THE BROWNIAN LIMIT OF SEPARABLE PERMUTATIONS

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This is a joint work with Frédérique Bassino, Valentin Féray, Lucas Gerin and Adeline Pierrot. The diagrams shown in Figure 1 are a courtesy of Carine Pivoteau.

The full paper is available at `http://arxiv.org/abs/1602.04960`.

Figure 1 shows two large separable permutations, drawn uniformly at random among separable permutations of the same size. The goal of this work is to explain these diagrams, by describing the "limit shape" of a uniform random separable permutation of size n, as n goes to infinity.

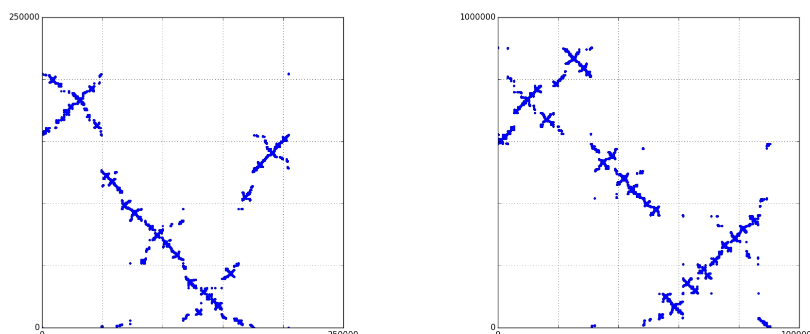

FIGURE 1. Two typical separable permutations of sizes respectively $n = 204\,523$ and $n = 903\,073$ (a permutation σ is represented here with its *diagram*: for every $i \leq n$, there is a dot at coordinates $(i, \sigma(i))$).

Context of our work. Describing typical properties of permutations in a given class is a question that has received quite a lot of attention in the last few years. Most of the attention is focused on permutation classes $\mathrm{Av}(\tau)$ for τ of size 3. See for instance the very precise description of the asymptotic shape of permutations in these classes by Miner and Pak [12], or the link between large random permutations in these classes and the Brownian excursion explained by Hoffman, Rizzolo and Slivken [5, 6]. Another approach, also of importance for our work, is that of Janson [9], following earlier works by several authors [1, 2, 3, 7, 10, 14]: it consists in studying the (normalized) number of occurrences of any given pattern π in a uniform random permutation σ avoiding τ, and in finding its limiting distribution.

Our work focuses on separable permutations, which are defined by the avoidance of the two patterns 2413 and 3142. The class of separable permutations is one of the most well-known permutation classes after $\mathrm{Av}(\tau)$ for τ of size 3. The popularity of separable permutation is maybe explained by their simple and robust structure. The class of separable permutations is indeed the smallest family of permutations closed under the sum $\oplus$ and skew-sum $\ominus$ operations. (And thus, they form the simplest non-trivial substitution-closed class.) Consequently, separable permutations may be encoded by signed Schröder trees. A signed Schröder tree of size n is a tree with n leaves whose internal vertices have degree at least 2 and are equipped with a + or − sign. In a nutshell, a signed Schröder tree associated with a separable permutation σ describes a sequence of nested $\oplus$ and $\ominus$ operations that produce σ. The tree structure of separable permutation is essential for our purpose. Note that the correspondence between separable permutations and signed Schröder trees can be made one-to-one by imposing the following constraint on the trees: on each path from the root to a leaf, the signs alternate.

Main result. Our main result is the description of the asymptotics in n of the number of occurrences of any fixed pattern π in a uniform separable permutation of size n. For any pattern π of size k, and any permutation σ of size n, we denote by

$$\widetilde{\mathrm{occ}}(\pi,\sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}}$$

the proportion of occurrences of π in σ. Our main theorem is the following:

Theorem 1. *Let $\boldsymbol{\sigma}_n$ be a uniform random separable permutation of size n. There exists a collection of random variables (Λ_π), π ranging over all permutations, such that for all π, $0 \leq \Lambda_\pi \leq 1$ and when $n \to +\infty$,*

$$\widetilde{\mathrm{occ}}(\pi,\boldsymbol{\sigma}_n) \stackrel{(d)}{\to} \Lambda_\pi,$$

where $\stackrel{(d)}{\to}$ denotes the convergence in distribution.

