary fields in a transversal direction, and equations where only tangential derivatives appear. The latter equations, called the constraints, must be imposed on the preboundary fields, but this enlarges the kernel of the presymplectic form, and the corresponding reduction has to be taken into account. To obtain the reduced phase space, it is advantageous to reformulate this procedure in terms of the geometric phase space we have introduced above.

Remark 4.2. An advantage of the Palatini–Cartan formulation is that it is formulated in terms of differential forms and, as a consequence, the constraints are readily available as the restriction to the boundary of Equations (4.2) [4]. One problem is that the constraints are not necessarily invariant under the transformations generated by X in the kernel of $\tilde{\tilde{\varpi}}$, i.e. translations of ω by v (and in fact they are not). There are two possible ways out: to select the v-invariant parts of the constraints and take the quotient by the v's or to look for a section to the v-translations. As in [CS19c], we will follow here the second strategy.

The first remark is that the constraint $eF_\omega + \frac{1}{3!}\Lambda e^3 = 0$ is indeed v-invariant upon using the first constraint $\mathrm{d}_\omega e = 0$.[5] Therefore, it is better to use a v-section that, in conjunction with the invariant part of $\mathrm{d}_\omega e = 0$, reproduces the whole constraint.

It is easy to check that the induced constraint

$$e\mathrm{d}_\omega e = 0 \tag{4.5}$$

(6 local components) is indeed v-invariant. We will call it the ***invariant constraint***. It turns out that Equation (4.5) determines the whole invariant part— under the condition that e is such that the boundary metric

$$g^\partial_{ij} := (e_i, e_j) \tag{4.6}$$

is non-degenerate,[6] where i, j are indices of boundary coordinates. From now on we will assume this condition.

[3] Observe that e and the remaining ω both have 12 local components, or *degrees of freedom*.

[4] Note that the relevant quantity is the zero locus generated by the constraints and not the actual functional form of them.

[5] If we denote by δ_v a variation along v, we get

$$\delta_v(eF_\omega) = e\mathrm{d}_\omega v = \mathrm{d}_\omega(ev) - \mathrm{d}_\omega e\, v.$$

The first term vanishes because $ev = 0$ and the second because we assume $\mathrm{d}_\omega e = 0$.

[6] This is for example the case when $M = B \times [0,1]$, where $[0,1]$ is an interval, and e is assumed to produce a metric for which $B \times \{0\}$ and $B \times \{1\}$ are space-like.

Since the remaining components of the constraint $d_\omega e = 0$, which we call the ***structural constraint***, are also 6, they can now be used to fix the v-translations completely. Note that the invariant constraints are canonically given, whereas the structural ones require a choice.

A few remarks are now in order (see [CS19c] for their proofs):

1. Since the structural constraint completely fixes the v-transformations, the space S of pre-boundary fields satisfying it is symplectomorphic to the space of boundary fields.
2. On S (12 local degrees of freedom) we still have to impose the 10 local constraints[7]

$$e d_\omega e = 0, \qquad eF_\omega + \frac{1}{3!}\Lambda e^3 = 0.$$

   These constraints are first class [CS19c], and we are then left with the expected 2 local physical degrees of freedom of four-dimensonal gravity (for a time- or space-like boundary).
3. The constraints may be written in terms of Lagrange multipliers c and μ as

$$\mathrm{L}_c = \int c e d_\omega e, \qquad J_\mu = \int \mu \left(eF_\omega + \frac{1}{3!}\Lambda e^3 = 0 \right). \tag{4.7}$$

   It turns out that "on shell," i.e., upon the constraints, L generates the internal gauge transformations and J the diffeomorphisms (including the remnant of the transversal ones).
4. One may also reduce by stages. One possibility is to impose $e d_\omega e = 0$ and to mod out by Lorentz gauge transformations. The resulting space, with 6 local degrees of freedom, is symplectomorphic to the phase space of the Einstein–Hilbert formulation (the cotangent bundle of the space of boundary metrics). The remaing constraints $eF_\omega + \frac{1}{3!}\Lambda e^3 = 0$ produce the momentum constraints. Another possibility is to split the Lie algebra of orthogonal transformations into two 3-dimensional subalgebras. The symplectic reduction with respect to one of the summands yields Ashtekar's formulation [Ash86].

Remark 4.3. Note that the Hamiltonian vector fields of L and J in (4.7) depend on the actual choice of the structural constraints and may be not very explicit if the choice is not optimal. This is not a serious problem for the classical considerations above, and for this reason no attempt to find an optimal choice was made in [CS19c]; however, a non-optimal choice is inconvenient for certain computations as well as for further considerations like, e.g., the explicit BFV description of the theory.

4.1.2 An optimal choice of structural constraints

The main result in this chapter is to present a good choice of structural constraints and how to use it to produce the BFV data associated to the reduced phase space of Palatini–Cartan theory (see Section 4.3). Another explicit choice, in the context of Dirac's formulation, has been presented in [MRC19], with its extension to higher dimension discussed in [Mon+20] (unrelated to BFV).

First of all we choose, locally, a section e_n of $\mathcal{V}$ that is a completion of the basis e_1, e_2, e_3 (here $1, 2, 3$ denote boundary coordinates, see Chapter 3 for more details). Note that in a neighborhood

[7] These constraints look like the restriction to the boundary of the Euler{Lagrange equations, except e^ cannot be eliminated from either expression. Note also that now part of the components of ω are constrained, so these constraints are only formally equal to the restriction of the Euler{Lagrange equations.

of a given e in the space of pre-boundary fields we may choose e_n once and for all independently of the es in the neighborhood. This done, we write the structural constraints as

$$e_n \mathrm{d}_\omega e = e\sigma \tag{4.8}$$

for some unspecified one-form σ taking values in V. Note that we have 18 equations with 12 unspecified parameters σ, so in total we have indeed 6 constraints.

We will show that this choice of structural constraint fixes the v-translations and that, together with the invariant constraint $e\mathrm{d}_\omega e = 0$, it produces the full constraint $\mathrm{d}_\omega e = 0$, which is necessary for the v-invariance of $eF_\omega + \frac{1}{3!}\Lambda e^3 = 0$. Moreover, we will show that this choice actually makes the Hamiltonian vector fields of L and J in (4.7) explicit enough to allow writing down the BFV action of the theory. Finally, it will allow us to extend the result in the presence of a cosmological term and to higher dimensions.

Remark 4.4. Observe that, although not necessary, one may interpret the linearly independent system (e_1, e_2, e_3, e_n) as a coframe in a neighborhood of Σ in M and the structural constraint as one of the remaining Euler–Lagrange equations, with σ interpreted as the transversal components of $\mathrm{d}_\omega e$. Viewed this way, the structural constraint (4.8) also immediately shows that the transversal Euler–Lagrange equations (the evolution equations) may actually be solved.

4.2 Constraint analysis of Palatini–Cartan theory in four dimensions

In this section we analyse the structural and invariant constraints of gravity in the Palatini–Cartan formulation for $N = 4$, as discussed in Section 4.1.1. In Section 4.4 we will extend this analysis to $N > 4$. We will assume henceforth that g^∂, as defined in (4.6), is non-degenerate.

4.2.1 An optimal structural constraint

The starting point of our analysis is the geometric phase space F^∂_{PC}, described in full detail in [CS19c] and recalled in Section 4.1.1. The classical fields of the theory are then $e \in \Omega^1_{nd}(\Sigma, V)$ — i.e $\Omega^{1,1}_\partial$ plus the non-degeneracy condition that the induced morphism $T\Sigma \to V$ should be injective— and the equivalence class of a connection $\omega \in \mathcal{A}(\Sigma)$ (where $\mathcal{A}(\Sigma)$ is the restriction of $\mathcal{A}(M)$ to the boundary) under the e-dependent relation $\omega \sim \omega + v$ for v such that $ev = 0$. We denote this equivalence class and the quotient space it belongs to by $[\omega] \in \mathcal{A}^{red}(\Sigma)$. The symplectic structure is given by

$$\varpi = \int e\delta e\delta[\omega]. \tag{4.9}$$

In this section we fix a convenient representative for this equivalence class.

As in Section 4.1.2 we choose a section of V completing the image of $e\colon T\Sigma \to V$ to a basis. Corollary 3.15 shows that the constraint $d_\omega e = 0$ splits into the ***invariant constraint*** $ed_\omega e = 0$ and the constraint

$$e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}, \tag{4.10}$$

which can then be taken as a choice of ***structural constraint***. We prove that Equation (4.10) does not impose any condition on $[\omega] \in \mathcal{A}^{red}(\Sigma)$— but fixes in a unique way a representative of the class. In particular we show that given $[\omega]$ there exists a unique $\omega \in [\omega]$ satisfying (4.10). Later on we will use such representative to define the constraint of the theory.

Theorem 4.5. *Suppose that g^∂, the metric induced on the boundary, is non-degenerate. Given any $\widetilde{\omega}$ 2 $\Omega^{1,2}$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v \tag{4.11}$$

with ω and v satisfying

$$ev = 0 \quad \textit{and} \quad e_n d_\omega e \text{ 2 } \operatorname{Im} W_1^{\partial,(1,1)}. \tag{4.12}$$

Proof. Let $\widetilde{\omega}$ 2 $\Omega_\partial^{1,2}$. From Lemma 3.17 we deduce that there exist unique σ 2 $\Omega_\partial^{1,1}$ and v 2 $\operatorname{Ker} W_1^{\partial,(1,2)}$ such that

$$e_n d_{\widetilde{\omega}} e = e\sigma + e_n[v, e].$$

We define $\omega := \widetilde{\omega} \quad v$. Then ω and v satisfy (4.11) and (4.12).

For uniqueness, suppose that $\widetilde{\omega} = \omega_1 + v_1 = \omega_2 + v_2$ with $ev_i = 0$ and $e_n d_{\omega_i} e$ 2 $\operatorname{Im} W_1^{\partial,(1,1)}$ for $i = 1, 2$. Hence

$$e_n d_{\omega_1} e \quad e_n d_{\omega_2} e = e_n[v_2 \quad v_1, e] \text{ 2 } \operatorname{Im} W_1^{\partial,(1,1)}.$$

Hence from Lemma 3.14 and 3.17, we deduce $v_2 \quad v_1 = 0$, since $v_2 \quad v_1$ 2 $\operatorname{Ker}(W)_1^{\partial,(1,2)}$. □

Remark 4.6. A decomposition similar to (4.11) was used in [CS19c, Remark 4.7], for a generic complement of $\operatorname{Ker} W_1^{\partial,(1,2)}$. Theorem 4.5 shows an explicit choice of a complement which will turn out to be particularly convenient in what follows.

Corollary 4.7. *The eld ω in the decomposition* (4.11) *depends only on the equivalence class* $[\omega]$ 2 $A^{red}(\Sigma)$.

Proof. Let $\widetilde{\omega}_1, \widetilde{\omega}_2$ 2 $[\omega]$. Hence $\widetilde{\omega}_1 \quad \widetilde{\omega}_2 = \tilde{v}$ 2 $\operatorname{Ker} W_1^{\partial,(1,2)}$. Applying Theorem 4.5 we get ω_1, v_1, ω_2, v_2 such that v_1, v_2 2 $\operatorname{Ker} W_1^{\partial,(1,2)}$ and

$$\begin{aligned} \widetilde{\omega}_1 &= \omega_1 + v_1 & \quad e_n d_{\omega_1} e \text{ 2 } \operatorname{Im} W_1^{\partial,(1,1)} \\ \widetilde{\omega}_2 &= \omega_2 + v_2 & \quad e_n d_{\omega_2} e \text{ 2 } \operatorname{Im} W_1^{\partial,(1,1)}. \end{aligned}$$

Subtracting these equations we get $\omega_2 \quad \omega_1 = v_1 \quad v_2 \quad \tilde{v}$ 2 $\operatorname{Ker} W_1^{\partial,(1,2)}$ together with $e_n[\omega_1 \quad \omega_2, e]$ 2 $\operatorname{Im} W_1^{\partial,(1,1)}$. Hence, from Lemma 3.17, we deduce $\omega_1 = \omega_2$. □

4.2.2 Poisson brackets of constraints

The restriction of the Euler–Lagrange equations to the boundary does not produce a well defined set of constraints in the geometric phase space F_{PC}, as they are not given by basic functions with respect to the pre-symplectic reduction $\pi_{PC} \colon \widetilde{F}_{PC}$! F_{PC} (see Remarks 4.1 and 4.2).

However, fixing a representative of the equivalence class of ω by imposing the structural constraint (4.10) in $\widetilde{F}_{PC}$ (thus constructing a section of the map π_{PC}) allows us to consider the restrictions of the Euler–Lagrange equations to the boundary and to construct a set of constraints on the geometric phase space. Moreover, we will see that these constraints turn out to be of first class (i.e. they define a coisotropic submanifold with respect to the symplectic form (4.9)).

Starting from the functions defined in (4.7), we consider the following functions by splitting R_μ into two separate constraints P_ξ and H_λ by expanding $\mu = \iota_\xi e + \lambda e_n$. Notice that with this choice of μ the cosmological term will appear only in the constraint with λ, since $\iota_\xi e^4 = 0$ on the boundary. We furthermore add to P_ξ a term proportional to the invariant constraint $ed_\omega e$ with the help of a reference connection ω_0 in order to simplify computations (see Remark 4.9):[8]

$$\mathrm{L}_c = \int ced_\omega e \tag{4.13a}$$

$$P_\xi = \int \iota_\xi e e F_\omega + \iota_\xi(\omega - \omega_0) e d_\omega e \tag{4.13b}$$

$$H_\lambda = \int \lambda e_n \left(eF_\omega + \frac{1}{3!}\Lambda e^3 \right) \tag{4.13c}$$

where $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega_\partial^{0,0}[1]$ are (odd) Lagrange multipliers and the notation $[1]$ denotes that the fields are shifted by 1 and are treated as odd variables.

Remark 4.8. We use odd Lagrange multipliers c, ξ and λ and we shift their degree by one, to be consistent with the subsequent construction of the BFV action, where we embed our space of fields into a graded manifold, but also in order to simplify the proof of Theorem 4.10 slightly. However, one could just as well formulate constraints (4.13) using even Lagrange multipliers, and the results of the following Theorem 4.10 would not change, upon antisymmetrisation of brackets: $\{\mathrm{L}_c, \mathrm{L}_{c'}\} = \mathbb{L}_c(\mathrm{L}_{c'}) - \mathbb{L}_{c'}(\mathrm{L}_c)$, where $\mathbb{L}$ denotes the Hamiltonian vector field of L_c (see [CS19c] for comparison), and similarly for the other constraints.

Remark 4.9. The second term in P_ξ does not change the constrained set but largely simplifies the computation of the Hamiltonian vector fields and, consequently, of the Poisson brackets. Indeed, one could just consider $P_\xi = \int \iota_\xi e e F_\omega$ and perform a similar analysis to the one presented in [CS19c], where the variation $\delta\omega$ is subject to some constraint. Indeed, in section 4.3.3 we will show how to build a covariant expression for the BFV action (4.29), which does not require the choice of a reference connection ω_0.

We denote with L_ξ^ω the covariant Lie derivative along the odd vector field ξ with respect to a connection ω:

$$L_\xi^\omega A = \iota_\xi d_\omega A - d_\omega \iota_\xi A \qquad A \in \Omega_\partial^{i,j}.$$

Theorem 4.10. *Let g^∂ be non-degenerate on Σ. Then, the functions L_c, P_ξ, H_λ are well defined on F_{PC}^∂ and define a coisotropic submanifold with respect to the symplectic structure ϖ_{PC}. In particular they satisfy the following relations*

$$\{\mathrm{L}_c, \mathrm{L}_c\} = -\frac{1}{2}\mathrm{L}_{[c,c]} \qquad \{P_\xi, P_\xi\} = \tfrac{1}{2}P_{[\xi,\xi]} - \tfrac{1}{2}\mathrm{L}_{\iota_\xi \iota_\xi F_{\omega_0}} \tag{4.14a}$$

$$\{\mathrm{L}_c, P_\xi\} = \mathrm{L}_{L_\xi^{\omega_0} c} \quad \{\mathrm{L}_c, H_\lambda\} = -P_{X^{(a)}} + \mathrm{L}_{X^{(a)}(\omega - \omega_0)_a} - H_{X^{(n)}} \tag{4.14b}$$

$$\{H_\lambda, H_\lambda\} = 0 \qquad \{P_\xi, H_\lambda\} = P_{Y^{(a)}} - \mathrm{L}_{Y^{(a)}(\omega - \omega_0)_a} + H_{Y^{(n)}} \tag{4.14c}$$

where $X = [c, \lambda e_n]$, $Y = L_\xi^{\omega_0}(\lambda e_n)$ and $Z^{(a)}$, $Z^{(n)}$ are the components of $Z \in \{X, Y\}$ with respect to the frame (e_a, e_n).

Remark 4.11. This result improves on the results of [CS19c, Theorem 4.22], since the constraints are manifestly independent of representatives of an equivalence class $[\omega]$, and because it allows us to present more explicit expressions for the Poisson brackets of constraints. Theorem 4.10 holds verbatim for higher dimensional generalisations of the theory as well (see Section 4.4).

[8] These constraints are a slightly modified but equivalent version of those proposed in [CS19c], defined on the geometric phase space using the $\omega \in [\omega]$ defined in Theorem 4.5, hence satisfying (4.10).

Proof. The constraints are well defined on F^{∂}_{PC} because of the definition and properties of ω coming from Theorem 4.5.

In order to compute their Poisson brackets, we should first find their Hamiltonian vector fields. We begin by varying the constraints. The variation of ω is constrained by (4.10). However, since (4.10) imposes a constraint only on the part of ω in the kernel of $W_1^{\partial,(1,2)}$, it does not impose any condition on $e\delta\omega$. In the following computation we can always express the variation of the constraints in terms of $e\delta\omega$; hence the Hamiltonian vector fields are well defined and no other restriction has to be taken into account. The variations of L_c, P_ξ, H_λ are respectively:

$$\delta \mathrm{L}_c = \int \quad \frac{1}{2}c[\delta\omega, ee] + \frac{1}{2}cd_\omega\delta(ee) = \int [c,e]e\delta\omega + d_\omega c e\delta e;$$

$$\begin{aligned}
\delta P_\xi = \int \iota_\xi(e\delta e)F_\omega \quad & \frac{1}{2}\iota_\xi(ee)d_\omega\delta\omega + \iota_\xi\delta\omega e d_\omega e \quad \frac{1}{2}\iota_\xi(\omega \quad \omega_0)[\delta\omega, ee]\\
&+ \frac{1}{2}\iota_\xi(\omega \quad \omega_0)d_\omega\delta(ee)\\
= \int \quad & e\delta e\iota_\xi F_\omega + \frac{1}{2}d_\omega\iota_\xi(ee)\delta\omega \quad \frac{1}{2}\delta\omega\iota_\xi d_\omega(ee) + \frac{1}{2}\delta\omega[\iota_\xi(\omega \quad \omega_0), ee]\\
&+ \frac{1}{2}d_\omega\iota_\xi(\omega \quad \omega_0)\delta(ee)\\
= \int \quad & e\delta e\iota_\xi F_\omega \quad (L_\xi^\omega e)e\delta\omega + e\delta\omega[\iota_\xi(\omega \quad \omega_0), e] + d_\omega\iota_\xi(\omega \quad \omega_0)e\delta e\\
= \int \quad & e\delta e(L_\xi^{\omega_0}(\omega \quad \omega_0) + \iota_\xi F_{\omega_0}) \quad (L_\xi^{\omega_0}e)e\delta\omega;
\end{aligned}$$

$$\begin{aligned}
\delta H_\lambda &= \int \lambda e_n\delta e F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\delta e \quad \lambda e_n e d_\omega\delta\omega\\
&= \int \lambda e_n\delta e F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\delta e + d_\omega(\lambda e_n)e\delta\omega + \lambda e_n d_\omega e\delta\omega\\
&= \int \lambda e_n\delta e F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\delta e + d_\omega(\lambda e_n)e\delta\omega + \lambda\sigma e\delta\omega.
\end{aligned}$$

In the last computation we used (4.10) with $\sigma := W_1^{\partial,(1,1)} {}^{\ 1}(e_n d_\omega e)$. Hence, the components of the Hamiltonian vector fields of L_c and P_ξ are

$$\mathbb{L}_e = [c, e] \qquad \mathbb{L}_\omega = d_\omega c \tag{4.15}$$

$$\mathbb{P}_e = \quad L_\xi^{\omega_0}e \quad \mathbb{P}_\omega = \quad L_\xi^{\omega_0}(\omega \quad \omega_0) \quad \iota_\xi F_{\omega_0} \tag{4.16}$$

where, e.g., $\mathbb{L}_e \quad \mathbb{L}(e)$, with $\iota_{\mathbb{L}}\varpi_{PC} = \delta\mathrm{L}_c$. The components of the Hamiltonian vector field of H_λ are described by

$$\mathbb{H}_e = d_\omega(\lambda e_n) + \lambda\sigma \qquad e\mathbb{H}_\omega = \lambda e_n F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2. \tag{4.17}$$

The second equation, together with the requirement that $\mathbb{H}$ preserves the structural constraint (4.10), uniquely determines $\mathbb{H}_\omega$. However, we do not need an explicit expression for it, since in the computations we will only need $e\mathbb{H}_\omega$. We can now compute the brackets between the constraints:

$$
\begin{aligned}
f\mathrm{L}_c, \mathrm{L}_c g &= \int [c,e]ed_\omega c = \int \frac{1}{2}[c,ee]d_\omega c \\
&= \int \frac{1}{4}d_\omega[c,c]ee = \int \quad \frac{1}{2}[c,c]ed_\omega e = \quad \frac{1}{2}\mathrm{L}_{[c,c]};
\end{aligned}
$$

$$
\begin{aligned}
f\mathrm{L}_c, P_\xi g &= \int \quad [c,e]e(L_\xi^{\omega_0}(\omega \quad \omega_0) + \iota_\xi F_{\omega_0}) \quad d_\omega c e L_\xi^{\omega_0} e \\
&= \int \frac{1}{2}\left(L_\xi^{\omega_0} c[\omega \quad \omega_0, ee] + c[\omega \quad \omega_0, L_\xi^{\omega_0}(ee)] \quad c[ee, \iota_\xi F_{\omega_0}] \quad d_\omega L_\xi^{\omega_0}(ee)c\right) \\
&= \int \frac{1}{2}L_\xi^{\omega_0} c[\omega, ee] \quad \frac{1}{2}dc\iota_\xi d(ee) + \frac{1}{2}[\iota_\xi\omega_0, d(ee)]c \\
&= \int \frac{1}{2}L_\xi^{\omega_0} c d_\omega(ee) = \int L_\xi^{\omega_0} c e d_\omega e = \mathrm{L}_{L_\xi^{\omega_0} c};
\end{aligned}
$$

$$
\begin{aligned}
fP_\xi, P_\xi g &= \int \frac{1}{2}L_\xi^{\omega_0}(ee)L_\xi^{\omega_0}(\omega \quad \omega_0) + \frac{1}{2}L_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} \\
&\overset{\wr\sqcup}{=} \int \frac{1}{4}L_{[\xi,\xi]}^{\omega_0}(ee)(\omega \quad \omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, ee](\omega \quad \omega_0) + \frac{1}{2}L_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} \\
&= \int \frac{1}{4}\iota_{[\xi,\xi]}d_{\omega_0}(ee)(\omega \quad \omega_0) + \frac{1}{4}d_{\omega_0}\iota_{[\xi,\xi]}(ee)(\omega \quad \omega_0) \\
&\quad + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, ee](\omega \quad \omega_0) + \frac{1}{2}L_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} \\
&\overset{\wr}{=} \int \frac{1}{4}\iota_{[\xi,\xi]}d_\omega(ee)(\omega \quad \omega_0) \quad \frac{1}{4}\iota_{[\xi,\xi]}[\omega \quad \omega_0, ee](\omega \quad \omega_0) \\
&\quad + \frac{1}{4}\iota_{[\xi,\xi]}(ee)d_{\omega_0}(\omega \quad \omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, ee](\omega \quad \omega_0) + \frac{1}{2}L_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} \\
&\overset{\sim}{=} \int \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega \quad \omega_0) \quad \frac{1}{4}[\omega \quad \omega_0, ee]\iota_{[\xi,\xi]}(\omega \quad \omega_0) \quad \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_{\omega_0} \\
&\quad + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega \quad \frac{1}{8}\iota_{[\xi,\xi]}(ee)[\omega_0 \quad \omega, \omega_0 \quad \omega] \\
&\quad + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0}, ee](\omega \quad \omega_0) + \frac{1}{2}L_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} \\
&\doteq \int \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega \quad \omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega + \frac{1}{4}d_{\omega_0}(ee)\iota_\xi\iota_\xi F_{\omega_0} \\
&\quad + \frac{1}{2}d_{\omega_0}\iota_\xi(ee)\iota_\xi F_{\omega_0} \quad \frac{1}{4}\iota_\xi\iota_\xi F_{\omega_0}[\omega \quad \omega_0, ee] \\
&\quad + \frac{1}{2}\left(\iota_\xi d_{\omega_0}(ee) \quad d_{\omega_0}\iota_\xi(ee)\right)\iota_\xi F_{\omega_0} \\
&= \int \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega \quad \omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega \quad \frac{1}{4}d_\omega(ee)\iota_\xi\iota_\xi F_{\omega_0} \\
&= \frac{1}{2}P_{[\xi,\xi]} \quad \frac{1}{2}\mathrm{L}_{\iota_\xi\iota_\xi F_{\omega_0}}.
\end{aligned}
$$

$$\begin{aligned}
\{\mathrm{L}_c, H_\lambda\} &= \int [c,e]\lambda e_n F_\omega + \frac{1}{2}[c,e]\Lambda\lambda e_n e^2 + d_\omega c e(d_\omega(\lambda e_n) + \lambda\sigma) \\
&= \int [c,e]\lambda e_n F_\omega + \frac{1}{3!}[c,e^3]\Lambda\lambda e_n + d_\omega c d_\omega(\lambda e_n e) \\
&\overset{\}}{=} \int - [c,\lambda e_n] e F_\omega - \frac{1}{3!}\Lambda[c,\lambda e_n]e^3 \\
&= \int - [c,\lambda e_n]^{(a)} e_a e F_\omega - [c,\lambda e_n]^{(n)} e_n e F_\omega - \frac{1}{3!}\Lambda[c,\lambda e_n]^{(n)} e_n e^3 \\
&= - P_{[c,\lambda e_n]^{(a)}} + \mathrm{L}_{[c,\lambda e_n]^{(a)}(\omega - \omega_0)_a} - H_{[c,\lambda e_n]^{(n)}};
\end{aligned}$$

In these computations we used integration by parts ($\}$) and the following identities (for a proof of the second see [CS19b, Lemma 18]):

$$(\bullet) \qquad \frac{1}{2}\iota_{[\xi,\xi]}A = \frac{1}{2}\iota_\xi\iota_\xi d_{\omega_0}A + \iota_\xi d_{\omega_0}\iota_\xi A - \frac{1}{2}d_{\omega_0}\iota_\xi\iota_\xi A \qquad \forall A \in \Omega_\partial^{i,j}$$

$$(|) \qquad L_\xi^{\omega_0} L_\xi^{\omega_0} B = \frac{1}{2}L_{[\xi,\xi]}^{\omega_0} B + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, B] \qquad \forall B \in \Omega_\partial^{i,j}$$

$$(\sim) \qquad d_{\omega_0}(\omega_0 - \omega) = F_{\omega_0} - F_\omega + \frac{1}{2}[\omega_0 - \omega, \omega_0 - \omega].$$

Finally we have

$$\begin{aligned}
\{H_\lambda, H_\lambda\} &= \int (d_\omega(\lambda e_n) + \lambda\sigma)\left(\lambda e_n F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\right) = \\
&= \int d_\omega \lambda e_n \left(\lambda e_n F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\right) - \lambda d_\omega e_n \left(\lambda e_n F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\right) = 0,
\end{aligned}$$

since $\lambda\lambda = 0$ and $e_n e_n = 0$, and

$$\begin{aligned}
\{P_\xi, H_\lambda\} = \int & - L_\xi^{\omega_0} e\lambda e_n F_\omega - \frac{1}{2}\Lambda L_\xi^{\omega_0} e\lambda e_n e^2 \\
& - \left(L_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}\right) e(d_\omega(\lambda e_n) + \lambda\sigma) \\
= \int & - L_\xi^{\omega_0} e\lambda e_n F_\omega - \frac{1}{3!}\Lambda L_\xi^{\omega_0} e^3 \lambda e_n \\
& - \left(L_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}\right) d_\omega(e\lambda e_n) \\
= \int & e L_\xi^{\omega_0}(\lambda e_n) F_\omega + \frac{1}{3!}\Lambda e^3 L_\xi^{\omega_0}(\lambda e_n) + e\lambda e_n L_\xi^{\omega_0} F_\omega \\
& + (d_\omega \iota_\xi(\omega - \omega_0) - \iota_\xi F_\omega)\, d_\omega(e\lambda e_n) \\
= \int & e L_\xi^{\omega_0}(\lambda e_n) F_\omega + \frac{1}{3!}\Lambda e^3 L_\xi^{\omega_0}(\lambda e_n) + e\lambda e_n L_\xi^{\omega_0} F_\omega \\
& - [F_\omega, \iota_\xi(\omega - \omega_0)] e\lambda e_n - L_\xi^\omega F_\omega e\lambda e_n \\
= \int & L_\xi^{\omega_0}(\lambda e_n) e F_\omega + \frac{1}{3!}\Lambda e^3 L_\xi^{\omega_0}(\lambda e_n) \\
= P&_{L_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{L_\xi^{\omega_0}(\lambda e_n)^{(n)}} - \mathrm{L}_{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega - \omega_0)_a},
\end{aligned}$$

where we used that $L_\xi^{\omega_0} F_\omega - L_\xi^{\omega} F_\omega = [\iota_\xi(\omega_0 - \omega), F_\omega]$. This shows that the relations (4.14) hold and, therefore, that the constraints are first class.

□

Remark 4.12. Theorem 4.10, in particular, shows that on time-like or space-like boundaries the constraints (4.13) are first class. Counting the number of components of the Lagrange multipliers c, ξ and λ we deduce that there are 10 local constraints, while the number of independent components of the conjugate fields e and ω is 12. Hence we recover the classical result of having 2 local physical degrees of freedom.

Remark 4.13. From the expressions of the Hamiltonian vector fields of the constraints (4.15), (4.16) and (4.17) we deduce that the constraints L_c parametrize the action of the gauge transformations of the theory, while the constraints P_ξ and H_λ parametrize the action of the diffeomorphisms, respectively tangent and transversal to the boundary.

4.3 Palatini–Cartan theory and its BFV data

This section is not required by Section 4.4.1 and can therefore be skipped by readers that are not interested in the BFV formalism but only wish to see the higher dimensional generalization of the construction of the reduced phase space. It will be however required by Section 4.4.2, where we will discuss the higher dimensional version of the BFV formalism.

4.3.1 From the reduced phase space to BFV

We start with a short overview of the construction that allows to pass from the KT description of the reduced phase space to the BFV version of the theory. The starting problem is the symplectic reduction of a coisotropic submanifold. For simplicity, as this is also the case at hand in this chapter, we will consider only the situation where the submanifold is defined in terms of global constraints (implicit function theorem). More precisely, the starting point are a symplectic manifold (M, ϖ) and a collection $\{\phi_i\}$ of independent, differentially independent constraints; their common zero locus $C = \{x \in M : \phi_i(x) = 0\ \forall i\}$ is then a submanifold.[9] In addition, the constraints are assumed to be of first class: i.e., their Poisson brackets vanish on C, or, equivalently, they satisfy

$$\{\phi_i, \phi_j\} = f_{ij}^k \phi_k, \tag{4.18}$$

where the $\{f_{ij}^k\}$s are functions on M (we assume a sum over repeated indices). The restriction of ϖ to C becomes degenerate, but one can easily show that its kernel, called the characteristic distribution, is spanned by the Hamiltonian vector fields X_i of the constraints ϕ_i. As a consequence of (4.18), the characteristic distribution is involutive. The symplectic reduction $\underline{C}$ of C is the quotient, which we temporarily assume to be smooth, of C by its characteristic distribution, endowed with the unique symplectic form $\underline{\varpi}$ whose pullback to C is the restriction of ϖ.

Since $\underline{C}$ is very often not smooth in applications, it is better to resort to a different, more flexible description. The first attempt is to work in terms of algebras of functions. We have that $C^\infty(\underline{C}) = C^\infty(C)^{\mathsf{inv}}$, where inv means invariant under the vector fields X_i. In turn, $C^\infty(C) = C^\infty(M)/I$, where $I = \mathrm{span}_{C^\infty(M)}\{\phi_i\}$ is the vanishing ideal of C. Therefore, we have $C^\infty(\underline{C}) = (C^\infty(M)/I)^{\mathsf{inv}}$. One can show that this algebra inherits a Poisson bracket which, in the smooth case, is also the one induced by $\underline{\varpi}$. The Poisson algebra $(C^\infty(M)/I)^{\mathsf{inv}}$ is defined also if $\underline{C}$ is

[9] For notational simplicity, we assume here a discrete family of constraints, even though in the case of field theory we will need a continuos family. In that case the sums will be replaced by integrals.

not smooth and it may be tempting to take it as a good replacement for $\underline{C}$. The problem is that often this algebra is very poor (for example, if $\underline{C}$ is not Hausdorff, this algebra is just $\mathbb{R}$).

A better way to proceed is to look for a cohomological description of the symplectic quotient. This is what is achieved by the BFV formalism. Namely, one first adds new odd variables c^i of degree (ghost number) $+1$, called the ghosts, one for each constraint, and their momenta $c_i^\dagger$ (a.k.a. the antighosts),which are also odd and have degree -1. One extends the original symplectic manifold (M, ϖ) to a graded symplectic manifold $M \times T^*W$, where W is the odd vector space whose coordinates are the c^is, with symplectic form

$$\varpi + \int \delta c_i^\dagger \, \delta c^i.$$

Next one introduces the BFV action, an odd function of degree 1,

$$S = \int c^i \phi_i + \frac{1}{2} f_{ij}^k c_k^\dagger c^i c^j + R,$$

where R is a function of higher degree in the ghost momenta $c_i^\dagger$ such that $\{S, S\} = 0$ (the BFV master equation). It has been proved [BV81; BF83; Sta97] that one can always find such a correction R. The Hamiltonian vector field Q of S is odd, of degree 1, and satisfies $[Q, Q] = 0$ (such a vector field is called cohomological because it acts as a differential on the algebra of functions). We have

$$\begin{aligned} Q c_i^\dagger &= \phi_i + \cdots, \\ Q f &= c^i X_i(f) + \cdots, \end{aligned}$$

where f is a function on M and $\cdots$ denotes terms depending on the ghost momenta. From this we see that, up to these higher terms, the image of Q contains the vanishing ideal I and the kernel of Q selects the invariant functions. One can actually show [BV81; BF83; Sta97] that in degree zero there is not more than this: The cohomology of Q in degree zero is isomorphic to $(C^\infty(M)/I)^{\mathsf{inv}}$ as a Poisson algebra. The idea of the BFV formalism is then to replace the original, possibly singular symplectic reduction with the "BFV manifold"

$$(M \times T^*W, \varpi + \int \delta c_i^\dagger \, \delta c^i, S).$$

The complex $(C^\infty(M \times T^*W), Q)$ is the sought for cohomological resolution of the symplectic reduction of C.[10]

Note that there is some freedom in the construction of the BFV data, but one can show [Sta97] that the solution is unique up to symplectomorphisms compatible with the BFV actions. A particularly good solution is when the correction term R vanishes. This is not always possible, but it is so in some cases. The most important one is when one can choose the constraints in such a way that the $\{f_{ij}^k\}$s are constant (this means that the constraints are assembled into an equivariant momentum map and that the reduction is actually an example of Marsden–Weinstein reduction). In this case the BFV construction goes often under the name of BRS [KS87].

It may however happen that the correction R vanishes beyond the BRS case. We will see that this is actually what occurs in the PC case at hand. A similar phenomenon was observed in the BFV treatment of Einstein–Hilbert gravity [CS16].

[10] Indeed, Q is a deformation, compatible with the symplectic structure, of a combination of the Koszul–Tate complex, which gives a cohomological resolution of C as a submanifold of M, and of the Chevalley–Eilenberg complex of the Lie algebroid naturally associated to the conormal bundle of C.

4.3.2 BFV Structure of Palatini–Cartan theory

From the constraints and their brackets it is possible extend the space of fields to a graded symplectic manifold by promoting the Lagrange multipliers to ghosts and adding ghost momenta. The following Theorem 4.15 shows that the naïve guess for a BFV action, containing only the constraints (constant term in the ghost momenta) and the information on their Poisson brackets (linear term in the ghost momenta) already satisfies the BFV master equation.

Remark 4.14. At a physical level, the Lagrange multipliers assume the meaning of symmetry generators of the system. In particular the field $c \in \Omega_\partial^{0,2}[1]$ represents the internal gauge symmetry (recall that we are using the identification $\mathfrak{so}(3,1) = \wedge^2 V$); the vector fields $\xi \in \mathfrak{X}[1](\Sigma)$ represent the vector fields parametrizing local diffeomorphisms tangent to the boundary; the scalar field $\lambda \in C^\infty[1](\Sigma)$ might in turn be thought of as the parameter representing the local diffeomorphisms in the transversal direction. This becomes evident when considering the classical part of the cohomological vector field Q (see Equation (4.24), below).

Theorem 4.15. *Let g^∂ be non-degenerate on Σ. Let F^∂ be the bundle*

$$F^\partial \to \Omega^1_{nd}(\Sigma, V), \tag{4.19}$$

with local trivialisation on an open $U \subset \Omega^1_{nd}(\Sigma, V)$

$$F^\partial \simeq U \times \mathcal{A}^{red}(\Sigma) \times T^*\left(\Omega_\partial^{0,2}[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma)\right), \tag{4.20}$$

and fields denoted by $e \in U$ and $\omega \in \mathcal{A}^{red}(\Sigma)$ in degree zero, $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega^{0,0}[1]$ in degree one, $c^\dagger \in \Omega_\partial^{3,2}[-1]$, $\lambda^\dagger \in \Omega_\partial^{3,4}[-1]$ and $\xi^\dagger \in \Omega_\partial^{1,0}[-1] \otimes \Omega_\partial^{3,4}$ in degree minus one, together with a fixed $e_n \in \Gamma(V)$, completing the image of elements $e \in U$ to a basis of V; define a symplectic form and an action functional on F respectively by

$$\varpi^\partial = \int e\delta e\delta\omega + \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger, \tag{4.21}$$

$$\begin{aligned} S^\partial = \int & ced_\omega e + \iota_\xi e e F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e + \lambda e_n\left(eF_\omega + \frac{1}{3!}\Lambda e^3\right) + \frac{1}{2}[c,c]c^\dagger \\ & - L_\xi^{\omega_0}cc^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}c^\dagger + [c,\lambda e_n]^{(a)}(\xi_a^\dagger - (\omega - \omega_0)_a c^\dagger) + [c,\lambda e_n]^{(n)}\lambda^\dagger \\ & - L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\xi_a^\dagger - (\omega - \omega_0)_a c^\dagger) - L_\xi^{\omega_0}(\lambda e_n)^{(n)}\lambda^\dagger + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger \end{aligned} \tag{4.22}$$

where e and ω satisfy the additional requirement $e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}$. Furthermore, let Q^∂ be the Hamiltonian vector field of S^∂ with respect to ϖ^∂. Then the quadruple $(F^\partial, \varpi^\partial, S^\partial, Q^\partial)$ defines a BFV structure on Σ.

Proof. We have to prove that the action S^∂ satisfies the classical master equation. By definition we have

$$\{S^\partial, S^\partial\} = \iota_{Q^\partial}\iota_{Q^\partial}\varpi^\partial.$$

In order to simplify the computation we can divide the action in two parts:

$$S^\partial = S_0 + S_1$$

where S_0 is independent of the ghost momenta and S_1 is linear in them. In particular S_0 is the sum of the constraints and S_1 is everything else. We divide the symplectic form too:

$$\varpi^\partial = \varpi_f + \varpi_g$$

where $\varpi_f = \int \quad e\delta e\delta\omega$ is the classical part and $\omega_g = \int \quad \delta c\delta c^y + \delta\lambda\delta\lambda^y + \iota_{\delta\xi}\delta\xi^y$ is the ghost part. Finally, we define Q_0 to be the part of Q^∂ satisfying $\iota_{Q_0}\varpi^\partial = \delta S_0$ and Q_1 to be the one satisfying $\iota_{Q_1}\varpi^\partial = \delta S_1$.

We can divide the master equation into the corresponding parts:

$$f S^\partial, S^\partial g = f S_0, S_0 g_f + 2 f S_0, S_1 g_f + 2 f S_0, S_1 g_g + f S_1, S_1 g_f + f S_1, S_1 g_g$$

where

$$f S_0, S_0 g_f = \iota_{Q_0}\iota_{Q_0}\varpi_f \quad f S_0, S_1 g_f = \iota_{Q_0}\iota_{Q_1}\varpi_f \tag{4.23a}$$
$$f S_0, S_0 g_g = \iota_{Q_0}\iota_{Q_0}\varpi_g \quad f S_0, S_1 g_g = \iota_{Q_0}\iota_{Q_1}\varpi_g \tag{4.23b}$$
$$f S_1, S_1 g_f = \iota_{Q_1}\iota_{Q_1}\varpi_f \quad f S_1, S_1 g_g = \iota_{Q_1}\iota_{Q_1}\varpi_g. \tag{4.23c}$$

This subdivision is particularly convenient, since we can exploit some properties of the action and prove the master equation piecewise. We first note that $f S_0, S_0 g_g = 0$ since S_0 has no antighost part. Furthermore, by Theorem 4.10 we have that

$$f S_0, S_0 g_f + 2 f S_0, S_1 g_g = 0.$$

The terms $f S_0, S_1 g_f$ and $f S_1, S_1 g_g$ are linear in the antighost while $f S_1, S_1 g_f$ is quadratic in the antighost. Hence we should prove separately that $2 f S_0, S_1 g_f + f S_1, S_1 g_g = 0$ and $f S_1, S_1 g_f = 0$. For these last two terms we have to do the computation explicitly. We start by computing δS^∂ in order to get Q^∂ from the equation $\iota_{Q^\partial}\varpi^\partial = \delta S^\partial$.

Note that for X odd, since $\delta e_n = 0$, we have

$$\begin{aligned} \delta X = \delta(X^{(\mu)}e_\mu) &= \delta(X^{(\mu)})e_\mu + X^{(a)}\delta(e_a) \\ \delta(X^{(\mu)}) &= (\delta X)^{(\mu)} \quad X^{(a)}\delta(e_a)^{(\mu)}. \end{aligned}$$

The variation of the action is

$$\begin{aligned} \delta S^\partial = \int & \delta c e d_\omega e \quad \frac{1}{2}c[\delta\omega, ee] + d_\omega c e\delta e + \frac{1}{2}\iota_{\delta\xi}(ee)F_\omega + \iota_{\delta\xi}(\omega \quad \omega_0)ed_\omega e \\ & e\delta e(L_\xi^{\omega_0}(\omega \quad \omega_0) + \iota_\xi F_{\omega_0}) \quad (L_\xi^{\omega_0}e)e\delta\omega + \delta\lambda e_n eF_\omega + \lambda e_n\delta eF_\omega + \\ & \frac{1}{3!}\Lambda\delta\lambda e_n e^3 + \frac{1}{2}\Lambda\lambda e_n e^2\delta e + d_\omega(\lambda e_n)e\delta\omega + \lambda\sigma e\delta\omega + [\delta c, c]c^y + \frac{1}{2}[c,c]\delta c^y \\ & \iota_{\delta\xi}d_{\omega_0}cc^y + \delta c d_{\omega_0}\iota_\xi c^y \quad L_\xi^{\omega_0}c\delta c^y + \iota_{\delta\xi}\iota_\xi F_{\omega_0}c^y + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}\delta c^y \\ & + \Big([\delta c, \lambda e_n]^{(a)} \quad [c, \delta\lambda e_n]^{(a)} \quad [c, \lambda e_n]^{(b)}\delta e_b^{(a)}\Big)(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \\ & + [c, \lambda e_n]^{(a)}(\delta\xi_a^y \quad \delta(\omega \quad \omega_0)_a c^y \quad (\omega \quad \omega_0)_a\delta c^y) + [\delta c, \lambda e_n]^{(n)}\lambda^y \\ & [c, \delta\lambda e_n]^{(n)}\lambda^y \quad [c, \lambda e_n]^{(b)}\delta e_b^{(n)}\lambda^y + [c, \lambda e_n]^{(n)}\delta\lambda^y \\ & \Big(\quad (\iota_{\delta\xi}d_{\omega_0}(\lambda e_n))^{(a)} + L_\xi^{\omega_0}(\delta\lambda e_n)^{(a)} + L_\xi^{\omega_0}(\lambda e_n)^{(b)}\delta e_b^{(a)}\Big)(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \\ & L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\delta\xi_a^y \quad \delta(\omega \quad \omega_0)_a c^y \quad (\omega \quad \omega_0)_a\delta c^y) \\ & (\iota_{\delta\xi}d_{\omega_0}(\lambda e_n))^{(n)}\lambda^y + L_\xi^{\omega_0}(\delta\lambda e_n)^{(n)}\lambda^y + L_\xi^{\omega_0}(\lambda e_n)^{(b)}\delta e_b^{(n)}\lambda^y \\ & L_\xi^{\omega_0}(\lambda e_n)^{(n)}\delta\lambda^y \quad \delta\xi^a(\partial_a\xi^b)\xi_b^y \quad \delta\xi^a\partial_b(\xi^b\xi_a^y) \quad \xi^a(\partial_a\xi^b)\delta\xi_b^y. \end{aligned}$$

This variation contains all the information necessary to construct the cohomological vector field Q^∂ (from now on we will drop the ∂ superscript). However δS^∂ contains some variation of $\delta\omega$ that are constrained by (4.10) and some other terms of difficult explicit inversion. For our purposes it is sufficient to have the explicit expressions of $Q_{0e}, Q_{0\omega}, Q_c, Q_\lambda, Q_\xi$ and some information about $Q_{1e}, Q_{1\omega}$ (recall that $Q_{0e}, Q_{0\omega}$ are the part of Q_e, Q_ω not containing antighosts, while $Q_{1e}, Q_{1\omega}$ contain everything else).

