On the BV Formalism

by
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In these brief notes we will describe the Batalin–Vilkovisky [1] formalism and apply it to the topological BF theories in three and four dimensions. Basic references for the first part are [3] and [4] (for some further details s. also [6] and [7]). For an interpretation of the BV cohomology, s. [8].

Please notice that this is a review for internal use only.

Here is a list of our notations:

- $\langle \cdot\,,\,\cdot \rangle$ denotes the scalar product
- Φ denotes the set of "fields", while $\Phi^\dagger$ or Φ^* denote the set of "antifields" [s. (12)]; Φ^i, $\Phi_i^\dagger$ and Φ_i^* denote a single field or antifield
- the index "cl" denotes the classical part of a functional, i.e., the functional evaluated at $\Phi^\dagger = 0$
- PS will denote the phase space, i.e., the space of fields and antifields with a supersymplectic structure; CS will denote the configuration space, i.e., the space of fields only; CS_0 will denote the configuration space modulo solutions of the equations of motion (stationary points of $S_{\rm cl}$)
- $*$ denotes the Hodge-$*$
- $\frac{\overrightarrow{\delta}}{\delta\Phi^i}$, $\frac{\overleftarrow{\delta}}{\delta\Phi^i}$ denote functional derivatives w.r.t. the field Φ^i acting to the right or to the left respectively; a similar notation is used for the antifield derivatives
- $\epsilon(\cdot)$ denotes the ghost number; ϵ_i will be used instead of $\epsilon(\Phi^i)$; since the physical fields we consider are bosonic, the Grassmann parity is simply given by $(-1)^\epsilon$.

Moreover, traces and sums over repeated indices are understood.

1 The BV formalism

1.1 The antibracket and the Laplacian

In the BV formalism one considers a set Φ of fields, which will be identified with the physical fields, the ghosts, the antighosts and the Lagrange multipliers (in the case of reducible algebras, all the generations of ghosts for ghosts with the corresponding antighosts and Lagrange multipliers will also be included).

An antifield $\Phi^\dagger_i$ is associated to each field Φ^i; Φ^i is a field of the same nature as Φ^i for all its properties but the ghost number, which satisfies

$$\epsilon(\Phi^\dagger_i) = -\epsilon_i - 1. \tag{1}$$

This also implies that the Grassmann parity is reversed.

A supersymplectic structure is defined over the *phase space* PS by considering the fields and antifields as Darboux coordinates [5]. Given two functionals X and Y over the phase space, the supersymplectic structure determines an antibracket

$$(X\,,\,Y) := X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger_i} \right\rangle Y - X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^i} \right\rangle Y \tag{2}$$

as well as a Laplacian

$$\Delta X := \sum_i (-1)^{\epsilon_i+1} X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}\,,\,\frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i} \right\rangle. \tag{3}$$

More generally, a supersymplectic structure over an $(n|n)$-dimensional supermanifold is provided by an odd symplectic matrix ω; by odd we mean that

$$p\,\omega\,p = -\omega, \tag{4}$$

where $p = (-1)^\epsilon$ is the Grassmann-parity operator. The metric is then provided by

$$g = \omega\,p. \tag{5}$$

If we call χ the coordinates over the phase space, then (2) and (3) read in general

$$(X\,,\,Y) \;\; = \;\; X \left\langle \frac{\overleftarrow{\delta}}{\delta\chi}\,,\,\omega\,\frac{\overrightarrow{\delta}}{\delta\chi} \right\rangle Y, \tag{6}$$

$$\Delta X = \frac{1}{2} X \left\langle \frac{\overleftarrow{\delta}}{\delta\chi}\, , \, g \frac{\overleftarrow{\delta}}{\delta\chi} \right\rangle. \tag{7}$$

The phase space is defined modulo *canonical transformations* [3], i.e., transformations of the fields and antifields that preserve the supersymplectic structure. A canonical tranformation can be obtained by introducing a generating functional $F(\Phi^i, \widetilde{\Phi_i^\dagger})$, with $\epsilon(F) = -1$, such that

$$\widetilde{\Phi^i} = \frac{\overrightarrow{\delta}}{\delta\widetilde{\Phi_i^\dagger}} F, \quad \Phi_i^\dagger = \frac{\overrightarrow{\delta}}{\delta\Phi^i} F. \tag{8}$$

Notice that the volume form $D\Phi D\Phi^\dagger$ is *not* preserved by general canonical trasformations. In the case of an infinitesimal canonical transformation generated by

$$F(\Phi, \widetilde{\Phi^\dagger}) = \left\langle \Phi^i\, , \, \Phi_i^\dagger \right\rangle + \lambda G(\Phi, \widetilde{\Phi^\dagger}), \tag{9}$$

we see that the Jacobian

$$J = \frac{\partial(\widetilde{\Phi}\, , \, \widetilde{\Phi}^\dagger)}{\partial(\Phi\, , \, \Phi^\dagger)} \tag{10}$$

