On the finite dimensional BV Formalism

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April 9, 2005

Introduction

We analyze the finite dimensional mathematical framework for Batalin Vilkovisiky (BV) formalism also called antifield formalism as suggested in [Schw]. In a general procedure for quantization of lagrangian field theories, called BV quantization one obtains an odd poisson algebra and an odd laplacian. But although the antibracket of the algebra is defined intrinsically there is no such coordinate independend definition of the laplacian. This leads to the general problem of describing all the generators of a fixed bracket and characterizing a special one.

The first three sections are the main part of the text. The fourth section tries to explain the physical backround. The first appendix is intended to be an elementary introduction to super mathematics and a second appendix lists some properties of the exponential function on supermanifolds.

This survey was written in the Summer Semester 2004 as a term project under guidance of Professor Alberto Cattaneo. The aim of the author was to understand the main ingredients of the BV formalism and more concretely certain results from the papers of Khudaverdian and Voronov [KV]. I would like to thank Prof. Cattaneo for helpful comments and explanations.

1 BV Algebras

Conventions: Linear differential operators are sometimes simply referred to as differential operators. The elements of a graded commutative algebra A are sometimes called functions in analogy to the situation where $A = C^\infty(M)$. Derivations of such algebras are sometimes called vector fields etc.

Let A be the function algebra of a supermanifold. By a **bracket** on A we will mean in the most general sense a $\mathbb{R}$ bilinear map

$$(\ , \ ) : A \times A \to A$$

A bracket is called **symmetric** if for homogeneous $f, g \in A$ one has

$$(f, g) = (-1)^{|f||g|}(g, f)$$

a bracket shall be called **co-symmetric** if

$$\{f, g\} = -(-1)^{(|f|+1)(|g|+1)}\{g, f\}$$

Symmetric and co-symmetric brackets can be easily related to one another: given a symmetric bracket $(\ , \ )$ one can define a co-symmetric bracket by $\{f, g\} := (-1)^{|f|}(f, g)$. Conversely given an co-symmetric bracket $\{ \ , \ \}$ one defines a symmetric one by $(f, g) := (-1)^{|f|}\{f, g\}$. These operations are inverse to another and so symmetric and co-symmetric brackets can be considered as the same notions by this correspondence.

Recall that there is also the notion of even and odd brackets, depending on wether the bilinear map is even or odd.

Definition 1.1. A graded algebra A supplied with an odd co-symmetric bracket $\{\ ,\ \}$ is called **odd Poisson algebra** or **Gerstenhaber algebra** if the bracket satisfies:

G1. $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|+1)(|g|+1)}\{g, \{f, h\}\}$

G2. $\{f, gh\} = \{f, g\}h + (-1)^{(|f|+1)|g|}g\{f, h\}$

Using the co-symmetry of the bracket one finds that property G1 is equivalent to the "Jacobi" identity:

$$(-1)^{(|f|+1)(|h|+1)}\{f, \{g, h\}\}+(-1)^{(|g|+1)(|f|+1)}\{g, \{h, f\}\}+(-1)^{(|h|+1)(|g|+1)}\{h, \{f, g\}\} = 0$$

The odd poisson bracket has several names in the literature like, Gerstenhaber bracket, Buttin bracket or antibracket.

Our main interest is in Gerstenhaber algebras supplied with an additional piece of data.

Definition 1.2. A Gerstenhaber algebra A, $\{\cdot, \cdot\}$ supplied with a $\mathbb{R}$ linear map

$$\Delta : A \to A$$

which squares to zero $\Delta^2 = 0$ is called a **BV-Algebra** if the bracket is generated from the operator in the following sense:

$$\{f, g\} = (-1)^{|f|}[[\Delta, f], g]1$$

This operator is sometimes called **(odd) Laplace operator**.

The bracket $[\ ,\ ]$ in the definition above is the graded commutator of linear operators, where f, g are considered as linear operators acting by multiplication from the left. The BV stands for Batalin and Vilkovisky and the name odd Laplacian will become clearer further on. Our motivations are the following questions (which will only be answered partially): Given a Gerstenhaber algebra
Does there always exist an operator Δ which makes the algebra into a BV algebra?
How many of these operators exist and how can we describe this set of generators?
Is there in some sense a canonical generator?

To make computations easier we first turn our bracket into a symmetric one and see how conditions G1 and G2 translate. The new bracket denoted by $(f, g) := (-1)^{|f|}\{f, g\}$ is then odd, symmetric and satisfies

G'1 $(-1)^{|f||h|}((f, g), h) + (-1)^{|g||f|}((g, h), f) + (-1)^{|h||g|}((h, f), g) = 0$

G'2 $(f, gh) = (f, g)h + (-1)^{(|f|+1)|g|}g(f, h)$

Note that G2 remained unchanged. We also note that the symmetrized bracket is now related to the odd laplacian by

$$(f, g) = [[\Delta, f], g]1$$

We shall sometimes speak of a symmetric Gerstenhaber or BV algebra. Let's also give the set of **S**quare **Z**ero **G**enerators of a fixed symmetric bracket the name

$$\mathrm{SZG}_{(,)} := \{\Delta \in \mathrm{End}_{\mathbb{R}-\mathrm{Vect}}(A) \mid \Delta^2 = 0, [\Delta, f], g]1 = (f, g)\}$$

2 Derived brackets and their generators

In a general setting a **generator** of some bracket $(\, , \,)$ is an $\mathbb{R}$ linear map $\Delta : A \to A$ which generates the bracket in the sense that:

$$(f, g) = [[\Delta, f], g]1$$

(No requirement of $\Delta^2 = 0$) Let's denote the set of all generators of a fixed bracket with

$$\mathrm{Gen}_{(,)} := \{\Delta \in \mathrm{End}_{\mathbb{R}-\mathrm{Vect}}(A) \mid [\Delta, f], g]1 = (f, g)\}$$

Given any linear operator Δ one can always construct a bracket by the above formula and calls it the **derived bracket** of the operator.

Proposition 2.1. *The derived bracket* $(\, , \,) : A \times A \to A$ *of a linear operator is always* $\mathbb{R}$ *bilinear and symmetric. If* Δ *is even (odd) then the derived bracket is even (odd). If two operators* Δ *and* $\tilde{\Delta}$ *differ by a differential operator of order less than 1 i.e.* $\Delta - \tilde{\Delta} \in \mathrm{Diff}_1(A)$ *then they generate the same bracket.*

Proof. The claim about the bilinearity and the parity of the bracket follows directly from the properties of the commutator $[\, , \,]$. For the symmetry note that if Δ is homogeneous its derived bracket can be written more explicitly as

$$(f, g) = \Delta(fg) - (\Delta f)g - (-1)^{|\Delta||f|} f(\Delta g) + (\Delta 1)fg$$

and here one can check the symmetry directly. For non-homogeneous Δ we split it into even and odd part $\Delta = \Delta_0 + \Delta_1$ and thereby split our bracket in even and odd part $(\, , \,) = (\, , \,)_0 + (\, , \,)_1$ and these homogenous parts are symmetric. Finally the last property is a direct consequence of the definition of first order differential operators, since if $\Delta - \tilde{\Delta}$ is a first order differential operator then $[[\Delta - \tilde{\Delta}, f], g] = 0$ for all $f, g \in A$ and so $(f, g)_\Delta = (f, g)_{\tilde{\Delta}}$. □

The next proposition shows that if we want to generate a bracket which is a bi-derivation the generator must to be a second order differential operator.

Proposition 2.2. *The derived bracket satisfies the derivation property G2:*

$$(f, gh) = (f, g)h + (-1)^{(|f|+|\Delta|)|g|} g(f, h)$$

if and only if Δ *is a differential operator of order less than* 2 *i.e.* $[[[\Delta, f], g], h] = 0, \forall f, g, h \in A$

Proof. Suppose Δ is a second order differential operator then $G := [\Delta, f]$ is a differential operator of order 1 and so $G - G(1)$ is a derivation. If Δ and f are homogeneous then the parity of G is $|f| + |\Delta|$. So we get:

$$\begin{aligned}
(f, gh) &= [[\Delta, f], gh]1 = [G, gh]1 \\
&= G(gh) - (-1)^{(|g|+|h|)|G|} ghG(1) = (G - G(1))gh \\
&= (G - G(1))(g)h + (-1)^{|g||G|} g(G - G(1))(h) \\
&= (f, g)h + (-1)^{|g|(|\Delta|+|f|)} g(f, h)
\end{aligned}$$

The converse statement is proven in [Kosz] □

So from now we restrict our search of generators to second order linear differential operators.

Proposition 2.3. *Two second order operators* Δ *and* $\tilde{\Delta}$ *generate the same bracket if and only if they have the same symbol (i.e. they differ by a first order operator).*

Proof. " $\Leftarrow$ " was already proven. For the other direction suppose two second order operators generate the same bracket then $[[\Delta - \tilde{\Delta}, f], g]1 = 0$ for all $f, g \in A$. Notice that $[[\Delta - \tilde{\Delta}, f], g]$ is a differential operator of order 0, and for any zero order operator $Z \in \mathrm{Diff}_0(A)$: $Z(f) = Z(1) \cdot f$, so it follows that $[[\Delta - \tilde{\Delta}, f], g] = 0$ which implies that $\Delta - \tilde{\Delta}$ is a differential operator of first order. □

So we could say that the bracket generated by a second order operator "is" its symbol.

Since we want to generate an odd bracket the symbol of our generator must be odd and so its even part is of order ≤ 1 and by the last proposition we may forget about it. So from now on we only look for purely odd second order operators. We will also narrow our search by demanding that $\Delta(1) = 0$ which only means that we drop any part of order 0 in our generator. So we redefine the set $\mathrm{SZG}_{(,)}$ to be

$$\mathrm{SZG}_{(,)} := \{\Delta \in \mathrm{Diff}_2(A) \mid \Delta^2 = 0,\ [\Delta, f], g]1 = (f, g),\ \Delta \text{ odd},\ \Delta(1) = 0\}$$

and

$$\mathrm{Gen}_{(,)} := \{\Delta \in \mathrm{Diff}_2(A) \mid [\Delta, f], g]1 = (f, g),\ \Delta \text{ odd},\ \Delta(1) = 0\}$$

It can also be shown that demanding the Jacobi identity for our derived bracket is equivalent to asking the square of our generator $\Delta \in \mathrm{Gen}_{(,)}$ to satisfy $\mathrm{ord}(\Delta^2) \leq 2$ see [Vor] for a proof. Note that in general the order of Δ^2 is less than 4 but since Δ is odd $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ and so the order is certainly less than 3.

A useful fact we shall prove is that $\mathrm{ord}(\Delta^2) \leq 1$ is equivalent to Δ being a derivation of the Gerstenhaber bracket.

Proposition 2.4. *Let $\Delta \in \mathrm{Gen}_{\{,\}}$ be a generator of an odd cosymmetric bracket. Δ is a derivation of the bracket:*

$$\Delta\{f, g\} = \{\Delta f, g\} + (-1)^{|f|+1}\{f, \Delta g\}$$

if and only if $\Delta^2 \in \mathrm{Der}(A)$

Proof. Using repeatedly the relation

$$(-)^{|f|}\{f, g\} = \Delta(fg) - (\Delta f)g - (-)^{|f|} f(\Delta g)$$

we get

$$\begin{aligned}(-)^{|f|}\Delta\{f, g\} &= \Delta^2(fg) - \Delta((\Delta f)g) - (-)^{|f|}\Delta(f(\Delta g))\\ (-)^{|f|+1}\{\Delta f, g\} &= \Delta((\Delta f)g) - (\Delta^2 f)g + (-)^{|f|}(\Delta f)(\Delta g)\\ \{f, \Delta g\} &= (-)^{|f|}\Delta(f(\Delta g)) - (-)^{|f|}(\Delta f)(\Delta g) - f(\Delta^2 g)\end{aligned}$$

adding the last two equation to the first one we have

$$(-)^{|f|}\Big(\Delta\{f, g\} - \{\Delta f, g\} - (-1)^{|f|+1}\{f, \Delta g\}\Big) = \Delta^2(fg) - (\Delta^2 f)g - f(\Delta^2 g)$$

□

Since we are interested in generators which square to zero (so Δ^2 is a differential operator of order 0) the last two properties are automatically fulfilled. Next we explore in which way two square zero generators of the same bracket can differ. But first some definitions.

