BOUNDARY STRUCTURE OF GENERAL RELATIVITY WITH EXTERNAL FIELDS

Master's Thesis

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Contents

Introduction

Despite being more than a hundred years old, General Relativity plays a central role in modern physics. It is inherently mathematical in its formulation, as it derives from the basic need of having a theory which is covariant. Indeed one of the fundamental assumption from which it arises is that equations must be independent of the reference frame in which the observer is placed.

The aim of this thesis, following [8], is to investigate GR in the Palatini–Cartan formulation with the coupling of external fields in the first order formalism. The result is a classical theory containing gravity and other fundamental forces, encoded in the language of field theory. Moreover, we assume our space–time manifold to have a boundary and investigate the symplectic structure of the space of fields on it.

It was first found by Cattaneo, Canepa and Schiavina [13, 8] that pure GR allows for a cohomological description of the observables on the space of boundary fields. In particular, they constructed a symplectic supermanifold $\mathcal{F}$ whose base manifold is the space of (pre)boundary fields. They showed that on $\mathcal{F}$ it is possible to define an operator Q whose degree-0 cohomology gives the space of observables on the boundary. Making use of the induced Poisson structure on $\mathcal{F}$, Q can be defined as $Q := \{S, \cdot\}$, where S is a function called BFV action and satisfies the Classical Master Equation (CME): $\{S, S\} = 0$. This procedure takes the name of BFV Formalism, and was first introduced by Batalim, Fradkin and Vilkovisky [3] as an extension to the boundary of the BV formalism [5][4], a generalization of BRST [7] used in gauge theory.

As it turns out, the BFV description is not only useful as a cohomological resolution (of symplectic reductions), but it is also used to perform quantization. In particular, it is in principle possible to quantize the supermanifold $\mathcal{F}$ to a graded Hilbert space and to find an operator $\hat{S}$ quantizing S. Furthermore, $\hat{S}$ satisfies the quantum master equation (QME) $[\hat{S}, \hat{S}] = 0$, which is reduced to the CME at the lowest order in $\hbar$. If one can prove that the QME is satisfied at all orders in $\hbar$, then the cohomology in degree 0 of $\hat{S}$ will define the Hilbert space quantizing the space of obervables on the boundary.

Outline of the Thesis

Chapter 1 is dedicated to presenting a discussion on principal fiber bundles and gauge theory, as we will reformulate General Relativity in this geometrical framework and because it is necessary to fully understand any field theory which can be a candidate to have physically interesting applications.

Chapter 2 is a review in a modern language of some well-known facts about field theory. In particular, we precisely define what a field is and describe its dynamics in a nice Lagrangian formalism, which is to this day one of the most natural and most useful descriptions available, especially when one is faced with the problem of defining a field theory on the boundary of space–time.

In Chapter 3 we look at some useful facts about Poisson and symplectic geometry, and discuss about symplectic reduction. In particular, we take into consideration the BRST formalism in the finite dimensional case, in order to be able to fully understand the BFV formalism, which is presented in the last section.

Chapter 4 starts with a thorough review of GR in Palatini-Cartan formalism. Then, we add the coupling of a scalar field and look in particular at the classical boundary structure, computing the Poisson brackets of the constraints (i.e. the Euler-Lagrange equations projected to the boundary). As it turns out, the structure is unaltered with respect to pure gravity, in fact we find the constraints to be first class, meaning that their Poisson brackets are a $\mathcal{C}^\infty(M)$-linear combination of the constraints. Finally, we define the BFV data on the boundary. In particular, we define a BFV action and prove that the classical master equation is satisfied.

Chapter 5 and 6 are applications of the same concepts that we saw in Chapter 4 respectively to the cases where we couple a Yang–Mills field and scalar electrodynamics to gravity. The introduction of a gauge field changes the boundary structure, but the constraints remain first class. In both cases, this allows to construct a well-defined BFV action satisfying the CME.

Finally, appendix A is a short review of some useful concepts of supergeometry, while appendix B contains some technical results and lengthy proofs that we left out from the body of the thesis in order to make it lighter.

Chapter 1

Mathematical Formalism of Gauge Theory

The theory of fiber bundles is of great importance in physics, especially when dealing with field theories with local symmetries (i.e. gauge theories). Fields themselves are usually defined as sections of a certain fiber bundle: i.e. as maps $\phi\colon M \to F$ which, composed with the projection $\pi\colon F \to M$, give the identity on the space–time manifold M.

If we look at the simple case when the fiber bundle is trivial, namely when $F = M \times W$ for some vector space W, we see that a section is equivalent to a global function $\phi\colon M \to W$. Locally, any fiber bundle can be trivialized (it can be given the structure of a trivial bundle). The spaces $\{W_x\}_{x\in M}$ are called the *fibers* of the fiber bundle, and they are all equivalent to a space W.

One particularly interesting example of fiber bundle is a *principal G-bundle*, namely a fiber bundle whose fiber is the Lie group G, which can furthermore act on the points of the total space by means of a right action.

It turns out that this object is particularly useful when one tries to geometrically describe field theories with local symmetries. In the following chapter we will give the fundamental definitions needed to discuss such constructions in a rigorous way, but not all the statements will be proved, since the scope of the chapter is to introduce the framework and the notation that will be fundamental to understand the rest of the thesis.

We refer to the book of C. Isham [18] for the proofs that we leave out, as well as the one of Kobayashi and Nomizu [21], while the notation used here follows [15], [32], [30] and [19].

1.1 Principal Bundles

Before diving into the discussion about principal fiber bundles, we recall some known facts about Lie groups and algebras.

Useful Facts about Lie Groups and Lie Algebras

Definition 1.1.1 (Lie group)**.** A **Lie group** is a group G which is also a differentiable manifold, such that the multiplication μ and the inversion i are smooth maps. They are defined as

- $\mu\colon G \times G \to G\colon (g, h) \mapsto \mu(g, h) := g \cdot h$, where we use the symbol $\cdot$ to indicate the group composition, we will omit it in the remainder of the chapter;
- $i : G \to G : g \mapsto i(g) := g^{-1}$.

Definition 1.1.2 (left translation map)**.** For any $g \in G$, the **left translation map** l_g is defined such that

$$\begin{aligned} l_g :& G \to G \\ & h \mapsto l_g(h) := gh. \end{aligned} \tag{1.1}$$

Remark 1.1.3. It can be proven that if G is a Lie group then l_g is a diffeomorphism. Furthermore, we can as well define a **right translation map** by

$$r_g : G \to G : h \mapsto hg. \tag{1.2}$$

Definition 1.1.4 (adjoint map)**.** For every $g \in G$ we can define the **adjoint map** $Ad_g : G \to G$ by

$$Ad_g := l_g \circ r_{g^{-1}} \quad Ad_g(h) := ghg^{-1}. \tag{1.3}$$

Now, since G is also a smooth manifold, it is possible to define the tangent bundle of G and vector fields on G in the usual way as sections of TG, namely $\mathfrak{X}(G) := \Gamma(TG)$.

With this definition, we can also define the push-forward of the left translation map as a map

$$\begin{aligned} (l_g)_* :& \mathfrak{X}(G) \to \mathfrak{X}(G) \\ & X \mapsto (l_g)_*(X). \end{aligned} \tag{1.4}$$

Then we have the following definition:

Definition 1.1.5 (left-invariant vector field)**.** A vector field X on G is **left-invariant** if $\forall g \in G$, $(l_g)_*(X) = X$. In particular, this means that $(l_g)_*(X_h) = X_{gh}$. We denote the set of all left-invariant vector fields as $L(G)$. Furthermore, $\forall g, h \in G$, we have

$$(l_g)_* \circ (l_h)_* = (l_{gh})_*. \tag{1.5}$$

Remark 1.1.6. As we know, the set of vector fields on G forms a Lie algebra with the usual Lie brackets of vector fields defined on smooth manifold. It can be proven that $L(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.

Lemma 1.1.7. *There is an isomorphism of the set of left-invariant vector fields on G and the tangent space at the identity T_eG. The isomorphism is given by*

$$\begin{aligned} L :& T_eG \xrightarrow{\sim} L(G) \\ & A \mapsto L^A, \end{aligned} \tag{1.6}$$

with $L^A_g := (l_g)_*(A)$, where by abuse of notation we indicated $(l_g)_* : T_eG \to T_{ge}G = T_gG$

Definition 1.1.8 (Lie algebra of a Lie group)**.** The **Lie algebra** $\mathfrak{g}$ of the Lie group G is defined to be the tangent space at the identity T_eG. The Lie bracket of elements in T_eG is defined by using the isomorphism defined above, which makes it a Lie algebra homorphism. In particular, for all $A, B \in \mathfrak{g}$, we find:

$$[A, B] := [L^A, L^B]_e, \tag{1.7}$$

which trivially satisfies $L[A, B] = [L^A, L^B]$

Theorem 1.1.9. *Every left-invariant vector field on a Lie group is complete, i.e. its integral curves are defined on $\mathbb{R}$.*

The previous theorem states that given any vector field on a Lie group, it is possible to construct an integral curve (such that the vector field is its tangent vector point by point on the curve) defined on all $\mathbb{R}$, hence allowing the following definition

Definition 1.1.10 (exponential map)**.** The **exponential map** exp: $T_eG \to G$ is defined for all $A \in T_eG$ by

$$\exp A := \exp(tA)|_{t=1}, \tag{1.8}$$

where $\exp(tA)$ is the integral curve of the left invariant vector field L^A such that it passes through $e \in G$ at $t = 0$, namely such that

$$\frac{d}{dt}\exp(tA) = L^A_{\exp(tA)} \qquad \exp(0) = e \in G. \tag{1.9}$$

Now we consider groups acting on manifolds, namely such that to each group element there is a diffeomorphism which acts on the point of the manifold, in particular we look for a homorphism of groups between the Lie group and the group of diffeomorphisms on a manifold.

Definition 1.1.11 (right action)**.** Let G be a Lie group and M be a smooth manifold, then a **right action** R of G on M is a defined to be a smooth map

$$\begin{aligned} R: M \times G &\longrightarrow M \\ : (x, g) &\longmapsto R_g(x) \in M \end{aligned} \tag{1.10}$$

satisfying

(i) $R_e(x) = x \; \forall x \in M$, where e is the identity in G;

(ii) $R_{g_1} \circ R_{g_2} = R_{g_1 g_2} \; \forall g_1, g_2 \in G$.

Since each R_g is a diffeomorphism, R can also be seen as a map $G \to Diff(M)$, which by the last point is a group homomorphism

Definition 1.1.12 (adjoint representation of G and $\mathfrak{g}$)**.** A **representation** of a Lie group G on a vector space V is a group homorphism $\phi : G \to \text{Aut}(V)$. When we consider $V = \mathfrak{g}$ as the Lie algebra of the Lie group, we have the **adjoint representation**, defined as

$$\begin{aligned} Ad : &G \to \text{Aut}(\mathfrak{g}) \\ &g \mapsto Ad_g, \end{aligned} \tag{1.11}$$

such that for every $A \in \mathfrak{g}$,

$$Ad_g(A) := \frac{d}{dt}\left(Ad_g(\exp(tA))\right)|_{t=0}. \tag{1.12}$$

If G is a matrix group, then we simply have $Ad_g(A) = gAg^{-1}$.

Definition of a Principal Bundle

Definition 1.1.13 (ρ-equivariance)**.** Let M, N be smooth manifolds, G, H Lie groups and $R : M \times G \to M$, $R' : N \times H \to N$ be smooth right actions. Then a map $f : M \to N$ is said to be **ρ-equivariant** if the diagram

$$\begin{array}{ccc} M \times G & \xrightarrow{f \times \rho} & N \times H \\ \downarrow R & & \downarrow R' \\ M & \xrightarrow{f} & N \end{array} \tag{1.13}$$

commutes. Here $(f \times \rho)(x, g) := (f(x), \rho(g))$ and $\rho : G \to H$ is a group homorphism.

Equivalently, one can require that $f(R_g(x)) = R'_{\rho(g)}(f(x))$.

Definition 1.1.14 (orbit of a point)**.** Given a group action $\Psi : M \times G \to M : (x, g) \mapsto \Psi_g(x)$, $\forall x \in M$ we define the **orbit** of x under the action of G as $O_x := \{y \in M | \exists g \in G, y = \Psi_g(x)\}$

Definition 1.1.15 (principal G-bundle)**.** Let G be a Lie group and M be a smooth manifold. A smooth fiber bundle (P, π, M) is called **Principal G-bundle** if

(i) P is equipped with a smooth right action $R : P \times G \to P$;

(ii) the right action is free, namely if $R_g(p) = p$ then necessarily $g = e$;

(iii) $P \xrightarrow{\pi} M$ is isomorphic as a bundle to $P \xrightarrow{\pi'} P/G$, namely there exists a diffeomorphism $\phi : M \to P/G$ such that the diagram

$$\begin{array}{ccc} & P & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\pi'} \\ M & \xrightarrow{\phi} & P/G \end{array} \tag{1.14}$$

commutes, where π' is the projection onto the quotient, and P/G is the orbit set. Equivalently we ask that $\pi' = \phi \circ \pi$.

Remark 1.1.16. If R is a free action then any orbit $O_p = \{q \in P \mid \exists g \in G, q = R_g(p)\}$ is diffeomorphic to the group G. Thus, if $[p] = \phi(x)$, $\pi^{-1}(x) = \pi'^{-1}([p]) = O_p \simeq G$, which means that every fiber is diffeomorphic to G.

Definition 1.1.17 (trivialization)**.** A **trivialization** of a principal G-bundle is a collection of diffeomorphisms (U_α, ϕ_α), where $\{U_\alpha\}$ is an open cover of M and

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \qquad (\pi = p_1 \circ t_\alpha), \tag{1.15}$$

where we set p_1 to be the projection onto the first factor $p_1 : U_\alpha \times G \to U_\alpha : (x, g) \mapsto x$.

We can also define **transition functions**, $\forall x \in U_1 \cap U_2$,

$$\phi_{12} := \phi_1 \circ \phi_2^{-1} : (U_1 \cap U_2) \times G \to (U_1 \cap U_2) \times G. \tag{1.16}$$

Defining $(\phi_\alpha)_x := \phi_\alpha(x, \cdot) : \pi^{-1}(x) \to G$, the fiber bundle can be covered by a set of **fibered coordinates** (x^μ, y^i) around $p \in P$, where x^μ is a chart around $x \in M$ and y^i is a chart around $(\phi_\alpha)_x(p) \in G$. Then transition functions between fibered coordinates are expressed for an arbitrary fiber bundle as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y'^i(x, y) \end{cases} \tag{1.17}$$

In the case of a principal fiber bundle though, the transition functions must be of the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ g' = \phi(x) \cdot g \end{cases} \tag{1.18}$$

which means that changes of coordinates on the fibers are obtained via a left translation by $\phi : U_\alpha \cap U_\beta \to G$

Definition 1.1.18 (associated bundle)**.** Let $(P, M, \pi; G)$ be a principal bundle, F a smooth manifold and $\lambda : G \times F \to F$ be a left action. We can induce a right action $\hat{\lambda}$ of G on $P \times F$ by

$$\begin{aligned} \hat{\lambda} : G \times (P \times F) &\to (P \times F) \\ (g, p, f) &\longmapsto \hat{\lambda}_g(p, f) := (R_g(p), \lambda_{g^{-1}}(f)). \end{aligned} \tag{1.19}$$

Then the **associated bundle** to P is defined as the quotient of $P \times F$ with respect to $\hat{\lambda}$, namely $P \times_\lambda F := P \times F/ \sim$, where $(p, f) \sim (q, h)$ if $\exists g \in G$ such that

$$\begin{aligned} q &= R_g(p), \\ h &= \lambda_{g^{-1}}(f). \end{aligned} \tag{1.20}$$

The projection on $P \times_\lambda F$ is defined by $\pi_\lambda : P \times_\lambda F \to M : [p, f] \mapsto \pi_\lambda(p)$

1.2 Connections on Principal Bundles

We now equip principal bundles with a geometrical object which is fundamental in gauge theory: a **principal connection**. There is more than one equivalent definition of connection, and we will present them in the following section.

We start by defining the Ehresmann connection:

Definition 1.2.1 (vertical subspace)**.** Let (P, π, M) be a principal G-bundle. The **vertical subspace** at $p \in M$ is defined as

$$V_pP := \{X \in T_pP \mid \pi_*(X) = 0\}. \tag{1.21}$$

The disjoint union of all vertical subspaces is called **vertical subbundle** VP

Definition 1.2.2 (horizontal subspace)**.** At any $p \in P$ the **horizontal subspace** $H_pP \subset T_pP$ is defined to be such that

$$H_pP \oplus V_pP = T_pP. \tag{1.22}$$

The choice of H_pP is not unique but, once the choice has been made, any vector $X_p \in T_pP$ uniquely decomposes into the sum of a horizontal and a vertical vector field as

$$X_p = X_{p,v} + X_{p,h} \qquad \text{with} \qquad X_{p,v} \in V_pP\ , X_{p,h} \in H_pP. \tag{1.23}$$

Definition 1.2.3 (Ehresmann connection)**.** An **Ehresmann connection** is the choice for every $p \in P$ of a horizontal subspace such that

(i) $\forall g \in G$, $\forall X_p \in H_pP$

$$(R_g)_\star X_p \in H_{R_g(p)}P; \tag{1.24}$$

(ii) every $X \in \Gamma(TP)$ uniquely decomposes into smooth horizontal and vertical vector fields $X_h \in \Gamma(HP)$ and $X_v \in \Gamma(VP)$

$$X = X_h + X_v. \tag{1.25}$$

In other words, we can see a connection as a distribution H on P that is complementary to the (naturally defined) distribution of vertical vector fields on P and such that it is G-invariant.

This was the first definition of a connection on a principal bundle. As it turns out, it carries the same data of a Lie algebra-valued 1-form obeying certain properties.

Definition 1.2.4 (connection 1-form)**.** A Lie algebra-valued 1-form ω over P is called **connection 1-form** if it is such that

$$\ker(\omega_p) = \{X \in T_pP \mid \omega_p(X) = 0\} = H_pP. \tag{1.26}$$

A choice of a connection one form automatically defines a horizontal distribution on P. One can easily check that it is also G-invariant.

Now we want to define a map which associates to any element of the Lie algebra $\mathfrak{g}$ of G an elements of the Lie algebra of vector fields on P, which is also a Lie algebra homomorphism.

Definition 1.2.5 (fundamental map)**.** The **fundamental map** $\# : \mathfrak{g} \to \Gamma(TP)$ is defined such that

$$\tilde{X}_p^A := \#_p(A) := \frac{d}{dt}\left[R_{\exp(tA)}(p)\right]\Big|_{t=0}. \tag{1.27}$$

In particular, it is easy to see that it is an algebra homomorphism, namely

$$[\tilde{X}^A, \tilde{X}^B] = \tilde{X}^{[A,B]}. \tag{1.28}$$

Remark 1.2.6. We notice that $\tilde{X}^A$ is a vertical vector field, in fact

$$\pi_*(\tilde{X}_p^A) = \frac{d}{dt}\pi_*\left[R_{\exp(tA)}(p)\right]\Big|_{t=0} = \frac{d}{dt}\pi(p)\Big|_{t=0} = 0. \tag{1.29}$$

Theorem 1.2.7. *[30] The following propositions are true:*

(a) $\forall p \in P$, we have $\omega_p(\tilde{X}_p^A) = A$, namely when acting on vertical vector fields the connection 1-form gives the element of the Lie algebra corresponding to that vertical field;

(b) $R_g^*(\omega) = (Ad_{g^{-1}})_*\omega$.

Definition 1.2.8 (curvature 2-form)**.** Let ω be a connection 1-form on a principal G-bundle P, then the **curvature 2-form** F is defined to be

$$F := d\omega + \frac{1}{2}[\omega, \omega], \tag{1.30}$$

where $[\cdot,\cdot]$ is defined by employing the Lie bracket in $\mathfrak{g}$ such that

$$[\omega, \eta](v, u) = \frac{1}{2}\{[\omega(u), \eta(v)] - [\omega(v), \eta(u)]\}, \tag{1.31}$$

or equivalently, it is defined by composing the wedge product with the Lie bracket on the Lie algebra $\mathfrak{g}$ of G. In particular, for all $\mathfrak{g}$-valued forms in $\Omega^\bullet(P, \mathfrak{g}) := \Gamma(\wedge^\bullet T^*P \otimes \mathfrak{g})$, we have

$$[\alpha \otimes A, \beta \otimes B] := (\alpha \wedge \beta)[A, B]. \tag{1.32}$$

Local Connection 1-form and Curvature 2-form

We now define yet another type of connection, the one that is obtained locally by pulling back the connection 1-form from the total space to the base manifold.

As we saw, we have canonical local sections $\sigma_\alpha : U_\alpha \to P$ associated to the trivialization of the principal bundle

Definition 1.2.9 (local connection)**.** The **local connection 1-form** ω_α is obtained by pulling back ω along a local section σ_α

$$\omega_\alpha := \sigma_\alpha^*(\omega_\alpha) \in \Omega^1(U_\alpha, \mathfrak{g}). \tag{1.33}$$

ω_α is also called **gauge field** and is the fundamental object in gauge theory.

The local connection is not really a global Lie-algebra-valued 1-form on M, since, as we will briefly see, it does not transform as a 1-form on the overlap of two patches on which two local sections are defined. In order to see why, frst we need the following

Definition 1.2.10 (Maurer–Cartan form)**.** Let G be a Lie group, the **Maurer–Cartan** form is the $\mathfrak{g}$-valued 1-form θ on G defined by

$$\theta_g := (l_{g^{-1}})_* : T_gG \to T_eG. \tag{1.34}$$

Remark 1.2.11. The Maurer–Cartan form satisfies the **Maurer-Cartan** equation

$$d\theta + \frac{1}{2}[\theta, \theta] = 0 \tag{1.35}$$

and is left-invariant

$$l_g^*(\theta_h) = \theta_{g^{-1}h}. \tag{1.36}$$

Furthermore, if G is a matrix group, then the Maurer-Cartan form is simply

$$\theta_g = g^{-1}dg. \tag{1.37}$$

Definition 1.2.12. (gauge map) Let $\sigma_\alpha : U_\alpha \to P$ and $\sigma_\beta : U_\beta \to P$ be two local sections such that $U_\alpha \cap U_\beta \neq \emptyset$. The map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is called **gauge map** and is such that $\forall x \in U_\alpha \cap U_\beta$

$$\sigma_\beta(x) = R_{g_{\alpha\beta(x)}}\sigma_\alpha(x). \tag{1.38}$$

Now we have all the ingredients we need to calculate how the gauge field transforms on the overlap of two patches.

Theorem 1.2.13. *[18] With the previous hypotheses, the gauge field $\omega_\beta := \sigma_{(\beta)}^*(\omega)$ is related to ω_α on $U_\alpha \cap U_\beta$ by*

$$\omega_\beta = ad_{g_{\alpha\beta}^{-1}} \circ \omega_\alpha + g_{\alpha\beta}^*(\theta), \tag{1.39}$$

where by $g_{\alpha\beta}^{-1}$ we intend the inverse of the group element $g_\alpha\beta$.

The theorem simplifies if G is a matrix Lie group, in which case the following equation holds

$$\omega_\beta = g_{\alpha\beta}\omega_\alpha g_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1}dg_{\alpha\beta}. \tag{1.40}$$

Remark 1.2.14. The content of the theorem above defines **gauge transformatios** of the gauge field from one patch of the base manifold to another. Physical observables are constructed starting from the gauge field in a way that makes them invariant under gauge transformation, hence defining a local symmetry called **gauge symmetry**. Since the "physics" is independent of the gauge, we are left with redundancies arising from this local degrees of freedom. In most cases it is useful to "fix the gauge" to make calculations easier, but we stress that the result will remain gauge-independent. Many fundamental physical properties arise from this redundancy.

Definition 1.2.15 (gauge field strength)**.** With the hypotheses of the previous definitions, we define the **field strenght** to be the pull-back of the curvature two-form $F \in \Omega(P, \mathfrak{g})$ by a local section $\sigma_\alpha : U_\alpha \to P$.

$$F_\alpha := \sigma_\alpha^* F = d\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha]. \tag{1.41}$$

Remark 1.2.16. The previous theorem stating the transformation law of the gauge field extends to the gauge field strength, giving the transformation rule

$$F_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ F_\alpha = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}. \tag{1.42}$$

In order to prove the above expression, one makes use of eq. (1.35) for the Maurer-Cartan form.

Example 1.2.17 (Yang–Mills action). Let us consider a principal G-bundle (P, G, π, M) over an N-dimensional pseudo-riemannian manifold M. We assume the Lie group G to be compact, and such that its Lie algebra $\mathfrak{g}$ is semisimple. Defining a principal connection $\omega \in \Omega^1(P, \mathfrak{g})$, we might consider its local form A_α on a patch U_α of M (from now on, we will get rid of the index α to lighten the notation).

In coordinates, assuming $\{T^I\}$ to be a basis for the Lie algebra $\mathfrak{g}$, we have

$$A(x) = A^I_\mu(x) T_I dx^\mu. \tag{1.43}$$

We now want to construct gauge invariant objects starting from the gauge field A. As we saw, the gauge field strength transforms in the adjoint representation of the gauge group G, which is a very useful property. In order to exploit it, we assume G to be a matrix Lie group, hence making $\mathfrak{g}$ a matrix Lie algebra. In coordinates we have

$$\begin{aligned} F_A = dA + \frac{1}{2}[A, A] &= \frac{1}{2} F^I_{\mu\nu}(x) T_I dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(\partial_\mu A^I_\nu - \partial_\nu A^I_\mu + f^I_{JK} A^I_\mu A^J_\nu) T_I dx^\mu \wedge dx^\nu. \end{aligned} \tag{1.44}$$

Where f^I_{JK} are the structure constants of $\mathfrak{g}$ defined by $[T_J, T_K] = f^I_{JK} T_I$. Now, being $\mathfrak{g}$ semi-simple, it has the property that

$$\mathrm{Tr}(T_I T_J) =: \gamma_{IJ} \tag{1.45}$$

defines a definite metric γ_{IJ} on $\mathfrak{g}$, called Killing metric.

We can now exploit the definition of Hodge star operator on M, $\star : \Omega^k(M) \to \Omega^{N-k}(M)$ given in coordinates for a generic $\alpha = \frac{1}{k!}\alpha_{\mu_1 \cdots \mu_k} dx^{\mu_1} \cdots dx^{\mu_k}$ by

$$\star\alpha := \frac{1}{k!(N-k)!}\sqrt{|\det(g_{\mu\nu})|}\alpha_{\nu_1\cdots\nu_k} g^{\nu_1\mu_1} \cdots g^{\nu_k\mu_k} \epsilon_{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_N} dx^{\mu_{k+1}} \cdots dx^{\mu^{N-k}}. \tag{1.46}$$

Then the minimal gauge invariant quantity that we can construct is given by

$$\mathrm{Tr}(F_A \wedge \star F_A) = \mathrm{Vol}_g F^I_{\mu\nu} F^J_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \gamma_{IJ}, \tag{1.47}$$

where $\mathrm{Vol}_g = \sqrt{|\det(g)|} d^N x$. We show that the above formula is invariant under gauge transformation using the cyclic properties of traces by

$$\begin{aligned} \mathrm{Tr}(F'_A \star F'_A) &= \mathrm{Tr}[g F_A g^{-1} g(\star F_A) g^{-1}] \\ &= \mathrm{Tr}[g(F_A \star F_A) g^{-1}] \\ &= \mathrm{Tr}(F_A \star F_A). \end{aligned} \tag{1.48}$$

The theory which arises from taking (1.47) as Lagrangian density is called Yang–Mills theory, which reduces to Electromagnetism if one chooses $G = U(1)$.

The Space of Connections

An important theorem states that it is always possible to define a connection on a principal G-bundle. We now would like to know how the space of connection is constructed. In particular we saw that a local connection does not transform according to the adjoint representation of the gauge group on the Lie algebra, but is also shifted by a term depending on the differential of the "gauge map". However, when considering the difference of two local connections, we see that it transforms in the adjoint representation of G.

In order to formalize this, first we consider a vector space V and a representation $\rho : G \to \mathrm{GL}(V)$. We denote by E the associated bundle to V $E := P \times_\rho V$.

Definition 1.2.18 (tensorial forms)**.** A V-valued k-form $\omega \in \Omega^k(P, V)$ is called

- **horizontal** if $\omega(v_1, \cdots, v_k) = 0$ if at least one v_i is vertical;
- **equivariant** if $R_g^*(\omega) = \rho(g^{-1}) \circ \omega$.

A form which is both equivariant and horizontal is called **tensorial**. The space of tensorial k-forms is denoted by $\Omega^k_{G,\rho}(P, V)$.

Remark 1.2.19. As we mentioned, the connection is not a tensorial form, because it is not horizontal. The non-tensoriality is "measured" by the Maurer-Cartan form

Theorem 1.2.20. *There is an isomorphism between tensorial forms on P and differential forms on M with values in the associated bundle $E := P \times_\rho V$*

$$\Omega^k_{G,\rho}(P, V) \simeq \Omega^k(M, E). \tag{1.49}$$

Proof. For any tensorial form $\xi \in \Omega^k_{G,\rho}(P, V)$ we define a $\eta_\xi \in \Omega^k(M, E)$ for all $x \in M$ by

$$\eta_{\xi x}(X_1, \cdots, X_k) := [p, \xi_p(Y_1, \cdots, Y_k)], \tag{1.50}$$

for $p \in \pi^{-1}(x)$, $X_i \in T_xM$ and $Y_j \in T_pP$ such that $\pi_*(Y_j) = X_j$. This definition is independent of the choice of Y_j, since for any two Y_j, Y_j' such that $\pi_*(Y_j) = \pi_*(Y_j') = X_j$ we have that $Y_j - Y_j'$ is vertical, but ξ is horizontal, therefore it only "sees" the equivalence class of horizontal vectors which is in one-to-one correspondence with X_j. Moreover, the definition of η is also independent of p because ξ is equivariant. Indeed, choosing $p' := R_g(p)$ and $\tilde{Y}_j \in T_{R_g(p)}P$ such that $\pi_*(\tilde{Y}_j) = X_j$ for all j's, we have

$$\begin{aligned}
[R_g(p), \xi_{R_g(p)}(\tilde{Y}_1, \cdots, \tilde{Y}_k)] &= [R_g(p), \xi_{R_g(p)}(R_{g*}Y_1, \cdots, R_{g*}Y_k)] \\
&= [R_g(p), \rho(g^{-1})\xi_p(Y_1, \cdots, Y_k)] \\
&= [p, \xi_p(Y_1, \cdots, Y_k)].
\end{aligned}$$

Vice-versa, let us consider a E-valued k-form η and define

$$\xi_{\eta p}(Y_1, \cdots, Y_k) := \iota_p^{-1}\eta_x(\pi_*(Y_1), \cdots, \pi_*(Y_k)), \tag{1.51}$$

where $\iota_p : V \to \tilde{\pi}^{-1}(x)$[1] is the inclusion of the fiber defined by $\iota_p(f) := [p, f]$ for any $p \in \pi^{-1}(x)$. We need to prove that ξ_η is tensorial.

[1] π and $\tilde{\pi}$ are respectively the projections of the bundles P and E.

Verticality is quickly checked by noticing that if at least one Y_i is vertical then $\pi_*(Y_i) = 0$ and therefore $\xi_\eta = 0$ For equivariance, since for any $v \in V$ we have $\iota_p(v) = \iota_{R_g(p)}(\rho(g^{-1})v)$ it follows that

$$\begin{aligned}(R_{g^*}\xi_\eta)_p(Y_1, \cdots, Y_k) &= \xi_{\eta R_g(p)}(R_{g*}Y_1, \cdots, R_{g*}Y_k) \\ &= \iota^{-1}_{R_g(p)}\eta_x((\pi \circ R_g)_*Y_1, \cdots, (\pi \circ R_g)_*Y_k) \\ &= \iota^{-1}_{R_g(p)}\eta_x(\pi_*Y_1, \cdots, \pi_*Y_k) \\ &= \iota^{-1}_{R_g(p)}\iota_p(\xi_{\eta p}(Y-1, \cdots, Y_k)) \\ &= \rho(g^{-1})\xi_{\eta p}(Y_1, \cdots, Y_k),\end{aligned}$$

where we used $\pi \circ R_g = \pi$. ✓

As an immediate consequence, since the difference of two connections transforms in the adjoint representation of the Lie group G, once we pick a reference connection ω_0, we can define the space of connections as $\mathcal{A}(M) := \Omega^1_{G,\mathrm{ad}}(P, \mathfrak{g})$

Corollary 1.2.21. *Let (P, M, π, G) be a principal bundle. The space of connections $\mathcal{A}(P)$ on P is an affine space modeled over the vector space $\Omega(M, \mathrm{ad}(P))$, where ad$(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$ is the **adjoint bundle**.*

Exterior Covariant Derivative

The usual de Rham differential on P does not send tensorial forms to tensorial forms, but we would like to define a kind of derivative that does exactly that.

Definition 1.2.22 (exterior covariant derivative)**.** Let P be a principal G-bundle and V a vector space. We define the **exterior covariant derivative** on vector valued forms $d^h : \Omega^k(P, V) \to \Omega^{k+1}(P, V)$ by

$$d^h\alpha(v_0, \cdots, v_k) := h^*d\alpha(v_0, \cdots, v_k) = d\alpha(hv_o, \cdots, hv_k), \tag{1.52}$$

where h is the horizontal projection, depending on the choice of horizontal subspace and hence on the connection.

Remark 1.2.23. If we restrict the covariant derivative only to horizontal forms $\Omega^k_{G,\rho}(P, V)$, then it will depend only on the choice of the connection ω, which allows the following definition

Definition 1.2.24 (covariant derivative d_ω)**.** We define the exterior covariant derivative d_ω : $\Omega^k_{G,\rho}(P, V) \to \Omega^{k+1}_{G,\rho}(P, V)$ on tensorial forms by

$$d_\omega\alpha := d\alpha + [\omega, \alpha], \tag{1.53}$$

where $[\omega, \alpha] := \rho(\omega)(\alpha)$ (the wedge product is assumed in the last expression but we omit the symbol $\wedge$ for a lighter notation)

By the previous theorem we are also able to extend the covariant derivative we just defined to the differential forms on M with values on the associated bundle $P \times_\rho V$. In particular

$$\begin{aligned}d_\omega :&\Omega^k(M, P \times_\rho V) \to \Omega^{k+1}(M, P \times_\rho V) \\ &\alpha \mapsto d_\omega\alpha := d\alpha + [\omega, \alpha].\end{aligned} \tag{1.54}$$

Proposition 1.2.25 (First Bianchi identity)**.**

$$d_\omega^2 \alpha = [F, \alpha]. \tag{1.55}$$

Therefore for a flat connection, for which $F = 0$*, we have* $d_\omega^2 = d^2 = 0$*.*

Proposition 1.2.26 (Second Bianchi identity)**.**

$$d_\omega F_\omega = 0. \tag{1.56}$$

Torsion

Definition 1.2.27 (soldering of a G-principal bundle)**.** Let (P, π, M, G) be a principal bundle, and $\rho : G \to \mathrm{Aut}(V)$ be an N-dimensional representation of G, with $\dim(M) = N$ and G a subgroup of $\mathrm{GL(N}, \mathbb{R})$. We define a **solder form** as the vector-valued one-form $e \in \Omega^1(M, P \times_\rho V)$ to be an isomorphism between the tangent bundle TM and $P \times_\rho V$

The soldering form will be of particular importance when dealing with General Relativity in the Palatini-Cartan formalism, since it will replace the metric as a dynamical field.

Definition 1.2.28 (torsion form)**.** The torsion form is defined to be

$$\Theta := d_\omega e \in \Omega^2(M, P \times_\rho V). \tag{1.57}$$

Of course, it can also be seen as a tensorial 2-form with values in V.

Chapter 2

Elements of Classical Lagrangian Field Theory

Given a space–time manifold M and a fiber bundle $\pi : F \to M$ over M, a field configuration is a smooth local section $\phi \in \Gamma(M, F)$, namely such that $\pi \circ \phi = id_M$. The set of all smooth sections $\Gamma(M, F)$ is called configuration space.

The dynamics of the field is encoded in an action functional, namely a local functional defined as the integral of a local (i.e. it depends on a finite number of derivatives of the fields) Lagrangian density L over the space–time manifold M.

The concept of locality is important to exclude any "action at a distance", implying that all fields propagate at a finite speed. For this reason, we will always deal with local sections of the fiber bundle $F \to M$, namely such that they are defined on open subsets $U \subset M$ and have the property that $\pi \circ \phi = id_U$.

Another consequence of locality is that the field configurations that describe a certain physical system must be solution to differential equations (which are at most of second order). A useful way to easily formulate such equations is to collect all the possible derivatives that a field may have at any given point into one big space of "field derivatives at spacetime points". This collection is called the jet bundle of the field bundle.

Furthermore, the equations of motion are obtained from a variational principle by minimizing the action, hence we need to define exactly what it means to "vary the action with respect to the field configurations".

In order to do so, we develop the theory of jets and define the variational bicomplex. As for the previous chapters, we omit most of the proofs, since our scope is just to present a self-consistent theory that will allow to concretely work on physical examples in the main chapters of this thesis. The most important references used in the theory of jet bundles and the variational bicomplex are [15], [1] and [22]. We refer to [33], [20] and [10] for the Lagrangian Field Theory part.

2.1 Jets and the Variational Bicomplex

Let $\pi : F \to M$ be a fiber bundle. It is useful to define an equivalence relation between local sections of F, that coincide on a neighborhood of a point up to a certain order k. We let $\sigma_1 : V_1 \to F$ and $\sigma_2 : V_2 \to F$ be two local sections in $\Gamma(M, f)$, where V_1 and V_2 are two neighborhoods of $x \in M$, then

$$\sigma_1(x) \sim \sigma_2(x) \quad \text{if} \quad \begin{cases} \partial_\mu \sigma_1|_x = \partial_\mu \sigma_2|_x \\ \cdots \\ \partial_{\mu 1, \cdots, \mu_k} \sigma_1|_x = \partial_{\mu 1, \cdots, \mu_k} \sigma_2|_x \end{cases} \tag{2.1}$$

namely, $\sigma_1 \sim \sigma_2$ at x if their partial derivatives at x agree up to order k.

Equivalently, if we consider all curves $\gamma : M \to \mathbb{R}$ such that $\gamma(0) = x$, and all functions $f \in \mathcal{C}^\infty(M)$, defining $T_0^k : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ to be the k-th Taylor polynomial at $s = 0$, then

$$\sigma_1(x) \sim \sigma_2(x) \quad \text{if} \quad T_0^k(f \circ \sigma_1 \circ \gamma) = T_0^k(f \circ \sigma_2 \circ \gamma). \tag{2.2}$$

Definition 2.1.1 (fiber bundle of k-jets)**.** We denote $j_x^k \sigma$ the equivalence class of local sections at x. Then the set of k-equivalent local sections at x is defined $J_x^k F$. The **fiber bundle of k-jets** is defined as

$$J^k F := \bigcup_{x \in M} J_x^k F. \tag{2.3}$$

The projection onto M is

$$\pi_M^k : J^k F \to M : j_x^k \sigma \mapsto x. \tag{2.4}$$

Furthermore, we may also define the projection onto F by

$$\pi_0^k : J^k F \to F : j_x^k \sigma \mapsto \sigma(x). \tag{2.5}$$

Considering a chart (U, ϕ) on F, where $\forall p \in \pi^{-1}(x)$ we have $\phi(p) = (x^\mu, y^i)$, we can express a local section in coordinates as $\phi(\sigma)(x) = (x^\mu, \sigma^i(x))$.

We can lift the chart (U, ϕ) on F to a chart $(\tilde{U}, \tilde{\phi})$ on $J^k F$ by

$$\tilde{U} := (\pi_0^k)^{-1}(U) \quad \text{and} \quad \tilde{\phi}(j_x^k \sigma) := (x^\mu, u^i, u_\mu^i, \cdots, u_{\mu_1 \cdots \mu_k}^i), \tag{2.6}$$

with

$$\begin{cases} u^i := \sigma^i(x) \\ u_\mu^i := \partial_\mu \sigma^i(x) \\ \vdots \\ u_{\mu_1 \cdots \mu_k}^i := \partial_{\mu_1 \cdots \mu_k} \sigma^i(x) \end{cases} . \tag{2.7}$$

One can show that transition functions between different charts are smooth functions[1], hence $J^k F$ is a smooth manifold.

For any $0 \leq l < k$ we can also define the projection

$$\pi_l^k : J^k F \to J^l F : j_x^k \sigma \mapsto j_x^l \sigma. \tag{2.8}$$

[1] in fact for a general change of coordinates

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i := Y^i(x, y) \end{cases}$$

we have

$$u'^i = Y'^i(x^\lambda, u^k) \qquad u'^i_{\mu_1 \cdots \mu_k} = A^{i\nu_1 \cdots \nu_k}_{j\mu_1 \cdots \mu_k}(x^\lambda, u^k) u^j_{\mu_1 \cdots \mu_k} + B^i_{\mu_1 \cdots \mu_k}(x^\lambda, u^k, u^k_\lambda, \cdots, u^k_{\lambda_1 \cdots \lambda_k}),$$

which are affine transformations

Definition 2.1.2 (infinite jet bundle)**.** The **infinite jet bundle** $J^\infty F$ is defined as the inverse limit of the finite order jet bundles under the projections π_n^{n+1}. Then we have projections

$$\pi_M^\infty : J^\infty F \to M \qquad \pi_k^\infty : J^\infty F \to J^k F \qquad \pi_0^\infty : J^\infty F \to J^0 F \simeq P. \tag{2.9}$$

Definition 2.1.3 (functions on k-jets)**.** Consider functions $J^k F \to \mathbb{R}$, denoted as $\mathcal{C}^\infty(J^k F)$. Then, for $l \geq k$ we can define connecting maps as the pullback by π_k^l

$$(\pi_k^l)^* : \mathcal{C}^\infty(J^k F) \to \mathcal{C}^\infty(J^l F) : f \mapsto (\pi_k^l)^*(f) := f \circ \pi_k^l. \tag{2.10}$$

The set of smooth functions on $J^\infty F$ is defined as the direct limit of the sequence $(\mathcal{C}^\infty(J^k F), (\pi_k^l)^*)$.