is given by

$$J = 1 - 2\lambda\Delta G + O(\lambda^2). \tag{11}$$

The definitions (2) and (3) of the antibracket and the Laplacian seem to require the introduction of a metric over the manifold M where the theory is defined. As a matter of fact, this spurious dependency can be removed by considering the antifields Φ^*, defined so that

$$\Phi_i^\dagger := *\Phi_i^*. \tag{12}$$

With this substitution, (2) and (3) read

$$(X\, , Y) = X \int_M \frac{\overrightarrow{\delta}}{\delta\Phi_i^*} \wedge \frac{\overleftarrow{\delta}}{\delta\Phi^i}\, Y - X \int_M \frac{\overleftarrow{\delta}}{\delta\Phi_i^*} \wedge \frac{\overrightarrow{\delta}}{\delta\Phi^i}\, Y, \tag{13}$$

$$\Delta X = \sum_i (-1)^{\epsilon_i + 1} X \int_M \frac{\overleftarrow{\delta}}{\delta\Phi_i^*} \wedge \frac{\overleftarrow{\delta}}{\delta\Phi^i}. \tag{14}$$

Notice that Φ_i and Φ_i^* commute if we are working in an odd-dimensional manifold or if Φ_i is an even-degree form; otherwise they anticommute.

The use of Φ^* instead of $\Phi^\dagger$ is convenient, and more natural, in the case of TQFTs, where the BV action, s. later, can be defined in terms of Φ and Φ^* without the need of a metric. However, the use of $\Phi^\dagger$ makes computations easier since one does not have to keep track of the form degree while commuting fields.

1.2 Basic properties

To prove the following useful identities, we have first to notice that by (1) the derivative w.r.t. a field commutes with the derivative w.r.t. the corresponding antifield and that

$$\frac{\overrightarrow{\delta}}{\delta\Phi^i} X = (-1)^{\epsilon_i\,(\epsilon(X)+1)} X \frac{\overleftarrow{\delta}}{\delta\Phi^i}, \quad \frac{\overrightarrow{\delta}}{\delta\Phi^\dagger_i} X = (-1)^{(\epsilon_i+1)\,(\epsilon(X)+1)} X \frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i}. \tag{15}$$

Using the definitions (2) and (3) together with (15), one can prove the following properties of the antibracket

$$\begin{aligned} \epsilon[(X\,,\,Y)] &= \epsilon(X) + \epsilon(y) + 1, & (16)\\ (X\,,\,Y) &= -(-1)^{(\epsilon(X)+1)\,(\epsilon(Y)+1)}\;(X\,,\,Y)\,, & (17)\\ 0 &= (-1)^{(\epsilon(X)+1)\,(\epsilon(W)+1)}\;(X\,,\,(Y\,,\,W)) + \text{cyclic permutations}, & (18) \end{aligned}$$

of the Laplacian

$$\begin{aligned} \epsilon(\Delta) &= 1, & (19)\\ \Delta^2 &= 0, & (20) \end{aligned}$$

and of the two together

$$\Delta\,(X\,,\,Y) = (X\,,\,\Delta Y) - (-1)^{\epsilon(Y)}\;(\Delta X\,,\,Y)\,. \tag{21}$$

One has moreover

$$\Delta(XY) = X\,\Delta Y + (-1)^{\epsilon(Y)}\Delta X\,Y + (-1)^{\epsilon(Y)}(X,Y). \tag{22}$$

By (17), we see that the antibracket of a fermionic functional with itself vanishes, while, if X is bosonic,

$$(X\,,\,X) = 2\,X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^i}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger_i} \right\rangle X = -2\,X \left\langle \frac{\overleftarrow{\delta}}{\delta\Phi^\dagger_i}\,,\,\frac{\overrightarrow{\delta}}{\delta\Phi^i} \right\rangle X. \tag{23}$$

Eventually, we consider a function f of a bosonic functional X; it is easy to prove that

$$\begin{aligned} (f(X)\,,\,Y) &= f'(x)\;(X\,,\,Y)\,, & (24)\\ \Delta f(X) &= f'(X)\,\Delta X + \frac{1}{2} f"(X)\;(X\,,\,X)\,,\,, & (25) \end{aligned}$$

where Y is a bosonic functional, and f' and $f"$ denote the first and second derivative of f w.r.t. X.

1.3 BV cohomology

The properties of the antibracket and the Laplacian allow the definition of interesting *coboundary operators*, i.e., of nilpotent operator of ghost number one. In all of these cohomologies, the form degree will be represented by the ghost number; this means that also negative-degree forms are allowed. An intepretation of them is given in [8].

1.3.1 The basic coboundary operator

The first operator we consider is the Laplacian, which is a coboundary operator because of (19) and (20). Notice that (25) implies that, in general, a function of a Δ-closed functional is *not* Δ-closed; the antibracket measures the failure of this property. We shall denote the Δ-cohomology by $H^*(PS, \Delta)$.