Definition 2.5. By a **Poisson vector field** of a Gerstenhaber algebra A we mean a derivation $D \in \mathrm{Der}(A)$ which in addition satisfies:

$$D\{f,g\} = \{Df,g\} + (-1)^{|D|(|f|+1)}\{f,Dg\}$$

Every element $f \in A$ generates a poisson vector field of opposite parity by

$$X_f := \{f,\ \}$$

this is called the **Hamiltonian vector field of** f. A function $f \in A$ is called a **Casimir** if $X_f = 0$.

From the last proposition we immediately get the following

Corollary 2.6. *If* $\{,\}$ *is Gerstenhaber bracket and* $\Delta,\ \tilde{\Delta} \in \mathrm{SZG}_{\{,\}}$ *then* $\Delta - \tilde{\Delta}$ *is a poisson vector field. Spoken out: square zero generators of a gerstenhaber bracket can at most differ by a poisson vector field. Further if we add an odd poisson vector field* X *to a square zero generator* Δ *then* $(\Delta + X)^2 = 0$ *if and only if*

$$[\Delta, X] + \frac{1}{2}[X,X] = 0$$

Proof. To show the last equation we use the fact that in general $[F,F] = 2F^2$ for odd operators:

$$(\Delta + X)^2 = \frac{1}{2}[\Delta + X, \Delta + X] = [\Delta, X] + \frac{1}{2}[X,X]$$

□

If we only consider hamiltonian vector fields instead of poisson vector fields the last result can be formulated as:

Theorem 2.7. *Let* Δ *be a square zero generators of a Gerstenhaber bracket and* $f \in A$ *an even fuction. Then* $\Delta + X_f$ *is of square zero if and only if the function*

$$\Delta f + \frac{1}{2}\{f,f\}$$

is a casimir

In particular if $f \in A$ satisfies the Maurer Cartan (or quantum master) equation

$$\Delta f + \frac{1}{2}\{f,f\} = 0$$

then $\Delta + X_f$ is also a generator.

Before we proof this we need a fact about X_f

Proposition 2.8. $X_{\{f,g\}} = [X_f, X_g]$

Proof.

$$\begin{aligned} X_{\{f,g\}}(h) &= \{\{f,g\},h\} \\ &= \{f,\{g,h\}\} - (-)^{(|f|+1)(|g|+1)}\{g,\{f,h\}\} \\ &= [X_f, X_g](h) \end{aligned}$$

□

Proof of the theorem. Using the derivation property of Δ with respect to the bracket we first get

$$[\Delta, X_f](h) = \Delta\{f, h\} + \{f, \Delta h\} = \{\Delta f, h\} = X_{\Delta f}(h)$$

and using this fact and the last proposition

$$[\Delta, X_f] + \frac{1}{2}[X_f, X_f] = X_{\Delta f + \frac{1}{2}\{f,f\}}$$

□

It will also be useful to note that

Proposition 2.9. *The map $A \to \mathrm{Der}(A)$, $f \mapsto X_f$ is a derivation i.e.*

$$X_{fg} = fX_g + (-1)^{|f||g|} gX_f$$

and therefore a differential operator of order less then 1

Proof.

$$\begin{aligned} X_{fg}(h) &= \{fg, h\} \\ &= f\{g, h\} + (-)^{|f||g|} g\{f, h\} = fX_g + (-1)^{|f||g|} gX_f \end{aligned}$$

□

3 Generators defined by Divergence operators and Berezinian sections

There is a natural method to get a generating operator on a Gerstenhaber Algebra if we are given a volume form (generator of the Berezinian bundle). This is explained in this section.

Definition 3.1. A **Divergence operator** is a even $\mathbb{R}$-linear map

$$\mathrm{div} : \mathrm{Der}(A) \to A$$

which satisfies:

$$\mathrm{div}(aX) = a \cdot \mathrm{div}(X) + (-1)^{|a||X|} X(a)$$

Proposition 3.2. *A divergence operator is a differential operator of order less then 1*

The proof is a straightforward computation

Proposition 3.3. *Given a Gerstenhaber bracket and a divergence operator then the operator*

$$\Delta(f) := (-1)^{|f|} \frac{1}{2} \mathrm{div}(X_f)$$

is a generator (not neccesarily square zero) of the bracket.

Proof. First note that Δ is odd and $\Delta(1) = 0$ since $X_1 = 0$. By the two previous propositions Δ is a differential operator of order less than 2. Next we compute

$$\begin{aligned} (-)^{|f|+|g|}\Delta(fg) &= \frac{1}{2}\mathrm{div}(X_{fg}) = \frac{1}{2}\mathrm{div}(fX_g + (-)^{|f||g|} gX_f) \\ &= (-)^{|g|} f\Delta g + (-)^{|f|+|g|}(\Delta f)g + \frac{1}{2}(-)^{|f|(|g|+1)}\{g, f\} + \frac{1}{2}(-)^{|g|}\{f, g\} \\ &= (-)^{|g|} f\Delta g + (-)^{|f|+|g|}(\Delta f)g + (-)^{|g|}\{f, g\} \end{aligned}$$

which proves the proposition. □

Note that divergence operators form an affine space over even one forms $\Lambda^1(A)$. So given a divergence operator div and an even function $f \in A$ we can construct a new divergence operator $\mathrm{div}' = \mathrm{div} + df$ where df is the differential of f.

Proposition 3.4. *If two divergence operators are related by*

$$\mathrm{div}' = \mathrm{div} + df$$

where f is an even function then the corresponding generating operators are related by the formula

$$\Delta' = \Delta + X_{\frac{1}{2}f}$$

Proof.

$$\begin{aligned}\Delta'(h) &= (-)^{|h|}\frac{1}{2}\left(\mathrm{div}(X_h) + df(X_h)\right)\\ &= (-)^{|h|}\frac{1}{2}\left(\mathrm{div}(X_h) + (-)^{|h|}\{f,h\}\right)\\ &= \Delta(h) + X_{\frac{1}{2}f}(h)\end{aligned}$$

□

Suppose now that the Berezinian Bundle of A is freely generated by one even element (this happens when the underlying even manifold is orientable). Then given a Berezinian Volume form ρ (i.e. an even generator of the Berezinian bundle) we can associate to every derivation $X \in \mathrm{Der}(A)$ a function (denoted by $\mathrm{div}_\rho(X) \in A$) by the formula:

$$L_X\rho = \mathrm{div}_\rho(X)\cdot\rho$$

where L_X is the Lie derivative of X.

Proposition 3.5. *The map*

$$\mathrm{div}_\rho : \mathrm{Der}(A) \to A$$

defined above is a divergence operator

For the proof we need some facts about the Lie derivative on the Berezin bundle. Proofs can be found in [Del]

Proposition 3.6. *For $X \in \mathrm{Der}(A)$, $f \in A$ and ρ a section of the Berezin bundle we have*

$$L_X(f\rho) = X(f)\rho + (-)^{|X||f|}fL_X\rho$$
$$L_{fX}(\rho) = (-)^{|X||f|}L_X(f\rho)$$

Now we prove the last proposition

Proof. It is clear that the map is even and $\mathbb{R}$ linear. Now

$$\begin{aligned}\mathrm{div}_\rho(fX)\rho = L_{fX}\rho &= (-)^{|f||X|}L_X(f\rho)\\ &= (-)^{|X||f|}X(f)\rho + fL_X\rho\\ &= \left((-)^{|X||f|}X(f) + f\mathrm{div}_\rho(X)\right)\rho\end{aligned}$$

□

We denote the generating operator induced by div_ρ with Δ_ρ. Now suppose we are given two Berezinian volume forms ρ and ρ'. Then $\rho' = f\rho$ for some even invertible $f \in A$. Recall that we can also write $f = \pm e^{2g}$ for some even $g \in A$ (see Appendix 2). For simplicity we assume that f has positive bosonic part (this situation appears when we work in local coordinates and make a orientation preserving chart change).

Proposition 3.7. *In the situation above the induced divergence operators are related by:*

$$\mathrm{div}_{\rho'} = \mathrm{div}_\rho + f^{-1}df$$

or equivalently

$$\mathrm{div}_{\rho'} = \mathrm{div}_\rho + 2dg$$

and the Laplacians by:

$$\Delta_{\rho'} = \Delta_\rho + X_g$$

Proof.

$$\begin{aligned}\mathrm{div}_{\rho'}(X)f\rho = \mathrm{div}_{\rho'}(X)\rho' &= L_X(\rho') \\ &= L_X(f\rho) = X(f)\rho + (-)^{|X||f|}fL_X\rho \\ &= (-)^{|X||f|}\left(df(X) + f\mathrm{div}_\rho(X)\right)\rho\end{aligned}$$

multiplying both sides with f^{-1} we get the result. For the second equation we use $de^{2g} = e^{2g}d(2g)$. And the last one follows from previous results. □

3.1 Laplacian on w-densities

In this section we shall work in local coordinates i.e. on the super domains $\mathbb{R}^{n|m}$, and the algebras $A = C^\infty(\mathbb{R}^{n|m})$. Denote the standard coordinates on $\mathbb{R}^{n|m}$ with x, ξ and the volume form induced by this coordinates with D. We will define densities of arbitrary weight $w \in \mathbb{R}$. For more details see [KV]. We first note that if $f \in A$ is invertible with positive bosonic part we can define f^w by first writing $f = e^g$ (see the second Appendix) and putting

$$f^w := e^{wg}$$

Definition 3.8. The module of densities of weight w is the free module generated by the one even element D^w, i.e. every w-density can be written as $\sigma = f(x,\xi)D^w$ where $f \in A$. Under orientation preserving cooordinate changes $x, \xi \mapsto \tilde{x}, \tilde{\xi}$ the new generator $\tilde{D}^w$ is related to the old one by the transformation formula

$$D = \left(\mathrm{Ber}J(\frac{x,\xi}{\tilde{x},\tilde{\xi}})\right)^w \tilde{D}^w$$

Here Ber is the Berezin determinant and $J(\frac{x,\xi}{\tilde{x},\tilde{\xi}})$ is the Jacobi matrix of the transformation.

So densities of weight 0 are functions and densities of weigt 1 are sections of the Berezin bundle. What is important for us is that if ρ is a berezinian volume form $\rho = fD$ where f is even invertible and has positive bosonic part then

$$\rho^w = f^w \cdot D^w$$

Khudaverdian and Voronov [KV] propose to extend the operator Δ_ρ of the previous section to densities of arbitrary weight as follows. Given a volume form ρ

any w-density σ can be uniquely written as $\sigma = f \cdot \rho^w$ with $f \in A$, we define the laplace operator $\Delta_{\rho,w}$ on w-densities by

$$\Delta_{\rho,w}(\sigma) := \Delta_\rho(f)\rho^w$$

Where Δ_ρ is the laplacian on functions. One sees immediately that the so defined operator is a second order linear differential operator on the module of densities of weight w.

Theorem 3.9 (Transformation formula of Khudaverdian). *If we have two Berezinian volume forms related by $\rho' := e^{2g}\rho$ with $g \in A$ even, then:*

$$\Delta_{\rho',w} = \Delta_{\rho,w} + (1-2w)L_g - 4w(1-w)\left(\Delta_\rho(g) + \frac{1}{2}\{g,g\}\right)$$

where the second term is the Lie derivative on w desities of X_g and the last term is a zero order operator (multiplication with the term).