Let us start from $Q_{1e}, Q_{1\omega}$. They are defined through the equation

$$\begin{aligned}\iota_{Q_1}(e\delta e\delta\omega) = \quad & [c, \lambda e_n]^{(b)}\delta e_b^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \quad [c, \lambda e_n]^{(a)}\delta(\omega \quad \omega_0)_a c^y \\ & [c, \lambda e_n]^{(b)}\delta e_b^{(n)}\lambda^y + L_\xi^{\omega_0}(\lambda e_n)^{(b)}\delta e_b^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \\ & + L_\xi^{\omega_0}(\lambda e_n)^{(a)}\delta(\omega \quad \omega_0)_a c^y + L_\xi^{\omega_0}(\lambda e_n)^{(b)}\delta e_b^{(n)}\lambda^y\end{aligned}$$

Since λ is a scalar function we have that $[c, \lambda e_n]^{(a)} = \lambda[c, e_n]^{(a)}$ and $L_\xi^{\omega_0}(\lambda e_n)^{(a)} = L_\xi^{\omega_0}(\lambda)e_n^{(a)}$ $\lambda L_\xi^{\omega_0}(e_n)^{(a)} = \quad \lambda L_\xi^{\omega_0}(e_n)^{(a)}$, since $e_n^{(a)} = 0$. We then deduce that every term in Q_{1e} and $Q_{1\omega}$ must be linear in λ. From (4.23) we have

$$f S_1, S_1 g_f = \iota_{Q_1}\iota_{Q_1}(e\delta e\delta\omega) = 2eQ_{1e}Q_{1\omega}$$

which contains only terms proportional to $\lambda^2 = 0$ since λ is an odd scalar function. This proves $f S_1, S_1 g_f = 0$.

From the above variation of S^∂ we can compute directly $Q_{0e}, Q_{0\omega}, Q_c, Q_\lambda, Q_\xi$:

$$\begin{aligned}Q_0 e &= [c, e] \quad L_\xi^{\omega_0}e + d_\omega(\lambda e_n) + \lambda\sigma \\ Q_0\omega &= d_\omega c \quad L_\xi^{\omega_0}(\omega \quad \omega_0) \quad \iota_\xi F_{\omega_0} + W_1^{\ 1}(\lambda e_n F_\omega) + \frac{1}{2}\Lambda\lambda e_n e\end{aligned} \tag{4.24}$$

$$\begin{aligned}Qc &= \frac{1}{2}[c, c] \quad L_\xi^{\omega_0}c + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0} \quad \left([c, \lambda e_n]^{(a)} \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)}\right)(\omega \quad \omega_0)_a \\ Q\lambda &= [c, \lambda e_n]^{(n)} \quad L_\xi^{\omega_0}(\lambda e_n)^{(n)} \\ Q_{\xi^a} &= [c, \lambda e_n]^{(a)} \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)} \quad \frac{1}{2}[\xi, \xi]^a\end{aligned} \tag{4.25}$$

where $W_1^{\ 1}(\lambda e_n F_\omega)$ is defined as in (4.17). The proof of $2f S_0, S_1 g_f + f S_1, S_1 g_g = 0$ is a lengthy computation fully detailed in Appendix 4.A. □

Remark 4.16. By setting $\lambda = 0$, we can read the action of Q^∂ on c and ξ as (a splitting by ω_0) of the Atiyah algebroid structure on $TP/O(N \quad 1, 1)$ [Mac87], where P is the orthogonal frame bundle of M restricted to Σ.

4.3.3 Alternative variables

The ξ-dependent part of S^∂ in (4.22) contains, in accordance with (4.13), a repetition of the invariant constraint $ed_\omega e = 0$ which we have added to simplify the computations. This term may actually be removed by using the following symplectomorphism (cf. with [CS19b]):

$$c^0 = c + \iota_\xi(\omega \quad \omega_0) \qquad \xi_a^{0y} = \xi_a^y \quad (\omega \quad \omega_0)_a c^y$$

The resulting expressions of the action and symplectic form are:

$$S^{\partial} = \int c' e d_\omega e + \iota_\xi e e F_\omega + \lambda e_n \left(e F_\omega + \frac{1}{3!}\Lambda e^3 \right) + \frac{1}{2}[c', c']c^\dagger - L_\xi^\omega c' c^\dagger + \frac{1}{2}\iota_\xi \iota_\xi F_\omega c^\dagger$$
$$+ [c', \lambda e_n]^{(a)} \xi_a^{\prime\dagger} + [c', \lambda e_n]^{(n)} \lambda^\dagger - L_\xi^\omega (\lambda e_n)^{(a)} \xi_a^{\prime\dagger}$$
$$- L_\xi^\omega (\lambda e_n)^{(n)} \lambda^\dagger + \frac{1}{2}\iota_{[\xi,\xi]} \xi^{\prime\dagger}, \tag{4.26}$$

$$\varpi^{\partial} = \int e \delta e \delta \omega + \delta c' \delta c^\dagger + \delta \omega \delta(\iota_\xi c^\dagger) + \delta \lambda \delta \lambda^\dagger + \iota_{\delta\xi} \delta \xi^{\dagger\prime}. \tag{4.27}$$

Note that the price for the simplication of the action is that *primed* chart is no longer a Darboux chart. We can further transform (4.26) and (4.27) in order to avoid using components. Since $\lambda^\dagger$ and $\xi^{\dagger\prime}$ both take value in $\wedge^4 V$ we can write them in terms of the basis (e_a, e_n):

$$\lambda^\dagger = \lambda^{\dagger(123n)} e_1 \wedge e_2 \wedge e_3 \wedge e_n;$$
$$\xi_a^{\dagger\prime} dx^a = \xi_a^{\dagger\prime(123n)} dx^a e_1 \wedge e_2 \wedge e_3 \wedge e_n, \; a = 1,2,3.$$

Now define the following fields:

$$x_a^{a\dagger} dx^a := \xi_a^{\dagger\prime(123n)} dx^a e_b \wedge e_c \wedge e_n \quad a,b,c \in \{1,2,3\}, \quad b,c \neq a$$
$$l^\dagger := \lambda^{\dagger(123n)} e_1 \wedge e_2 \wedge e_3 \qquad y^\dagger := l^\dagger + \sum_{a=1}^{3} (-1)^a x_a^{a\dagger}.$$

Multiplying $y^\dagger$ by e_a and e_n gives back the original fields $\lambda^\dagger$ and $\xi^{\dagger\prime}$: $e_n y^\dagger = -\lambda^\dagger$, $e_a y^\dagger = -\xi_a^{\dagger\prime}$. Using these properties it is easy to show that we can express the action S^∂ and the symplectic form ϖ^∂ on the new space of fields given by the bundle

$$F'^{\partial} \to \Omega_{nd}^1(\Sigma, V), \tag{4.28}$$

with local trivialisation on an open $U \subset \Omega_{nd}^1(\Sigma, V)$

$$F^\partial \simeq U \times A^{red}(\Sigma) \times \left(\Omega_{4,\partial}^{0,2}[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma) \right)$$
$$\times \Omega_\partial^{3,2}[-1] \times \Omega_\partial^{3,3}[-1],$$

where all the fields are denoted as in Theorem 4.15 but $y^\dagger \in \Omega_\partial^{3,3}[-1]$:

$$S^{\partial} = \int c' e d_\omega e + \iota_\xi e e F_\omega + \lambda e_n \left(e F_\omega + \frac{1}{3!}\Lambda e^3 \right) + \frac{1}{2}[c', c']c^\dagger - L_\xi^\omega c' c^\dagger + \frac{1}{2}\iota_\xi \iota_\xi F_\omega c^\dagger$$
$$- [c', \lambda e_n] y^\dagger + L_\xi^\omega (\lambda e_n) y^\dagger + \frac{1}{2}\iota_{[\xi,\xi]} e y^\dagger, \tag{4.29}$$

$$\varpi^{\partial} = \int e \delta e \delta \omega + \delta c' \delta c^\dagger + \delta \omega \delta(\iota_\xi c^\dagger) - \delta \lambda e_n \delta y^\dagger + \iota_{\delta\xi} \delta(e y^\dagger). \tag{4.30}$$

It is a simple computation to show that this two form is actually non-degenerate.

Remark 4.17. Equation (4.29) is again a covariant version of the BFV action functional. Moreover, it has the advantage of not including the implicit terms of (4.26) and satisfies by construction the classical master equation. It hence provides a good starting point for the AKSZ construction and the analysis of higher codimension strata, carried out in Chapter 6 and 7 respectively.

4.4 Generalization to $\dim(M) > 4$

In this section we generalize the results of the previous sections to dimensions $N = \dim(M) > 4$. The construction is substantially unchanged while a few details have to be fixed. We recall the main steps and adapt them to the generalization.

4.4.1 Extension of the reduced phase space to higher dimensions

The classical fields of the theory are as in the $N = 4$ case: a non-degenerate coframe $e \in \Omega_\partial^{1,1}$ restricted to the boundary and an equivalence class of connections $[\omega] \in \mathcal{A}^{red}(\Sigma)$ where $\mathcal{A}^{red}(\Sigma)$ is the quotient under $\omega \mapsto \omega + v$ for v such that $e^{N-3} \wedge v = 0$. The symplectic structure of the geometric phase space is given by

$$\varpi = \int e^{N-3}\delta e \delta[\omega]. \tag{4.31}$$

Let now e_n be a fixed section of $\mathcal{V}$ completing the image of $e : T\Sigma \to \mathcal{V}$ to a basis of $\mathcal{V}$. The structural constraint is

$$e_n e^{N-4} d_\omega e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}. \tag{4.32}$$

Theorem 4.18. *Suppose that the boundary metric g^∂ is non-degenerate. Given any $\widetilde{\omega} \in \Omega(\Sigma, \wedge^2\mathcal{V})$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v \tag{4.33}$$

with ω and v satisfying

$$e^{N-3} v = 0 \quad \text{and} \quad e_n e^{N-4} d_\omega e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}. \tag{4.34}$$

Proof. Let $\widetilde{\omega} \in \Omega(\Sigma, \wedge^2\mathcal{V})$. From Lemma 3.17 we deduce that there exist unique $\sigma \in \Omega(\Sigma, \mathcal{V})$ and $v \in \operatorname{Ker} W_{N-3}^{\partial,(1,2)}$ such that

$$e_n e^{N-4} d_{\widetilde{\omega}} e = e^{N-3}\sigma + e_n e^{N-4}[v, e].$$

We define $\omega := \widetilde{\omega} - v$. Then ω and v satisfy (4.11) and (4.12).

To prove uniqueness, suppose that $\widetilde{\omega} = \omega_1 + v_1 = \omega_2 + v_2$ with $e^{N-3} v_i = 0$ and $e_n e^{N-4} d_{\omega_i} e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}$ for $i = 1, 2$. Hence

$$e_n e^{N-4} d_{\omega_1} e - e_n e^{N-4} d_{\omega_2} e = e_n e^{N-4}[v_2 - v_1, e] \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}.$$

Hence from Lemma 3.14 and 3.17, we deduce $v_2 - v_1 = 0$, since $v_2 - v_1 \in Ker W_{N-3}^{\partial,(1,2)}$. □

Corollary 4.19. *The field ω in the decomposition* (4.11) *depends only on the equivalence class $[\omega] \in \mathcal{A}^{red}(\Sigma)$.*

Proof. Let $\widetilde{\omega}_1, \widetilde{\omega}_2 \in [\omega]$. Hence $\widetilde{\omega}_1 - \widetilde{\omega}_2 = \tilde{v} \in \operatorname{Ker} W_{N-3}^{\partial,(1,2)}$. Applying Theorem 4.18 we get ω_1, v_1, ω_2, v_2 such that $v_1, v_2 \in \operatorname{Ker} W_{N-3}^{\partial,(1,2)}$ and

$$\begin{aligned} \widetilde{\omega}_1 &= \omega_1 + v_1 \qquad e_n e^{N-4} d_{\omega_1} e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)} \\ \widetilde{\omega}_2 &= \omega_2 + v_2 \qquad e_n e^{N-4} d_{\omega_2} e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}. \end{aligned}$$

Subtracting these equations we get $\omega_2 - \omega_1 = v_1 - v_2 - \tilde{v} \in \operatorname{Ker} W_{N-3}^{\partial,(1,2)}$ and $e_n e^{N-4}[\omega_1 - \omega_2, e] \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}$. Hence, from (3.7), we deduce $\omega_1 = \omega_2$. □

As for $N = 4$, we consider the following constraints defined on F^{∂}_{PC} using $\omega \in [\omega]$ defined in Theorem 4.18, hence satisfying (4.32):

$$L_c = \int c e^{N-3} d_\omega e \tag{4.35a}$$

$$P_\xi = \int \iota_\xi e e^{N-3} F_\omega + \iota_\xi(\omega - \omega_0) e^{N-3} d_\omega e \tag{4.35b}$$

$$H_\lambda = \int \lambda e_n \left(e^{N-3} F_\omega + \frac{1}{(N-1)!} \Lambda e^{N-1} \right), \tag{4.35c}$$

and Theorem 4.10 holds verbatim for these constraints too.

4.4.2 Extension of BFV data to higher dimensions

Since the the brackets between the constraints are the same as in the $N = 4$ case (Theorem 4.10), the BFV action will have a similar expression too. For reference purposes, below we write the general version of Theorem 4.15:

Theorem 4.20. *Let g^∂ be non-degenerate on Σ. Let F be the bundle*

$$F^\partial \longrightarrow \Omega^1_{nd}(\Sigma, V), \tag{4.36}$$

with local trivialisation on an open $U \subset \Omega^1_{nd}(\Sigma, V)$

$$F^\partial \simeq U \times A^{red}(\Sigma) \times T^* \left(\Omega^{0,2}_\partial[1] \oplus \mathfrak{X}[1](\Sigma) \oplus \Omega^{0,0}_\partial[1] \right), \tag{4.37}$$

and fields denoted by $e \in U$ and $\omega \in A^{red}(\Sigma)$ in degree zero, $c \in \Omega^{0,2}_\partial[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega^{0,0}[1]$ in degree one, $c^\dagger \in \Omega^{N-1,N-2}_\partial[-1]$, $\lambda^\dagger \in \Omega^{N-1,N}_\partial[-1]$ and $\xi^\dagger \in \Omega^{1,0}[-1] \otimes \Omega^{N-1,N}_\partial$ in degree minus one, together with a fixed $e_n \in \Gamma(V)$, completing the image of elements $e \in U$ to a basis of V; define a symplectic form and an action functional on F^∂ respectively by

$$\varpi^\partial = \int e^{N-3} \delta e \delta \omega + \delta c \delta c^\dagger + \delta \lambda \delta \lambda^\dagger + \iota_{\delta \xi} \delta \xi^\dagger, \tag{4.38}$$

$$\begin{aligned} S^\partial = \int & c e^{N-3} d_\omega e + \iota_\xi e e^{N-3} F_\omega + \iota_\xi(\omega - \omega_0) e^{N-3} d_\omega e + \lambda e_n e^{N-3} F_\omega \\ & + \frac{1}{(N-1)!} \Lambda \lambda e_n e^{N-1} + \frac{1}{2}[c,c] c^\dagger - L^{\omega_0}_\xi c c^\dagger + \frac{1}{2} \iota_\xi \iota_\xi F_{\omega_0} c^\dagger + [c, \lambda e_n]^{(n)} \lambda^\dagger \\ & + [c, \lambda e_n]^{(a)} (\xi^\dagger_a - (\omega - \omega_0)_a c^\dagger) - L^{\omega_0}_\xi (\lambda e_n)^{(a)} (\xi^\dagger_a - (\omega - \omega_0)_a c^\dagger) \\ & - L^{\omega_0}_\xi (\lambda e_n)^{(n)} \lambda^\dagger - \frac{1}{2} \iota_{[\xi,\xi]} \xi^\dagger \end{aligned} \tag{4.39}$$

where ω satisfies the additional requirement $e_n e^{N-4} d_\omega e \in \operatorname{Im} W^{\partial,(1,1)}_{N-3}$. Furthermore, let Q^∂ be the Hamiltonian vector field of S^∂ with respect to ϖ^∂. Then the quadruple $(F^\partial, \varpi^\partial, S^\partial, Q^\partial)$ forms a BFV structure on Σ.

The covariant version of the action and symplectic form are

$$\begin{aligned} S^\partial = \int & c' e^{N-3} d_\omega e + \iota_\xi e e^{N-3} F_\omega + \lambda e_n \left(e^{N-3} F_\omega + \frac{1}{(N-1)!} \Lambda e^{N-1} \right) + \frac{1}{2}[c', c'] c^\dagger \\ & - L^\omega_\xi c' c^\dagger + \frac{1}{2} \iota_\xi \iota_\xi F_\omega c^\dagger + [c', \lambda e_n] y^\dagger - L^\omega_\xi (\lambda e_n) y^\dagger - \frac{1}{2} \iota_{[\xi,\xi]} e y^\dagger, \end{aligned} \tag{4.40}$$

$$\varpi^\partial = \int e^{N-3} \delta e \delta \omega + \delta c' \delta c^\dagger - \delta \omega \delta (\iota_\xi c^\dagger) + \delta \lambda e_n \delta y^\dagger + \iota_{\delta \xi} \delta (e y^\dagger). \tag{4.41}$$

Appendix

4.A Lengthy proofs of Section 4.3

We complete the proof of Theorem 4.15. Namely we prove here explicitly that $2 f S_0, S_1 g_f + f S_1, S_1 g_g = 0$. From the expression of Q and $f S_0, S_1 g_f = \iota_{Q_0}\iota_{Q_1}\varpi_f$, we get:

$$
\begin{aligned}
& f S_0, S_1 g_f \qquad (4.42)\\
= \; & \underline{[c,\lambda e_n]^{(b)}([c,e])_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_1 \quad \underline{[c,\lambda e_n]^{(b)}([c,e])_b^{(n)}\lambda^{\mathsf{y}}}_2 \\
& + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}([c,e])_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_3 + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}([c,e])_b^{(n)}\lambda^{\mathsf{y}}}_4 \\
& + \underline{[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0} e)_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_5 + \underline{[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0} e)_b^{(n)}\lambda^{\mathsf{y}}}_6 \\
& \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(L_\xi^{\omega_0} e)_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_7 \quad \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(L_\xi^{\omega_0} e)_b^{(n)}\lambda^{\mathsf{y}}}_8 \\
& \underline{[c,\lambda e_n]^{(b)}(d_\omega(\lambda e_n))_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_9 \quad \underline{[c,\lambda e_n]^{(b)}(d_\omega(\lambda e_n))_b^{(n)}\lambda^{\mathsf{y}}}_{10} \\
& + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(d_\omega(\lambda e_n))_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{11} + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(d_\omega(\lambda e_n))_b^{(n)}\lambda^{\mathsf{y}}}_{12} \\
& \underline{[c,\lambda e_n]^{(b)}(\lambda\sigma)_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{13} \quad \underline{[c,\lambda e_n]^{(b)}(\lambda\sigma)_b^{(n)}\lambda^{\mathsf{y}}}_{14} \\
& + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(\lambda\sigma)_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{15} + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(b)}(\lambda\sigma)_b^{(n)}\lambda^{\mathsf{y}}}_{16} \\
& \underline{[c,\lambda e_n]^{(a)}(d_\omega c)_a c^{\mathsf{y}}}_{17} + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(d_\omega c)_a c^{\mathsf{y}}}_{18} + \underline{[c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0))_a c^{\mathsf{y}}}_{19} \\
& \underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0))_a c^{\mathsf{y}}}_{20} \quad \underline{[c,\lambda e_n]^{(a)}(W_1^{\quad 1}(\lambda e_n F_\omega))_a c^{\mathsf{y}}}_{21} \\
& + \underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(W_1^{\quad 1}(\lambda e_n F_\omega))_a c^{\mathsf{y}}}_{22} + \underline{[c,\lambda e_n]^{(a)}(\iota_\xi F_{\omega_0})_a c^{\mathsf{y}}}_{23} \\
& \underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\iota_\xi F_{\omega_0})_a c^{\mathsf{y}}}_{24} \quad \underline{\frac{1}{2}\Lambda[c,\lambda e_n]^{(a)} e_a \lambda e_n c^{\mathsf{y}}}_{25} + \underline{\frac{1}{2}\Lambda L_\xi^{\omega_0}(\lambda e_n)^{(a)} e_a \lambda e_n c^{\mathsf{y}}}_{26}.
\end{aligned}
$$

From $f S_1, S_1 g_g = \iota_{Q_1}\iota_{Q_1}\varpi_g$ we get:

$$
\begin{aligned}
& \frac{1}{2} f S_1, S_1 g_g \qquad (4.43)\\
= \; & \underline{\frac{1}{2}[[c,c],c]c^{\mathsf{y}}}_1 \quad \underline{\frac{1}{2}[c,c]L_\xi^{\omega_0} c^{\mathsf{y}}}_2 + \underline{\frac{1}{2}[[c,c],\lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_3 + \underline{\frac{1}{2}[[c,c],\lambda e_n]^{(n)}\lambda^{\mathsf{y}}}_4 \\
& \underline{[L_\xi^{\omega_0} c, c]c^{\mathsf{y}}}_5 + \underline{L_\xi^{\omega_0} c L_\xi^{\omega_0} c^{\mathsf{y}}}_6 \quad \underline{[L_\xi^{\omega_0} c, \lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} + (\omega \quad \omega_0)_a c^{\mathsf{y}})}_7 \\
& \underline{[L_\xi^{\omega_0} c, \lambda e_n]^{(n)}\lambda^{\mathsf{y}}}_8 \quad \underline{[[c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a, c]c^{\mathsf{y}}}_9 + \underline{[c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0} c^{\mathsf{y}}}_{10} \\
& \underline{[[c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a, \lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} + (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{11} \quad \underline{[[c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a, \lambda e_n]^{(n)}\lambda^{\mathsf{y}}}_{12} \\
& + \underline{[L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a, c]c^{\mathsf{y}}}_{13} \quad \underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0} c^{\mathsf{y}}}_{14} \\
& + \underline{[L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a, \lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{15} + \underline{[L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a, \lambda e_n]^{(n)}\lambda^{\mathsf{y}}}_{16} \\
& \underline{[c,[c,\lambda e_n]^{(n)} e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{17} \quad \underline{[c,[c,\lambda e_n]^{(n)} e_n]^{(n)}\lambda^{\mathsf{y}}}_{18} \\
& + \underline{L_\xi^{\omega_0}([c,\lambda e_n]^{(n)} e_n)^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{19} + \underline{L_\xi^{\omega_0}([c,\lambda e_n]^{(n)} e_n)^{(n)}\lambda^{\mathsf{y}}}_{20} \\
& + \underline{[c, L_\xi^{\omega_0}(\lambda e_n)^{(n)} e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{21} + \underline{[c, L_\xi^{\omega_0}(\lambda e_n)^{(n)} e_n]^{(n)}\lambda^{\mathsf{y}}}_{22}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{23} \quad \underline{L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(n)}\lambda^{\mathsf{y}}}_{24}\\
&\underline{[c,\lambda e_n]^{(a)}d_{\omega_0 a}cc^{\mathsf{y}}}_{25} \quad \underline{([c,\lambda e_n]^{(b)}d_{\omega_0 b}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{26}\\
&\underline{([c,\lambda e_n]^{(a)}d_{\omega_0 a}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}}}_{27} \quad \underline{[c,\lambda e_n]^{(a)}(\partial_a\xi^b)\xi_b^{\mathsf{y}}}_{28} \quad \underline{[c,\lambda e_n]^{(a)}\partial_b(\xi^b\xi_a^{\mathsf{y}})}_{29}\\
&+\underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}d_{\omega_0 a}cc^{\mathsf{y}}}_{30}+\underline{(L_\xi^{\omega_0}(\lambda e_n)^{(b)}d_{\omega_0 b}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{31}\\
&+\underline{(L_\xi^{\omega_0}(\lambda e_n)^{(a)}d_{\omega_0 a}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}}}_{32}+\underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\partial_a\xi^b)\xi_b^{\mathsf{y}}}_{33}+\underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}\partial_b(\xi^b\xi_a^{\mathsf{y}})}_{34}\\
&+\underline{\frac{1}{2}\iota_{[\xi,\xi]}d_{\omega_0}cc^{\mathsf{y}}}_{35}+\underline{\frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{36}+\underline{\frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}}}_{37}\\
&+\underline{\frac{1}{2}[\xi,\xi]^a(\partial_a\xi^b)\xi_b^{\mathsf{y}}}_{38}+\underline{\frac{1}{2}[\xi,\xi]^a\partial_b(\xi^b\xi_a^{\mathsf{y}})}_{39}+\underline{\frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0},c]c^{\mathsf{y}}}_{40} \quad \underline{\frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}L_\xi^{\omega_0}c^{\mathsf{y}}}_{41}\\
&+\underline{\frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0},\lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})}_{42}+\underline{\frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0},\lambda e_n]^{(n)}\lambda^{\mathsf{y}}}_{43}\\
&\underline{[c,\lambda e_n]^{(a)}(\iota_\xi F_{\omega_0})_a c^{\mathsf{y}}}_{44}+\underline{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\iota_\xi F_{\omega_0})_a c^{\mathsf{y}}}_{45}+\underline{\frac{1}{2}\iota_{[\xi,\xi]}\iota_\xi F_{\omega_0}c^{\mathsf{y}}.}_{46}
\end{aligned}
$$

We now check term by term that the sum $2 f S_0, S_1 g_f + f S_1, S_1 g_g$ is zero. We have:

(4.43.1) =0 using (graded) Jacobi identity.

(4.43.2) and (4.43.5):

$$\frac{1}{2}L_\xi^{\omega_0}([c,c]c^{\mathsf{y}}) = \quad \frac{1}{2}[c,c]L_\xi^{\omega_0}c^{\mathsf{y}} \quad [L_\xi^{\omega_0}c,c]c^{\mathsf{y}}.$$

(4.43.3), (4.43.17), (4.42.1): using (graded) Jacobi identity:

$$
\begin{aligned}
\frac{1}{2}[[c,c],\lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \quad & [c,[c,\lambda e_n]^{(n)}e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})\\
& [c,\lambda e_n]^{(b)}([c,e])_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) = 0
\end{aligned}
$$

(4.43.4), (4.43.18), (4.42.2): as before.

(4.43.6), (4.43.35) and (4.43.40) :

$$L_\xi^{\omega_0}cL_\xi^{\omega_0}c^{\mathsf{y}}+\frac{1}{2}\iota_{[\xi,\xi]}d_{\omega_0}cc^{\mathsf{y}}+\frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0},c]c^{\mathsf{y}} = \quad d_{\omega_0}(\iota_\xi d_{\omega_0}c\iota_\xi c^{\mathsf{y}}).$$

(4.43.7), (4.43.21) and (4.42.3):

$$
\begin{aligned}
[L_\xi^{\omega_0}c,\lambda e_n]^{(a)}(\xi_a^{\mathsf{y}}+(\omega \quad \omega_0)_a c^{\mathsf{y}}) &+ [c,L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})\\
&+L_\xi^{\omega_0}(\lambda e_n)^{(b)}([c,e])_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})\\
&= \quad (L_\xi^{\omega_0}[c,\lambda e_n])^{(a)}(\xi_a^{\mathsf{y}}+(\omega \quad \omega_0)_a c^{\mathsf{y}})
\end{aligned}
$$

We have $(L_\xi^{\omega_0}\omega)_a = L_\xi^{\omega_0}(\omega \quad \omega_0)_a + \partial_a\xi^c\omega_c$ and $(L_\xi^{\omega_0}e)_a = L_\xi^{\omega_0}e_a + \partial_a\xi^c e_c$. (4.42.5), (4.43.28), (4.42.19):

$$
\begin{aligned}
&[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0}e)_b^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \quad [c,\lambda e_n]^{(a)}(\partial_a\xi^b)\xi_b^y + [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}\omega)_a c^y\\
=&[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0}e_b)^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) + [c,\lambda e_n]^{(b)}(\partial_b\xi^c e_c)^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y)\\
&\quad [c,\lambda e_n]^{(b)}(\partial_b\xi^c)\xi_c^y + [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^y + [c,\lambda e_n]^{(a)}(\partial_a\xi^c\omega_c)c^y\\
=&[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0}e_b)^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) + [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^y
\end{aligned}
$$

(4.43.29): $[c,\lambda e_n]^{(a)}\partial_b(\xi^b\xi_a^y) = \partial_b([c,\lambda e_n]^{(a)}\xi^b\xi_a^y) + L_\xi^{\omega_0}[c,\lambda e_n]^{(a)}\xi_a^y$
(4.43.19), (4.43.10) and previous relations:

$$
\begin{aligned}
&[c,\lambda e_n]^{(b)}(L_\xi^{\omega_0}e_b)^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) + [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^y\\
&+ L_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) + L_\xi^{\omega_0}[c,\lambda e_n]^{(a)}\xi_a^y\\
&+ [c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0}c^y\\
=&L_\xi^{\omega_0}([c,\lambda e_n])^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) \quad L_\xi^{\omega_0}([c,\lambda e_n]^{(a)})(\xi_a^y \quad (\omega \quad \omega_0)_a c^y)\\
&+ L_\xi^{\omega_0}[c,\lambda e_n]^{(a)}\xi_a^y + [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^y + [c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0}c^y\\
=&L_\xi^{\omega_0}([c,\lambda e_n])^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y) + L_\xi^{\omega_0}([c,\lambda e_n]^{(a)})(\omega \quad \omega_0)_a c^y\\
&+ [c,\lambda e_n]^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^y + [c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0}c^y\\
=&L_\xi^{\omega_0}([c,\lambda e_n])^{(a)}(\xi_a^y \quad (\omega \quad \omega_0)_a c^y)
\end{aligned}
$$

This last term cancels out with the one resulting from the first computation.

(4.43.8), (4.43.22) and (4.42.4):

$$
\begin{aligned}
&[L_\xi^{\omega_0}c,\lambda e_n]^{(n)}\lambda^y + [c, L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n]^{(n)}\lambda^y + L_\xi^{\omega_0}(\lambda e_n)^{(b)}([c,e])_b^{(n)}\lambda^y\\
= \quad &(L_\xi^{\omega_0}[c,\lambda e_n])^{(n)}\lambda^y
\end{aligned}
$$

(4.43.20), (4.42.6): since $e_a^{(n)} = 0$ we have

$$
\begin{aligned}
&L_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(n)}\lambda^y + [c,\lambda e_n]^{(b)}(L_\xi^{\omega_0}e)_b^{(n)}\lambda^y\\
=&L_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(n)}\lambda^y + L_\xi^{\omega_0}([c,\lambda e_n]^{(a)}e_a)^{(n)}\lambda^y\\
=&(L_\xi^{\omega_0}[c,\lambda e_n])^{(n)}\lambda^y
\end{aligned}
$$

(4.43.9), (4.43.25) and (4.42.17):

$$
\begin{aligned}
&[[c,\lambda e_n]^{(a)}(\omega \quad \omega_0)_a, c]c^y \quad [c,\lambda e_n]^{(a)}d_{\omega_0\,a}cc^y\\
= \quad &[c,\lambda e_n]^{(a)}d_{(\omega \quad \omega_0)_a}cc^y = [c,\lambda e_n]^{(a)}(d_\omega c)_a c^y.
\end{aligned}
$$

(4.43.13), (4.43.30) and (4.42.18): as before.

(4.43.11), (4.43.26) and (4.42.9): as before.

(4.43.12), (4.43.27) and (4.42.10): as before.

(4.43.15), (4.43.31) and (4.42.11): as before.

(4.43.16), (4.43.32) and (4.42.12): as before.

(4.43.36) and (4.43.42) :

$$\frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) + \frac{1}{2}[\iota_\xi \iota_\xi F_{\omega_0}, \lambda e_n]^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \\ = (L_\xi^{\omega_0} L_\xi^{\omega_0}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}})$$

(4.43.14) and (4.42.20):

$$\begin{aligned} & L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a L_\xi^{\omega_0} c^{\mathsf{y}} \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)}(L_\xi^{\omega_0}\omega)_a c^{\mathsf{y}} \\ = & L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega \quad \omega_0)_a)c^{\mathsf{y}} \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)}(L_\xi^{\omega_0}(\omega \quad \omega_0)_a)c^{\mathsf{y}} \\ & L_\xi^{\omega_0}(\lambda e_n)^{(a)}\partial_a \xi^c \omega_c c^{\mathsf{y}} \\ = & L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(a)})(\omega \quad \omega_0)_a c^{\mathsf{y}} \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)}\partial_a \xi^c \omega_c c^{\mathsf{y}} \end{aligned}$$

(4.43.33), (4.43.34) and previous relation:

$$\begin{aligned} & L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\partial_a \xi^b)\xi_b^{\mathsf{y}} + L_\xi^{\omega_0}(\lambda e_n)^{(a)}\partial_b(\xi^b \xi_a^{\mathsf{y}}) + L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(a)})(\omega \quad \omega_0)_a c^{\mathsf{y}} \\ & \quad L_\xi^{\omega_0}(\lambda e_n)^{(a)}\partial_a \xi^c \omega_c c^{\mathsf{y}} \\ = & \left[L_\xi^{\omega_0}(\lambda e_n)^{(b)}(\partial_b \xi^c e_c)^{(a)} \quad L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(b)})e_b^{(a)}\right](\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \end{aligned}$$

(4.42.7) and previous relation:

$$\begin{aligned} & \left[\quad L_\xi^{\omega_0}(\lambda e_n)^{(b)}(L_\xi^{\omega_0} e)_b^{(a)} + L_\xi^{\omega_0}(\lambda e_n)^{(b)}(\partial_b \xi^c e_c)^{(a)}\right](\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \\ & \quad L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(b)})e_b^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \\ = & \quad (L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \end{aligned}$$

(4.43.23) and previous relation:

$$\begin{aligned} & \left[\quad L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(a)} \quad (L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b))^{(a)}\right](\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \\ = & \quad (L_\xi^{\omega_0} L_\xi^{\omega_0}(\lambda e_n))^{(a)}(\xi_a^{\mathsf{y}} \quad (\omega \quad \omega_0)_a c^{\mathsf{y}}) \end{aligned}$$

This last term cancels out with the one resulting from the first computation.

(4.43.37) and (4.43.43):

$$\frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}} + \frac{1}{2}[\iota_\xi \iota_\xi F_{\omega_0}, \lambda e_n]^{(n)}\lambda^{\mathsf{y}} = (L_\xi^{\omega_0} L_\xi^{\omega_0}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}}$$

(4.43.24) and (4.42.8):

$$\begin{aligned} & L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(n)}\lambda^{\mathsf{y}} \quad L_\xi^{\omega_0}(\lambda e_n)^{(b)}(L_\xi^{\omega_0} e)_b^{(n)}\lambda^{\mathsf{y}} \\ = & \quad L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(n)}\lambda^{\mathsf{y}} \quad L_\xi^{\omega_0}(L_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b)^{(n)}\lambda^{\mathsf{y}} \\ = & \quad (L_\xi^{\omega_0} L_\xi^{\omega_0}(\lambda e_n))^{(n)}\lambda^{\mathsf{y}} \end{aligned}$$

This last term cancels out with the one resulting from the first computation.

(4.42.13), (4.42.14), (4.42.15), (4.42.16), (4.42.21), (4.42.22), (4.42.25) and (4.42.26): Everything vanishes since $\lambda\lambda = 0$ and $e_n^{(b)} = 0$.

(4.43.38) and (4.43.39):

$$\begin{aligned} &+\frac{1}{2}[\xi,\xi]^a(\partial_a\xi^b)\xi_b^{\mathsf{y}} + \frac{1}{2}[\xi,\xi]^a\partial_b(\xi^b\xi_a^{\mathsf{y}}) = \xi^c\partial_c\xi^a(\partial_a\xi^b)\xi_b^{\mathsf{y}} + \xi^c\partial_c\xi^a\partial_b(\xi^b\xi_a^{\mathsf{y}}) \\ &= \partial_b(\xi^c\partial_c\xi^a\xi^b\xi_a^{\mathsf{y}}) \quad \xi^c\partial_c\partial_b\xi^a\xi^b\xi_a^{\mathsf{y}} \end{aligned}$$

where the last term vanishes since it is symmetric and antisymmetric in the indexes b and c.

(4.43.44) = (4.42.23).

(4.43.45) = (4.42.24).

(4.43.41) and (4.43.46):

$$\frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0} L_\xi^{\omega_0} c^{\mathsf{y}} + \frac{1}{2}\iota_{[\xi,\xi]}\iota_\xi F_{\omega_0} c^{\mathsf{y}} = \frac{1}{2} d_{\omega_0}(\iota_\xi\iota_\xi F_{\omega_0}\iota_\xi c^{\mathsf{y}})$$

Chapter 5

Palatini–Cartan Boundary structure, degenerate case

After exploring the boundary structure of General Relativity in the coframe formalism on space-like and time-like boundaries in Chapter 4 through the construction of a presentation of the reduced phase space, in this chapter we study the reduced phase space of General Relativity in the coframe formulation in the case where the boundary has a light-like induced metric.

The differences between the light-like and the time- and space-like cases are given by the signature of the restriction of the metric to the boundary. Indeed it turns out that there are major differences between the cases when the metric is ***space-like*** or ***time-like*** —respectively with signature as a symmetric bilinear form $(N \quad 1, 0, 0)$ or $(N \quad 2, 1, 0)$ where the first index denotes positive eigenvalues, the second negative ones and the third zero ones— and when the metric is ***light-like*** —with signature $(N \quad 2, 0, 1)$ where the last entry refers to the transversal direction. Note that, since the metric in the bulk is Lorentzian, the metric on the boundary can only be non-degenerate or have a unique direction along which it is degenerate.

Following the same scheme of the construction outlined in Chapter 4, the boundary structure is recovered also in this case through the method that was firstly described by Kijowski and Tulczijew in [KT79]. This problem has also been approached with the method proposed by Dirac [Dir58]. This latter approach to the problem at hand has been developed in [AS15].

In this chapter we will describe the geometric structure of the boundary fields by adapting the results of Chapter 4 to the case of a degenerate boundary metric. In $3+1$ dimensions, this results in a reduced phase space with two local degrees of freedom (in good agreement with the literature [AS15]) instead of four in the non-degenerate case[1].

As we have seen in the non-degenerate case, one of the greatest challenges of the constraint analysis of PC theory comes from the quotient structure of the symplectic form of the ***true*** space of boundary fields also called geometric phase space. To avoid the use of equivalence classes, usually more unwieldy, it is useful to introduce a ***structural constraint*** to fix a representative, as in the non-degenerate case. However, in the light-like case, such a choice is more complicated. In this chapter we propose a solution that leads to the emergence of second class constraints, as opposed to the non-degenerate case where all constraints are first class[2]. The key point is to

[1] By number of local degrees of freedom we mean the rank of the phase space as a C^{1}-module (ignoring global degrees). In the space-like or time-like case, one also usually speaks of the number of local physical degrees of freedom meaning by this half the rank of the reduced phase space (i.e., the rank of the con guration space).

[2] Note that the fact that not all the constraints are rst class implies that the reduced phase space is not a coisotropic submanifold and hence the initial value problem (de ned on) is not well posed

adapt the ***structural constraint*** (4.10) to encompass the differences of the light-like case. This results in a slightly more complicated, but still very interesting, structure of the constraints. This case is carried out in full generality for every dimension $N \quad 4$.[3]

A linearized version of the theory has been proposed in [Tec19], and in [CCT20, Appendix B] where the theory is expanded around a reference solution of the Euler-Lagrange equations. In this case it can be shown that the quotient space of the space of fields is isomorphic to a space of sections of some bundle, thus not involving quotients. This leads to a large simplification of the computations still retaining the most important properties of the general theory (e.g. the number of physical local degrees of freedom) and can be therefore thought of as an interesting toy model of the latter. This result has only been developed for $N = 4$, but the extension to higher dimensions is straightforward.

This chapter is structured as follows: Section 5.1 is devoted to reviewing the results of this chapter. In Section 5.2 we consider the general case and illustrate in full detail the boundary structure of the degenerate case. The main results are collected in Theorem 5.11.

5.1 Overview

We present here the problem and the results of the chapter at a qualitative level (and for $N = 4$) and refer to the subsequent sections for a more precise treatment. We follow here the notations and conventions fixed in Chapter 3.

The solution proposed in Chapter 4 for the convenient choice of the structural constraint requires that the induced metric $g^\partial = e \ \eta$ on the boundary Σ is non-degenerate. Hence it needs to be adapted in the degenerate case.

Remark 5.1. Here, we address the problem assuming that in the boundary manifold there exists a light-like subset and we assume to be working only in an open subset of the light-like one. The general case of a boundary with points of different types (lightlike, spacelike and timelike) can be recovered as explained below in Remark 5.2.

The main difference in the degenerate case is the impossibility of finding a representative of the equivalence class $[\omega]$ satisfying the structural constraint. The idea is to modify this equation by subtracting the problematic part and impose a weakened structural constraint as follows:

$$e_n d_\omega e \quad e_n p_{\mathcal{T}} (d_\omega e) \ 2 \ \mathrm{Im} \, W_1^{\partial,(1,1)} \tag{5.1}$$

where $p_{\mathcal{T}}$ is the projection to an appropriately defined subspace (see (3.5); see also Section 3.1 for the notation and Theorem 5.3 for more details). This weakened structural constraint no longer fixes the representative in the equivalence class uniquely and hence it has to be supplemented with another set of equations, though of little importance for the construction. Furthermore this weakened constraint does not guarantee the equivalence between the constraint L_c and $d_\omega e = 0$. Indeed, an important feature that was a key point in the non-degenerate case was the fact that the equation $e d_\omega e = 0$, after imposing the structural constraint $e_n d_\omega e \ 2 \ \mathrm{Im}(e \wedge \ )$, defines the same zero locus as $d_\omega e = 0$. As a consequence, in order to get the correct reduced phase space, in the degenerate case one has to add an additional constraint accounting for the missing part in the weakened structural constraint: namely,

$$R_\tau = \int \tau d_\omega e$$

[3] For $N = 3$, the results presented in Chapter 2 do not depend on the degeneracy of the boundary metric, hence we do not have to study the theory separately.

with τ belonging to an appropriate space S (see (3.5c) for the definition). We will call this constraint the *degeneracy constraint*[4]. This construction is made precise in the first part of Section 5.2 where we also analyse the structure of this new set of constraints (Theorem 5.11 and Corollary 5.15).

By computing the Poisson brackets of the constraints, we show that all the constraints are first class except the degeneracy constraint R_τ which is second class. Finally we also compute the number of local physical degrees of freedom of the theory. In dimension 3+1 we obtain that the reduced phase space has two local degrees of freedom.

Remark 5.2. This construction can be extended to the general case of a boundary only part of which is allowed to be light-like. In this case the field $\tau \in S$ defining the degeneracy constraint has support in the closure of the light-like points. Furthermore, since the equations defining $\tau \in S$ are algebraic, by continuity we also have that τ vanishes on the boundary (if present) of the closed light-like subset.

	Nondegenerate case	Light-like case
Geometric phase space	$(F^\partial, \varpi^\partial)$	$(F^\partial, \varpi^\partial)$
Structural constraint	(4.8)	(5.1)
Constraints	L_c, P_ξ, H_λ	$L_c, P_\xi, H_\lambda, R_\tau$

Table 5.1: Differences between the nondegenerate case and the light-like one

We conclude the overview with the Table 5.1 showing the differences between the nondegenerate case and the light-like one.

5.2 Degenerate boundary structure

In this section we give a description of the *geometric phase space* and of the constraints defined on it. Recall that the geometric phase space is the natural space of fields associated to the bulk before performing the (symplectic) reduction with respect to the constraints, the result of which is the reduced phase space (hence the name).

The analysis of the degenerate case will start from the results of the non-degenerate case of Chapter 4 and will address the various issues arising with the generalisation of the technical results of Section 3.2 to the degenerate metric scenario.corresponding technical results for a degenerate metric. In particular the crucial difference will come from the different outcome of Lemma 3.6.

Let now g^∂ be degenerate, i.e. it admits a vector field X such that $\iota_X g^\partial = 0$.

5.2.1 Fixing a representative

In this section we describe a possible way for fixing the freedom of the choice of the connection $\omega \in [\omega]$, adapting the non-degenerate case. The main difference is that in the degenerate case,

[4]We thank M. Schiavina for the helpful discussion about the form of this constraint (and its name).

because of the different outcome of Lemma 3.6, it is no longer possible to find an $\omega \in [\omega]$ such that $e_n e^{N-4} d_\omega e \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}$. Indeed, in contrast to the non-degenerate case, the map

$$v \in \operatorname{Ker}(W_{N-3}^{\partial,(1,2)}) \mapsto e_n e^{N-4}[v,e] \in \Omega_\partial^{N-2,N-2}$$

is not injective on $\Omega_\partial^{(1,1)}$ (Lemma 3.6). The workaround is to separately consider the components of $d_\omega e$ in $\mathcal{T}$ and the components of ω in $\mathcal{K}$ (where $\mathcal{T}$ and $\mathcal{K}$ have been introduced in (3.5)). Indeed in the following theorem we consider a weaker version of the structural constraint (4.32) that generalizes it for a degenerate metric. This theorem is the generalization of Theorem 4.18.