Therefore all $f \in \mathcal{C}^\infty(J^\infty F)$ must be obtained from a function $\hat{f} \in \mathcal{C}^\infty(J^k F)$ for some k by

$$f := \hat{f} \circ \pi_k^\infty. \tag{2.11}$$

Then we say that f has order k.

Vector Fields

Definition 2.1.4 (tangent bundle of infinite jet bundle)**.** The tangent space $T_{[u]} J^\infty F$ of $J^\infty F$ at $[u] \in J^\infty F$ is defined as the set of derivations on $\mathcal{C}^\infty(J^\infty F)$ at $[u]$. Hence we are able to define **tangent bundle of** $J^\infty F$ as

$$TJ^\infty F = \coprod_{[u]\in J^\infty F} T_{[u]} J^\infty F. \tag{2.12}$$

Equivalently, we can consider the inverse system of tangent bundles $TJ^k F$ with the push forward of the projections $(\pi_k^l)_* : TJ^l F \to TJ^k F : X \mapsto (\pi_k^l)_*(X)$ for all $l \geq k$, such that $\forall f \in \mathcal{C}^\infty(J^k F)$

$$(\pi_k^l)_*(X)(f) := (\pi_k^l)^*(X((\pi_k^l)^*(f))) = (\pi_k^l)^*(X(f \circ \pi_k^l)). \tag{2.13}$$

Then for any vector $X \in TJ^\infty F$ we can determine a sequence of derivations on $\mathcal{C}^\infty(J^k F)$ by

$$X_k := (\pi_k^\infty)_*(X). \tag{2.14}$$

Definition 2.1.5 (vector fields)**.** A **vector field** X on $J^\infty F$ is defined to be a $\mathcal{C}^\infty(J^\infty F)$-valued, $\mathbb{R}$-linear derivation on $\mathcal{C}^\infty(J^\infty F)$

In local coordinates on $J^\infty(U)$, we can represent a vector field as

$$a^\mu \partial_\mu + b^i \partial_i + \Sigma_{p=1}^\infty b^i_{\mu_1\cdots\mu_p} \partial_i^{\mu_1\cdots\mu_p}, \tag{2.15}$$

defining l_ν as the number of times the index ν appears amongst $\mu_1 \cdots \mu_k$, we impose

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \partial_i = \frac{\partial}{\partial u^i} \qquad \partial_i^{\mu_1\cdots\mu_k} = \frac{l_1! \cdots l_m!}{k!} \frac{\partial}{\partial u^i_{\mu_1\cdots\mu_k}} \tag{2.16}$$

and a^μ, b^i and $b^i_{\mu_1\cdots\mu_k} = b^i_{(\mu_1\cdots\mu_k)}$ are smooth functions in $J^\infty(U)$.

We can also easily see that

$$\partial_i^{\mu_1\cdots\mu_k}(u^j_{\nu_1\cdots\nu_k}) = \delta_j^i \delta_{\nu_1}^{(\mu_1} \cdots \delta_{\nu_k}^{\mu_k)}. \tag{2.17}$$

We define the multi-index $I := \mu_1 \cdots \mu_k$, where the indices are taken to be symmetric[2], and $|I| = k$ the order of the index.

Then

$$X = a^\mu \partial_\mu + b^i_I \partial_i^I \qquad \text{and} \qquad X(u^i_I) = b^i_I. \tag{2.18}$$

The set of vector fields on $J^\infty F$ is denoted as $\mathfrak{X}(J^\infty F)$.

[2] we recall that $u^i_{(\mu_1\cdots\mu_k)} = u^i_{\mu_1\cdots\mu_k}$

Contact Structure and Variational Bicomplex

We now need to define differential forms on the jet bundle, as they represent one of the main tools used in Lagrangian field theory, allowing to define the variational bicomplex.

Definition 2.1.6 (differential forms)**.** We define a **p-form** on $J^\infty F$ starting from a p-form on J^kF for some $k \geq 0$. Then $\omega \in \Omega^p(J^\infty F)$ iff there exists $\hat{\omega} \in \Omega^p(J^kF)$ such that

$$\omega = (\pi_k^\infty)^*\hat{\omega}, \tag{2.19}$$

where, considering $X_1, \cdots, X_p \in \mathfrak{X}(J^\infty F)$, we see for the pullback of a differential form

$$\omega(X_1, \cdots, X_p) = (\pi_k^\infty)^*\hat{\omega}((\pi_k^\infty)_*X_1, \cdots, (\pi_k^\infty)_*X_p). \tag{2.20}$$

Any form on $J^\infty F$ that can be represented by a form on J^kF is said to have *order k*.

Definition 2.1.7 (k-jet prolongation)**.** Let $\sigma \in \Gamma(P)$ be a local section, the **k-jet prolongation** of σ is defined as

$$\begin{aligned} j^k\sigma : U &\to J^kF \\ : x &\mapsto j^k_x\sigma \end{aligned}. \tag{2.21}$$

k-prolongations of sections in $\Gamma(M, F)$ are a subset of $\Gamma(J^kF)$, and they are called **holonomic sections**.

This definition can be extended to $J^\infty F$.

Definition 2.1.8 (contact form)**.** A p-form $\omega \in \Omega^\bullet(J^kF)$ is called a **contact form** iff for all holonomic sections $j^k\sigma$

$$(j^k\sigma)^*(\omega) = 0. \tag{2.22}$$

Remark 2.1.9. We see that $(j^k\sigma)^*(\omega)$ is a p-form on M, by definition of the pullback.

The set of all contact forms $\Omega_K(J^kF)$ is an ideal in the exterior algebra $\Omega(J^kF)$. To find a basis for such ideal, we first consider an example:

Let $\omega \in \Omega^1(J^1F)$, then in coordinates it can be written as

$$\omega = \omega_\mu dx^\mu + \omega_i du^i + \omega_i^\mu du^i_\mu. \tag{2.23}$$

Considering any section σ of F, ω is contact if the pullback of ω by the 1-prolongation vanishes, in coordinates

$$(j^1\sigma)^*(\omega) = (\omega_\mu + \omega_i\sigma^i_\mu + \omega^\nu_i\sigma^i_{\nu\mu})dx^\mu = 0. \tag{2.24}$$

Since this has to hold for any σ, we find the condition

$$\omega_\mu + \omega_i u^i_\mu + \omega^\nu_i u^i_{\nu\mu} = 0 \qquad \Rightarrow \qquad \begin{cases} \omega^\nu_i = 0 \\ \omega_i u^i_\mu + \omega_\mu = 0 \end{cases} \tag{2.25}$$

Therefore any contact 1-form in $\Omega^1_K(J^1F)$ can be written locally as

$$\omega = \omega_i \underbrace{(du^i - u^i_\mu dx^\mu)}_{=:\theta^i}. \tag{2.26}$$

Then we have found a basis θ^i for $\Omega^1_K(J^1F)$. Continuing with this construction, we find that for $\Omega^1_K(J^kF)$ a suitable basis is given by

$$\theta^i = du^i - u^i_\mu dx^\mu \qquad \theta^i_\mu = du^i_\mu - u^i_{\mu\nu}dx^\nu \qquad \cdots \qquad \theta^i_{\mu_1\cdots\mu_{k-1}} = du^i_{\mu_1\cdots\mu_{k-1}} - u^i_{\mu_1\cdots\mu_{k-1}\nu}dx^\nu. \tag{2.27}$$

A local basis for contact forms on J^kF is given by θ^i_I, defining $|I| = k-1$ to be the order of the contact form θ^i_I [3]

[3] we point out that the contact forms of order $|I|$ are defined on the $(|I|+1)$-th jet bundle over U

Contact forms on $J^\infty F$ are such that $\forall \sigma \in \Gamma(M, F)$ local sections, $j^\infty \sigma^*(\omega) = 0$. In a local patch $J^\infty U$ we can use the previous construction, then $\Omega^1_K(J^\infty F)$ is spanned by

$$\theta^i_I := du^i_I - u^i_{I\nu} dx^\nu, \tag{2.28}$$

where I is taken at all orders.

Definition 2.1.10. We may also define the exterior differential on $\Omega(J^\infty F)$ as

$$d : \Omega^p(J^\infty P) \to \Omega^{p+1}(J^\infty F), \tag{2.29}$$

where if $\omega \in \Omega(J^\infty F)$ is represented by $\hat{\omega} \in \Omega(J^k F)$, then $d\omega = d((\pi^\infty_k)^* \hat{\omega}) = (\pi^\infty_k)^*(d\hat{\omega})$. For a generic function $\in \mathcal{C}^\infty(J^\infty U)$ in coordinates we see

$$\begin{aligned} df &= \partial_\mu f dx^\mu + \partial_i f du^i + \cdots + \partial_i^{\mu_1 \cdots \mu_k} f du^i_{\mu_1 \cdots \mu_k} + \cdots \\ &= \partial_\mu f dx^\mu + \partial^I_i f du^i_I. \end{aligned} \tag{2.30}$$

Definition 2.1.11 (Lie derivative)**.** The **Lie derivative** of a differential form on $J^\infty F$ along a vector field $X \in \mathfrak{X}(J^\infty F)$ is defined through the Cartan formula as

$$L_X \omega = d(\iota_X \omega) + \iota_X d\omega. \tag{2.31}$$

Definition 2.1.12 (infinite prolongation)**.** x Let $\pi : F \to M$ and $\pi' : E \to N$ be 2 fiber bundles and $\phi : F \to E$ a fibered morphism with $\phi_0 : M \to N$ the restriction on the base manifold, we define the **infinite prolongation** of ϕ

$$\begin{aligned} j^\infty \phi &: J^\infty F \to J^\infty E \\ &: j^\infty_x \sigma \mapsto j^\infty \phi(j^\infty_x \sigma) := j^\infty_{\phi_0(x)}(\phi \circ \sigma \circ \phi_0^{-1}). \end{aligned} \tag{2.32}$$

Then it is easy to extend the definition of the ideal of contact forms on $J^\infty F$, we denote it by $\mathcal{C}$. It has the additional property of being a differential ideal, since $d\mathcal{C} \subset \mathcal{C}$. Now we define two kinds of differentials on functions on $J^\infty F$

Definition 2.1.13. (horizontal and vertical differential) The **horizontal differential** of a function $f \in \mathcal{C}^\infty(J^\infty F)$ is defined as

$$d_H f := \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{l_1! \cdots l_m!}{k!} \frac{\partial f}{\partial u^i_I} u^i_{I\mu} dx^\mu, \tag{2.33}$$

with $|I| = k$ and l_ν the number of times ν appears in I. Defining $D_\mu := \partial_\mu + u^i_{I\mu} \partial^I_i$, then it is easy to see that

$$d_H f := D_\mu f dx^\mu. \tag{2.34}$$

The **vertical differential** is simply given as

$$d_V := d - d_H. \tag{2.35}$$

In particular, $d_V f = (d - d_H) f = \partial^I_i f (d - d_H) u^i_I = \partial^I_i f d_V u^i_I = \partial^I_i \theta^i_I$

Remark 2.1.14. It is interesting to notice that $d_H f$ is parallel to differentials dx^μ of the coordinates on the base manifold M, while $d_V f$ is simply the remaining terms in df. This distinction will become useful when considering a local functional depending both on some field configuration $\{\phi\}$ and on the spacetime coordinates $\{x^\mu\}$, since we would like to be able to distinguish variations with respect to ϕ from simple coordinate transformations. To nicely encode this into a rigorous formalism, we need the following

Definition 2.1.15 (vertical forms)**.** Let $\mathcal{C}^s$ be the s-th wedge product of the contact ideal. Then $\forall s = 0, 1, \cdots, p+1$ we define (s, p) **vertical forms** to be

$$\Omega_V^{(s,p)} := \mathcal{C}^s \cap \Omega^p. \tag{2.36}$$

$\Omega_V^{(s,p)}$ consists of terms containing at least s contact 1-forms. Clearly there are inclusions

$$\Omega^p = \Omega_V^{(0,p)} \subset \Omega_V^{(1,p)} \subset \cdots \subset \Omega_V^{(p,p)} \subset \Omega_V^{(p+1,p)} = \{0\} \tag{2.37}$$

and, since $\mathcal{C}$ is a differential ideal,

$$d\Omega_V^{(s,p)} \subset \Omega_V^{(s,p+1)}. \tag{2.38}$$

Definition 2.1.16 (horizontal forms)**.** Let $\omega \in \Omega^p$ be a differential form and let $Y \in \mathfrak{X}$ be a vertical vector field, namely such that $(\pi_M^\infty)_*(Y) = 0$. Then ω is said to be **horizontal** if

$$\iota_Y \omega = 0 \tag{2.39}$$

More generally $\forall r = 0, 1, \cdots, p+1$, we define $\Omega_H^{(r,p)}$as the set of (r, p)-horizontal forms, which are given by differential forms ω such that

$$\omega(X_1, \cdots, X_p) = 0, \tag{2.40}$$

whenever at least $p - r + 1$ tangent vectors X_i are vertical. Then we have inclusions

$$\begin{aligned} &\Omega^p = \Omega_H^{(0,p)} \supset \cdots \supset \Omega_H^{(p,p)} \supset \Omega_H^{(p+1,p)} = \{0\} \\ &d\Omega_H^{(r,p)} \subset \Omega_H^{(r,p+1)}. \end{aligned} \tag{2.41}$$

Definition 2.1.17 (variational bicomplex)**.** The space of forms of degree (r, s) is the intersection

$$\Omega^{(r,s)}(J^\infty F) := \Omega_H^{(r,p)}(J^\infty F) \cap \Omega_V^{(s,p)}(J^\infty F), \tag{2.42}$$

where r is the horizontal degree, s is the vertical degree and $p = r + s$ is the form degree. It follows easily from the definition that

$$\Omega^p(J^\infty F) = \bigoplus_{r+s=p} \Omega^{(r,s)}(J^\infty F), \tag{2.43}$$

with projection maps to each summands $\pi^{(r,s)}\Omega^p(J^\infty F) \to \Omega^{(r,s)}(J^\infty F)$.

Furthermore, as we saw, we have the splitting of the differential $d = d_H + d_V$, with

$$\begin{aligned} d_H &: \Omega^{(r,s)}(J^\infty F) \to \Omega^{(r+1,s)}(J^\infty F) \\ d_V &: \Omega^{(r,s)}(J^\infty F) \to \Omega^{(r,s+1)}(J^\infty F). \end{aligned} \tag{2.44}$$

The condition $d^2 = 0$ obviously implies that

$$d_H^2 = 0 \qquad d_V^2 = 0 \qquad d_V \circ d_H = -d_H \circ d_V, \tag{2.45}$$

with the above considerations, we can see that $\Omega^{(r,s)}(J^\infty F)$ defines a bicomplex with respect to the two differentials, called the **variational bicomplex**.

Definition 2.1.18 (horizontal and vertical vector fields)**.** As we saw, on $J^\infty F$ we have defined vector fields $\mathfrak{X}(J^\infty F)$. As it turns out, they split into **horizontal** and **vertical vector fields**.

$$\mathfrak{X}(J^\infty F) = \mathfrak{X}_H(J^\infty F) \oplus \mathfrak{X}_V(J^\infty F), \tag{2.46}$$

where $\mathfrak{X}_H(J^\infty F)$ is the subspace of vector fields whose contraction with a horizontal form vanishes, and $\mathfrak{X}_V(J^\infty F)$ is the subspace of vector fields whose contraction with a vertical form vanishes.

In particular, it is possible to show that

$$\begin{gathered} X = X^\mu \frac{\partial}{\partial x^\mu} + \Sigma_{|I|=0}^\infty X_I^i \frac{\partial}{\partial u_I^i} \\ \text{with} \qquad X_H = X^\mu D_\mu \qquad X_V = \Sigma_{|I|=0}^\infty (X_I^i - X^\mu u_{I\mu}^i) \frac{\partial}{\partial u_I^i}. \end{gathered} \tag{2.47}$$

2.2 Lagrangian Field Theory

We define the space of fields $\mathcal{F} := \Gamma(M, F)$ as the set of local sections of F. It is an infinite-dimensional smooth manifold, which inherits the structure of a Fréchet space. We do not give the details of the definition and properties of Fréchet spaces, but it is important to note that they allow the usual Cartan calculus of differential geometry, which is enough to us.

Now we consider the local calculus on $\mathcal{F} \times M$. In particular, we first look at sections of the infinite jet bundle $\Gamma(M, J^\infty F)$, which can also be obtained from the jet prolongation

$$j^\infty : \Gamma(M, F) \to \Gamma(M, J^\infty F). \tag{2.48}$$

If we precompose j^∞ with the evaluation map ev $: M \times \mathcal{F} \to F : (x, \phi) \mapsto \phi(x)$, then we can define a smooth map

$$e_\infty : M \times \mathcal{F} \xrightarrow{(\mathrm{id}, j^\infty)} M \times \Gamma(M, J^\infty F) \xrightarrow{\mathrm{ev}} J^\infty F. \tag{2.49}$$

Definition 2.2.1 (local forms)**.** We define (r, s) **local forms** on $X \times \mathcal{F}$ to be the pullback by e_∞ of (r, s) forms on $J^\infty F$, namely

$$\Omega_{\mathrm{loc}}^{(r,s)}(X \times \mathcal{F}) := e_\infty^*(\Omega^{(r,s)}(J^\infty F)). \tag{2.50}$$

$\Omega_{\mathrm{loc}}^{(r,s)}(X \times \mathcal{F})$ is the true **variational bicomplex** in the field theoretic formalism, bigraded with respect to two differentials $\boldsymbol{d} = d + \delta$, defined $\forall \alpha \in \Omega^{(r,s)}(J^\infty F)$ such that

$$\begin{aligned} \delta(e_\infty^*(\alpha)) &:= e_\infty^*(d_V \alpha) \\ d(e_\infty^*(\alpha)) &:= e_\infty^*(d_H \alpha). \end{aligned} \tag{2.51}$$

In particular δ is the vertical differential, also called "variational differential", that measures how much a local form changes when the field configuration varies, and d is the usual de Rham differential on the differential forms on M, which in this case measures how much a local form changes when the space–time point at which it is evaluated varies. Of course we obtain properties

$$\delta^2 = 0 \qquad d^2 = 0 \qquad d\delta = -\delta d. \tag{2.52}$$

Local Lagrangian Field Theory

Now we assume that the space–time manifold M is a manifold[4] of dimension N with boundary, which we denote by $\Sigma = \partial M$.

A **field theory** is the assignment of a space of fields F_M and a local action functional S_M to an N-dimensional manifold with boundary $(M, \partial M)$ [10]. As we saw, we consider a smooth fiber bundle $\pi : F \to M$ and define the space of fields to be the local sections on F, $F_M := \Gamma(M, F)$.

A **Lagrangian** is a $(N, 0)$ local form $L \in \Omega^{(N,0)}_{\text{loc}}(F_M \times M)$. It is represented by (L, L^∞) with $L^\infty \in \Omega^{(N,0)}(J^\infty F)$ such that $L^\infty = L(x^\mu, u^i, u^i_\mu, \cdots, u^i_{\mu_1,\cdots,\mu_k})dx^1 \wedge \cdots \wedge dx^d$.

When evaluated at ϕ, we call $L(\phi)$ **Lagrangian density** which is simply a top form on M, therefore it can be integrated and give rise to the action functional:

$$S_M = \int_M L(\phi) = \int_M L(x, \phi, \partial\phi, \cdots, \partial^k\phi)dx^1 \wedge \cdots \wedge dx^d. \tag{2.53}$$

Now we are all set to deal with variational problems.

Definition 2.2.2 (source form)**.** A **source form** is any local $(N, 1)$ form $\alpha \in \Omega^{(N,1)}_{\text{loc}}(F_M \times M)$ such that it only depends on differential dx^μ and δu^i but not on δu^i_I. Equivalently, it is such that $\alpha_\phi(\delta\phi)$ only depends on the 0-jet of ϕ.

The set of source forms is denoted by $\Omega^{(\bullet,\bullet)}_s(F_M \times M)$.

Now we have a fundamental lemma

Lemma 2.2.3. *[33]*

$$\Omega^{(N,1)}_{loc} = \Omega^{(N,1)}_s \oplus d\Omega^{(N-1,1)}_{loc}. \tag{2.54}$$

In particular, this means that $\forall K \in \Omega^{(N,1)}_{loc}$*, there exists a unique source form* $G \in \Omega^{(N,1)}_s$ *and a* $H \in \Omega^{(N-1,1)}_{loc}$ *(unique up to d-exact terms) such that*

$$K = G + dH. \tag{2.55}$$

From this lemma, it is very easy to recover the fundamental variational formula

Proposition 2.2.4. *[33] Let* $L \in \Omega^{(N,1)}_{loc}$ *be a Lagrangian. Then there exist a unique source form* $E(L) \in \Omega^{(N,1)}_s$ *and a* $(N-1, 1)$ *local form* $\alpha \in \Omega^{(N-1,1)}_{loc}$ *(unique up to d-exact terms) such that*

$$\delta L = E(L) - d\alpha. \tag{2.56}$$

Moreover, $E(L)$ *is independent of changes of* L *by d-exact terms,*

$$E(L + dK) = E(L). \tag{2.57}$$

E is called ***Euler-Lagrange*** *operator*

This proposition is a nice way to rephrase the well known variational principle, in particular we see that asking $E(L)(\phi) = 0$ is exactly equivalent to imposing the Euler-Lagrange equations.

Now we can define an $(N - 1, 2)$ form as

$$\omega := \delta\alpha. \tag{2.58}$$

Clearly, since $\delta^2 = 0$, we can see from (2.56) that

$$\delta^2 L = 0 = \delta E(L) + \delta d\alpha \qquad \Rightarrow \qquad d\omega = -\delta E(L). \tag{2.59}$$

Furthermore, $\delta\omega = 0$.

This $(N - 1, 2)$ local form is very important because, as we will now see, it is essential to define a closed "2-form" on the space of (pre)boundary fields.

[4]For simplicity we assume M to be oriented

2.3 Boundary Structure and the Reduced Phase Space

We now want to study the structure of the fields on the boundary Σ of the space–time manifold M. We let EL_M be the subspace of F_M of fields that satisfy the Euler-Lagrange equations. In the usual description of Lagrangian field theory one would now consider an N-dimensional cylinder $\tilde{\Sigma}$ defined as $\tilde{\Sigma} := \Sigma \times [0, \epsilon]$, for ϵ arbitrarily close to 0. Now, the variational calculus on $\tilde{\Sigma}$ produces the following data:

(1) the space C_Σ of Cauchy data given by the values of the fields and their derivatives on Σ that produce a unique solution to the EL equation on $\Sigma \times [0, \epsilon]$ for some ϵ

(2) a local 1-form α_Σ on C_Σ given by the $\Sigma \times \{0\}$-boundary contribution of the 1-form $\alpha_{\Sigma\times[0,\epsilon]}$ found after the variation of the action $S_{\Sigma\times[0,\epsilon]}$

This induced structure can be further developed by considering the canonical surjective submersion

$$\pi_M : F_M \to C_{\partial M}, \tag{2.60}$$

which can be thought of as a projection of the space of fields on M to the space of fields on the boundary that define a unique solution to the EL equations in the cylinder $\tilde{\Sigma}$.

Now we notice that the terms in (2.56) are all horizontal N forms on M, therefore they can be integrated. Doing so, we find

$$\delta S_M = E(L)_M - \pi_M^*(\alpha_{\partial M}), \tag{2.61}$$

where, using Stokes' theorem, we define $\alpha_{\partial M} := \int_M d\alpha = \int_{\partial M} \alpha$ as a one-form on $C_{\partial M}$. In the same way we can define a two form on $C_{\partial M}$ as $\varpi_{\partial M} := \delta\alpha_{\partial M}$. Clearly, since $\delta^2 = 0$, $\delta\varpi_{\partial M} = 0$, hence defining a closed two form. We also assume $\varpi_{\partial M}$ to have a regular kernel (forms with such properties are called presymplectic). If $\varpi_{\partial M}$ is also non-degenerate, then $C_{\partial M}$ becomes a symplectic manifold.

At this point, assuming L_M to be smooth, one considers the projection of the EL locus to the boundary, defining the submanifold $L_M := \pi_M(EL_M) \subset C_{\partial M}$. Then L_M is clearly isotropic, since it is an immediate consequence of (2.61) that α vanishes on L_M.

The Alternative Approach

The following section follows [10]. The description we are about to develop, instead of starting with a space of Cauchy data C_Σ, allows to derive it in a more natural way. It might not coincide with the one which we previously defined, but it will produce the same structure after reduction.

The idea is to associate to the boundary $\Sigma = \partial M$ the space $\tilde{F}_\Sigma$ of germs of fields at $\Sigma \times \{0\}$ on $\Sigma \times [0, \epsilon]$. $\tilde{F}_\Sigma$ is called the space of **preboundary** fields and, as above, we can define a one form $\tilde{\alpha}_\Sigma$ on it arising from the variation of the action

$$\delta S_M = E(L)_M - \tilde{\pi}_M^*(\tilde{\alpha}_\Sigma), \tag{2.62}$$

where $\tilde{\pi}_M : F_M \to \tilde{F}_\Sigma$ is the natural surjective submersion to the space of preboundary fields. Introducing $\tilde{\varpi}_\Sigma := \delta\tilde{\alpha}_\Sigma$, this defines a closed two-form on $\tilde{F}_\Sigma$, but in general will be degenerate. In order to have a symplectic two-form, we need to assume that $\ker(\tilde{\varpi}_\Sigma)$ defines a smooth distribution on $\tilde{F}_\Sigma$. Then, by Frobenius' theorem, it induces a regular foliation on the space of fields on the boundary, hence we are able to consider the **symplectic reduction** $F_\Sigma := \tilde{F}_\Sigma/\sim$ defined as the leaf space of the foliation. It is a well known fact that F_Σ inherits a symplectic structure ϖ_Σ induced by $\tilde{\varpi}_\Sigma$. We will call F_Σ the **geometric phase space** of the theory.

Assuming the induced one form α_Σ to be well defined, and considering yet another surjective submersion $\pi_M : F_M \to F_\Sigma$, we obtain

$$\delta S_M = E(L)_M - \pi_M^*(\alpha_\Sigma). \tag{2.63}$$

As before, we find that $L_\Sigma := \pi_M(EL_M)$ is isotropic. At this point we define C_Σ as the submanifold of the geometric phase space that can be completed to a pair belonging to $L_{\Sigma\times[0,\epsilon]}$ for any ϵ. The space C_Σ of Cauchy data is in general a coisotropic submanifold of the geometric phase space. Its symplectic reduction $\underline{C}_\Sigma$ is called **reduced phase space**, which is what we are interested in. In the next chapter we will see how to find it in a cohomological way.

Chapter 3

Reduction and the BFV Formalism

The following chapter is dedicated to the cohomological resolution of constrained Poisson algebras. In particular, we want to find the algebra of functions on the symplectic reduction of a coisotropic submanifold of a symplectic manifold. We will see how this is found as the degree-0 cohomology of an odd operator Q acting on the functions on a symplectic supermanifold[1] whose body is given by the starting symplectic manifold.

This procedure is called **BFV formalism** [3], it is a generalization of the BRST formalism to arbitrary coisotropic submanifolds. It turns out [10],[11] that it has a very important application in field theories on manifolds with boundary, as it allows to find the algebra of functions on the reduced phase space even when it is not smooth. Furthermore, it allows to define the first steps towards the quantization of field theories.

In order to proceed with the description of the BFV formalism, we will first revise some known fact about symplectic and Poisson geometry, we will then discuss symplectic reduction à la Marsden-Weinstein [24] and study in detail the particular case of the BRST formalism (a subcase of BFV). Finally we will quickly present the general construction of the BFV formalism, with a particular interest for the case of the reduction of a coisotropic submanifold arising from constraints.

The main references used for this chapter are [14], [29] and [31].

3.1 Reduction in Symplectic Geometry

Definition 3.1.1 (symplectic vector space)**.** A **symplectic vector space** (V, ω) is a vector space V together with a non-degenerate skew symmetric bilinear form ω. Non-degenerate means that $\omega^{\#} : V \to V^* : v \mapsto \omega(v, \cdot)$ is invertible[2].

Definition 3.1.2 (symplectic form and manifold)**.** A **symplectic form** ω on a smooth manifold F is a non-degenerate closed 2-form. Any smooth manifold which can be endowed with such structure is called **symplectic**. In other words, $d\omega = 0$ and the map

$$\flat : \Omega^1(F) \to \mathfrak{X}(F) \tag{3.1}$$

is an isomorphism with inverse given by

$$\begin{aligned} \# : \mathfrak{X}(F) &\to \Omega^1(F) \\ \xi &\mapsto \iota_\xi \omega. \end{aligned} \tag{3.2}$$

[1]see appendix A

[2]In the case where we consider an infinite dimensioanl symplectic manifold, we only require $\omega^{\#}$ to be injective (weak symplectic form)

Definition 3.1.3 (Poisson structure)**.** Let P be a smooth manifold. A **Poisson Structure** on P is an $\mathbb{R}$-bilinear skew-symmetric bracket $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(P)$ satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad \forall f, g \in \mathcal{C}^\infty(P) \tag{3.3}$$

and the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{3.4}$$

These requirements are equivalent to the existence of a **Poisson bivector field** $\pi \in \Gamma(\wedge^2 TP)$ such that

$$\{f, g\} = \pi(df, dg). \tag{3.5}$$

Remark 3.1.4. Every symplectic manifold is also Poisson. Indeed, since the symplectic form is non-degenerate, it is possible to define the components of the Poisson bivector field by

$$\pi^{\mu\nu} := (-\omega_{\mu\nu})^{-1}. \tag{3.6}$$

Hence the map $\# : \Omega^1(F) \to \mathfrak{X}(F)$ is given for all $\alpha \in \Omega^1(F)$ by $\#(\alpha) := \pi(\alpha, \cdot)$

Definition 3.1.5 (Schouten-Nijenhuis bracket)**.** The **Schouten-Nijenhuis bracket** is a graded bracket of multivector fields which extends the Lie brackets of vector fields to multivector fields. It is defined by

$$\begin{aligned}[X_1 \wedge \cdots \wedge X_m, Y_1 \wedge \cdots \wedge Y_n] = \Sigma_{i=1}^m \Sigma_{j=1}^n (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge [X_i] \wedge \cdots \wedge X_m \\ \wedge Y_i \wedge \cdots \wedge [Y_j] \wedge \cdots \wedge Y_n,\end{aligned} \tag{3.7}$$

where $[X_i]$ indicates that we removed X_i from the wedge product

Proposition 3.1.6. *The Schouten-Nijenhuis bracket satisfies the graded Jacobi identity. Furthermore,* $[\pi, \pi] = 0$ *for all Poisson bivector fields* π*.*

Remark 3.1.7. If P is also a symplectic manifold with symplectic structure given by ω, then $[\pi, \pi] = 0$ is equivalent to $d\omega = 0$.

Definition 3.1.8 (Hamiltonian vector field)**.** Let F be symplectic. A vector field X on F is said to be **Hamiltonian** if there exists a function $f \in \mathcal{C}^\infty(F)$ such that $\iota_X \omega = df$.

Remark 3.1.9. For every function $f \in \mathcal{C}^\infty(F)$ we can obtain the corresponding Hamiltonian vector field by $X_f := \{f, \cdot\}$. Thanks to the Leibniz rule $\{f, \cdot\} = \pi(df, d\cdot)$ is a derivation, hence defining a smooth vector field on F.

Furthermore, $\iota_{X_f} \omega = \omega(\pi(df, \cdot)) = df$

Proposition 3.1.10. *It is easy to see that the Lie bracket of two Hamiltonian vector fields is a Hamiltonian vector field, in particular,* $\forall f, g \in \mathcal{C}^\infty(F)$*, the Lie bracket of* X_f *and* X_g *gives the Hamiltonian vector field associated to* $\{f, g\}$*,*

$$[X_f, X_g] = X_{\{f,g\}}. \tag{3.8}$$

Proof. Using the Jacobi identity

$$[X_f, X_g] = \{f, \{g, \cdot\}\} - \{g, \{f, \cdot\}\} = \{\{f, g\}, \cdot\} = X_{\{f,g\}}.$$

✓

Momentum Maps

Definition 3.1.11 (symplectic action)**.** Let (F, ω) be a symplectic manifold, G a Lie group and $\Psi : G \to \mathrm{Diff}(F)$ a left action of G on F. Ψ is said to be **symplectic** if it maps to symplectomorphisms (i.e. those diffeomorphisms $\psi \in$Diff(F) such that they preserve the symplectic structure: $\Psi^*(\omega) = \omega$)

$$\begin{aligned} \Psi : G &\to \mathrm{Sympl}(F) \subset \mathrm{Diff}(F) \\ g &\mapsto \Psi_g. \end{aligned} \tag{3.9}$$

Now let $\mathfrak{g}$ be the Lie algebra of the group G. For any $X \in \mathfrak{g}$ we can define a vector field on F by

$$\xi_X := \frac{d}{dt} \Psi_{\exp(tX)}\big|_{t=0}. \tag{3.10}$$

It acts on ω via the Lie derivative, in particular, if Ψ is a symplectic action

$$L_{\xi_X} \omega = \frac{d}{dt} (\Psi^*_{\exp(tX)} \omega)\big|_{t=0} = 0, \tag{3.11}$$

which means that $\iota_{\xi_X} d\omega + d\iota_{\xi_X} \omega = d\iota_{\xi_X} \omega = 0$

We define the set of **symplectic vector fields** as those fields such that $\iota_\xi \omega$ is closed; $\mathfrak{sym}(F) := \{\xi \in \mathfrak{X}(F) \mid d\iota_\xi \omega = 0\}$. In particular, it is made up by those fields that are in the image under $\#$ of closed one-forms

$$\mathfrak{sym}(F) = \#(\Omega^1_{\mathrm{closed}}(F)). \tag{3.12}$$

If $\xi^\flat = \iota_\xi \omega$ is not only closed, but also exact, then ξ is a Hamiltonian vector field, since there must be an f such that

$$\iota_\xi \omega = df. \tag{3.13}$$

The previous statements can be summarized by the following exact sequence

$$0 \longrightarrow H^0_{\mathrm{dR}}(F) \xrightarrow{i} \mathcal{C}^\infty(F) \xrightarrow{\# \circ d} \mathfrak{sym}(F) \xrightarrow{\flat} H^1_{\mathrm{dR}}(F) \longrightarrow 0, \tag{3.14}$$

where $H^k_{\mathrm{dR}}(F)$ indicates the k-th cohomology group on F. In particular, it is given as the quotient of closed k-forms with respect to exact k-forms.

Definition 3.1.12 (Hamiltonian action and momentum map)**.** An action Ψ of G on F is said to be **Hamiltonian** if there exist a map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, such that

- For every $X \in \mathfrak{g}$, we let $\mu^X : F \to \mathbb{R} : q \mapsto \mu(q)(X)$ be the component of μ along X, then μ^X is the Hamiltonian associated to the vector field ξ_X, namely
$$d\mu^X = \iota_{\xi_X} \omega; \tag{3.15}$$
- μ is equvariant with respect to the action Ψ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$,[3], namely
$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu. \tag{3.17}$$

[3]the coadjoint action is defined such that

$$\begin{aligned} \mathrm{Ad}^* : G &\to \mathrm{GL}(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*_g \end{aligned}, \tag{3.16}$$

and $\forall X \in \mathfrak{g}$, $\forall \eta \in \mathfrak{g}^*$,

$$\langle \mathrm{Ad}^*_g \eta, X \rangle = \langle \eta, \mathrm{Ad}_{g^{-1}} X \rangle$$

Such a map μ is called **momentum map**

Remark 3.1.13. The map assigning $\mu^X \in \mathcal{C}^\infty(F)$ to every $X \in \mathfrak{g}$ is a Lie algebra homomorphism, in particular

$$\mu^{[X,Y]} = \{\mu^X, \mu^Y\}, \tag{3.18}$$

where $[,\cdot,\cdot]$ is the Lie bracket of $\mathfrak{g}$ and $\{\cdot,\cdot\}$ is the canonical Poisson structure on F induced by the symplectic form. If the group G is connected, this condition is equivalent to the equivariance condition, indeed it is sufficient to prove equivariance under the action of the Lie algebra, but this is simply the fact that

$$\xi_X \mu^Y = \{\mu^X, \mu^Y\} = \mu^{[X,Y]}. \tag{3.19}$$

Symplectic Reduction

The equivariance condition implies that, if 0 is a regular value of μ, then the level set

$$C := \{x \in F \mid \mu(x) = 0\} \tag{3.20}$$

is preserved by the group action, i.e. $\Psi_g|_C : C \to C$ for all $g \in G$. We now want to study the orbit set C/G.

First, we recall the definition of **group orbit**: $O_q := \{x \in F \mid \exists g \in G,\ \Psi_g(q) = x\}$ and of the **stabilizer subgroup** of a point $q \in F$ as the set $G_q := \{g \in G \mid \Psi_g(q) = q\}$.

Definition 3.1.14 (transitive/free action)**.** An action Ψ of G on F is said to be

- **free** if all stabilizers are trivial $G_q = \{e\}$ for every $q \in F$
- **transitive** if it has just one orbit

Now consider the following equivalence relation: $q \sim p$ iff they belong to the same orbit. Then we can define $F/G := F/\sim$ as the orbit set. It is endowed with the quotient topology with respect to the projection $\pi : F \to F/G : q \mapsto O_q$.

Theorem 3.1.15. *Let G be a compact Lie group that acts freely on a smooth manifold F. Then F/G is a smooth manifold and $\pi : M \to F/G$ is a principal G-bundle*

Theorem 3.1.16 (Marsden-Weinstein)**.** *Let (F, ω) be a symplectic manifold, G a compact Lie group and μ an equivariant momentum mapping such that 0 is a regular value.*

Let $i : \mu^{-1}(0) = C \hookrightarrow F$ be the inclusion map. Then

- *the orbit space $\hat{C} := C/G$ is a smooth manifold*
- *$\pi : F \to \hat{C}$ is a principal G-bundle*
- *there is a unique symplectic form ω_{red} on $\hat{C}$ satisfying $i^*\omega = \pi^*\omega_{red}$*

The main result of this chapter will be to find a cohomological resolution for the functions on the reduced space $\hat{C}$, which in the context of field theory on a manifold with boundary will represent the observables on the space of boundary fields.

Coisotropic Reduction

Definition 3.1.17. Given a subspace W of a symplectic vector space (V, ω), we define $W^{\perp} := \{v \in V \mid \omega(v, w) = 0 \; \forall w \in W\}$. Then W is said to be

- **isotropic** if $W \subset W^{\perp}$
- **coisotropic** if $W^{\perp} \subset W$
- **Lagrangian** if $W^{\perp} = W$
- **symplectic** if $W^{\perp} \cap W = \{0\}$

Analogously, when F is a symplectic manifold and $N \subset F$ a smooth submanifold, then we say N is **isotropic** (resp. **coisotropic**, **Lagrangian**) if for all $p \in N$, T_pN is isotropic (resp. coisotropic, Lagrangian) with respect to T_pF.

Coisotropic sumbanifolds are our main interest. In the context of Poisson geometry, since we do not have a symplectic form at our disposal, they are defined in terms of the conormal bundle

Definition 3.1.18 (conormal bundle)**.** Let F be a manifold and $S \subset F$ be a submanifold. The **conormal bundle** N^*S is given by

$$N_q^*S := \{\eta \in T_q^*S \mid \forall X \in T_qS \; \eta(X) = 0\} \tag{3.21}$$

for all $q \in S$. In other words, N_q^*S is the annihilator of T_q^*S.

Definition 3.1.19 (coisotropic submanifold)**.** Let (F, π) be a Poisson manifold and $S \subset F$ be a submanifold. S is **coisotropic** if the restriction of the map

$$\begin{aligned} \pi^{\#} :& T^*F \to TF \\ & \alpha \mapsto \pi(\alpha, \cdot) \end{aligned} \tag{3.22}$$

to N^*S has image in TS.

We need to prove that this definition agrees with the previous one when F is symplectic. In order to do so, we simply notice that $\forall \alpha \in N_q^*S$, $\forall X \in T_qS$, then

$$\omega(\pi^{\#}(\alpha), X) = \alpha(X) = 0, \tag{3.23}$$

which proves that $\pi^{\#}(\alpha) \in T_qS^{\perp}$. Clearly, since $\pi^{\#}(N^*S) \subset TS$, we have $T_qS^{\perp} \subset T_qS$ for all $q \in S$, proving that S is coisotropic in the usual sense. The converse can easily be checked as well.

Proposition 3.1.20. *Let F be symplectic, G be a connected compact Lie group and μ be an equivariant momentum map with 0 a regular value. Then the zero-locus $C \subset M$ is a coisotropic submanifold.*

Proof. In order to prove this, we first need to show that for all $p \in C$, $(T_pC)^{\perp} \subset T_pC$.

