Explicit bounds on automorphic and canonical Green functions of Fuchsian groups

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Abstract. We study the automorphic Green function gr_Γ on quotients of the hyperbolic plane by cofinite Fuchsian groups Γ, and the canonical Green function $\mathrm{gr}_X^{\mathrm{can}}$ on the standard compactification X of such a quotient.

We use a limiting procedure, starting from the resolvent kernel, and lattice point estimates for the action of Γ on the hyperbolic plane to prove an "approximate spectral representation" for gr_Γ. Combining this with bounds on Maaß forms and Eisenstein series for Γ, we prove explicit bounds on gr_Γ. From these results on gr_Γ and new explicit bounds on the canonical $(1,1)$-form of X, we deduce explicit bounds on $\mathrm{gr}_X^{\mathrm{can}}$.

1. Introduction and statement of results

1.1. Automorphic Green functions

Let $\mathbf{H} = \{z \in \mathbf{C} \mid \Im z > 0\}$ be the hyperbolic plane, and let Γ be a cofinite Fuchsian group. The *automorphic Green function* gr_Γ for the Laplace operator Δ_Γ on $\Gamma \backslash \mathbf{H}$ is an important object in the theory of automorphic forms. The first goal of this paper is to study gr_Γ quantitatively, and in particular to obtain explicit upper and lower bounds. One result that can be stated without introducing too much notation is the following corollary of Theorem 5.1. The function $u(z, w)$ is the hyperbolic cosine of the hyperbolic distance, and $L_\delta(z, w)$, defined in (2.2) below, is a real-valued function outside the diagonal on $\mathbf{H} \times \mathbf{H}$ with a logarithmic singularity of the form $-\frac{1}{2\pi} \log|z - w|$ as $z \to w$.

Theorem 1.1. *Let Γ_0 be a cofinite Fuchsian group, let Y_0 be a compact subset of $\Gamma_0 \backslash \mathbf{H}$, and let $\delta > 1$ and $\eta > 0$ be real numbers. There exist real numbers A and B such that the following holds. Let Γ be a subgroup of finite index in Γ_0 such that all non-zero eigenvalues of $-\Delta_\Gamma$ are at least η. Then for all $z, w \in \mathbf{H}$ whose images in $\Gamma_0 \backslash \mathbf{H}$ lie in Y_0, we have*

$$A \le \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma \in \Gamma \\ u(z, \gamma w) \le \delta}} L_\delta(z, \gamma w) \le B.$$

1.2. Canonical Green functions

Every compact connected Riemann surface of positive genus has a *canonical Green function* $\mathrm{gr}_X^{\mathrm{can}}$. This is a fundamental object in the intersection theory on arithmetic surfaces developed by Arakelov [1], Faltings [8] and others, where it is used to define local intersection numbers of horizontal divisors at Archimedean places.

Let X be the standard compactification of $\Gamma \backslash \mathbf{H}$ obtained by adding the cusps. We assume that X has positive genus. The second goal of this paper is to derive explicit bounds on $\mathrm{gr}_X^{\mathrm{can}}$ from our bounds on gr_Γ. The following theorem illustrates our results. For simplicity, we only give an upper bound; see Theorem 7.1 for more precise results.

Theorem 1.2. *Let Γ be a congruence subgroup of level n of $\mathrm{SL}_2(\mathbf{Z})$ such that the compactification X of $\Gamma \backslash \mathbf{H}$ has positive genus. Then the canonical Green function $\mathrm{gr}_X^{\mathrm{can}}$ satisfies*

$$\sup_{X \times X} \mathrm{gr}_X^{\mathrm{can}} \le 1.6 \cdot 10^4 + 7.7n + 0.088n^2.$$

1.3. Overview

In Sections 2 and 3, we collect the necessary results about Fuchsian groups, making them more explicit where necessary. In Section 4, we give a construction of the automorphic Green function gr_Γ involving the resolvent kernel and use this to "sandwich" gr_Γ (with the logarithmic singularity removed) between two functions that, unlike gr_Γ itself, admit spectral representations. In Section 5, we bound these functions in a way that lends itself to explicit evaluation, obtaining results implying Theorem 1.1. We then extend our bounds on $\mathrm{gr}_\Gamma(z, w)$ to the case where Y_0 is obtained by cutting out discs around the cusps in $\Gamma_0 \backslash \mathbf{H}$ and where one or both of z and w lie in such a disc.

Let X be the compactification of $\Gamma \backslash \mathbf{H}$; we assume that X has positive genus. We give new explicit bounds on the canonical $(1, 1)$-form of X in Section 6 and on the canonical Green function $\mathrm{gr}_X^{\mathrm{can}}$ in Section 7. We apply our results to congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ in Section 8. Finally, a number of bounds on Legendre functions that we will need have been collected in an appendix.

1.4. Remarks

Our results are valid for any cofinite Fuchsian group, although we are motivated by the case of arithmetic groups, and in particular congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$. Bounds on the canonical Green functions of the modular curves $\mathrm{X}_1(n)$, i.e. in the case $\Gamma = \Gamma_1(n)$, are relevant to recent work of Edixhoven, Couveignes et al. [5] and of the author [3], where Arakelov theory is employed to obtain a polynomial-time algorithm for computing Galois representations attached to Hecke eigenforms over finite fields.

This article may be compared with earlier work of Jorgenson and Kramer on bounding canonical and automorphic Green functions of compact Riemann surfaces [14, especially Theorem 4.5]. Jorgenson and Kramer consider compact Riemann surfaces X of genus at least 2, which are of the form $X = \Gamma \backslash \mathbf{H}$ for a cofinite Fuchsian group Γ without elliptic and parabolic elements. They obtain bounds on the automorphic Green function by comparing it to the heat kernel on X. They also find an interesting expression for the canonical Green function in terms of data associated to the hyperbolic metric. The work of Jorgenson and Kramer will be generalised to cofinite Fuchsian groups with parabolic elements by A. Aryasomayajula in his forthcoming thesis [2].

Our method starts likewise with comparing gr_Γ to a kernel that can be written as a sum over Γ. However, the subsequent arguments are rather different and altogether less involved. Let us note some of the differences. First, we allow arbitrary cofinite Fuchsian groups, which is the natural setting for modular curves. Second, the procedure that we apply in § 4.1 to construct gr_Γ as a limit of a family of kernels K_a for $a \to 1$ leads to bounds that are independent of the specific family. We take K_a to be the resolvent kernel with parameter $a \to 1$, but the heat kernel with parameter $t \to \infty$ could have been used with the same result; see [3, § II.5.2]. Finally, our bounds are much easier to make explicit than those in [14]; this is illustrated in Section 8.

2. Fuchsian groups and Green functions

2.1. Hyperbolic geometry

The hyperbolic plane $\mathbf{H}$ is the unique two-dimensional, complete, connected and simply connected Riemannian manifold with constant Gaussian curvature -1. We identify $\mathbf{H}$ with the complex upper half-plane; this gives $\mathbf{H}$ a complex structure. In terms of the standard coordinate $z = x + iy$, the Riemannian metric is

$$\frac{dz\, d\bar{z}}{(\Im z)^2} = \frac{dx^2 + dy^2}{y^2},$$

and the associated volume form is

$$\mu_{\mathbf{H}} = \frac{i\, dz \wedge d\bar{z}}{2(\Im z)^2} = \frac{dx \wedge dy}{y^2}.$$

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Instead of using the geodesic distance $r(z, w)$ on $\mathbf{H}$ directly, we use the more convenient function

$$u(z, w) = \cosh r(z, w)$$

$$= 1 + \frac{|z - w|^2}{2(\Im z)(\Im w)}.$$

Let Δ denote the Laplace–Beltrami operator on $\mathbf{H}$, given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2).$$

The Green function for Δ is the unique smooth real-valued function $\mathrm{gr}_{\mathbf{H}}$ outside the diagonal on $\mathbf{H} \times \mathbf{H}$ satisfying

$$\mathrm{gr}_{\mathbf{H}}(z, w) = \frac{1}{2\pi} \log |z - w| + O(1) \quad \text{as } z \to w,$$

$$\Delta \mathrm{gr}_{\mathbf{H}}(\ , w) = \delta_w \quad \text{for all } w \in \mathbf{H},$$

$$\mathrm{gr}_{\mathbf{H}}(z, w) = O(u(z, w)^{-1}) \quad \text{as } u(z, w) \to \infty,$$

where Δ is taken with respect to the first variable. It is given by

$$\mathrm{gr}_{\mathbf{H}}(z, w) = -L(u(z, w)),$$

where

$$L(u) = \frac{1}{4\pi} \log \frac{u + 1}{u - 1}. \tag{2.1}$$

For later use, we define

$$L_\delta(z, w) = L(u(z, w)) - L(\delta) \quad \text{for } \delta > 1 \text{ and } z, w \in \mathbf{H}. \tag{2.2}$$

The group $\mathrm{SL}_2(\mathbf{R})$ acts on $\mathbf{H}$ by isometries. Under the identification of $\mathbf{H}$ with the complex upper half-plane, this action on $\mathbf{H}$ is the restriction of the action on $\mathbf{P}^1(\mathbf{C})$ by Möbius transformations. Elements of $\mathrm{SL}_2(\mathbf{R}) \setminus \{\pm 1\}$ are classified according to their fixed points in $\mathbf{P}^1(\mathbf{C})$ as elliptic (two conjugate fixed points in $\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R})$), parabolic (a unique fixed point in $\mathbf{P}^1(\mathbf{R})$), and hyperbolic (two distinct fixed points in $\mathbf{P}^1(\mathbf{R})$).

2.2. Fuchsian groups

A *Fuchsian group* is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$. A Fuchsian group Γ is *cofinite* if $\Gamma \backslash \mathbf{H}$ has finite volume with respect to the measure induced by $\mu_{\mathbf{H}}$. We will exclusively consider cofinite Fuchsian groups, and for such a group Γ we write

$$\mathrm{vol}_\Gamma = \int_{\Gamma \backslash \mathbf{H}} \mu_{\mathbf{H}}.$$

Let Γ be a cofinite Fuchsian group. We define the quotient $\Gamma \backslash \mathbf{H}$ in a stack-like way. The terms *smooth function on $\Gamma \backslash \mathbf{H}$* and *smooth Γ-invariant functions on $\mathbf{H}$* will have the same meaning. Furthermore, the value of an integral over $\Gamma \backslash \mathbf{H}$ is $1/\#(\Gamma \cap \{\pm 1\})$ times the integral over the corresponding Riemann surface. This implies that if f is a Γ-invariant function on $\mathbf{H}$ and Γ' is a subgroup of finite index in Γ, then

$$\int_{\Gamma' \backslash \mathbf{H}} f \mu_{\mathbf{H}} = (\Gamma : \Gamma') \cdot \int_{\Gamma \backslash \mathbf{H}} f \mu_{\mathbf{H}}.$$

Furthermore, this definition justifies the method of "unfolding": if f is a smooth function with compact support on $\mathbf{H}$ and F is the function on $\Gamma \backslash \mathbf{H}$ defined by

$$F(z) = \sum_{\gamma \in \Gamma} f(\gamma z),$$

then

$$\int_{\Gamma \backslash \mathbf{H}} F \mu_{\mathbf{H}} = \int_{\mathbf{H}} f \mu_{\mathbf{H}}.$$

Let X be the standard compactification of $\Gamma \backslash \mathbf{H}$. It is useful to keep in mind that the space of cusp forms of weight 2 for Γ, the space of holomorphic differentials on X, and the space of holomorphic differentials on the coarse moduli space of X are all isomorphic. We write g_X for the dimension of these spaces, and call it the genus of X. To avoid any subtleties, the reader can restrict himself to groups containing neither -1 nor any elliptic elements and with all cusps regular, such as $\Gamma_1(n)$ for $n \geq 5$.

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2.3. Cusps

Let Γ be a cofinite Fuchsian group. The cusps of Γ correspond to the conjugacy classes of non-trivial maximal parabolic subgroups in Γ. Every such subgroup has a unique fixed point in $\mathbf{P}^1(\mathbf{R})$. Let $\mathfrak{c}$ be a cusp of Γ. We choose a representative $\Gamma_{\mathfrak{c}}$ of the corresponding conjugacy class and an element $\sigma_{\mathfrak{c}} \in \mathrm{SL}_2(\mathbf{R})$ such that $\sigma_{\mathfrak{c}}\infty \in \mathbf{P}^1(\mathbf{R})$ is the unique fixed point of $\Gamma_{\mathfrak{c}}$ in $\mathbf{P}^1(\mathbf{R})$ and such that

$$\{\pm 1\}\sigma_{\mathfrak{c}}^{-1}\Gamma_{\mathfrak{c}}\sigma_{\mathfrak{c}} = \{\pm 1\}\left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbf{Z} \right\}.$$

Such a $\sigma_{\mathfrak{c}}$ exists and is unique up to multiplication from the right by a matrix of the form $\pm\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x \in \mathbf{R}$; see Iwaniec [12, § 2.2]. We define

$$q_{\mathfrak{c}} \colon \mathbf{H} \to \mathbf{C}$$
$$z \mapsto \exp(2\pi i \sigma_{\mathfrak{c}}^{-1} z)$$

and

$$y_{\mathfrak{c}} \colon \mathbf{H} \to (0, \infty)$$
$$z \mapsto \Im \sigma_{\mathfrak{c}}^{-1} z = -\frac{\log |q_{\mathfrak{c}}(z)|}{2\pi}.$$

For all $\gamma \in \Gamma$, we write

$$C_{\mathfrak{c}}(\gamma) = |c| \quad \text{if } \sigma_{\mathfrak{c}}^{-1}\gamma\sigma_{\mathfrak{c}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$\Gamma_{\mathfrak{c}} = \{\gamma \in \Gamma \mid C_{\mathfrak{c}}(\gamma) = 0\}.$$

It is known that the set $\{C_{\mathfrak{c}}(\gamma) \mid \gamma \in \Gamma, \gamma \notin \Gamma_{\mathfrak{c}}\}$ is bounded from below by a positive number, and that if ϵ is a real number satisfying the inequality

$$0 < \epsilon \leq \min_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma), \tag{2.3}$$

then for all $z \in \mathbf{H}$ and $\gamma \in \Gamma$ one has the implication

$$y_{\mathfrak{c}}(z) > 1/\epsilon \text{ and } y_{\mathfrak{c}}(\gamma z) > 1/\epsilon \implies \gamma \in \Gamma_{\mathfrak{c}}.$$

For any ϵ satisfying (2.3), the image of the strip

$$\{x + iy \mid 0 \leq x < 1 \text{ and } y > 1/\epsilon\} \subset \mathbf{H}$$

under the map

$$\mathbf{H} \xrightarrow{\sigma_{\mathfrak{c}}} \mathbf{H} \longrightarrow \Gamma\backslash\mathbf{H}$$

is an open disc $D_{\mathfrak{c}}(\epsilon)$ around $\mathfrak{c}$, and the map $q_{\mathfrak{c}}$ induces a chart on $\Gamma\backslash\mathbf{H}$ identifying $D_{\mathfrak{c}}(\epsilon)$ with the punctured disc $\{z \in \mathbf{C} \mid 0 < |z| < \exp(-2\pi/\epsilon)\}$. A compactification of $\Gamma\backslash\mathbf{H}$ can be obtained by adding a point for every cusp $\mathfrak{c}$ in such a way that $q_{\mathfrak{c}}$ extends to a chart with image equal to the disc $\{z \in \mathbf{C} \mid |z| < \exp(-2\pi/\epsilon)\}$. Let $\bar{D}_{\mathfrak{c}}(\epsilon)$ denote the compactification of $D_{\mathfrak{c}}(\epsilon)$ obtained by adding the boundary $\partial \bar{D}_{\mathfrak{c}}(\epsilon)$ in $\Gamma\backslash\mathbf{H}$ and the cusp $\mathfrak{c}$.

Remark. Let us fix a point $w \in \mathbf{H}$ and write Γ_w for the stabiliser of w in Γ. The behaviour of $\mathrm{gr}_{\Gamma}(z, w)$ as $z \to w$ is

$$\mathrm{gr}_{\Gamma}(z, w) = \frac{\#\Gamma_w}{2\pi} \log |z - w| \text{ as } z \to w.$$

Furthermore, the behaviour of $\mathrm{gr}_{\Gamma}(z, w)$ as z moves toward a cusp $\mathfrak{c}$ of Γ is

$$\mathrm{gr}_{\Gamma}(z, w) = \frac{1}{\mathrm{vol}_{\Gamma}} \log y_{\mathfrak{c}}(z) + O(1) \text{ as } y_{\mathfrak{c}}(z) \to \infty.$$

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2.4. The automorphic Green function

Let Γ be a cofinite Fuchsian group. The restriction of the Laplace operator Δ to the space of smooth and bounded functions on $\Gamma\backslash\mathbf{H}$ can be extended to an (unbounded, densely defined) self-adjoint operator on the Hilbert space $L^2(\Gamma\backslash\mathbf{H})$, which we denote by Δ_Γ.

The operator Δ_Γ is invertible on the orthogonal complement of the constant functions in the following sense: there exists a unique bounded self-adjoint operator G_Γ on $L^2(\Gamma\backslash\mathbf{H}, \mu_\mathbf{H})$ such that for all smooth and bounded functions f on $\Gamma\backslash\mathbf{H}$ the function $G_\Gamma f$ satisfies

$$\Delta_\Gamma G_\Gamma f = f - \frac{1}{\mathrm{vol}_\Gamma}\int_{\Gamma\backslash\mathbf{H}} f\mu_\mathbf{H} \quad \text{and} \quad \int_{\Gamma\backslash\mathbf{H}} G_\Gamma f\mu_\mathbf{H} = 0.$$

There exists a unique function gr_Γ on $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$ that satisfies $\mathrm{gr}_\Gamma(z,w) = \mathrm{gr}_\Gamma(w,z)$, is smooth except for logarithmic singularities at points of the form $(z, \gamma z)$, and has the property that if f is a smooth and bounded function on $\Gamma\backslash\mathbf{H}$, then the function $G_\Gamma f$ is given by

$$G_\Gamma f(z) = \int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w) f(w)\mu_\mathbf{H}(w).$$

The function gr_Γ is called the *automorphic Green function* of the Fuchsian group Γ.

2.5. The canonical $(1,1)$-form

Let X be a compact connected Riemann surface of genus $g_X \geq 1$. The $\mathbf{C}$-vector space $\Omega^1(X)$ of global holomorphic differentials on X has dimension g_X and is equipped with the inner product

$$\langle \alpha, \beta \rangle = \frac{i}{2}\int_X \alpha \wedge \bar{\beta}.$$

The *canonical $(1,1)$-form on X* is

$$\mu_X^{\mathrm{can}} = \frac{i}{2g_X}\sum_{\alpha\in B} \alpha \wedge \bar{\alpha},$$

where B is any orthonormal basis of $\Omega^1(X)$ with respect to $\langle \ , \ \rangle$. The form μ_X^{can} is independent of the choice of B.