Theorem 5.3. *Given any $\widetilde{\omega} \in \Omega_\partial^{1,2}$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v \tag{5.2}$$

with ω and v satisfying

$$e^{N-3} v = 0, \tag{5.3a}$$

$$e_n e^{N-4} d_\omega e - e_n e^{N-4} p_{\mathcal{T}}(d_\omega e) \in \operatorname{Im} W_{N-3}^{\partial,(1,1)}, \tag{5.3b}$$

$$p_{\mathcal{K}} v = 0. \tag{5.3c}$$

The proof is based on Lemmas 3.18 and 3.19 generalizing Lemmas 3.14 and 3.17 respectively.

Proof of Theorem 5.3. Let $\widetilde{\omega} \in \Omega_\partial^{1,2}$. From Lemma 3.19 we deduce that there exist $\sigma \in \Omega_\partial^{1,1}$, $v \in \operatorname{Ker} W_1^{\partial,(1,2)}$ and $\theta \in \mathcal{T}$ such that

$$e_n e^{N-4} d_{\widetilde{\omega}} e = e^{N-3}\sigma + e_n e^{N-4}[v,e] + e_n e^{N-4}\theta.$$

We define $\omega := \widetilde{\omega} - v$. Then ω and v satisfy (5.2) and (5.3). □

In contrast with the non-degenerate case, this theorem does not fix completely the freedom of $\omega \in [\omega]$. Hence we require the following additional equation[5]:

$$p_{\mathcal{K}} \omega = 0. \tag{5.4}$$

5.2.2 Independence from the choices

In this section we explore the independence of the analysis from the choices that we have made in the construction. We prove it through the following general theorem.

Theorem 5.4. *Let (P, ϖ) be a presymplectic manifold with kernel distribution K, smooth leaf space $(\underline{P}, \underline{\varpi})$ and canonical projection $\pi : P \to \underline{P}$. Let Q be a submanifold of P such that*

$$\rho := \pi|_Q : Q \to \underline{P}$$

is a diffeomorphism. Then $(Q, \varpi|_Q)$ is a symplectic manifold and ρ is a symplectomorphism.

[5] Starting from the definition of $\mathcal{K}$ in (3.5), it is a straightforward check that this last equation fixes the components of $\omega \in \operatorname{Ker}(W_{N-3}^{\partial,(1,2)})$ not included in (5.3b). Indeed, the elements of $\omega \in \operatorname{Ker}(\rho)$ are the ones that no longer appear in the structural constraint in the degenerate case as opposed to the non-degenerate one.

Proof. For every $x \in P$ we have that the exact sequence

$$0 \to K_x \to T_xP \xrightarrow{d_x\pi} T_{\pi(x)}\underline{P} \to 0.$$

For $x \in Q$ we have the splitting $d_{\rho(x)} : T_{\pi(x)}\underline{P} \to T_xP$ with image T_xQ which gives $T_xM = T_xQ \oplus K_x$. Let now $v \in (T_xQ)^{\perp}$, then $\varpi_x(v, w) = 0\ \forall w \in T_xQ$. Furthermore $\varpi_x(v, w) = \varpi_x(v, w + \widetilde{w})$ for all $\widetilde{w} \in K_x$. From the previous result we get that $\varpi_x(v, \widehat{w}) = 0$ for all $\widehat{w} \in T_xP$. This implies that $v \in (T_xP)^{\perp} = K_x$. Therefore $(T_xQ)^{\perp} \subset K_x$ and

$$(T_xQ)^{\perp} \cap T_xQ \subset K_x \cap T_xQ = \emptyset.$$

Hence $(Q, \varpi|_Q)$ is symplectic.

From the definition of leaf space we have that

$$\varpi_x(v, w) = \underline{\varpi}_{\pi(x)}([v], [w]) \qquad \forall x \in P\ \forall v, w \in T_xP$$

Restricted to Q this becomes

$$\varpi_x(v, w) = \underline{\varpi}_{\rho(x)}([v], [w]) \qquad \forall x \in Q\ \forall v, w \in T_xQ.$$

Since ρ is a diffeomorphism and $(Q, \varpi|_Q)$ is a symplectic manifold, this last equation proves that ρ is a symplectomorphim. □

Corollary 5.5. *If Q and Q' are submanifolds of P such that $\pi|_Q$ and $\pi|_{Q'}$ are diffeomorphisms with $\underline{P}$, then $(Q, \varpi|_Q)$ and $(Q', \varpi|_{Q'})$ are canonically simplectomorphic.*

Remark 5.6. In our case P is the space of restrictions to the boundary $\widetilde{F}_{PC}$ with presymplectic form $\tilde{\varpi}$, Q is the subspace of $\widetilde{F}_{PC}$ where ω satisfies the constraints (5.3b) and (5.4), while $\underline{P}$ is the geometric phase space F^{∂}_{PC} with simplectic form ϖ^{∂}_{PC}. The map π is given by π_{PC} defined in (4.4) and ρ is its restriction to Q. The inverse of ρ is given by the map $(e, [\omega]) \mapsto (e, \omega')$ where ω' is the unique representative of the class $[\omega]$ satisfying (5.3b) and (5.4).

The existence of a canonical symplectomorphism between the constructions corresponding to different possible choices of the representative in the equivalence class of $[\omega]$ guarantees the independence of the construction on such choices. In particular the choice of the projection that leads to (5.4) is immaterial in the construction since we do not use this constraints anywhere else.

5.2.3 Constraints of the theory

Let us now turn to the constraints of the theory. In the degenerate case we will still adopt the approach of the non-degenerate case. However we will adapt it to encompass the differences between Lemma 3.14 and Lemma 3.18. The main difference is that now the constraint L_c together with the new structural constraint (5.3b) is no longer equivalent to $d_\omega e = 0$ (one set of the Euler-Lagrange equations in the bulk) since we are missing the third equation in (3.8). Indeed we have to add an additional constraint that, thanks to Lemma 3.11, we can express as

$$R_\tau = \int \tau d_\omega e \tag{5.5}$$

by means of an odd Lagrange multiplier $\tau \in \mathcal{S}[1]$[6]. Furthermore, to simplify the computation of the brackets between the constraints, it is useful to modify the constraint H_λ by adding to it a

[6] As in Chapter 4 the notation [1] denotes that τ is an odd quantity.

term proportional to R_τ :

$$H_\lambda = \int \lambda e_n \left(\frac{1}{(N-3)} e^{N-3} F_\omega - e^{N-4}(\omega - \omega_0) p_{\mathcal{T}} (d_\omega e) + \frac{1}{(N-1)!} \Lambda e^{N-1} \right). \tag{5.6}$$

Note that we can as well express the second term in this constraint as

$$\lambda p_{\mathcal{S}}(e_n e^{N-4}(\omega - \omega_0)) d_\omega e$$

to make it explicitly in the form of (5.5).

Remark 5.7. The additional part in H_λ proportional to R_τ has been added only to ease the computation of the Hamiltonian vector field of the constraint H_λ itself. Such a linear combination does not affect the constrained set and the structure of the constraints, i.e. the distinction between first and second class constraints (see Proposition 5.18 and Remark 5.20 in Appendix 5.A). Similar considerations hold also for the part of the constraint P_ξ proportional to L_c, as already mentioned in Remark 4.9 and [CS19c, Remark 4.24].

Before analysing the structure of these constraints and their Poisson brackets we need some additional results concerning the variation of elements in $\mathcal{S}$ which are constrained and are thus depending on e.

Lemma 5.8. *The variation of an element $\tau \in \mathcal{S}$ is constrained by the following equations:*

$$p^0_{\widetilde{\rho}} \delta\tau = \widetilde{\rho}^{-1} \left(\frac{\delta \widetilde{\rho}}{\delta e}(\tau) \delta e \right),$$
$$p^0_{W} \delta\tau = W_e^{-1}(\tau \delta e)$$

where the inverses are defined on their images and $p^0_{\widetilde{\rho}}$ and p^0_W are respectively the projections to a complement of the kernel of $\widetilde{\rho}$ and $W_1^{\partial,(N-3,N-1)}$.

Remark 5.9. Different choices of projections lead to different terms in the kernel of the two maps. Nonetheless these additional terms are in $\mathcal{S}$ where the variation is free. Hence they will not play any role in the computations.

Proof. From (3.5c) we know the elements $\tau \in \mathcal{S}$ must satisfy the following equations:

$$\tau \wedge e = 0; \quad \widetilde{\rho}(\tau) = 0.$$

Hence varying each equation we obtain some constraints for the variation $\delta\tau$:

$$\delta\tau \wedge e - \tau \wedge \delta e = 0; \quad \widetilde{\rho}(\delta\tau) + \frac{\delta\widetilde{\rho}}{\delta e}(\tau)\delta e = 0.$$

We can invert these equations using the inverses of W and $\widetilde{\rho}$ on their images. Denoting with p^0_W and $p^0_{\widetilde{\rho}}$ the projections to some complements of the kernel of W and $\widetilde{\rho}$ in $\Omega_\partial^{N-3,N-1}$ respectively, we obtain

$$p^0_W \delta\tau = W_1^{-1}(\tau \wedge \delta e); \quad p^0_{\widetilde{\rho}} \delta\tau = \widetilde{\rho}^{-1} \left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)\delta e \right).$$

These relations fix the constrained part of the variation of $\tau \in \mathcal{S}$ in terms of the variation of e. □

Lemma 5.10. *The following identities hold:*

$$\widetilde{\rho}^{-1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)[c,e]\right) = p^{0}_{\widetilde{\rho}}[c,\tau], \qquad \widetilde{\rho}^{-1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)L_{\xi}^{\omega_0}e\right) - p^{0}_{\widetilde{\rho}}L_{\xi}^{\omega_0}\tau.$$

Proof. We start by making more explicit the expression $\widetilde{\rho}^{-1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)\delta e\right)$. By definition, if $\tau \in S$, then $[\tau,\widetilde{e}] = 0$. Hence

$$0 = \delta[\tau,\widetilde{e}] = [\delta\tau,\widetilde{e}] + [\tau,\delta\widetilde{e}].$$

We now compute $\delta\widetilde{e}$ in terms of δe:

$$\delta\widetilde{e} = \delta e - \delta\widehat{e} = \delta e - \delta(\beta\iota_X e) = \delta e - \delta\beta\iota_X e + \beta\iota_{\delta X} e - \beta\iota_X\delta e.$$

We have then to compute the variation δX and $\delta\beta$. We start from the first: from the defining equation $\iota_X g^{\partial} = 0$ we get

$$\iota_{\delta X} g^{\partial} - \iota_X\delta g^{\partial} = 0$$

and hence, inverting g^{∂} on its image, we get $\delta X = g^{\partial\,-1}(\iota_X\delta g^{\partial})$. Since g^{∂} can be written in terms of e and η as $g^{\partial} = \eta(e,e)$, we can write this part of δX in terms of δe. The remaining part of δX not fixed by this equation is such that $\iota_{\delta X} g^{\partial} = 0$, hence

$$\delta X = 2g^{\partial\,-1}(\iota_X\eta(\delta e,e)) + \lambda X$$

for some function λ.

Let us now pass to $\delta\beta$. Its value is completely determined by the equations $\iota_X\delta\beta - \iota_{\delta X}\beta = 0$ and

$$\begin{aligned}\iota_{Y_0^1}\dots\iota_{Y_0^{N-2}}&\left(\iota_{\delta X}(\beta e^{N-3})v - \iota_X(\delta\beta e^{N-3})v\right)\\ &+ \iota_{Y_0^1}\dots\iota_{Y_0^{N-2}}\left((N-3)\iota_X(\beta\delta e e^{N-4})v + \iota_X(\beta e^{N-3})\delta v\right) = 0.\end{aligned}$$

This last equation must hold for every v and δv that satisfy respectively $e^{N-3}\wedge v = 0$ and $(N-3)\delta e e^{N-4}v + e^{N-3}\delta v = 0$.

We can now plug the values $\delta e = [c,e]$ and $\delta e = L_{\xi}^{\omega_0}e$ in the first formula of Lemma 5.8 using the above results. In the first case we get

$$\delta X = 2g^{\partial\,-1}(\iota_X[[c,e],e]) + \lambda X = 2g^{\partial\,-1}(\iota_X[c,[e,e]]) + \lambda X = \lambda X$$

and $\delta\beta = \lambda\beta$. Consequently

$$\begin{aligned}\widetilde{\rho}^{-1}([\tau,[c,e] - \beta\iota_X[c,e]]) &= \widetilde{\rho}^{-1}([\tau,[c,e] - [c,\beta\iota_X e]]) = \widetilde{\rho}^{-1}([\tau,[c,\widetilde{e}])\\ &= \widetilde{\rho}^{-1}([[\tau,c],\widetilde{e}] + [c,[\tau,\widetilde{e}]]) = p^{0}_{\widetilde{\rho}}[\tau,c].\end{aligned}$$

In the second case we have

$$\delta X = 2g^{\partial\,-1}(\iota_X[L_{\xi}^{\omega_0}e,e]) + \lambda X = g^{\partial\,-1}(\iota_X L_{\xi}^{\omega_0}g^{\partial}) + \lambda X.$$

and $\delta\beta = L_{\xi}^{\omega_0}\beta + \lambda\beta$. In coordinates we obtain the following expressions

$$\begin{aligned}\delta X^{\mu} &= X^{\rho}\partial_{\rho}\xi^{\mu} + \xi^{\rho}\partial_{\rho}X^{\mu} + \lambda X^{\mu}\\ \iota_X L_{\xi}^{\omega_0}e &= X^{\rho}\xi^{\mu}d_{\omega_0\,\mu}e_{\rho} - X^{\rho}e_{\mu}d_{\rho}\xi^{\mu}.\end{aligned}$$

Hence

$$\iota_X L_\xi^{\omega_0} e + \iota_{\delta X} e = \iota_\xi d_\omega(\iota_X e) + \lambda \iota_X e,$$

and collecting all these formulas we get

$$\begin{aligned}\widetilde{\rho}^{-1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau) L_\xi^{\omega_0} e\right) &= \widetilde{\rho}^{-1}\left([\tau, L_\xi^{\omega_0} e - L_\xi^{\omega_0}(\beta \iota_X e)]\right) = \widetilde{\rho}^{-1}\left([\tau, L_\xi^{\omega_0}\widetilde{e}]\right) \\ &= \widetilde{\rho}^{-1}\left(L_\xi^{\omega_0}[\tau, \widetilde{e}] - [L_\xi^{\omega_0}\tau, \widetilde{e}]\right) = p^0_{\widetilde{\rho}} L_\xi^{\omega_0}\tau.\end{aligned}$$

□

The addition of the constraint R_τ to compensate the different structure of the lightlike case has important consequences on the structure of the set of constraints.

Theorem 5.11. *Let g^∂ be degenerate on Σ. Then the structure of the Poisson brackets of the constraints L_c, P_ξ, H_λ and R_τ is given by the following expressions:*

$$\begin{aligned}
&\{L_c, L_c\} = -\frac{1}{2} L_{[c,c]} && \{P_\xi, P_\xi\} = \frac{1}{2} P_{[\xi,\xi]} - \frac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}} \\
&\{L_c, P_\xi\} = L_{L_\xi^{\omega_0} c} && \{H_\lambda, H_\lambda\} \approx F_{\tau^0 \tau^0} \\
&\{L_c, R_\tau\} = -R_{p_S[c,\tau]} && \{P_\xi, R_\tau\} = R_{p_S L_\xi^{\omega_0}\tau}. \\
&\{R_\tau, H_\lambda\} \approx F_{\tau\tau^0} + G_{\lambda\tau} && \{R_\tau, R_\tau\} \approx F_{\tau\tau} \\
&\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} + R_{p_S(X^{(a)} e_a e^{N-4}(\omega-\omega_0) - \lambda e_n d_{\omega_0} c)} \\
&\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}} - R_{p_S(Y^{(a)} e_a e^{N-4}(\omega-\omega_0) - \lambda e_n \iota_\xi F_{\omega_0})}
\end{aligned}$$

where $\tau^0 = p_S(\lambda e_n e^{N-4}(\omega - \omega_0))$, $X = [c, \lambda e_n]$, $Y = L_\xi^{\omega_0}(\lambda e_n)$ and $Z^{(a)}$, $Z^{(n)}$ are the components of $Z \in \{X, Y\}$ with respect to the frame (e_a, e_n). Furthermore $F_{\tau\tau}$, $F_{\tau\tau^0}$, $F_{\tau^0\tau^0}$ and $G_{\lambda\tau}$ are functions of e, ω, τ (or τ^0) and λ defined in the proof that are not proportional to any other constraint.

Remark 5.12. In Theorem 5.11 we use the symbol $\approx$ to denote the fact that the result can be obtained only working on shell, i.e. imposing the constraints. Here we want to stress that the brackets are not proportional to the constraints (hence not first-class, see Corollary 5.15) while in the other cases (the ones with the = sign) we get an exact result. Equivalently we could have written e.g. $\{L_c, L_c\} \approx 0$.

Proof. We first compute the variation of the constraints in order to find their Hamiltonian vector fields. Using the results of Theorem 4.10 for L_c and P_ξ, we have:

$$\delta L_c = \int -\frac{1}{N-2} c[\delta\omega, e^{N-2}] + \frac{1}{N-2} c d_\omega \delta(e^{N-2}) = \int [c, e] e^{N-3}\delta\omega + d_\omega c e^{N-3}\delta e;$$

$$\begin{aligned}\delta P_\xi = \int & \iota_\xi(e^{N-3}\delta e) F_\omega - \frac{1}{N-2}\iota_\xi(e^{N-2}) d_\omega \delta\omega + \iota_\xi \delta\omega e^{N-3} d_\omega e \\ & - \frac{1}{N-2}\iota_\xi(\omega - \omega_0)[\delta\omega, e^{N-2}] + \frac{1}{N-2}\iota_\xi(\omega-\omega_0) d_\omega \delta(e^{N-2}) \\ = \int & -e^{N-3}\delta e(L_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}) - (L_\xi^{\omega_0} e) e^{N-3}\delta\omega;\end{aligned}$$

$$\begin{aligned}\delta R_\tau &= \int \delta_e \tau d_\omega e \quad \tau[\delta\omega, e] + \tau d_\omega \delta e \\ &= \int \delta e g(\tau,\omega,e) + [\tau, e]\delta\omega + d_\omega \tau \delta e\end{aligned}$$

where $g(\tau,\omega,e)$ is a formal expression that encodes the dependence of $\delta\tau$ on δe i.e. such that

$$\delta e g(\tau,\omega,e) = p^0_{\widetilde{\rho}}\widetilde{\rho}^{\ 1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)\delta e\right) d_\omega e + p^0_W W_e^{\ 1}(\tau\delta e) d_\omega e \quad p^0_X \widetilde{\rho}^{\ 1}\left(\frac{\delta\widetilde{\rho}}{\delta e}(\tau)\delta e\right) d_\omega e$$

as shown in Lemma 5.8 where p^0_X is the projection to the intersection of the complement of the kernel of $\widetilde{\rho}$ and $W_1^{\partial,(N\ \ 3,N\ \ 1)}$. Using this last computation we can compute the variation of the Hamiltonian constraint H_λ:

$$\begin{aligned}\delta H_\lambda = \int & \lambda e_n e^{N\ \ 4}\delta e F_\omega + \frac{1}{(N\ \ 2)!}\Lambda\lambda e_n e^{N\ \ 2}\delta e \quad \frac{1}{(N\ \ 3)}\lambda e_n e^{N\ \ 3} d_\omega \delta\omega \\ & \lambda p_{\mathsf{S}}(e_n e^{N\ \ 4}\delta\omega) d_\omega e \quad (N\ \ 4)\lambda p_{\mathsf{S}}(e_n e^{N\ \ 5}\delta e(\omega\ \ \omega_0)) d_\omega e \\ & \delta_e \tau^0 d_\omega e + \tau^0[\delta\omega, e] \quad \tau^0 d_\omega \delta e \\ = \int & \lambda e_n e^{N\ \ 4}\delta e F_\omega + \frac{1}{(N\ \ 2)!}\Lambda\lambda e_n e^{N\ \ 2}\delta e + \frac{1}{(N\ \ 3)} d_\omega(\lambda e_n) e^{N\ \ 3}\delta\omega \\ & + \lambda e_n e^{N\ \ 4} d_\omega e \delta\omega \quad \lambda e_n e^{N\ \ 4}\delta\omega p_{\mathcal{T}}(d_\omega e) \\ & (N\ \ 4)\lambda e_n e^{N\ \ 5}\delta e(\omega\ \ \omega_0) p_{\mathcal{T}}(d_\omega e) \quad \delta e g(\tau^0,\omega,e) + \tau^0[\delta\omega, e] \quad \tau^0 d_\omega \delta e \\ = \int & \lambda e_n e^{N\ \ 4}\delta e F_\omega + \frac{1}{(N\ \ 2)!}\Lambda\lambda e_n e^{N\ \ 2}\delta e + \frac{1}{(N\ \ 3)} d_\omega(\lambda e_n) e^{N\ \ 3}\delta\omega \\ & + \lambda\sigma e^{N\ \ 3}\delta\omega \quad (N\ \ 4)\lambda e_n e^{N\ \ 5}\delta e(\omega\ \ \omega_0) p_{\mathcal{T}}(d_\omega e) \\ & \delta e g(\tau^0,\omega,e) + \tau^0[\delta\omega, e] \quad \tau^0 d_\omega \delta e\end{aligned}$$

where $\tau^0 = p_{\mathsf{S}}(\lambda e_n e^{N\ \ 4}(\omega\ \ \omega_0))$ and we used (5.3b). From the expressions of the variation of the constraints we can deduce their Hamiltonian vector fields. Let X be a generic constraint, then we denote with $\mathbb{X}$ the corresponding Hamiltonian vector field $\iota_{\mathbb{X}}\varpi^\partial_{PC} = \delta X$ and with $\mathbb{X}_e\ \mathbb{X}_\omega$ its components, i.e.

$$\mathbb{X} = \int \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega}.$$

Hence we have

$$\begin{aligned}&\mathbb{L}_e = [c, e] && \mathbb{L}_\omega = d_\omega c \\ &\mathbb{P}_e = \quad L^{\omega_0}_\xi e && \mathbb{P}_\omega = \quad L^{\omega_0}_\xi(\omega\ \ \omega_0) \quad \iota_\xi F_{\omega_0} \\ &e^{N\ \ 3}\mathbb{R}_e = [\tau, e] && e^{N\ \ 3}\mathbb{R}_\omega = g(\tau,\omega,e) + d_\omega\tau\end{aligned}$$

$$e^{N\ \ 3}\mathbb{H}_e = \frac{1}{(N\ \ 3)} e^{N\ \ 3} d_\omega(\lambda e_n) + \lambda e^{N\ \ 3}\sigma \quad [\tau^0, e]$$

$$\begin{aligned}e^{N\ \ 3}\mathbb{H}_\omega = {} & \lambda e_n e^{N\ \ 4} F_\omega + \frac{1}{(N\ \ 2)!}\Lambda\lambda e_n e^{N\ \ 2} \quad (N\ \ 4)\lambda e_n e^{N\ \ 5}(\omega\ \ \omega_0) p_{\mathcal{T}}(d_\omega e) \\ & g(\tau^0,\omega,e) \quad d_\omega\tau^0.\end{aligned}$$

The components $\mathbb{R}_\omega$ and $\mathbb{H}_\omega$ are uniquely determined requiring the structural constraint (5.3b). The components $\mathbb{R}_e$ and $\mathbb{H}_e$ are recovered by inversion of $W_{N\ \ 3}^{\partial,(1,1)}$ (which is possible thanks to

Lemma 3.12). Following these we compute the Poisson brackets between the constraints and analyse their structure. The brackets between L_c and P_ξ are the same as in the non-degenerate case presented in Theorem 4.10:

$$\{\mathrm{L}_c, \mathrm{L}_c\} = -\frac{1}{2}\mathrm{L}_{[c,c]}; \quad \{\mathrm{L}_c, P_\xi\} = \mathrm{L}_{L_\xi^{\omega_0} c}; \quad \{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}\mathrm{L}_{\iota_\xi\iota_\xi F_{\omega_0}}.$$

Let us now compute the brackets between L_c, P_ξ and R_τ. In both computations we use the results of Lemmas 5.8 and 5.10 and the properties of τ.

$$\begin{aligned}
\{\mathrm{L}_c, R_\tau\} &= \int [c,e]g(\tau,\omega,e) + [c,e]d_\omega\tau + d_\omega c[\tau,e]\\
&= \int [c,e]g(\tau,\omega,e) - [c,\tau]d_\omega e\\
&= \int p'_S[c,\tau]d_\omega e - [c,\tau]d_\omega e = \int -p_S[c,\tau]d_\omega e = -R_{p_S[c,\tau]};
\end{aligned}$$

$$\begin{aligned}
\{P_\xi, R_\tau\} &= \int -[\tau,e]L_\xi^{\omega_0}(\omega-\omega_0) - [\tau,e]\iota_\xi F_{\omega_0} - L_\xi^{\omega_0}eg(\tau,\omega,e) - L_\xi^{\omega_0}ed_\omega\tau\\
&= \int -L_\xi^{\omega_0}eg(\tau,\omega,e) + L_\xi^{\omega_0}\tau d_\omega e\\
&= \int -p'_S L_\xi^{\omega_0}\tau d_\omega e + L_\xi^{\omega_0}\tau d_\omega e = \int p_S L_\xi^{\omega_0}\tau d_\omega e = R_{p_S L_\xi^{\omega_0}\tau}.
\end{aligned}$$

We now compute the brackets between L_c, P_ξ and H_λ.

$$\begin{aligned}
\{\mathrm{L}_c, H_\lambda\} = \int & [c,e]e^{N-4}\lambda e_n F_\omega + \frac{1}{(N-2)!}[c,e]\Lambda\lambda e_n e^{N-2} - [c,e]g(\tau',\omega,e)\\
&- [c,e]d_\omega\tau' - (N-4)[c,e]\lambda e_n e^{N-5}(\omega-\omega_0)p_T(d_\omega e)\\
&+ \frac{1}{(N-3)}e^{N-3}d_\omega c d_\omega(\lambda e_n) + e^{N-3}d_\omega c\lambda\sigma - d_\omega c[\tau',e]\\
= \int & -\frac{1}{(N-3)}[c,\lambda e_n]e^{N-3}F_\omega - \frac{1}{(N-1)!}\Lambda[c,\lambda e_n]e^{N-1}\\
&+ p_S([c,\tau'] - \lambda e_n e^{N-4}d_\omega c - [c,e^{N-4}]\lambda e_n(\omega-\omega_0))d_\omega e\\
= \int & -\frac{1}{(N-3)}\left([c,\lambda e_n]^{(a)}e_a e^{N-3}F_\omega - [c,\lambda e_n]^{(n)}e_n e^{N-3}F_\omega\right)\\
&- \frac{1}{(N-1)!}\Lambda[c,\lambda e_n]^{(n)}e_n e^{N-1} - p_S(\lambda e_n e^{N-4}d_{\omega_0}c)d_\omega e\\
&+ p_S([c,\lambda e_n]^{(a)}e_a e^{N-4}(\omega-\omega_0) + [c,\lambda e_n]^{(n)}e_n e^{N-4}(\omega-\omega_0))d_\omega e\\
= & -P_{[c,\lambda e_n]^{(a)}} + \mathrm{L}_{[c,\lambda e_n]^{(a)}(\omega-\omega_0)_a} - H_{[c,\lambda e_n]^{(n)}}\\
&+ R_{p_S([c,\lambda e_n]^{(a)}e_a e^{N-4}(\omega-\omega_0))} - R_{p_S(\lambda e_n e^{N-4}d_{\omega_0}c)};
\end{aligned}$$

$$\begin{aligned}
\{P_\xi, H_\lambda\} = \int & -L_\xi^{\omega_0}e\lambda e_n e^{N-4}F_\omega - \frac{1}{(N-2)!}\Lambda L_\xi^{\omega_0}e\lambda e_n e^{N-2} + L_\xi^{\omega_0}eg(\tau',\omega,e)\\
&+ L_\xi^{\omega_0}ed_\omega\tau' + (N-4)L_\xi^{\omega_0}e\lambda e_n e^{N-5}(\omega-\omega_0)p_T(d_\omega e)\\
&- \left(L_\xi^{\omega_0}(\omega-\omega_0) + \iota_\xi F_{\omega_0}\right)\left(\frac{e^{N-3}d_\omega(\lambda e_n)}{N-3} + \lambda e^{N-3}\sigma - [\tau',e]\right)
\end{aligned}$$

$$
\begin{aligned}
=\int & \frac{1}{(N-3)} L_\xi^{\omega_0}(\lambda e_n) e^{N-3} F_\omega + \frac{1}{(N-1)!} \Lambda e^{N-1} L_\xi^{\omega_0}(\lambda e_n) \\
& + p_{\mathcal{S}} \left(-L_\xi^{\omega_0} \tau^\partial + \lambda e_n e^{N-4} \left(L_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0} \right) \right) d_\omega e \\
& + p_{\mathcal{S}} (L_\xi^{\omega_0}(e^{N-4}) \lambda e_n (\omega - \omega_0)) d_\omega e \\
=\int & \frac{1}{(N-3)} \left(L_\xi^{\omega_0}(\lambda e_n)^{(a)} e_a e^{N-3} F_\omega + L_\xi^{\omega_0}(\lambda e_n)^{(n)} e_n e^{N-3} F_\omega \right) \\
& + \frac{1}{(N-1)!} \Lambda e^{N-1} L_\xi^{\omega_0}(\lambda e_n)^{(n)} e_n + p_{\mathcal{S}}(\lambda e_n e^{N-4} \iota_\xi F_{\omega_0}) d_\omega e \\
& - p_{\mathcal{S}} \left(L_\xi^{\omega_0}(\lambda e_n)^{(n)} e_n e^{N-4} (\omega - \omega_0) + L_\xi^{\omega_0}(\lambda e_n)^{(a)} e_a e^{N-4} (\omega - \omega_0) \right) d_\omega e \\
= & P_{L_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{L_\xi^{\omega_0}(\lambda e_n)^{(n)}} - \mathrm{L}_{L_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega - \omega_0)_a} \\
& - R_{p_{\mathcal{S}}(L_\xi^{\omega_0}(\lambda e_n)^{(a)} e_a e^{N-4}(\omega - \omega_0))} + R_{p_{\mathcal{S}}(\lambda e_n e^{N-4} \iota_\xi F_{\omega_0})}.
\end{aligned}
$$

We now compute the remaining brackets $\{R_\tau, R_\tau\}$, $\{R_\tau, H_\lambda\}$ and $\{H_\lambda, H_\lambda\}$. Since H_λ contains terms proportional to R_τ (for $\tau = p_{\mathcal{S}}(\lambda e_n e^{N-4}(\omega - \omega_0))$) we first compute the brackets between two R_τ and then the others:

$$\{R_\tau, R_\tau\} = \int W_N^{-1}{}_3([\tau, e]) g(\tau, \omega, e) + W_N^{-1}{}_3([\tau, e]) d_\omega \tau.$$

The first term is proportional to $d_\omega e$ by construction, so it will be 0 on shell. Let us concentrate on the second term. We want to prove, using normal geodesic coordinates, that it is not proportional to any of the constraints and not 0. Let us fix a point $p \in \Sigma$ and consider an open neighbourhood U of it. From Proposition 3.9 we deduce that the unique components at the point p with respect to the standard basis that compose τ are $X_{\mu_2}^{\mu_1}, Y_\mu$ for $\mu, \mu_1, \mu_2 = 1 \dots N-2$ subject to

$$\sum_{\mu=1}^{N-2} Y_\mu = 0 \text{ and } X_{\mu_1}^{\mu_2} = -X_{\mu_2}^{\mu_1}.$$

The first equation holds also on the whole neighborhood while the second set holds only on the point p. From Corollary 3.13 we know that the non-zero components in $W_N^{-1}{}_3([\tau, e])$ are

$$
\begin{aligned}
[W_N^{-1}{}_3([\tau, e])]_{\mu_1}^{\mu_2} &\propto X_{\mu_1}^{\mu_2} \\
[W_N^{-1}{}_3([\tau, e])]_{\mu}^{\mu} &\propto Y_\mu
\end{aligned}
$$

such that $\sum_{\mu=1}^{N-2} [W_N^{-1}{}_3([\tau, e])]_\mu^\mu = 0$ and $[W_N^{-1}{}_3([\tau, e])]_{\mu_1}^{\mu_2} = -[W_N^{-1}{}_3([\tau, e])]_{\mu_2}^{\mu_1}$.

Furthermore, from Proposition 3.9 we also know that the non-zero components of τ are Y_μ and $X_{\mu_1}^{\mu_2}$ such that

$$\sum_{\mu=1}^{N-2} Y_\mu = 0 \text{ and } X_{\mu_1}^{\mu_2} = f(\widetilde{g}^\partial, X_{\mu_2}^{\mu_1}, Y_\mu)$$

for $\mu_1 < \mu_2$ and some linear function f. Remembering that $W_{N-3}^{-1}([\tau, e])d_\omega\tau$ should be a volume form, we deduce that, on shell,

$$\begin{aligned} W_{N-3}^{-1}([\tau, e])d_\omega\tau &= \left(\sum_{\mu=1}^{N-2} Y_\mu \partial_{N-1} Y_\mu + \sum_{\mu_1,\mu_2=1}^{N-2} X_{\mu_1}^{\mu_2}\partial_{N-1} X_{\mu_2}^{\mu_1}\right)\mathbf{V} \\ &= \left(\sum_{\mu=1}^{N-2} Y_\mu \partial_{N-1} Y_\mu + \sum_{\mu_1<\mu_2 \, \mu_1,\mu_2=1}^{N-2} X_{\mu_1}^{\mu_2}\partial_{N-1} f(\widetilde{g}^\partial, X_{\mu_1}^{\mu_2}, Y_\mu)\right)\mathbf{V} \\ &=: F_{\tau\tau} \end{aligned}$$

where $\mathbf{V} = e_1 \dots e_{N-1} e_n dx^1 \dots dx^{N-1}$. This quantity is for generic τ different from zero, on shell. Hence

$$\{R_\tau, R_\tau\} - F_{\tau\tau} \approx 0.$$

With this result we can more easily compute the last two brackets:

$$\begin{aligned} \{H_\lambda, H_\lambda\} = \int & \left(\frac{1}{(N-3)} d_\omega(\lambda e_n) + \lambda\sigma - W_{N-3}^{-1}([\tau^0, e])\right)\lambda e_n e^{N-4} F_\omega \\ &+ \left(\frac{1}{(N-3)} d_\omega(\lambda e_n) + \lambda\sigma - W_{N-3}^{-1}([\tau^0, e])\right)\frac{1}{(N-2)!}\Lambda\lambda e_n e^{N-2} \\ &- \left(\frac{1}{(N-3)} d_\omega(\lambda e_n) + \lambda\sigma - W_{N-3}^{-1}([\tau^0, e])\right) g(\tau^0, \omega, e) \\ &- \left(\frac{1}{(N-3)} d_\omega(\lambda e_n) + \lambda\sigma - W_{N-3}^{-1}([\tau^0, e])\right) d_\omega\tau^0 \\ &- (N-4)\frac{1}{(N-3)} d_\omega(\lambda e_n)\lambda e_n e^{N-5}(\omega-\omega_0)p_T(d_\omega e) \\ &- (N-4)\left(\lambda\sigma - W_{N-3}^{-1}([\tau^0, e])\right)\lambda e_n e^{N-5}(\omega-\omega_0)p_T(d_\omega e). \end{aligned}$$

Since λ and e_n are odd quantities and $\tau^0 = \lambda p_S(e_n e^{N-4}(\omega-\omega_0))$, the terms in the first two lines and in the last two vanish. Furthermore the last terms of the third and fourth lines are the one composing the brackets $\{R_{\tau^0}, R_{\tau^0}\}$. Expanding the first and the second term of the third line we get

$$\widetilde{\rho}^{-1}([\tau^0, d_\omega(\lambda e_n)])d_\omega e + W_e^{-1}(\tau^0 d_\omega(\lambda e_n))d_\omega e + \widetilde{\rho}^{-1}([\tau^0, \lambda\sigma])d_\omega e + W_e^{-1}(\tau^0\lambda\sigma)d_\omega e.$$

All these terms are zero since they encompass terms with either $\lambda\lambda = 0$ or $e_n e_n = 0$. We can draw the same conclusion also for the following term:

$$d_\omega(\lambda e_n)d_\omega\tau^0 = [F_\omega, \lambda e_n]\tau^0 = 0.$$

The same holds also for the term $\lambda\sigma d_\omega\tau^0$ since both σ and τ^0 contain e_n.[7] Hence

$$\{H_\lambda, H_\lambda\} = \{R_{\tau^0}, R_{\tau^0}\} - F_{\tau^0\tau^0} \approx 0.$$

[7] Using the lemmas in Section 3.1 it is possible to prove that all the non-zero components of σ are in the direction of e_n.

The last bracket that we have to compute is $\{R_\tau, H_\lambda\}$. From the expression of the Hamiltonian vector fields we get

$$
\begin{aligned}
\{R_\tau, H_\lambda\} = \int & \frac{1}{(N-3)} d_\omega(\lambda e_n)\,(g(\tau,\omega,e) + d_\omega\tau) + \lambda\sigma g(\tau,\omega,e) + \lambda\sigma d_\omega\tau \\
& + W_{N-3}^{-1}([\tau,e])\lambda e_n e^{N-4} F_\omega - W_{N-3}^{-1}([\tau^0,e])\,(g(\tau,\omega,e) + d_\omega\tau) \\
& + \frac{1}{(N-2)!}\Lambda\lambda e_n e[\tau,e] - W_{N-3}^{-1}([\tau,e])\,(g(\tau^0,\omega,e) + d_\omega\tau^0) \\
& - (N-4) W_{N-3}^{-1}([\tau,e])\lambda e_n e^{N-5}(\omega - \omega_0) p_{\mathcal{T}}(d_\omega e).
\end{aligned}
$$

The last two terms of the second and third lines are the one composing the brackets $\{R_\tau, R_{\tau^0}\}$, and the first term of the third line vanishes because $e\tau = 0$ and $[e,e] = 0$. We want to prove that $\{R_\tau, H_\lambda\} \not\approx 0$. Using coordinate expansion one can prove that the second and the fifth term have the same expression and read:

$$
\begin{aligned}
d_\omega(\lambda e_n) d_\omega\tau + W_e^{-1}([\tau,e])\lambda e_n F_\omega &= -[F_\omega, \lambda e_n]\tau + W_e^{-1}([\tau,e])\lambda e_n F_\omega \\
&= 2\lambda \sum_{\mu=1}^{N-2} Y_\mu (F_\omega)^{\mu N-1}_{\mu N-1} + \lambda \sum_{\mu_1,\mu_2=1}^{N-2} X^{\mu_2}_{\mu_1} (F_\omega)^{\mu_1 N-1}_{\mu_2 N-1} =: G_{\lambda\tau}.
\end{aligned}
$$

These terms are not proportional to any of the constraints and not proportional to $\{R_\tau, R_{\tau^0}\}$. The term in the fourth line is proportional to R_τ so we can discard it. Let us now consider the fourth term: since $d_\omega\tau$ is in the image of W_e we can invert it and get

$$
\begin{aligned}
\lambda\sigma d_\omega\tau &= \lambda e^{N-3}\sigma W^{-1}(d_\omega\tau) \\
&= \lambda e_n e^{N-4} d_\omega e W^{-1}(d_\omega\tau) - \lambda e_n e^{N-4} p_{\mathcal{T}}(d_\omega e) W_e^{-1}(d_\omega\tau).
\end{aligned}
$$

The second term is again proportional to R_τ so we can discard it as well. Let us now consider the first term of this expression and $d_\omega(\lambda e_n) g(\tau,\omega,e) + \lambda\sigma g(\tau,\omega,e)$ — the last two remaining terms. By expanding these terms using the definition of f, integrating by parts and using $\tau \wedge e_n = 0$ we get that these three terms add up to zero. Collecting these results we get

$$
\{R_\tau, H_\lambda\} \approx \{R_\tau, R_{\tau^0}\} + G_{\lambda\tau} \approx F_{\tau\tau^0} + G_{\lambda\tau} \not\approx 0.
$$

□

Remark 5.13. For $N = 4$ some of the previous computations simplify. In particular it is possible to give a compact explicit expression for the function $F_{\tau\tau}$. This coincides with the corresponding one of the linearized theory $\widetilde{F}_{\tau\tau}$ expressed in [CCT20, eq. (32)]:

$$
\begin{aligned}
\{R_\tau, R_\tau\} &= \int \left(-\tau_1^{134}\partial_3\tau_2^{234} - \tau_2^{234}\partial_3\tau_1^{134} + \tau_2^{134}\partial_3\tau_1^{234} + \tau_1^{234}\partial_3\tau_2^{134} \right) \mathbf{V} \\
&= \int \left(2\tau_1^{134}\partial_3\tau_1^{134} + \tau_1^{234}\partial_3\tau_1^{234} + \tau_1^{234}\partial_3 \frac{\tau_1^{234} g_{22} - 2\tau_1^{134} g_{12}}{g_{11}} \right) \mathbf{V}
\end{aligned}
$$

where $\mathbf{V}$ was defined above as $\mathbf{V} = e_1 \dots e_{N-1} e_n dx^1 \dots dx^{N-1}$. As a consequence it is also possible to give an explicit expression for the other brackets not proportional to the constraints.

Remark 5.14. As outlined in [CCT20], in the linearized case we can identify some first class zero modes inside the second class constraint. In the non-linearized case such identification is more complicated but such modes should anyway be present. This will be object of future studies.

Corollary 5.15. *The constraints* L_c*,* P_ξ*,* H_λ *and* R_τ *do not form a first class system. In particular* R_τ *is a second class constraint while the others are first class (as defined in Remark 5.19 in Appendix 5.A).*

Proof. Throughout the proof we use the notation and terminology established in Appendix 5.A. Since the bracket between R_τ and itself is not zero on shell the system contains constraints that are second class. We want now to establish which constraints are of second class and which are of first class. The constraints L_c and P_ξ commute —on shell— with themselves and all the other constraints, hence they are of first class. Let us now consider R_τ and H_λ. We want to prove that R_τ is of second class while, using a linear transformation of the constraints H_λ is of first class. Using the result of Proposition 5.18, if we call D the matrix representing the bracket $\{R_{\tau'}, R_\tau\}$, B the one representing the bracket $\{R_\tau, H_\lambda\}$, and C the one representing the bracket $\{H_\lambda, H_\lambda\}$, we have to prove that $B^T D^{-1} B = -C$.

From the proof of Theorem 5.11 we can deduce the expressions of the matrices B, D and C. All the components of such matrices contain a derivative in the *lightlike* direction, apart from the terms coming from $G_{\lambda\tau}$ in B. Hence all components of D^{-1} will contain the inverse of such derivative. Since λ is an odd quantity, all the terms contained in $B^T D^{-1} B$ without a derivative vanish because of Lemma 5.21. Hence the only surviving elements in $B^T D^{-1} B$ come from the multiplication of the elements containing a derivative in B. We denote such terms by B'. It is then a straightforward computation to check that the coefficients of such combination are actually equal to those of C. Indeed, since these matrices have the same functional form $(F_{\tau\tau})$, we can express the matrices B' and C respectively as $B' = Dp_S(e_n e^{N-4}(\omega - \omega_0))$ and $C = p_S(e_n e^{N-4}(\omega - \omega_0)) D p_S(e_n e^{N-4}(\omega - \omega_0))^T$. Hence we have

$$
\begin{aligned}
B'^T D^{-1} B' &= p_S(e_n e^{N-4}(\omega - \omega_0))^T D D^{-1} D p_S(e_n e^{N-4}(\omega - \omega_0)) \\
&= -p_S(e_n e^{N-4}(\omega - \omega_0))^T D p_S(e_n e^{N-4}(\omega - \omega_0)) = -C.
\end{aligned}
$$

□

Remark 5.16. Using the formula of Remark 5.20 we can reconstruct a constraint H' that Poisson commutes with itself and all other constraints. However, in doing so, we loose the locality of H.

We can now count the degrees of freedom of the reduced phase space. From the definition given in Section 5.A we can deduce that the correct number of physical degrees of freedom is given by the following formula [HT92, (1.60)]:

$$
r = p - 2f - s.
$$

where r is the number of degrees of freedom of the reduced phase space, p the number of degrees of freedom of the geometric phase space, f the number of first class constraints and s the number of second class constraints.

In our case these quantities have the following values: the geometric phase space has $2N(N-1)$ degrees of freedom. From Corollary 5.15 we have that there are $\frac{N(N-1)}{2} + N = \frac{N(N+1)}{2}$ first class constraints and $\frac{N(N-3)}{2}$ second class constraints (see Proposition 3.9 for the number of degrees of freedom of τ). We can deduce that the correct number of local degrees of freedom is given by

$$
2N(N-1) - N(N+1) - \frac{N(N-3)}{2} = \frac{N(N-3)}{2}.
$$

In the case $N = 4$ this computation produces two local degrees of freedom. This result agrees with the previous works in the literature (e.g. [AS15]).

Appendix

5.A First and Second class constraints

An important distinction between the constraints of a system is the one provided by the difference between first and second class constraints. In this section we review the definition and prove a result to easily distinguish the two classes.

Roughly speaking, a constraint is of second class if its Poisson brackets with other constraints do not vanish on the constrained surfaces. However, this definition is not precise since it is always possible to take linear combinations of the constraints without modifying the reduced phase space of the theory. Furthermore first and second class constraints correspond to different physical interpretations: the first ones are in one to one correspondence with the generators of *residual gauge transformations* of the theory, while the second ones are just identities through which we can express some canonical variables in terms of the others. Hence, to correctly encompass these differences, we need a more sophisticated definition. Starting from the results presented in [HT92, Chapter 1] we can give the following definition:

Definition 5.17. Let F be a symplectic manifold and let $\phi_i \in C^\infty(F)$ be a set of smooth functions on it. Denote with $C_{ij} = \{\phi_i, \phi_j\}$ the matrix of the Poisson brackets of the functions. Then the number of second class functions of the set is the rank[8] of the matrix C_{ij} on the zero locus of the functions. In particular if $C_{ij} \equiv 0$ then we say that all the functions are first class.