1.3.2 The quantum coboundary operator

The second interesting coboundary operator is

$$\Omega X := (X\,,\,\Sigma) - i\hbar\Delta X, \tag{26}$$

where the *quantum action* Σ is a bosonic functional satisfying the *quantum master equation*

$$(\Sigma\,,\,\Sigma) - 2i\hbar\Delta\Sigma = 0. \tag{27}$$

The nilpotency of Ω is ensured by (18), (20), (21) and (27). Notice that, by (25),

$$\Omega f(X) = f'(X)\,\Omega X - \frac{i\hbar}{2} f"(X)\;(X\,,\,X)\,; \tag{28}$$

therefore, a function of an Ω-closed functional is *not* Ω-closed in general.

The Ω-cohomology, which we shall denote by $H^*(PS, \Omega)$, is isomorphic to the Δ-cohomology. Indeed, if we set

$$W = e^{\frac{i}{\hbar}\Sigma}, \tag{29}$$

we see that (25) and the quantum master equation imply that W is Δ-closed; therefore, by (22), (24) and (17), we have

$$\Delta(WX) = \frac{i}{\hbar} W\,\Omega X. \tag{30}$$

If we see Δ as an exterior derivative, then we can see Ω as the covariant derivative for a flat connection.

The parameter $\hbar$ in (26),(27) and (28) can be removed by rescaling $\Sigma \to \Sigma' = \Sigma/\hbar$ and $\Omega \to \Omega' = \Omega/\hbar$; however, it is useful to show it explicitly and look for perturbative solutions of (27) by setting

$$\Sigma = S + \sum_{n=1}^{\infty} \hbar^n\, \Sigma_n. \tag{31}$$

1.3.3 The classical coboundary operator

The third coboundary operator we consider is

$$\sigma X := (X\,,\, S) \tag{32}$$

where the *action* S is a bosonic functional satisfying the *master equation*

$$(S\,,\, S) = 0. \tag{33}$$

The nilpotency of σ is ensured by (18) and (33). We shell denote the σ-cohomology by $H^*(PS, \sigma)$. Notice that by (24)

$$\sigma f(X) = f'(X)\,\sigma X, \tag{34}$$

which implies that a function of a σ-closed functional *is* σ-closed.

A quantum action Σ defines an action S by (31), the master equation (33) being the limit $\hbar \to 0$ of the quantum master equation (27). However, to an action S can correspond no quantum action; in this case, one says that the theory is *anomalous*. As a matter of fact, (27) and (31) show that we can find a first-order correction Σ_1 only if ΔS is σ-exact [by (21) we only see that it is σ-closed].

Moreover, in the same limit $\hbar \to 0$, an Ω-closed (Ω-exact) functional X gives a σ-closed (σ-exact) functional X_0. However, to a σ-closed functional X_0 can correspond no Ω-closed functional. As a matter of fact, if we expand $X = X_0 + \hbar X_1 + \cdots$, we see that, to find a first-order correction X_1 that makes $\Omega X = 0$, we need

$$i\Delta X_0 - (X_0\,,\,\Sigma_1) \tag{35}$$

to be σ-exact and not only σ-closed.

One can usually expand the action S in a power series in the antifields (in this subsection, integrations over space–time are understood),

$$S(\Phi, \Phi^\dagger) = S_{\mathrm{cl}}(\varphi) + \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n} \Phi^\dagger_{i_1} \cdots \Phi^\dagger_{i_n} \, S^{i_n,\dots,i_i}(\Phi), \tag{36}$$

where φ (the *physical fields*) is the subset of Φ on which S_{cl} turns out to depend.

The master equation (33) gives then a (possibly infinite) set of equations, one for each order in $\Phi^\dagger$. At order zero, we have

$$S_{\mathrm{cl}} \, \frac{\overleftarrow{\delta}}{\delta\varphi^\alpha} S^\alpha = 0, \tag{37}$$

where the index α runs on the set of physical fields φ. This equation shows the symmetries of the classical action S_{cl}. At order one, one has

$$S^j \, \frac{\overleftarrow{\delta}}{\delta\Phi^i} S^i + S_{\mathrm{cl}} \, \frac{\overleftarrow{\delta}}{\delta\varphi^\alpha} \, S^{j\alpha} = 0. \tag{38}$$

This equation gives the commutator of two symmetry transformations (when j corresponds to a physical field) and the Jacobi identity (otherwise).

When S is linear in $\Phi^\dagger$, (37) and (38) are the only structure equations, and the last term in (38) is absent. In this case, one says that the algebra of symmetries is closed.

1.3.4 The BRST operator

The last operator is the BRST operator [2] obtained by restricting σ to the configuration space Φ, viz.,

$$sX := (\sigma X)\big|_{\Phi^\dagger=0}, \tag{39}$$

where X is a functional of the fields only (notice that one first evaluates σ and then sets $\Phi^\dagger = 0$). By (36), we can also write

$$sX = X \frac{\overleftarrow{\delta}}{\delta\Phi^i} S^i. \tag{40}$$

By (38), one gets

$$s^2 X = X \frac{\overleftarrow{\delta}}{\delta\Phi^i} \left(S_{\rm cl} \frac{\overleftarrow{\delta}}{\delta\varphi^\alpha} \right) S^{i\alpha}. \tag{41}$$