One sees from this formula that half densities are privileged since the laplacian on them only changes by a zero order term. Before we proof it we need

Lemma 3.10. *Let $g \in A$ be even. Then:*

$$\Delta(e^g) = \left(\Delta(g) + \frac{1}{2}\{g,g\}\right)e^g$$

Proof. By induction one can easily prove that:

$$\Delta(g^n) = ng^{n-1}\Delta g + \frac{n(n-1)}{2}g^{n-2}\{g,g\}$$

(use the relation $\Delta(fg) = \Delta(f)g + (-)^{|f|}f\Delta g + (-)^{|f|}\{f,g\}$) Then the proposition follows from this. □

We also need some facts on the Lie derivative on w-densities, namely

$$L_X(f\rho^w) = X(f)\rho^w + (-)^{|f||X|}fL_X(\rho^w)$$

and if ρ is a Berezinian volume form and $g \in A$ is even then

$$L_g(\rho^w) = w\rho^{w-1}L_g(\rho) = w\mathrm{div}_\rho(X_g)\rho^w = 2w\Delta_\rho(g)\rho^w$$

from these equations we get that if $\sigma = f \cdot \rho^w$ is any density of weight w then

$$L_g(\sigma) = X_g(f)\rho^w + (-)^{|f|}2wf\Delta_\rho(g)\rho^w$$

Proof of the theorem. Let $\sigma = f \cdot \rho^w$ with f even. Then

$$\sigma = (fe^{-w2g})\rho'^w$$

and so

$$
\begin{aligned}
&\Delta_{\rho',w}(\sigma) - \Delta_{\rho,w}(\sigma) - L_g(\sigma) = \\
&\quad =\Delta_{\rho'}(fe^{-w2g})\rho'^w - \Delta_{\rho,w}(\sigma) - L_g(\sigma) \\
&\quad = \left(\Delta_\rho(fe^{-w2g}) + X_g(fe^{-w2g})\right)\rho'^w - \Delta_\rho(f)\rho^w - L_g(\sigma) \\
&\quad = \left(f\Delta_\rho(e^{-w2g}) + \{f, e^{-2wg}\} + fX_g(e^{-2wg})\right)\rho'^w - 2wf\Delta_\rho(g)\rho^w \\
&\quad = \left(f\Delta_\rho(-2wg) + \frac{1}{2}f\{2wg, 2wg\} - 2w\{f,g\} - 2wf\{g,g\} - 2wf\Delta_\rho(g)\right)\rho^w \\
&\quad = -2w\left(f\Delta_\rho(g) - wf\{g,g\} + X_g(f) + f\{g,g\} + f\Delta_\rho(g)\right)\rho^w \\
&\quad = -2w\left(X_g(f) + f\Delta_\rho(g) + f\Delta_\rho(g) + (1-w)\{g,g\}f\right)\rho^w \\
&\quad = -2w\left(X_g(f) + 2w\Delta_\rho(g)f + ((2-2w)\Delta_\rho(g) + (1-w)\{g,g\})\,f\right)\rho^w \\
&\quad = -2w\left(X_g(f) + 2wf\Delta_\rho(g) + 2(1-w)\left(\Delta_\rho(g) + \frac{1}{2}\{g,g\}\right)f\right)\rho^w \\
&\quad = -2wL_g(\sigma) - 4w(1-w)\left(\Delta_\rho(g) + \frac{1}{2}\{g,g\}\right)\sigma
\end{aligned}
$$

□

4 Some physical background to Batalin-Vilkovisky formalism

This section is intended to explain (to a small extent) how the Batalin-Vilkovisky formalism appears in physics.

4.1 Faddeev-Popov quantization and BRST symmetry

In field theory one is given a set of fields M, an action functional $S : M \to \mathbb{R}$ on the set of fields and a group of symmetries G acting on M. The action S is invariant under the operation of the group: $S[\psi] = S[g\psi]$ for $\psi \in M$, $g \in G$, and one thinks of two fields as having the same physical content if they lie in the same orbit of the group operation. In classical (not quantized) field theory, the evolution of the system is given by critical "points" of the action. In quantum field theory one is interested in computing the expectation value of observables. An observable is a function on the set of fields $O : M \to \mathbb{R}$ invariant under the operation of the group. The expectation is computed (following an idea of Feynman) as

$$\langle O \rangle = | \int_M O e^{\frac{i}{\hbar}S} D\psi |$$

where the integral $D\psi$ over all fields is not defined as in measure theory but is computed by complicated methods depending on the specific field theory, and is maybe the central mystery surrounding quantum field theory from a mathematicians point of view. Intuitively one should think of the functional integral as *summing up* the contributions $O[\psi]e^{\frac{i}{\hbar}S[\psi]}$ for every field ψ.

One common method for computing it goes under the name of perturbative expansion. One starts by learning to compute the integral when the action is quadratic (i.e. $S[\psi] = \langle \psi, A\psi \rangle$, where A is a positive definite symmetric operator on the set of fields and $\langle , \rangle$ is a scalar product). These integrals are called Gaussian integrals. The next case is to allow more general actions S which, as the quadratic ones only have one nondegenerate critical point. In that case the action and the observable are

Taylor expanded around that critical point and each term is computed using gaussian integrals. The method is called perturbative expansion because S is considered to be a perturbation of its quadratic part d^2S.

But in the case the action has symmetries the critical points are certainly degenerate. The first idea would be to compute the integral

$$I := \int_{M/G} \overline{O} e^{\frac{i}{\hbar}\overline{S}} D[\psi]$$

Where M/G is the orbit space and $\overline{O}$, $\overline{S}$ are the functions O, S considered as functions on M/G. This is also physically more plausible since two fields in the same orbit are actually physically equivalent. But the space M/G is very difficult to handle, so physicists devised a method called Faddeev-Popov quantization which shall be sketched below.

To capture the idea we shall work with at a toy model where M is a finite dimensional smooth manifold equipped with a volume form μ representing $D\psi$. The action shall be a smooth function on our manifold $S \in C^\infty(M)$ and the symmetries are given by the operation of a Lie Algebra $\mathfrak{g} \to \mathrm{Vect}(M)$, $\gamma \mapsto X_\gamma$. The action being invariant under the operation translates to the condition.

$$X_\gamma(S) = 0$$

for $\gamma \in \mathfrak{g}$.

Instead of computing the functional integral over the orbit space one tries to choose a set of representatives of the orbits and compute the integral over the resulting "submanifold" of M. Such a set of representatives is in practice given by the set of zeroes of a function

$$F : M \to \mathfrak{g}$$

the function is called gauge fixing function. The integral can then be expressed as

$$I = \int_M O e^{\frac{i}{\hbar}S} \delta_0(F) \det(A)\mu$$

where δ_0 is the delta function at $0 \in \mathfrak{g}$ and $A : M \to \mathrm{End}(\mathfrak{g})$ is given by $A(p)\gamma = dF_p(X_\gamma(p))$ This integral can again be written differently as the integral over a super manifold. Let's define this super manifold by the function algebra

$$\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes_{\mathbb{R}} C^\infty(M \times \mathfrak{g}^*)$$

as a supermanifold it is written as $\tilde{M} = \Pi\mathfrak{g}^* \times \Pi\mathfrak{g} \times M \times \mathfrak{g}^*$ it evidently contains M as a submanifold but there is also a projection onto M making it possible to consider the action S and all observables on M as functions on $\tilde{M}$. We define a new measure on $\tilde{M}$ by

$$D = \tilde{\omega} \otimes \mu \otimes \omega$$

where we chose a $\omega \in \Lambda^{\mathrm{top}}(\mathfrak{g}^*)$ to give a measure $\mu \otimes \omega$ on $M \times \mathfrak{g}^*$. And where $\tilde{\omega}$ acts by first projecting $\Lambda(\mathfrak{g}^* \oplus \mathfrak{g})$ canonically to $\Lambda^{\mathrm{top}}(\mathfrak{g}^* \oplus \mathfrak{g})$ and then projecting onto $\mathbb{R}$ by the canonical (up to a sign) identification. Now note that $\mathfrak{g}^* \otimes \mathfrak{g}$ can be canonically imbedded in $\Lambda^2(\mathfrak{g}^* \oplus \mathfrak{g})$ and thereby A can be considered an element of $\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(M \times \mathfrak{g}^*)$. The gauge fixing function F is considered as an element of $C^\infty(M \times \mathfrak{g}^*)$ by $F(p, g^*) = g^*(F(p))$. Now using the fact that

$$\int_{\Pi\mathfrak{g} \times \Pi\mathfrak{g}^*} e^A = \det(A)$$

(here the exponential function is defined by the infinite sum and A is considered as an element of the function algebra $\Lambda(\mathfrak{g}^* \oplus \mathfrak{g})$) and using the Fourier transform of the delta function the integral becomes.

$$I = \int_{\tilde{M}} O e^{\frac{i}{\hbar} S + F + A} D$$

we can define the **gauge fixed action**

$$S_F := S + \frac{\hbar}{i} F + \frac{\hbar}{i} A \in C^\infty(\tilde{M})$$

it can be considered a action on the superspace. The integral is now written

$$I = \int_{\tilde{M}} O e^{\frac{i}{\hbar} S_F} D$$

and looks just like doing QFT on "supermanifolds". One has the possibility now to allow even more observables since we have expanded $C^\infty(M)$ to the function algebra on the super space. But what happened with the symmetries? This is where the BRST symmetry comes in (which will also be useful for answering the question whether the above expression is independent of the chosen gauge fixing).

Definition 4.1. We shall define a derivation on $C^\infty(\tilde{M})$

$$\delta : \Lambda(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(M \times \mathfrak{g}^*) \to \Lambda(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(M \times \mathfrak{g}^*)$$

called the **BRST operator**. To this end first fix a basis $e_1, \dots e_m$ in $\mathfrak{g}$ and let e^i be the dual basis, let f^i_{jk} denote corresponding structure constants. With this basis we introduce coordinates c^i and $\bar{c}_i$ in $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ which become odd functions on $\tilde{M}$. A point $(p, \gamma^*) \in M \times \mathfrak{g}^*$ shall be written as $(p, \lambda_1, \dots \lambda_m)$ where λ_i are coordinates on $\mathfrak{g}^*$ induced by the basis e^i. With these choices a function on $\tilde{M}$ can be written as

$$f = \sum f_{I,\bar{I}}(p, \lambda_1, \dots \lambda_m) c^I \ \bar{c}_{\bar{I}}$$

where I and $\bar{I}$ are ordered subsets of $\{1, \dots, m\}$ and $f_{I,\bar{I}} \in C^\infty(M \times \mathfrak{g}^*)$. We define δ in these coordinate by

$$\delta c^i = -\frac{1}{2} f^i_{jk} c^j c^k$$

$$\delta f(p) = \sum_i X_{e_i}(f) c^i$$

$$\delta \bar{c}_i = \lambda_i, \ \delta \lambda_i = 0$$

and one immediately checks that the definition is independent of the choice of basis in $\mathfrak{g}$

By the definition it is evident that δ is odd and one verifies that $\delta^2 = 0$. Another easily recognized fact is that $f \in C^\infty(M)$ is invariant under the operation of the Lie algebra (i.e. is an observable in the original sense) iff $\delta(f) = 0$. For the next central theorem it is useful to write S_F differently. Namely the information contained in the gauge fixing F can also be represented differently as an element of $C^\infty(\tilde{M})$ by defining the function

$$\Psi_F = F^i(p) \bar{c}_i$$

called the gauge-fixing fermion. One verifies that $S_F = S + \delta \Psi_F$

Theorem 4.2. *If δ is divergence free i.e. $div(\delta) = 0$ and $g \in C^\infty(\tilde{M})$ satisfies $\delta(g) = 0$ then*

$$\int_{\tilde{M}} e^{\frac{i}{\hbar}S_F} gD$$

is gauge fixing independent. If moreover $g = \delta f$ for some $f \in C^\infty(\tilde{M})$ then

$$\int_{\tilde{M}} e^{\frac{i}{\hbar}S_F} gD = 0$$

Skechy Proof. Let F_0 and F_1 be two gauge fixing functions and let F_t be a family of gauge fixing functions connecting the two. Then (using $S_F = S + \delta\Psi_F$) we get

$$\begin{aligned}
\frac{d}{dt}\int_{\tilde{M}} e^{\frac{i}{\hbar}S_{F_t}} gD &= \int_{\tilde{M}} \frac{d}{dt} e^{\frac{i}{\hbar}S_{F_t}} gD \\
&= \frac{i}{\hbar}\int_{\tilde{M}} (\frac{d}{dt}\delta\Psi_{F_t}) e^{\frac{i}{\hbar}S_{F_t}} gD = \frac{i}{\hbar}\int_{\tilde{M}} \delta(\frac{d}{dt}\Psi_{F_t}) e^{\frac{i}{\hbar}S_{F_t}} gD \\
&= \frac{i}{\hbar}\int_{\tilde{M}} \delta(\frac{d}{dt}\Psi_{F_t} e^{\frac{i}{\hbar}S_{F_t}} g) D = \frac{i}{\hbar}\int_{\tilde{M}} \mathrm{div}(\delta)(\frac{d}{dt}\Psi_{F_t} e^{\frac{i}{\hbar}S_{F_t}} g) D \\
&= 0
\end{aligned}$$

□

The last theorem suggests to define the *extended observables* of our physical theory as the cohomology of the BRST operator $\ker(\delta)/\mathrm{im}(\delta)$.