This definition clearly coincides with the standard one in case all the constraints are first class, i.e. all the constraints commute with every other one. However, it allows us to treat the general case, since it is invariant under rearranging the constraints by linear combinations. We now state a pair of results that are helpful in assessing the number of second class constraints in a system.

Proposition 5.18. *Let F be a symplectic manifold and let $\psi_i, \phi_j \in C^\infty(F)$, $i = 1 \dots n$, $j = 1 \dots m$. Denote with $C_{jj'}, B_{ij}, D_{ii'}$ respectively the matrices representing the Poisson brackets $\{\phi_j, \phi_{j'}\}$, $\{\psi_i, \phi_j\}$ and $\{\psi_i, \psi_{i'}\}$, with $i, i' = 1 \dots n$, $j, j' = 1 \dots m$. Then, if D is invertible and $C = -B^T D^{-1} B$, the number of second class constraints is n, i.e. the rank of the matrix D.*

Remark 5.19. In this case, we will say that the ϕ's are the first class constraints and the ψ's the second class constraints of the system.

Proof. The matrix representing the Poisson brackets has the form

$$P = \begin{pmatrix} C & B^T \\ B & D \end{pmatrix}$$

where the blocks are as in the statement. We want to prove that this matrix is congruent to one of rank n i.e. that there exists an invertible matrix Q such that $Q^T P Q$ has rank n. Since D is invertible, we can build Q as follows:

$$Q = \begin{pmatrix} 1 & 0 \\ -D^{-1}B & 1 \end{pmatrix}.$$

An easy computation shows that

$$Q^T P Q = \begin{pmatrix} C + B^T D^{-1} B & 0 \\ 0 & D \end{pmatrix}.$$

Hence, using the second hypothesis $C = -B^T D^{-1} B$ we get the claim. □

[8] We assume the rank to be constant on the zero locus.

Remark 5.20. This result shows explicitly that a naive definition of first class constraint as the one commuting with everything else is not sufficient to correctly consider more involved cases where the constraints do not commute (under the Poisson brackets) on the nose, but there are linear combinations of them that do. In this specific setting, from the proof of the Proposition, we gather that we can consider the set of functions

$$\widetilde{\phi}_j = \phi_j + \sum_{i,i^0} B_{ij} D^{ii^0} \psi_{i^0}; \qquad \widetilde{\psi}_i = \psi_i$$

and conclude that the functions $\widetilde{\phi}_j$ are first class (in the classical sense) and $\widetilde{\psi}_i$ are second class.

Lemma 5.21. *Let D be an invertible matrix such that the inverse does not contain derivatives and let B some matrix proportional to an odd parameter λ and not containing derivatives. Then $BD^{\ 1}B^T = 0$.*

Proof. The key point of the proof is that every term containing λ^2 vanishes since λ is an odd quantity. Now, by hypothesis every term in $BD^{\ 1}B^T$ does not contain derivatives and since this expression is quadratic in λ vanishes because of the previous consideration. □

Chapter 6

The AKSZ construction and cylindrical gravity

This chapter is devoted to the application of the Alexandrov–Kontsevich–Schwarz–Zaboronski (AKSZ) construction and its application to Einstein–Hilbert and Palatini–Cartan BFV theories. These have been described in [CS17] (for N 3) and in Chapters 2 (for $N = 3$) and 4 (for N 4) respectively. In this chapter we will assume that the metric encoded in the BFV data F^∂ is non-degenerate (i.e. assuming that the manifold Σ is either spacelike or timelike but not lightlike).

As we already described in the Introduction, the AKSZ construction produces a BV theory $\mathit{AKSZ}(\mathsf{F}^\partial)$ on Σ I given a BFV theory F^∂ associated to a manifold Σ. Moreover, the resulting BV theory can automatically be extended to a BV-BFV theory, reproducing the original BFV theory on Σ seen as (one component of) the boundary of Σ I.

Sometimes, the boundary BFV theory F^∂ can be constructed directly from a BV theory F, hence we can compare the two theories F and $\mathit{AKSZ}(\mathit{BFV}(\mathsf{F}))$. This is the case of EH theory and of PC theory in three dimension. For the first, we show that F and $\mathit{AKSZ}(\mathit{BFV}(\mathsf{F}))$ are effectively equivalent, with the former being actually the first-order formulation of the latter. In the case of PC in three dimensions, where we know from Chapter 2 that $\mathit{BFV}(\mathsf{F}) = \mathsf{F}^\partial$ holds, we show that $\mathit{AKSZ}(\mathit{BFV}(\mathsf{F}))$ and F are strongly BV equivalent, which is not unexpected, since PC is strongly BV equivalent to BF theory [CSS18], and the latter is a topological AKSZ theory (i.e. satisfying some conditions inducing simplifications in the construction).

Instead, for higher dimensional PC theory we show that $\mathit{AKSZ}(\mathsf{F}^\partial)$ and F are classically equivalent, with F^∂ the BFV data constructed from the reduced phase space of PC theory (see Chapter 4). This case is particularly interesting because the BV-BFV construction for PC is obstructed in dimension 4 (and presumably higher, see Section 1.2.6 for a brief recap). The data $\mathit{AKSZ}(\mathsf{F}^\partial)$ resulting from the AKSZ construction is a new BV theory defined on cylinders that is still classically equivalent to EH, but also compatible with the BV-BFV formalism (by construction via the AKSZ procedure). Classically, it is simply PC on a smaller space of fields, where part of the torsion-free condition is imposed a priori instead of through the Euler–Lagrange equations.

The result of this chapter addresses the problem presented in [CS19b], where it was pointed out that PC theory in dimension greater than three must be complemented with requirements on field configurations at the boundary in order to induce a well-defined BV-BFV structure. This is a necessary requirement for the quantisation scheme of BV theories with boundary [CMR18]. The fact that the boundary-compatible AKSZ version of PC theory is only classically equivalent to the original PC formulation reinforces the idea that care must be placed when attempting BV quantisation of the latter.

A related approach is the 'parent formulation' by Barnich and Grigoriev [BG11; Gri11] which derives an AKSZ construction of the BV theory from the jet space formalism (trivariational complex). What is crucially different in our construction is that we consider, as a target, a symplectic description of the classical boundary states. This involves a careful symplectic reduction of the naively associated boundary spaces[1]. The result of our construction is not only a BV reformulation of the original bulk theory, but a reformulation that is compatible with the boundary as a 1-extended BV-BFV theory (see Definition 5), which is the starting point for quantum (or at least semiclassical) considerations for a theory with boundary [CMR18].

For the same reason, unlike the presymplectic AKSZ formulation presented by Grigoriev et Alkalaev in [AG14] and [Gri16], our BV-BFV description of PC gravity is based on a symplectic structure, which is essential for quantization. This does not arise directly from a reduction of the natural presymplectic BFV structure derived from BV in the bulk, which is impossible for N 4 as shown in [CS19b], but it is the symplectic BFV structure (the one described in Chapter 4) that resolves the reduced phase space of the theory.

Finally, note that in this chapter we consider two separate applications of the AKSZ "reconstruction" of a parametrization-invariant bulk BV theory from its boundary BFV structure, respectively for two formulations of GR (EH and PC). We do not discuss the equivalence between EH and PC, but we investigate the appropriate BV equivalence between each formulation and its own AKSZ "reconstruction."

These considerations do not exclude, however, some deeper connection between our construction and the ones mentioned above, which are definitely worth exploring.

The chapter is organised as follow. In Sections 1.1 and 6.1.1 we will outline the BV-BFV and AKSZ constructions, while Section 6.2 is a brief review of the construction of the BFV data for Einstein–Hilbert and Palatini–Cartan theories of gravity.

Finally, in Sections 6.3 and 6.4 we will apply the AKSZ construction to the BFV data of EH and PC gravity, respectively, and compare it with the BV data for the two formulations as presented in [CS16] and [CS19b].

6.1 Background

In this section we introduce formally the AKSZ formalism and recap the information of the BFV theories on which we will apply the construction.

6.1.1 The AKSZ construction

Let X be a graded manifold and N an ordinary manifold.

Definition 6.1 (Transgression map)**.** Consider the map

$$\mathsf{T}_N^{(\ )} : \Omega\ (X) \to \Omega\ (\mathrm{Map}(T[1]N, X)) \tag{6.1}$$

defined by $\mathsf{T}_N^{(\ )} := p\ \mathrm{ev}$, where

$$\begin{array}{ccc} \mathrm{Map}(T[1]N, X) & T[1]N & \xrightarrow{\mathrm{ev}} X \\ \Big\downarrow p & & \\ \mathrm{Map}(T[1]N, X) & & \end{array} \tag{6.2}$$

[1] This association is the natural restriction of fields and normal jets to the boundary, see [CMR14].

and we set $p_* = \int_N \mu_N$ where μ_N is the canonical Berezinian on $T[1]N$. We will call $\mathsf{T}_N^{(\bullet)}$ the transgression map, and its evaluation a transgression.

We endow the graded manifold X with a function S of degree n and parity n mod 2, together with a one-form α of degree $n-1$ and parity $n-1$ mod 2, such that $\varpi = d\alpha$ is non-degenerate and $\{S, S\} = 0$ with respect to the Poisson structure defined by ϖ. Then we say that X has dg Hamiltonian structure, with differential $\{S, \cdot\}$.

Observing that the de Rham differential d_N on N can be seen as a degree 1 vector field on $\mathrm{Map}(T[1]N, X)$ we have

Theorem 6.2 ([Ale+97]). *Let (X, S, α) a Hamiltonian dg-manifold as described above. Consider the data*

$$\mathsf{F}^{AKSZ}(N; X, S, \alpha) := (F^{AKSZ}, S^{AKSZ}, \varpi^{AKSZ}, Q^{AKSZ})$$

with $F^{AKSZ} = \mathrm{Map}(T[1]N, X)$, $\varpi^{AKSZ} := \mathsf{T}_N^{(2)}(\varpi)$, the functional $S^{AKSZ} \colon F^{AKSZ} \to \mathbb{R}$,

$$S^{AKSZ} := \mathsf{T}_N^{(0)}(S) + \iota_{d_N} \mathsf{T}_N^{(1)}(\alpha). \tag{6.3}$$

and the cohomological vector field Q^{AKSZ} such that $\iota_{Q^{AKSZ}} \varpi^{AKSZ} = \delta S^{AKSZ}$. Then, $\mathsf{F}^{AKSZ}(N; X, S, \alpha)$ defines a BV theory.

We will call $F^{AKSZ} := \mathrm{Map}(T[1]N, X)$ the ***AKSZ space of fields***. Introducing Darboux coordinates $\{p_i, q^i\}$ in X so that $\alpha = p_i dq^i$, the space of AKSZ fields is composed of inhomogeneous differential forms P, Q on N. Then, if we consider X to be the space of sections of a bundle $E \to \Sigma$, that is to say $X = T^*[n-1]C^\infty(\Sigma, E)$, we can write

$$\varpi^{AKSZ} = \int_N \langle \delta \mathsf{P}, \delta \mathsf{Q} \rangle^{\mathrm{top}} \equiv \int_N \left[\delta \mathsf{P}_i \delta \mathsf{Q}^i\right]^{\mathrm{top}}$$

where we have denoted by δ the deRham differential on spaces of maps and C^∞-sections, and top denotes the top-form parts of the inhomogeneous differential forms within brackets. We will drop the superscript top in what follows.

Consider this elementary fact:

Lemma 6.3. *Let A, B, C be graded manifolds, $\phi \colon B \to C$ an isomorphism of graded manifolds, and μ_A a measure on A. Consider the diagram*

$$\begin{array}{ccc} A \times B & \xrightarrow{\mathrm{id} \times \phi} & A \times C \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_C} \\ B & \xrightarrow{\phi} & C \end{array}$$

Then, setting $\pi_{B*} = \int \mu_A$ and $\pi_{C*} = \int \mu_A$, we have $\phi^* \circ \pi_{C*} = \pi_{B*} \circ (\mathrm{id} \times \phi)^*$.

Theorem 6.4. *Let (X, S_X, α_X) and (Y, S_Y, α_Y) be equivalent Hamiltonian dg-manifolds, i.e. there exists a diffeomorphism $\phi \colon X \to Y$ such that $\varpi_X = \phi^* \varpi_Y$, and $S_X = \phi^* S_Y$. Then $\mathsf{F}^{AKSZ}(N; X, S_X, \alpha_X)$ and $\mathsf{F}^{AKSZ}(N; Y, S_Y, \alpha_Y)$ are strongly equivalent BV(-BFV) theories for every manifold N.*

Proof. $\phi \colon X \to Y$ induces an isomorphism

$$\tilde{\phi} \colon \mathrm{Maps}(T[1]N, X) \to \mathrm{Maps}(T[1]N, Y)$$

by precomposing maps with ϕ or ϕ^{-1}. Then, we can apply Lemma 6.3 with $A = T[1]N$, $B = \mathrm{Maps}(T[1]N, X)$ and $C = \mathrm{Maps}(T[1]N, Y)$. □

6.1.2 One-dimensional AKSZ construction

Let $I \quad \mathbb{R}$ be an interval, and F^∂ an exact BFV theory. We can construct a BV theory by applying Theorem 6.2 on the Hamiltonian dg-manifold underlying an exact BFV theory:

$$\mathsf{F}^{\text{AKSZ}}(I;\mathsf{F}^\partial) := \mathsf{F}^{\text{AKSZ}}(I;F^\partial,S^\partial,\alpha^\partial).$$

The resulting space of fields reads

$$F^{\text{AKSZ}} = \mathrm{Map}(T[1]I, F^\partial).$$

Since the target space F^∂ is (locally) a graded vector space, we identify the space of AKSZ fields with

$$F^{\text{AKSZ}} = \Omega\ (I) \quad F^\partial.$$

In particular, when F^∂ is modeled on sections of some bundle over a $(N \quad 1)$-dimensional manifold Σ, we can view F^{AKSZ} as the space of sections of some (graded) bundle over $\Sigma \quad I$. The space $\Omega\ (I)$ splits into:

$$\Omega\ (I) = C^{1}(I) \quad \Omega^1(I)[\ 1]$$

hence, to each field in F^∂ we associate two new fields. For simplicity we denote the field in $C^{1}(I) \quad F^\partial$ with the same letter as the old one, and use another letter for the fields in $\Omega^1[\ 1](I)$ F^∂.

Proposition 6.5 ([CMR18] and [CMR14])**.** *Let* $\mathsf{F}^\partial(\Sigma) = (F^\partial(\Sigma), S^\partial(\Sigma), \varpi^\partial(\Sigma), Q^\partial(\Sigma))$ *be an exact BFV theory, with* $F^\partial(\Sigma) := \Gamma(E\ !\ \Sigma)$*, and* $\varpi^\partial(\Sigma) = \delta\alpha^\partial(\Sigma)$*. Then, if* $I := [0,1]$ *we have that* $\mathsf{F}^{\text{AKSZ}}(I;\mathsf{F}^\partial(\Sigma))$ *is a 1-extended BV-BFV theory over* $\mathsf{F}^\partial(\Sigma)$ *(see De nition 1.5).*

Proof. Theorem 6.2 tells us that $\mathsf{F}^{\text{AKSZ}}(I;\mathsf{F}^\partial(\Sigma))$ is a BV theory (up to boundary terms). If we parametrise fields in F^{AKSZ} as

$$\mathsf{P} = p(t) + q^y(t)dt \qquad \mathsf{Q} = q(t) + p^y(t)dt$$

we get

$$\varpi^{\text{AKSZ}} = \int_I h\delta\mathsf{P}, \delta\mathsf{Q}i = \int_I \left\{h\delta p, \delta p^y i + (\ 1)^{jqj+1}h\delta q, \delta q^y i\right\} dt$$

and

$$S^{\text{AKSZ}} = \int_I hp, d_I q i + [\mathsf{T}_I^{(0)}(S^\partial(\Sigma))]^{\text{top}}.$$

The transgressed integrand needs to be first-order in dt, which leaves us with

$$[\mathsf{T}_I^{(0)}(S^\partial(\Sigma))]^{\text{top}} \quad S^\partial(\Sigma)[p + q^y dt, q + p^y dt] = \frac{\delta S^\partial(\Sigma)}{\delta p}(q^y dt) + \frac{\delta S^\partial(\Sigma)}{\delta q}(p^y dt)$$

Then Q^{AKSZ} splits in a transversal part plus a ***tangential*** one: $Q^{\text{AKSZ}} = Q^T + \hat{Q}$, where $Q^T q^y = \ \dot{p}$ and $Q^T p^y = \dot{q}$ is essentially just deRham differential on I, and $\hat{Q}$ is easily obtained:

$$\hat{Q}p = \frac{\delta S^\partial(\Sigma)}{\delta q} \quad Q^\partial p \qquad \hat{Q}q = \frac{\delta S^\partial(\Sigma)}{\delta p} \quad Q^\partial q$$

$$\hat{Q}p^{y} = \frac{\delta^2 S^\partial(\Sigma)}{\delta q \delta p}(p^{y}) + \frac{\delta^2 S^\partial(\Sigma)}{\delta p^2}(q^{y}) \qquad \hat{Q}q^{y} = \frac{\delta^2 S^\partial(\Sigma)}{\delta p \delta q}(q^{y}) + \frac{\delta^2 S^\partial(\Sigma)}{\delta q^2}(p^{y}).$$

The boundary terms are easily seen in the given local chart, in fact:

$$\iota_{Q^{\mathsf{AKSZ}}} \varpi^{\mathsf{AKSZ}} = \delta S^{\mathsf{AKSZ}} + \check{\alpha}$$

but $\check{\alpha}$ only sees contributions from hp, dqi and, up to sign, we get $\check{\alpha} = \alpha^\partial(\Sigma)$, with $\delta\alpha^\partial(\Sigma) = \varpi^\partial$. Then, the projection of Q^{AKSZ} along the natural projection map from F^{AKSZ} to the space of boundary fields, which coincides with $F^\partial(\Sigma)$, is precisely $Q^\partial(\Sigma)$, concluding the argument. □

Remark 6.6. A similar statement to Proposition 6.5 is presented in Henneaux–Bunster [HT92, Theorem 18.4.5], where one identifies the output of the above AKSZ construction with the (first-order) BV theory obtained by embedding in the BV formalism the generalised Hamiltonian formulation of a given field theory (see also [DGH90]). An analogous construction, already in the context of AKSZ theories, was presented in [GD00; BG05]. The added observation of Proposition 6.5 is the compatibility between BV for the bulk and BFV on the boundary, viz., what we call a 1-extended BV-BFV theory in Definition 1.5.

This construction behaves well under equivalences of the relevant BFV data.

Corollary 6.7 (Theorem 6.4)**.** *Let F_1^∂ and F_2^∂ be two strongly BFV-equivalent (exact) theories, then $\mathsf{F}^{AKSZ}(I;\mathsf{F}_1^\partial)$ and $\mathsf{F}^{AKSZ}(I;\mathsf{F}_2^\partial)$ are strongly BV-equivalent.*

Proof. A strong BV equivalence induces an isomorphism of the underlying dg-manifolds. □

6.2 BFV theories of gravity

We recap here the data composing the Palatini–Cartan BFV theory. The corresponding ones composing the Einstein–Hilbert BFV theory are fully detailed in Section 1.2.3.

In the Palatini–Cartan case the BFV theory is described in Chapter 2 for $N = 3$ and in Chapter 4 for N 4. We further assume that the boundary metric (4.6) is non-degenerate.

It is possible to give the structure in a unified way. We collect the relevant information in the following definition:

Definition 6.8. We define BFV Palatini–Cartan theory to be the assignment

$$\Sigma \quad \mathsf{F}^\partial_{PC}(\Sigma) = (F^\partial_{PC}(\Sigma), S^\partial_{PC}(\Sigma), \varpi^\partial_{PC}(\Sigma), Q^\partial_{PC}(\Sigma))$$

where the space of fields is

$$F^\partial_{PC}(\Sigma) \quad ! \;\; \Omega^1_{nd}(\Sigma, V),$$

with local trivialisation on an open $U \quad \Omega^1_{nd}(\Sigma, V)$

$$F^\partial_{PC}(\Sigma) \;\; ' \;\; U \quad A^{red}(\Sigma) \quad T \;\left(\Omega^{0,2}_\partial[1] \quad \mathsf{X}[1](\Sigma) \quad C^{\,1}[1](\Sigma)\right),$$

the symplectic form is

$$\varpi^\partial_{PC}(\Sigma) = \int e^{N \quad 3}\delta e \delta\omega + \delta c \delta c^{y} + \delta\omega\delta(\iota_\xi c^{y}) \quad \delta\lambda\epsilon_n \delta y^{y} + \iota_{\delta\xi}\delta(e y^{y}), \tag{6.4}$$

the action is

$$S^{\partial}_{PC}(\Sigma) = \int c e^{N-3} d_\omega e + \iota_\xi e e^{N-3} F_\omega + \epsilon_n \lambda e^{N-3} F_\omega + \frac{1}{2}[c,c]c^\dagger - L^\omega_\xi c c^\dagger$$
$$+ \frac{1}{2}\iota_\xi \iota_\xi F_\omega c^\dagger - [c, \epsilon_n \lambda] y^\dagger + L^\omega_\xi(\epsilon_n \lambda) y^\dagger + \frac{1}{2}\iota_{[\xi,\xi]} e y^\dagger, \tag{6.5}$$

and $Q^{\partial}_{PC}(\Sigma)$ the Hamiltonian vector field of $S^{\partial}_{PC}(\Sigma)$ with respect to $\varpi^{\partial}_{PC}(\Sigma)$. Furthermore, for $N \geq 4$, ω and e must satisfy the following equation

$$(N-3)\epsilon_n e^{N-4} d_\omega e \in \mathrm{Im} W_{N-3}^{\partial,(1,1)} \tag{6.6}$$

Remark 6.9. In the case $N = 3$, the additional constraint (6.6) is void. Furhtermore the signs and the name of the fields in (6.5) and in (6.4) differ from those in Proposition 2.10. In order to make contact between the formulas one has to perform the following change of variables:

$$e^\dagger \mapsto y^\dagger \qquad \omega^\dagger \mapsto c^\dagger$$
$$c \mapsto -c \qquad \xi \mapsto -\xi \qquad \xi^n \mapsto -\lambda.$$

In the case $N \geq 4$ some signs differ from the ones in Theorem 4.20 due to a flip in the sign of λ.

6.3 AKSZ Einstein–Hilbert

We explore here the idea of reconstructing the $(N+1)$-dimensional BV extension of Einstein–Hilbert theory by means of the AKSZ construction, with target the BFV data for Einstein–Hilbert theory (as presented in Section 1.2.3, based on [CS16]). In order to do this one looks at the space $\mathcal{F}^{\mathsf{AKSZ}}_{EH} := \mathrm{Maps}(T[1]I, \mathcal{F}^{\partial}_{EH}(\Sigma))$, with I an interval. In a chart, to consider the transgression map of Equation (6.1) means to look at fields composed of a 0-form and a 1-form on the interval I, with values in $\mathcal{F}^{\partial}_{EH}(\Sigma)$ and fixed total degree. In the case at hand we will then have

$$\mathsf{G} = \gamma(t) + \Pi^\dagger(t)dt \qquad \mathsf{Z}^n = \xi^n(t) + \eta(t)dt \qquad \mathsf{Z}^a = \xi^a(t) + \beta^a(t)dt$$
$$\mathsf{P} = \Pi(t) + \gamma^\dagger(t)dt \qquad \mathsf{H}_n = \varphi_n(t) + \xi^\dagger_n(t)dt \qquad \mathsf{H}_a = \varphi_a(t) + \xi^\dagger_a(t)dt$$

where, for all $t \in I$, we parametrise $\mathcal{F}^{\mathsf{AKSZ}}$ with fields[2]

$$\begin{aligned}
\gamma(t) &\in \mathrm{Map}(I, S^2_{nd}(T\Sigma)), & \gamma^\dagger(t) &\in \mathrm{Map}(I, S^2[-1](T^*\Sigma)),\\
\Pi(t) &\in \mathrm{Map}(I, S^2(T^*\Sigma)), & \Pi^\dagger(t) &\in \mathrm{Map}(I, S[-1]^2(T\Sigma)),\\
\eta(t), \beta^a(t) &\in \mathrm{Map}(I, C^\infty(\Sigma)), & \varphi_n(t), \varphi_a(t) &\in \mathrm{Map}(I, \mathrm{Dens}[-1](\Sigma)),\\
\xi^n(t), \xi^a(t) &\in \mathrm{Map}(I, C^\infty[1](\Sigma)), & \xi^\dagger_n(t), \xi^\dagger_a(t) &\in \mathrm{Map}(I, \mathrm{Dens}[-2](\Sigma)),
\end{aligned}$$

where we required $\gamma(t)$ to be non-degenerate for all $t \in I$. Now, observe that η, ξ^n are functions on Σ whereas β^a, ξ^a can be considered as the components of vector (fields) tangent to Σ, which we will denote by β and ξ^∂. Similarly, we can promote φ_a and $\xi^\dagger_a$ into Σ-density valued one forms, which we will denote by $\varphi_\partial, \xi^\dagger_\partial$. For simplicity of notation we will often use a unified index $\rho \in \{n, a\}$, so that $\varphi_\rho \in \{\varphi_n, \varphi_a\}$ and $\xi^\dagger_\rho \in \{\xi^\dagger_n, \xi^\dagger_a\}$.

In what follows it will be useful to denote the *kinetic* part of the Hamiltonian functional (Equation (1.13)) as

$$K := \frac{1}{\sqrt{\gamma}}\left(\mathrm{Tr}_\gamma[\Pi^2] - \frac{1}{N-1}\mathrm{Tr}_\gamma[\Pi]^2\right), \tag{6.7}$$

[2] The motivation for this particular choice of notation will be manifest very soon.

and the ***cosmological Einstein tensor*** with respect to a metric γ with cosmological constant Λ will be

$$\mathbf{G}[\gamma,\Lambda] = R[\gamma] + \left(\Lambda \quad \frac{1}{2}\mathrm{Tr}_\gamma R[\gamma]\right)\gamma, \tag{6.8}$$

where $R[\gamma]$ is the Ricci–Riemann tensor of γ. We also introduce a tensor-valued second order operator $\mathbf{D}_\gamma$ that on functions $\phi\ 2\ C^{1}(\Sigma)$, in a coordinate chart, acts as

$$\mathbf{D}_\gamma\phi = [\mathbf{D}_\gamma\phi]_{ab} = \gamma_{ab}\gamma^{cd}\, r_c\, r_d\phi \quad r_a\, r_b\phi. \tag{6.9}$$

Notice that we are using (non-degenerate) sections of $S^2(TM)$ instead of actual metrics. Occasionally we will need to raise/lower indices using γ and its "inverse" which we will denote by $\gamma\ ^1$. With this parametrisation in mind we can state the following:

Theorem 6.10. *The AKSZ data* $\mathsf{F}^{\mathsf{AKSZ}}_{EH}(I;\mathsf{F}^{\partial}_{EH}(\Sigma))$ *is comprised of the* (1)*-shifted symplectic manifold*

$$F^{\mathsf{AKSZ}}_{EH}\ '\ T\ [\ 1]\left(\mathsf{Map}(I, S^2_{nd}(T\ )\quad S^2(T\ )\quad C^1(\ )\quad \mathsf{X}(\ )\quad \mathsf{X}[1](\ )\quad C^1[1](\ ))\right)$$

$$\varpi^{\mathsf{AKSZ}}_{EH} = \int_I \left\{\ h\delta\gamma, \delta\gamma^y i + h\delta\Pi, \delta\Pi^y i + \delta\xi^\rho\delta\xi^y_\rho + \delta\eta\delta\varphi_n + \delta\beta^a\delta\varphi_a\right\} dt \tag{6.10}$$

and the AKSZ action functional :

$$\begin{aligned} S^{\mathsf{AKSZ}}_{EH} = \int_I \Big\{ & h\Pi,\dot\gamma i \quad h\varphi_\rho,\dot\xi^\rho i + H_n\eta + H_\partial(\beta) \quad h\gamma^y, L_{\xi^\partial}\gamma i + h\Pi^y, L_{\xi^\partial}\Pi i \\ & \left(\frac{\delta K}{\delta\gamma}(\Pi^y) + \frac{\delta K}{\delta\Pi}(\gamma^y)\right)\xi^n \quad p\overline{\gamma} h\Pi^y, \mathbf{G}[\gamma,\lambda]\xi^n + \mathbf{D}_\gamma(\xi^n) i \\ & + h\varphi_\partial,\ r_\gamma\eta\xi^n \quad \eta\, r_\gamma\xi^n + L_{\xi^\partial}\beta i \quad \varphi_n L_\beta\xi^n + \varphi_n L_{\xi^\partial}\eta \\ & + \left\langle \xi^y_\partial, \frac{1}{2}[\xi^\partial,\xi^\partial] + \xi^n\, r_\gamma\xi^n \right\rangle + \Pi^y(\varphi_\partial, d\xi^n)\xi^n + \xi^y_n L_{\xi^\partial}\xi^n \Big\} dt, \end{aligned} \tag{6.11}$$

together with its Hamiltonian vector eld Q^{AKSZ}_{EH}*.*

Proof. The prescription of Theorem 6.2, suggests that to construct the data in $\mathsf{F}^{\mathsf{AKSZ}}(I;\mathsf{F}^{\partial}_{EH})$ we need to compute

$$\varpi^{\mathsf{AKSZ}}_{EH} = \mathsf{T}^{(2)}_I\varpi^\partial_{EH}(\Sigma) = \int_I h\delta\mathsf{P}, \delta\mathsf{G} i + h\delta\mathsf{H}_\rho, \delta\mathsf{Z}^\rho i.$$

By selecting the top-form part of the integrand and observing that $jdtj = 1$ we get

$$\varpi^{\mathsf{AKSZ}}_{EH} = \int_I \left\{\ h\delta\gamma^y, \delta\gamma i + h\delta\Pi, \delta\Pi^y i + \delta\xi^y_\rho\delta\xi^\rho + \delta\varphi_n\delta\eta + \delta\varphi_a\delta\beta^a\right\} dt$$

where the sign comes from $h\delta(\gamma^y dt), \delta\gamma i = \quad h\delta\gamma^y, \delta\gamma i dt$, since $j\delta\gamma j = 1$, while $\delta\xi^y_\rho dt\delta\xi^\rho = \delta\xi^y_\rho\delta\xi^\rho dt$, since $j\delta\xi^\rho j = 2$. ϖ^{AKSZ} is a (1)-symplectic structure on $\mathrm{Maps}(T[1]I, F^\partial_{EH}(\Sigma))$, a BV 2-form.

Now, from $\alpha^\partial_{EH}(\Sigma)$ we can construct a degree-0 functional on F^{AKSZ}_{EH} by first applying the transgression map, which yields the 1-form

$$\mathsf{T}^{(1)}_I\alpha^\partial\ 2\ \Omega^1(\mathrm{Maps}(T[1]I, F^\partial_{EH}(\Sigma))),$$

and then contracting it with the de Rham differential on I seen as an odd cohomological vector field d_I. In a local chart this is tantamount to replacing $\delta \quad d_I := dt\frac{d}{dt}$, so that

$$\iota_{d_I}\mathsf{T}_I^{(1)}\alpha_{EH}^{\partial}(\Sigma) = \int_I \left\{\langle\Pi, \dot{\gamma}\rangle \quad \langle\varphi_\rho, \dot{\xi}^\rho\rangle\right\} dt.$$

where the sign comes from the fact that $\langle\varphi_\rho dt, \dot{\xi}^\rho\rangle = \quad \langle\varphi_\rho, \dot{\xi}^\rho\rangle dt$ Finally, we want to compute the AKSZ action functional

$$S_{EH}^{\mathsf{AKSZ}} := \mathsf{T}_I^{(0)}(S_{EH}^{\partial}(\Sigma)) + \iota_{d_I}\mathsf{T}_I^{(1)}(\alpha_{EH}^{\partial}(\Sigma)).$$

This calculation is completely analogous to the previous ones, and it is mostly straightforward. One needs to pay attention to the signs, so it is worthwhile to stress that

$$\Pi^{y} dt\,(\varphi_\partial, d\xi^n)\xi^n = [\Pi^{y}]^{ab} dt\varphi_a\partial_b\xi^n\xi^n = [\Pi^{y}]^{ab}\varphi_a\partial_b\xi^n\xi^n dt = \Pi^{y}(\varphi_\partial, d\xi^n)\xi^n dt$$

while

$$\gamma(\xi_\partial^{y} dt, d\xi^n)\xi^n \quad \gamma(\varphi_\partial, d\eta\, dt\,\xi^n + d\xi^n\eta dt) = \left(\langle\xi_\partial^{y}, \xi^n\, r_\gamma\xi^n\rangle + \langle\varphi_\partial,\, r_\gamma\eta\xi^n \quad \eta\, r_\gamma\xi^n\rangle\right) dt$$

Finally, at first order in dt,

$$H_n(\gamma + \Pi^{y} dt, \Pi + \gamma^{y})\xi^n = \frac{\delta(H_n\xi^n)}{\delta\gamma}(\Pi^{y} dt) + \frac{\delta(H_n\xi^n)}{\delta\Pi}(\gamma^{y} dt),$$

and $dt\,\xi^n = \quad \xi^n dt$. We write the formulas above as derivatives of the functional $H_n\xi^n$ to stress that total derivatives will appear, due to the term R in H_n. Recalling the expression for H_n of equation (1.13) and the definition of K, $\mathbf{G}[\gamma,\Lambda]$ and $\mathbf{D}_\gamma$ of Equations (6.7),(6.8) and (6.9), the variation of $H_n\xi^n$ with respect to γ yields

$$\frac{\delta(H_n\xi^n)}{\delta\gamma} = \frac{\delta K}{\delta\gamma}\xi^n + \frac{\delta(\sqrt{\gamma}R\ \xi^n)}{\delta\gamma} = \frac{\delta K}{\delta\gamma}\xi^n + \sqrt{\gamma}\,(\mathbf{G}[\gamma,\Lambda]\xi^n + \mathbf{D}_\gamma(\xi^n)) + d(\dots).$$

The total derivative term is exact with respect to the tangent differential d . It can be discarded, provided Σ has no boundary (which we are assuming throughout), so:

$$H_n(\gamma + \Pi^{y} dt, \Pi + \gamma^{y})\xi^n = \left(\frac{\delta K}{\delta\Pi}(\gamma^{y}) + \frac{\delta K}{\delta\gamma}(\Pi^{y})\right)\xi^n dt \quad \sqrt{\gamma}\left\langle\Pi^{y}, \mathbf{G}[\gamma,\Lambda]\xi^n + \mathbf{D}_\gamma(\xi^n)\right\rangle dt.$$

□

Remark 6.11. In order to compute the cohomological vector field Q_{EH}^{AKSZ} we enforce the Hamiltonian condition $\iota_{Q_{EH}^{\mathsf{AKSZ}}}\varpi_{EH}^{\mathsf{AKSZ}} = \delta S_{EH}^{\mathsf{AKSZ}}$ dropping all possible boundary terms. It reads (we omit the expression for $Q_{EH}^{\mathsf{AKSZ}}\xi^{y}$ and $Q_{EH}^{\mathsf{AKSZ}}\varphi$):

$$Q_{EH}^{\mathsf{AKSZ}}\gamma = \frac{\delta H_n}{\delta\Pi}\xi^n + L_{\xi^\partial}\gamma \tag{6.12a}$$

$$Q^{\mathsf{AKSZ}}_{EH}\Pi = \quad \frac{\delta K}{\delta\gamma}\xi^n \quad p\overline{\gamma}\,(\mathbf{G}[\gamma,\Lambda]\xi^n + \mathbf{D}_\gamma(\xi^n)) + L_{\xi^\partial}\Pi + \varphi \quad d\xi^n\xi^n \tag{6.12b}$$

$$Q^{\mathsf{AKSZ}}_{EH}\eta = \quad \dot{\xi}^n + L_{\xi^\partial}\eta \quad L_\beta\xi^n \tag{6.12c}$$

$$Q^{\mathsf{AKSZ}}_{EH}\beta = \quad \dot{\xi}^\partial + L_{\xi^\partial}\beta + r_\gamma\eta\,\xi^n \quad \eta\, r_\gamma\xi^n \quad r\ {}_y\xi^n\xi^n \tag{6.12d}$$

$$Q^{\mathsf{AKSZ}}_{EH}\xi^\partial = \frac{1}{2}[\xi^\partial,\xi^\partial] + \xi^n\, r_\gamma\xi^n \tag{6.12e}$$

$$Q^{\mathsf{AKSZ}}_{EH}\xi^n = L_{\xi^\partial}\xi^n \tag{6.12f}$$

$$Q^{\mathsf{AKSZ}}_{EH}\gamma^y = \dot{\Pi} + \frac{\delta K}{\delta\gamma}\eta + p\overline{\gamma}\,(\mathbf{G}[\gamma,\Lambda]\eta + \mathbf{D}_\gamma(\eta)) + L_\beta\Pi + L_{\xi^\partial}\gamma^y \tag{6.12g}$$

$$+\,\xi^y_\partial \quad d\xi^n\xi^n \quad \varphi_\partial \quad d\eta\xi^n + \eta\varphi_\partial \quad d\xi^n \tag{6.12h}$$

$$+\left[\frac{\delta^2 K}{\delta\gamma^2}(\Pi^y) + \frac{\delta^2 H_n}{\delta\gamma\delta\Pi}(\gamma^y)\right]\xi^n \quad \frac{1}{2}\gamma \quad {}^1 h\Pi^y, \mathbf{G}[\gamma,\lambda]\xi^n + \mathbf{D}_\gamma(\xi^n)i \tag{6.12i}$$

$$p\overline{\gamma}\left\langle \Pi^y, \frac{\delta\mathbf{G}[\gamma,\lambda]}{\delta\gamma}\xi^n + \frac{\delta\mathbf{D}_\gamma(\xi^n)}{\delta\gamma}\right\rangle \tag{6.12j}$$

$$Q^{\mathsf{AKSZ}}_{EH}\Pi^y = \dot{\gamma} + \frac{\delta K}{\delta\Pi}\eta + L_\beta\gamma + L_{\xi^\partial}\Pi^y \quad \left[\frac{\delta^2 K}{\delta\Pi^2}(\gamma^y) + \frac{\delta^2 K}{\delta\gamma\delta\Pi}(\Pi^y)\right]\xi^n. \tag{6.12k}$$

Remark 6.12. We notice that the term $p\overline{\gamma}\mathbf{D}_\gamma(\ )$ is the contribution to the field equations for a metric due to the presence of a Brans–Dicke "dilaton" field, whose role is played by the ghost ξ^n in the BFV action $S^\partial_{EH}(\Sigma)$ and by η in S^{AKSZ}_{EH}.

6.3.1 Pushforward

We would like to compute the BV pushforward (see Section 1.3.2) of $\mathsf{F}^{\mathsf{AKSZ}}$ along the symplectic submanifold (Π,Π^y) 2 $F^{00} = T\ [\ 1]\mathrm{Map}(I, S^2(T\ \Sigma))\quad F^{\mathsf{AKSZ}}_{EH}$.

Remark 6.13. This is the same as evaluating S_{e} from equation (1.24). Since S^{AKSZ}_{EH} is only quadratic in Π, the calculation reduces to computing the Batalin–Vilkovisky–Legendre transform S_{BVL} of S^{AKSZ}_{EH} with respect to L, as in Definition 1.41, plus a correction in the integration measure for the remaining (second-order) effective theory. Note that Equation (1.23) is equivalent to setting to zero the r.h.s. of Equation (6.12k), together with $\Pi^y = 0$.

Recall that we are assuming $\gamma(t)$ to be a non-degenerate section of $S^2(T\Sigma)$ for every t, i.e. it represents the inverse of a metric, and dually $\Pi(t)$ 2 $S^2(T\ \Sigma)$. We can use γ and its inverse (denoted $\gamma^\flat$) to raise/lower indices: explicitly, if $\gamma = \gamma^{ab}\partial_a\quad\partial_b$, we have $\gamma^\flat = \gamma_{ab}dx^a\quad dx^b$, with $\gamma^{ab}\gamma_{bc} = \delta^a_c$. Then, for X 2 $S^2(T\ \Sigma)$, Y 2 $S^2(T\Sigma)$ we define $(X^\sharp)^{ab} := \gamma^{ac}\gamma^{bd}X_{bc}$ and $(Y^\flat)_{ab} = \gamma_{ac}\gamma_{bd}Y^{cd}$.

Definition 6.14. Consider the space of fields $F_R(\Sigma\quad I)\quad F^{\mathsf{AKSZ}}$ as

$$F_R(\Sigma\quad I) := T\ [\ 1]\left(\mathrm{Map}\left(I, S^2_{nd}(T\Sigma)\quad T[1]\,(C^1(\Sigma)\quad \mathsf{X}(\Sigma))\right)\right)$$

parametrised by $(\gamma,\eta,\beta,\xi^n,\xi^\partial,\varphi_n,\varphi_\partial,\xi^y_n,\xi^y_\partial)$, with $\iota_{EH}\colon F_R(\Sigma\quad I)\ !\ F^{AKSZ}$ the inclusion map.

Theorem 6.15. *The BV-pushforward of* $\mathsf{F}^{\mathsf{AKSZ}}(I;\mathsf{F}^\partial_{EH}(\Sigma))$ *with respect to the Lagrangian submanifold* $L = f(\Pi,\Pi^y)$ 2 F^{00} j $\Pi^y = 0g$ *is the BV theory given by*

$$\mathsf{F}_R(\Sigma\quad I) := (F_R(\Sigma\quad I), S_R(\Sigma\quad I), \varpi_R(\Sigma\quad I))$$

where

$$\begin{aligned}S_R(\Sigma\times I) = \int_{\mathbb{R}} dt \int & \eta\sqrt{\gamma}\left[\left(\langle K^\sharp, K\rangle - \mathrm{Tr}(K)^2\right) + R - 2\Lambda\right] \\ & -\langle\gamma^\vee, L_{\xi^\partial}\gamma\rangle - 2\langle K^\sharp, \gamma^\vee\rangle\xi^n - \langle\varphi_\rho, \dot{\xi}^\rho\rangle \\ & + \langle\varphi_\partial, \nabla_\gamma\eta\xi^n - \eta\nabla_\gamma\xi^n + L_{\xi^\partial}\beta\rangle + \varphi_n\left(-L_\beta\xi^n + L_{\xi^\partial}\eta\right) \\ & + \left\langle \xi^\vee_\partial, \xi^n\nabla_\gamma\xi^n + \frac{1}{2}[\xi^\partial,\xi^\partial]\right\rangle + \xi^\vee_n L_{\xi^\partial}\xi^n\end{aligned}$$

with $K := \frac{\eta^{-1}}{2}\left(\dot{\gamma} + L_\beta\gamma\right)^\flat$, *and*

$$\varpi_R(\Sigma\times I) = \iota_{EH}\varpi_{EH}^{AKSZ}.$$

Proof. As discussed in Remark 6.13, we are interested in finding the effective action one obtains by means of the perturbative expansion of the integral

$$\exp\left(\frac{i}{\hbar}S_{e}\right) := \int_{L\subset F^{\infty}} \exp\left(\frac{i}{\hbar}S_{EH}^{AKSZ}\right)$$

because $S_{EH}^{AKSZ}|_L$ is quadratic in Π, through the term $K(\Pi)\eta$. Observing that

$$\begin{aligned}\frac{\delta K}{\delta\Pi} &= \frac{2}{\sqrt{\gamma}}\left(\Pi^\sharp - \frac{\gamma}{N-1}\mathrm{Tr}(\Pi)\right)\\ \frac{\delta^2 K}{\delta\Pi^2}(\gamma^\vee) &= \frac{2}{\sqrt{\gamma}}\left((\gamma^\vee)^\sharp - \frac{\gamma}{N-1}\mathrm{Tr}(\gamma^\vee)\right),\end{aligned}$$

we have that Equation (6.12k) reads

$$\frac{2}{\sqrt{\gamma}}\left(\Pi^\sharp - \frac{\gamma}{N-1}\mathrm{Tr}(\Pi)\right) = \eta^{-1}\left(\dot{\gamma} + L_\beta\gamma - \frac{2}{\sqrt{\gamma}}\left((\gamma^\vee)^\sharp - \frac{\gamma}{N-1}\mathrm{Tr}(\gamma^\vee)\right)\xi^n\right) + F(\Pi^\vee)$$

where $\mathrm{Tr}(X) = \gamma^{ab}X_{ab}$. We will use the symbol $\approx$ to denote the enforcing of Equation (6.12k) and of $\Pi^\vee = 0$. Then, requiring $\Pi^\vee = 0$ and defining

$$K := \frac{\eta^{-1}}{2}\left(\dot{\gamma} + L_\beta\gamma\right)^\flat, \tag{6.13}$$

we obtain that

$$\Pi \approx \sqrt{\gamma}\left(K - \mathrm{Tr}(K)\gamma^\flat\right) + \eta^{-1}\gamma^\vee\xi^n.$$

It is easy to compute now

$$H_n \approx \sqrt{\gamma}\left[\left(\langle K^\sharp, K\rangle - \mathrm{Tr}(K)^2\right) + R - 2\Lambda\right] - 2\eta^{-1}\langle K^\sharp, \gamma^\vee\rangle\xi^n$$

which, together with

$$\langle\Pi, \dot{\gamma} + L_\beta\gamma\rangle \approx 2\sqrt{\gamma}\left[\left(\langle K^\sharp, K\rangle - \mathrm{Tr}(K)^2\right) + R - 2\Lambda\right] + 2\eta^{-1}\langle K^\sharp, \gamma^\vee\rangle\xi^n$$

and

$$\frac{\delta H_n}{\delta\Pi}(\gamma^\vee)\xi^n \approx +2\langle K^\sharp, \gamma^\vee\rangle\xi^n,$$

yields

$$S_{EH}^{\mathsf{AKSZ}} = \int_{\mathbb{R}} dt \int \eta \sqrt{\gamma}\left[\left(\langle K^\sharp, K\rangle - \mathrm{Tr}(K)^2\right) + R - 2\Lambda\right] \tag{6.14}$$

$$\begin{aligned} &\langle \gamma^\vee, L_{\xi^\partial}\gamma\rangle + 2\langle K^\sharp, \gamma^\vee\rangle \xi^n - \langle \varphi_\rho, \dot{\xi}^\rho\rangle \\ &+ \langle \varphi_\partial, \nabla_\gamma \eta \xi^n - \eta \nabla_\gamma \xi^n + L_{\xi^\partial}\beta\rangle + \varphi_n\left(-L_\beta \xi^n + L_{\xi^\partial}\eta\right) \\ &+ \left\langle \xi^\vee_\partial, \xi^n \nabla_\gamma \xi^n + \frac{1}{2}[\xi^\partial, \xi^\partial]\right\rangle + \xi^\vee_n L_{\xi^\partial}\xi^n =: S_R(\Sigma \times I). \end{aligned}$$

So, formula (6.14) shows that $S_{\mathsf{e}} = S_R(\Sigma \times I) + O(\hbar)$. The $\hbar$ correction is the (logarithm of the) determinant of the operator defining the quadratic form K, i.e. the determinant of the deWitt super metric[3] [DeW67]

$$\mathsf{W}_\gamma^{ijkl} = \frac{1}{\sqrt{\gamma}}\left(\gamma^{ik}\gamma^{jl} - \frac{1}{N-1}\gamma^{ij}\gamma^{kl}\right),$$

or, more invariantly

$$\mathsf{W}_\gamma(\Pi, \Pi) = \frac{1}{\sqrt{\gamma}}\left(\langle \Pi^\sharp, \Pi\rangle - \frac{1}{N-1}\mathrm{Tr}_\gamma[\Pi]^2\right),$$

and it will have the effect of correcting the overall measure on the residual BV space of fields $\mathcal{F}_R(\Sigma \times I)$. □

Remark 6.16. Up to boundary, we can compute $Q_R(\Sigma \times I)$ (denoted hereinafter by Q_R) to be:

$$Q_R\gamma = L_{\xi^\partial}\gamma - \eta^{-1}(\dot{\gamma} + L_\beta\gamma)\xi^n \tag{6.15a}$$
$$Q_R\eta = -L_\beta\xi^n + L_{\xi^\partial}\eta \tag{6.15b}$$
$$Q_R\beta = \nabla_\gamma\eta\xi^n - \eta\nabla_\gamma\xi^n + L_{\xi^\partial}\beta \tag{6.15c}$$
$$Q_R\xi^n = L_{\xi^\partial}\xi^n \tag{6.15d}$$
$$Q_R\xi^\partial = \xi^n\nabla_\gamma\xi^n + \frac{1}{2}[\xi^\partial, \xi^\partial] \tag{6.15e}$$

and similarly for antifields.