Note that we can refine the above statement to prove that the convergence holds jointly for patterns $\pi_1, \dots, \pi_r$. Moreover, we can also prove that if π is a separable permutation of size at least 2, Λ_π is a non-deterministic random variable. (Clearly, $\Lambda_\pi = 0$ or 1 otherwise.) These additional results, although interesting, are not further discussed in this abstract.

Theorem 1 is not just an existential result: we give for any pattern π a construction of Λ_π.

Construction of Λ_π. An *excursion* is a continuous function $f : [0,1] \to [0,+\infty)$ with $f(0) = f(1) = 0$. From an excursion f and a set of points $\mathbf{x} = \{x_1, \ldots, x_k\}$ in $[0,1]$, there is a classical construction which builds a Schröder tree, looking at the local minima of f between the x_i's. This construction is illustrated by Figure 2 (forgetting about the signs for the moment). We define a *signed excursion* as an excursion where the local minima are given a sign $+$ or $-$. As shown by Figure 2 (with the signs), the previous construction naturally extends to signed excursions, producing then signed Schröder trees. We denote by $\mathrm{Tree}_\pm(f, s, \mathbf{x})$ the signed Schröder tree associated with the signed excursion (f, s) and the set of points $\mathbf{x}$.

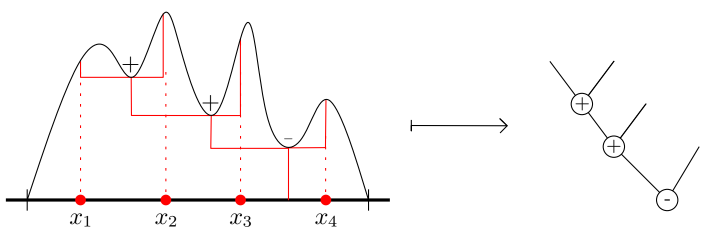

FIGURE 2. Extracting a (signed) tree from a (signed) excursion.

Let π be any pattern of size k. For any signed excursion (f, s) we define $\Psi_\pi(f, s)$ as the probability that $\mathrm{Tree}_\pm(f, s, \mathbf{X})$ is a signed Schröder tree of π when $\mathbf{X}$ consists of k uniform and independent points in $[0,1]$.

Finally, we define the *signed Brownian excursion* as the pair (e, S), where e is the Brownian excursion, and S assigns signs to the local minima of e in a balanced and independent manner. Then, we set $\Lambda_\pi = \Psi_\pi(e, S)$.

Hints of the proof. Why should signed trees extracted from the signed Brownian excursion be related to patterns in separable permutations?

First, there is a deep (and well-known) connection between trees and excursion: the *contour* of a tree is an excursion. This construction adapts immediately to *signed* trees and *signed* excursions. This is illustrated by Figure 3. Note that the leaves of the tree correspond to the peaks (*i.e.*, local maxima) of its contour. It is known[1] from [11] or [13] that the contours of Schröder trees converge to the Brownian excursion.

Second, it is easy to see that extracting a pattern π from a separable permutation σ corresponds to extracting a subtree (induced by a set of leaves ℓ) from a signed Schröder tree of σ. Going one step further, and considering the signed contour (f, s) associated to this signed Schröder tree, we see that π is the permutation corresponding to $\mathrm{Tree}_\pm(f, s, \mathbf{x})$ for $\mathbf{x}$ a set of abscissa of peaks of f (those corresponding to the leaves in ℓ).

These observations (vaguely) explain why Λ_π is related to patterns in separable permutations. The main difficulty that we have in proving Theorem 1

[1]This is known in the unsigned case only. Proving a signed analogue is a hard problem, that we have rather bypassed than solved.

