EXAM
GALOIS REPRESENTATIONS AND AUTOMORPHIC FORMS

12 January 2017 10:00–13:00

- You only have to answer *two* questions of your choice. You may hand in answers to more questions, but your mark will be based on up to two questions on which you did best.
- The use of books, notes and electronic devices is not allowed.
- You may use results from the lecture notes (but not from the exercises) without proof. Clearly state which results you use.
- If you are unable to answer a subquestion, you may still use the result in the remainder of the question.

Question 1. (a) Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$, and let $\mathbb{A}$ be the adèle ring of $\mathbb{Q}$. Consider the quotients $(\mathbb{R}\times\widehat{\mathbb{Z}})/\mathbb{Z}$ and $\mathbb{A}/\mathbb{Q} = (\mathbb{R}\times\mathbb{A}^\infty)/\mathbb{Q}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are diagonally embedded in $\mathbb{R}\times\widehat{\mathbb{Z}}$ and $\mathbb{R}\times\mathbb{A}^\infty$, respectively. Show that the map

$$\begin{aligned}(\mathbb{R}\times\widehat{\mathbb{Z}})/\mathbb{Z} &\longrightarrow (\mathbb{R}\times\mathbb{A}^\infty)/\mathbb{Q}\\ (x,y) \bmod \mathbb{Z} &\longmapsto (x,y) \bmod \mathbb{Q}\end{aligned}$$

is an isomorphism of groups.

(b) Let $\mathbb{Q}(i) = \{a+bi \mid a,b\in\mathbb{Q}\}\subset\mathbb{C}$ be the field of Gauss numbers, and let $\mathbb{A}_{\mathbb{Q}(i)}^\times = \mathbb{C}^\times\times\mathbb{A}_{\mathbb{Q}(i)}^{\infty,\times}$ be the idèle group of $\mathbb{Q}(i)$. The ring of integers $\mathbb{Z}[i]$ of $\mathbb{Q}(i)$ is known to be a principal ideal domain with unit group $\langle i\rangle = \{\pm1,\pm i\}$. Use this to prove that the map

$$\begin{aligned}\langle i\rangle\backslash\bigl(\mathbb{C}^\times\times\widehat{\mathbb{Z}[i]}^\times\bigr) &\longrightarrow \mathbb{Q}(i)^\times\backslash\mathbb{A}_{\mathbb{Q}(i)}^\times\\ \langle i\rangle\cdot(x,y) &\longmapsto \mathbb{Q}(i)^\times\cdot(x,y)\end{aligned}$$

is an isomorphism of groups, where $\langle i\rangle$ and $\mathbb{Q}(i)^\times$ are embedded diagonally in $\mathbb{C}^\times\times\widehat{\mathbb{Z}[i]}^\times$ and $\mathbb{A}_{\mathbb{Q}(i)}^\times$, respectively.

Question 2. Let F be a number field, and let E be an elliptic curve over F.

(a) The group law on E can be given by rational functions with coefficients in F, and the n-torsion subgroup $E[n](\overline{F})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. Use these facts to show that the action of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ on $E[n](\overline{F})$ induces a group homomorphism $\mathrm{Gal}(\overline{F}/F)\to\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ that is well-defined up to conjugacy by elements of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

(b) Let ℓ be a prime number. Define the ℓ-adic Tate module $T_\ell(E)$, including the action of $\mathrm{Gal}(\overline{F}/F)$ on $T_\ell(E)$.

(c) Let ℓ be a prime number, and let v be a finite place of F with $v\nmid\ell$ and such that E has good reduction at v. Express the number of points on E over the residue field $\kappa(v)$ in terms of the Galois representation $T_\ell(E)$.

Question 3. Let F be a number field, and let E be an elliptic curve over F. For every finite place v of F at which E has good reduction, let $\kappa(v)$ be the residue field, and let $a_v(E) = 1 - \#E(\kappa(v)) + \#\kappa(v)$.

