TOPOLOGY

PROF. DR. A. S. CATTANEO

HS 2025

Program

(1) Introduction: topology, classification of surfaces, categories. Examples of topologies.
(2) Metric spaces and induced topology. Topology generated by a basis. Continuity.
(3) Continuous maps and their properties. Definition of a (locally small) category and examples. The opposite category. Invertible morphisms.
(4) Isomorphisms and homeomorphisms. Topological properties and examples. Metrizability. Pullback and pushforward. Functors and examples thereof.
(5) Full/faithful/fully faithful functors. Natural transformations. Digression: Yoneda's lemma. Injective, left cancellative, left invertible maps.
(6) Surjective, right cancellative, right invertible maps. Initial and terminal objects. Producs and coproducts.
(7) Exponentiation. The product–hom adjunction. The Sierpiński two-point space. The subspace topology. Quotients.
(8) The quotient topology. Examples of quotients. The product topology. The coproduct topology.
(9) The coproduct topology. Homotopies. The homotopy category. Homotopy invariance. Contractible spaces. Homotopy of paths. Connected and path-connected spaces. Connected and path-connected components.
(10) Theorems on (path) connected spaces. The topologist's sine curve. Intervals. Path-connected spaces are connected. Homotopy invariance.
(11) 1D Brouwer's fixed point theorem. The functor π_0. Product of (path) connected spaces. Decomposition into connected components. The functor $\mathsf{Top}(X, -)$ and connectedness. Locally (path) connected spaces and their properties.
(12) Hausdorff spaces: properties and examples. Compact spaces and their properties. Closed bounded intervals. Limit points. The Bolzano–Weierstrass theorem. The finite intersection property (FIP). Closed subsets of compacts spaces.

(13) Compact subsets of Hausdorff spaces. Properties of maps from a compact space to a Hausdorff space. Finite products of compact spaces. The Heine–Borel theorem. The maximum principle. The Tube Lemma. Local compactness.
(14) Closure, interior, limit points, boundary points. Sequences. T0, T1, and T2 spaces. Neighborhood bases. First- and second-countable spaces. Properties of first-countable spaces.
(15) Manifolds and examples. Filters and examples. Convergence of a filter. Pushforward of a filter. Hausdorff, closure, continuity via filters. Ultrafilters and equivalent characterizations. Principal filters.
(16) Prime filters. Pushforward of prime filters. The Ultrafilter Lemma. Nonprincipal ultrafilters. Compactness in terms of ultrafilters. Tychonoff's theorem.
(17) Diagrams. Cones and cocones. Limits and colimits. Products and coproducts. Pullbacks and pushouts.
(18) Inverse and direct limits. Equalizers and coequalizers. Completeness and cocompleteness. (Co)continuous functors. Isomorphisms of categories. Equivalences of categories.
(19) Adjunctions and examples. The product–hom adjunction. Unit and counit, universal properties. Free vector spaces as adjunction.
(20) Free–forgetful adjunctions. Top–Set adjunctions. Right adjoints preserve limits (RAPL), and left adjoints preserve colimits (LAPC). Splitting, conjoining, and exponential topologies.
(21) The compact–open topology and examples. The exponential topology for locally compact, Hausdorff spaces. Homotopy equivalence between path spaces and cylinders. Groupoids.
(22) Path homotopies and path concatenation. The fundamental groupoid and its functorial property. The fundamental group and its relation to the fundamental groupoid. Functorial property of the fundamental group. The fundamental group(oid) of a convex set.
(23) Higher homotopy groups. The category of pointed topological spaces: products, exponentials, the forgetful adjunction, the wedge product, the smash product. The smash–hom adjunction. The suspension–loop adjunction.
(24) The exponential map and lifts of paths on the circle. The loop space of the circle. The homotopy groups of the circle. The fundamental group of the torus.
(25) Brouwer's Fixed Point Theorem. The Perron–Frobenius Theorem. The degree of a map. The Fundamental Theorem of Algebra.
(26) The Seifert–van Kampen Theorem (for fundamental groupoids and for fundamental groups). Amalgamated products. Examples: sphere, wedge of circles, torus, Klein bottle.
(27) Topological surfaces. Examples. Connected sum. Triangulations. The Euler characteristic. Orientability and orientation. Polygonal presentation and its code.

(28) Normal form of the polygonal presentation of a compact, orientable surface. Genus. Classification of compact, orientable surfaces. Fundamental groups of spheres and applications. Borsuk and Borsuk-Ulam theorems.

References

[1] T.-D. Bradeley, T. Bryson, and J. Terilla, *Topology: A Categorical Approach*, MIT Press, 2020.

Further suggested literature:

[2] S. Waldmann, *Topology: An Introduction*, Springer, 2014.

[3] W. Fulton, *Algebraic Topology: A First Course*, Springer, 1995