6.3.2 Reconstruction of Einstein–Hilbert theory

In this section we wish to show that the BV pushforward of the AKSZ theory constructed in Section 6.3.1 is strongly equivalent to Einstein–Hilbert theory in the BV formalism.

To do this, we begin by considering the following definitions:

$$\widetilde{\zeta} = -\eta^{-1}\xi^n(\partial_t + \beta) + \xi^\partial \tag{6.16a}$$
$$\widetilde{g} = -\eta^{-2}\partial_t \otimes \partial_t - 2\eta^{-2}\beta \odot \partial_t + \gamma - \eta^{-2}\beta \otimes \beta \tag{6.16b}$$

Lemma 6.17. *We have the following relations*

$$\frac{1}{2}[\widetilde{\zeta}, \widetilde{\zeta}] = Q_R\widetilde{\zeta},$$
$$L_{\widetilde{\zeta}}\widetilde{g} = Q_R\widetilde{g}.$$

[3] To be precise, W_γ is the inverse of the metric introduced by deWitt, due to our choice of working with inverse metrics γ. Note that this metric is called *super*, but this terminology does not refer to the supermanifold notion.

Proof. It is a straightforward calculation to show

$$\begin{aligned}
Q_R\widetilde{\zeta} &= \eta \;\; ^2(Q_R\eta)\xi^n(\partial_t+\beta) + \eta \;\; ^1 Q_R\xi^n(\partial_t+\beta) + Q_R\xi^\partial + \eta \;\; ^1\xi^n Q_R\beta \\
&= \Big(\;\; \eta \;\; ^2\dot{\xi}^n\xi^n \quad \eta \;\; ^2 L_\beta\xi^n\xi^n + L_{\xi^\partial}(\;\; \eta \;\; ^1\xi^n)\Big)(\partial_t+\beta) \\
&\quad \eta \;\; ^1\xi^n\dot{\xi}^\partial + \eta \;\; ^1\xi^n L_{\xi^\partial}\beta + \frac{1}{2}[\xi^\partial,\xi^\partial].
\end{aligned}$$

Observe that the "algebroid term" (see Remark 6.18, below) $\xi^n\, r_\gamma\xi^n$ in $Q_R\xi^\partial$ cancels out with part of $\eta \;\; ^1\xi^n Q_R\beta$. On the other hand this coincides with

$$\begin{aligned}
\frac{1}{2}[\widetilde{\zeta},\widetilde{\zeta}] &= \frac{1}{2}[\;\; \eta \;\; ^1\xi^n(\partial_t+\beta)+\xi^\partial, \;\; \eta \;\; ^1\xi^n(\partial_t+\beta)+\xi^\partial] \\
&= \left(\eta \;\; ^2\xi^n(\partial_t+\beta)\xi^n + L_{\xi^\partial}(\;\; \eta \;\; ^1\xi^n)\right)(\partial_t+\beta) \\
&\quad \eta \;\; ^1\xi^n\dot{\xi}^\partial + \eta \;\; ^1\xi^n L_{\xi^\partial}\beta + \frac{1}{2}[\xi^\partial,\xi^\partial],
\end{aligned}$$

proving the first claim. We compute

$$\begin{aligned}
L_{\widetilde{\zeta}}\widetilde{g} = &\quad 2\eta \;\; ^3\dot{\xi}^n\partial_t\partial_t + 2\eta \;\; ^3 L_{\xi^\partial}\eta\partial_t\partial_t + 2\eta \;\; ^2\dot{\xi}\partial_t \quad 2\eta \;\; ^4\xi^n L_\beta\eta\partial_t\partial_t \\
&\quad 2\eta \;\; ^2\partial_t\left(\eta \;\; ^1\xi^n\beta\right)\partial_t \quad 4\eta \;\; ^4\xi^n\partial_t\eta\beta\partial_t + 4\eta \;\; ^3 L_{\xi^\partial}\eta\beta\partial_t \quad 4\eta \;\; ^4 L_\beta\eta\xi^n\beta\partial_t \\
&\quad 2\eta \;\; ^2 L_{\xi^\partial}\beta\partial_t \quad 2\eta \;\; ^2 L_\beta\left(\eta \;\; ^1\xi^n\right)\partial_t\partial_t + 2\eta \;\; ^2\partial_t\left(\xi^\partial \quad \eta \;\; ^1\xi^n\beta\right)\beta \\
&\quad 2\eta \;\; ^2 L_\beta\left(\eta \;\; ^1\xi^n\right)\beta\partial_t \quad 2\eta \;\; ^2\partial_t\left(\eta \;\; ^1\xi^n\right)\beta\partial_t + 2\eta \;\; ^2\xi^n\dot{\beta}\partial_t \quad \eta \;\; ^1\xi^n\dot{\gamma}^{ab}\partial_a\partial_b \\
&+ L_{\xi^\partial}\left(\gamma^{ab}\partial_a\partial_b\right) \quad \eta \;\; ^1\xi^n L_\beta(\gamma^{ab})\partial_a\partial_b + 2\, r_\gamma\left(\eta \;\; ^1\xi^n\right)\partial_t + 2\, r_\gamma\left(\eta \;\; ^1\xi^n\beta\right)\partial_c \\
&\quad 2\eta \;\; ^4\xi^n\dot{\eta}\beta\beta + 2\eta \;\; ^3 L_\xi\eta\beta\beta \quad 2\eta \;\; ^4\xi^n L_\beta\eta\beta\beta + 2\eta \;\; ^3\xi^n\dot{\beta}\beta \quad \eta \;\; ^2 L_{\xi^\partial}(\beta\beta) \\
&\quad 2\eta \;\; ^2 L_\beta\left(\eta \;\; ^1\xi^n\right)(\beta\partial_t+\beta\beta)
\end{aligned}$$

where we recall that expressions like $L_{\xi^\partial}(\beta)$ denote the Lie derivative of the vector field $\beta=\beta^a\partial_a$ along ξ^∂. On the other hand we have

$$\begin{aligned}
Q_R(\;\; \eta \;\; ^2)\partial_t\partial_t &= \Big(\;\; 2\eta \;\; ^3\dot{\xi}^n + 2\eta \;\; ^3 L_{\xi^\partial}\eta \quad 2\eta \;\; ^3 L_\beta\xi^n\Big)\partial_t\partial_t \\
Q_R(\;\; 2\eta \;\; ^2\beta)\partial_t &= \Big(\;\; 4\eta \;\; ^3\dot{\xi}^n\beta + 4\eta \;\; ^3 L_{\xi^\partial}\eta \quad 4\eta \;\; ^3 L_\beta\xi^n\beta\Big)\partial_t \\
&\quad + \Big(2\eta \;\; ^2\dot{\xi}^\partial \quad 2\eta \;\; ^2 L_{\xi^\partial}\beta \quad 2\eta \;\; ^2\, r_\gamma\eta\xi^n + 2\eta \;\; ^1\, r_\gamma\xi^n\Big)\partial_t \\
Q_R(\gamma^{ab})\partial_a\partial_b &= \;\; \eta \;\; ^1\xi^n\dot{\gamma} \quad \eta \;\; ^1\xi^n L_\beta\gamma + L_{\xi^\partial}\gamma \\
Q_R(\;\; \eta \;\; ^2\beta\beta) &= \;\; 2\eta \;\; ^3\dot{\xi}^n\beta\beta + 2\eta \;\; ^3 L_{\xi^\partial}\eta\beta\beta \quad 2\eta \;\; ^3 L_\beta\xi^n\beta\beta \\
&\quad + 2\eta \;\; ^2\beta\dot{\xi}^\partial \quad 2\eta \;\; ^2 L_{\xi^\partial}\beta\beta \quad 2\eta \;\; ^2\left(\, r_\gamma\eta\xi^n \quad \eta\, r_\gamma\xi^n\right)\beta
\end{aligned}$$

And it is a matter of a straightforward, but lengthy computation to show that the two expressions coincide. Indeed, subtracting one from the other we obtain

$$\begin{aligned}
L_{\widetilde{\zeta}}\widetilde{g} \quad Q_R\widetilde{g} = &\; 2\eta \;\; ^3(\;\; \eta \;\; ^1\xi^n(\dot{\eta}+L_\beta(\eta))\beta\beta + 2\eta \;\; ^3\xi^n\dot{\beta}\beta \quad 2\eta \;\; ^2 L_\beta(\eta \;\; ^1)\xi^n(\beta\partial_t+\beta\beta) \\
&\quad 2\eta \;\; ^4\xi^n L_\beta(\eta)\partial_t^2 + 2\eta \;\; ^4\dot{\eta}\xi^n\beta\partial_t \quad 2\eta \;\; ^3\xi^n\dot{\beta}\xi^n\partial_t \quad 4\eta \;\; ^4\xi^n(\dot{\eta}+L_\beta\eta)\beta\partial_t \\
&+ 2\eta \;\; ^3\xi^n\dot{\beta}\partial_t + 2\eta \;\; ^4 L_\beta\eta(\partial_t^2+\beta\partial_t) + 2\eta \;\; ^4\dot{\eta}\xi^n(\beta\partial_t+\beta\beta) \quad 2\eta \;\; ^3\xi^n\dot{\beta}\beta \quad 0
\end{aligned}$$

□

Remark 6.18. Using Lemma 6.17 we wish to interpret (6.16) as a map of Lie algebroids. Consider the (trivial) vector bundle over

$$\mathrm{Map}(I, S^2_{nd}(T\Sigma) \times C^\infty(\Sigma) \times \mathfrak{X}(\Sigma)) \simeq \mathcal{PR}(\Sigma \times I),$$

where $\mathcal{PR}(\Sigma \times I)$ denotes pseudo-Riemennian metrics on $\Sigma \times I$ such that their restriction to Σ is non-degenerate, with fibre

$$\mathrm{Map}(I, C^\infty(\Sigma) \oplus \mathfrak{X}(\Sigma)) \simeq \mathfrak{X}(\Sigma \times I).$$

We consider two different Lie algebroid structures on this vector bundle. One is the action algebroid with bracket given by the bracket of $(N+1)$-vector fields, and anchor given by Lie derivatives on metrics. The other algebroid structure is given by formulas (6.15), with (6.15a),(6.15b) and (6.15c) defining the anchor map, and (6.15d) and (6.15e) specifying the bracket of sections. Observe that the morphism of algebroids (6.16) does not preserve constant sections, as the splitting of a generic vector field $\widetilde{\zeta}$ depends on the so-called *lapse* η and *shift* β, which are coordinates on the base of the fibre bundle. The latter algebroid encodes the algebraic relations of the constraints of Einstein–Hilbert theory[4], and was carefully studied by other means in [BFW13]. It was also mentioned as a motivating example for the notion of Hamiltonian Lie Algebroid, introduced in [BW18]. It is an interesting question to check whether this construction satisfies the Hamiltonian requirements for an algebroid.

To proceed, we need to recall the BV data associated with Einstein–Hilbert theory, in the ADM formalism. Given a pseudo-Riemannian (inverse) metric $\widetilde{g}$ on a manifold M, we can perform a $N+1$ decomposition and rewrite it as[5]

$$\widetilde{g}^{\mu\nu} = \begin{pmatrix} -\eta^{-2} & \eta^{-2}\beta^b \\ \eta^{-2}\beta^a & \gamma^{ab} - \eta^{-2}\beta^a\beta^b \end{pmatrix}$$

In the case where M has a boundary, we can define the second fundamental form of the boundary submanifold K_{ab} and its trace K by means of the boundary covariant derivative ∇^∂ (the Levi-Civita connection of γ) as follows

$$K_{ab} = \frac{1}{2}\eta^{-1}(-2\nabla^\partial_{(a}\beta_{b)} + \partial_t\gamma_{ab}) \qquad K = \gamma^{ab}K_{ab} \tag{6.17}$$

where t denotes a coordinate transverse to the boundary ∂M. Finally, notice that

$$(L_\beta\gamma)^{cd}\gamma_{ac}\gamma_{bd} = -2\nabla^\partial_{(a}\beta_{b)} \qquad (\dot{\gamma})^{cd}\gamma_{ac}\gamma_{bd} = -\partial_t\gamma_{ab}.$$

Definition 6.19. Let $(\mathcal{F}_{EH}(M), \varpi_{EH}(M))$ be the symplectic manifold

$$\mathcal{F}_{EH}(M) \coloneqq T^*[-1]\Big(\mathcal{PR}^{\partial M}(M) \times \mathfrak{X}[1](M)\Big)$$

with its canonical symplectic structure, and $\mathcal{PR}^{\partial M}(M)$ denotes pseudo-Riemennian metrics on M such that their restriction to ∂M is non-degenerate. Consider the functional

$$S_{EH}(M) = \int_M \Big\{ -\eta\sqrt{\gamma}(\epsilon(K_{ab}K^{ab} - K^2) + R - 2\Lambda)\Big\} + \widetilde{g}^\dagger L_{\widetilde{\zeta}}\widetilde{g} + \frac{1}{2}\iota_{[\widetilde{\zeta},\widetilde{\zeta}]}\widetilde{\zeta}^\dagger \tag{6.18}$$

[4] We stress that, as it is, the structure one can extract from the BFV differential Q^∂ is that of a curved L_∞ algebroid, due to the dependency on fields of negative degree. We thank A. Weinstein, C. Blohmann and N. L. Delgado for enlightening discussions on this matter.

[5] In this chapter we will assume that the manifold M has a global product structure $M = \Sigma \times \mathbb{R}$, and the induced metric on Σ will be Riemannian, i.e. the leaves Σ_t are spacelike submanifolds of M. It is straightforward to generalise this to the *timelike* case. The relevant formulas for EH theory in the BV-BFV formalism have been given in [CS16].

and denote by $Q_{EH}(M)$ the Hamiltonian vector field of $S_{EH}(M)$, up to boundary terms. Then, the assignment of the tuple

$$\mathsf{F}_{EH} = (F_{EH}(M), S_{EH}(M), \varpi_{EH}(M), Q_{EH}(M)))$$

to every $(N+1)$-dimensional manifold M that admits a Lorentzian structure will be called Einstein–Hilbert theory in the BV formalism.

Remark 6.20. The sign convention used above is obtained from the standard ADM decomposition by redefining (η, β) ! $(\ \eta,\ \ \beta)$. This matches our conventions below. This change is due to the choice of using inverse metrics for the first order formulation, instead of metrics (in fact $\Pi_{ab}\partial_t\gamma^{ab} = \ \ \Pi^{ab}\partial_t\gamma_{ab}$).

Theorem 6.21. *Einstein{Hilbert theory in the BV formalism* $\mathsf{F}_{EH}(\Sigma \quad I)$ *is strongly equivalent to* $\mathsf{F}_R(\Sigma \quad I)$*. Explicitly, the isomorphism of the underlying symplectic dg-manifolds reads:*

$$\begin{aligned}
\widetilde{g} &= \ \ \eta^{\ \ 2}\partial_t\partial_t \quad 2\eta^{\ \ 2}\beta\partial_t + \gamma \quad \eta^{\ \ 2}\beta\beta \\
\widetilde{\zeta} &= \ \ \eta^{\ \ 1}\xi^n\partial_t + \xi^\partial \quad \eta^{\ \ 1}\xi^n\beta \\
\widetilde{\zeta}^y &= \xi^y_\partial \quad \left(\eta\xi^y_n + \iota_\beta\xi^y_\partial\right) dt \\
\widetilde{g}^y &= \left(\frac{1}{2}\eta^3\varphi_n \quad \eta^2\varphi_a\beta^a \quad \gamma^y_{ab}\beta^a\beta^b + \eta\beta^a\xi^y_a\xi^n + \frac{1}{2}\eta^{\ \ 1}\xi^y_n\xi^n\right) dt^2 \\
&\quad + \left(\frac{1}{2}\eta^2\varphi_a + \gamma^y_{ab}\beta^b \quad \frac{1}{2}\eta\xi^y_a\xi^n\right) dx^a dt \quad \gamma^y_{ab}dx^a dx^b
\end{aligned}$$

with inverse:

$$\begin{aligned}
\eta &= [\ \ \widetilde{g}^{tt}]^{\ \ \frac{1}{2}} \\
\beta^a &= \ \ [\ \ \widetilde{g}^{tt}]^{\ \ 1}\widetilde{g}^{ta} \\
\gamma^{ab} &= [\ \ \widetilde{g}^{tt}]^{\ \ 1}\widetilde{g}^{ta}\widetilde{g}^{tb} \\
\xi^n &= \ \ [\ \ \widetilde{g}^{tt}]^{\ \ \frac{1}{2}}\widetilde{\zeta}^t \\
\xi^a &= \widetilde{\zeta}^a + [\ \ \widetilde{g}^{tt}]\widetilde{g}^{ta}\widetilde{\zeta}^t \\
\gamma^y_{ab} &= \widetilde{g}^y_{ab} \\
\varphi_a &= 2[\ \ \widetilde{g}^{tt}]\widetilde{g}^y_{at} + 2\widetilde{g}^y_{ab}\widetilde{g}^{tb} + \widetilde{\zeta}^y_a\widetilde{\zeta}^t \\
\varphi_n &= 2[\ \ \widetilde{g}^{tt}]^{\frac{3}{2}}\widetilde{g}^y_{tt} \quad 4[\ \ \widetilde{g}^{tt}]^{\frac{1}{2}}\widetilde{g}^y_{ta}\widetilde{g}^{ta} + 2[\ \ \widetilde{g}^{tt}]^{\ \ \frac{1}{2}}\widetilde{g}^y_{ab}\widetilde{g}^{ta}\widetilde{g}^{tb} \\
&\quad + [\ \ \widetilde{g}^{tt}]^{\frac{1}{2}}\widetilde{\zeta}^y_n\widetilde{\zeta}^t \quad [\ \ \widetilde{g}^{tt}]^{\ \ \frac{1}{2}}g^{ta}\widetilde{\zeta}^y_a\widetilde{\zeta}^a \\
\xi^y_n &= \ \ [\ \ \widetilde{g}^{tt}]^{\frac{1}{2}}\widetilde{\zeta}^y_n + [\ \ \widetilde{g}^{tt}]^{\ \ \frac{1}{2}}\widetilde{\zeta}^y_a\widetilde{g}^{ta} \\
\xi^y_a &= \widetilde{\zeta}^y_a
\end{aligned}$$

Proof. We begin observing that the definitions of K in (6.13) and K in (6.17) coincide up to sign, after identifying $\widetilde{g}$ with the expression of Equation (6.16b). Since the expression $S_{ADM} := \eta\sqrt{\gamma}(\epsilon(K_{ab}K^{ab} \quad K^2) + R \quad 2\Lambda)$ is quadratic in K, we conclude that the degree-zero part of (6.18) and (6.14) coincide. This means that the two theories are classically equivalent, and (6.16b) is the map between second-order and first-order Einstein–Hilbert theory.

We endeavour now to find the explicit expression for $\widetilde{g}^y$ and $\widetilde{\zeta}^y$ so that

$$\phi\ \ (h\widetilde{g}^y, \delta\widetilde{g}i + h\widetilde{\zeta}^y, \delta\widetilde{\zeta}i) = \ \ h\delta\gamma^y, \delta\gamma i + h\Pi, \delta\Pi^y i + \xi^y_\rho\delta\xi^\rho + \varphi_n\delta\eta + \varphi_a\delta\beta^a.$$

It is straightforward to compute

$$\begin{aligned}\phi\ (\hbar\widetilde{g}^{y}, \delta\widetilde{g}\,i + \hbar\widetilde{\zeta}^{y}, \delta\widetilde{\zeta}\,i) = &\ \ \left[(\phi\ \widetilde{\zeta}^{y})_n\eta\ \ ^{1} + (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{1}\beta^a\right]\delta\xi^n + (\phi\ \widetilde{\zeta}^{y})_a\delta\xi^a \\ &+ \left[2(\phi\ \widetilde{g}^{y})_{tt}\eta\ \ ^{3} + 4(\phi\ \widetilde{g}^{y})_{at}\eta\ \ ^{3}\beta^a + 2(\phi\ \widetilde{g}^{y})_{ab}\eta\ \ ^{3}\beta^a\beta^b\right]\delta\eta \\ &\ \ \left[(\phi\ \widetilde{\zeta}^{y})_t\eta\ \ ^{2}\xi^n\ \ \ (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{2}\beta^a\xi^n\right]\delta\eta + (\phi\ \widetilde{g}^{y})_{ab}\delta\gamma^{ab} \\ &+ \left[2(\phi\ \widetilde{g}^{y})_{at}\eta\ \ ^{2}\ \ \ 2(\phi\ \widetilde{g}^{y})_{ab}\eta\ \ ^{2}\beta^b + (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{1}\xi^n\right]\delta\beta^a\end{aligned}$$

which leaves us with the intermediate expression:

$$\xi^{y}_a = (\phi\ \widetilde{\zeta}^{y})_a \tag{6.19a}$$

$$\xi^{y}_n = \ \ \left[(\phi\ \widetilde{\zeta}^{y})_t\eta\ \ ^{1} + (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{1}\beta^a\right] \tag{6.19b}$$

$$\gamma^{y}_{ab} = \ \ (\phi\ \widetilde{g}^{y})_{ab} \tag{6.19c}$$

$$\varphi_a = \ \ 2(\phi\ \widetilde{g}^{y})_{at}\eta\ \ ^{2}\ \ \ 2(\phi\ \widetilde{g}^{y})_{ab}\eta\ \ ^{2}\beta^b + (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{1}\xi^n \tag{6.19d}$$

$$\begin{aligned}\varphi_n = &\ 2(\phi\ \widetilde{g}^{y})_{tt}\eta\ \ ^{3} + 4(\phi\ \widetilde{g}^{y})_{at}\eta\ \ ^{3}\beta^a + 2(\phi\ \widetilde{g}^{y})_{ab}\eta\ \ ^{3}\beta^a\beta^b \\ &\ \ (\phi\ \widetilde{\zeta}^{y})_t\eta\ \ ^{2}\xi^n\ \ \ (\phi\ \widetilde{\zeta}^{y})_a\eta\ \ ^{2}\beta^a\xi^n\end{aligned} \tag{6.19e}$$

Starting from the top and solving downwards, we easily get

$$\begin{aligned}(\phi\ \widetilde{\zeta}^{y})_a &= \xi^{y}_a \\ (\phi\ \widetilde{\zeta}^{y})_n &= \ \ \eta\xi^{y}_n\ \ \ \xi^{y}_a\beta^a \\ (\phi\ \widetilde{g}^{y})_{ab} &= \ \ \gamma^{y}_{ab} \\ (\phi\ \widetilde{g}^{y})_{at} &= \ \ \frac{1}{2}\eta^2\varphi_a + \gamma^{y}_{ab}\beta^b + \frac{1}{2}\eta\xi^{y}_a\xi^n \\ (\phi\ \widetilde{g}^{y})_{tt} &= \frac{1}{2}\eta^3\varphi_n + \eta^2\varphi_a\beta^a\ \ \ \gamma^{y}_{ab}\beta^a\beta^b\ \ \ \eta\xi^{y}_a\beta^a\xi^n\ \ \ \frac{1}{2}\eta\xi^{y}_n\xi^n\end{aligned}$$

Alternatively, from (6.19), observing that the assignment (6.16) can be inverted to yield

$$\phi\ \ ^{1}\ \eta = [\ \ \widetilde{g}^{tt}]\ \ ^{\frac{1}{2}}, \qquad \phi\ \ ^{1}\ \beta^a = \ \ [\ \ \widetilde{g}^{tt}]\ \ ^{1}\widetilde{g}^{ta}, \qquad \phi\ \ ^{1}\ \gamma^{ab} = [\ \ \widetilde{g}^{tt}]\ \ ^{1}\widetilde{g}^{ta}\widetilde{g}^{tb}$$

together with

$$\phi\ \ ^{1}\ \xi^n = \ \ [\ \ \widetilde{g}^{tt}]\ \ ^{\frac{1}{2}}\widetilde{\zeta}^t; \qquad \phi\ \ ^{1}\ \xi^a = \widetilde{\zeta}^a + [\ \ \widetilde{g}^{tt}]\ \ ^{1}\widetilde{g}^{ta}\widetilde{\zeta}^t$$

we can similarly solve bottom-down to obtain the inverse:

$$\begin{aligned}\phi\ \ ^{1}\ \eta &= [\ \ \widetilde{g}^{tt}]\ \ ^{\frac{1}{2}} \\ \phi\ \ ^{1}\ \beta^a &= \ \ [\ \ \widetilde{g}^{tt}]\ \ ^{1}\widetilde{g}^{ta} \\ \phi\ \ ^{1}\ \gamma^{ab} &= [\ \ \widetilde{g}^{tt}]\ \ ^{1}\widetilde{g}^{ta}\widetilde{g}^{tb} \\ \phi\ \ ^{1}\ \xi^n &= \ \ [\ \ \widetilde{g}^{tt}]\ \ ^{\frac{1}{2}}\widetilde{\zeta}^t \\ \phi\ \ ^{1}\ \xi^a &= \widetilde{\zeta}^a + [\ \ \widetilde{g}^{tt}]\widetilde{g}^{ta}\widetilde{\zeta}^t \\ \phi\ \ ^{1}\ \gamma^{y}_{ab} &= \ \ \widetilde{g}^{y}_{ab}\end{aligned}$$

$$
\begin{aligned}
\phi^{\ \ 1}\ \xi^{y}_{a} &= \widetilde{\zeta}^{y}_{a} \\
\phi^{\ \ 1}\ \xi^{y}_{n} &= \quad \widetilde{\zeta}^{y}_{n}[\quad \widetilde{g}^{tt}]^{\frac{1}{2}} + \widetilde{\zeta}^{y}_{a}[\quad \widetilde{g}^{tt}]^{\quad \frac{1}{2}}\widetilde{g}^{ta} \\
\phi^{\ \ 1}\ \varphi_{a} &= \quad 2\widetilde{g}^{y}_{at}[\quad \widetilde{g}^{tt}] + 2\widetilde{g}^{y}_{ab}\widetilde{g}^{tb} \quad \widetilde{\zeta}^{y}_{a}\widetilde{\zeta}^{t} \\
\phi^{\ \ 1}\ \varphi_{n} &= 2\widetilde{g}^{y}_{tt}[\quad \widetilde{g}^{tt}]^{\frac{3}{2}} \quad 4\widetilde{g}^{y}_{ta}[\quad \widetilde{g}^{tt}]^{\frac{1}{2}}\widetilde{g}^{ta} + 2\widetilde{g}^{y}_{ab}[\quad \widetilde{g}^{tt}]^{\quad \frac{1}{2}}\widetilde{g}^{ta}\widetilde{g}^{tb} \\
&\quad + [\quad \widetilde{g}^{tt}]^{\frac{1}{2}}\widetilde{\zeta}^{y}_{t}\widetilde{\zeta}^{t} \quad [\quad \widetilde{g}^{tt}]^{\quad \frac{1}{2}}g^{ta}\widetilde{\zeta}^{y}_{a}\widetilde{\zeta}^{t}.
\end{aligned}
$$

Now, using again the intermediate expressions (6.19) let us consider the following terms, coming from Equation (6.14):

$$
\begin{aligned}
\xi^{y}_{n}L_{\xi^{\partial}}\xi^{n} &= \quad (\phi\ \widetilde{\zeta}^{y})_{t}\eta^{\ \ 1}L_{\xi^{\partial}}\xi^{n} \quad (\phi\ \widetilde{\zeta}^{y})_{a}\eta^{\ \ 1}\beta^{a}L_{\xi^{\partial}}\xi^{n} \\
h\xi^{y}_{\partial}, (\xi^{n}\ r_{\gamma}\xi^{n} + \frac{1}{2}[\xi^{\partial},\xi^{\partial}]i &= h(\phi\ \widetilde{\zeta}^{y})_{\partial}, (\xi^{n}\ r_{\gamma}\xi^{n} + \frac{1}{2}[\xi^{\partial},\xi^{\partial}]i
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_{n}\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \dot{\xi}^{n}\right) \\
&\quad = \eta^{\ \ 3}\left[2(\phi\ \widetilde{g}^{y})_{tt} + 4(\phi\ \widetilde{g}^{y})_{at}\beta^{a} + (\phi\ \widetilde{g}^{y})_{ab}\beta^{a}\beta^{b}\right]\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \dot{\xi}^{n}\right) \\
&\qquad \eta^{\ \ 2}\left[(\phi\ \widetilde{\zeta}^{y})_{n}\xi^{n} + (\phi\ \widetilde{\zeta}^{y})_{a}\beta^{a}\xi^{n}\right]\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \dot{\xi}^{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&h\varphi_{\partial}, \left(\ r_{\gamma}\xi^{n} \quad \eta\, r_{\gamma}\xi^{n} + L_{\xi^{\partial}}\beta\right)i \\
&\quad = \quad 2\eta^{\ \ 2}\left((\phi\ \widetilde{g}^{y})_{at} + (\phi\ \widetilde{g}^{y})_{ab}\beta^{b}\right)\left((\ r_{\gamma}\eta)^{a}\xi^{n} \quad \eta(\ r_{\gamma}\xi^{n})^{a} + (L_{\xi^{\partial}}\beta)^{a} \quad \dot{\xi}^{a}\right) \\
&\qquad + (\phi\ \widetilde{\zeta}^{y})_{a}\eta^{\ \ 1}\xi^{n}\left((L_{\xi^{\partial}}\beta)^{a} \quad \eta(\ r_{\gamma}\xi^{n})^{a} \quad \dot{\xi}^{a}\right)
\end{aligned}
$$

$$
h\gamma^{y}, \eta^{\ \ 1}(\dot{\gamma} + L_{\beta}\gamma)\,\xi^{n} \quad L_{\xi^{\partial}}\gamma i = \quad (\phi\ \widetilde{g}^{y})_{ab}\left(\eta^{\ \ 1}(\dot{\gamma} + L_{\beta}\gamma)\,\xi^{n} \quad L_{\xi^{\partial}}\gamma\right)^{ab}.
$$

Then, summing all terms on the left hand side and factoring $(\phi\ \widetilde{\zeta}^{y})_{t}$, $(\phi\ \widetilde{\zeta}^{y})_{a}$ and $(\phi\ \widetilde{g}^{y})$, we obtain

$$
\begin{aligned}
&(\phi\ \widetilde{\zeta}^{y})_{t}\left[\quad L_{\xi^{\partial}}(\eta^{\ \ 1}\xi^{n}) + \eta^{\ \ 2}\xi^{n}\left(L_{\beta}\xi^{n} + \xi^{n}\right)\right] \\
&+ (\phi\ \widetilde{g}^{y})_{tt}\left[2\eta^{\ \ 3}\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \xi^{n}\right)\right] + \left\langle(\phi\ \widetilde{\zeta}^{y})_{\partial}, \frac{1}{2}[\xi^{\partial},\xi^{\partial}]\right\rangle \\
&+ \left\langle(\phi\ \widetilde{\zeta}^{y})_{\partial}, \quad L_{\xi^{\partial}}(\eta^{\ \ 1}\xi^{n})\beta + \eta^{\ \ 1}L_{\xi^{\partial}}\beta\xi^{n} + \eta^{\ \ 2}\beta\xi^{n}L_{\beta}\xi^{n} + \eta^{\ \ 2}\beta\xi^{n}\xi^{n} \quad \eta^{\ \ 1}\xi^{n}\xi^{\partial}\right\rangle \\
&+ (\phi\ \widetilde{g}^{y})_{ab}\left[\quad \eta^{\ \ 1}\gamma^{ab}\xi^{n} \quad \eta^{\ \ 1}(L_{\beta}\gamma)^{ab}\xi^{n} + (L_{\xi^{\partial}}\gamma)^{ab} + 4\eta^{\ \ 3}\beta^{a}\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \xi^{n}\right)\right] \\
&+ (\phi\ \widetilde{g}^{y})_{ab}\left[\quad 2\eta^{\ \ 2}\left((\ r_{\gamma}\eta)^{a}\xi^{n} \quad \eta(\ r_{\gamma}\xi^{n})^{a} + (L_{\xi^{\partial}}\beta)^{a} \quad \xi^{a}\right)\right] \\
&+ (\phi\ \widetilde{g}^{y})_{ab}\left[2\eta^{\ \ 3}\beta^{a}\beta^{b}\left(L_{\xi^{\partial}}\eta \quad L_{\beta}\xi^{n} \quad \xi^{n}\right)\right] \\
&+ (\phi\ \widetilde{g}^{y})_{ab}\left[\quad 2\eta^{\ \ 2}\beta^{b}\left((\ r_{\gamma}\eta)^{a}\xi^{n} \quad \eta(\ r_{\gamma}\xi^{n})^{a} + (L_{\xi^{\partial}}\beta)^{a} \quad \xi^{a}\right)\right]
\end{aligned}
$$

Which, using Lemma 6.17, can be shown to be

$$\phi^* \left(\widetilde{g}^\dagger L_{\widetilde{\zeta}}\widetilde{g} + \iota_{[\widetilde{\zeta},\widetilde{\zeta}]}\widetilde{\zeta}^\dagger\right)$$

leading to

$$\phi^* S_{EH}(\Sigma \times I) = S_R(\Sigma \times I).$$

□

Remark 6.22. We would like to stress here that the results in this section are a “strictification” of the general construction of a solution of the classical master equation for the extended Hamiltonian, as presented by Henneaux and Bunster in [HT92, Theorem 18.8]. Indeed, the Hamiltonian analysis for a field theory relies on a (possibly) non-reduced version of the strict BFV data we consider, where strict indicates that we require all spaces of fields to be smooth symplectic manifolds. The AKSZ construction yields a BV theory (Theorem 6.10) which is effectively equivalent to the natural BV extension of Einstein–Hilbert theory (Theorems 6.15 and 6.21). It could be argued that this effective equivalence preserves the BV cohomology [DGH90; Hen90; BBH95]. However, note that the quantisation procedure outlined in [CMR18] does indeed require the strict version of a BV-BFV structure[6], and its existence is not to be taken for granted, as was shown in [CS19b] and [CS17].

6.4 AKSZ Palatini–Cartan

Following the construction outlined in Section 6.1.2, starting from the BFV theory of Palatini–Cartan gravity (see Section 6.2), we can construct the AKSZ space of fields F_{PC}^{AKSZ}. We will use the following notation:

$$\mathsf{e} = e + f^\dagger \qquad \mathsf{w} = \omega + u^\dagger \tag{6.20a}$$
$$\mathsf{c} = c + w \qquad \mathsf{x} = \xi + z \tag{6.20b}$$
$$\mathsf{l} = \lambda + \mu \qquad \mathsf{c}^\dagger = k^\dagger + c^\dagger \tag{6.20c}$$
$$\mathsf{y}^\dagger = e^\dagger + y^\dagger \tag{6.20d}$$

where

$$\begin{aligned}
&e \in C^\infty(I) \otimes \Omega^1_{nd}(\Sigma, V) && f^\dagger \in \Omega^1[-1](I) \otimes \Omega^1(\Sigma, V)\\
&\omega \in C^\infty(I) \otimes \mathcal{A}^{red}(\Sigma) && u^\dagger \in \Omega^1[-1](I) \otimes \mathcal{A}^{red}(\Sigma)\\
&c \in \Omega^0[1](I \times \Sigma, \wedge^2 V) && w \in \Omega^1[-1](I) \otimes \Omega^0[1](\Sigma, \wedge^2 V)\\
&\xi \in C^\infty(I) \otimes \mathfrak{X}[1](\Sigma) && z \in \Omega^1[-1](I) \otimes \mathfrak{X}[1](\Sigma)\\
&\lambda \in C^\infty[1](I \times \Sigma) && \mu \in \Omega^1[-1](I) \otimes C^\infty[1](\Sigma)\\
&k^\dagger \in C^\infty(I) \otimes \Omega^3[-1](\Sigma, \wedge^2 V) && c^\dagger \in \Omega^1[-1](I) \otimes \Omega^3[-1](\Sigma, \wedge^2 V)\\
&e^\dagger \in C^\infty(I) \otimes \Omega^3[-1](\Sigma, \wedge^3 V) && y^\dagger \in \Omega^1[-1](I) \otimes \Omega^3[-1](\Sigma, \wedge^3 V).
\end{aligned}$$

Theorem 6.23. *The AKSZ data $\mathsf{F}_{PC}^{\mathsf{AKSZ}}(I; \mathsf{F}_{PC}^{\partial})$ are given by the quadruple*

$$\mathsf{F}_{PC}^{\mathsf{AKSZ}}(I; \mathsf{F}_{PC}^{\partial}) = (F_{PC}^{\mathsf{AKSZ}}, S_{PC}^{\mathsf{AKSZ}}, \varpi_{PC}^{\mathsf{AKSZ}}, Q_{PC}^{\mathsf{AKSZ}})$$

[6] See [MSW19] for the comparison between strict and lax BV-BFV structures.

where:

$$\mathsf{F}^{\mathsf{AKSZ}}_{PC} \ ' \ T[\ 1](\mathrm{Map}(I, \mathsf{F}^{\partial}_{PC}))$$

$$\begin{aligned}\varpi^{\mathsf{AKSZ}}_{PC} = \int_I \ & \delta(e^{N\ 3} f^{y})\delta\omega + e^{N\ 3}\delta e \delta u^{y} + \delta w \delta k^{y} + \delta c \delta c^{y} + \delta u^{y} \delta(\iota_\xi k^{y}) \\ & + \delta\omega\delta(\iota_z k^{y}) + \delta\omega\delta(\iota_\xi c^{y}) \quad \delta\mu\epsilon_n \delta e^{y} \quad \delta\lambda\epsilon_n\delta y^{y} \\ & + \iota_{\delta z}\delta(e e^{y}) + \iota_{\delta\xi}\delta(f^{y} e^{y}) + \iota_{\delta\xi}\delta(e y^{y});\end{aligned}$$

$$\begin{aligned}S^{\mathsf{AKSZ}}_{PC} = \int_I \ & w e^{N\ 3} d_\omega e + (N\ \ 3) c e^{N\ 4} f^{y} d_\omega e + c e^{N\ 3}[u^{y}, e] + c e^{N\ 3} d_\omega f^{y} \\ & + \iota_z e e^{N\ 3} F_\omega + \iota_\xi (e^{N\ 3} f^{y}) F_\omega + \iota_\xi e e^{N\ 3} d_\omega u^{y} + \epsilon_n \mu e^{N\ 3} F_\omega \\ & + (N\ \ 3)\epsilon_n \lambda e^{N\ 4} f^{y} F_\omega + \epsilon_n \lambda e^{N\ 3} d_\omega u^{y} + [w, c]k^{y} + \frac{1}{2}[c, c]c^{y} \\ & \quad \iota_z d_\omega c k^{y} \quad [\iota_\xi u^{y}, c]k^{y} \quad \iota_\xi d_\omega w k^{y} \quad \iota_\xi d_\omega c c^{y} + \iota_z \iota_\xi F_\omega k^{y} \\ & + \frac{1}{2}\iota_\xi\iota_\xi d_\omega u^{y} k^{y} + \frac{1}{2}\iota_\xi\iota_\xi F_\omega c^{y} \quad [w, \epsilon_n\lambda]e^{y} \quad [c, \epsilon_n \mu]e^{y} \quad [c, \epsilon_n\lambda]y^{y} \\ & + \iota_z d_\omega(\epsilon_n\lambda)e^{y} + [\iota_\xi u^{y}, \epsilon_n\lambda]e^{y} + \iota_\xi d_\omega(\epsilon_n\mu)e^{y} + \iota_\xi d_\omega(\epsilon_n\lambda)y^{y} \\ & + \iota_{[z,\xi]} e e^{y} + \frac{1}{2}\iota_{[\xi,\xi]} f^{y} e^{y} + \frac{1}{2}\iota_{[\xi,\xi]} e y^{y} \\ & + \frac{1}{N\ \ 2} e^{N\ 2} d_I \omega + c d_I k^{y} + d_I \omega \iota_\xi k^{y} \quad \iota_{d_I \xi} e e^{y} + d_I \lambda \epsilon_n e^{y}.\end{aligned}$$

Remark 6.24. The explicit expression of the vector field Q^{AKSZ}_{PC} is rather complicated due to the presence of the constraint (6.6) (for $N\ \ 4$). Since we will not need it in what follows, we omit it. It can be computed from the equation

$$\iota_{Q^{\mathsf{AKSZ}}_{PC}} \varpi^{\mathsf{AKSZ}}_{PC} = \delta S^{\mathsf{AKSZ}}_{PC},$$

up to boundary terms.

Proof. This is a straightforward application of the AKSZ prescription outlined in Section 6.1.2. Using the transgression map we can build a symplectic form $\mathsf{F}^{\mathsf{AKSZ}}_{PC}$

$$\varpi^{\mathsf{AKSZ}}_{PC} = \int_I \ \mathsf{e}^{N\ 3}\delta\mathsf{e}\delta\mathsf{w} + \delta\mathsf{c}\delta\mathsf{c}^{y} + \delta\mathsf{w}\delta(\iota_{\mathfrak{x}}\mathsf{c}^{y}) \quad \delta\mathsf{l}e_n\delta\mathsf{y}^{y} + \iota_{\delta\mathfrak{x}}\delta(\mathsf{e}\mathsf{y}^{y}) \tag{6.21}$$

from which we obtain the claimed expression using (6.20). Analogously the AKSZ action can be constructed using the transgression map from the boundary one-form α^∂ and from the boundary action S^∂. Namely we have

$$\begin{aligned}S^{\mathsf{AKSZ}}_{PC} = \int_I \ & \frac{1}{N\ \ 2}\mathsf{e}^{N\ 2} d_I \mathsf{w} + \mathsf{c} d_I \mathsf{c}^{y} + d_I \mathsf{w}\iota_{\mathfrak{x}}\mathsf{c}^{y} \quad \iota_{d_I \mathfrak{x}}\mathsf{e}\mathsf{y}^{y} + d_I \mathsf{l} e_n \mathsf{y}^{y} \\ & \quad \mathsf{c}\mathsf{e} d_{\mathfrak{w}}\mathsf{e} + \iota_{\mathfrak{x}}\mathsf{e}\mathsf{e}F_{\mathfrak{w}} + e_n \mathsf{l}\mathsf{e} F_{\mathfrak{w}} + \frac{1}{2}[\mathsf{c}, \mathsf{c}]\mathsf{c}^{y} \quad L^{\mathfrak{w}}_{\mathfrak{x}}\mathsf{c}\mathsf{c}^{y} + \frac{1}{2}\iota_{\mathfrak{x}}\iota_{\mathfrak{x}}F_{\mathfrak{w}}\mathsf{c}^{y} \\ & \quad [\mathsf{c}, e_n \mathsf{l}]\mathsf{y}^{y} + L^{\mathfrak{w}}_{\mathfrak{x}}(e_n\mathsf{l})\mathsf{y}^{y} + \frac{1}{2}\iota_{[\mathfrak{x},\mathfrak{x}]}\mathsf{e}\mathsf{y}^{y}.\end{aligned} \tag{6.22}$$

Again the claimed expression can be obtained straightforwardly from (6.20). □

From Theorem 6.2 we know that $\mathsf{F}_{PC}^{\mathsf{AKSZ}}$ yields a BV theory on the manifold $I \quad \Sigma$. Furthermore, by Proposition 6.5 these data satisfy also the BV-BFV axioms of Definition 1.5.