Let us now assume that X is (the coarse moduli space associated to) the compactification of $\Gamma\backslash\mathbf{H}$ with Γ a cofinite Fuchsian group. We define a smooth and bounded function F_Γ on $\Gamma\backslash\mathbf{H}$ by

$$F_\Gamma(z) = \sum_{f\in B} (\Im z)^2 |f(z)|^2,$$

where B is any orthonormal basis for the space of holomorphic cusp forms of weight 2 for Γ. The $(1,1)$-forms μ_X^{can} and $\mu_\mathbf{H}$ are related by

$$\mu_X^{\mathrm{can}} = \frac{1}{g_X} F_\Gamma \mu_\mathbf{H}. \tag{2.4}$$

2.6. The canonical Green function of a Riemann surface

Let X be a compact, connected Riemann surface of positive genus. Let $*$ denote the star operator on smooth 1-forms, given with respect to any local holomorphic coordinate $z = x + iy$ by

$$*dx = dy, \quad *dy = -dx.$$

If we identify X locally with the hyperbolic plane, the operator $d*d$ sending functions to $(1,1)$-forms is related to the Laplace operator Δ as follows: if f is any smooth function on X, then

$$d*df = \Delta f \cdot \mu_\mathbf{H}.$$

For every smooth $(1,1)$-form α on X, there exists a unique smooth function h_α on X such that

$$d*dh_\alpha = \alpha - \left(\int_X \alpha\right)\mu_X^{\mathrm{can}} \quad \text{and} \quad \int_X h_\alpha \mu_X^{\mathrm{can}} = 0.$$

There exists a unique function $\mathrm{gr}_X^{\mathrm{can}}$ on $X \times X$ that satisfies $\mathrm{gr}_X^{\mathrm{can}}(z,w) = \mathrm{gr}_X^{\mathrm{can}}(w,z)$, is smooth except for a logarithmic singularity along the diagonal, and has the property that if α is a smooth $(1,1)$-form on X, then the function h_α is given by

$$h_\alpha(z) = \int_{w\in X} \mathrm{gr}_X^{\mathrm{can}}(z,w)\alpha(w).$$

The function $\mathrm{gr}_X^{\mathrm{can}}$ is called the *canonical Green function* of X.

2.7. Comparison of automorphic and canonical Green functions

There is a standard way to relate the automorphic and canonical Green functions, which we will use to find explicit bounds on the canonical Green function. Let Γ be a cofinite Fuchsian group, and let X be the compactification of $\Gamma\backslash\mathbf{H}$. We define a function $h_\Gamma\colon\Gamma\backslash\mathbf{H}\to\mathbf{R}$ by

$$
\begin{aligned}
h_\Gamma(z) &= \int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w)\mu_X^{\mathrm{can}}(w) \\
&= \frac{1}{g_X}\int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w)F_\Gamma(w)\mu_{\mathbf{H}}(w).
\end{aligned}
\tag{2.5}
$$

By the defining properties of gr_Γ, the function h_Γ satisfies

$$
\begin{aligned}
\Delta h_\Gamma &= \frac{1}{g_X}F_\Gamma - \frac{1}{g_X\,\mathrm{vol}_\Gamma}\int_{\Gamma\backslash\mathbf{H}}F_\Gamma\mu_{\mathbf{H}} \\
&= \frac{1}{g_X}F_\Gamma - \frac{1}{\mathrm{vol}_\Gamma}.
\end{aligned}
$$

This implies that the canonical Green function $\mathrm{gr}_X^{\mathrm{can}}$ can be expressed as

$$
\mathrm{gr}_X^{\mathrm{can}}(z,w) = \mathrm{gr}_\Gamma(z,w) - h_\Gamma(z) - h_\Gamma(w) + \int_{\Gamma\backslash\mathbf{H}}h_\Gamma\mu_X^{\mathrm{can}}.
\tag{2.6}
$$

3. Tools

3.1. The Selberg–Harish-Chandra transform

Let P_ν denote the Legendre function of the first kind of degree ν; see for example Iwaniec [12]equation 1.43 or Erdélyi et al. [6, §3.6.1].

Let $\theta\colon[1,\infty)\to\mathbf{R}$ be a smooth function with compact support. The *Selberg–Harish-Chandra transform*, or *Mehler–Fock transform*, of θ is defined by

$$
h_\theta(s) = 2\pi\int_1^\infty \theta(u)P_{s-1}(u)du,
\tag{3.1}
$$

see for example Iwaniec [12, equation 1.62′]. The function θ can be recovered from h_θ.

Let $f\colon\mathbf{H}\to\mathbf{C}$ be an eigenfunction of $-\Delta$ with eigenvalue $\lambda = s(1-s)$. Then we have

$$
-\Delta f = s(1-s)f \implies \int_{w\in\mathbf{H}}\theta(u(z,w))f(w)\mu_{\mathbf{H}}(w) = h_\theta(s)f(z).
\tag{3.2}
$$

In particular, taking $f = 1$, we see that

$$
h_\theta(0) = h_\theta(1) = 2\pi\int_1^\infty \theta(u)du.
\tag{3.3}
$$

The identity (3.2) holds more generally than just for smooth functions θ with compact support; see Selberg [19, pages 60–61]. It will be enough for us to state a slightly weaker, but more convenient sufficient condition (cf. Selberg [19, page 72] or Iwaniec [12, equation 1.63]). Let $\epsilon > 0$ and $\beta > 1$, and let h be a holomorphic function on the strip $\{s\in\mathbf{C}\mid -\epsilon < \Re s < 1+\epsilon\}$ such that $h(s) = h(1-s)$ and such that $s\mapsto |h(s)||s(1-s)|^\beta$ is bounded on this strip. Then h is the Selberg–Harish-Chandra transform of a suitable function θ, and (3.2) holds for the pair (θ,h).

More generally, let k be a real number, and let $\Delta_k = y^2(\partial_x^2 + \partial_y^2) - iky\partial_x$ denote the Laplace operator of weight k on $\mathbf{H}$. Let $\theta\colon[1,\infty)\to\mathbf{R}$ be a piecewise smooth function with compact support. We define

$$
\theta^{(k)}(z,w) = \left(\frac{w-\bar z}{z-\bar w}\right)^{k/2}\theta(u(z,w)).
$$

6

Let $P_{s,k}$ be the generalisation of the Legendre function $P_{s-1}(u)$ given by Fay [9, §1] (note that our definition of weight is twice that of [9]):

$$P_{s,k} = \left(\frac{2}{u+1}\right)^s F\left(s - \frac{k}{2}, s + \frac{k}{2}; 1; \frac{u-1}{u+1}\right). \tag{3.4}$$

We define the *Selberg–Harish-Chandra transform* of weight k of the function θ as

$$h_\theta^{(k)}(s) = 2\pi \int_1^\infty \theta(u) P_{s,k}(u) du. \tag{3.5}$$

If f is an eigenfunction of $-\Delta_k$ with eigenvalue $s(1-s)$, then we have

$$\int_{w\in\mathbf{H}} \theta^{(k)}(z,w) f(w) dw = h_\theta^{(k)}(s) f(z); \tag{3.6}$$

see Fay [9, Theorem 1.5].

3.2. Automorphic forms

For simplicity, we take $k \in \{0, 2\}$ from now on. We write

$$\nu^{(k)}(\gamma, z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{k/2} = \frac{(cz+d)^k}{|cz+d|^k} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}) \text{ and } z \in \mathbf{H}.$$

We recall that an *automorphic form (of Maaß)* of weight k for Γ is a smooth function $f \colon \mathbf{H} \to \mathbf{C}$ with the following properties:

(1) the function f satisfies the transformation formula

$$f(\gamma z) = \nu^{(k)}(\gamma, z) f(z) \quad \text{for all } \gamma \in \Gamma \text{ and } z \in \mathbf{H};$$

(2) for every cusp $\mathfrak{c}$ of Γ, there exists $\kappa \in \mathbf{R}$ such that $|f(z)| = O(y_{\mathfrak{c}}(z)^\kappa)$ as $y_{\mathfrak{c}}(z) \to \infty$.

A *cusp form* of weight k for Γ is a function f satisfying (1) and the following strengthening of (2):

(2′) for every cusp $\mathfrak{c}$ of Γ there exists $\epsilon > 0$ such that $|f(z)| = O(\exp(-\epsilon y_{\mathfrak{c}} z))$ as $y_{\mathfrak{c}}(z) \to \infty$.

Let $L^2(\Gamma \backslash \mathbf{H}, k)$ denote the Hilbert space of square-integrable automorphic forms of weight k for Γ, equipped with the Petersson inner product.

Let $\theta \colon [1, \infty) \to \mathbf{R}$ be a smooth function with compact support. Then we have

$$\theta^{(k)}(\gamma z, \gamma w) = \frac{\nu^{(k)}(\gamma, z)}{\nu^{(k)}(\gamma, w)} \theta^{(k)}(z, w) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbf{R}) \text{ and } z, w \in \mathbf{H}.$$

Let Γ be a cofinite Fuchsian group. We define

$$K_{\Gamma,\theta}^{(k)}(z,w) = \sum_{\gamma\in\Gamma} \nu^{(k)}(\gamma, w) \theta^{(k)}(z, \gamma w). \tag{3.7}$$

This function satisfies

$$K_{\Gamma,\theta}^{(k)}(w,z) = \overline{K_{\Gamma,\theta}^{(k)}(z,w)}$$

and, for all $\gamma \in \Gamma$,

$$K_{\Gamma,\theta}^{(k)}(\gamma z, w) = \nu^{(k)}(\gamma, z) K_{\Gamma,\theta}^{(k)}(z, w),$$

$$K_{\Gamma,\theta}^{(k)}(z, \gamma w) = \nu^{(k)}(\gamma, w)^{-1} K_{\Gamma,\theta}^{(k)}(z, w).$$

Now (3.6) implies that if f is an automorphic form of weight k for Γ satisfying $-\Delta_k f = s(1-s)f$, then

$$\int_{w\in\Gamma\backslash\mathbf{H}} K_{\Gamma,\theta}^{(k)}(z,w) f(w) \mu_{\mathbf{H}}(w) = h_\theta^{(k)}(s) f(z). \tag{3.8}$$

3.3. Spectral theory of the Laplace operator for Fuchsian groups

Let Γ be a cofinite Fuchsian group. The spectrum of $-\Delta_\Gamma$ on $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$ consists of a discrete part and a continuous part.

The discrete spectrum consists of eigenvalues of $-\Delta_\Gamma$ and is of the form $\{\lambda_j\}_{j=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

Let $\{\phi_j\}_{j=0}^\infty$ be a corresponding orthonormal set of eigenfunctions. For each $j \ge 0$, we define $s_j \in \mathbf{C}$ by

$$\lambda_j = s_j(1 - s_j),$$

with $s_j \in [1/2, 1]$ if $\lambda_j \le 1/4$. For $\lambda_j > 1/4$, the s_j are only determined up to $s_j \leftrightarrow 1 - s_j$.

The continuous part of the spectrum of $-\Delta_\Gamma$ is the interval $[1/4, \infty)$ with multiplicity equal to the number of cusps of Γ. The continuous spectrum does not correspond to eigenfunctions, but to "wave packets" constructed from the non-holomorphic Eisenstein series introduced by Maaß [16]. These series are defined as follows: for every cusp $\mathfrak{c}$ of Γ the series

$$E_\mathfrak{c}(z, s) = \sum_{\gamma \in \Gamma_\mathfrak{c} \backslash \Gamma} (y_\mathfrak{c}(\gamma z))^s \quad (z \in \mathbf{H}, s \in \mathbf{C} \text{ with } \Re s > 1)$$

converges uniformly on sets of the form $K \times \{s \in \mathbf{C} \mid \Re s \ge \delta\}$ with K a compact subset of $\mathbf{H}$ and $\delta > 1$. In particular, $E_\mathfrak{c}(z, s)$ is a holomorphic function of s for $\Re s > 1$. The Eisenstein–Maaß series admit a meromorphic continuation and functional equation (Selberg [20]; cf. Faddeev [7, §4], Hejhal [10, §VI.11 or Appendix F] or Iwaniec [12, Chapter 6]). The function $z \mapsto E_\mathfrak{c}(z, s)$, for s not a pole, satisfies the differential equation

$$-\Delta_\Gamma E_\mathfrak{c}(\ , s) = s(1 - s) E_\mathfrak{c}(\ , s).$$

For $s \in \mathbf{C}$ with $\Re s = 1/2$, the Eisenstein–Maaß series $z \mapsto E_\mathfrak{c}(z, s)$ are integrable, but not square-integrable, on $\Gamma\backslash\mathbf{H}$. In contrast, the "wave packets" mentioned above are square-integrable.

In the remainder of the article, we will often consider integrals over the line $\Re s = 1/2$. For this we need an orientation; we fix one by requiring that the map $t \mapsto 1/2 + it$ from $\mathbf{R}$ with the usual orientation to the line $\Re s = 1/2$ preserves orientations.

It is known (see Iwaniec [12, Theorems 4.7 and 7.3]; cf. Faddeev [7, Theorem 4.1]) that every smooth and bounded function $f\colon \Gamma\backslash\mathbf{H} \to \mathbf{C}$ has the spectral representation

$$f(z) = \sum_{j=0}^\infty b_j \phi_j(z) + \sum_\mathfrak{c} \frac{1}{4\pi i} \int_{\Re s = 1/2} b_\mathfrak{c}(s)\, E_\mathfrak{c}(z, s)ds, \tag{3.9}$$

where $\mathfrak{c}$ runs over the cusps of Γ and the coefficients b_j and $b_\mathfrak{c}(s)$ are given by

$$b_j = \int_{\Gamma\backslash\mathbf{H}} f\bar{\phi}_j \mu_\mathbf{H} \quad \text{and} \quad b_\mathfrak{c}(s) = \int_{\Gamma\backslash\mathbf{H}} f\bar{E}_\mathfrak{c}(\ , s)\mu_\mathbf{H}.$$

The right-hand side of (3.9) converges to f in the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$. If in addition the smooth function $\Delta f\colon \Gamma\backslash\mathbf{H} \to \mathbf{C}$ is bounded, the convergence is uniform on compact subsets of $\mathbf{H}$.

With regard to these spectral representations, the effect of the operator G_Γ from the introduction is as follows: if f has the spectral representation (3.9), then $G_\Gamma f$ has the corresponding spectral representation

$$G_\Gamma f(z) = -\sum_{j=1}^\infty \frac{b_j}{\lambda_j}\phi_j(z) - \sum_\mathfrak{c} \frac{1}{4\pi i} \int_{\Re s = 1/2} \frac{b_\mathfrak{c}(s)}{s(1 - s)} E_\mathfrak{c}(z, s)ds. \tag{3.10}$$

(Note the absence of the eigenvalue $\lambda_0 = 0$.)

There is an analogous result (see Iwaniec [12, Theorem 7.4]) for functions on $\mathbf{H} \times \mathbf{H}$ of the form $\sum_{\gamma \in \Gamma} \theta(u(z, \gamma w))$, where $\theta \colon [1, \infty) \to \mathbf{R}$ is a function whose Selberg–Harish-Chandra transform h_θ exists and satisfies the conditions of § 3.1. In this situation, the function

$$K_\theta \colon \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{R}$$
$$(z, w) \longmapsto \sum_{\gamma \in \Gamma} \theta(u(z, \gamma w))$$

is Γ-invariant with respect to both variables and admits the spectral representation

$$K_\theta(z, w) = \sum_{j=0}^{\infty} h_\theta(s_j) \phi_j(z) \bar{\phi}_j(w) + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} h_\theta(s) E_{\mathfrak{c}}(z, s) \bar{E}_{\mathfrak{c}}(w, s) ds, \qquad (3.11)$$

where the expression on the right-hand side converges uniformly to K_θ for (z, w) in compact subsets of $\Gamma \backslash \mathbf{H} \times \Gamma \backslash \mathbf{H}$, and also with respect to the L^2-norm in the variable w, uniformly for z in compact subsets of $\Gamma \backslash \mathbf{H}$.

3.4. A point counting function

We fix a real number $U \geq 1$, and define

$$\theta_U \colon [1, \infty) \longrightarrow \mathbf{R}$$
$$u \longmapsto \begin{cases} 1 & \text{if } u \leq U; \\ 0 & \text{if } u > U. \end{cases} \qquad (3.12)$$

From (3.1) and the formula for $\int_1^z P_\nu(w) dw$ found in Erdélyi et al. [6, § 3.6.1, equation 8], we see that the Selberg–Harish-Chandra transform of θ_U is

$$h_U(s) = 2\pi \sqrt{U^2 - 1} \, P_{s-1}^{-1}(U). \qquad (3.13)$$

Here P_ν^μ is the Legendre function of the first kind of degree ν and order μ; see [6, § 3.2].

Now let Γ be a cofinite Fuchsian group. We introduce the following point counting function. For any two points z, w in $\mathbf{H}$ and any $U \geq 1$, we denote by $N_\Gamma(z, w, U)$ the number of translates of w by elements of Γ lying in a disc around z of radius r given by $\cosh(r) = U$, i.e.

$$N_\Gamma(z, w, U) = \#\{\gamma \in \Gamma \mid u(z, \gamma w) \leq U\}$$
$$= \sum_{\gamma \in \Gamma} \theta_U(u(z, \gamma w)). \qquad (3.14)$$

This is Γ-invariant in z and w separately.

Lemma 3.1. *Let $U \in [1, 3]$, and let $s \in \mathbf{C}$ be such that $s(1 - s)(U - 1) \in [0, 1/2]$. Then $h_U(s)$ is a real number satisfying*

$$(4\pi - 8)(U - 1) \leq h_U(s) \leq 8(U - 1).$$

Proof. This follows from (3.13) and Lemma A.1. $\qquad \square$

3.5. Bounds on eigenfunctions

The convergence of the spectral representation (3.11) can be deduced from suitable bounds on the function

$$\Phi_\Gamma \colon \mathbf{H} \times [0, \infty) \longrightarrow [0, \infty)$$
$$(z, \lambda) \longmapsto \sum_{j \colon \lambda_j \leq \lambda} |\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\substack{\Re s = 1/2 \\ s(1-s) \leq \lambda}} |E_{\mathfrak{c}}(z, s)|^2 ds. \qquad (3.15)$$

We will prove a bound on Φ_Γ which holds uniformly for all subgroups Γ of finite index in a given cofinite Fuchsian group Γ_0. This will give a similar uniformity in Section 5.