Therefore, s is *not* nilpotent in general; however, it is always nilpotent *on shell*, i.e., modulo the solutions of

$$S_{\rm cl} \frac{\overleftarrow{\delta}}{\delta\varphi^\alpha} = 0. \tag{42}$$

We shall denote the BRST cohomology by $H^*(CS_0, s)$. Notice that, in the case of a closed algebra, we can define the BRST cohomology on all the configuration space [we shall denote it by $H^*(CS, s)$]. Notice that we have in particular

$$sS_{\rm cl} = 0 \quad \text{on shell}. \tag{43}$$

Deriving (43) w.r.t. φ, we get the set of classical symmetries,

$$S_{\rm cl} \frac{\overleftarrow{\delta}}{\delta\varphi^\alpha} R^{\alpha i} = 0 \quad \text{on shell}, \tag{44}$$

where

$$R^{\alpha i} = S^\alpha \frac{\overleftarrow{\delta}}{\delta\Phi^i}. \tag{45}$$

If X is a functional over the phase space, we can expand it as a power series in the antifields:

$$X(\Phi, \Phi^\dagger) = X_{\rm cl}(\Phi) + X^i(\Phi)\, \Phi^\dagger_i + \cdots \tag{46}$$

(notice that in general $X_{\rm cl}$ will not depend only on the physical fields φ). Then, by (36) and (40), we have

$$\sigma X = s X_{\rm cl} - X^\alpha \,\frac{\overrightarrow{\delta}}{\delta\varphi^\alpha}\, S_{\rm cl}. \tag{47}$$

Thus, we see that the classical part of a σ-closed (σ-exact) functional is s-closed (s-exact) on shell. This means that $H^*(PS,\sigma)$ projects on $H^*(CS_0,s)$. Notice that this is true also in the case of closed algebras; i.e., more general observables than the usual s-closed functionals on CS can be considered.

There are some theorems about the BV extension of a classical action $S_{\rm cl}$ [9]. Under some reasonable assumptions, a classical action $S_{\rm cl}$ always determines an action S satisfying the master equation (33). This assumptions include the *regularity condition* which states that the Hessian

$$H_{\alpha\beta} = \frac{\overrightarrow{\delta}}{\delta\varphi^\alpha}\, S_{\rm cl}\, \frac{\overleftarrow{\delta}}{\delta\varphi^\beta} \tag{48}$$

evaluated on a solution of (42) has a rank equal to the *number of degrees of freedom*, i.e., the number of phisical fields φ minus the symmetries (44). We shall see in the next section that the regularity condition is necessary to quantize the theory. Moreover, one can prove that the action S is unique, modulo canonical transformations, if it is *proper*, i.e., if the Hessian

$$\mathcal{H}_{ij} = \frac{\overrightarrow{\delta}}{\delta\chi i}\, S\, \frac{\overleftarrow{\delta}}{\delta\chi j} \tag{49}$$

(where $\chi = (\Phi,\Phi^\dagger)$ are coordinates of the phase space) evaluated on a solution of

$$S\,\frac{\overleftarrow{\delta}}{\delta\chi} = 0, \tag{50}$$

has rank equal to the number of fields Φ.

1.3.5 Left- and right-acting operators

With our definitions (3), (26), (32) and (39), the operators Δ, Ω, σ and s act from the right. It is however possible to define the corresponding operators acting from the left

$$
\begin{aligned}
\Delta_l X &= \sum_i (-1)^{\epsilon_i} \left\langle \frac{\overrightarrow{\delta}}{\delta \Phi^\dagger_i}\,,\, \frac{\overrightarrow{\delta}}{\delta \Phi^i} \right\rangle X, && (51)\\
\Omega_l X &= (\Sigma\,,\, X) - i\hbar \Delta_l X, && (52)\\
\sigma_l X &= (S\,,\, X)\,, && (53)\\
s_l X &= (S\,,\, X)\Big|_{\Phi^\dagger = 0}\,. && (54)
\end{aligned}
$$

By (15) and (17) one simply has

$$
\begin{aligned}
\Delta_l X &= (-1)^{\epsilon(X)} \Delta X, && (55)\\
\Omega_l X &= (-1)^{\epsilon(X)} \Omega X, && (56)\\
\sigma_l X &= (-1)^{\epsilon(X)} \sigma X, && (57)\\
s_l X &= (-1)^{\epsilon(X)} s X. && (58)
\end{aligned}
$$

The operator s_l is the ordinary BRST operator [2]; by (40), it turns out that

$$
s_l X = \sum_i (-1)^{\epsilon_i}\; S^i \, \frac{\overrightarrow{\delta}}{\delta \Phi^i}\, X, \tag{59}
$$

with S^i defined in (36).

1.4 Conclusions

In this section we have considered three coboundary operators on the phase space, viz., Δ, Ω, σ, and one coboundary operator, s, on the configuration space modulo solutions of the classical equations of motion.

Since all these cohomologies are defined in terms of antibrackets and/or the Laplacian, they are preserved by canonical transformations. [Notice moreover that, by (11), volume-preserving canonical transformations are generated by Δ-closed (-1)-forms.] Notice moreover that the Ω- and σ-cohomologies are preserved by smooth changes of Σ and S (e.g., by redefinitions of the coupling constants).