In [CS19b] two of the authors proved that, using the natural symmetries of PC theory, the resulting BV theory F_{PC} does not satisfy the BV-BFV axioms (it is not a 1-extended BV theory) unless additional requirements on the fields are enforced. Next section will be devoted to the comparison between $\mathsf{F}_{PC}(\Sigma \quad I)$ and $\mathsf{F}^{\mathsf{AKSZ}}(I; \mathsf{F}_{PC}^{\partial}(\Sigma))$.

6.4.1 Comparison of BV data for Palatini–Cartan theory

We want to compare the AKSZ BV theory of Theorem 6.23 with the one proposed for PC-gravity in [CS19b] and recalled in Section 1.2.5, which we briefly recap here. Let M be an N-dimensional manifold.

Definition 6.25. We call standard BV theory for PC gravity the BV theory

$$\mathsf{F}_{PC}(M) = (\mathcal{F}_{PC}(M), S_{PC}(M), \varpi_{PC}(M), Q_{PC}(M))$$

where

$$\mathcal{F}_{PC}(M) := T \;[\quad 1] \left(\Omega^1_{nd}(M, \mathcal{V}) \quad \mathcal{A} \quad \mathfrak{X}[1](M) \quad \Omega^0[1](M, \mathrm{ad}P)\right)$$

and the fields in the base are denoted by $(\widetilde{e}, \widetilde{\omega}, \widetilde{\zeta}, \widetilde{c})$,[7] while the corresponding variables in the cotangent fibre are denoted by $(\widetilde{e}^{y}, \widetilde{\omega}^{y}, \widetilde{\zeta}^{y}, \widetilde{c}^{y})$;

$$\varpi_{PC}(M) = \int_M \delta\widetilde{e}\delta\widetilde{e}^{y} + \delta\widetilde{\omega}\delta\widetilde{\omega}^{y} + \delta\widetilde{c}\delta\widetilde{c}^{y} + \iota_{\delta\widetilde{\zeta}}\widetilde{\zeta}^{y};$$

$$\begin{aligned} S_{PC}(M) = &\int_M \frac{1}{N \quad 2}\widetilde{e}^{N \quad 2}F_{\widetilde{\omega}} + \left(\iota_{\widetilde{\zeta}}F_{\widetilde{\omega}} \quad d_{\widetilde{\omega}}\widetilde{c}\right)\widetilde{\omega}^{y} \quad \left(L_{\widetilde{\zeta}}^{\widetilde{\omega}}\widetilde{e} \quad [\widetilde{c}, \widetilde{e}]\right)\widetilde{e}^{y} \\ &+ \int_M \frac{1}{2}\left(\iota_{\widetilde{\zeta}}\iota_{\widetilde{\zeta}}F_{\widetilde{\omega}} \quad [\widetilde{c}, \widetilde{c}]\right)\widetilde{c}^{y} + \frac{1}{2}\iota_{[\widetilde{\zeta}, \widetilde{\zeta}]}\widetilde{\zeta}^{y}. \end{aligned}$$

The explicit expression of the cohomological vector field Q_{PC}, defined by the equation $\iota_{Q_{PC}}\varpi_{PC} = \delta S_{PC}$, will be useful in the following:

$$\begin{aligned} Q_{PC}\widetilde{e} &= L_{\widetilde{\zeta}}^{\widetilde{\omega}}\widetilde{e} \quad [\widetilde{c}, \widetilde{e}] \\ Q_{PC}\widetilde{\omega} &= \iota_{\widetilde{\zeta}}F_{\widetilde{\omega}} \quad d_{\widetilde{\omega}}\widetilde{c} \\ Q_{PC}\widetilde{c} &= \frac{1}{2}\iota_{\widetilde{\zeta}}\iota_{\widetilde{\zeta}}F_{\widetilde{\omega}} \quad \frac{1}{2}[\widetilde{c}, \widetilde{c}] \\ Q_{PC}\widetilde{\zeta} &= \frac{1}{2}[\widetilde{\zeta}, \widetilde{\zeta}] \\ Q_{PC}\widetilde{e}^{y} &= \widetilde{e}^{N \quad 3}F_{\widetilde{\omega}} + L_{\widetilde{\zeta}}^{\widetilde{\omega}}\widetilde{e}^{y} \quad [\widetilde{c}, \widetilde{e}^{y}] \\ Q_{PC}\widetilde{\omega}^{y} &= \widetilde{e}^{N \quad 3}d_{\widetilde{\omega}}\widetilde{e} \quad d_{\widetilde{\omega}}\iota_{\widetilde{\zeta}}\widetilde{\omega}^{y} \quad [\widetilde{c}, \widetilde{\omega}^{y}] + \iota_{\widetilde{\zeta}}[\widetilde{e}, \widetilde{e}^{y}] \quad \frac{1}{2}d_{\widetilde{\omega}}\iota_{\widetilde{\zeta}}\iota_{\widetilde{\zeta}}\widetilde{c}^{y} \\ Q_{PC}\widetilde{c}^{y} &= \quad d_{\widetilde{\omega}}\widetilde{\omega}^{y} \quad [\widetilde{e}, \widetilde{e}^{y}] \quad [\widetilde{c}, \widetilde{c}^{y}] \\ Q_{PC}\widetilde{\zeta}^{y} &= F_{\widetilde{\omega}} \;\; \widetilde{\omega}^{y} \quad (d_{\widetilde{\omega}} \;\; \widetilde{e})\widetilde{e}^{y} + \iota_{\widetilde{\zeta}}F_{\widetilde{\omega}} \;\; \widetilde{c}^{y} + L_{\widetilde{\zeta}}^{\widetilde{\omega}}\widetilde{\zeta}^{y} + (d_{\widetilde{\omega}}\iota_{\widetilde{\zeta}}\widetilde{\zeta}^{y}) \;\; . \end{aligned}$$

[7] Here we use tilde-variables for the elds of the *classical* BV formulation of PC theory. These are not to be confused with the similar ones used for the boundary eld of the BFV theory of PC in three dimensions.

Here we used the symbol   to remind the reader that $\widetilde{\zeta}^\dagger$ is a one-form with values in densities, and on the right hand side we highlight the one-form part of the expression.

Remark 6.26. Throughout the analysis we should always keep in mind that, while Definition 6.25 is valid for any manifold M (possibly with boundary), the AKSZ theory obtained in Theorem 6.23 is by construction defined on a manifold diffeomorphic to a cylinder. Furthermore, as we will see later in this section in more detail, the fields of the AKSZ theory must satisfy the additional constraint (6.6) not required in Definition 6.25. A more detailed analysis of a BV theory satisfying this additional constraints with symmetries is postponed to future work.

Theorem 6.27. *If $\mu \neq 0$, there exists a BV inclusion $\mathsf{F}^{\mathsf{AKSZ}}_{PC} \to \mathsf{F}_{PC}$.*

Remark 6.28. Exploiting Proposition 1.40 we can prove Theorem 6.27 by showing that there exists a a BV manifold $\mathsf{F}_R = (F_R, S_R, \varpi_R, Q_R)$, a strong BV equivalence φ and a BV inclusion ι

$$F^{\mathsf{AKSZ}}_{PC} \xrightarrow{\varphi} F_R \xrightarrow{\iota} F_{PC}.$$

In particular we will show an explicit expression for the diffeomorphism φ and use a notation in the target F_R so that the inclusion ι becomes evident.

Proof. We apply the following scheme: first we define the BV manifold F_R as BV-subspace of F_{PC} and then we prove that F_R is strongly BV-equivalent to $\mathsf{F}^{\mathsf{AKSZ}}$. Let now F_R be the BV theory defined by the following data:

$$F_R := T^*[-1]\left(\Omega^1_{nd}(M,V) \oplus \mathcal{A}^{red} \oplus \mathfrak{X}[1](M) \oplus \Omega^0[1](M,\mathrm{ad}P)\right)$$

where $\mathcal{A}^{red}$ is the quotient of the space of connections $\mathcal{A}$ with respect to the kernel of the map $W^{\partial,(1,2)}_{N-3}$ defined in Chapter 3 as $W^{\partial,(1,2)}_{N-3}(X) = e|^{N-3} \wedge X$. Keeping the same notation for the fields that we used for F_{PC}, the expressions for ϖ_R, S_R and Q_R coincide with those of Definition 6.25. The only thing that one needs to check is that $Q^2 = 0$, but this is exactly the same computation done for F_{PC} in [CS19b]. By construction, it is clear that the natural inclusion $F_R \hookrightarrow F_{PC}$ is also a BV-inclusion.

We go on to show that there exists a strong BV-equivalence between $\mathsf{F}^{\mathsf{AKSZ}}$ and F_R. We proceed as follows: we first show that there exists a symplectomorphism $(F^{\mathsf{AKSZ}}_{PC}, \varpi^{\mathsf{AKSZ}}_{PC}) \to (F_R, \varpi_R)$, and then we prove that such a map preserves the action functionals.

We define the following map $\varphi : F^{\mathsf{AKSZ}}_{PC} \to F_R$:

$$\mathbf{e} = \widetilde{e} + \underline{\widetilde{e}}_n \quad \boldsymbol{\omega} = \widetilde{\omega} + \underline{\widetilde{\omega}}_n \quad \mathbf{e}^\dagger = \widetilde{e}^\dagger + \underline{\widetilde{e}}_n{}^\dagger \tag{6.23a}$$

$$\boldsymbol{\omega}^\dagger = \widetilde{\omega}^\dagger + \underline{\widetilde{\omega}}^\dagger_n \qquad \mathbf{c} = \widetilde{c} \qquad \mathbf{c}^\dagger = \underline{\widetilde{c}}^\dagger \tag{6.23b}$$

$$\boldsymbol{\xi} = \widetilde{\zeta} \quad \boldsymbol{\xi}^\dagger = \underline{\widetilde{\zeta}^\dagger} + \underline{\lambda}^\dagger \tag{6.23c}$$

where the underlined variables contain the one form dx^n in the direction of the interval[8] and

$$\widetilde{e} = e + \lambda\epsilon_n^{[n]} f^\dagger \qquad \underline{\widetilde{e}}_n = \epsilon_n \underline{\mu} + \iota_{\underline{z}} e + \lambda\epsilon_n^{[a]} \underline{f}^\dagger_a \tag{6.24a}$$

$$\widetilde{\omega} = \omega - \lambda\epsilon_n^{[n]} u^\dagger \qquad \underline{\widetilde{\omega}}_n = \underline{w} - \iota_\xi \underline{u}^\dagger - \lambda\epsilon_n^{[a]} \underline{u}^\dagger_a \tag{6.24b}$$

[8] This notation is useful to identify the variables where such one form is contracted with the corresponding vector field representing the diffeomorphisms. An example of the use of such notation is in (6.24) where the definition of $\widetilde{e}$ does not contain dx^n while the one of $\underline{\widetilde{e}}_n$ does.

$$\widetilde{e}^\mathsf{y} = e^\mathsf{y} - \lambda\epsilon_n^{[n]} y_n^\mathsf{y} \qquad \underline{\widetilde{e}}_n^\mathsf{y} = e^{N-3}\underline{u}^\mathsf{y} + \iota_{\underline{z}} e^\mathsf{y} - \lambda\epsilon_n^{[a]}\underline{y}_a^\mathsf{y} + (N-3)e^{N-4}\lambda\epsilon_n^{[n]} f^\mathsf{y}\underline{u}^\mathsf{y} \tag{6.24c}$$

$$\widetilde{\omega}^\mathsf{y} = k^\mathsf{y} \qquad \underline{\widetilde{\omega}}_n^\mathsf{y} = e^{N-3}\underline{f}^\mathsf{y} + \iota_{\underline{z}} k^\mathsf{y} + \iota_\xi \underline{c}^\mathsf{y} \tag{6.24d}$$

$$\widetilde{c} = c - \iota_\xi \lambda\epsilon_n^{[n]} u^\mathsf{y} \qquad \widetilde{\underline{c}}^\mathsf{y} = \underline{c}^\mathsf{y} \tag{6.24e}$$

$$\widetilde{\zeta}^\alpha = \xi^\alpha + \lambda\epsilon_n^{[\alpha]} \qquad \underline{\widetilde{\zeta}}^\mathsf{y} = e\underline{y}^\mathsf{y} + \underline{f}^\mathsf{y} e^\mathsf{y} - \underline{u}^\mathsf{y} k^\mathsf{y} + \underline{c}^\mathsf{y}\lambda\epsilon_n^{[n]} u^\mathsf{y} \tag{6.24f}$$

$$\underline{\lambda}^\mathsf{y} = \widetilde{e}_n \underline{y}^\mathsf{y} + e^{N-3} f^\mathsf{y}\underline{u}^\mathsf{y} + f^\mathsf{y}\iota_{\underline{z}} e^\mathsf{y} + u^\mathsf{y}\iota_{\underline{z}} k^\mathsf{y} + c^\mathsf{y}\lambda\epsilon_n^{[a]}\underline{u}_a^\mathsf{y} \tag{6.24g}$$

where $\epsilon_n^{[\alpha]}$ $(\alpha = 1, 2, \dots, N-1, n)$ are the components of ϵ_n with respect to e and $\underline{e_n} := \epsilon_n \underline{\mu} + \iota_{\underline{z}} e$ i.e. $\epsilon_n = \epsilon_n^{[a]} e_a + \epsilon_n^{[n]} e_n$. The proof that this transformation is actually a symplectomorphism that preserves the action is quite long and hence postponed to Appendix 6.A. □

Remark 6.29. The condition $\mu \neq 0$ in Theorem 6.27 is necessary in order to make $\tilde{e}$ non-degenerate on the bulk, and to build the symplectomorphism (6.24). In particular note that $\epsilon_n^{[n]} = \mu^{-1}$.

Remark 6.30. The number of free components of $\tilde{\omega}$ is $\frac{3N(N-1)}{2}$, since ω and w have respectively $N(N-1)$ and $\frac{N(N-1)}{2}$ free components. The $\frac{N(N-1)(N-3)}{2}$ missing components are those corresponding to the quotient on the boundary and are fixed by the condition $\epsilon_n e^{N-4} d_\omega e \in \mathrm{Im} W_{N-3}^{\partial,(1,1)}$. Correspondingly, also $\tilde{\omega}^\mathsf{y}$ has $\frac{3N(N-1)}{2}$ independent components: $\frac{N(N-1)}{2}$ coming from k^y and $N(N-1)$ from f^y.

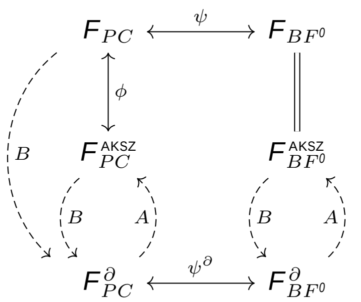
6.4.2 Three dimensional case

When $N = 3$ some simplification occur. Indeed, in this case the inclusion is actually an identity since there are no additional constraints on the field. Furthermore we know that the theory is strongly BV-equivalent, both in the bulk and on the boundary, to the topological BF theory, denoted here by $\mathsf{F}_{BF^0}^{\mathsf{AKSZ}}$. Hence we can summarize the results in the following theorem.

Corollary 6.31. *The theories F_{PC}^{AKSZ} and $\mathsf{F}_{BF^0}^{AKSZ}$ are strongly BV equivalent.*

Proof. The claim follows directly from Theorem 6.4 given the results of Theorem 6.27 and of Chapter 2, which proves the strong equivalence (at all codimensions) of non-degenerate BF theory and PC gravity in three dimensions. □

Pictorially we can describe the content of Corollary 6.31 as follows



(6.25)

where the arrows A represent the AKSZ constructions, the arrows B represent the BV-BFV reductions, while ψ, ψ^∂ and ϕ are the symplectomorphisms mentioned above.

Appendix

6.A Lengthy calculations

We prove here that the transformation (6.24) is a symplectomorphism between F^{AKSZ} and F_R that preserves the action. In the computation we will use multiple times the following useful relation:

$$\epsilon_n^{[a]} = \quad z^a \epsilon_n^{[n]}, \qquad \epsilon_n^{[n]} = \mu \quad ^{1}.$$

We now prove that this is a symplectomorphism, i.e. $\varpi_R = \varphi \quad \varpi_{PC}^{\mathsf{AKSZ}}$.

$$\begin{aligned}
\varpi_R = &\int_M \delta \mathbf{e} \delta \mathbf{e}^{\mathsf{y}} + \delta \boldsymbol{\omega} \delta \boldsymbol{\omega}^{\mathsf{y}} + \delta \mathbf{c} \delta \mathbf{c}^{\mathsf{y}} + \iota_{\delta \boldsymbol{\xi}} \boldsymbol{\xi}^{\mathsf{y}} \\
= &\int_M \delta \widetilde{e} \delta \underline{\widetilde{e}}^{\mathsf{y}}_n + \delta \underline{\widetilde{e}}_n \delta \widetilde{e}^{\mathsf{y}} + \delta \widetilde{\omega} \delta \underline{\widetilde{\omega}}^{\mathsf{y}}_n + \delta \underline{\widetilde{\omega}}_n \delta \widetilde{\omega}^{\mathsf{y}} + \delta \widetilde{c} \delta \underline{\widetilde{c}}^{\mathsf{y}} + \delta \widetilde{\zeta}^a \underline{\widetilde{\zeta}^{\mathsf{y}}}_a + \delta \widetilde{\zeta}^n \underline{\widetilde{\zeta}^{\mathsf{y}}}_n \qquad (6.26)\\
= &\int_M \underline{\delta e \delta (e^{N \quad 3} \underline{u}^{\mathsf{y}})}_1 + \underline{\delta e \delta (\iota_{\underline{z}} e^{\mathsf{y}})}_2 \quad \underline{\delta e \delta (\lambda \epsilon_n^{[a]} \underline{y}^{\mathsf{y}}_a)}_3 + \underline{(N \quad 3) \delta e \delta (e^{N \quad 4} \lambda \epsilon_n^{[n]} f^{\mathsf{y}} \underline{u}^{\mathsf{y}})}_4 \\
&+ \underline{\delta (\lambda \epsilon_n^{[n]} f^{\mathsf{y}}) \delta (e^{N \quad 3} \underline{u}^{\mathsf{y}})}_5 + \underline{\delta (\lambda \epsilon_n^{[n]} f^{\mathsf{y}}) \delta (\iota_{\underline{z}} e^{\mathsf{y}})}_6 \quad \underline{\delta (\lambda \epsilon_n^{[n]} f^{\mathsf{y}}) \delta (\lambda \epsilon_n^{[a]} \underline{y}^{\mathsf{y}}_a)}_7 \\
&+ \underline{(N \quad 3) \delta (\lambda \epsilon_n^{[n]} f^{\mathsf{y}}) \delta (e^{N \quad 4} \lambda \epsilon_n^{[n]} f^{\mathsf{y}} \underline{u}^{\mathsf{y}})}_8 + \underline{\delta (\iota_{\underline{z}} e) \delta e^{\mathsf{y}}}_9 \quad \underline{\delta (\iota_{\underline{z}} e) \delta (\lambda \epsilon_n^{[n]} y^{\mathsf{y}})}_{10} \\
&+ \underline{\delta (\epsilon_n \underline{\mu}) \delta e^{\mathsf{y}}}_{11} \quad \underline{\delta (\epsilon_n \underline{\mu}) \delta (\lambda \epsilon_n^{[n]} y^{\mathsf{y}})}_{12} + \underline{\delta (\lambda \epsilon_n^{[a]} \underline{f}^{\mathsf{y}}_a) \delta e^{\mathsf{y}}}_{13} \quad \underline{\delta (\lambda \epsilon_n^{[a]} \underline{f}^{\mathsf{y}}_a) \delta (\lambda \epsilon_n^{[n]} y^{\mathsf{y}})}_{14} \\
&+ \underline{\delta \omega \delta (e^{N \quad 3} \underline{f}^{\mathsf{y}})}_{15} + \underline{\delta \omega \delta (\iota_{\underline{z}} k^{\mathsf{y}})}_{16} + \underline{\delta \omega \delta (\iota_\xi \underline{c}^{\mathsf{y}})}_{17} \quad \underline{\delta (\lambda \epsilon_n^{[n]} u^{\mathsf{y}}) \delta (e^{N \quad 3} \underline{f}^{\mathsf{y}})}_{18} \\
&\quad \underline{\delta (\lambda \epsilon_n^{[n]} u^{\mathsf{y}}) \delta (\iota_{\underline{z}} k^{\mathsf{y}})}_{19} \quad \underline{\delta (\lambda \epsilon_n^{[n]} u^{\mathsf{y}}) \delta (\iota_\xi \underline{c}^{\mathsf{y}})}_{20} + \underline{\delta \underline{w} \delta k^{\mathsf{y}}}_{21} \quad \underline{\delta (\iota_\xi \underline{u}^{\mathsf{y}}) \delta k^{\mathsf{y}}}_{22} \\
&\quad \underline{\delta (\lambda \epsilon_n^{[a]} \underline{u}^{\mathsf{y}}_a) \delta k^{\mathsf{y}}}_{23} + \underline{\delta c \delta \underline{c}^{\mathsf{y}}}_{24} \quad \underline{\delta (\iota_\xi \lambda \epsilon_n^{[n]} u^{\mathsf{y}}) \delta \underline{c}^{\mathsf{y}}}_{25} + \underline{\iota_{\delta \xi} \delta (e \underline{y}^{\mathsf{y}})}_{26} \\
&+ \underline{\iota_{\delta \xi} \delta (\underline{f}^{\mathsf{y}} e^{\mathsf{y}})}_{27} \quad \underline{\iota_{\delta \xi} \delta (\underline{u}^{\mathsf{y}} k^{\mathsf{y}})}_{28} + \underline{\iota_{\delta \xi} \delta (\underline{c}^{\mathsf{y}} \lambda \epsilon_n^{[n]} u^{\mathsf{y}})}_{29} + \underline{\delta (\lambda \epsilon_n^{[a]}) \delta (e_a \underline{y}^{\mathsf{y}})}_{30} \\
&+ \underline{\delta (\lambda \epsilon_n^{[a]}) \delta (\underline{f}^{\mathsf{y}}_a e^{\mathsf{y}})}_{31} + \underline{\delta (\lambda \epsilon_n^{[a]}) \delta (\underline{u}^{\mathsf{y}}_a k^{\mathsf{y}})}_{32} + \underline{\delta (\lambda \epsilon_n^{[a]}) \delta (\underline{c}^{\mathsf{y}}_a \lambda \epsilon_n^{[n]} u^{\mathsf{y}})}_{33} \\
&+ \underline{\delta (\lambda \epsilon_n^{[n]}) \delta (\widetilde{e}_n \underline{y}^{\mathsf{y}})}_{34} + \underline{\delta (\lambda \epsilon_n^{[n]}) \delta (e^{N \quad 3} f^{\mathsf{y}} \underline{u}^{\mathsf{y}})}_{35} + \underline{\delta (\lambda \epsilon_n^{[n]}) \delta (f^{\mathsf{y}} \iota_{\underline{z}} e^{\mathsf{y}})}_{36} \\
&+ \underline{\delta (\lambda \epsilon_n^{[n]}) \delta (u^{\mathsf{y}} \iota_{\underline{z}} k^{\mathsf{y}})}_{37} + \underline{\delta (\lambda \epsilon_n^{[n]}) \delta (c^{\mathsf{y}} \lambda \epsilon_n^{[a]} \underline{u}^{\mathsf{y}}_a)}_{38}
\end{aligned}$$

This expression should be compared with the symplectic form coming from the AKSZ construction:

$$\begin{aligned}
\varpi_{PC}^{\mathsf{AKSZ}} = \int_I \quad & \underline{\delta (e^{N \quad 3} f^{\mathsf{y}}) \delta \omega}_1 + \underline{e^{N \quad 3} \delta e \delta u^{\mathsf{y}}}_2 + \underline{\delta w \delta k^{\mathsf{y}}}_3 + \underline{\delta c \delta c^{\mathsf{y}}}_4 + \underline{\delta u^{\mathsf{y}} \delta (\iota_\xi k^{\mathsf{y}})}_5 \\
&+ \underline{\delta \omega \delta (\iota_z k^{\mathsf{y}})}_6 + \underline{\delta \omega \delta (\iota_\xi c^{\mathsf{y}})}_7 \quad \underline{\delta \mu \epsilon_n \delta e^{\mathsf{y}}}_8 \quad \underline{\delta \lambda \epsilon_n \delta y^{\mathsf{y}}}_9 \\
&+ \underline{\iota_{\delta z} \delta (e e^{\mathsf{y}})}_{10} + \underline{\iota_{\delta \xi} \delta (f^{\mathsf{y}} e^{\mathsf{y}})}_{11} + \underline{\iota_{\delta \xi} \delta (e y^{\mathsf{y}}).}_{12} \qquad (6.27)
\end{aligned}$$

Almost all the terms in (6.27) can be direclty found in ϖ_R:
(6.27.1)= (6.26.15); (6.27.2)= (6.26.1); (6.27.3)= (6.26.21);
(6.27.4)= (6.26.24); (6.27.6)= (6.26.16); (6.27.7)= (6.26.17);
(6.27.8)= (6.26.11); (6.27.11)= (6.26.27); (6.27.12)= (6.26.26).
The remaining terms can be identified using the following relations:

(6.27.5)= (6.26.22)+(6.26.28):

$$\delta\underline{u}^{y}\delta(\iota_\xi k^{y}) = -\delta(\iota_\xi \underline{u}^{y})\delta k^{y} - \iota_{\delta\xi}\delta(\underline{u}^{y}k^{y});$$

(6.27.9)=(6.26.3)+(6.26.10)+(6.26.12)+(6.26.30)+(6.26.34):

$$\begin{aligned}\delta(\lambda\epsilon_n)\delta\underline{y}^{y} = & -\delta(\lambda\epsilon_n^{[a]}e_a)\delta\underline{y}^{y} - \delta(\lambda\epsilon_n^{[n]}\widetilde{e}_n)\delta\underline{y}^{y} \\ & = \delta(\lambda\epsilon_n^{[a]})\delta(e_a\underline{y}^{y}) - \delta e\delta(\lambda\epsilon_n^{[a]}\underline{y}_a^{y}) \\ & + \delta(\lambda\epsilon_n^{[n]})\delta(\widetilde{e}_n\underline{y}^{y}) - \delta\underline{\widetilde{e}}_n\delta(\lambda\epsilon_n^{[n]}y^{y}).\end{aligned}$$

(6.27.10)=(6.26.2)+(6.26.9):

$$\iota_{\delta\underline{z}}\delta(ee^{y}) = \delta e\delta(\iota_{\underline{z}}e^{y}) + \delta(\iota_{\underline{z}}e)\delta e^{y}$$

All the other terms in (6.26) sum to zero because of the following identities:

(6.26.4)+(6.26.5)+(6.26.18)+(6.26.35)=0:

$$\begin{aligned}&\delta(\lambda\epsilon_n^{[n]}f^{y})\delta(e^{N-3}\underline{u}^{y}) + (N-3)\delta e\delta(e^{N-4}\lambda\epsilon_n^{[n]}f^{y}\underline{u}^{y}) \\ &-\delta(\lambda\epsilon_n^{[n]}u^{y})\delta(e^{N-3}\underline{f}^{y}) + \delta(\lambda\epsilon_n^{[n]})\delta(e^{N-3}f^{y}\underline{u}^{y}) = 0;\end{aligned}$$

(6.26.6)+(6.26.13)+(6.26.31)+(6.26.36)=0:

$$\delta(\lambda\epsilon_n^{[n]}f^{y})\delta(\iota_{\underline{z}}e^{y}) + \delta(\lambda\epsilon_n^{[a]}\underline{f}_a^{y})\delta e^{y} + \delta(\lambda\epsilon_n^{[a]})\delta(\underline{f}_a^{y}e^{y}) + \delta(\lambda\epsilon_n^{[n]})\delta(f^{y}\iota_{\underline{z}}e^{y}) = 0;$$

(6.26.19)+(6.26.23)+(6.26.32)+(6.26.37)=0:

$$\delta(\lambda\epsilon_n^{[n]}u^{y})\delta(\iota_{\underline{z}}k^{y}) - \delta(\lambda\epsilon_n^{[a]}\underline{u}_a^{y})\delta k^{y} + \delta(\lambda\epsilon_n^{[a]})\delta(\underline{u}_a^{y}k^{y}) + \delta(\lambda\epsilon_n^{[n]})\delta(u^{y}\iota_{\underline{z}}k^{y}) = 0;$$

(6.26.20)+(6.26.25)+(6.26.29)=0:

$$\delta(\lambda\epsilon_n^{[n]}u^{y})\delta(\iota_\xi\underline{c}^{y}) - \delta(\iota_\xi\lambda\epsilon_n^{[n]}u^{y})\delta\underline{c}^{y} + \iota_{\delta\xi}\delta(\underline{c}^{y}\lambda\epsilon_n^{[n]}u^{y}) = 0;$$

(6.26.33)+(6.26.38)=0:

$$\delta(\lambda\epsilon_n^{[a]})\delta(\underline{c}_a^{y}\lambda\epsilon_n^{[n]}u^{y}) + \delta(\lambda\epsilon_n^{[n]})\delta(c^{y}\lambda\epsilon_n^{[a]}\underline{u}_a^{y}) = 0;$$

(6.26.7)+(6.26.14)=0:

$$\delta(\lambda\epsilon_n^{[n]}f^{y})\delta(\lambda\epsilon_n^{[a]}\underline{y}_a^{y}) - \delta(\lambda\epsilon_n^{[a]}\underline{f}_a^{y})\delta(\lambda\epsilon_n^{[n]}y^{y}) = 0;$$

(6.26.8)=0 since both $\lambda\lambda = 0$ and $f^{y}f^{y} = 0$:

$$(N-3)\delta(\lambda\epsilon_n^{[n]}f^{y})\delta(e^{N-4}\lambda\epsilon_n^{[n]}f^{y}\underline{u}^{y}) = 0.$$

We have now to show that this symplectomorphism preserves the action i.e. $S_R = S_{PC}^{AKSZ}$. We do it by direct inspection:

$$S_R = \int_M \frac{1}{N-2} e^{N-2} F_! + (d_! c - F_!)! ^y + L^! e - [c;e] \, e^y \tag{6.28}$$

$$+ \frac{1}{2}([c;c] - F_!) c^y + \frac{1}{2}[\;;\;]^y$$

$$= \int_M e^{N-3} \underline{e}_n F_\flat + \frac{1}{N-2} e^{N-2} \underline{F_{\flat_n}} - {}_e F_\flat + F_{\flat_n} e^n - d_\flat e \, \underline{\flat}_n^y - {}_e \underline{F_{\flat_n}} - \underline{d_{\flat_n}} e - \flat^y$$

$$+ L_e^\flat e + d_{\flat_n} e e^n + e_n d e^n - [e;e] \, \underline{e}_n^y - \frac{1}{2} {}_e \, {}_e F_\flat + {}_e F_{\flat_n} e^n - \frac{1}{2}[e;e] \, \underline{e}^y$$

$$+ {}_e d_\flat \underline{e}_n + \underset{n}{@} {}_e e - \underline{d_{\flat_n}} (e_n e^n) - [e; \underline{e}_n] \, e^y + \frac{1}{2} [e, e] \underline{e}^y + \frac{1}{2} [e, e]^n \underline{}^y$$

$$= \int_M \underline{e^N \; 3 \; {}_n \underline{} F_!}_1 + \underline{e^{N-3} \underline{z} e F_!}_2 + \underline{e^{N-3} \; {}^{[a]}_n f^y \underline{}_a F_!}_3 + \underline{(N-3) e^{N-4} \; {}^{[n]}_n f^y \underline{e_n} F_!}_4$$

$$\underline{e^N \; {}^3 \underline{e_n} d_! (\; {}^{[n]}_n u^y)}_5 \quad \underline{e^{N-3} \; {}^{[a]}_n f^y \underline{}_a d_! (\; {}^{[n]}_n u^y)}_6 \quad \underline{(N-3) e^{N-4} \; {}^{[n]}_n f^y \underline{e_n} d_! (\; {}^{[n]}_n u^y)}_7$$

$$+ \underline{\frac{1}{N-2} e^{N-2} \underset{n}{@} !}_8 \quad \underline{\frac{1}{N-2} e^{N-2} \underset{n}{@} \; {}^{[n]}_n u^y}_9 + \underline{\frac{1}{N-2} e^{N-2} d_! \underline{w}}_{10} \quad \underline{\frac{1}{N-2} e^{N-2} d_! (\; \underline{u}^y)}_{11}$$

$$\underline{\frac{1}{N-2} e^{N-2} d_! (\; {}^{[a]}_n \underline{u}_a^y)}_{12} \quad \underline{\frac{1}{N-2} e^{N-2} [\; {}^{[n]}_n u^y; \underline{w}}_{13} \quad \underline{\underline{u}^y]}_{14}$$

$$+ \underline{e^{N-3} \; {}^{[n]}_n f^y \underline{F_{\flat_n}}}_{15}$$

$$\underline{F_! (e^{N-3} f^y \underline{}}_{16} + \underline{\underline{z} k^y}_{17} + \underline{\underline{c}^y)}_{18} + \underline{d_! (\; {}^{[n]}_n u^y)(e^{N-3} f^y \underline{}}_{19} + \underline{\underline{z} k^y}_{20} + \underline{\underline{c}^y)}_{21}$$

$$\underline{F_{!\,a} \; {}^{[a]}_n (e^{N-3} f^y \underline{}}_{22} + \underline{\underline{z} k^y}_{23} + \underline{\underline{c}^y)}_{24} + \underline{d_! (\; {}^{[n]}_n u^y)_a \; {}^{[a]}_n (e^{N-3} f^y \underline{}}_{25} + \underline{\underline{z} k^y}_{26} + \underline{\underline{c}^y)}_{27}$$

$$\underline{F_{\flat_n} \; {}^{[n]}_n e^{N-3} f^y \underline{}}_{28} \quad \underline{F_{\flat_n} \; {}^{[n]}_n \underline{z} k^y}_{29} \quad \underline{F_{\flat_n} \; {}^{[n]}_n \underline{c}^y}_{30}$$

$$+ \underline{d_! c (e^{N-3} f^y \underline{}}_{31} + \underline{\underline{z} k^y}_{32} + \underline{\underline{c}^y)}_{33} \quad \underline{[\; {}^{[n]}_n u^y; c](e^{N-3} f^y \underline{}}_{34} + \underline{\underline{z} k^y}_{35} + \underline{\underline{c}^y)}_{36}$$

$$\underline{d_! (\; {}^{[n]}_n u^y)(e^{N-3} f^y \underline{}}_{37} + \underline{\underline{z} k^y}_{38} + \underline{\underline{c}^y)}_{39}$$

$$\underline{\underset{n}{@} ! k^y}_{40} + \underline{\underset{n}{@} (\; {}^{[n]}_n u^y) k^y}_{41} \quad \underline{d_! \underline{w} k^y}_{42} + \underline{d_! (\; \underline{u}^y) k^y}_{43}$$

$$+ \underline{d_! (\; {}^{[a]}_n \underline{u}_a^y) k^y}_{44} + \underline{[\; {}^{[n]}_n u^y; \underline{w}}_{45} \quad \underline{\underline{u}^y] k^y}_{46} \quad \underline{\underline{F_{\flat_{an}}} \; {}^{[a]}_n k^y}_{47} + \underline{\underset{n}{@} c k^y}_{48}$$

$$\underline{\underset{n}{@} (\; {}^{[n]}_n u^y) k^y}_{49} + \underline{[\underline{w}; c] k^y}_{50} \quad \underline{[\; \underline{u}^y; c] k^y}_{51} \quad \underline{[\; {}^{[a]}_n \underline{u}_a^y; c] k^y}_{52}$$

$$+ [\underline{w}_{53} \quad \underline{\underline{u}^y; \; {}^{[n]}_n u^y] k^y}_{54}$$

$$+ \underline{L^! e e^{N-3} \underline{u}^y}_{55} + \underline{L^! e \, \underline{z} e^y}_{56} \quad \underline{L^! e \; {}^{[a]}_n \underline{y}_a^y}_{57} + \underline{(N-3) L^! e e^{N-4} \; {}^{[n]}_n f^y \underline{u}^y}_{58} + \underline{((d_! e)_a \; {}^{[a]}_n}_{59}$$

$$\underline{d_! (e_a \; {}^{[a]}_n)) e^{N-3} \underline{u}^y}_{60} + \underline{((d_! e)_a \; {}^{[a]}_n}_{61} \quad \underline{d_! (e_a \; {}^{[a]}_n)) \, \underline{z} e^y}_{62} + \underline{d_! (e_a \; {}^{[a]}_n) \; {}^{[a]}_n \underline{y}_a^y}_{63}$$

$$\underline{(N-3) d_! (e_a \; {}^{[a]}_n) e^{N-4} \; {}^{[n]}_n f^y \underline{u}^y}_{64} + \underline{L^! (\; {}^{[n]}_n f^y) e^{N-3} \underline{u}^y}_{65} + \underline{L^! (\; {}^{[n]}_n f^y) \, \underline{z} e^y}_{66}$$

$$\underline{L^! (\; {}^{[n]}_n f^y) \; {}^{[a]}_n \underline{y}_a^y}_{67} + \underline{(d_! (\; {}^{[n]}_n f^y))_a \; {}^{[a]}_n e^{N-3} \underline{u}^y}_{68} + \underline{(d_! (\; {}^{[n]}_n f^y))_a \; {}^{[a]}_n \, \underline{z} e^y)}_{69}$$

$$\underline{[\; {}^{[n]}_n u^y; e](e^{N-3} \underline{u}^y}_{70} + \underline{\underline{z} e^y)}_{71} + \underline{\underset{n}{@} e \; {}^{[n]}_n e^{N-3} \underline{u}^y}_{72} + \underline{\underset{n}{@} e \; {}^{[n]}_n \, \underline{z} e^y}_{73}$$

$$+ [\underline{w}_{74} \quad \underline{\underline{u}^y; e] \; {}^{[n]}_n e^{N-3} \underline{u}^y}_{75} + [\underline{w}_{76} \quad \underline{\underline{u}^y; e] \; {}^{[n]}_n \, \underline{z} e^y}_{77}$$

$$
\begin{aligned}
&+ \underline{\partial_n(\lambda\epsilon_n^{[n]} f^y)\lambda\epsilon_n^{[n]}(e^{N\quad 3}\underline{u}^y}_{78} + \underline{\iota_{\underline{z}} e^y)}_{79} + \underline{\underline{e_n} d(\lambda\epsilon_n^{[n]}) e^{N\quad 3}\underline{u}^y}_{80} + \underline{\underline{e_n} d(\lambda\epsilon_n^{[n]})\iota_{\underline{z}} e^y}_{81} \\
&+ \underline{\lambda\epsilon_n^{[a]} f_a^y d(\lambda\epsilon_n^{[n]}) e^{N\quad 3}\underline{u}^y}_{82} + \underline{\lambda\epsilon_n^{[a]} f_a^y d(\lambda\epsilon_n^{[n]})\iota_{\underline{z}} e^y}_{83} \quad \underline{\underline{e_n} d(\lambda\epsilon_n^{[n]})\lambda\epsilon_n^{[a]}\underline{y}_a^y}_{84} \\
&+ \underline{(N \quad 3)\underline{e_n} d(\lambda\epsilon_n^{[n]}) e^{N\quad 4}\lambda\epsilon_n^{[n]} f^y \underline{u}^y}_{85} \quad \underline{[c,e] e^{N\quad 3}\underline{u}^y}_{86} \quad \underline{[c,e]\iota_{\underline{z}} e^y}_{87} + \underline{[c,e]\lambda\epsilon_n^{[a]}\underline{y}_a^y}_{88} \\
&\quad \underline{(N \quad 3)[c,e] e^{N\quad 4}\lambda\epsilon_n^{[n]} f^y \underline{u}^y}_{89} \quad \underline{[c,\lambda\epsilon_n^{[n]} f^y] e^{N\quad 3}\underline{u}^y}_{90} \quad \underline{[c,\lambda\epsilon_n^{[n]} f^y]\iota_{\underline{z}} e^y}_{91} \\
&+ \underline{[\iota_\xi\lambda\epsilon_n^{[n]} u^y, e](e^{N\quad 3}\underline{u}^y}_{92} + \underline{\iota_{\underline{z}} e^y)}_{93} \\
&+ \underline{\iota_\xi d_\omega(\epsilon_n\underline{\mu}) e^y}_{94} + \underline{\iota_\xi d_\omega(\iota_{\underline{z}} e) e^y}_{95} + \underline{\iota_\xi d_\omega(\lambda\epsilon_n^{[a]}\underline{f}_a^y) e^y}_{96} + \underline{d_{\omega_a}(\underline{e_n})\lambda\epsilon_n^{[a]} e^y}_{97} \\
&+ \underline{d_{\omega_a}(\lambda\epsilon_n^{[a]}\underline{f}_a^y)\lambda\epsilon_n^{[a]} e^y}_{98} \quad \underline{\iota_\xi[\lambda\epsilon_n^{[n]} u^y, \underline{e_n}] e^y}_{99} \quad \underline{\iota_\xi d_\omega(\underline{e_n})\lambda\epsilon_n^{[n]} y_n^y}_{100} \\
&\quad \underline{\iota_\xi d_\omega(\lambda\epsilon_n^{[a]}\underline{f}_a^y)\lambda\epsilon_n^{[n]} y_n^y}_{101} \quad \underline{\iota_{\underline{\partial}_n\xi} e e^y}_{102} + \underline{e_a\underline{\partial}_n(\lambda\epsilon_n^{[a]}) e^y}_{103} \quad \underline{\iota_{\underline{\partial}_n\widetilde{\zeta}}\lambda\epsilon_n^{[n]} f^y e^y}_{104} \\
&+ \underline{\iota_{\underline{\partial}_n\widetilde{\zeta}} e\lambda\epsilon_n^{[n]} y_n^y}_{105} \quad \underline{\underline{\partial}_n(\widetilde{e}_n\lambda\epsilon_n^{[n]}) e^y}_{106} \quad \underline{[\underline{w}}_{107} \quad \underline{\iota_\xi\underline{u}^y, \widetilde{e}_n\lambda\epsilon_n^{[n]}] e^y}_{108} \\
&+ \underline{\underline{\partial}_n(\widetilde{e}_n\lambda\epsilon_n^{[n]})\lambda\epsilon_n^{[n]} y_n^y}_{109} \quad \underline{[c,\epsilon_n\underline{\mu}] e^y}_{110} \quad \underline{[c,\iota_{\underline{z}} e] e^y}_{111} \quad \underline{[c,\lambda\epsilon_n^{[a]}\underline{f}_a^y] e^y}_{112} \\
&+ \underline{[\iota_\xi\lambda\epsilon_n^{[n]} u^y, \underline{e_n}] e^y}_{113} + \underline{[c,\underline{e_n}]\lambda\epsilon_n^{[n]} y_n^y}_{114} \\
&\quad \underline{\frac{1}{2}\iota_\xi\iota_\xi F_\omega\underline{c}^y}_{115} + \underline{\frac{1}{2}\iota_\xi\iota_\xi d_\omega(\lambda\epsilon_n^{[n]} u^y)\underline{c}^y}_{116} \quad \underline{\iota_\xi F_{\omega_a}\lambda\epsilon_n^{[a]}\underline{c}^y}_{117} \\
&+ \underline{\iota_\xi d_\omega(\lambda\epsilon_n^{[n]} u^y)_a\lambda\epsilon_n^{[a]}\underline{c}^y}_{118} \quad \underline{\iota_\xi F_{\widetilde{\omega}_n}\lambda\epsilon_n^{[n]}\underline{c}^y}_{119} + \underline{\frac{1}{2}[c,c]\underline{c}^y}_{120} \quad \underline{[\iota_\xi\lambda\epsilon_n^{[n]} u^y, c]\underline{c}^y}_{121} \\
&\quad \underline{\xi^b\partial_b\xi^a e_a\underline{y}^y}_{122} \quad \underline{\xi^b\partial_b(\lambda\epsilon_n^{[a]}) e_a\underline{y}^y}_{123} \quad \underline{\lambda\epsilon_n^{[b]}\partial_b\xi^a e_a\underline{y}^y}_{124} \quad \underline{\lambda\epsilon_n^{[b]}\partial_b(\lambda\epsilon_n^{[a]}) e_a\underline{y}^y}_{125} \\
&\quad \underline{\lambda\epsilon_n^{[n]}\partial_n\widetilde{\zeta} a e_a\underline{y}^y}_{126} \quad \underline{\xi^b\partial_b\xi^a\underline{f}_a^y e^y}_{127} \quad \underline{\xi^b\partial_b(\lambda\epsilon_n^{[a]})\underline{f}_a^y e^y}_{128} \quad \underline{\lambda\epsilon_n^{[b]}\partial_b\xi^a\underline{f}_a^y e^y}_{129} \\
&\quad \underline{\lambda\epsilon_n^{[b]}\partial_b(\lambda\epsilon_n^{[a]})\underline{f}_a^y e^y}_{130} \quad \underline{\lambda\epsilon_n^{[n]}\partial_n\widetilde{\zeta} a\underline{f}_a^y e^y}_{131} \quad \underline{\xi^b\partial_b\xi^a\underline{u}_a^y k^y}_{132} \quad \underline{\xi^b\partial_b(\lambda\epsilon_n^{[a]})\underline{u}_a^y k^y}_{133} \\
&\quad \underline{\lambda\epsilon_n^{[b]}\partial_b\xi^a\underline{u}_a^y k^y}_{134} + \underline{\lambda\epsilon_n^{[b]}\partial_b(\lambda\epsilon_n^{[a]})\underline{u}_a^y k^y}_{135} \quad \underline{\lambda\epsilon_n^{[n]}\partial_n\xi^a\underline{u}_a^y k^y}_{136} \\
&\quad \underline{\lambda\epsilon_n^{[n]}\partial_n(\lambda\epsilon_n^{[a]})\underline{u}_a^y k^y}_{137} \quad \underline{\xi^b\partial_b\xi^a\underline{c}_a^y\lambda\epsilon_n^{[n]} u^y}_{138} \quad \underline{\xi^b\partial_b(\lambda\epsilon_n^{[a]})\underline{c}_a^y\lambda\epsilon_n^{[n]} u^y}_{139} \\
&\quad \underline{\xi^a\partial_a(\lambda\epsilon_n^{[n]})\widetilde{e}_n\underline{y}^y}_{140} \quad \underline{\lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]})\widetilde{e}_n\underline{y}^y}_{141} \quad \underline{\lambda\epsilon_n^{[n]}\partial_n(\lambda\epsilon_n^{[n]})\widetilde{e}_n\underline{y}^y}_{142} \\
&\quad \underline{\xi^a\partial_a(\lambda\epsilon_n^{[n]}) f^y e^{N\quad 3}\underline{u}^y}_{143} \quad \underline{\lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]}) f^y e^{N\quad 3}\underline{u}^y}_{144} \quad \underline{\lambda\epsilon_n^{[n]}\partial_n(\lambda\epsilon_n^{[n]}) f^y e^{N\quad 3}\underline{u}^y}_{145} \\
&\quad \underline{\xi^a\partial_a(\lambda\epsilon_n^{[n]}) f^y\iota_{\underline{z}} e^y}_{146} \quad \underline{\lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]}) f^y\iota_{\underline{z}} e^y}_{147} \quad \underline{\lambda\epsilon_n^{[n]}\partial_n(\lambda\epsilon_n^{[n]}) f^y\iota_{\underline{z}} e^y}_{148} \\
&+ \underline{\xi^a\partial_a(\lambda\epsilon_n^{[n]}) u^y\iota_{\underline{z}} k^y}_{149} + \underline{\lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]}) u^y\iota_{\underline{z}} k^y}_{150} + \underline{\lambda\epsilon_n^{[n]}\partial_n(\lambda\epsilon_n^{[n]}) u^y\iota_{\underline{z}} k^y}_{151} \\
&\quad \underline{\xi^a\partial_a(\lambda\epsilon_n^{[n]}) c^y\lambda\epsilon_n^{[a]}\underline{u}_a^y}_{152}
\end{aligned}
$$

We want to compare this with the AKSZ action:

$$
\begin{aligned}
S_{PC}^{\text{AKSZ}} = \int_I \quad & \underline{\underline{w} e^{N\quad 3} d_\omega e}_1 + \underline{(N \quad 3) c e^{N\quad 4}\underline{f}^y d_\omega e}_2 + \underline{c e^{N\quad 3}[\underline{u}^y, e]}_3 + \underline{c e^{N\quad 3} d_\omega\underline{f}^y}_4 + \underline{\iota_{\underline{z}} e e^{N\quad 3} F_\omega}_5 \\
& + \underline{\iota_\xi(e^{N\quad 3}\underline{f}^y) F_\omega}_6 + \underline{\iota_\xi e e^{N\quad 3} d_\omega\underline{u}^y}_7 + \underline{\epsilon_n\underline{\mu} e^{N\quad 3} F_\omega}_8 + \underline{(N \quad 3)\epsilon_n\lambda e^{N\quad 4}\underline{f}^y F_\omega}_9 \\
& + \underline{\epsilon_n\lambda e^{N\quad 3} d_\omega\underline{u}^y}_{10} + \underline{[\underline{w}, c] k^y}_{11} + \underline{\frac{1}{2}[c,c]\underline{c}^y}_{12} \quad \underline{\iota_{\underline{z}} d_\omega c k^y}_{13} \quad \underline{[\iota_\xi\underline{u}^y, c] k^y}_{14} \quad \underline{\iota_\xi d_\omega\underline{w} k^y}_{15}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{\iota_\xi d_\omega c\underline{c}^{y}}_{16} + \underline{\iota_{\underline{z}}\iota_\xi F_\omega k^{y}}_{17} + \underline{\frac{1}{2}\iota_\xi\iota_\xi d_\omega \underline{u}^{y} k^{y}}_{18} + \underline{\frac{1}{2}\iota_\xi\iota_\xi F_\omega \underline{c}^{y}}_{19} \quad \underline{[\underline{w}, \epsilon_n\lambda]e^{y}}_{20} \\
&\underline{[c, \epsilon_n\underline{\mu}]e^{y}}_{21} \quad \underline{[c, \epsilon_n\lambda]\underline{y}^{y}}_{22} + \underline{\iota_{\underline{z}} d_\omega(\epsilon_n\lambda)e^{y}}_{23} + \underline{[\iota_\xi\underline{u}^{y}, \epsilon_n\lambda]e^{y}}_{24} + \underline{\iota_\xi d_\omega(\epsilon_n\underline{\mu})e^{y}}_{25} \\
&+\underline{\iota_\xi d_\omega(\epsilon_n\lambda)\underline{y}^{y}}_{26} + \underline{\iota_{[\underline{z},\xi]}ee^{y}}_{27} + \underline{\frac{1}{2}\iota_{[\xi,\xi]}\underline{f}^{y}e^{y}}_{28} + \underline{\frac{1}{2}\iota_{[\xi,\xi]}e\underline{y}^{y}}_{29} + \underline{\frac{1}{N\quad 2}e^{N\quad 2}\underline{\partial_n}\omega}_{30} \\
&+\underline{c\underline{\partial_n}k^{y}}_{31} + \underline{\underline{\partial_n}\omega\iota_\xi k^{y}}_{32} \quad \underline{\iota_{\underline{\partial_n}\xi}ee^{y}}_{33} + \underline{\underline{\partial_n}\lambda\epsilon_n e^{y}}_{34}.
\end{aligned} \tag{6.29}
$$

We proceed as follows: we first check that all terms in (6.29) can be found in (6.28), then we show that all other terms in (6.28) sum to zero.
We can easily recognized the following terms identically repeated in both expressions:

(6.29.1)= (6.28.10);	(6.29.3)= (6.28.86);	(6.29.5)= (6.28.2);
(6.29.6)= (6.28.16);	(6.29.8)= (6.28.1);	(6.29.11)= (6.28.50);
(6.29.12)= (6.28.120);	(6.29.13)= (6.28.32);	(6.29.14)= (6.28.51);
(6.29.15)= (6.28.42);	(6.29.16)= (6.28.33);	(6.29.17)= (6.28.17);
(6.29.21)= (6.28.110);	(6.29.25)= (6.28.94);	(6.29.28)= (6.28.127);
(6.29.29)= (6.28.122);	(6.29.30)= (6.28.8);	(6.29.31)= (6.28.48);
(6.29.32)= (6.28.40);	(6.29.33)= (6.28.102).	