Lemma 3.2. *Let Γ be a cofinite Fuchsian group. Then the function $\Phi_\Gamma(z, \lambda)$ satisfies*

$$\Phi_\Gamma(z, \lambda) \leq \frac{\pi}{(2\pi - 4)^2} N_\Gamma(z, z, 17)\lambda \quad \text{for all } z \in \mathbf{H} \text{ and all } \lambda \geq 1/4.$$

Proof. Let $z \in \mathbf{H}$ and $\lambda \geq 1/4$. We put

$$U = 1 + \frac{1}{2\lambda} \in (1, 3].$$

From Bessel's inequality one can deduce (see Iwaniec [12, § 7.2]) that

$$\sum_{j:\, \lambda_j \leq \lambda} |h_U(s_j)\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\substack{\Re s = 1/2 \\ s(1-s) \leq \lambda}} \left| h_U(s) E_{\mathfrak{c}}(z, s) \right|^2 ds \leq \int_{w \in \Gamma \backslash \mathbf{H}} N_\Gamma(z, w, U)^2 \mu_{\mathbf{H}}(w).$$

From the definition (3.15) of Φ_Γ and the bound $h_U(s) \geq (2\pi - 4)/\lambda$ given by Lemma 3.1, we deduce

$$\Phi_\Gamma(z, \lambda) \leq \frac{\lambda^2}{(2\pi - 4)^2} \int_{w \in \Gamma \backslash \mathbf{H}} N_\Gamma(z, w, U)^2 \mu_{\mathbf{H}}(w).$$

We rewrite the integral on the right-hand side by partial unfolding as follows [12, page 109]:

$$\int_{w \in \Gamma \backslash \mathbf{H}} N_\Gamma(z, w, U)^2 \mu_{\mathbf{H}}(w) = \sum_{\gamma, \gamma' \in \Gamma} \int_{w \in \Gamma \backslash \mathbf{H}} \theta_U(z, \gamma' w)\theta_U(\gamma z, \gamma' w)\mu_{\mathbf{H}}(w)$$

$$= \sum_{\gamma \in \Gamma} \int_{w \in \mathbf{H}} \theta_U(z, w)\theta_U(\gamma z, w)\mu_{\mathbf{H}}(w).$$

The last integral can be interpreted as the area of the intersection of the discs of radius r around the points z and γz of $\mathbf{H}$, where $\cosh r = U$. By the triangle inequality for the hyperbolic distance, this intersection is empty unless

$$u(z, \gamma z) \leq \cosh(2r) = 2U^2 - 1;$$

furthermore, the area of this intersection is at most $2\pi(U - 1) = \pi/\lambda$. From this we deduce that

$$\int_{w \in \Gamma \backslash \mathbf{H}} N_\Gamma(z, w, U)^2 \mu_{\mathbf{H}}(w) \leq \frac{\pi}{\lambda} N_\Gamma(z, z, 2U^2 - 1)$$

Since $2U^2 - 1 \leq 17$, this proves the lemma. $\qquad\square$

3.6. The hyperbolic lattice point problem

Let Γ be a cofinite Fuchsian group. The hyperbolic lattice point problem for Γ is the following question: what is the asymptotic behaviour of the point counting function $N_\Gamma(z, w, U)$ from (3.14) as $U \to \infty$? This question has been studied intensively, for example by Delsarte [4], Huber [11, Satz B], Patterson [18, Theorem 2] and Selberg, using spectral theory on $\Gamma \backslash \mathbf{H}$; see Iwaniec [12, Chapter 12]. We consider functions

$$\theta_U^+, \theta_U^- : [1, \infty) \to \mathbf{R}$$

of the form

$$\theta_U^+(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq U, \\ \frac{V - u}{V - U} & \text{if } U \leq u \leq V, \\ 0 & \text{if } V \leq u, \end{cases}$$

$$\theta_U^-(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq T, \\ \frac{U - u}{U - T} & \text{if } T \leq u \leq U, \\ 0 & \text{if } U \leq u \end{cases}$$

for certain T, V, depending on U, with $1 \leq T < U < V$; these will be chosen below.
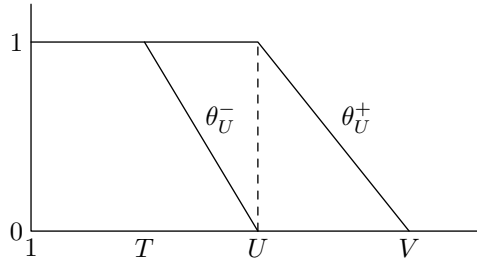

Figure 1: The functions θ_U^+ and θ_U^-.

By (3.1), the Selberg–Harish-Chandra transforms of $\theta_U^\pm$ are

$$h_U^+(s) = 2\pi \int_1^V P_{s-1}(u) \frac{V-u}{V-U} du - 2\pi \int_1^V P_{s-1}(u) \frac{U-u}{V-U} du,$$

$$h_U^-(s) = 2\pi \int_1^U P_{s-1}(u) \frac{U-u}{U-T} du - 2\pi \int_1^T P_{s-1}(u) \frac{T-u}{U-T} du.$$

Integrating by parts and applying the integral relation between the Legendre functions P_ν and P_ν^{-2} given in Erdélyi et al. [6, §3.6.1, equation 8], we get

$$h_U^+(s) = 2\pi \frac{(V^2-1)P_{s-1}^{-2}(V) - (U^2-1)P_{s-1}^{-2}(U)}{V-U},$$

$$h_U^-(s) = 2\pi \frac{(U^2-1)P_{s-1}^{-2}(U) - (T^2-1)P_{s-1}^{-2}(T)}{U-T}. \tag{3.16}$$

In particular, it follows from (3.3) or the formula $P_0^{-2}(u) = (u-1)/(2u+2)$ that

$$h_U^+(1) = 2\pi(U-1) + \pi(V-U) \quad \text{and} \quad h_U^-(1) = 2\pi(U-1) - \pi(U-T). \tag{3.17}$$

We define functions K_U^+ and K_U^- on $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$ by the following sums, which are finite because the functions $\theta_U^\pm$ have compact support:

$$K_U^\pm(z,w) = \sum_{\gamma\in\Gamma} \theta_U^\pm(u(z,\gamma w)).$$

Our choice of $\theta_U^\pm$ implies the inequalities

$$K_U^-(z,w) \leq N_\Gamma(z,w,U) \leq K_U^+(z,w) \quad \text{for all } z,w \in \mathbf{H} \text{ and } U > 1. \tag{3.18}$$

The functions $h_U^\pm$ satisfy the conditions of §3.1, so the functions $K_U^\pm$ have spectral representations

$$K_U^\pm(z,w) = \sum_{j=0}^\infty h_U^\pm(s_j)\phi_j(z)\bar\phi_j(w) + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s=1/2} h_U^\pm(s) E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)\,ds. \tag{3.19}$$

We now explain how to choose T and V as functions of U such that (3.19) gives good estimates for $N_\Gamma(z,w,U)$ as $U \to \infty$. Let $\delta \geq 1$ be given. We fix parameters α^+, α^-, β^+ and β^- satisfying

$$\alpha^\pm \in (0,1/2), \quad \beta^\pm > 0, \quad \beta^- \leq \frac{\delta^{1+\alpha^-}}{\delta+1}. \tag{3.20}$$

We choose T and V as functions of U as follows:

$$T(U) = U - \beta^- U^{-1-\alpha^-}(U^2-1), \qquad V(U) = U + \beta^+ U^{-1-\alpha^+}(U^2-1). \tag{3.21}$$

The last inequality in (3.20) ensures that if $U \geq \delta$, then $T(U) \geq 1$.

For later use, we will keep the parameters $\alpha^\pm$ and $\beta^\pm$ variable for greater flexibility. To obtain the best known error bound in the hyperbolic lattice point problem, the right choice is $\alpha^\pm = 1/3$, so that

$$V - U \sim \beta^+ U^{2/3} \quad \text{and} \quad U - T \sim \beta^- U^{2/3} \quad \text{as } U \to \infty.$$

This choice leads to the estimate

$$N_\Gamma(z,w,U) = \sum_{j:\, 2/3 < s_j \leq 1} 2^{s_j}\sqrt{\pi}\frac{\Gamma\left(s_j - \frac{1}{2}\right)}{\Gamma(s_j+1)}\phi_j(z)\bar\phi_j(w)U^{s_j} + O(U^{2/3}) \quad \text{as } U \to \infty, \tag{3.22}$$

with an implied constant depending on Γ and the points z and w.

The dominant term in (3.19) as $U \to \infty$ comes from the eigenvalue $\lambda_0 = 0$, corresponding to $s_0 = 1$. Since $|\phi_0|^2$ is the constant function $1/\mathrm{vol}_\Gamma$, the number of lattice points inside a disc of radius r, where $\cosh r = U$, is asymptotically equivalent to the area $2\pi(U-1)$ of this disc divided by the area of a fundamental domain for the action of Γ, as one would intuitively expect.

4. An approximate spectral representation of the automorphic Green function

Let Γ be a cofinite Fuchsian group. The automorphic Green function gr_Γ *formally* has the spectral representation

$$\mathrm{gr}_\Gamma(z,w) \overset{?}{=} -\sum_{j=1}^{\infty} \frac{1}{\lambda_j} \phi_j(z)\bar{\phi}_j(w) - \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} \frac{1}{s(1-s)} E_{\mathfrak{c}}(z,s)\bar{E}_{\mathfrak{c}}(w,s)ds.$$

The problem is that this expansion does not converge. Neither should one be tempted to write gr_Γ by "averaging" $\mathrm{gr}_{\mathbf{H}}$ as a (likewise divergent) sum

$$\mathrm{gr}_\Gamma(z,w) \overset{?}{=} \sum_{\gamma \in \Gamma} \mathrm{gr}_{\mathbf{H}}(z, \gamma w).$$

However, both of these divergent expressions have at least some value as guiding ideas for what follows. In fact, we will bound $\mathrm{gr}_\Gamma(z,w)$ by means of certain functions $R_{\Gamma,\delta}^{\pm}(z,w)$, defined in (4.5) below, that reflect the above formal spectral representation of gr_Γ.

4.1. A construction of gr_Γ using the resolvent kernel

We will give a construction of the automorphic Green function gr_Γ using the family of auxiliary functions

$$g_a \colon (1, \infty) \longrightarrow [0, \infty)$$
$$u \longmapsto \frac{1}{2\pi} Q_{a-1}(u)$$

for $a \geq 1$, where Q_ν is the Legendre function of the second kind of degree ν; see Erdélyi et al. [6, §3.6.1]. By [6, §3.6.2, equation 20], we have

$$Q_0(u) = \frac{1}{2} \log \frac{u+1}{u-1},$$

which shows that g_1 equals the function L from (2.1). By (3.1) and [6, §3.12, equation 4], the Selberg–Harish-Chandra transform of g_a is

$$\begin{aligned} h_a(s) &= \int_1^{\infty} P_{s-1}(u)Q_{a-1}(u)du \\ &= \frac{1}{(a-s)(a-1+s)} \\ &= \frac{1}{s(1-s) + a(a-1)}. \end{aligned} \tag{4.1}$$

For all $a > 1$, the sum $\sum_{\gamma \in \Gamma} g_a(u(z, \gamma w))$ converges uniformly on compact subsets of $\mathbf{H} \times \mathbf{H}$ not containing any points of the form $(z, \gamma z)$ and defines a continuous function that is square-integrable in each variable; see Fay [9, Theorem 1.5]. We can therefore define

$$\begin{aligned} K_a^\Gamma \colon \{(z,w) \in \mathbf{H} \times \mathbf{H} \mid z \notin \Gamma w\} &\longrightarrow \mathbf{R} \\ (z,w) &\longmapsto \sum_{\gamma \in \Gamma} g_a(u(z, \gamma w)) - c_a, \end{aligned} \tag{4.2}$$

where

$$c_a = \frac{2\pi}{\mathrm{vol}_\Gamma} \int_1^{\infty} g_a(u)du = \frac{1}{\mathrm{vol}_\Gamma} h_a(1) = \frac{1}{\mathrm{vol}_\Gamma \, a(a-1)}.$$

The constant c_a is such that the integral of K_a^Γ over $\Gamma \backslash \mathbf{H}$ with respect to each of the variables vanishes. Up to this constant, K_a^Γ is the resolvent kernel with parameter a.

The resolvent kernel admits a meromorphic continuation in the variable a, and the constant term in the Laurent expansion at $a = 1$ equals $-\mathrm{gr}_\Gamma$. The following proposition makes this precise.

Proposition 4.1. *The family of functions* $\{K_a^\Gamma\}_{a>1}$ *converges uniformly to* $-\mathrm{gr}_\Gamma$ *on compact subsets of* $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$, *and also with respect to the* L^2-*norm in the variable* w, *uniformly for* z *in compact subsets of* $\Gamma\backslash\mathbf{H}$.

Proof. This is presumably well known (cf. Fay [9, Theorem 2.3] or Iwaniec [12, Theorem 7.5]), but lacking a reference for this precise result, we sketch a proof.

For all $a, b > 1$, one shows using (4.1) that $h_a(s) - h_b(s)$ satisfies the conditions of §3.1, so that the function

$$(K_a^\Gamma - K_b^\Gamma)(z, w) = \sum_{\gamma \in \Gamma} \big(g_a(u(z, \gamma w)) - g_b(u(z, \gamma w)) \big) - c_a + c_b$$

has the spectral representation (without the eigenvalue $\lambda_0 = 0$, because of the definition of c_a)

$$
\begin{aligned}
(K_a^\Gamma - K_b^\Gamma)(z, w) = &\sum_{j=1}^\infty (h_a(s_j) - h_b(s_j))\phi_j(z)\bar\phi_j(w) \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} (h_a(s) - h_b(s)) E_{\mathfrak{c}}(z, s) \bar{E}_{\mathfrak{c}}(w, s)\,ds.
\end{aligned}
\tag{4.3}
$$

We claim that $\{K_a^\Gamma - K_b^\Gamma\}_{a,b>1}$ converges to 0 in the desired sense as $a, b \searrow 1$. In particular, $\{K_a^\Gamma\}_{a>1}$ converges to a symmetric continuous function outside the diagonal on $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$ that is square-integrable with respect to each variable separately. We fix $\sigma \in (0, 1/2)$ such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\sigma(1-\sigma), \infty)$.

Again using (4.1), one finds real numbers $C_{a,b,\sigma}$, with $C_{a,b,\sigma} \to 0$ as $a, b \searrow 1$, such that the following holds. We have

$$
\begin{aligned}
\big|K_a^\Gamma - K_b^\Gamma\big|(z, w) \leq &\sum_{j=1}^\infty |h_a(s_j) - h_b(s_j)| \cdot |\phi_j(z)\bar\phi_j(w)| \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} |h_a(s) - h_b(s)| \cdot \big|E_{\mathfrak{c}}(z, s)\bar{E}_{\mathfrak{c}}(w, s)\big|\,ds \\
\leq\ &C_{a,b,\sigma}\bigg(\sum_{j=1}^\infty \big(s_j(1 - s_j)\big)^{-2}|\phi_j(z)\bar\phi_j(w)| \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} \big(s(1 - s)\big)^{-2}\big|E_{\mathfrak{c}}(z, s)\bar{E}_{\mathfrak{c}}(w, s)\big|\,ds \bigg).
\end{aligned}
$$

By the Cauchy–Schwarz inequality and Lemma 3.2, the right-hand side converges to 0 uniformly on compact subsets of $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$. Furthermore, by (4.3) and Plancherel's theorem, we have

$$
\begin{aligned}
\int_{w \in \Gamma\backslash\mathbf{H}} \big|K_a^\Gamma - K_b^\Gamma\big|^2(z, w)\mu_{\mathbf{H}}(w) = &\sum_{j=1}^\infty \big|h_a(s_j) - h_b(s_j)\big|^2 |\phi_j(z)|^2 \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} \big|h_a(s) - h_b(s)\big|^2 \big|E_{\mathfrak{c}}(z, s)\big|^2 ds \\
\leq\ &C_{a,b,\sigma}^2 \bigg(\sum_{j=1}^\infty \big(s_j(1 - s_j)\big)^{-4} |\phi_j(z)|^2 \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} \big(s(1 - s)\big)^{-4}\big|E_{\mathfrak{c}}(z, s)\big|^2 ds \bigg).
\end{aligned}
$$

By Lemma 3.2, the last factor is bounded on compact subsets of $\Gamma\backslash\mathbf{H}$. This implies that the right-hand side converges to 0 uniformly on compact subsets of $\Gamma\backslash\mathbf{H}$ as $a, b \searrow 1$, and hence $\{K_a^\Gamma - K_b^\Gamma\}_{a,b>1}$ converges to 0 with respect to the L^2-norm in the variable w, uniformly for z in compact subsets of $\Gamma\backslash\mathbf{H}$.

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The fundamental property (3.2) of the Selberg–Harish-Chandra transform implies that if f is a smooth and bounded function on $\Gamma\backslash\mathbf{H}$, with spectral representation (3.9), then

$$\int_{w\in\Gamma\backslash\mathbf{H}} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w) = \sum_{j=1}^\infty b_j h_a(s_j)\phi_j(z) + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} b_{\mathfrak{c}}(s)h_a(s)E_{\mathfrak{c}}(z,s)ds.$$

Taking the limit, using the L²-convergence that we just proved and applying (3.10), we get

$$\int_{w\in\Gamma\backslash\mathbf{H}} \lim_{a\searrow 1} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w) = \lim_{a\searrow 1}\int_{w\in\Gamma\backslash\mathbf{H}} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w)$$

$$= \sum_{j=1}^\infty \frac{b_j}{s_j(1-s_j)}\phi_j(z) + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}\frac{b_{\mathfrak{c}}(s)}{s(1-s)}E_{\mathfrak{c}}(z,s)ds$$

$$= -G_\Gamma f(z)$$

$$= -\int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w).$$

Since the set of smooth and bounded functions is dense in $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$, this proves that the limit of the convergent family of functions $\{K_a^\Gamma\}_{a>1}$ equals $-\mathrm{gr}_\Gamma$. $\qquad\square$

4.2. An approximate spectral representation

We now exploit the estimates for the hyperbolic lattice point problem given in §3.6. We choose parameters $\alpha^\pm$ and $\beta^\pm$ satisfying (3.20). Using these, we define functions $T(U)$, $V(U)$, $\theta_U^\pm(u)$, $h_U^\pm(s)$ and $K_U^\pm(z,w)$ as in §3.6. Furthermore, we fix a real number $\delta > 1$. In the following, we will treat elements $\gamma\in\Gamma$ differently depending on whethere $u(z,\gamma w) \le \delta$ or $u(z,\gamma w) > \delta$.