Consider any tangent vector $v \in T_pF$, then v is tangent to C if and only if $d\mu(v) = 0$ (since μ is constantly 0 on C). However, given any $X \in \mathfrak{g}$

$$\langle d\mu(v), X \rangle = d\mu^X(v) = \omega(v, \xi_X), \tag{3.24}$$

which shows that $T_pC^{\perp}$ is spanned by the $\xi_X(p)$. In other words, $T_pC^{\perp}$ is the tangent space to the orbit O at p. The action of G preserves C, then $O \subset C$, hence showing $T_pC^{\perp} = T_pO \subset T_pC$. ✓

Now we are interested in the coisotropic reduction of a symplectic manifold. In order to understand it, we recall the following

Definition 3.1.21 (k-dimensional distribution)**.** Given a smooth n-dimensional manifold F, a **k-dimensional distribution** D on F is a collection $\{D_q\}_{q\in F}$ of k-dimensional subspaces $D_q \subset T_qF$ for all $q \in F$. The distribution is **smooth** if for every point $q \in F$ there is a neighborhood $U \subset F$ such that it is possible to find k smooth vector fields on U spanning D_q for all $q \in U$.

Given any coisotropic submanifold $C \subset F$, we let $i : C \to F$ denote the inclusion and $\omega_0 := i^*(\omega)$ the restriction of the symplectic form on the submanifold C.

In general ω_0 will be degenerate. The kernel of ω_0 at $p \in C$ is defined to be $\ker(\omega_0)_p := \{X \in T_pC \mid \iota_X\omega_0 = 0\}$, hence $\ker(\omega_0)_p = \cap(T_pC)$.

Assuming that the dimension of the kernel of ω_0 does not change when p is changed, then we obtain a smooth distribution called the **characteristic distribution of** ω_0.

Proposition 3.1.22. *Let (F, ω) be a smooth symplectic manifold, C a submanifold such that $ker(\omega_0)_p$ is k-dimensional for any $p \in C$. In particular, this condition is satisfied if C is coisotropic. Then the characteristic distribution is **involutive**, i.e. for all $X, Y \in \Gamma(\ker\omega_0)$ local sections, the commutator is still a local section in the kernel of ω_0, $[X, Y] \in \Gamma(\ker\omega_0)$.*

Proof. Clearly ω_0 is closed, since the differential commutes with the pullback. Then, given Z any local section of TC, we have

$$\begin{aligned} 0 &= d\omega_0(Z, X, Y) \\ &= Z\omega_0(X, Y) - X\omega_0(Z, Y) + Y\omega_0(Z, X) \\ &\quad - \omega_0([Z, X], Y) + \omega_0([Z, Y], X) - \omega_0([X, Y], Z). \end{aligned}$$

Most of the terms vanish because X, Y are sections of $\ker\omega_0$. We are then left with $\omega_0([X, Y], Z) = 0$, hence proving that $[X, Y]$ is a section of $\ker\omega_0$. ✓

Definition 3.1.23 (integral manifold)**.** An immersed submanifold N of F is an **integral manifold** of the distribution D if $T_qN = D_q$ for all $q \in N$, and D is **integrable** if each point of F is contained in an integral manifold of D.

Definition 3.1.24 (foliation)**.** A **k-dimensional foliation** on an n-manifold F is a collection of disjoint, connected, immersed k-dimensional submanifolds of F (the **leaves** of the foliation) such that

(i) the union of all the leaves gives F

(ii) there is a parametrization ϕ around each $q \in U \subset F$ such that $\phi(U)$ is a product of connected open sets in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and each leaf intersects U in the empty set or a countable union of k-dimensional slices of the form $x_{k+1} = c_{k+1}, \cdots, x_n = c_n$.

Theorem 3.1.25 (Frobenius)**.** *If D is an involutive distribution on F, then the collection of all maximal connected integral manifolds of D forms a foliation of F.*

We are finally ready to define the coistropic reduction

Theorem 3.1.26. *Let (F, ω) be a symplectic manifold and $i : C \to F$ be a coisotropic submanifold. By Frobenius theorem, from the characteristic distribution we obtain a foliation (the **characteristic foliation**). Then the space of leaves $\underline{C}$ of the characteristic foliation inherits a unique symplectic form $\underline{\omega}$ such that $\pi^*(\underline{\omega}) = i^*(\omega)$, where $\pi : F \to \underline{C}$ denotes the projection to the quotient $\underline{C} = C/\sim$.*

Proof. C is foliated into connected submanifold such that their tangent space at each point is given by $TC^\perp$. $\underline{C}$ is defined to be the space of leaves. Locally, the tangent space at q of $\underline{C}$ is given by $T_qC/T_qC^\perp$. We can then define a unique symplectic form $\underline{\omega}$ on $\underline{C}$ by requiring $\pi^*\underline{\omega} = \omega_0$. In particular, given two vectors $\bar{X}, \bar{Y}$ tangent to $\underline{C}$ at a leaf, we define $\underline{\omega}(\bar{X}, \bar{Y})$ by choosing a point p on the leaf and raising $\bar{X}, \bar{Y}$ to vectors $X, Y \in T_pC$ and declaring $\underline{\omega}(\bar{X}, \bar{Y}) := \omega_0(X, Y)$

We now need to prove that $\underline{\omega}$ is independent on the choice of p on the leaf and on the choice of X, Y on T_pC. The second statement is clear since T_pC is a coisotropic subspace of T_pF, hence $T_pC/T_pC^\perp$ inherits a symplectic strucuture. Independence on the point follows from uniqueness.

Finally, we just need to show that $\underline{\omega}$ is smooth and closed. Smoothness comes from the fact that $\pi^*(\underline{\omega})$ is smooth, whilst the closure requirement is found easily by noticing that

$$\pi^* d\underline{\omega} = d\pi^*\underline{\omega} = d\omega_0 = 0. \tag{3.25}$$

and that π is a submersion. ✓

3.2 The BRST Formalism as a Motivating Example

Until now, we have looked at reduction from a purely geometric point of view. We will now present an equivalent algebraic construction in terms of cohomology, which is called BRST formalism. It allows to find the algebra of functions of the reduced space $\underline{C}$ as the degree-0 cohomology of an operator called the BRST operator.

We start by considering the algebra of functions $\mathcal{C}^\infty(F)$. It inherits the structure of a **Poisson algebra**, since it is endowed with a Poisson bracket[4] and since for all $f \in \mathcal{C}^\infty(F)$ we obtain a derivation $X_f := \{f, \cdot\} \in \mathrm{Der}\mathcal{C}^\infty(F)$ given by the Hamiltonian vector field associated to f.

Given a function on F, we can always restrict it to the closed submanifold C. Viceversa, we can extend a function on C to F, but not uniquely. Indeed $\forall f, g \in \mathcal{C}^\infty(F)$, $f = g$ on C iff their difference vanishes on C. This motivates the following definition:

Definition 3.2.1 (Vanishing ideal)**.** The **vanishing ideal** $\mathcal{I} \subset \mathcal{C}^\infty(F)$ of C is given by

$$\mathcal{I} := \{f \in \mathcal{C}^\infty(F) \mid f|_C = 0\}. \tag{3.26}$$

Hence it is easy to see that we can define an equivalence relation on $\mathcal{C}^\infty(F)$ by requiring $f \sim g$ if $f - g \in \mathcal{I}$, therefore we can recover the functions on C algebraically as

$$\mathcal{C}^\infty(C) = \frac{\mathcal{C}^\infty(F)}{\mathcal{I}}. \tag{3.27}$$

The following theorem allows us to characterize a coisotropic sumbanifold algebraically

Theorem 3.2.2. *Let (F, ω) be symplectic and C be a sumbanifold with $\mathcal{I}$ its vanishing ideal. Then C is coisotropic if and only if $\mathcal{I}$ is a Lie subalgebra of the functions on F, i.e.*

$$\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}. \tag{3.28}$$

Proof. Observe that we just need to check (3.28) locally.

First, we assume C to be coisotropic and let $f, g \in \mathcal{I}$. Clearly, df and dg are sections of the conormal bundle N^*C, since $df(X) = 0$ for all X section of TC. Now, we just compute

$$\{f, g\}|_q = X_f(g)|_q = \iota_{X_f} dg|_q =< \pi^\#(df)|_q, dg|_q >= 0,$$

[4]In particular, $\mathcal{C}^\infty(F)$ is a Lie algebra with $\{\cdot, \cdot\}$ as a Lie bracket

since $\pi^{\#}(df)|_q$ is an element of T_qS and $dg|_q$ leis in the annihilator of T_qS.

Now assume that (3.28) holds. Any element $\alpha \in N_q^*C$ can locally be written as the differential of a function vanishing on C, $\alpha = df|_q$. By definition

$$\pi^{\#}(\alpha) = \pi^{\#}(df)|_q = X_f|_q. \tag{3.29}$$

For any element $\lambda \in N_q^*C$, we have $\lambda = dg|_q$ for some locally defined smooth function g vanishing on C, hence

$$< \pi^{\#}(\alpha), \lambda >=< \pi^{\#}(df)|_q, dg|_q >= \{f, g\}|_q = 0, \tag{3.30}$$

since f, g are locally defined functions living in the vanishing ideal of C. Hence $\pi^{\#}(\alpha)$ is annihilated by all elements of N_q^*C, which means that $\pi^{\#}(\alpha)$ is in the annihilator of the annihilator of T_qC, implying $\pi^{\#}(\alpha) \in T_qC$. ✓

Remark 3.2.3. In the infinite dimensional case, it is still true that the vanishing ideal of a coisotropic submanifold satisfies (3.28). The converse is however in general not true.

The functions on the reduced space $\underline{C}$ are those functions on C that are constant on the leaves. Again, since the leaves are connected, constant functions on the leaves are the ones left invariant by the vector fields tangent to the leaves, i.e. the Hamiltonian vector fields associated to the generators of the vanishing ideal. Hence we obtain

$$\mathcal{C}^\infty(\underline{C}) = \{f \in \mathcal{C}^\infty(C) \mid \{\mathcal{I}, f\} = 0\}. \tag{3.31}$$

Extending f to a function on F, we find

$$\mathcal{C}^\infty(\underline{C}) = \{f \in \mathcal{C}^\infty(F) \mid \{\mathcal{I}, f\} \subset \mathcal{I}\}/\mathcal{I} = N(\mathcal{I})/\mathcal{I}, \tag{3.32}$$

where $N(\mathcal{I})$ is the **Lie normalizer** of $\mathcal{I}$. Also notice that $N(\mathcal{I})/\mathcal{I}$ is a Poisson algebra (after all $\underline{C}$ is itself a symplectic manifold). We will now construct a complex of Poisson graded algebras with a differential whose degree-0 cohomology is isomorphic as a Poisson algebra to $N(\mathcal{I})/\mathcal{I}$.

The BRST complex of a group action

Recall that we are mainly interested in the reduction of a coisotropic submanifold arising as the zero-locus of some constraints (the projection of the E.L. equations on to the boundary of the spacetime manifold). We start by analizing the BRST formalism in the case where the constraints are simply the components of an equivariant momentum map $\mu : F \to \mathfrak{g}^*$, where F is an N-dimensional symplectic manifold and $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G.

We assume that $X_i \in \mathfrak{g}$ for $i = 1, \cdots, \dim(\mathfrak{g})$ is a basis of $\mathfrak{g}$. Then we define the components of the momentum map as

$$\phi_i := \mu^{X_i} =< \mu, X_i > . \tag{3.33}$$

The ϕ_i's represent the "constraints". Now let $C := \mu^{-1}(0)$ for 0 a regular value of μ, and let $\pi : C \to \underline{C}$ be the projection onto the quotient $\underline{C} := C/G$. By the pullback $\pi^* : \mathcal{C}^\infty(\underline{C}) \to \mathcal{C}^\infty(C)$ we can lift functions on $\underline{C}$ to functions on C, since they simply are the constant functions on the leaves, which are the G-orbits.

Since G is connected, constancy along the leaves is equivalent to the invariance along the vector fields ξ_X, hence the algebra of functions on $\underline{C}$ is found as the $\mathfrak{g}$-invariant functions on C

$$\mathcal{C}^\infty(\underline{C}) = \mathcal{C}^\infty(C)^{\mathfrak{g}}. \tag{3.34}$$

Recalling $\mathcal{C}^\infty(C) = \mathcal{C}^\infty(F)/\mathcal{I}$, we can then see

$$\mathcal{C}^\infty(\underline{C}) = (\mathcal{C}^\infty(F)/\mathcal{I})^{\mathfrak{g}}\,, \tag{3.35}$$

where $\mathcal{I}$ is the vanishing ideal on C.

Lemma 3.2.4. *The vanishing ideal $\mathcal{I}$ on $C = \mu^{-1}(0)$ is spanned by the components ϕ_i of μ.*

Proof. All the components ϕ_i clearly belong to $\mathcal{I}$ by definition. We just need to prove that there are no other independent functions generating $\mathcal{I}$. In particular, we prove (locally) that any function vanishing on C is given as a $\mathcal{C}^\infty(F)$-linear combination of the ϕ_i's.

First of all, let $N = \dim(F)$ and $k = \dim(\mathfrak{g})$. Since C is an embedded manifold, for all $q \in C$ it is possible to find an open neighborhood U of q and a local chart $\phi : U \to \mathbb{R}^{N-k} \times \mathbb{R}^k : q \mapsto (x, y)$ such that $y^i(q) := \phi_i(q)$.

Now let $f \in \mathcal{C}^\infty(F)$ vanish on C, i.e. $f \in \mathcal{I}$. Restricting to U, we might define $\hat{f} := \phi_* f = \phi \circ f|_U$. Then clearly $\hat{f}(x, 0) = 0$. Hence

$$\begin{aligned}\hat{f}(x,y) &= \int_0^1 dt \frac{d}{dt}\hat{f}(x,yt)\\ &= \int_0^1 dt\; y^i \partial_i \hat{f}(x,ty)\\ &= \Sigma_i \hat{\phi}_i \int_0^1 dt\; \partial_i \hat{f}(x,yt)\end{aligned}$$

Pulling it back to U, it is then clear that there exist functions h^i_U on U such that for all $f \in \mathcal{I}$ $f|_U = \phi_i|_U h^i_U$.

Globally, we just cover F with charts defined as before and use a partition of unity subordinate to it to patch them together, which ensures that $f = h^i\phi_i$ for some functions $h^i \in \mathcal{C}^\infty(F)$ and for all $f \in \mathcal{I}$. ✓

Now, as stated earlier, we would like to find the algebra $\mathcal{C}^\infty(C)^{\mathfrak{g}}$ in a cohomological fashion. In order to do so, we will combine the Koszul complex of the vanishing ideal $\mathcal{I}$ with the Chevalley–Eilenberg complex of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathcal{C}^\infty(F)$ (the module structure is given by the Lie algebra action)[23].

Definition 3.2.5 (Koszul complex)**.** The **Koszul complex** $(K^\bullet, \delta)$ is given by the graded vector space $K^\bullet := \wedge^\bullet\mathfrak{g} \otimes \mathcal{C}^\infty(F)$ and the operator $\delta : K^p \to K^{p-1}$ which is defined on K^1 by

$$\delta_K X = \mu^X \qquad \delta_K f = 0 \tag{3.36}$$

for any $f \in \mathcal{C}^\infty(M)$ and for any $X \in \mathfrak{g}$. Clearly $\delta_K^2 = 0$. It is furthermore extended to K^p as a derivation. For example, $\delta_K : K^2 \to K^1$ is given by

$$\delta_K(X \wedge Y \otimes f) = Y \otimes \mu^X f - X \otimes \mu^Y f. \tag{3.37}$$

Lemma 3.2.6. *The homology in degree 0 of the Koszul complex is isomorphic to the algebra of functions on C, i.e.*

$$H^0_{\delta_K} = \frac{\mathcal{C}^\infty(F)}{\mathcal{I}} = \mathcal{C}^\infty(C). \tag{3.38}$$

Proof. First we notice that $\ker(\delta_0)$ is simply given by $\mathcal{C}^\infty(F)$ while $\mathrm{Im}(\delta_1)$ is given by $\mathcal{I}$. In fact, letting X_i be a basis of $\mathfrak{g}$, any element of K^1 is written as $X_i \otimes f^i$, then $\delta(X_i \otimes f^i) = \phi_i f^i \in \mathcal{I}$, hence

$$H^0_{\delta_K} = \frac{\mathcal{C}^\infty(F)}{\mathcal{I}} = \mathcal{C}^\infty(C). \tag{3.39}$$

✓

Remark 3.2.7. Actually, the homology of the Koszul complex is concentrated in degree-0, meaning that it vanishes in every other degree[5], hence the previous lemma defines a projective resolution of $\mathcal{C}^\infty(C)$ in terms of $\mathcal{C}^\infty(F)$-modules, i.e. an exact sequence

$$\cdots \xrightarrow{\delta} K^2 \xrightarrow{\delta} K^1 \xrightarrow{\delta} \mathcal{C}^\infty(F) \longrightarrow \mathcal{C}^\infty(C) \longrightarrow 0, \tag{3.40}$$

which is called **Koszul resolution**.

We have already defined the Chevalley-Eileberg complex in the appendix A, but now we give a more refined definition:

Definition 3.2.8 (C.E. complex of $\mathfrak{g}$ with values in a $\mathfrak{g}$-module)**.** Let us consider a representation ρ of $\mathfrak{g}$ on $\mathfrak{M}$, $\rho : \mathfrak{g} \to \mathrm{End}(\mathfrak{M})$. $\mathfrak{M}$ together with ρ is called a **$\mathfrak{g}$-module**[6].

The **Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathfrak{M}$** is given by the graded space of linear maps

$$C^p(\mathfrak{g};\mathfrak{M}) := \mathrm{Hom}(\wedge^p\mathfrak{g}, \mathfrak{M}) = \wedge^p\mathfrak{g}^* \otimes \mathfrak{M}, \tag{3.41}$$

together with a differential $d_{CE} : C^p(\mathfrak{g};\mathfrak{M}) \to C^{p+1}(\mathfrak{g};\mathfrak{M})$ defined as follows:

- for all $m \in \mathfrak{M}$, for all $X \in \mathfrak{g}$, $d_{CE}m(X) := \rho(X)m$;
- for all $\alpha \in \mathfrak{g}^*$, we define $d_{CE}\alpha(X,Y) := -\alpha([X,Y])$ for every $X, Y \in \mathfrak{g}$;
- we extend it to $\wedge^\bullet\mathfrak{g}^*$ as a graded derivation, i.e. $d_{CE}(\alpha\wedge\beta) = d_{CE}\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d_{CE}\beta$;
- we extend it to $\wedge^\bullet\mathfrak{g}^* \otimes \mathfrak{M}$ analogously by $d_{CE}(\alpha\otimes m) = d_{CE}\alpha\otimes m + (-1)^{|\alpha|}\alpha\otimes d_{CE}m$.

It follows from the algebra homomorphism property of the representation ρ and from the Jacobi identity that $d_{CE}^2 = 0$.

Of course, we can define the cohomology of the Chevalley-Eilenberg complex by

$$H^p(\mathfrak{g},\mathfrak{M}) := \frac{\ker d_{CE} : C^p \to C^{p+1}}{\mathrm{Im} d_{CE} : C^{p-1} \to C^p} =: \frac{Z^n(\mathfrak{g},\mathfrak{M})}{B^n(\mathfrak{g},\mathfrak{M})}. \tag{3.42}$$

It is called the **Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{M}$**. In particular, elements in $Z^n(\mathfrak{g},\mathfrak{M})$ are called **cocycles** and elements in $B^n(\mathfrak{g},\mathfrak{M})$ are called **coboundaries**.

Remark 3.2.9. As in the example in appendix A, we can find an alternative expression for the Chevalley-Eilenberg differential. In our case, letting X_i be a basis of $\mathfrak{g}$ and α^i be the dual on $\mathfrak{g}^*$ and, defining $\forall\alpha \in \mathfrak{g}^*$ $\epsilon(\alpha) : \wedge^p\mathfrak{g} \to \wedge^{p+1}\mathfrak{g} : \omega \mapsto \alpha\wedge\omega$, we obtain

$$d_{CE} = \epsilon(\alpha^i)\rho(X_i) - \frac{1}{2}\epsilon(\alpha^i)\epsilon(\alpha^j)\iota_{[X_i,X_j]}. \tag{3.43}$$

[5] for a proof of this statement see [14]

[6] in our example $\mathfrak{M}$ will be simply $\mathcal{C}^\infty(C)$, while ρ is given by the action of $\mathfrak{g}$ on it.

Introducing the ghosts $c^i := \epsilon(\alpha^i)$ and the ghost momenta $b_i := \iota_{X_i}$, the differential then becomes

$$d_{CE} = c^i \rho(X_i) - \frac{1}{2} f_{ij}^k c^i c^j b_k. \tag{3.44}$$

With this expression, clearly

$$dm = \alpha^i \otimes \rho(X_i) m \qquad d\alpha^k = -\frac{1}{2} f_{ij}^k \alpha_i \wedge \alpha_j, \tag{3.45}$$

where the f_{ij}^k are the structure constants of $\mathfrak{g}$

Now we look at the degree-0 cohomology of the C.E. complex. It is easy to see that it is given by the $\mathfrak{g}$-invariant elements of $\mathfrak{M}$

$$H^0(\mathfrak{g}, \mathfrak{M}) = \mathfrak{M}^{\mathfrak{g}} = \{m \in \mathfrak{M} \mid \rho(X)m = 0 \ \forall X \in \mathfrak{g}\}. \tag{3.46}$$

Recalling that $\mathcal{C}^\infty(\underline{C})$ is exactly given by the $\mathfrak{g}$-invariant functions on C, this suggests constructing a bi-complex that will mix the Koszul and C.E. complexes, namely we define the graded double complex

$$C^{p,q} := \wedge^p \mathfrak{g}^* \otimes \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(F), \tag{3.47}$$

with the two commuting differentials d_{CE} and δ_K. In particular, the following square commutes

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d_{CE}} & C^{p+1,q} \\ \big\downarrow \delta_K & & \big\downarrow \delta_K \\ C^{p,q-1} & \xrightarrow{d_{CE}} & C^{p+1,q-1}. \end{array} \tag{3.48}$$

We then define the **BRST complex** as the graded complex $(\mathcal{C}^\bullet, D)$ with

$$\mathcal{C}^n := \bigoplus_{p-q=n} C^{p,q},$$
$$D := d_{CE} + (-1)^p \delta_K =: d + \delta,$$

where we defined $d := d_{CE}$ and $\delta = (-1)^p \delta_K$ The main result is:

Theorem 3.2.10. *Under the above regularity assumptions, we have*

$$H^n(\mathcal{C}^\bullet) \simeq H^n(\mathfrak{g}, \mathcal{C}^\infty(C)).$$

Therefore, in degree-0, we have that

$$H^0(\mathcal{C}^\bullet) \simeq H^0(\mathfrak{g}, \mathcal{C}^\infty(C)) = \mathcal{C}(C)^{\mathfrak{g}} = \mathcal{C}^\infty(\underline{C}).$$

Proof. Our strategy is to find a map between cohomologies $H^n(\mathcal{C}^\bullet)$ and $H^n(\mathfrak{g}, \mathcal{C}^\infty(C))$ and its inverse. In order to do so, first we express cocycles and coboundaries in $C^n(\mathfrak{g}, \mathcal{C}^\infty(C)) := \wedge^n \mathfrak{g}^* \otimes \mathcal{C}^\infty(C)$ in terms of objects of $C^{\bullet,\bullet}$.

As we saw, $\mathcal{C}^\infty(C) = \mathcal{C}^\infty(F)/\mathcal{I} = H^0_{\delta_k} = K^0/\delta K^1$. Also recall that $H^n(\mathfrak{g}, \mathcal{C}^\infty(C)) = Z^n(\mathfrak{g}, \mathcal{C}^\infty(C))/B^n(\mathfrak{g}, \mathcal{C}^\infty(C))$, with

$$Z^n(\mathfrak{g}, \mathcal{C}^\infty(C)) := \ker d : \wedge^n \mathfrak{g}^* \otimes \mathcal{C}^\infty(C) \to \wedge^{n+1} \mathfrak{g}^* \otimes \mathcal{C}^\infty(C),$$
$$B^n(\mathfrak{g}, \mathcal{C}^\infty(C)) := \mathrm{Im} d : \wedge^{n-1} \mathfrak{g}^* \otimes \mathcal{C}^\infty(C) \to \wedge^n \mathfrak{g}^* \otimes \mathcal{C}^\infty(C).$$

From d on $C^{n,0}$ we can induce d on $C^n(\mathfrak{g}, \mathcal{C}^\infty(C))$ projecting modulo $\delta C^{n+1,1}$, therefore we obtain

$$Z^n(\mathfrak{g}, \mathcal{C}^\infty(C)) \simeq \frac{\{\omega \in C^{n,0} \mid d\omega \in \delta C^{n+1,1}\}}{\delta C^{n,1}},$$
$$B^n(\mathfrak{g}, \mathcal{C}^\infty(C)) \simeq \frac{dC^{n-1,0} + \delta C^{n,1}}{\delta C^{n,1}}.$$

Now we need to define a map sending cocycles to cocycles and coboundaries to coboundaries, and its converse. First of all, consider a cocycle $\omega \in \mathcal{C}^n$. In particular, we have $\omega = \omega_0 + \omega_1 + \cdots$, with $\omega_i \in \mathcal{C}^{n+i,i}$. The condition $D\omega = 0$ implies

$$\begin{aligned} &\delta\omega_0 = 0, \\ &d\omega_0 + \delta\omega_1 = 0, \\ &d\omega_1 + \delta\omega_2 = 0, \\ &\vdots \\ &d\omega_{\text{top}} = 0. \end{aligned} \tag{3.49}$$

The second equation shows that $d\omega_0 \in \delta C^{n+1,1}$. Therefore ω_0 defines a cocycle $\in Z^n(\mathfrak{g}, \mathcal{C}^\infty(C))$, which means we found a map sending cocycles in $\mathcal{C}^n$ to cocycles in $C^n(\mathfrak{g}, \mathcal{C}^\infty(C))$. We just need to show that coboundaries are also sent to coboundaries.

Assume ω is such that there exists a $\varphi = \varphi_0 + \varphi_1 + \cdots \in \mathcal{C}^{n-1}$ satisfying $D\varphi = \omega$, with $\varphi_i \in \mathcal{C}^{n-1+i,i}$. Then

$$\omega_0 = d\varphi_0 + \delta\varphi_1 \in dC^{n-1,0} + \delta C^{n,1}, \tag{3.50}$$

which defines a coboundary in $B^n(\mathfrak{g}, \mathcal{C}^\infty(C))$. Hence the map $\omega \mapsto \omega_0$ gives rise to a map in cohomology

$$H^n(\mathcal{C}^\bullet) \to H^n(\mathfrak{g}, \mathcal{C}^\infty(C)). \tag{3.51}$$

Conversely, assume that $\omega_0 \in C^{n,0}$ defines a cocycle in $Z^n(\mathfrak{g}, \mathcal{C}^\infty(C))$. This means that $d\omega_0 \in \delta C^{n+1,1}$, hence there must exist a $\omega_1 \in C^{n+1,1}$ such that $d\omega_0 + \delta\omega_1 = 0$. Now

$$\delta d\omega_1 = -d\delta\omega_1 = d^2\omega_0 = 0 \tag{3.52}$$

implies that $d\omega \in C^{n+2,1}$ is a δ-cocycle. However, since δ is the differential of a resolution, it has no cohomology[7], meaning that $d\omega_1$ is also a δ-coboundary, i.e. there exists a $\omega_2 \in C^{n+1,2}$ such that $d\omega_1 + \delta\omega_2 = 0$. This procedure can be iterated until we arrive at $\omega = \omega_0 + \omega_1 + \cdots \in \mathcal{C}^n$ satisfying (3.49), which implies that the newly found ω is a D-cocycle.

Now assume that ω_0 is a coboundary, then $\exists \varphi_0 \in C^{n-1,0}, \varphi_1 \in C^{n,1}$ such that $\omega_0 = d\varphi_0 + \delta\varphi_1$. We see that

$$\delta d\varphi_1 = -d\delta\varphi_1 = -d(\omega_0 - d\varphi_0) = -d\omega_0 = \delta\omega_1, \tag{3.53}$$

therefore $\delta(\omega_1 - d\varphi_1) = 0$. But since δ is acyclic there must exist $\varphi_2 \in C^{n+1,2}$ such that $\omega_1 - d\varphi_1 = \delta\varphi_2$. Carrying on with this procedure, we can construct $\varphi = \varphi_0 + \varphi_1 + \cdots \in \mathcal{C}^{n-1}$ such that $\omega = D\varphi$, hence showing that ω is a D-coboundary. We are then able to define a map in cohomology

$$H^n(\mathfrak{g}, \mathcal{C}^\infty(C)) \to H^n(\mathcal{C}^\bullet), \tag{3.54}$$

which is the inverse of (3.51). ✓

[7] recall that the Koszul resolution (3.40) is an exact sequence

We have just shown that the algebra of functions on the reduced space is found as the degree-0 cohomology of a larger and more complicated complex which is given in terms of the functions on M. However, we have not proved that the newly found space is isomorphic to $\mathcal{C}^\infty(\underline{C})$ also as a Poisson algebra.

In order to do so, we need to define a Poisson structure on $\mathcal{C}^\bullet$:

- The canonical pairing $< \cdot, \cdot >$ of $\mathfrak{g}$ with $\mathfrak{g}^*$ defines a Poisson structure on $\mathfrak{g}^* \otimes \mathfrak{g}$. It is extended to the whole $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ as a graded derivation. This is an example of a **Poisson superalgebra**;

- Being $(\mathcal{C}^\infty(F), \{\cdot, \cdot\})$ a Poisson algebra, we can define a Poisson structure on $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{C}^\infty(F)$ by considering the bracket

$$[\![\cdot, \cdot]\!] := \{\cdot, \cdot\} + < \cdot, \cdot > \tag{3.55}$$

  One can check that, with this definition, $\mathcal{C}^\bullet$ inherits the structure of a **graded Poisson algebra**.

Futhermore, since there is a differential D on $\mathcal{C}^\bullet$, it actually becomes a **differential graded Poisson algbra**. Now we proceed to find an explicit expression for the differential, since one can check that D is compatible with the Poisson bracket, as we now show.

Proposition 3.2.11. *There exist an (odd) element θ of $\mathcal{C}^1$ called the* ***BRST charge*** *which generates the differential as*

$$D := [\![\theta, \cdot]\!]. \tag{3.56}$$

It is given explicitly by

$$\theta = \alpha^i \phi_i - \frac{1}{2} f^i_{jk} \alpha^j \wedge \alpha^k \wedge X_i, \tag{3.57}$$

where we chose X_i and α^i respectively as bases of $\mathfrak{g}$ and $\mathfrak{g}^*$

Proof. That $\{\theta, \cdot\}$ is a derivation is shown easily from the definition of the Poisson bracket. Now we need to show that it acts as D on $\mathcal{C}^\bullet$. We just need to show it on its generators, i.e. functions $f \in \mathcal{C}^\infty(MF)$ and elements $Y \in \mathfrak{g}$, $\beta \in \mathfrak{g}^*$

$$\begin{aligned}
\{\theta, f\} &= \alpha^i \{\phi_i, f\} \in \mathfrak{g}^* \otimes \mathcal{C}^\infty(F), \\
\{\theta, \beta\} &= -\frac{1}{2} \beta_i f^i_{jk} \alpha^j \wedge \alpha^k \in \wedge^2 \mathfrak{g}^*, \\
\{\theta, Y\} &= Y^i \phi_i + f^i_{jk} Y^k \alpha^j \wedge X_i.
\end{aligned}$$

In particular, we notice that $\{\theta, f\} = d_{CE} f$, $\{\theta, \beta\} = d_{CE} \beta$ and $\{\theta, Y\} = \delta_K Y + dY$ where $dY(Z) :=< Z, Y >$ for all $Z \in \mathfrak{g}^*$.

✓

It is customary in physics to introduce ghosts c^i and ghost momenta b_i and rewrite the BRST charge as

$$\theta = c^i \phi_i - \frac{1}{2} f^i_{jk} c^j c^k b_i. \tag{3.58}$$

3.3 The BFV Formalism

We have presented a powerful procedure to compute the symplectic reduction of a coisotropic submanifold arising from an equivariant momentum map. It turns out that this procedure can be generalized to arbitrary coisotropic submanifolds of Poisson manifolds (F, π).

It has been proven in [29] that for any coisotropic submanifold $C \subset F$ that there is a vector bundle $E \to C$ such that F is its total space is a neighborhood of C in F and, it is possible to construct a complex $BFV(E)$ endowed with a **BFV charge** Ω and a Poisson bracket $[\cdot,\cdot]_{BFV}$ such that the cohomology in degree zero with respect to the differential $[\Omega,\cdot]_{BFV}$ of E gives exactly the functions on the reduced space $\underline{C}$.

We restrict ourselves to the case where F is symplectic and we are given constraints $\phi_i : F \to \mathbb{R}$ such that they are first class, i.e.

$$\{\phi_i, \phi_j\} = f_{ij}^k \phi_k, \tag{3.59}$$

with the $f_{jk}^i \in \mathcal{C}^\infty(F)$. The case where the f_{jk}^i are constants is exactly BRST. Notice that the ϕ_i might also be interpreted as the components of a map $\phi : F \to W$, with $i = 1, \cdots, \dim(W)$.

The submanifold C is found as usual as the zero-locus of ϕ and hence it is coisotropic. Then we have the following theorem:

Theorem 3.3.1. *[10] Let $\phi : F \to W$ be a map to a vector space W such that 0 is a regular value and let $C := \phi^{-1}(0)$ be a coisotropic submanifold of a finite-dimensional symplectic manifold F with smooth reduction $\underline{C}$. Then it is possible to embed F as the body of a supermanifold $\mathcal{F}$ with an additional $\mathbb{Z}$-grading endowed with an even symplectic form $\omega_{\mathcal{F}}$ of degree zero and an even function S of degree +1 such that its Hamiltonian vector field Q squares to zero and its cohomology in degree zero is isomorphic as a Poisson algebra to the algebra of functions on $\tilde{C} := C/\sim$.*

Proof. First of all, we look at functions on $\mathcal{C}^\infty(F \times T^*W[1])$. Setting the parity of the coordinates on $T^*W[1]$ equal to degree modulo 2, it is easy to see that $\mathcal{C}^\infty(F \times T^*W[1]) \simeq \wedge^\bullet W^* \otimes \wedge^\bullet W \otimes \mathcal{C}^\infty(F)$, in fact, letting x^μ be coordinates on F, c^i and $c_i^\dagger$ be coordinates on $T^*W[1]$ respectively of ghost degree 1 and -1, we see that a generic function on $F \times T^*W[1]$ is given by

$$f(x, c, c^\dagger) = f_0(x) + f_i(x)c^i + f^j(x)c_j^\dagger + f_{ij}(x)c^i c^j + f^{ij}(x)c_i^\dagger c_j^\dagger + \cdots. \tag{3.60}$$

But since the c's and the $c^\dagger$'s anticommute, setting $\{e_i\}$ and $\{\epsilon^i\}$ as dual bases of W and W^*, we can rewrite f as a map

$$f = f_0 + f_i\epsilon^i + f^j e_j + f_{ij}\epsilon^i \wedge \epsilon^j + f^{ij} e_i \wedge e_j \in \mathcal{C}^\infty(F) \otimes \wedge^\bullet W^* \otimes \wedge^\bullet W, \tag{3.61}$$

with each coefficient $f_{\cdots}^{\cdots} \in \mathcal{C}^\infty(F)$.

We now let

$$\mathcal{C}^n := \bigoplus_{n=p-q} \wedge^p W^* \otimes \wedge^q W \otimes \mathcal{C}^\infty(F) \tag{3.62}$$

be the functions on $\mathcal{F}$ with degree n. $S \in \mathcal{C}^1$ has degree 1, hence

$$S = S_0 + S_1 + \cdots \qquad S_j \in \wedge^{j+1} W^* \otimes \wedge^j W \otimes \mathcal{C}^\infty(F). \tag{3.63}$$

We fix $S_0 = c^i\phi_i$, since, $\forall f \in \mathcal{C}^\infty(F)$, $Q(f) = \{S, f\} = c^i\{\phi_i, f\} + \cdots$ will select the functions constant along the leaves in the degree-0 cohomology. We now need to show that there are $S_j \in \wedge^{j+1} W^* \otimes \wedge^j W \otimes \mathcal{C}^\infty(F)$ such that $S = S_0 + \Sigma_{j>0} S_j$ satisfies $\{S, S\} = 0$.

Setting $k := \dim(W)$, we define the filtration

$$F^n := \bigoplus_{q\geq 0}\bigoplus_{p\geq n} \wedge^p W^* \otimes \wedge^q W \otimes \mathcal{C}^\infty(F) \tag{3.64}$$

such that

$$\wedge W^* \otimes \wedge W \otimes \mathcal{C}^\infty(F) = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0. \tag{3.65}$$

We build S by induction. First, we let $R_j := S_0 + S_1 + \cdots + S_j$, then we will construct S_{j+1} in terms of $\{R_j, R_j\}$ and show by induction that $\{R_j, R_j\} \in F^{j+2}$, which implies that at some point $\{R_j, R_j\} = 0$, hence showing that the process of constructing S is finite. We start by looking at the zeroth order case. $R_0 = S_0$ and

$$\{S_0, S_0\} = c^i c^j \{\phi_i, \phi_j\} \in F^2.$$

We also prove the first order case by hand in order to show how to construct S_1. Using the Jacobi identity, we see $\{S_0, \{S_0, S_0\}\} = \delta_K\{S_0, S_0\} = 0$, which implies that $\{S_0, S_0\}$ is δ_K-closed. It also δ_K-exact, in fact

$$\{S_0, S_0\} = c^i c^j \{\phi_i, \phi_j\} = f^k_{ij} c^i c^j \phi_k = \delta_k(f^k_{ij} c^i c^i c^\dagger_k). \tag{3.66}$$

This suggests defining $S_1 = -\frac{1}{2} f^k_{ij} c^i c^j c^\dagger_k$ so that $R_1 = S_0 + S_1$ satisfies $\{R_1, R_1\} \in F^3$.

Now, assuming that $\{R_j, R_j\} \in F^{j+2}$, we need to show that $\{R_{j+1}, R_{j+1}\} \in F^{j+3}$. In order to do so, we need the following lemma

Lemma 3.3.2. *Assuming $\{R_j, R_j\} \in F^{j+2}$, then $\delta_K\{R_j, R_j\} = 0$*

Proof. We notice

$$\{S_0, \{R_j, R_j\}\} = \delta_K\{R_j, R_j\} + \mathrm{mod} F^{j+3}. \tag{3.67}$$

Then we just need to show that $\{S_0, \{R_j, R_j\}\} \in F^{j+3}$.

Using Jacobi, we see $\{R_j, \{R_j, R_j\}\} = 0$, which implies

$$\{S_0, \{R_j, R_j\}\} = -\{S_1 + \cdots + S_j, \{R_j, R_j\}\} \in F^{j+3}, \tag{3.68}$$

since $\{R_j, R_j\} \in F^{j+2}$ and $\{S_{i>0}, \cdot\}$ has positive filtration degree ✓

Now, since all the homology groups H^n_K of the Koszul differential vanish except when $n = 0$, there must exist a S_{j+1} such that

$$2\delta_K S_{j+1} + \{R_j, R_j\} = 0, \tag{3.69}$$

implying that $2\{R_j, S_{j+1}\} + \{R_j, R_j\} \in F^{j+3}$. Defining $R_{j+1} = R_j + S_{j+1}$, we have

$$\{R_{j+1}, R_{j+1}\} = \{R_j, R_j\} + 2R_j, S_{j+1} + \{S_{j+1}, S_{j+1}\} \in F^{j+3}, \tag{3.70}$$

hence proving the existence of S. Furthermore, if $\underline{C}$ is smooth, the cohomology in degree 0 of Q is isomorphic (as a Poisson algebra) to the functions on $\underline{C}$. ✓

Remark 3.3.3. In the general case, when the reduction $\underline{C}$ is not smooth, it is still possible to define functions on it through the BFV complex.

The BFV Formalism in Field Theory

We are interested in the use of this formalism in field theory. In particular, we saw in the previous chapter that if we let M be the N-dimensional space–time manifold with boundary given by $\Sigma := \partial M$ and we consider an action functional S_M, then the variation of the action δS_M produces a local 1-form $\tilde{\alpha}_\Sigma$ on the boundary.

The variation of this one form is a closed two-form $\tilde{\omega}_\Sigma$. As we saw, assuming that $\ker \tilde{\varpi}_\Sigma$ has constant dimension defines a smooth distribution on the space of pre-boundary fields $\tilde{F}_\Sigma$. The symplectic reduction of $\tilde{F}_\Sigma$ gives the geometric phase space F_Σ, which is indeed a symplectic manifold (we assume that it is smooth). The projection $\pi_M : F_M \to F_\Sigma$ from the space of fields in the bulk to the space of fields on the boundary is a surjective submersion. Defining EL_M to be the critical locus of the action S_M, we see that the projection $L_M := \pi_M(EL_M)$ is isotropic in F_Σ. We also assume that the space C_Σ of Cauchy data, defined as the points in F_Σ that can be completed to a pair belonging to $L_{\Sigma\times[0,\epsilon]}$ for any ϵ, is a coisotropic submanifold of F_Σ. C_Σ is again not symplectic, hence the corresponding symplectic reduction $\underline{C}_\Sigma$ will have to be taken into consideration. The functions on $\underline{C}_\Sigma$ are the observables of the theory and it is exactly what we want to compute using the BFV formalism.