4.2 BV formalism for quantization

In the BV quantization one extends the space $\tilde{M}$ into an even bigger super space N which has a odd poisson bracket on it. This is achieved by taking for example the space $\Pi T^*\tilde{M}$, whose function algebra are the multivectorfields with their canonical Schouten bracket (see [Vall] for a general definition of this space and the bracket when $\tilde{M}$ is a supermanifold, there one also finds a constructions of generating operators for the bracket starting from a divergence operator on $\tilde{M}$). The action S on M is extended to an action $\tilde{S}$ on N and a gauge fixing fermion Ψ_F from before now defines a lagrangian submanifold L_F of N. One gets that

$$S_F = \tilde{S}|_{L_\psi}$$

One can also show that $d\pi(X_{\tilde{S}}) = \delta$ where $\pi : N \to \tilde{M}$ is the canonical projection We assume that we are given a berezinian on N extending the one on $\tilde{M}$ so that the associated laplacian satisfies $\Delta^2 = 0$. The expectation value of an observable can then be written as an integral over the Lagrangian submanifold

$$\int_{L_F} e^{\frac{i}{\hbar}\tilde{S}} g\tilde{D_L}$$

Where $\tilde{D_L}$ is the volume form on the lagrangian submanifold. One can show as before that the integral is independent of the gauge fixing if in addition $\Delta(e^{\frac{i}{\hbar}\tilde{S}}g) = 0$. Schwarz shows [Schw] that more generally if one is given two lagrangian submanifolds of an odd *symplectic* manifold whose bosonic parts are homologuos, then $\int_{L_1} H\tilde{D}_1 = \int_{L_2} H\tilde{D}_2$ if $\Delta(H) = 0$.

Proposition 4.3. *Let A, Δ be a smooth BV algebra and $\Sigma \in A$, then $\Delta e^{\frac{i}{\hbar}\Sigma} = 0$ is equivalent to the quantum master equation*

$$\{\Sigma, \Sigma\} - 2i\hbar\Delta\Sigma = 0$$

Proof. This follows from the equation $\Delta(e^f) = (\Delta(f) + \frac{1}{2}\{f, f\})e^f$ □

Definition 4.4. An even element $S \in \mathbb{C} \otimes A =: A_{\mathbb{C}}$ satisfying the **quantum master equation**:

$$\{S, S\} - 2i\hbar\Delta S = 0$$

is called a **quantum action**. Given a quantum action we define a **quantum coboundary operator** $\Omega : A_{\mathbb{C}} \to A_{\mathbb{C}}$ associated to S by

$$\Omega(f) := \{S, f\} - i\hbar\Delta f$$

Proposition 4.5. *Let S be a quantum action and Ω the associatet coboundary operator then for $f \in A$: $\Delta(fe^{\frac{i}{\hbar}S}) = 0 \Leftrightarrow \Omega(f) = 0$*

Proof.

$$\begin{aligned}
\Delta(fe^{\frac{i}{\hbar}S}) =& \Delta(f)e^{\frac{i}{\hbar}S} + f\underbrace{\Delta(e^{\frac{i}{\hbar}S})}_{=0} + (-)^{|f|}\{f, e^{\frac{i}{\hbar}S}\} \\
=& \left(\Delta(f) + (-)^{|f|}\frac{i}{\hbar}\{f, S\}\right)e^{\frac{i}{\hbar}S} \\
=& \frac{i}{\hbar}\left(\{S, f\} - i\hbar\Delta f\right)e^{\frac{i}{\hbar}S}
\end{aligned}$$

□

Proposition 4.6. *The quantum coboundary operator squares to zero:*

$$\Omega^2 = 0$$

Proof.

$$\begin{aligned}
\Omega^2 f &= \{S, (\{S, f\} - i\hbar\Delta f)\} - i\hbar\Delta(\{S, f\} - i\hbar\Delta f) \\
&= \{S, \{S, f\}\} - i\hbar\{S, \Delta f\} - i\hbar\Delta\{S, f\} - \hbar^2\underbrace{\Delta^2(f)}_{=0} \\
&= \frac{1}{2}\{\{S, S\}, f\} - i\hbar\{S, \Delta f\} - i\hbar\Delta\{S, f\} \\
&= i\hbar\{\Delta S, f\} - i\hbar\{S, \Delta f\} - i\hbar\Delta\{S, f\} \\
&= 0
\end{aligned}$$

□

The elements in the homology of Ω are called **quantum observables**.

5 Appendix: Introduction to Supermanifolds

The main notions of differential geometry can be considered as a part of commutative algebra by replacing a manifold M with the algebra $C^\infty(M)$ and a vector bundle $\pi : E \to M$ with the $C^\infty(M)$ module of sections $\Gamma(E)$. See [Nes] for this approach. In theoretical physics the notion of "functions" which anticommute arose i.e. $fg = -gf$. This among other things gave birth to super- or $\mathbb{Z}_2$ graded mathematics [Ber] [Bar]. One might say that the idea here is to replace the commutative algebra $C^\infty(M)$ by a "weakly" non commutative algebra A (specifically a $\mathbb{Z}_2$ graded commutative algebra) which is to be thought of as the "function algebra" of a superspace. Of course no space has functions with values in $\mathbb{R}$ which don't commute, so this space does not exist in the same sense as a manifold. Yet it seems that all the main notions known from differential geometry like vector fields, differential forms, linear differential operators, integration etc. may all be defined for these specific non commutative algebras similarly as in the commutative case. So we will only work with the "functions" on the superspace, and will never try to construct this space. Roughly speaking the whole theory might be summarized as commutative algebra with the addition of the *"sign rule"*: Whenever exchanging two adjacent objects in an expression which are both odd one catches a minus sign.

The difference of this presentation with the usual ones is that I tried to formulate the theory in a purely algebraic way in contrast to the usual approach through sheafs. That means we only work with the algebra of global functions. The belief is that this approach is as powerful as the usual one without the extra layer of machinery, and it may be extendible to the "infinite dimensional" case, i.e. when we are dealing with spaces of functions. The purely algebraic approach to supermathematics is not new [Verb] and is suggested by the mathematicians around the Diffiety institute. From the algebraic viewpoint one might say that the central notion of calculus is that of a linear differential operator. Most other concepts are seen to follow "functorially" from this one.

If this is the first contact of the reader with supermathematics its recommended to read the first section parallel to the second one. Proofs which consist of direct computations are skipped in this appendix.

5.1 The purely algebraic framework for Supermathematics

Conventions: As base field for linear spaces or other algebraic constructions we will always use $\mathbb{R}$. When I speak of a graded algebraic object I mean $\mathbb{Z}_2$-graded unless otherwise stated. The elements of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ are denoted by their representatives 0 and 1. The decomposition of an element a of some graded space into even and odd part is usually written as $a = a_0 + a_1$.

In this section we summarize main notions of "super commutative" algebra. These will be graded commutative algebras (functions of our superspace), graded modules (sections of vectorbundles), and graded Lie algebras (infinitesimal symmetries). But we will start by recalling some details of graded linear spaces. These are of importance since all the objects mentioned before have the underlying structure of graded linear spaces.

5.1.1 Graded Linear Algebra

Definition 5.1. A $\mathbb{Z}_2$**-graded Vector space** consist of a real vector space V together with a linear map $P : V \to V$ called **parity operator** which satisfies $P^2 = id_V$.

It is a fact that such an operator P decomposes our vector space into a direct sum $V = V_0 \oplus V_1$ (the indices should be thought of as the elements of $\mathbb{Z}_2$) where V_0 is the eigenspace to the eigenvalue 1 and its elements are called **even**, while V_1 is the eigenspace to the eigenvalue -1 and its elements are called **odd**. An element of $V_0 \cup V_1$ is called **homogenous**. If $x \in V\backslash\{0\}$ is homogeneous we denote its **parity** (or degree) by

$$|x| := \begin{cases} 0 & \text{if } x \in V_0 \\ 1 & \text{if } x \in V_1 \end{cases}$$

Equivalently we could have defined a graded vector space as a space given with a decomposition into a direct sum and a choice of parity.

Let V and W be to graded vector spaces. The vector space $\mathrm{Hom}_{\mathbb{R}}(V, W)$ of linear maps is then naturally supplied with a $\mathbb{Z}_2$ grading, namely $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, W)$

is called even if it respects the parity: $\phi(V_i) \subset \phi(W_i)$ where $i \in \mathbb{Z}_2$ (equivalently $\phi \circ P_V = P_W \circ \phi$). It is called odd if it reverses parity: $\phi(V_i) \subset \phi(W_{i+1})$ (equivalently $\varphi \circ -P_V = P_W \circ \varphi$), and one checks that every linear map can be decomposed uniquely into even and odd part $\varphi = \varphi_0 + \varphi_1$.

It is natural to consider two graded vector spaces as isomorphic if there is a *parity preserving* isomorphism of vector spaces. This leads us to consider the following definition.

Definition 5.2. A **morphism of graded vector spaces** is a even linear map. The set of these morphisms is denoted by $\mathrm{Hom}_{\mathbb{R}}(V,W)_0$

Having defined this category, common notions of linear algebra such as subspace, quotien, product of graded spaces, multi-linear algebra and tensor-product of graded linear spaces follow naturally, and we will have a look at some of them now.

Definition 5.3. A **graded subspace** of a graded linear space $V = V_0 \oplus V_1$ is a subspace $U \subset V$ of the form $U = U_0 + U_1$ where U_0 is a subspace of V_0 and U_1 is a subspace of V_1. The grading of U is then the one induced by V.

Note that not all subspaces are graded subspaces.

Proposition 5.4. *If U is a graded subspace of V then the* ***quotient*** *V/U is a graded linear space and the canonical projection $\pi : V \to V/U$ is a morphism of graded spaces. Given any morphism $\varphi : V \to W$ graded spaces, then* $\ker(\varphi)$ *and* $\mathrm{im}(\varphi)$ *are graded subspaces and the morphism φ factors uniquely through the projection: $\varphi = \psi \circ \pi$ with ψ even.*

Proof. We define the even resp. odd elements of V/U by $\pi(V_0)$ resp. $\pi(V_1)$. These two spaces span V/U since $v + U = (v_0 + v_1) + U = (v_0 + U) + (v_1 + U)$. They are disjoint since from $v_0 + U = v_1 + U$ follows $v_0 - v_1 \in U$ an therefore $v_0 \in U_0, v_1 \in U_1$ so $v_0 + U = v_1 + U = 0$. By definition of this grading π is parity preserving.

The other statements follow from noting that $\ker(\varphi) = \ker(\varphi|_{V_0}) \oplus \ker(\varphi|_{V_1})$ and $\mathrm{im}(\varphi) = \mathrm{im}(\varphi|_{V_0}) \oplus \mathrm{im}(\varphi|_{V_1})$. The unique factorization follows analogous to common linear algebra. □

Definition 5.5. The **graded sum** $V \oplus W$ of two graded spaces V and W is given by the usual sum, and the grading is given by $(V \oplus W)_0 := V_0 \oplus W_0$ and $(V \oplus W)_1 := V_1 \oplus W_1$. As usual there are the canonical projections π_V, π_W and injections i_V, i_W which are even mappings.