We define

$$I_\delta^\pm(s) = \frac{1}{2\pi}\int_\delta^\infty \frac{h_U^\pm(s)}{U^2-1}dU \quad \text{for } 0 < \Re s < 1, \tag{4.4}$$

$$R_{\Gamma,\delta}^\pm(z,w) = \sum_{j=1}^\infty I_\delta^\pm(s_j)\phi_j(z)\bar\phi_j(w) + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2} I_\delta^\pm(s)E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)ds, \tag{4.5}$$

$$q_{\Gamma,\delta}^+ = \frac{1}{\mathrm{vol}_\Gamma}\left(\frac{\beta^+}{2\alpha^+\delta^{\alpha^+}} - \log\frac{\delta+1}{2}\right), \quad q_{\Gamma,\delta}^- = -\frac{1}{\mathrm{vol}_\Gamma}\left(\frac{\beta^-}{2\alpha^-\delta^{\alpha^-}} + \log\frac{\delta+1}{2}\right). \tag{4.6}$$

The intuition behind the following theorem is that although the automorphic Green function gr_Γ does not admit a spectral representation, it can be bounded (after removing the logarithmic singularity) by functions that do admit spectral representations. The terms $q_{\Gamma,\delta}^\pm$ below correspond to the eigenvalue 0, while the terms $R_{\Gamma,\delta}^\pm(z,w)$ correspond to the non-zero part of the spectrum.

Theorem 4.2. *Let Γ be a cofinite Fuchsian group. For all $\delta > 1$ and for every choice of the parameters $\alpha^\pm$ and $\beta^\pm$ satisfying (3.20), the automorphic Green function of Γ satisfies the inequalities*

$$-q_{\Gamma,\delta}^+ - R_{\Gamma,\delta}^+(z,w) \le \mathrm{gr}_\Gamma(z,w) + \sum_{\substack{\gamma\in\Gamma \\ u(z,\gamma w)\le\delta}} L_\delta(z,\gamma w) \le -q_{\Gamma,\delta}^- - R_{\Gamma,\delta}^-(z,w).$$

Proof. For any $U \ge \delta$, the inequality (3.18) implies that the number of elements $\gamma \in \Gamma$ with $\delta < u(z,\gamma w) \le U$ can be bounded as

$$A(U) \le \#\{\gamma\in\Gamma \mid \delta < u(z,\gamma w) \le U\} \le B(U), \tag{4.7}$$

where the functions $A, B\colon [\delta,\infty) \to \mathbf{R}$ are defined by

$$A(U) = K_U^-(z,w) - N_\Gamma(z,w,\delta) \quad \text{and} \quad B(U) = K_U^+(z,w) - N_\Gamma(z,w,\delta).$$

The functions A and B are continuous and increasing. By the estimates from §3.6, they are bounded linearly in U as $U \to \infty$, with an implied constant depending on the group Γ, the points z and w and the functions T and V.

Let $\{h_a\}_{a>1}$, $\{g_a\}_{a>1}$ and $\{K_a^\Gamma\}_{a>1}$ be as in §4.1. For all $a > 1$, applying partial summation and (4.7) gives

$$-\int_\delta^\infty g_a'(U)A(U)dU \leq \sum_{\substack{\gamma \in \Gamma \\ u(z,\gamma w) > \delta}} g_a(u(z,\gamma w)) \leq -\int_\delta^\infty g_a'(U)B(U)dU.$$

Using the definition (4.2) of K_a^Γ, we deduce the upper bound

$$K_a^\Gamma(z,w) \leq \sum_{\substack{\gamma \in \Gamma \\ u(z,\gamma w) \leq \delta}} g_a(u(z,\gamma w)) - \int_\delta^\infty g_a'(U)B(U)dU - \frac{2\pi}{\mathrm{vol}_\Gamma}\int_1^\infty g_a(u)du.$$

The definition of B implies

$$\int_\delta^\infty g_a'(U)B(U)dU = \int_\delta^\infty g_a'(U)K_U^+(z,w)dU - N_\Gamma(z,w,\delta)\int_\delta^\infty g_a'(U)dU$$

$$= \int_\delta^\infty g_a'(U)\Big(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Big)dU + \frac{2\pi}{\mathrm{vol}_\Gamma}\int_\delta^\infty g_a'(U)(U-1)dU$$

$$+ N_\Gamma(z,w,\delta)g_a(\delta).$$

Using integration by parts, we rewrite the second integral in the last expression as follows:

$$\int_\delta^\infty g_a'(U)(U-1)dU = \int_1^\infty g_a'(U)(U-1)dU - \int_1^\delta g_a'(U)(U-1)dU$$

$$= -\int_1^\infty g_a(U)dU - \int_1^\delta g_a'(U)(U-1)dU.$$

We can now rewrite our upper bound for $K_a^\Gamma(z,w)$ as

$$K_a^\Gamma(z,w) \leq \sum_{\substack{\gamma \in \Gamma \\ u(z,\gamma w) \leq \delta}} \big(g_a(u(z,\gamma w)) - g_a(\delta)\big) - \int_\delta^\infty g_a'(U)\Big(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Big)dU$$

$$+ \frac{2\pi}{\mathrm{vol}_\Gamma}\int_1^\delta g_a'(U)(U-1)dU.$$

Lemma A.2 implies

$$\frac{1}{2\pi}\Big(\frac{2}{u+1}\Big)^{a-1}\frac{1}{u^2-1} \leq g_a'(u) \leq 0,$$

and equality holds for $a = 1$. By the dominated convergence theorem, we may take the limit $a \searrow 1$ inside the integrals. Together with Proposition 4.1, this leads to

$$\mathrm{gr}_\Gamma(z,w) + \sum_{\substack{\gamma \in \Gamma \\ u(z,\gamma w) \leq \delta}} L_\delta(z,\gamma w) \geq -\frac{1}{2\pi}\int_\delta^\infty \Big(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Big)\frac{dU}{U^2-1} + \frac{1}{\mathrm{vol}_\Gamma}\log\frac{\delta+1}{2}.$$

In the integral, we insert the spectral representation (3.19) of K_U^+, the formula (3.17) for $h_U^+(1)$ and the fact that $|\phi_0|^2 = 1/\mathrm{vol}_\Gamma$. We then interchange the resulting sums and integrals with the integral over U; this is permitted because the double sums and integrals converge absolutely, as one deduces from Lemma 3.2 and (3.22). This yields

$$\frac{1}{2\pi}\int_\delta^\infty \Big(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Big)\frac{dU}{U^2-1} = R_{\Gamma,\delta}^+(z,w) + \frac{1}{2\,\mathrm{vol}_\Gamma}\int_\delta^\infty \frac{V-U}{U^2-1}dU.$$

Finally, we note that

$$\int_\delta^\infty \frac{V-U}{U^2-1}dU = \beta^+\int_\delta^\infty U^{-1-\alpha^+}dU$$

$$= \frac{\beta^+}{\alpha^+\delta^{\alpha^+}}.$$

This proves the lower bound of the theorem. The proof of the upper bound is similar. $\qquad\square$

Remark. The only inequality responsible for the fact that the inequalities in Theorem 4.2 are not equalities is (4.7).

5. Bounds on the automorphic Green function

5.1. Bounds on $h_U^\pm(s)$ and $I_\delta^\pm(s)$

We keep the notation of § 4.2. In addition, we choose real numbers $\sigma^\pm$ such that

$$0 < \alpha^+ < \sigma^+ < 1/2 \quad \text{and} \quad 0 < \alpha^- < \sigma^- < 1/2.$$

Given a real number $\sigma < 1/2$, we consider the strip

$$S_\sigma = \{s \in \mathbf{C} \mid \sigma \le \Re s \le 1 - \sigma\}. \tag{5.1}$$

Let $s \in S_{\sigma^+}$, and let $p_{\sigma^+}(u)$ be the elementary function defined by (A.6) below. From (3.16), Corollary A.6 and (3.21), we obtain

$$
\begin{aligned}
|h_U^+(s)| &\le 2\pi \frac{(V^2 - 1)\big|P_{s-1}^{-2}(V)\big| + (U^2 - 1)\big|P_{s-1}^{-2}(U)\big|}{V - U} \\
&\le 2\pi \big|s(1-s)\big|^{-5/4} \frac{p_{\sigma^+}(V) + p_{\sigma^+}(U)}{V - U} \\
&= 2\pi \big|s(1-s)\big|^{-5/4} \frac{\big(p_{\sigma^+}(V) + p_{\sigma^+}(U)\big)U^{1+\alpha^+}}{\beta^+(U^2 - 1)}.
\end{aligned}
$$

Similarly, for $s \in S_{\sigma^-}$,

$$|h_U^-(s)| \le 2\pi \big|s(1-s)\big|^{-5/4} \frac{\big(p_{\sigma^-}(U) + p_{\sigma^-}(T)\big)U^{1+\alpha^-}}{\beta^-(U^2 - 1)}.$$

Substituting this in the definition (4.4) of I, we obtain

$$|I_\delta^+(s)| \le D_\delta^+ \big|s(1-s)\big|^{-5/4} \quad \text{and} \quad |I_\delta^-(s)| \le D_\delta^- \big|s(1-s)\big|^{-5/4}, \tag{5.2}$$

where

$$
\begin{aligned}
D_\delta^+ &= \frac{1}{\beta^+} \int_\delta^\infty \frac{\big(p_{\sigma^+}(V) + p_{\sigma^+}(U)\big)U^{1+\alpha^+}}{(U^2 - 1)^2} \, dU, \\
D_\delta^- &= \frac{1}{\beta^-} \int_\delta^\infty \frac{\big(p_{\sigma^-}(U) + p_{\sigma^-}(T)\big)U^{1+\alpha^-}}{(U^2 - 1)^2} \, dU.
\end{aligned}
\tag{5.3}
$$

5.2. Bounds on gr_Γ

Theorem 5.1. *Let Γ be a cofinite Fuchsian group. Let $\delta > 1$ and $\eta \in (0, 1/4]$ be real numbers such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\eta, \infty)$. Let σ^+, σ^-, α^+, α^-, β^+, β^- be real numbers satisfying (3.20) and the inequalities*

$$0 < \alpha^+ < \sigma^+ < 1/2, \quad 0 < \alpha^- < \sigma^- < 1/2 \quad \text{and} \quad \sigma^\pm(1 - \sigma^\pm) \le \eta.$$

Then the automorphic Green function gr_Γ satisfies the inequalities

$$A(z, w) \le \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma \in \Gamma \\ u(z, \gamma w) \le \delta}} L_\delta(z, \gamma w) \le B(z, w) \quad \text{for all } z, w \in \mathbf{H},$$

where

$$
\begin{aligned}
A(z, w) &= -q_{\Gamma, \delta}^+ - D_\delta^+ \frac{\pi}{(2\pi - 4)^2} \left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2} \right) \frac{N_\Gamma(z, z, 17) + N_\Gamma(w, w, 17)}{2}, \\
B(z, w) &= -q_{\Gamma, \delta}^- + D_\delta^- \frac{\pi}{(2\pi - 4)^2} \left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2} \right) \frac{N_\Gamma(z, z, 17) + N_\Gamma(w, w, 17)}{2}.
\end{aligned}
$$

Proof. In view of Theorem 4.2, we have to bound the absolute values of the functions $R_{\Gamma,\delta}^{\pm}(z,w)$ from (4.5). Applying the triangle inequality and the Cauchy–Schwarz inequality, we see that

$$\left|R_{\Gamma,\delta}^{\pm}(z,w)\right| \leq \frac{S^{\pm}(z) + S^{\pm}(w)}{2},$$

where S^{+} and S^{-} are defined by

$$S^{\pm}(z) = \sum_{j=1}^{\infty} |I_{\delta}^{\pm}(s_j)||\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} |I_{\delta}^{\pm}(s)| |E_{\mathfrak{c}}(z,s)|^2 ds.$$

Let $\Phi_{\Gamma}(z,\lambda)$ be as in (3.15). Applying (5.2), we obtain (with $\partial\Phi_{\Gamma}/\partial\lambda$ taken in a distributional sense)

$$\begin{aligned}
S^{\pm}(z)/D_{\delta}^{\pm} &\leq \sum_{j=1}^{\infty} \lambda_j^{-5/4}|\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} (s(1-s))^{-5/4} |E_{\mathfrak{c}}(z,s)|^2 ds \\
&\leq \eta^{-5/4} \sum_{j:\,\lambda_j \leq 1/4} |\phi_j(z)|^2 + \int_{1/4}^{\infty} \lambda^{-5/4} \frac{\partial\Phi_{\Gamma}}{\partial\lambda}(z,\lambda) d\lambda \\
&= \eta^{-5/4}\Phi_{\Gamma}(z,1/4) + \left[\lambda^{-5/4}\Phi_{\Gamma}(z,\lambda)\right]_{\lambda=1/4}^{\infty} + \frac{5}{4} \int_{1/4}^{\infty} \lambda^{-9/4}\Phi_{\Gamma}(z,\lambda) d\lambda \\
&= \left(\eta^{-5/4} - 2^{5/2}\right)\Phi_{\Gamma}(z,1/4) + \frac{5}{4} \int_{1/4}^{\infty} \lambda^{-9/4}\Phi_{\Gamma}(z,\lambda) d\lambda.
\end{aligned}$$

The bound on $\Phi_{\Gamma}(z,\lambda)$ given by Lemma 3.2 implies

$$\begin{aligned}
S^{\pm}(z) &\leq D_{\delta}^{\pm} \frac{\pi}{(2\pi-4)^2} N_{\Gamma}(z,z,17)\left(\left(\eta^{-5/4} - 2^{5/2}\right) \cdot \frac{1}{4} + \frac{5}{4} \int_{1/4}^{\infty} \lambda^{-9/4}\lambda\, d\lambda\right) \\
&= D_{\delta}^{\pm} \frac{\pi}{(2\pi-4)^2} N_{\Gamma}(z,z,17)\left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2}\right).
\end{aligned}$$

This proves the theorem. $\qquad\square$

We emphasise that the choice of the parameters $\delta, \eta, \sigma^{\pm}, \alpha^{\pm}, \beta^{\pm}$ (satisfying the conditions of the theorem) only has a quantitative influence on the bounds. In principle, the same values can be taken simultaneously for all groups satisfying the condition that η imposes on the spectrum. The optimal choice depends on the behaviour of the function $N_{\Gamma}(z,z,17)$; see Section 8.

Proof of Theorem 1.1. Let the notation be as in the theorem; we may assume $\eta \leq 1/4$. We apply Theorem 5.1 to Γ, with parameters $\sigma^{\pm}, \alpha^{\pm}$ and $\beta^{\pm}$ depending only on η and not on Γ. It is clear that the factor $1/\mathrm{vol}_{\Gamma}$ occurring in the definition (4.6) is bounded by $1/\mathrm{vol}_{\Gamma_0}$, and that $N_{\Gamma}(z,z,17)$ is bounded by $N_{\Gamma_0}(z,z,17)$. It remains to remark that $N_{\Gamma_0}(z,z,17)$ is bounded on Y_0. $\qquad\square$

The bounds given by Theorem 5.1 are easy to make explicit. First, the real numbers $D_{\delta}^{\pm}$ from (5.3) can be bounded in elementary ways or approximated by numerical integration. Second, a straightforward computation shows that for $z = x + iy \in \mathbf{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$, we have

$$u(z,\gamma z) = \frac{1}{2}\left((a-cx)^2 + \left(\frac{b+(a-d)x - cx^2}{y}\right)^2 + (cy)^2 + (d+cx)^2\right). \qquad (5.4)$$

This can be used for concrete groups Γ to find an upper bound on $N_{\Gamma}(z,z,U)$ for $U > 1$, as we will show in Section 8.

5.3. Extension to neighbourhoods of the cusps

The bounds given by Theorem 5.1 do not have the right asymptotic behaviour when z or w are near the cusps of Γ. This means that we have to do some more work to find suitable bounds on the automorphic Green function $\mathrm{gr}_\Gamma(z,w)$ in this case.