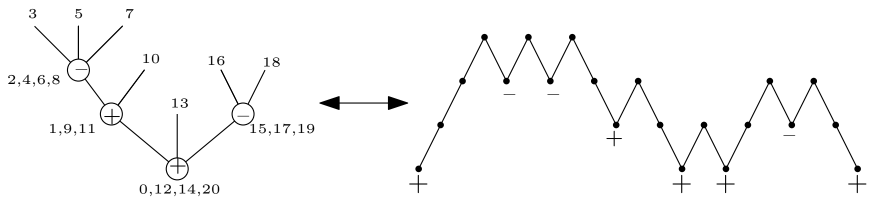

FIGURE 3. A (signed) tree and its (signed) contour.

is handling the signs, the rest following nicely from known results or adaptations of such. Instead of attacking the difficulty related to signs head-on (see also footnote 1), our proof takes a detour in the unsigned world.

The first part is to show that Theorem 1 is *equivalent* to the seemingly weaker statement that $\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)$ converges to Λ_π in expectation. Indeed, our random variables are bounded, so convergence in distribution is equivalent to the convergence of all moments; moreover, all moments of Λ_π can be expressed with expectations of Λ_ρ, for larger permutations ρ, with a similar statement holding in the limit for $\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)$. In passing, this gives a combinatorial and relatively efficient way of computing all moments of Λ_π.

Second, proving the convergence in expectation of *unsigned trees* extracted from contours of unsigned Schröder trees to *unsigned trees* extracted from the Brownian excursion is not too hard, and uses known techniques. What we need to do next is to re-introduce signs on these excursions and on the trees extracted from them.

Recall that the signs on the local minima of the signed Brownian excursion are independent and balanced. Consequently, in a signed tree extracted from it, signs of the internal vertices also are balanced and independent. The last key ingredient to prove Theorem 1 is to show that this also holds in the limit when n tends to infinity, for trees of any fixed size extracted from the signed contours of the signed Schröder trees associated with uniform random separable permutations of size n.

Permuton interpretation of our result. Theorem 1 also has an interpretation in terms of *permutons* [8, 4]. A permuton is a measure on the square $[0,1]^2$ with uniform marginals. The permutation diagram of any permutation σ can be seen as a permuton μ_σ, up to normalization: with n denoting the size of σ, just rescale the diagram of σ so that it fits in the square $[0,1]^2$, and replace every point $(i, \sigma(i))$ by a square $[(i-1)/n, i/n] \times [(\sigma(i)-1)/n, \sigma(i)/n]$ having weight $1/n$. The weak convergence provides a good notion of convergence for permutons, as discussed in [8].

Our Theorem 1, combined with Skorohod's representation theorem and Theorem 1.6(i) of [8], implies the following:

Theorem 2. *Let $\boldsymbol{\sigma}_n$ be a uniform random separable permutation of size n. There exists a random permuton $\boldsymbol{\mu}$ such that $\mu_{\boldsymbol{\sigma}_n}$ tends to $\boldsymbol{\mu}$ in distribution in the weak convergence topology.*

This limit permuton $\boldsymbol{\mu}$ describes the "limit shape" of the diagrams of uniform random separable permutations that we were looking for.

It is important to note that $\boldsymbol{\mu}$ is *not deterministic*, because Λ_{12} (or any Λ_π for a separable pattern π) is not deterministic. This is contrast with all classes $\mathrm{Av}(\tau)$ for τ of size 3 studied earlier, where the limit shape of the diagrams is deterministic.

From our work, we can prove the existence of $\boldsymbol{\mu}$, but we have no explicit description of $\boldsymbol{\mu}$. The construction of $\boldsymbol{\mu}$ is a problem that we have recently solved with J. Bertoin and V. Féray.

Finally, we believe that $\boldsymbol{\mu}$ (or maybe rather a one-parameter deformation $\boldsymbol{\mu}(p)$ of $\boldsymbol{\mu}$, for $p \in [0, 1]$) could describe the limit shape of permutations in other permutation classes, and more precisely in substitution-closed classes containing a finite number of simple permutations. The reason is that the encoding of separable permutations by trees, which is essential to our work, extend naturally and nicely to those classes.

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