Consider the set S of finite places v of F with $v \nmid 3$ and such that E has good reduction at v and the integer $a_v(E)$ is divisible by 3. In this question we investigate the density δ_S of the set S in the set of all places of F.

(a) Let Z be the normal subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$ consisting of the matrices $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ with $x \in \mathbb{F}_3^\times$. Let $\mathrm{PGL}_2(\mathbb{F}_3)$ be the quotient $\mathrm{GL}_2(\mathbb{F}_3)/Z$. Show that $\mathrm{PGL}_2(\mathbb{F}_3)$ is isomorphic to the fourth symmetric group S_4.
Hint: consider the action of the group $\mathrm{PGL}_2(\mathbb{F}_3)$ on the set $\mathbb{P}^1(\mathbb{F}_3)$ of lines in the 2-dimensional space $(\mathbb{F}_3)^2$ passing through the origin.

(b) Assume that F and E are such that the Galois representation

$$\rho_{E,3} \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{F}_3}(E[3](\bar{F})) \simeq \mathrm{GL}_2(\mathbb{F}_3)$$

on the 3-torsion of E is surjective. Show that the density δ_S equals $3/8$.

(c) It is known that the group S_4 has 11 subgroups up to conjugacy. Show that as F and E vary (and $\rho_{E,3}$ is not necessarily surjective), there are at most 11 possible values for the density δ_S.

Question 4. Let F be a p-adic local field, and let $\mathcal{O}_F$ be its ring of integers. We write $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O}_F)$.

(a) State what it means for an irreducible admissible smooth representation (π, V) of G to be *unramified*.

(b) Give the definition of the (*unramified* or *spherical*) *Hecke algebra* $\mathcal{H}(G, K)$, including the algebra structure.

(c) Give the definition of the *Satake transform*

$$\mathcal{S} \colon \mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]^{S_2}.$$

(Here S_2 acts on $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]$ by permuting the variables.)

(d) Explain that using $\mathcal{S}$ one can attach to any unramified irreducible smooth representation π of $\mathrm{GL}_2(F)$ a semi-simple conjugacy class ϕ_π in $\mathrm{GL}_2(\mathbb{C})$ (the *Satake parameter* of π).

Hint: For (b) and (c), you will need the subgroups $B, N, T \subset G$, where B consists of the upper triangular matrices, N consists of the upper triangular matrices with ones on the diagonal, and T consists of the diagonal matrices. If H is one of the groups G, B, N, T, write μ_H for the left Haar measure on H such that the compact open subgroup $H \cap K$ of H has measure 1. It is known that μ_G, μ_T and μ_N are also right Haar measures, but μ_B is not; you will need the modular function $\delta_B \colon B \to \mathbb{R}_{>0}$.

Question 5. [1] In this question we investigate some consequences of the local Langlands theorem and the global Langlands conjecture. Let F be a number field, let ℓ be a prime number, and fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. The *symmetric square representation*

$$\mathrm{Sym}^2\colon\ \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$$

is a 3-dimensional irreducible representation obtained from the action of $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ on the space of homogenous polynomials $f \in \overline{\mathbb{Q}}_\ell[X_1, X_2]$ of degree 2 defined by

$$g \cdot f := f\left(aX_1 + cX_2, bX_1 + dX_2\right) \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell), f \in \overline{\mathbb{Q}}_\ell[X_1, X_2].$$

We have seen in the practice exam that for all $g \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, writing $t = \mathrm{Tr}(g)$ and $d = \det(g)$, we have

$$\mathrm{charpol}(\mathrm{Sym}^2(g)) = X^3 - (t^2 - d)X^2 + d(t^2 - d)X - d^3 \in \overline{\mathbb{Q}}_\ell[X].$$

In the solutions below, you may use this as a fact that you do not need to justify.