Let D and $\bar{D}$ denote the open and closed unit discs in $\mathbf{C}$, respectively. We recall that the Poisson kernel on D is defined by

$$P(\zeta) = \frac{1 - |\zeta|^2}{|1 - \zeta|^2}$$

$$= 1 + \sum_{n=1}^{\infty} \zeta^n + \sum_{n=1}^{\infty} \bar{\zeta}^n.$$

We will use the notation

$$\tilde{P}(t,\zeta) = P(\exp(2\pi i t)\zeta).$$

Lemma 5.2. *The Poisson kernel satisfies*

$$\int_0^1 \tilde{P}(a,\zeta)\tilde{P}(-a,\eta)da = P(\zeta\eta) \quad \text{for all } \zeta,\eta \in D \tag{5.5}$$

and

$$\tilde{P}(t,\zeta) = \frac{d}{dt}\left(t + \frac{1}{2\pi i}\left(\log\frac{1 - \exp(-2\pi i t)\bar{\zeta}}{1 - \exp(2\pi i t)\zeta} - \log\frac{1 - \bar{\zeta}}{1 - \zeta}\right)\right) \quad \text{for all } \zeta \in D. \tag{5.6}$$

Proof. The first claim can be verified in several ways, for example using the residue theorem, Fourier series, or the fact that the Poisson kernel solves the Laplace equation with Dirichlet boundary conditions. The second claim is straightforward to check. $\qquad\square$

Let $\mathrm{gr}_{\bar{D}}$ denote the Green function for the Laplace operator on $\bar{D}$; this is an integral kernel for the Poisson equation $\Delta f = g$ with boundary condition $f = 0$ on $\partial\bar{D}$. It is given explicitly by

$$\mathrm{gr}_{\bar{D}}(\zeta,\eta) = \frac{1}{2\pi}\log\left|\frac{\zeta - \eta}{1 - \zeta\bar{\eta}}\right| \quad \text{for all } \zeta,\eta \in D \text{ with } \zeta \neq \eta.$$

For all $\xi \in D$ and $t \in \mathbf{R}$, we write

$$\lambda(\xi,t) = \frac{1}{2\pi i}\bigl(\log(1 - \exp(-2\pi i t)\xi) - \log(1 - \exp(2\pi i t)\xi)\bigr).$$

Lemma 5.3. *The function $\lambda(\xi,t)$ satisfies*

$$\left|\int_0^t \frac{\lambda(\xi,y)}{y}dy + \frac{1}{2}\log(1-\xi)\right| \leq \frac{1}{12t} \quad \text{for all } t > 0.$$

Proof. We expand $\lambda(\xi,t)$ for $\xi \in D$ in a Fourier series:

$$\lambda(\xi,t) = \frac{1}{2\pi i}\left(\sum_{n=1}^{\infty}\frac{\xi^n \exp(2\pi i n t)}{n} - \sum_{n=1}^{\infty}\frac{\xi^n \exp(-2\pi i n t)}{n}\right)$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n \sin(2\pi n t)}{n}.$$

This implies

$$\int_0^t \frac{\lambda(\xi,y)}{y}dy = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}\int_0^t\frac{\sin(2\pi n y)}{y}dy$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}\int_0^{2\pi n t}\frac{\sin x}{x}dx$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}(\mathrm{si}(0) - \mathrm{si}(2\pi n t)).$$

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Here si(y) is the sine integral function normalised such that $\lim_{y\to\infty}$ si$(y) = 0$:

$$\text{si}(y) = \int_y^\infty \frac{\sin x}{x}dx.$$

It is known that

$$\text{si}(0) = \frac{\pi}{2} \quad \text{and} \quad |\text{si}(x)| \leq \frac{1}{x} \quad \text{for all } x > 0.$$

From this we get

$$\int_0^t \frac{\lambda(\xi,y)}{y}dy = \frac{1}{2}\sum_{n=1}^\infty \frac{\xi^n}{n} - \frac{1}{\pi}\sum_{n=1}^\infty \frac{\xi^n \text{ si}(2\pi nt)}{n}$$

$$= -\frac{1}{2}\log(1-\xi) - \frac{1}{\pi}\sum_{n=1}^\infty \frac{\xi^n \text{ si}(2\pi nt)}{n}$$

and

$$\left|\sum_{n=1}^\infty \frac{\xi^n \text{ si}(2\pi nt)}{n}\right| \leq \sum_{n=1}^\infty \frac{1}{n(2\pi nt)}$$

$$= \frac{1}{2\pi t}\sum_{n=1}^\infty \frac{1}{n^2}$$

$$= \frac{1}{2\pi t}\frac{\pi^2}{6}.$$

This proves the claim. $\qquad\square$

For $\delta > 1$ and $u > 1$, we write

$$J_\delta(u) = \max\{0, L(u) - L(\delta)\}.$$

For $\xi \in D$, $\delta > 1$ and $\epsilon > 0$, we write

$$N_{\delta,\epsilon}(\xi) = \int_{t\in\mathbf{R}} J_\delta\left(1 + \frac{(\epsilon t)^2}{2}\right)P(\exp(2\pi it)\xi)dt.$$

Lemma 5.4. *The function $N_{\delta,\epsilon}$ satisfies*

$$\left|N_{\delta,\epsilon}(\xi) - \frac{1}{\epsilon}\cdot\frac{2}{\pi}\arctan\sqrt{\frac{\delta-1}{2}} + \frac{1}{2\pi}\log|1-\xi|\right| \leq \epsilon r_\delta \quad \text{for all } \xi \in D,$$

where

$$r_\delta = \frac{1}{24\pi}\left(\sqrt{\frac{2}{\delta-1}} + \arctan\sqrt{\frac{\delta-1}{2}}\right). \tag{5.7}$$

Proof. We note that

$$1 + \frac{(\epsilon t)^2}{2} \leq \delta \iff |t| \leq \tau,$$

where

$$\tau = \frac{\sqrt{2\delta-2}}{\epsilon}.$$

By the definition of J_δ, this gives

$$N_{\delta,\epsilon}(\xi) = \int_{-\tau}^\tau \left(L\left(1 + \frac{(\epsilon t)^2}{2}\right) - L(\delta)\right)\tilde{P}(t,\xi)dt.$$

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Using (5.6), integrating by parts, and taking the contributions for positive and negative t together, we obtain

$$N_{\delta,\epsilon}(\xi) = -\int_{-\tau}^{\tau} \epsilon^2 t L'(1 + (\epsilon t)^2/2)\left(t + \frac{1}{2\pi i}\left(\log\frac{1 - \exp(-2\pi it)\bar\xi}{1 - \exp(2\pi it)\xi} - \log\frac{1 - \bar\xi}{1 - \xi}\right)\right)dt$$

$$= -\int_{0}^{\tau} \epsilon^2 t L'(1 + (\epsilon t)^2/2)\left(2t + \frac{1}{2\pi i}\left(\log\frac{1 - \exp(-2\pi it)\bar\xi}{1 - \exp(2\pi it)\xi} - \log\frac{1 - \exp(2\pi it)\bar\xi}{1 - \exp(-2\pi it)\xi}\right)\right)dt$$

$$= -\int_{0}^{\tau} \epsilon^2 t L'(1 + (\epsilon t)^2/2)\left(2t + \lambda(\xi,t) + \lambda(\bar\xi,t)\right)dt.$$

Using the definition (2.1) of L and rearranging gives

$$N_{\delta,\epsilon}(\xi) = \frac{1}{2\pi}\int_{0}^{\tau} \epsilon^2 t\left(\frac{1}{(\epsilon t)^2} - \frac{1}{4 + (\epsilon t)^2}\right)\left(2t + \lambda(\xi,t) + \lambda(\bar\xi,t)\right)dt$$

$$= \frac{1}{2\pi}\int_{0}^{\tau}\left(2 - \frac{2(\epsilon t)^2}{4 + (\epsilon t)^2}\right)dt + \frac{1}{2\pi}\int_{0}^{\tau}\left(1 - \frac{(\epsilon t)^2}{4 + (\epsilon t)^2}\right)\frac{\lambda(\xi,t) + \lambda(\bar\xi,t)}{t}dt \qquad (5.8)$$

$$= \frac{1}{2\pi}\int_{0}^{\tau}\frac{8}{4 + (\epsilon t)^2}dt + \frac{1}{2\pi}\int_{0}^{\tau}\frac{4}{4 + (\epsilon t)^2}\frac{\lambda(\xi,t) + \lambda(\bar\xi,t)}{t}dt.$$

We consider the two integrals in the last expression one by one. As for the first integral, we have

$$\int_{0}^{\tau}\frac{8}{4 + (\epsilon t)^2}dt = \frac{2}{\epsilon}\int_{0}^{\epsilon\tau/2}\frac{2}{1 + x^2}dx$$

$$= \frac{4}{\epsilon}\arctan\frac{\epsilon\tau}{2} \qquad (5.9)$$

$$= \frac{4}{\epsilon}\arctan\sqrt{\frac{\delta - 1}{2}}.$$

As for the second integral in (5.8), let us write for convenience

$$I_\xi = \int_{0}^{\tau}\frac{4}{4 + (\epsilon t)^2}\frac{\lambda(\xi,t) + \lambda(\bar\xi,t)}{t}dt$$

and

$$\Lambda_\xi(t) = \int_{0}^{t}\frac{\lambda(\xi,y) + \lambda(\bar\xi,y)}{y}dy + \log|1 - \xi|.$$

Then we have

$$\Lambda_\xi'(t) = \frac{\lambda(\xi,t) + \lambda(\bar\xi,t)}{t} \quad\text{and}\quad \Lambda_\xi(0) = \log|1 + \xi|.$$

Integration by parts gives

$$I_\xi = -\log|1 - \xi| + \frac{4}{4 + (\epsilon\tau)^2}\Lambda_\xi(\tau) + \int_{0}^{\tau}\frac{8\epsilon^2 t}{(4 + (\epsilon t)^2)^2}\Lambda_\xi(t)dt.$$

By Lemma 5.3, it follows that

$$\left|I_\xi + \log|1 - \xi|\right| \le +\frac{4}{4 + (\epsilon\tau)^2}\frac{1}{6\tau} + \int_{0}^{\tau}\frac{8\epsilon^2 t}{(4 + (\epsilon t)^2)^2}\frac{1}{6t}dt.$$

The integral can be evaluated by elementary means, and the result is

$$\left|I_\xi + \log|1 - \xi|\right| \le \frac{1}{6\tau} + \frac{\epsilon}{12}\arctan\frac{\epsilon\tau}{2}$$

$$= \frac{\epsilon}{12}\left(\sqrt{\frac{2}{\delta - 1}} + \arctan\sqrt{\frac{\delta - 1}{2}}\right).$$

Combining this with (5.8) and (5.9) proves the claim. $\qquad\square$

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Lemma 5.5. *Let Γ be a cofinite Fuchsian group, and let $\delta > 1$ and $\epsilon' > \epsilon > 0$ be real numbers satisfying the inequalities*

$$\left(\delta + \sqrt{\delta^2 - 1}\right)^{1/2}\epsilon' \le \min_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma) \quad \text{and} \quad \left(\delta + \sqrt{\delta^2 - 1}\right)\epsilon \le \epsilon'.$$

(a) *For all $z, w \in \mathbf{H}$ with $y_{\mathfrak{c}}(z) \ge 1/\epsilon'$ and $y_{\mathfrak{c}}(w) \ge 1/\epsilon'$ and all $\gamma \in \Gamma$, we have*

$$u(z, \gamma w) < \delta \implies \gamma \in \Gamma_{\mathfrak{c}}.$$

(b) *For all $z, w \in \mathbf{H}$ such that $y_{\mathfrak{c}}(z) \ge 1/\epsilon$ and such that the image of w in $\Gamma \backslash \mathbf{H}$ lies outside $D_{\mathfrak{c}}(\epsilon')$, and for all $\gamma \in \Gamma$, we have $u(z, \gamma w) \ge \delta$.*

Proof. Let z, w and γ be as in (a). We write

$$\sigma_{\mathfrak{c}}^{-1}\gamma\sigma_{\mathfrak{c}} = \gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose $\gamma \notin \Gamma_{\mathfrak{c}}$. Then our assumptions imply

$$|c|/\epsilon' \ge \left(\delta + \sqrt{\delta^2 - 1}\right)^{1/2}. \tag{5.10}$$

We have

$$
\begin{aligned}
u(z, \gamma w) &= u(\sigma_{\mathfrak{c}}^{-1} z, \sigma_{\mathfrak{c}}^{-1}\gamma w) \\
&= u(\sigma_{\mathfrak{c}}^{-1} z, \gamma' \sigma_{\mathfrak{c}}^{-1} w) \\
&= 1 + \frac{|\sigma_{\mathfrak{c}}^{-1} z - \gamma' \sigma_{\mathfrak{c}}^{-1} w|^2}{2(\Im \sigma_{\mathfrak{c}}^{-1} z)(\Im \gamma' \sigma_{\mathfrak{c}}^{-1} w)} \\
&\ge 1 + \frac{(\Im \sigma_{\mathfrak{c}}^{-1} z - \Im \gamma' \sigma_{\mathfrak{c}}^{-1} w)^2}{2(\Im \sigma_{\mathfrak{c}}^{-1} z)(\Im \gamma' \sigma_{\mathfrak{c}}^{-1} w)} \\
&= \frac{1}{2}\left(\frac{\Im \sigma_{\mathfrak{c}}^{-1} z}{\Im \gamma' \sigma_{\mathfrak{c}}^{-1} w} + \frac{\Im \gamma' \sigma_{\mathfrak{c}}^{-1} w}{\Im \sigma_{\mathfrak{c}}^{-1} z} \right) \\
&= \frac{1}{2}\left(\frac{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2}{y_{\mathfrak{c}}(w)} + \frac{y_{\mathfrak{c}}(w)}{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2} \right).
\end{aligned}
$$

From (5.10), we deduce

$$
\begin{aligned}
\frac{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2}{y_{\mathfrak{c}}(w)} &\ge \frac{y_{\mathfrak{c}}(z)(cy_{\mathfrak{c}}(w))^2}{y_{\mathfrak{c}}(w)} \\
&= c^2 y_{\mathfrak{c}}(z) y_{\mathfrak{c}}(w) \\
&\ge (|c|/\epsilon')^2 \\
&\ge \delta + \sqrt{\delta^2 - 1}.
\end{aligned}
$$

Using the fact that the function $x \mapsto x + x^{-1}$ is increasing for $x \ge 1$, we obtain

$$
\begin{aligned}
u(z, \gamma w) &\ge \frac{1}{2}\left(\left(\delta + \sqrt{\delta^2 - 1}\right) + \frac{1}{\delta + \sqrt{\delta^2 - 1}} \right) \\
&= \delta.
\end{aligned}
$$

This proves (a).

Now let z, w and γ be as in (b). Our assumption that the image of w in $\Gamma \backslash \mathbf{H}$ lies outside $D_{\mathfrak{c}}(\epsilon')$ implies

$$y_{\mathfrak{c}}(\gamma w) \le 1/\epsilon'$$

and hence

$$
\begin{aligned}
\frac{y_{\mathfrak{c}}(z)}{y_{\mathfrak{c}}(\gamma w)} &\ge \frac{\epsilon'}{\epsilon} \\
&\ge \delta + \sqrt{\delta^2 - 1}.
\end{aligned}
$$

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Using the fact that the function $x \mapsto x^{-1}$ is increasing for $x \geq 1$ as in the proof of (a), we get

$$
\begin{aligned}
u(z, \gamma w) &= u(\sigma_{\mathfrak{c}}^{-1} z, \sigma_{\mathfrak{c}}^{-1} \gamma w) \\
&\geq \frac{1}{2}\left(\frac{y_{\mathfrak{c}}(z)}{y_{\mathfrak{c}}(\gamma w)} + \frac{y_{\mathfrak{c}}(\gamma w)}{y_{\mathfrak{c}}(z)}\right) \\
&\geq \frac{1}{2}\left(\left(\delta + \sqrt{\delta^2 - 1}\right) + \frac{1}{\delta + \sqrt{\delta^2 - 1}}\right) \\
&= \delta.
\end{aligned}
$$

This proves (b). □

In the following proposition, we extend our bounds on gr_Γ to the neighbourhoods $D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ of the cusps. We will commit the following abuse of notation: for $z \in \mathbf{H}$ and S a subset of $\Gamma \backslash \mathbf{H}$, we write $z \in S$ if the image of z in $\Gamma \backslash \mathbf{H}$ lies in S.

Proposition 5.6. *Let Γ be a cofinite Fuchsian group, and let δ be a real number with $\delta > 1$. For every cusp $\mathfrak{c}$ of Γ, let $\epsilon_{\mathfrak{c}}' > \epsilon_{\mathfrak{c}} > 0$ be real numbers satisfying the inequalities*

$$
\epsilon_{\mathfrak{c}}'\left(\delta + \sqrt{\delta^2 - 1}\right)^{1/2} \leq \min_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma), \quad \text{and} \quad \left(\delta + \sqrt{\delta^2 - 1}\right)\epsilon_{\mathfrak{c}} \leq \epsilon_{\mathfrak{c}}'
$$

and small enough such that the discs $D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}')$ are pairwise disjoint. Let

$$
Y = (\Gamma \backslash \mathbf{H}) \setminus \bigsqcup_{\mathfrak{c}} D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}).
$$

Let A and B be real numbers satisfying

$$
A \leq \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma \in \Gamma \\ u(z, \gamma w) \leq \delta}} L_\delta(z, \gamma w) \leq B \quad \text{for all } z, w \in Y. \tag{5.11}
$$

(a) *If $\mathfrak{c}$ is a cusp such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $w \in Y$ and $w \notin D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}')$, we have*

$$
A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) \leq B.
$$

(a') *If $\mathfrak{c}$ is a cusp such that $w \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $z \in Y$ and $z \notin D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}')$, we have*

$$
A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(w)) \leq B.
$$

(b) *If $\mathfrak{c}, \mathfrak{d}$ are two distinct cusps such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ and $w \in D_{\mathfrak{d}}(\epsilon_{\mathfrak{d}})$, we have*

$$
A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(w)) \leq B.
$$

(c) *If $\mathfrak{c}$ is a cusp such that $z, w \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}')$, we have*

$$
\begin{aligned}
\tilde{A}_{\mathfrak{c}} \leq \mathrm{gr}_\Gamma(z, w) &- \#(\Gamma \cap \{\pm 1\}) \cdot \frac{1}{2\pi} \log|q_{\mathfrak{c}}(z) - q_{\mathfrak{c}}(w)| \\
&- \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}}' y_{\mathfrak{c}}(z)) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}}' y_{\mathfrak{c}}(w)) \leq \tilde{B}_{\mathfrak{c}},
\end{aligned}
$$

where $\tilde{A}_{\mathfrak{c}}$ and $\tilde{B}_{\mathfrak{c}}$ are defined using the function r_δ from (5.7) by

$$
\tilde{A}_{\mathfrak{c}} = A + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon_{\mathfrak{c}}'}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) - \epsilon_{\mathfrak{c}}' r_\delta\right],
$$

$$
\tilde{B}_{\mathfrak{c}} = B + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon_{\mathfrak{c}}'}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) + \epsilon_{\mathfrak{c}}' r_\delta\right].
$$

Proof. In view of §2.3 (or Lemma 5.5(a)), the discs $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) \supset D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ are well defined. Furthermore, the assumption that the discs $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ are pairwise disjoint implies that for every cusp $\mathfrak{c}$, the boundaries of $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ and $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ are contained in Y.

Let us prove part (a). We keep $w \in Y$ fixed and consider $\mathrm{gr}_{\Gamma}(z, w)$ as a function of $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. The defining properties of gr_{Γ} imply

$$\mathrm{gr}_{\Gamma}(z, w) = \frac{1}{\mathrm{vol}_{\Gamma}} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + h_w(z) \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}),$$

with h_w a real-valued harmonic function on $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. By construction, $h_w(z)$ coincides with $\mathrm{gr}_{\Gamma}(z, w)$ for z on the boundary of $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. This implies

$$h_w(z) = \int_0^1 \mathrm{gr}_{\Gamma}(\sigma_{\mathfrak{c}}(a + i/\epsilon_{\mathfrak{c}}), w) \tilde{P}(-a, q_{\mathfrak{c}}(z) \exp(2\pi/\epsilon_{\mathfrak{c}})) da.$$

By Lemma 5.5(b), there are no $\gamma \in \Gamma$ such that $u(z, \gamma w) < \delta$. By the assumption (5.11) on A and B, we conclude

$$A \le h_w(z) \le B \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon).$$

This proves (a). Part (a′) is equivalent to (a) by symmetry, and (b) is proved in a similar way.

It remains to prove part (c). We identify $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ with the closed unit disc $\bar{D}$ via the map

$$\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) \xrightarrow{\sim} \bar{D}$$
$$z \longmapsto \zeta_z = q_{\mathfrak{c}}(z) \exp(2\pi/\epsilon'_{\mathfrak{c}}).$$

Let $\mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}$ be the Green function for the Laplace operator on $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, given in terms of gr_D by

$$\mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z, w) = \#(\Gamma \cap \{\pm 1\}) \cdot \mathrm{gr}_D(\zeta_z, \zeta_w).$$

The factor $\#(\Gamma \cap \{\pm 1\})$ arises because of how we defined integration on $\Gamma \backslash \mathbf{H}$ in Section 1.