To do so, we introduce a symplectic supermanifold $\mathcal{F}$ whose body is given by F_Σ, containing odd additional fields (the coordinates of $\mathcal{F}$) c^i (the ghosts) and $c_i^\dagger$ (the ghost momenta)[8]. The new symplectic form on $\mathcal{F}$ is given by

$$\varpi_{BFV} := \varpi_\Sigma + \int_\Sigma \delta c_i^\dagger \delta c^i. \tag{3.71}$$

In the cases presented in the following chapters, C_Σ will be presented in terms of first class constraints (i.e. local functions on the space of boundary fields which are in involution with respect to the canonical Poisson structure on F_Σ induced by the symplectic one). For the sake of simplicity, we denote them again with ϕ_i.

The BFV charge S, which will now be called the **BFV action**, is an odd local function of degree 1 given by

$$S := \int_\Sigma c^i\phi_i - \frac{1}{2} f^i_{jk} c_i^\dagger c^j c^k + R. \tag{3.72}$$

Also in this case it is possible to show [31] that R can be found such that $\{S, S\} = 0$, namely the **classical master equation** is fulfilled[9].

The Hamiltonian vector field Q associated to S is given by $\iota_Q \varpi_{BFV} = \delta S$ and is cohomological. Furthermore, it can be proved (under some regularity assumptions that are not usually met) that the cohomology in degree 0 of Q represents the functions on the reduced phase space $\underline{C}_\Sigma$: the "observables" of the theory on the boundary. In the general case, the reduction is then defined as the BFV complex.

Remark 3.3.4. The BFV formalism is not only useful to cohomologically resolve symplectic reductions (even in the singular case), but, in the context of field theory, it is interlaced with the **BV formalism**, which is yet another generalization of the BRST formalism in the bulk of the spacetime manifold. If one is able to prove that the two formalisms produce nicely agreeing theories (i.e. the BFV data on the boundary and the BV ones in the bulk respect some compatibility conditions), then it is possible to define a BV/BFV theory [9][10][11].

This is the first step towards a possible quantization. The idea is to associate a graded Hilbert space to the symplectic supermanifold $\mathcal{F}$ and to find an operator $\hat{S}$ which satifies $[\hat{S}, \hat{S}]$ and that quantizes the (BV)BFV action S. At lowest order in $\hbar$ this is certainly possible because of the

[8] The index i is for simplicity, it might be a continuous one

[9] In the examples that we will see later, we will have $R = 0$

classical master equation $\{S, S\} = 0$. If one can prove that $[\hat{S}, \hat{S}]$ is fulfilled at any order in $\hbar$, then the cohomology in degree 0 of $\hat{S}$ will produce a Hilbert space quantizing the symplecict reduction (the space of observables of the theory).

Chapter 4

Palatini-Cartan Theory with Scalar Coupling

In this chapter we look at the free theory of gravity in the Palatini-Cartan formalism and introduce the coupling to a massless scalar field. Most of the results will be a generalization of [8]. We do not present the case of pure gravity because it can be easily recovered by setting the scalar field (and its momentum) to zero.

4.1 The Vielbein Formalism

Physical Motivation

The Equivalence Principle is the starting point of General Relativity. It states that at any point in space–time we cannot locally (i.e. in a small neighborhood around the point) determine whether gravitational accelerations felt by falling bodies are caused by a gravitational field or they are simply a consequence of the non-inertial character of the chosen reference frame. Indeed local accelerations and forces acting on a point x are given by the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$.

The essence of the equivalence principle is the existence of a frame (locally inertial frame) in which the coefficients of the Levi-Civita connections vanish locally. This is a result of the following lemma:

Lemma 4.1.1. *Let (M,g) be a Lorentzian manifold of dimension N and $x \in M$ an arbitrary point of M. Let U_x be an open neighborhood of x in M whose points are labeled by coordinates x^μ. Then we can construct a new set of coordinates $\overline{x}^\mu$ on U_x such that*

(i) $\overline{x}^\mu(x) = 0$;

(ii) the value of the metric tensor at x in the new coordinates is equal to the Minkowski metric

$$\overline{g}_{\mu\nu}(x) = \eta_{\mu\nu}; \tag{4.1}$$

(iii) the coefficients of the Levi-Civita connection in the new coordinates vanish at x .

$$\overline{\Gamma}^{\rho}_{\mu\nu}(x) = 0 \tag{4.2}$$

Consider any open subset $U \subset M$. Let us construct a family of coordinate transformations which, point by point $x \in U$ realize the three conditions in Lemma 4.1.1. It is parametrized by a multiplet of N bi-local functions $\xi^a(x,y)$, where the x^μ's denote the coordinates of the point x and the y^μ's the coordinates of an arbitrary point in U. By construction we have that $\xi^a(x,x) = 0$, and defining

$$\begin{aligned} e^a_\mu(x) &:= \frac{\partial \xi^a(x,y)}{\partial y^\mu}\bigg|_{y=x}, \\ e^\mu_a(x) &:= (e(x)^{-1})^\mu_a \qquad \Leftrightarrow \qquad e^a_\mu(x) e^\mu_b(x) = \delta^a_b. \end{aligned} \tag{4.3}$$

Then it is clear that the e^μ_a correspond to the jacobian of the change of variables that sends to the coordinates in which

$$\overline{g}_{ab}(x) = \eta_{ab} = e^\mu_a(x) e^\nu_b(x) g_{\mu\nu}(x). \tag{4.4}$$

Equivalently, we have that

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}. \tag{4.5}$$

The last identity shows that the metric tensor can be recovered from the Minkowski metric in terms of a more fundamental object $e^a_\mu(x)$.

Élie Cartan was the first to notice this property and came up with the following idea: instead of using the metric tensor $g_{\mu\nu}(x)$, he introduced a non-degenerate $N \times N$ matrix $e^a_\mu(x)$ depending smoothly on x, such that (4.5) becomes the definition of the metric.

One of the advantages of this formalism is that the components of the $N \times N$ matrix $e^a_\mu(x)$ can be recasted into N differential 1-forms

$$e^a(x) := e^a_\mu(x) dx^\mu. \tag{4.6}$$

Furthermore, it is easy to notice that for any orthogonal transformation $\Lambda(x) \in$SO$(N-1,1)$, the trasnformed matrix $\tilde{e}^a_\mu(x) := \Lambda(x)^a{}_b e^a_\mu(x)$ produces the same metric tensor.

Let us now see how these properties can be geometrically encoded into the theory of principal bundles we developed in the first chapter.

Geometrical Definition of the Vielbein

We start by considering (M, g) as before (N-dimensional pseudo-riemannian). We have

Definition 4.1.2 (frame bundle)**.** For any $x \in M$ the **frame space** is defined as

$$L_x M := \{e_a = (e_1, \cdots, e_N) \mid e_a \text{ is a basis of } T_x M\}$$

A frame is an ordered basis in $L_x M$. Then the **frame bundle** LM over M is simply defined to be

$$\bigcup_{x \in M} L_x M.$$

Proposition 4.1.3. *LM is a* GL$(N, \mathbb{R})$*-principal fiber bundle with projection $\pi : LM \to M$.*

Proof. We know that a chart $\{x^\mu\}$ induces canonically a basis ∂_μ, then any arbitrary basis of $T_x M$ is given in terms of $\{\partial_\mu\}$ by a matrix e^μ_a in GL$(N, \mathbb{R})$ such that $e_a = e^\mu_a \partial_\mu$. A local trivialization can be simply given by

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{GL}(N, \mathbb{R}) : (x, e_a) \to (x^\mu, e^\nu_a), \tag{4.7}$$

with transition functions

$$\begin{cases} x'^\mu = x'^\mu(x) \\ e'^\mu_a = \frac{\partial x'^\mu}{\partial x^\nu} e^\nu_a, \end{cases} \tag{4.8}$$

which proves that e_a transforms by a left action of an element $\frac{\partial x'^\mu}{\partial x^\nu}$ of $\mathrm{GL}(N,\mathbb{R})$.

Furthermore, we can define a right action of $\mathrm{GL}(N,\mathbb{R})$ on LM for all $A \in \mathrm{GL}(N,\mathbb{R})$ by

$$\begin{aligned} R_A :& LM \to LM \\ & (x, e_a) \mapsto (x, e_b A^b{}_a). \end{aligned} \tag{4.9}$$

✓

Remark 4.1.4. The set $L_x M$ of all possible bases of the tangent space $T_x M$ is isomorphic $GL(N,\mathbb{R})$, therefore the dimension of $\dim(LM) = \dim(M) + \dim(GL(N,\mathbb{R})) = N + N^2$

At this point, assuming by simplicity that M is oriented, given a metric g of signature $(N-1,1)$ on M, we can define the **orthonormal frame bundle** $\mathrm{SO}(M,g)$ as the subbundle of LM containing orthonormal frames, such that $g(e_a,e_b) = \eta_{ab}$ for all $x \in M$. In particular, $SO(M,g)$ is a principal subbundle of LM with structure group $SO(N-1,1)$.

It is important to point out that one metric g has more than one orthonormal basis. If e_a is one ON basis, $\forall \Lambda \in \mathrm{SO}(r,s)$ then also $e'_a := e_b \Lambda^b{}_a$ is an ON basis. However, given any basis e_a, there is a unique metric g for which it is ON, namely $g(e_a,e_b) = \eta_{ab}$.

The last condition can as well be expressed in terms of **orthonormal coframes**, which are defined to be dual bases with respect to frames. This means that if e_a defines a local frame, then e^b is the corresponding coframe if

$$e^b(e_a) = \delta^b_a. \tag{4.10}$$

Therefore the condition $g(e_a,e_b) = \eta_{ab}$ can be expressed as

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \quad \Leftrightarrow \quad g = e^*(\eta). \tag{4.11}$$

Remark 4.1.5. Coframes and orthonormal coframes respectively define the **coframe bundle** L^*M and the **orthonormal conframe bundle** $\mathrm{SO}^*(M,g)$, which are both principal bundles with respect to their structure groups.

A basis e_a is a point in LM. The bundle LM might not have global sections (if it has, then M is said to be parallelizable); however we might consider a local section $e_a : M \to LM, x \mapsto e_a(x) = e^\mu_a(x)\partial_\mu$, which is called a **moving frame**.

We can easily define a metric g once a moving frame is given by equation 4.11, but the problem is that there is more than one moving frame defining one. In fact, if we assume that two observers (in intersecting neighborhoods) set their moving frames respectively to e_a and e'_a, they define the same metric g if the local frames in the overlap are related as

$$e'^\mu_a(x') = \frac{\partial x'^\mu}{\partial x^\nu} e^\nu_b (\Lambda^{-1})^b{}_a, \tag{4.12}$$

for some local map $\Lambda^a{}_b : U \to \mathrm{SO}(N-1,1)$ which represents a gauge transformation. Clearly, the moving frames are not just natural objects transforming by the action of the jacobian of the change of coordinates, but also by the action of a gauge transformation in $\mathrm{SO}(N-1,1)$.

We then abandon the idea of using moving frames as good candidates for an equivalent description of GR which does not involve the direct use of a metric tensor as a dynamical field. Furthermore, for non-parallelizable space–time manifolds the existence of global moving frames is not even guaranteed.

We therefore need a different description, which will be given in terms of the vielbein.

Definition 4.1.6 (vielbein map)**.** Consider an $\mathrm{SO}(N-1,1)$-principal bundle $(P, \mathrm{SO}(N-1,1), p, M)$ on M. We define the **vielbein map** to be a principal bundle morphism $e : P \to LM$, i.e. such that the following diagrams commute

$$(verticality): \quad \pi \circ e = p \qquad \begin{array}{ccc} P & \xrightarrow{\quad e \quad} & LM \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}, \tag{4.13}$$

$$(equivariance): \quad R_{i(g)} \circ e = e \circ R_g \qquad \begin{array}{ccc} P & \xrightarrow{e} & LM \\ \downarrow{\scriptstyle R_g} & & \downarrow{\scriptstyle R_{i(g)}} \\ P & \xrightarrow{e} & LM \end{array}. \tag{4.14}$$

where $i : \mathrm{SO}(1, N-1) \to \mathrm{GL}(N, \mathbb{R})$ denotes the canonical embedding.

We now need to check that we are dealing with the correct objects, which amounts to showing that the map e is equivalently described by a family of frames which differ by an orthogonal transformation on the overlaps.

To do so, we consider a family of local sections $\sigma_{(\alpha)} : U_{(\alpha)} \to P$. The domains $U_{(\alpha)}$ form an open covering of M and on the overlaps there are transition functions $\phi_{(\alpha\beta)}$ such that

$$\sigma_{(\beta)} = \sigma_{(\alpha)} \cdot \phi_{(\alpha\beta)}, \tag{4.15}$$

namely sections are mapped into one another by the canonical right action on P. If we set $e(\sigma_{(\alpha)}(x)) = (x, V_a^{(\alpha)})$ (where $V_a^{(\alpha)}$ is a frame in LM), we have that if one knows the map e on the section $\sigma_{(\alpha)}$, then one knows it on the whole tube $p^{-1}(U_{(\alpha)})$, since for any $V \in p^{-1}(U_{(\alpha)})$, $V = \sigma_{(\alpha)} \cdot g$ for some g in $\mathrm{SO}(N-1,1)$ by definition. Therefore

$$e(V) = e(\sigma_{(\alpha)} \cdot g) = e(\sigma_{(\alpha)}) \cdot i(g) = (x, V_b^{(\alpha)} g^b{}_a). \tag{4.16}$$

Setting $i(g) = g^b{}_a \in \mathrm{GL}(N, \mathbb{R})$. Hence all the information about the vielbein map e is encoded in a family of moving frames $V_a^{(\alpha)}$. On the overlaps, we have

$$V_a^{(\beta)} = e(\sigma_{(\beta)}(x)) = e(\sigma_{(\alpha)}(x) \cdot \phi_{(\alpha\beta)}(x)) = e(\sigma_{(\alpha)}(x)) \cdot i(\phi_{(\alpha\beta)}(x)) = V_b^{(\alpha)} (\phi_{(\alpha\beta)})^b{}_a. \tag{4.17}$$

Thus vielbein maps are in one-to-one correspondence with families of frames which differ by orthonormal transformations on the overlaps. And that is exactly what one needs to define a global metric.

At this point, one possible route is to define the "gauge-natural" bundle whose sections are in one to one correspondence with global vielbein maps $e : P \to LM$, but the definition of gauge-natural objects is quite involved and we do not strictly need it now.

Equivalently, we can consider any N-dimensional real vector space V on which we let $\mathrm{SO}(N-1,1)$ act via the fundamental representation $\rho : \mathrm{SO}(N-1,1) \to \mathrm{End}(V)$. Consequently, we can define the associated bundle

$$\mathcal{V} := P \times_\rho V, \tag{4.18}$$

which is also called **Minkowski bundle** or *fake tangent bundle*. In fact, the vielbein map induces a vector bundle isomorphism $\mathcal{V} \to TM : (x^\mu, v^a) \mapsto v^a e_a$ which is the dual construction of the soldering form as defined in (1.2.27). By abuse of notation, we call this soldering form **vielbein** or **vierbein** (tetrad in English) when $N = 4$ and denote it by e.[1] In particular, e provides an isomorphism between the tangent bundle and the Minkowski bundle $\mathcal{V}$, and will be the dynamical field we choose to consider in the next chapters.

[1] This is the usual notation found in literature [17], [16]. Throughout the following chapter we will never refer to e as the bundle morphism defined by the vielbein map

4.2 The Palatini-Cartan Formalism

Having proved that the vielbein $e \in \Gamma(T^*M \otimes \mathcal{V})$ is the correct object replacing the metric, we consider the other dynamical fields in the theory.

Having defined an SO$(N-1,1)$-principal bundle suggests the obvious idea of introducing a principal connection on P. Let $\Omega \in \Omega^1(P, \mathfrak{so}(N-1,1))$ be a connection 1-form. We want to consider the gauge field as a dynamical field of the theory. Let $\sigma_\alpha : U_\alpha \subset M \to P$ be a local section, then the local representation of the connection (i.e. the gauge field) is

$$\omega_\alpha := (\sigma_\alpha)^*(\Omega) \in \Omega^1(U, \mathfrak{so}(N-1,1)). \tag{4.19}$$

The Lie algebra $\mathfrak{so}(N-1,1)$ is made by matrices ${\omega^a}_b$ satisfying

$$ {\omega^a}_c \eta_{da} + \eta_{ac} {\omega^a}_d = 0. \tag{4.20}$$

Defining $\omega^{ab} := {\omega^a}_c \eta^{cb}$, then the previous condition implies that $\omega^{ab} = -\omega^{ba}$. Now recall that we have defined V as an N-dimensional vector space on which we have ρ :SO$(N-1,1)$ →Aut(V) as the fundamental representation of SO$(N-1,1)$.

Therefore, choosing a basis $\{u_a\}$ of V, we might identify $\mathfrak{so}(N-1,1) \simeq \bigwedge^2 V$ by raising indices with η.

Then it is easy to see that ω_α is an element of $\Omega^1(U_\alpha, \bigwedge^2 \mathcal{V})$. Now consider ω_α and ω_β two local connections associated to sections σ_α and σ_β. As we saw, $\forall U_\alpha, U_\beta \subset M$ such that $U_\alpha \cap U_\beta \neq \emptyset$, there exists a unique $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{SO}(N-1,1) : x \mapsto \Lambda(x)$ for which

$$\sigma_\beta(x) = R_{h_{\alpha\beta}(x)} \sigma_\alpha(x), \tag{4.21}$$

with R_g right action of SO$(N-1,1)$ on P. Then

$$\omega_\beta = (Ad_{h^{-1}_{\alpha\beta}})_*(\omega_\alpha) + h^*_{\alpha\beta}\theta. \tag{4.22}$$

In our notation this simply becomes

$$\omega^{ab}_\beta = {\Lambda^a}_c(x){\Lambda^b}_d(x)\omega^{cd}_\alpha(x) + \eta^{cd}{\Lambda^a}_c(x)d{\Lambda^b}_d(x). \tag{4.23}$$

Finally, recalling corollary (1.2.21), the space of connections on M is $\mathcal{A}(M) := \Omega^1(M, \wedge^2 \mathcal{V})$.

Classical Space of Fields and the PC Action

We define the space of (i,j)-forms to be the differential i-forms with values in the j-th exterior power of V, namely $\Omega^{(i,j)}(M) := \Omega^i(M, \bigwedge^j \mathcal{V})$.

The space of fields of our theory is then defined to be $\mathcal{F}_{PC} := \Omega^{(1,1)}_{nd} \times \mathcal{A}(M)$, where $\Omega^{(1,1)}_{nd}$ is the space of vielbeins as non-degenerate one-forms with values in V. It was Palatini who first proposed to consider the connection ω as a dynamical field, with the idea of writing an action whose equations of motion fixed the connection to be Levi-Civita. This formalism has the further advantage that all the fields are expressed as differential forms, and hence can easily be restricted to a suitable submanifold of M (e.g. its boundary, if it has one).

At this point one might be tempted to consider General Relavity as a Yang–Mills type gauge theory. However, recall that the Yang–Mills action is quadratic in the curvature field strength so that it leads to second order differential equations for the gauge field A, but in the case of gravity we have to exclude a qudratic action in the curvature tensor, since it would lead to fourth order differential equations for the gravitational field (which is the metric tensor $g_{\mu\nu}$ or the vielbein e^a_μ). Such a difference streams from the fact that the basic dynamical degrees of freedom of the gravitational field are not encoded in the connection itself (the Levi Civita connection) but rather in the metric from which it derives.

We are looking for an action functional that gives the same EL locus as Einstein–Hilbert. The PC action is

$$S_{PC} := \int_M \frac{1}{(N-2)!} e^{N-2} \wedge F_\omega + \frac{\Lambda}{N!} e^N, \tag{4.24}$$

where $e^n := \underbrace{e \wedge e \wedge \cdots \wedge e}_{\text{n times}}$ and $F_\omega := d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature associated to ω.

We can find equations of motion by varying the action,

$$\begin{aligned} \delta S_{PC} &= \int_M \frac{1}{(N-3)!} e^{N-3} \delta e F_\omega - \frac{1}{(N-2)!} e^{N-2} d_\omega(\delta\omega) + \frac{\Lambda}{(N-1)!} e^{N-1} \delta e \\ &= \int_M \left[\frac{1}{(N-3)!} e^{N-3} F_\omega + \frac{\Lambda}{(N-1)!} e^{N-1} \right] \delta e + \frac{1}{(N-2)!} d_\omega(e^{N-2}) \delta\omega - \frac{1}{(N-2)!} d(e^{N-2}\delta\omega), \end{aligned} \tag{4.25}$$

where we used integration by parts and that $\delta_\omega F_\omega = -d_\omega(\delta\omega)$.[2] The last term in (4.25) will produce a boundary term if $\partial M \neq \emptyset$, due to Stokes theorem.

Then we find equations of motion

$$e^{N-3} d_\omega e = 0; \tag{4.26}$$

$$\frac{1}{(N-3)!} e^{N-3} F_\omega + \frac{\Lambda}{(N-1)!} e^{N-1} = 0. \tag{4.27}$$

Equation (4.26) is equivalent to $d_\omega e = 0$ because of the non-degeneracy condition (and because e^{N-3} is injective in this case [8]). Furthermore, it fixes ω to be torsionless, and since it is compatible with η, then $d_\omega e = 0$ implies the metricity condition $d_{e^*(\omega)} g = 0$, which is uniquely solved by the Levi-Civita metric connection.

After imposing (4.26), we find that equation 4.27 is equivalent to Einstein's field equation, with the addition of a cosmological constant Λ.

Remark 4.2.1. It is important to notice that, even if e is an isomorphism, $e \wedge \cdot$ might not be, indeed $e^{N-3} \wedge F_\omega = 0$ is not equivalent to the flatness condition $F_\omega = 0$

Remark 4.2.2. There are two ways of showing that the PC and EH theories are equivalent. The first one is to rewrite equation (4.27) after imposing (4.26) and see that it actually yelds Einstein's field equation. The other way is to use (4.26) and rewrite the action S_{PC} in terms of the metric tensor, to see that it is equivalent to the Einstein–Hilbertt action. This is seen very easily by noticing that

$$\frac{e^N}{N!} = \sqrt{-\det(g)} d^N x = \mathrm{Vol}_g, \qquad \frac{e^{N-2}}{(N-2)!} F_\omega = \mathrm{Vol}_g R, \tag{4.28}$$

where R is the Ricci scalar.

[2]

$$\begin{aligned} \delta\omega F_\omega &= \delta_\omega(d\omega + \frac{1}{2}[\omega, \omega]) = -d\delta\omega + \frac{1}{2}[\delta\omega, \omega] - \frac{1}{2}[\omega, \delta\omega] \\ &= -d(\delta\omega) - [\omega, \delta\omega] = -d_\omega(\delta\omega). \end{aligned}$$

4.3 Real Scalar Field in the First Order Formalism

We now consider a scalar field $\phi \in \mathcal{C}^\infty(M)$ as a smooth function on space–time. In order to couple the scalar field to gravity in the Palatini-Cartan formalism, it is useful to consider the first order formulation introducing a new field $\Pi \in \Omega^{(0,1)}(M)$, which is a section of the "Poincaré" bundle $\mathcal{V}$. The idea behind the introduction of this new field is to avoid to explicitly consider the term

$$\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, \tag{4.29}$$

which usually appears in the Klein-Gordon Lagrangian on an arbitrary background, because it involves the inverse $g^{\mu\nu}$ of the metric tensor, which is hard to deal with in calculations in terms of the vielbein.

The new field Π is a priori independent of ϕ, but after the equations of motion are found, it will assume the role of the momentum associated to the scalar field.

The minimal coupling condition (in the massless case) implies that the action will be of the form

$$\begin{aligned} S &= S_{PC} + S_{scal} \qquad \text{with} \\ S_{PC} &= \int_M \frac{1}{(N-2)!} e^{N-2} \wedge F_\omega + \frac{\Lambda}{N!} e^N \\ S_{scal} &= \int_M \frac{1}{(N-1)!} e^{(N-1)} \wedge \Pi \wedge d\phi + \frac{1}{2N!} e^N (\Pi, \Pi), \end{aligned} \tag{4.30}$$

where $(\cdot, \cdot)$ is defined to be the pairing in V. In an orthonormal (with respect to the Minkowski metric) basis $\{v_a\}$ of V, $\forall A = A^a v_a, B = B^b v_b \in V$ it reads:

$$(A, B) := A^a B^b \eta_{ab}. \tag{4.31}$$

The variation of the action yelds

$$\begin{aligned} \delta S = \int_M & \left[\frac{1}{(N-3)!} e^{N-3} F_\omega + \frac{\Lambda}{(N-1)!} e^{N-1} + \frac{1}{(N-2)!} e^{N-2} \Pi d\phi + \frac{1}{2(N-1)!} e^{N-1} (\Pi, \Pi) \right] \delta e + \\ & + \frac{1}{(N-2)!} d_\omega (e^{N-2}) \delta\omega + \frac{1}{(N-1)!} e^{N-1} d\phi \delta\Pi + \frac{1}{N!} e^d (\Pi, \delta\Pi) + \frac{1}{(N-1)!} d(e^{N-1}\Pi) \delta\phi + \\ & - d\left(\frac{1}{(N-2)!} e^{N-2} \delta\omega + \frac{1}{(N-1)!} e^{N-1} \Pi \delta\phi \right). \end{aligned} \tag{4.32}$$

We notice that the variation of the action produces a boundary term which, applying Stokes' theorem, is given by

$$\tilde{\alpha} := \int_{\partial M} \frac{1}{(N-2)!} e^{N-2} \delta\omega + \frac{1}{(N-1)!} e^{N-1} \Pi \delta\phi \tag{4.33}$$

This is the term corresponding to the local 1-form on the space of preboundary fields defined in Chapter 2. Its variation will produce the pre-symplectic form which will be essential to construct the reduced phase space in the next section.

From the variation of the action we also find the equations of motion, which are given by

$$d_\omega e = 0; \tag{4.34}$$

$$\tfrac{1}{(N-3)!} e^{N-3} F_\omega + \tfrac{\Lambda}{(N-1)!} e^{N-1} + \tfrac{1}{(N-2)!} e^{N-2} \Pi d\phi + \tfrac{1}{2(N-1)!} e^{N-1} (\Pi, \Pi) = 0; \tag{4.35}$$

$$d(e^{N-1}\Pi) = 0; \tag{4.36}$$

$$e^{N-1}(d\phi - (e, \Pi)) = 0, \tag{4.37}$$

where, to find the equation of motion corresponding to $\delta\Pi$, we used the following identity[3], which holds for every $A, B \in \Omega^{(0,1)}$:

$$\frac{1}{N} e^N(A,B) = (-1)^{|A|+|B|} e^{N-1}(e,A)B. \tag{4.38}$$

We can further simplify equations (4.36), in fact, using $d(e^{N-1}\Pi) = d_\omega(e^{N-1}\Pi)$ because top forms transform trivially under the action of the Lie algebra, then $d(e^{N-1}\Pi) = d_\omega(e^{N-1})\Pi + e^{N-1}\Pi = 0$, but $d_\omega e = 0$, therefore we find

$$e^{N-1} d_\omega \Pi = 0. \tag{4.39}$$

Furthermore, we can also simplify (4.37), since $W_{N-1}^{(1,0)} : \Omega^{(1,0)}(M) \to \Omega^{(N,N-1)}(M) : A \mapsto e^{N-1}A$ is injective[4]. Therefore we obtain

$$d\phi - (e, \Pi) = 0. \tag{4.40}$$

Eq. (4.40) fixes Π in terms of the derivatives of ϕ, while (4.39) is then just the usual Klein-Gordon equation for a massless scalar field on an arbitrary background. To see this, we compute the scalar field part of the Lagrangian after having imposed the constraint and plug it into the action, showing that we recover the usual Klein-Gordon Lagrangian on a curved background.

First of all, we recall the definition of the determinant of an $N \times N$ matrix P_μ^a, it is given by $\det(P) = \epsilon_{a_1 \cdots a_N} P_1^{a_1} \cdots P_N^{a_N}$, where $\epsilon_{a_1 \cdots a_N}$ is the completely anti-symmetric Levi-Civita symbol. Then we notice

$$\begin{aligned} \frac{e^N}{N!} &= \frac{1}{N!} \epsilon_{a_1 \cdots a_N} e_{\mu_1}^{a_1} \cdots e_{\mu_N}^{a_N} dx^{\mu_1} \cdots dx^{\mu_N} = \epsilon_{a_1 \cdots a_N} e_1^{a_1} \cdots e_N^{a_N} d^N x \\ &= \det(e) d^N x. \end{aligned} \tag{4.41}$$

Then, since $\det(g) = \det(e^\eta e) = -\det(e)^2$, we obtain $e^N/N! = \sqrt{-\det(g)}d^N x = \mathrm{Vol}_g$ as the canonical volume form. In coordinates (with respect to the local basis $\{e_\mu\}$ of V), assuming that the metric is non-degenerate, eq. (4.40) reads

$$\pi^\mu = -g^{\mu\nu}\partial_\nu \phi. \tag{4.42}$$

Finally we can compute the term in the scalar part of the action, using the previous identity

$$S_{\text{scal}} = \int_M \frac{1}{(N-1)!} e^{N-1}\Pi d\phi + \frac{1}{2N!} e^N(\Pi, \Pi) = -\int_M \frac{1}{2} (\mathrm{Vol}_g) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{4.43}$$

which is exactly the Klein-Gordon Lagrangian, once we notice that $\nabla_\mu \phi = \partial_\mu \phi$.

4.4 Classical Boundary Structure in $N = 4$

We now assume our space–time manifold M to be 4-dimensional and with boundary $\Sigma := \partial M$ and we study the fields on Σ, constructing the reduced phase space of boundary fields as defined at the end of Chapter 2 and showing that the constraints (given by the projection of the equations of motion to the boundary) are first class, thus defining a coisotropic submanifold as their zero locus. As mentioned before, we show that the scalar field coupling does not modify the boundary structure of pure gravity.

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[3] proved in Lemma B.1.3.(1)

[4] see Lemma B.1.6.(1)

The Reduced Phase Space

Many of the results which we will present in the following section have been shown in [8], we will recall some of their proofs and refer to that paper for the ones we leave out.

We start by considering the boundary term (4.33) that is found after the variation of the action, for $N = 4$ it reads

$$\tilde{\alpha} := \int_\Sigma \frac{1}{2} e^2 \delta\omega + \frac{1}{3!} e^3 \Pi \delta\phi. \tag{4.44}$$

The classical fields on the boundary will again be indicated by (e, ω, ϕ, Π). The inclusion $\iota : \Sigma \hookrightarrow M$ of Σ in M induces the bundles $P|_\Sigma := \iota^*(P)$ and $\mathcal{V}|_\Sigma := \iota^*(\mathcal{V})$. The fields are respectively defined as

- e is a non-degenerate section of $T^*\Sigma \otimes \mathcal{V}|_\Sigma$, meaning that (i) at each point the three components are linearly independent and (ii) the underlying metric g, defined by $g := e^*(\eta)$, is non-degenerate;
- ω is an element of the space of connections $\mathcal{A}_\Sigma$, locally modeled by $\Gamma(T^*\Sigma \otimes \bigwedge^2 \mathcal{V}|_\Sigma)$;
- $\phi \in \mathcal{C}^\infty(\Sigma)$ is a smooth function on Σ;
- Π is an element of $\Omega_\partial^{(0,1)} := \Omega^{(0,1)}(\Sigma)$, where we define $\Omega^{(i,j)}(\Sigma) := \Gamma(\bigwedge^i T^*\Sigma \otimes \bigwedge^j \mathcal{V}|_\Sigma)$.

We denote the space of (pre)boundary fields as $\tilde{F}_\partial = \Omega_{\partial,\text{n.d.}}^{(1,1)} \times \mathcal{A}_\Sigma \times \mathcal{C}^\infty(\Sigma) \times \Omega_\partial^{(0,1)}$.

We note that $\tilde{\alpha}$ is the integral of a local (top, 1) form on $\tilde{F}_\partial \times \Sigma$ as defined in (2.2.1) and therefore is a 1-form on $\tilde{F}_\partial$. By taking its variation (the variational vertical differential), we obtain a two-form on $\tilde{F}_\partial$

$$\tilde{\varpi} := \delta\alpha = \int_\Sigma e\delta e\delta\omega + \frac{1}{3!}\delta(e^3\Pi)\delta\phi. \tag{4.45}$$

By construction, $\tilde{\varpi}$ is closed on $\tilde{F}_\partial$. However, it is degenerate, namely $\ker(\tilde{\varpi}) := \{X \in T\tilde{F}_\partial \mid \iota_X \tilde{\varpi} = 0\} \neq \{0\}$. In order to get rid of this degeneracy, we perform a symplectic reduction[5]. The quotient space F_∂ will be called the **geometric phase space** of the theory

$$F_\partial := \frac{\tilde{F}_\partial}{\ker(\tilde{\varpi})}, \tag{4.46}$$

with the canonical projection $\pi_\partial : \tilde{F}_\partial \to F_\partial$. Considering a generic vector field $X = \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta\omega} + \mathbb{X}_\phi \frac{\delta}{\delta\phi} + \mathbb{X}_\Pi \frac{\delta}{\delta\Pi}$[6], we explicitly find the kernel of $\tilde{\varpi}$ as those vector fields satisfying $\iota_X \tilde{\varpi} = 0$, which is equivalent to the following system of equations:

$$e\mathbb{X}_e = 0; \tag{4.47}$$

$$e\mathbb{X}_\omega + \frac{1}{2}e^2\Pi\mathbb{X}_\phi = 0; \tag{4.48}$$

$$\frac{1}{2}e^2\Pi\mathbb{X}_e + \frac{1}{3!}e^3\mathbb{X}_\Pi = 0; \tag{4.49}$$

$$e^3\mathbb{X}_\phi = 0. \tag{4.50}$$

[5] The vector fields in the kernel of the presymplectic form span a smooth involutive distribution, which, if regular, by Frobenius' theorem defines a foliation. The quotient space $\tilde{F}_\partial/\ker(\tilde{\varpi})$ is the set of leaves in the foliation induced by $\ker(\tilde{\varpi})$. In our case, the vector fields in the kernel only act as translations of the field, therefore it is easy to see that the quotient space is still a smooth manifold.

[6] of course the components of the vector fields are such that $\mathbb{X}_e \in \Omega_{\partial\text{n.d}}^{(1,1)}$, $\mathbb{X}_\omega \in \mathcal{A}_\Sigma$, $\mathbb{X}_\phi \in \mathcal{C}^\infty(\Sigma)$ and $\mathbb{X}_\Pi \in \Omega_\partial^{(0,1)}$

Defining $W_k^{\partial(i,j)} := e^k \wedge : \Omega_\partial^{(i,j)} \to \Omega_\partial^{(i+k,j+k)}$, by Lemmas B.1.8.(2) and B.1.8.(4) $W_1^{\partial(1,1)}$ and $W_3^{\partial(0,0)}$ are both injective, therefore (4.47) and (4.50) are solved respectively by $\mathbb{X}_e = 0$ and $\mathbb{X}_\phi = 0$. (4.48) and (4.49) reduce to $e\mathbb{X}_\omega = 0$ and $e^3\mathbb{X}_\Pi = 0$. The geometric phase space is then found to be $F_\partial = \Omega_{\partial,\text{n.d.}}^{(1,1)} \times (\mathcal{A}_\Sigma/\sim) \times \mathcal{C}^\infty(\Sigma) \times (\Omega_\partial^{(0,1)}/\sim)$, where

$$\omega \sim \tilde{\omega} \quad \Leftrightarrow \quad \omega - \tilde{\omega} = v \;\text{ with }\; ev = 0 \tag{4.51}$$

$$\Pi \sim \tilde{\Pi} \quad \Leftrightarrow \quad \Pi - \tilde{\Pi} = \gamma \;\text{ with }\; e^3\gamma = 0, \tag{4.52}$$

$$\tag{4.53}$$

because a vector field $X \in \ker(\tilde{\varpi})$ acts on the space of fields as a translation of ω and Π respectively by $\mathbb{X}_\omega$ and $\mathbb{X}_\Pi$. We also define $\mathcal{A}_\Sigma^{red} := \mathcal{A}_\Sigma/\sim$ and $\Omega_{red}^{\partial(0,1)} := \Omega_\partial^{(0,1)}/\sim$.

F_∂ is thus a symplectic manifold with symplectic form

$$\varpi = \int_\Sigma e\delta e\delta[\omega] + \frac{1}{3!}\delta(e^3[\Pi])\delta\phi. \tag{4.54}$$

Remark 4.4.1. Instead of Π, we might define a new boundary field $p := \frac{1}{3!}e^3\Pi$. In this way the prefactor e^3 automatically selects the physical part in Π without the need of a further symplectic reduction. Furthermore, we obtain a sympletic 2-form whose "scalar field part" is written in Darboux coordinates: $\varpi = \int_\Sigma e\delta e\delta[\omega] + \delta p\delta\phi$.

The geometric phase space F_∂ is however not yet the "physical phase space" of the theory, i.e. the space of admissible initial conditions on a Cauchy surface Σ, which is called the reduced phase space. Indeed, the Euler-Lagrange equations (4.34) split into evolution equations and constraints. The evolution equations contain the derivatives of the (pre)boundary fields in a transversal direction with respect to the boundary Σ, while only the tangential derivatives appear in the constraints. Imposing the constraints on the preboundary fields enlarges the kernel of the presymplectic form, therefore the corresponding reduction has to be performed. We first reduce the space $\tilde{F}_\partial$ to the geometric phase space, then apply the constraints (which can easily be found as the restriction of the equations of motion to the boundary of M), then perform a reduction on the zero-locus of the constraints, which, as we saw in the previous chapter, is a coisotropic submanifold of F_∂. The reduced phase space is the coisotropic reduction of C, which allows us exploit what we have seen in the previous chapter to define the BFV formalism and find the algebra of functions on the reduced phase space.

Remark 4.4.2. Notice that the constraints (as the restrictions of the EL equations from the bulk to the boundary) are not necessarily invariant under v-translations and γ-translations, therefore we fix a convenient set of representatives of the equivalence classes $[\omega]$ and $[\Pi]$. The next subsection deals with choosing such representatives in the ideal way. In order to do so, we choose a section e_n of $\mathcal{V}|_\Sigma$ that is a completion of the linearly independent system $\{e_1, e_2, e_3\}$, where $i = 1, 2, 3$ are the indices of the boudary coordinates. Note that in a neighborhood of a given e in the space of pre-boundary fields we may choose e_n once and for all independently of the e's in the neighborhood.

Choice of Representatives via Constraints

As mentioned, we need to fix convenient representatives of the classes $[\omega] \in \mathcal{A}_\Sigma^{red}$ and $[\Pi] \in \Omega_{red}^{\partial(0,1)}$. The idea is to take advantage of the constraints to fix the representatives, in particular we will use parts of the dynamical constraints. The constraints to be imposed on the space of preboundary fields are

$$d_\omega e = 0; \tag{4.55}$$

$$eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi) = 0; \tag{4.56}$$

$$d\phi + (e,\Pi) = 0. \tag{4.57}$$

We do not impose $e^{d-1}d_\omega\Pi = 0$ because it is an evolution equation. Furthermore, it is a top form on M, therefore it cannot be restricted to Σ.

Choice of Representative of $[\omega]$

We start by considering (4.55). As already mentioned, we will assume throughout the paper that $g^\partial_{ij} := (e_i, e_j)$ is invertible. Then we have the following proposition

Proposition 4.4.3. *The constraint $d_\omega e = 0$ splits into two further constraints*

$$d_\omega e = 0 \qquad \Longleftrightarrow \qquad \begin{cases} ed_\omega e = 0 \\ e_n d_\omega e \in \operatorname{Im} W_1^{\partial(1,1)} \end{cases}, \tag{4.58}$$

*where $ed_\omega e = 0$ is called **invariant constraint**[7] and $e_n d_\omega e \in \operatorname{Im} W_1^{\partial(1,1)}$ **structural constraint***

Proof. It is an immediate consequence of Lemma (B.1.9) ✓

We have just seen that (4.55) splits into two equations, each of which has 6 components. This procedure is useful because we can now use the structural constraint to fix the representative of $[\omega]$, in particular we use it to fix the 6 components of $v \in\ker(W_1^{\partial(1,2)})$

Theorem 4.4.4. *[8] Suppose that g^∂, the metric induced on the boundary, is non-degenerate. Given any $\widetilde{\omega} \in \Omega^{1,2}$, there is a unique decomposition*

$$\widetilde{\omega} = \omega + v, \tag{4.59}$$

with ω and v satisfying

$$ev = 0 \quad \text{and} \quad e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}. \tag{4.60}$$

Proof. Let $\widetilde{\omega} \in \Omega^{1,2}_\partial$. From Lemma B.1.10 we deduce that there exist unique $\sigma \in \Omega^{1,1}_\partial$ and $v \in \mathrm{Ker} W_1^{\partial,(1,2)}$ such that

$$e_n d_{\widetilde{\omega}} e = e\sigma + e_n[v, e].$$

We define $\omega := \widetilde{\omega} - v$. Then ω and v satisfy (4.59) and (4.60).