We have shown that the Δ- and Ω-cohomologies are isomorphic. Moreover, we have shown that the Ω-cohomology induces the σ-cohomology that in turn projects to the s-cohomology on shell:

$$\begin{array}{ccc} H^*(PS,\Delta)\sim H^*(PS,\Omega) & \hookrightarrow & H^*(PS,\sigma) \\ & & \downarrow \\ & & H^*(CS_0,s) \end{array} \tag{60}$$

We see then that the BV formalism is more general than the BRST one, since not only does it allow considering open algebras, but, even in the case of closed algebras, it permits to consider more general observables, i.e., functionals that are s-closed only on shell.

In the next section we will see that the cohomology relevant to the quantization of a theory is the Ω-cohomology. However, one usually starts with a set of physical fields and a classical action S_{cl}. One then extends it to an action S that satisfies the master equation (33) and defines a σ-cohomology. The next step is to find an extension Σ satisfying the quantum master equation (27). When this is not possible, one says that the theory is anomalous. We have also seen that a necessary condition for a theory not to be anomalous is that ΔS be a trivial element in $H^1(PS,\sigma)$.

If the theory is not anomalous, we have then a quantum action Σ that defines an Ω-cohomology generalizing the σ-cohomology. Notice that not all the elements of $H^p(PS,\sigma)$ correspond to an element of $H^p(PS,\Omega)$. A necessary condition for this to happen is that the expression (35) be a trivial element of $H^{p+1}(PS,\sigma)$.

We conclude this section by noticing that in dimensional regularization the Laplacian of a local functional always vanishes because of the peculiar property "$\delta(0)=0$" (where δ is Dirac's delta). A local functional is a functional that can be written as a space–time integral of a density containing only a finite number of derivatives (i.e., it is the integral of a local function on the jet space V^j with finite j). In this context, a local action is automatically a quantum local action (i.e., $\Sigma=S$); moreover, the limit $\hbar\to 0$ defines a map from $H^*(PS,\Omega)$ onto $H^*(PS,\sigma)_{\mathrm{local}}$. Notice, however, that some symmetries (like the chiral symmetry) are peculiar of some dimensionalities, so the dimensional regularization explicitly breaks them.

Problem The mathematicians that have explicitly studied the perturbative expansion of the Chern–Simons theory have realized that a suitable formulation of the Feynman diagrams consists in replacing copies of the manifold with the Fulton–McPherson compactification of the related configuration space. It would be nice to find a regularization scheme mimicking this procedure; the perturbation theory of the resulting quantum action should then give the previous Feynman diagrams directly and shed some light on Bott and Taubes's "anomaly."

2 The BV quantization

2.1 The functional integration

In this section we describe functional integration over the configuration space CS. All the arguments are of course only formal, unless we are working with a finite-dimensional model. Since the functional integrals diverge in general, renormalization is needed to obtain finite results.

We start considering functional integration over a Lagrangian submanifold $\mathcal{L}_{\Psi,K^\dagger}$ defined by the equations

$$\Phi^\dagger_i = \frac{\overrightarrow{\delta}}{\delta\Phi^i}\,\Psi(\Phi) + K^\dagger_i, \tag{61}$$

where $K^\dagger$ are external sources satisfying

$$\epsilon(K^\dagger_i) = \epsilon(\Phi^\dagger_i), \tag{62}$$

while Ψ is a functional on the configuration space only that has ghost number -1; this shows that one must consider theories containing fields with negative ghost number. It is easily shown [7] that the form of (61) is preserved by canonical transformations that are connected to the identity; viz., if $\widetilde{\Phi}$ and $\widetilde{\Phi^\dagger}$ are the canonically transformed variables, then

$$\widetilde{\Phi^\dagger_i} = \frac{\overrightarrow{\delta}}{\delta\widetilde{\Phi^\dagger_i}}\,\widetilde{\Psi}(\widetilde{\Phi}, K^\dagger) + K^\dagger_i; \tag{63}$$

in general, $\widetilde{\Psi}$ differs from Ψ and depends also on $K^\dagger$. Then, assuming that there are no boundary contributions, we have the following two basic theorems:

Theorem 1 *If X is a Δ-exact integrable functional then*

$$\int_{\mathcal{L}_{\Psi,K^\dagger}} X = 0,$$

for any Ψ

Theorem 2 *If X is an integrable functional, under an infinitesimal variation $\Psi \to \Psi + \delta\Psi$ we have*

$$\delta \int_{\mathcal{L}_{\Psi,K^\dagger}} X = \int_{\mathcal{L}_{\Psi,K^\dagger}} \Delta X \, \delta\Psi$$

Thus, we have the following

Corollary 1 *If X is Δ-closed and integrable, then $\int_{\mathcal{L}_{\Psi,K^\dagger}} X$ is invariant under infinitesimal deformations of Ψ.*

This shows that we can intepret the Laplacian as a sort of external derivative. Notice, however, that $\int_{\mathcal{L}_{\Psi,K^\dagger}} X$ is not invariant under changes of the sources $K^\dagger$; this means that we cannot see translated surfaces as homologically equivalent.