Fixing w and considering gr_{Γ} as a function of z, we have

$$\mathrm{gr}_{\Gamma}(z, w) = \mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z, w) + \frac{1}{\mathrm{vol}_{\Gamma}} \log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + h_w(z) \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}),$$

where h_w is a real-valued harmonic function on $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$. By construction, $h_w(z)$ coincides with $\mathrm{gr}_{\Gamma}(z, w)$ for z on the boundary of $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$. This implies

$$h_w(z) = \int_0^1 \mathrm{gr}_{\Gamma}(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), w) \tilde{P}(-a, \zeta_z) da.$$

Applying the same argument to $\mathrm{gr}_{\Gamma}(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), w)$ as a function of w, we obtain

$$\mathrm{gr}_{\Gamma}(z, w) = \mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z, w) + \frac{1}{\mathrm{vol}_{\Gamma}} \log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + \frac{1}{\mathrm{vol}_{\Gamma}} \log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(w)) + K(z, w), \tag{5.12}$$

where K is the function on $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) \times D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ defined by

$$K(z, w) = \int_{a=0}^1 \int_{b=0}^1 \mathrm{gr}_{\Gamma}(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), \sigma_{\mathfrak{c}}(b + i/\epsilon'_{\mathfrak{c}})) \tilde{P}(-b, \zeta_w) \tilde{P}(-a, \zeta_z) db\, da.$$

In (5.11), we may replace Γ by $\Gamma_{\mathfrak{c}}$ in view of Lemma 5.5(a), i.e., we have

$$A \le \mathrm{gr}_{\Gamma}(z, w) + \sum_{\substack{\gamma \in \Gamma_{\mathfrak{c}} \\ u(z, \gamma w) \le \delta}} L_{\delta}(z, \gamma w) \le B \quad \text{for all } z, w \in \partial \bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}).$$

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Substituting this in the definition of $K(z,w)$ and "unfolding" the action of $\Gamma_{\mathfrak{c}}$, we get

$$A \leq K(z,w) + \#(\Gamma \cap \{\pm 1\})M(\zeta_z, \zeta_w) \leq B, \tag{5.13}$$

where M is the function on $D \times D$ defined by

$$M(\zeta, \eta) = \int_{a=0}^{1} \int_{b \in \mathbf{R}} J_\delta(u(\sigma_{\mathfrak{c}}(a + i/\epsilon_{\mathfrak{c}}'), \sigma_{\mathfrak{c}}(b + i/\epsilon_{\mathfrak{c}}')))\tilde{P}(-b, \eta)\tilde{P}(-a, \zeta)db\,da.$$

Making the change of variables

$$b = a + t$$

and noting that

$$u(\sigma_{\mathfrak{c}}(a + i/\epsilon_{\mathfrak{c}}'), \sigma_{\mathfrak{c}}(b + i/\epsilon_{\mathfrak{c}}')) = u(a + i/\epsilon_{\mathfrak{c}}', b + i/\epsilon_{\mathfrak{c}}')$$

$$= 1 + \frac{(b-a)^2}{2/\epsilon_{\mathfrak{c}}'^2}$$

$$= 1 + \frac{(\epsilon_{\mathfrak{c}}'t)^2}{2},$$

we obtain

$$M(\zeta, \eta) = \int_{a=0}^{1} \int_{t \in \mathbf{R}} J_\delta\left(1 + \frac{(\epsilon_{\mathfrak{c}}'t)^2}{2}\right)\tilde{P}(-a-t, \eta)\tilde{P}(-a, \zeta)dt\,da.$$

Interchanging the order of integration, noting that

$$\tilde{P}(-a-t, \eta) = \tilde{P}(a, \exp(2\pi i t)\bar{\eta})$$

and using (5.5), we simplify this to

$$M(\zeta, \eta) = \int_{t \in \mathbf{R}} J_\delta\left(1 + \frac{(\epsilon_{\mathfrak{c}}'t)^2}{2}\right)\tilde{P}(t, \zeta\bar{\eta})dt$$

$$= N_{\delta, \epsilon_{\mathfrak{c}}'}(\zeta\bar{\eta}).$$

Applying Lemma 5.4, we conclude from (5.13) that

$$K(z,w) \leq B + \#(\Gamma \cap \{\pm 1\})\left(\frac{1}{2\pi} \log|1 - \zeta_z\bar{\zeta}_w| - \frac{1}{\epsilon_{\mathfrak{c}}'} \cdot \frac{2}{\pi} \arctan\sqrt{\frac{\delta - 1}{2}} + \epsilon_{\mathfrak{c}}'r_\delta\right),$$

$$K(z,w) \geq A + \#(\Gamma \cap \{\pm 1\})\left(\frac{1}{2\pi} \log|1 - \zeta_z\bar{\zeta}_w| - \frac{1}{\epsilon_{\mathfrak{c}}'} \cdot \frac{2}{\pi} \arctan\sqrt{\frac{\delta - 1}{2}} - \epsilon_{\mathfrak{c}}'r_\delta\right).$$

We now note that

$$\mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}')}(z,w) = \#(\Gamma \cap \{\pm 1\}) \cdot \frac{1}{2\pi} \log\left|\frac{\zeta_z - \zeta_w}{1 - \zeta_z\bar{\zeta}_w}\right|$$

$$= \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{2\pi} \log|q_{\mathfrak{c}}(z) - q_{\mathfrak{c}}(w)| + \frac{1}{\epsilon_{\mathfrak{c}}'} - \frac{1}{2\pi} \log|1 - \zeta_z\bar{\zeta}_w|\right].$$

Combining this with (5.12) and the above bounds on $K(z,w)$ yields the proposition. $\square$

6. Bounds on the canonical $(1,1)$-form

Let Γ be a cofinite Fuchsian group. In this section we find bounds on the function F_Γ that are easy to evaluate explicitly in concrete cases. We essentially adapt the methods of Iwaniec [12, §7.2] from weight 0 to weight 2. This method is more elementary than that of Jorgenson and Kramer [13], and our bounds are easy to make explicit, as the example in Section 8 shows.

Proposition 6.1. *For every cofinite Fuchsian group Γ, all $z \in \mathbf{H}$ and all $a > 1$, we have*

$$F_\Gamma(z) \leq \frac{(a-1)N_\Gamma(z, z, 2a^2-1)}{8\pi\left(\log\frac{a+1}{2}\right)^2}.$$

Proof. Let $(f_1, \ldots, f_g)$ be an orthonormal basis of the space of holomorphic cusp forms of weight 2 for Γ. We write

$$\phi_j(z) = (\Im z)f_j(z).$$

Then the ϕ_j are annihilated by the operator Δ_2, and $(\phi_1, \ldots, \phi_g)$ is an orthonormal system in the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H}, 2)$ of automorphic forms of weight 2 for Γ.

Let $z \in \mathbf{H}$ and $a > 1$ be given. We apply §§ 3.1 and 3.2 with

$$\theta(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq a, \\ 0 & \text{if } u > a. \end{cases}$$

In the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H}, 2)$, we consider $\overline{K_{\Gamma,\theta}^{(2)}(z, w)}$, as a function of w, and the orthonormal system $(\phi_1, \ldots, \phi_g)$. From Bessel's inequality and (3.8), we obtain

$$\sum_{j=1}^{g}\left|h_\theta^{(2)}(0)\phi_j(z)\right|^2 \leq \int_{w\in\Gamma\backslash\mathbf{H}}\left|K_{\Gamma,\theta}^{(2)}(z, w)\right|^2\mu_{\mathbf{H}}(w).$$

The left-hand side is equal to $\left|h_\theta^{(2)}(0)\right|^2 F_\Gamma(z)$. The right-hand side is a function $\Gamma\backslash\mathbf{H} \to [0, \infty)$, which we denote by $\kappa(z)$. The definition (3.7) gives

$$\kappa(z) = \sum_{\gamma_1, \gamma_2 \in \Gamma}\nu^{(2)}(\gamma_1, w)\theta^{(2)}(z, \gamma_1 w)\overline{\nu^{(2)}(\gamma_2, w)\theta^{(2)}(z, \gamma_2 w)}\mu_{\mathbf{H}}(w).$$

Putting $\gamma = \gamma_1\gamma_2^{-1}$, we obtain after a straightforward computation

$$\kappa(z) = \sum_{\gamma\in\Gamma}\nu^{(2)}(\gamma, z)\int_{w\in\mathbf{H}}\theta^{(2)}(z, w)\overline{\theta^{(2)}(\gamma z, w)}\mu_{\mathbf{H}}(w).$$

This implies

$$\kappa(z) \leq \sum_{\gamma\in\Gamma}\int_{w\in\mathbf{H}}\theta(u(z, w))\theta(u(\gamma z, w))\mu_{\mathbf{H}}(w).$$

By the definition of θ, the integral on the right-hand side can be interpreted as the area of the intersection of the discs of area $2\pi(a-1)$ around z and γz, respectively. By the triangle inequality for the hyperbolic distance, this intersection is empty unless

$$u(z, \gamma z) \leq 2a^2 - 1.$$

This implies

$$\kappa(z) \leq 2\pi(a-1)N_\Gamma(z, z, 2a^2-1),$$

and hence

$$|h_\theta^{(2)}(0)|^2 F_\Gamma(z) \leq 2\pi(a-1)N_\Gamma(z, z, 2a^2-1).$$

We evaluate $h_\theta^{(2)}(0)$ using (3.4) and (3.5). The hypergeometric series terminates after two terms and gives

$$P_{0,2}(u) = \frac{2}{u+1}.$$

This implies

$$h_\theta^{(2)}(0) = 4\pi\log\frac{u+1}{2}.$$

This finishes the proof. $\qquad\square$

The above proposition does not give the correct asymptotic behaviour of $F_\Gamma(z)$ for z close to a cusp of Γ. The following result extends our bounds to neighbourhoods of the cusps.

Lemma 6.2 (cf. Jorgenson and Kramer [13, Theorem 3.1]). *Let Γ be a cofinite Fuchsian group, let $\mathfrak{c}$ be a cusp of Γ, and let ϵ be a real number satisfying (2.3). Then for all $z \in D_\mathfrak{c}(\epsilon)$, we have*

$$F_\Gamma(z) \leq (\epsilon y_\mathfrak{c}(z))^2 \exp(4\pi/\epsilon - 4\pi y_\mathfrak{c}(z)) \sup_{\partial \bar{D}_\mathfrak{c}(\epsilon)} F_\Gamma$$

$$\leq \begin{cases} \sup\limits_{\partial \bar{D}_\mathfrak{c}(\epsilon)} F_\Gamma & \text{if } \epsilon \leq 2\pi, \\ \left(\dfrac{\epsilon}{2\pi} \exp(2\pi/\epsilon - 1)\right)^2 \sup\limits_{\partial \bar{D}_\mathfrak{c}(\epsilon)} F_\Gamma & \text{if } \epsilon > 2\pi. \end{cases}$$

Proof. Every holomorphic cusp form f of weight 2 for Γ has a q-expansion of the form

$$f(z)dz = \sum_{n=1}^{\infty} a_{\mathfrak{c},n}(f)q_\mathfrak{c}(z)^n \cdot d(\sigma_\mathfrak{c}^{-1}z) \quad \text{with } a_{\mathfrak{c},n}(f) \in \mathbf{C}.$$

This implies

$$(\Im z)^2|f(z)|^2 = y_\mathfrak{c}(z)^2 \left| \sum_{n=1}^{\infty} a_{\mathfrak{c},n}(f)q_\mathfrak{c}(z)^n \right|^2.$$

Applying this to an orthonormal basis of the space of holomorphic cusp forms of weight 2, we see that the function

$$y_\mathfrak{c}(z)^{-2} \exp(4\pi y_\mathfrak{c}(z))F_\Gamma(z) = \sum_{f \in B} \left| \frac{f(z)}{q_\mathfrak{c}(z)} \right|^2$$

extends to a subharmonic function on $\bar{D}_\mathfrak{c}(\epsilon)$. By the maximum principle for subharmonic functions, the function assumes its maximum on the boundary. This implies the first inequality. The second inequality follows from the easily checked fact that the function $(\epsilon y_\mathfrak{c}(z))^2 \exp(4\pi/\epsilon - 4\pi y_\mathfrak{c}(z))$ for $y \geq 1/\epsilon$ assumes its maximum at $y = 1/(2\pi)$ if $\epsilon > 2\pi$, and at $y = 1/\epsilon$ if $\epsilon \leq 2\pi$. $\square$

7. Bounds on the canonical Green function

If Γ is a Fuchsian group and X is the compactification of $\Gamma\backslash\mathbf{H}$ obtained by adding the cusps, we write

$$\zeta_\Gamma = \frac{1}{g_X} \int_{\Gamma\backslash\mathbf{H}} F_\Gamma \mu_X^{\mathrm{can}} - \frac{1}{\mathrm{vol}_\Gamma}. \tag{7.1}$$

It follows from Lemma 3.2 that for any compact subset Y of $\Gamma\backslash\mathbf{H}$, there exists $C > 0$ such that the function $\Phi_\Gamma(z, \lambda)$ defined by (3.15) satisfies

$$\Phi_\Gamma(z, \lambda) \leq C\lambda \quad \text{for all } z \in Y \text{ and } \lambda \geq 1/4. \tag{7.2}$$

Theorem 7.1. *Let Γ be a cofinite Fuchsian group, and let X be the compactification of $\Gamma\backslash\mathbf{H}$ obtained by adding the cusps. Let δ be a real number with $\delta > 1$. For every cusp $\mathfrak{c}$ of Γ, let $\epsilon'_\mathfrak{c} > \epsilon_\mathfrak{c} > 0$ be real numbers satisfying the inequalities*

$$\epsilon'_\mathfrak{c}\left(\delta + \sqrt{\delta^2 - 1}\right)^{1/2} \leq \min_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_\mathfrak{c}}} C_\mathfrak{c}(\gamma) \quad \text{and} \quad \left(\delta + \sqrt{\delta^2 - 1}\right)\epsilon_\mathfrak{c} \leq \epsilon'_\mathfrak{c}$$

and small enough such that the discs $D_\mathfrak{c}(\epsilon'_\mathfrak{c})$ are pairwise disjoint. Let Y be the compact subset of $\Gamma\backslash\mathbf{H}$ defined by

$$Y = (\Gamma\backslash\mathbf{H}) \setminus \bigsqcup_{\mathfrak{c} \text{ cusp}} D_\mathfrak{c}(\epsilon_\mathfrak{c}).$$

Let A and B be real numbers such that the automorphic Green function gr_Γ satisfies (5.11), and let $C > 0$ be such that the function $\Phi_\Gamma(z, \lambda)$ satisfies (7.2). Let $\eta \in (0, 1/4]$ be such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\eta, \infty)$. With the notation

$$S = \sqrt{\left(\frac{1}{4\eta^2} + 4\right) C \zeta_\Gamma},$$

$$T(\epsilon) = \frac{\sup_Y F_\Gamma\left(\frac{\epsilon}{4\pi}\right)^2}{g_X} \quad \text{for all } \epsilon > 0,$$

$$r_\delta = \frac{1}{24\pi}\left(\sqrt{\frac{2}{\delta - 1}} + \arctan\sqrt{\frac{\delta - 1}{2}}\right),$$

$$\tilde{A}_{\mathfrak{c}} = A + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon'_{\mathfrak{c}}}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) - \epsilon'_{\mathfrak{c}} r_\delta\right],$$

$$\tilde{B}_{\mathfrak{c}} = B + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon'_{\mathfrak{c}}}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) + \epsilon'_{\mathfrak{c}} r_\delta\right],$$

we have the following bounds on the canonical Green function $\mathrm{gr}_X^{\mathrm{can}}(z, w)$:

(a) If $z, w \in Y$, we have

$$A - 2S - \zeta_\Gamma/\eta \leq \mathrm{gr}_X^{\mathrm{can}}(z, w) + \sum_{\substack{\gamma \in \Gamma \\ u(z, \gamma w) \leq \delta}} L_\delta(z, \gamma w) \leq B + 2S.$$

(b) If $\mathfrak{c}$ is a cusp such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $w \in Y$ and $w \notin D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, or such that $w \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $z \in Y$ and $z \notin D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, then we have

$$A - 2S - \zeta_\Gamma/\eta \leq \mathrm{gr}_X^{\mathrm{can}}(z, w) \leq B + 2S + T(\epsilon_{\mathfrak{c}}).$$

(c) If $\mathfrak{c}, \mathfrak{d}$ are two distinct cusps such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ and $w \in D_{\mathfrak{d}}(\epsilon_{\mathfrak{d}})$, we have

$$A - 2S - \zeta_\Gamma/\eta \leq \mathrm{gr}_X^{\mathrm{can}}(z, w) \leq B + 2S + T(\epsilon_{\mathfrak{c}}) + T(\epsilon_{\mathfrak{d}}).$$

(d) If $\mathfrak{c}$ is a cusp such that $z, w \in D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, we have

$$\tilde{A}_{\mathfrak{c}} - 2S - \zeta_\Gamma/\eta \leq \mathrm{gr}_X^{\mathrm{can}}(z, w) - \#(\Gamma \cap \{\pm 1\}) \cdot \frac{1}{2\pi}\log|q_{\mathfrak{c}}(z) - q_{\mathfrak{c}}(w)| \leq \tilde{B}_{\mathfrak{c}} + 2S + 2T(\epsilon'_{\mathfrak{c}}).$$

Our proof of Theorem 7.1 is based on the equation (2.6), the bounds on the automorphic Green function from §5, and on bounds on the function h_Γ defined by (2.5). The proof of the latter bounds occupies most of this section; Theorem 7.1 then follows without difficulties.