Proposition 5.6. *$V \oplus W, \pi_V, \pi_W$ satisfies the universal property of the product in the category of graded spaces. $V \oplus W, i_V, i_W$ satisfies the universal property of the coproduct.*

Proof. Let U be another graded vector space and $\psi_1 : U \to V$, $\psi_2 : U \to W$ even morphisms. Since we can consider everything as ungraded objects, then by the universal property of the sum of vector spaces there is a unique vector space morphism $\phi : U \to V \oplus W$ such that $\pi_i \circ \phi = \psi_i$. So all that remains to check is that ϕ is parity preserving. But this follows by recalling that $\phi(v, w) = \psi_1(v) + \psi_2(w)$. The result about the coproduct follows analogously. □

Lets also recall that bilinear maps $\beta : V \times W \to U$, where V, W, U are classical vector spaces, correspond bijectively to linear maps $\tilde{\beta} : V \to \mathrm{Hom}_{\mathbb{R}}(W,U)$, i.e. there is a natural isomorphims:

$$\mathrm{Bil}_{\mathbb{R}}(V,W;U) \simeq \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{R}}(W,U))$$

Now if V, W, U are graded then the r.h.s is naturally graded and so we get a grading on the bilinear maps. Concretely on verifies that β is even iff

$$\beta(V_i, W_j) \subset U_{i+j} \ i, j \in \mathbb{Z}_2$$

and odd iff

$$\beta(V_i, W_j) \subset U_{i+j+1} \ i, j \in \mathbb{Z}_2$$

5.1.2 Supermanifolds

Now we turn to graded commutative algebras. These should be thought of as "function algebras" $C^\infty(S)$ of our "super space" S. The space S will not be defined but it is useful to think in analogy to the ungraded situation where S is a manifold.

Definition 5.7. By a **graded commutative Algebra** A we shall mean an associative $\mathbb{R}$-algebra with unit, for which the underlying vector space is graded $A = A_0 \oplus A_1$. The multiplication of the algebra is demanded to satisfy:

i) $A_i A_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$ (even operation)

ii) $ab = (-1)^{|a||b|} ba$ for homogeneous elements (graded commutative)

The morphisms of graded Algebras are even algebra morphisms.

Note that the non-commutativity of the algebra is in a sense very mild. We shall sometimes call an element $f \in A$ a **function**.

Definition 5.8. A **graded subalgebra** (for short subalgebra) is a subalgebra of A (containing the unit) which is also a graded subspace.

Subalgebras should be thought of as foliations (or quotients) of our space, i.e. as those functions which are constant on each leaf of the foliation.

Definition 5.9. By a **graded Ideal** we mean a left ideal that is a graded subspace.

Note that a left ideal of this form is also a right ideal and conversely (check it first for homogeneous elements, and decompose general elements). We can consider the "subspace" of our superspace given by the quotient algebra A/I. So the ideal is thought of as those functions which vanish on the subspace.

Proposition 5.10. *If $\varphi : A \to B$ is a morphism of graded algebras, then* $\mathrm{im}(\varphi)$ *is a graded subalgebra of B and* $\ker\varphi$ *is a graded ideal of A. Also the canonical inclusion map of a subalgebra is a morphism of graded algebras, and if I is a graded Ideal then the quotient A/I is a graded algebra and the canonical projection is a morphism of graded algebras.*

Proof. For the first statement, it is clear that $\mathrm{im}(\varphi)$ is a subalgebra of B, it remains to check that it is a graded one. So let $b = b_0 + b_1 \in \mathrm{im}(\varphi)$ then there is a $a = a_0 + a_1 \in A$ with $\varphi(a) = b$ but since φ is even and the decomposition of b in even and odd part is unique it follows that $\varphi(a_0) = b_0$ and $\varphi(a_1) = b_1$ and so $b_0, b_1 \in \mathrm{im}(\varphi)$. In the same manner one deduces that the kernel of a even morphism is a graded ideal. The remark concerning the inclusion map of a subalgebra is also clear. Let's now check that the quotient of a graded algebra by a graded ideal is again a graded algebra. From the section above it follows that A/I is a graded vector space. We define the multiplication as usual through representatives $[a] \cdot [b] := [a \cdot b]$. It is well defined and one easily verifies that it satisfies the axioms of a graded algebra. □

Let A be a graded commutative $\mathbb{R}$ algebra. Denote by

$$\mathcal{N} := \{f \in A \mid \exists n \in \mathbb{N} : f^n = 0\}$$

the set of nilpotent elements of the algebra.

Lemma 5.11. *$\mathcal{N}$ is a graded Ideal of A which contains the odd elements.*

First note that in general if $a = a_0 + a_1 \in A$ then $a^n = a_0^n + na_0^{n-1}a_1$. This follows by applying the binomial formula to $(a_0 + a_1)^n$ (which is allowed since even elements commute with odd ones) and using the fact that $a_1^2 = 0$. Now we proof the Lemma.

Proof. It is evident that the odd elements are nilpotent. Now suppose $a = a_0 + a_1 \in \mathcal{N}$ with $a^n = 0$ using the above formula we get $a_0^n = -na_0^{n-1}a_1$ so $a_0^{2n} = 0$ and we get that $\mathcal{N}$ is graded. Now let $a, b \in \mathcal{N}$ with $a^n = b^n = 0$ then $(a+b)^{4n+1} = (a_0+b_0)^{4n+1} + (4n+1)(a_0+b_0)^{4n}(a_1+b_1) = 0$ where we used the binomial formula for the last equality. So we get that the sum of two nilpotent elements is nilpotent. So it remains to check that $A \cdot \mathcal{N} \subset \mathcal{N}$. But it suffices to check this for homogeneous elements in $\mathcal{N}$ and for those it is evident. □

Now we define:

$$\mathcal{N}^k := \{\sum_{i=1}^{n} a_{i_1} a_{i_2} \cdots a_{i_k} \mid a_{i_j} \in \mathcal{N}\}$$

It is easily verified that $\mathcal{N}^k$ is a graded ideal in A and that

$$A \supset \mathcal{N} \supset \mathcal{N}^2 \supset \mathcal{N}^3 \ldots$$

this is a $\mathbb{Z}$ filtration on A which is compatible with the product, i.e.

$$\mathcal{N}^k \cdot \mathcal{N}^l \subset \mathcal{N}^{k+l}$$

This makes it possible to define the $\mathbb{Z}_2 \times \mathbb{Z}$ graded algebra (for a general definition of G-graded commutative algebras, where G is an abelian group, see [Verb]):

$$A/\mathcal{N} \otimes \mathcal{N}/\mathcal{N}^2 \otimes \mathcal{N}^2/\mathcal{N}^3 \otimes \ldots$$

where an element has degree $(l, k) \in \mathbb{Z}_2 \times \mathbb{Z}$ if it is contained in $\mathcal{N}_l^k/\mathcal{N}_l^{k+1}$, the lower index refers to the $\mathbb{Z}_2$ grading of A. The product in this algebra is defined by representatives, and it is graded commutative with respect to the commutation factor $(-1)^{ll'}$ (see [Verb]). The next lemma assures that this construction is functorial. More specifically we get a covariant functor from the $\mathbb{Z}_2$ graded commutative algebras to the $\mathbb{Z}_2 \times \mathbb{Z}$ graded commutative (with the commutation factor given before) algebras.

Lemma 5.12. *Let $\varphi : A \to B$ be a morphism of graded algebras then $\varphi(\mathcal{N}_i^k) \subset \mathcal{N}_i^k$*

Proof. The statement is true for $k = 1$ and follows almost immediately from this case for the situation where $k \geq 1$ □

Lemma 5.13. *The quotient $\tilde{A} := A/\mathcal{N}$ is a reduced commutative algebra, with the universal property that every algebra morphism $A \to B$ into a commutative reduced algebra B factors uniquely through the projection $A \to \tilde{A}$*

The proof is standard.

Remark 5.14. So $\tilde{A}$ may be considered the biggest classical submanifold contained in the supermanifold. Every other puerly even submanifold is contained in that one. Also note that we may speak of the points of this submanifold but not of a "point" in the superspace outside of $\tilde{A}$. Dan Freed [Freed] propses that one should picture a supermanifold as a classical manifold surrounded by some sort of superfuzz.

Now we can give a definition of Supermanifold. First note that if

$$\pi : E \to M$$

is a smooth vector bundle over a classical manifold M then $\Gamma(\Lambda E)$ is naturally a $\mathbb{Z}_2$ graded algebra by giving it the grading $\Gamma(\Lambda E)_i := \bigoplus_n \Lambda^{2n+i} E$.

Definition 5.15. A graded Algebra A shall be called a **smooth superalgebra** (or the function algebra of a supermanifold) if
i) $\tilde{A}$ is a smooth algebra (i.e. it is a function algebra of a smooth manifold [Nes])
ii) $\mathcal{N}/\mathcal{N}^2$ is finitely generated and projective as $\tilde{A}$-module
iii) A is isomorphic to $\Lambda_{\tilde{A}}\mathcal{N}/\mathcal{N}^2$ as $\mathbb{Z}_2$ graded Algebras.

The **dimension** of a super-manifold is the pair $n|d$ where n is the dimension of the manifold associated to $\tilde{A}$ and d is the rank of the vector-bundle associated to $\mathcal{N}/\mathcal{N}^2$. The classical manifold associated to $\tilde{A}$ is sometimes called the **body** or the **bosonic** part of the supermanifold.

Note that we only demand the existence of such an isomorphism between A and $\Lambda_{\tilde{A}}\mathcal{N}/\mathcal{N}^2$, so it is not canonical in any sense. The full subcategory of smooth graded algebras in the category of graded commutative algebras shall be called the **category of Smooth Superalgebras**, it is so to speak the dual to the category of Supermanifolds. We shall sometimes refer to the algebra A itself as the supermanifold, and even worse shall we after a while forget the prefix super.

Example 5.16. The most common example of smooth superalgebras are alternating forms $\Gamma(\Lambda T^*M)$ and multi-vectorfields $\Gamma(\Lambda TM)$ of a classical manifold. The supermanifold "associated" to the first one is denoted by ΠTM the one associated to the second one by ΠT^*M. The reason for this notation is that in local coordinates (see section 3) the transformation behavior for the standard coordinates in these manifolds is just as the transformation behavior for the bundle TM resp. T^*M with the parity of the fiber coordinates reversed.

A closed submanifold of a supermanifold may be defined as follows

Definition 5.17. Let I be a graded ideal of a smooth super algebra A. If A/I is a smooth super algebra then A/I together with the projection $A \to A/I$ is called a submanifold of A

When applying this definition to the purely commutative case one sees that it is more general than the usual definition of (closed) submanifold.

Let U_q be a superdomain (see section: local superanalysis) of the same dimension as our supermanifold S. A morphism $C^\infty(S) \to C^\infty(U_q)$ shall be called a **chart** if there exists an isomorphism $\Gamma(\Lambda(E)) \to C^\infty(S)$ such that the composition with the map above gives a restricion morphism of the sections of $\Lambda(E)$ to open subset of M.

5.1.3 Super Vectorbundles

The objects which replace the notion of "vector bundle" (actually sections of a vector bundle) are given next:

Definition 5.18. By a **graded right A-Module** P we mean a right A-module whose underlying abelian group is graded $P = P_0 \oplus P_1$ and the scalar multiplication satisfies the additional condition:

$$P_j \cdot A_i \subset P_{i+j}$$

Right modules together with right module morphisms (i.e. $\varphi(pf) = \varphi(p)f$ for $p \in P$ and $f \in A$) form a category denoted by $mod - A$. The set of morphisms in this category is a graded abelian group $\mathrm{Hom}_{mod-A}(P,Q) = \mathrm{Hom}_{mod-A}(P,Q)_0 + \mathrm{Hom}_{mod-A}(P,Q)_1$ (but no right module with the evident multiplication). The restricted category of graded right modules with purely *even* morphisms shall be denoted by $mod_0 - A$.

Be aware that the P_i are not submodules of P. The definition of left graded A-module is analogous and this category shall be denoted by $A - mod$. It may be convenient to write left module morphisms as operating from the right

$$(fp)\varphi = f(p)\varphi$$

(and so the composition of two arrows

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} S$$

is written as $\varphi \circ \psi$ in this category) The usefulness of this notation will become clear in a moment.