For uniqueness, suppose that $\widetilde{\omega} = \omega_1 + v_1 = \omega_2 + v_2$ with $ev_i = 0$ and $e_n d_{\omega_i} e \in \operatorname{Im} W_1^{\partial,(1,1)}$ for $i = 1, 2$. Hence

$$e_n d_{\omega_1} e - e_n d_{\omega_2} e = e_n[v_2 - v_1, e] \in \operatorname{Im} W_1^{\partial,(1,1)}.$$

Hence from Lemma B.1.9 and Lemma B.1.10 (for which we need nondegeneracy of g^∂), we deduce $v_2 - v_1 = 0$, since $v_2 - v_1 \in Ker W_1^{\partial,(1,2)}$. ✓

Corollary 4.4.5. *[8] The field ω in the decomposition* (4.59) *depends only on the equivalence class $[\omega] \in \mathcal{A}^{red}(\Sigma)$.*

[7] Indeed $\delta_v(ed_\omega e) = e[v, e] = 0$

Proof. Let $\widetilde{\omega}_1, \widetilde{\omega}_2 \in [\omega]$. Hence $\widetilde{\omega}_1 - \widetilde{\omega}_2 = \tilde{v} \in \mathrm{Ker} W_1^{\partial,(1,2)}$. Applying Theorem 4.4.4 we get ω_1, v_1, ω_2, v_2 such that $v_1, v_2 \in \mathrm{Ker} W_1^{\partial,(1,2)}$ and

$$\widetilde{\omega}_1 = \omega_1 + v_1 \qquad e_n d_{\omega_1} e \in \mathrm{Im}\, W_1^{\partial,(1,1)}$$
$$\widetilde{\omega}_2 = \omega_2 + v_2 \qquad e_n d_{\omega_2} e \in \mathrm{Im}\, W_1^{\partial,(1,1)}.$$

Subtracting these equations we get $\omega_2 - \omega_1 = v_1 - v_2 - \tilde{v} \in \mathrm{Ker} W_1^{\partial,(1,2)}$ together with $e_n[\omega_1 - \omega_2, e] \in \mathrm{Im}\, W_1^{\partial,(1,1)}$. Hence, from Lemma B.1.10, we deduce $\omega_1 = \omega_2$. ✓

Choice of Representative of $[\Pi]$

We now replicate the procedure of the last paragraph and use a constraint to fix the representative of $[\Pi]$. It is easy to see that there is a unique decomposition $\Pi = \pi^n e_n + \tilde{\pi}$, with $\tilde{\pi} \in \ker W_3^{\partial(0,1)}$. This means that equivalence classes of Π are in 1-to-1 correspondence with π^n, hence fixing $\tilde{\pi}$ fixes the representative.

Theorem 4.4.6. *Suppose that g^∂ is non-degenerate, then $\tilde{\pi}$ in the decomposition $\Pi = \pi^n e_n + \tilde{\pi}$ is uniquely fixed in terms of ϕ and π^n by imposing*

$$(e, \Pi) - d\phi = 0. \tag{4.61}$$

Proof. We define $g_{na} := (e_n, e_a)$. We also drop the subscript ∂ in g^∂ to have a lighter notation. Then in coordinates we have that $d\phi + (e, \Pi) = 0$ implies $\partial_a \phi + \pi^n g_{na} + \pi^b g_{ab} = 0$, which yelds the following equation on the boundary:

$$\pi^a = -g^{ab}(\partial_b \phi + g_{nb}\pi^n). \tag{4.62}$$

✓

Poisson Brackets of the Constraints

We still have to impose the constraints on the space of pre-boundary fields. In order to do so, we recast them into local forms by means of Lagrangian multipliers. In particular, we have

$$J_{\tilde{\mu}} := \int_\Sigma \tilde{\mu}\left(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi)\right); \tag{4.63}$$

$$L_c := \int_\Sigma c e d_\omega e. \tag{4.64}$$

As it turns out, when acting on shell (i.e. on the zero-locus of the constraints), L_c generates the internal gauge transformations and $J_{\tilde{\mu}}$ the diffeomorphisms.

Furthermore, if we split $\tilde{\mu} = \iota_\xi e + \lambda e_n$, from $J_{\tilde{\mu}}$ we obtain two functions

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \frac{1}{3!}\iota_\xi(e^3\Pi)d\phi + \iota_\xi(\omega - \omega_0)e d_\omega e; \tag{4.65}$$

$$H_\lambda = \int_\Sigma \lambda e_n\left(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2}e^2\Pi d\phi + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi)\right). \tag{4.66}$$

Remark 4.4.7. It is important to notice that the Lagrange multiplier have the role of the generators of the symmetry. In particular, $c \in \Omega_\partial^{(0,2)}$ generates the internal gauge symmetry[8], $\xi \in \mathfrak{X}(\Sigma)$

[8] Recall that we identify $\mathfrak{so}(3,1) \simeq \wedge^2 \mathcal{V}$

represent the vector field parametrizing the local diffeomorphisms in the direction tangential to the boundary, while $\lambda \in \mathcal{C}^\infty(\Sigma)$ is the generator of the local diffeomorphism normal to the boundary.

For future advantage we added a term in P_ξ proportional to $ed_\omega e$, depending also on a reference connection. The addition of this term does not change the constrained set. It is also important to notice that the terms in $J_{\tilde{\mu}}$ containing e^3 disappear in P_ξ because $\iota_\xi(e^4) = 0$.

Furthermore, we assume the Lagrange multipliers to be odd, namely we consider $c \in \Omega_\partial^{0,2}[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega_\partial^{0,0}[1]$, and we denote with L_ξ^ω the covariant Lie derivative along the odd vector field ξ with respect to a connection ω:

$$\mathrm{L}_\xi^\omega A = \iota_\xi d_\omega A - d_\omega \iota_\xi A \qquad A \in \Omega_\partial^{i,j}. \tag{4.67}$$

Theorem 4.4.8. *With the usual hypothesis that g^∂ is non-degenerate, the functions L_c, P_ξ, H_λ define a coisotropic submanifold with respect to the symplectic structure ϖ_{PC}. Their Poisson brackets read*

$$\{L_c, L_c\} = -\frac{1}{2} L_{[c,c]} \qquad \{P_\xi, P_\xi\} = \tfrac{1}{2} P_{[\xi,\xi]} - \tfrac{1}{2} L_{\iota_\xi \iota_\xi F_{\omega_0}} \tag{4.68a}$$

$$\{L_c, P_\xi\} = L_{\mathrm{L}_\xi^{\omega_0} c} \quad \{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} \tag{4.68b}$$

$$\{H_\lambda, H_\lambda\} = 0 \quad \{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}}, \tag{4.68c}$$

where $X = [c, \lambda e_n]$, $Y = \mathrm{L}_\xi^{\omega_0}(\lambda e_n)$ and $Z^{(a)}$, $Z^{(n)}$ are the components of $Z \in \{X, Y\}$ with respect to the frame (e_a, e_n).

Remark 4.4.9. As said before, this theorem is the same as in [8], where the Palatini-Cartan theory without the scalar coupling is analyzed.

Proof. Theorem 4.4.4 allows to have well defined constraints, because of the uniqueness of the representative ω of $[\omega]$.

In order to compute the brackets of the constraints, we first compute the Hamiltonian vector fields associated to the constraints, defined for a function f on the space of boundary fields as $\mathbb{X}_f$ such that $\iota_{\mathbb{X}_f}\varpi = \delta f$.

Before explicitly computing the vector fields, we recall Remark (4.4.1) and notice that P_ξ can also be written as

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi(e^2)F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e + \iota_\xi(p)d\phi. \tag{4.69}$$

Then the variations of the constraints are

$$\delta L_c = \int_\Sigma -\frac{1}{2}c[\delta\omega, ee] + \frac{1}{2}cd_\omega\delta(ee) = \int_\Sigma [c, e]e\delta\omega + d_\omega ce\delta e;$$

$$
\begin{aligned}
\delta P_\xi &= \int_\Sigma \iota_\xi(e\delta e)F_\omega - \frac{1}{2}\iota_\xi(ee)d_\omega\delta\omega + \iota_\xi\delta\omega e d_\omega e - \frac{1}{2}\iota_\xi(\omega-\omega_0)[\delta\omega, ee] \\
&\quad + \frac{1}{2}\iota_\xi(\omega-\omega_0)d_\omega\delta(ee) + \iota_\xi(\delta p)d\phi + \iota_\xi p d(\delta\phi) \\
&\overset{\diamondsuit}{=} \int_\Sigma -e\delta e\iota_\xi F_\omega + \frac{1}{2}d_\omega\iota_\xi(ee)\delta\omega - \frac{1}{2}\delta\omega\iota_\xi d_\omega(ee) + \frac{1}{2}\delta\omega[\iota_\xi(\omega-\omega_0), ee] \\
&\quad + \frac{1}{2}d_\omega\iota_\xi(\omega-\omega_0)\delta(ee) + \delta p\iota_\xi(d\phi) + d(\iota_\xi p)\delta\phi \\
&= \int_\Sigma -e\delta e\iota_\xi F_\omega - (\mathrm{L}_\xi^\omega e)e\delta\omega + e\delta\omega[\iota_\xi(\omega-\omega_0), e] + d_\omega\iota_\xi(\omega-\omega_0)e\delta e \\
&\quad - \xi(\phi)\delta p - L_\xi^{\omega_0}(p)\delta\phi \\
&= \int_\Sigma -e\delta e(\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) + \iota_\xi F_{\omega_0}) - (\mathrm{L}_\xi^{\omega_0} e)e\delta\omega - \xi(\phi)\delta p - L_\xi^{\omega_0}(p)\delta\phi.
\end{aligned}
$$

In the last computation the symbol ($\diamondsuit$) indicates that we used integration by parts.

$$
\begin{aligned}
\delta H_\lambda &= \int_\Sigma \lambda e_n\delta e F_\omega + \frac{1}{2}\Lambda\lambda e_n e^2\delta e - \lambda e_n e d_\omega\delta\omega + \delta\left[\frac{1}{2\cdot 3!}\lambda e_n e^3(\Pi,\Pi) + \frac{1}{2}\lambda e_n e^2\Pi d\phi\right] \\
&= \int_\Sigma \lambda e_n\left[\left(F_\omega + \frac{\Lambda}{2}e^2 + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right)\delta e + \frac{e^3}{3!}(\Pi,\delta\Pi) + \frac{e^2}{2}d\phi\delta\Pi + \frac{e^2}{2}\Pi d\delta\phi\right] \\
&\quad + d_\omega(\lambda e_n e)\delta\omega \\
&\overset{\bigstar\diamondsuit}{=} \int_\Sigma \lambda e_n\left[\left(F_\omega + \frac{\Lambda}{2}e^2 + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right)\delta e + \frac{1}{2}d_\omega(\lambda e_n e^2\Pi)\delta\phi\right. \\
&\quad + \frac{\lambda e_n}{2}e^2\left[d\phi - (e,\Pi)\right]\delta\Pi - \lambda\frac{e^3}{3!}(e_n,\Pi)\delta\Pi + d_\omega(\lambda e_n e)\delta\omega \\
&= \int_\Sigma \left[\lambda e_n\left(F_\omega + \frac{\Lambda}{2}e^2 + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) - \frac{\lambda}{2}e^2(e_n,\Pi)\Pi\right]\delta e \\
&\quad - \lambda(e_n,\Pi)\delta p + \frac{1}{2}d_\omega(\lambda e_n e^2\Pi)\delta\phi + d_\omega(\lambda e_n e)\delta\omega,
\end{aligned}
$$

where we used $\forall A, B \in \Omega_\partial^{(0,1)}$ the following identity[9]:

$$
(\bigstar)\qquad e_n\frac{e^{N-1}}{(N-1)!}(A,B) = (-1)^{|A|+|B|}\left[\frac{1}{(N-2)!}e_n e^{N-2}(e,A)B + \frac{e^{N-1}}{(N-1)!}(e_n,A)B\right]
$$

The components of the Hamiltonian vector fields of L_c and P_ξ are

$$
\begin{aligned}
&\mathbb{L}_e = [c,e] && \mathbb{L}_p = 0 \\
&\mathbb{L}_\omega = d_\omega c + \mathbb{V}_L && \mathbb{L}_\phi = 0 \\
&\mathbb{P}_e = -\mathrm{L}_\xi^{\omega_0} e && \mathbb{P}_p = -\mathrm{L}_\xi^{\omega_0}(p) \\
&\mathbb{P}_\omega = -\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) - \iota_\xi F_{\omega_0} + \mathbb{V}_P && \mathbb{P}_\phi = -\xi(\phi),
\end{aligned}
$$

where, e.g., $\mathbb{L}_e \equiv \mathbb{L}(e)$, with $\iota_{\mathbb{L}}\varpi_{PC} = \delta L_c$, and $\mathbb{V}_L, \mathbb{V}_P \in \ker(W_1^{\partial,(1,2)})$.

[9] a proof can be found in B.1.7

The components of the Hamiltonian vector field of H_λ are described by

$$\begin{aligned}
\mathbb{H}_e &= d_\omega(\lambda e_n) + \lambda\sigma \\
e\mathbb{H}_\omega &= \lambda e_n \left(F_\omega + \frac{1}{2}\Lambda e^2 + e\Pi d\phi + \frac{1}{4}e^2(\Pi,\Pi) \right) - \frac{\lambda}{2}e^2\Pi(\Pi, e_n) \\
\mathbb{H}_p &= d_\omega \left(\frac{\lambda e_n}{2} e^2 \Pi \right) \\
\mathbb{H}_\phi &= -\lambda(\Pi, e_n).
\end{aligned} \tag{4.70}$$

As one may see, we are not able to fully extrapolate $\mathbb{H}_\omega$ from the variation of H_λ, but we do not need an explicit expression for it, since in the computations we will only need $e\mathbb{H}_\omega$. A similar argument holds for $\mathbb{L}_\omega$ and $\mathbb{P}_\omega$, which are defined up to an element in $\ker(W_1^{\partial,(1,2)})$ that will be irrelevant in the following arguments.

Remark 4.4.10. We argued that λ is the parameter generating the local diffeomorphisms normal to the boundary. We now also see in (4.70) that $\mathbb{H}_\phi$ depends on (Π, e_n). In the cylider $\Sigma \times [0, \epsilon]$ we can apply the equation of motion $(\Pi, e_n) = \partial_n\phi$, hence showing that the (infinitesimal) gauge transformation generated by $\mathbb{H}$ on ϕ depends on the transversal component of ϕ, as predictable.

We now proceed to compute the Poisson brackets of the constraints. In the following computations we use integration by parts ($\diamondsuit$) and the following identities (for a proof of the second see [13]):

$$\begin{aligned}
(\spadesuit) &\qquad \frac{1}{2}\iota_{[\xi,\xi]}A = -\frac{1}{2}\iota_\xi\iota_\xi d_{\omega_0}A + \iota_\xi d_{\omega_0}\iota_\xi A - \frac{1}{2}d_{\omega_0}\iota_\xi\iota_\xi A \qquad \forall A \in \Omega_\partial^{i,j} \\
(\clubsuit) &\qquad \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0} B = \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^{\omega_0} B + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, B] \qquad \forall B \in \Omega_\partial^{i,j} \\
(\heartsuit) &\qquad d_{\omega_0}(\omega_0 - \omega) = F_{\omega_0} - F_\omega + \frac{1}{2}[\omega_0 - \omega, \omega_0 - \omega];
\end{aligned}$$

$$\begin{aligned}
\{L_c, H_\lambda\} &= \int_\Sigma [c,e]\lambda e_n F_\omega + \frac{\lambda e_n}{4}e^2(\Pi,\Pi)[c,e] + \lambda e_n e\Pi d\phi[c,e] - \frac{\lambda}{2}e^2\Pi(\Pi,e_n)[c,e] \\
&\quad + \frac{1}{2}[c,e]\Lambda\lambda e_n e^2 + d_\omega c e(d_\omega(\lambda e_n) + \lambda\sigma) \\
&= \int_\Sigma \lambda e_n \left([c,e]F_\omega + \frac{1}{3!}[c,e^3]\Lambda + \frac{1}{2\cdot 3!}[c,e^3](\Pi,\Pi) + \frac{1}{2}[c,e^2]\Pi d\phi \right) \\
&\quad + d_\omega c d_\omega(\lambda e_n e) - \frac{\lambda}{3!}[c,e^3]\Pi(\Pi,e_n) \\
&\overset{\diamondsuit}{=} \int_\Sigma -[c,\lambda e_n]\left(eF_\omega - \frac{1}{3!}e^3 - \frac{1}{2\cdot 3!}e^2(\Pi,\Pi) - \frac{1}{2}e^2\Pi d\phi \right) \\
&\quad - \frac{\lambda e_n}{2}e^2[c,\Pi]d\phi - \frac{\lambda}{3!}e^3[c,\Pi](\Pi,e_n) \\
&= \int_\Sigma -[c,\lambda e_n]^{(a)}e_a e F_\omega - [c,\lambda e_n]^{(n)}e_n e F_\omega - \frac{1}{3!}\Lambda[c,\lambda e_n]^{(n)}e_n e^3 \\
&= -P_{[c,\lambda e_n]^{(a)}} + L_{[c,\lambda e_n]^{(a)}(\omega-\omega_0)_a} - H_{[c,\lambda e_n]^{(n)}};
\end{aligned}$$

In the missing step we used that

$$\begin{aligned}-\frac{\lambda e_n}{2}e^2[c,\Pi]d\phi - \frac{\lambda}{3!}e^3[c,\Pi](\Pi,e_n) &= \frac{\lambda e_n}{3!}e^3\left([c,\Pi]^{(a)}(\Pi,e_a)+[c,\Pi]^{(n)}(\Pi,e_n)\right)\\ &= \frac{\lambda e_n}{3!}e^3([c,\Pi],\Pi) = \frac{\lambda e_n}{2\cdot 3!}e^3[c,(\Pi,\Pi)] = 0.\end{aligned}$$

$$\begin{aligned}\{P_\xi,P_\xi\} &= \int_\Sigma \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0) + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} + \xi(\phi)\mathrm{L}_\xi^{\omega_0}(p)\\ &\overset{\diamondsuit\clubsuit}{=} \int_\Sigma \frac{1}{4}\mathrm{L}_{[\xi,\xi]}^{\omega_0}(ee)(\omega-\omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0},ee](\omega-\omega_0) + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} + d_{\omega_0}(\iota_\xi(p))\iota_\xi(d\phi)\\ &= \int_\Sigma \frac{1}{4}\iota_{[\xi,\xi]}d_{\omega_0}(ee)(\omega-\omega_0) + \frac{1}{4}d_{\omega_0}\iota_{[\xi,\xi]}(ee)(\omega-\omega_0)\\ &\quad + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0},ee](\omega-\omega_0) + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} + \iota_\xi(d_{\omega_0}\iota_\xi(p))d\phi\\ &\overset{\diamondsuit\spadesuit}{=} \int_\Sigma \frac{1}{4}\iota_{[\xi,\xi]}d_\omega(ee)(\omega-\omega_0) - \frac{1}{4}\iota_{[\xi,\xi]}[\omega-\omega_0,ee](\omega-\omega_0)\\ &\quad + \frac{1}{4}\iota_{[\xi,\xi]}(ee)d_{\omega_0}(\omega-\omega_0) + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0},ee](\omega-\omega_0) + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} + \frac{1}{2}\iota_{[\xi,\xi]}(p)d\phi\\ &\overset{\heartsuit}{=} \int_\Sigma \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega-\omega_0) - \frac{1}{4}[\omega-\omega_0,ee]\iota_{[\xi,\xi]}(\omega-\omega_0) - \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_{\omega_0}\\ &\quad + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega - \frac{1}{8}\iota_{[\xi,\xi]}(ee)[\omega_0-\omega,\omega_0-\omega]\\ &\quad + \frac{1}{4}[\iota_\xi\iota_\xi F_{\omega_0},ee](\omega-\omega_0) + \frac{1}{2}\mathrm{L}_\xi^{\omega_0}(ee)\iota_\xi F_{\omega_0} + \frac{1}{2}\iota_{[\xi,\xi]}(p)d\phi\\ &\overset{\spadesuit}{=} \int_\Sigma \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega-\omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega + \frac{1}{4}d_{\omega_0}(ee)\iota_\xi\iota_\xi F_{\omega_0}\\ &\quad + \frac{1}{2}d_{\omega_0}\iota_\xi(ee)\iota_\xi F_{\omega_0} - \frac{1}{4}\iota_\xi\iota_\xi F_{\omega_0}[\omega-\omega_0,ee]\\ &\quad + \frac{1}{2}\left(\iota_\xi d_{\omega_0}(ee) - d_{\omega_0}\iota_\xi(ee)\right)\iota_\xi F_{\omega_0} + \frac{1}{2}\iota_{[\xi,\xi]}(p)d\phi\\ &= \int_\Sigma \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega-\omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega + \frac{1}{2}\iota_{[\xi,\xi]}(p)d\phi - \frac{1}{4}d_\omega(ee)\iota_\xi\iota_\xi F_{\omega_0}\\ &= \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}};\end{aligned}$$

$$\begin{aligned}\{L_c,L_c\} &= \int_\Sigma [c,e]ed_\omega c = \int_\Sigma \frac{1}{2}[c,ee]d_\omega c\\ &= \int_\Sigma \frac{1}{4}d_\omega[c,c]ee = \int_\Sigma -\frac{1}{2}[c,c]ed_\omega e = -\frac{1}{2}L_{[c,c]};\end{aligned}$$

$$
\begin{aligned}
\{L_c, P_\xi\} &= \int_\Sigma -[c,e]e(\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0)+\iota_\xi F_{\omega_0}) - d_\omega c e \mathrm{L}_\xi^{\omega_0} e \\
&= \int_\Sigma \frac{1}{2}\left(\mathrm{L}_\xi^{\omega_0} c[\omega-\omega_0, ee] + c[\omega-\omega_0, \mathrm{L}_\xi^{\omega_0}(ee)] - c[ee, \iota_\xi F_{\omega_0}] - d_\omega \mathrm{L}_\xi^{\omega_0}(ee)c\right) \\
&= \int_\Sigma \frac{1}{2}\mathrm{L}_\xi^{\omega_0} c[\omega, ee] - \frac{1}{2} dc\iota_\xi d(ee) + \frac{1}{2}[\iota_\xi\omega_0, d(ee)]c \\
&= \int_\Sigma \frac{1}{2}\mathrm{L}_\xi^{\omega_0} c d_\omega(ee) = \int_\Sigma \mathrm{L}_\xi^{\omega_0} c e d_\omega e = L_{\mathrm{L}_\xi^{\omega_0} c};
\end{aligned}
$$

$$
\begin{aligned}
\{P_\xi, H_\lambda\} &= \int_\Sigma -\mathrm{L}_\xi^{\omega_0} e\lambda e_n F_\omega - \frac{1}{2}\Lambda \mathrm{L}_\xi^{\omega_0} e\lambda e_n e^2 - \frac{\lambda e_n}{2\cdot 3!}(\Pi,\Pi)\mathrm{L}_\xi^{\omega_0}(e^3) - \frac{\lambda e_n}{2}\Pi d\phi \mathrm{L}_\xi^{\omega_0}(e^2) \\
&\quad + \frac{\lambda}{3!}\Pi(\Pi, e_n)\mathrm{L}_\xi^{\omega_0}(e^3) - \left(\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0)+\iota_\xi F_{\omega_0}\right) e(d_\omega(\lambda e_n)+\lambda\sigma) \\
&\quad + \lambda \mathrm{L}_\xi^{\omega_0}(p)(\Pi, e_n) - \frac{1}{2} d_\omega\left(\lambda e_n e^2 \Pi\right)\iota_\xi d\phi \\
&= \int_\Sigma -\mathrm{L}_\xi^{\omega_0} e\lambda e_n F_\omega - \frac{1}{3!}\Lambda \mathrm{L}_\xi^{\omega_0} e^3 \lambda e_n - \left(\mathrm{L}_\xi^{\omega_0}(\omega-\omega_0)+\iota_\xi F_{\omega_0}\right) d_\omega(e\lambda e_n) \\
&\quad + \mathrm{L}_\xi^{\omega_0}\left(\frac{\lambda e_n}{2\cdot 3!}\right) e^3(\Pi,\Pi) + \frac{\lambda e_n}{2\cdot 3!} e^3 \mathrm{L}_\xi^{\omega_0}(\Pi,\Pi) - \frac{\lambda e_n}{2}\Pi d\phi \mathrm{L}_\xi^{\omega_0}(e^2) \\
&\quad + \frac{\lambda}{3!}\Pi(\Pi, e_n)\mathrm{L}_\xi^{\omega_0}(e^3) + \frac{\lambda}{3!}\mathrm{L}_\xi^{\omega_0}(\pi^n e_n)e^3(\Pi, e_n) - \frac{\lambda e_n}{3!}\mathrm{L}_\xi^{\omega_0}(e^3)\pi^n \\
&\quad + \mathrm{L}_\xi^{\omega_0}\left(\frac{\lambda e_n}{2}\right) e^2 \Pi d\phi + \frac{\lambda e_n}{2}\mathrm{L}_\xi^{\omega_0}(e^2)\Pi d\phi + \frac{\lambda e_n}{2} e^2 \mathrm{L}_\xi^{\omega_0}(\Pi) d\phi \\
&= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(eF_\omega + \frac{e^2}{2}\Lambda + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) \\
&\quad + e\lambda e_n \mathrm{L}_\xi^{\omega_0} F_\omega + \left(d_\omega \iota_\xi(\omega-\omega_0) - \iota_\xi F_\omega\right) d_\omega(e\lambda e_n) \\
&\quad + \lambda e_n\left[\frac{e^3}{3!}(\Pi, \mathrm{L}_\xi^{\omega_0}(\Pi)) + \frac{e^3}{2}(e,\Pi)\mathrm{L}_\xi^{\omega_0}(\Pi)\right] + \frac{\lambda}{3!}e^3(\Pi, e_n)\mathrm{L}_\xi^{\omega_0}(\Pi) \\
&\overset{\bigstar}{=} \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(eF_\omega + \frac{e^2}{2}\Lambda + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) \\
&\quad + e\lambda e_n \mathrm{L}_\xi^{\omega_0} F_\omega + \left(d_\omega \iota_\xi(\omega-\omega_0) - \iota_\xi F_\omega\right) d_\omega(e\lambda e_n) \\
&\quad - \frac{\lambda e_n}{2} e^3(e,\Pi)\mathrm{L}_\xi^{\omega_0}(\Pi) - \frac{\lambda}{3!}e^3(\Pi, e_n)\mathrm{L}_\xi^{\omega_0}(\Pi) \\
&\quad + \frac{\lambda e_n}{2} e^3(e,\Pi)\mathrm{L}_\xi^{\omega_0}(\Pi) + \frac{\lambda}{3!}e^3(\Pi, e_n)\mathrm{L}_\xi^{\omega_0}(\Pi) \\
&= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(eF_\omega + \frac{e^2}{2}\Lambda + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) \\
&= P_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}} - L_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega-\omega_0)_a},
\end{aligned}
$$

where we used that $(e, \Pi) = d\phi$

Finally,

$$\begin{aligned}\{H_\lambda, H_\lambda\} &= \int_\Sigma \left[\lambda e_n \left(F_\omega + \frac{\Lambda}{2}e^2 + \frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) - \frac{\lambda}{2}e^2(e_n,\Pi)\Pi\right](d_\omega(\lambda e_n) + \lambda\sigma) \\ &\quad - \lambda(e_n,\Pi)d_\omega\left(\frac{\lambda e_n}{2}e^2\Pi\right) \\ &= \int_\Sigma \frac{\lambda}{2}d\lambda e_n e^2(e_n,\Pi)\Pi - \frac{\lambda}{2}d\lambda e_n e^2(e_n,\Pi)\Pi = 0,\end{aligned}$$

since most of the terms vanish because $e_n^2 = 0$ and $\lambda^2 = 0$

✓

4.5 BFV Formalism

In this section we apply the content of Chapter 3. In particular, we embed F_∂ as the body of a supermanifold $\mathcal{F}$, whose odd coordinates are given by taking the Lagrange multipliers as fields (the ghosts) and adding their momenta (ghost momenta).

Theorem 4.5.1. *Let $\mathcal{F}$ be the bundle*

$$\mathcal{F} \longrightarrow \Omega^1_{nd}(\Sigma,\mathcal{V}), \tag{4.71}$$

with local trivialisation on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma,\mathcal{V})$

$$\mathcal{F} \simeq \mathcal{U}_\Sigma \times \mathcal{A}(\Sigma) \times \Omega^{(0,1)}_{\partial,red} \times \mathcal{C}^\infty(\Sigma) \oplus T^*\left(\Omega^{0,2}_\partial[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma)\right), \tag{4.72}$$

The fields are denoted by $e \in \mathcal{U}_\Sigma$, $\omega \in \mathcal{A}(\Sigma)$, $\Pi \in \omega^{(0,1)}_{\partial,red}$ and $\phi \in \mathcal{C}^\infty(\Sigma)$. They are in degree zero and satisfy the structural constraints $e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}$ and $(e,\Pi) = d\phi$. The ghost fields are denoted by $c \in \Omega^{0,2}_\partial[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $\lambda \in \Omega^{0,0}[1]$ in degree one, $c^\dagger \in \Omega^{3,2}_\partial[-1]$, $\lambda^\dagger \in \Omega^{3,4}_\partial[-1]$ and $\xi^\dagger \in \Omega^{1,0}_\partial[-1] \otimes \Omega^{3,4}_\partial$ in degree minus one, together with a fixed $e_n \in \Gamma(\mathcal{V})$, completing the image of elements $e \in \mathcal{U}_\Sigma$ to a basis of $\mathcal{V}$. The symplectic form and the action functional on $\mathcal{F}$ are defined by

$$\varpi = \int_\Sigma e\delta e\delta\omega + \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger, \tag{4.73}$$

$$S = \int_\Sigma c e d_\omega e + \iota_\xi e e F_\omega + \iota_\xi(\omega - \omega_0)e d_\omega e + \frac{1}{3!}\iota_\xi(e^3\Pi)d\phi \tag{4.74}$$

$$\begin{aligned}&+ \lambda e_n\left(eF_\omega + \frac{1}{3!}\Lambda e^3 + \frac{1}{2\cdot 3!}e^3(\Pi,\Pi) + \frac{1}{2}e^2\Pi d\phi\right) + \frac{1}{2}[c,c]c^\dagger \\ &- \mathrm{L}_\xi^{\omega_0} c c^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}c^\dagger + [c,\lambda e_n]^{(a)}(\xi^\dagger_a - (\omega-\omega_0)_a c^\dagger) + [c,\lambda e_n]^{(n)}\lambda^\dagger \\ &- \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\xi^\dagger_a - (\omega-\omega_0)_a c^\dagger) - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}\lambda^\dagger - \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger.. \end{aligned} \tag{4.75}$$

*Then the triple $(\mathcal{F},\varpi,S)$ defines a **BFV structure** on Σ.*

Proof. We follow the same strategy of [8], from which we also borrow the notation. We need to prove that the BFV action satisfies the classical master equation

$$\{S,S\} = \iota_Q\iota_Q\varpi = 0, \tag{4.76}$$

where Q is the Hamiltonian vector field of F, defined by $\iota_Q \varpi = \delta S$. In order to make the calculations easier, we define

$$S = S_0 + S_1. \tag{4.77}$$

S_0 is the part of S which is independent of the ghost momenta, while S_1 is the part that is linear in them. We do the further splitting

$$S_0 = S_0^0 + S_0^1,$$

where S_0^0 indicates the part independent of the scalar field and S_0^1 the part depending on it. We further simplify the calculations by defining

$$\varpi = \varpi_f + \varpi_g. \tag{4.78}$$

$\varpi_f = \int_\Sigma e\delta e\delta\omega + \delta p\delta\phi$ is the classical part and $\omega_g = \int_\Sigma \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger$ is the ghost part. Finally, we define Q_0 to be the part of Q satisfying $\iota_{Q_0}\varpi = \delta S_0$ and Q_1 to be the one satisfying $\iota_{Q_1}\varpi = \delta S_1$. We can of course refine the splitting by defining

$$Q_0 = Q_0^0 + Q_0^1 \qquad \iota_{Q_0^i}\varpi = \delta S_0^i. \tag{4.79}$$

We can divide the master equation into the corresponding parts:

$$\{S,S\} = \{S_0,S_0\}_f + 2\{S_0,S_1\}_f + 2\{S_0,S_1\}_g + \{S_1,S_1\}_f + \{S_1,S_1\}_g,$$

where

$$\{S_0,S_0\}_f = \iota_{Q_0}\iota_{Q_0}\varpi_f \quad \{S_0,S_1\}_f = \iota_{Q_0}\iota_{Q_1}\varpi_f \tag{4.80a}$$
$$\{S_0,S_0\}_g = \iota_{Q_0}\iota_{Q_0}\varpi_g \quad \{S_0,S_1\}_g = \iota_{Q_0}\iota_{Q_1}\varpi_g \tag{4.80b}$$
$$\{S_1,S_1\}_f = \iota_{Q_1}\iota_{Q_1}\varpi_f \quad \{S_1,S_1\}_g = \iota_{Q_1}\iota_{Q_1}\varpi_g. \tag{4.80c}$$

We first note that $\{S_0,S_0\}_g = 0$ since S_0 has no antighost part. Furthermore, by definition of the BFV action, we can exploit the results of Theorem 4.4.8 and find that

$$\{S_0,S_0\}_f + 2\{S_0,S_1\}_g = 0.$$

Indeed, $\{S_0,S_0\}_f$ gives exactly the brackets of the constraints and S_1 is constructed such that the previous equation is satisfied. The terms $\{S_0,S_1\}_f$ and $\{S_1,S_1\}_g$ are linear in the antighost while $\{S_1,S_1\}_f$ is quadratic in the antighost. Hence the two following equations need to be proved separately

$$2\{S_0,S_1\}_f + \{S_1,S_1\}_g = 0$$
$$\{S_1,S_1\}_f = 0.$$

Using the further splitting of S_0 and S_1, we find

$$2\{S_0^0,S_1\}_f + 2\{S_0^1,S_1\}_f + \{S_1,S_1\}_g = 0.$$

It has been proven in [8] that $S_0^0 + S_1$ satisfies the CME, which means that $2\{S_0^0,S_1\}_f + \{S_1^0,S_1\}_g = 0$ and $\{S_1,S_1\}_f = 0$. we are then left with

$$\{S_0^1,S_1\}_f = 0. \tag{4.81}$$

To show that this holds, first we calculate Q_0^1 by imposing $\iota_{Q_0^1}\varpi = \delta S_0^1$

$$\begin{aligned}\delta S_0^1 = \int_\Sigma & \left[\lambda e_n \left(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) - \frac{\lambda}{2}e^2(e_n,\Pi)\Pi\right]\delta e \\ & - \lambda(e_n,\Pi)\delta p + \frac{1}{2}d_\omega(\lambda e_n e^2\Pi)\delta\phi \\ & - \mathrm{L}_\xi^{\omega_0}\phi\delta p - \mathrm{L}_\xi^{\omega_0}(p)\delta\phi.\end{aligned}$$

Hence

$$Q_{0,e}^1 = 0 \qquad eQ_{0,\omega}^1 = \lambda e_n \left(\frac{e^2}{4}(\Pi,\Pi) + e\Pi d\phi\right) - \frac{\lambda}{2}e^2(e_n,\Pi)\Pi$$

$$Q_{0,\phi}^1 = -\lambda(e_n,\Pi) - \mathrm{L}_\xi^{\omega_0}(\phi) \qquad Q_{0,p}^1 = -\mathrm{L}_\xi^{\omega_0}(p) + \frac{1}{2}d_\omega(\lambda e_n e^2\Pi).$$

From [8] we know that $Q_{1,e} \propto \lambda$, then we obtain

$$\{S_0^1, S_1\}_f = \iota_{Q_1}\iota_{Q_0^1}\varpi_f = \int_\Sigma eQ_{0,\omega}^1 Q_{1,e} = 0,$$

since $Q_{0,\omega}^1 \propto \lambda$ and $\lambda^2 = 0$. ✓

Chapter 5

Yang–Mills in the PC Theory

We now move our attention to the more complicated (but also more physically interesting) case of the coupling of a Yang–Mills field to gravity. Also in this case it is useful to work in the first order formalism.

We start by considering a principal bundle (P, G, π, M) over the N-dimensional space–time manifold M. We assume G to be a compact Lie group with Lie algebra $\mathfrak{g}$.

As we saw in example 1.2.17, the gauge field is defined to be the local connection 1-form A. Choosing $\{T_I\}$ to be a basis for $\mathfrak{g}$, then we express A as

$$A = A^I(x)T_I = A^I_\mu T_I dx^\mu. \tag{5.1}$$

In particular, the gauge fields are in a space locally modeled by $\Gamma(T^*M \otimes \mathfrak{g})$, we will denote it by $\mathcal{A}_{\mathrm{YM}}$. The connection two-form is as usual defined to be $F_A := dA + \frac{1}{2}[A, A]$. In coordinates, it reads

$$F_A = \left(dA^I + \frac{1}{2} f^I_{JK} A^J A^K \right) T_I = F^I T_I, \tag{5.2}$$

where $F^I = \frac{1}{2} F^I_{\mu\nu} dx^\mu \wedge dx^\nu$.

We have seen in the example that the gauge invariant quantity that we can construct starting from A is $\mathrm{Tr}(F_A \wedge \star F_A)$, where $\star$ denotes the Hodge dual. However, in order to define it, we need to use the metric tensor, which as we know is not the fundamental object of our field theoretical description and is found in terms of the vielbein. As in the case of the scalar field, we then need to find a way to encode the dynamics of the Yang–Mills field in an action functional containing the vielbein. To do so, we introduce an independent field $B \in \Gamma(\wedge^2 \mathcal{V} \otimes \mathfrak{g})$, which is a $\mathfrak{g}$-valued section of the second exterior power of the Minkwoski bundle $\mathcal{V}$. In coordinates, it reads $B = B^{\mu\nu|I} e_\mu e_\nu T_I$, where we used $\{e_\mu\}$ as a local basis for V.

The Yang–Mills action in the first order formalism is

$$S_{\mathrm{YM}} := \int_M \frac{1}{(N-2)!} e^{N-2} \mathrm{Tr}(BF_A) + \frac{1}{2N!} e^N \mathrm{Tr}(B, B), \tag{5.3}$$

where $(\cdot, \cdot)$ is the canonical pairing in $\wedge^2 V$ defined in coordinates for all $C, D \in \wedge^2 V$ by $(C, D) := C^{ab} D^{cd} \eta_{ac} \eta_{bd}$ with respect to an orthonormal basis $\{u_a\}$ of V.

We compute the variation of the action $S = S_{PC} + S_{\mathrm{YM}}$ and find

$$\begin{aligned} \delta S = \int_M \left[\frac{e^{N-3}}{(N-3)!} (F_\omega + \mathrm{Tr}(BF_A)) + \frac{e^{N-1}}{(N-1)!} \left(\Lambda + \frac{1}{2} \mathrm{Tr}(B,B) \right) \right] \delta e \\ + \frac{1}{(N-2)!} d_\omega (e^{N-2}) \delta\omega + \frac{e^{N-2}}{(N-2)!} \mathrm{Tr} \left[\left(F_A + \frac{1}{2}(e^2, B) \right) \delta B \right] \\ + \mathrm{Tr} \left[d_A \left(\frac{e^{N-2}}{(N-2)!} B \right) \delta A \right] - d \left\{ \frac{e^{N-2}}{(N-2)!} \left[\delta\omega + \mathrm{Tr}\,(B\delta A) \right] \right\}, \end{aligned} \tag{5.4}$$

where to extrapolate δB out of the bracket we used the following identity[1], holding for all $C \in \Omega^{(0,2)}$ and $D \in \Omega^{0,2}[1]$ (the fact that they might also have values in $\mathfrak{g}$ is here irrelevant):

$$\frac{e^N}{N!}(C, D) = \frac{e^{N-2}}{2(N-2)!}(e^2, C)D. \tag{5.5}$$

First of all, we notice that the variation of the action produces a boundary term, which will be the local 1-form on the space of preboundary fields whose vertical differential will give rise to the presymplectic two form on the boundary. It is given by

$$\tilde{\alpha}_{\mathrm{YM}} = \int_{\partial M} \frac{e^{N-2}}{(N-2)!} \delta\omega + \frac{e^{N-2}}{(N-2)!} \mathrm{Tr}\,(B\delta A). \tag{5.6}$$

The equations of motion are found to be

$$d_\omega e = 0; \tag{5.7}$$

$$\tfrac{e^{N-3}}{(N-3)!} \left(F_\omega + \mathrm{Tr}(BF_A) \right) + \tfrac{e^{N-1}}{(N-1)!} \left(\Lambda + \tfrac{1}{2} \mathrm{Tr}(B,B) \right); \tag{5.8}$$

$$e^{N-2} \left(F_A + \tfrac{1}{2}(e^2, B) \right) = 0; \tag{5.9}$$

$$d_A(e^{N-2}B) = 0. \tag{5.10}$$

Equation (5.9) can be further simplified by noticing that $W_{N-2}^{(2,0)}$ is injective[2]. Therefore we obtain

$$F_A + \frac{1}{2}(e^2, B) = 0, \tag{5.11}$$

which in coordinates gives $B^{\mu\nu} = (-1)^N g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$ (omitting the Lie algebra indices). With this definition, using corollary B.1.5 we then find

$$\frac{e^{N-2}}{(N-2)!} BF_A + \frac{e^N}{2N!}(B, B) = -\frac{1}{2} \mathrm{Vol}_g F_{\mu\nu} F^{\mu\nu}, \tag{5.12}$$

giving (up to factors) the standard Yang–Mills term in the action.

In the next section we will analyze the boundary structure of the Yang–Mills field.