Lemma 7.2. *Let Γ, Y, η and C be as in Theorem 7.1. Then the function*

$$M_\Gamma(z) = \sum_{j \geq 1} \frac{1}{\lambda_j^2}|\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} \frac{1}{(s(1-s))^2}|E_{\mathfrak{c}}(z, s)|^2 ds$$

satisfies

$$M_\Gamma(z) \leq \left(\frac{1}{4\eta^2} + 4\right) C \quad \text{for all } z \in Y.$$

Proof. Separating the terms with $\lambda_j \leq 1/4$, we get (with $\partial\Phi_\Gamma/\partial\lambda$ taken in a distributional sense)

$$M_\Gamma(z) = \sum_{j: 0 < \lambda_j \leq 1/4} \frac{1}{\lambda_j^2}|\phi_j(z)|^2 + \int_{1/4}^{\infty} \frac{1}{\lambda^2}\frac{\partial\Phi_\Gamma(z, \lambda)}{\partial\lambda}d\lambda$$

$$\leq \frac{1}{\eta^2}\Phi(z, 1/4) + \left[\frac{1}{\lambda^2}\Phi_\Gamma(z, \lambda)\right]_{\lambda = 1/4}^{\infty} + 2\int_{1/4}^{\infty}\lambda^{-3}\Phi_\Gamma(z, \lambda)d\lambda$$

$$= \left(\frac{1}{\eta^2} - 16\right)\Phi_\Gamma(z, 1/4) + 2\int_{1/4}^{\infty}\lambda^{-3}\Phi_\Gamma(z, \lambda)d\lambda$$

$$\leq \left(\frac{1}{\eta^2} - 16\right)\frac{C}{4} + 2C\int_{1/4}^{\infty}\lambda^{-2}d\lambda$$

$$= \left(\frac{1}{\eta^2} - 16\right)\frac{C}{4} + 8C$$

$$= \left(\frac{1}{4\eta^2} + 4\right)C,$$

where the second inequality follows from (7.2). ☐

Lemma 7.3. *Let* Γ, Y, η *and* C *be as in Theorem 7.1, and let* ζ_Γ *be as in (7.1). Then we have*

$$|h_\Gamma(z)|^2 \le \left(\frac{1}{4\eta^2} + 4\right)C\zeta_\Gamma \quad \text{for all } z \in Y.$$

Proof. Let X be the compactification of $\Gamma\backslash\mathbf{H}$. Since the function F_Γ is smooth and bounded, we may consider its spectral representation, say

$$\frac{1}{g_X}F_\Gamma(z) = \sum_{j \ge 0} b_j\phi_j(z) + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} b_{\mathfrak{c}}(s)E_{\mathfrak{c}}(z,s)ds. \tag{7.3}$$

Then the definition of h_Γ implies that it has the spectral representation

$$h_\Gamma(z) = -\sum_{j \ge 1} \frac{b_j}{\lambda_j}\phi_j(z) - \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} \frac{b_{\mathfrak{c}}(s)}{s(1-s)}E_{\mathfrak{c}}(z,s)ds; \tag{7.4}$$

note the absence of the term corresponding to $j = 0$. Now the Cauchy–Schwarz inequality implies

$$h_\Gamma(z)^2 \le M_\Gamma(z)\left(\sum_{j \ge 1} |b_j|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} |b_{\mathfrak{c}}(s)|^2\right).$$

Next, it follows from (7.3), the identity $|a_0^2| = 1/\mathrm{vol}_\Gamma$ and (2.4) that

$$\begin{aligned}
\sum_{j \ge 1} |b_j|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} |b_{\mathfrak{c}}(s)|^2 &= \int_{z \in \Gamma\backslash\mathbf{H}} \left(\frac{1}{g_X}F_\Gamma(z) - \frac{1}{\mathrm{vol}_\Gamma}\right)^2 \mu_{\mathbf{H}}(z) \\
&= \frac{1}{g_X^2}\int_{z \in \Gamma\backslash\mathbf{H}} F_\Gamma(z)^2 \mu_{\mathbf{H}}(z) - \frac{1}{\mathrm{vol}_\Gamma} \\
&= \frac{1}{g_X}\int_{\Gamma\backslash\mathbf{H}} F_\Gamma \mu_X^{\mathrm{can}} - \frac{1}{\mathrm{vol}_\Gamma}.
\end{aligned} \tag{7.5}$$

Together with Lemma 7.2 and the definition of ζ_Γ, this finishes the proof. ☐

We now extend our bounds on h_Γ to the discs around the cusps.

Lemma 7.4. *Let* Γ *be a cofinite Fuchsian group, and let* X *be the compactification of* $\Gamma\backslash\mathbf{H}$. *Let* $\mathfrak{c}$ *be a cusp of* Γ, *and let* ϵ *be a real number satisfying (2.3). For all* $z \in D_{\mathfrak{c}}(\epsilon)$, *we have*

$$h_{\Gamma,\mathfrak{c}}^-(z) \le h_\Gamma(z) \le h_{\Gamma,\mathfrak{c}}^+(z),$$

where

$$h_{\Gamma,\mathfrak{c}}^+(z) = \sup_{\partial\bar{D}_{\mathfrak{c}}(\epsilon)} h_\Gamma + \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon y_{\mathfrak{c}}(z))$$

and

$$h_{\Gamma,\mathfrak{c}}^-(z) = \inf_{\partial\bar{D}_{\mathfrak{c}}(\epsilon)} h_\Gamma - \frac{\sup_{\partial\bar{D}_{\mathfrak{c}}(\epsilon)} F_\Gamma}{g_X}\left(\frac{\epsilon}{4\pi}\right)^2\left(1 - \exp(4\pi/\epsilon - 4\pi y_{\mathfrak{c}}(z))\right) + \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon y_{\mathfrak{c}}(z)).$$

Proof. We note that

$$\Delta h_{\Gamma,\mathfrak{c}}^+(z) = -\frac{1}{\mathrm{vol}_\Gamma}$$

and

$$\Delta h_{\Gamma,\mathfrak{c}}^-(z) = \frac{\sup_{\partial\bar{D}_{\mathfrak{c}}(\epsilon)} F_\Gamma}{g_X}(\epsilon y_{\mathfrak{c}}(z))^2 \exp(4\pi/\epsilon - 4\pi y_{\mathfrak{c}}(z)) - \frac{1}{\mathrm{vol}_\Gamma}.$$

By the non-negativity of F_Γ and Lemma 6.2, this implies

$$\Delta h_{\Gamma,\mathfrak{c}}^+(z) \le \Delta h_\Gamma(z) \le \Delta h_{\Gamma,\mathfrak{c}}^-(z).$$

Therefore $h_{\Gamma,\mathfrak{c}}^+ - h_\Gamma$ and $h_\Gamma - h_{\Gamma,\mathfrak{c}}^-$ are subharmonic functions on $\bar{D}_{\mathfrak{c}}(\epsilon)$. By the maximum principle for subharmonic functions, each of these functions assumes its maximum on the boundary. The definitions of $h_{\Gamma,\mathfrak{c}}^\pm$ imply that these maxima are non-negative. ☐

Finally, we prove bounds on the integral $\int_{\Gamma \backslash \mathbf{H}} h_\Gamma \mu_X^{\mathrm{can}}$.

Lemma 7.5. *Let Γ be a cofinite Fuchsian group, let X be the compactification of $\Gamma \backslash \mathbf{H}$, and let $\eta > 0$ be such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\eta, \infty)$. Then we have*

$$-\zeta_\Gamma / \eta \leq \int_{\Gamma \backslash \mathbf{H}} h_\Gamma \mu_X^{\mathrm{can}} \leq 0.$$

Proof. We use the spectral representations (7.3) and (7.4). We obtain

$$\int_{\Gamma \backslash \mathbf{H}} h_\Gamma \mu_X^{\mathrm{can}} = \int_{z \in \Gamma \backslash \mathbf{H}} h_\Gamma(z) \cdot \frac{1}{g_X} F_\Gamma(z) \mu_{\mathbf{H}}(z)$$

$$= -\sum_{j \geq 1} \frac{|b_j|^2}{\lambda_j} - \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} \frac{|b_{\mathfrak{c}}(s)|^2}{s(1-s)}.$$

We note that the right-hand side is non-positive. Next, the assumption that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\} \cup [\eta, \infty)$ implies

$$\int_{\Gamma \backslash \mathbf{H}} h_\Gamma \mu_X^{\mathrm{can}} \geq -\frac{1}{\eta} \left(\sum_{j \geq 1} |b_j|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i} \int_{\Re s = 1/2} |b_{\mathfrak{c}}(s)|^2 ds \right).$$

Together with (7.5), this proves the claim. $\qquad \square$

Proof of Theorem 7.1. Part (a) of the theorem follows from the comparison formula (2.6), the bound (5.11) for gr_Γ, the bound on for h_Γ given by Lemma 7.3, and the bound on $\int_{\Gamma \backslash \mathbf{H}} h_\Gamma \mu_X^{\mathrm{can}}$ given by Lemma 7.5.

The proof of parts (b)–(d) is similar. We first note that, by Lemmata 7.3 and 7.4,

$$-S - T(\epsilon_{\mathfrak{c}}) \leq h_\Gamma(z) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) \leq S \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}),$$

and similarly with $\epsilon_{\mathfrak{c}}'$ in place of $\epsilon_{\mathfrak{c}}$. Instead of (5.11) we now invoke Proposition 5.6, which gives bounds for the function gr_Γ when one or both variables are near a cusp. As in the proof of (a), it remains to apply the formula (2.6) and Lemma 7.5. $\qquad \square$

8. Example: congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$

8.1. The automorphic Green function

Let us consider the case where $\Gamma_0 = \mathrm{SL}_2(\mathbf{Z})$. We will make the bounds from Theorem 1.1 explicit for congruence subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$. By our convention for integration on $\Gamma \backslash \mathbf{H}$ if $-1 \in \Gamma$, we have

$$\frac{1}{\mathrm{vol}_\Gamma} \leq \frac{1}{\mathrm{vol}_{\mathrm{SL}_2(\mathbf{Z})}} = \frac{6}{\pi}.$$

We start by fixing the various parameters. We choose

$$\delta = 2.$$

Selberg conjectured in [21] that the least non-zero eigenvalue λ_1 of $-\Delta_\Gamma$ is at least $1/4$, and he proved that $\lambda_1 \geq 3/16$. The sharpest bound so far, due to Kim and Sarnak [15, Appendix 2], is $\lambda_1 \geq (25/64)(1 - 25/64) = 975/4096$. We may therefore take

$$\eta = 975/4096.$$

We now consider the point counting function $N_{\mathrm{SL}_2(\mathbf{Z})}(z, z, U)$ defined by (3.14) on a rectangle of the form

$$R = \{x + iy \in \mathbf{H} \mid x_{\min} \leq x \leq x_{\max}, y_{\min} \leq y \leq y_{\max}\}$$

for given real numbers $x_{\min} < x_{\max}$ and $0 < y_{\min} < y_{\max}$. The function $z \mapsto N_{\mathrm{SL}_2(\mathbf{Z})}(z, z, U)$ on R is clearly bounded from above by the number of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbf{Z})$ such that for some $z \in R$, the inequality

$$u(z, \gamma z) \leq U \tag{8.1}$$

holds. We now show how to enumerate these matrices. We distinguish the cases $c = 0$ and $c \neq 0$. We assume (after multiplying by -1 if necessary) that $a = d = 1$ in the first case, and that $c > 0$ in the second case. The total number of matrices γ as above is then twice the number produced by our enumeration.

In the case $c = 0$, by (5.4), the inequality (8.1) reduces to

$$1 + \tfrac{1}{2}(b/y)^2 \leq U.$$

This implies

$$|b| \leq y_{\max}\sqrt{2U - 2}.$$

In the case $c > 0$, it follows from (5.4) and (8.1) that

$$|c| \leq \sqrt{2U}/y_{\min},$$
$$-\sqrt{2U} + cx_{\min} \leq a \leq \sqrt{2U} + cx_{\max},$$
$$-\sqrt{2U} - cx_{\max} \leq d \leq \sqrt{2U} - cx_{\min}.$$

Since $c \neq 0$, the coefficients a, c, d and the condition $ad - bc = 1$ determine b. If γ is a matrix obtained in this way, we compute the minimum of $u(z, \gamma z)$ for $z \in R$ using (5.4) to decide whether there exists a point $z \in R$ satisfying (8.1).

Let Y_0 denote the compact subset in $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}$ which is the image of the rectangle

$$\{x + iy \in \mathbf{H} \mid -1/2 \leq x \leq 1/2 \text{ and } \sqrt{3}/2 \leq y \leq 2\}.$$

This is the complement of a disc around the unique cusp of $\mathrm{SL}_2(\mathbf{Z})$. Dividing this rectangle into 100×100 small rectangles and bounding $N_{\mathrm{SL}_2(\mathbf{Z})}(z, z, U)$ on each of them as described above, we get

$$N_{\Gamma_0}(z, z, 17) \leq 216 \quad \text{for all } z \in Y_0.$$

Given this upper bound for $N_{\mathrm{SL}_2(\mathbf{Z})}(z, z, 17)$, we fix the the remaining parameters experimentally to optimise the bounds in the theorem below. This leads to the following values:

$$\alpha^+ = 0.0366, \quad \beta^+ = 2.72, \quad \sigma^+ = 0.306,$$
$$\alpha^- = 2.96 \cdot 10^{-3}, \quad \beta^- = 0.668, \quad \sigma^- = 0.250.$$

With these choices, a numerical calculation gives

$$q_{\Gamma,\delta}^+ < 69.0, \quad q_{\Gamma,\delta}^- > -216, \quad D_\delta^+ < 18.5, \quad D_\delta^- < 9.61.$$

This implies the following explicit bounds on automorphic Green functions of congruence subgroups.

Theorem 8.1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Then for all $z, w \in \mathbf{H}$ whose images in $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}$ lie in Y_0, we have*

$$-2.87 \cdot 10^4 \leq \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma \in \Gamma \\ u(z, \gamma w) \leq 2}} L_2(z, \gamma w) \leq 1.51 \cdot 10^4.$$

8.2. The canonical Green function

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$ such that the corresponding modular curve X has positive genus. Let n be the level of Γ, i.e. the minimal positive integer with the property that Γ contains the kernel of the reduction map $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/n\mathbf{Z})$.

We define

$$\epsilon = (\delta + \sqrt{\delta^2 - 1})^{-3/2} \approx 0.139 \quad \text{and} \quad \epsilon' = (\delta + \sqrt{\delta^2 - 1})^{-1/2} \approx 0.518.$$

Let Y_0 denote the compact subset of $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}$ which is the image of the strip

$$\{x + iy \in \mathbf{H} \mid |x| \leq 1/2 \text{ and } \sqrt{3}/2 \leq y \leq 1/\epsilon\}.$$

For every cusp $\mathfrak{c}$ of Γ, we let $m_{\mathfrak{c}}$ denote the ramification index of $\mathfrak{c}$ over the unique cusp ∞ of $\mathrm{SL}_2(\mathbf{Z})$; this equals the index of the corresponding maximal parabolic subgroups considered modulo $\{\pm 1\}$. For the parameters $\epsilon_{\mathfrak{c}}$ and $\epsilon'_{\mathfrak{c}}$, we take

$$\epsilon_{\mathfrak{c}} = m_{\mathfrak{c}}\epsilon \quad \text{and} \quad \epsilon'_{\mathfrak{c}} = m_{\mathfrak{c}}\epsilon'.$$

Using the definition of $C_{\mathfrak{c}}(\gamma)$, it is not hard to show that

$$\min_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma) \geq m_{\mathfrak{c}}.$$

This implies that the parameters $\epsilon_{\mathfrak{c}}$ and $\epsilon'_{\mathfrak{c}}$ satisfy the conditions in Theorem 7.1. As in Theorem 7.1, let Y be the complement of the discs $D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. Then Y is the inverse image of Y_0 in $\Gamma \backslash \mathbf{H}$.

We will need an upper bound on the point counting function $N_{\Gamma}(z, z, 17)$ for $z \in Y_0$. It is clear from the definition of $N_{\Gamma}(z, z, U)$ that

$$\sup_{z \in Y} N_{\Gamma}(z, z, U) \leq \sup_{z \in Y_0} N_{\mathrm{SL}_2(\mathbf{Z})}(z, U).$$

Using the methods of § 8.1, we have

$$N_{\mathrm{SL}_2(\mathbf{Z})}(z, 17) \leq 226 \quad \text{for all } z \in Y_0. \tag{8.2}$$

We next compute suitable A and B satisfying (5.11). For this we use (8.2) and the remaining part of § 8.1, with the same parameters $\alpha^{\pm}$, $\beta^{\pm}$ and $\gamma^{\pm}$. The result is

$$A = -3.00 \cdot 10^4 \quad \text{and} \quad B = 1.58 \cdot 10^4.$$

We next find a suitable value of the parameter C. We use Lemma 3.2, which says

$$\Phi_{\Gamma}(z, \lambda) \leq \frac{\pi}{(2\pi - 4)^2} N_{\mathrm{SL}_2(\mathbf{Z})}(z, 17)\lambda \quad \text{for all } z \in \mathbf{H} \text{ and all } \lambda \geq 1/4.$$

The inequality (8.2) implies that we can take

$$C = 137.$$

We continue with explicit bounds on the canonical $(1, 1)$-form. For the parameter a from Proposition 6.1, we take

$$a = 1.44.$$

Again using the method from § 8.1, we compute an upper bound for $N_{\mathrm{SL}_2(\mathbf{Z})}(z, 2a^2 - 1)$ for $z \in Y_0$. The result is

$$N_{\mathrm{SL}_2(\mathbf{Z})}(z, 2a^2 - 1) \leq 58 \quad \text{for all } z \in Y_0.$$

Substituting this in the bound from Proposition 6.1, we see that

$$\sup_Y F_{\Gamma} \leq 25.7.$$

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For every cusp $\mathfrak{c}$, Lemma 7.4 implies

$$\sup_{D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})} F_{\Gamma} \le \max\left\{1, \left(\frac{\epsilon_{\mathfrak{c}}}{2\pi}\right)^2\right\} \sup_Y F_{\Gamma}.$$

From the definition of $\epsilon_{\mathfrak{c}}$ and the fact that all ramification indices $m_{\mathfrak{c}}$ are bounded by the level n of Γ, we conclude

$$\sup_X F_{\Gamma} \le \max\left\{1, \left(\frac{n\epsilon}{2\pi}\right)^2\right\} \sup_Y F_{\Gamma}$$
$$\le \max\{25.7, 0.0126n^2\}.$$

Finally, we consider the invariant ζ_{Γ}. Using $\int_X \mu_X^{\mathrm{can}} = 1$ and $g_X \ge 1$, we make the rather coarse estimate

$$\zeta_{\Gamma} \le \sup_X F_{\Gamma} \le \max\{25.7, 0.0126n^2\}.$$

Proof of Theorem 1.2. With the above estimates, we obtain the following bounds on the various constants in the theorem:

$$S \le \max\{172, 3.79n\},$$
$$T(\epsilon_{\mathfrak{c}}) \le 0.00313n^2,$$
$$T(\epsilon_{\mathfrak{c}}') \le 0.0436n^2,$$
$$\tilde{A}_{\mathfrak{c}} \ge -3.00 \cdot 10^4 - 0.0279n,$$
$$\tilde{B}_{\mathfrak{c}} \le 1.58 \cdot 10^4 + 0.0279n.$$

The theorem follows from Theorem 7.1 and the above bounds. □

Appendix: Bounds on Legendre functions

In this appendix, we prove a number of bounds on the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ that are used in the rest of the paper.