To prove Thms. (1) and (2), consider now the particular canonical transformation generated by

$$F(\Phi, \widetilde{\Phi^\dagger}) = \Psi(\Phi) + \left\langle \Phi^i \,,\, \Phi_i^\dagger \right\rangle \tag{64}$$

that gives

$$\widetilde{\Phi^i} = \Phi^i, \tag{65}$$

$$\widetilde{\Phi_i^\dagger} = \Phi_i^\dagger - \frac{\overrightarrow{\delta}}{\delta\Phi^i}\,\Psi(\Phi). \tag{66}$$

The Lagrangian manifold $\mathcal{L}_{\Psi,K^\dagger}$ transforms into $\mathcal{L}_{0,K^\dagger}$ defined by

$$\widetilde{\Phi_i^\dagger} = K_i^\dagger; \tag{67}$$

thus,

$$\int_{\mathcal{L}_{\Psi,K^\dagger}} X = \int_{\mathcal{L}_{0,K^\dagger}} \widetilde{X}_\Psi, \tag{68}$$

where the transformed functional is given by

$$\widetilde{X}_\Psi(\widetilde{\Phi}, \widetilde{\Phi^\dagger}) = X(\widetilde{\Phi}, \widetilde{\Phi^\dagger} + \frac{\overrightarrow{\delta}}{\delta\Phi}\,\Psi). \tag{69}$$

The proof of Thm. (1) is immediate on $\mathcal{L}_{0,K^\dagger}$; however, since the Laplacian is preserved by canonical transformations, it holds for $\mathcal{L}_{\Psi,K^\dagger}$ as well. To prove Thm. (2), we notice that

$$\delta \int_{\mathcal{L}_{\Psi,K^\dagger}} X = - \int_{\mathcal{L}_\Psi} (X \, , \, \delta\Psi) \, ,$$

which can is easily proved to hold on $\mathcal{L}_{0,K^\dagger}$ and then extended to $\mathcal{L}_{\Psi,K^\dagger}$ since the antibracket is preserved by canonical transformations. By (22) and Thm. (1), we get to prove Thm. (2).

2.2 The partition function and the vacuum expectation values

Since the Δ-cohomology is isomorphic to the Ω-cohomology generated by a quantum action Σ satisfying the quantum master equation (27), Thm. (2) implies that the *partition function*

$$Z_\Psi := \int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar}\Sigma}, \tag{70}$$

is invariant under infinitesimal variations of the *gauge-fixing fermion* Ψ; by $\mathcal{L}_\Psi$ we mean the Lagrangian submanifold (61) with no sources (i.e., $K^\dagger = 0$).

However, for (70) to be meaningful, we must require that at least the saddle-point approximation be computable. This requires that the action S, determined by Σ as $\hbar \to 0$, should be a *proper* solution of the master equation (33), and that the classical action S_{cl}, given by $S(\Phi, 0)$, should satisfy the *regularity condition* (for a definition of proper solution and regularity condition, s. p. 9).

The structure of the quantum action Σ is then in general

$$\Sigma(\Phi, \Phi^\dagger) = \Sigma'(\varphi, C; \varphi^\dagger, C^\dagger) + \sum_{i=1}^{\#C} \left\langle \bar{C}_i^\dagger \, , \, h^i \right\rangle , \tag{71}$$

where

- φ are the *physical fields* appearing in the classical action; $\epsilon(\varphi^\alpha) = 0$
- C are the *ghosts* (we also include the ghosts for ghosts and so on when necessary); $\epsilon(C^i) > 0$

- $\bar{C}$ are the *antighosts* (one for each ghost); $\epsilon(\bar{C}^i) = -\epsilon(C^i)$
- h^i are the *Lagrange multipliers* (one for each ghost); $\epsilon(h^i) = \epsilon(\bar{C}^i) + 1$

The antighosts are needed to get a (-1)-ghost-number gauge-fixing fermion. Notice that one cannot simply choose $\Psi = 0$; as a matter of fact, $\Sigma(\Phi, 0)$ depends only on φ; since among the C's there are Grassmann variables, with this gauge-fixing the partition function vanishes.

In the case of a closed algebra, the action S is linear in the antifields; this implies that

$$S\left(\Phi, \frac{\overrightarrow{\delta}}{\delta\Phi}\,\Psi\right) = S_{\mathrm{cl}}(\varphi) + s\Psi, \tag{72}$$

as is familiar in BRST quantization.