5.1 Boundary Structure in $N = 4$

We assume the spacetime manifold M to be 4-dimensional with boundary $\Sigma := \partial M$. Unlike the case of the scalar field, we will see that the equations of motion produce an additional constraint,

[1] see Lemma B.1.3

[2] see Lemma B.1.6.(2)

hence modifying the boundary structure (but still preserving the first class condition) and the BFV description.

The boundary term in (5.6) reads

$$\tilde{\alpha}_{\text{em}} = \frac{1}{2}\int_{\Sigma} e^2\delta\omega + \text{Tr}(e^2 B\delta A). \tag{5.13}$$

Here B and A are the fields restricted to the boundary, while e and ω are in the previous chapter, in particular

- B is an element of $\Omega_{\partial,\mathfrak{g}}^{(0,2)} = \Omega_{\partial}^{(0,2)} \otimes \mathfrak{g}$;
- A is an element of $\mathcal{A}_{\partial}^{\text{YM}}$, locally represented by $\Omega_{\partial}^{(1,0)} \otimes \mathfrak{g}$.

The space of preboundary fields is denoted by $\tilde{F}_{\partial}^{\text{YM}} = \Omega_{\text{n.d.}}^{(1,1)} \times \mathcal{A}_{\partial} \times \mathcal{A}_{\partial}^{\text{YM}} \times \Omega_{\partial,\mathfrak{g}}^{(0,2)}$. The presymplectic form on $\tilde{F}_{\partial}^{\text{YM}}$ is defined as the variation of $\tilde{\alpha}_{\text{YM}}$

$$\tilde{\varpi}_{\text{YM}} := \int_{\Sigma} e\delta e\delta\omega + \text{Tr}(eB\delta e\delta A) + \frac{1}{2}\text{Tr}(e^2\delta B\delta A). \tag{5.14}$$

We are interested in computing the kernel of $\tilde{\varpi}_{\text{YM}}$ defined as $\ker(\tilde{\varpi}_{\text{YM}}) := \{X \in T\tilde{F}_{\partial}^{\text{YM}} \mid X \lrcorner \tilde{\varpi}_{\text{YM}} = 0\}$. Considering a generic vector field $\mathbb{X} = \mathbb{X}_e\frac{\delta}{\delta e} + \mathbb{X}_{\omega}\frac{\delta}{\delta\omega} + \mathbb{X}_A\frac{\delta}{\delta A} + \mathbb{X}_B\frac{\delta}{\delta B}$, we find $\ker(\tilde{\varpi}_{\text{YM}})$ as the vector fields satisfying

$$e\mathbb{X}_e = 0; \tag{5.15}$$
$$e\mathbb{X}_{\omega} + eB\mathbb{X}_A = 0; \tag{5.16}$$
$$eB\mathbb{X}_e + \frac{1}{2}e^2\mathbb{X}_B = 0; \tag{5.17}$$
$$e^2\mathbb{X}_A = 0. \tag{5.18}$$

We now see by the previous section that (5.15) is solved by $\mathbb{X}_e = 0$, while (5.18) is solved by $\mathbb{X}_A = 0$ by Lemma (7), therefore we are left with $e\mathbb{X}_{\omega} = 0$ and $e^2\mathbb{X}_B = 0$.

As usual, we define the geometric space F_{∂}^{YM} to be the symplectic reduction of $\tilde{F}_{\partial}^{\text{YM}}$, namely $F_{\partial}^{\text{YM}} := \Omega_{\text{n.d.}}^{(1,1)} \times \mathcal{A}_{\partial}^{\text{red}} \times \mathcal{A}_{\partial}^{\text{YM}} \times \Omega_{\partial,\text{red}}^{(0,2)}$, where $\mathcal{A}_{\partial}^{\text{red}} := \mathcal{A}_{\partial}/\sim$, $\Omega_{\partial,\text{red}}^{(0,2)} := \Omega_{\partial,\mathfrak{g}}^{(0,2)}/\sim$ and

$$\omega \sim \tilde{\omega} \quad \Leftrightarrow \quad \omega - \tilde{\omega} = v \text{ with } ev = 0 \tag{5.19}$$
$$B \sim \tilde{B} \quad \Leftrightarrow \quad B - \tilde{B} = C \text{ with } e^2C = 0, \tag{5.20}$$

because a vector field $X \in \ker(\tilde{\varpi})$ acts on the space of fields as a translation of ω and Π respectively by $\mathbb{X}_{\omega}$ and $\mathbb{X}_{\Pi}$.

F_{∂}^{YM} is thus a symplectic manifold with symplectic form

$$\tilde{\varpi}_{\text{YM}} = \int_{\Sigma} e\delta e\delta[\omega] + \frac{1}{2}\text{Tr}(\delta(e^2[B])\delta A). \tag{5.21}$$

Remark 5.1.1. As one can easily notice, we can rewrite the part of ϖ_{YM} depending on A and B in Darboux form, by defining $\rho := \frac{1}{2}e^2B$, since in this way the components of B which are in the kernel of e^2 are automatically suppressed. Therefore we obtain the symplectic form as

$$\tilde{\varpi}_{\text{YM}} = \int_{\Sigma} e\delta e\delta[\omega] + \text{Tr}(\delta\rho\delta A). \tag{5.22}$$

We can as well consider a generic vector field $\mathbb{X} = \mathbb{X}_e \frac{\delta}{\delta e} + \mathbb{X}_\omega \frac{\delta}{\delta \omega} + \mathbb{X}_\rho \frac{\delta}{\delta \rho} + \mathbb{X}_A \frac{\delta}{\delta A}$, then it will be useful to consider $\iota_{\mathbb{X}} \varpi_{\mathrm{YM}}$

$$\iota_{\mathbb{X}} \varpi_{\mathrm{YM}} = \int_\Sigma e \mathbb{X}_e \delta\omega + e\delta e \mathbb{X}_\omega + \mathrm{Tr}(\mathbb{X}_\rho \delta A) + \mathrm{Tr}(\delta\rho \mathbb{X}_A). \tag{5.23}$$

As we saw in the previous section, to obtain the physical space of fields on the boundary (i.e. the reduced phase space) we need to impose constraints on F_∂^{YM}. Recall that the equations of motion split into evolution equations (containing the derivatives of the fields in the transversal direction with respect to the boundary) and in the constraints, which contain only derivatives tangential to the boundary. The latter are readily obtained as the restriction of the equations of motion to the boundary.

Choice of Representative via Constraints

We now fix the representatives of the fields in the geometric phase space. In order to do so, we make use of the constraints, which in $N = 4$ are

$$d_\omega e = 0; \tag{5.24}$$

$$eF_\omega + \frac{\Lambda}{3!} e^3 + \mathrm{Tr}\left[eBF_A + \frac{1}{2 \cdot 3!} e^3 (B, B) = 0\right]; \tag{5.25}$$

$$d_A(e^2 B) = 0; \tag{5.26}$$

$$F_A + \frac{1}{2}(e^2, B) = 0. \tag{5.27}$$

The choice of the representative of $[\omega]$ is performed exactly as in section (4.4).

To fix the representative of $[B]$ we use (5.27). In particular,it is easy to see that there is a unique decomposition $B = B^{ab} e_a e_b + 2B^{an} e_a e_n$, with $B^{ab} e_a e_b$ lying in the kernel of $W_2^{\partial(0,2)}$, therefore fixing B^{ab} is equivalent to fixing the representative of $[B]$, or in other words, the equivalence class of B is uniquely determined by B^{an}.

Theorem 5.1.2. *Suppose that g^∂ is non-degenerate, then the components B^{ab} in the decomposition $B = B^{ab} e_a e_b + 2B^{an} e_a e_n$ are uniquely fixed in terms of B^{an} and A by imposing*

$$F_A + \frac{1}{2}(e^2, B) = 0. \tag{5.28}$$

Proof. In coordinates $F_A + \frac{1}{2}(e^2, B) = 0$ implies $(e_a e_b, B) = F_{ab}$, which yelds

$$B^{ab|I} = g^{ac} g^{bd} F^I_{cd} - 2B^{an|I} g^{bd} g_{nd}, \tag{5.29}$$

where F^I_{ab} are the components of F_A ✓

Poisson Brackets of the Constraints

Having defined a symplectic manifold, it is of course possible to define the induced Poisson structure. In this section we will show that also in the case of an Yang–Mills field coupled to gravity the boundary structure is such that it produces first class constraints, namely a set of functions on the space of fields on the boundary which is algebraically closed with respect to the Poisson bracket.

As in the case of the scalar field, we use Lagrange multipliers, and we split the constraint (5.25) (the projection of Einstein's equations to the boundary) into two independent ones. We are left with four constraints:

$$L_c := \int_\Sigma c e d_\omega e; \tag{5.30}$$

$$M_\mu := \int_\Sigma \frac{1}{2}\mathrm{Tr}(\mu d_A(e^2 B)); \tag{5.31}$$

$$\begin{aligned} P_\xi := \int_\Sigma & \frac{1}{2}\iota_\xi e^2 F_\omega + \frac{1}{2}\iota_\xi e^2 \mathrm{Tr}(BF_A) + \iota_\xi(\omega-\omega_0)ed_\omega e \\ &+ \frac{1}{2}\mathrm{Tr}\{\iota_\xi(A-A_0)d_A(e^2B)\}; \end{aligned} \tag{5.32}$$

$$H_\lambda := \int_\Sigma \lambda e_n \left(eF_\omega + \frac{\Lambda}{3!}e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!}e^3\mathrm{Tr}(B,B)\right). \tag{5.33}$$

$$\tag{5.34}$$

Remark 5.1.3. Notice that the constraint M_μ can be rewritten in terms of the fields in Darboux form simply as

$$M_\mu = \int_\Sigma Tr(\mu d_A \rho). \tag{5.35}$$

Concerning P_ξ, we added the term $\frac{1}{2}\mathrm{Tr}\{\iota_\xi(A-A_0)d_A(e^2B)\}$ with respect to a reference connection A_0. Again, this addition does not change the properties of the boundary structure (we are simply adding a term that vanishes on the submanifold defined as the zero-locus of the constraints), but it largely simplifies the calculations, since it allows to find a more explicit form of the Hamiltonian vector field. We might as well rewrite P_ξ in terms of ρ as

$$P_\xi = \int_\Sigma \frac{1}{2}\iota_\xi e^2 F_\omega + \frac{1}{2}\iota_\xi \mathrm{Tr}(\rho F_A) + \iota_\xi(\omega-\omega_0)ed_\omega e + \mathrm{Tr}\{\iota_\xi(A-A_0)d_A\rho\}. \tag{5.36}$$

The Lagrange multipliers are again chosen to be odd, in particular we have $\lambda \in \mathcal{C}^\infty[1](\Sigma)$, $\mu \in \Gamma(\mathfrak{g})[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$ and $c \in \Omega_\partial^{(0,2)}[1]$.

.

Theorem 5.1.4. *The constraints L_c, M_μ, P_ξ, H_λ define a coisotropic submanifold with respect to the symplectic structure ϖ_{YM}. Their Poisson brackets*[3] *read*

$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{A_0}} \qquad \{H_\lambda, H_\lambda\} = 0$$

$$\{M_\mu, M_\mu\} = -\frac{1}{2}M_{[\mu,\mu]} \qquad \{M_\mu, L_c\} = 0$$

$$\{M_\mu, H_\lambda\} = 0 \qquad \{M_\mu, P_\xi\} = M_{\mathrm{L}_\xi^{A_0}\mu}$$

$$\{L_c, P_\xi\} = L_{\mathrm{L}_\xi^{\omega_0}c} \qquad \{L_c, L_c\} = -\frac{1}{2}L_{[c,c]};$$

$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} + M_{X^{(a)}(A-A_0)_{(a)}}$$

$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}} - M_{Y^{(a)}(A-A_0)_{(a)}},$$

[3]We point out that one should not confuse L with L, which respectively indicate the constraint and the Lie derivative

where $X = [c, \lambda e_n]$, $Y = \mathrm{L}_\xi^{\omega_0}(\lambda e_n)$ *and* $Z^{(a)}$, $Z^{(n)}$ *are the components of* $Z \in \{X, Y\}$ *with respect to the frame* (e_a, e_n)*.*

Remark 5.1.5. The new constraint M_μ is associated with the G gauge symmetry of the Yang–Mills field. In particular, we will see that the Hamiltonian vector field associated to M_μ exactly generates the G gauge transformations. Furthermore, we notice an analogy between M and L, which is not surprising since they both encode the gauge symmetry of the fields, respectively given by a compact Lie group G and by $SO(3,1)$.

Proof. We start by computing the Hamiltonian vector fields associated to the constraints. Many of the calculations will be exactly the same as in the previous section, therefore we refer to (4.4) for the parts that we leave out.

$$\delta L_c = \int_\Sigma [c, e]e\delta\omega + d_\omega c e \delta e;$$

$$\begin{aligned}
\delta P_\xi &= \int_\Sigma -e\delta e(\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}) - (\mathrm{L}_\xi^{\omega_0} e)e\delta\omega + \mathrm{Tr}\left[\delta(\iota_\xi \rho F_A) + \delta(\iota_\xi(A - A_0)d_A\rho)\right] \\
&= \int_\Sigma (\cdots) - \mathrm{Tr}\left\{\iota_\xi\delta\rho F_A - \iota_\xi\rho d_A\delta A - \iota_\xi(\delta A)d_A\rho - \iota_\xi(A - A_0)[\delta A, \rho] + \iota_\xi(A - A_0)d_A\delta\rho\right\} \\
&= \int_\Sigma (\cdots) - \mathrm{Tr}\left\{\delta\rho(\iota_\xi F_A - d_A\iota_\xi(A - A_0)) + (-\iota_\xi d_A\rho + d_A\iota_\xi\rho + [\iota_\xi(A - A_0), \rho])\delta A\right\} \\
&= \int_\Sigma -e\delta e(\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}) - (\mathrm{L}_\xi^{\omega_0} e)e\delta\omega - \mathrm{Tr}\left\{\delta\rho(\mathrm{L}_\xi^{A_0}(A - A_0) + \iota_\xi F_{A_0}) + \mathrm{L}_\xi^{A_0}\rho\delta A\right\};
\end{aligned}$$

$$\begin{aligned}
\delta M_\mu &= \int_\Sigma \mathrm{Tr}\left[\mu\delta(d_A\rho)\right] = \int_\Sigma \mathrm{Tr}\left[-\mu([\delta A, \rho] + d_A(\delta\rho))\right] \\
&= \int_\Sigma \mathrm{Tr}(\delta A[\mu, \rho] + d_A\mu\delta\rho);
\end{aligned}$$

$$\begin{aligned}
\delta H_\lambda &= \int_\Sigma (\cdots) + \delta\mathrm{Tr}\left[\lambda e_n B F_A + \frac{\lambda e_n}{2 \cdot 3!}e^3(B, B)\right] \\
&= \int_\Sigma (\cdots) + \mathrm{Tr}\left\{\lambda e_n\left[BF_A + \frac{e^2}{4}(B, B)\right]\delta e + \lambda e_n e\delta B F_A - \lambda e_n e b d_A(\delta A) + \frac{\lambda e_n}{3!}e^3(B, \delta B)\right\} \\
&\overset{\bigstar}{=} (\cdots) + \int_\Sigma \mathrm{Tr}\left\{\lambda e_n\left[BF_A + \frac{e^2}{4}(B, B)\right]\delta e + \lambda e_n e\delta B F_A + d_A(\lambda e_n e b)\delta A + \frac{\lambda e_n}{2}e(e^2, B)\delta B\right\} \\
&\quad + \mathrm{Tr}\left\{\frac{\lambda}{2}(B, e_n e)e^2\delta B\right\} \\
&= \int_\Sigma (\cdots) + \mathrm{Tr}\left\{\left[\lambda e_n(BF_A + \frac{e^2}{4}(B, B)) - \lambda e B(B, e_n e)\right]\delta e + d_A(\lambda e_n e B)\delta A + \lambda(B, e_n e)\delta\rho\right\},
\end{aligned}$$

where we used a generalization of (5.5) to the boundary in $N = 4$. Assuming $C \in \Omega_\partial^{(0,2)}$ and $D \in \Omega_\partial^{(0,2)}$, we find the following useful identity[4]

$$(\bigstar) \qquad \frac{\lambda e_n}{3!}e^3(C, D) = \frac{\lambda}{2}(C, e_n e)e^2 D + \frac{\lambda e_n}{2}e(e^2, C)D. \qquad (5.37)$$

[4]See Lemma B.1.7 in $N = 4$

The components of the Hamiltonian vector fields therefore are

$$e\mathbb{H}_\omega = \lambda e_n \left(F_\omega + \frac{\Lambda}{2}e^2 + \frac{1}{4}e^2\mathrm{Tr}(B,B) + \mathrm{Tr}(BF_A) \right) - \lambda e \mathrm{Tr}(B(B, e_n e)) \tag{5.38}$$

$$\mathbb{H}_e = d_\omega(\lambda e_n) + \lambda\sigma \tag{5.39}$$

$$\mathbb{H}_\rho = d_A(\lambda e_n e B) \tag{5.40}$$

$$\mathbb{H}_A = \lambda(B, ee_n) \tag{5.41}$$

$$\mathbb{L}_e = [c, e] \qquad \mathbb{L}_A = 0 \tag{5.42}$$

$$\mathbb{L}_\omega = d_\omega c + \mathbb{V}_L \qquad e^2\mathbb{L}_B = e^2[c, B] \tag{5.43}$$

$$\mathbb{M}_e = 0 \qquad \mathbb{M}_A = d_A\mu \tag{5.44}$$

$$\mathbb{M}_\omega = 0 \qquad \mathbb{M}_\rho = [\mu, \rho] \tag{5.45}$$

$$\mathbb{P}_e = -\mathrm{L}_\xi^{\omega_0}(e) \qquad \mathbb{P}_\omega = -\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) - \iota_\xi(F_{\omega_0}) + \mathbb{V}_P \tag{5.46}$$

$$\mathbb{P}_\rho = -\mathrm{L}_\xi^{A_0}(\rho) \qquad \mathbb{P}_A = -\mathrm{L}_\xi^{A_0}(A - A_0) - \iota_\xi(F_{A_0}). \tag{5.47}$$

We can now start computing the Poisson brackets of the constraints. We notice that since $L_c(\rho) = 0$ and $L_c(A) = 0$, the brackets $\{L_c, L_c\}$ and $\{L_c, P_\xi\}$ will be computed exactly as in section (4.4). Also $\{M_\mu, L_c\} = 0$ is seen very easily without the need of any calculation

$$\begin{aligned}
\{M_\mu, M_\mu\} &= \int_\Sigma \mathrm{Tr}\left(d_A\mu[\mu,\rho]\right) = \int_\Sigma -\mathrm{Tr}\left([\mu, d_A\mu]\rho\right) \\
&= \frac{1}{2}\int_\Sigma \mathrm{Tr}\left(d_A[\mu,\mu]\rho\right) = -\frac{1}{2}\int_\Sigma \mathrm{Tr}\left([\mu,\mu]d_A\rho\right) \\
&= -\frac{1}{2}M_{[\mu,\mu]};
\end{aligned}$$

$$\begin{aligned}
\{M_\mu, P_\xi\} &= \int_\Sigma -\mathrm{Tr}\left\{[\mu,\rho]\left(\mathrm{L}_\xi^{A_0}(A - A_0) + \iota_\xi F_{A_0}\right) + \mathrm{L}_\xi^{A_0}\rho d_A\mu\right\} \\
&= \int_\Sigma \mathrm{Tr}\left\{\mathrm{L}_\xi^{A_0}\mu[A - A_0, \rho] + \mu[A - A_0, \mathrm{L}_\xi^{A_0}(\rho)] - \mu[\rho, \iota_\xi F_{A_0}] - d_A\mathrm{L}_\xi^{A_0}(\rho)\mu\right\} \\
&= \int_\Sigma \mathrm{Tr}\left\{\mathrm{L}_\xi^{A_0}\mu[A,\rho] - d\mu\iota_\xi d\rho + [\iota_\xi A_0, d\rho]\mu\right\} \\
&= \int_\Sigma \mathrm{Tr}\left\{\mathrm{L}_\xi^{A_0}(\mu)d_A\rho\right\} = M_{\mathrm{L}_\xi^{A_0}\mu};
\end{aligned}$$

$$\begin{aligned}
\{M_\mu, H_\lambda\} &= \mathrm{Tr}\int_\Sigma [\mu,\rho]\lambda(B, ee_n) + d_A(\lambda e_n e B)d_A\mu \\
&= \mathrm{Tr}\int_\Sigma d(\lambda e_n e B)[A,\mu] + [A, \lambda e_n e B]d_\mu + [A, \lambda e_n e B][A,\mu] + \frac{\lambda}{2}e^2(ee_n, B)[\mu, B] \\
&\overset{\bigstar}{=} \mathrm{Tr}\int_\Sigma -\lambda e_n e B[dA,\mu] + \lambda e_n e B[A, d\mu - \lambda e_n e B[A, d\mu] + \frac{\lambda e_n}{2}eB[\mu,[A,A]] \\
&\quad \frac{\lambda e_n}{3!}e^3(B,[\mu,B]) - \frac{\lambda e_n}{2}e(B,e^2)[\mu,B] \\
&= \mathrm{Tr}\int_\Sigma \lambda e_n e B\left([\mu, F_A] + \frac{1}{2}[\mu,(e^2,B)]\right) + \frac{\lambda e_n}{2\cdot 3!}e^3[\mu,(B,B)] = 0,
\end{aligned}$$

where in the last passage we used that $\frac{(B,e^2)}{2} + F_A = 0$ and that $\mathrm{Tr}[\mu, (B,B)] = 0$

The computation of the YM part of $\{P_\xi, P_\xi\}$ depending only on ρ and A is exactly equivalent to the computation of the free part of $\{P_\xi, P_\xi\}$ (i.e. the one depending only on e and ω), as one can easily notice by substituting $\frac{1}{2}e^2 \mapsto \rho$ and $\omega_{(0)} \mapsto A_{(0)}$, then we obtain

$$\begin{aligned}\{P_\xi, P_\xi\} &= \int_\Sigma \frac{1}{4} d_\omega(ee)\iota_{[\xi,\xi]}(\omega - \omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega - \frac{1}{4} d_\omega(ee)\iota_\xi\iota_\xi F_{\omega_0} \\ &\quad + \mathrm{Tr}\left\{\frac{1}{2} d_A(\rho)\iota_{[\xi,\xi]}(A - A_0) + \frac{1}{2}\iota_{[\xi,\xi]}(\rho)F_A - \frac{1}{2} d_A(\rho)\iota_\xi\iota_\xi F_{A_0}\right\} \\ &= \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{A_0}};\end{aligned}$$

$$\begin{aligned}\{H_\lambda, H_\lambda\} &= \int_\Sigma (\cdots) - \lambda e B(B, e_n e) d_\omega(\lambda e_n) + \lambda(B, e_n e) d_A(\lambda e_n e B) \\ &= \int_\Sigma (\cdots) - \lambda e B(B, e_n e) d\lambda e_n + \lambda e B(B, e_n e) d\lambda e_n = 0;\end{aligned}$$

$$\begin{aligned}\{P_\xi, H_\lambda\} &= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma -\frac{\lambda e_n}{4} e^2 (B,B)\mathrm{L}_\xi^{\omega_0}(e) - \lambda e_n B F_A \mathrm{L}_\xi^{\omega_0}(e) - \lambda e B(B, e_n e)\mathrm{L}_\xi^{\omega_0}(e) \\ &\quad + \lambda(B, e_n e)\mathrm{L}_\xi^{A_0}(\rho) + d_A(\lambda e_n e B)(-\iota_\xi F_A + d_A\iota_\xi(A - A_0)) \\ &= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma -\frac{\lambda e_n}{2\cdot 3!}(B,B)\mathrm{L}_\xi^{\omega_0}(e^3) - \lambda e_n B F_A \mathrm{L}_\xi^{\omega_0}(e) - \frac{\lambda}{2} B(B, e_n e)\mathrm{L}_\xi^{\omega_0}(e^2) \\ &\quad + \frac{\lambda}{2} B(B, e_n e)\mathrm{L}_\xi^{\omega_0}(e^2) + \frac{\lambda}{2} e^2 (B, e_n e)\mathrm{L}_\xi^{A_0}(B) - \lambda e_n e B d_A(-\iota_\xi F_A + d_A\iota_\xi(A - A_0)) \\ &\overset{\bigstar\blacktriangle}{=} \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\frac{e^3}{2\cdot 3!}(B,B) + \frac{\lambda e_n}{3!} e^3 (B, \mathrm{L}_\xi^{A_0} B) + \mathrm{L}_\xi^{\omega_0}(\lambda e_n) e B F_a \\ &\quad + \lambda e_n e \mathrm{L}_\xi^{A_0}(B) F_A + \lambda e_n e B \mathrm{L}_\xi^{A_0}(F_A) - \frac{\lambda e_n}{3!} e^3 (B, \mathrm{L}_\xi^{A_0} B) - \frac{\lambda e_n}{2} e(B, e^2)\mathrm{L}_\xi^{A_0} B \\ &\quad - \lambda e_n e B\left\{-d_A\iota_\xi F_A + [F_A, \iota_\xi(A - A_0)]\right\} \\ &= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma +\mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{1}{2\cdot 3!} e^3 (B,B) + e B F_A\right) + \lambda e_n e B \mathrm{L}_\xi^{A_0}(F_A) \\ &\quad - \lambda e_n e B\left\{-d_{A_0}\iota_\xi F_A + \iota_\xi[A - A_0, F_A]\right\} \\ &= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma +\mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{1}{2\cdot 3!} e^3 (B,B) + e B F_A\right) + \lambda e_n e B \mathrm{L}_\xi^{A_0}(F_A) \\ &\quad - \lambda e_n e B\left(\iota_\xi d_{A_0} F_A - d_{A_0}\iota_\xi F_A\right) \\ &= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(e F_\omega + \frac{\Lambda}{3!} e^3 + \mathrm{Tr}\left[\frac{1}{2\cdot 3!} e^3 (B,B) + e B F_A\right]\right) \\ &= P_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}} - L_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega - \omega_0)_{(a)}} - M_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega - \omega_0)_{(a)}},\end{aligned}$$

where we also used the Bianchi identities

$$(\blacktriangle) \qquad d_A^2\alpha = [F_A, \alpha] \qquad d_A F_A = 0. \tag{5.48}$$

$$\begin{aligned}
\{L_c, H_\lambda\} &= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma \lambda e_n \left(\frac{1}{4}e^2(B,B)[c,e] + BF_A[c,e]\right) + \lambda e B(B, e_n e)[c,e] \\
&= \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma -[c,\lambda e_n]\left(\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\right) - \lambda e_n e[c,B]FA + \frac{\lambda}{2}e^2(B,e_n e)[c,B] \\
&\overset{\star}{=} \int_\Sigma (\cdots) + \mathrm{Tr}\int_\Sigma -[c,\lambda e_n]\left(\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\right) - \lambda e_n e F_A[c,B] \\
&\quad + \frac{\lambda e_n}{2\cdot 3!}e^3[c,(B,B)] - \frac{\lambda e_n}{2}e(e^2,B)[c,B] \\
&= \int_\Sigma -[c,\lambda e_n]\left(eF_\omega + \frac{\Lambda}{3!}e^3 + \mathrm{Tr}\left\{\frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\right\}\right) \\
&= -P_{[c,\lambda e_n]^{(a)}} + L_{[c,\lambda e_n]^{(a)}(\omega-\omega_0)_a} - H_{\lambda e_n^{(n)}} + M_{[c,\lambda e_n]^{(a)}(A-A_0)_{(a)}}.
\end{aligned}$$

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5.2 The BFV Formalism in the YMPC Theory

As we did for the case of the scalar field, we replicate the discussion about the BFV formalism applied to the space of boundary fields, which is now promoted to a graded symplectic manifold by considering the Lagrange multipliers as ghost fields and adding ghost momenta.

Theorem 5.2.1. *Let $\mathcal{F}^{\mathrm{YM}}$ be the bundle*

$$\mathcal{F}^{\mathrm{YM}} \longrightarrow \Omega^1_{nd}(\Sigma,\mathcal{V}), \tag{5.49}$$

with local trivialisation on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma,\mathcal{V})$

$$\mathcal{F}^{\mathrm{YM}} \simeq \mathcal{U}_\Sigma \times \mathcal{A}^{red}_\partial \times \mathcal{A}^{YM}_\partial \times \Omega^{(0,2)}_{\partial,red} \oplus T^*\left(\Omega^{0,2}_\partial[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma) \oplus \Gamma[1](\mathfrak{g})\right), \tag{5.50}$$

The fields are in degree zero, we denote them by $e \in \mathcal{U}_\Sigma$, $\omega \in \mathcal{A}_\partial$, $A \in \mathcal{A}^{YM}_\partial$ and $B \in \Omega^{(0,2)}_\partial$. They are defined such that they satisfy the structural constraints $e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}$ and $1/2(e^2,B) + F_A = 0$. The ghost fields are denoted by $c \in \Omega^{0,2}_\partial[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$, $\lambda \in \Omega^{0,0}[1]$ and $\mu \in \Gamma[1](\mathfrak{g})$ in degree one, $c^\dagger \in \Omega^{3,2}_\partial[-1]$, $\lambda^\dagger \in \Omega^{3,4}_\partial[-1]$, $\xi^\dagger \in \Omega^{1,0}_\partial[-1] \otimes \Omega^{3,4}_\partial$ and $\mu^\dagger \in \Gamma[-1](\wedge^3 T^*\Sigma \otimes \wedge^4\mathcal{V} \otimes \mathfrak{g})$ in degree minus one, together with a fixed $e_n \in \Gamma(\mathcal{V})$, completing the image of elements $e \in \mathcal{U}_\Sigma$ to a basis of $\mathcal{V}$.

We define a symplectic form and an action functional on $\mathcal{F}$ by

$$\begin{aligned}
S_{YM} = \int_\Sigma & c e d_\omega e + \frac{1}{2}\iota_\xi e^2 F_\omega + \mathrm{Tr}(\iota_\xi \rho F_A) + \iota_\xi(\omega-\omega_0)e d_\omega e + \mathrm{Tr}\{\iota_\xi(A-A_0)d_A\rho\} \\
&+ \lambda e_n\left(eF_\omega + \frac{\Lambda}{3!}e^3 + e\mathrm{Tr}(BF_A) + \frac{1}{2\cdot 3!}e^3\mathrm{Tr}(B,B)\right) + \mathrm{Tr}(\mu d_A\rho) \\
&+ \frac{1}{2}[c,c]c^\dagger - \mathrm{L}^{\omega_0}_\xi c c^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}c^\dagger + [c,\lambda e_n]^{(a)}(\xi^\dagger_a - (\omega-\omega_0)_a c^\dagger) + [c,\lambda e_n]^{(n)}\lambda^\dagger \\
&- \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(a)}(\xi^\dagger_a - (\omega-\omega_0)_a c^\dagger) - \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(n)}\lambda^\dagger - \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger \\
&+ \mathrm{Tr}\left\{\frac{1}{2}[\mu,\mu]\mu^\dagger - \mathrm{L}^{A_0}_\xi(\mu)\mu^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\mu^\dagger + \left[\mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(a)} - [c,\lambda e_n]^{(a)}\right](A-A_0)_a\mu^\dagger\right\}.
\end{aligned} \tag{5.51}$$

$$\varpi_{\text{YM}} = \int_\Sigma e\delta e\delta\omega + \text{Tr}(\delta\rho\delta A) + \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger, +\text{Tr}(\delta\mu\delta\mu^\dagger) \tag{5.52}$$

$$\tag{5.53}$$

Then the triple $(\mathcal{F}^{\text{YM}}, \varpi_{\text{YM}}, S_{\text{YM}})$ defines a BFV structure on Σ.

Proof. We need to prove that $\{S_{\text{YM}}, S_{\text{YM}}\} = 0$.

As in the previous chapter, we split the symplectic form into the classical part and the ghost part

$$\varpi_{\text{YM},f} = \textstyle\int_\Sigma e\delta e\delta\omega + \text{Tr}(\delta\rho\delta A); \tag{5.54}$$

$$\varpi_{\text{YM},g} = \textstyle\int_\Sigma \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger + \text{Tr}(\delta\mu\delta\mu^\dagger). \tag{5.55}$$

Furthermore it is useful to employ the already known results and split $S_{\text{YM}} = S_0^{\text{YM}} + S_1^{\text{YM}}$, with $S_0^{\text{YM}} = S_0^0 + S_0^1$ and $S_1^{\text{YM}} = S_1^0 + S_1^1$ defined such that

$$S_0^0 = \int_\Sigma ced_\omega e + \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi(\omega - \omega_0)ed_\omega e + \lambda e_n\left(eF_\omega + \frac{\Lambda}{3!}e^3\right); \tag{5.56}$$

$$S_0^1 = \text{Tr}\int_\Sigma \iota_\xi\rho F_A + \iota_\xi(A - A_0)d_A\rho + \lambda e_n\left(eBF_A + \frac{e^3}{2\cdot 3!}e^3(B,B)\right) + \mu d_A\rho; \tag{5.57}$$

$$\begin{aligned} S_1^1 = \int_\Sigma &\frac{1}{2}[c,c]c^\dagger - \mathrm{L}_\xi^{\omega_0}cc^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0}c^\dagger + [c,\lambda e_n]^{(a)}(\xi_a^\dagger - (\omega-\omega_0)_a c^\dagger) \\ &+ [c,\lambda e_n]^{(n)}\lambda^\dagger - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\xi_a^\dagger - (\omega-\omega_0)_a c^\dagger) - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}\lambda^\dagger \\ &- \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger; \end{aligned} \tag{5.58}$$

$$\begin{aligned} S_1^1 = \text{Tr}\int_\Sigma &\frac{1}{2}[\mu,\mu]\mu^\dagger - \mathrm{L}_\xi^{A_0}(\mu)\mu^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\mu^\dagger + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(A-A_0)_a\mu^\dagger \\ &- [c,\lambda e_n]^{(a)}(A-A_0)_a\mu^\dagger. \end{aligned} \tag{5.59}$$

The cohomological vector field Q splits into $Q = Q_0^0 + Q_0^1 + Q_1^0 + Q_1^1$, such that $\iota_{Q_j^i}\varpi_{\text{YM}} = \delta S_j^i$.

The classical master equation reads

$$\{S,S\} = \{S_0,S_0\}_f + 2\{S_0,S_1\}_f + 2\{S_0,S_1\}_g + \{S_1,S_1\}_f + \{S_1,S_1\}_g.$$

Of course we have $\{S_0,S_0\}_f + 2\{S_0,S_1\}_g = 0$ by "definition" and $\{S_0,S_0\}_g = 0$ since S_0 has no antighost part. Again we should prove separately that $2\{S_0,S_1\}_f + \{S_1,S_1\}_g = 0$ and $\{S_1,S_1\}_f = 0$. This means

$$\{S_0^1, S_1^0\}_f + \{S_0^0, S_1^1\}_f + \{S_0^1, S_1^1\}_f + \{S_1^0, S_1^1\}_g + \frac{1}{2}\{S_1^1, S_1^1\}_g = 0; \tag{5.60}$$

$$\{S_1^0, S_1^1\}_f + \frac{1}{2}\{S_1^1, S_1^1\}_f = 0. \tag{5.61}$$

We compute them explicitly. In order to do so, we first need to find Q_1^1

$$\begin{aligned} \delta S_1^1 = \text{Tr}\int_\Sigma &\iota_{\delta\xi}\iota_\xi F_{A_0}\mu^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\delta\mu\dagger - \delta\mu[\mu,\mu^\dagger] - \iota_{\delta\xi}d_{A_0}\mu\mu^\dagger - \delta\mu\mathrm{L}_\xi^{A_0}(\mu^\dagger) \\ &- \mathrm{L}_\xi^{A_0}(\mu)\delta\mu^\dagger + \{(\iota_{\delta\xi}d_{\omega_0}(\lambda e_n))^{(a)} - \mathrm{L}_\xi^{\omega_0}(\delta\lambda e_n)^{(a)} - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}\delta e_b^{(a)} \\ &- [\delta c,\lambda e_n]^{(a)} + [c,\delta\lambda e_n]^{(a)} + [c,\lambda e_n]^{(b)}\delta e_b^{(a)}\}(A-A_0)_a\mu^\dagger + \\ &(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)} - [c,\lambda e_n]^{(a)})\delta A_a\mu^\dagger + (\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)} - [c,\lambda e_n]^{(a)})(A-A_0)_a\delta\mu^\dagger. \end{aligned}$$

From this variation we find that $Q^1_{1A}, Q^1_{1e}, Q^1_{1\lambda}, Q^1_{1c}, Q^1_{1\xi}$ vanish. In particular, we are also able to explicitly compute $Q^1_{1\mu}$ and $Q^1_{1\mu^\dagger}$

$$\begin{aligned}
Q^1_{1\mu} &= \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\mu^\dagger + \frac{1}{2}[\mu,\mu] - \mathrm{L}^{A_0}_\xi(\mu) + (\mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(a)} - [c,\lambda e_n]^{(a)})(A-A_0)_a \\
Q^1_{1\mu^\dagger} &= -[\mu,\mu^\dagger] - \mathrm{L}^{A_0}_\xi(\mu^\dagger).
\end{aligned}$$

The components of Q^0_0 and Q^1_0 are recovered from the Hamiltonian vector fields in the previous sections, while Q^0_1 is the same as in [8].

We now prove eq. (5.61) and we leave the other for the appendix. First, we notice that $\{S^1_1, S^1_1\}_f = 0$ because $Q^1_{1A} = 0$ and $Q^1_{1e} = 0$. Furthermore

$$\begin{aligned}
\{S^0_1, S^1_1\}_f &= \iota_{Q^0_1}\iota_{Q^1_1}\int_\Sigma e\delta e\delta\omega + \mathrm{Tr}(\delta\rho\delta A) \\
&= \iota_{Q^0_1}\int_\Sigma([c,\lambda e_n]^{(b)} - \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(b)})\delta^{(a)}_b(A-A_0)_a\mu^\dagger \\
&= \int_\Sigma \underbrace{([c,\lambda e_n]^{(b)} - \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(b)})}_{\propto\lambda}\underbrace{(Q^0_{1e})^{(a)}_b}_{\propto\lambda}(A-A_0)_a\mu^\dagger \propto \int_\Sigma \lambda^2 = 0.
\end{aligned}$$

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Chapter 6

Scalar Electrodynamics in the PC Theory

We now consider the signifcant example of the coupling of a charged scalar field and an electromagnetic field to gravity.

Let P be a $U(1)$-principal bundle. We define the associated bundle $E_\lambda := P \times_\lambda \mathbb{C}$ where $\lambda : U(1) \times \mathbb{C} \to \mathbb{C} : (e^{i\alpha}, \phi) \to e^{i\alpha}\phi$ is the canonical action of $U(1)$ on $\mathbb{C}$. A **charged scalar field** is defined to be a section of E_λ, $\phi \in \Gamma(E_\lambda)$. Working in the first order formalism, we also define $\Pi \in \Gamma(\mathcal{V} \otimes E_\lambda)$.

Since we are dealing with complex fields, we can as well define the complex conjugates of ϕ and Π by $\bar{\phi} \in \Gamma(\bar{E_\lambda})$ and $\bar{\Pi} \in \Gamma(\mathcal{V} \otimes \bar{E}_\lambda)$, where $\bar{E}_\lambda$ is defined as the associated bundle with respect to the the conjugate representation $\bar{\lambda} : U(1) \times \mathbb{C} \to \mathbb{C} : (e^{i\alpha}, \bar{\phi}) \to e^{-i\alpha}\bar{\phi}$

We also have the gauge field A and the corresponding momentum B defined as in the previous section. In this case we notice that $\mathfrak{u}(1) \simeq \mathbb{R}$, which simplifies the fields, in particular $\mathcal{A}_{\text{em}}(M)$ is modeled by $\Omega^{(1,0)}(M)$, and $B \in \Omega^{(0,2)}(M)$. This also means that we will not need the trace operator in the action.