Some relations are proved using Leibniz rule and Cartan calculus. We use the symbol $\}$ to indicate that we are neglecting a boundary term arising from an integration by parts.

(6.29.2)+ (6.29.4)= (6.28.31):

$$(N\quad 3)ce^{N\quad 4}\underline{f}^{y}d_\omega e + c^{N\quad 3}d_\omega\underline{f}^{y} = cd_\omega(e^{N\quad 3}\underline{f}^{y}) \overset{\}}{=} d_\omega c(e^{N\quad 3}\underline{f}^{y});$$

(6.29.7)= (6.28.55)+(6.28.11):

$$
\begin{aligned}
\frac{1}{N\quad 2}e^{N\quad 2}d_\omega(\iota_\xi\underline{u}^{y}) + L_\xi^\omega ee^{N\quad 3}\underline{u}^{y} \overset{\}}{=} &\quad \frac{1}{N\quad 2}e^{N\quad 2}d_\omega(\iota_\xi\underline{u}^{y}) \quad \frac{1}{N\quad 2}e^{N\quad 2}L_\xi^\omega\underline{u}^{y} \\
&= +\iota_\xi ee^{N\quad 3}d_\omega\underline{u}^{y};
\end{aligned}
$$

(6.29.18)= (6.28.43)+(6.28.132):

$$\xi^b\partial_b\xi^a\underline{u}_a^{y}k^{y} = \quad \frac{1}{2}\iota_{[\xi,\xi]}\underline{u}^{y}k^{y} = \quad \iota_\xi d_\omega\iota_\xi\underline{u}^{y}k^{y} + \frac{1}{2}\iota_\xi\iota_\xi d_\omega\underline{u}^{y}k^{y};$$

(6.29.19)= (6.28.18)+(6.28.115):

$$\frac{1}{2}\iota_\xi\iota_\xi F_\omega c^{y} = \quad \frac{1}{2}\iota_\xi\iota_\xi F_\omega c^{y} \quad \iota_\xi F_\omega\iota_\xi c^{y};$$

(6.29.27)=(6.28.56)+(6.28.95):

$$\iota_{[\underline{z},\xi]}ee^{y} = \iota_{\underline{z}}d_\omega\iota_\xi ee^{y} + \iota_\xi d_\omega\iota_{\underline{z}}ee^{y} \quad \iota_{\underline{z}}\iota_\xi d_\omega ee^{y} = \iota_\xi d_\omega\iota_{\underline{z}}ee^{y} + L_\xi^\omega e\iota_{\underline{z}}e^{y}$$

All the other relations involving terms of (6.28) are based on the expansion

$$\epsilon_n = e_a\epsilon_n^{(a)} + e_n\epsilon_n^{(n)}.$$

More in detail we have the following relations:

(6.29.9)= (6.28.3)+(6.28.4)+(6.28.22):

$$\begin{aligned}(N\;\;3)\epsilon_n\lambda e^{N\;\;4}f^{y}F_\omega &= (N\;\;3)e^{N\;\;4}e_a\lambda\epsilon_n^{(a)}\underline{f}^{y}F_\omega + (N\;\;3)e^{N\;\;4}e_n\lambda\epsilon_n^{(n)}\underline{f}^{y}F_\omega\\ &= e^{N\;\;3}\lambda\epsilon_n^{(a)}\underline{f}^{y}_{a}F_\omega\;\;\; e^{N\;\;3}\underline{f}^{y}F_{\omega_a}\lambda\epsilon_n^{(a)} + (N\;\;3)e^{N\;\;4}\lambda\epsilon_n^{(n)}f^{y}\underline{e_n}F_\omega\end{aligned}$$

(6.29.10)= (6.28.5)+(6.28.12)+(6.28.59)+(6.28.60)+(6.28.80):

$$\begin{aligned}&\epsilon_n\lambda e^{N\;\;3}d_\omega u^{y} = e_a\lambda\epsilon_n^{(a)}e^{N\;\;3}d_\omega\underline{u}^{y} + e_n\lambda\epsilon_n^{(n)}e^{N\;\;3}d_\omega\underline{u}^{y}\\ &\overset{\wr}{=}\;\; d_\omega(e_a\lambda\epsilon_n^{(a)})e^{N\;\;3}\underline{u}^{y}\;\;\; e_a\lambda\epsilon_n^{(a)}d_\omega e^{N\;\;3}\underline{u}^{y}\;\;\; e_ne^{N\;\;3}d_\omega(\lambda\epsilon_n^{(n)}\underline{u}^{y}) + e_ne^{N\;\;3}d_\omega(\lambda\epsilon_n^{(n)})\underline{u}^{y}\\ &\overset{\wr}{=}\;\; d_\omega(e_a\lambda\epsilon_n^{(a)})e^{N\;\;3}\underline{u}^{y} + e^{N\;\;3}(d_\omega e)_a\lambda\epsilon_n^{(a)}\underline{u}^{y}\;\;\; \frac{1}{N\;\;2}e^{N\;\;2}d_\omega(\lambda\epsilon_n^{(a)}\underline{u}^{y}_{a})\\ &\qquad \underline{e}_ne^{N\;\;3}d_\omega(\lambda\epsilon_n^{(n)}u^{y}) + e_ne^{N\;\;3}d_\omega(\lambda\epsilon_n^{(n)})\underline{u}^{y};\end{aligned}$$

(6.29.20)+ (6.29.24)= (6.28.76)+(6.28.77)+(6.28.107)+(6.28.108):

$$\begin{aligned}[\underline{w}\;\;\;\iota_\xi\underline{u}^{y},\epsilon_n\lambda]e^{y} &= \;\;[\underline{w}\;\;\;\iota_\xi\underline{u}^{y},e_a\lambda\epsilon_n^{(a)}]e^{y}\;\;\;[\underline{w}\;\;\;\iota_\xi\underline{u}^{y},e_n\lambda\epsilon_n^{(n)}]e^{y}\\ &= [\underline{w}\;\;\;\iota_\xi\underline{u}^{y},e]e^{y}_a\lambda\epsilon_n^{(a)}\;\;\;[\underline{w}\;\;\;\iota_\xi\underline{u}^{y},e_n\lambda\epsilon_n^{(n)}]e^{y}\\ &= [w\;\;\;\iota_\xi u^{y},e]\lambda\epsilon_n^{(n)}\iota_{\underline{z}}e^{y}\;\;\;[\underline{w}\;\;\;\iota_\xi\underline{u}^{y},e_n\lambda\epsilon_n^{(n)}]e^{y};\end{aligned}$$

(6.29.22)= (6.28.88)+(6.28.114):

$$[c,\epsilon_n\lambda]\underline{y}^{y} = \;\;[c,e_a\lambda\epsilon_n^{(a)}]\underline{y}^{y}\;\;\;[c,e_n\lambda\epsilon_n^{(n)}]\underline{y}^{y} = [c,e]\lambda\epsilon_n^{(a)}\underline{y}^{y}_{a} + [c,\underline{e_n}]\lambda\epsilon_n^{(n)}y^{y};$$

(6.29.23)= (6.28.62)+(6.28.81)+(6.28.97):

$$\begin{aligned}\iota_{\underline{z}}d_\omega(\epsilon_n\lambda)e^{y} &= \iota_{\underline{z}}d_\omega(e_a\lambda\epsilon_n^{(a)})e^{y} + \iota_{\underline{z}}d_\omega(e_n\lambda\epsilon_n^{(n)})e^{y}\\ &= \;\;\; d_\omega(e_a\lambda\epsilon_n^{(a)})\iota_{\underline{z}}e^{y} + e_nd_\omega(\lambda\epsilon_n^{(n)})\iota_{\underline{z}}e^{y}\;\;\; d_\omega e_n\lambda\epsilon_n^{(n)}\iota_{\underline{z}}e^{y}\\ &= \;\;\; d_\omega(e_a\lambda\epsilon_n^{(a)})\iota_{\underline{z}}e^{y} + e_nd_\omega(\lambda\epsilon_n^{(n)})\iota_{\underline{z}}e^{y} + d_{\omega_a}\underline{e}_n\lambda\epsilon_n^{(a)}e^{y};\end{aligned}$$

(6.29.26)=(6.28.57)+(6.28.100)+(6.28.123)+(6.28.124)+(6.28.140):

$$\begin{aligned}\iota_\xi d_\omega(\epsilon_n\lambda)\underline{y}^{y} &= \iota_\xi d_\omega(e_a\lambda\epsilon_n^{(a)})\underline{y}^{y} + \iota_\xi d_\omega(e_n\lambda\epsilon_n^{(n)})\underline{y}^{y}\\ &= \iota_\xi d_\omega e_a\lambda\epsilon_n^{(a)}\underline{y}^{y}\;\;\;\iota_\xi d_\omega(\lambda\epsilon_n^{(a)})e_a\underline{y}^{y}\;\;\;\iota_\xi d_\omega\underline{e}_n\lambda\epsilon_n^{(n)}y^{y}\;\;\;\iota_\xi d_\omega(\lambda\epsilon_n^{(n)})e_n\underline{y}^{y}\end{aligned}$$

and

$$\iota_\xi d_\omega e_a\lambda\epsilon_n^{(a)}\underline{y}^{y} = \;\;\; L_\xi^\omega e\lambda\epsilon_n^{(a)}\underline{y}^{y}_{a}\;\;\;\lambda\epsilon_n^{(a)}\partial_a\xi^be_b\underline{y}^{y};$$

(6.29.34)= (6.28.73)+(6.28.103)+(6.28.106)

$$\begin{aligned}\underline{\partial_n}(\lambda\epsilon_n)e^{y} &= \;\;\;\underline{\partial_n}(e_a\lambda\epsilon_n^{(a)})e^{y}\;\;\;\underline{\partial_n}(e_n\lambda\epsilon_n^{(n)})e^{y}\\ &= e_a\underline{\partial_n}(\lambda\epsilon_n^{(a)})e^{y}\;\;\;\underline{\partial_n}e_a\lambda\epsilon_n^{(a)}e^{y}\;\;\;\underline{\partial_n}(e_n\lambda\epsilon_n^{(n)})e^{y}\\ &= e_a\underline{\partial_n}(\lambda\epsilon_n^{(a)})e^{y}\;\;\;\partial_ne\lambda\epsilon_n^{(n)}\iota_{\underline{z}}e^{y}\;\;\;\underline{\partial_n}(e_n\lambda\epsilon_n^{(n)})e^{y}\end{aligned}$$

All the other terms sum to zero. We list here the details of the computations, starting from the trivial ones: (6.28.23)=0 and (6.28.61)=0 because they contain terms of the form ${}^{[a]}_{n}\ {}^{[b]}_{n}X_{ab}$ which vanishes by antisymmetry;

(6.28.9)+(6.28.72)=0; (6.28.15)+(6.28.28)=0; (6.28.29)+(6.28.47)=0;
(6.28.30)+(6.28.119)=0; (6.28.78)+(6.28.145)=0; (6.28.79)+(6.28.148)=0;
(6.28.104)+(6.28.131)=0; (6.28.105)+(6.28.126)=0; (6.28.109)+(6.28.142)=0;
(6.28.137)+(6.28.151)=0; (6.28.13)+(6.28.74)=0; (6.28.14)+(6.28.75)=0;
(6.28.35)+(6.28.52)=0; (6.28.36)+(6.28.121)=0; (6.28.45)+(6.28.53)=0;
(6.28.46)+(6.28.54)=0; (6.28.70)+(6.28.92)=0; (6.28.71)+(6.28.93)=0;
(6.28.87)+(6.28.111)=0; (6.28.91)+(6.28.112)=0; (6.28.99)+(6.28.113)=0;
(6.28.24)+(6.28.117)=0; (6.28.63)+(6.28.125)=0; (6.28.84)+(6.28.141)=0;
(6.28.7)+(6.28.85)=0; (6.28.67)+(6.28.101)=0; (6.28.98)+(6.28.130)=0.

Furthermore we have the following:

(6.28.41)+(6.28.49)+(6.28.136)=0: Using Leibniz rule we get

$$\underline{@}_n(\ {}^{[n]}_{n}u^y)k^y \quad \underline{@}_n(\ {}^{[n]}_{n}u^y)k^y \quad {}^{[n]}_{n}@_a(\ {}^{[a]}_{n})\underline{u}^y_a k^y = 0;$$

(6.28.34)+(6.28.89)+(6.28.90)=0:

$$[\ {}^{[n]}_{n}u^y; c]e^{N\ 3}\underline{f}^y \quad (N\ \ 3)[c; e]e^{N\ 4}\ {}^{[n]}_{n}f^y\underline{u}^y \quad [c;\ {}^{[n]}_{n}f^y]e^{N\ 3}\underline{u}^y$$
$$= [c; e^{N\ 3}\ {}^{[n]}_{n}f^y\underline{u}^y] = 0;$$

(6.28.19)+(6.28.37)+(6.28.58)+(6.28.65)+(6.28.143)=0:

$$d_!(\ {}^{[n]}_{n}u^y)e^{N\ 3}\underline{f}^y \quad d_!(\ {}^{[n]}_{n}u^y)e^{N\ 3}\underline{f}^y + (N\ \ 3)L^!\,ee^{N\ 4}\ {}^{[n]}_{n}f^y\underline{u}^y$$
$$+ L^!(\ {}^{[n]}_{n}f^y)e^{N\ 3}\underline{u}^y \quad {}^a@_a(\ {}^{[n]}_{n})f^y e^{N\ 3}\underline{u}^y$$
$$= L^!(\ {}^{[n]}_{n}u^y)e^{N\ 3}\underline{f}^y + L^!\,e^{N\ 3}\ {}^{[n]}_{n}f^y\underline{u}^y \quad {}^{[n]}_{n}L^!\,f^y e^{N\ 3}\underline{u}^y = 0;$$

(6.28.20)+(6.28.38)+(6.28.44)+(6.28.133)+(6.28.134)+(6.28.149)=0:

$$d_!(\ {}^{[n]}_{n}u^y)\ {}_{\underline{z}}k^y \quad d_!(\ {}^{[n]}_{n}u^y)\ {}_{\underline{z}}k^y + {}^a@_a(\ {}^{[n]}_{n})u^y\ {}_{\underline{z}}k^y$$
$${}^b@_b(\ {}^{[a]}_{n})\underline{u}^y_a k^y \quad {}^{[b]}_{n}@_b\ {}^a\underline{u}^y_a k^y + \ d_!(\ {}^{[a]}_{n}\underline{u}^y_a)k^y$$
$$= L^!(\ {}^{[n]}_{n}u^y)\ {}_{\underline{z}}k^y + {}^a@_a(\ {}^{[n]}_{n})u^y\ {}_{\underline{z}}k^y \quad (L^!\,\underline{u}^y)_a\ {}^{[a]}_{n}k^y$$
$$= \quad L^!(u^y)\ {}^{[n]}_{n}\ {}_{\underline{z}}k^y \quad (L^!\,\underline{u}^y)_a\ {}^{[a]}_{n}k^y = 0;$$

(6.28.66)+(6.28.96)+(6.28.128)+(6.28.129)+(6.28.146)=0:

$$L^!(\ {}^{[n]}_{n}f^y)\ {}_{\underline{z}}e^y \quad {}^a@_a(\ {}^{[n]}_{n})f^y\ {}_{\underline{z}}e^y + \ d_!(\ {}^{[a]}_{n}\underline{f}^y_a)e^y \quad {}^b@_b(\ {}^{[a]}_{n})\underline{f}^y_a e^y$$
$${}^{[b]}_{n}@_b\ {}^a\underline{f}^y_a e^y$$
$$= \quad L^!\,f^y\ {}^{[n]}_{n}\ {}_{\underline{z}}e^y \quad L^!\,\underline{f}^y_a\ {}^{[a]}_{n}e^y \quad {}^{[b]}_{n}@_b\ {}^a\underline{f}^y_a e^y$$
$$= L^!\,\underline{f}^y\ {}^{[a]}_{n}e^y_a + (L^!\,\underline{f}^y)_a\ {}^{[a]}_{n}e^y = 0;$$

(6.28.21)+(6.28.39)+(6.28.116)+(6.28.138)=0:

$$d_!(\ {}^{[n]}_{n}u^y)\ \underline{c}^y \quad d_!(\ {}^{[n]}_{n}u^y)\ \underline{c}^y + \frac{1}{2}\ \ d_!(\ {}^{[n]}_{n}u^y)\underline{c}^y \quad {}^b@_b\ {}^a\underline{c}^y_a\ {}^{[n]}_{n}u^y$$
$$= \quad \frac{1}{2}\ \ d_!(\ {}^{[n]}_{n}u^y)\underline{c}^y + \ d_!(\ {}^{[n]}_{n}u^y)\underline{c}^y \quad \frac{1}{2}\,{}_{[\ ;\ ]}(\ {}^{[n]}_{n}u^y)\underline{c}^y = 0;$$

(6.28.6)+(6.28.25)+(6.28.64)+(6.28.68)+(6.28.82)+(6.28.144)=0:

$$e^{N\;\;3}\lambda\epsilon_n^{[a]}\underline{f}^{\mathsf{y}}_a d_\omega(\lambda\epsilon_n^{[n]}u^{\mathsf{y}}) + d_\omega(\lambda\epsilon_n^{[n]}u^{\mathsf{y}})_a\lambda\epsilon_n^{[a]}e^{N\;\;3}\underline{f}^{\mathsf{y}} \quad (N \quad 3)d_\omega(e_a\lambda\epsilon_n^{[a]})\lambda\epsilon_n^{[n]}e^{N\;\;4}f^{\mathsf{y}}\underline{u}^{\mathsf{y}}$$
$$+\,(d_\omega(\lambda\epsilon_n^{[n]}f^{\mathsf{y}}))_a\lambda\epsilon_n^{[a]}e^{N\;\;3}\underline{u}^{\mathsf{y}} + \lambda\epsilon_n^{[a]}f^{\mathsf{y}}_a d(\lambda\epsilon_n^{[n]})e^{N\;\;3}\underline{u}^{\mathsf{y}} \quad \lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]})f^{\mathsf{y}}e^{N\;\;3}\underline{u}^{\mathsf{y}} = 0;$$

(6.28.26)+(6.28.135)+(6.28.150)=0:

$$d_\omega(\lambda\epsilon_n^{[n]}u^{\mathsf{y}})_a\lambda\epsilon_n^{[a]}\iota_{\underline{z}}k^{\mathsf{y}} + \lambda\epsilon_n^{[b]}\partial_b(\lambda\epsilon_n^{[a]})\underline{u}^{\mathsf{y}}_a k^{\mathsf{y}} + \lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]})u^{\mathsf{y}}\iota_{\underline{z}}k^{\mathsf{y}} = 0;$$

(6.28.27)+(6.28.118)+(6.28.139)+(6.28.152)=0:

$$d_\omega(\lambda\epsilon_n^{[n]}u^{\mathsf{y}})_a\lambda\epsilon_n^{[a]}\iota_\xi\underline{c}^{\mathsf{y}} + \iota_\xi d_\omega(\lambda\epsilon_n^{[n]}u^{\mathsf{y}})_a\lambda\epsilon_n^{[a]}\underline{c}^{\mathsf{y}}$$
$$\xi^b\partial_b(\lambda\epsilon_n^{[a]})\underline{c}^{\mathsf{y}}_a\lambda\epsilon_n^{[n]}u^{\mathsf{y}} \quad \xi^a\partial_a(\lambda\epsilon_n^{[n]})c^{\mathsf{y}}\lambda\epsilon_n^{[a]}\underline{u}^{\mathsf{y}}_a = 0;$$

(6.28.69)+(6.28.83)+(6.28.147)=0:

$$(d_\omega(\lambda\epsilon_n^{[n]}f^{\mathsf{y}}))_a\lambda\epsilon_n^{[a]}\iota_{\underline{z}}e^{\mathsf{y}} \quad \lambda\epsilon_n^{[a]}f^{\mathsf{y}}_a d(\lambda\epsilon_n^{[n]})\iota_{\underline{z}}e^{\mathsf{y}} \quad \lambda\epsilon_n^{[a]}\partial_a(\lambda\epsilon_n^{[n]})f^{\mathsf{y}}\iota_{\underline{z}}e^{\mathsf{y}} = 0.$$

Chapter 7

Codimension 2 structure

The goal of this chapter is to analyse the codimension 2 structure in the BV-BFV formalism of General Relativity in dimension greater than or equal to 3. We analyse the problem in both the Palatini–Cartan (in dimension greater than 3, since the three dimensional case is already treated in Chapter 2) and the Einstein–Hilbert formalism. The upshot of the construction, carried out with the standard BV-BFV procedure described in Section 1.1.2, is that the boundary BFV theories do not induce exact BF^{2}V theories on corners (i.e. codimension 2 submanifolds). It is however possible to construct such exact theories under additional hypotheses on the fields.

This is part of the program of casting GR into the BV-BFV formalism. The BF^{2}V structure can be considered as part of the data necessary for quantisation, and in particular for gluing, of theories on manifolds with boundary and corners.

As proved in Chapter 2, in the three-dimensional case the extension to codimension 2 is possible in the Palatini–Cartan formalism. The difference in the behaviour in different dimensions can be traced back to a similar problem to the one that appears in the passage from bulk to boundary, i.e. the presence of a kernel for $N \quad 4$ in the classical pre-boundary two form (see Section 1.2.6 for more details). Technically the failure is connected to the non-injectivity of some of the maps $W^{\partial\partial,(\ , \ )}$ defined in Chapter 3, in the same way as the non-injectivity of the map $W_1^{\partial,(1,2)}$ does not allow the reduction in the bulk to boundary case.

The results presented in this chapter are only preliminary and aimed at giving an overview of the developments in this direction. More work has to be done to fully understand the corner BF^{2}V structure and in particular to assess if it possible to adopt a strategy similar to the one used to construct a BFV structure on the boundary to circumvent the difficulties of a direct approach. Nonetheless we present here the full pre-corner structure, in particular the action and the cohomological vector field from which it is possible to deduce interesting information, for example by comparing them to the three-dimensional counterparts.

The chapter is divided in two parts: in section 7.1 we analyse the problem in the Palatini–Cartan case, while in section 7.2 we analyse it in the Einstein–Hilbert case.

7.1 Palatini–Cartan 2-extended theory

The starting point for the construction of the BF^{2}V structure is the BFV boundary structure. In the Palatini–Cartan formalism this is described in Section 4.3.2. We recall here the relevant quantities of this construction. The boundary structure is given by the following action and symplectic form

$$S^{\partial} = \int c e d_{\omega} e + \iota_{\xi} e e F_{\omega} + \lambda e_n e F_{\omega} + \frac{1}{3!}\lambda e_n \Lambda e^3 + \frac{1}{2}[c,c]\gamma^{y} \quad L_{\xi}^{\omega} c \gamma^{y} + \frac{1}{2}\iota_{\xi}\iota_{\xi} F_{\omega}\gamma^{y}$$
$$+ [c, \lambda e_n] y^{y} \quad L_{\xi}^{\omega}(\lambda e_n) y^{y} \quad \frac{1}{2}\iota_{[\xi,\xi]} e y^{y},$$

$$\varpi^{\partial} = \int e \delta e \delta \omega + \delta c \delta \gamma^{y} \quad \delta\omega\delta(\iota_{\xi}\gamma^{y}) + \delta\lambda e_n \delta y^{y} + \iota_{\delta\xi}\delta(e y^{y})$$

for $\mu = 1, 2, 3, n$. with the additional structural constraint

$$e_n d_{\omega} e = e\sigma \tag{7.1}$$

for some $\sigma\ 2\ \Omega_{\partial}^{1,1}$.

Remark 7.1. For simplicity we consider here only the case $N = 4$. However, the considerations of this chapter can be extended to the higher-dimensional cases. This can be done in the same way in which we can extend the boundary results on the boundary from the case $N = 4$ to a generic $N \quad 4$ (see Section 4.4).

7.1.1 Corner induced structure

From the boundary BFV action we can now induce a corner structure. We follow here the same procedure that we used in the three dimensional case. This is outlined in detail in Section 1.1.2. From now on we assume that the manifold Σ has a non-empty boundary $\partial\Sigma = \Gamma$[1]. In the three dimensional case we have seen in Proposition 2.15 that it is possible to induce a BF^{2}V theory on the corner. Nonetheless, contrary to this case, for $N \quad 4$ this construction fails.

Proposition 7.2. *The BFV theory* $\mathsf{F}_{PC}^{(1)} = (F_{PC}^{\partial}, S_{PC}^{\partial}, \varpi_{PC}^{\partial}, Q_{PC}^{\partial})$ *is not 1-extendable.*

Remark 7.3. On cylindrical manifolds we can reformulate Proposition 7.2 in the following way: The BV theory $\mathsf{F}_{PC}^{\mathsf{AKSZ}}$ is not 2-extendable.

Proof. From the variation of the boundary action, using the formula

$$\delta S^{\partial} = \iota_{Q^{\partial}}\varpi^{\partial} + \check{\pi}^{\partial}\check{\alpha}^{\partial}$$

we can deduce the pre-corner (or pre-codimension-2) one form:

$$\check{\alpha}^{\partial} = \int c e \delta e \quad \iota_{\xi} e e \delta\omega \quad e_m \xi^m e \delta\omega \quad \lambda e_n e \delta\omega \quad \delta c \gamma_m^{y} \xi^m \quad \delta\omega \iota_{\xi}\gamma_m^{y}\xi^m$$
$$\delta(\lambda e_n)\iota_{\xi} y^{y} \quad \delta(\lambda e_n) y_m^{y} \xi^m \quad \iota_{\delta\xi} e y_m^{y} \xi^m + e_m \delta\xi^m y_m^{y} \xi^m.$$

Taking its variation, we obtain the pre-corner two-form:

$$\check{\varpi}^{\partial} = \delta\check{\alpha}^{\partial} = \int \delta c e \delta e \quad \iota_{\delta\xi} e e \delta\omega \quad \iota_{\xi}(e\delta e)\delta\omega \quad \delta e_m \xi^m e \delta\omega + e_m \delta\xi^m e \delta\omega \quad e_m \xi^m \delta e \delta\omega \tag{7.2}$$
$$\delta\lambda e_n e \delta\omega \quad \lambda e_n \delta e \delta\omega \quad \delta c \gamma_m^{y} \delta\xi^m \quad \delta c \delta\gamma_m^{y} \xi^m \quad \delta\omega\delta(\iota_{\xi}\gamma_m^{y}\xi^m)$$
$$+ \delta(\lambda e_n)\delta y_m^{y} \xi^m + \delta(\lambda e_n) y_m^{y} \delta\xi^m + \iota_{\delta\xi}\delta e y_m^{y}\xi^m + \iota_{\delta\xi} e \delta y_m^{y} \xi^m \quad \iota_{\delta\xi} e y_m^{y} \delta\xi^m$$
$$+ \delta e_m \delta\xi^m y_m^{y}\xi^m \quad e_m \delta\xi^m \delta y_m^{y} \xi^m + e_m \delta\xi^m y_m^{y} \delta\xi^m$$

[1] I thank Odül Tetik for the suggestion on the notation.

In order to proceed we have to check if this two form is pre-symplectic, i.e. if the kernel of the corresponding application

$$\begin{aligned} \check{\varpi}^{\partial\sharp} : T\check{F}^{\partial} &\to T^* \check{F}^{\partial} \\ X &\mapsto \check{\varpi}^{\partial\sharp}(X) = \check{\varpi}^{\partial}(X, \cdot). \end{aligned}$$

is regular. The equations defining the kernel are:

$$\begin{aligned}
\delta c: &\quad eX_e + X_{\gamma_m^y}\xi^m - \gamma_m^y X_{\xi^m} = 0 && (7.3a)\\
\delta e: &\quad eX_c - e\iota_\xi X_\omega - \lambda e_n X_\omega - e_m \xi^m X_\omega - \iota_{X_\xi} y_m^y \xi^m = 0 && (7.3b)\\
\delta \xi: &\quad e\, eX_\omega - X_\omega c_m^y\, \xi^m + (X_e)\, y_m^y \xi^m + e\, X_{y_m^y}\xi^m - e\, y_m^y X_{\xi^m} = 0 && (7.3c)\\
\delta \omega: &\quad -\iota_{X_\xi} ee - \iota_\xi(eX_e) - X_{e_m}\xi^m e + e_m X_{\xi^m} e - e_m \xi^m X_e\\
&\quad - X_\lambda e_n e - \lambda e_n X_e - X_{(\iota_\xi \gamma_m^y \xi^m)} = 0 && (7.3d)\\
\delta e_m: &\quad \xi^m e X_\omega + X_{\xi^m} y_m^y \xi^m = 0 && (7.3e)\\
\delta \xi^m: &\quad e_m e X_\omega - X_c \gamma_m^y - X_\omega \iota_\xi \gamma_m^y + X_\lambda e_n y_m^y - \iota_{X_\xi} e y_m^y + X_{e_m} y_m^y \xi^m\\
&\quad - e_m X_{y_m^y}\xi^m + 2 e_m y_m^y X_{\xi^m} = 0 && (7.3f)\\
\delta \lambda: &\quad e_n e X_\omega + e_n X_{y_m^y}\xi^m + e_n y_m^y X_{\delta\xi^m} = 0 && (7.3g)\\
\delta \gamma_m^y: &\quad X_c \xi^m + \iota_\xi X_\omega \xi^m = 0 && (7.3h)\\
\delta y_m^y: &\quad + X_\lambda e_n \xi^m + \iota_{X_\xi} e \xi^m - e_m X_{\xi^m}\xi^m = 0 && (7.3i)
\end{aligned}$$

Let us consider (7.3a) and (7.3b). They can be solved only if the functions $W_1^{\partial\partial(1,1)}$ and $W_1^{\partial\partial(0,2)}$ are invertible (at least on the space generated by the remaining terms, not proportional to e). From Lemma 3.5 we gather that both $W_1^{\partial\partial(0,2)}$ and $W_1^{\partial\partial(1,1)}$ are neither injective nor surjective. In particular $\dim \operatorname{Im} W_1^{\partial\partial(1,1)} = \dim \operatorname{Im} W_1^{\partial\partial(0,2)} = 5$ while the respective domains $\Omega_{\partial\partial}^{1,1}$ and $\Omega_{\partial\partial}^{0,2}$ have dimension 6 each. Hence we deduce that these two equations are singular and so is the kernel. Hence it is not possible to perform a symplectic reduction and the BFV data do not induce a 1-extended BFV theory. □

Remark 7.4. It might be interesting to work out a procedure to produce such a BF^{2}V theory similar to the one used in the boundary to bypass the failure of the BFV procedure between bulk and boundary. In this case we would have to consider a *classical* version of the data induced on the corner. We postpone this analysis to future work.

With reference to diagram (1.7) we can represent Proposition 7.2 as follows, where the first vertical arrow represents the restriction to the boundary and the second would represent the reduction with respect to the kernel of $\check{\varpi}^{\partial}$:

$$\begin{array}{c} (F^{\partial}, \varpi^{\partial}) \\ \downarrow \pi^{\partial} \\ (\check{F}^{\partial}, \check{\varpi}^{\partial}) \\ \downarrow \pi^{\partial}_{red} \\ (F^{\partial\partial}, \varpi^{\partial\partial}). \end{array}$$

7.1.2 Pre-corner data

The failure of the standard procedure does not allow us to construct a BF^{2}V theory. It is however still possible to analyse the pre-corner structure. To complete the picture, along the pre-corner two form (7.2) we have to find the pre-corner action $\check{S}^\partial$ an expression for a ***pre-Hamiltonian*** vector field. Even if the two-form is degenerate, we can still get a pair $\check{Q}^\partial$ and $\check{S}^\partial$ satisfying $\iota_{Q^\partial}\check{\varpi}^\partial = \delta\check{S}^\partial$, out of the boundary data.

The first step towards this goal is to compute the boundary cohomological vector field Q^∂ as the Hamiltonian vector field of S^∂ with $\partial\Sigma = \emptyset$. Since the symplectic form ϖ^∂ in (6.4) is not in the Darboux form, the computation is non-trivial and leads to the following expression (in components):

$$
\begin{aligned}
Q^\partial e &= [c,e] + L_\xi^\omega e + d_\omega(\lambda e_n) + \lambda(W_1^{\partial,(1,1)})^{-1}(e_n d_\omega e) && \text{(7.4a)}\\
&\quad + (W_1^{\partial,(1,1)})^{-1}\Big([c,\lambda e_n]^{(a)}\gamma_a^y + (L_\xi^\omega \lambda e_n)^{(a)}\gamma_a^y\Big)\\
Q^\partial \omega &= d_\omega c - \iota_\xi F_\omega + \lambda(W_1^{\partial,(1,2)})^{-1}(e_n F_\omega) + \frac{1}{2}\lambda e_n \Lambda e && \text{(7.4b)}\\
&\quad + (W_1^{\partial,(1,2)})^{-1}\Big([c,\lambda e_n]^{(a)} y_a^y + (L_\xi^\omega \lambda e_n)^{(a)} y_a^y\Big)\\
Q^\partial c &= \frac{1}{2}[c,c] + \frac{1}{2}\iota_\xi\iota_\xi F_\omega + \lambda\iota_\xi(W_1^{\partial,(1,2)})^{-1}(e_n F_\omega) && \text{(7.4c)}\\
&\quad + \iota_\xi(W_1^{\partial,(1,2)})^{-1}\Big([c,\lambda e_n]^{(a)} y_a^y + (L_\xi^\omega \lambda e_n)^{(a)} y_a^y\Big)\\
Q^\partial \lambda &= [c,\lambda e_n]^{(n)} + (L_\xi^\omega \lambda e_n)^{(n)} && \text{(7.4d)}\\
Q^\partial \xi^a &= [c,\lambda e_n]^{(a)} + (L_\xi^\omega \lambda e_n)^{(a)} + \frac{1}{2}[\xi,\xi]^a && \text{(7.4e)}\\
Q^\partial \gamma^y &= e d_\omega e + [c,\gamma^y] + L_\xi^\omega \gamma^y + [\lambda e_n, y^y] && \text{(7.4f)}\\
Q^\partial y^y &= [c,y^y] + L_\xi^\omega y^y + eF_\omega + \frac{1}{3!}\Lambda e^3 + p_a^y(\gamma_a^y \lambda(W_1^{\partial,(1,2)})^{-1}(e_n F_\omega) && \text{(7.4g)}\\
&\quad + y_a\lambda(W_1^{\partial,(1,1)})^{-1}(e_n d_\omega e))^{(a)}
\end{aligned}
$$

Remark 7.5. Notice that $W_1^{\partial,(i,j)}$ is not invertible in general as we proved in Lemma 3.4. Nonetheless the $(W_1^{\partial,(1,1)})^{-1}$ appearing in the expression of Q_e are well defined either directly because of the constraint or because the corresponding variation of $\delta\omega$ appearing is not free, since the six component in the kernel of $W_1^{\partial,(1,2)}$ are fixed (these components correspond by duality to the ones not in the image of $W_1^{\partial,(1,1)}$ in the expression of eQ_e). Comparing this expression with the corresponding one of the three-dimensional theory (Equation (2.8)) we also note that the terms containing the inverse of the function $W_1^{\partial,(i,j)}$ constitute exactly the difference between the two expressions.

Since we have the boundary cohomological vector field, we can let $\partial\Sigma = \Gamma \neq \emptyset$, and, using the equation $\iota_{Q^\partial}\iota_{Q^\partial}\varpi^\partial = 2d\check{S}^\partial$ find an expression for the pre-corner action. After a long but straightforward computation we get

$$
\begin{aligned}
\check{S}^\partial = \int & \frac{1}{4}[c,c]ee + \frac{1}{2}\iota_\xi(ee)d_\omega c + ee_m\xi^m d_\omega c + \lambda e_n e d_\omega c \\
&+\frac{1}{4}\iota_\xi\iota_\xi(ee)F_\omega + \frac{1}{2}\iota_\xi e e_m \xi^m F_\omega + \frac{1}{2}\iota_\xi e e_n\lambda F_\omega + \frac{1}{2}e_m\xi^m e_n\lambda F_\omega
\end{aligned} \tag{7.5}
$$

$$+\frac{1}{2}[c,c]\gamma_m^{y}\xi^m + L_\xi^\omega c\gamma_m^{y}\xi^m + \frac{1}{2}\iota_\xi\iota_\xi F_\omega\gamma_m^{y}\xi^m$$
$$+\frac{1}{2}\iota_{[\xi,\xi]}ey_m^{y}\xi^m + L_\xi^\omega(\lambda e_n)y_m^{y}\xi^m + L_\xi^\omega(e_m\xi^m)y_m^{y}\xi^m + [c,\lambda e_n]y_m^{y}\xi^m.$$

The starting data on the boundary were constrained by (7.1), hence this constraint will also descend to the pre-corner. However it will split into two separate equations:

$$e_n d_\omega e = e\sigma \qquad e_n d_{\omega_m} e + e_n d_\omega e_m = e_m\sigma + e\sigma_m.$$

These pair of equations still correctly fix the same components of ω: the first are 6 equations, fixing the value of 5 components of σ and one constraint, the second are 12 equations fixing the value of the remaining 3 components of σ, the 4 components of σ_m and the remaining 5 constraints.

The last bit of information that is missing is the pre-corner cohomological vector field. Since it not possible to perform a symplectic reduction, it is also difficult to find a cohomological vector field corresponding to the action (7.5). One possible way to find a vector field is to push forward manually the one on the boundary to the corner.

Since the expression of Q^∂ (7.4) contains non-explicit parts involving the function $(W_1^{\partial,(i,j)})^{\ 1}$ we must find a way to invert it.

Lemma 7.6. *Let* $\widetilde{\gamma}\ 2\ \omega_\partial^{i,j}$ *and* $\widetilde{X}\ 2\ \omega_\partial^{i+1,j+1}$ *such that* $\widetilde{\gamma} = (W_1^{\partial,(i,j)})^{\ 1}(\widetilde{X})$. *Then if we let* $\widetilde{e} = e\boldsymbol{j}\ \ + e_m dx^m$, $\widetilde{\gamma} = \gamma\boldsymbol{j}\ \ + \gamma_m dx^m$ *and* $\widetilde{X} = X\boldsymbol{j}\ \ + X_m dx^m$ *we have*

$$\gamma\boldsymbol{j}\ \ = (W_1^{\partial\partial,(i,j)})^{\ 1}(\pi_I(X\boldsymbol{j}\ \ ))$$
$$\gamma_m = (W_1^{\partial\partial,(i\ \ 1,j)})^{\ 1}(\pi_I(\ \ e_m(W_1^{\partial\partial,(i,j)})^{\ 1}(\pi_I(X\boldsymbol{j}\ \ )) + X_m))$$

Proof. Omitting the restriction to the corner, we have that

$$\widetilde{e}\widetilde{\gamma} = (e + e_m dx^m)(\gamma + \gamma_m dx^m) = X + X_m dx^m = \widetilde{X}.$$

This equation splits into two subequations, containing dx^m or not:

$$e\gamma = X, \qquad e\gamma_m + e_m\gamma = X_m.$$

From the first we deduce $\gamma = (W_1^{\partial\partial,(i,j)})^{\ 1}(\pi_I(X))$ while from the second we find

$$\gamma_m = (W_1^{\partial\partial,(i\ \ 1,j)})^{\ 1}(\pi_I(\ \ e_m(W_1^{\partial\partial,(i,j)})^{\ 1}(\pi_I(X)) + X_m))$$

where π_I stands for the projection to the image of the map $W_1^{\partial\partial,(i,j)}$. □

Remark 7.7. One has to be careful here since the map $W_1^{\partial\partial,(i,j)}$ can be non-invertible. Hence technically here we are finding the values of γ and γ_m up to terms in the kernel of the map $W_1^{\partial,(i,j)}$ and we need to keep the projection π_I at all times.