Example 5.19 *(Free Modules).* If A is a graded commutative algebra then the abelian group $P := A$ is evidently an (left or right) A module. But we can choose essentially two different gradings on this Module. The first one is the one already given on A

$$P_0 := A_0,\ P_1 := A_1$$

but the second one swaps the parity:

$$P_0 := A_1,\ P_1 := A_0$$

The first one can also be considered as the free graded module *generated by an even* element (in this case $1 \in P$) and the second one as the free module *generated by an odd* element (also $1 \in P$). We shall denote the first one by $A^{1|0}$ an the second one by $A^{0|1}$. Note that the parity of the generator is essential: $A^{1|0}$ can not be generated by an odd element and $A^{0|1}$ not by an even one. More generally we can consider $P = \underbrace{A \times A \times \ldots \times A}_{r \text{ times}}$ as an A module, and if $n + m = r$ then we can give it the grading

$$P_0 = A_0^n \oplus A_1^m,\ P_1 = A_1^n \oplus A_0^m$$

This module is denoted by $A^{n|m}$. It is free generated by n even and m odd elements.

Example 5.20. Let $E \to M$ be a classical vectorbundle over a manifold and $A = \Gamma(\Lambda E)$. Suppose $F = F_0 \oplus F_1 \to M$ is a graded vector bundle over M then $A \otimes_{C^\infty(M)} \Gamma(F)$ is a graded A module.

Definition 5.21. A graded module P over a smooth graded algebra is called a module of sections of a smooth super vectorbundle (for short a smooth module, or smooth vectorbundle) if it is finitely generated and projective.

We may sometimes refers to the elements of a smooth graded module P as "sections".

There are two omnipresent functors for graded modules which cause some subtle differences to the purely commutative world. The first one turns left modules into right ones and has an inverse (and no name).

Definition 5.22. Given a left A-module P it becomes a right A-module by defining

$$p \cdot f := (-1)^{|f||p|} f \cdot p$$

for homogenous $p \in P$ and $f \in A$, and extending to general elements by decomposing them into even and odd part. Conversely a right module becomes a left one by the same construction and these construction are inverse to one another. It may be convenient to denote the module by $\overleftarrow{P}$ if considered as right and by $\overrightarrow{P}$ if considered as left module

Note that one may also consider P as a bi-module since both multiplications commute: $f(pg) = (fp)g$. How does the *functor* operate on morphisms? For the restricted categories $mod_0 - A$ and $A - mod_0$ one can define it as the identity:

$$\overrightarrow{\varphi}(p) := (p)\varphi$$

One is tempted to extend this covariant functor to the whole set $\mathrm{Hom}_{A-mod}(P,Q)$ by

$$\overrightarrow{\varphi}(p) := (-1)^{|p||\varphi|}(p)\varphi$$

this indeed defines a bijection between $\mathrm{Hom}_{A-mod}(P,Q)$ and $\mathrm{Hom}_{mod-A}(P,Q)$ but it is not functorial since for the composition of two morphisms

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$$

we get

$$\overrightarrow{\varphi \circ \psi}(p) = (-1)^{|p|(|\psi|+|\varphi|)}((p)\varphi)\psi = (-1)^{|\psi||\varphi|}\overrightarrow{\psi}(\overrightarrow{\varphi}(p))$$

in other words we get a functor up to a sign (satisfying again the sign rule)

$$\overrightarrow{\varphi \circ \psi} = (-1)^{|\psi||\varphi|}\overrightarrow{\psi} \circ \overrightarrow{\varphi}$$

Corollary 5.23. *If P, Q are two graded modules then $\mathrm{Hom}_{mod-A}(P,Q)$ is a left A-module by $(f\varphi)(p) := f(\varphi(p))$, and by the general construction above it becomes a right module. Analogously $\mathrm{Hom}_{A-mod}(P,Q)$ becomes a right module etc. The bijection between these two Hom sets as defined above is actually an (even) isomorphism of graded modules.*

This makes it possible to define the dual P^* of a graded module as either $\mathrm{Hom}_{A-mod}(P,A)$ or $\mathrm{Hom}_{mod-A}(P,A)$ since these are now canonically isomorphic.

Now we turn to the second functor mentioned earlier which is called the parity change functor Π. The phenomenon could already be observed in the example of free modules. Namely given a graded module P we can define a new graded module ΠP which as a module is exactly the same as P, only with the grading swapped:

$$\Pi P_i := P_{i+1}$$

This functor goes from either of the categories of A-Modules to itself and operates without any change on morphism.

Definition 5.24. If P and Q are graded A-modules then the set $P \times Q$ has a natural structure of a graded A-module denoted $P \oplus Q$. Where the grading is given by

$$(P \oplus Q)_i := P_i \times Q_i$$

Proposition 5.25. *The graded module $P \oplus Q$ with the canonical projections onto the factors satisfies the universal property of a product in the category of graded A modules with even morphisms. Together with the canonical injections it satisfies the universal property of a coproduct.*

Now we define the graded tensor product

Definition 5.26. Let P, Q be graded A-modules. Consider P as a right and Q as a left module then the usual tensor product $P \otimes_A Q$ (the abelian group generated by the elements of $P \times Q$ modulo the relations $(p+p') \otimes q = p \otimes q + p' \otimes q$, $p \otimes (q+q') = p \otimes q + p \otimes q'$ and $pf \otimes q = p \otimes fq$) can be given the structure of a graded A module by giving it the grading

$$(P \otimes_A Q)_0 := \left\{ \sum p_i \otimes q_i \mid |p_i| = 0,\ |q_i| = 0 \right\} \oplus \left\{ \sum p_i \times q_i \mid |p_i| = 1,\ |q_i| = 1 \right\}$$

$$(P \otimes_A Q)_1 := \left\{ \sum p_i \otimes q_i \mid |p_i| = 1,\ |q_i| = 0 \right\} \oplus \left\{ \sum p_i \times q_i \mid |p_i| = 0,\ |q_i| = 1 \right\}$$

in other words $|p \otimes q| = |p| + |q|$. Then the scalar multiplication can be given by:

$$f(p \otimes q) = (fp) \otimes q$$

which in accordance with the general construction of the right multiplication gives the formula $p \otimes qf = (p \otimes q)f$

This graded tensor product satisfies universal properties as in the commutative case. Which we will not list now. See for example [Bar].

Definition 5.27. The **symmetric algebra** $\mathrm{Sym}(P)$ of a graded module P is the quotient

$$TP/I$$

where

$$TP = \bigoplus_{n=0}^{\infty} \underbrace{P \otimes \ldots \otimes P}_{n \text{ times}}$$

and I is the graded Ideal generated elements of the form

$$s_1 \otimes s_2 - (-)^{|s_1||s_2|} s_2 \otimes s_1$$

with $s_1, s_2 \in P$ homogenous. The **exterior algebra** $\Lambda(P)$ is the quotient

$$TP/J$$

where J is the ideal generated by elements of the form

$$s_1 \otimes s_2 + (-)^{|s_1||s_2|} s_2 \otimes s_1$$

5.1.4 Super Lie algebras

Definition 5.28. A **graded Lie Algebra** consists of a graded vector space L endowed with a even bilinear map $[\ ,\ ] : L \times L \to L$ called (graded) **bracket** which satisfies:

i) $[x, y] = -(-1)^{(|x||y|)}[y, x]$
(graded anti-commutative)

ii) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$
(graded Jacobi identity)

where the equalities are demanded for homogeneous elements. The morphisms of graded Lie Algebras are even linear maps that respect the brackets:

$$\phi[x, y]_L = [\phi(x), \phi(y)]_{\tilde{L}}$$

Using property i) of the Lie bracket one finds that property ii) is equivalent to the derivation property

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$$

Example 5.29. For any graded linear space V the graded space $\mathrm{End}_{\mathbb{R}}(V)$ of $\mathbb{R}$ linear endomorphisms is naturally supplied with a graded lie bracket called the **graded commutator**. On homogeneous elements $G, H \in \mathrm{End}_{\mathbb{R}}(V)$ it is given by:

$$[G,H] := GH - (-1)^{|G||H|}HG$$

And it is extended to general elements by linearity. Note also that $\mathrm{End}_{\mathbb{R}}(V)$ is a graded Algebra with the usual composition $FG = F \circ G$ but this composition is *not* graded commutative. Still the bracket is a derivation with respect to the composition:

$$[F, GH] = [F,G]H + (-1)^{|F||G|}G[F,H]$$

Since any graded commutative algebra A is also a graded linear space the above construction applies to $\mathrm{End}_{\mathbb{R}}(A)$. This is part of what we will do in the next section to define vector fields, differential operators etc. . .

5.2 Calculus on Supermanifolds

5.2.1 Derivations

Remember that in the classical situation the vector fields of a manifold are the derivations of the function algebra $C^\infty(M)$. In the graded case the analog is the following.

Definition 5.30. Let A be a graded commutative algebra. An even (odd) element $X \in \mathrm{End}_{\mathbb{R}}(A)$ is called a **even (odd) derivation** of the graded algebra A if:

$$X(fg) = X(f)g + (-1)^{|X||f|}fX(g)$$

for homogeneous $f, g \in A$. The set of even derivations is denoted by $\mathrm{Der}_0(A)$ the one of odd derivations by $\mathrm{Der}_1(A)$. They are both $\mathbb{R}$ linear subspaces. We define $\mathrm{Der}(A) := \mathrm{Der}_0(A) \oplus \mathrm{Der}_1(A)$ and call the elements **derivations** of the algebra.

Actually this is what is sometimes called a left derivation in the literature, there is also the notion of right derivation which is a $\mathbb{R}$ linear operator X (better written on the right of the function they operate on $(f)X$) which satisfies:

$$(fg)X = (-1)^{|g||X|}(f)X \cdot g + f \cdot (g)X$$

One writes $\overrightarrow{X}$ for left and $\overleftarrow{X}$ for right derivations to distinguish them easier.

Right derivations don't contribute anything essentially new since left and right derivations are canonically in bijection: given a left derivation $\overrightarrow{X}$ we can construct a right one by $f\overleftarrow{X} := (-1)^{|f||\overrightarrow{X}|}\overrightarrow{X}f$. Of course the inverse of this construction is given by $\overrightarrow{X}f = (-1)^{|f||\overleftarrow{X}|}f\overleftarrow{X}$. So when I speak of derivation I will always mean left derivation unless otherwise stated.

The following result is the graded analog to the fact that vector fileds form a Lie algebra and are the sections of a bundle over the manifold.

Proposition 5.31. $\mathrm{Der}(A)$ *is a graded Lie algebra since it is closed under the bracket given on* $\mathrm{End}_{\mathbb{R}}(A)$*. It is also a graded A-module since if X is a derivation and $f \in A$ then $f \cdot X$ is also a derivation.*

Proof. Let ∇ and $\tilde{\nabla}$ be homogeneous derivations and $f, g \in A$ homogenous. Then direct computation yields

$$\begin{aligned}
[\nabla, \tilde{\nabla}](f \cdot g) =& \nabla\tilde{\nabla}(fg) - (-1)^{|\tilde{\nabla}||\nabla|}\tilde{\nabla}\nabla(fg) \\
=& \nabla\left(\tilde{\nabla}(f)g + (-1)^{|f||\tilde{\nabla}|}f\tilde{\nabla}g\right) - (-1)^{|\tilde{\nabla}||\nabla|}\tilde{\nabla}\left(\nabla(f)g + (-1)^{|f||\nabla|}f\nabla g\right) \\
=& \nabla(\tilde{\nabla}f)g + (-1)^{|\nabla|(|f|+|\tilde{\nabla}|)}\tilde{\nabla}f\nabla g \\
&+ (-1)^{|f||\tilde{\nabla}|}\nabla f\tilde{\nabla}g + (-1)^{|f|(|\tilde{\nabla}|+|\nabla|)}f\nabla\tilde{\nabla}g \\
&- (-1)^{|\tilde{\nabla}||\nabla|}\tilde{\nabla}(\nabla f)g - (-1)^{|\tilde{\nabla}||f|}\nabla f\tilde{\nabla}g \\
&- (-1)^{(|f|+|\tilde{\nabla}|)|\nabla|}\tilde{\nabla}f\nabla g - (-1)^{|f|(|\nabla|+|\tilde{\nabla}|)-|\nabla||\tilde{\nabla}|}f\tilde{\nabla}\nabla g \\
=& [\nabla, \tilde{\nabla}](f) \cdot g + (-1)^{|f|(|\nabla|+|\tilde{\nabla}|)}f \cdot [\nabla, \tilde{\nabla}](g)
\end{aligned}$$

so the bracket of two derivations is again a derivation. To check that derivations are closed under multiplication from the left with elements of A it suffices again to verify the case where $f \in A$ and $\nabla \in \mathrm{Der}(A)$ are both homogenous.

$$\begin{aligned}
f\nabla(gh) &= f\nabla(g)h + (-1)^{|\nabla||g|}fg\nabla(h) \\
&= f\nabla(g)h + (-1)^{(|\nabla|+|f|)|g|}gf\nabla(h)
\end{aligned}$$

The general situation follows by decomposing $f = f_0 + f_1$ and $\nabla = \nabla_0 + \nabla_1$ into homogenous parts. □

Note that $D(A)$ is not closed under multiplication from the right (where I mean the multiplication $X \cdot f(g) := X(g)f$). So if we consider $D(A)$ as a right A module we mean the formal multiplication defined above for general A Modules. This multiplication satisfies the (by now natural) property $Xf(g) = (-1)^{|g||f|}X(g)f$.