Lemma A.1. *Let $u \in [1, 3]$ and $\lambda \ge 0$ be such that $\lambda(u - 1) \le \frac{1}{2}$, and let $s \in \mathbf{C}$ be such that $s(1 - s) = \lambda$. Then the real number $P_{s-1}^{-1}(u)$ satisfies the inequalities*

$$(2 - 4/\pi)\sqrt{\frac{u-1}{u+1}} \le P_{s-1}^{-1}(u) \le (4/\pi)\sqrt{\frac{u-1}{u+1}}.$$

Proof. We start by expressing the Legendre function P_ν^μ in terms of Gauß's hypergeometric function $F(a, b; c; z)$. Because of the many symmetries satisfied by the hypergeometric function (see Erdélyi et al. [6, Chapter II]), there are lots of ways to do this. Using [6, §3.2, equation 3] gives

$$P_{s-1}^{-1}(u) = \sqrt{\frac{u-1}{u+1}} F\left(s, 1 - s; 2; \frac{1-u}{2}\right).$$

Next we use the hypergeometric series for $F(a, b; c; z)$ with $z < 1$ (see [6, §2.1, equation 2]):

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{A.1}$$

where

$$(y)_n = \Gamma(y + n)/\Gamma(y) = y(y + 1) \cdots (y + n - 1).$$

Putting $x = \frac{u-1}{2}$ for a moment, using (A.1) and applying the triangle inequality, we get the bound

$$\left|F(s, 1 - s; 2; -x) - 1\right| \le \sum_{n \ge 1} \left|\frac{(s)_n (1 - s)_n}{(2)_n n!} (-x)^n\right|$$

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The assumption $\lambda(u-1) \leq \frac{1}{2}$ is equivalent to $\lambda x \leq \frac{1}{4}$. Therefore the n-th term in the series on the right-hand side can be bounded as follows:

$$\left| \frac{(s)_n(1-s)_n}{(2)_n n!}(-x)^n \right| = \frac{\prod_{k=0}^{n-1}\left(s(1-s)x + k(k+1)x\right)}{(2)_n n!}$$

$$\leq \frac{\prod_{k=0}^{n-1}\left(\frac{1}{4} + k(k+1)\right)}{(2)_n n!}$$

$$= \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n n!}.$$

This implies that

$$\left| F(s, 1-s; 2; -x) - 1 \right| \leq F(\tfrac{1}{2}, \tfrac{1}{2}; 2; 1) - 1$$
$$= 4/\pi - 1,$$

where the last equality follows from the formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \Re c > 0 \text{ and } \Re c > \Re(a+b) \tag{A.2}$$

(see Erdélyi et al. [6, § 2.1.3, equation 14] or Iwaniec [12, equation B.20]) and the fact that $\Gamma(3/2) = \sqrt{\pi}/2$. We conclude that

$$\left| P_{s-1}^{-1} - \sqrt{\frac{u-1}{u+1}} \right| = \sqrt{\frac{u-1}{u+1}} \left| F(s, 1-s; 2; -x) - 1 \right|$$

$$\leq \sqrt{\frac{u-1}{u+1}}(4/\pi - 1)$$

$$= (4/\pi - 1)\sqrt{\frac{u-1}{u+1}},$$

which is equivalent to the inequalities in the statement of the lemma. $\qquad \square$

Lemma A.2. *For all real numbers $\nu \geq 0$ and $u > 1$, the real number $Q'_\nu(u)$ satisfies*

$$\left(\frac{2}{u+1}\right)^\nu \frac{1}{u^2-1} \leq Q'_\nu(u) \leq 0.$$

Proof. We express Q'_ν in terms of the hypergeometric function using [6, § 3.6.1, equation 5, and § 3.2, equation 36]:

$$Q'_\nu(u) = -\left(\frac{2}{u+1}\right)^\nu \frac{1}{u^2-1} \frac{\Gamma(1+\nu)\Gamma(2+\nu)}{\Gamma(2+2\nu)} F\left(\nu, 1+\nu; 2+2\nu; \frac{2}{u+1}\right).$$

Since $2/(u+1) < 1$, the hypergeometric function is given by the series (A.1). The non-negativity of all the arguments gives the bounds

$$0 \leq F\left(\nu, 1+\nu; 2+2\nu; \frac{2}{u+1}\right)$$

$$\leq F(\nu, 1+\nu; 2+2\nu; 1)$$

$$= \frac{\Gamma(2+2\nu)\Gamma(1)}{\Gamma(2+\nu)\Gamma(1+\nu)};$$

the last equality follows from (A.2). Combining this with the above formula for $Q'_\nu(u)$ yields the claim. $\qquad \square$

Lemma A.3. *For all real numbers a, b and y with $b \geq a > 0$, we have*

$$\exp\left(-\frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right)\frac{(a^2 + y^2)^{a/2-1/4}}{(b^2 + y^2)^{b/2-1/4}} \leq \left|\frac{\Gamma(a + iy)}{\Gamma(b + iy)}\right|$$

$$\leq \exp\left(b - a + \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right)\frac{(a^2 + y^2)^{a/2-1/4}}{(b^2 + y^2)^{b/2-1/4}}.$$

Proof. We use Binet's formula for $\log\Gamma$ (see Erdélyi et al. [6, § 1.9, equation 4]):

$$\log\Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log 2\pi + \int_0^\infty B(t)\exp(-zt)dt \quad \text{for } \Re z > 0,$$

where

$$B(t) = \frac{(\exp(t) - 1)^{-1} - t^{-1} + \frac{1}{2}}{t}$$

$$= \frac{\frac{t}{2}\coth\frac{t}{2} - 1}{t^2}.$$

We write

$$M(a, y) = \Re\int_0^\infty B(t)\exp(-(a + iy)t)dt$$

$$= \int_0^\infty B(t)\exp(-at)\cos(yt)dt.$$

Then we have

$$\log\left|\frac{\Gamma(a + iy)}{\Gamma(b + iy)}\right| = \Re\Big((a + iy - 1/2)\log(a + iy) - (b + iy - 1/2)\log(b + iy)\Big)$$

$$- a + b + M(a, y) - M(b, y)$$

$$= (a - 1/2)\log|a + iy| - (b - 1/2)\log|b + iy|$$

$$- y\arg(a + iy) + y\arg(b + iy) - a + b + M(a, y) - M(b, y).$$

We note that

$$y\arg(a + iy) - y\arg(b + iy) = y\arctan\frac{(b - a)y}{ab + y^2} \in [0, b - a].$$

Using $b \geq a$, we conclude

$$\log\left|\frac{\Gamma(a + iy)}{\Gamma(b + iy)}\right| \leq (a - 1/2)\log|a + iy| - (b - 1/2)\log|b + iy| - a + b + M(a, y) - M(b, y)$$

and

$$\log\left|\frac{\Gamma(a + iy)}{\Gamma(b + iy)}\right| \geq (a - 1/2)\log|a + iy| - (b - 1/2)\log|b + iy| + M(a, y) - M(b, y).$$

It remains to bound $M(a, y) - M(b, y)$. The function B satisfies

$$0 < B(t) \leq \lim_{x\to 0} B(x) = 1/12 \quad \text{for all } t > 0.$$

Using this and the positivity of $\exp(-at) - \exp(-bt)$, we bound $M(a, y) - M(b, y)$ as follows:

$$\big|M(a, y) - M(b, y)\big| \leq \int_0^\infty B(t)\big(\exp(-at) - \exp(-bt)\big)|\cos(yt)|\,dt$$

$$= \int_0^\infty B(t)\big(\exp(-at) - \exp(-bt)\big)dt$$

$$\leq \frac{1}{12}\int_0^\infty \big(\exp(-at) - \exp(-bt)\big)dt$$

$$= \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right).$$

This implies the inequality we wanted to prove. $\qquad\square$

Corollary A.4. *For $b \geq a > 0$, $b \geq 1/2$, and $y \in \mathbf{R}$, we have*

$$\left|\frac{\Gamma(a+iy)}{\Gamma(b+iy)}\right| \leq \exp\left(b - a + \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right)(a^2 + y^2)^{-(b-a)/2}.$$

Given a real number $\sigma \in (0, 1/2)$, we consider the strip S_σ defined by (5.1). We put

$$
\begin{aligned}
C_\sigma &= \max\{1, \tan \pi\sigma\}(\sigma^{-1} - 1)^{1/4} \exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2} + \sigma)}\right), \\
C'_\sigma &= \max\{1, \tan \pi\sigma\}(\sigma^{-1} - 1)^{1/4} \exp\left(\frac{1}{2} + \frac{1}{24(1 - \sigma)(\frac{3}{2} - \sigma)}\right).
\end{aligned}
\tag{A.3}
$$

Proposition A.5. *Let m be an even non-negative integer, and let $\sigma \in (0, 1/2)$. For all $s \in S_\sigma$ and all $u > 1$, we have*

$$\left|P_{s-1}^{-m}(u)\right| \leq \left|s(1-s)\right|^{-(2m+1)/4} \frac{C_\sigma x^{m-\sigma} + C'_\sigma x^{m-1+\sigma}}{\sqrt{\pi}(4(u^2 - 1))^{m/2}} \sum_{n=0}^{\infty} \frac{|(\frac{1}{2} - m)_n|}{n!} x^{-2n},$$

where

$$x = u + \sqrt{u^2 - 1}, \quad u = \frac{x + x^{-1}}{2}.$$

Proof. We use the following expression for P_{s-1}^{-m} (see Erdélyi et al. [6, §3.2, equation 27]):

$$
\begin{aligned}
P_{s-1}^{-m}(u) = {} & \frac{\Gamma(-\frac{1}{2} + s)}{\sqrt{\pi}\Gamma(m + s)} \frac{x^{m+s-1}}{(x - x^{-1})^m} F\left(\tfrac{1}{2} - m, 1 - m - s; \tfrac{3}{2} - s; x^{-2}\right) \\
& + \frac{\Gamma(\frac{1}{2} - s)}{\sqrt{\pi}\Gamma(m + 1 - s)} \frac{x^{m-s}}{(x - x^{-1})^m} F\left(\tfrac{1}{2} - m, -m + s; \tfrac{1}{2} + s; x^{-2}\right).
\end{aligned}
$$

Using the hypergeometric series (A.1) and the functional equation

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z},$$

we get

$$
\begin{aligned}
\sqrt{\pi}(x - x^{-1})^m P_{s-1}^{-m}(u) = {} & \frac{\Gamma(-\frac{1}{2} + s)}{\Gamma(m + s)} x^{m-1+s} \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n(-m + 1 - s)_n}{(\frac{3}{2} - s)_n} \frac{x^{-2n}}{n!} \\
& + \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(m + 1 - s)} x^{m-s} \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n(-m + s)_n}{(\frac{1}{2} + s)_n} \frac{x^{-2n}}{n!} \\
= {} & \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n}{n!} \left\{ \frac{\Gamma(-\frac{1}{2} + s)}{\Gamma(m + s)} \frac{\Gamma(\frac{3}{2} - s)\Gamma(n - m + 1 - s)}{\Gamma(n + \frac{3}{2} - s)\Gamma(-m + 1 - s)} x^{m-1+s-2n} \right. \\
& \left. + \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(m + 1 - s)} \frac{\Gamma(\frac{1}{2} + s)\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)\Gamma(-m + s)} x^{m-s-2n} \right\} \\
= {} & \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n}{n!} \left\{ \frac{\sin \pi(m + s)}{\sin \pi(-\frac{1}{2} + s)} \frac{\Gamma(n - m + 1 - s)}{\Gamma(n + \frac{3}{2} - s)} x^{m-1+s-2n} \right. \\
& \left. + \frac{\sin \pi(m + 1 - s)}{\sin \pi(\frac{1}{2} - s)} \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} x^{m-s-2n} \right\}.
\end{aligned}
$$

Basic trigonometric manipulations simplify this to

$$
\begin{aligned}
P_{s-1}^{-m}(u) = {} & \frac{(-1)^m \tan(\pi s)}{\sqrt{\pi}(x - x^{-1})^m} \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n}{n!} \left\{ \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} x^{m-s-2n} \right. \\
& \left. - \frac{\Gamma(n - m + 1 - s)}{\Gamma(n + \frac{3}{2} - s)} x^{m-1+s-2n} \right\}.
\end{aligned}
\tag{A.4}
$$

On the right-hand side, the pole of $\tan(\pi s)$ at $s = 1/2$ is cancelled by a corresponding zero of the function defined by the sum.

For fixed $u > 1$, we consider the holomorphic function

$$H(s) = \left(s(1-s)\right)^{(2m+1)/4} \tan(\pi s) \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - m)_n}{n!} \left\{ \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} x^{m-s-2n} \right.$$
$$\left. - \frac{\Gamma(n - m + 1 - s)}{\Gamma(n + \frac{3}{2} - s)} x^{m-1+s-2n} \right\} \tag{A.5}$$

on S_σ, where we have fixed a branch of $s \mapsto (s(1-s))^{(2m+1)/4}$. Because $H(s) = H(1-s)$, the Phragmén–Lindelöf principle gives

$$\sup_{s \in S_\sigma} |H(s)| \leq \sup_{y \in \mathbf{R}} |H(\sigma + iy)|.$$

Together with (A.4), this implies

$$|P_{s-1}^{-m}(u)| \leq \frac{|s(1-s)|^{-(2m+1)/4} \sup_{y \in \mathbf{R}} |H(\sigma + iy)|}{\sqrt{\pi}(4(u^2 - 1))^{m/2}} \quad \text{for all } s \in S_\sigma.$$

Let $y \in \mathbf{R}$ and $s = \sigma + iy$. Then we have

$$\left| s(1-s) \right|^{(2m+1)/4} = (\sigma^2 + y^2)^{(2m+1)/8}((1-\sigma)^2 + y^2)^{(2m+1)/8}.$$

A straightforward calculation gives

$$|\tan \pi s| = |\tan \pi(\sigma + iy)| \leq \max\{1, \tan \pi\sigma\}.$$

Using Corollary A.3 and the assumption that m is even, we bound the quotients of Γ-functions appearing on the right-hand side of (A.5) independently of n:

$$\left| \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} \right| = \frac{|\Gamma(n + s)|}{|\Gamma(n + \frac{1}{2} + s)|} \prod_{j=n-m}^{n-1} \frac{1}{|j + s|}$$

$$\leq \exp\left(\frac{1}{2} + \frac{1}{24(n + \sigma)(n + \frac{1}{2} + \sigma)} \right) |n + \sigma + iy|^{-1/2} \frac{1}{|\sigma + iy|^{m/2}|1 - \sigma + iy|^{m/2}}$$

$$\leq \exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2} + \sigma)} \right) \frac{1}{(\sigma^2 + y^2)^{(m+1)/4}((1 - \sigma)^2 + y^2)^{m/4}}.$$

This implies

$$\left| s(1-s) \right|^{(2m+1)/4} \left| \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} \right| \leq \exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2} + \sigma)} \right)$$
$$\cdot \frac{(\sigma^2 + y^2)^{(2m+1)/8}((1 - \sigma)^2 + y^2)^{(2m+1)/8}}{(\sigma^2 + y^2)^{(m+1)/4}((1 - \sigma)^2 + y^2)^{m/4}}$$

$$= \exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2} + \sigma)} \right) \frac{((1 - \sigma)^2 + y^2)^{1/8}}{(\sigma^2 + y^2)^{1/8}}$$

$$\leq \exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2} + \sigma)} \right) \frac{(1 - \sigma)^{1/4}}{\sigma^{1/4}}$$

and hence

$$\left| s(1-s) \right|^{(2m+1)/4} |\tan \pi s| \left| \frac{\Gamma(n - m + s)}{\Gamma(n + \frac{1}{2} + s)} \right| \leq C_\sigma.$$

Similarly,

$$\left| \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} \right| \le \exp\left(\frac{1}{2} + \frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)} \right) \frac{1}{((1-\sigma)^2+y^2)^{(m+1)/4}(\sigma^2+y^2)^{m/4}}$$

and

$$\left| s(1-s) \right|^{(2m+1)/4} \left| \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} \right| \le \exp\left(\frac{1}{2} + \frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)} \right)$$

$$\cdot \frac{(\sigma^2+y^2)^{(2m+1)/8}((1-\sigma)^2+y^2)^{(2m+1)/8}}{((1-\sigma)^2+y^2)^{m/4}(\sigma^2+y^2)^{(m+1)/4}}$$

$$= \exp\left(\frac{1}{2} + \frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)} \right) \frac{((1-\sigma)^2+y^2)^{1/8}}{(\sigma^2+y^2)^{1/8}}$$

$$\le \exp\left(\frac{1}{2} + \frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)} \right) \frac{(1-\sigma)^{1/4}}{\sigma^{1/4}}.$$

This implies

$$\left| s(1-s) \right|^{(2m+1)/4} |\tan \pi s| \left| \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} \right| \le C'_\sigma.$$

We conclude that

$$\sup_{y \in \mathbf{R}} |H(\sigma+iy)| \le \sum_{n=0}^\infty \frac{|(\frac{1}{2}-m)_n|}{n!} \left(C_\sigma x^{m-\sigma-2n} + C'_\sigma x^{m-1+\sigma-2n} \right)$$

$$= \left(C_\sigma x^{m-\sigma} + C'_\sigma x^{m-1+\sigma} \right) \sum_{n=0}^\infty \frac{|(\frac{1}{2}-m)_n|}{n!} x^{-2n}.$$

This finishes the proof. □

With C_σ and C'_σ as in (A.3), we define an elementary function $p_\sigma \colon [1,\infty) \to \mathbf{R}$ by

$$p_\sigma(u) = \frac{C_\sigma x^{2-\sigma} + C'_\sigma x^{1+\sigma}}{4\sqrt{\pi}} \left((1-x^{-2})^{3/2} + 3x^{-2} \right), \quad \text{where } x = u + \sqrt{u^2-1}. \tag{A.6}$$

Corollary A.6. *For all $\sigma \in (0,1/2)$, $s \in S_\sigma$ and $u > 1$, we have*

$$|P^{-2}_{s-1}(u)| \le |s(1-s)|^{-5/4} \frac{p_\sigma(u)}{u^2-1}.$$

Proof. We note that

$$|(-\tfrac{3}{2})_n| = \begin{cases} -(-\tfrac{3}{2})_n & \text{for } n=1, \\ (-\tfrac{3}{2})_n & \text{otherwise.} \end{cases}$$

This implies that for $z \in (0,1)$, we have

$$\sum_{n=0}^\infty \frac{|(-\tfrac{3}{2})_n|}{n!} z^n = \sum_{n=0}^\infty \frac{(-\tfrac{3}{2})_n}{n!} z^n - 2\frac{(-\tfrac{3}{2})_1}{1!} z^1$$

$$= (1-z)^{3/2} + 3z.$$

The claim immediately follows from this identity and Proposition A.5. □

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