The covariant derivative[1] on the fields is defined by

$$d_A\phi = d\phi + [A, \phi] = d\phi + iA\phi; \tag{6.1}$$

$$d_A\Pi = d\Pi + iA\Pi; \tag{6.2}$$

$$d_A B = dB + [A, B] = dB + AB - BA = dB. \tag{6.3}$$

The scalar electrodynamics action is

$$S_{\text{sed}} = \int_M \frac{e^{N-1}}{2(N-1)!}\left[\bar{\Pi} d_A\phi + \Pi d_A\bar{\phi}\right] + \frac{e^N}{2\cdot N!}(\bar{\Pi}, \Pi) + \frac{e^{N-2}}{(N-2)!}BF_A + \frac{e^N}{2\cdot N!}(B, B). \tag{6.4}$$

The full action is given by $S = S_{PC} + S_{\text{sed}}$, producing equations of motion[2]

$$\begin{aligned} &\frac{e^{N-3}}{(N-3)!}(F_\omega + BF_A) + \frac{e^{N-2}}{2(N-2)!}(\bar{\Pi} d_A\phi + \Pi d_A\bar{\phi}) \\ &+ \frac{e^{N-1}}{(N-1)}\left[\Lambda + \frac{1}{2}(\Pi, \bar{\Pi}) + \frac{1}{2}(B, B)\right] = 0; \end{aligned} \tag{6.5}$$

[1] we assume that the scalar field has a unit charge $q = 1$

[2] The variation with respect to fields and their complex conjugate produce the same equations of motion, up to conjugation

$$d_\omega e = 0; \tag{6.6}$$
$$d_A\phi - (e,\Pi) = 0; \tag{6.7}$$
$$d_A(e^{N-1}\Pi) = 0; \tag{6.8}$$
$$d_A\left(\frac{e^{N-2}}{(N-2)!}B\right) - \frac{e^{N-1}}{2(N-1)!}(i\bar{\Pi}\phi - i\Pi\bar{\phi}) = 0; \tag{6.9}$$
$$F_A + \frac{1}{2}(e^2, B) = 0. \tag{6.10}$$

We now quickly check that the Lagrangian that we wrote actually represents the usual charged scalar field Lagrangian. Assuming e to be non-degenerate, we see that eq. (6.7) gives $\pi^\mu = -g^{\mu\nu}(\partial_\mu\phi + iA_\mu\phi)$, hence

$$\begin{aligned}
&\frac{e^{N-1}}{2(N-1)!}\left(\bar{\Pi}d_A\phi + \Pi d_A\bar{\phi}\right) + \frac{e^N}{2\cdot N!}(\bar{\Pi},\Pi) = \\
&= -\frac{e^{N-1}}{2(N-1)!}\left((e,\Pi)\bar{\Pi} + (e,\bar{\Pi}\Pi)\right) + \frac{e^N}{2\cdot N!}(\bar{\Pi},\Pi) \\
&= -\frac{e^N}{2\cdot N!}[(\Pi,\bar{\Pi}) + (\bar{\Pi},\Pi)] + \frac{e^N}{2\cdot N!}(\bar{\Pi},\Pi) \\
&= -\frac{e^N}{2\cdot N!}(\bar{\Pi},\Pi) = -\frac{\mathrm{Vol}_g}{2}g^{\mu\nu}[\partial_\mu\phi\partial_\nu\phi + iA_\mu(\phi\partial_\nu\bar{\phi} - \partial_\nu\phi\bar{\phi})] \\
&= -\frac{\mathrm{Vol}_g}{2}g^{\mu\nu}\nabla_\mu\bar{\phi}\nabla_\nu\phi.
\end{aligned}$$

We also obtain a boundary term

$$\tilde{\alpha}_{\text{sed}} = \int_{\partial M}\frac{e^{N-2}}{(N-2)!}\delta\omega + \frac{e^{N-2}}{(N-2)!}B\delta A + \frac{e^{N-1}}{2(N-1)!}\left[\bar{\Pi}\delta\phi + \Pi\delta\bar{\phi}\right]. \tag{6.11}$$

6.1 Boundary Structure in $N = 4$

As in the previous chapters, we look at the boundary structure of the fields under the same assumptions that we had earlier. We will see that the constraints will be modified with respect to the YMPC theory, but the Poisson brackets will remain unchanged, which will largely simplify the BFV discussion.

The boundary term in $N = 4$ is

$$\tilde{\alpha}_{\text{sed}} = \frac{1}{2}\int_\Sigma e^2\delta\omega + e^2B\delta A + \frac{1}{3!}(\bar{\Pi}\delta\phi + \Pi\delta\bar{\phi}), \tag{6.12}$$

where the fields in the previous expression are restricted to the boundary, namely $B \in \Omega_\partial^{(0,2)}$, $A \in \mathcal{A}_{\text{em}}(\Sigma)$, $\Pi \in \Gamma(E_\lambda|_\Sigma \otimes \mathcal{V}|_\Sigma)$ and $\phi \in \Gamma(E_\lambda|_\Sigma)$. The space of preboundary fields s denoted by $\tilde{F}_\partial^{\text{sed}} = \Omega_{\text{n.d.}}^{(1,1)} \times \mathcal{A}_\partial \times \mathcal{A}_\partial^{\text{em}} \times \Omega_\partial^{(0,2)} \times \Gamma(E_\lambda|_\Sigma \otimes \mathcal{V}|_\Sigma) \times \Gamma(E_\lambda|_\Sigma)$. The presymplectic form on $\tilde{F}_\partial^{\text{sed}}$ is defined as

$$\begin{aligned}
\tilde{\varpi}_{\text{sed}} = \int_\Sigma &\frac{e^3}{2\cdot 3!}(\delta\bar{\Pi}\delta\phi + \delta\Pi\delta\bar{\phi}) + \frac{e^2}{4}\delta e(\bar{\Pi}\delta\phi + \Pi\delta\bar{\phi}) \\
&+ e\delta e\delta\omega + eB\delta e\delta A + \frac{e^2}{2}\delta B\delta A.
\end{aligned} \tag{6.13}$$

The kernel of the presymplectic form is found as the vector fields in $T\tilde{F}_{\partial}^{\text{sed}}$ satisfying the following system of equations

$$e\mathbb{X}_e = 0; \tag{6.14}$$
$$e\mathbb{X}_\omega + eB\mathbb{X}_A + \frac{e^2}{4}(\bar{\Pi}\mathbb{X}_\phi + \Pi\mathbb{X}_{\bar{\Pi}}) = 0 \tag{6.15}$$
$$eB\mathbb{X}_e + \frac{e^2}{2}\mathbb{X}_B = 0; \tag{6.16}$$
$$e^2\mathbb{X}_A = 0; \tag{6.17}$$
$$\frac{e^3}{2\cdot 3!}\mathbb{X}_\Pi + \frac{e^2}{4}\mathbb{X}_e = 0; \tag{6.18}$$
$$e^3\mathbb{X}_\phi = 0. \tag{6.19}$$

We obtain $\mathbb{X}_e = 0, \mathbb{X}_\phi = 0, \mathbb{X}_A = 0$, therefore we are left with

$$e\mathbb{X}_\omega = 0; \tag{6.20}$$
$$e^2\mathbb{X}_B = 0; \tag{6.21}$$
$$e^3\mathbb{X}_\Pi = 0. \tag{6.22}$$

The geometric phase space is then $F_{\partial}^{\text{sed}} = \tilde{F}_{\partial}^{\text{sed}}/\sim = \Omega_{\text{n.d.}}^{(1,1)} \times \mathcal{A}_{\partial}^{\text{red}} \times \mathcal{A}_{\partial}^{\text{em}} \times \Omega_{\partial}^{(0,2),\text{red}} \times \Gamma(E_\lambda|_\Sigma \otimes \mathcal{V}|_\Sigma)_{\text{red}} \times \Gamma(E_\lambda|_\Sigma)$ where

$$\begin{aligned} \omega \sim \tilde{\omega} \quad &\Leftrightarrow \quad \omega - \tilde{\omega} = v \text{ with } ev = 0 \\ B \sim \tilde{B} \quad &\Leftrightarrow \quad B - \tilde{B} = C \text{ with } e^2C = 0 \\ \Pi \sim \tilde{\Pi} \quad &\Leftrightarrow \quad \Pi - \tilde{\Pi} = e^3\gamma. \end{aligned}$$

Remark 6.1.1. As usual we can also work in Darboux coordinates by defining $\rho := \frac{1}{2}e^2[B]$ and $p := \frac{1}{3!}e^3[\Pi]$. Then we obtain the symplectic form on the geometric phase space as

$$\varpi_{\text{sed}} = \int_\Sigma e\delta e\delta[\omega] + \delta\rho\delta A + \frac{1}{2}(\delta p\delta\bar{\phi} + \delta\bar{p}\delta\phi). \tag{6.23}$$

Choice of Representatives via Constraint

The representative of $[\omega]$ is chosen as in the previous chapters, while the representatives of the other fields in the geometric phase space are found using constraints

Theorem 6.1.2. *Assuming g^∂ is non-degenerate, the representatives of $[\Pi]$ and $[B]$ are respectively uniquely found in terms of the transversal ("physical") components of Π and B and in terms of ϕ and A by imposing*

$$F_A + \frac{1}{2}(e^2, B) = 0; \tag{6.24}$$
$$d_A\phi + (e, \Pi) = 0. \tag{6.25}$$

Proof. It is a simple application of the theorems 4.4.6 and 5.1.2. ✓

Poisson Brackets of the Constraints

We consider the constraints directly in Darboux coordinates, since it will make calculations easier

$$L_c := \int_\Sigma c e d_\omega e; \tag{6.26}$$

$$M_\mu := \int_\Sigma \mu \left[d_A \rho - \frac{i}{2}(\bar{p}\phi - p\bar{\phi}) \right]; \tag{6.27}$$

$$\begin{aligned} P_\xi : = \int_\Sigma & \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi \rho F_A + \frac{1}{2}\left(\iota_\xi \bar{p} d_A \phi + \iota_\xi p d_A \bar{\phi}\right) \\ & + \iota_\xi(\omega - \omega_0) e d_\omega e + \iota_\xi (A - A_0) d_A \rho \\ & - \frac{i}{2}\iota_\xi (A - A_0)(\bar{p}\phi - p\bar{\phi}); \end{aligned} \tag{6.28}$$

$$\begin{aligned} H_\lambda := \int_\Sigma \lambda e_n \Bigg[& e F_\omega + e B F_A + \frac{e^2}{4}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi) \\ & + \frac{e^3}{3!}\left(\Lambda + \frac{1}{2}(\Pi, \bar{\Pi}) + \frac{1}{2}(B, B)\right)\Bigg], \end{aligned} \tag{6.29}$$

with the usual choice of Lagrange multipliers $\mu, \lambda \in \mathcal{C}^\infty[1](\Sigma)$, $c \in \Omega_\partial^{(0,2)}[1]$ and $\xi \in \mathfrak{X}[1](\Sigma)$.

Theorem 6.1.3. *The functions L_c, M_μ, P_ξ, H_λ define a coisotropic submanifold with respect to the symplectic structure ϖ_{sed}. Their Poisson brackets read*

$$\{P_\xi, P_\xi\} = \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi \iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi \iota_\xi F_{A_0}} \qquad \{H_\lambda, H_\lambda\} = 0$$

$$\{M_\mu, M_\mu\} = -\frac{1}{2}M_{[\mu,\mu]} \qquad \{M_\mu, L_c\} = 0$$

$$\{M_\mu, H_\lambda\} = 0 \qquad \{M_\mu, P_\xi\} = M_{\mathrm{L}_\xi^{A_0}\mu}$$

$$\{L_c, P_\xi\} = L_{\mathrm{L}_\xi^{\omega_0} c} \qquad \{L_c, L_c\} = -\frac{1}{2}L_{[c,c]}$$

$$\{L_c, H_\lambda\} = -P_{X^{(a)}} + L_{X^{(a)}(\omega-\omega_0)_a} - H_{X^{(n)}} + M_{X^{(a)}(A-A_0)_{(a)}}$$
$$\{P_\xi, H_\lambda\} = P_{Y^{(a)}} - L_{Y^{(a)}(\omega-\omega_0)_a} + H_{Y^{(n)}} - M_{Y^{(a)}(A-A_0)_{(a)}},$$

where $X = [c, \lambda e_n]$, $Y = \mathrm{L}_\xi^{\omega_0}(\lambda e_n)$ and $Z^{(a)}$, $Z^{(n)}$ are the components of $Z \in \{X, Y\}$ with respect to the frame (e_a, e_n).

Proof. We compute the Hamiltonian vector fields of the constraints.

$$\delta L_c = \int_M [c, e] e \delta\omega + d_\omega c e \delta e;$$

$$\begin{aligned} \delta M_\mu &= \int_\Sigma d_A \mu \delta\rho - \frac{i}{2}\delta[\mu \bar{p}\phi - p\bar{\phi}] \\ &= \int_\Sigma d_A \mu \delta\rho + \frac{i\mu}{2}[\delta\bar{p}\phi - \bar{p}\delta\phi - \delta p\bar{\phi} + p\delta\bar{\phi}]; \end{aligned}$$

$$
\begin{aligned}
\delta P_\xi =& \int_\Sigma (\cdots) + \frac{1}{2}\delta \left\{\iota_\xi \bar{p} d_A \phi - i\iota_\xi (A - A_0)\bar{p}\phi + \text{c.c.}\right\} \\
=& \int_\Sigma (\cdots) + \frac{1}{2}\left\{\iota_\xi \delta\bar{p} d_A \phi - i\iota_\xi \bar{p}\delta A \phi + d_A(\iota_\xi \bar{p})\delta\phi - i\iota_\xi(\delta A)\bar{p}\phi + i\iota_\xi (A - A_0)(\delta\bar{p}\phi - \bar{p}\delta\phi)\right\} + \text{c.c.} \\
=& \int_\Sigma -e\delta e(\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) + \iota_\xi F_{\omega_0}) - (\mathrm{L}_\xi^{\omega_0} e)e\delta\omega - \delta\rho(\mathrm{L}_\xi^{A_0}(A - A_0) + \iota_\xi F_{A_0}) + \mathrm{L}_\xi^{A_0}\rho\delta A \\
&+ \frac{1}{2}\left\{-\delta\bar{p}\mathrm{L}_\xi^{A_0}\phi - \mathrm{L}_\xi^{A_0}\bar{p}\delta\phi + \text{c.c.}\right\};
\end{aligned}
$$

$$
\begin{aligned}
\delta H_\lambda =& \int_\Sigma (\cdots) + \lambda e_n \delta \left\{\frac{e^2}{4}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi) + \frac{e^3}{2\cdot 3!}(\bar{\Pi}, \Pi)\right\} \\
=& \int_\Sigma (\cdots) + \lambda e_n \left[\frac{e}{2}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi) + \frac{e^2}{4}(\bar{\Pi}, \Pi)\right]\delta e + \frac{\lambda e_n}{4} e^2 (e, \bar{\Pi})\delta\Pi \\
&+ i\frac{\lambda e_n}{4} e^2 \Pi \delta A \bar{\phi} + d_A\left(\frac{\lambda e_n}{4} e^2 \Pi\right)\delta\bar{\phi} + \frac{\lambda e_n}{2\cdot 3!} e^3 (\bar{\Pi}, \delta\Pi) + \text{c.c.} \\
\overset{\bigstar}{=}& \int_\Sigma (\cdots) + \lambda e_n \left[\frac{e}{2}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi) + \frac{e^2}{4}(\bar{\Pi}, \Pi)\right]\delta e + \delta A \left[\frac{\lambda e_n}{4} e^2 (i\Pi\bar{\phi} + \text{c.c.})\right] \\
&+ d_A\left(\frac{\lambda e_n}{4} e^2 \Pi\right)\delta\bar{\phi} - \frac{\lambda}{2\cdot 3!} e^3 (e_n, \bar{\Pi})\delta\Pi + \text{c.c.} \\
=& \int_\Sigma \left\{\lambda e_n \left[e F_\omega + B F_A + \frac{e^2}{4}\left(2\Lambda + (B, B) + (\Pi, \bar{\Pi})\right) + \frac{e}{2}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi)\right]\right. \\
&\left. - \lambda e B(B, e_n e) - \frac{\lambda}{4}(e_n, \Pi)e^2\bar{\Pi} - \frac{\lambda}{4}(e_n, \bar{\Pi})e^2\Pi\right\}\delta e \\
&+ \left[d_A(\lambda e_n e B) + \frac{\lambda e_n}{4} e^2 (i\Pi\bar{\phi} - i\bar{\Pi}\phi)\right]\delta A + \lambda(B, e_n e)\delta\rho \\
&+ d_A\left(\frac{\lambda e_n}{4} e^2 \bar{\Pi}\right)\delta\phi - \frac{\lambda}{2}(e_n, \bar{\Pi})\delta p + \text{c.c.},
\end{aligned}
$$

where we used

$$
(\bigstar) \qquad \frac{e_n}{3!} e^3 (A, B) = (-1)^{|A|+|B|}\left[\frac{e_n}{2} e^2 (e, A) B + \frac{e^3}{3!}(e_n, A) B\right]
$$

Hence we obtain

$$
\begin{aligned}
e\mathbb{H}_\omega =& \lambda e_n \left[e F_\omega + B F_A + \frac{e^2}{4}\left(2\Lambda + (B, B) + (\Pi, \bar{\Pi})\right) + \frac{e}{2}(\Pi d_A \bar{\phi} + \bar{\Pi} d_A \phi)\right] \\
&- \lambda e B(B, e_n e) - \frac{\lambda}{4}(e_n, \Pi)e^2\bar{\Pi} - \frac{\lambda}{4}(e_n, \bar{\Pi})e^2\Pi
\end{aligned}
\tag{6.30}
$$

$$
\mathbb{H}_e = d_\omega(\lambda e_n) + \lambda\sigma \tag{6.31}
$$

$$\mathbb{H}_\rho = d_A(\lambda e_n e B) - \tfrac{\lambda}{4} e^2 (i\Pi\bar{\phi} - i\bar{\Pi}\phi) \qquad \mathbb{H}_A = \lambda(B, e e_n) \tag{6.32}$$
$$\mathbb{H}_p = d_A\left(\tfrac{\lambda e_n}{2} e^2 \Pi\right) \qquad \mathbb{H}_\phi = -\lambda(e_n, \Pi) \tag{6.33}$$
$$\mathbb{L}_e = [c, e] \qquad \mathbb{L}_A = 0 \tag{6.34}$$
$$\mathbb{L}_\omega = d_\omega c + \mathbb{V}_L \qquad \mathbb{L}_\rho = 0 \tag{6.35}$$
$$\mathbb{L}_\phi = 0 \qquad \mathbb{L}_p = 0 \tag{6.36}$$
$$\mathbb{M}_e = 0 \qquad \mathbb{M}_A = d_A\mu \tag{6.37}$$
$$\mathbb{M}_\omega = 0 \qquad \mathbb{M}_\rho = 0 \tag{6.38}$$
$$\mathbb{M}_\phi = i\mu\phi \qquad \mathbb{M}_p = i\mu p \tag{6.39}$$
$$\mathbb{P}_e = -\mathrm{L}_\xi^{\omega_0}(e) \qquad \mathbb{P}_\omega = -\mathrm{L}_\xi^{\omega_0}(\omega - \omega_0) - \iota_\xi(F_{\omega_0}) + \mathbb{V}_P \tag{6.40}$$
$$\mathbb{P}_\rho = -\mathrm{L}_\xi^{A_0}(\rho) \qquad \mathbb{P}_A = -\mathrm{L}_\xi^{A_0}(A - A_0) - \iota_\xi(F_{A_0}) \tag{6.41}$$
$$\mathbb{P}_\phi = -\mathrm{L}_\xi^{(A_0)}\phi \qquad \mathbb{P}_p = -\mathrm{L}_\xi^{A_0}(p). \tag{6.42}$$

Before computing the Poisson brackets of the constraints, we notice that $\{L_c, L_c\}$ and $\{L_c, P_\xi\}$ are as in the previous chapter. Also $\{M_\mu, L_c\} = 0$ is very easily seen because $L_\Phi = 0$ for all the fields Φ which are not e and ω.

$$\{M_\mu, M_\mu\} \propto \int_\Sigma \mu^2 = 0;$$

$$\begin{aligned}
\{M_\mu, P_\xi\} &= \int_\Sigma (\cdots) - \frac{1}{2}\left\{-i\mu\bar{p}\mathrm{L}_\xi^{A_0}(\phi) + \mathrm{L}_\xi^{A_0}\bar{p}\phi + \text{c.c.}\right\} \\
&= \int_\Sigma (\cdots) - \frac{i}{2}\left\{\mathrm{L}_\xi^{A_0}(\mu\bar{p})\phi + \mu\mathrm{L}_\xi^{A_0}(\bar{p})\phi + \text{c.c.}\right\} \\
&= \int_\Sigma \mathrm{L}_\xi^{A_0}(\mu)\left\{d_A\rho - \frac{i}{2}\left[\bar{p}\phi - p\bar{\phi}\right]\right\} = M_{\mathrm{L}_\xi^{A_0}\mu};
\end{aligned}$$

$$\begin{aligned}
\{M_\mu, P_\xi\} &= \int_\Sigma \left[d_A(\lambda e_n e B) + \frac{\lambda e_n}{4} e^3\left(i\Pi\bar{\phi} - i\bar{\Pi}\phi\right)\right] d_A\mu \\
&\quad + d_A(\frac{\lambda e_n}{4} e^2\bar{\Pi}) i\mu\phi - d_A(\frac{\lambda e_n}{4} e^2 \Pi) i\mu\bar{\phi} \\
&\quad - i\frac{\lambda}{2}(e_n, \bar{\Pi})\mu p + i\frac{\lambda}{2}(e_n, \Pi)\mu\bar{p} \\
&= \int_\Sigma d_A\left[\frac{\lambda e_n}{4} e^3\left(i\Pi\bar{\phi} - i\bar{\Pi}\phi\right)\right]\mu + \frac{i}{4} d_A(\lambda e_n e^2\bar{\Pi})\phi\mu \\
&\quad - \frac{i}{4} d_A(\lambda e_n e^2 \Pi)\bar{\phi}\mu + i\frac{\lambda}{2\cdot 3!} e^3(e_n, \bar{\Pi})\Pi\mu - i\frac{\lambda}{2\cdot 3!} e^3(e_n, \Pi)\bar{\Pi}\mu \\
&\overset{\star}{=} \int_\Sigma \frac{i}{4} d_A(\lambda e_n e^2\Pi)\bar{\phi}\mu - i\frac{\lambda e_n}{4} e^2 \Pi d_A\bar{\phi}\mu - \frac{i}{4} d_A(\lambda e_n e^2\bar{\Pi})\phi\mu \\
&\quad + i\frac{\lambda e_n}{4} e^2\bar{\Pi} d_A\phi\mu - \frac{i}{4} d_A(\lambda e_n e^2\Pi)\bar{\phi}\mu + \frac{i}{4} d_A(\lambda e_n e^2\bar{\Pi})\phi\mu \\
&\quad - i\frac{\lambda e_n}{4} e^2(e, \bar{\Pi})\Pi\mu + i\frac{\lambda e_n}{4} e^2(e, \Pi)\bar{\Pi}\mu = 0;
\end{aligned}$$

$$\begin{aligned}
\{P_\xi, P_\xi\} &= \int_\Sigma (\cdots) + [\mathrm{L}_\xi^{A_0}\bar{p}\mathrm{L}_\xi^{A_0}\phi + \text{c.c.}] \\
&= \int_\Sigma (\cdots) + (\bar{p}\mathrm{L}_\xi^{A_0}\mathrm{L}_\xi^{A_0}\phi + \text{c.c.}) \\
&\overset{\clubsuit}{=} \int_\Sigma (\cdots) + \frac{1}{2}[\bar{p}\mathrm{L}_{[\xi,\xi]}^{A_0}\phi + i\bar{p}\iota_\xi\iota_\xi F_{A_0} + \text{c.c.}] \\
&= \int_\Sigma (\cdots) + \frac{1}{2}\left[-\frac{1}{2}\bar{p}\iota_\xi\iota_\xi d_{A_0}d_{A_0}\phi + i\bar{p}\iota_\xi\iota_\xi F_{A_0} + \text{c.c.}\right] \\
&\overset{\triangle}{=} \int_\Sigma \frac{1}{4}d_\omega(ee)\iota_{[\xi,\xi]}(\omega-\omega_0) + \frac{1}{4}\iota_{[\xi,\xi]}(ee)F_\omega - \frac{1}{4}d_\omega(ee)\iota_\xi\iota_\xi F_{\omega_0} \\
&\quad + \frac{1}{2}d_A(\rho)\iota_{[\xi,\xi]}(A-A_0) + \frac{1}{2}\iota_{[\xi,\xi]}(\rho)F_A - \frac{1}{2}d_A(\rho)\iota_\xi\iota_\xi F_{A_0} \\
&\quad + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\left(-\frac{i}{2}\right)[\bar{p}\phi - p\bar{\phi}] \\
&= \frac{1}{2}P_{[\xi,\xi]} - \frac{1}{2}L_{\iota_\xi\iota_\xi F_{\omega_0}} - \frac{1}{2}M_{\iota_\xi\iota_\xi F_{A_0}},
\end{aligned}$$

where we used[3]

$$(\clubsuit) \qquad \mathrm{L}_\xi^{\omega_0}\mathrm{L}_\xi^{\omega_0}B = \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^{\omega_0}B + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, B] \qquad \forall B \in \Omega_\partial^{i,j}$$

$$(\triangle) \qquad d_A^2\alpha = [F_A, \alpha] \qquad \forall \alpha \in \Omega_\partial^{(i,j)}.$$

$$\begin{aligned}
\{L_c, H_\lambda\} &= \int_\Sigma (\dots) + \left\{\lambda e_n \left[\frac{e}{2}(\Pi d_A\bar{\phi} + \bar{\Pi}d_A\phi) + \frac{e^2}{4}(\Pi,\bar{\Pi})\right]\right. \\
&\quad \left. - \frac{\lambda e^2}{4}\left[(e_n,\Pi)\bar{\Pi} + (e_n,\bar{\Pi})\Pi\right]\right\}[c,e] \\
&= \int_\Sigma (\cdots) - [c,\lambda e_n]\left\{\frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi}d_A\phi) + \frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi})\right\} \\
&\quad - \frac{\lambda e_n}{4}e^2\left([c,\Pi]d_A\bar{\phi} + \text{c.c.}\right) - \frac{\lambda}{2\cdot 3!}e^3\left[(e_n,\bar{\Pi})\Pi + \text{c.c.}\right] \\
&\overset{\bigstar}{=} \int_\Sigma (\cdots) - [c,\lambda e_n]\left\{\frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi}d_A\phi) + \frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi})\right\} \\
&\quad + \frac{\lambda e_n}{2\cdot 3!}e^3\left((\Pi,[c,\bar{\Pi}]) + ([c,\Pi],\bar{\Pi})\right) \\
&= \int_\Sigma -[c,\lambda e_n]\left\{eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2\cdot 3!}e^3(B,B) + eBF_A\right. \\
&\quad \left. + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi}d_A\phi) + \frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi})\right\} \\
&= -P_{[c,\lambda e_n]^{(a)}} + L_{[c,\lambda e_n]^{(a)}(\omega-\omega_0)_a} - H_{\lambda e_n^{(n)}} + M_{[c,\lambda e_n]^{(a)}(A-A_0)_{(a)}};
\end{aligned}$$

[3]the proofs of this identities are found in [13]

$$
\begin{aligned}
\{P_\xi, H_\lambda\} &= \int_\Sigma (\cdots) + \lambda e_n \left[\frac{e^2}{4}(\Pi,\bar{\Pi}) + \frac{e}{2}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A \phi) - \frac{\lambda}{4}(e_n,\Pi)e^2\bar{\Pi}\frac{\lambda}{4}(e_n,\bar{\Pi})e^2\Pi\right](-\mathrm{L}_\xi^{\omega_0} e) \\
&\quad - \frac{\lambda e_n}{4} e^2 (i\Pi\bar{\phi} - i\bar{\Pi}\phi)[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}] - d_A\left(\frac{\lambda e_n}{4}e^2\bar{\Pi}\right)\mathrm{L}_\xi^{A_0}\phi \\
&\quad + \frac{\lambda}{2}(e_n,\bar{\Pi})\mathrm{L}_\xi^{A_0} p - d_A\left(\frac{\lambda e_n}{4}e^2\Pi\right)\mathrm{L}_\xi^{A_0}\bar{\phi} + \frac{\lambda}{2}(e_n,\Pi)\mathrm{L}_\xi^{A_0}\bar{p} \\
&= \int_\Sigma (\cdots) + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi}) + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi)\right) + \frac{\lambda e_n}{2\cdot 3!}e^3[(\mathrm{L}_\xi^{A_0}\Pi,\bar{\Pi}) + (\Pi,\mathrm{L}_\xi^{A_0}\bar{\Pi})] \\
&\quad + \frac{\lambda e_n}{4}e^2\left[\mathrm{L}_\xi^{A_0}\Pi d_A\bar{\phi} - \Pi\mathrm{L}_\xi^{A_0} d_A\bar{\phi} + \mathrm{L}_\xi^{A_0}\bar{\Pi} d_A\phi - \bar{\Pi}\mathrm{L}_\xi^{A_0} d_A\phi\right] - \frac{\lambda}{2\cdot 3!}(e_n,\Pi)\mathrm{L}_\xi^{\omega_0}(e^3)\bar{\Pi} \\
&\quad - \frac{\lambda}{2\cdot 3!}(e_n,\bar{\Pi})\mathrm{L}_\xi^{\omega_0}(e^3)\Pi + \frac{\lambda e_n}{4}e^2(i\bar{\Pi}\phi - i\Pi\bar{\phi})[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}] - d_A\left(\frac{\lambda e_n}{4}e^2\bar{\Pi}\right)\mathrm{L}_\xi^{A_0}\phi \\
&\quad - d_A\left(\frac{\lambda e_n}{4}e^2\Pi\right)\mathrm{L}_\xi^{A_0}\bar{\phi} + \frac{\lambda}{2\cdot 3!}(e_n,\bar{\Pi})\mathrm{L}_\xi^{\omega_0}(e^3)\Pi + \frac{\lambda}{2\cdot 3!}(e_n,\bar{\Pi})e^3\mathrm{L}_\xi^{A_0}\Pi + \frac{\lambda}{2\cdot 3!}(e_n,\Pi)\mathrm{L}_\xi^{\omega_0}(e^3)\bar{\Pi} \\
&\quad + \frac{\lambda}{2\cdot 3!}(e_n,\Pi)e^3\mathrm{L}_\xi^{A_0}\bar{\Pi} \\
&\overset{\bigstar}{=} \int_\Sigma (\cdots) + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi}) + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi)\right) + \frac{\lambda e_n}{2\cdot 3!}e^3[(\mathrm{L}_\xi^{A_0}\Pi,\bar{\Pi}) + (\Pi,\mathrm{L}_\xi^{A_0}\bar{\Pi})] \\
&\quad + \frac{\lambda e_n}{4}^2\left[(e,\bar{\Pi})\mathrm{L}_\xi^{A_0}\Pi + (e,\Pi)\mathrm{L}_\xi^{A_0}\bar{\Pi} - \Pi\mathrm{L}_\xi^{A_0}(d_A\bar{\phi}) - \bar{\Pi}\mathrm{L}_\xi^{A_0}(d_A\phi)\right] \\
&\quad + \frac{\lambda e_n}{4}^2(i\bar{\Pi}\phi - i\Pi\bar{\phi})[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}] - d_A\left(\frac{\lambda e_n}{4}e^2\bar{\Pi}\right)\mathrm{L}_\xi^{A_0}\phi \\
&\quad - d_A\left(\frac{\lambda e_n}{4}e^2\Pi\right)\mathrm{L}_\xi^{A_0}\bar{\phi} - \frac{\lambda e_n}{4}e^2(e,\bar{\Pi})\mathrm{L}_\xi^{A_0}(\Pi) - \frac{\lambda e_n}{2\cdot 3!}e^3(\bar{\Pi},\mathrm{L}_\xi^{A_0}(\Pi)) \\
&\quad - \frac{\lambda e_n}{4}e^2(e,\Pi)\mathrm{L}_\xi^{A_0}(\bar{\Pi}) - \frac{\lambda e_n}{2\cdot 3!}e^3(\Pi,\mathrm{L}_\xi^{A_0}(\bar{\Pi})) \\
&= \int_\Sigma (\cdots) + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi}) + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi)\right) \\
&\quad + \frac{\lambda e_n}{4}^2\left\{-\Pi\mathrm{L}_\xi^{A_0}(d_A\bar{\phi}) - \bar{\Pi}\mathrm{L}_\xi^{A_0}(d_A\phi) + (i\bar{\Pi}\phi - i\Pi\bar{\phi})[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}]\right\} \\
&\quad - \frac{\lambda e_n}{4}^2\left[d_A\mathrm{L}_\xi^{A_0}\phi + d_A\mathrm{L}_\xi^{A_0}\bar{\phi}\right] \\
&= \int_\Sigma (\cdots) + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(\frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi}) + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi)\right) \\
&\quad + \frac{\lambda e_n}{4}e^2\Pi\left\{-\mathrm{L}_\xi^{A_0}\bar{\phi} + i\left[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}\right]\bar{\phi} - d_A\mathrm{L}_\xi^{A_0}\bar{\phi}\right\} + \text{c.c.} \\
&= \int_\Sigma \mathrm{L}_\xi^{\omega_0}(\lambda e_n)\left(eF_\omega + \frac{\Lambda}{3!}e^3 + \frac{1}{2\cdot 3!}e^3(B,B) + eBF_A + \frac{e^3}{2\cdot 3!}(\Pi,\bar{\Pi}) + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi)\right) \\
&= P_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}} + H_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}} - L_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega-\omega_0)_{(a)}} - M_{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\omega-\omega_0)_{(a)}},
\end{aligned}
$$

where in the second to last passage we used that

$$
-\mathrm{L}_\xi^{A_0} d_A\phi + i[-\mathrm{L}_\xi^{A_0}(A-A_0) - \iota_\xi F_{A_0}]\phi - d_A\mathrm{L}_\xi^{A_0}\phi = 0.
$$

$$
\begin{aligned}
\{H_\lambda, H_\lambda\} &= \int_\Sigma -\frac{\lambda}{4} e^2 \left\{(e_n, \Pi)\bar{\Pi}\right\} d\lambda e_n - \lambda (e_n, \Pi) d_A \left(\frac{\lambda e_n}{4} e^2 \bar{\Pi}\right) + \text{c.c.} \\
&= \int_\Sigma \frac{\lambda d\lambda}{4} e_n e^2 \left\{(e_n, \Pi)\bar{\Pi} - (e_n, \Pi)\bar{\Pi}\right\} + \text{c.c.} \\
&= 0.
\end{aligned}
$$

✓

6.2 The BFV Formalism in Scalar Electrodynamics

Theorem 6.2.1. *Let $\mathcal{F}^{\mathrm{YM}}$ be the bundle*

$$\mathcal{F}^{\mathrm{YM}} \longrightarrow \Omega^1_{nd}(\Sigma, \mathcal{V}), \tag{6.43}$$

with local trivialisation on an open $\mathcal{U}_\Sigma \subset \Omega^1_{nd}(\Sigma, \mathcal{V})$

$$
\begin{aligned}
\mathcal{F}^{\mathrm{YM}} \simeq & \mathcal{U}_\Sigma \times \mathcal{A}^{red}_\partial \times \mathcal{A}^{YM}_\partial \times \Omega^{(0,2)}_{\partial, red} \times \Gamma(\bar{E}_\lambda) \times \Gamma(\bar{E}_\lambda \otimes \mathcal{V}) \\
& \oplus T^* \left(\Omega^{0,2}_\partial[1] \oplus \mathfrak{X}[1](\Sigma) \oplus C^\infty[1](\Sigma) \oplus \Gamma[1](\mathfrak{g})\right),
\end{aligned} \tag{6.44}
$$

The fields are in degree zero, we denote them by $e \in \mathcal{U}_\Sigma$, $\omega \in \mathcal{A}_\partial$, $\phi \in \Gamma(\bar{E}_\lambda)$, $\Pi \in \Gamma(\mathcal{V} \otimes \bar{E}_\lambda)$, $A \in \mathcal{A}^{YM}_\partial$ and $B \in \Omega^{(0,2)}_\partial$. They are defined such that they satisfy the structural constraints $e_n d_\omega e \in \operatorname{Im} W_1^{\partial,(1,1)}$, $1/2(e^2, B) + F_A = 0$ and $(e, \Pi) = d_A\phi$. The ghost fields are denoted by $c \in \Omega^{0,2}_\partial[1]$, $\xi \in \mathfrak{X}[1](\Sigma)$, $\lambda \in \Omega^{0,0}[1]$ and $\mu \in \mathcal{C}^\infty(\Sigma)[1]$ in degree one, $c^\dagger \in \Omega^{3,2}_\partial[-1]$, $\lambda^\dagger \in \Omega^{3,4}_\partial[-1]$, $\xi^\dagger \in \Omega^{1,0}_\partial[-1] \otimes \Omega^{3,4}_\partial$ and $\mu^\dagger \in \Gamma[-1](\wedge^3 T^*\Sigma \otimes \wedge^4 \mathcal{V})$ in degree minus one, together with a fixed $e_n \in \Gamma(\mathcal{V})$, completing the image of elements $e \in \mathcal{U}_\Sigma$ to a basis of $\mathcal{V}$.

We define a symplectic form and an action functional on $\mathcal{F}$ by

$$\varpi_{\mathrm{sed}} = \int_\Sigma e\delta e\delta\omega + \delta\rho\delta A + \delta c\delta c^\dagger + \delta\lambda\delta\lambda^\dagger + \iota_{\delta\xi}\delta\xi^\dagger, + \delta\mu\delta\mu^\dagger \tag{6.45}$$

$$
\begin{aligned}
S_{sed} = \int_\Sigma & c e d_\omega e + \mu \left[d_A\rho - \frac{i}{2}(\bar{p}\phi - p\bar{\phi})\right] + \frac{1}{2}\iota_\xi e^2 F_\omega + \iota_\xi \rho F_A + \frac{1}{2}\left(\iota_\xi \bar{p} d_A\phi + \iota_\xi p d_A\bar{\phi}\right) \\
& + \iota_\xi(\omega - \omega_0) e d_\omega e + \iota_\xi(A - A_0) d_A\rho - \frac{i}{2}\iota_\xi(A - A_0)(\bar{p}\phi - p\bar{\phi}) \\
& + \lambda e_n \left[e F_\omega + e B F_A + \frac{e^2}{4}(\Pi d_A\bar{\phi} + \bar{\Pi} d_A\phi) + \frac{e^3}{3!}\left(\Lambda + \frac{1}{2}(\Pi, \bar{\Pi}) + \frac{1}{2}(B, B)\right)\right] \\
& + \frac{1}{2}[c, c]c^\dagger - \mathrm{L}^{\omega_0}_\xi c c^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{\omega_0} c^\dagger + [c, \lambda e_n]^{(a)}(\xi^\dagger_a - (\omega - \omega_0)_a c^\dagger) + [c, \lambda e_n]^{(n)}\lambda^\dagger \\
& - \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(a)}(\xi^\dagger_a - (\omega - \omega_0)_a c^\dagger) - \mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(n)}\lambda^\dagger - \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger \\
& - \mathrm{L}^{A_0}_\xi(\mu)\mu^\dagger + \frac{1}{2}\iota_\xi\iota_\xi F_{A_0}\mu^\dagger + \left[\mathrm{L}^{\omega_0}_\xi(\lambda e_n)^{(a)} - [c, \lambda e_n]^{(a)}\right](A - A_0)_a\mu^\dagger.
\end{aligned} \tag{6.46}
$$

Then the triple $(\mathcal{F}^{\mathrm{sed}}, \varpi_{\mathrm{sed}}, S_{\mathrm{sed}})$ defines a BFV structure on Σ.

Proof. As usual, we define $\varpi_{\text{sed}} = \varpi_{\text{sed},f} + \varpi_{\text{sed},g}$, with

$$\varpi_{\text{sed},f} = \int_\Sigma e\delta e\delta\omega + \delta\rho\delta A + \frac{1}{2}\left(\delta\bar{p}\delta\phi + \delta p\delta\bar{\phi}\right);$$
$$\varpi_{\text{sed},g} = \int_\Sigma \delta\lambda\delta\lambda^\dagger + \delta c\delta c^\dagger + \iota_{\delta\xi}\delta\xi^\dagger + \delta\mu\delta\mu^\dagger.$$

Furthermore, we split $S_{\text{sed}} := S_0 + S_1$, where S_0 is the part that does not contain the ghosts and S_1 is the part linear in them. We need to prove that $\{S_{\text{sed}}, S_{\text{sed}}\} = 0$. Similarly to 4.5, we define the further splitting $S_0 = S_0^0 + S_0^1$ where

$$S_0^0 = S_0^{\text{YM}};$$
$$S_0^1 = \int_\Sigma -\frac{i}{2}\mu\bar{p}\phi + \frac{1}{2}\iota_\xi\bar{p}d_A\phi - \frac{i}{2}\iota_\xi(A-A_0)\bar{p}\phi + \lambda e_n\left[\frac{e^2}{4}\bar{\Pi}d_A\phi + \frac{e^3}{4\cdot 3!}(\bar{\Pi},\Pi)\right] + \text{c.c.}\ .$$

Thanks to theorem (5.2.1) the CME reduces to proving that

$$\{S_0^1, S_1\}_f = 0. \tag{6.47}$$

First, we compute δS_0^1:

$$\delta S_0^1 = \int_\Sigma \frac{\delta\bar{p}}{2}\left[i\mu\phi - \mathrm{L}_\xi^{A_0}\phi - \lambda(e_n,\Pi)\right] + \frac{1}{2}\left[-i\mu\bar{p} - \mathrm{L}_\xi^{A_0}(\bar{p}) + d_A\left(\frac{\lambda e_n}{2}e^2\bar{\Pi}\right)\right]\delta\phi$$
$$+ \delta\mu\left(-\frac{i}{2}\bar{p}\phi\right) + \frac{1}{2}\iota_{\delta\xi}\bar{p}d_A\phi - \frac{i}{2}\iota_{\delta\xi}(A-A_0)\bar{p}\phi + \delta\lambda e_n\left[\frac{e^2}{4}\bar{\Pi}d_A\phi + \frac{e^3}{4\cdot 3!}(\bar{\Pi},\Pi)\right] + \text{c.c.}\ .$$

We notice immediately that $Q_{0,\rho}^1$, $Q_{0,\omega}^1$, $Q_{0,e}^1$ and $Q_{0,A}^1$ vanish, where Q_0^1 is defined such that $\iota_{Q_0^1}\varpi_{\text{sed}} = \delta S_0^1$. Hence

$$\{S_0^1, S_1\}_f = \iota_{Q_0^1}\iota_{Q_1}\int_\Sigma e\delta e\delta\omega + \delta\rho\delta A + \frac{1}{2}(\delta\bar{p}\delta\phi + \delta p\delta\bar{\phi}) = 0, \tag{6.48}$$

because $Q_{1,\phi}$ and $Q_{1,p}$ vanish (as well as their complex conjugates), hence proving the theorem. ✓

Appendix A

Introduction to Supergeometry

A.1 Motivation

In this section we give the basic definition of supergeometrical objects which are useful throughout the thesis. We will be dealing essentially with the equivalent of the usual tools of differential geometry with the addition of parity and possibily grading. This rather involved theory of differential geometry becomes essential when one is faced with the problem of defining the space of observables on a theory with local symmetries, if they decide to resort to cohomological methods such as BRST, where the space of observables[1] is recovered as the degree 0 cohomology of a certain operator. In this formalism it is necessary to introduce *ghost fields*, which are unphysical fictitious fields with an additional degree (the ghost degree) also defining their parity. Supergeometry is also very useful as an alternative way of studying some geometrical objects, for example it is possible to show that there is a uniform description of Poisson manifolds and Courant algebroids in terms of supermanifolds. Redefining the usual structures in this alternative setting may help describe some aspects which are hard to deal with in the original formalism. Finally, supergeometry is used in the classical description of fermions.