A useful generalization of the concept above is

Definition 5.32. A derivation of a graded algebra A to a graded module P is a $\mathbb{R}$ linear map

$$D : A \to P$$

that satisfies

$$D(fg) = fD(g) + (-)^{|D||f|}fD(g)$$

The set of derivations from A to P is denoted with $\mathrm{Der}(P)$.

One can show that they form a graded A module as in the previous case

5.2.2 Linear Differential Operators

Derivations are a special case of the more general notion of linear partial differential operator to which we turn to now.

We shall at once give the more general definition of linear differential operators between two A modules P and Q. First note that the set of $\mathbb{R}$ linear maps $\mathrm{Hom}_{\mathbb{R}}(P, Q)$ can be given two *left* A-module structures. For the first one we define the product of $f \in A$ and $\Delta \in \mathrm{Hom}_{\mathbb{R}}(P, Q)$ by:

$$f\Delta : P \to Q, \; p \mapsto f \cdot \Delta(p)$$

In the second module structure the product of f and Δ (now denoted by $f^+\Delta$) is given by

$$f^+\Delta : P \to Q, \; p \mapsto (-)^{|\Delta||f|}\Delta(fp)$$

Notice that in the first case the multiplication occurs in Q while for the second one it occurs in P. These multiplications commute (in the super sense):

$$f(g^+\Delta) = (-)^{|f||g|}g^+(f\Delta)$$

To distinguish these different structures we denote the $\mathbb{R}$ linear space $\mathrm{Hom}_{\mathbb{R}}(P,Q)$ by $\mathrm{Hom}^{(\ )}_{\mathbb{R}}(P,Q)$ if it is considered as the left A-module in the first sense, and by $\mathrm{Hom}^{+}_{\mathbb{R}}(P,Q)$ if it is considered as A module in the second sense. Finally it is written as $\mathrm{Hom}^{(+)}_{\mathbb{R}}(P,Q)$ if it is considered as bimodule.

Next we define

$$\delta_f(\Delta) := f^+\Delta - f\Delta$$

it is evidently again in $\mathrm{Hom}_{\mathbb{R}}(P,Q)$.

Proposition 5.33. *The map*

$$\delta_f : \mathrm{Hom}_{\mathbb{R}}(P,Q) \to \mathrm{Hom}_{\mathbb{R}}(P,Q)$$

is a (right) module morphism (i.e.: $\delta_f(g\Delta) = (-)^{|f||g|}g\delta_f(\Delta)$) for both module structures, and has degree $|f|$.

Proposition 5.34. *The map*

$$\delta : A \to \mathrm{End}_A\left(\mathrm{Hom}^{(+)}(P,Q)\right)$$

is even $\mathbb{R}$ linear and satisfies:

$$\delta_{f\cdot g}(\Delta) = \delta_f(g^+\Delta) + f(\delta_g\Delta)$$

We also define

$$\delta_{f_0,\ldots,f_k} := \delta_{f_0}\circ\ldots\circ\delta_{f_k}$$

note that $\delta_{f,g} = (-)^{|f||g|}\delta_{g,f}$

Now we come to the central definition.

Definition 5.35. An even (odd) linear **Differential operator** from P to Q of order less than k is an even (odd) linear map $\Delta : P \to Q$ which satisfies:

$$\delta_{f_0,f_1\ldots,f_k}\Delta = 0$$

for all $f_0,\ldots,f_k \in A$. The set of even DO of order $\leq k$ is denoted by $\mathrm{Diff}^0_k(P,Q)$ the set of odd ones by $\mathrm{Diff}^1_k(P,Q)$, they are both $\mathbb{R}$ vector spaces. The direct sum of even and odd differential operators of order $\leq k$ is denoted by $\mathrm{Diff}_k(P,Q)$ and its elements are called differential operators of order less than k.

Note that $\mathrm{Diff}_k(P,Q)\subset\mathrm{Diff}_{k+1}(A)$. The elements of $\mathrm{Diff}(P,Q) := \bigcup_{k\in\mathbb{N}}\mathrm{Diff}_k(P,Q)$ are called differential operators. We say that $\Delta\in\mathrm{Diff}(P,Q)$ has **order k** if $\Delta\in\mathrm{Diff}_k(P,Q)\backslash\mathrm{Diff}_{k-1}(P,Q)$. But one should remember that differential operators of (highest) order k do not form a vector space. We also write $\mathrm{Diff}_k(P) := \mathrm{Diff}_k(A,P)$.

Proposition 5.36. $\mathrm{Diff}_k(P,Q)$ *is closed under both module structures mentioned above and so* $\mathrm{Diff}^{(+)}(P,Q)$ *is a $\mathbb{N}$ filtered, $\mathbb{Z}_2$ graded A-bimodule.*

The proof is straightforward applying the previous formulas. Note that zero order operators are module morphism.

Definition 5.37. We define the **module of symbols** as

$$\mathrm{Symb}_k(P,Q) := \mathrm{Diff}_k(P,Q)/\mathrm{Diff}_{k-1}(P,Q)$$

Proposition 5.38. *If* $\Delta \in \mathrm{Diff}_k(R, Q)$ *and* $\tilde{\Delta} \in \mathrm{Diff}_l(P, R)$ *then* $\Delta \circ \tilde{\Delta} \in \mathrm{Diff}_{k+l}(P, Q)$

Proof. Use

$$\delta_f(\Delta \circ \tilde{\Delta}) = \delta_f(\Delta) \circ \tilde{\Delta} + (-)^{|f||\Delta|}\Delta \circ \delta_f(\tilde{\Delta})$$

□

Proposition 5.39. *Let* $\Delta \in \mathrm{Diff}_k(A)$ *and* $\tilde{\Delta} \in \mathrm{Diff}_l(A)$*, then* $[\Delta, \tilde{\Delta}] \in \mathrm{Diff}_{k+l-1}(A)$*. So* $\mathrm{Diff}(A)$ *is actually a* $\mathbb{Z}$ *filtered and* $\mathbb{Z}_2$ *graded Lie algebra, with a Lie bracket of degree* -1

Proof. ? □

Proposition 5.40. *Let* $\Delta : A \to P$ *be* $\mathbb{R}$ *linear. Then*

$$\Delta \in \mathrm{Diff}_1(P) \Leftrightarrow \Delta - \Delta(1) \in \mathrm{Der}(P)$$

So we have canocnical splitting

$$\mathrm{Diff}_1(P) = \mathrm{Der}(P) \oplus P$$

Proof. Note that if D is a derivation then $\delta_f(D)(a) = (-)^{|f||D|}D(f)a$. So $\delta_f D$ is a zero order operator, hence D is first order. So if $\Delta - \Delta(1)$ is a derivation then Δ is a first order differential operator. Suppose now conversely that Δ is first oder, then $\tilde{\Delta} := \Delta - \Delta(1)$ is also first order and additionally $\tilde{\Delta}(1) = 0$. So

$$\begin{aligned} 0 =& \delta_{f,g}\tilde{\Delta}(1) = f^+g^+\tilde{\Delta}(1) - f^+g\tilde{\Delta}(1) - fg^+\tilde{\Delta}(1) \\ =& (-)^{|\tilde{\Delta}|(|f|+|g|)}\tilde{\Delta}(fg) - (-)^{|\tilde{\Delta}|(|f|+|g|)}\tilde{\Delta}(f)g - (-)^{|g||\tilde{\Delta}|}f\tilde{\Delta}(g) \end{aligned}$$

□

5.2.3 Super differential forms

Let A be a graded algebra, we will now define the analog of differential forms and the exterior derivative.

Definition 5.41. The A-dual of the module of vector fields is called the module of **1-forms** and denoted by:

$$\Lambda^1 := \mathrm{Hom}_A(D(A), A)$$

The differential df of a function $f \in A$ is the 1-form

$$df(X) := (-1)^{|X||f|}X(f)$$

This gives us an even map

$$d : A \to \Lambda^1$$

Some authors define the module of one forms with the parity chaged. The algebra of forms of arbitrary degree is defined as the exterior algebra of one forms. One can extend d in a unique way to a derivation of this algebra satisfying the property $d^2 = 0$. The cohomology of the resulting complex is denoted with $H(A)$.

5.2.4 Berenzinian Bundle, Integration

Definition 5.42. Let A be a graded algebra, we shall write $\Lambda^k = \Lambda^k(A)$. Consider the complex

$$0 \to \mathrm{Diff}^+(\Lambda^0) \xrightarrow{w} \mathrm{Diff}^+(\Lambda^1) \xrightarrow{w} \mathrm{Diff}^+(\Lambda^2) \ldots$$

where $w(\Delta) := d \circ \Delta$. The cohomology of this complex at the term k is denoted with $\hat{A}_k$. We call the module

$$\bigoplus_{k=0}^{\infty} \hat{A}_k$$

the Berezinian of A and denote it with $\mathrm{Ber}(A)$. The integral is defined as the map

$$\int : \mathrm{Ber}(A) \to H(A)$$
$$[\Delta] \mapsto [\Delta(1)]$$

5.3 The local description and index notation in supermathematics

In this chapter we analyze the super analogs to the algebras $C^\infty(U)$ where U is a domain in $\mathbb{R}^p$. Here we also use the index notation as in classical differential geometry which is a useful tool for computations.

Let U be an open subset of $\mathbb{R}^p$ with standard coordinates denoted by $x_1, \ldots, x_p$, and let Λ_q be the Grassmann algebra generated by the elements $\xi_1, \ldots, \xi_q$. The algebra of interest is then $A := C^\infty(U) \otimes_{\mathbb{R}} \Lambda_q$. The elements of these algebras will be referred to as functions on the **superdomain** U_q and the elements

$$x_1, \ldots, x_p, \xi_1, \ldots, \xi_q \in A$$

are referred to as the **standard coordinates** on the superdomain U_q, we also write $C^\infty(U_q) := A$ (we stress again that the superdomian U_q is not defined as some set of points, it is just an expression). The elements of such an algebra can be uniquely described by expressions of the form:

$$f = \sum_{k \geq 0} \sum_{1 \leq i_1, \ldots i_k \leq q} f_{i_1, \ldots, i_k} \xi_{i_1} \cdots \xi_{i_k}$$

where the $f_{i_1, \ldots, i_k}$ are in $C^\infty(U)$ and are antisymmetric in their indices. Becaouse of this one also writes $f = f(x, \xi)$. More geometrically one can consider $f \in C^\infty(U)$ as a smooth section of the trivial bundle $U \times \Lambda_q \to U$. The algebra A is naturally graded: an element is even if it is of the form

$$f = \sum_{k \geq 0} \sum_{1 \leq i_1, \ldots i_{2k} \leq q} f_{i_1, \ldots, i_{2k}} \xi_{i_1} \cdots \xi_{i_{2k}}$$

and odd if:

$$f = \sum_{k \geq 0} \sum_{1 \leq i_1, \ldots i_{2k+1} \leq q} f_{i_1, \ldots, i_{2k+1}} \xi_{i_1} \cdots \xi_{i_{2k+1}}$$

The superdomains $\mathbb{R}^p_q$ are sometimes written as $\mathbb{R}^{p|q}$.

5.3.1 Morphisms between superdomains

In this short section we will sketch a useful formalism that allows to handle elements of these graded algebras as if they were "functions depending on commuting and anti-commuting variables".