(a) We first consider the local case. Let v be a finite place of F, let F_v be the completion of F at v, and let q_v be the cardinality of the residue field of F_v. Let π_v be an unramified smooth admissible irreducible representation of $\mathrm{GL}_2(F_v)$, let ϕ_{π_v} be the Satake parameter of π_v (a conjugacy class in $\mathrm{GL}_2(\mathbb{C})$), and write $d_v = \det(\phi_{\pi_v})$ and $t_v = \mathrm{Tr}(\phi_{\pi_v})$. Show that to π_v we may attach a smooth admissible irreducible representation Π_v of $\mathrm{GL}_3(F_v)$, in such a way that the L-function of Π_v equals

$$\frac{1}{1 - (t_v^2 - d_v)q_v^{-s} + d_v(t_v^2 - d_v)q_v^{-2s} - d_v^3 q_v^{-3s}}.$$

A Galois representation $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ that is isomorphic to a subquotient of the ℓ-adic étale cohomology of an algebraic variety X over F is called *geometric*. In number theory and algebraic geometry, it is the geometric Galois representations that we are interested in, and a great deal of research is done on these representations. In particular, there is a famous conjecture of Fontaine and Mazur that characterizes the irreducible Galois representations ρ_π arising from some algebraic cuspidal automorphic representation π as those which are geometric.

In the questions below you may use that the class of geometric Galois representations is stable under all possible linear algebra constructions, meaning that if V and W are geometric ℓ-adic Galois representations, then representations such as $V \oplus W$, $V \otimes W$, $V^{\otimes 5} \otimes W^\vee$, $\wedge^3 V \otimes W^{\otimes 2}$, and so on, are all geometric as well. In particular, if $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}_\ell^2$ is a 2-dimensional geometric Galois representation, then the symmetric square of ρ, i.e. the representation

$$\mathrm{Sym}^2(\rho) := \mathrm{Sym}^2 \circ \rho\colon \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}_\ell^3,$$

is geometric as well.

In the rest of this question, let π be an algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. We call π *non-special* if there exists no strict algebraic[2]

[1] Students who choose this question will receive a bonus point on their exam score.

[2] "Algebraic" means that H is equal to the vanishing locus of a collection of polynomials.

subgroup $H \subset \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that $\rho_\pi(\mathrm{Gal}(\overline{F}/F)) \subset H$. For these non-special π, the composition $r \circ \rho_\pi$ is irreducible for any irreducible algebraic representation $r\colon \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$. Specializing to $r = \mathrm{Sym}^2$, the representation $\mathrm{Sym}^2(\rho_\pi)$ is irreducible as well. We will assume from now on that π is non-special.

(b) Assuming the global Langlands conjecture and the Fontaine-Mazur conjecture, prove that there exists an automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ such that at all F-places v where π_v is unramified, Π_v is unramified as well, and the Satake parameter ϕ_{Π_v} of Π_v equals $\mathrm{Sym}^2(\phi_{\pi_v})$, where ϕ_{π_v} is the Satake parameter of π_v.

(c) Prove that the automorphic representation Π is uniquely characterized (up to isomorphism) by this property.

(d) Let E be an elliptic curve over F without complex multiplication, so that the Galois representation on its Tate module $V_\ell(E) := \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(E)$ is non-special in the sense that the image of $\mathrm{Gal}(\overline{F}/F)$ is not contained in any strict algebraic subgroup of $\mathrm{GL}_{\overline{\mathbb{Q}}_\ell}(V_\ell(E)) \cong \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$. Let S be the set of finite places of F where E has good reduction. For all $v \in S$, let $\kappa(v)$ be the residue field of F at v, let $q_v = \#\kappa(v)$, and let $a_v(E) = 1 - \#E(\kappa(v)) + q_v$. Prove that the Euler product

$$\prod_{v \in S} \frac{1}{1 - (a_v(E)^2 - q_v)q_v^{-s} + q_v(a_v(E)^2 - q_v)q_v^{-2s} - q_v^3 q_v^{-3s}}$$

converges for $\Re s$ sufficiently large and has an analytic continuation to the whole complex plane that satisfies a functional equation (which you do not need to specify).

Good luck!

Thank you for following our course, it has been a pleasure.