As an example we consider $Q^\partial\omega$: it contains a term of the form $\lambda(W_1^{\partial,(1,2)})^{\ 1}(e_nF_\omega)$. Here $X = e_nF_\omega$. Hence we will have

$$\check{Q}^{\partial\partial}\omega = \quad + (W_1^{\partial\partial,(1,2)})^{\ 1}(e_nF_\omega)$$
$$\check{Q}^{\partial\partial}\omega_m = \quad + (W_1^{\partial\partial,(0,2)})^{\ 1}(\pi_I(\ \ e_m(W_1^{\partial\partial,(1,2)})^{\ 1}(e_nF_\omega) + e_nF_{\omega_m})) + K$$

where $eK = 0$. Notice that since $W_1^{\partial\partial,(1,2)}$ is surjective on $\Omega_{\partial\partial}^{1,2}$ we do not need the projection on $e_n F_\omega$, while since the map $W_1^{\partial\partial,(0,2)}$ is neither surjective nor injective on $\Omega_{\partial\partial}^{0,2}$ we need the projection π_I on the second expression and we still miss something in the kernel of $W_1^{\partial\partial,(0,2)}$, denoted by K.

A similar procedure is needed also for $Q^\partial y^y$. On the boundary we have

$$\widetilde{e}_i Q^{\partial} y^y = \lambda \widetilde{\sigma}_i \widetilde{y}^y + \widetilde{\mu} \widetilde{c}_i^{\,y}$$

for $i = a, m$. Hence since y_m^y is a top form on the boundary we get

$$\begin{aligned} e_m Q^\partial y_m^y dx^m &= \lambda \sigma_m y_m^y dx^m + \mu_m dx^m \gamma_m^y \\ e_a Q^\partial y_m^y dx^m &= \lambda \sigma_a y_m^y dx^m + \mu \gamma_{am}^y dx^m \end{aligned}$$

from which we can easily deduce the expression of $\check{Q}^\partial$ on the pre-corner.

Collecting all the above information and pushing forward to the pre-corner all other components we get the following expression for the pre-corner cohomological vector field $\check{Q}^\partial$:

$$\begin{aligned} \check{Q}^\partial e = {}& [c, e] + L_\xi^\omega e + \xi^m d_{\omega_m} e + e_m d\xi^m + d_\omega(\lambda e_n) \\ & + \lambda \sigma + (W_1^{\partial\partial,(1,1)})^{\ 1} \left(Y^{(m)} \gamma_m^y \right) \end{aligned} \tag{7.6a}$$

$$\begin{aligned} \check{Q}^\partial e_m = {}& [c, e_m] + L_\xi^\omega e_m + \iota_{\partial_m \xi} e + d_{\omega_m}(e_m \xi^m) + d_{\omega_m}(\lambda e_n) \\ & + \lambda \sigma_m + (W_1^{\partial\partial,(0,1)})^{\ 1} \left(Y^{(a)} \gamma_{am}^y \right) + (W_1^{\partial\partial,(0,1)})^{\ 1} \left(e_m (W_1^{\partial\partial,(1,1)})^{\ 1} \left(Y^{(m)} \gamma_m^y \right) \right) \end{aligned} \tag{7.6b}$$

$$\check{Q}^\partial \omega = d_\omega c \quad \iota_\xi F_\omega \quad F_{\omega_m} \xi^m + \lambda \mu + \frac{1}{2} \lambda e_n \Lambda e + (W_1^{\partial\partial,(1,2)})^{\ 1} \left(Y^{(m)} y_m^y \right) \tag{7.6c}$$

$$\begin{aligned} \check{Q}^\partial \omega_m = {}& d_{\omega_m} c \quad \iota_\xi F_{\omega_m} + \lambda \mu_m + \frac{1}{2} \lambda e_n \Lambda e_m + (W_1^{\partial\partial,(0,2)})^{\ 1} \left(Y^{(a)} y_{am}^y \right) \\ & + (W_1^{\partial\partial,(0,2)})^{\ 1} \left(e_m (W_1^{\partial\partial,(1,2)})^{\ 1} \left(Y^{(m)} y_m^y \right) \right) \end{aligned} \tag{7.6d}$$

$$\begin{aligned} \check{Q}^\partial c = {}& \frac{1}{2}[c, c] + \frac{1}{2} \iota_\xi \iota_\xi F_\omega + \iota_\xi F_{\omega_m} \xi^m + \lambda \iota_\xi \mu + \lambda \mu_m \xi^m + \iota_\xi (W_1^{\partial\partial,(1,2)})^{\ 1} \left(Y^{(m)} y_m^y \right) \\ & + (W_1^{\partial\partial,(0,2)})^{\ 1} \left(Y^{(a)} y_{am}^y \right) \xi^m + (W_1^{\partial\partial,(0,2)})^{\ 1} \left(e_m (W_1^{\partial\partial,(1,2)})^{\ 1} \left(Y^{(m)} y_m^y \right) \right) \xi^m \end{aligned} \tag{7.6e}$$

$$\check{Q}^\partial \lambda = Y^{(n)} \tag{7.6f}$$

$$\check{Q}^\partial \xi^a = Y^{(a)} + \frac{1}{2}[\xi, \xi]^a \tag{7.6g}$$

$$\check{Q}^\partial \xi^m = Y^{(m)} + \frac{1}{2}[\xi, \xi]^m \tag{7.6h}$$

$$\check{Q}^\partial \gamma^y = e_m d_\omega e + e d_{\omega_m} e + e d_\omega e_m + [c, \gamma_m^y] + L_\xi^\omega \gamma_m^y + d_{\omega_m}(\gamma_m^y \xi^m) + [\lambda e_n, y_m^y] \tag{7.6i}$$

$$\begin{aligned} \check{Q}^\partial y^y = {}& [c, y_m^y] + L_\xi^\omega y_m^y + d_{\omega_m}(y_m^y \xi^m) + e_m F_\omega + e F_{\omega_m} + + \frac{1}{2} \Lambda e_m e^2 \\ & + \lambda (\sigma_m y_m^y)^{(m)} + \lambda (\mu_m \gamma_m^y)^{(m)} + \lambda (\sigma_a y_m^y)^{(a)} + \lambda (\mu \gamma_{am}^y)^{(m)} \end{aligned} \tag{7.6j}$$

where

$$Y = [c, \lambda e_n] + L_\xi^\omega(\lambda e_n) + \xi^m d_{\omega_m}(\lambda e_n), \qquad \sigma = (W_1^{\partial\partial,(1,1)})^{\ 1}(e_n d_\omega e),$$

$$\sigma_m = (W_1^{\partial\partial,(0,1)})\quad ^1(e_m\sigma + e_n d_{\omega_m} e + e_n d_\omega e_m), \qquad \mu = (W_1^{\partial\partial,(1,2)})\quad ^1(e_n F_\omega),$$
$$\text{and} \qquad \mu_m = (W_1^{\partial\partial,(0,2)})\quad ^1(e_m \mu + e_n F_{\omega_m}).$$

It is a long check, but without difficulties, to show that $\iota_{Q^\partial}\check{\varpi}^\partial = \delta\check{S}^\partial$.

7.1.3 Particular cases

We can now consider some particular cases and see under which additional hypothesis we can obtain a reduction to a symplectic space under stronger hypothesis. Since we have to solve the degeneracy of (7.3a) we have to assume that the two last terms vanish. Indeed as we will see below the equation $eX_e = 0$ can be solved. One possibility is to assume $\xi^m = 0$.

Case $\xi^m = 0$

Assuming $\xi^m = 0$ the pre-corner two form becomes

$$\check{\varpi}^\partial_{part} = \int \delta c e \delta e \quad \iota_{\delta\xi} e e \delta\omega \quad \iota_\xi(e\delta e)\delta\omega \quad \delta\lambda e_n e \delta\omega \quad \lambda e_n \delta e \delta\omega$$

As before the equations defining the kernel of the corresponding application $(\check{\varpi}^\partial_{part})^\sharp$ are:

$$\delta c: \quad eX_e = 0 \tag{7.7a}$$
$$\delta e: \quad eX_c \quad e\iota_\xi X_\omega \quad \lambda e_n X_\omega = 0 \tag{7.7b}$$
$$\delta\xi: \quad e \;\; eX_\omega = 0 \tag{7.7c}$$
$$\delta\omega: \quad \iota_{X_\xi} ee \quad \iota_\xi(eX_e) \quad X_\lambda e_n e \quad \lambda e_n X_e = 0 \tag{7.7d}$$
$$\delta\lambda: \quad e_n e X_\omega = 0. \tag{7.7e}$$

This system is still singular. Indeed the third element of the second equation might not be proportional to e and the map $W_1^{\partial\partial,(0,2)}$ is not surjective.

Case $\xi^m = 0$, $\lambda = 0$

We now add another condition and consider the case $\xi^m = 0$, $\lambda = 0$, i.e. we require both transversal components of ξ (on the bulk) to vanish. The pre-corner two-form now reads:

$$\check{\varpi}^\partial_{part} = \int \delta c e \delta e \quad \iota_{\delta\xi} e e \delta\omega \quad \iota_\xi(e\delta e)\delta\omega$$

and the equations defining the kernel of the corresponding application $(\check{\varpi}^\partial_{part})^\sharp$ are:

$$\delta c: \quad eX_e = 0 \tag{7.8a}$$
$$\delta e: \quad eX_c \quad e\iota_\xi X_\omega = 0 \tag{7.8b}$$
$$\delta\xi: \quad e \;\; eX_\omega = 0 \tag{7.8c}$$
$$\delta\omega: \quad \iota_{X_\xi} ee \quad \iota_\xi(eX_e) = 0. \tag{7.8d}$$

This system is not singular. We can define the following space

$$\tilde{F}^{\partial\partial} = T\;\,[1]\left(\Omega^{2,2}_{\partial\partial} \quad (\Omega^{2,4}_{\partial\partial} \quad \Omega^1(\Gamma))\right) \tag{7.9}$$

with corresponding symplectic form

$$\widetilde{\varpi}^{\partial\partial} = \int \delta\widetilde{c}\delta\widetilde{E} - \iota_{\delta\widetilde{\xi}}\delta\widetilde{P}.$$

Furthermore, we can define an embedding of the space of pre-corner fields $\widetilde{\pi_{red}} : \check{F}^{\partial} \to \widetilde{F}^{\partial\partial}$:

$$\widetilde{\pi_{red}} := \begin{cases} \widetilde{E} = \frac{1}{2}ee \\ \widetilde{c} = c + \iota_{\xi}(\omega - \omega_0) \\ \widetilde{\xi}^i = \xi^i \\ \widetilde{P}_i = \frac{1}{2}ee(\omega_i - \omega_{0i}). \end{cases}$$

Notice that here we are assuming to work around a connection ω_0. It is a short computation to show that this map is compatible with the two-forms (respectively the pre-corner form $\check{\varpi}^{\partial}_{part}$ on $\check{F}^{\partial}$ and $\widetilde{\varpi}^{\partial\partial}$ on $\widetilde{F}^{\partial\partial}$).

Define now the submanifold $\mathcal{E} \subset \widetilde{F}^{\partial\partial}$, such that $(E, P, c, \xi) \in \mathcal{E}$ if E is *pure* tensor. This condition may be translated to requiring that the Pfaffian of E vanishes. In these cases we drop the tilde. Note that $\mathcal{E}$ coincides with the image of $\widetilde{\pi_{red}}$.

Let now $p' : \Omega^{0,2}_{\partial\partial} \to \Omega^{0,2}_{\partial\partial}$ a projection to the complement of the kernel of the map $W_1^{\partial\partial,(0,2)} : \Omega^{0,2}_{\partial\partial} \to \Omega^{1,3}_{\partial\partial}$. Then, the characteristic distribution of $\mathcal{E}$ is given by the vector fields $X_{p'c}$. Hence we have the following:

Proposition 7.8. *The BF^2V space of fields $F^{\partial\partial}$ is symplectomorphic to the symplectic reduction of $\widetilde{F}^{\partial\partial}$ with respect to the coisotropic submanifold $\mathcal{E}$.*

We can express the symplectic form on the space of corner fields as

$$\varpi^{\partial\partial} = \int \delta[c]\delta E - \iota_{\delta\widetilde{\xi}}\delta\widetilde{P}$$

where E is a pure tensor and $[c]$ denotes the equivalence class of elements $c \in \Omega^{0,2}_{\partial\partial}[1]$ under the equivalence $c + d \sim c$ for $d \in \Omega^{0,2}_{\partial\partial}[1]$ such that $ed = 0$.

From the expression of the pre-corner action in this particular case

$$\check{S}^{\partial} = \int \frac{1}{4}[c,c]ee + \frac{1}{2}\iota_{\xi}(ee)d_{\omega}c + \frac{1}{4}\iota_{\xi}\iota_{\xi}(ee)F_{\omega} \tag{7.10}$$

we can deduce the corresponding action on the corner:

$$S^{\partial\partial}_{\omega_0} = \int \frac{1}{2}[[c],[c]]E + \iota_{\widetilde{\xi}}(E)d_{\omega_0}[c] - \frac{1}{2}\iota_{[\widetilde{\xi},\widetilde{\xi}]}\widetilde{P} + \frac{1}{2}E\iota_{\widetilde{\xi}}\iota_{\widetilde{\xi}}F_{\omega_0}.$$

This expression is invariant under the quotient map above: $\frac{1}{2}[c,c]ee = [ce,c]e - [e,c]ec = [ce,ce]$, $\iota_{\xi}(ee)dc = -d\iota_{\xi}eec = L_{\xi}(ee)c = 2(L_{\xi}e)ec$.

Remark 7.9. The map π_{red} is not strictly speaking the reduction with respect to the kernel of the pre-corner two form but it does satisfy the BV-BFV axioms.

Remark 7.10. From this corner data it is possible to extract the structure of an Atiyah algebroid. A detailed analysis of this is postponed to future work.

7.2 Einstein–Hilbert 2-extended theory

In this section we explore the extendability of BV-BFV Einstein–Hilbert theory to a 2-extended theory. This is done in a similar way to Palatini–Cartan theory with similar results as well. Indeed, we will see that it is not possible in general to induce a BF^{2}V theory on the corner because the kernel of the pre-corner two-form is not regular. However in the EH case, along the non-regularity issue, the extremely lengthy computations constitute a challenge in their own. As a consequence we prefer to just claim the results instead of stating them, leaving the complete check to future work.

7.2.1 Pre-corner structure

The starting point of our analysis is the BV structure of Einstein-Hilbert theory (or 1-extended BV theory) described in Section 1.2.3.

In order to construct the pre-corner structure, it is better to start from an expression of the boundary action in coordinates[2]:

$$S^{\partial}_{EH} = \int \left\{ \frac{1}{\sqrt{\gamma}} \left(\Pi^{ab}\Pi_{ab} - \frac{1}{d-1}\Pi^2 \right) + \sqrt{\gamma}\left(R - 2\Lambda\right) + \partial_a\left(\xi^a\varphi_n\right) - \gamma^{ab}\varphi_b\partial_a\xi^n \right\}\xi^n \quad (7.11)$$
$$+ \int \left\{ -\partial_c\left(\gamma^{cd}\Pi_{da}\right) - (\partial_a\gamma^{cd})\Pi_{cd} + \partial_c\left(\xi^c\varphi_a\right)\right\}\xi^a.$$

The corresponding symplectic form is:

$$\varpi^{\partial}_{EH} = \int \delta\gamma^{ab}\delta\Pi_{ab} + \delta\xi^{\rho}\delta\varphi_{\rho}.$$

Let us now suppose that the boundary manifold Σ has a non-empty boundary $\partial\Sigma = \Gamma$. In order to keep things simple in this preliminary analysis we can suppose to have a space-like boundary Σ, corresponding to $\epsilon = 1$ in the boundary action (7.11).

In the same way as the bulk to boundary case, we can use the ADM decomposition on the boundary metric γ^{ab} to compute the variation of the boundary Ricci scalar appearing in the boundary action and write it in terms of a corner metric h_{ij}, (boundary) lapse b_i and (boundary) shift α. We can write the boundary metric and its inverse as

$$\gamma_{ab} = \begin{pmatrix} -(\alpha^2 - b^i b_i) & b_i \\ b_j & h_{ij} \end{pmatrix}$$
$$\gamma^{ab} = \alpha^{-2} \begin{pmatrix} -1 & b^i \\ b^j & \alpha^2 h^{ij} - b^i b^j \end{pmatrix}.$$

Furthermore we will denote by h the determinant of the corner metric h_{ij}. With this notation the Ricci curvature of the boundary metric can be expressed in terms of the corner metric h_{ij}, its Ricci curvature R and the second fundamental form H, defined as

$$H_{ij} = \frac{1}{2}\alpha^{-1}\left(\nabla_{(i}b_{j)} - \partial_m h_{ij}\right)$$

where ∇ is the corner covariant derivative associated to h_{ij} and m is an index denoting the normal direction between Γ and Σ. Hence we have:

$$\sqrt{h}R = \alpha\sqrt{h}\left(H_{ij}H^{ij} - H^2 + R - 2\Lambda\right)$$
$$- 2\partial_m\left(\sqrt{h}H\right) + 2\partial_i\left(\sqrt{h}Hb^i - \sqrt{h}h^{ij}\partial_j\alpha\right).$$

[2] This can be found in [CS17].

Claim 7.11. *The BV theory F_{EH} is not 2-extendable.*

Sketch of the proof. In order to construct the corner theory we start by the variation of the boundary action and obtain a *pre-corner* one form from which we will deduce a pre-symplectic form through the equation

$$\delta S^\partial = EL_\partial + \pi^* \check{\alpha}^\partial$$

where EL_∂ is the part defining the boundary Euler-Lagrange equation and π is the restriction to the corner. We now compute the variation of the part of the boundary action S^∂ containing R and retain only the terms having corner derivatives. Note that on the boundary the term in the action corresponding to R has the form of a Brans–Dicke theory, due to the presence of the (odd) scalar field ξ^n [BD61]. We split the computation in multiple parts, starting from the one coming from the *Hamiltonian part* of the boundary action:

$$\begin{aligned}\delta \int \sqrt{\gamma} R\, \xi^n = & -\delta \int \sqrt{h}\partial_m h_{ij} H^{ij}\xi^n - \sqrt{h} H \partial_m h_{ij} h^{ij} + 2\partial_m \left(\sqrt{h} H\right)\xi^n \\ = & +\int \sqrt{h}\delta h_{ij} H^{ij}\xi^n - \sqrt{h} H\delta h_{ij} h^{ij}\xi^n + \delta\left(\sqrt{h}H\right)\xi^n.\end{aligned}$$

Similarly the *momentum part* gives the following contributions:

$$\begin{aligned}\int & -2\delta(\alpha^{-2}\Pi_{mi})\xi^i - 2\delta(\alpha^{-2}b^j\Pi_{ji})\xi^i + 2\delta(\alpha^{-2}\Pi_{mm})\xi^m + 2\delta(\alpha^{-2}b^i\Pi_{im})\xi^m \\ & + \delta(h^{ij})\Pi_{ij}\xi^m - \delta(\alpha^{-2}b^i b^j)\Pi_{ij}\xi^m + 2\delta(\alpha^{-2}b^i)\Pi_{im}\xi^m + \delta\alpha^{-2}\Pi_{mm}\xi^m\end{aligned}$$

Additionally we have the following other terms from the variation of the antifield part of the boundary action (7.11):

$$\int -\delta(\xi^m\varphi_n)\xi^n - \alpha^{-2}\varphi_m\delta\xi^n\xi^n + \alpha^{-2}b^i\varphi_i\delta\xi^n\xi^n - \delta(\xi^m\varphi_i)\xi^i - \delta(\xi^m\varphi_m)\xi^m$$

Consequently, the pre-corner two form $\check{\varpi}^\partial = \delta\check{\alpha}^\partial$ has the following expression:

$$\begin{aligned}\check{\varpi}^\partial = \int & \delta h_{ij}\delta(\sqrt{h}H^{ij}\xi^n) - \delta(\sqrt{h}Hh^{ij}\xi^n)\delta h_{ij} + \delta\left(\sqrt{h}H\right)\delta\xi^n \\ & -2\delta(\alpha^{-2}\Pi_{mi})\delta\xi^i - 2\delta(\alpha^{-2}b^j\Pi_{ji})\delta\xi^i + 2\delta(\alpha^{-2}\Pi_{mm})\delta\xi^m \\ & + 2\delta(\alpha^{-2}b^i\Pi_{im})\delta\xi^m + \delta(h^{ij})\delta(\Pi_{ij}\xi^m) - \delta(\alpha^{-2}b^ib^j)\delta(\Pi_{ij}\xi^m) \\ & + 2\delta(\alpha^{-2}b^i)\delta(\Pi_{im}\xi^m) + \delta\alpha^{-2}\delta(\Pi_{mm}\xi^m) - \delta(\xi^m\varphi_n)\delta\xi^n \\ & -\delta\xi^n\delta(\alpha^{-2}\varphi_m\xi^n) + \delta\xi^n\delta(\alpha^{-2}b^i\varphi_i\xi^n) - \delta(\xi^m\varphi_i)\delta\xi^i - \delta(\xi^m\varphi_m)\delta\xi^m\end{aligned}$$

To assess the extedability of the BV-BFV theory to a 2-extended theory on a corner we have now to consider the regularity of the kernel of the associated map

$$\begin{aligned}\check{\varpi}^{\partial\sharp} : T\check{F}^\partial & \to T^*\check{F}^\partial \\ X & \mapsto \check{\varpi}^{\partial\sharp}(X) = \check{\varpi}^\partial(X,\cdot).\end{aligned}$$

The equations defining the kernel are the following:

$$
\begin{aligned}
\delta J_{ij}: &\quad \frac{1}{2}X_{h_{ij}}\sqrt{h}\alpha^{-1}\xi^n - \frac{1}{2}X_{h_{kl}}h^{kl}h^{ij}\sqrt{h}\alpha^{-1}\xi^n + \frac{1}{2}h^{ij}\sqrt{h}\alpha^{-1}X_{\xi^n} = 0 && \text{(7.12a)}\\
\delta\varphi_n: &\quad -\xi^m X_{\xi^n} = 0 && \text{(7.12b)}\\
\delta\varphi_m: &\quad \xi^m X_{\xi^m} - \xi^n\alpha^{-2}X_{\xi^n} = 0 && \text{(7.12c)}\\
\delta\varphi_i: &\quad \xi^m X_{\xi^i} - \xi^n\alpha^{-2}b^i X_{\xi^n} = 0 && \text{(7.12d)}\\
\delta\xi^n: &\quad \frac{1}{2}X_{h_{ij}}H^{ij}\sqrt{h} - \sqrt{h}Hh^{ij}X_{h_{ij}} + X_{\sqrt{h}H} - X_{\xi^m}\varphi_n - \xi^m X_{\varphi_n} && \text{(7.12e)}\\
&\quad -2X_{\xi_n}\alpha^{-2}\varphi_m - 2X_\alpha\alpha^{-3}\varphi_m\xi^n - \alpha^{-2}X_{\varphi_m}\xi^n - 2X_{\xi_n}\alpha^{-2}b^i\varphi_i\\
&\quad -2X_\alpha\alpha^{-3}b^i\varphi_i\xi^n - \alpha^{-2}X_{b^i}\varphi_i\xi^n - \alpha^{-2}b^iX_{\varphi_i}\xi^n = 0\\
\delta\xi^i: &\quad 4X_\alpha\alpha^{-3}\Pi_{mi} - 2\alpha^{-2}X_{\;mi} + 4X_\alpha\alpha^{-3}b^j\Pi_{ji} - 2\alpha^{-2}X_{b^j}\Pi_{ji} && \text{(7.12f)}\\
&\quad -2\alpha^{-2}b^jX_{\;ji} - X_{\xi^m}\varphi_i - \xi^m X_{\varphi_i} = 0\\
\delta\xi^m: &\quad -4X_\alpha\alpha^{-3}\Pi_{mm} + 2\alpha^{-2}X_{\;mm} - 4X_\alpha\alpha^{-3}b^i\Pi_{im} + 2\alpha^{-2}X_{b^i}\Pi_{im} && \text{(7.12g)}\\
&\quad + 2\alpha^{-2}b^iX_{\;im} + X_{h^{ij}}\Pi_{ij} - 2X_\alpha\alpha^{-3}b^ib^j\Pi_{ij} - 2\alpha^{-2}X_{b^i}b^j\Pi_{ij}\\
&\quad -4X_\alpha\alpha^{-3}b^i\Pi_{im} + 2\alpha^{-2}X_{b^i}\Pi_{im} - 2X_\alpha\alpha^{-3}\Pi_{mm} - \varphi_nX_{\xi^n}\\
&\quad -\varphi_iX_{\xi^i} - 2\varphi_mX_{\xi^m} - \xi^mX_{\varphi_m} = 0
\end{aligned}
$$

$$
\begin{aligned}
\delta\Pi_{mm}: &\quad -2\alpha^{-2}X_{\xi^m} - 2X_\alpha\alpha^{-3}\xi^m = 0 && \text{(7.12h)}\\
\delta\Pi_{mi}: &\quad 2\alpha^{-2}X_{\xi^i} + 2\alpha^{-2}b^iX_{\xi^m} - 4X_\alpha\alpha^{-3}b^i\xi^m + 2\alpha^{-2}X_{b^i}\xi^m = 0 && \text{(7.12i)}\\
\delta\Pi_{ij}: &\quad 2\alpha^{-2}b^jX_{\xi^i} + X_{h^{ij}}\xi^m - 2X_\alpha\alpha^{-3}b^ib^j\xi^m - 2\alpha^{-2}X_{b^i}b^j\xi^m = 0 && \text{(7.12j)}
\end{aligned}
$$

$$
\begin{aligned}
\delta b_i: &\quad 2\alpha^{-2}\Pi_{ij}X_{\xi^j} + 2\alpha^{-2}\Pi_{im}X_{\xi^m} - 2\alpha^{-2}b^jX_{\;ij}\xi^m - 2\alpha^{-2}b^j\Pi_{ij}X_{\xi^m} && \text{(7.12k)}\\
&\quad + 2\alpha^{-2}X_{\;im}\xi^m + 2\alpha^{-2}\Pi_{im}X_{\xi^m} + X_{\xi^n}\alpha^{-2}\varphi_i\xi^n\\
&\quad + \frac{1}{2}\nabla_j\left(X_{h^{ij}}\sqrt{h}\alpha^{-1}\xi^n\right) - \frac{1}{2}\nabla_j\left(X_{h_{kl}}h^{kl}h^{ij}\sqrt{h}\alpha^{-1}\xi^n\right)\\
&\quad + \frac{1}{2}\nabla_j\left(h^{ij}\sqrt{h}\alpha^{-1}X_{\xi^n}\right) = 0\\
\delta\alpha: &\quad 2X_{h_{ij}}\sqrt{h}\alpha^{-2}S^{ij}\xi^n + 2\sqrt{h}\alpha^{-2}Sh^{ij}\xi^nX_{h_{ij}} - 2\sqrt{h}\alpha^{-2}SX_{\xi^n} && \text{(7.12l)}\\
&\quad + 4\alpha^{-3}\Pi_{mi}X_{\xi^i} + 4\alpha^{-3}b^j\Pi_{ji}X_{\xi^i} - 4\alpha^{-3}\Pi_{mm}X_{\xi^m}\\
&\quad -4\alpha^{-3}b^i\Pi_{im}X_{\xi^m} + 2\alpha^{-3}b^ib^jX_{\;ij}\xi^m + 2\alpha^{-3}b^ib^j\Pi_{ij}X_{\xi^m}\\
&\quad -4\alpha^{-3}b^iX_{\;im}\xi^m - 4\alpha^{-3}b^i\Pi_{im}X_{\xi^m} - 2\alpha^{-3}X_{\;mm}\xi^m\\
&\quad -2\alpha^{-3}\Pi_{mm}X_{\xi^m} + 2X_{\xi^n}\alpha^{-3}\varphi_m\xi^n - 2X_{\xi^n}\alpha^{-3}b^i\varphi_i\xi^n = 0\\
\delta h_{ij}: &\quad X_{\sqrt{h}H^{ij}\xi^n} + \frac{1}{2}X_{h_{kl}}\sqrt{h}h_{ij}H^{kl}\xi^n) - X_{\sqrt{h}Hh^{ij}\xi^n} && \text{(7.12m)}\\
&\quad -\frac{1}{2}\sqrt{h}h_{ij}Hh^{kl}\xi^nX_{h_{kl}} + \sqrt{h}h^{ij}HX_{\xi^n} + X_{\;ij}\xi^m + \Pi_{ij}X_{\xi^m} = 0
\end{aligned}
$$

where $S_{ij} = \alpha H_{ij}$. This system is singular. Indeed the we can try to solve it as follows and get an impossible equation. We can solve (7.12f) and find an expression for $X_{\;im}$, (7.12g) and find $X_{\;mm}$. Same holds for (7.12h) for X_{ξ^m} and (7.12i) for X_{ξ^i}. Taking the trace of (7.12a) would give an expression for X_{ξ^n} and (7.12e) gives an expression for the trace of $X_{J_{ij}}$. Substituting these results into the other equations we get that the remaining are singular because they are proportional either to ξ^m or ξ^n.[3] □

In principle, as in the Palatini–Cartan case, it is possible to compute the pre-corner action and cohomological vector field. However the complexity of the computation is once again so big that at the current point of the research we cannot explore this direction and we have to postpone it to future work.

7.2.2 Corner structure in special conditions

Given the failure of the procedure in the general case, we can consider some particular cases and assess if we can overcome the singularity of the kernel by imposing some additional constraint on the fields, as in the Palatini–Cartan case.

Case $\xi^n = 0$

The first case that we consider is $\xi^n = 0$. Note that we are losing all the hamiltonian part of the boundary action. With this assumption the pre-corner two form is:

$$\begin{aligned}\check{\varpi}^{\partial} = \int \quad & 2\delta(\alpha \ ^2\Pi_{mi})\delta\xi^i \quad 2\delta(\alpha \ ^2 b^j \Pi_{ji})\delta\xi^i + 2\delta(\alpha \ ^2\Pi_{mm})\delta\xi^m \\ & + 2\delta(\alpha \ ^2 b^i\Pi_{im})\delta\xi^m + \delta(h^{ij})\delta(\Pi_{ij}\xi^m) \quad \delta(\alpha \ ^2 b^i b^j)\delta(\Pi_{ij}\xi^m) \\ & + 2\delta(\alpha \ ^2 b^i)\delta(\Pi_{im}\xi^m) + \delta\alpha \ ^2\delta(\Pi_{mm}\xi^m) \quad \delta(\xi^m\varphi_i)\delta\xi^i \quad \delta(\xi^m\varphi_m)\delta\xi^m.\end{aligned}$$

The equations defining the kernel are the following:

$$\begin{aligned}
\delta\varphi_m : \quad & \xi^m X_{\xi^m} = 0 \\
\delta\varphi_i : \quad & \xi^m X_{\xi^i} = 0 \\
\delta\xi^i : \quad & 4X_\alpha\alpha \ ^3\Pi_{mi} \quad 2\alpha \ ^2 X_{\ \ mi} + 4X_\alpha\alpha \ ^3 b^j\Pi_{ji} \quad 2\alpha \ ^2 X_{b^j}\Pi_{ji} \\
& 2\alpha \ ^2 b^j X_{\ \ ji} \quad X_{\xi^m}\varphi_i \quad \xi^m X_{\varphi_i} = 0 \\
\delta\xi^m : \quad & 4X_\alpha\alpha \ ^3\Pi_{mm} + 2\alpha \ ^2 X_{\ \ mm} \quad 4X_\alpha\alpha \ ^3 b^i\Pi_{im} + 2\alpha \ ^2 X_{b^i}\Pi_{im} \\
& + 2\alpha \ ^2 b^i X_{\ \ im} + X_{h^{ij}}\Pi_{ij} \quad 2X_\alpha\alpha \ ^3 b^i b^j\Pi_{ij} \quad 2\alpha \ ^2 X_{b^i} b^j\Pi_{ij} \\
& 4X_\alpha\alpha \ ^3 b^i\Pi_{im} + 2\alpha \ ^2 X_{b^i}\Pi_{im} \quad 2X_\alpha\alpha \ ^3\Pi_{mm} \\
& \varphi_i X_{\xi^i} \quad 2\varphi_m X_{\xi^m} \quad \xi^m X_{\varphi_m} = 0
\end{aligned}$$

$$\begin{aligned}
\delta\Pi_{mm} : \quad & 2\alpha \ ^2 X_{\xi^m} \quad 2X_\alpha\alpha \ ^3\xi^m = 0 \\
\delta\Pi_{mi} : \quad & 2\alpha \ ^2 X_{\xi^i} + 2\alpha \ ^2 b^i X_{\xi^m} \quad 4X_\alpha\alpha \ ^3 b^i\xi^m + 2\alpha \ ^2 X_{b^i}\xi^m = 0 \\
\delta\Pi_{ij} : \quad & 2\alpha \ ^2 b^j X_{\xi^i} + X_{h^{ij}}\xi^m \quad 2X_\alpha\alpha \ ^3 b^i b^j\xi^m \quad 2\alpha \ ^2 X_{b^i} b^j\xi^m = 0
\end{aligned}$$

[3] In doing this computation we are assuming that r_j commutes with δ. This is in general not true, but the system should anyway be singular in the general case. This is postponed to future work.

$$
\begin{aligned}
\delta b_i : \quad & \quad 2\alpha \ {}^2\Pi_{ij}X_{\xi^j} + 2\alpha \ {}^2\Pi_{im}X_{\xi^m} \quad 2\alpha \ {}^2 b^j X \ {}_{ij}\xi^m \quad 2\alpha \ {}^2 b^j \Pi_{ij}X_{\xi^m} \\
& + 2\alpha \ {}^2 X \ {}_{im}\xi^m + 2\alpha \ {}^2\Pi_{im}X_{\xi^m} = 0 \\
\delta\alpha : \quad & + 4\alpha \ {}^3\Pi_{mi}X_{\xi^i} + 4\alpha \ {}^3 b^j\Pi_{ji}X_{\xi^i} \quad 4\alpha \ {}^3\Pi_{mm}X_{\xi^m} \\
& \quad 4\alpha \ {}^3 b^i\Pi_{im}X_{\xi^m} + 2\alpha \ {}^3 b^i b^j X \ {}_{ij}\xi^m + 2\alpha \ {}^3 b^i b^j \Pi_{ij}X_{\xi^m} \\
& \quad 4\alpha \ {}^3 b^i X \ {}_{im}\xi^m \quad 4\alpha \ {}^3 b^i\Pi_{im}X_{\xi^m} \quad 2\alpha \ {}^3 X \ {}_{mm}\xi^m \\
& \quad 2\alpha \ {}^3\Pi_{mm}X_{\xi^m} = 0 \\
\delta h_{ij} : \quad & X \ {}_{ij}\xi^m + \Pi_{ij}X_{\xi^m} = 0
\end{aligned}
$$

As in the general case, for the same reasons this system is singular. Namely we cured only the singularities coming from the field ξ^n but not the ones coming from ξ^m. Hence in the next case we will try to solve this issue.

Case $\xi^n, \xi^m = 0$

If we consider the case $\xi^n, \xi^m = 0$, many simplifications can be taken into account. The pre-corner two-form reads

$$\check{\varpi}^\partial = \int \quad 2\delta(\alpha \ {}^2\Pi_{mi})\delta\xi^i \quad 2\delta(\alpha \ {}^2 b^j\Pi_{ji})\delta\xi^i.$$

Hence the system of equations defining the kernel becomes:

$$\delta\alpha : \quad + 4\alpha \ {}^3\Pi_{mi}X_{\xi^i} + 4\alpha \ {}^3 b^j\Pi_{ji}X_{\xi^i} = 0 \tag{7.13a}$$
$$\delta b_i : \quad 2\alpha \ {}^2\Pi_{ij}X_{\xi^j} = 0 \tag{7.13b}$$
$$\delta\Pi_{mi} : \quad 2\alpha \ {}^2 X_{\xi^i} = 0 \tag{7.13c}$$
$$\delta\Pi_{ij} : \quad 2\alpha \ {}^2 b^j X_{\xi^i} = 0 \tag{7.13d}$$
$$\delta\xi^i : \quad 4X_\alpha\alpha \ {}^3\Pi_{mi} \quad 2\alpha \ {}^2 X \ {}_{mi} + 4X_\alpha\alpha \ {}^3 b^j\Pi_{ji} \quad 2\alpha \ {}^2 X_{b^j}\Pi_{ji} \quad 2\alpha \ {}^2 b^j X \ {}_{ji} = 0. \tag{7.13e}$$

From (7.13c) we deduce $X_{\xi^i} = 0$ and from (7.13e) we find

$$X \ {}_{mi} = 2X_\alpha\alpha \ {}^1\Pi_{mi} + 2X_\alpha\alpha \ {}^2 b^j\Pi_{ji} \quad X_{b^j}\Pi_{ji} \quad b^j X \ {}_{ji}.$$

This completely solves the system. After the reduction the new variables are:

$$
\begin{aligned}
\xi^{i\,0} &= \xi^i \\
\Pi^0_{mi} &= 2\alpha \ {}^2\Pi_{mi} + 2\alpha \ {}^2 b^j\Pi_{ji}.
\end{aligned}
$$

We can now compute the corner action in this particular case using the formula $\iota_{Q^\partial}\iota_{Q^\partial}\varpi^\partial = 2\pi \ \check{S}^\partial$. From the boundary action S^∂ we get (in the case ξ^n):

$$
\begin{aligned}
Q^\partial \ {}_{ab} &= \Pi_{cb}\partial_a\xi^c + \partial_c(\Pi_{ab}\xi^c) \\
Q^\partial_{\gamma^{ab}} &= \gamma^{cb}\partial_c\xi^a + \partial_c(\gamma^{ab}\xi^c) \\
Q^\partial_{\varphi_a} &= \quad 2\partial_c(\gamma^{cd}\Pi_{da}) \quad \partial_a(\gamma^{cd}\Pi_{cd}) + \varphi_c\partial_a\xi^c + \partial_c(\varphi_a\xi^c) \\
Q^\partial_{\xi^a} &= \xi^c\partial_c\xi^a.
\end{aligned}
$$

From the expression of the cohomological vector field it is possible to construct the pre-corner action through the above mentioned formula. A brief computation shows that

$$\check{S}^{\partial} = \int \varphi_c \partial_a \xi^c \xi^a \xi^m + 2\gamma^{md}\Pi_{cd}\xi^a \partial_a \xi^c + 2\gamma^{bd}\Pi_{cd}\xi^m \partial_b \xi^c + \Pi_{ab}\xi^m \partial_d \gamma^{ab}\xi^d.$$

In the special case $\xi^m = 0$ we get

$$\check{S}^{\partial} = \int 2\gamma^{md}\Pi_{jd}\xi^i \partial_i \xi^j = \int 2\alpha \quad {}^{2}\Pi_{jm}\xi^i \partial_i \xi^j + 2\alpha \quad {}^{2}b^i \Pi_{ij}\xi^i \partial_i \xi^j.$$

Reducing it to the corner variables we get the corner action in this particular case:

$$S^{\partial\partial} = \int \Pi^{0}_{jm}\xi^{i^0}\partial_i \xi^{j^0}.$$

The corresponding symplectic form is

$$\varpi^{\partial\partial} = \int \delta\Pi^{0}_{im}\delta\xi^{i^0}$$

Case $\xi^n = 0, b^i = 0$

We can consider yet another case. In this case we only require the metric to be block diagonal and $\xi^n = 0$. The pre-corner two form is:

$$\begin{aligned}\check{\varpi}^{\partial} = \int \quad & 2\delta(\alpha \quad {}^{2}\Pi_{mi})\delta\xi^i + 2\delta(\alpha \quad {}^{2}\Pi_{mm})\delta\xi^m + \delta(h^{ij})\delta(\Pi_{ij}\xi^m) \\ & + \delta\alpha \quad {}^{2}\delta(\Pi_{mm}\xi^m) \quad \delta(\xi^m\varphi_i)\delta\xi^i \quad \delta(\xi^m\varphi_m)\delta\xi^m.\end{aligned}$$

The equations defining the kernel are the following:

$$\begin{aligned}
\delta\varphi_m : &\quad \xi^m X_{\xi^m} = 0 \\
\delta\varphi_i : &\quad \xi^m X_{\xi^i} = 0 \\
\delta\xi^i : &\quad 4X_\alpha\alpha \quad {}^{3}\Pi_{mi} \quad 2\alpha \quad {}^{2}X_{\quad mi} \quad X_{\xi^m}\varphi_i \quad \xi^m X_{\varphi_i} = 0 \\
\delta\xi^m : &\quad 4X_\alpha\alpha \quad {}^{3}\Pi_{mm} + 2\alpha \quad {}^{2}X_{\quad mm} + X_{h^{ij}}\Pi_{ij} \\
&\quad 2X_\alpha\alpha \quad {}^{3}\Pi_{mm} \quad \varphi_i X_{\xi^i} \quad 2\varphi_m X_{\xi^m} \quad \xi^m X_{\varphi_m} = 0 \\
\delta\Pi_{mm} : &\quad 2\alpha \quad {}^{2}X_{\xi^m} \quad 2X_\alpha\alpha \quad {}^{3}\xi^m = 0 \\
\delta\Pi_{mi} : &\quad 2\alpha \quad {}^{2}X_{\xi^i} = 0 \\
\delta\Pi_{ij} : &\quad + X_{h^{ij}}\xi^m = 0 \\
\\
\delta\alpha : &\quad + 4\alpha \quad {}^{3}\Pi_{mi}X_{\xi^i} \quad 4\alpha \quad {}^{3}\Pi_{mm}X_{\xi^m} \quad 2\alpha \quad {}^{3}X_{\quad mm}\xi^m \\
&\quad 2\alpha \quad {}^{3}\Pi_{mm}X_{\xi^m} = 0 \\
\delta h_{ij} : &\quad X_{\quad ij}\xi^m + \Pi_{ij}X_{\xi^m} = 0
\end{aligned}$$

This system is singular, in particular due to the last equation. We can however give a possible singular reduction. Solving the equation labelled by $\delta\Pi_{mm}$ and by $\delta\Pi_{mi}$ we get respectively:

$$\begin{aligned} X_{\xi^m} &= X_\alpha \alpha \quad {}^{1}\xi^m \\ X_{\xi^i} &= 0. \end{aligned}$$

Hence the first two equations are automatically solved. From the third and the fourth we get:

$$
\begin{aligned}
X_{\quad mi} &= 2X_\alpha\alpha \quad {}^1\Pi_{mi} \quad \frac{1}{2}X_\alpha\alpha\xi^m\varphi_i \quad \frac{1}{2}\alpha^2\xi^m X_{\varphi_i} \\
X_{\quad mm} &= X_\alpha\alpha \quad {}^1\Pi_{mm} \quad \frac{1}{2}\alpha^2 X_{h^{ij}}\Pi_{ij} \quad \varphi_m X_\alpha\alpha\xi^m \quad \frac{1}{2}\alpha^2\xi^m X_{\varphi_m}.
\end{aligned}
$$

The last equation is singular but we can interpret it as defining a new odd field of the form $P^0_{ij} = \Pi_{ij}\xi^m$. Computing the reduction with respect to the other equations we get the following variables:

$$
\begin{aligned}
\xi^{m0} &= \alpha \quad {}^1\xi^m \\
\xi^{i^0} &= \xi^i \\
\Pi^0_{mi} &= \alpha \quad {}^2\Pi_{mi} + \frac{1}{2}\varphi_i\xi^m \\
\Pi^0_{mm} &= \alpha \quad {}^1\Pi_{mm} + \frac{1}{2}\alpha\varphi_m\xi^m \\
h^{ij^0} &= h^{ij}
\end{aligned}
$$

Hence we have

$$
\varpi^{\partial\partial} = \int \delta\Pi^0_{im}\delta\xi^{i^0} + \delta\Pi^0_{mm}\delta\xi^{m0} + \delta P^0_{ij}\delta h^{ij^0}.
$$

From the pre-corner action, imposing $b^i = 0$, we get:

$$
\begin{aligned}
\check{S}^\partial = \int & \varphi_m\partial_i\xi^m\xi^i\xi^m + \varphi_i\partial_j\xi^i\xi^j\xi^m + 2\alpha \quad {}^2\Pi_{im}\xi^j\partial_j\xi^i + 2\alpha \quad {}^2\Pi_{mm}\xi^j\partial_j\xi^m \\
& + 2h^{ij}\Pi_{mj}\xi^m\partial_i\xi^m + 2h^{ij}\Pi_{kj}\xi^m\partial_i\xi^k + \Pi_{mm}\xi^m\partial_i\alpha \quad {}^2\xi^i + \Pi_{ij}\xi^m\partial_k h^{ij}\xi^k.
\end{aligned}
$$

After the reduction, the corner action is:

$$
\begin{aligned}
S^{\partial\partial} = \int & \Pi^0_{mi}\xi^{j^0}\partial_j\xi^{i^0} + \Pi^0_{mm}\xi^{j^0}\partial_j\xi^{m0} + h^{ij^0}\Pi^0_{mi}\xi^{m0}\partial_j\xi^{m0} \\
& + h^{ij^0}P^0_{kj}\partial_i\xi^{k^0} + P^0_{ij}\xi^{k^0}\partial_k h^{ij^0}.
\end{aligned}
$$

These (partial) expression can carry some information about physically relevant quantities, for example (possibly asymptotic) charges and related BMS algebras [BT11]. This will be object of future work.

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