Whenever the partition function is well defined (and not vanishing), we can define the vacuum expecation value of a functional X over the phase space as

$$\langle X \rangle_{\Psi} = \frac{1}{Z_{\Psi}} \int_{\mathcal{L}_{\Psi}} e^{\frac{i}{\hbar}\Sigma}\, X. \tag{73}$$

With all the above precisations in mind, we see that Thms. (1) and (2) can be restated as the following

Theorem 3 *If Σ satisfies the quantum master equation (27), then*

1. *the partion function Z and the expectation values of Ω-closed functionals do not change under infinitesimal variations of the gauge-fixing fermion Ψ, and*

2. *the expectation value an Ω-exact functional vanishes.*

2.3 The effective action

To discuss renormalization, we have to introduce the effective action. To do this, we have first to generalize the partition function to include sources. We have already considered the "sources for BRST-variations" $K^\dagger$ [with $\epsilon(K_i^\dagger) =$

$-\epsilon_i - 1$]; the name comes from the fact that, in the case of a closed algebra, (72) generalizes to

$$S\left(\Phi, \frac{\overrightarrow{\delta}}{\delta\Phi}\,\Psi + K^\dagger\right) = S_{\rm cl}(\varphi) + s\Psi + \left\langle K_i^\dagger\,,\, s\Phi_i\right\rangle. \tag{74}$$

We want to consider also "sources for the fields" J [with $\epsilon(J_i) = -\epsilon_i$], i.e., define the generating functional

$$Z_\Psi(J, K^\dagger) = \int_{\mathcal{L}_{\Psi,K^\dagger}} e^{\frac{i}{\hbar}\Sigma}\, e^{\frac{i}{\hbar}\left\langle J_i\,,\,\Phi^i\right\rangle}. \tag{75}$$

Notice that the generating functional is *not* invariant under infinitesimal variations of Ψ.

It is not difficult to show (starting from $\mathcal{L}_{0,K^\dagger}$ and then extending the result to $\mathcal{L}_\Psi$) that

$$\left\langle J_i\,,\, \frac{\overrightarrow{\delta}}{\delta K_i^\dagger}\right\rangle Z_\Psi(J, K^\dagger) = 0. \tag{76}$$

One then defines the "generating functional of connected diagrams" W as

$$Z_\Psi(J, K^\dagger) = e^{\frac{i}{\hbar}W_\Psi(J,K^\dagger)}; \tag{77}$$

W satisfyes the same equation (76) as Z. Eventually one introduces the *effective action* Γ as the Legendre transform of W

$$\Gamma_\Psi(K, K^\dagger) = W_\Psi(J(K, K^\dagger), K^\dagger) - \left\langle J_i(K, K^\dagger)\,,\, K^i\right\rangle, \tag{78}$$

with

$$K^i = \frac{\overrightarrow{\delta}}{\delta J_i}\, W_\Psi \tag{79}$$

[notice that $\epsilon(K^i) = \epsilon_i$]. By using the property

$$J_i = -\Gamma_\Psi\, \frac{\overleftarrow{\delta}}{\delta K^i}, \tag{80}$$

one can prove that (76) implies

$$(\Gamma_\Psi\,,\, \Gamma_\Psi) = 0; \tag{81}$$

i.e., Γ_Ψ satisfies the master equation in the phase space $K, K^\dagger$ (which isomorphic to the phase space $\Phi, \Phi^\dagger$). Notice, however, that Γ_Ψ does *not* have to satisfy to all the conditions to which S does (i.e., to be a proper solution and to define a classical action satisfying the regularity condition); moreover, Γ_Ψ is *not* local. However, S is contained in Γ_Ψ since it is possible to show that

$$\lim_{\hbar\to 0} \Gamma_\Psi(K, K^\dagger) = S(K, K^\dagger). \tag{82}$$

Notice that, since Γ_Ψ satisfies the master equation, it defines a coboundary operator γ_Ψ acting as

$$\gamma_\Psi \cdot = (\cdot\,,\, \Gamma_\Psi)\,. \tag{83}$$

One can also consider vacuum expectation values in presence of sources, viz.,

$$\langle X \rangle_\Psi (J, K^\dagger) = \frac{1}{Z_\Psi(J, K^\dagger)} \int_{\mathcal{L}_\Psi} e^{\frac{i}{\hbar}\Sigma}\, e^{\frac{i}{\hbar}\langle J_i\,,\,\Phi^i\rangle}\, X. \tag{84}$$

If one then defines

$$E_\Psi[X](K, K^\dagger) = \langle X \rangle_\Psi (J(K, K^\dagger), K^\dagger), \tag{85}$$

it is possible to prove [3] that, if X is Ω-closed (Ω-exact), then $E_\Psi(X)$ is γ_Ψ-closed (γ_Ψ-exact); i.e., E_Ψ is a map from $H^*(\Omega, PS)$ to $H^*(\gamma_\Psi, PS)$.

2.4 Renormalization

The renormalization procedure descends from equation (81). In this section we consider Σ as a function of Φ and $K^\dagger$, since we suppose we have fixed the gauge as in (61). (We should write Σ_Ψ, but we drop all the subscripts Ψ in this section for simplicity). The canonical transformation (64) shows that we can think of $K^\dagger$ as the new antifields; moreover, $\Sigma_\Psi(\Phi, K^\dagger)$ satisfies the quantum master equation (27) on the phase space with coordinates $\Phi, K^\dagger$.

The result of this section are contained in [3] and [4].