Suppose again $U \subset \mathbb{R}^p$ open, with standard coordinates denoted by $y_1, \ldots, y_p$, and $V \subset \mathbb{R}^{p'}$ open, with coordinates $x_1, \ldots, x_{p'}$. Recall that in the classical situation a smooth map: $G : V \to U$ is given by p functions $g_1(x_1, \ldots x_{p'}), \ldots, g_p(x_1, \ldots x_{p'})$ (the components of G) which are nothing other than the pullback of the functions $y_1, \ldots y_p$, i.e. the images of these elements under $G^* : C^\infty(U) \to C^\infty(V)$. To obtain $G^*(f)$ for any other element $f(y_1, \ldots y_p) \in C^\infty(U)$ one simply composes:

$$f(g_1(x_1, \ldots x_{p'}), \ldots, g_p(x_1, \ldots x_{p'}))$$

The same fact is true for superdomains. For a proof see [Ber], here we only show how the composition $f(g)$ can be defined for functions on superdomains. The idea is to use Taylor expansion, since sufficiently high powers of terms with odd generators vanish.

Definition 5.43. Let $f \in C^\infty(\mathbb{R}^p_q)$ where the standard coordinates on $\mathbb{R}^p_q$ shall be denoted by y_i, η_j, $f(y, \eta) = \sum_{I \subset \{1,\ldots q\}} f_I(y_1, \ldots, y_p)\eta^I$. Suppose we are further given functions $g_1 \ldots g_p, \varphi_1 \ldots \varphi_q \in C^\infty(\mathbb{R}^{p'}_{q'})$ where the g_i are even and the φ_i are odd. Lets denote the standard coordinates on $\mathbb{R}^{p'}_{q'}$ by x_i, ξ_j. Then we define $f(g, \varphi) \in C^\infty(\mathbb{R}^{p'}_{q'})$ by:

$$f(g_1, \ldots, g_p, \varphi_1, \ldots, \varphi_q) := \sum_{I \subset \{1,\ldots,q} f_I(g_1, \ldots g_p)\varphi^I$$

Where we still have to define $f_I(g_1, \ldots, g_p)$. For this first write $g_i = \tilde{g}_i + n_i$ where $\tilde{g}_i$ is the term of g_i containing no odd elements and n_i is the rest. Then we put:

$$f_I(g_1, \ldots, g_p) := \sum_{\lambda \in \mathbb{N}^p} \frac{\partial^{|\lambda|}}{\partial y^\lambda} f_I(\tilde{g}_1 \ldots \tilde{g}_p) n^\lambda$$

Note that the sum is finite since the n_i are nilpotent and also note that the product n^λ is independent of the order of the factors since the g_i are even.

5.3.2 Derivations and differential operators on superdomains

Given a superdomain U_q with standard coordinates $x_1, \ldots, x_p, \xi_1, \ldots, \xi_q$ we define the even derivations $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p} \in \mathrm{Der}(C^\infty(U_q))$ as those which operate on functions $f \in C^\infty(U) \subset C^\infty(U_q)$ in the usual way known from calculus

$$\frac{\partial}{\partial x_i} f = \frac{\partial}{\partial x_i} f(x_1, \ldots, x_p)$$

and on the odd coordinates by

$$\frac{\partial}{\partial x_i}\xi_j = 0$$

We also define the odd derivations $\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_q} \in \mathrm{Der}(C^\infty(U_q))$ by the properties:

$$\frac{\partial}{\partial \xi_j} f = 0, \ \frac{\partial}{\partial \xi_j}\xi_i = \delta_{ij}$$

where $f \in C^\infty(U)$. One verifies easily that these derivations exist and are uniquely characterized by the properties given above.

Proposition 5.44. *Every derivation $\nabla \in \mathrm{Der}(C^\infty(U_q))$ can be written uniquely as*

$$\nabla = \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i} + \sum_{j=1}^q \beta_j \frac{\partial}{\partial \xi_j}$$

where α_i and β_j are elements of $C^\infty(U_q)$.

Proof. The uniqueness follows by applying the operator on the r.h.s to standard coordinates $x_1, \ldots, x_p, \xi_1, \ldots, \xi_q$. This also gives us the clue as how to construct the α's and β's, namely by

$$\alpha_i := \nabla(x_i),\ \beta_j := \nabla(\xi_j)$$

It suffices to check that ∇ and $\sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} \beta_j \frac{\partial}{\partial \xi_j}$ operate in the same way on functions $f \in C^\infty(U)$ and on the elements $\xi_1, \ldots, \xi_q$. By linearity and the derivation property it then follows that they operate in the same way on any superfunction. That they operate in the same fashion on odd elements follows by construction, and that they operate in the same way on bosinic functions can be proven as in the commutative case. □

From this proposition follows that the module of derivations on a superdomain is a free, finitely generated module with even generators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p}$ and odd generators $\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_q}$. We also get that the module of one forms is generated by the even one forms

$$dx_1 \ldots dx_p$$

and the odd ones

$$d\xi_1 \ldots d\xi_q$$

Proposition 5.45. *Every differential operator $\Delta \in \mathrm{Diff}_k(C^\infty(U_q))$ can be uniquely written as:*

$$\Delta = \sum_{0 \le s \le \min(q,k)} \sum_{1 \le i_1 < \ldots < i_s \le q} \sum_{\substack{\gamma \in \mathbb{N}^p \\ |\gamma| \le k-s}} f_{\gamma, i_1, \ldots, i_s} \frac{\partial^\gamma}{\partial x^\gamma} \circ \frac{\partial}{\partial \xi_{i_1}} \circ \ldots \circ \frac{\partial}{\partial \xi_{i_s}}$$

where the $f_{\gamma,i_1,\ldots,i_s}$ are in $C^\infty(U_q)$ and by $\frac{\partial^\gamma}{\partial x^\gamma}$ we mean the operator $\frac{\partial^{\gamma_1}}{\partial x_{\gamma_1}} \circ \ldots \circ \frac{\partial^{\gamma_p}}{\partial x_{\gamma_p}}$

In other words $\mathrm{Diff}_k(C^\infty(U_q))$ is again a free and finitely generated $C^\infty(U_q)$ module with the generators

$$\frac{\partial^\gamma}{\partial x^\gamma} \circ \frac{\partial}{\partial \xi_{i_1}} \circ \ldots \circ \frac{\partial}{\partial \xi_{i_s}}$$

where the total degree is smaller than k i.e. $s + |\gamma| \le k$.

5.3.3 Berezin integration in local coordinates

One can show (see [Verb]) that if $A = C^\infty(\mathbb{R}^{p|q})$ all the cohomologies $\hat{A}^k$ vanish except the term $\hat{A}^p$ and so

$$\mathrm{Ber}(C^\infty(R^{p|q}) = \hat{A}^p$$

Further if $x_1, \ldots x_p, \xi_1, \ldots \xi_q$ are local coordinates then every section of $\mathrm{Ber}(C^\infty(R^{p|q}))$ can be written by

$$[f^+ \frac{\partial^q}{\partial \xi_1 \cdots \partial \xi_q} dx_1 \cdots dx_p]$$

with $f \in A$ and the differential operator $\frac{\partial^q}{\partial \xi_1 \cdots \partial \xi_q} \cdot dx_1 \cdots dx_p$ acts as

$$a \mapsto \left(\frac{\partial^q}{\partial \xi_1 \cdots \partial \xi_q} a \right) \cdot dx_1 \cdots dx_p$$

Further if y_i, η_j are new local coordinates then

$$dx_1 \cdots dx_p \frac{\partial^q}{\partial \xi_1 \cdots \partial \xi_q} = \mathrm{Ber}(J \left(\frac{x, \xi}{y, \eta} \right)) dy_1 \cdots dy_p \frac{\partial^q}{\partial \eta_1 \cdots \partial \eta_q} + T$$

Where

$$J\left(\frac{x,\xi}{y,\eta}\right)$$

is the Jacobi Matrix of the chart change

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \det(A - BD^{-1}C)(\det D)^{-1}$$

is the Berezin determinant, and T is a differential operator cohomologous to zero. One also sees that the integral is given by

$$\int [f^{+}\frac{\partial^q}{\partial\xi_1\cdots\partial\xi_q}dx_1\cdots dx_p] = [f_{1,\dots,q}dx_1\cdots dx_p]$$

here $f_{1,\dots,q} \in C^\infty(\mathbb{R}^p)$ is the coefficient of $\xi_1\dots\xi_q$ in the representation

$$f = \sum_{k\geq 0}\sum_{1\leq i_1,\dots i_k\leq q} f_{i_1,\dots,i_k}\xi_{i_1}\cdots\xi_{i_k}$$

6 APPENDIX: Some facts about the exponential on the function algebra of a supermanifold

In this section A denotes a superalgebra of the form $A = \Gamma(\Lambda E)$ where $E \to M$ is a smooth vector bundle. For $f \in A$ we split $f = f_b + r$ according to decomposition $\Gamma(\Lambda E) = C^\infty(M) \oplus \bigoplus_{i>0}\Lambda^i(\Gamma(E))$. So f_b is a function on the base manifold called the bosonic part of f and r is the nilpotent rest.

Proposition 6.1. *Let R be a commutative ring. $f \in R$ possesses an inverse with respect to multiplication if and only if $\tilde{f} \in \tilde{R}$ is invertible (here $\tilde{R}$ is the reduced ring). For $f \in \Gamma(\Lambda E)$ this means that f_b vanishes nowhere as a function on the base manifold.*

Proof. It is evident that if f is invertible then so is $\tilde{f}$. Suppose now that $\tilde{f}$ is invertible then there exists a $g \in R$ such that $fg = 1 + n$ where n nilpotent. But every element of the form $1 + n$ is invertible since we can multiply it with $1 - n$ to get $1 - n^2$, now if $n^2 \neq 0$ we multiply again this time with $1 + n^2$ and so on, until at one point the second term vanishes because of nilpotency. □

Definition 6.2. Consider a graded algebra $A = \Gamma(\Lambda E)$. For $f \in A$ we define the exponential (as usual) by

$$e^f := e^{f_b}\left(\sum_{k=0}^{\infty}\frac{1}{k!}r^k\right)$$

it is well defined since the sum is finite

Remark 6.3. We note that if A is the function algebra for a smooth supermanifold then we can define the exponential e^f by using an isomorphism to an algebra $\Gamma(\Lambda E)$. One can probably show that this definition is independent of the choice of isomorphism.

Proposition 6.4. *If $f, g \in A$ satisfy $fg = gf$ then*

$$e^f e^g = e^{f+g}$$

Proof. Note that if f, g commute then so do their nilpotent parts. Then we can follow the same proof as in a standard analysis course □

Corollary 6.5. *e^f is invertible with $(e^f)^{-1} = e^{-f}$. We also have $e^{nf} = (e^f)^n$ for $n \in \mathbb{N}$*

Proposition 6.6. *The image of $\exp : A \to A$ are the invertible elements of A whose bosonic part is a positive function.*

Proof. That e^f is invertible was shown above, and that its bosonic part is positive can be seen in the definition of the exponential. To show that every g with strictly positive bosonic part is in the image write $g = g_b(1 + u)$ where $u \in \Gamma(\Lambda E)$ is nilpotent. Then $g_b = \pm e^{f_b}$ for some smooth function $f_b \in C^\infty(M)$ and $1 + u = e^{\log(1+u)}$ where

$$\log(1 + u) := \sum_{k=1}^{\infty} (-)k - 1 \frac{u^k}{k}$$

and the sum is finite because of nilpotency. □

Proposition 6.7. *The map $\exp : A \to A$ is injective*

Proof. Suppose first that $f \in A$ is even and satisfies

$$e^f = 1$$

then considering the degrees of the terms on the right hand side one concludes that $f = 0$. From this, and using $e^{f+g} = e^f e^g$ we get that for even functions $e^f = e^g$ implies $f = g$. Now for general functions we split $f = f_0 + f_1$ and use

$$e^f = e^{f_0}(1 + f_1)$$

□

Proposition 6.8. *Let $X \in \mathrm{Der}(A)$ be a derivation and $f \in A$ even, then*

$$X(f^n) = nX(f)f^{n-1}$$

$$X(e^f) = X(f)e^f$$

$$d(e^f) = e^f df$$

Proof. The first equation is an easy induction and the second one follows from the first one. The last one ist the second one rewritten. □

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