A.2 Graded Spaces

Before diving into the definition of supermanifold, we look at the linear case where we introduce some odd anticommuting coordinates and some grading

Definition A.2.1 (graded vector space)**.** A **$\mathbb{Z}$-graded vector space** is a collection of vector spaces $\{V_k\}_{k\in\mathbb{Z}}$. The total space is indicated with $V := \oplus_{k\in\mathbb{Z}} V_k$.

We might also define the k-**shift** of a graded vector space as $(V[k])_n := V_{k+n}$ for every $k \in \mathbb{Z}$.

Definition A.2.2 (superspace)**.** Given any graded vector space V, we might define an additional $\mathbb{Z}_2$ grading called **parity** which is simply given for any $x \in V_k$ by the $\mathbb{Z}$-grading modulo 2. We denote by $|\cdot|$ the parity of an element, hence

$$|x| = \begin{cases} 0 & k \text{ even} \\ 1 & k \text{ odd} \end{cases} . \tag{A.1}$$

Therefore we might also see $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ and $V_{\bar{1}}$ are respectively the even and odd parts of V. Any space with such splitting is called **superspace**, and it is said to have dimension $\dim(V) := (\dim(V_{\bar{0}})|\dim(V_{\bar{1}}))$

[1] In this setting we are still interested in classical obserbables, but the BRST formalism originates from QFT

Remark A.2.3. In the context of physics odd coordinates are used to consider fermionic quantities, whereas even coordinates describe bosonic ones. In particular:

- parity and degree are equivalent (modulo 2) in the description of unphysical fermions (ghosts);
- physical fermions are described by coordinates with independent parity and degree, indeed they have degree zero but they are odd.

Definition A.2.4 (morphism of superspaces)**.** A **morphism of superspaces** is a morphism of spaces preserving the grading. In particular, it is a linear map.

Definition A.2.5 (parity reversing operator)**.** Given any vector space V we define its **parity reversed space** to be ΠV, whose coordinates are now anticommuting and of parity 1.

The extension to a superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is given by $\Pi(V_{\overline{0}}) = V_{\overline{1}}$ and $\Pi(V_{\overline{0}}) = V_{\overline{1}}$.

The idea is that we can take these superspaces to be "homeomorphic" to the local patches of a supermanifold. To do so, we first need to consider their local description.

Let us consider two vector spaces U and V, then clearly $W := U \times \Pi V$ is a superspace in the sense defined above. If we let $\{x^i\}$ and $\{\theta^\mu\}$ be coordinates respectively on U and ΠV, then they have the properties that

$$x^i x^j = x^j x^i \tag{A.2}$$

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu \tag{A.3}$$

$$x^i \theta^\mu = \theta^\mu x^i. \tag{A.4}$$

Furthermore, we define the algebra of functions on ΠV as $\mathcal{C}^\infty(\Pi V) := V[\theta]/(\theta^\mu\theta^\nu + \theta^\nu\theta^\mu)$, namely it is the algebra of polynomials with coefficients in V quotiented with respect to the equivalence defined by $\theta^\mu\theta^\nu \sim -\theta^\nu\theta^\mu$. It is interesting to notice that this definition is in accordance with $\mathcal{C}^\infty(\Pi V) = \bigwedge V^*$.

It is then possible to define following algebra of functions on the superspace W as

$$\mathcal{C}^\infty(W) := \mathcal{C}^\infty(U) \otimes \bigwedge V^*. \tag{A.5}$$

We are now ready to see the definition of a supermanifold

A.3 Supermanifolds and Graded Manifolds

Definition A.3.1 (supermanifold)**.** A **supermanifold** $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to

$$(U, \mathcal{C}^\infty(U) \otimes \wedge V^*), \tag{A.6}$$

where U is an open subset of $\mathbb{R}^n$ and V is some m-dimensional real vector space. We call M **body** of the supermanifold $\mathcal{M}$, which has dimension $(n|m)$, while $\mathcal{O}_M$ is its structure sheaf.

The aforementioned isomorphism is a morphism of sheaves of superalgebras. In particular we require the parity

$$\begin{aligned} |\cdot| : \mathcal{C}^\infty(U) \otimes \wedge V^* &\longrightarrow \mathbb{Z}_2 \\ f \otimes \theta &\longmapsto |f \otimes \theta| := |\theta| = k \bmod 2 \end{aligned}$$

to be preserved. This definition extends to the global functions on $\mathcal{M}$, therefore $\mathcal{C}^\infty(\mathcal{M})$ is a $\mathbb{Z}_2$-graded algebra (a superalgebra) such that, for two homogeneous $f, g \in \mathcal{C}^\infty(\mathcal{M})$, we have $fg = (-1)^{|f||g|} gf$.

Intuitively, a supermanifold is the gluing of patches that locally look like open subsets of $\mathbb{R}^n$ together with some odd coordinates. In particular, a diffeomorphism $\phi : U \times \Pi V \to U' \times \Pi V'$ between patches is defined such that the induced morphism $\phi^* : \mathcal{C}^\infty(U' \times \Pi V') \to \mathcal{C}^\infty(U \times \Pi V)$ is a superalgebra morphism (preserving parity).

Example A.3.2 (split supermanifold). Let $E \to M$ be a rank m vector bundle over M, then we can construct a **split** $(n|m)$ supermanifold ΠE with body M and structure sheaf $\mathcal{O}_{\Pi E} := \Gamma(M, \bigwedge E^*)$.

Example A.3.3 (odd (co)tangent bundle). As a subexample of the previous one, the can consider a smooth manifold M and define (split) supermanifolds ΠTM and ΠT^*M, respectively called odd tangent and odd cotangent bundles. Their algebras of functions are given by

$$\mathcal{C}^\infty(\Pi TM) = \Gamma(\wedge^\bullet T^*M) = \Omega(M), \tag{A.7}$$

$$\mathcal{C}^\infty(\Pi T^*M) = \Gamma(\wedge^\bullet TM) = \mathfrak{X}_{mult}(M), \tag{A.8}$$

so they respectively are differential forms on M and multivector fields on M.

Theorem A.3.4 (Batchelor). *[6] Any smooth supermanifold with body M is (non-canonically) ismorphic to a split supermanifold ΠE for some vector bundle $E \to M$.*

Now we can generalize the features of a supermanifold to those of a graded manifold. In particular, if V is a graded vector space, then we define the algebra of polynomials on V to be the **graded symmetric algebra** $S(V^*)$, defined such that, for any graded vector space W, $S(W) := T(W)/(v \otimes w - (-1)^{|v||w|} w \otimes v)$, where $T(W)$ is the tensor algebra of W. We also define $V_i^* := (V_{-i})^*$.

Definition A.3.5 (graded manifold). A **graded manifold** $\mathcal{M}$ is a manifold which locally looks like $(U, \mathcal{C}^\infty(U) \otimes S(V^*))$ where V is a graded vector space. Of course transition maps will be compatible with the grading.

Remark A.3.6. In the context of field theory, when dealing with the BFV formalism, the space of fields is a (infinite-dimensional) graded manifold. The $\mathbb{Z}$-grading is called **ghost** grading and is denoted by gh.

We now formulate the equivalent of Batchelor theorem for graded supermanifolds

Proposition A.3.7. *Any smooth graded manifold is isomorphic to a graded manifold associated to a graded vector bundle, which is a collection of ordinary vector bundles $\oplus_{k \in \mathbb{Z}} E_k$ on M, where the structure sheaf is given by $\mathcal{O}_{\mathcal{M}} := \Gamma(M, S(E^*))$*

A.4 Vector Fields, Differential Forms and Cartan Calculus

Definition A.4.1 (graded vector fields)**.** A **graded vector field** X on a graded manifold $\mathcal{M}$ is a graded linear map $\mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[k]$ such that it satifies the *graded Leibniz rule*:

$$X(fg) = X(f)g + (-1)^{|f|k} f X(g). \tag{A.9}$$

The space of graded vector fields is denoted by $\mathfrak{X}(\mathcal{M})$ and is endowed with a **graded Lie bracket** $[\cdot,\cdot] : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ defined by

$$[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X, \tag{A.10}$$

satisfying the **graded Jacobi identity**

$$(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|X||Y|}[Y, [Z, X]] + (-1)^{|Y||Z|}[Z, [X, Y]] = 0. \tag{A.11}$$

Therefore $\mathfrak{X}(\mathcal{M})$ is endowed with the structure of a **graded Lie algebra**

Considering local coordinates $\{x^i, \theta^\mu\}$ of a smooth graded manifold $\mathcal{M}$, we can locally define the **graded Euler vector field** E as

$$E := \Sigma_i |x^i| x^i \frac{\partial}{\partial x^i} + \Sigma_\mu |\theta^\mu| \theta^\mu \frac{\partial}{\partial \theta^\mu}. \tag{A.12}$$

It has the property that for every homogeneous $f \in \mathcal{C}^\infty(\mathcal{M})$ of degree k, $E(f) = kf$, and for all vector fields $X \in \mathfrak{X}(\mathcal{M})_k$ of degree k, we have $[E, X] = kX$.

Definition A.4.2 (differential forms)**.** Considering again local coordinates on a graded manifold, we can simply define **differential forms** to be locally generated by the differentials of the coordinates dx^i and $d\theta^\mu$.

It is interesting to notice that the de Rham differential on $\mathcal{M}$ can actually be seen as a vector field on the odd tangent bundle $\Pi T\mathcal{M}$ with local coordinates $\{x^i, \theta^\mu, dx^i, d\theta^\mu\}$, in particular

$$d = dx^i \partial_i + d\theta^\mu \partial_\mu, \tag{A.13}$$

with the property that $[d, d] = 0$.

Definition A.4.3 (cohomological)**.** Any vector field Q of degree 1 on a graded manifold $\mathcal{M}$ such that $[Q, Q] = 0$ is called **cohomological vector field**

Lemma A.4.4. *Every cohomological vector field on a supermanifold corresponds to a differential on the graded algebra of smooth functions* $\mathcal{C}^\infty(\mathcal{M})$

Proof. It is a simple consequence of $[Q, Q] = 2Q \circ Q = 0$ and that Q is of degree 1. ✓

Then the **de Rham complex** $(\Omega(\mathcal{M}), d)$ is given by $\mathcal{C}^\infty(T[1]\mathcal{M})$ equipped with a cohomological vector field $Q : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[1]$.

Definition A.4.5 (dg-manifold)**.** A **differential graded manifold** (a.k.a. Q-manifold) is a pair $(\mathcal{M}, Q)$ where $\mathcal{M}$ is a graded manifold and Q a cohomological vector field on $\mathcal{M}$.

Example A.4.6 (Chevalley-Eilenberg). Let $\mathfrak{g}$ be a Lie algebra. Consider the graded manifold $\mathcal{M} := \mathfrak{g}[1]$. This supermanifold can also be seen as the shifted vector bundle $\mathfrak{g}[1]$ over a point (the body of $\mathcal{M}$). The algebra of functions is given by $\mathcal{C}^\infty(\mathcal{M}) = \wedge^\bullet \mathfrak{g}^*$.

We define a cohomological vector field Q on $\mathcal{M}$ to be the Chevalley-Eilenberg differential $Q := d_{CE} : \wedge^k \mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$ given by the extension to $\wedge^\bullet \mathfrak{g}^*$ of the dual of the Lie bracket $[\cdot,\cdot]^* : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$. If we let f^a_{bc} be the structure constants of the Lie algebra $\mathfrak{g}$ and $\{c^a\}$ coordinates on $\mathcal{M}$, then d_{CE} is given by

$$d_{CE} := \frac{1}{2} f^a_{bc} c^b c^c \frac{\partial}{\partial c^a}. \tag{A.14}$$

The cohomological condition $d^2_{CE} = 0$ is just equivalent to the Jacobi identity, whereas the degree 1 is easily checked.

Definition A.4.7 (Lie derivative)**.** The definition of **Lie derivative** extends to the graded (and super) case. We simply use Cartan's magic formula as a definition. With respect to any vector field $X \in \mathfrak{X}(\mathcal{M})$, the Lie derivative is defined as a derivation on $\mathcal{M}$, therefore can be regarded as an element of $\mathfrak{X}(\Pi T\mathcal{M})$

$$\mathrm{L}_X := [\iota_X, d] \in \mathfrak{X}(\Pi T\mathcal{M}). \tag{A.15}$$

It has properties $[\mathrm{L}_X, \mathrm{L}_Y] = \mathrm{L}_{[X,Y]}$ and $[\iota_X, \mathrm{L}_Y] = \pm\iota_{[X,Y]}$.

Remark A.4.8. The Lie derivative with respect to the Euler vector field E acts on homogeneous differential objects (functions, vector fields, differential forms) as the identity multiplied by the grade of the object. For all homogeneous $f \in \mathcal{C}^\infty(M)$, $X \in \mathfrak{X}(\mathcal{M})$, $\alpha \in \Omega(\mathcal{M})$, we have

$$\begin{aligned} \mathrm{L}_E(f) &= E(f) = |f|f \\ \mathrm{L}_E(X) &= [E, X] = |X|X \\ \mathrm{L}_E(\alpha) &= |\alpha|\alpha. \end{aligned} \tag{A.16}$$

Appendix B

Technical Results and Lenghty Proofs

B.1 Technical Result

In this section we present a collection of results that are useful throughout the paper, especially in the constraint analysis of the theories and in some calculations. We refer to [8] and [13] for the proofs that we leave out.

First we present a rigorous definition of the brackets $(\cdot, \cdot)$ we employed in the previous chapters.

Definition B.1.1 (internal product)**.** Let $A, B \in \Omega^{(0,1)}$ and $C, D \in \Omega^{(0,2)}$, then, expanding them in the bases $\{e_\mu\}$ and $\{e_\mu e_\nu\}$, we obtain

$$\begin{aligned}(A, B) &= g_{\mu\nu} A^\mu B^\nu, \\ (C, D) &= g_{\mu\rho} g_{\nu\sigma} C^{\mu\nu} B^{\rho\sigma},\end{aligned}$$

which is a simple consequence of the fact that $(e_\mu, e_\nu) = g_{\mu\nu}$ by definition of the vielbein.

We also notice that

$$\begin{aligned}(e, A) &= (e_\mu dx^\mu, A) = -dx^\mu g_{\mu\nu} A^\nu; \\ (e^2, C) &= -dx^\mu dx^\nu g_{\mu\rho} g_{\nu\sigma} C^{\rho\sigma}.\end{aligned}$$

Lemma B.1.2. *For all $N > 0$, define*

$$\left[\frac{N}{2}\right] := \begin{cases} \frac{N}{2} & \text{if } N \text{ even} \\ \frac{N-1}{2} & \text{if } N \text{ odd} \end{cases} . \tag{B.1}$$

Then

$$e^n = (-1)^{\left[\frac{n}{2}\right]} e_{\mu_1} \cdots e_{\mu_n} dx^{\mu_1} \cdots dx^{\mu_n}$$

.

Proof. We proceed by induction. In the case $n = 1$ we have $e^n = e_\mu dx^\mu = (-1)^{\lfloor \frac{n}{2} \rfloor} e_\mu dx^\mu$.

Assuming that the identity holds for n, we prove that it is true also for $n + 1$, in fact

$$\begin{aligned}e^{n+1} = e^n e_\mu dx^\mu &= (-1)^{\left[\frac{n}{2}\right]} e_{\mu_1} \cdots e_{\mu_n} dx^{\mu_1} \cdots dx^{\mu_n} e_\mu dx^\mu \\ &= (-1)^{\left[\frac{n}{2}\right]+n} e_{\mu_1} \cdots e_{\mu_{n+1}} dx^{\mu_1} \cdots dx^{\mu_{n+1}} \\ &= (-1)^{\left[\frac{n+1}{2}\right]} e_{\mu_1} \cdots e_{\mu_{n+1}} dx^{\mu_1} \cdots dx^{\mu_{n+1}}.\end{aligned}$$

In fact $(-1)^{\left[\frac{n+1}{2}\right]} = (-1)^{\left[\frac{n}{2}\right]+n}$:

- if $n = even$, then $(-1)^{\left[\frac{n+1}{2}\right]} = (-1)^{\left[\frac{n}{2}\right]} = (-1)^{\left[\frac{n}{2}\right]+n}$;
- if $n = odd$, then $(-1)^{\left[\frac{n+1}{2}\right]} = (-1)^{\left[\frac{n}{2}\right]+1} = (-1)^{\left[\frac{n}{2}\right]+n}$.

✓

In the Bulk

Lemma B.1.3. *Let $C, D \in \Omega^{(0,2)}$ and $A, B \in \Omega^{(0,1)}$. Then the following identities hold*

(1) $\frac{e^N}{N}(A, B) = (-1)^{|A|+|B|} e^{N-1}(e, A)B$;

(2) $\frac{e^{N-2}}{2(N-2)!}(e^2, C)D = \frac{e^N}{N!}(C, D)$.

Proof. (1) We use e_μ as a basis for $\Omega^{(0,1)}$. Then

$$\begin{aligned} e^{N-1}(e, A)B &= (-1)^{\left[\frac{N-1}{2}\right]+1} e_{\mu_1} \cdots e_{\mu_{N-1}} dx^{\mu_1} \cdots dx^{\mu_{N-1}} dx^\mu g_{\mu\nu} A^\nu B^\rho e_\rho \\ &= (-1)^{\left[\frac{N-1}{2}\right]+|A|+|B|+N+1} e_{\mu_1} \cdots e_{\mu_{N-1}} e_\rho dx^{\mu_1} \cdots dx^{\mu_{N-1}} dx^\mu g_{\mu\nu} A^\nu B^\rho \\ &= (-1)^{\left[\frac{N-1}{2}\right]+|A|+|B|+N+1} (N-1)! e_1 \cdots e_N d^N x g_{\mu\nu} A^\mu B^\nu \\ &= (-1)^{\left[\frac{N-1}{2}\right]+|A|+|B|+N+1+\left[\frac{N}{2}\right]} \frac{e^N}{N}(A, B) \\ &= (-1)^{|A|+|B|} \frac{e^N}{N}(A, B); \end{aligned}$$

(2) We now use $e_\mu e_\nu$ as a basis for $\Omega^{(0,2)}$. Then

$$\begin{aligned} e^{N-2}(e^2, C)D &= (-1)^{\left[\frac{N-2}{2}\right]+1} e_{\mu_1} \cdots e_{\mu_{N-2}} e_\rho e_\sigma dx^{\mu_1} \cdots dx^{\mu_{N-2}} dx^\mu dx^\nu g_{\mu\alpha} g_{\nu\beta} C^{\alpha\beta} D^{\rho\sigma} \\ &= (-1)^{\left[\frac{N-2}{2}\right]+1} 2(N-2)! e_1 \cdots e_N dx^1 \cdots dx^N g_{\mu\alpha} g_{\nu\beta} C^{\alpha\beta} D^{\mu\nu} \\ &= (-1)^{\left[\frac{N-2}{2}\right]+\left[\frac{N}{2}\right]+1} \frac{2(N-2)!}{N!} e^N(C, D) \\ &= (-1)^{2\left[\frac{N}{2}\right]} \frac{2(N-2)!}{N!} e^N(C, D) \\ &= \frac{2(N-2)!}{N!} e^N(C, D). \end{aligned}$$

✓

Lemma B.1.4. *Let $\varrho_n := (e^n, \cdot) : \Omega^{(0,n)} \to \Omega^{(n,0)}$ and let e be non-degenerate. Then for $N \geq 2$ we have that ϱ_n is bijective for $n = 1, 2$.*

Proof. It is a simple consequence of the fact that the metric $g_{\mu\nu}$ is invertible. ✓

Corollary B.1.5. *We then have a corollary of the two previous lemmas. Let $\alpha \in \Omega^{(1,0)}$, $\pi \in \Omega^{(0,1)}$, $\omega \in \Omega^{(2,0)}$ and $C \in \Omega^{(0,2)}$, then*

(1) $e^{N-1}\alpha B = (-1)^{|B|+1} \frac{e^N}{N} \alpha_\mu B^\mu$;

(2) $e^{N-2}\omega D = -\frac{2(N-2)!}{N!} e^N \omega_{\mu\nu} D^{\mu\nu}$.

Proof. By B.1.4 there have to exist $B \in \Omega^{(0,1)}$ and $D \in \Omega^{(0,2)}$ such that $\alpha = (e, A)$ and $\omega = (e^2, C)$[1]. In particular, this means

$$C^{\rho\sigma} = -g^{\rho\mu} g^{\sigma\nu} \omega_{\mu\nu},$$
$$A^{\nu} = (-1)^{1+|\alpha|} g^{\nu\mu} \alpha_{\mu}.$$

We then simply apply Lemma B.1.3

✓

Lemma B.1.6. *Let $W_k^{(i,j)}$ be such that $W_k^{(i,j)} : \Omega^{(i,j)} \to \Omega^{(i+k,j+k)} : \alpha \mapsto e^k \wedge \alpha$. Then the following propositions are true*

(1) $W_{N-1}^{(1,0)}$ is injective;

(2) $W_{N-2}^{(2,0)}$ is injective;

Proof. We prove the statements locally. Choosing as usual a local basis $\{e_\mu\}$ of $\mathcal{V}$, we have that

(1) $\ker(W_{N-1}^{(1,0)}) := \{\alpha \in \Omega^{(1,0)} \mid e^{N-1}\alpha = 0\}$. In particular, this means

$$e_{\mu_1} \cdots e_{\mu_{N-1}} \alpha_\nu dx^{\mu_1} \cdots dx^{\mu_{N-1}} dx^\nu \propto e_{[1} \cdots e_{N-1} \alpha_{N]} d^N x = 0 \quad \Leftrightarrow \quad \alpha_\mu = 0, \tag{B.2}$$

hence proving that $\ker(W_{N-1}^{(1,0)}) = \{0\}$;

(2) $\ker(W_{N-2}^{(2,0)}) := \{\omega \in \Omega^{(2,0)} \mid e^{N-2}\omega = 0\}$. Similarly as before, we find:

$$e_{\mu_1} \cdots e_{\mu_{N-2}} \omega_{\nu\rho} dx^{\mu_1} \cdots dx^{\mu_{N-2}} dx^\nu dx^\rho \propto e_{[1} \cdots e_{N-2} \omega_{N_1,N]} d^N x = 0 \quad \Leftrightarrow \quad \omega_{\mu\nu} = 0, \tag{B.3}$$

hence proving that $\ker(W_{N-2}^{(2,0)}) = \{0\}$.

✓

On the Boundary

We now generalize Lemma B.1.3 to the boundary. We can simply do this by setting $e_N dx^N \longrightarrow e_n$. Then it is easy to see that

$$e^N \longrightarrow N e_n e^{(N-1)}. \tag{B.4}$$

Hence we have

Lemma B.1.7. *Let $C, D \in \Omega_\partial^{(0,2)}$ and $A, B \in \Omega_\partial^{(0,1)}$. Then the follwing identities hold*

(1) $e_n \frac{e^{N-1}}{(N-1)!}(A, B) = (-1)^{|A|+|B|} \left[\frac{1}{(N-2)!} e_n e^{N-2}(e, A)B + \frac{e^{N-1}}{(N-1)!}(e_n, A)B \right];$

(2) $e_n \frac{e^{N-1}}{(N-1)!}(C, D) = \left[e_n \frac{e^{N-3}}{2(N-3)!}(e^2, C)D + \frac{e^{N-2}}{(N-2)!}(e_n e, C) \right].$

Proof. We simply impose the substitution defined in equation (B.4), noticing also that $(A, B) \longrightarrow (A, B)$ and $(C, D) \to (C, D)$.

[1] Of course $|\alpha| = |A|$ and $|\omega| = |C|$

(1)

$$\frac{e^N}{N!}(A,B) \longrightarrow e_n\frac{e^{N-1}}{(N-1)!}(A,B);$$

$$\begin{aligned}\frac{e^{N-1}}{(N-1)!}(e,A)B &\longrightarrow \frac{N}{(N-1)!}\left\{\frac{N-1}{N}e_n e^{N-2}(e,A)B + \frac{e^{N-1}}{N}(e_n,A)B\right\}\\ &\longrightarrow \frac{1}{(N-2)!}e_n e^{N-2}(e,A)B + \frac{1}{(N-1)!}e^{N-1}(e_n,A)B;\end{aligned}$$

(2)

$$\frac{e^N}{N!}(C,D) \longrightarrow e_n\frac{e^{N-1}}{(N-1)!}(C,D);$$

$$\begin{aligned}\frac{e^{N-2}}{2(N-2)!}(e^2,C)D &\longrightarrow N\left\{\frac{N-2}{N}e_n e^{N-3}(e^2,C)D + \frac{2}{N}e^{N-2}(e_n e,C)D\right\}\\ &\longrightarrow \frac{1}{2(N-3)!}e_n e^{N-3}(e^2,C)D + \frac{1}{(N-2)!}e^{N-2}(e_n e,C)D.\end{aligned}$$

✓

We recall

$$\begin{aligned}W_k^{\partial(i,j)} &: \Omega_\partial^{(i,j)} \longrightarrow \Omega_\partial^{(i+k,j+k)}\\ &: \alpha \longmapsto e^k \wedge \alpha.\end{aligned} \tag{B.5}$$

Then we have

Lemma B.1.8. *The maps $W_k^{\partial,(i,j)}$ have the following properties for $N \geq 4$:*

(1) $W_{N-3}^{\partial,(2,1)}$ is surjective;

(2) $W_{N-3}^{\partial,(1,1)}$ is injective;

(3) $W_{N-3}^{\partial,(1,2)}$ is surjective;

(4) $W_k^{\partial(0,0)}$ is injective;

(5) $\dim \mathrm{Ker} W_{N-3}^{\partial,(1,2)} = \dim \mathrm{Ker} W_{N-3}^{\partial,(2,1)}$;

(6) $W_{N-4}^{\partial,(2,1)}$ is injective. ($N \geq 5$) ;

(7) $W_{N-2}^{\partial,(1,0)}$ is injective.

Proof. The proofs of the statements (1) – (6) can be found in [8] and [13], with the exception of (4), which is easily seen since any $\phi \in \Omega^{(0,0)}$ is a function, hence it e^k just acts as a multiplication. We just need to prove 7. Considering $A \in \Omega_\partial^{(1,0)}$, then

$$e^{N-2}A = e_{\mu_1}\cdots e_{\mu_{N-2}}A_\rho dx^{\mu_1}\cdots dx^{\mu_{N-2}}dx^\rho = 0$$

is satisfied if and only if $A_\rho = 0$ for all $\rho = 1, \cdots, N$, hence showing that $A = 0$. ✓

Lemma B.1.9. *Let* $\alpha \in \Omega_\partial^{2,1}$. *Then*

$$\alpha = 0 \qquad \Longleftrightarrow \qquad \begin{cases} e^{N-3}\alpha = 0 \\ e_n e^{N-4}\alpha \in \operatorname{Im} W_{N-3}^{\partial,(1,1)} \end{cases} . \tag{B.6}$$

Lemma B.1.10. *Let* $\beta \in \Omega_\partial^{N-2,N-2}$. *If* g^∂ *is nondegenerate, there exist a unique* $v \in \operatorname{Ker} W_{N-3}^{\partial,(1,2)}$ *and a unique* $\gamma \in \Omega_\partial^{1,1}$ *such that*

$$\beta = e^{N-3}\gamma + e_n e^{N-4}[v, e].$$

Proof. The proofs of the previous two lemmas are found in [8] ✓

B.2 Lenghty Proofs of section 5.2

In this section we show explicitly equation

$$\{S_0^1, S_1^0\}_f + \{S_0^0, S_1^1\}_f + \{S_0^1, S_1^1\}_f + \{S_1^0, S_1^1\}_g + \frac{1}{2}\{S_1^1, S_1^1\}_g = 0. \tag{B.7}$$

$$\begin{aligned} \{S_0^1, S_1^0\}_f &= \iota_{Q_0^1}\iota_{Q_1^0}\varpi_f \\ &= \iota_{Q_0^1}\int_\Sigma \iota_{Q_1^0}(e\delta e\delta\omega) + \operatorname{Tr}\iota_{Q_1^0}(\delta\rho\delta A) \\ &= \int_\Sigma eQ_{1e}^0 Q_{0\omega}^1 \propto \int_\Sigma \lambda^2 = 0, \end{aligned} \tag{B.8}$$

because $Q_{1\rho}^0 = 0$, $Q_{1A}^0 = 0$ and both Q_{1e}^0 and $eQ_{0\omega}^1$ are proportional to λ.

$$\begin{aligned} &\{S_0^0, S_1^1\}_f = \\ &= \operatorname{Tr}\int_\Sigma \underline{[c,[c,\lambda e_n]^{(b)}e_b]^{(a)}(A-A_0)_a\mu^\dagger}_1 - \underline{[c,\lambda e_n]^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)}(A-A_0)_a\mu^\dagger}_2 \\ &\quad - \underline{[c,\lambda e_n]^{(b)}\partial_b\xi^c e_c^{(a)}(A-A_0)_a\mu^\dagger}_3 - \underline{[c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b]^a(A-A_0)_a\mu^\dagger}_4 \\ &\quad + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)}(A-A_0)_a\mu^\dagger}_5 + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}\partial_b\xi^c e_c^{(a)}(A-A_0)_a\mu^\dagger}_6, \end{aligned} \tag{B.9}$$

where we used $\mathrm{L}_\xi^{\omega_0}(e)_b = \mathrm{L}_\xi^{\omega_0}(e_b) + \partial_b\xi^c e_c$.

$$\begin{aligned} &\{S_0^1, S_1^1\}_f = \\ &= \operatorname{Tr}\int_\Sigma \underline{[c,\lambda e_n]^{(a)}(\mathrm{L}_\xi^{\omega_0}(A-A_0))_a\mu^\dagger}_1 - \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\mathrm{L}_\xi^{\omega_0}(A-A_0))_a\mu^\dagger}_2 \\ &\quad + \underline{[c,\lambda e_n]^{(a)}(\iota_\xi F_{A_0})_a\mu^\dagger}_3 - \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\iota_\xi F_{A_0})_a\mu^\dagger}_4 \\ &\quad - \underline{[c,\lambda e_n]^{(a)}(d_A\mu)_a\mu^\dagger}_5 + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(d_A\mu)_a\mu^\dagger}_6; \end{aligned} \tag{B.10}$$

$$\begin{aligned}
&\{S_1^0, S_1^1\}_g = \\
&= \mathrm{Tr}\int_\Sigma \Big\{ -\underline{\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(a)}}_1 + \underline{\mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(a)}}_2 + \underline{[\mathrm{L}_\xi^{\omega_0}(c),\lambda e_n]^{(a)}}_3 - \underline{\frac{1}{2}[[c,c],\lambda e_n]^{(a)}}_4 \\
&\quad - \underline{[c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n]^{(a)}}_5 + \underline{[c,[c,\lambda e_n]^{(n)}e_n]^{(a)}}_6 - \underline{\frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0},\lambda e_n]^{(a)}}_7 \Big\}(A-A_0)_a\mu^\dagger \\
&\quad - \underline{[c,\lambda e_n]^{(a)}(\iota_\xi F_{A_0})_a\mu^\dagger}_8 + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}(\iota_\xi F_{A_0})_a\mu^\dagger}_9 + \underline{\frac{1}{2}\iota_{[\xi,\xi]}\iota_\xi F_{A_0}\mu^\dagger}_{10} \\
&\quad - \underline{[c,\lambda e_n]^{(a)}d_{A_{0a}}\mu\mu^\dagger}_{11} + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}d_{A_{0a}}\mu\mu^\dagger}_{12} + \underline{\frac{1}{2}\iota_{[\xi,\xi]}d_{A_0}\mu\mu^\dagger}_{13} - \underline{\frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}(\lambda e_n))^{(a)}(A-A_0)_a\mu^\dagger}_{14} \\
&\quad + \Big\{\underline{([c,\lambda e_n]^{(b)}(d_{\omega_0}(\lambda e_n))_b)^{(a)}}_{15} - \underline{(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}(d_{\omega_0}(\lambda e_n))_b)^{(a)}}_{16}\Big\}(A-A_0)_{(a)}\mu^\dagger;
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
&\frac{1}{2}\{S_1^1, S_1^1\}_g = \\
&= \mathrm{Tr}\int_\Sigma \underline{\frac{1}{2}\mathrm{L}_\xi^{A_0}(\iota_\xi\iota_\xi F_{A_0})\mu^\dagger}_1 + \underline{[\mu,\mathrm{L}_\xi^{A_0}(\mu)]\mu^\dagger}_2 - \underline{\mathrm{L}_\xi^{A_0}\mathrm{L}_\xi^{A_0}\mu\mu^\dagger}_3 \\
&\quad \underline{\mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)})(A-A_0)_a\mu^\dagger}_4 - \underline{\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n]^{(a)})(A-A_0)_a\mu^\dagger}_5 \\
&\quad + \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}\mathrm{L}_\xi^{A_0}((A-A_0)_a)\mu^\dagger}_6 - \underline{[c,\lambda e_n]^{(a)}\mathrm{L}_\xi^{A_0}((A-A_0)_a)\mu^\dagger}_7 \\
&\quad + \underline{\frac{1}{2}[\iota_\xi\iota_\xi F_{A_0},\mu]\mu^\dagger}_8 + \underline{\frac{1}{2}[[\mu,\mu],\mu]\mu^\dagger}_9 - \underline{[\mu,\mathrm{L}_\xi^{A_0}(\mu)]\mu^\dagger}_{10} \\
&\quad \underline{\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}[(A-A_0)_a,\mu]\mu^\dagger}_{11} - \underline{[c,\lambda e_n]^{(a)}[(A-A_0)_a,\mu]\mu^\dagger}_{12}.
\end{aligned} \tag{B.12}$$

Now we check term by term that the sum is zero

- (B.9.1), (B.11.6) and (B.11.4) give

  $$\begin{aligned}
  &[c,[c,\lambda e_n]^{(b)}e_b]^{(a)} + [c,[c,\lambda e_n]^{(n)}e_n]^{(a)} - \frac{1}{2}[[c,c]\lambda e_n]^{(a)} \\
  &= [c,[c,\lambda e_n]]^{(a)} - \frac{1}{2}[[c,c]\lambda e_n]^{(a)} = 0,
  \end{aligned}$$

  because of graded Jacobi identity

- (B.9.2), (B.9.4), (B.11.1),(B.11.3), (B.11.5) and (B.12.5) sum to zero, in fact

  $$\begin{aligned}
  -\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n])^{(a)} &= -\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n + [c,\lambda e_n]^{(b)}e_b)^{(a)} \\
  &= -\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(a)} - \mathrm{L}_\xi^{\omega_0}([c,\lambda e_n])^{(a)} - [c,\lambda e_n]^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)} \\
  &= -[\mathrm{L}_\xi^{\omega_0}(c),\lambda e_n]^{(a)} + [c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b]^{(a)} + [c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n]^{(a)} \\
  \Rightarrow &-\mathrm{L}_\xi^{\omega_0}([c,\lambda e_n]^{(n)}e_n)^{(a)} - \mathrm{L}_\xi^{\omega_0}([c,\lambda e_n])^{(a)} - [c,\lambda e_n]^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)} \\
  &- [\mathrm{L}_\xi^{\omega_0}(c),\lambda e_n]^{(a)} + [c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b]^{(a)} + [c,\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n]^{(a)} = 0;
  \end{aligned}$$

- (B.9.5), (B.11.2), (B.11.7), (B.11.14), (B.12.4) sum to zero, in fact

$$\begin{aligned}
\mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n))^{(a)} &= \mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}e_b + \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}e_n)^{(a)}\\
&= \mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}) - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)} - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}\mathrm{L}_\xi^{\omega_0}(e_n)^{(a)}\\
&= \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^{\omega_0}(\lambda e_n)^{(a)} + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, \lambda e_n]^{(a)}\\
&= \frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}\lambda e_n)^{(a)} + \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, \lambda e_n]^{(a)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &\mathrm{L}_\xi^{\omega_0}(\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(a)}) - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}\mathrm{L}_\xi^{\omega_0}(e_b)^{(a)} - \mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(n)}\mathrm{L}_\xi^{\omega_0}(e_n)^{(a)}\\
&- \frac{1}{2}(\iota_{[\xi,\xi]}d_{\omega_0}\lambda e_n)^{(a)} - \frac{1}{2}[\iota_\xi\iota_\xi F_{\omega_0}, \lambda e_n]^{(a)} = 0;
\end{aligned}$$

- Now consider the following identity: $(\mathrm{L}_\xi^{(A_0)}(A-A_0))_a = \mathrm{L}_\xi^{(A_0)}(A-A_0)_a + \partial_a\xi^b(A-A_0)_b$. Then, considering the terms (B.9.3), (B.10.1) and (B.12.7) we find

$$\begin{aligned}
&-[c,\lambda e_n]^{(b)}\partial_b\xi^a + [c,\lambda e_n]^{(a)}(\mathrm{L}_\xi^{(A_0)}(A-A_0))_a - [c,\lambda e_n]^{(a)}\mathrm{L}_\xi^{A_0}(A-A_0)_a =\\
&= -[c,\lambda e_n]^{(b)}\partial_b\xi^a + [c,\lambda e_n]^{(a)}\mathrm{L}_\xi^{A_0}(A-A_0)_a - [c,\lambda e_n]^{(b)}\partial_b\xi^a - [c,\lambda e_n]^{(a)}\mathrm{L}_\xi^{A_0}(A-A_0)_a = 0;
\end{aligned}$$

- the same can be done with the terms (B.9.6), (B.10.2) and (B.12.6);
- the following pairs of terms simply cancel each other out
  - (B.10.3) and (B.11.8);
  - (B.10.4) and (B.11.9);
  - (B.12.2) and (B.12.10);
- the terms (B.11.15) and (B.11.16) vanish because they are proportional to λ^2. They are separately zero because both $\mathrm{L}_\xi^{\omega_0}(\lambda e_n)^{(b)}$ and $[c,\lambda e_n]^{(b)}$ are proportional to λ, and

$$\begin{aligned}
(d_{\omega_0}(\lambda e_n)_{(b)})^{(a)} &= \partial_b\lambda e_n^{(a)} - \lambda(d_{\omega_0}e_n)^{(a)}_{(b)}\\
&= -\lambda(d_{\omega_0}e_n)^{(a)}_{(b)};
\end{aligned}$$

- (B.12.9) vanishes because of the graded Jacobi identity;
- Considering (B.11.11) and (B.12.12) we find

$$\begin{aligned}
&-[[c,\lambda e_n]^{(a)}(A-A_0)_a,\mu]\mu^\dagger - [c,\lambda e_n]^{(a)}d_{A_{0a}}\mu\mu^\dagger =\\
&= -[c,\lambda e_n]^{(a)}d_{(A-A_0)_a}\mu\mu^\dagger = [c,\lambda e_n]^{(a)}(d_A\mu)_a\mu^\dagger
\end{aligned}$$

  , which cancels out (B.10.5);
- the same holds also for (B.10.5) (B.11.12) and (B.12.11);
- the terms (B.11.10) and (B.12.1) sum to a boundary term, in fact

$$\begin{aligned}
&\frac{1}{2}\iota_{[\xi,\xi]}\iota_\xi F_{A_0}\mu^\dagger + \frac{1}{2}\mathrm{L}_\xi^{A_0}(\iota_\xi\iota_\xi F_{A_0})\mu^\dagger =\\
&= \frac{1}{2}\iota_{[\xi,\xi]}\iota_\xi F_{A_0}\mu^\dagger - \frac{1}{2}\iota_\xi\iota_\xi F_{A_0} + \mathrm{L}_\xi^{A_0}(\mu^\dagger)\\
&= \frac{1}{2}d_{A_0}(\iota_\xi\iota_\xi F_{A_0}\iota_\xi\mu^\dagger)
\end{aligned}$$

  ;

- Finally, we are left with (B.11.13), (B.12.3) and (B.12.8), they sum up to zero, in fact

$$\begin{aligned} \mathrm{L}_\xi^{A_0}\mathrm{L}_\xi^{A_0}\mu &= \frac{1}{2}\mathrm{L}_{[\xi,\xi]}^{A_0}(\mu) + \frac{1}{2}[\iota_\xi\iota_\xi F_{A_0},\mu] \\ &= \frac{1}{2}\iota_{[\xi,\xi]}d_{A_0}\mu + \frac{1}{2}[\iota_\xi\iota_\xi F_{A_0},\mu], \end{aligned}$$

  which proves equation (B.7).

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