2.4.1 The additional hypothesis

We suppose that the action S depends on some (possibly infinite) parameters λ. We shall show that, *under the hypothesis that the functionals*

$$\mathcal{S}^i = \frac{\partial S}{\partial \lambda_i} \tag{86}$$

are a basis for $H^0(\sigma, PS)$, the theory is renormalizable (this does not mean that it is predictive); more precisely, we shall show that, by a step-by-step redefinition of the parameters λ (*renormalization-group flow*) and a canonical transformation of the quantum action (this generalizing the *wave-function renormalization*), it is possible to make the effective action Γ finite at any order in $\hbar$.

Notice that the additional hypothesis is by no means automatically verified. As a matter of fact, the master equation (33) implies that the $\mathcal{S}$'s are σ-closed; however, we require that they constitue a basis for $H^0(\sigma, PS)$. This has been proved true in some particular cases, as in Yang–Mills theory, but can be wrong in general (i.e., there could be theories that are not renormalizable even in this generalized way).

2.4.2 The renormalization algorithm and the proof of renormalizability

To start the inductive proof, we suppose that we have been able to redefine the quantum action as $\Sigma^{(n-1)}$, so that the corresponding effective action $\Gamma^{(n-1)}$ is finite up to order $\hbar^{n-1}$; moreover, we suppose that $\Sigma^{(n-1)}$ is local, satisfies the quantum master equation (27) and its limit for $\hbar \to 0$ is the action S modulo canonical transformations. Notice that all these hypotheses are satisfied at $n = 1$ by $\Sigma^{(0)} = \Sigma$.

As a consequence, $\Gamma^{(n-1)}$ satisfies the master equation (81) and the limiting condition (82). At order $\hbar^n$, $\Gamma^{(n-1)}$ will in general diverge; call $\Gamma^{(n-1)}_{n,\mathrm{div}}$ its divergent part at this order. The master equation (81), the limiting condition (82) and the finiteness of $\Gamma^{(n-1)}$ up to order $\hbar^{n-1}$ imply

$$\sigma\Gamma^{(n-1)}_{n,\mathrm{div}} = 0. \tag{87}$$

Then, by the additional hypothesis at the beginning of this section, we get

$$\Gamma^{(n-1)}_{n,\mathrm{div}} = \sum_i \delta_n\lambda_i\, \mathcal{S}^i + \sigma R^{(n)}. \tag{88}$$

Thus, by redefining the parameters as $\lambda \to \lambda - \delta_n\lambda$, we can remove the first term of the r.h.s. of (88); to remove the second term, i.e., $\sigma R^{(n)}$, we perform a suitable canonical transformation. Since the combination of this two operations yields a new action $\Sigma^{(n)}$ that is local, satisfies the quantum master equation (27) and, in the limit $\hbar \to 0$, reduces to the action S modulo canonical transformations, we have proved the starting hypotheses for the further induction step.

2.4.3 The gauge-fixing independence

Thm. (3) is not enough to guarantee the gauge-fixing independence of the renormalized theory since, in principle, the renormalization of the coupling constants λ could be gauge-fixing dependent. In [3] and [4], it is shown that this is not the case. It is supposed there that the gauge-fixing fermion depends on some parameters $\kappa \in \mathcal{M}_{\text{g.f.}}$ (where $\mathcal{M}_{\text{g.f.}}$ could as well be infinite dimensional). It is then proved that

$$d\,\delta_m \lambda_i = 0, \tag{89}$$

where d is the exterior derivative on $\mathcal{M}_{\text{g.f.}}$.

Moreover, it is proved that at the nth step of the renormalization algorithm, it is possible to define a local functional $\chi^{(n)}$ that takes values in $\Omega^1(\mathcal{M}_{\text{g.f.}})$ and has ghost number -1 such that

$$d\Sigma^{(n)} - \left(\chi^{(n)}\,,\,\Sigma^{(n)}\right) = 0. \tag{90}$$

At order zero we have

$$\chi^{(0)} = d\Psi; \tag{91}$$

If one then defines

$$\omega^{(n)} = E(\chi^{(n)}), \tag{92}$$

one has

$$d\Gamma^{(n)} - \left(\omega^{(n)}\,,\,\Gamma^{(n)}\right) = 0. \tag{93}$$

If we work in dimensional regularization, the locality of $\chi^{(n)}$ together with (90) ensures that $\chi^{(n)}$ is $\Omega^{(n)}$-exact. Since the vacuum expectation value of an $\Omega^{(n)}$-exact functional is $\gamma^{(n)}$-exact, (93) follows. Notice however that (93) can be proved in general (i.e., with no reference to a particular regularization scheme). If we now set $K = 0,\ K^\dagger = 0$, (93) ensures that $d\Gamma^{(n)} = 0$. This is the expected gauge-fixing indepndence of the renormalized theory.

To conlude we note that, in [3] and [4], (90) and (93) are given a further intepretation; i.e., χ and ω are seen as a sort of connections on $\mathcal{M}_{\text{g.f.}}$ defining the covariant derivatives

$$D\cdot = d\cdot + (\chi\,,\,\cdot)\,, \tag{94}$$

$$\mathcal{D}\cdot = d\cdot + (\omega\,,\,\cdot)\,; \tag{95}$$

(90) and (93) then mean that Σ is D-closed while Γ is $\mathcal{D}$-closed.

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