SYMPLECTIC AND POISSON GEOMETRY I

PROF. DR. A. S. CATTANEO

Wintersemester 2005/2006

1. Symplectic linear algebra

(1) Symplectic forms and presymplectic forms
(2) Normal form theorem
(3) Weak and strong infinite-dimensional symplectic spaces
(4) The symplectic orthogonal space
(5) Symplectic, isotropic, coisotropic, and Lagrangian subspaces
(6) Lagrangian splittings
(7) Linear symplectic reduction
(8) Canonical relations, their composition and the extended linear symplectic category.
(9) Kähler structures
(10) The Lagrangian Grassmannian

2. Symplectic manifolds

(1) Generalities (definitions, symplectomorphisms, Liouville's volume form)
(2) Lagrangian and Hamiltonian mechanics; the Legendre transformation of hyperregular Lagrangians
(3) Moser's trick
(4) Darboux and Darboux–Weinstein theorems
(5) Classification of compact symplectic surfaces
(6) Presymplectic (sub)manifolds and reduction
(7) Symplectic, isotropic, coisotropic, and Lagrangian submanifolds
(8) Example: The space of solution of a Hamiltonian system
(9) Reduction of Lagrangian submanifolds intersecting coisotropic submanifolds
(10) Canonical relations; composition thereof; the extended symplectic "category"
(11) Generating functions and Morse families
(12) Example: (deformed) conormal bundles in cotangent bundles
(13) The Legendre mapping; the generalized Legendre transform
(14) (Almost) Kähler structures

3. Distributions and foliations

(1) Involutive and integrable distributions
(2) Frobenius theorem
(3) The leaf space
(4) Basic forms
(5) Singular distributions (after Stefan–Sussmann)
(6) Lie algebroids
(7) Lie groupoids

4. Reduction and symmetry

(1) Algebraic description of reduction
(2) Symplectic and Hamiltonian vector fields
(3) Symplectic actions of Lie groups
(4) Symplectic, Hamiltonian ad Poisson actions of Lie algebras; moment maps; obstructions (Digression: Lie algebra cohomology)
(5) Marsden–Weinstein reduction
(6) Noether's Theorem
(7) Infinite-dimensional examples: BF theories, Chern–Simons theory, Maxwell's equations

5. Poisson geometry

(1) Motivations, definitions and examples; Poisson algebras
(2) Digression: The Schouten–Nijenhuis bracket
(3) Poisson cohomology and its interpretation up to second degree
(4) Canonical actions of Lie groups and quotients
(5) Poisson and Hamiltonian vector fields
(6) Coisotropic submanifolds and reduction; algebraic description ("coisotropes"); examples
(7) Symplectic groupoids and the integration of Poisson manifolds
(8) Twisted symplectic groupoids and twisted Poisson manifolds

6. Canonical quantization

(1) Schrödinger's quantization of $T^*\mathbb{R}^n$
(2) Schrödinger's equation; position and momentum operators; the ordering problem
(3) Expectation values; Ehrenfest's Theorem; the "correspondence principle"
(4) Heisenberg's uncertainty principle
(5) Schrödinger's and Heisenberg's pictures
(6) The momentum description

(7) The harmonic oscillator; creation and annihilation operators
(8) Problems of canonical quantization; "quantization as a functor"; Dirac's dream; the Groenewald–van Howe Theorem

7. Geometric quantization

(1) Prequantization and the integrality condition (Digression: line bundles, connection, curvature, Chern class)
(2) The integral Hall effect
(3) Prequantization and representations of Lie algebras
(4) Prequantization of the Poisson algebra of functions
(5) Polarizations
(6) Holomorphic quantization and the Riemann–Roch Theorem
(7) The Schrödinger equation
(8) Half densities; distributional states; the metaplectic correction
(9) Quantization and reduction; introduction to the $[Q, R]$-conjecture
(10) Lagrangian submanifolds and states; Weinstein's dream
(11) Problems of geometric quantization

8. Path-integral quantization

(1) Motivations
(2) From the Schrödinger equation to the path integral
(3) From the path integral to the Schrödinger equation
(4) The semiclassical limit
(5) Perturbation theory
(6) Introduction to field theory, functional integrals, regularization, and renormalization
(7) Degenerate critical points

9. Deformation quantization

(1) Motivations and definitions
(2) The Moyal product
(3) The Gutt product
(4) Path integrals and deformation quantization of cotangent bundles
(5) Introduction to the A-model and to the Poisson sigma model
(6) The star-Schrödinger equation
(7) Fourier–Dirichlet expansion, spectra, and dependence on the star product
(8) Introduction to the BRST method in deformation quantization

References

[1] S. Bates, A. Weinstein, *Lectures on the Geometry of Quantization*, Berkeley Mathematics Lecture Notes **8** (AMS, 1997).

[2] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics **1764** (Springer-Verlag, Berlin, 2001).

[3] A. Cannas da Silva, A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes **10** (AMS, 1999).

[4] R. P. Feynman, A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[5] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids* (Cambridge University Press, Cambridge, 2005).

[6] J.-P. Ortega, T. Ratiu, *Moment Maps and Hamiltonian Reduction*, Progress in Mathematics **222** (Birkhäuser Boston, Inc., Boston, MA, 2004).

[7] N. M. J. Woodhouse, *Geometric quantization*, Second edition, Oxford Mathematical Monographs, Oxford Science Publications (The Clarendon Press, Oxford University Press, New York, 1992).